Lecture 2

Growth Model: Dynamic Optimization in Discrete Time

Macroeconomics EC417

Jane Olmstead-Rumsey

London School of Economics, Fall 2022

Growth Model: Setup

- Preferences: a single household with preferences defined by

$$\sum_{t=0}^{\infty} \beta^t u(c_t, 1 - h_t)$$

  with $u : \mathbb{R}_+ \times [0, 1] \to \mathbb{R}$

- Technology:

$$\begin{aligned} y_t &= F(k_t, h_t), \quad F : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+ \\ c_t + i_t &= y_t \\ k_{t+1} &= i_t + (1 - \delta) k_t \\ c_t &\geq 0, \quad i_t \geq -(1 - \delta) k_t \end{aligned}$$

- Endowments:
  - 1 unit of time each period
  - $\hat{k}_0$ units of capital at time 0

Assumptions

- Preferences: $0 < \beta < 1$ and u is
  - strictly increasing
  - strictly concave
  - C^2 (twice continuously differentiable)
- Technology: $0 < \delta \leq 1$ and F is
  - constant returns to scale
  - strictly increasing
  - weakly concave in (k, h) jointly, strictly concave in each argument individually
  - $F(0, h) = 0$ for all h.
  - C^2
  - ("Inada conditions")

$$\lim_{k \to 0} F_k(k, h) = \infty, \quad \forall h > 0,$$

$$\lim_{k \to \infty} F_k(k, h) = 0, \quad \forall h > 0,$$

Comments

- Tradeoffs in the model
  - consumption today c_t vs. consumption tomorrow c_{t+1}
  - consumption c_t vs. leisure $1 - h_t$
- Model assumes "representative household" and "representative firm" (jointly = "representative agent")
- When is this justified? If at least one of following 3 conditions are satisfied
  1. all individuals in economy are identical
  2. particular assumptions on preferences ("homotheticity", "Gorman aggregation")
  3. perfect markets
     - representative firm $\Leftrightarrow$ perfect factor markets (capital, labor), equalize marginal products
     - representative HH $\Leftrightarrow$ perfect insurance markets, equalize MUs
- Do we believe these conditions are satisfied? No, but...

General Comment: Modeling in (Macro)economics

Objective is not to build one big model we use to address all issues

- descriptive realism is not the objective
- instead make modeling choices that are dependent on the issue
- whether a model is "good" is context dependent

Approach to modeling in macro(economics) well summarized by following statements

- "All models are false; some are useful"
  https://en.wikipedia.org/wiki/All_models_are_wrong
- "If you want a model of the real world, look out the window" (kidding, but only half kidding)
- "The map is not the territory"
  https://en.wikipedia.org/wiki/Map-territory_relation

Crucial or Critical Assumptions

A CONTRIBUTION TO THE THEORY OF ECONOMIC GROWTH

By ROBERT M. SOLOW

I. INTRODUCTION

All theory depends on assumptions which are not quite true. That is what makes it theory. The art of successful theorizing is to make the inevitable simplifying assumptions in such a way that the final results are not very sensitive.[1] A "crucial" assumption is one on which the conclusions do depend sensitively, and it is important that crucial assumptions be reasonably realistic. When the results of a theory seem to flow specifically from a special crucial assumption, then if the assumption is dubious, the results are suspect.

Source: Solow (1956) "A Contribution to the Theory of Economic Growth"

General Comment: Modeling in (Macro)economics

Ten commandments for non-economists
1. Economics is a collection of models with no predetermined conclusions; do not let anyone tell you otherwise.
2. Do not criticize an economist's model because of its assumptions; ask how the results would be changed if the assumptions that seem problematic were more realistic.
3. Analysis requires simplicity; beware of incoherence that passes itself off as complexity.
4. Do not be put off by the math; remember economists use math not because they are smart, but because they are not smart enough.
5. When an economist makes a recommendation, ask what makes him/her sure the underlying model applies to the case at hand.
6. When an economists uses the term "economic welfare," ask what s/he means by it.
7. Do not assume what an economist says in public is the same as what he says in the seminar room.
8. Economists don't (all) worship markets; if they seem like they do, it's probably because they know better how they work than you do.
9. If you think all economists think alike, do attend one of their seminars.
10. If you think economists are especially rude to non-economists, do attend one of their seminars.

Source: Rodrik (2015) "Economics Rules"
http://www.centreformacroeconomics.ac.uk/pdf/Events/Slides/1510-RodrikD.pdf

Navigating Among Models

- Chapter 3 “Navigating Among Models” of “Economics Rules” by Dani Rodrik (2015)

- “Eminent Economists: Their Life and Work Philosophies” by Szenberg (1993/2014) c/o Beatrice Cherrier

- Ed Prescott’s (Nobel laureate 2004) advice via Gianluca Violante: “start with the (historically) workhorse model and try to explain your fact with it. Only if you can’t you go on and amend it.”

General Comment: Modeling in (Macro)economics

- Growth model is "the" benchmark model of macro
- Why is this the benchmark model?
  - minimal model of y where $y = F(k, h)$
- Also, growth model = great laboratory for teaching you tools of macro...
- ... and many other models in macroeconomics build on growth model. Examples:
  - Real business cycle (RBC) model = growth model with aggregate productivity shocks
  - New Keynesian model = RBC model + sticky prices
  - Incomplete markets model (Aiyagari-Bewley-Huggett) = growth model + heterogeneity in form of uninsurable idiosyncratic shocks

What issues is growth model useful for?

- Growth model is designed to be model of capital accumulation process
- Growth model is not a "good" model of
  - growth (somewhat ironically given its name)
  - income and wealth distribution (given rep. agent assumption)
  - inflation and monetary policy
  - unemployment
  - financial crises
- But some of growth model's extensions (e.g. those mentioned on previous slide) are "good" models of these issues

Some Concepts

Definition: A feasible allocation for the growth model is a list of sequences $\{c_t, h_t, k_t\}_{t=0}^{\infty}$ such that

$$c_t + k_{t+1} = F(k_t, h_t) + (1 - \delta)k_t$$

$$0 \leq h_t \leq 1, \quad c_t \geq 0, \quad k_t \geq 0, \quad k_0 = \hat{k}_0$$

Analysis of Growth Model

- Consistent with there being two key tradeoffs, captured by the model, there are two choices to be made each period
  - c_t vs. k_{t+1}
  - c_t vs. h_t
- Will analyze
  1. Pareto efficient allocations
  2. decentralized equilibrium allocations
- Start with Pareto efficient allocations

Solow Model

- Historically, much interest in allocations that resulted from specific “ad hoc” decision rules

$$c_t = sy_t$$
$$h_t = \bar{h}$$

- = “Solow model” you may know from your undergraduate courses

Pareto Efficient Alloc. in Growth Model

- To simplify analysis and focus on dynamic considerations, begin with extreme case: leisure not valued, or (with slight abuse of notation)

$$u(c_t, 1 - h_t) = u(c_t)$$

- Assume (Inada condition akin to those on F)

$$\lim_{c \to 0} u'(c) = \infty$$

- Also define

$$f(k_t) = F(k_t, 1)$$

Social Planner's Problem

- Only one person in economy ⇒ our life is simple.
- Pareto efficient allocation = max. utility of household subject to feasibility
- Think of this as problem of fictitious "social planner":

$$
\begin{aligned}
V(\hat{k}_0) = \max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t) \quad \text{s.t.}& \\
c_t + k_{t+1} = f(k_t) + (1-\delta)k_t& \\
c_t \geq 0, \quad k_{t+1} \geq 0, \quad k_0 = \hat{k}_0.&
\end{aligned}
$$

- Alternatively, substitute resource constraint into objective

$$
\begin{aligned}
V(\hat{k}_0) = \max_{\{k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(f(k_t) + (1-\delta)k_t - k_{t+1}) \quad \text{s.t.}& \\
0 \leq k_{t+1} \leq f(k_t) + (1-\delta)k_t, \quad k_0 = \hat{k}_0.&
\end{aligned}
$$

Dynamic Optimization: General Theory

- There's a general theory for solving these types of problems
    - let's first work out more general theory
    - then apply to growth model
    - purpose: teach you some tools that are also applicable for solving other models

- In general will encounter two different formulations of dynamic optimization problems
    1. control-state formulation
    2. state-only formulation

Dynamic Optimization: General Theory

Control-State Formulation

- Pretty much all deterministic optimal control problems in discrete time can be written as

$$V(\hat{x}_0) = \max_{\{\alpha_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t r(x_t, \alpha_t)$$

  subject to the law of motion for the state

$$x_{t+1} = g(x_t, \alpha_t) \text{ and } \alpha_t \in A, \quad x_0 = \hat{x}_0.$$

- $\beta \in (0, 1)$: discount factor
- $x \in X \subseteq \mathbb{R}^m$: state vector
- $\alpha \in A \subseteq \mathbb{R}^k$: control vector
- $r : X \times A \to \mathbb{R}$: instantaneous return function

Example: Growth Model

$$V(\hat{k}_0) = \max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t) \quad \text{s.t.}$$

$$c_t + k_{t+1} = f(k_t) + (1-\delta)k_t$$

$$c_t \geq 0, \quad k_{t+1} \geq 0, \quad k_0 = \hat{k}_0.$$

- Here the state is $x_t = k_t$ and the control $\alpha_t = c_t$
- $r(x, \alpha) = u(\alpha)$
- $g(x, \alpha) = f(x) + (1-\delta)x - \alpha$

Dynamic Optimization: General Theory

State-only Formulation

- Alternatively, can write the same problem in terms of states only

$$V(\hat{x}_0) = \max_{\{x_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(x_t, x_{t+1}) \quad \text{s.t.}$$
$$x_{t+1} \in \Gamma(x_t), \quad x_0 = \hat{x}_0.$$

- $\beta \in (0,1)$: discount factor
- $x \in X \subseteq \mathbb{R}^m$: state vector
- $U : X \times X \to \mathbb{R}$: instantaneous return function
- $\Gamma : X \to X$: correspondence describing feasible values for state

Example: Growth Model

$$V(\hat{k}_0) = \max_{\{k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(f(k_t) + (1-\delta)k_t - k_{t+1}) \quad \text{s.t.}$$

$$k_{t+1} \in [0, f(k_t) + (1-\delta)k_t], \quad k_0 = \hat{k}_0.$$

- Here the state is $x_t = k_t$
- $U(x, y) = u(f(x) + (1-\delta)x - y)$
- $\Gamma(x) = [0, f(x) + (1-\delta)x]$

Dynamic Optimization: General Properties

- Existence of a solution
  - Extreme Value Theorem (or "Weierstrass Theorem"): continuous function on compact set has a maximum
- Satisfied in growth model?
  - objective continuous? Yes
  - constraint set compact? Yes. Result: there exists a "maximum maintainable capital stock" $\hat{k}$ s.t. $k_t > \hat{k} \Rightarrow k_{t+1} - k_t < 0$, and we can restrict attention to $k_t \in [0, \hat{k}]$.
    - Inada conditions $\Rightarrow f'(k_t) - \delta < 0$ for k_t large enough $\Rightarrow$ there exists $\hat{k}$ satisfying $0 = f(\hat{k}) - \delta\hat{k}$ and $f(k_t) - \delta k_t < 0$, $k_t > \hat{k}$
    - $k_t > \hat{k} \Rightarrow k_{t+1} - k_t = f(k_t) - \delta k_t - c_t \leq f(k_t) - \delta k_t < 0$
  - $\Rightarrow$ in growth model, there exists an optimal $\{k_{t+1}\}_{t=0}^{\infty}$
- Uniqueness of a solution
  - strictly concave objective & convex constraint set $\Rightarrow$ unique solution
- Satisfied in growth model? Yes

Overview: Solution Methods

- Different methods for solving dynamic optimization problems
  - not only deterministic ones ...
  - ... but also stochastic ones (= with uncertainty)
  - Table provides an overview of different solution methods

	Discrete Time		Continuous Time	
	sequence	recursive	sequence	recursive
deterministic	"classical"	Bellman eqn	Hamiltonian	HJB eqn
stochastic		Bellman eqn		HJB eqn

- blue = 1st half, red = 2nd half
- recursive approach also called "dynamic programming"
- blank box
  - can solve stochastic problems using sequence formulation...
  - ... but recursive/dynamic programming approach superior

Classical Solution Method of Sequence Problem

- Recall general dynamic optimization problem

$$V(\hat{x}_0) = \max_{\{x_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(x_t, x_{t+1}) \quad \text{s.t.} \tag{P}$$

$$x_{t+1} \in \Gamma(x_t), \quad x_0 = \hat{x}_0.$$

- For now: consider only $\{x_{t+1}\}_{t=0}^{\infty}$ in the interior of $\Gamma(x_t)$
- Later: generalization to binding constraints (Kuhn-Tucker)
- Necessary and sufficient conditions for $\{x_{t+1}\}_{t=0}^{\infty}$ to be optimal:

$$U_y(x_t, x_{t+1}) + \beta U_x(x_{t+1}, x_{t+2}) = 0, \quad \forall t \tag{EE}$$

$$\lim_{T\to\infty} \beta^T U_y(x_T, x_{T+1}) \cdot x_{T+1} = 0 \tag{TC}$$

  and $x_0 = \hat{x}_0$.

- (EE) together with (TC) and initial condition $x_0 = \hat{x}_0$ fully characterizes optimal $\{x_{t+1}\}_{t=0}^{\infty}$

Derivation/Interpretation

- (EE) is called "Euler equation"
  - simply first-order condition (FOC) w/ respect to x_{t+1}
  - derivation: differentiate problem (P) with respect to x_{t+1}
  - "Euler equation" simply means "intertemporal FOC"
  - Parker (2016): Tintner (1937) makes link to Euler-Lagrange
    https://en.wikipedia.org/wiki/Euler-Lagrange_equation
- (TC) is called "transversality condition"
  - understanding it is harder than (EE), let's postpone this for now and revisit in a few slides
  - Note: some books (e.g. Stokey-Lucas-Prescott) write (TC) as
    $$\lim_{T\to\infty} \beta^T U_x(x_T, x_{T+1}) \cdot x_T = 0 \tag{TC2}$$
  - To see that (TC2) is equivalent to (TC), substitute (EE) into (TC), and evaluate at T rather than $T+1$

Example: Growth Model

- Recall social planner's problem in growth model

$$V(\hat{k}_0) = \max_{\{k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(f(k_t) + (1-\delta)k_t - k_{t+1}) \quad \text{s.t.} \qquad \text{(P')}$$

$$k_{t+1} \in [0, f(k_t) + (1-\delta)k_t], \quad k_0 = \hat{k}_0.$$

- (EE) and (TC) are

$$\begin{aligned} &- u'(f(k_t) + (1-\delta)k_t - k_{t+1}) && \text{(EE')} \\ &+ \beta u'(f(k_{t+1}) + (1-\delta)k_{t+1} - k_{t+2})(f'(k_{t+1}) + 1 - \delta) = 0 \end{aligned}$$

$$\lim_{T\to\infty} \beta^T u'(f(k_T) + (1-\delta)k_T - k_{T+1})k_{T+1} = 0 \qquad \text{(TC')}$$

- Get (EE') simply by differentiating (P') w.r.t. k_{t+1} (or by applying formula on previous slide)

Example: Growth Model

- (EE') can be written more intuitively as

$$\underbrace{\frac{u'(c_t)}{\beta u'(c_{t+1})}}_{MRS} = \underbrace{f'(k_{t+1}) + 1 - \delta}_{MRT}$$

  MRS between c_t and c_{t+1} = MRT between c_t and c_{t+1}

- Same logic as in static utility maximization problems, e.g.

$$\max_{c_A, c_B} u(c_A, c_B) \quad \text{s.t.} \quad c_A = f(\ell_A), \quad c_B = f(\ell_B), \quad \ell_A + \ell_B \leq 1$$

  where A=apples, B=bananas

$$\Rightarrow \quad \frac{u_{c_A}(c_A, c_B)}{u_{c_B}(c_A, c_B)} = \frac{f'(\ell_B)}{f'(\ell_A)}$$

- growth model: different dates = different goods

Example: Growth Model

- Summarizing all necessary conditions

$$
\begin{aligned}
u'(c_t) &= \beta u'(c_{t+1})(f'(k_{t+1}) + 1 - \delta) \\
k_{t+1} &= f(k_t) + (1-\delta)k_t - c_t
\end{aligned}
\tag{DE}
$$

for all t, with

$$
k_0 = \hat{k}_0
$$

$$
\lim_{T\to\infty} \beta^T u'(c_T)k_{T+1} = 0 \tag{TC'}
$$

- (DE) is system of two difference equations in (c_t, k_t) ...
- ... needs two boundary conditions
  1. initial condition for capital stock: $k_0 = \hat{k}_0$
  2. transversality condition, plays role of boundary condition

Where does (TC) come from?

- Transversality condition is a bit mysterious
- Best treatments are in various papers by Kamihigashi
    - most intuitive "Transversality Conditions and Dynamic Economic Behavior," New Palgrave Dict. of Economics, 2008
      `http://www.dictionaryofeconomics.com/download/pde2008_T000217.pdf`
    - "A simple proof of the necessity of the transversality condition," Economic Theory, 2002
    - "Necessity of transversality conditions for infinite horizon problems," Econometrica, 2001
- Next slide: intuitive but "fake" derivation from finite horizon problem
- Afterwards: necessity proof from Kamihigashi (2002)

Where does (TC) come from?

- Consider finite horizon problem:

$$V(\hat{k}_0, T) = \max_{\{k_{t+1}\}_{t=0}^{T}} \sum_{t=0}^{T} \beta^t u(c_t) \quad \text{s.t.}$$

$$k_{t+1} = f(k_t) + (1-\delta)k_t - c_t, \quad k_{t+1} \geq 0.$$

- Lagrangian

$$\mathcal{L} = \sum_{t=0}^{T} \beta^t u(c_t) + \sum_{t=0}^{T} \lambda_t (f(k_t) + (1-\delta)k_t - c_t - k_{t+1}) + \sum_{t=0}^{T} \mu_t k_{t+1}$$

- Necessary conditions at $t = T$

$$\begin{aligned} \beta^T u'(c_T) &= \lambda_T \\ \lambda_T &= \mu_T \qquad \Rightarrow \quad \beta^T u'(c_T) k_{T+1} = 0 \\ \mu_T k_{T+1} &= 0 \end{aligned}$$

Where does (TC) come from?

- From previous slide: in finite horizon problem

$$\beta^T u'(c_T) k_{T+1} = 0 \qquad (*)$$

- $(*)$ is really two conditions in one
  1. $\beta^T u'(c_T) > 0$: need $k_{T+1} = 0$
  2. $\beta^T u'(c_T) = 0$: k_{T+1} can be > 0
- Intuition for case 1: if my marginal utility of consumption at T is positive, I want to eat up all my wealth before I die
- (TC) is same condition as $(*)$ in economy with $T \to \infty$

$$\lim_{T \to \infty} \beta^T u'(c_T) k_{T+1} = 0 \qquad \text{(TC)}$$

- Intuition: capital should not grow too fast compared to marginal utility
  - e.g. with $u(c) = \log c$: $\beta^T k_{T+1}/c_{T+1} \to 0$
  - if I save too much/spend too little, I'm not behaving optimally
- (TC) rules out overaccumulation of wealth

Proof of Necessity of (TC), Kamihigashi (2002)

- Consider general optimal control problem

$$V(\hat{x}_0) = \max_{\{x_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(x_t, x_{t+1}) \quad \text{s.t.} \qquad \text{(P)}$$
$$x_{t+1} \in \Gamma(x_t), \quad x_0 = \hat{x}_0.$$

- Assumptions:
  1. $x_t \in X \subset \mathbb{R}_+^m$ (i.e. $x_t \geq 0$)
  2. $Gr(\Gamma) = \{(y, x) : x \in X, y \in \Gamma(x)\}$ is convex, $(0, 0) \in Gr(\Gamma)$
  3. $U : Gr(\Gamma) \to \mathbb{R}$ is C^1 and concave
  4. $\forall (x, y) \in Gr(\Gamma), U_y(x, y) \leq 0$
  5. For any feasible path $\{x_t\}$ $\lim_{T\to\infty} \sum_{t=0}^{T} \beta^t U(x_t, x_{t+1})$ exists (i.e. it is bounded)
- (TC) can also be derived under weaker assumptions. But above assumptions yield straightforward proof.

Proof of Necessity of (TC), Kamihigashi (2002)

- Definition: A feasible path $\{x_t^*\}$ is optimal if

$$\sum_{t=0}^{\infty} \beta^t U(x_t^*, x_{t+1}^*) \geq \sum_{t=0}^{\infty} \beta^t U(x_t, x_{t+1})$$

  for any feasible path $\{x_t\}$

- i.e. $\{x_t^*\}$ attains the maximum of (P)

- Theorem: Under Assumptions 1 to 5, for any interior optimal path $\{x_t^*\}$

$$\lim_{T\to\infty} \beta^T U_y(x_T^*, x_{T+1}^*) \cdot x_{T+1}^* = 0$$

Proof of Necessity of (TC), Kamihigashi (2002)

- Useful preliminary fact: Let $f : [0,1] \to \mathbb{R}$ be a concave function with $f(1) > -\infty$. Then

$$\frac{f(1) - f(\lambda)}{1 - \lambda} \leq f(1) - f(0) \qquad (*)$$

- Follows immediately from definition of a concave function:

$$f(\lambda x + (1-\lambda)y) \geq \lambda f(x) + (1-\lambda) f(y) \quad \forall\ 0 \leq \lambda \leq 1,\ x, y$$

- Letting $x = 1$ and $y = 0$

$$f(1) - f(\lambda) \leq f(1) - (\lambda f(1) + (1-\lambda) f(0))$$

- Rearranging yields $(*)$

Proof of Necessity of (TC), Kamihigashi (2002)

- Let x_t^* be an interior optimal path. Consider alternative path
$$\{x_0^*, x_1^*, ..., x_T^*, \lambda x_{T+1}^*, \lambda x_{T+2}^*, ...\}, \quad \lambda \in [0,1)$$
- path is feasible by interiority and convexity of constraint set
- By optimality

$$\beta^T[U(x_T^*, \lambda x_{T+1}^*) - U(x_T^*, x_{T+1}^*)] + \sum_{t=T+1}^{\infty} \beta^t \left[U(\lambda x_t^*, \lambda x_{t+1}^*) - U(x_t^*, x_{t+1}^*)\right] \leq 0$$

- Dividing through by $1 - \lambda$

$$\begin{aligned} \beta^T \frac{U(x_T^*, \lambda x_{T+1}^*) - U(x_T^*, x_{T+1}^*)}{1-\lambda} &\leq \sum_{t=T+1}^{\infty} \beta^t \frac{U(x_t^*, x_{t+1}^*) - U(\lambda x_t^*, \lambda x_{t+1}^*)}{1-\lambda} \\ &\leq \sum_{t=T+1}^{\infty} \beta^t [U(x_t^*, x_{t+1}^*) - U(0,0)] \end{aligned}$$

where the last inequality follows from A3 (concavity of U) and $(*)$

Proof of Necessity of (TC), Kamihigashi (2002)

- Applying $\lim_{\lambda \to 1}$ to the LHS

$$0 \leq -\beta^T U_y(x_T^*, x_{T+1}^*) \cdot x_{T+1}^* \leq \sum_{t=T+1}^{\infty} \beta^t [U(x_t^*, x_{t+1}^*) - U(0,0)]$$

  where the first inequality follows from A4 ($U_y(x, y) \leq 0$) and A1 ($x_t \geq 0$)

- Applying $\lim_{T \to \infty}$ to both sides

$$\begin{aligned} 0 &\leq -\lim_{T \to \infty} \beta^T U_y(x_T^*, x_{T+1}^*) \cdot x_{T+1}^* \\ &\leq \lim_{T \to \infty} \sum_{t=T+1}^{\infty} \beta^t [U(x_t^*, x_{t+1}^*) - U(0,0)] = 0 \end{aligned}$$

  where the equality follows from A5 (boundedness)

- (TC) now follows.$\square$

(TC) in Practice

- In practice, often don't have to impose (TC) exactly
- Instead, just have to make sure trajectories "don't blow up."
- E.g. consider growth model: since $\beta < 1$, easy to see that

$$\lim_{T \to \infty} \beta^T u'(c_T) k_{T+1} = 0$$

whenever

$$\lim_{T \to \infty} c_T = c^*, \quad \lim_{T \to \infty} k_T = k^*$$

with $0 < c^*, k^* < \infty$ which is satisfied if $\{c_t, k_t\}$ converge to steady state.

Steady State

- Definition: a steady state is a point in the state space x^* such that $x_0 = x^*$ implies $x_t = x^*$ for all $t \geq 1$. ("if you start there, you stay there")
- Steady state in general model: any $x^* \in X$ such that
$$U_y(x^*, x^*) + \beta U_x(x^*, x^*) = 0$$
- Steady state in growth model: (c^*, k^*) satisfying
$$\begin{aligned} 1 &= \beta(f'(k^*) + 1 - \delta) \\ c^* &= f(k^*) - \delta k^* \end{aligned} \tag{$*$}$$
comes from (DE) with $c_{t+1} = c_t = c^*$ and $k_{t+1} = k_t = k^*$
- For example, if $f(k) = Ak^\alpha, \alpha < 1$. Then
$$k^* = \left(\frac{\alpha A}{\beta^{-1} - 1 + \delta}\right)^{\frac{1}{1-\alpha}}$$

Dynamics

- What else can we say about dynamics of $\{c_t\}_{t=0}^{\infty}$ and $\{k_{t+1}\}_{t=0}^{\infty}$?
- Turns out answering this is easier in continuous time
  - phase diagram
  - can also do discrete-time phase diagram, but a bit awkward
  - so rather do it properly
- See next lecture

Appendix

Representative firm $\Leftarrow$ perfect factor markets

Economy with heterogeneous firms

- N firms indexed by $i = 1, ..., N$
- Firm-specific production functions $y_i = f_i(k_i, \ell_i)$, no specific functional form assumptions
- Aggregate capital and labor endowments K and L

What's efficient allocation of capital and labor?

$$F(K, L) = \max_{\{k_i, \ell_i\}_{i=1}^N} \left\{ \sum_{i=1}^N f_i(k_i, \ell_i) \quad \text{s.t.} \quad \sum_{i=1}^N k_i \leq K, \quad \sum_{i=1}^N \ell_i \leq L \right\}$$

Write Lagrangean with multipliers R and w

$$\mathcal{L} = \sum_{i=1}^N f_i(k_i, \ell_i) + R\left[K - \sum_{i=1}^N k_i\right] + w\left[L - \sum_{i=1}^N \ell_i\right]$$

Representative firm $\Leftarrow$ perfect factor markets

- First-order conditions

$$\frac{\partial f_i(k_i, \ell_i)}{\partial k_i} = R \quad \text{and} \quad \frac{\partial f_i(k_i, \ell_i)}{\partial \ell_i} = w \quad \text{for all } i = 1, ..., N \qquad (*)$$

- Efficient allocation equalizes marginal products
- Also exactly what happens in competitive equilibrium with perfect factor markets
  - perfect factor markets = firms can rent/hire as much k_i, ℓ_i as they want at fixed R, w
- Lagrange multipliers R and w are equilibrium prices!
  - from firm profit maximization

$$\max_{k_i, \ell_i} f_i(k_i, \ell_i) - Rk_i - w\ell_i$$

Representative firm $\Leftarrow$ perfect factor markets

$F(K, L)$ from two slides ago is the aggregate production function

$$F(K, L) = \max_{\{k_i, \ell_i\}_{i=1}^N} \left\{ \sum_{i=1}^N f_i(k_i, \ell_i) \quad \text{s.t.} \quad \sum_{i=1}^N k_i \leq K, \quad \sum_{i=1}^N \ell_i \leq L \right\}$$

Have shown: if factor markets are perfect so that $(*)$ holds, then there exists an aggregate production function given by F above

Deviations from $(*)$: "misallocation"

Some references:

- Banerjee & Duflo (2005) "Growth Theory Through the Lens of Development Economics", Section 1.1. "The Aggregate Production Function" `https://www.sciencedirect.com/science/article/pii/S1574068405010075`
- Houthakker (1955) "The Pareto Distribution and the Cobb-Douglas Production Function in Activity Analysis" `https://academic.oup.com/restud/article/23/1/27/1522606`
- Hsieh & Klenow (2009) "Misallocation and Manufacturing TFP in China and India" `http://klenow.com/MMTFP.pdf`