Lecture 7

Adding Growth to the Growth Model

Macroeconomics EC417

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Adding Growth to the Growth Model

- Version of growth model we studied so far predicts that growth dies out relatively quicky
- In reality, economies like U.S. have growth at $\approx 2\%$ per year for more than a century

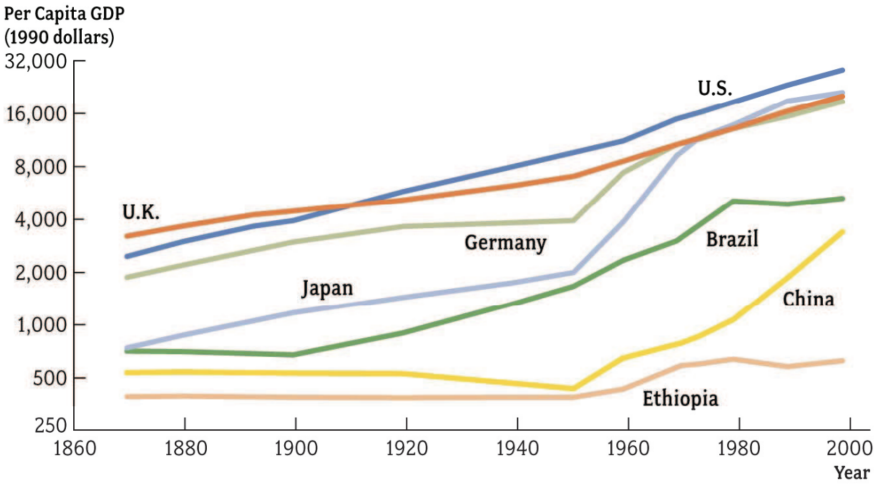


FIGURE 1.1 **Per Capita GDP in Seven Countries, 1870–2000**

Macroeconomics, Charles I. Jones

Adding Growth to the Growth Model

- What is missing?
- Consensus: technological progress
- Today: consequences of adding technological progress to growth model
- Deeper and important issue: how to model the process that leads to technological progress
    - "endogenous growth models": see Acemoglu if interested
    - what we do today will be very "reduced form," taking technological progress as exogenous to agents' choices

Adding Growth: Choices

- Previously
$$y_t = F(k_t, h_t)$$
- Let A_t = index of technology
  - increase in A_t = technological progress
- 3 different ways to "append" A_t into our existing model
$$\begin{aligned} y_t &= A_t F(k_t, h_t) && \text{neutral} \\ y_t &= F(A_t k_t, h_t) && \text{capital augmenting} \\ y_t &= F(k_t, A_t h_t) && \text{labor augmenting} \end{aligned}$$
- Note: if F is Cobb-Douglas, all three are isomorphic
- Result: to generate balanced growth, require that technological progress be labor augmenting
- Note: assumption is that tech. progress can be modeled as one-dimensional
  - simplifying assumption, tech. change takes many forms
  - recent work goes beyond this

Growth Model with Tech. Progress

- Preferences:

$$\sum_{t=0}^{\infty} \beta^t u(c_t)$$

- Technology:

$$\begin{aligned} y_t &= F(k_t, A_t h_t), \quad \{A_t\}_{t=0}^{\infty} \text{ given} \\ c_t + i_t &= y_t \\ k_{t+1} &= i_t + (1-\delta)k_t \end{aligned}$$

- Endowment: $k_0 = \hat{k}_0$, one unit of time each period
- Assumption: path of technological change is known
    - can extend to stochastic growth model
- Now redo everything we did before

Social Planner's Problem

$$\max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t) \quad \text{s.t.}$$

$$k_{t+1} = F(k_t, A_t) + (1-\delta)k_t - c_t$$

$$c_t \geq 0, \quad k_t \geq 0, \quad k_0 = \hat{k}_0$$

- proceed as before $\Rightarrow$ necessary and sufficient conditions

$$u'(c_t) = \beta u'(c_{t+1})(F_k(k_{t+1}, A_{t+1}) + 1 - \delta)$$

$$k_{t+1} = F(k_t, A_t) + (1-\delta)k_t - c_t$$

$+$ TVC $+ k_0 = \hat{k}_0$.

Asymptotic Behavior

- Looking for steady state as before does not really make sense
- Consider special case: A_t grows at constant rate

$$A_{t+1} = (1+g)A_t, \quad A_0 \text{ given}, \; 0 < g < \bar{g}$$

  where $\bar{g}$ is an upper bound (more on this later)

- Idea is not that A_t literally grows at constant rate ...
- ... rather that trend growth is constant
  - what would things look like if trend growth were the only component?

Balanced Growth Path

- Definition: a balanced growth path (BGP) solution to the SP problem is a solution in which all quantities grow at constant rates
- In principle different variables could grow at different rates
- But rates turn out to be the same. To see this, consider

$$c_t = F(k_t, A_t) + (1-\delta)k_t - k_{t+1}$$

- For RHS to grow at constant rate, k_t has to grow at same rate as $A_t \Rightarrow c_t$ also grows at same rate

Balanced Growth Path

- Now return to full necessary conditions for growth model

$$\frac{u'(c_t)}{\beta u'(c_{t+1})} = F_k(k_{t+1}, A_{t+1}) + 1 - \delta \qquad (*)$$
$$c_t = F(k_t, A_t) + (1-\delta)k_t - k_{t+1}$$

  \+ TVC + initial condition

- Looking for solution of form

$$k_t^* = (1+g)^t k_0^* \qquad (**)$$

  i.e. need to find k_0^* such that this condition holds for all t

- Important: similar to steady state, a BGP is a k_0 such that "if you start there, you stay there" (up to trend $1+g$)
  - "balanced growth" a.k.a. "steady state growth"
  - put differently: steady state in previous version of growth model = BGP with $g = 0$

Balanced Growth Path

- Now return to full necessary conditions for growth model

$$\frac{u'(c_t)}{\beta u'(c_{t+1})} = F_k(k_{t+1}, A_{t+1}) + 1 - \delta \qquad (*)$$
$$c_t = F(k_t, A_t) + (1-\delta)k_t - k_{t+1}$$

  $+$ TVC $+$ initial condition

- If $(**)$ holds, then RHS of $(*)$ is constant
  (because CRS $\Rightarrow F_k(k_t, A_t) = F_k(k_t/A_t, 1)$)

- $\Rightarrow$ LHS of $(*)$ must also be constant

- But $c^*_{t+1} = (1+g)c^*_t$. So how can we guarantee that $\frac{u'(c_t)}{\beta u'(c_{t+1})}$ is constant with $c^*_{t+1} = (1+g)c^*_t$? See next slide.

Balanced Growth Path

- Suppose

$$u(c_t) = \frac{c_t^{1-\sigma} - 1}{1-\sigma} \tag{CRRA}$$

- Then $u'(c_t) = c_t^{-\sigma}$ and

$$\frac{u'(c_t^*)}{\beta u'(c_{t+1}^*)} = \frac{1}{\beta}\left(\frac{c_t^*}{c_{t+1}^*}\right)^{-\sigma} = \frac{1}{\beta}(1+g)^{\sigma}$$

- $\Rightarrow$ if u satisfies (CRRA), LHS of $(*)$ is constant
- Still need to find k_0^*
  - We said LHS is constant, RHS is constant
  - still need to make them equal $\Rightarrow$

$$\frac{1}{\beta}(1+g)^{\sigma} = F_k(k_0^*, A_0) + 1 - \delta$$

Balanced Growth Path

- Last slide: if u satisfies (CRRA), then there is a BGP solution
- Turns out that (CRRA) is the only choice of utility function that works
- i.e. there is a BGP solution if and only if u satisfies (CRRA)

Balanced Growth Path

- Only if part: note that we require

$$\frac{u'(c)}{u'(c(1+g))} = \text{constant for all } c$$

- Differentiate w.r.t. c and require numerator is 0:

$$u''(c) = (1+g)u''(c(1+g))\frac{u'(c)}{u'(c(1+g))}$$

$$\frac{u''(c)c}{u'(c)} = \frac{u''(c(1+g))c(1+g)}{u'(c(1+g))}$$

$$\frac{u''(c)c}{u'(c)} = a \text{ (= constant)}$$

$$\frac{d \log u'(c)}{d \log c} = a \quad \Rightarrow \quad \log u'(c) = b + a \log c$$

Hence $u'(c) = e^b c^a$ = monotone transformation of (CRRA)

Balanced Growth Path

- From now on restrict preferences to (CRRA)
- Need ("-1 term" in (CRRA) doesn't matter)

$$\sum_{t=0}^{\infty} \beta^t \frac{(c_t^*)^{1-\sigma}}{1-\sigma} = \frac{(c_0^*)^{1-\sigma}}{1-\sigma} \sum_{t=0}^{\infty} (\beta(1+g)^{1-\sigma})^t < \infty$$

- Need $\beta(1+g)^{1-\sigma} < 1$
- If $\sigma < 1$, need upper bound $g < \bar{g} = \beta^{\frac{1}{\sigma-1}} - 1$

Balanced Growth Path

- Note: along a BGP, have c_t, k_t, y_t all growing at same rate
- And
$$\frac{i_t}{y_t} = \frac{k_t}{y_t} = \text{constant}$$
- Same property as steady state in version without growth

Transform Model w/ Growth into Model w/o Growth

- Know how to solve for BGP = generalization of steady state
- But what about transition dynamics? Turns out this is easy:
  - transform model with growth into model without growth
  - analysis of transformed model same as before
- Preferences:

$$\sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma} - 1}{1-\sigma}$$

- Technology:

$$c_t + k_{t+1} = F(k_t, A_t) + (1-\delta)k_t$$

- Define detrended consumption and capital

$$\tilde{c}_t = \frac{c_t}{(1+g)^t}, \quad \tilde{k}_t = \frac{k_t}{(1+g)^t}$$

Transform Model w/ Growth into Model w/o Growth

- $\Rightarrow$ Preferences:

$$\sum_{t=0}^{\infty} (\beta(1+g)^{1-\sigma})^t \frac{\tilde{c}_t^{1-\sigma} - 1}{1-\sigma} + \text{additive term}$$

- $\Rightarrow$ Technology:

$$\tilde{c}_t(1+g)^t + \tilde{k}_{t+1}(1+g)^{t+1} = F(\tilde{k}_t(1+g)^t, A_0(1+g)^t) + (1-\delta)\tilde{k}_t(1+g)^t$$

$$\tilde{c}_t + \tilde{k}_{t+1}(1+g) = f(\tilde{k}_t) + (1-\delta)\tilde{k}_t$$

where we normalized $A_0 = 1$ and used that CRS $\Rightarrow$

$$F(\tilde{k}_t(1+g)^t, (1+g)^t) = (1+g)^t F(\tilde{k}_t, 1) = (1+g)^t f(\tilde{k}_t)$$

Transform Model w/ Growth into Model w/o Growth

- Hence it is sufficient to solve (drop $\sim$'s for simplicity)

$$\max_{\{c_t, k_{t+1}\}} \sum_{t=0}^{\infty} \tilde{\beta}^t \frac{c_t^{1-\sigma} - 1}{1-\sigma} \quad \text{s.t.}$$

$$c_t + k_{t+1}(1+g) = f(k_t) + (1-\delta)k_t$$

where $\tilde{\beta} = \beta(1+g)^{1-\sigma}$

  - need $\beta(1+g)^{1-\sigma} < 1$
  - same restriction as before

Transform Model w/ Growth into Model w/o Growth

- Everything else just like before. E.g. Euler equation

$$c_t^{-\sigma} = \tilde{\beta} c_{t+1}^{-\sigma} \frac{f'(k_{t+1}) + 1 - \delta}{1 + g}$$

- Steady state

$$\frac{1}{\tilde{\beta}} = \frac{f'(k^*) + 1 - \delta}{1 + g} \quad \Leftrightarrow \quad \frac{1}{\beta}(1 + g)^\sigma = f'(k^*) + 1 - \delta$$

- Steady state in transformed economy = BGP in orig. economy
    - transformed economy: plot $\log \tilde{k}_t$ against t
    - original economy: plot $\log k_t$ against t: BGP = linear slope

$$\log k_{t+1} - \log k_t = \log\left(\frac{k_{t+1}}{k_t}\right) = \log(1 + g) \approx g$$

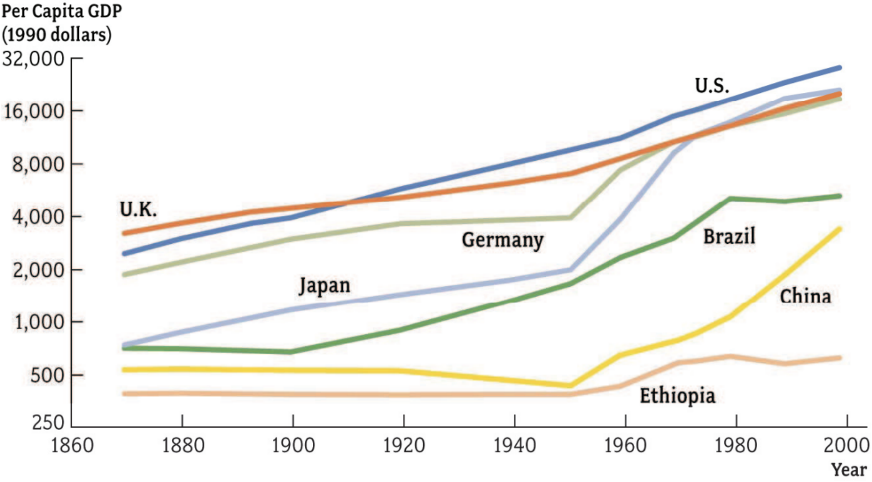
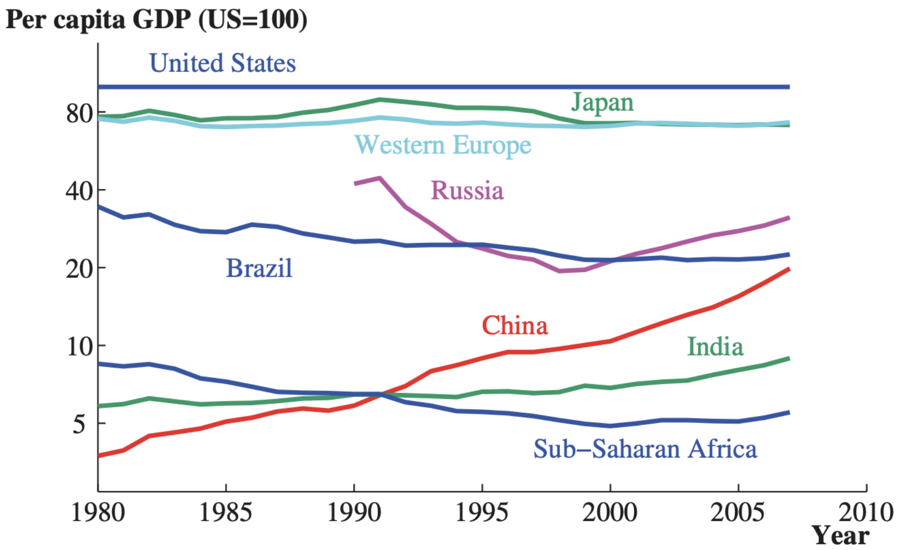


FIGURE 1.1 Per Capita GDP in Seven Countries, 1870–2000

Macroeconomics, Charles I. Jones


Per capita GDP (US=100)
United States
Japan
Western Europe
Russia
Brazil
China
India
Sub–Saharan Africa
80
40
20
10
5
1980
1985
1990
1995
2000
2005
2010
Year

Prevailing Paradigm

- Most countries share a long run growth rate
  - for these countries, policy differences have level effects
  - countries "transition around" in world BGP
- In terms of growth model
  - countries $i = 1, \ldots, n$, each runs a growth model
  - productivities satisfy (note: no i subscript on g)

$$A_{it} = A_{i0}(1+g)^t e^{\varepsilon_{it}}$$

  - interpret A_{it} more broadly than technology, also include institutions, policy
  - every now and then, country gets ε_{it} shock, triggers transition
- Is prevailing paradigm = right paradigm?
  - hard to say given data span only $\approx$ 100 years

Competitive Equilibria and BGP Prices

- Both ADCE and SOMCE can be defined just like before
- Prices along BGP

$$w_t^* = A_t F_h(k_t^*, A_t) \quad \text{grows at rate } g$$

$$r_t^* = F_k(k_t^*, A_t) - \delta \qquad \text{constant}$$

- Easy to show: interest rate r_t^* satisfies

$$1 + r_t^* = \frac{1}{\beta}(1+g)^\sigma$$

- Will often see this written in terms of $\rho = 1/\beta - 1$

$$1 + r_t^* = (1+\rho)(1+g)^\sigma$$
$$r_t^* \approx \rho + \sigma g$$

where $\approx$ uses $\log(1+x) \approx x$ for x small

- In continuous time, $r_t^* = \rho + \sigma g$ exactly