An Introduction to Quantitative Dynamic Spatial Models

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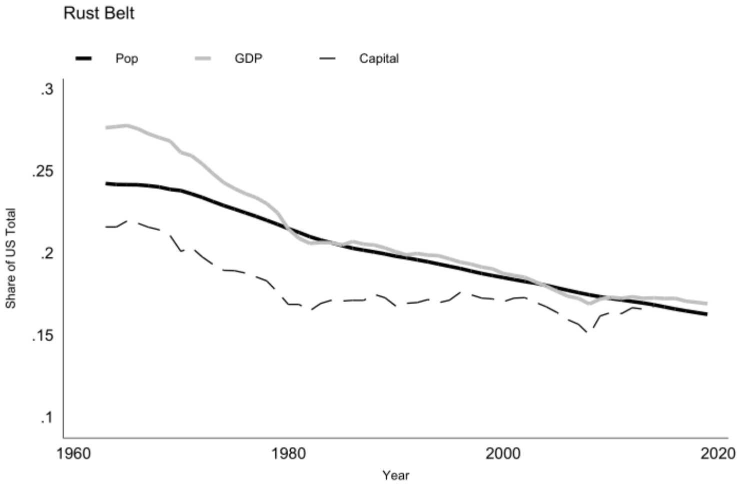
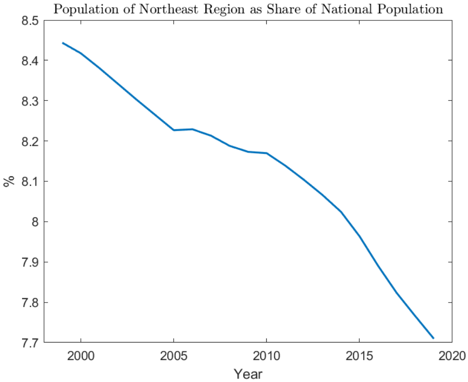
Motivation: accounting for the decline of US Rust Belt



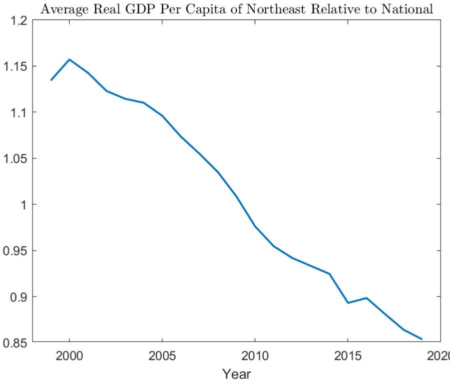
Rust Belt: Illinois, Indiana, Michigan, New York, Ohio, Pennsylvania, West Virginia and Wisconsin

Source: Kleinman, Liu and Redding (2021)

Motivation: accounting for the decline of Northeast China



(a) Population



(b) Real GDP Per Capita

- How much do the following factors account for the decline of Northeast China: (1) reduction in trade costs due to expressway expansion; (2) reduction in domestic migration costs due to Hukou reform
- This lecture guides you to provide an answer to this question

Logistics

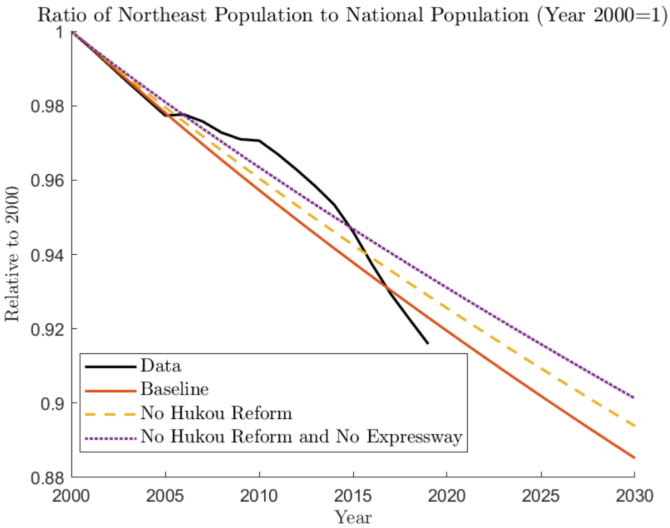
- Instructor: Wenlan Luo, School of Economics and Management, Tsinghua University. luowl@sem.tsinghua.edu.cn
- Slides and codes (in MATLAB): http://www.sem.tsinghua.edu.cn/en/lwl20171009 - Projects or download from:
- Recommended Preparation: Advanced Microeconomics, Multivariate Calculus, Linear Algebra

Outline

- Caliendo, Dvorkin and Parro (2019)
  - A spatial model with dynamic migration decisions
  - Solve the model using Dynamic Hat Algebra[1]
  - Accounting for the decline of Northeast China
- Kleinman, Liu and Redding (2021)
  - Solve the model with log linearization
  - Spectral analysis based on linearized models
- Discussions on the literature and research questions
- Principle: will be pedagogical and detailed on algebra and coding; but do work them out on your own!

[1] Note this nests the exact hat algebra (Caliendo and Parro, 2015) and linearization of static models (Kleinman, Liu and Redding, 2020)

Spoiler: accounting for the decline of the Northeast

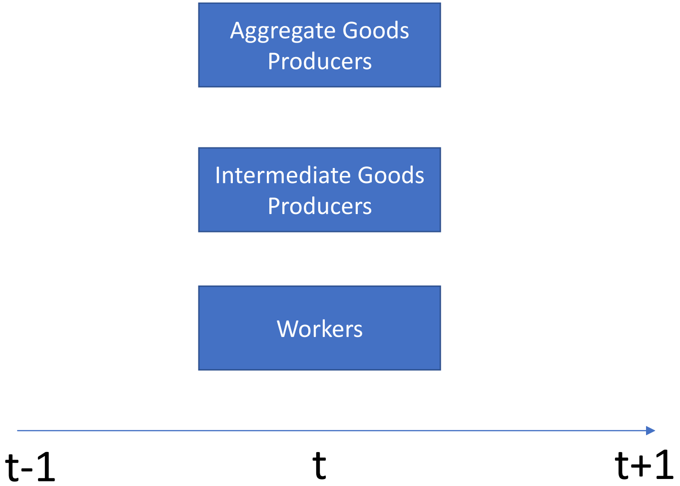


Model

Model overview

- Stripped-down version of Caliendo et al. (2019) taking out (1) land; (2) unemployment; (3) multi sectors (but keep roundabout production). Add shocks to migration costs
- N regions denoted by (n, i). Infinite horizon. All markets competitive
- One sector. Sectoral aggregate goods are non-tradable
- Intermediate goods are tradable and produced by heterogeneous firms combining workers and sectoral aggregate goods
- Sectoral aggregate goods are produced by aggregating varieties of intermediate goods from different regions
- Workers derive utility from aggregate goods produced locally and make dynamic migration decisions subject to idiosyncratic preference shocks

Model overview



Intermediate goods production

- The output of a variety for a producer with efficiency z is given by

$$z(A_t^n l_t^n)^{\gamma^n}(M_t^n)^{\tilde{\gamma}^n},$$

where A_t^n is the labor productivity shifter of region n at time t, l_t^n is the use of labor, M_t^n are the inputs of local sectoral aggregate goods; γ^n and $\tilde{\gamma}^n$ are the labor and material input share: $\gamma^n + \tilde{\gamma}^n = 1$.

- The price of an input bundle is thus

$$x_t^n = B^n(w_t^n)^{\gamma^n}(P_t^n)^{\tilde{\gamma}^n}, \tag{1}$$

with B^n being a sectoral specific constant; w_t^n is the wage and P_t^n is the price of sectoral aggregate goods in region n

- Denote $\kappa_t^{n,i}$ the trade cost from i to n at t. Competitive pricing $\Rightarrow$ the producer with efficiency z at region i selling to region d charges

$$\tilde{p}_t^{ni}(z) = \frac{\kappa_t^{n,i} x_t^i}{z[A_t^i]^{\gamma^i}}$$

Aggregate goods production

- Representative aggregate goods producer at region n combine varieties of intermediate goods from different regions according to

$$Q_t^n = \left(\int_0^1 (q(\nu))^{1-1/\eta^n} \, \mathrm{d}\nu \right)^{\eta^n/(\eta^n-1)},$$

  where $q(\nu)$ is the quantity of variety ν
- For each variety ν, the producer draws one intermediate goods firm from each country and chooses the one with the lowest price
- Denote the $\tilde{q}_t^n(\boldsymbol{z})$ the optimal quantity purchased from the firm with the lowest price given a draw $\boldsymbol{z} = (z^1, z^2, ..., z^N)$
- We can rewrite the integral as

$$Q_t^n = \left(\int (\tilde{q}_t^n(\boldsymbol{z}))^{1-1/\eta^n} \, \mathrm{d}\phi(\boldsymbol{z}) \right)^{\eta^n/(\eta^n-1)},$$

  where $\phi(\boldsymbol{z})$ is the joint distribution function of vector $\boldsymbol{z}$

$$Q_t^n = \left(\int \left(\tilde{q}_t^n \left(\boldsymbol{z} \right) \right)^{1-1/\eta^n} \mathrm{d}\phi \left(\boldsymbol{z} \right) \right)^{\eta^n/(\eta^n-1)}$$

- Assuming $\boldsymbol{z}$ follows i.i.d. Fréchet, i.e., $\phi(\boldsymbol{z}) = \exp(-\sum_{n=1}^{N}(z^n)^{-\theta})$, with θ a parameter, we have the standard results from EK
  - Unit price (marginal cost) of Q_t^n:

$$P_t^n = \Gamma^n \left(\sum_{i=1}^{n} (x_t^i \kappa_t^{n,i})^{-\theta} (A_t^i)^{\theta\gamma^i} \right)^{-1/\theta}, \tag{2}$$

    with Γ^n a constant
  - Trade share (expenditure share $i \to n$):

$$\pi_t^{n,i} = \frac{(x_t^i \kappa_t^{n,i})^{-\theta} (A_t^i)^{\theta\gamma^i}}{\sum_m (x_t^m \kappa_t^{n,m})^{-\theta} (A_t^m)^{\theta\gamma^i}}, \tag{3}$$

    (note the order of the index)

Workers

- Workers are infinitely-lived. They supply labor inelastically and are hand-to-mouth. They derive utility from consuming local aggregate goods. The flow utility:

$$U_t^n = \log(C_t^n),$$
$$s.t. \quad C_t^n = \frac{w_t^n}{P_t^n},$$

  for workers living in region n at time t

- At the end of every period, a worker choose to migrate to a region that gives him the highest utility. The migration decision is described by

$$v_t^n = U_t^n + \max_i \{\beta \mathbb{E}[v_{t+1}^i] - \tau_t^{n,i} - \nu \epsilon_t^i\},$$

  where v_t^n is the discounted life time value, β is the discount factor, $\tau_t^{n,i}$ is the migration cost from n to i, ϵ_t^i is an *idiosyncratic* preference shock, and ν is a parameter; $\mathbb{E}$ is the expectation operator

$$v_t^n = U_t^n + \max_i \{\beta \mathbb{E}[v_{t+1}^i] - \bar{\tau}_t^{n,i} - \nu \epsilon_t^i\}$$

- Assuming ϵ_t^i is i.i.d. drawn over time and across individuals, and follows the standardized Gumbel distribution, i.e., $F(\epsilon) = exp(-exp(-(\epsilon - \gamma_\epsilon)))$. Then we have the aggregation
  - Expected value:

$$V_t^n \equiv \mathbb{E}[v_t^n] = \log\left(\frac{w_t^n}{P_t^n}\right) + \nu \log\left(\sum_{i=1}^{N} \{\exp(\beta V_{t+1}^i - \bar{\tau}_t^{n,i})\}^{1/\nu}\right) \tag{4}$$

  - Out-migration share ($n \rightarrow i$):

$$\mu_t^{n,i} = \frac{\{\exp(\beta V_{t+1}^i - \bar{\tau}_t^{n,i})\}^{1/\nu}}{\sum_{m=1}^{N} \{\exp(\beta V_{t+1}^m - \bar{\tau}_t^{n,m})\}^{1/\nu}} \tag{5}$$

(Note the order of index; discuss on the convention of index order)

Discussions on the Gumbel distribution

- Used in dynamic discrete choice models back to McFadden (1978)
- Same derivation as in your textbook Logit regression model
- $X \sim \text{Frechet} \Rightarrow \log(X) \sim \text{Gumbel}$

Remaining equilibrium conditions

- Denote X_t^n total expenditure on aggregate goods of region n at time t
- Assuming trade balance:

$$X_t^n = w_t^n L_t^n + \underbrace{\tilde{\gamma}^n \overbrace{\sum_{i=1}^{N} \pi_t^{i,n} X_t^i}^{\text{Expenditure on region-}n\text{ intermediate goods}}}_{\text{Expenditure on aggregate goods as material inputs}} \tag{6}$$

(note the order of index in $\pi_t^{i,n}$)

- Labor market clearing

$$L_t^n = \frac{\gamma^n \sum_{i=1}^{N} \pi_t^{i,n} X_t^i}{w_t^n} \tag{7}$$

- Transitions of labor:

$$L_{t+1}^n = \sum_{i=1}^{N} \mu_t^{i,n} L_t^i \tag{8}$$

Definition of equilibrium

- Given *parameters* $\bar{\Theta} \equiv (\theta, \nu, \beta, \eta^n, \gamma^n, \tilde{\gamma}^n)$, time-varying *fundamentals* $\Theta_t \equiv (\boldsymbol{A}_t, \boldsymbol{\kappa}_t, \boldsymbol{\tau}_t)$ and initial labor allocation L_0, a competitive equilibrium is is a time sequence of *variables* $\{L_t, \mu_t, V_t, w_t, P_t, \pi_t, x_t, X_t\}_{t=0}^{\infty}$ that solves *Equations* (1) to (8)

- **Q**: why defining an equilibrium?
  A: eventually you are solving a system of equations; at minimum you need to know what are parameters, the unknowns and the equations

- **Q**: how to define an equilibrium?
  A: find all endogenous variables as unknowns; find the *same number* of equations given by model conditions (optimality; market clearing). definition not unique (e.g., can define based on quantities instead of expenditure here); define in a way that you find convenient to solve

- As a first pass, always conclude the model by defining an equilibrium!

A prelim on dynamic systems

- A dynamic system is in general much harder to solve than a static one
- Usually it involves a coupled backward equation that characterizes the decision problem (Equation 4) and forward equation characterizes the transition of "states" (Equation 5 and 8)
- Not obvious for the infinite-horizon case but should be clear using a finite horizon analogy, there is one missing condition on $\lim_{t\to\infty} \boldsymbol{V}_t$
- Usually given by imposing some stability property of the solution (e.g., assuming the economy converges to a steady state)
- $\boldsymbol{L}_t$ are state variables with values predetermined from last periods, whereas other endogenous variables can "jump" responding to shocks
- Discuss on notions of steady state, transition path, and the assumption of rational expectation

Dynamic Hat Algebra

"Exact hat algebra" - why?

- Usually the model is only meaningful when it's taken to data. This involves finding fundamentals (e.g, Θ_t) under which the equilibrium variables (such as $\pi_t^{n,i}, \mu_t^{n,i}$) match their data counterparts
- This is called "calibration". Calibration in this model is hard as it involves solving many fundamentals matching many targets
- Exact hat algebra addresses this issue by noticing that by writing equilibrium conditions in changes, only the *changes* in fundamentals show up but not their *levels*, saving the hassle of finding levels of Θ_t
  - Some equilibrium variables enter the conditions in changes but they are directly observables
- first hat algebra used for static models: Dekle, Eaton and Kortum (2008); for dynamic models: Caliendo et al. (2019)

"Exact hat algebra" - how?

By recursively applying the following three basic rules.
Define $\hat{Y} = Y'/Y$, for Y' denoting next-period or counterfactual variable

1. Suppose $Y = X^{\theta}$, then
$$\hat{Y} = \hat{X}^{\theta}.$$
2. Suppose $Y = \prod_{i=1}^{N} X_i$, then
$$\hat{Y} = \prod_{i=1}^{N} \hat{X}_i.$$
3. Suppose $Y = \sum_{i=1}^{N} X_i$, then
$$\hat{Y} = \frac{\sum_{i=1}^{N} X_i'}{\sum_{i=1}^{N} X_i} = \sum_{i=1}^{N} \frac{X_i'}{\sum_{m=1}^{N} X_m} = \sum_{i=1}^{N} \frac{X_i}{\sum_{m=1}^{N} X_m} \frac{X_i'}{X_i} = \sum_{i=1}^{N} \pi_i \hat{X}_i,$$
where $\pi_i \equiv \frac{X_i}{\sum_{m=1}^{N} X_m}$.

"Exact hat algebra" - a simple example

Start with Equation (1)

$$x_t^n = B^n (w_t^n)^{\gamma^n} (P_t^n)^{\tilde{\gamma}^n}$$

Denote $\dot{x}_{t+1}^n$ as the change *over time*, i.e., $\dot{x}_{t+1}^n \equiv x_{t+1}^n / x_t^n$. (For change over time, we usually use notation *dot* instead of *hat* variable)

Apply (2) then (1) we have

$$\dot{x}_{t+1}^n = (\dot{w}_{t+1}^n)^{\gamma^n} (\dot{P}_{t+1}^n)^{\tilde{\gamma}^n}$$

"Exact hat algebra" - a harder example

Consider Equation (2)

$$P_t^n = \Gamma^n \left(\sum_{i=1}^{N} (x_t^i \kappa_t^{n,i})^{-\theta} (A_t^i)^{\theta \gamma^i} \right)^{-1/\theta}$$

First, apply rule (1) we have

$$\dot{P}_{t+1}^n = \dot{\Xi}_{t+1}^{-1/\theta},$$

where $\Xi_t \equiv \sum_{i=1}^{N} (x_t^i \kappa_t^{n,i})^{-\theta} (A_t^i)^{\theta \gamma^i}$. Then apply rule (3) we have

$$\dot{\Xi}_{t+1} = \sum_{i=1}^{N} \frac{\xi_t^i}{\sum_{m=1}^{N} \xi_t^m} \dot{\xi}_{t+1}^i,$$

where $\xi_t^i = (x_t^i \kappa_t^{n,i})^{-\theta} (A_t^i)^{\theta \gamma^i}$

Finally, apply rule (2) then (1) we have

$$\dot{\xi}^i_{t+1} = (\dot{x}^i_{t+1}\dot{\kappa}^{n,i}_{t+1})^{-\theta}(\dot{A}^i_{t+1})^{\theta\gamma^i}.$$

To summarize we have

$$\dot{P}^n_{t+1} = \left(\sum_{i=1}^{N} \frac{(x^i_t\kappa^{n,i}_t)^{-\theta}(A^i_t)^{\theta\gamma^i}}{\sum_{m=1}^{N}(x^m_t\kappa^{n,m}_t)^{-\theta}(A^m_t)^{\theta\gamma^i}} \times (\dot{x}^i_{t+1}\dot{\kappa}^{n,i}_{t+1})^{-\theta}(\dot{A}^i_{t+1})^{\theta\gamma^i}\right)^{-1/\theta},$$

but $\frac{(x^i_t\kappa^{n,i}_t)^{-\theta}(A^i_t)^{\theta\gamma^i}}{\sum_{m=1}^{N}(x^m_t\kappa^{n,m}_t)^{-\theta}(A^m_t)^{\theta\gamma^i}} = \pi^{n,i}_t$ so we have

$$\dot{P}^n_{t+1} = \left(\sum_{i=1}^{N} \pi^{n,i}_t \times (\dot{x}^i_{t+1}\dot{\kappa}^{n,i}_{t+1})^{-\theta}(\dot{A}^i_{t+1})^{\theta\gamma^i}\right)^{-1/\theta}.$$

Observation: (1) $\dot{\boldsymbol{A}}_{t+1}$ enters the system but not its levels; (2) $\pi^{n,i}_t$ enters the system but are observables

"Exact hat algebra" - on dynamic equations

Consider Equation (4)

$$V_t^n = \log\left(\frac{w_t^n}{P_t^n}\right) + \nu \log\left(\sum_{i=1}^{N}\{\exp(\beta V_{t+1}^i - \bar{\tau}_t^{n,i})\}^{1/\nu}\right)$$

Denote $u_t^n \equiv \exp(V_t^n)$ and $\tau_t^{n,i} \equiv \exp(\bar{\tau}_t^{n,i})$ we have

$$u_t^n = \left(\frac{w_t^n}{P_t^n}\right)\left(\sum_{i=1}^{N}(u_{t+1}^i)^{\frac{\beta}{\nu}}(\tau_t^{n,i})^{-\frac{1}{\nu}}\right)^{\nu}.$$

Apply the rules iteratively and recall $\mu_t^{n,i} = \frac{(u_{t+1}^i)^{\frac{\beta}{\nu}}(\tau_t^{n,i})^{-\frac{1}{\nu}}}{\sum_{i=1}^{N}(u_{t+1}^m)^{\frac{\beta}{\nu}}(\tau_t^{n,m})^{-\frac{1}{\nu}}}$ we have

$$\dot{u}_{t+1}^n = \frac{\dot{w}_{t+1}^n}{\dot{P}_{t+1}^n}\left(\sum_{i=1}^{N}\mu_t^{n,i}(\dot{u}_{t+2}^i)^{\frac{\beta}{\nu}}(\dot{\tau}_{t+1}^{n,i})^{-\frac{1}{\nu}}\right)^{\nu}.$$

Observation: again $\dot{\boldsymbol{\tau}}_{t+1}$ enter the system but not its levels

All equilibrium conditions in change over time

We have transformed the original system into one solving for $\{\dot{w}_t, \dot{P}_t, \pi_t, \dot{x}_t, X_t, \dot{u}_t, \mu_t, L_t\}_{t=0}^{\infty}$:

$$\dot{x}_{t+1}^n = (\dot{w}_{t+1}^n)^{\gamma^n} (\dot{P}_{t+1}^n)^{\tilde{\gamma}^n} \tag{9a}$$

$$\dot{P}_{t+1}^n = \left(\sum_{i=1}^{N} \pi_t^{n,i} \times (\dot{x}_{t+1}^i \dot{\kappa}_{t+1}^{n,i})^{-\theta} (\dot{A}_{t+1}^i)^{\theta\gamma^i} \right)^{-1/\theta} \tag{9b}$$

$$\frac{\pi_{t+1}^{n,i}}{\pi_t^{n,i}} = \left(\frac{\dot{x}_{t+1}^i \dot{\kappa}_{t+1}^{n,i}}{\dot{P}_{t+1}^n} \right)^{-\theta} (\dot{A}_{t+1}^i)^{\theta\gamma^i} \tag{9c}$$

$$X_{t+1}^n = w_t^n L_t^n \dot{w}_{t+1}^n \dot{L}_{t+1}^n + \tilde{\gamma}^n \sum_{i=1}^{N} \pi_{t+1}^{i,n} X_{t+1}^i \tag{9d}$$

$$w_t^n L_t^n \dot{w}_{t+1}^n \dot{L}_{t+1}^n = \gamma^n \sum_{i=1}^{N} \pi_{t+1}^{i,n} X_{t+1}^i . \tag{9e}$$

$$\dot{u}^n_{t+1} = \frac{\dot{w}^n_{t+1}}{\dot{P}^n_{t+1}} \left(\sum_{i=1}^{N} \mu^{n,i}_t (\dot{u}^i_{t+2})^{\frac{\beta}{\nu}} (\dot{\tau}^{n,i}_{t+1})^{-\frac{1}{\nu}} \right)^{\nu}, \tag{10a}$$

$$\mu^{n,i}_{t+1} = \frac{\mu^{n,i}_t (\dot{u}^i_{t+2})^{\frac{\beta}{\nu}} (\dot{\tau}^{n,i}_{t+1})^{-\frac{1}{\nu}}}{\sum_{m=1}^{N} \mu^{n,m}_t (\dot{u}^m_{t+2})^{\frac{\beta}{\nu}} (\dot{\tau}^{n,m}_{t+1})^{-\frac{1}{\nu}}}, \tag{10b}$$

$$L^n_{t+1} = \sum_{i=1}^{N} \mu^{i,n}_t L^i_t. \tag{10c}$$

- **Q**: what are we solving?
  A: a *transition path* given fundamentals that include their *initial conditions* and *anticipated changes*
- **Q**: how did their changes enter the system?
  A: captured by their over-time changes $\{\dot{\boldsymbol{A}}_{t+1}, \dot{\boldsymbol{\kappa}}_{t+1}, \dot{\boldsymbol{\tau}}_{t+1}\}$
- **Q**: how did their initial conditions enter the system?
  A: captured by *observables* at the initial period $\{\boldsymbol{\pi}_0, \boldsymbol{\mu}_{-1}, \boldsymbol{X}_0, \boldsymbol{L}_0\}$
- This also makes clear the data requirements for solving the system

Designing an algorithm in solving the transition path

- One could ask the computer to solve the large system directly
- You can do better by partitioning the system into blocks, as you know better about the "sparsity" of the problem than the computer
- **Observation 1**: given $\dot{\boldsymbol{\Theta}}_{t+1}$, $\dot{\boldsymbol{L}}_{t+1}$ and $(\boldsymbol{w}_t, \boldsymbol{\pi}_t, \boldsymbol{L}_t)$ solved from the t-th time step, (9a)-(9e) are a system of equations for $(\dot{w}_{t+1}, \dot{P}_{t+1}, \pi_{t+1}, \dot{x}_{t+1}, X_{t+1})$ at the $t+1$-th time step
  - i.e., the linkage between the dynamic and static parts is via $\dot{\boldsymbol{L}}_{t+1}$
- **Observation 2**: the solution to (9a)-(9e) enters the system (10a)-(10c) via $\frac{\dot{w}^n_{t+1}}{\dot{P}^n_{t+1}}$
- These observations about the sparsity motivate a nested algorithm, with the inner nest solving (9a)-(9e) and outer nest solving (10a)-(10c)

Algorithm 1 Shooting algorithm for solving the transition path in changes

1 Choose T large enough. Guess $\{\dot{u}^n_{t+1,old}\}_{t=0}^T$
2 Set flag_converged to **false**
while flag_converged is false **do**
    3 Solve out the sequence of $\{\mu^{n,i}_{t+1}\}_{t=-1}^T$,$\{L^n_{t+1}\}_{t=0}^T$ according to (10b) and (10c) based on $\{\dot{u}^n_{t+1,old}\}_{t=0}^T$. Construct $\{\dot{L}^n_{t+1}\}_{t=0}^T$.
    4 Solve the temporary equilibrium characterized by (9a)-(9e) and construct $\dot{w}_{t+1}/\dot{P}_{t+1}$.
    5 Construct $\{\dot{u}^n_{t+1}\}_{t=0}^T$ backwardly according to (10a) assuming $\dot{u}^n_{T+1} = 1$.
    6 Set flag_converged to **true** if distance between $\{\dot{u}^n_{t+1,old}\}_{t=0}^T$ and $\{\dot{u}^n_{t+1}\}_{t=0}^T$ is small enough
    7 Update $\{\dot{u}^n_{t+1,old}\}_{t=0}^T$ according to $\{\dot{u}^n_{t+1}\}_{t=0}^T$
end while
8 Check if $\dot{u}^n_T$ is close to one. If not, increase T and go to Step 1.

Toward an efficient algorithm in solving the static problem

The static problem given for $(\dot{w}_{t+1}, \dot{P}_{t+1}, \pi_{t+1}, \dot{x}_{t+1}, X_{t+1})$ restated here:

$$\dot{x}^n_{t+1} = (\dot{w}^n_{t+1})^{\gamma^n} (\dot{P}^n_{t+1})^{\tilde{\gamma}^n} \tag{9a}$$

$$\dot{P}^n_{t+1} = \left(\sum_{i=1}^{N} \pi^{n,i}_t \times (\dot{x}^i_{t+1} \dot{\kappa}^{n,i}_{t+1})^{-\theta} (\dot{A}^i_{t+1})^{\theta\gamma^i} \right)^{-1/\theta} \tag{9b}$$

$$\frac{\pi^{n,i}_{t+1}}{\pi^{n,i}_t} = \left(\frac{\dot{x}^i_{t+1} \dot{\kappa}^{n,i}_{t+1}}{\dot{P}^n_{t+1}} \right)^{-\theta} (\dot{A}^i_{t+1})^{\theta\gamma^i} \tag{9c}$$

$$X^n_{t+1} = w^n_t L^n_t \dot{w}^n_{t+1} \dot{L}^n_{t+1} + \tilde{\gamma}^n \sum_{i=1}^{N} \pi^{i,n}_{t+1} X^i_{t+1} \tag{9d}$$

$$w^n_t L^n_t \dot{w}^n_{t+1} \dot{L}^n_{t+1} = \gamma^n \sum_{i=1}^{N} \pi^{i,n}_{t+1} X^i_{t+1}. \tag{9e}$$

Observ.: given others, (9d) and (9e) form a *linear* system for $(X^n_{t+1}, \dot{w}^n_{t+1})$

Algorithm 2 Algorithm for solving the temporary equilibrium (9a)-(9e)

1 Guess $(\dot{w}^n_{t+1,old}, \dot{P}^n_{t+1,old})$
2 Set flag_converged to **false**
while flag_converged is false **do**
  3 Construct $\pi^{n,i}_{t+1}$ according to (9a),(9b),(9c) based on $(\dot{w}^n_{t+1,old}, \dot{P}^n_{t+1,old})$
  4 Solve the system of linear equations (9d) and (9e) for $(\dot{w}^n_{t+1}, X^n_{t+1})$ (with GPUs)[2]
  5 Construct $\dot{P}^n_{t+1}$ according to (9b)
  6 Set flag_converged to **true** if distance between $(\dot{w}^n_{t+1,old}, \dot{P}^n_{t+1,old})$ and $(\dot{w}^n_{t+1}, \dot{P}^n_{t+1})$ is small enough
  7 Update $(\dot{w}^n_{t+1,old}, \dot{P}^n_{t+1,old})$ according to $(\dot{w}^n_{t+1}, \dot{P}^n_{t+1})$
end while

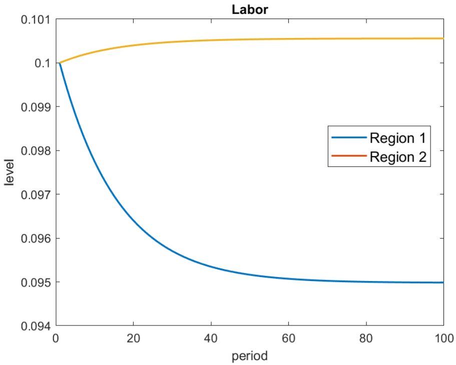
[2] Pay attention to how to do normalization

Implementations for solving the transition path in changes

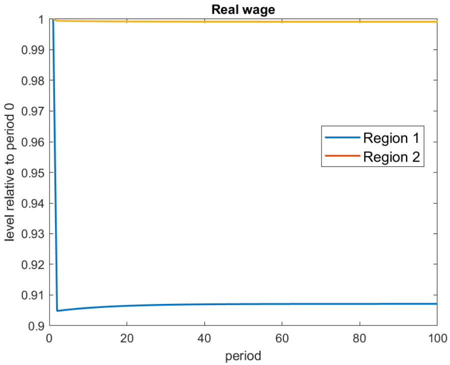
- Information (data) required: $\{\boldsymbol{\pi}_0, \boldsymbol{\mu}_{-1}, \boldsymbol{X}_0, \boldsymbol{L}_0\}$, $\{\dot{\boldsymbol{A}}_{t+1}, \dot{\boldsymbol{\kappa}}_{t+1}, \dot{\boldsymbol{\tau}}_{t+1}\}$
- `test_baseline.m` loads a test dataset and solves the transition path by calling `solve_trans.m`
- `solve_trans.m` implements Algorithm 1 which involves calling `solve_temporary_eq` that solves the static problem[3]
- By default it sets symmetric initial matrix and no shocks, so the economy is initially at a steady state
- Play around by modifying the shock to see more dynamic actions (the code provides an example)

[3] Setting $\dot{\boldsymbol{L}}_t = 0$ in calling the procedure nests the exact hat algebra for static models

Example: transition path after an anticipated change in fundamentals



(a) Population



(b) Real wage

An *anticipated* change in fundamentals that lowers productivity of Region 1 by 10% at period 1

Recap and motivating the Dynamic Hat Algebra

- So far we have solved the transition path given certain *initial fundamentals* with potentially time-varying fundamentals

- Note this is NOT about the counterfactual question we are interested
  - e.g, what would be the dynamics if there were no expressway expansion or Hukou policy reform

- Answer the counterfactual question is essentially comparing two transition-path equilibria with two difference paths of shocks

- (**Important**) But the two transition paths should be conditional on the same initial fundamentals, not the same initial observables
  - the equilibrium variables at the initial period may change after the counterfactual path of shocks, which are not observable

- The counterfactual initial equilibrium variables that are not observable can be canceled out when we apply the hat algebra to the conditions for *dot* variables

Dynamic Hat Algebra

- **Observ.**: The equilibrium conditions under change over time (i.e., for the dot variables) are also mostly equations involving (1) power, (2) product, (3) summation that the hat algebra rules can directly apply

- Denote $\hat{Y}_{t+1} \equiv \dot{Y}'_{t+1} / \dot{Y}_{t+1}$, where Y'_t denotes the variable under a *counterfactual* equilibrium.
  - In other words, $\hat{Y}_{t+1}$ is the change in $\dot{Y}_{t+1}$ from the baseline to the counterfactual

- Then we apply the hat algebra to the equilibrium conditions for *dot* variables to arrive at equilibrium conditions for *hat* variables

Dynamic Hat Algebra - the simple example again

Starting with the condition

$$\dot{x}^n_{t+1} = (\dot{w}^n_{t+1})^{\gamma^n} (\dot{P}^n_{t+1})^{\tilde{\gamma}^n} \tag{9a}$$

Apply the hat algebra we have

$$\hat{x}^n_{t+1} = (\hat{w}^n_{t+1})^{\gamma^n} (\hat{P}^n_{t+1})^{\tilde{\gamma}^n}$$

Dynamic Hat Algebra - the harder example again

Starting with the condition

$$\dot{P}^n_{t+1} = \left(\sum_{i=1}^{N} \pi^{n,i}_t \times (\dot{x}^i_{t+1} \dot{\kappa}^{n,i}_{t+1})^{-\theta} (\dot{A}^i_{t+1})^{\theta\gamma^i} \right)^{-1/\theta} \qquad (9b)$$

Apply the hat algebra we have

$$\hat{P}^n_{t+1} = \left(\sum_{i=1}^{N} \frac{\pi^{n,i}_t \times (\dot{x}^i_{t+1} \dot{\kappa}^{n,i}_{t+1})^{-\theta} (\dot{A}^i_{t+1})^{\theta\gamma^i}}{\sum_{m=1}^{N} \pi^{n,m}_t \times (\dot{x}^m_{t+1} \dot{\kappa}^{n,m}_{t+1})^{-\theta} (\dot{A}^m_{t+1})^{\theta\gamma^i}} \frac{{\pi'}^{n,i}_t}{\pi^{n,i}_t} (\hat{x}^i_{t+1} \hat{\kappa}^{n,i}_{t+1})^{-\theta} (\hat{A}^i_{t+1})^{\theta\gamma^i} \right)^{-\frac{1}{\theta}}$$

But from (9c) we have

$$\pi^{n,i}_{t+1} = \frac{\pi^{n,i}_t \times (\dot{x}^i_{t+1} \dot{\kappa}^{n,i}_{t+1})^{-\theta} (\dot{A}^i_{t+1})^{\theta\gamma^i}}{\sum_{m=1}^{N} \pi^{n,m}_t \times (\dot{x}^m_{t+1} \dot{\kappa}^{n,m}_{t+1})^{-\theta} (\dot{A}^m_{t+1})^{\theta\gamma^i}}$$

So we arrive at

$$\hat{P}^n_{t+1} = \left(\sum_{i=1}^{N} \dot{\pi}^{n,i}_{t+1} {\pi'}^{n,i}_t (\hat{x}^i_{t+1} \hat{\kappa}^{n,i}_{t+1})^{-\theta} (\hat{A}^i_{t+1})^{\theta\gamma^i} \right)^{-\frac{1}{\theta}}$$

$$\hat{P}^n_{t+1} = \left(\sum_{i=1}^{N} \dot{\pi}^{n,i}_{t+1} \pi'^{n,i}_t (\hat{x}^i_{t+1} \hat{\kappa}^{n,i}_{t+1})^{-\theta} (\hat{A}^i_{t+1})^{\theta\gamma^i} \right)^{-\frac{1}{\theta}}$$

Some observations:

- $\dot{\pi}^{n,i}_{t+1}$ is known either (1) inferred a baseline transition path solved (by assuming a path of $\dot{\Theta}_{t+1}$), or (2) $\{\pi^{n,i}_{t+1}\}_{t=0}^{\infty}$ are directly observed
- Only $\hat{\boldsymbol{\Theta}}_{t+1}$ needs to be observed, but not $\dot{\boldsymbol{\Theta}}_{t+1}$: for counterfactual questions like following we do not even need to know the underlying $\dot{\boldsymbol{\Theta}}_{t+1}$ that generates the factual economy:
  - What would happen if productivities/trade costs increase by xx%?
- Note to do this, one does need extra data (such as $\{\pi^{n,i}_{t+1}\}_{t=0}^{\infty}$)
- Key: $\dot{\boldsymbol{\Theta}}_{t+1}$ are implicitly inferred from the time series of observables of the factual economy

All equilibrium conditions under the Dynamic Hat Algebra

Equilibrium conditions for unknowns $\{\hat{w}_t, \hat{P}_t, \pi'_t, \hat{x}_t, X'_t, \hat{u}_t, \hat{\mu}_t, L'_t\}_{t=0}^{\infty}$:

$$\hat{x}^n_{t+1} = (\hat{w}^n_{t+1})^{\gamma^n} (\hat{P}^n_{t+1})^{\tilde{\gamma}^n} \tag{11a}$$

$$\hat{P}^n_{t+1} = \left(\sum_{i=1}^{N} \dot{\pi}^{n,i}_{t+1} \pi'^{n,i}_t (\hat{x}^i_{t+1} \hat{\kappa}^{n,i}_{t+1})^{-\theta} (\hat{A}^i_{t+1})^{\theta\gamma^i} \right)^{-\frac{1}{\theta}} \tag{11b}$$

$$\pi'^{n,i}_{t+1} = \pi'^{n,i}_t \pi^{n,i}_{t+1} \left(\frac{\hat{x}^i_{t+1} \hat{\kappa}^{n,i}_{t+1}}{\hat{P}^n_{t+1}} \right)^{-\theta} (\hat{A}^i_{t+1})^{\theta\gamma^i} \tag{11c}$$

$$X'^n_{t+1} = w^n_{t+1} \hat{L}^n_{t+1} w'^n_t L'^n_t \dot{w}^n_{t+1} \dot{L}^n_{t+1} + \tilde{\gamma}^n \sum_{i=1}^{N} \pi'^{i,n}_{t+1} X'^i_{t+1} \tag{11d}$$

$$\hat{w}^n_{t+1} \hat{L}^n_{t+1} w'^n_t L'^n_t \dot{w}^n_{t+1} \dot{L}^n_{t+1} = \gamma^n \sum_{i=1}^{N} \pi'^{i,n}_{t+1} X'^i_{t+1}. \tag{11e}$$

$$\hat{u}^n_{t+1} = \frac{\hat{w}^n_{t+1}}{\hat{P}^n_{t+1}} \left(\sum_{i=1}^{N} \mu'^{n,i}_t \dot{\mu}^{n,i}_{t+1} (\hat{u}^i_{t+2})^{\frac{\beta}{\nu}} (\hat{\tau}^{n,i}_{t+1})^{-\frac{1}{\nu}} \right)^{\nu}, \tag{12a}$$

$$\mu'^{n,i}_{t+1} = \frac{\mu'^{n,i}_t \dot{\mu}^{n,i}_{t+1} (\hat{u}^i_{t+2})^{\frac{\beta}{\nu}} (\hat{\tau}^{n,i}_{t+1})^{-\frac{1}{\nu}}}{\sum_{m=1}^{N} \mu'^{n,m}_t \dot{\mu}^{n,m}_{t+1} (\hat{u}^m_{t+2})^{\frac{\beta}{\nu}} (\hat{\tau}^{n,m}_{t+1})^{-\frac{1}{\nu}}}, \tag{12b}$$

$$L'^n_{t+1} = \sum_{i=1}^{N} \mu'^{i,n}_t L'^i_t. \tag{12c}$$

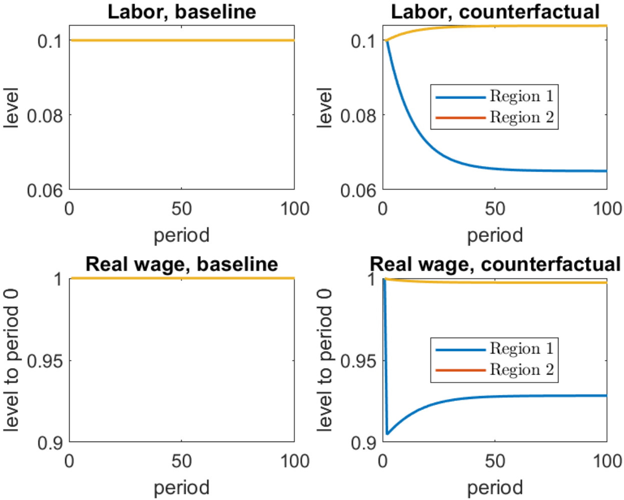
Algorithm for solving the counterfactual in changes

- Again, one can design a nested algorithm: solving (11a)-(11e) in the inner nest given $\{\hat{L}^n_{t+1}\}$, and solving (12a)-(12c) in the outer nest
- Note that all *dot* variables are unknown after you have solved the factual transition path or directly observe them in the data

Implementation of the dynamic hat algebra

- Information (data) required: $\{\dot{\boldsymbol{\pi}}_t, \dot{\boldsymbol{\mu}}_t, \dot{\boldsymbol{X}}_t, \dot{\boldsymbol{L}}_t\}$, $\{\hat{\boldsymbol{A}}_{t+1}, \hat{\boldsymbol{\kappa}}_{t+1}, \hat{\boldsymbol{\tau}}_{t+1}\}$
- We assume $\{\dot{\boldsymbol{\pi}}_t, \dot{\boldsymbol{\mu}}_t, \dot{\boldsymbol{X}}_t, \dot{\boldsymbol{L}}_t\}$ coming from the transition path of the factual economy
- `test_cf.m` loads a test dataset, solves the baseline transition path by calling `solve_trans.m`, and then solves the counterfactual transition by calling `solve_cf_trans.m`
- Example in the code: the initial baseline is already at the steady state and the counter-factual is an unexpected shock at period 1 that lowers productivity of region 1 by 10% (i.e., $\hat{A}_1^1 = 0.9$)
- (**Important**) How is "setting $\dot{A}_1^1 = 0.9$ and solving a baseline transition" different from the current example?

Example: transition path after a shock to fundamentals



Shock that lowers productivity of Region 1 by 10% at period 1

Accounting for the Decline of Northeast China

A short summary on empiricalc components

- Need to determine parameters $\bar{\Theta} \equiv (\theta, \nu, \beta, \gamma^n, \tilde{\gamma}^n)$.
  - They come from micro-founded models and are associated with explicit statistics
    - θ: trade elasticity
    - ν: migration elasticity (i.e., how people respond changes in relative wages within a model period)
    - β: discount factor
    - $\gamma^n, \tilde{\gamma}^n$: input shares that can be constructed from input-output tables
  - Estimate your own or cite others' estimates
- Thanks to the hat algebra, we do not need to calibrate the level of Θ_t
  - but need to observe $\{\boldsymbol{\pi}_0, \boldsymbol{\mu}_{-1}, \boldsymbol{X}_0, \boldsymbol{L}_0\}$
  - need to observe $\dot{\Theta}_t$ if we rely on $\{\dot{\boldsymbol{\pi}}_t, \dot{\boldsymbol{\mu}}_t, \dot{\boldsymbol{X}}_t, \dot{\boldsymbol{L}}_t\}$ from the baseline equilibrium...
  - or need to observe $\{\dot{\boldsymbol{\pi}}_t, \dot{\boldsymbol{\mu}}_t, \dot{\boldsymbol{X}}_t, \dot{\boldsymbol{L}}_t\}$ from the data directly.
  - need to construct $\hat{\Theta}_t$ that corresponds to the counterfactual experiment

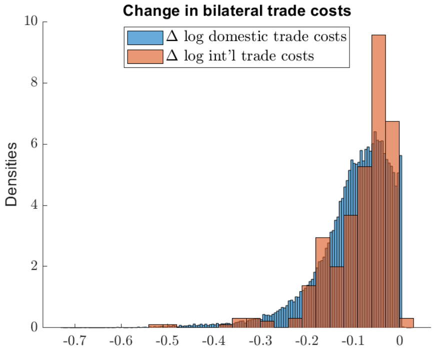
Calibration for the current exercise

- Take estimates of $\bar{\Theta}$ from existing studies

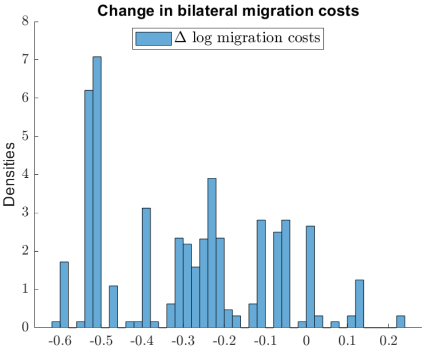
Parameter	Description	Value	Source
θ	Trade elasticity	4	Simonovska and Waugh (2014)
β	Discount factor	0.96	Annual interest rate 4%
ν	Migration elasticity	3β	
γ^n	Labor input share	0.5	Input-output table
$\tilde{\gamma}^n$	Material input share	0.5	Input-output table

- Calibrated at the city level. Set $\dot{\boldsymbol{\tau}}_1$ to the change in domestic migration costs due to the Hukou reform estimated by Fan (2019) (i.e., Hukou reform as a permanent shock that hits unexpectedly at period 1)
- Set $\dot{\boldsymbol{\kappa}}_1$ to the change in domestic and international trade costs due to domestic expressway expansion, estimated by Fan, Lu and Luo (2021)
- Set $\hat{\boldsymbol{\tau}}_1$ and $\hat{\boldsymbol{\kappa}}_1$ to revert the changes
- Set $\{\boldsymbol{\pi}_0, \boldsymbol{\mu}_{-1}, \boldsymbol{X}_0, \boldsymbol{L}_0\}$ inferred from the two models

Inspecting the changes in migration and trade costs

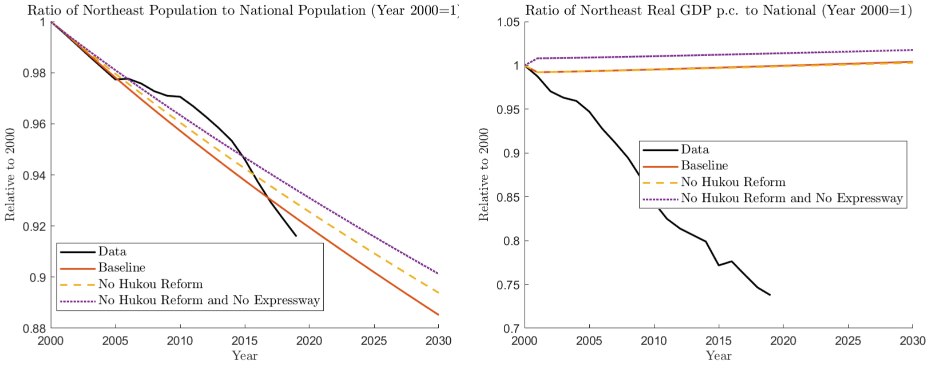


(a) Change in trade costs



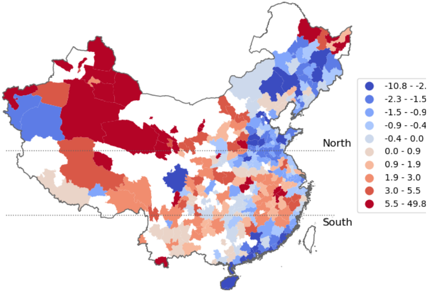
(b) Change in migration costs

Accounting for the change in population and real wage of Northeast China

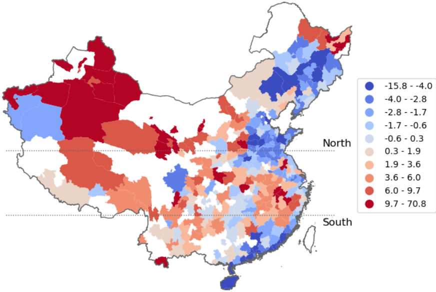


(a) Population (b) Real wage

- baseline accounts well for the dynamics of population but not of income
- Hukou and Expressway Expansion each accounts for $\sim$10% of population decline as of 2019



(a) as of 2010



(b) as of 2020

Number are the difference from the baseline to the counterfactual, in percentage point of baseline population at each period

How far are we from a publishable paper?

Easier part (from easy to hard):

- introduce multi sectors and non-traded goods
- introduce heterogeneous migration costs of skilled/unskilled or young/old labor; different complementarity between labor and fundamental productivity
- more serious quantitative exercise
  - observe *bilateral* trade shares and migration shares period by period; would help discipline the declining speed of the baseline

Harder part:

- introduce spill-over
  - will encounter multiple-equilibrium issue. See Allen and Donaldson (2020) for an example
- introduce capital accumulation
  - examples of shortcuts: Caliendo and Parro (2019), Kleinman et al. (2021); did not capture realistic capital investment, e.g., irreversibility
- introduce consumer saving and risk sharing

Summary of literature on dynamic spatial models

- Static decision problems: Desmet and Rossi-Hansberg (2014), Allen and Donaldson (2020) (decision based on one-period forward, essentially static)
- Dynamic decision problems: Caliendo et al. (2019), Caliendo and Parro (2019), Kleinman et al. (2021)
- Agglomeration: Allen and Donaldson (2020)
- Static trade + dynamic investment: Eaton Kortum, Neiman, and Romalis (2010), Ravikumar, Santacreu, and Sposi (2019), Fan and Luo (2019)
- Macro models with regional linkages: Beraja et al. (2019) (business cycle accounting); Flynn et al. (2021) (fiscal policy; US regions); Devereux et al. (2020) (fiscal and monetary policy interaction; EU)

Homework

- Go over the derivation of the hat and dynamic hat algebra on your own
- Code your own version of solving the static problem, the transition path, and the counterfactual
- Play with the code by changing parameter values β, ν, θ and inspect the transition speed of the dynamics. What do you find?
- Extend the model to multi sectors, and multiple types of labor with heterogeneous migration costs and complementarity
- Calibrate $\dot{\Theta}$ to have a baseline that is closer to the data
- completing the last two will get the paper publishable

Method of Log Linearization

Linearization - why?

- The non-linear model is still slow to solve - e.g., took around 10 minutes for solving the 318-region one sector economy in our example
- Need to do many counterfactuals to open the black box of model mechanisms; consider questions such as
  - among shocks to trade costs of many bilateral city pairs, which shocks are most important in explaining the decline of Northeast
- The model is likely close to (log) linear already:
  - with trade elasticity=1 and appropriate labor market clearing, the model is exactly log-linear
  - estimated trade elasticities are usually small (in the range of 3-12); and trade shocks and changes in trade/migration costs are usually small
- It's been a standard method in macro with a lot of methods ready

Log linearizing equilibrium conditions by hand - why?

- For many models you don't need to linearize them by hand; can do automatically using existing tools
  - e.g., Dynare; any automatic differentiation routine
- But for trade models you do want to linearize them by hand because
  - the system is large (but symmetric) so you want to assist the computer by already writing things into compact coefficient matrices derived by hand
  - these coefficients are usually trade/migration shares that are observables - saving the need for calibrating deep parameters conditional on these observables, similar to the hat algebra
- Why coefficients of linearized system are observable shares?
  - log linearizing market clearing conditions directly gives you equations in shares
  - a micro envelope theorem (Shephard's lemma) applies that associates the marginal effects of productivities to expenditure share, when no distortions
  - recognized long by macro economists, see Baqaee and Farhi (2019) for review

Log linearizing equilibrium conditions by hand - how?

Goal: write equilibrium conditions for *log changes* of variables

How?: First apply the following three rules for total differentiation

1. Suppose $Y = X^{\theta}$, then

$$\mathrm{d}\ln(Y) = \theta \cdot \mathrm{d}\ln(X)$$

2. Suppose $Y = \prod_{i=1}^{N} X_i$, then

$$\mathrm{d}\ln(Y) = \sum_{i=1}^{N} \mathrm{d}\ln(X_i)$$

3. Suppose $Y = \sum_{i=1}^{N} X_i$, then

$$\mathrm{d}\ln(Y) = \sum_{i=1}^{N} \frac{\mathrm{d}X_i}{\sum_{m=1}^{N} X_m} = \sum_{i=1}^{N} \frac{X_i}{\sum_{m=1}^{N} X_m} \mathrm{d}\ln(X_i)$$

Total differentiation - the simple example again

Consider Equation (1)

$$x_t^n = B^n (w_t^n)^{\gamma^n} (P_t^n)^{\tilde{\gamma}^n}$$

Apply rule (2) then rule (1) we have

$$\mathrm{d} \ln x_t^n = \gamma^n \mathrm{d} \ln w_t^n + \tilde{\gamma}^n \mathrm{d} \ln P_t^n$$

Total differentiation - the harder example again

Consider Equation (2)

$$P_t^n = \Gamma^n \left(\sum_{i=1}^{N} (x_t^i \kappa_t^{n,i})^{-\theta} (A_t^i)^{\theta\gamma^i} \right)^{-1/\theta}$$

Apply rule (1), rule (3), then rule (2), then rule (1) in sequence we have

$$\mathrm{d} \ln P_t^n = -\frac{1}{\theta}\Big(\sum_{i=1}^{N} \pi_t^{n,i} \Big[-\theta \mathrm{d} \ln x_t^i - \theta \mathrm{d} \ln \kappa_t^{n,i} + \theta\gamma^i \mathrm{d} \ln A_t^i \Big] \Big)$$

Discuss Shephard's lemma behind this result

Differentiate the expenditure share equation

From Equation (3)

$$\pi_t^{n,i} = \frac{(x_t^i \kappa_t^{n,i})^{-\theta} (A_t^i)^{\theta\gamma^i}}{\sum_m (x_t^m \kappa_t^{n,m})^{-\theta} (A_t^m)^{\theta\gamma^i}} = \left(\frac{x_t^i \kappa_t^{n,i}}{P_t^i}\right)^{-\theta} (A_t^i)^{\theta\gamma^i}$$

Apply rule (1) and rule (2) we have

$$\mathrm{d} \ln \pi_t^{n,i} = -\theta\Big(\mathrm{d} \ln x_t^i + \mathrm{d} \ln \kappa_t^{n,i} - \mathrm{d} \ln P_t^i + \gamma^i \mathrm{d} \ln A_t^i\Big)$$

Differentiate the market clearing condition

From Equation (6) and Equation (7)

$$X_t^n = \sum_{i=1}^{N} \pi_t^{i,n} X_t^i$$

Apply rule (3) then (2) we have

$$\mathrm{d} \ln X_t^n = \sum_{i=1}^{N} \frac{\pi_t^{i,n} X_t^i}{\sum_{m=1}^{N} \pi_t^{m,n} X_t^m} (\mathrm{d} \ln \pi_t^{i,n} + \mathrm{d} \ln X_t^i).$$

Denote $T_t^{n,i} \equiv \frac{\pi_t^{i,n} X_t^i}{\sum_{m=1}^{N} \pi_t^{m,n} X_t^m}$. $T_t^{n,i}$ is the income share of n from i at time t

Q: Why have we reversed the order of index in $T_t^{n,i}$?

Since we have imposed trade balance and there is only one sector, from Equation (7) we have:

$$w_t^n L_t^n = \gamma^n \sum_{i=1}^{N} \pi_t^{i,n} X_t^i = \gamma^n X_t^n$$

and apply rule (2) to the left hand side we have

$$\mathrm{d} \ln w_t^n + \mathrm{d} \ln L_t^n = \mathrm{d} \ln X_t^n$$

Total differentiation - the dynamic equations

Consider the transformed Equation (4)

$$u_t^n = \left(\frac{w_t^n}{P_t^n}\right)\left(\sum_{i=1}^{N}(u_{t+1}^i)^{\frac{\beta}{\nu}}(\tau_t^{n,i})^{-\frac{1}{\nu}}\right)^{\nu}.$$

Apply rule (2), rule (1), rule (3) and rule (2) we have

$$\mathrm{d}\ln u_t^n = \mathrm{d}\ln w_t^n - \mathrm{d}\ln P_t^n + \nu\sum_{i=1}^{N}\mu_t^{n,i}\left(\frac{\beta}{\nu}\mathrm{d}\ln u_{t+1}^i - \frac{1}{\nu}\mathrm{d}\ln \tau_t^{n,i}\right) \qquad (13)$$

Total differentiation - the dynamic equations

Start with Equation (5)

$$\mu_t^{n,i} = \frac{\{\exp(\beta V_{t+1}^i - \bar{\tau}_t^{n,i})\}^{1/\nu}}{\sum_{m=1}^N \{\exp(\beta V_{t+1}^m - \bar{\tau}_t^{n,m})\}^{1/\nu}}$$

Notice

$$\mathrm{d}\ln\left(\sum_{m=1}^N \{\exp(\beta V_{t+1}^m - \tau_t^{n,m})\}^{1/\nu}\right) = \sum_{m=1}^N \mu_t^{n,m}\left(\frac{\beta}{\nu}\mathrm{d}\ln u_{t+1}^m - \frac{1}{\nu}\mathrm{d}\ln \tau_t^{n,m}\right)$$

Therefore,

$$\mathrm{d}\ln \mu_t^{n,i} = \frac{\beta}{\nu}\mathrm{d}\ln u_{t+1}^i - \frac{1}{\nu}\mathrm{d}\ln \tau_t^{n,i} - \frac{1}{\nu}\sum_{m=1}^N \mu_t^{n,m}\left(\beta\mathrm{d}\ln u_{t+1}^m - \mathrm{d}\ln \tau_t^{n,m}\right)$$

Total differentiation - the transitions of labor

Start with Equation (8)

$$L_{t+1}^{n} = \sum_{i=1}^{N} \mu_t^{i,n} L_t^{i}$$

Apply rule (3) and then (2) we have

$$\mathrm{d}\ln L_{t+1}^{n} = \sum_{i=1}^{N} \frac{\mu_t^{i,n} L_t^{i}}{\sum_{m=1}^{N} \mu_t^{m,n} L_t^{m}} (\mathrm{d}\ln \mu_t^{i,n} + \mathrm{d}\ln L_t^{i})$$

Denote $E_t^{n,i} \equiv \frac{\mu_t^{i,n} L_t^{i}}{\sum_{m=1}^{N} \mu_t^{m,n} L_t^{m}}$. $E_t^{n,i}$ is the in-migration share of n from i at time t (recall $\mu_t^{n,i}$ is the out-migration share)

From differentiation to difference

- Differentiation is difference in infinitedesimal steps
- We can use differentiation to approximate difference in larger steps
- Formally, this is through the first order Taylor expansion
- As a reminder, for $y = F(x)$, the first order Taylor expansion states that

$$y - y^* = F_x(x^*)(x - x^*) + o(x - x^*),$$

where $\lim_{x \to x^*} \frac{o(x-x^*)}{x-x^*} = 0$ and $F_x(x^*)$ satisfies

$$dy = F_x(x^*)dx$$

- Apply the approximation to total differentiation derived before, e.g.,

$$\begin{aligned} \mathrm{d}\ln x_t^n &= \gamma^n \mathrm{d}\ln w_t^n + \tilde{\gamma}^n \mathrm{d}\ln P_t^n \\ \Rightarrow \ln x_t^n - \ln x^{*n} &\approx \gamma^n(\ln w_t^n - \ln w^{*n}) + \tilde{\gamma}^n(\mathrm{d}\ln P_t^n - \ln P^{*n}) \end{aligned}$$

- Denote $\tilde{x}_t^n \equiv \ln x_t^n - \ln x_t^{n*}$ the log deviation and replace $\approx$ with $=$:

$$\tilde{x}_t^n = \gamma^n \tilde{w}_t^n + \tilde{\gamma}^n \tilde{P}_t^n$$

- **Q**: Does the choice of the point to linearize $(x^{*n}, w^{*n}, \text{etc.})$ matter?
  A: (1) not necessarily, as we are often interested in the change from the baseline to the counterfactual and the constants would be canceled out
  (2) yes, as some coefficients depend on the point at which linearization is done, e.g., rewrite (13) as

$$\tilde{P}_t^n = -\frac{1}{\theta}\left(\sum_{i=1}^{N} \pi^{*n,i}\left[-\theta\tilde{x}_t^i - \theta\tilde{\kappa}_t^{n,i} + \theta\gamma^i\tilde{A}_t^i\right]\right).$$

  But it turns out these coefficients are going to be replaced by observables

We summarize all conditions in log deviation below

$$
\begin{aligned}
\tilde{x}_t^n &= \gamma^n \tilde{w}_t^n + \tilde{\gamma}^n \tilde{P}_t^n, \\
\tilde{P}_t^n &= \sum_{i=1}^N S^{n,i} \tilde{x}_t^i - \sum_{i=1}^N S^{n,i} \gamma^i \tilde{A}_t^i + \sum_{i=1}^N S^{n,i} \tilde{\kappa}_t^{n,i} \\
\tilde{\pi}_t^{n,i} &= -\theta \tilde{x}_t^i - \theta \tilde{\kappa}_t^{n,i} + \theta \tilde{P}_t^n + \theta \gamma^i \tilde{A}_t^i \\
\tilde{X}_t^n &= \sum_{i=1}^N T^{n,i} (\tilde{\pi}_t^{i,n} + \tilde{X}_t^i), \\
\tilde{w}_t^n + \tilde{L}_t^n &= \sum_{i=1}^N T^{n,i} \tilde{X}_t^n,
\end{aligned}
$$

where $S^{n,i} \equiv \pi^{*n,i}$ is the expenditure share and $T^{n,i} \equiv T^{*n,i}$ is the income share at the point to be linearized

Observation: Given $\tilde{L}_t^n$ and log deviations of shocks, this is a static linear system in $(\tilde{\boldsymbol{x}}_t, \tilde{\boldsymbol{P}}_t, \tilde{\boldsymbol{\pi}}_t, \tilde{\boldsymbol{X}}_t, \tilde{\boldsymbol{w}}_t)$

$$
\begin{aligned}
\tilde{u}_t^n &= \tilde{w}_t^n - \tilde{P}_t^n + \nu \sum_{i=1}^{N} D^{n,i} \Big(\frac{\beta}{\nu} \tilde{u}_{t+1}^i - \frac{1}{\nu} \tilde{\tau}_t^{n,i} \Big), \\
\tilde{\mu}_t^{n,i} &= \frac{\beta}{\nu} \tilde{u}_{t+1}^i - \frac{1}{\nu} \tilde{\tau}_t^{n,i} - \frac{1}{\nu} \sum_{m=1}^{N} D^{n,m} \Big(\beta \tilde{u}_{t+1}^m - \tilde{\tau}_t^{n,m} \Big), \\
\tilde{L}_{t+1}^n &= \sum_{i=1}^{N} E^{n,i} (\tilde{\mu}_t^{i,n} + \tilde{L}_t^i),
\end{aligned}
$$

where $D^{n.i} \equiv \mu^{*n,i}$ is the out-migration share and $E^{n,i} \equiv E^{*n,i}$ is the in-migration share at the point to be linearized

Write the equations in matrix form

- One important advantage/challenge of linearization is to organize the equations in matrix forms
- The first goal is to write the linearized temporary equilibrium conditions into

$$\boldsymbol{\Omega} \begin{pmatrix} \tilde{\boldsymbol{x}}_t \\ \tilde{\boldsymbol{P}}_t \\ \tilde{\boldsymbol{X}}_t \\ \tilde{\boldsymbol{w}}_t \end{pmatrix} = \boldsymbol{\Lambda} \tilde{\boldsymbol{L}}_t + \boldsymbol{\Xi} \tilde{\boldsymbol{f}}_t,$$

  where $\tilde{\boldsymbol{x}}_t = (\tilde{x}_t^1, \tilde{x}_t^2, ..., \tilde{x}_t^N)'$ etc., and $\tilde{\boldsymbol{f}}_t$ is a vector of log deviation of shocks, to be detailed below.

- We are going to find coefficient matrices $\boldsymbol{\Omega}, \boldsymbol{\Lambda}, \boldsymbol{\Xi}$.

Write the equation in matrix form - the simple example

- You already get most of the conversion done. Consider

$$\tilde{x}_t^n = \gamma^n \tilde{w}_t^n + \tilde{\gamma}^n \tilde{P}_t^n$$

In matrix form this is simply

$$\tilde{\boldsymbol{x}}_t = diag(\boldsymbol{\gamma}^n)\tilde{\boldsymbol{w}}_t + diag(\tilde{\gamma}^n)\tilde{\boldsymbol{P}}_t,$$

where $diag(\boldsymbol{\gamma}^n)$ is a diagonal matrix with diagonal element $(\gamma^1, \gamma^2, ..., \gamma^N)$.

Write the equation in matrix form - the difficult part

From

$$\tilde{P}_t^n = \sum_{i=1}^{N} S^{n,i} \tilde{x}_t^i - \sum_{i=1}^{N} S^{n,i} \gamma^i \tilde{A}_t^i + \sum_{i=1}^{N} S^{n,i} \tilde{\kappa}_t^{n,i}$$

we have

$$\tilde{\boldsymbol{P}}_t = \boldsymbol{S} \tilde{\boldsymbol{x}}_t - \boldsymbol{S} diag(\boldsymbol{\gamma}) \tilde{\boldsymbol{A}}_t + \tilde{\boldsymbol{\kappa}}_{in,t},$$

where we define $\tilde{\kappa}_{in,t}^n \equiv \sum_{i=1}^N S^{n,i} \tilde{\kappa}_t^{n,i}$ to be the average log deviation of import costs of region n at time t.

Write the equation in matrix form - the more difficult part

Starting from

$$\tilde{\pi}_t^{n,i} = -\theta\tilde{x}_t^i - \theta\tilde{\kappa}_t^{n,i} + \theta\tilde{P}_t^n + \theta\gamma^i\tilde{A}_t^i$$

$$\tilde{X}_t^n = \sum_{i=1}^N T^{n,i}(\tilde{\pi}_t^{i,n} + \tilde{X}_t^i)$$

Substitute the first line to the second to eliminate $\tilde{\boldsymbol{\pi}}_t$

$$X_t^n = \sum_{i=1}^N T^{n,i}(-\theta\tilde{x}_t^n - \theta\tilde{\kappa}_t^{i,n} + \theta\tilde{P}_t^i + \theta\gamma^n\tilde{A}_t^n + \tilde{X}_t^i)$$

$$= -\theta\tilde{x}_t^n - \theta\sum_{i=1}^N T^{n,i}\tilde{\kappa}_t^{i,n} + \theta\sum_{i=1}^N T^{n,i}\tilde{P}_t^{i,n} + \theta\gamma^n\tilde{A}_t^n + \sum_{i=1}^N T^{n,i}\tilde{X}_t^i$$

Denote $\tilde{\kappa}_{out,t}^n \equiv \sum_{i=1}^N T^{n,i}\tilde{\kappa}_t^{i,n}$ the average deviation of out-migration costs, we have

$$\tilde{\boldsymbol{X}}_t = -\theta\tilde{\boldsymbol{x}}_t - \theta\tilde{\boldsymbol{\kappa}}_{out,t} + \theta\boldsymbol{T}\tilde{\boldsymbol{P}}_t + \theta diag(\boldsymbol{\gamma})\tilde{\boldsymbol{A}}_t + \boldsymbol{T}\tilde{\boldsymbol{X}}_t$$

The temporary equilibrium conditions in matrix forms

In summary, we have

$$\begin{aligned}
\tilde{\boldsymbol{x}}_t &= diag(\gamma)\tilde{\boldsymbol{w}}_t + diag(\tilde{\gamma})\tilde{\boldsymbol{P}}_t,\\
\tilde{\boldsymbol{P}}_t &= \boldsymbol{S}\tilde{\boldsymbol{x}}_t - \boldsymbol{S}diag(\gamma)\tilde{\boldsymbol{A}}_t + \tilde{\boldsymbol{\kappa}}_{in,t},\\
\tilde{\boldsymbol{X}}_t &= -\theta\tilde{\boldsymbol{x}}_t - \theta\tilde{\boldsymbol{\kappa}}_{out,t} + \theta\boldsymbol{T}\tilde{\boldsymbol{P}}_t + \theta diag(\gamma)\tilde{\boldsymbol{A}}_t + \boldsymbol{T}\tilde{\boldsymbol{X}}_t,\\
\tilde{\boldsymbol{w}}_t + \tilde{\boldsymbol{L}}_t &= \tilde{\boldsymbol{X}}_t.
\end{aligned}$$

Denote $\tilde{\boldsymbol{f}}_t = (\tilde{A}_t^1, \tilde{A}_t^2, ..., \tilde{A}_t^n; \tilde{\kappa}_{in,t}^1, ..., \tilde{\kappa}_{in,t}^n; \tilde{\kappa}_{out,t}^1, ..., \tilde{\kappa}_{out,t}^n; \tilde{\boldsymbol{\tau}}_{in,t}; \tilde{\boldsymbol{\tau}}_{out,t})'$, we arrive at

$$\boldsymbol{\Omega}\begin{pmatrix}\tilde{\boldsymbol{x}}_t\\ \tilde{\boldsymbol{P}}_t\\ \tilde{\boldsymbol{X}}_t\\ \tilde{\boldsymbol{w}}_t\end{pmatrix} = \boldsymbol{\Lambda}\tilde{\boldsymbol{L}}_t + \boldsymbol{\Xi}\tilde{\boldsymbol{f}}_t.$$

Solving the temporary equilibrium in matrix forms

- You can simplify the system further (like in Kleinman et al., 2021), but you don't have to
- Just keep in mind the coefficients $\boldsymbol{\Omega}, \boldsymbol{\Gamma}, \boldsymbol{\Xi}$ only contain parameters θ and observables share matrices $\boldsymbol{S}, \boldsymbol{T}; \boldsymbol{D}, \boldsymbol{E}$
- Then with the above information you can already construct the coefficients, and solve the temporary equilibrium as

$$\begin{pmatrix} \tilde{\boldsymbol{x}}_t \\ \tilde{\boldsymbol{P}}_t \\ \tilde{\boldsymbol{X}}_t \\ \tilde{\boldsymbol{w}}_t \end{pmatrix} = \boldsymbol{\Omega}^{-1}\boldsymbol{\Lambda}\tilde{\boldsymbol{L}}_t + \boldsymbol{\Omega}^{-1}\boldsymbol{\Xi}\tilde{\boldsymbol{f}}_t.$$

Notice need to add an normalization equation to $\boldsymbol{\Omega}$ such as $\boldsymbol{e}'\tilde{\boldsymbol{X}}_t = 0$.

Solving the temporary equilibrium - implementation

- **Q**: how to form the large coefficient matrices
- **A**: by partition the matrix into e.g.,

$$\boldsymbol{\Omega} = \begin{pmatrix} \boldsymbol{\Omega}_{1,x}, \boldsymbol{\Omega}_{1,P}, \boldsymbol{\Omega}_{1,X}, \boldsymbol{\Omega}_{1,w} \\ \dots \\ \boldsymbol{\Omega}_{4,x}, \boldsymbol{\Omega}_{4,P}, \boldsymbol{\Omega}_{4,X}, \boldsymbol{\Omega}_{4,w} \end{pmatrix}$$

$$\boldsymbol{\Omega}_{1,x} = \boldsymbol{I}_N, \boldsymbol{\Omega}_{1,P} = -diag(\tilde{\gamma}^n), \boldsymbol{\Omega}_{1,X} = \boldsymbol{0}_N, \boldsymbol{\Omega}_{1,w} = -diag(\gamma^n)$$

corresponding to $\tilde{\boldsymbol{x}}_t = diag(\gamma)\tilde{\boldsymbol{w}}_t + diag(\tilde{\gamma})\tilde{\boldsymbol{P}}_t$.

- A sanity check: the rank of $\boldsymbol{\Omega}$ needs to be $4N - 1$ (why?)
- See the code

The linearized dynamic equations in matrix forms

$$\tilde{u}_t^n = \tilde{w}_t^n - \tilde{P}_t^n + \nu \sum_{i=1}^{N} D^{n,i} \left(\frac{\beta}{\nu} \tilde{u}_{t+1}^i - \frac{1}{\nu} \tilde{\tau}_t^{n,i} \right)$$

In matrix form this is

$$\tilde{\boldsymbol{u}}_t = \tilde{\boldsymbol{w}}_t - \tilde{\boldsymbol{P}}_t + \beta \boldsymbol{D} \tilde{\boldsymbol{u}}_{t+1} - \tilde{\boldsymbol{\tau}}_{out,t},$$

where $\tilde{\tau}_{out,t}^n \equiv \sum_{m=1}^{N} D^{n,m} \mathrm{d} \ln \tau_t^{n,m}$ is the average log deviations of out-migration costs

Starting from

$$\tilde{\mu}_t^{n,i} = \frac{\beta}{\nu}\tilde{u}_{t+1}^{i} - \frac{1}{\nu}\tilde{\tau}_t^{n,i} - \frac{1}{\nu}\sum_{m=1}^{N} D^{n,m}\Big(\beta\tilde{u}_{t+1}^{m} - \tilde{\tau}_t^{n,m}\Big),$$

$$\tilde{L}_{t+1}^{n} = \sum_{i=1}^{N} E^{n,i}(\tilde{\mu}_t^{i,n} + \tilde{L}_t^{i}).$$

Substitute first line in to second to eliminate $\mu_t^{n,i}$

$$
\begin{aligned}
\tilde{L}_{t+1}^{n} &= \sum_{i=1}^{N} E^{n,i}\Big(\frac{\beta}{\nu}\tilde{u}_{t+1}^{n} - \frac{1}{\nu}\tilde{\tau}_{t}^{i,n} - \sum_{m} D^{i,m}\Big(\frac{\beta}{\nu}\mathrm{d}\ln u_{t+1}^{m} - \frac{1}{\nu}\tilde{\tau}_{t}^{i,m}\Big) + \tilde{L}_{t}^{i}\Big) \\
&= \frac{\beta}{\nu}\tilde{u}_{t+1}^{n} - \frac{1}{\nu}\sum_{i=1}^{N} E^{n,i}\tilde{\tau}_{t}^{i,n} - \frac{\beta}{\nu}\sum_{i=1}^{N}\sum_{m=1}^{N} E^{n,i}D^{i,m}\mathrm{d}\ln u_{t+1}^{m} \\
&+ \frac{1}{\nu}\sum_{i=1}^{N} E^{n,i}\sum_{m=1}^{N} D^{i,m}\tilde{\tau}_{t}^{i,m} + \sum_{i=1}^{N} E^{n,i}\tilde{L}_{t}
\end{aligned}
$$

Denote $\tilde{\tau}_{in,t}^{n} \equiv \sum_{i=1}^{N} E^{n,i}\tilde{\tau}_{t}^{i,n}$ the average log deviations of in-migration cost, in matrix form we have

$$
\tilde{\boldsymbol{L}}_{t+1} = \frac{\beta}{\nu}(\boldsymbol{I} - \boldsymbol{E}\boldsymbol{D})\tilde{\boldsymbol{u}}_{t+1} - \frac{1}{\nu}\tilde{\boldsymbol{\tau}}_{in,t} + \frac{1}{\nu}\boldsymbol{E}\tilde{\boldsymbol{\tau}}_{out,t} + \boldsymbol{E}\tilde{\boldsymbol{L}}_{t}.
$$

Form the linear difference equation in $(\tilde{\boldsymbol{L}}_t, \tilde{\boldsymbol{u}}_t)$

- The solution to the temporary equilibrium gives

$$\tilde{\boldsymbol{w}}_t - \tilde{\boldsymbol{P}}_t = \boldsymbol{B}_\omega \tilde{\boldsymbol{L}}_t + \boldsymbol{C}_\omega \tilde{\boldsymbol{f}}_t$$

- Plug this into the dynamic equations in matrix form we have

$$\begin{aligned} \tilde{\boldsymbol{L}}_{t+1} &= \frac{\beta}{\nu}(\boldsymbol{I} - \boldsymbol{E}\boldsymbol{D})\tilde{\boldsymbol{u}}_{t+1} - \frac{1}{\nu}\tilde{\boldsymbol{\tau}}_{in,t} + \frac{1}{\nu}\boldsymbol{E}\tilde{\boldsymbol{\tau}}_{out,t} + \boldsymbol{E}\tilde{\boldsymbol{L}}_t \\ \tilde{\boldsymbol{u}}_t &= \boldsymbol{B}_\omega \tilde{\boldsymbol{L}}_t + \boldsymbol{C}_\omega \tilde{\boldsymbol{f}}_t + \beta \boldsymbol{D}\tilde{\boldsymbol{u}}_{t+1} - \tilde{\boldsymbol{\tau}}_{out,t} \end{aligned}$$

or compactly

$$\underbrace{\begin{pmatrix} \boldsymbol{I}_N, -\frac{\beta}{\nu}(\boldsymbol{I} - \boldsymbol{E}\boldsymbol{D}) \\ \boldsymbol{0}_N, \beta\boldsymbol{D} \end{pmatrix}}_{\boldsymbol{\Gamma}} \begin{pmatrix} \tilde{\boldsymbol{L}}_{t+1} \\ \tilde{\boldsymbol{u}}_{t+1} \end{pmatrix} = \underbrace{\begin{pmatrix} \boldsymbol{E}, \boldsymbol{0}_N \\ -\boldsymbol{B}_\omega, \boldsymbol{I}_N \end{pmatrix}}_{\boldsymbol{\Theta}} \begin{pmatrix} \tilde{\boldsymbol{L}}_t \\ \tilde{\boldsymbol{u}}_t \end{pmatrix} + \boldsymbol{\Psi}\tilde{\boldsymbol{f}}_t$$

Summary ont the linearized system

$$\boldsymbol{\Gamma}\begin{pmatrix}\tilde{\boldsymbol{L}}_{t+1}\\ \tilde{\boldsymbol{u}}_{t+1}\end{pmatrix}=\boldsymbol{\Theta}\begin{pmatrix}\tilde{\boldsymbol{L}}_{t}\\ \tilde{\boldsymbol{u}}_{t}\end{pmatrix}+\boldsymbol{\Psi}\tilde{\boldsymbol{f}}_t$$

- This is a first-order difference equations in $(\tilde{\boldsymbol{L}}_t, \tilde{\boldsymbol{u}}_t)$
- Notice coefficients matrices depend on parameters θ, β, ν and **observable** share matrices $\boldsymbol{S}, \boldsymbol{T}; \boldsymbol{D}, \boldsymbol{E}$ only and can be constructed without knowing the levels of deep parameters such as $\boldsymbol{A}_t, \boldsymbol{\kappa}_t, \boldsymbol{\tau}_t$
- This is where these parameters and share matrices get the name "sufficient statistics" in Kleinman et al. (2021)
- Notice this is approximation equations to the original system. How well do these equations approximate depend on how much these share matrices change over-time or responding to shocks

A prelim on solving linear difference equations

$$\boldsymbol{\Gamma}\begin{pmatrix}\tilde{\boldsymbol{L}}_{t+1}\\ \tilde{\boldsymbol{u}}_{t+1}\end{pmatrix}=\boldsymbol{\Theta}\begin{pmatrix}\tilde{\boldsymbol{L}}_{t}\\ \tilde{\boldsymbol{u}}_{t}\end{pmatrix}+\boldsymbol{\Psi}\tilde{\boldsymbol{f}}_t$$

- There are standard tools in solving the linear difference equations
- Intuitively, there 2*N time-series variables and 2*N equations, so you need 2*N *boundary conditions* to solve the system
  - give a single-variable analogy to establish an intuition of a boundary condition
- N boundary conditions are given by the initial conditions of the state $\tilde{\boldsymbol{L}}_0$
- The remaining N conditions are given by the behavior at the infinite-far future (e.g., $\lim_{t\to 0}\beta^t\tilde{\boldsymbol{u}}_t=0$)
- Discuss briefly on saddle-path stability, eigen values of $\boldsymbol{\Gamma}$, HOD1 of $\boldsymbol{\Gamma}$
- Consult any textbook on advanced macro (e.g., Sargent; Miao)

A practical guide in solving the linear difference equations

$$\boldsymbol{\Gamma}\begin{pmatrix}\tilde{\boldsymbol{L}}_{t+1}\\ \tilde{\boldsymbol{u}}_{t+1}\end{pmatrix}=\boldsymbol{\Theta}\begin{pmatrix}\tilde{\boldsymbol{L}}_{t}\\ \tilde{\boldsymbol{u}}_{t}\end{pmatrix}+\boldsymbol{\Psi}\tilde{\boldsymbol{f}}_t$$

- **Goal**: looking for solutions characterized by

$$\tilde{\boldsymbol{L}}_{t+1}=\boldsymbol{H}\tilde{\boldsymbol{L}}_t+\boldsymbol{R}\tilde{\boldsymbol{f}}_t$$
$$\tilde{\boldsymbol{u}}_{t}=\boldsymbol{H}_u\tilde{\boldsymbol{L}}_t+\boldsymbol{R}_u\tilde{\boldsymbol{f}}_t$$

- **Theorem**: a unique stable solution exists if (i) $\boldsymbol{S}$ and $\boldsymbol{D}$ are strongly connected; (2) $\boldsymbol{S}_{ii}>0$ and $\boldsymbol{D}_{ii}>0$ for all i; (3) $\boldsymbol{S}$ and $\boldsymbol{D}$ are of rank N-1
  - give examples when these conditions are violated
- Well-known methods to solve this: e.g., method of undetermined coefficients (Uhlig, 1999); Chris Sim's `gensys`; Klein's `solab`

- Notice the time paths of $\tilde{\boldsymbol{L}}_t$ can be derived combining initial conditions $\tilde{\boldsymbol{L}}_0$ and paths of shocks $\tilde{\boldsymbol{f}}_t$
- Other variables can be recovered. Recall $\begin{pmatrix} \tilde{\boldsymbol{x}}_t \\ \tilde{\boldsymbol{P}}_t \\ \tilde{\boldsymbol{X}}_t \\ \tilde{\boldsymbol{w}}_t \end{pmatrix} = \boldsymbol{\Omega}^{-1}\boldsymbol{\Lambda}\tilde{\boldsymbol{L}}_t + \boldsymbol{\Omega}^{-1}\boldsymbol{\Xi}\tilde{\boldsymbol{f}}_t$

Solving the log-linearized system - implementations

- `solve_linearized.m` construct the coefficient matrices, solve the temporary equilibrium, and the difference equations
- `test_linearized.m` calls `solve_linearized.m` to solve the model with the test data and compares it with the solutions from dynamic hat algebra
- `main_linearized.m` calls `solve_linearized.m` to solve the model in accounting for the decline of the Northeast

A note on using solab

- Solab gives a state space representation of a stable solution to system

$$\boldsymbol{\Gamma}\begin{pmatrix}\tilde{\boldsymbol{L}}_{t+1}\\ \tilde{\boldsymbol{u}}_{t+1}\end{pmatrix} = \boldsymbol{\Theta}\begin{pmatrix}\tilde{\boldsymbol{L}}_{t}\\ \tilde{\boldsymbol{u}}_{t}\end{pmatrix}$$

- Therefore, first solve for

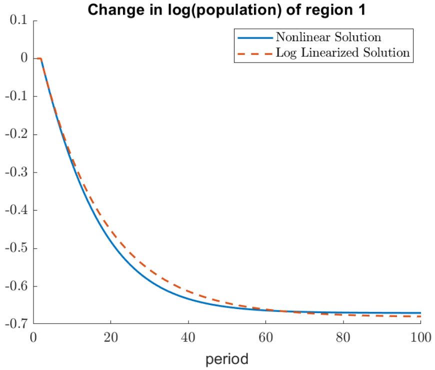
$$\tilde{\boldsymbol{L}}_{t+1} = \boldsymbol{H}\tilde{\boldsymbol{L}}_t + \boldsymbol{R}\tilde{\boldsymbol{f}}_t$$
$$\tilde{\boldsymbol{u}}_{t} = \boldsymbol{H}_u\tilde{\boldsymbol{L}}_t + \boldsymbol{R}_u\tilde{\boldsymbol{f}}_t$$

  with $\tilde{\boldsymbol{f}}_t$ set to zero, and use method of undetermined coefficients to determine $\boldsymbol{R}$ and $\boldsymbol{R}_u$

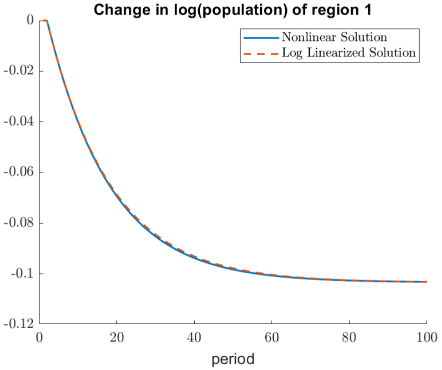
- Specifically, can show

$$\tilde{\Theta}_{u,u}R_u + \Psi_{u,:} = R_u + P_u(\tilde{\Theta}_{L,u}R_u + \Psi_{L,:})$$
$$\Rightarrow R_u = (I + P_u\tilde{\Theta}_{L,u} - \tilde{\Theta}_{u,u})^{-1}(\Psi_{u,:} - P_u\Psi_{L,:})$$
$$\Rightarrow R = \tilde{\Theta}_{L,u}R_u + \Psi_{L,:}$$

How well does the linearized model approximate?



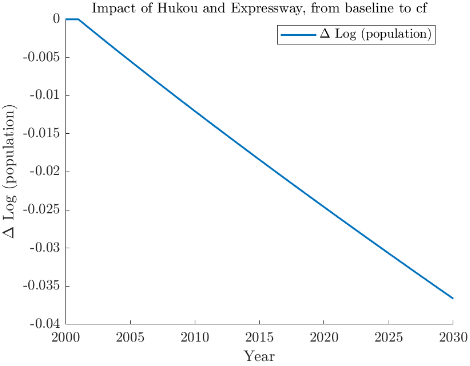
(a) Shock size=50%



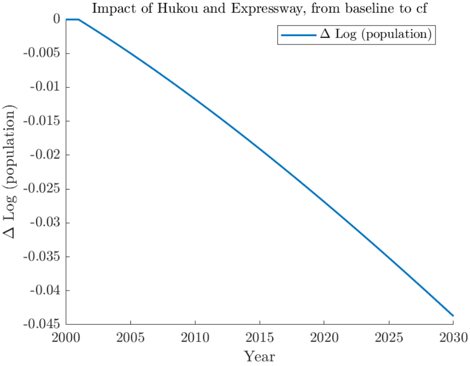
(b) Shock size=10%

The shock that hits at period 1 lowers the productivity of region 1 permanently

Accounting for decline of Northeast with linearized models



(a) Nonlinear solution



(b) Log-linearized solution

The figure plots the change in log population from the baseline (with expressway and Hukou) to the counterfactual (without)

Lessons

- Linearized models are very fast and satisfactorily accurate
- Not a surprise considering all model blocks are sums of log-linear components and the trade/migration elasticities are relatively small
- Suggestions: do not work on models that are close to linear (including the current ones)...

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