Lecture 5

Competitive Equilibrium in the Growth Model

Macroeconomics EC417

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Competitive Eqm in the Growth Model

- There are two issues we are interested in regarding resource allocation problems
  1. efficient allocations
  2. decentralized equilibrium allocations
- So far did (1). Now consider (2).
- Focus on particular decentralized equilibrium concept: competitive equilibrium
  - benchmark notion of decentralized eqm, but not only one

Competitive Eqm (CE) in the Growth Model

- Static economies: one way to formulate CE
- Dynamic economies: three ways to formulate CE
  1. "Arrow-Debreu CE" (ADCE)
  2. "Sequence of Markets CE" (SOMCE)
  3. "Recursive CE" (RCE)
- Outcomes same for all three. Just different representations.
- Begin with ADCE
  - extension of static CE
  - but defining commodities as pairs of goods $\times$ time

Preliminaries: Ownership of Capital

- Detail to consider in economy with capital: who owns capital?
    - households who then rent it to firms?
    - firms who own capital are in turn owned by households?
    - reality: see some of each
- Turns out this is of no substantive importance in this setting
    - lecture: assume capital owned by HH and rented to firms
    - exercise: other extreme
- Also assume single "stand-in" firm
    - exercise: show that this is harmless
- We also go back to discrete-time formulation

Characterizing Competitive Equilibria

- First Welfare Theorem applies (perfect info., perfect comp., complete markets), so could simply use fact that CE allocation = planner's allocation
- But will later consider environments with various distortions (financial frictions, incomplete insurance, ...) in which this fails
- $\Rightarrow$ want to know how to solve for CE even when First Welfare Theorem fails. Consider such method now.
- General idea:
  - max problem for hh's and firms $\Rightarrow$ necessary conditions
  - + market clearing

Arrow-Debreu CE

ADCE

- Definition: An ADCE for the growth model are sequences $\{c_t^h, h_t^h, k_t^h, k_t^f, h_t^f, p_t, w_t, R_t\}_{t=0}^{\infty}$ s.t.
  1. (HH max) Taking $\{p_t, w_t, R_t\}$ as given, $\{c_t^h, h_t^h, k_t^h\}$ solves
  $$\max_{\{c_t^h, h_t^h, k_t^h\}} \sum_{t=0}^{\infty} \beta^t u(c_t) \quad \text{s.t.}$$
  $$\sum_{t=0}^{\infty} p_t (c_t^h + k_{t+1}^h - (1-\delta)k_t^h) \leq \sum_{t=0}^{\infty} (R_t k_t^h + w_t h_t^h)$$
  $$c_t^h \geq 0, \quad 0 \leq h_t^h \leq 1, \quad k_{t+1}^h \geq 0, \quad k_0^h = \bar{k}_0$$
  2. (Firm max) Taking $\{p_t, w_t, R_t\}$ as given, $\{k_t^f, h_t^f\}$ solves
  $$\max_{\{k_t^f, h_t^f\}} \sum_{t=0}^{\infty} (p_t F(k_t^f, h_t^f) - w_t h_t^f - R_t k_t^f) \quad k_t^f \geq 0, \quad h_t^f \geq 0.$$
  3. (Market clearing) For each t:
  $$k_t^h = k_t^f, \quad h_t^h = h_t^f, \quad c_t^h + k_{t+1}^h - (1-\delta)k_t^h = F(k_t^f, h_t^f)$$

Comments

- Single budget constraint for HH
- Prices take care of discounting implicitly
- Everything happens at $t = 0$

Simplifying

- An ADCE are sequences $\{c_t, h_t, k_t, p_t, w_t, R_t\}_{t=0}^{\infty}$ s.t.
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$$\max_{\{c_t, h_t, k_t\}} \sum_{t=0}^{\infty} \beta^t u(c_t) \quad \text{s.t.}$$

$$\sum_{t=0}^{\infty} p_t(c_t + k_{t+1} - (1-\delta)k_t) \leq \sum_{t=0}^{\infty} (R_t k_t + w_t h_t)$$

$$c_t \geq 0, \quad 0 \leq h_t \leq 1, \quad k_{t+1} \geq 0, \quad k_0 = \bar{k}_0$$

  2. (Firm max) Taking $\{p_t, w_t, R_t\}$ as given, $\{k_t, h_t\}$ solves

$$\max_{\{k_t, h_t\}} \sum_{t=0}^{\infty} (p_t F(k_t, h_t) - w_t h_t - R_t k_t) \quad k_t \geq 0, \quad h_t \geq 0.$$

  3. (Market clearing) For each t:

$$c_t + k_{t+1} - (1-\delta)k_t = F(k_t, h_t)$$

  (k and h markets clear implicitly)

Characterizing ADCE

- Necessary conditions for consumer problem ($h_t = 1$ wlog)

$$c_t: \quad \beta^t u'(c_t) = \lambda p_t, \qquad \lambda = \text{multiplier on b.c.} \tag{1}$$

$$k_{t+1}: \quad \lambda p_t + \lambda[-p_{t+1}(1-\delta) - R_{t+1}] = 0 \tag{2}$$

$$\sum_{t=0}^{\infty} p_t(c_t + k_{t+1} - (1-\delta)k_t) \leq \sum_{t=0}^{\infty}(R_t k_t + w_t) \tag{3}$$

$$\text{TVC}: \quad \lim_{T\to\infty} \beta^T u'(c_T) k_{T+1} = 0 \tag{4}$$

$$\text{initial}: \quad k_0 = \bar{k}_0 \tag{5}$$

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- Necessary conditions for firm problem

$$p_t F_k(k_t, h_t) = R_t \tag{6}$$

$$p_t F_h(k_t, h_t) = w_t \tag{7}$$

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- Market clearing

$$c_t + k_{t+1} - (1-\delta)k_t = F(k_t, h_t) \tag{8}$$

Characterizing ADCE: TVC?

- As before, can think of TVC (4) as coming from finite horizon problem

$$\max_{\{c_t, h_t, k_t\}} \sum_{t=0}^{T} \beta^t u(c_t) \quad \text{s.t.}$$

$$\sum_{t=0}^{T} p_t(c_t + k_{t+1} - (1-\delta)k_t) \leq \sum_{t=0}^{T} (R_t k_t + w_t h_t), \quad k_{t+1} \geq 0$$

- Denote multipliers by λ, μ_t, necessary conditions at $t = T$ are

$$\begin{aligned} \beta^T u'(c_T) &= \lambda p_T \\ \lambda p_T &= \mu_T \qquad \Rightarrow \quad \beta^T u'(c_T) k_{T+1} = 0 \\ \mu_T k_{T+1} &= 0 \end{aligned}$$

Characterizing ADCE

- Use (1) at t and $t+1$

$$\frac{u'(c_t)}{\beta u'(c_{t+1})} = \frac{p_t}{p_{t+1}} \tag{9}$$

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$$\begin{aligned} \frac{p_t}{p_{t+1}} &= \frac{R_{t+1}}{p_{t+1}} + 1 - \delta \\ \Rightarrow \frac{u'(c_t)}{\beta u'(c_{t+1})} &= \frac{R_{t+1}}{p_{t+1}} + 1 - \delta \end{aligned}$$

Characterizing ADCE

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- From (2)

$$\frac{p_t}{p_{t+1}} = \frac{R_{t+1}}{p_{t+1}} + 1 - \delta$$
$$\Rightarrow \frac{u'(c_t)}{\beta u'(c_{t+1})} = \frac{R_{t+1}}{p_{t+1}} + 1 - \delta$$

- From (6)

$$\frac{R_t}{p_t} = F_k(k_t, 1) = f'(k_t)$$
$$\Rightarrow \quad \frac{u'(c_t)}{\beta u'(c_{t+1})} = f'(k_{t+1}) + 1 - \delta \tag{10}$$

Characterizing ADCE

- Recall
$$\frac{u'(c_t)}{\beta u'(c_{t+1})} = f'(k_{t+1}) + 1 - \delta \tag{10}$$
- (10) + TVC (4) + initial condition (5) + market clearing (8) = same set of equations as for SP problem
- Hence: ADCE allocation is the same for the SP problem

Characterizing ADCE

- Recall
$$\frac{u'(c_t)}{\beta u'(c_{t+1})} = f'(k_{t+1}) + 1 - \delta \tag{10}$$
- (10) + TVC (4) + initial condition (5) + market clearing (8) = same set of equations as for SP problem
- Hence: ADCE allocation is the same for the SP problem
- How get prices?
    - can always normalize one price to unity: wlog set $p_0 = 1$
    - get R_0, w_0 from (6) and (7) at $t = 0$
    - get p_1 from (9) given c_0, c_1, p_0
    - get R_1, w_1 from (6) and (7) at $t = 1$ given p_1
    - ...

Aside: Walras' Law

- Q: Why didn't we use budget constraint (3)?
- A: because it is implied by the other equations, in particular firm's problem + market clearing (8) $\Rightarrow$ (3)
- Firm's problem and market clearing are

$$\sum_{t=0}^{\infty}(p_t F(k_t, h_t) - w_t h_t - R_t k_t) = 0 \tag{11}$$

$$c_t + k_{t+1} - (1-\delta)k_t = F(k_t, h_t) \tag{12}$$

Substituting (12) into (11) gives (3):

$$\sum_{t=0}^{\infty}[p_t(c_t + k_{t+1} - (1-\delta)k_t) - w_t h_t - R_t k_t] = 0$$

- This is "Walras' Law" `http://en.wikipedia.org/wiki/Walras'_law`
    - very general: all budget constraints $\Rightarrow$ resource constraint
    - useful check when writing models: if Walras' Law doesn't hold, you did something wrong (e.g. forgot term in mkt clearing)

Steady State (SS) ADCE

- Definition: A SS ADCE is a value of k^* and an ADCE for the economy with $\bar{k}_0 = k^*$ s.t. $k_t = k^*$ (and $c_t = c^*$) for all t.
- Clearly from (10)
$$\frac{1}{\beta} = f'(k^*) + 1 - \delta$$
- $\Rightarrow k^*$ same as in SP problem
- Question: what do you think prices look like in a SS ADCE?
  - p_t = constant?
  - R_t = constant?
  - w_t = constant?

Steady State ADCE

- Let's work it out
- Have

$$\frac{u'(c^*)}{\beta u'(c^*)} = \frac{p_t}{p_{t+1}} \quad \Rightarrow \quad \frac{p_{t+1}}{p_t} = \beta$$

- normalizing $p_0 = 1$

$$p_t = \beta^t$$

- Also

$$\frac{R_t}{p_t} = F_k(k^*, 1), \quad \frac{w_t}{p_t} = F_h(k^*, 1)$$

- Summary:
  - $R_t/p_t, w_t/p_t$ constant
  - R_t, w_t, p_t decreasing at rate β
- So prices are not constant in SS ADCE
  - prices implicitly reflect discounting of future values
  - price of future output is lower
  - return to future work is lower

Alternative Pricing Convention

- Denote factor prices relative to output price in each period
- Write budget constraint as

$$\sum_{t=0}^{\infty} p_t(c_t + k_{t+1} - (1-\delta)k_t) \leq \sum_{t=0}^{\infty} p_t(\tilde{R}_t k_t + \tilde{w}_t h_t)$$

- This formulation is useful for thinking about real rates of return and interest rates in ADCE
  - no explicit credit market in ADCE
  - but can infer implicit real interest rate on one-period ahead borrowing and lending

Alternative Pricing Convention

- Denote real interest rate by r_t
- Definition: $1 + r_{t+1}$ = amount of consumption you can get tomorrow by giving up one unit of consumption today
  - giving up one unit today saves p_t
  - with this you buy p_t/p_{t+1} tomorrow

$$\Rightarrow 1 + r_{t+1} = \frac{p_t}{p_{t+1}}$$

- In steady state, $1 + r_t = 1/\beta$
- Real rate of return on capital: from HH max. w.r.t. k_{t+1}

$$p_t = p_{t+1}\tilde{R}_{t+1} + p_{t+1}(1-\delta)$$

  - buy 1 unit of k today, get $p_{t+1}\tilde{R}_{t+1} + p_{t+1}(1-\delta)$ tomorrow
  - must equal cost of doing so p_t

$$1 + r_{t+1} = \tilde{R}_{t+1} + 1 - \delta \quad \Rightarrow \quad \tilde{R}_t = r_t + \delta$$

- Terminology: rental rate $\tilde{R}_t$ = "user cost of capital" $r_t + \delta$

Sequence of Markets CE

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- Arrow-Debreu CE
  - period 0: markets for everything
- Sequence of Markets CE: particular markets at particular points in time

Period 0	Period 1	Period 2	...
market for period 0 capital,	...	...	...
period 0 labor,	...	...	...
period 0 output,	...	...	...
1 period ahead borrowing/lending	...	...	...

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period 0 labor,	...	...	...
period 0 output,	...	...	...
1 period ahead borrowing/lending	...	...	...

- Individ. formulates plan at $t = 0$, but executes it in real time
  - in contrast, in ADCE everything happens in period 0
- SOMCE features explicit borrowing & lending
  - riskless one-period bond that pays real interest rate r_t

Sequence of Market CE

- Definition: A SOMCE = sequences $\{c_t, h_t, k_t, a_t, w_t, R_t, r_t\}_{t=0}^{\infty}$ s.t.
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$$c_t + k_{t+1} - (1-\delta)k_t + a_{t+1} \leq R_t k_t + w_t h_t + (1+r_t)a_t$$

$$c_t \geq 0, \quad 0 \leq h_t \leq 1, \quad k_{t+1} \geq 0, \quad k_0 = \bar{k}_0, \quad a_0 = 0$$

$$\lim_{T\to\infty} \left(\prod_{t=0}^{T} \frac{1}{1+r_t} \right) a_{T+1} \geq 0 \qquad (*)$$

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$$\max_{k_t, h_t} F(k_t, h_t) - w_t h_t - R_t k_t \quad k_t \geq 0, \quad h_t \geq 0 \quad \forall t.$$

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  3. (Market clearing) For each t:

$$c_t + k_{t+1} - (1-\delta)k_t = F(k_t, h_t)$$

$$a_{t+1} = 0 \qquad (**)$$

Comments

- a_t = HH bond holdings
  - $a_t > 0$: HH saves, $a_t < 0$: HH borrows
  - period-t price of bond that pays off at $t+1$: $q_t = 1/(1+r_t)$
  - some people like to write

$$c_t + k_{t+1} - (1-\delta)k_t + q_t b_{t+1} \leq R_t k_t + w_t h_t + b_t$$

  - this is equivalent with $b_t = (1+r_t)a_t$ and $q_t = 1/(1+r_t)$

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  - this is equivalent with $b_t = (1+r_t)a_t$ and $q_t = 1/(1+r_t)$
- Interpretation of bond market clearing condition $(**)$
  - bonds are in zero net supply
  - more generally, in economy with individuals $i = 1, ..., N$

$$\sum_{i=1}^{N} a_{i,t+1} = 0$$

  - for every dollar borrowed, someone else saves a dollar
  - here only one type, so $a_{t+1} = 0$.
  - Q: since $a_t = 0$, why not eliminate? A: need to know eq. r_t

Comments

- Could have written firm's problem as

$$\max_{\{k_t, h_t\}} \sum_{t=0}^{\infty} \left(\prod_{s=0}^{t} \frac{1}{1+r_s} \right) (F(k_t, h_t) - w_t h_t - R_t k_t) \quad k_t \geq 0, \quad h_t \geq 0$$

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- $(*)$ is a so-called "no-Ponzi condition"
    - with period budget constraints only, individuals could choose time paths with $a_t \to -\infty$
    - no-Ponzi condition $(*)$ rules out such time paths: a_t cannot become too negative
    - implies that sequence of budget constraints can be written as present-value (or time-zero) budget constraint
    - return to this momentarily

Sequence BC + no-Ponzi $\Rightarrow$ PVBC

- Result: If $\{c_t, i_t, h_t\}$ satisfy the sequence budget constraint

$$c_t + i_t + a_{t+1} = R_t k_t + w_t h_t + (1 + r_t)a_t$$

and if the no-Ponzi condition $(*)$ holds with equality, then $\{c_t, i_t, h_t\}$ satisfy the present value budget constraint

$$\sum_{t=0}^{\infty}\left(\prod_{s=0}^{t}\frac{1}{1+r_s}\right)(c_t + i_t) = \sum_{t=0}^{\infty}\left(\prod_{s=0}^{t}\frac{1}{1+r_s}\right)(R_t k_t + w_t h_t)$$

- Proof: next slide

Proof

- Write period t budget constraint as

$$\frac{1}{1+r_t}a_{t+1} = \frac{1}{1+r_t}\left(R_t k_t + w_t h_t - c_t - i_t\right) + a_t$$

Proof

- Write period t budget constraint as

$$\frac{1}{1+r_t} a_{t+1} = \frac{1}{1+r_t} \left(R_t k_t + w_t h_t - c_t - i_t \right) + a_t$$

- At $t = 0, t = 1, \ldots$

$$\begin{aligned} \frac{1}{1+r_0} a_1 &= \frac{1}{1+r_0} \left(R_0 k_0 + w_0 h_0 - c_0 - i_0 \right) + a_0 \\ \frac{1}{1+r_0} \frac{1}{1+r_1} a_2 &= \frac{1}{1+r_0} \frac{1}{1+r_1} \left(R_1 k_1 + w_1 h_1 - c_1 - i_1 \right) \\ &\quad + \frac{1}{1+r_0} \left(R_0 k_0 + w_0 h_0 - c_0 - i_0 \right) + a_0 \end{aligned}$$

Proof

- Write period t budget constraint as

$$\frac{1}{1+r_t}a_{t+1} = \frac{1}{1+r_t}\left(R_t k_t + w_t h_t - c_t - i_t\right) + a_t$$

- At $t = 0, t = 1, \ldots$

$$\frac{1}{1+r_0}a_1 = \frac{1}{1+r_0}\left(R_0 k_0 + w_0 h_0 - c_0 - i_0\right) + a_0$$

$$\begin{aligned}\frac{1}{1+r_0}\frac{1}{1+r_1}a_2 =& \frac{1}{1+r_0}\frac{1}{1+r_1}\left(R_1 k_1 + w_1 h_1 - c_1 - i_1\right)\\ &+ \frac{1}{1+r_0}\left(R_0 k_0 + w_0 h_0 - c_0 - i_0\right) + a_0\end{aligned}$$

- By induction/repeated substitution

$$\left(\prod_{t=0}^{T}\frac{1}{1+r_t}\right)a_{T+1} = \sum_{t=0}^{T}\left(\prod_{s=0}^{t}\frac{1}{1+r_s}\right)\left(R_t k_t + w_t h_t - i_t - c_t\right)$$

- Result follows from taking $T \to \infty$ and imposing $(*)$

Why no-Ponzi Condition?

- Expression also provides some intuition for no-Ponzi condition

$$\left(\prod_{t=0}^{T} \frac{1}{1+r_t}\right) a_{T+1} = \sum_{t=0}^{T} \left(\prod_{s=0}^{t} \frac{1}{1+r_s}\right) (R_t k_t + w_t h_t - i_t - c_t)$$

- Suppose for the moment this were a finite horizon economy
  - would impose: die without debt, i.e.

$$a_{T+1} \geq 0$$

  - in fact, HH's would always choose $a_{T+1} = 0$

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- Suppose for the moment this were a finite horizon economy
    - would impose: die without debt, i.e.

$$a_{T+1} \geq 0$$

    - in fact, HH's would always choose $a_{T+1} = 0$
- Right analogue for infinite horizon economy

$$\lim_{T \to \infty} \left(\prod_{t=0}^{T} \frac{1}{1+r_t}\right) a_{T+1} \geq 0$$

  and HH's choose $\{a_t\}$ so that this holds with equality
- no-Ponzi condition not needed for physical capital because natural constraint $k_t \geq 0$.

Characterizing SOMCE

- Necessary conditions for consumer problem ($h_t = 1$ wlog)

$$c_t : \quad \beta^t u'(c_t) = \lambda_t = \text{multiplier on period } t \text{ b.c.} \tag{13}$$

$$k_{t+1} : \quad \lambda_t = \lambda_{t+1}(R_{t+1} + 1 - \delta) \tag{14}$$

$$a_{t+1} : \quad \lambda_t = \lambda_{t+1}(1 + r_{t+1}) \tag{15}$$

$$c_t + k_{t+1} - (1-\delta)k_t + a_{t+1} = R_t k_t + w_t h_t + (1 + r_t)a_t \tag{16}$$

$$\text{no-Ponzi: } \lim_{T\to\infty} \left(\prod_{t=0}^{T} \frac{1}{1+r_t} \right) a_{T+1} \geq 0 \tag{17}$$

$$\text{TVC on } k : \quad \lim_{T\to\infty} \beta^T u'(c_T) k_{T+1} = 0 \tag{18}$$

$$\text{TVC on } a : \quad \lim_{T\to\infty} \beta^T u'(c_T) a_{T+1} = 0 \tag{19}$$

$$\text{initial} : \quad k_0 = \bar{k}_0, \quad a_0 = 0 \tag{20}$$

Characterizing SOMCE

- Necessary conditions for firm problem

$$F_k(k_t, h_t) = R_t, \quad F_h(k_t, h_t) = w_t \tag{21}$$

Characterizing SOMCE

- Necessary conditions for firm problem

$$F_k(k_t, h_t) = R_t, \quad F_h(k_t, h_t) = w_t \tag{21}$$

- Market clearing

$$c_t + k_{t+1} - (1-\delta)k_t = F(k_t, h_t), \quad a_{t+1} = 0 \tag{22}$$

Characterizing SOMCE

- (13), (15) and (17)

$$\beta^T u'(c_T) = \lambda_T = \prod_{t=0}^{T} \frac{1}{1+r_t}$$

$$\Rightarrow \lim_{T\to\infty} \beta^T u'(c_T) a_{T+1} \geq 0$$

- No-Ponzi condition looks very similar to TVC on $\{a_t\}$
- But no-Ponzi and TVC are different conditions
- Kamihigashi (2008) "A no-Ponzi-game condition is a constraint that prevents overaccumulation of debt, while a typical transversality condition is an optimality condition that rules out overaccumulation of wealth. They place opposite restrictions, and should not be confused."

Characterizing SOMCE

- (14) and (15)
$$1 + r_{t+1} = R_{t+1} + 1 - \delta$$
i.e. rate of return on bonds = rate of return on capital
    - arbitrage condition
    - if this holds, HH is indifferent between a and k

Characterizing SOMCE

- (14) and (15)
$$1 + r_{t+1} = R_{t+1} + 1 - \delta$$
i.e. rate of return on bonds = rate of return on capital
    - arbitrage condition
    - if this holds, HH is indifferent between a and k
- (13), (14) and (21) $\Rightarrow$
$$\frac{u'(c_t)}{\beta u'(c_{t+1})} = f'(k_{t+1}) + 1 - \delta \tag{23}$$
- (23) + TVC (18) + initial condition (20) + market clearing (22) = same set of equations as for SP problem
- Hence: SOMCE allocation is same as social planner's allocation
    - this is actually somewhat surprising, see next slide

Why is SOMCE allocation =SP's alloc.?

- Relative to ADCE, we closed down many markets
- Q: Why do we still get SP solution even though we closed down many markets?
- A: We only closed down markets that didn't matter
- In fact, ADCE and SOMCE are equivalent

Equivalence of SOMCE and ADCE

- Recall HH's problem in ADCE (first part of lecture):

$$\max_{\{c_t, h_t, k_t\}} \sum_{t=0}^{\infty} \beta^t u(c_t) \quad \text{s.t.}$$

$$\sum_{t=0}^{\infty} p_t(c_t + k_{t+1} - (1-\delta)k_t) \leq \sum_{t=0}^{\infty} p_t(R_t k_t + w_t h_t)$$

Equivalence of SOMCE and ADCE

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$$\max_{\{c_t,h_t,k_t\}} \sum_{t=0}^{\infty} \beta^t u(c_t) \quad \text{s.t.}$$

$$\sum_{t=0}^{\infty} p_t(c_t + k_{t+1} - (1-\delta)k_t) \leq \sum_{t=0}^{\infty} p_t(R_t k_t + w_t h_t)$$

- Have shown earlier: HH's problem in SOMCE is same with present-value budget constraint

$$\sum_{t=0}^{\infty} \left(\prod_{s=0}^{t} \frac{1}{1+r_s} \right) (c_t + k_{t+1} - (1-\delta)k_t) = \sum_{t=0}^{\infty} \left(\prod_{s=0}^{t} \frac{1}{1+r_s} \right) (R_t k_t + w_t h_t)$$

- Clearly these are equivalent
  - ADCE is SOMCE with $p_t = \prod_{s=0}^{t} \frac{1}{1+r_s}$
  - SOMCE is ADCE with $1 + r_{t+1} = p_t / p_{t+1}$
- Firm's problems are also equivalent.

Why is SOMCE allocation =SP's alloc.?

- riskless one-period bond is surprisingly powerful
- one period ahead borrowing and lending $\Rightarrow$ arbitrary period ahead borrowing and lending

Why is SOMCE allocation =SP's alloc.?

- riskless one-period bond is surprisingly powerful
- one period ahead borrowing and lending $\Rightarrow$ arbitrary period ahead borrowing and lending
- When is SOMCE allocation with one-period bonds $\neq$ SP's allocation? That is, when do the welfare theorems fail?
  - risk (idiosyncratic or aggregate)
    - welfare theorems may hold if sufficiently rich insurance markets
  - "financial frictions." Examples:
    - interest rate = $r_t(a_t)$ with $r_t' \neq 0$.
    - in more general environments: borrowing constraint $-a_t \leq 0$ or collateral constraints (need to back debt with collateral)

$$-a_{t+1} \leq \theta k_{t+1}$$

  - ...

Infinitely-Elastic Long-Run Capital Supply

Infinitely-elastic steady state capital supply

- Recall condition for k^* (as usual $\rho = 1/\beta - 1$)

$$\frac{1}{\beta} = f'(k^*) + 1 - \delta \quad \Leftrightarrow \quad f'(k^*) = \rho + \delta$$

- Can think of this in terms of demand and supply of capital
- Will draw demand-supply diagram with k on x-axis and r on y-axis

Infinitely-elastic steady state capital supply

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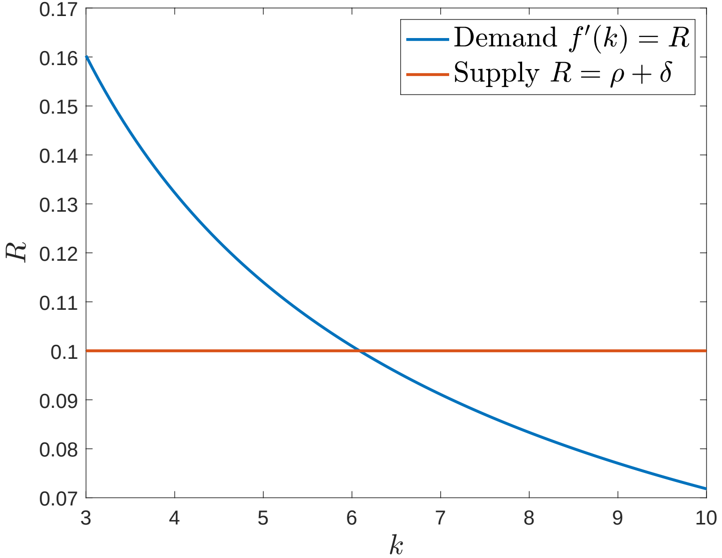
$$\frac{1}{\beta} = f'(k^*) + 1 - \delta \quad \Leftrightarrow \quad f'(k^*) = \rho + \delta$$
- Can think of this in terms of demand and supply of capital
- Will draw demand-supply diagram with k on x-axis and r on y-axis
- Demand: from firm's problem, capital demand $k^d(R)$ satisfies
$$f'(k) = R$$
This is a nice, well-behaved downward-sloping demand curve
- Supply: capital supply $k^s(R)$ satisfies
$$R = \rho + \delta$$
This is an infinitely-elastic supply curve! Intuition in 3 slides.
- This infinite elasticity = important property of growth model

Infinitely-elastic steady state capital supply



Capital Demand: Derivation

- Recall representative firm's optimality condition

$$F_k(k_t, h_t) = R_t$$

- Defining $f(k) := F(k, 1)$ and using $h_t = 1$

$$f'(k_t) = R_t$$

- And in particular in steady state:

$$f'(k^*) = R$$

- This defines a downward-sloping capital demand curve $k^d(R)$

Capital Supply: Derivation

- Euler equation for capital

$$u'(c_t) = \beta(R_{t+1} + 1 - \delta)u'(c_{t+1})$$

- In steady state

$$1 = \beta(R + 1 - \delta)$$

- Therefore the steady state rental rate must equal

$$R = \frac{1}{\beta} - 1 + \delta = \rho + \delta \qquad (*)$$

- This is an infinitely-elastic supply curve! Intuition:
  - if $\beta(R + 1 - \delta) > 1$, households would accumulate $k = \infty$

$$\beta(R_{t+1} + 1 - \delta) > 1 \quad \Rightarrow \quad c_{t+1} > c_t$$

  - if $\beta(R + 1 - \delta) < 1$, households would accumulate 0

$$\beta(R_{t+1} + 1 - \delta) < 1 \quad \Rightarrow \quad c_{t+1} < c_t$$

  - any equilibrium with $0 < k^* < \infty$ has to feature $(*)$

Supply and demand in terms of interest rate r

- Sometimes people also write this in terms of the steady-state interest rate r rather than rental rate R
  - recall that alternative "decentralization" = firms own and accumulate capital (and firms in turn owned by households)
- Demand: capital demand $k^d(r)$ satisfies

$$f'(k) = r + \delta$$

  This is a nice, well-behaved downward-sloping demand curve
- Supply: capital supply $k^s(r)$ satisfies

$$r = \rho$$

  This is an infinitely-elastic supply curve! Intuition:
  - if $r > \rho$, households would accumulate $k = \infty$
  - if $r < \rho$, households would accumulate 0
  - any equilibrium with $0 < k^* < \infty$ has to feature $r = \rho$

Supply and demand in terms of interest rate r