Lecture 3

Growth Model: Dynamic Optimization in Continuous Time

Macroeconomics EC417

Jane Olmstead-Rumsey

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Growth Model in Continuous Time

- Preferences: representative household with utility function

$$\int_0^\infty e^{-\rho t} u(c(t))dt$$

$\rho \geq 0$ = discount *rate* (as opposed to β = discount *factor*)

- Technology:

$$y(t) = f(k(t)), \quad c(t) + i(t) = y(t)$$
$$\dot{k}(t) = i(t) - \delta k(t), \quad c(t) \geq 0, \quad k(t) \geq 0$$

where $\dot{k}(t) = \frac{\partial k(t)}{\partial t}$

- Endowments: $\hat{k}_0$ of capital at $t = 0$

- Pareto optimal allocation solves

$$V(\hat{k}_0) = \max_{c(t)_{t=0}^\infty} \int_0^\infty e^{-\rho t} u(c(t))dt \quad \text{s.t.}$$
$$\dot{k}(t) = f(k(t)) - \delta k(t) - c(t), \quad k(0) = \hat{k}_0$$

Hamiltonians

- Pretty much all deterministic optimal control problems in continuous time can be written as

$$V\left(\hat{x}_0\right) = \max_{\alpha(t)_{t=0}^{\infty}} \int_0^{\infty} e^{-\rho t} r\left(x\left(t\right), \alpha\left(t\right)\right) dt$$

subject to the law of motion for the state

$$\dot{x}\left(t\right) = g\left(x\left(t\right), \alpha\left(t\right)\right) \text{ and } \alpha\left(t\right) \in A$$

for $t \geq 0$, $x(0) = \hat{x}_0$ given.

- $\rho \geq 0$: discount rate
- $x \in X \subseteq \mathbb{R}^m$: state vector
- $\alpha \in A \subseteq \mathbb{R}^k$: control vector
- $r : X \times A \to \mathbb{R}$: instantaneous return function

Example: Growth Model

$$V\left(\hat{k}_0\right) = \max_{c(t)_{t=0}^{\infty}} \int_0^{\infty} e^{-\rho t} u(c(t)) dt \quad \text{s.t.}$$

$$\dot{k}(t) = f(k(t)) - \delta k(t) - c(t), \quad k(0) = \hat{k}_0$$

- Here the state is $x = k$ and the control $\alpha = c$
- $r(x, \alpha) = u(\alpha)$
- $g(x, \alpha) = f(x) - \delta x - \alpha$

Hamiltonian: General Formulation

- Consider the general optimal control problem two slides back.
- Can obtain necessary and sufficient conditions for an optimum using the following procedure ("cookbook")
- Current-value Hamiltonian

$$\mathcal{H}(x, \alpha, \lambda) = r(x, \alpha) + \lambda g(x, \alpha).$$

- $\lambda \in \mathbb{R}^m$: "co-state"

Hamiltonian: General Formulation

Necessary and sufficient conditions:

$$\mathcal{H}_{\alpha}\left(x\left(t\right),\alpha\left(t\right),\lambda\left(t\right)\right)=0 \tag{1}$$

$$\dot{x}\left(t\right)=g\left(x\left(t\right),\alpha\left(t\right)\right) \tag{2}$$

$$\dot{\lambda}\left(t\right)=\rho\lambda\left(t\right)-\mathcal{H}_{x}\left(x\left(t\right),\alpha\left(t\right),\lambda\left(t\right)\right) \tag{3}$$

for all $t \geq 0$.

- Initial value for state variable(s): $x(0)=\hat{x}_0$.
- Boundary condition (transversality condition) for co-state $\lambda\left(t\right)$

$$\lim_{T\to\infty} e^{-\rho T}\lambda\left(T\right)x\left(T\right)=0.$$

- Note: initial value of the co-state $\lambda\left(0\right)$ not predetermined.

Where Does This Come From?

- Eq. 1 just says that $\alpha(t)$ maximizes $\mathcal{H}$ for all t.
- Eq. 2 is the law of motion for the state.
- Eq. 3 is the law of motion for the co-state. [1]

Interpretation of the law of motion of the co-state (Acemoglu):

- $\lambda(t)$ is the value of the stock $x(t)$.
- $-\dot{\lambda}(t)$ is the depreciation in the value of the stock due to small increase in $x(t)$.
- The value of increasing $x(t)$ is the negative of the RHS of 3.
- Eq. 3 says the two should be equal.

[1] Comes from Pontryagin's Maximum Principle. To see where it comes from, can set up a "Lagrangian" for the entire problem, use integration by parts on the resource constraint, and take FOCs for $\alpha(t)$ *and* $x(t)$.

Example: Neoclassical Growth Model

- Recall: $r(x, \alpha) = u(\alpha)$ and $g(x, \alpha) = f(x) - \delta x - \alpha$
- Using the "cookbook"

$$\mathcal{H}(k, c, \lambda) = u(c) + \lambda[f(k) - \delta k - c]$$

- We have

$$\begin{aligned} \mathcal{H}_c(k, c, \lambda) &= u'(c) - \lambda \\ \mathcal{H}_k(k, c, \lambda) &= \lambda(f'(k) - \delta) \end{aligned}$$

- Therefore conditions for optimum are:

$$\begin{aligned} \dot{\lambda} &= \lambda(\rho + \delta - f'(k)) \\ \dot{k} &= f(k) - \delta k - c \\ u'(c) &= \lambda \end{aligned} \tag{ODE}$$

with $k(0) = \hat{k}_0$ and $\lim_{T\to\infty} e^{-\rho T}\lambda(T)k(T) = 0$.

Example: Neoclassical Growth Model

- Interpretation: continuous time Euler equation
- In discrete time

$$
\begin{aligned}
\lambda_t &= \beta \lambda_{t+1}(f'(k_{t+1}) + 1 - \delta) \\
k_{t+1} &= f(k_t) + (1-\delta)k_t - c_t \\
u'(c_t) &= \lambda_t
\end{aligned}
$$

- (ODE) is continuous-time analogue

Phase Diagrams

- How to analyze (ODE)? In one-dimensional case (scalar x): use phase-diagram
- Two possible phase-diagrams:
  (i) in (λ, k)-space: more general strategy.
  (ii) in (c, k)-space: nicer in terms of the economics.
- For (i), use $u'(c) = \lambda$ or $c = (u')^{-1}(\lambda)$ to write (ODE) as

$$\begin{aligned} \dot{\lambda} &= \lambda(\rho + \delta - f'(k)) \\ \dot{k} &= f(k) - \delta k - (u')^{-1}(\lambda) \end{aligned} \tag{ODE'}$$

  with $k(0) = k_0$ and $\lim_{T\to\infty} e^{-\rho T}\lambda(T)k(T) = 0$.
- Exercise: draw phase-diagram in (λ, k)-space.

Phase Diagrams

- For (ii), assume CRRA utility, not nec. but simplifies algebra:

$$u(c) = \frac{c^{1-\sigma}}{1-\sigma}$$

- To substitute $\dot{\lambda}$ out of ODE take logs and diff. wrt t:

$$c^{-\sigma} = \lambda \quad \Rightarrow \quad -\sigma \log c(t) = \log \lambda(t) \quad \Rightarrow \quad -\sigma \frac{\dot{c}}{c} = \frac{\dot{\lambda}}{\lambda}$$

- Therefore write (ODE) as

$$\begin{aligned} \frac{\dot{c}}{c} &= \frac{1}{\sigma}(f'(k) - \rho - \delta) \\ \dot{k} &= f(k) - \delta k - c \end{aligned} \qquad \text{(ODE”)}$$

with $k(0) = k_0$ and $\lim_{T\to\infty} e^{-\rho T} c(T)^{-\sigma} k(T) = 0$.

Steady State

- In steady state $\dot{k} = \dot{c} = 0$. Therefore

$$f'(k^*) = \rho + \delta$$

$$c^* = f(k^*) - \delta k^*$$

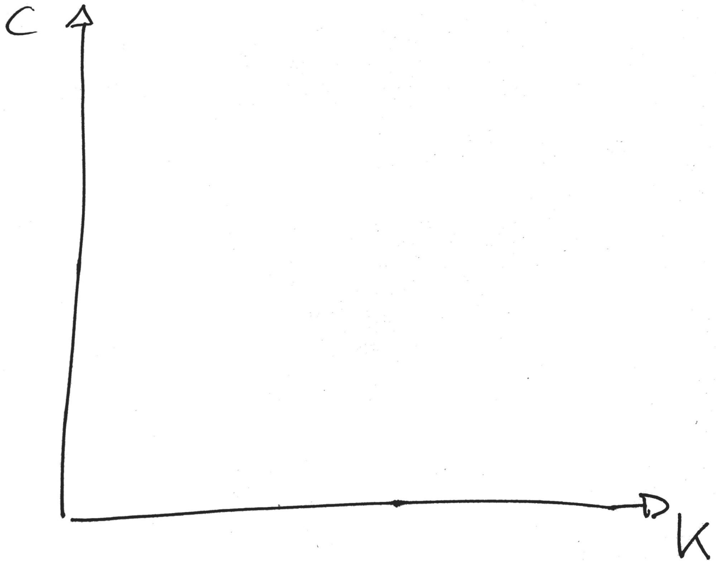
- Same as in discrete time with $\beta = 1/(1+\rho)$.
- For example, if $f(k) = Ak^{\alpha}, \alpha < 1$. Then

$$k^* = \left(\frac{\alpha A}{\rho + \delta}\right)^{\frac{1}{1-\alpha}}$$

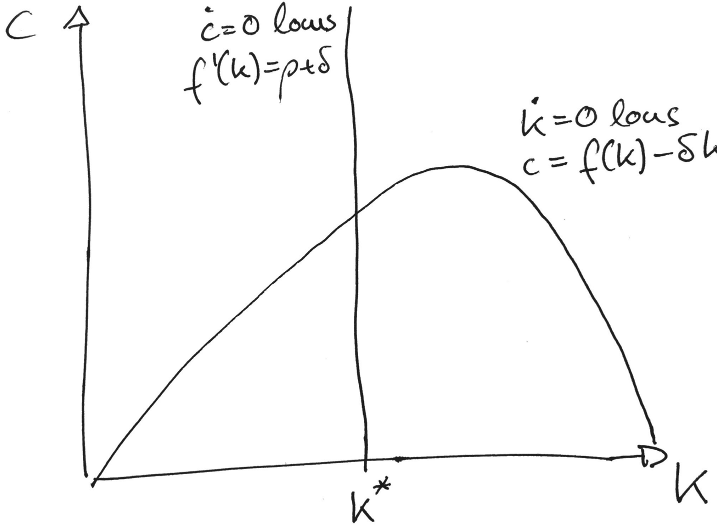
Phase Diagram

- See graph drawn by hand or Figure 8.1 in Acemoglu's textbook
- Steps for drawing phase diagrams:
  1. draw figure in phase plane, here in (c, k) space
  2. draw $\dot{c} = 0$ and $\dot{k} = 0$ loci (loci = plural of locus)
  3. dynamics away from $\dot{c} = 0$ and $\dot{k} = 0$ loci
  4. use loci to split phase plane into areas and find dynamics in these areas using step 3
  5. draw arrows indicating dynamics
  6. draw trajectories for different initial conditions and find saddle path (if there is one)

Step 1: draw figure in phase plane



Step 2: draw $\dot{c} = 0$ and $\dot{k} = 0$ loci



Step 3: dynamics away from $\dot{c} = 0$ and $\dot{k} = 0$ loci

- Recall

$$\frac{\dot{c}}{c} = \frac{1}{\sigma}(f'(k) - \rho - \delta)$$
$$\dot{k} = f(k) - \delta k - c \qquad \text{(ODE'')}$$

  with $k(0) = k_0$ and $\lim_{T\to\infty} e^{-\rho T} c(T)^{-\sigma} k(T) = 0$.

- Dynamics of k away from $\dot{k} = 0$ locus:

$$\dot{k} > 0 \quad \Leftrightarrow \quad f(k) - \delta k - c > 0 \quad \Leftrightarrow \quad c < f(k) - \delta k$$

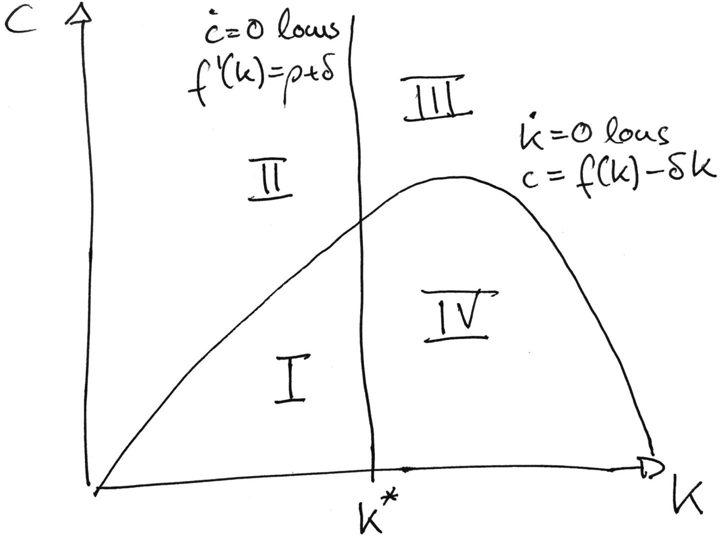
  $\Rightarrow$ $\dot{k} > 0$ below $\dot{k} = 0$ locus and $\dot{k} < 0$ above $\dot{k} = 0$ locus

- Dynamics of c away from $\dot{c} = 0$ locus:

$$\dot{c} > 0 \quad \Leftrightarrow \quad f'(k) - \rho - \delta > 0 \quad \Leftrightarrow \quad f'(k) > \rho + \delta \quad \Leftrightarrow \quad k < k^*$$

  $\Rightarrow$ $\dot{c} > 0$ to left of $\dot{c} = 0$ locus and $\dot{c} < 0$ to right of $\dot{c} = 0$ locus

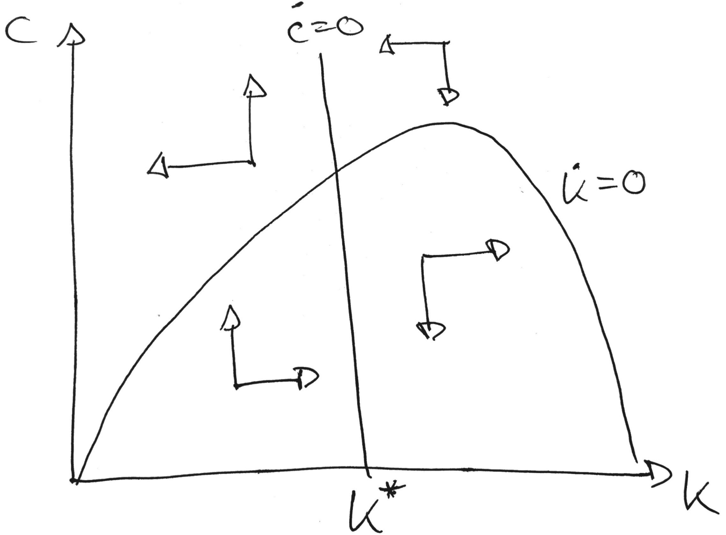
Step 4: use loci to split phase plane into areas ...



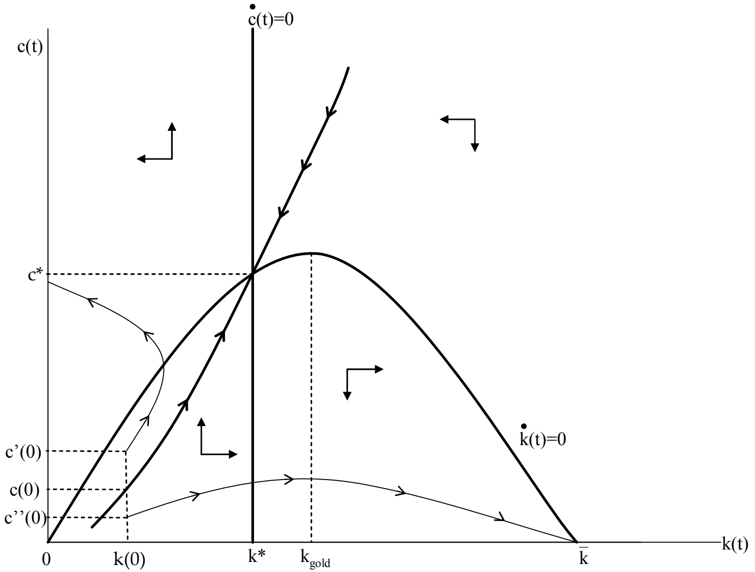
Step 4: ... and find dynamics in areas using step 3

- Area I: left of $\dot{c} = 0$ locus and below $\dot{k} = 0$ locus: $\dot{c} > 0, \dot{k} > 0$
- Area II: left of $\dot{c} = 0$ locus and above $\dot{k} = 0$ locus: $\dot{c} > 0, \dot{k} < 0$
- Area III: right of $\dot{c} = 0$ locus and below $\dot{k} = 0$ locus: $\dot{c} < 0, \dot{k} > 0$
- Area IV: right of $\dot{c} = 0$ locus and above $\dot{k} = 0$ locus: $\dot{c} < 0, \dot{k} < 0$

Step 5: draw arrows indicating dynamics



Step 6: trajectories & saddle path (Acemoglu Fig 8.1)



Phase Diagram

- Obtain saddle path.
- Prove stability of steady state.
- Important: saddle path is not a "knife edge" case in the sense that the system only converges to steady state if $(c(0), k(0))$ happens to lie on the saddle path and diverges for all other initial conditions.
- In contrast to the state variable $k(t)$, $c(t)$ is a "jump variable." That is, $c(0)$ is free and always adjusts so as to lie on the saddle path.

Violations of Transversality Condition

- Question: how do you know that trajectories with $c(0)$ off the saddle path violate the transversality condition?
- See Acemoglu, chapter 8 "The Neoclassical Growth Model" section 5 "Transitional Dynamics"
  - if $c(0)$ below saddle path, $k(t) \to k_{\max}$ and $c(t) \to 0$
  - if $c(0)$ above saddle path, $k(t) \to 0$ in finite time while $c(t) > 0$. Violates feasibility.
  - local analysis/linearization gives same answer. Next lecture.
  - notes that most rigorous and straightforward way is to use the fact that concave problems have unique solution (his Theorem 7.14)

Numerical Solution: Finite-Diff. Methods

- By far the simplest and most transparent method for numerically solving differential equations.
- Approximate $k(t)$ and $c(t)$ at N discrete points in the time dimension, $t^n, n = 1, ..., N$. Denote distance between grid points by Δt.
- Use short-hand notation $k^n = k(t^n)$.
- Approximate derivatives

$$\dot{k}(t^n) \approx \frac{k^{n+1} - k^n}{\Delta t}$$

- Approximate (ODE”) as

$$\frac{c^{n+1} - c^n}{\Delta t} \frac{1}{c^n} = \frac{1}{\sigma}(f'(k^n) - \rho - \delta)$$
$$\frac{k^{n+1} - k^n}{\Delta t} = f(k^n) - \delta k^n - c^n$$

Finite-Diff. Methods/Shooting Algorithm

- Or

$$
\begin{aligned}
c^{n+1} &= \Delta t c^n \frac{1}{\sigma}(f'(k^n) - \rho - \delta) + c^n \\
k^{n+1} &= \Delta t(f(k^n) - \delta k^n - c^n) + k^n
\end{aligned}
\tag{FD}
$$

  with $k^0 = k_0$ given.
- Exercise: draw phase diagram/saddle path in MATLAB.
- Assume $f(k) = Ak^\alpha$, $A = 1$, $\alpha = 0.3$, $\sigma = 2$, $\rho = \delta = 0.05$, $k_0 = \frac{1}{2}k^*$, $\Delta t = 0.1$, $N = 700$.
- Algorithm:
  (i) guess c^0
  (ii) obtain (c^n, k^n), $n = 1, ..., N$ by running (FD) forward in time.
  (iii) If the sequence converges to (c^*, k^*), then you have obtained the correct saddle path. If not, back to (i) and try different c^0.
- This is called a "shooting algorithm"