Supplementary Material: R^{2}LIVE: A Robust, Real-time, LiDAR-Inertial-Visual tightly-coupled state Estimator and mapping

A. Perturbation on $SO(3)$

In this appendix, we will use the following approximation of perturbation $\delta\mathbf{r} \rightarrow \mathbf{0}$ on $SO(3)$ [25, 26]:

$$\begin{aligned}
\texttt{Exp}(\mathbf{r}+\delta\mathbf{r}) &\approx \texttt{Exp}(\mathbf{r})\texttt{Exp}(\mathbf{J}_r(\mathbf{r})\delta\mathbf{r}) \\
\texttt{Exp}(\mathbf{r})\texttt{Exp}(\delta\mathbf{r}) &\approx \texttt{Exp}(\mathbf{r}+\mathbf{J}_r^{-1}(\mathbf{r})\delta\mathbf{r}) \\
\mathbf{R}\cdot\texttt{Exp}(\delta\mathbf{r})\cdot\mathbf{u} &\approx \mathbf{R}\left(\mathbf{I}+[\delta\mathbf{r}]_\times\right)\mathbf{u} = \mathbf{R}\mathbf{u} - \mathbf{R}[\mathbf{u}]_\times\delta\mathbf{r}
\end{aligned}$$

where $\mathbf{u}\in\mathbb{R}^3$ and we use $[\cdot]_\times$ denote the skew-symmetric matrix of vector $(\cdot)$; $\mathbf{J}_r(\mathbf{r})$ and $\mathbf{J}_r^{-1}(\mathbf{r})$ are called the *right Jacobian* and the *inverse right Jacobian* of $SO(3)$, respectively.

$$\begin{aligned}
\mathbf{J}_r(\mathbf{r}) =& \mathbf{I} - \frac{1-\cos||\mathbf{r}||}{||\mathbf{r}||^2}[\mathbf{r}]_\times + \frac{||\mathbf{r}||-\sin(||\mathbf{r}||)}{||\mathbf{r}||^3}[\mathbf{r}]_\times^2 \\
\mathbf{J}_r^{-1}(\mathbf{r}) =& \mathbf{I} + \frac{1}{2}[\mathbf{r}]_\times + \left(\frac{1}{||\mathbf{r}||^2} - \frac{1+\cos(||\mathbf{r}||)}{2||\mathbf{r}||\sin(||\mathbf{r}||)}\right)[\mathbf{r}]_\times^2
\end{aligned}$$

B. Computation of $\mathbf{F}_{\delta\mathbf{x}}$ and $\mathbf{F}_{\mathbf{w}}$

Combing (4) and (6) , we have:

$$\begin{aligned}
&\delta\hat{\mathbf{x}}_{i+1} = \mathbf{x}_{i+1} \boxminus \hat{\mathbf{x}}_{i+1} \\
&= (\mathbf{x}_i \boxplus (\Delta t\cdot\mathbf{f}(\mathbf{x}_i,\mathbf{u}_i,\mathbf{w}_i))) \boxminus (\hat{\mathbf{x}}_i \boxplus (\Delta t\cdot\mathbf{f}(\hat{\mathbf{x}}_i,\mathbf{u}_i,\mathbf{0}))) \\
&= \begin{bmatrix}
\mathbf{Log}\left(\left({}^G\hat{\mathbf{R}}_{I_i}\texttt{Exp}(\hat{\boldsymbol{\omega}}_i\Delta t)\right)^T\cdot\left({}^G\hat{\mathbf{R}}_{I_i}\texttt{Exp}\left({}^G\delta\hat{\mathbf{r}}_{I_i}\right)\texttt{Exp}(\boldsymbol{\omega}_i\Delta t)\right)\right) \\
{}^G\delta\hat{\mathbf{p}}_{I_i} + {}^G\delta\hat{\mathbf{v}}_i\Delta t \\
{}^I\delta\hat{\mathbf{r}}_{C_i} \\
{}^I\delta\hat{\mathbf{p}}_{C_i} \\
{}^G\delta\hat{\mathbf{v}}_i + \left({}^G\hat{\mathbf{R}}_{I_i}\texttt{Exp}\left({}^G\delta\hat{\mathbf{r}}_{I_i}\right)\right)\mathbf{a}_i\Delta t - {}^G\hat{\mathbf{R}}_{I_i}\hat{\mathbf{a}}_i\Delta t \\
\delta\hat{\mathbf{b}}_{\mathbf{g}_i} + \mathbf{n}_{\mathbf{b}\mathbf{g}_i}\Delta t \\
\delta\hat{\mathbf{b}}_{\mathbf{a}_i} + \mathbf{n}_{\mathbf{b}\mathbf{a}_i}\Delta t
\end{bmatrix}
\end{aligned}$$

with:

$$\hat{\boldsymbol{\omega}}_i = \boldsymbol{\omega}_{m_i} - \hat{\mathbf{b}}_{\mathbf{g}_i},\ \ \boldsymbol{\omega}_i = \hat{\boldsymbol{\omega}}_i - \delta\hat{\mathbf{b}}_{\mathbf{g}_i} - \mathbf{n}_{\mathbf{g}_i} \tag{S1}$$

$$\hat{\mathbf{a}}_i = \mathbf{a}_{m_i} - \hat{\mathbf{b}}_{\mathbf{a}_i},\ \ \mathbf{a}_i = \hat{\mathbf{a}}_i - \delta\hat{\mathbf{b}}_{\mathbf{a}_i} - \mathbf{n}_{\mathbf{a}_i} \tag{S2}$$

And we have the following simplification and approximation from Section. A.

$$\begin{aligned}
&\texttt{Log}\left(\left({}^G\hat{\mathbf{R}}_{I_i}\texttt{Exp}(\hat{\boldsymbol{\omega}}_i\Delta t)\right)^T\cdot\left({}^G\hat{\mathbf{R}}_{I_i}\texttt{Exp}\left({}^G\delta\hat{\mathbf{r}}_{I_i}\right)\texttt{Exp}(\boldsymbol{\omega}_i\Delta t)\right)\right) \\
=&\texttt{Log}\left(\texttt{Exp}(\hat{\boldsymbol{\omega}}_i\Delta t)^T\cdot\left(\texttt{Exp}\left({}^G\delta\hat{\mathbf{r}}_{I_i}\right)\cdot\texttt{Exp}(\boldsymbol{\omega}_i\Delta t)\right)\right) \\
\approx&\texttt{Log}\left(\texttt{Exp}(\hat{\boldsymbol{\omega}}_i\Delta t)^T\texttt{Exp}\left({}^G\delta\hat{\mathbf{r}}_{I_i}\right)\texttt{Exp}(\hat{\boldsymbol{\omega}}_i\Delta t)\cdot\right. \\
&\qquad\left.\texttt{Exp}\left(-\mathbf{J}_r(\hat{\boldsymbol{\omega}}_i\Delta t)\left(\delta\hat{\mathbf{b}}_{g_i}+\mathbf{n}_{\mathbf{g}_i}\right)\Delta t\right)\right) \\
\approx&\texttt{Exp}(\hat{\boldsymbol{\omega}}_i\Delta t)\cdot{}^G\delta\hat{\mathbf{r}}_{I_i} - \mathbf{J}_r(\hat{\boldsymbol{\omega}}_i\Delta t)\delta\hat{\mathbf{b}}_{\mathbf{g}_i}\Delta t - \mathbf{J}_r(\hat{\boldsymbol{\omega}}_i\Delta t)\mathbf{n}_{\mathbf{g}_i}\Delta t \\
&\left({}^G\hat{\mathbf{R}}_{I_i}\texttt{Exp}\left({}^G\delta\hat{\mathbf{r}}_{I_i}\right)\right)\mathbf{a}_i\Delta t \\
\approx&\left({}^G\hat{\mathbf{R}}_{I_i}\left(\mathbf{I}+[{}^G\delta\hat{\mathbf{r}}_{I_i}]_\times\right)\right)\left(\hat{\mathbf{a}}_i-\delta\hat{\mathbf{b}}_{\mathbf{a}_i}-\mathbf{n}_{\mathbf{a}_i}\right)\Delta t \\
\approx&{}^G\hat{\mathbf{R}}_{I_i}\hat{\mathbf{a}}_i\Delta t - {}^G\hat{\mathbf{R}}_{I_i}\delta\hat{\mathbf{b}}_{\mathbf{a}_i}\Delta t - {}^G\hat{\mathbf{R}}_{I_i}\mathbf{n}_{\mathbf{a}_i}\Delta t - {}^G\hat{\mathbf{R}}_{I_i}[\hat{\mathbf{a}}_i]_\times{}^G\delta\hat{\mathbf{r}}_{I_i}\Delta t
\end{aligned}$$

To conclude, we have the computation of $\mathbf{F}_{\delta\hat{\mathbf{x}}}$ and $\mathbf{F}_{\mathbf{w}}$ as follow:

$$\begin{aligned}
\mathbf{F}_{\delta\hat{\mathbf{x}}} &= \left.\frac{\partial(\delta\hat{\mathbf{x}}_{i+1})}{\partial\delta\hat{\mathbf{x}}_i}\right|_{\delta\hat{\mathbf{x}}_i=\mathbf{0},\mathbf{w}_i=\mathbf{0}} \\
&= \begin{bmatrix}
\texttt{Exp}(-\hat{\boldsymbol{\omega}}_i\Delta t) & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{J}_r(\hat{\boldsymbol{\omega}}_i\Delta t)\Delta t & \mathbf{0} \\
\mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{I}\Delta t & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
-{}^G\hat{\mathbf{R}}_{I_i}[\hat{\mathbf{a}}_i]_\times\Delta t & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} & -{}^G\hat{\mathbf{R}}_{I_i}\Delta t \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}
\end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
\mathbf{F}_{\mathbf{w}} &= \left.\frac{\partial(\delta\hat{\mathbf{x}}_{i+1})}{\partial\mathbf{w}_i}\right|_{\delta\hat{\mathbf{x}}_i=\mathbf{0},\mathbf{w}_i=\mathbf{0}} \\
&= \begin{bmatrix}
-\mathbf{J}_r(\hat{\boldsymbol{\omega}}_i\Delta t)\Delta t & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & -{}^G\hat{\mathbf{R}}_{I_i}\Delta t & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{I}\Delta t & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}\Delta t
\end{bmatrix}
\end{aligned}$$

C. The computation of $\boldsymbol{\mathcal{H}}$

Recalling (15), we have:

$$\begin{aligned}
\boldsymbol{\mathcal{H}} &= \frac{(\check{\mathbf{x}}_{k+1}\boxplus\delta\check{\mathbf{x}}_{k+1})\boxminus\hat{\mathbf{x}}_{k+1}}{\partial\delta\check{\mathbf{x}}_{k+1}}|_{\delta\check{\mathbf{x}}_{k+1}=\mathbf{0}} \\
&= \begin{bmatrix}
\mathbf{A} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}_{3\times9} \\
\mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0}_{3\times9} \\
\mathbf{0} & \mathbf{0} & \mathbf{B} & \mathbf{0} & \mathbf{0}_{3\times9} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0}_{3\times9} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_{9\times9}
\end{bmatrix}
\end{aligned}$$

with the 3×3 matrix $\mathbf{A} = \mathbf{J}_r^{-1}(\texttt{Log}({}^G\hat{\mathbf{R}}_{I_{k+1}}{}^T\,{}^G\check{\mathbf{R}}_{I_{k+1}}))$ and $\mathbf{B} = \mathbf{J}_r^{-1}(\texttt{Log}({}^I\hat{\mathbf{R}}_{C_{k+1}}{}^T\,{}^I\check{\mathbf{R}}_{C_{k+1}}))$.

D. The computation of $\mathbf{H}_j^l$

Recalling (12) and (15), we have:

$$\begin{aligned}
&\mathbf{r}_l(\check{\mathbf{x}}_{k+1}\boxplus\delta\check{\mathbf{x}}_{k+1},{}^L\mathbf{p}_j) = \mathbf{u}_j^T\left({}^G\check{\mathbf{p}}_{I_{k+1}} + {}^G\delta\check{\mathbf{p}}_{I_{k+1}} - \right. \\
&\left.\mathbf{q}_j + {}^G\check{\mathbf{R}}_{I_{k+1}}\texttt{Exp}({}^G\delta\check{\mathbf{r}}_{I_{k+1}})\left({}^I\mathbf{R}_L{}^L\mathbf{p}_j + {}^I\mathbf{p}_L\right)\right)
\end{aligned} \tag{S3}$$

And with the small perturbation approximation, we get:

$$\begin{aligned}
&{}^G\check{\mathbf{R}}_{I_{k+1}}\texttt{Exp}({}^G\delta\check{\mathbf{r}}_{I_{k+1}})\mathbf{P}_{\mathbf{a}} \\
\approx&{}^G\check{\mathbf{R}}_{I_{k+1}}\left(\mathbf{I}+\left[{}^G\delta\check{\mathbf{r}}_{I_{k+1}}\right]_\times\right)\mathbf{P}_{\mathbf{a}} \\
=&{}^G\check{\mathbf{R}}_{I_{k+1}}\mathbf{P}_{\mathbf{a}} - {}^G\check{\mathbf{R}}_{I_{k+1}}[\mathbf{P}_{\mathbf{a}}]_\times{}^G\delta\check{\mathbf{r}}_{I_{k+1}}
\end{aligned} \tag{S4}$$

where $\mathbf{P}_{\mathbf{a}} = {}^I\mathbf{R}_L{}^L\mathbf{p}_j + {}^I\mathbf{p}_L$. Combining (S3) and (S4) together we can obtain:

$$\mathbf{H}_j^l = \mathbf{u}_j^T\left[-{}^G\check{\mathbf{R}}_{I_{k+1}}[\mathbf{P}_{\mathbf{a}}]_\times \quad \mathbf{I}_{3\times3} \quad \mathbf{0}_{3\times15}\right]$$

E. The computation of $\mathbf{H}_s^c$ and $\mathbf{F}_{\mathbf{P}_s}$

Recalling (16), we have:

$${}^C\mathbf{P}_s = \mathbf{P}_{\mathbf{C}}(\check{\mathbf{x}}_{k+1},{}^G\mathbf{P}_s) = \left[{}^CP_{sx}\ {}^CP_{sy}\ {}^CP_{sz}\right]^T$$

where the function $\mathbf{P}_{\mathbf{C}}(\check{\mathbf{x}}_{k+1},{}^G\mathbf{P}_s)$ is:

$${}^C\mathbf{P}_s = \left({}^G\check{\mathbf{R}}_{I_{k+1}}{}^I\check{\mathbf{R}}_{C_{k+1}}\right)^T{}^G\mathbf{P}_s - {}^I\check{\mathbf{R}}_{C_{k+1}}^T{}^I\check{\mathbf{p}}_{C_{k+1}} \tag{S5}$$

$$- \left({}^G\check{\mathbf{R}}_{I_{k+1}}{}^I\check{\mathbf{R}}_{C_{k+1}}\right)^T{}^G\check{\mathbf{p}}_{I_{k+1}} \tag{S6}$$

From (20), we have:

$$\mathbf{r}_c\left(\check{\mathbf{x}}_{k+1},{}^C\mathbf{p}_s,{}^G\mathbf{P}_s\right) = {}^C\mathbf{p}_s - \boldsymbol{\pi}({}^C\mathbf{P}_s)$$

$$\boldsymbol{\pi}({}^C\mathbf{P}_s) = \left[f_x\frac{{}^CP_{sx}}{{}^CP_{sz}} + c_x \quad f_y\frac{{}^CP_{sy}}{{}^CP_{sz}} + c_y\right]^T \tag{S7}$$

where f_x and f_y are the focal length, c_x and c_y are the principal point offsets in image plane.

For conveniently, we omit the $(\cdot)|_{\delta\check{\mathbf{x}}_{k+1}=\mathbf{0}}$ in the following derivation, and we have:

$$\mathbf{H}_s^c = -\frac{\partial \boldsymbol{\pi}({}^C\mathbf{P}_s)}{\partial {}^C\mathbf{P}_s} \cdot \frac{\partial \mathbf{P_C}(\check{\mathbf{x}}_{k+1} \boxplus \delta\check{\mathbf{x}}_{k+1}, {}^G\mathbf{P}_s)}{\partial \delta\check{\mathbf{x}}_{k+1}} \tag{S8}$$

$$\mathbf{F}_{\mathbf{P}_s} = -\frac{\partial \boldsymbol{\pi}({}^C\mathbf{P}_s)}{\partial {}^C\mathbf{P}_s} \cdot \frac{\partial \mathbf{P_C}(\check{\mathbf{x}}_{k+1}, {}^G\mathbf{P}_s)}{\partial {}^G\mathbf{P}_s} \tag{S9}$$

where:

$$\frac{\partial \boldsymbol{\pi}({}^C\mathbf{P}_s)}{\partial {}^C\mathbf{P}_s} = \frac{1}{{}^C P_{sz}} \begin{bmatrix} f_x & 0 & -f_x \dfrac{{}^C P_{sx}}{{}^C P_{sz}} \\ 0 & f_y & -f_y \dfrac{{}^C P_{sy}}{{}^C P_{sz}} \end{bmatrix} \tag{S10}$$

$$\frac{\partial \mathbf{P_b}(\check{\mathbf{x}}_{k+1}, {}^G\mathbf{P}_s)}{\partial {}^G\mathbf{P}_s} = \left({}^G\check{\mathbf{R}}_{I_{k+1}} {}^I\check{\mathbf{R}}_C\right)^T \tag{S11}$$

According to Section. A, we have the following approximation of $\mathbf{P_C}(\check{\mathbf{x}}_{k+1} \boxplus \delta\check{\mathbf{x}}_{k+1}, {}^G\mathbf{P}_s)$:

$$\begin{aligned}
&\mathbf{P_C}(\check{\mathbf{x}}_{k+1} \boxplus \delta\check{\mathbf{x}}_{k+1}, {}^G\mathbf{P}_s) \\
&= \left({}^G\check{\mathbf{R}}_{I_{k+1}} \mathtt{Exp}\left({}^G\delta\check{\mathbf{r}}_{I_{k+1}}\right) {}^I\check{\mathbf{R}}_{C_{k+1}} \mathtt{Exp}\left({}^I\delta\check{\mathbf{r}}_{C_{k+1}}\right)\right)^T {}^G\mathbf{P}_s \\
&- \left({}^I\check{\mathbf{R}}_{C_{k+1}} \mathtt{Exp}\left({}^I\delta\check{\mathbf{r}}_{C_{k+1}}\right)\right)^T \left({}^I\check{\mathbf{p}}_{C_{k+1}} + {}^I\delta\check{\mathbf{p}}_{C_{k+1}}\right) \\
&- \left({}^G\check{\mathbf{R}}_{I_{k+1}} \mathtt{Exp}\left({}^G\delta\check{\mathbf{r}}_{I_{k+1}}\right) {}^I\check{\mathbf{R}}_{C_{k+1}} \mathtt{Exp}\left({}^I\delta\check{\mathbf{r}}_{C_{k+1}}\right)\right)^T \left({}^G\check{\mathbf{p}}_{I_{k+1}} + {}^G\delta\check{\mathbf{p}}_{I_{k+1}}\right) \\
&\approx \mathbf{P_b}(\check{\mathbf{x}}_{k+1}, {}^G\mathbf{P}_s) - \left({}^I\check{\mathbf{R}}_{C_{k+1}}\right)^T \left({}^G\check{\mathbf{R}}_{I_{k+1}}\right)^T {}^G\delta\check{\mathbf{r}}_{I_{k+1}} \\
&+ \left({}^I\check{\mathbf{R}}_{C_{k+1}}\right)^T \left[{}^G\check{\mathbf{R}}_{I_{k+1}}^T ({}^G\mathbf{P}_s - {}^G\check{\mathbf{p}}_{I_{k+1}})\right]_\times {}^G\delta\check{\mathbf{p}}_{I_{k+1}} \\
&+ \left[\mathbf{P_b}(\check{\mathbf{x}}_{k+1}, {}^G\mathbf{P}_s)\right]_\times {}^I\delta\check{\mathbf{r}}_{C_{k+1}} - \left({}^I\check{\mathbf{R}}_C\right)^T {}^I\delta\check{\mathbf{p}}_{C_{k+1}}
\end{aligned} \tag{S12}$$

With this, we can derive:

$$\frac{\partial \mathbf{P_C}(\check{\mathbf{x}}_{k+1} \boxplus \delta\check{\mathbf{x}}_{k+1}, {}^G\mathbf{P}_s)}{\partial \delta\check{\mathbf{x}}_{k+1}} = \begin{bmatrix}\mathbf{M_A} & \mathbf{M_B} & \mathbf{M_C} & \mathbf{M_D} & \mathbf{0}_{3\times 12}\end{bmatrix} \tag{S13}$$

$$\begin{aligned}
&\mathbf{M_A} = -\left({}^I\check{\mathbf{R}}_{C_{k+1}}\right)^T \left({}^G\check{\mathbf{R}}_{I_{k+1}}\right)^T \\
&\mathbf{M_B} = \left({}^I\check{\mathbf{R}}_{C_{k+1}}\right)^T \left[{}^G\check{\mathbf{R}}_{I_{k+1}}^T ({}^G\mathbf{P}_s - {}^G\check{\mathbf{p}}_{I_{k+1}})\right]_\times \\
&\mathbf{M_C} = \left[{}^C\mathbf{P}_s\right]_\times \\
&\mathbf{M_D} = -\left({}^I\check{\mathbf{R}}_C\right)^T
\end{aligned} \tag{S14}$$

Substituting (S10), (S11) and (S13) into (S8) and (S9), we finish the computation of $\mathbf{H}_s^c$ and $\mathbf{F}_{\mathbf{P}_s}$.