Equilibrium Properties

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July, 2021

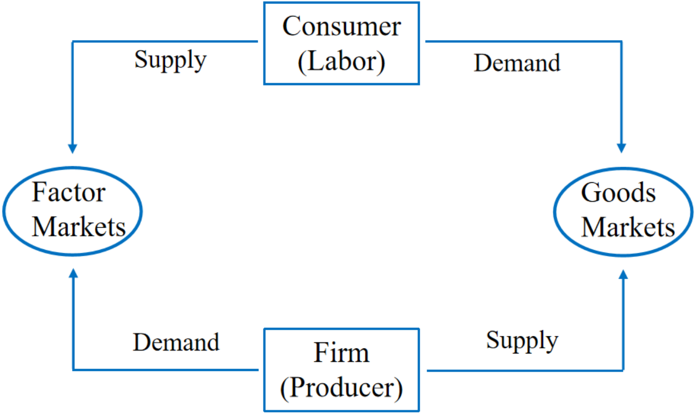
- Caliendo Parro, 2015, Estimates of the Trade and Welfare Effects of NAFTA, 82(1),1-44

Definition 1. Given L_n, D_n, λ_n^j and d_{ni}^j, an equilibrium under tariff structure τ is a wage vector $\mathbf{w} \in \mathbf{R}_{++}^N$ and prices $\left\{P_n^j\right\}_{j=1,n=1}^{J,N}$ that satisfy equilibrium conditions $(2), (4), (6), (7)$, and (9) for all j, n.

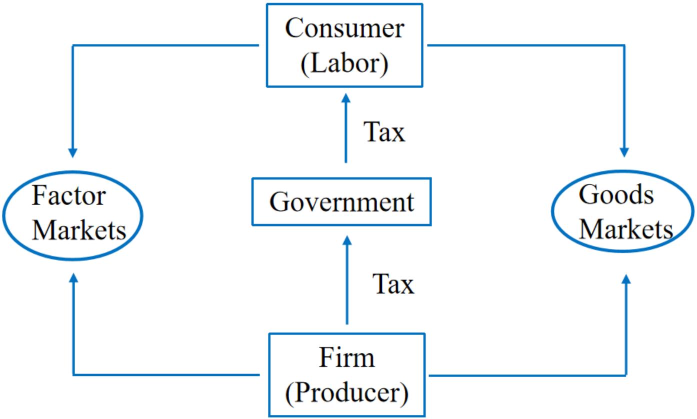
Outline

- Equilibrium Conditions
- Equilibrium in Trade Models
- Equilibrium in Economic Geography Models
- Equilibrium in General Network Models

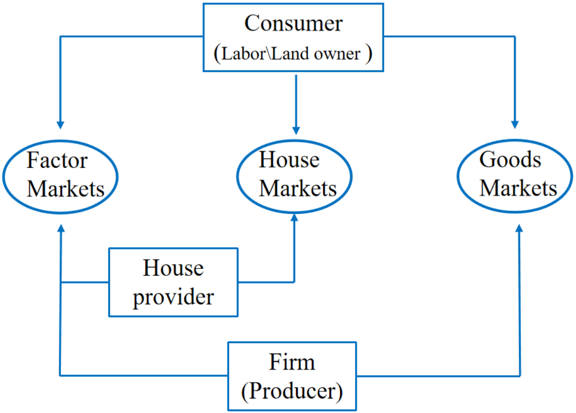
- Basic Elements in General Equilibrium Model



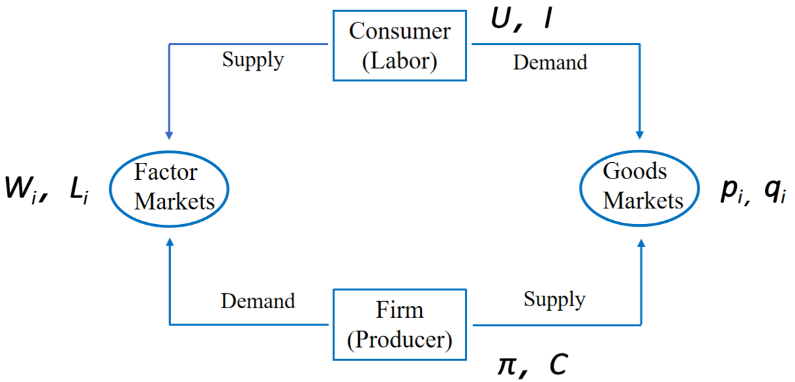
- Adding Elements in General Equilibrium Model



- Adding Elements in General Equilibrium Model



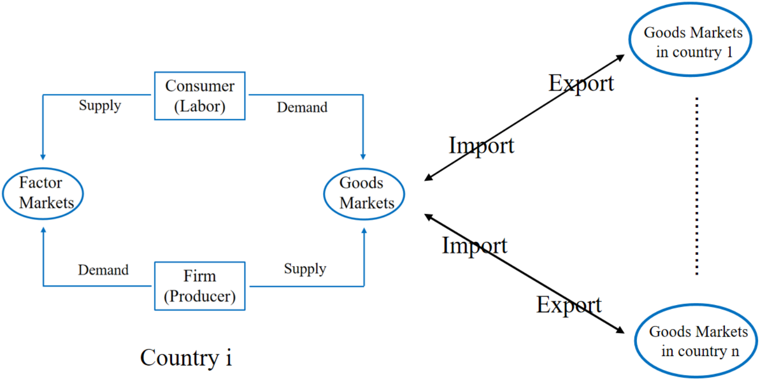
- Basic Elements in General Equilibrium Model



- Equilibrium Conditions

1. Utility maximization $X = X(p, E)\ ;\ E = wL$
2. Profit maximization $p = c(w)$
3. Good market clear condition $Y = X$
4. Labor market clear condition $Y = wL$

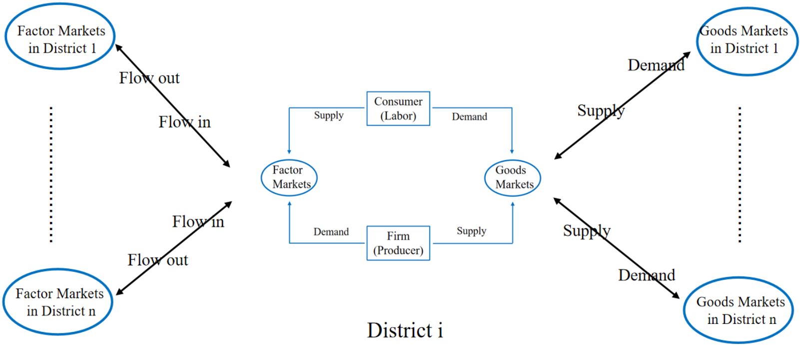
- Trade Model


Consumer (Labor)
Supply
Demand
Factor Markets
Goods Markets
Demand
Supply
Firm (Producer)
Country i
Export
Import
Import
Export
Goods Markets in country 1
Goods Markets in country n

- Equilibrium conditions in trade model

1. Utility maximization $X_{ji} = X_{ji}(p, E_i)\ ;\ E_i = w_i L_i$
2. Profit maximization $p_i = c_i(w)$
3. Good market clear condition $Y_i = \sum_{j \in S} X_{ij}$
4. Balanced trade condition $Y_i = E_i = \sum_{j \in S} X_{ji}$
5. Labor market clear condition $Y_i = w_i L_i$

▶ Geography Model



- Equilibrium conditions in spatial model

1. Utility maximization $X_{ji} = X_{ji}(p, E_i)\ ;\ E_i = w_i L_i$
2. Profit maximization $p_i = c_i(w)$
3. Good market clear condition $Y_i = \sum_{j \in S} X_{ij}$
4. Balanced trade condition $Y_i = E_i = \sum_{j \in S} X_{ji}$
5. Labor market clear condition $Y_i = w_i L_i\ ;\ \bar{L} = \sum_{i \in S} L_i$

   $W_i = W > 0$, for $L_i > 0$

Equilibrium in Trade Models(Labor Is Immobile)

Utility Maximization + Profit Maximization

- Anderson (1979)

$$X_{ij} = \tau_{ij}^{1-\sigma} w_i^{1-\sigma} A_i^{\sigma-1} E_j P_j^{1-\sigma}, \quad P_j \equiv \Big(\sum_{i\in S} \tau_{ij}^{1-\sigma} w_i^{1-\sigma} A_i^{\sigma-1}\Big)^{\frac{1}{\sigma-1}}$$

$$X_{ij} = \frac{\tau_{ij}^{1-\sigma} A_i^{\sigma-1} w_i^{1-\sigma}}{\sum_{k\in S} \tau_{kj}^{1-\sigma} A_k^{\sigma-1}{}_k^{1-\sigma}} E_j$$

- Krugman (1980)

$$X_{ij} = \frac{1}{\sigma} \tau_{ij}^{1-\sigma} w_i^{1-\sigma} A_i^{\sigma-1} \frac{L_i}{f_i^e} E_j P_j^{\sigma-1}, P_j \equiv \frac{1}{\sigma}\Big(\sum_{i\in S} \tau_{ij}^{1-\sigma} \frac{L_i}{f_i^e} w_i^{1-\sigma} A_i^{\sigma-1}\Big)^{\frac{1}{1-\sigma}}$$

$$X_{ij} = \frac{\tau_{ij}^{1-\sigma} w_i^{1-\sigma} A_i^{\sigma-1} \frac{L_i}{f_i^e}}{\sum_{k\in S} \tau_{kj}^{1-\sigma} \frac{L_k}{f_k^e} w_k^{1-\sigma} A_k^{\sigma-1}} E_j$$

Utility Maximization + Profit Maximization

- Melitz (2003) with a Pareto distribution and exogenous entry

$$X_{ij} = \tau_{ij}^{-\theta_i} w_i^{-\theta_i} f_{ij}^{\frac{\sigma-\theta_i-1}{\sigma-1}} M_i E_j^{\frac{\theta_i}{\sigma-1}} P_j^{\theta}, P_j \equiv \left(E_j^{-\frac{\sigma-\theta_i-1}{\sigma-1}} \sum_{i \in S} M_i w_i^{-\theta_i} \tau_{ij}^{-\theta} f_{ij}^{\frac{\sigma-\theta_i-1}{\sigma-1}} \right)^{-\frac{1}{\theta_i}}$$

$$X_{ij} = \frac{\tau_{ij}^{-\theta_i} f_{ij}^{\frac{\sigma-\theta_i-1}{\sigma-1}} M_i w_i^{-\theta_i}}{\sum_{k \in S} \tau_{kj}^{-\theta_i} f_{kj}^{\frac{\sigma-\theta_i-1}{\sigma-1}} M_k w_k^{-\theta_i}} E_j$$

- Eaton and Kortum (2002)

$$X_{ij} = \tau_{ij}^{-\theta} w_i^{-\theta} T_i E_j P_j^{\theta}, \qquad P_j \equiv \left(\sum_{j \in S} T_i w_i^{-\theta} \tau_{ij}^{-\theta}\right)^{-\frac{1}{\theta}}$$

$$X_{ij} = \frac{\tau_{ij}^{-\theta} T_i w_i^{-\theta}}{\sum_{k \in S} \tau_{kj}^{-\theta} T_k w_k^{-\theta}} E_j$$

Utility Maximization + Profit Maximization

$$X_{ij} = \frac{\tau_{ij}^{1-\sigma} A_i^{\sigma-1} w_i^{1-\sigma}}{\sum_{k\in S} \tau_{kj}^{1-\sigma} A_k^{\sigma-1} w_k^{1-\sigma}} E_j \quad X_{ij} = \frac{\tau_{ij}^{1-\sigma} w_i^{1-\sigma} A_i^{\sigma-1} \frac{L_i}{f_i^e}}{\sum_{k\in S} \tau_{kj}^{1-\sigma} \frac{L_k}{f_k^e} w_k^{1-\sigma} A_k^{\sigma-1}} E_j$$

$$X_{ij} = \frac{\tau_{ij}^{-\theta_i} f_{ij}^{\frac{\sigma-\theta_i-1}{\sigma-1}} M_i w_i^{-\theta_i}}{\sum_{k\in S} \tau_{kj}^{-\theta_i} f_{kj}^{\frac{\sigma-\theta_i-1}{\sigma-1}} M_k w_k^{-\theta_i}} E_j \qquad X_{ij} = \frac{\tau_{ij}^{-\theta} T_i w_i^{-\theta}}{\sum_{k\in S} \tau_{kj}^{-\theta} T_k w_k^{-\theta}} E_j$$

We can have:

$$X_{ij} = \frac{K_{ij} w_i^{\alpha}}{\sum_{k\in S} K_{kj} w_k^{\alpha}} E_j$$

Exogenous variables : $\mathrm{K}_{ij} > 0, \alpha < 0$

Endogenous variables: w_i, E_j

- Equilibrium conditions

1. Utility maximization+Profit maximization $X_{ij} = \frac{K_{ij} w_i^{\alpha}}{\sum_{k \in S} K_{kj} w_k^{\alpha}} E_j$

2. Good market clear condition $Y_i = \sum_{j \in S} X_{ij}$

3. Balanced trade condition $Y_i = E_i = \sum_{j \in S} X_{ji}$

4. Labor market clear condition $w_i L_i = Y_i$

- **The Excess Demand Function**

$$Z_i(\mathbf{w}) \equiv \frac{1}{w_i} \left(\sum_{j \in S} \frac{K_{ij} w_i^{\alpha}}{\sum_{k \in S} K_{kj} \omega_k^{\alpha}} w_j L_j - w_i L_i \right)$$

- Existence

Proposition

$$Z_i(\mathbf{w}) \equiv \frac{1}{w_i}\left(\sum_{j\in S}\frac{K_{ij}w_i^{\alpha}}{\sum_{k\in S}K_{kj}w_k^{\alpha}}w_jL_j - w_iL_i\right)$$

1. For all $\mathbf{w} \gg 0$ (i.e. for all W such that $\mathbf{w}_i > 0$ for all $i \in S$), $Z_i(.)$ is continuous.
2. For all $i \in S$, $Z_i(.)$ is homogeneous of degree zero.To see this, note that for any $\beta > 0$:

$$\begin{aligned}
Z_i(\beta\mathbf{w}) &= \frac{1}{\beta w_i}\left(\sum_{j\in S}\frac{K_{ij}(\beta w_i)^{\alpha}}{\sum_{k\in S}K_{kj}(\beta w_k)^{\alpha}}\beta w_jL_j - \beta w_iL_i\right)\\
&= \frac{1}{w_i}\left(\sum_{j\in S}\frac{K_{ij}w_i^{\alpha}}{\sum_{k\in S}K_{kj}w_k^{\alpha}}w_jL_j - w_iL_i\right)\\
&= Z_i(\mathbf{w})
\end{aligned}$$

- Existence

$$Z_i(\mathbf{w}) \equiv \frac{1}{w_i}\left(\sum_{j\in S}\frac{K_{ij}w_i^{\alpha}}{\sum_{k\in S}K_{kj}w_k^{\alpha}}w_jL_j - w_iL_i\right)$$

3. For all $\mathbf{w} \gg 0$,we have:$\sum_{i\in S} w_iZ_i(\mathbf{w}) = 0$

$$\begin{aligned}
\sum_{i\in S} w_iZ_i(\mathbf{w}) &= \sum_{i\in S} w_i\frac{1}{w_i}\left(\sum_{j\in S}\frac{K_{ij}w_i^{\alpha}}{\sum_{k\in S}K_{kj}w_k^{\alpha}}w_jL_j - w_iL_i\right) \\
&= \sum_{i\in S}\left(\sum_{j\in S}\frac{K_{ij}w_i^{\alpha}}{\sum_{k\in S}K_{kj}w_k^{\alpha}}w_jL_j - w_iL_i\right) \\
&= \sum_{i\in S}\sum_{j\in S}\frac{K_{ij}w_i^{\alpha}}{\sum_{k\in S}K_{kj}w_k^{\alpha}}w_jL_j - \sum_{i\in S}w_iL_i \\
&= \sum_{i\in S}\sum_{j\in S}X_{ij} - \sum_{i\in S}Y_i \\
&= \sum_{i\in S}Y_i - \sum_{i\in S}Y_i \\
&= 0
\end{aligned}$$

- Existence

$$Z_i(\mathbf{w}) \equiv \frac{1}{w_i}\left(\sum_{j\in S}\frac{K_{ij}w_i^{\alpha}}{\sum_{k\in S}K_{kj}w_k^{\alpha}}\omega_j L_j - w_i L_i\right)$$

4. For all $\mathbf{w} \gg 0$,there exists an $s > 0$ such that $Z_i(\mathbf{w}) > -s$ for all $i \in S$.

$$\begin{aligned}Z_i(\mathbf{w}) &\equiv \frac{1}{w_i}\left(\sum_{j\in S}\frac{K_{ij}w_i^{\alpha}}{\sum_{k\in S}K_{kj}w_k^{\alpha}}w_j L_j - w_i L_i\right)\\ &= \frac{1}{w_i}\sum_{j\in S}\frac{K_{ij}w_i^{\alpha}}{\sum_{k\in S}K_{kj}w_k^{\alpha}}w_j L_j - L_i\\ &> -L_i\end{aligned}$$

- Existence

$$Z_i(\mathbf{w}) \equiv \frac{1}{w_i}\left(\sum_{j\in S}\frac{K_{ij}w_i^{\alpha}}{\sum_{k\in S}K_{kj}w_k^{\alpha}}w_jL_j - w_iL_i\right)$$

5. Consider any $\mathbf{w} \in R^{||s||}$ such that there exists an $l \in S$ where $w_l = 0$ and an $l' \in S$ where $w' > 0$. Consider any sequence of wages such that $\mathbf{w}^n \to \mathbf{w}$ as $n \to +\infty$. Then:

$$\underset{i\in S}{max}\, Z_i(\mathbf{w}^n) \to +\infty$$

- Existence

$$\max_{i\in S} Z_i(\mathbf{w}) = \max_{i\in S} \frac{1}{w_i}\left(\sum_{j\in S} \frac{K_{ij}w_i^{\alpha}}{\sum_{k\in S} K_{kj}w_k^{\alpha}} w_j L_j - w_i L_i\right)$$

$$\max_{i\in S} Z_i(\mathbf{w}^n) = \max_{i\in S} \left(\frac{1}{w_i}\sum_{j\in S} \frac{K_{ij}w_i^{\alpha}}{\sum_{k\in S} K_{kj}w_k^{\alpha}} w_j L_j - w_i L_i\right)$$

$$> \max_{i\in S} \frac{1}{w_i} \max_{j\in S} \frac{K_{ij}w_i^{\alpha}}{\sum_{k\in S} K_{kj}w_k^{\alpha}} w_j L_j - \max_{i\in S} L_i$$

$$= \max_{i,j\in S} \frac{w_j}{w_i} \frac{K_{ij}w_i^{\alpha}}{\sum_{k\in S} K_{kj}w_k^{\alpha}} L_j - \max_{i\in S} L_i$$

- Existence

$$\max_{i,j\in S} \frac{w_j}{w_i} \frac{K_{ij} w_i^{\alpha}}{\sum_{k\in S} K_{kj} w_k^{\alpha}} L_j > (\min_{l\in S} L_s) \max_{i,j\in S} w_j \frac{K_{ij} w_i^{\alpha-1}}{\sum_{k\in S} K_{kj} w_k^{\alpha}}$$

$$= C \max_{i\in S} \frac{K_{ij} w_i^{\alpha-1}}{\sum_{k\in S} K_{kj} w_k^{\alpha}}$$

$$> C \max_{i\in S} \frac{K_{ij} w_i^{\alpha-1}}{\sum_{k\in S} K_{kj} (min_{l\in S} w_l)^{\alpha}}$$

$$> C \frac{K_{ij} (min_{l\in S} w_l)^{\alpha-1}}{\sum_{k\in S} K_{kj} (min_{l\in S} w_l)^{\alpha}}$$

$$= C \times \frac{K_{ij}}{\sum_{k\in S} K_{kj}} \times (\min_{l\in S} w_l)^{-1}$$

Where, $C \equiv (\min_{l\in S} L_s)(\min_{j\in S} w_j) > 0$

- Uniqueness

Definition. A (differentiable) excess demand function is said to satisfies the **gross substitution property** if for all $i \in S$:

$$\frac{\partial Z_i(\mathbf{w})}{\partial w_j} > 0 \qquad \text{for all } j \neq i$$

Theorem. If a function $\{Z_i(\cdot)\}_{i \in S}$ satisfies the gross substitute property (and is homogenous of degree zero), then the equilibrium $\mathbf{w}^* \gg 0$ such that $Z_i\left(\mathbf{w}^*\right) = 0$ for all $i \in S$ is unique (to scale).

- Uniqueness

$$\frac{\partial Z_i(\mathbf{w})}{\partial w_j} = \frac{\partial}{\partial w_j}\frac{1}{w_i}\left(\sum_{l\in S}\frac{K_{il}w_i^{\alpha}}{\sum_{k\in S}K_{kl}w_k^{\alpha}}w_l L_l - w_i L_i\right)$$

$$= \frac{1}{w_i}\left(\frac{K_{ij}w_i^{\alpha}}{\sum_{k\in S}K_{kj}w_k^{\alpha}}L_j - \alpha w_j^{\alpha-1}\sum_{k\in S}\frac{K_{jk}K_{ik}w_i^{\alpha}w_k L_k}{(\sum_{j\in S}K_{jk}w_j^{\alpha})^2}\right) > 0$$

- A "universal" approach

Theorem. Consider any trade model that yield the following equilibrium conditions:

1. Utility and Profit maximization $X_{ji} = K_{ij}\gamma_i\delta_j$
2. Good market clearing $Y_i = \sum_{j \in S} X_{ij}$
3. Balanced trade condition $Y_i = \sum_{j \in S} X_{ji}$
4. Labor market clearing $Y_i = \bar{B}_i \gamma_i^{\alpha}$

When a<0, there exists a unique (to scale) equilibrium.

- A "universal" approach

$$Y_j = \sum_{i \in S} X_{ij} = \sum_{i \in S} k_{ij}\gamma_i\delta_j = \delta_j \sum_{i \in S} k_{ij}\gamma_i \Rightarrow \delta_j = \frac{Y_j}{\sum_{i \in S} k_{ij}\gamma_i}$$

$$X_{ij} = k_{ij}\gamma_i\delta_j = \frac{k_{ij}\gamma_i Y_j}{\sum_{i \in S} k_{ij}\gamma_i} \qquad \Rightarrow$$

$$\sum_{j \in S} X_{ij} = \sum_{j \in S} \frac{k_{ij}\gamma_i Y_j}{\sum_{i \in S} k_{ij}\gamma_i} \qquad \Rightarrow \qquad Y_i = \sum_{j \in S} \frac{k_{ij}\gamma_i Y_j}{\sum_{i \in S} k_{ij}\gamma_i}$$

$$\overline{B}_i\gamma_i^{\alpha} = \sum_{j \in S} \frac{k_{ij}\gamma_i\overline{B}\gamma_j^{\alpha}}{\sum_{i \in S} k_{ij}\gamma_i} \qquad \Rightarrow \qquad \overline{B}_i\widetilde{\gamma}_i = \sum_{j \in S} \frac{k_{ij}\widetilde{\gamma}_i^{\frac{1}{\alpha}}\overline{B}_j\widetilde{\gamma}_j}{\sum_{i \in S} k_{ij}\widetilde{\gamma}_i^{\frac{1}{\alpha}}}$$

Equilibrium in Economic Geography Models

(Labor Is Mobile)

- Setup (Armington set-up)
  - Identical CES preference
  - $i = 1, ..., S$ regions, each produces one variety
  - $\overline{A} \geq 0$, exogenous productivity; $\overline{u_i} \geq 0$, exogenous amenity
  - $\overline{L}$, total labor
  - Agglomeration and dispersion forces :

$$A_i = \overline{A} L_i^{\alpha} \qquad u_i = \overline{u} L_i^{\beta}$$

Equilibrium Conditions

1. Utility Maximization+Profit Maximization

$$X_{ij} = \tau_{ij}^{1-\sigma}\omega_i^{1-\sigma}A_i^{\sigma-1}E_jP_j^{\sigma-1}$$
$$P_j \equiv \left(\sum_{i\in S}\tau_{ij}^{1-\sigma}w_i^{1-\sigma}A_i^{\sigma-1}\right)^{\frac{1}{1-\sigma}}$$

2. Good market condition $\quad Y_i = \sum_{j\in S} X_{ij}$

3. Balanced trade condition $\quad Y_i = \sum_{j\in S} X_{ji}$

4. Labor market condition $\quad Y_i = w_iL_i\,;\,\overline{L} = \sum_{i\in S} L_i$

$$W_i = \frac{w_i}{p_i}u_i = W\text{ , for } L_i > 0$$

- Solving the equilibrium-equilibrium without spillovers

$$Y_i = \sum_{j\in S} X_{ij} \iff w_i L_i = \sum_{j\in S} \tau_{ij}^{1-\sigma} w_i^{1-\sigma} A_i^{\sigma-1} P_j^{\sigma-1} Y_j \iff$$

$$w_i{}^{\sigma} L_i = \sum_{j\in S} \tau_{ij}^{1-\sigma} A_i^{\sigma-1} (W_j^{1-\sigma} w_j^{\sigma-1} u_j^{\sigma-1}) w_j L_j \iff$$

$$w_i{}^{\sigma} L_i = W^{1-\sigma} \sum_{j\in S} \tau_{ij}^{1-\sigma} A_i^{\sigma-1} u_j^{\sigma-1} w_j^{\sigma} L_j$$

$$W = \frac{w_i}{P_i} u_i \iff \mathrm{w}_i^{1-\sigma} = W^{1-\sigma} P_i^{1-\sigma} u^{1-\sigma} \iff$$

$$w_i{}^{1-\sigma} = W^{1-\sigma} \sum_{j\in S} \tau_{ji}^{1-\sigma} A_j^{\sigma-1} u_i^{\sigma-1} w_j^{1-\sigma}$$

$$\Downarrow$$

$$\mathbf{x} = \lambda \mathbf{A} \mathbf{x}, \quad \mathbf{y} = \lambda \mathbf{A}^{\mathbf{T}} \mathbf{y}$$

$$\mathbf{x} = [x_i] = [w_i^{\sigma} L_i], \quad \mathbf{y} = [y_i] = [w_i^{1-\sigma}]$$

$$\lambda = W^{1-\sigma}, \quad \mathbf{A} = [a_{ij}] = [\tau_{ij}^{1-\sigma} A_i^{\sigma-1} u_j^{\sigma-1}]$$

Perron-Frobenius

- Solving the equilibrium-equilibrium without spillovers

Proposition 1. For any regular geography when $\alpha = \beta = 0$, there exists a unique spatial equilibrium. Furthermore, that spatial equilibrium is regular.

- Solving the equilibrium-equilibrium with spillovers

$$w_i{}^{\sigma} L_i = W^{1-\sigma} \sum_{j \in S} \tau_{ij}^{1-\sigma} A_i^{\sigma-1} u_j^{\sigma-1} w_j^{\sigma} L_j$$

$$w_i{}^{1-\sigma} = W^{1-\sigma} \sum_{j \in S} \tau_{ji}^{1-\sigma} A_j^{\sigma-1} u_i^{\sigma-1} w_j^{1-\sigma}$$

$$\Downarrow$$

$$w_i{}^{\sigma} L_i^{1-\alpha(\sigma-1)} = W^{1-\sigma} \sum_{j \in S} \tau_{ij}^{1-\sigma} \overline{A}_i^{\sigma-1} \overline{u}_j^{\sigma-1} w_j^{\sigma} L_j^{1+\beta(\sigma-1)}$$

$$w_i{}^{1-\sigma} L_i^{\beta(1-\sigma)} = W^{1-\sigma} \sum_{j \in S} \tau_{ji}^{1-\sigma} \overline{A}_j^{\sigma-1} \overline{u}_i^{\sigma-1} w_j^{1-\sigma} L_j^{\alpha(\sigma-1)}$$

- Quasi-symmetry trade cost

Definition. Trade costs are **quasi-symmetric** when for all $i, j \in S$, we have:

$$\tau_{ij} = \tilde{\tau}_{ij} \tau_i^A \tau_j^B$$

where $\tilde{\tau}_{ij} = \tilde{\tau}_{ji}$ .Note that symmetric trade cost (where $\tau_{ij} = \tau_{ji}$) are quasi-symmetric, but there are also exist non-symmetric trade cost that are quasi-symmetric.

- Quasi-symmetry trade cost

Proposition 2. For any gravity trade model with quasi-symmetric trade cost, if there exist strictly positive and bounded origin and destination fixed effects such that the goods market clears and trade is balanced, there it must be the case that:

$$\tau_i^A \gamma_i = \kappa \tau_i^B \delta_i$$

$$\sum_{j\in S} X_{ij} = \sum_{j\in S} X_{ji} \qquad \Leftrightarrow$$

$$\sum_{j\in S} \tilde{\tau}_{ij}\tau_i^A\tau_j^B\gamma_i\delta_j = \sum_{j\in S} \tilde{\tau}_{ij}\tau_j^A\tau_i^B\gamma_j\delta_i \qquad \Leftrightarrow$$

$$\tau_i^A\gamma_i \sum_{j\in S} \tilde{\tau}_{ij}\tau_j^B\delta_j = \tau_i^B\delta_i \sum_{j\in S} \tilde{\tau}_{ij}\tau_j^A\gamma_j \qquad \Leftrightarrow$$

$$\frac{\tau_i^A\gamma_i}{\tau_i^B\delta_i} = \frac{\sum_{j\in S} \tilde{\tau}_{ij}\tau_j^A\gamma_j}{\sum_{j\in S} \tilde{\tau}_{ij}\tau_j^B\delta_j} = \sum_{j\in S} \frac{\tilde{\tau}_{ij}\tau_j^B\delta_j}{\sum_{j\in S} \tilde{\tau}_{ij}\tau_j^B\delta_j} \frac{\tau_j^A\gamma_j}{\tau_j^B\delta_j}$$

$$\Downarrow$$

$$\phi_i = \lambda \sum_{j\in S} F_{ij}\phi_j \qquad\qquad \phi_i = \frac{\tau_i^A\gamma_i}{\tau_i^B\delta_i}$$

$$\Downarrow$$

$$\phi_i = \phi_j = \kappa$$

$$\Downarrow$$

$$\tau_i^A\gamma_i = \kappa\tau_i^B\delta_i$$

- Quasi-symmetry trade cost Economic Geography

$$\tau_i^A \gamma_i = \kappa \tau_i^B \delta_i \iff \tau_i^A (w_i^{1-\sigma} A_i^{\sigma-1}) = \kappa \tau_i^B (P_i^{\sigma-1} w_i L_i)$$

$$\iff \tau_i^A (w_i^{1-\sigma} A_i^{\sigma-1}) = \kappa \tau_i^B (W^{\sigma-1} u_i^{\sigma-1} w_i^{\sigma-1} w_i L_i)$$

$$\iff w_i = (\kappa \frac{\tau_i^A}{\tau_i^B} W^{1-\sigma} u_i^{\sigma-1} A_i^{\sigma-1} L_i)^{\frac{1}{1-2\sigma}}$$

$$w_i{}^\sigma L_i = W_j^{1-\sigma} \sum_{j \in S} \tau_{ij}^{1-\sigma} A_i^{\sigma-1} u_j^{\sigma-1} w_j^\sigma L_j$$

$$\iff (\kappa \frac{\tau_i^A}{\tau_i^B} W^{1-\sigma} u_i^{\sigma-1} A_i^{\sigma-1} L_i)^{\frac{\sigma}{1-2\sigma}} L_i$$

$$= W^{1-\sigma} \sum_{j \in S} \tau_{ij}^{1-\sigma} A_i^{\sigma-1} u_j^{\sigma-1} (\kappa \frac{\tau_j^A}{\tau_j^B} W^{1-\sigma} A_j^{\sigma-1} u_j L_j)^{\frac{\sigma}{1-2\sigma}} L_j$$

$$\iff \mathrm{L}_i^{\tilde{\sigma}\gamma_1} = W^{1-\sigma} \sum_{j \in S} \tau_{ij}^{1-\sigma} (\frac{\tau_j^B \tau_i^A}{\tau_j^A \tau_i^B})^{\frac{\sigma}{2\sigma-1}} \overline{A}_i^{(\sigma-1)\tilde{\sigma}} \overline{A}_j^{\sigma\tilde{\sigma}} \overline{u}_j^{(\sigma-1)\tilde{\sigma}} \overline{u}_i^{\sigma\tilde{\sigma}} L_j^{\tilde{\sigma}\gamma_2}$$

$$\tilde{\sigma} \equiv \frac{\sigma-1}{2\sigma-1}, \gamma_1 \equiv 1 - \alpha(\sigma - 1) - \beta\sigma, \gamma_2 \equiv 1 + \alpha\sigma + \beta(\sigma - 1)$$

- Quasi-symmetry trade cost Economic Geography

$$L_i^{\tilde{\sigma}\gamma_1} = W^{1-\sigma} \sum_{j \in S} \tau_{ij}^{1-\sigma} \left(\frac{\tau_j^B \tau_i^A}{\tau_j^A \tau_i^B}\right)^{\frac{\sigma}{2\sigma-1}} \overline{A}_i^{(\sigma-1)\tilde{\sigma}} \overline{A}_j^{\sigma\tilde{\sigma}} \overline{u}_j^{(\sigma-1)\tilde{\sigma}} \overline{u}_i^{\sigma\tilde{\sigma}} L_j^{\tilde{\sigma}\gamma_2}$$

$$\Downarrow$$

$$x_i = \lambda \sum_{j \in S} F_{ij} x_j^{\frac{\gamma_2}{\gamma_1}}$$

Theorem 1. Consider a system of equation $z_i = \lambda \sum_{j \in S} F_{ij} z_j^{\alpha}$ for all $i \in S$ and $\sum_i z_i = C$. Then there exists a unique equilibrium if $|\alpha| < 1$.

- Quasi-symmetry trade cost Economic Geography Banach's Theorem

$$x_i = \lambda \sum_{j \in S} F_{ij} x_j^{\frac{\gamma_2}{\gamma_1}} \iff z_i = \lambda \sum_{j \in S} F_{ij} z_j^{\alpha} \iff h_i = \sum_{j \in S} F_{ij} h_j^{\alpha}$$

Define function $T : \mathbb{R}^N \to \mathbb{R}^N$

$$T(m)_i \equiv \log \left(\sum_{j\{1,..,N\}} F_{ij} \exp\left(\alpha m_j\right) \right)$$

For any two $m \in \mathbb{R}^N$ and $n \in \mathbb{R}^N$,define the metric to be the maxium metric:

$$d(m,n) \equiv \max_{i \in \{1,..,N\}} |m_i - n_i|$$

If

$$d(T(m), T(n)) \leq c \times d(m,n), 0 \leq c < 1$$

Then there exists a unique $m^* \in \mathbb{R}^N$, such that $m^* = T(m^*)$. This also indicates that there is exist a unique $h^* \in R^N_{++}$ such that $h_i^* = \sum_{j \in S} F_{ij} h_j^{*\alpha}$ for all $i \in \{1, \ldots, N\}$.

- Quasi-symmetry trade cost Economic Geography

$$d(T(m),T(n)) = \max_i \left|\log\left(\sum_j F_{ij}\exp\left(\alpha m_j\right)\right) - \log\left(\sum_j F_{ij}\exp\left(\alpha n_j\right)\right)\right|$$

$$= \max_i \left|log\left(\sum_j C_i \exp(\alpha(m_j - n_j))\right)\right|$$

Where $C_{ij} = \frac{F_{ij}exp(\alpha n_j)}{\sum\limits_k F_{ik}exp(\alpha n_k)}, \sum\limits_j C_{ij} = 1.$

Then we have:

$$\sum_j C_{ij}exp(\alpha(m_j - n_j) \leq \sum_j C_{ij}exp(|\alpha| \max_j |m_j - n_j|) = exp(|\alpha| \max_j |m_j - n_j|)$$

$$d(T(m),T(n)) = \max_i |log(\sum_j C_{ij}exp(\alpha(m_j - n_j))| \leq \max_i |(|\alpha| \max_j |m_j - n_j|)$$

$$d(T(m),T(n)) \leq |\alpha| d(m,n)$$

- Quasi-symmetry trade cost & Economic Geography

$$|\alpha| < 1 \iff d(T(m), T(n)) \leq |\alpha| d(m, n) \iff$$

There exists unique $m^* \in \mathbb{R}^N$,such that $m^* = T(m^*)$.

Corollary 1. For any regular geography with quasi-symmetric trade costs, there exists a regular spatial equilibrium. Furthermore, if $\frac{\gamma_2}{\gamma_1} \in [-1, 1] \Leftrightarrow \alpha + \beta \leq 0$, that equilibrium is unique.

- A “universal” approach

  Theorem. Consider any trade model that yield the following equilibrium conditions:

1. Utility and Profit maximization $X_{ij} = K_{ij}\gamma_i\delta_j$
2. Good market clearing $Y_i = \sum_{j \in S} X_{ij}$
3. Balanced trade condition $Y_i = \sum_{j \in S} X_{ji}$
4. Labor market clearing $Y_i = \bar{B}_i \gamma_i^{\alpha} \delta_i^{\beta}$

When $\alpha + \beta \geq 2$ or $\alpha + \beta \leq 0$, there exists a unique (to scale) equilibrium.

- A “universal” approach

$$Y_i = \sum_{j \in S} X_{ij} = \sum_{j \in S} K_{ij} \gamma_i \delta_j \quad \Rightarrow \quad \overline{B}_i \gamma_i^{\alpha} \delta_i^{\beta} = \sum_{j \in S} K_{ij} \gamma_i \delta_j \quad \Rightarrow$$

$$\overline{B}_i \gamma_i^{\alpha-1} \delta_i^{\beta} = \sum_{j \in S} K_{ij} \delta_j \qquad \Rightarrow$$

$$\overline{B}_i \gamma_i^{\alpha-1} \left(\frac{1}{\kappa} \frac{K_i^A}{K_i^B} \gamma_i\right)^{\beta} = \sum_{j \in S} K_{ij} \left(\frac{1}{\kappa} \frac{K_j^A}{K_j^B} \gamma_j\right) \qquad \Rightarrow$$

$$\gamma_i^{\alpha+\beta-1} = \kappa^{\beta-1} \sum_{j \in S} K_{ij} \left(\frac{K_j^A}{K_j^B}\right) \left(\frac{K_i^A}{K_i^B}\right)^{-\beta} \frac{1}{\overline{B}_i} \gamma_j \quad \Rightarrow$$

$$\widetilde{\gamma}_i = \lambda \sum_{j \in S} F_{ij} \widetilde{\gamma}_i^{\frac{1}{\alpha+\beta-1}}$$

Equilibrium Properties in General Network Models

- **Theorem.** Consider a system of N×K equations:

$$y_{i,k} = \sum_{j=1}^{N} K_{ij,k} \prod_{l=1}^{K} y_{j,l}^{\alpha_{k,j}}, \quad \text{for } i \in \{1, ..., N\}, k \in \{1, ..., K\}$$

Where $K_{ij,k} > 0$ are the given **bilateral spatial linkage** and $\alpha_{k,l}$ are the given **model elastisticities.** Define the non-negative K×K matrix $A = [|\alpha_{kl}|]$. Let $c \equiv [c_k]$ be the K×1 eigenvector of A corresponding to the largest eigenvector of the matrix. From the Perro-Frobenius theorem, $c_k > 0$ and can be nomalized so that $\sum_{k=1}^{K} c_k = 1$. Let $\rho(A)$ be the largest eigenvalue of **A**. If $\rho(A) < 1$,there exists a unique strictly postive solution to the systems of eguation and an iterative producer based on the right hand side of equation from any inital guess of will uniformly converge to the solution.

Define function $T : \mathbb{R}^{N\times K} \to \mathbb{R}^{N\times K}$

Multi dimensional

$$T(\mathbf{x})_{ik} \equiv \log\left(\sum_{j(1,\dots,N)} F_{ij,k} \prod_{l=1}^{K} exp(\alpha_{k,l}x_{j,l})\right)$$

For any two x and y in $\mathbb{R}^{N\times K}_{++}$, define the metric to be:

$$d(x,y) \equiv \sum_{k=1}^{K} c_k \max_i |x_{i.k} - y_{i.k}|$$

$$d(T(\mathbf{x}), T(\mathbf{y})) = \sum_{k=1}^{K} c_k \max_i \left|\log\left(\sum_{j\{1,\dots,N\}} F_{ij,k} \prod_{l=1}^{K} \exp\left(\alpha_{k,l}x_{j,l}\right)\right) - \log\left(\sum_{j\{1,\dots,N\}} F_{ij,k} \prod_{l=1}^{K} \exp(\alpha_{k,l}y_{j,l})\right)\right|$$

$$= \sum_{k=1}^{K} c_k \max_i \left|\log\left(\frac{\sum_{j\{1,\dots,N\}} F_{ij,k} \prod_{l=1}^{K} exp(\alpha_{k,l}x_{j,l})}{\sum_{j\{1,\dots,N\}} F_{ij,k} \prod_{l=1}^{K} exp(\alpha_{k,l}y_{j,l})}\right)\right|$$

$$= \sum_{k=1}^{K} c_k \max_i \left|log\left(\sum_{j\{1,\dots,N\}} \frac{F_{ij,k} \prod_{l=1}^{K} exp(\alpha_{k,l}y_{j,l})}{\sum_{j\{1,\dots,N\}} F_{ij,k} \prod_{l=1}^{K} exp(\alpha_{k,l}y_{j,l})} \frac{F_{ij,k} \prod_{l=1}^{K} exp(\alpha_{k,l}x_{j,l})}{F_{ij,k} \prod_{l=1}^{K} exp(\alpha_{k,l}y_{j,l})}\right)\right|$$

$$= \sum_{k=1}^{K} c_k \max_i \left|log\left(\sum_{j\{1,\dots,N\}} C_{ij,k} \prod_{l=1}^{K} exp(\alpha_{k,l}(x_{j,l} - y_{j,l}))\right)\right|$$

$$d(T(x),T(y)) = \sum_{k=1}^{K} c_k \max_i \left| log\left(\sum_{j\{1,...,N\}} C_{ij,k} \prod_{l=1}^{K} exp(\alpha_{k,l}(x_{j,l} - y_{j,l})) \right) \right|$$

$$\sum_{j(1,...,N)} C_{ij,k} \prod_{l=1}^{K} exp(\alpha_{k,l}(x_{j,l} - y_{j,l})) \leq \sum_{j\{1,...,N\}} C_{ij,k} \prod_{l=1}^{K} exp\left(|\alpha_{k,l}| \max_i |(x_{i,l} - y_{i,l})| \right)$$

$$= \prod_{l=1}^{K} exp\left(|\alpha_{k,l}| \max_i |(x_{i,l} - y_{i,l})| \right)$$

$$\text{d(T(x),T(y))} \leq \sum_{k=1}^{K} \sum_{l=1}^{K} c_k |\alpha_{k,l}| \max_i |(x_{i,l} - y_{i,l})| = \mathbf{m}'\mathbf{A}\mathbf{c} = m'\rho(A)\mathbf{c}$$

$$= \rho(\mathbf{A}) \sum_{l=1}^{K} c_k \max_i |x_{i,l} - y_{i,l}| = \rho(\mathbf{A}) d(\mathbf{x},\mathbf{y})$$

$$\mathbf{m} \equiv \max_i |(x_{i,l} - y_{i,l})|$$

So we have a contraction as long as $\rho(A) < 1$.

If $\rho(A) = 1$, the solution is column-wise up-to-scale unique, i.e. for any $h \in H$ and solutions x and x' it must be $x'_h = c_h x_h$ for some scalar $c_h > 0$.

- Applications

$$\prod_{h'\in H} x_{ih'}^{\gamma_{hh'}} = \sum_{j\in N} K_{ijh} \prod_{h'\in H} x_{ih}^{k_{hh'}} x_{jh'}^{\beta_{hh'}}$$

$$\Downarrow$$

$$\widetilde{x}_{ih} = \sum_{j\in N} K_{ijh} \prod_{h'\in H} \widetilde{x}_{jh'}^{\alpha_{hh'}}$$

$$\widetilde{x}_{ih} = \prod_{h'\in H} x_{ih'}^{\gamma_{hh'}-k_{hh'}} \qquad (B(\Gamma-K)^{-1})_{hh'} = (A)_{hh'} = |\alpha_{hh'}|$$

- Applications

Collatz Wielandt Formula

$$\mathrm{L}_i A_i^{1-\sigma} w_i^{\sigma} = W^{1-\sigma} \sum_{j=1}^{N} T_{ij}^{1-\sigma} \bar{u}_j^{\sigma-1} L_j^{1+\beta(\sigma-1)} w_j^{\sigma}$$

$$\mathrm{L}_i^{\beta(1-\sigma)} w_i^{1-\sigma} = W^{1-\sigma} \sum_{j=1}^{N} T_{ji}^{1-\sigma} \bar{u}_i^{\sigma-1} A_j^{\sigma-1} w_j^{1-\sigma}$$

$$\mathrm{A}_i = \bar{A}_i \sum_{j=1}^{N} K_{ij}^{A} L_j^{\alpha}$$

$$\mathrm{B}\Gamma^{-1} = \begin{pmatrix} 1+\beta(\sigma-1) & \beta\sigma & \sigma-1+\beta(\sigma-1)^2 \\ \beta(\sigma-1)^2 & \beta\sigma(\sigma-1)+1 & \sigma-1+\beta(\sigma-1)^3 \\ \alpha & \frac{\alpha\sigma}{\sigma-1} & \alpha(\sigma-1) \end{pmatrix}$$

$$\alpha+\beta(\sigma-1)(2-\sigma) \leq 0 \quad \frac{\alpha\sigma}{\sigma-1}+\beta\sigma(2-\sigma) \leq 0 \quad \alpha+\beta\sigma(\sigma-1) \leq \frac{1}{\sigma-1}-2$$

Kakutani's Fixed Point Theorem

(角谷静夫不动点定理)

Theorem 2.4. Kakutani's Fixed Point Theorem 1: If $x \to \Phi(x)$ is an upper semi-continuous point-to-set mapping of an r -dimensional closed simplex S into $\Omega(S)$, then there exists an $x_0 \in S$ such that $x_0 \in \Phi\left(x_0\right)$.

- **Proposition 17.C.1 in Mas-Colell et al. (1995)** Existence

Suppose that z(p) is a function defined for all strictly positive vectors $p \in R^L_{++}$, and satisfying following conditions. Then the system of equations $\mathrm{z(p)} = 0$ has a solution:

1. z(.) is continuous.
2. z(.) is homogeneous of degree zero.
3. $\mathrm{p.z(.)} = 0$ for all p
4. There is an $\mathrm{s} > 0$ such that $z_l(p) > -s$, for every commodity and all p.
5. If $p^n \to p$, where $p \neq 0$ and $p_l = 0$,for some l, then

$$\mathrm{Max}\, \mathrm{Z_i}(p^n) \to \infty$$

Step 1: Construct a closed simplex $\Delta = \{p \in R^L_+ : \sum_f p_f = 1\}$

Step 2: Construction a correspondence;

Step 3: Fixed point of the correspondence is an equilibrium;

Step 4: The correspondence is convex and upper hemicontinous.

Step 5: A fixed point exist.

Perron - Frobenius Theorem (佩龙-弗罗贝尼乌斯定理)

equilibrium-equilibrium

Suppose all entries in $A \in R^{n \times n}$ are positive, then:

- Spectral radius $\rho(A) > 0$
- Associated eigenvector is unique up to scale.
- All entries in eigenvector is positive.

Banach's Fixed Point Theorem (巴拿赫不动点定理)

Definition. Let (X, d) be a metric space. A mapping $T : X \to X$ is a **contraction mapping**, or contraction, if there exist a constant c, with $0 \leq c < 1$, such that

$$d(T(x), T(y)) \leq cd(x, y)$$

for all $x, y \in X$.

Theorem (Contraction mapping). If $T : X \to X$ is a contraction mapping on a complete metric space (X, d), then there is exactly one solution $x \in X$ of $T(x) = x$.

Multi-dimensional extension of the Standard Contract Mapping theorem

Standard Contract Mapping theorem

Let $\{(X_h, d_h)\}_{h=1,2,\dots H}$ be H metric spaces where X_h is a set and d_h is its corresponding metric. Define $X \equiv X_1 \times X_2 \times \ldots \times X_n$, and $d : X \times X \to \mathbb{R}_+^H$ such that for $x = (x_1, \ldots, x_H)\,, x' = (x'_1, \ldots, x'_H) \in X \quad , d(x, x') = \begin{pmatrix} d_1(x_1, x'_1) \\ \ldots \\ d_H(x_H, x'_H) \end{pmatrix}$.

Given operator $T : X \to X$, support for any $x, x' \in X$

$$d(T(x), T(x')) \leq A d(x, x')$$

Where A is a non-negative matrix and the inequality is entry-wise. Denote $\rho(A)$ as the spectral radius (largest eigenvalue in absolute value) of A.

If $\rho(A) < 1$ and for all $h = 1, 2, \ldots, H$, (X_h, d_h) is complete, there exists a unique fixed point of T, for any $x \in X$, the sequence of $x, T(x), T(T(x)), \ldots$ converge to the fixed point of T.

- Proposition 17.C.1 in Mas-Colell et al. (1995)

  Suppose that z(p) is a function defined for all strictly positive vectors $p \in R^L_{++}$, and satisfying following conditions. Then the system of equations $\mathrm{z(p)} = 0$ has a solution.

1. z(.) is continuous.
2. z(.) is homogeneous of degree zero.
3. $\mathrm{p.z(.)} = 0$ for all p
4. There is an $\mathrm{s} > 0$ such that $z_l(p) > -s$, for every commodity and all p.
5. If $p^n \to p$, where $p \neq 0$ and $p_l = 0$, for some l, then

$$\mathrm{Max\,Z_i}\,(p^n) \to \infty$$

- Collatz-Wielandt Formula

Applications

If the summation of each row (or column) of A is less than 1, then $\rho(A) \leq 1$.

Reference

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