Lecture 4

Analysis of Optimal Trajectories, Transition Dynamics in Growth Model

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Macroeconomics EC417

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Plan of Lecture

1. Linearization around steady state, speed of convergence, slope of saddle path
2. Some transition experiments in the growth model

Linearization around Steady State

- Two questions:
  - can we say more than "there exists a unique steady state and the economy converges to it"? Speed of convergence?
  - How analyze stability if two- or N-dimensional state x (so that cannot draw phase diagram)?

- Can answer these questions by analyzing local dynamics close to steady state

Linearization around Steady State

- Let $y \in \mathbb{R}^n$ and the function $m : \mathbb{R}^n \to \mathbb{R}^n$ define a dynamical system:
$$\dot{y}(t) = m(y(t)) \text{ for } t \geq 0,$$
- Let y^* be a steady state, i.e. $0 = m(y^*)$

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$$\dot{y}\,(t) = m(y(t)) \text{ for } t \geq 0,$$
- Let y^* be a steady state, i.e. $0 = m\,(y^*)$
- Consider a first order approximation of m around y^* :
$$\dot{y} \approx m(y^*) + m'\,(y^*)\,(y - y^*)$$
where $m'(y^*)$ is the $n \times n$ Jacobian of m evaluated at y^*, i.e. the matrix with entries $\partial m_i(y^*)/\partial y_j$

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where $m'(y^*)$ is the $n \times n$ Jacobian of m evaluated at y^*, i.e. the matrix with entries $\partial m_i(y^*)/\partial y_j$
- Equivalently
$$\dot{\hat{y}} \approx A\hat{y}, \quad \hat{y} = y - y^*, \quad A = m'(y^*)$$
- Idea is then to analyze this linear differential equation.
- Analysis is valid globally (i.e. for all $\mathbb{R}^n$) if the system is indeed linear.
- Alternatively it is valid in a neighborhood of the steady state.

Linearized Growth Model

- Recall system of two ODEs

$$\dot{c} = \frac{1}{\sigma}(f'(k) - \rho - \delta)c$$
$$\dot{k} = f(k) - \delta k - c \qquad \text{(ODE")}$$

- Let $y = (c, k)$ and do analysis on previous slide

$$\dot{\hat{y}} \approx A\hat{y}, \quad A = \begin{bmatrix} \partial \dot{c}/\partial c & \partial \dot{c}/\partial k \\ \partial \dot{k}/\partial c & \partial \dot{k}/\partial k \end{bmatrix}, \quad \hat{y} = \begin{bmatrix} c - c^* \\ k - k^* \end{bmatrix}$$

  where the partial derivatives are evaluated at (c^*, k^*)
- Have

$$A = \begin{bmatrix} \partial \dot{c}/\partial c & \partial \dot{c}/\partial k \\ \partial \dot{k}/\partial c & \partial \dot{k}/\partial k \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{\sigma} f''(k^*)c^* \\ -1 & \rho \end{bmatrix}$$

  where we used $\partial \dot{k}/\partial k = f'(k^*) - \delta = \rho$

Properties of Linear Systems

- Theorem (see e.g. Acemoglu, Theorem 7.18) Consider the following linear differential equation system

$$\dot{\hat{y}}(t) = A\hat{y}(t), \quad \hat{y} = y - y^* \tag{$*$}$$

  with initial value $\hat{y}(0)$, and where A is an $n \times n$ matrix. Suppose that $\ell \leq n$ of the eigenvalues of A have negative real parts. Then, there exists an ℓ-dimensional subspace L of $\mathbb{R}^n$ such that starting from any $\hat{y}(0) \in L$, the differential equation $(*)$ has a unique solution with $\hat{y}(t) \to 0$.
- Proof: next slide
- Interpretation: important thing is to compare number of negative eigenvalues ℓ and number of pre-determined state variables m
  - if $\ell = m$ (standard case): "saddle-path stable", unique optimal trajectory. Neg. eigenvalues govern speed of convergence.
  - if $\ell < m$: unstable, $y(t)$ does not converge to steady state.
  - if $\ell > m$: multiple optimal trajectories ("indeterminacy")

Proof of Theorem

- First step is to solve $(*)$. Also see
  http://en.wikipedia.org/wiki/Matrix_differential_equation
- Denote the eigenvalues of A by $\lambda_1, ..., \lambda_n$ and the corresponding eigenvectors by $v_1, ..., v_n$.
- Diagonalizing the matrix A we obtain:

$$A = P\Lambda P^{-1}$$

- Λ is diagonal matrix with eigenvalues of A, possibly complex, on its diagonal.
- Matrix P contains the eigenvectors of A, i.e. $P = (v_1, ..., v_n)$ (to see this write $AP = P\Lambda$ or $Av_i = \lambda_i v_i$) and is invertible
  - ignores some technicalities discussed in ch. 6 of SLP book
- Write system as

$$P^{-1}\dot{\hat{y}}(t) = \Lambda P^{-1}\hat{y}(t)$$
$$\Leftrightarrow \quad \dot{z}(t) = \Lambda z(t), \quad z(t) = P^{-1}\hat{y}(t)$$

Proof of Theorem

- Since Λ is diagonal $\dot{z}_i = \lambda_i z_i(t)$, $i = 1, ..., n$, i.e. it can be solved element by element

$$z_i(t) = a_i e^{\lambda_i t}$$

  where d_i are constants of integration

- We have that $\hat{y}(t) = Pz(t)$. Using $P = (v_1, ..., v_n)$ we have

$$\hat{y}(t) = \sum_{i=1}^{n} d_i e^{\lambda_i t} v_i \qquad (**)$$

- For now, assume all λ_i are real
- Let λ_i be such that for $i = 1, 2, .., \ell$ we have $\lambda_i < 0$ and for $i = \ell + 1, \ell + 2, ..., n$ we have $\lambda_i \geq 0$. That is, eigenvalues of A are ordered so that first ℓ are negative.

Proof of Theorem

- Q: when does $\hat{y}(t) \to 0$? A: initial condition needs to satisfy

$$\hat{y}(0) = \sum_{i=1}^{n} d_i v_i, \quad d_i = 0, i = \ell + 1, \ell + 2, \ldots, n$$

- That is, $\hat{y}(t) \to 0$ only if $\hat{y}(0)$ lies in particular subspace of $\mathbb{R}^n$. Dimension of subspace = # of negative eigenvalues ℓ.

- Exercise: how extend to case where λ_i can be complex?

Linearized Growth Model

- Recall

$$\dot{\hat{y}} \approx Ay, \quad A = \begin{bmatrix} 0 & \frac{1}{\sigma} f''(k^*)c^* \\ -1 & \rho \end{bmatrix}, \quad \hat{y} = \begin{bmatrix} c - c^* \\ k - k^* \end{bmatrix}$$

- Let's look at eigenvalues of A. These satisfy

$$0 = \det(A - \lambda I) = -\lambda(\rho - \lambda) + \frac{1}{\sigma} f''(k^*)c^*$$
$$0 = \lambda^2 - \rho\lambda + \frac{1}{\sigma} f''(k^*)c^*$$

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$$0 = \det(A - \lambda I) = -\lambda(\rho - \lambda) + \frac{1}{\sigma} f''(k^*) c^*$$
$$0 = \lambda^2 - \rho\lambda + \frac{1}{\sigma} f''(k^*) c^*$$

- This is a simple quadratic with two solutions ("roots")

$$\lambda_{1/2} = \frac{\rho \pm \sqrt{\rho^2 - 4\frac{1}{\sigma} f''(k^*) c^*}}{2}$$

- $f''(k^*) < 0$ so both eigenvalues are real, and $\lambda_1 < 0 < \lambda_2$
- Have one pre-determined state variable.
- Theorem says: $\ell = m = 1 \Rightarrow$ saddle-path stable

Linearized Growth Model

- What does this tell us about the time path of capital $k(t)$?
- From $(**)$, solution to matrix differential equation for growth model is
$$\begin{bmatrix} \hat{y}_1(t) \\ \hat{y}_2(t) \end{bmatrix} \approx d_1 e^{\lambda_1 t} \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} + d_2 e^{\lambda_2 t} \begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix}$$
where $\hat{y}_1 = c(t) - c^*, \hat{y}_2 = k(t) - k^*$ and v_{i1}, v_{i2} denote elements of v_i
- $\lambda_2 > 0 \Rightarrow$ need $d_2 = 0$
$$c(t) - c^* \approx d_1 e^{\lambda_1 t} v_{11}, \quad k(t) - k^* \approx d_1 e^{\lambda_1 t} v_{12}$$
- Have initial condition for $k(0) = k_0 \Rightarrow d_1 v_{12} = k_0 - k^*$
$$c(t) - c^* \approx \frac{v_{11}}{v_{12}} e^{\lambda_1 t}(k_0 - k^*) \tag{1}$$
$$k(t) - k^* \approx e^{\lambda_1 t}(k_0 - k^*) \tag{2}$$
- From (2), know approximate time path for $k(t)$
- (1) pins down initial consumption (v_{11} and v_{12} are known)

Linearization: Speed of Convergence

- Negative eigenvalue λ_1 governs speed of convergence

$$k(t) - k^* \approx e^{-|\lambda_1| t}(k_0 - k^*)$$

- Half-life for convergence to steady state

$$k(t_{1/2}) - k^* = \frac{1}{2}(k_0 - k^*) \quad \Rightarrow \quad t_{1/2} = \frac{\ln(2)}{|\lambda_1|}$$

Linearization: Speed of Convergence

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- Half-life for convergence to steady state

$$k(t_{1/2}) - k^* = \frac{1}{2}(k_0 - k^*) \quad \Rightarrow \quad t_{1/2} = \frac{\ln(2)}{|\lambda_1|}$$

- Formula from previous slide:

$$\lambda_1 = \frac{\rho - \sqrt{\rho^2 - 4\frac{1}{\sigma}f''(k^*)c^*}}{2}$$

- Convergence fast ($|\lambda_1|$ large) if
  - high f'' (strongly diminishing returns)
  - low σ (high "intertemp. elas. of substit.")
  - low ρ (more patient)
- Later: for reasonable parameter values, neoclassical growth model features very fast convergence, e.g. about $t_{1/2} = 5$ years.

Linearization: Speed of Convergence

- Insights also go through with general utility function $u(c)$
- can show: with general utility function $u(c)$, formula generalizes to

$$\lambda_1 = \frac{\rho \pm \sqrt{\rho^2 - 4\frac{f''(k^*)c^*}{\sigma(c^*)}}}{2}$$

where

$$\sigma(c) \equiv -\frac{u''(c)c}{u'(c)}$$

- check: $u(c) = \frac{c^{1-\sigma}}{1-\sigma} \Rightarrow \sigma(c) = \sigma$

Slope of Saddle Path

- Recall conditions for optimum:

$$\frac{\dot{c}}{c} = \frac{1}{\sigma}(f'(k) - \rho - \delta)$$
$$\dot{k} = f(k) - \delta k - c \qquad \text{(ODE'')}$$

  with $k(0) = k_0$ and $\lim_{T\to\infty} e^{-\rho T} c(T)^{-\sigma} k(T) = 0$.

- Saddle path $c(k)$ defines optimal consumption for each k
  - a.k.a. consumption policy function

- For many questions, useful to know slope of saddle path

Slope of Saddle Path

- Slope of saddle path satisfies

$$c'(k) = \frac{dc}{dk} = \frac{dc/dt}{dk/dt}$$

$$c'(k) = \frac{\frac{1}{\sigma}(f'(k) - \rho - \delta)c(k)}{f(k) - \delta k - c(k)} \tag{$*$}$$

- Digression: $(*)$ is a non-linear ODE in $c(k)$ that can be solved numerically
  - once solved, know entire dynamics $\dot{k} = f(k) - \delta k - c(k)$
  - alternative to shooting algorithm, no issues with transversality
  - $(*)$ can also be derived from continuous-time Bellman equation (HJB equation)

Slope of Saddle Path

- Now consider slope of saddle path at steady state $c'(k^*)$

$$c'(k^*) = \lim_{k\to k^*} c'(k) = \lim_{k\to k^*} \frac{\frac{1}{\sigma}(f'(k)-\rho-\delta)c(k)}{f(k)-\delta k - c(k)} =$$

$$= \frac{\frac{1}{\sigma}f''(k^*)c^*}{f'(k^*)-\delta-c'(k^*)} = \frac{\frac{1}{\sigma}f''(k^*)c^*}{\rho - c'(k^*)}$$

where the third equality follows from L'Hopital's rule

(`http://en.wikipedia.org/wiki/L'H%C3%B4pital's_rule`)

- Rearranging, we see that $\lambda = c'(k^*)$ satisfies

$$-\lambda(\rho-\lambda) + \frac{1}{\sigma}f''(k^*)c^* = 0$$

- Same quadratic as before. Two solutions ("roots")

$$\lambda_{1/2} = \frac{\rho \pm \sqrt{\rho^2 - 4\frac{1}{\sigma}f''(k^*)c^*}}{2}, \qquad \lambda_1 < 0 < \lambda_2$$

Slope of Saddle Path

- Know $c'(k^*) > 0 \Rightarrow$ slope of saddle path = positive root

$$c'(k^*) = \frac{\rho + \sqrt{\rho^2 - 4\frac{1}{\sigma}f''(k^*)c^*}}{2}$$

Slope of Saddle Path

- Know $c'(k^*) > 0 \Rightarrow$ slope of saddle path = positive root

$$c'(k^*) = \frac{\rho + \sqrt{\rho^2 - 4\frac{1}{\sigma}f''(k^*)c^*}}{2}$$

- In growth model
  - negative eigenvalue of linearized system = speed of convergence
  - positive eigenvalue of linearized system = slope of saddle path
  - seems to be a coincidence, same not true in more general models. Instead slope of saddle path related to eigenvectors.

Linearization and Perturbation: Relation

- Popular method in economics: perturbation methods
- Some useful references:
  - Judd (1996) "Approximation, perturbation, and projection methods in economic analysis", Judd's (1998) book
  - Schmitt-Grohe and Uribe (2004)
  - Fernandez-Villaverde lecture notes
    `http://economics.sas.upenn.edu/~jesusfv/Chapter_2_Perturbation.pdf`
  - Original references from math literature: Fleming (1971), Fleming and Souganidis (1986)
- Takeaway: linearization around steady state is particular application of first-order perturbation method
- Why perturbation? Perturbation methods are more general and there are more powerful theorems

Linearization and Perturbation: Relation

- Recall dynamical system from beginning of lecture

$$\dot{y}(t) = m(y(t)), \quad y(0) = y_0 \tag{$*$}$$

- In general, to apply perturbation method need:
  1. some known solution of equation, call it $y^0(t)$
  2. to express equation as a perturbed version of known solution in terms of scalar "perturbation parameter", call it ε
- Application to our system ($*$):
  1. know solution if initial condition is steady state, $y_0 = y^*$
  2. view ($*$) as

$$\dot{y}(t,\varepsilon) = m(y(t,\varepsilon)), \quad y(0,\varepsilon) = y^* + \varepsilon\hat{y}_0 \tag{$**$}$$

     where $\varepsilon\hat{y}_0$ is initial deviation from steady state
- Key idea of perturbation: look for solution of ($**$) of form

$$y(t,\varepsilon) = y^* + \varepsilon y_1(t) + \varepsilon^2 y_2(t) + \dots$$

- First-order perturbation: $y(t,\varepsilon) \approx y^* + \varepsilon y_1(t)$

Linearization and Perturbation: Relation

- Let's implement first-order perturbation: look for solution of $(**)$ of form

$$y(t,\varepsilon) \approx y^* + \varepsilon y_1(t)$$

- Taylor's theorem: set

$$y_1(t) = \frac{\partial y(t,0)}{\partial \varepsilon}$$

- Find by differentiating $(**)$ with respect to ε

$$\frac{\partial}{\partial \varepsilon}\dot{y}(t,0) = m'(y(t,0))\frac{\partial y(t,0)}{\partial \varepsilon}$$

$$\text{i.e.} \quad \dot{y}_1(t) = m'(y^*)y_1(t)$$

- Recall linearized system from beginning of lecture

$$\dot{\hat{y}} = A\hat{y}, \quad A = m'(y^*)$$

- Hence $y_1(t)$ solves same equation as $\hat{y}(t)$. Linearization is 1st-order perturbation with $\varepsilon = 1$ so that $y(t) = y^* + y_1(t)$

Linearization: Discrete Time

- Similar results apply for discrete-time optimal control problems
- Main difference: it's about whether eigenvalues are < 1 rather than < 0.
- See Stokey-Lucas-Prescott chapter 6.
- Let $y \in \mathbb{R}^n$ and the function $m : \mathbb{R}^n \to \mathbb{R}^n$ define a dynamical system:
$$y_{t+1} = m(y_t)$$
- Let y^* be a steady state, i.e. $y^* = m(y^*)$
- Consider a first order approximation of m around y^* :

$$y_{t+1} \approx m(y^*) + m'(y^*)(y_t - y^*)$$
$$\hat{y}_{t+1} \approx A\hat{y}_t, \quad \hat{y}_t = y_t - y^*, \quad A = m'(y^*),$$

Linearization: Discrete Time

- Theorem Consider the following linear difference equation system

$$\hat{y}_{t+1} = A\hat{y}_t, \quad \hat{y}_t = y_t - y^* \qquad (*)$$

  with initial value $\hat{y}(0)$, and where A is an $n \times n$ matrix. Suppose that $\ell \leq n$ of the eigenvalues of A have real parts that are less than one. Then, there exists an ℓ-dimensional subspace L of $\mathbb{R}^n$ such that that starting from any $\hat{y}_0 \in L$, the difference equation $(*)$ has a unique solution with $\hat{y}_t \to 0$.

- Proof is exact analogue

- My advice: if you want to linearize a model/do stability analysis, do it in continuous time
  - always works out more nicely
  - but be my guest and linearize growth model in discrete time

Transition Experiments

- Consider growth model with utility and production functions

$$u(c) = \frac{c^{1-\sigma}}{1-\sigma}, \quad f(k) = \varepsilon k^{\alpha}$$

- Consider following thought experiment
  - up until $t = 0$, economy in steady state
  - at $t = 0$, ε increases permanently to $\varepsilon' > \varepsilon$

- Question: what can we say about time paths of $k(t), i(t)$ and particularly $c(t)$ as model converges to new steady state?
  - consumption increases in long-run, but what about short-run?

Steady State Effects

- For given ε, steady state capital and consumption are

$$k^* = \left(\frac{\alpha\varepsilon}{\rho+\delta}\right)^{\frac{1}{1-\alpha}}, \quad c^* = \varepsilon(k^*)^\alpha - \delta k^*$$

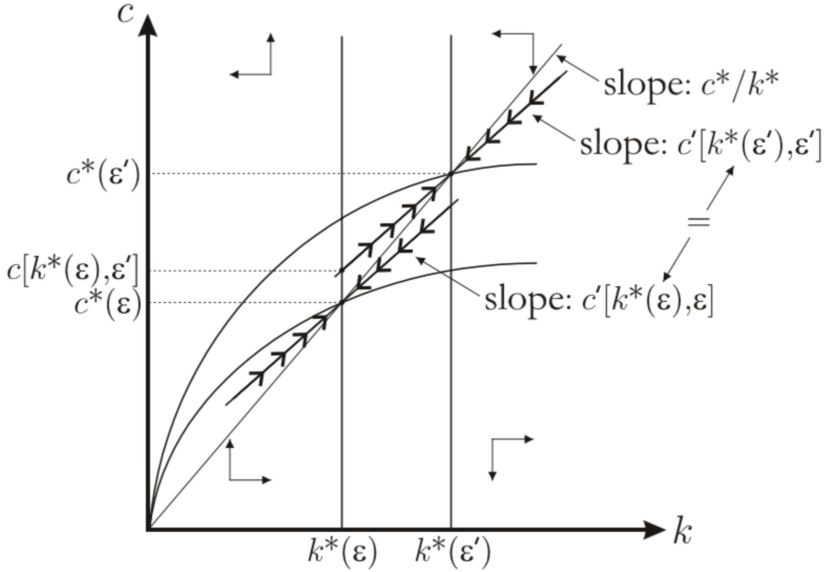
- So both k^* and c^* increase. Note also that

$$\frac{c^*}{k^*} = \varepsilon(k^*)^{\alpha-1} - \delta = \frac{\rho+\delta}{\alpha} - \delta$$

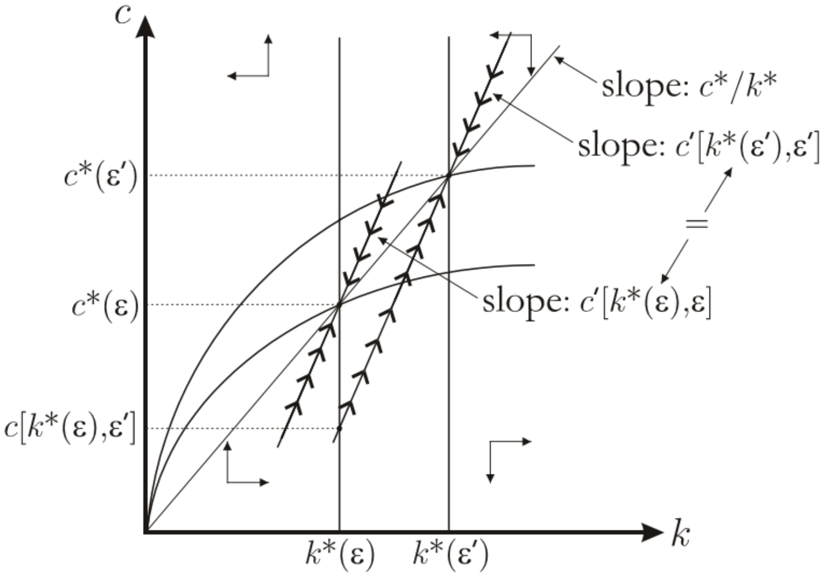
  which is independent of ε

- But what about transition?
  - see phase diagrams on next slides

Case 1: slope of new saddle path $< c^*/k^*$



Case 2: slope of new saddle path > c^*/k^*



Intuition: income vs. substitution effect

- Why can consumption decrease on impact?
- Offsetting income and substitution effects
  - income effect: $\varepsilon \uparrow \Rightarrow$ wealthier (PDV of earnings higher) $\Rightarrow$ eat more
  - substitution effect: $\varepsilon \uparrow \Rightarrow \text{MPK} \uparrow =$ return to saving $\uparrow$ $\Rightarrow$ eat less
- In general, overall effect is ambiguous
- "Income vs. substitution effect" is always the right answer!

When does $c(t)$ decrease on impact?

- Have formula for slope of saddle path

$$c'(k^*) = \frac{\rho + \sqrt{\rho^2 - 4\frac{1}{\sigma}f''(k^*)c^*}}{2}$$

$\Rightarrow$ can say more

- Consider special case $\delta = 0$ (makes algebra easier). Have

$$\alpha\varepsilon'(k^*)^{\alpha-1} = \rho, \quad \frac{c^*}{k^*} = \frac{\rho}{\alpha}$$

$$\begin{aligned} c'(k^*) &= \frac{\rho + \sqrt{\rho^2 + 4\frac{1-\alpha}{\sigma}\alpha\varepsilon'(k^*)^{\alpha-2}c^*}}{2} \\ &= \frac{\rho + \sqrt{\rho^2 + 4\frac{1-\alpha}{\sigma}\alpha\varepsilon'(k^*)^{\alpha-1}\frac{c^*}{k^*}}}{2} = \frac{\rho}{2}\left(1 + \sqrt{1 + \frac{4}{\sigma}\frac{1-\alpha}{\alpha}}\right) \end{aligned}$$

When does $c(t)$ decrease on impact?

- So question is when

$$c'(k^*) = \frac{\rho}{2}\left(1 + \sqrt{1 + \frac{4}{\sigma}\frac{1-\alpha}{\alpha}}\right) > \frac{\rho}{\alpha} = \frac{c^*}{k^*}$$

$$\sqrt{\alpha^2 + \frac{4}{\sigma}(1-\alpha)\alpha} > 2 - \alpha$$

$$\alpha > \sigma$$

- Summary:
  - $\sigma > \alpha$: income effect dominates $\Rightarrow c(t)$ increases
  - $\sigma < \alpha$: subst. effect dominates $\Rightarrow c(t)$ decreases
  - $\sigma = \alpha$: income and subst. effects cancel $\Rightarrow c(t)$ constant

- Exercise: what is the cutoff in general case $\delta > 0$

Transition Experiments

- Exercise: analogous experiments for other parameter values
  - increase in ρ
  - increase in δ