Lecture 4

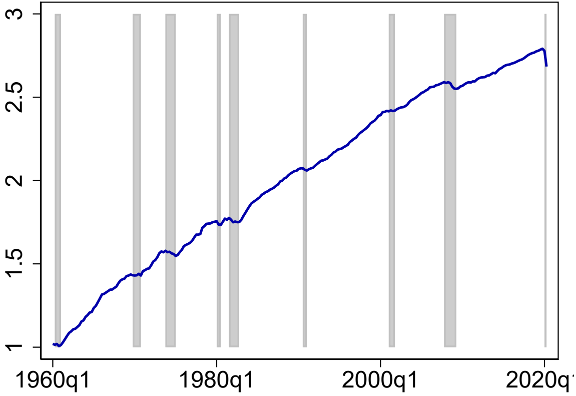
Introduction to the New Keynesian DSGE Model

Maarten De Ridder

London School of Economics

EC417

Business Cycles



Real Gross Domestic Product for the U.S. 1960-2020

Source: FRED

This term

Part I: Shocking theory of the business cycle (weeks 1-6)

- Introduction to business cycles ✓
- Real Business Cycle (RBC) Model ✓
- New Keynesian DSGE Models $\Leftarrow$

Part II: Perspectives on business cycles and steady states (weeks 7-10)

- Persistent effects of recessions
- Aggregate shocks? Firm-heterogeneity and the business cycle
- Interesting steady states: firms, productivity, market power

Previous lecture

RBC models present microfounded theories of the business cycle. But:

- Unclear what ‘productivity shocks’ represent
- Underestimate volatility of employment, require high Frish elasticity
- No role for involuntary unemployment, little endog. propagation
- Price level does not matter: no analysis of **inflation**
- No role for, e.g., monetary policy

In the news

Cost of living crisis

'Anxiety is at its maximum level': cost of living crisis hits UK businesses

Jem Bartholomew, Rachel Obordo *and* Jillian Ambrose

Tue 1 Feb 2022 08.00 GMT

UK inflation jumped to its highest level in almost 30 years at 5.4% in December. Composite: Guardian

Five bosses on how inflation has sparked worry, potential job layoffs and financial losses

- **How are European countries tackling energy crisis?**
- **Why the price of basic food items is rising**

The UK's cost of living crisis escalated in December, as inflation jumped to its highest level in almost 30 years at 5.4%.

The Bank of England expects the consumer prices index will climb to about 6% by April, and the International Monetary Fund warned inflation and

Inflation

	GDP	Consum.	Invest.	Employment
Inflation (t)	0.04	-0.18	0.01	0.26
Inflation (t+*1*)	0.19	-0.05	0.15	0.36
Inflation (t+*2*)	0.31	0.08	0.26	0.42
Inflation (t+*3*)	0.41	0.21	0.33	0.41

Correlation matrix for the U.S. 1960-2019 - Deviations from HP Trend, quarterly

Source: Fred

In the news

The Telegraph News Business Sport Opinion Politics World Money Life Style Travel Culture

Economy Companies Markets Tech Alex

Bank of England should have predicted inflation and raised rates, says ex-Treasury minister

Household spending falls sharply as cost of living crisis grips Britain

By Tom Rees
8 February 2022 • 6:00am

Related Topics
Inflation, UK economy

18

Former Treasury minister Lord O'Neill has hit out at the Bank of England's record of low interest rates and said central bankers should have seen the current wave of inflation coming.

In the news

Support the Guardian

Available for everyone, funded by readers

Support us →

The Guardian

News website of the year

News | Opinion | Sport | Culture | Lifestyle | More

Business ▶ Economics Banking Money Markets Project Syndicate B2B Retail

Interest rates

Bank of England could be forced to raise interest rates again, says policymaker

Inflation could become entrenched as Covid, energy shock and Brexit hit economy, says Catherine Mann

Richard Partington

@RJPartington

Mon 6 Feb 2023 11.17 GMT

Some evidence

Effect of monetary policy on real economy:

- Endogeneity problem: central bank responds to forecasts
- Romer and Romer (2004): identify deviations from usual CB's response

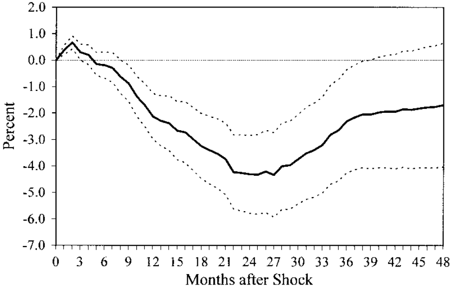
$$\begin{aligned} \Delta i_m &= \alpha + \beta i_m^b + \sum_{j=-1}^{2} [\gamma_j \Delta \tilde{y}_{mj} + \lambda_j (\Delta \tilde{y}_{mj} - \Delta \tilde{y}_{m-1j}) \\ &+ \varphi_j \tilde{\pi}_{mj} + \theta_j (\tilde{\pi}_{mj} - \tilde{\pi}_{m-1j})] + \rho \tilde{u}_{m0} + \varepsilon_m \end{aligned}$$

  $\Delta \tilde{y}_{mj}, \tilde{\pi}_{mj}, \tilde{u}_{mj}$ are forecast growth, inflation, unemployment, quart. j, meeting m

- Relationship between output and interest rate shocks:

$$\Delta y_t = a_0 + \sum_{k=1}^{11} a_k D_{kt} + \sum_{j=1}^{36} b_j \varepsilon_{kt-j} + \sum_{j=1}^{24} c_i \Delta y_{kt-j}$$

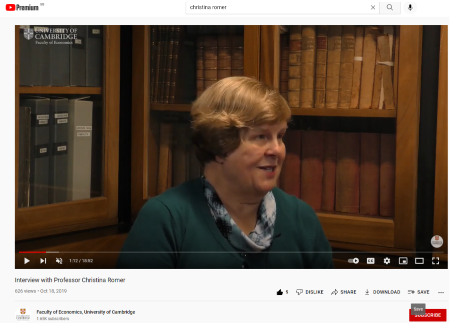
Some evidence



Impulse response to monetary policy shock of 1 percentage point (monthly)

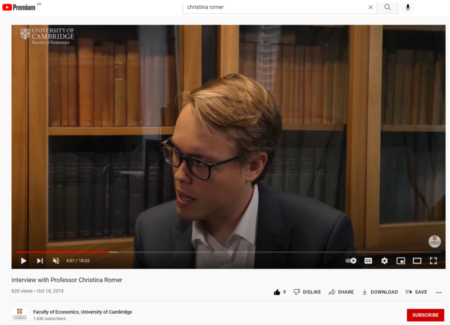
Source: Romer and Romer (2004)

Christina Romer



Source: https://www.youtube.com/watch?v=psLIQekHAfo

Christina Romer AND ME



Source: https://www.youtube.com/watch?v=psLIQekHAfo

New Keynesian DSGE lectures

- Lecture 1: Introduction to nominal rigidity, set up NK-DSGE model
- Lecture 2: Solve model with sticky prices, determinacy, analysis
- Lecture 3: Unemployment in NK-DSGE, extensions, critiques

This lecture

- Introduction to nominal rigidities
- Solve the canonical New Keynesian model under flexible prices
- Derive optimal price setting with nominal rigidity

Reference

Gali (2008) *Monetary Policy, Inflation and the Business Cycle*, Ch 1 and 3

This lecture

- **Introduction to nominal rigidities**
- Set up the canonical New Keynesian model under flexible prices
- Derive optimal price setting with nominal rigidity

Nominal rigidities

New Keynesian DSGE add **nominal rigidities** to the RBC model

- **Price rigidity**: price-adjustments are less frequent than expected
- Wage (..): wage-adjustments are very infrequent, esp. downwards

Key conceptual difference: business cycle is **inefficient**

- Output and employment are lower (or higher) than optimal
- Model can allow for **involuntary unemployment**

Are prices sticky?

Price stickiness is straightforward to observe

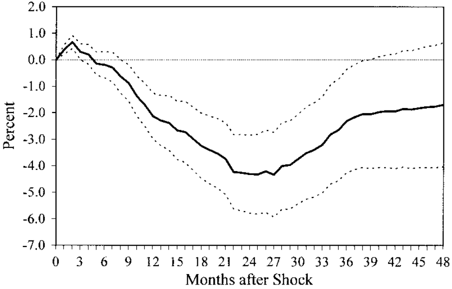
- Supply and demand vary constantly $\Rightarrow$ consumer prices do not
- What about Uber?

Significant evidence in the literature. E.g.:

- Median price duration is 8 to 11 months in U.S. CPI micro data (Nakamura and Steinsson 2006)
- Some evidence that prices are even **more sticky** in the Euro Area (Dhyne et al. 2006)
- Although there is significant heterogeneity across products (Taylor 1999, Dhyne et al. 2006, Cavallo 2018 - ‘Billion Prices Project’)

Are prices sticky?

Prices also respond slower than output after shocks:

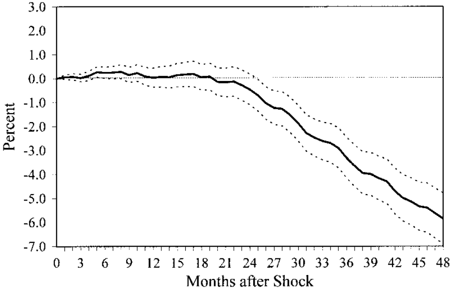


Impulse response to monetary policy shock of 1 percentage point (monthly)

Source: Romer and Romer (2004)

Are prices sticky?

Prices also respond slower than output after shocks:

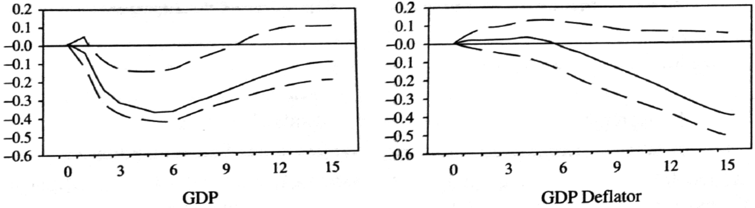


Impulse response to monetary policy shock of 1 percentage point (monthly)

Source: Romer and Romer (2004)

Are prices sticky?

Prices also respond slower than output after shocks:



Impulse response to monetary policy shock of 1 percentage point (quarterly)

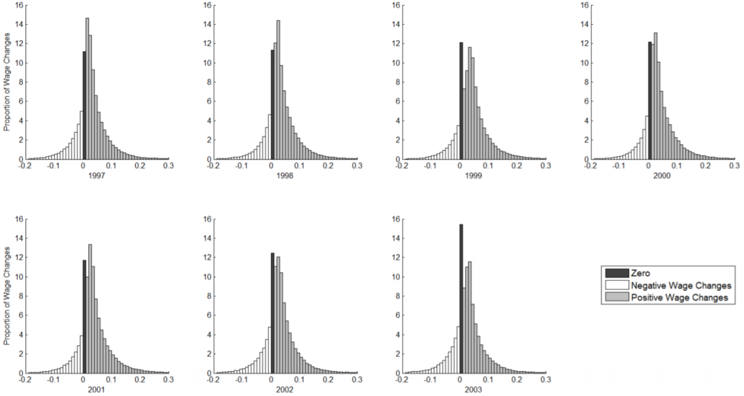
Source: Christiano Eichenbaum Evans (1999)

Why are prices sticky?

Many theories... (see Blinder 1994)

- Menu costs: price changes are too costly
- Prices fixed by contracts
- Implicit contracts: price changes 'unfair', risk of losing customers
- Cost-based pricing rules: costs may be sticky $\rightarrow$ don't change price
- Sticky information: don't know you should change price (Mankiw and Reis 2002)

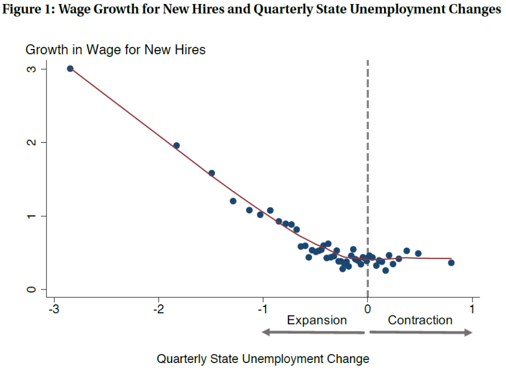
Sticky wages?



Distribution of nominal wage changes in Germany

Source: Ehrlich and Montes (2020)

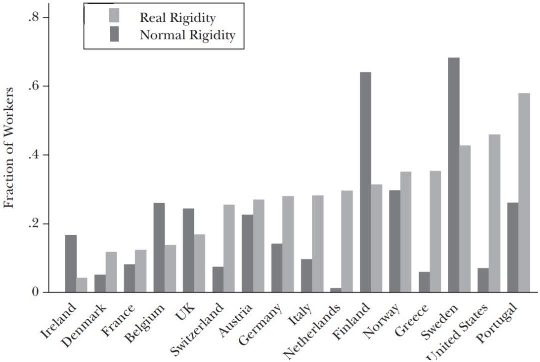
Sticky wages?



Relation between Δ unemployment and Δ wage for vacancies (U.S.)

Source: Hazell (2020)

Sticky wages?



Rigidity in wages is heterogeneous across countries (using spike at 0)

Source: Dickens et al. (2007)

Why are wages sticky?

Reference: **Bewley (1999)** ‘Why Wages Don’t Fall During Recessions’

- Interviews with 300 managers during 1990s recession

Rigidity is mainly driven by:

- Morale
  - Pay cuts hit everyone, layoffs only the laid off
  - Increases staff turnover, reduces productivity
- Distributional effect: best staff leaves (layoff: least productive staff)

This lecture

- Introduction to nominal rigidities
- **Set up the canonical New Keynesian model, flexible prices**
- Derive optimal price setting with nominal rigidity

Overview

Households:

- Consume a **basket of goods** and supply labor to firms
- Save in the form of a risk-free government **bond**
- Own firms and receive **dividends** if they make profits

Firms:

- Produce **differentiated** goods
- Choose the price at which they sell their variety; **staggered** (Calvo)

Central Bank:

- Set the **nominal interest rate** on government bonds

Representative household: problem

$$\max_{C_t, N_t, B_t} \quad \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U(C_t, L_t),$$

subject to

$$\int_0^1 P_{i,t} C_{i,t} di + Q_t B_t \leq B_{t-1} + W_t L_t + Profits_t, \quad \text{and no-ponzi}$$

- B_t: one-period, riskless, bonds maturing in $t+1$
- Q_t: price of bond paying one unit of money at maturity
- Consumption is an aggregate of individual goods i:

Representative household: changes

1. Prices now feature in the budget constraint
   - Bonds carry risk-free **nominal** interest rate $Q_t^{-1} - 1$
2. Consumption goods are no longer perfect substitutes
   - There is a continuum of **varieties** of measure 1
   - Households derive utility from consuming a basket of goods

$$C_t = \left[\int_0^1 C_{i,t}^{\frac{\epsilon - 1}{\epsilon}} di \right]^{\frac{\epsilon}{\epsilon - 1}}$$

   - Constant Elasticity of Substitution (CES) aggregator, elasticity $\epsilon > 1$

Representative household: consumption bundle

Bundle of varieties C_t is chosen to minimize expenditures.

$$
\begin{aligned}
\mathfrak{L} &= \int_0^1 P_{i,t} C_{i,t} di - \lambda \left(\left[\int_0^1 C_{i,t}^{1-1/\epsilon} di \right]^{\frac{\epsilon}{\epsilon-1}} - C_t \right) \\
\frac{\partial \mathfrak{L}}{\partial C_{i,t}} &= P_{i,t} - \lambda \left(\frac{\epsilon - 1}{\epsilon} \frac{\epsilon}{\epsilon - 1} C_{i,t}^{-1/\epsilon} \right) \left[\int_0^1 C_{i,t}^{1-1/\epsilon} di \right]^{\frac{\epsilon}{\epsilon-1} - 1} = 0 \\
&\Rightarrow \quad C_{i,t} = \lambda^{\epsilon} p_{i,t}^{-\epsilon} C_t
\end{aligned}
$$

Hence relative demand for any two goods i and j:

$$
C_{i,t}/C_{j,t} = (P_{i,t}/P_{j,t})^{-\epsilon}
$$

As a function of total expenditure:

$$
\int_0^1 P_{i,t} C_{i,t} di = \int_0^1 P_{i,t} \left(\frac{P_{i,t}}{P_{j,t}} \right)^{-\epsilon} C_{j,t} di \Rightarrow C_{jt} = \frac{P_{j,t}^{-\epsilon} \int_0^1 P_{i,t} C_{i,t} di}{\int_0^1 P_{i,t}^{1-\epsilon} di}
$$

Representative household: consumption bundle and prices

Definition: price index P_t is expenditure required to purchase 1 basket

$$P_t = \frac{\int_0^1 P_{i,t} C_{i,t} di}{C_t}$$

To find the index, insert the demand for individual goods i :

$$\begin{aligned}
P_t &= \frac{\int_0^1 P_{i,t} C_{i,t} di}{\left[\int_0^1 C_{i,t}^{1-1/\epsilon} di\right]^{\frac{\epsilon}{\epsilon-1}}} \\
&= \frac{\int_0^1 P_{i,t} \left(\frac{P_{i,t}^{-\epsilon} \int_0^1 P_{i,t} C_{i,t} di}{\int_0^1 P_{i,t}^{1-\epsilon} di} \right) di}{\left[\int_0^1 \left(\frac{P_{i,t}^{-\epsilon} \int_0^1 P_{i,t} C_{i,t} di}{\int_0^1 P_{i,t}^{1-\epsilon} di} \right)^{1-1/\epsilon} di\right]^{\frac{\epsilon}{\epsilon-1}}} \\
&= \left[\int_0^1 P_{i,t}^{1-\epsilon} di\right]^{\frac{1}{1-\epsilon}}
\end{aligned}$$

Representative household: optimality

Optimal consumption and bond holdings (Euler):

$$Q_t = \beta \mathbb{E}_t \left[\frac{U'_{C,t+1}}{U'_{C,t}} \frac{P_t}{P_{t+1}} \right]$$

Static labor vs consumption optimization:

$$-\frac{U'_{L,t}}{U'_{C,t}} = \frac{W_t}{P_t}$$

Optimal expenditure allocation

$$C_{i,t} = \left(\frac{P_{i,t}}{P_t} \right)^{-\epsilon} C_t$$

Representative household: optimality

Remainder of the lecture:

$$U(C_t, L_t) = \frac{C_t^{1-\sigma} - 1}{1-\sigma} - \frac{L_t^{1+\varphi}}{1+\varphi}$$

Euler equation:

$$Q_t = \beta \mathbb{E}_t \left[\frac{U'_{C,t+1}}{U'_{C,t}} \frac{P_t}{P_{t+1}} \right] \Rightarrow Q_t = \beta \mathbb{E}_t \left[\left(\frac{C_t}{C_{t+1}} \right)^{\sigma} \frac{P_t}{P_{t+1}} \right]$$

Static labor vs consumption optimization:

$$-\frac{U'_{L,t}}{U'_{C,t}} = \frac{W_t}{P_t} \Rightarrow C_t^{\sigma} L_t^{\varphi} = \frac{W_t}{P_t}$$

Representative household: log-linearized

Notation:

$$x_t \equiv \log X_t$$

Euler equation:

$$c_t = \mathbb{E}_t\left(c_{t+1}\right) - \frac{1}{\sigma}\left(\underbrace{i_t}_{-\log Q_t} - \underbrace{E_t\left[\pi_{t+1}\right]}_{\log P_{t+1}/P_t} - \underbrace{\rho}_{-\log \beta}\right)$$

Static labor vs consumption:

$$w_t - p_t = \sigma c_t + \varphi l_t$$

(note: add log-steady state values)

Derivation?

Firms

- There is no longer a representative firm: firms produce a **variety**
- Firm index *i*, continuum of measure 1, **monopolist** in production of *i*
- All firms have the same production function, same productivity:

$$Y_{i,t} = A_t L_{i,t}^{1-\alpha}$$

- As monopolist in production of *i*, they have **pricing power**
- But sticky prices: firms can choose price only with some probability

Firms: flexible price

Say prices were flexible and firms could set them every t:

- Firm maximizes present value of dividends for owners (households)
- Households discount **utility** at rate β but **income** at rate $Q_{t,t+k}$
- $Q_{t,t+k}$: inv. gross nominal interest between today (t) and $t+k$
- Intuition: if household saves 1 today, expects $1 \cdot Q_{t,t+k}^{-1}$ at $t+k$
  - From Euler: $Q_{t,t+\mathbf{1}} = \beta \left[\frac{U'_{C,t+\mathbf{1}}}{U'_{C,t}} \frac{P_t}{P_{t+\mathbf{1}}} \right] \Rightarrow Q_{t,t+k} = \beta^k \left[\frac{U'_{C,t+k}}{U'_{C,t}} \frac{P_t}{P_{t+k}} \right]$
  - $Q_{t,t+k}$ is also known as the 'stochastic discount factor'
- Hence, firms maximize:

$$\mathbb{E}_t \sum_{k=0}^{\infty} Q_{t,t+k} \left(P_{i,t+k} Y_{i,t+k} - W_{t+k} L_{i t+k} \right)$$

Firms: flexible price

Optimization problem:

$$\max_{P_{i,t}} \mathbb{E}_t \sum_{k=0}^{\infty} Q_{t,t+k} \left(P_{i,t+k} Y_{i,t+k} - W_{t+k} L_{t+k}\right)$$

$$\text{s.t. } Y_{it} = C_{it} = \left(P_{it}/P_t\right)^{-\epsilon} C_t \text{ and } Y_{it} = A_t L_{it}^{1-\alpha}$$

Inserting the constraints:

$$\max_{P_{i,t}} \mathbb{E}_t \sum_{k=0}^{\infty} Q_{t,t+k} \left(P_{i,t+k} \left(\frac{P_{i,t+k}}{P_{t+k}}\right)^{-\epsilon} C_{t+k} - W_{t+k} \left[\left(\frac{P_{i,t+k}}{P_{t+k}}\right)^{-\epsilon} \frac{C_{t+k}}{A_{t+k}} \right]^{\frac{1}{1-\alpha}} \right)$$

Firms: flexible price

$$\max_{P_{i,t}} \mathbb{E}_t \sum_{k=0}^{\infty} Q_{t,t+k} \left(P_{i,t+k} \left(\frac{P_{i,t+k}}{P_{t+k}} \right)^{-\epsilon} C_{t+k} - W_{t+k} \left[\left(\frac{P_{i,t+k}}{P_{t+k}} \right)^{-\epsilon} \frac{C_{t+k}}{A_{t+k}} \right]^{\frac{1}{1-\alpha}} \right)$$

1. Take first order condition with respect to $P_{i,t}$

$$\mathbb{E}_t \left([1-\epsilon] \left(\frac{P_{i,t}}{P_t} \right)^{-\epsilon} C_t + \frac{1}{1-\alpha} \epsilon W_t P_{i,t}^{-1} \left[\left(\frac{P_{i,t}}{P_t} \right)^{-\epsilon} \frac{C_t}{A_t} \right]^{\frac{1}{1-\alpha}} \right) = 0$$

2. **Symmetric equilibrium**: all firms have same FOC s.t. $P_{i,t} = P_t$

$$P_t^{FLEX} = \frac{\epsilon}{\epsilon - 1} W_t \left(\frac{1}{1-\alpha} \right) \left(\frac{1}{A_t} \right)^{\frac{1}{1-\alpha}} C_t^{\frac{\alpha}{1-\alpha}}$$

Standard result for CES competition with flexible prices :

- Price is **constant markup** $\frac{\varepsilon}{\varepsilon-1}$ over marginal cost
- Note: marginal cost is W_t/A_t if $\alpha = 0$

Aggregate variables

Wages:

- First order condition for pricing:

$$P_{it} = \frac{\epsilon}{\epsilon - 1} W_t \left(\frac{1}{1-\alpha} \right) \left(\frac{1}{A_t} \right)^{\frac{1}{1-\alpha}} C_t^{\frac{\alpha}{1-\alpha}}$$

- Divide by price index, impose symmetry such that $P_{it} = P_t$

$$1 = \left(\frac{\epsilon}{\epsilon - 1} \frac{W_t}{P_t} \left(\frac{1}{1-\alpha} \right) \left(\frac{1}{A_t} \right)^{\frac{1}{1-\alpha}} C_t^{\frac{\alpha}{1-\alpha}} \right)$$

$$\frac{W_t}{P_t} = \frac{\epsilon - 1}{\epsilon} \left(\underbrace{A_t^{\frac{1}{1-\alpha}} C_t^{-\frac{\alpha}{1-\alpha}} (1-\alpha)}_{MPL_t} \right)$$

Aggregate variables

Wage is **marked down** because of market power in product market

- 'Aggregate Demand externality'
- Lower wage reduces labor supply:

$$
\begin{aligned}
L_t &= \left[C_t^{-\sigma} \left(\frac{W_t}{P_t} \right) \right]^{1/\varphi} \\
&= \left[C_t^{-\sigma} \left(\frac{\epsilon - 1}{\epsilon} A_t^{\frac{1}{1-\alpha}} C_t^{-\frac{\alpha}{1-\alpha}} (1-\alpha) \right) \right]^{1/\varphi}
\end{aligned}
$$

Aggregate variables

GDP:

- From the goods market equilibrium $C_t = Y_t$, $C_{it} = Y_{it}$:

$$Y_t = \left[\int_0^1 Y_{i,t}^{1-1/\epsilon} di\right]^{\frac{\epsilon}{\epsilon-1}}$$

- Production function $Y_{it} = A_t L_{it}^{1-\alpha}$ and using symmetry:

$$Y_t = \left[\int_0^1 \left(A_t L_t^{1-\alpha}\right)^{1-1/\epsilon} di\right]^{\frac{\epsilon}{\epsilon-1}} = A_t L_t^{1-\alpha}$$

- Insert labor supply, isolate output:

$$Y_t = A_t^{\frac{\varphi+1}{\zeta}} \left(\frac{\epsilon-1}{\epsilon}\right)^{\frac{1-\alpha}{\zeta}} (1-\alpha)^{\frac{1-\alpha}{\zeta}}$$

where $\zeta = \sigma(1-\alpha) + \alpha + \varphi$

Flexible price symmetric equilibrium

Definition: sequence for the combination of quantities and prices $\{L_t, W_t/P_t, Y_t\}$ such that:

- Households first order condition for optimal labor supply
- Firm's first order condition for optimal prices (subject to demand)
- Technology constraint: production function

Flexible price symmetric equilibrium

Definition: sequence for the combination of quantities and prices $\{L_t, W_t/P_t, Y_t, C_t\}$ such that:

- Households first order condition for optimal labor supply
- Firm's first order condition for optimal prices (subject to demand)
- Technology constraint: production function
- Resource constraint: $C_t = Y_t$

Flexible price symmetric equilibrium

Definition: sequence for the combination of quantities and prices $\{L_t, W_t/P_t, Y_t, C_t, \mathbb{E}_t(P_t/P_{t+1})Q_t^{-1}\}$ such that:

- Households first order condition for optimal labor supply
- Firm's first order condition for optimal prices (subject to demand)
- Technology constraint: production function
- Resource constraint: $C_t = Y_t$
- Ex-ante real interest rate from Euler equation

Note: we don't have determinacy for nominal interest rate, inflation

Efficiency

Compared to the social efficient level, there is too little production

- Firms raise prices to maximize profits
- Higher prices $\Rightarrow$ lower output, lower labor demand
- Lower wages reduce labor supply **if labor is supplied elastically**

Social planner would set markup to 1:

$$Y_t^* = A_t^{\frac{\varphi+1}{\zeta}} (1-\alpha)^{\frac{1-\alpha}{\zeta}}$$

$$\frac{Y_t^{Flex}}{Y_t^*} = \left(\frac{\epsilon - 1}{\epsilon}\right)^{\frac{1-\alpha}{\zeta}} < 1$$

This lecture

- Introduction to nominal rigidities
- Set up the canonical New Keynesian model under flexible prices
- **Derive optimal price setting with nominal rigidity**

Calvo pricing

The Calvo Fairy

- Firms change price with probability $(1-\theta)$
- Expected price duration: $(1-\theta)^{-1}$

Firms: sticky prices

- Firms will no longer be in a **symmetric equilibrium**
- At time t, set price P_{it}^* to maximize present value of dividends:

$$P_{it}^* = \arg\max_{P_{i,t}} \sum_{k=0}^{\infty} \theta^k \mathbb{E}_t Q_{t,t+k} \left(P_{i,t} Y_{i,t+k} - W_{t+k} L_{it+k} \right)$$

$$\text{s.t. } Y_{it+k} = \left(P_{it} / P_{t+k} \right)^{-\epsilon} C_{t+k} \text{ and } Y_{i,t+k} = A_{t+k} L_{it+k}^{1-\alpha}$$

- Define $\Psi_{t+k}\left(Y_{t+k|t}\right)$ as costs at $t+k$ for firm that set prices at t

$$\max \sum_{k=0}^{\infty} \theta^k \mathbb{E}_t Q_{t,t+k} \left(P_t^* Y_{t+k|t} - \Psi_{t+k}\left(Y_{t+k|t}\right)\right) \text{ s.t. } Y_{t+k|t} = \left(\frac{P_t^*}{P_{t+k}} \right)^{-\epsilon} C_{t+k}$$

Firms: sticky prices

First order condition:

$$\sum_{k=0}^{\infty} \theta^k \mathbb{E}_t Q_{t,t+k} \left(Y_{t+k|t} + P_t^* \frac{\partial Y_{t+k|t}}{\partial P_t^*} - \frac{\partial \Psi_{t+k}\left(Y_{t+k|t}\right)}{\partial Y_{t+k|t}} \frac{\partial Y_{t+k|t}}{\partial P_t^*} \right) = 0$$

where:

$$\frac{\partial Y_{t+k|t}}{\partial P_t^*} = -\left(\frac{\epsilon}{P_t^*}\right)\left(\frac{P_t^*}{P_{t+k}}\right)^{-\epsilon} C_{t+k} = -\epsilon\left(\frac{Y_{t+k|t}}{P_t^*}\right)$$

$$\frac{\partial \Psi_{t+k}\left(Y_{t+k|t}\right)}{\partial Y_{t+k|t}} = \psi_{t+k|t} \Rightarrow \textbf{nominal marginal cost}$$

such that:

$$\sum_{k=0}^{\infty} \theta^k \mathbb{E}_t Q_{t,t+k} \left[Y_{t+k|t}(1-\varepsilon) - \psi_{t+k|t}(-\epsilon)\left(\frac{Y_{t+k|t}}{P_t^*}\right)\right] = 0$$

$$\sum_{k=0}^{\infty} \theta^k \mathbb{E}_t Q_{t,t+k} Y_{t+k|t} \left[P_t^* - \underbrace{\left(\frac{\epsilon}{\epsilon-1}\right)\psi_{t+k|t}}_{\text{flex price m.u. x mar. costs}} \right] = 0$$

Firms: sticky prices log-linearized

Rewrite the first order condition in terms with well-defined steady state:

$$\sum_{k=0}^{\infty} \theta^k \mathbb{E}_t Q_{t,t+k} Y_{t+k|t} \left[P_t^* - \left(\frac{\epsilon}{\epsilon - 1} \right) \psi_{t+k|t} \right] = 0$$

$$\sum_{k=0}^{\infty} \theta^k \mathbb{E}_t Q_{t,t+k} Y_{t+k|t} \left[\frac{P_t^*}{P_{t-1}} - \left(\frac{\epsilon}{\epsilon - 1} \right) \underbrace{MC_{t+k|t}}_{\psi_{t+k|t}/P_{t+k}} \underbrace{\Pi_{t-1,t+k}}_{P_{t+k}/P_{t-1}} \right] = 0$$

To log-linearize around the steady state, use:

- Zero inflation: $P_t^*/P_{t-1} = 1$ and $\Pi_{t-1,t+k} = 1$
- Symmetry: $Y_{t,t+k} = Y$, $MC_{t+k|t} = MC$, $P^* = P_{t+k}$
- No inflation, growth: same discounting for income and utility $Q_{t,t+k} = \beta^k$

Firms: sticky prices log-linearized

Left-hand side of the equation:

$$X_t = \sum_{k=0}^{\infty} \theta^k \mathbb{E}_t Q_{t,t+k} Y_{t+k|t} \left[\frac{P_t^*}{P_{t-1}} \right]$$

Step 1: steady state using $P_t^*/P_{t-1} = 1$ and $\Pi_{t-1,t+k} = 1$, $Q_{t,t+k} = \beta^k$, symmetry:

$$X = \frac{Y}{1 - \beta\theta}$$

Step 2: log-linearize the left hand side equation:

- Write the function in exponential terms in deviations from the steady state:

$$X_t = \sum_{k=0}^{\infty} \theta^k \mathbb{E}_t Q_{t,t+k} Y \left(e^{\widehat{q}_{t,t+k} + \widehat{y}_{t+k|t} + p_t^* - p_{t-1}} \right)$$

- First-order Taylor approximation:

$$\begin{aligned} x_t \quad &\approx \quad X + \sum_{k=0}^{\infty} \mathbb{E}_t \left[\frac{\partial X_t}{\partial \widehat{q}_{t,t+k}} \Big|_{X_t = X} \left(\widehat{q}_{t,t+k} \right) + \frac{\partial X_t}{\partial \widehat{y}_{t+k|t}} \Big|_{X_t = X} \left(\widehat{y}_{t,t+k} \right) \right] + \frac{\partial X_t}{\partial p_t^* - p_{t-1}} \Big|_{X_t = X} \left(p_t^* - p_{t-1} \right) \\ &= \quad X + Y \sum_{k=0}^{\infty} (\theta\beta)^k \mathbb{E}_t (\widehat{q}_{t,t+k} + \widehat{y}_{t+k|t}) + \frac{Y}{1 - \beta\theta} \left(p_t^* - p_{t-1} \right) \end{aligned}$$

Firms: sticky prices log-linearized

Right-hand side of the equation:

$$X_t = \sum_{k=0}^{\infty} \theta^k \mathbb{E}_t Q_{t,t+k} Y_{t+k|t} \left(\frac{\epsilon}{\epsilon - 1} \right) MC_{t+k|t} \Pi_{t-\mathbf{1},t+k}$$

Step 1: steady state using $P_t^*/P_{t-1} = 1$ and $\Pi_{t-\mathbf{1},t+k} = 1$, $Q_{t,t+k} = \beta^k$, symmetry:

$$X = \frac{Y}{1-\beta\theta} \frac{\epsilon}{\epsilon - 1} MC$$

Step 2: log-linearize the left hand side equation:

- Write the function in exponential terms in deviations from the steady state:

$$X_t = \sum_{k=0}^{\infty} \theta^k \mathbb{E}_t Q_{t,t+k} Y(MC) \left(\frac{\epsilon}{\epsilon - 1} \right) \left(e^{\widehat{q}_{t+k|t} + \widehat{y}_{t+k|t} + \widehat{mc}_{t+k|t} + \pi_{t-\mathbf{1},t+k}} \right)$$

- First-order Taylor approximation:

$$\begin{aligned} X_t \quad &\approx \quad X + \sum_{k=\mathbf{0}}^{\infty} \mathbb{E}_t \left[\frac{\partial X_t}{\partial \widehat{q}_{t+k|t}} \Big|_{X_t = X} \left(\widehat{q}_{t+k|t} \right) + \frac{\partial X_t}{\partial \widehat{y}_{t+k|t}} \Big|_{X_t = X} \left(\widehat{y}_{t+k|t} \right) + \frac{\partial X_t}{\partial \widehat{mc}_{t+k|t}} \Big|_{..} \left(\widehat{mc}_{t+k|t} \right) \right] \\ &\quad + \frac{\partial X_t}{\partial \widehat{\pi}_{t-\mathbf{1},t+k}} \Big|_{.} \left(\pi_{t-\mathbf{1},t+k} \right) \\ &= \quad X + MC \left(\frac{\epsilon}{\epsilon - \mathbf{1}} \right) Y \sum_{k=\mathbf{0}}^{\infty} (\theta\beta)^k \mathbb{E}_t \left(\widehat{q}_{t+k|t} + \widehat{y}_{t+k|t} + \widehat{mc}_{t+k|t} + \pi_{t-\mathbf{1},t+k} \right) \end{aligned}$$

Firms: sticky prices log-linearized

Left-hand side log-linearized:

$$X + Y \sum_{k=0}^{\infty} (\theta\beta)^k \mathbb{E}_t(\widehat{q}_{t,t+k} + \widehat{y}_{t+k|t}) + \frac{Y}{1-\beta\theta}(p_t^* - p_{t-1})$$

Right-hand side log-linearized:

$$X + \underbrace{MC\left(\frac{\epsilon}{\epsilon-1}\right)}_{=1 \text{ given } P=MC\cdot P\frac{\epsilon}{\epsilon-1}} Y \sum_{k=0}^{\infty} (\theta\beta)^k \mathbb{E}_t\left(\widehat{q}_{t,t+k} + \widehat{y}_{t+k|t} + \widehat{mc}_{t+k|t} + \pi_{t-1,t+k}\right)$$

Equate and solve for $p_t^* - p_{t-1}$ to get:

$$\begin{aligned} p_t^* - p_{t-1} &= (1-\beta\theta) \sum_{k=0}^{\infty} (\theta\beta)^k \mathbb{E}_t\left(\widehat{mc}_{t+k|t} + \pi_{t-1,t+k}\right) \\ &= (1-\beta\theta) \sum_{k=0}^{\infty} (\theta\beta)^k \, \mathbb{E}_t(\widehat{mc}_{t+k|t} + p_{t+k} - p_{t-1}) \end{aligned}$$

Price index

Advantage of log-linearization: straightforward expression for inflation

- How does price index develop?

$$
\begin{aligned}
P_t &= \left[\int_0^1 P_{i,t}^{1-\epsilon} di\right]^{\frac{1}{1-\epsilon}} \\
&= \left[\theta \underbrace{\int_0^1 P_{i,t-1}^{1-\epsilon} di}_{P_{t-1}^{1-\epsilon}} + (1-\theta) \underbrace{\int_0^1 (P_t^*)^{1-\epsilon} di}_{(P_t^*)^{1-\epsilon}}\right]^{\frac{1}{1-\epsilon}} \\
&= \left[\theta P_{t-1}^{1-\epsilon} + (1-\theta)(P_t^*)^{1-\epsilon}\right]^{\frac{1}{1-\epsilon}}
\end{aligned}
$$

- Note: continuum of firms, **law of large numbers** applies

Price index

- Index:

$$P_t = \left[\theta P_{t-1}^{1-\epsilon} + (1-\theta)(P_t^*)^{1-\epsilon}\right]^{\frac{1}{1-\epsilon}}$$

- **Define:** gross inflation rate is

$$\Pi_t \equiv \frac{P_t}{P_{t-1}} \Rightarrow \Pi_t^{1-\epsilon} = \theta + (1-\theta)\left(\frac{P_t^*}{P_{t-1}}\right)^{1-\epsilon}$$

- Log-linearized:

$$\pi_t = (1-\theta)(p_t^* - p_{t-1})$$

Next week

- Dynamic IS Equation

$$\widehat{y_t} = -\frac{1}{\sigma}\left(i_t - \mathbb{E}_t(\pi_{t+1}) - \rho\right) + \mathbb{E}_t\left(\widehat{y_{t+1}}\right)$$

- New Keynesian Philips Curve

$$\pi_t = \beta\mathbb{E}_t\left(\pi_{t+1}\right) + \kappa\widehat{y_t}$$

- Monetary policy rule:

$$i_t = \rho + \phi_\pi \pi_t + \phi_y \widehat{y_t} + v_t$$

What have we done?

1. Empirical evidence: nominal rigidity, real effect of monetary policy ✓
2. Setup of the New Keynesian Model
   - Constant Elasticity of Substitution Aggregator (CES) ✓
   - Derive first order conditions for sticky-price firm problem ✓
   - Linearize household and firm first order conditions ✓
3. Derive the equilibrium under flexible prices ✓