Solving Quantitative Model

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UIBE

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Road Map

- Model-1: EK model with no intermediate input
  - Eaton and Kortum (2002)
- Model-2: EK model with intermediate input
  - Eaton and Kortum (2002)
- Model-3: EK model & migration
  - Redding(2016)
- Model-4: EK model with Input-Output linkage
  - Caliendo and Parro(2015)

Model-1

Model-1 Setup

- utility function: $U_i = \left[\int_0^1 q(j)^{\frac{\sigma-1}{\sigma}} dj\right]^{\frac{\sigma}{\sigma-1}}$
- production function: $q_i(j) = z_i(j)L_i(j)$
- random productivity: $z_i(j) \sim F_i(z) = e^{-T_i z^{-\theta}}$
- unit cost of input: $c_i = w_i$
- ice-berg cost: d_{ni}
- trade price: $p_{ni}(j) = \frac{c_i}{z_i(j)} d_{ni}$

Model-1 Trade Gravity Equation

- trade share

$$\pi_{ni} = \frac{T_i \left(c_i d_{ni}\right)^{-\theta}}{\sum_{i'} T_{i'} \left(c_{i'} d_{ni'}\right)^{-\theta}}$$

- price index

$$P_n = \Gamma \left[\sum_i T_i \left(c_i d_{ni}\right)^{-\theta}\right]^{-\frac{1}{\theta}}$$

where $c_i = w_i$

Model-1 Equilibrium Condition

- consumers maximize utility, producers maximize profit (c_i, π_{ni})
- labor clear condition (Y_i is total income)

$$Y_i = w_i L_i \tag{1}$$

- goods clear condition (X_n is total expenditure)

$$Y_i = \sum_n X_{ni} = \sum_n \pi_{ni} X_n \tag{2}$$

- balance trade condition

$$Y_i = X_i \tag{3}$$

- condition(1)(2)(3) implies

$$Y_i = w_i L_i = \sum_n \pi_{ni} w_n L_n == \frac{T_i \left(w_i d_{ni}\right)^{-\theta}}{\sum_{i'} T_{i'} \left(w_{i'} d_{ni'}\right)^{-\theta}} w_n L_n \tag{4}$$

Model-1 Solve equilibrium

$$w_i L_i = \frac{T_i \left(w_i d_{ni}\right)^{-\theta}}{\sum_{i'} T_{i'} \left(w_{i'} d_{ni'}\right)^{-\theta}} w_n L_n$$

- Parameters and fundamentals θ, $\{T_i\}$,$\{L_i\}$, $\{d_{ni}\}$
- Endogenous variables $\{w_i\}$,...

Model-1 Solve $\{w_i\}$ in Matlab

$$w_i L_i = \frac{T_i \left(w_i d_{ni}\right)^{-\theta}}{\sum_{i'} T_{i'} \left(w_{i'} d_{ni'}\right)^{-\theta}} w_n L_n$$

- 'fsolve' approach $\Psi(\{\mathbf{w_i}\}) = \mathbf{0}$
- iteration approach
  - step-1 initial guess of $\{\mathbf{w_i}\}$
  - step-2 calculate the two sides of equation above
  - step-3 update the guess of $\{\mathbf{w_i}\}$ and return to step-1 until the two sides of equation get closed enough (convergence condtion)

Model-1 Matlab code for Solving $\{w_i\}$

Model-2

Model-2 Setup

- utility function: $U_i = \left[\int_0^1 q(j)^{\frac{\sigma-1}{\sigma}} dj\right]^{\frac{\sigma}{\sigma-1}}$
- production function: $q_i(j) = z_i(j)\left(\frac{L_i(j)}{\alpha}\right)^{\alpha}\left(\frac{M_i(j)}{1-\alpha}\right)^{1-\alpha}$
- random productivity: $z_i(j) \sim F_i(z) = e^{-T_i z^{-\theta}}$
- unit cost of input: $c_i = w_i^{\alpha} P_n^{1-\alpha}$
- ice-berg cost: d_{ni}
- trade price: $p_{ni}(j) = \frac{c_i}{z_i(j)} d_{ni}$

Model-2 Trade Gravity Equation

- trade share

$$\pi_{ni} = \frac{T_i \left(c_i d_{ni}\right)^{-\theta}}{\sum_{i'} T_{i'} \left(c_{i'} d_{ni'}\right)^{-\theta}}$$

- price index

$$P_n = \Gamma \left[\sum_i T_i \left(c_i d_{ni}\right)^{-\theta}\right]^{-\frac{1}{\theta}}$$

where $c_i = w_i^{\alpha} P_n^{1-\alpha}$

Model-2 Equilibrium condition

- consumers maximize utility, producers maximize profit (c_i, π_{ni})
- labor clear condition (Y_i is total income)

$$Y_i = w_i L_i \tag{5}$$

- goods clear condition (X_i is total expenditure)

$$\frac{1}{\alpha} Y_i = \sum_n X_{ni} = \sum_n \pi_{ni} X_n \tag{6}$$

- balance trade condition

$$X_i = Y_i + \frac{1-\alpha}{\alpha} w_i L_i \tag{7}$$

- condition(5)(6)(7) implies

$$X_i = \frac{1}{\alpha} Y_i = \sum_n \pi_{ni} X_n = \sum_n \pi_{ni} \frac{1}{\alpha} Y_n \tag{8}$$

Model-2 Equilibrium condition

- Equation (8) implies

$$w_i L_i = \frac{T_i \left(c_i d_{ni}\right)^{-\theta}}{\sum_{i'} T_{i'} \left(c_{i'} d_{ni'}\right)^{-\theta}} w_n L_n$$

- price index

$$P_n = \Gamma \left[\sum_i T_i \left(c_i d_{ni}\right)^{-\theta} \right]^{-\frac{1}{\theta}}$$

where $c_i = w_i^{\alpha} P_n^{1-\alpha}$

Model-2 Solve $\{w_n\}$, $\{P_n\}$ in Matlab

$$P_n = \Gamma \left[\sum_i T_i \left(c_i d_{ni} \right)^{-\theta} \right]^{-\frac{1}{\theta}} \tag{9}$$

$$w_i L_i = \frac{T_i \left(c_i d_{ni} \right)^{-\theta}}{\sum_{i'} T_{i'} \left(c_{i'} d_{ni'} \right)^{-\theta}} w_n L_n \tag{10}$$

where $c_i = w_i^{\alpha} P_n^{1-\alpha}$

- step-1 initial guess of $\{\mathbf{w_n}\}$ and $\{\mathbf{P_n}\}$
- step-2 calculate $\{\mathbf{c_i}\}$ and then new $\{\mathbf{P_n}\}$ using equation (9), update guess of $\{\mathbf{P_n}\}$ until equation (9) converge
- step-3 calculate the two sides of equation (10)
- step-4 update the guess of $\{\mathbf{w_i}\}$ and return to step-1 until equation (10) converge

Model-2 Matlab code for Solving $\{w_i\}$,$\{P_i\}$

Model-3

Model-3 Setup

- consumption and production is the same as model-2
- utility function of individual ζ migrating from i to n

$$W_i(\zeta) = U_n(\zeta)\tau_{ni}^{-1}a_n(\zeta)$$

where U_n is real income in n, τ_{ni} is migration cost from i to n

- idiosyncratic location taste

$$a_n(\zeta) \sim G_n(a) = e^{-A_n a^{-\epsilon}}$$

- migration share

$$\lambda_{ni} = \frac{A_n(U_n\tau_{ni}^{-1})^{\epsilon}}{\sum_n A_{n'}(U_{n'}\tau_{n'i}^{-1})^{\epsilon}}$$

Model-3 Equilibrium equation

- trade block: given $\{\mathbf{L_n}\}$ (given labor supply after migration)

$$P_n = \Gamma \left[\sum_i T_i \left(c_i d_{ni} \right)^{-\theta} \right]^{-\frac{1}{\theta}} \tag{11}$$

$$w_i L_i = \frac{T_i \left(c_i d_{ni} \right)^{-\theta}}{\sum_{i'} T_{i'} \left(c_{i'} d_{ni'} \right)^{-\theta}} w_n L_n \tag{12}$$

- migration block (labor market clear condition)

$$L_n = \sum_i \lambda_{ni} L_{i0} \tag{13}$$

where L_{i0} is initial labor distribution and

$$\lambda_{ni} = \frac{A_n (U_n \tau_{ni}^{-1})^{\epsilon}}{\sum_n A_{n'} (U_{n'} \tau_{n'i}^{-1})^{\epsilon}}$$

Model-3 Solve Equilibrium

- step-1: initial guess of any $\{\mathbf{L_n}\}$
- step-2: take $\{\mathbf{L_n}\}$ as given, solve trade block using (11)(12)
- step-3: calculate the migration share based on real consumption from trade block and then the two sides of equation (13)
- step-4: update the guess of $\{\mathbf{L_n}\}$ and return to step-1 until equation (13) converge

Model-3 Matlab code for Solving $\{w_i\}$,$\{P_i\}$,$\{L_i\}$

Model-4

Model-4 Setup

- N countries and J sectors
- utility function: $U_j = \prod_{j=1}^{J} \left(c_n^j\right)^{\alpha_n^j}$
- unit cost of input: $c_n^j = w_n^{\gamma_n^j} \prod_{k=1}^{J} \left(P_n^k\right)^{\gamma_n^{k,j}}$
- trade share: $\pi_{ni}^j = \frac{T_i^j \left[c_i^j d_{ni}^j\right]^{-\theta^j}}{\sum_{i'=1} T_{i'}^j \left[c_{i'}^j d_{ni'}^j\right]^{-\theta^j}}$

Model-4 Equilibrium condition

solve $\{\mathbf{P_n}\}$, $\{\mathbf{X_n}\}$ and $\{\mathbf{w_n}\}$ using the following three equations

$$P_n^j = \Gamma^j \left[\sum_i T_i^j \left(c_i^j d_{ni}^j \right)^{-\theta^j} \right]^{-\frac{1}{\theta^j}} \tag{14}$$

$$X_i^j = \alpha_i^j w_i L_i + \sum_k \gamma_i^{jk} \sum_n \pi_{ni}^k X_n^k \tag{15}$$

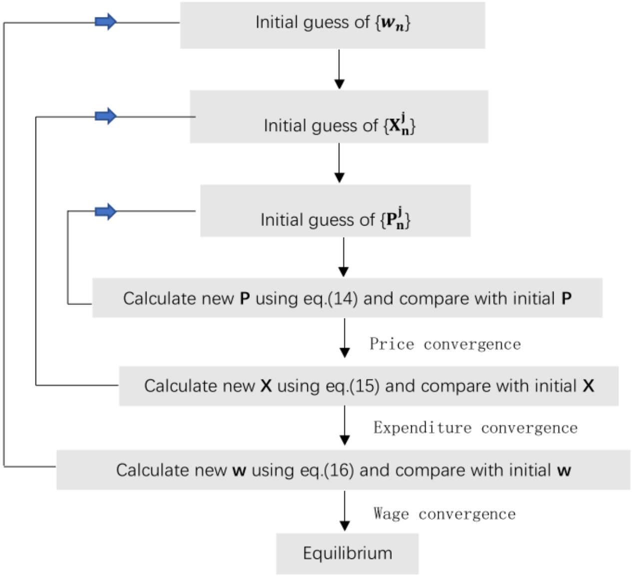
$$w_i L_i = \sum_j \gamma_i^j \sum_n \pi_{ni}^j X_n^j \tag{16}$$

where

$$c_n^j = w_n^{\gamma_n^j} \prod_{k=1}^{J} \left(P_n^k \right)^{\gamma_n^{k,j}}$$

$$\pi_{ni}^j = \frac{T_i^j \left[c_i^j d_{ni}^j \right]^{-\theta^j}}{\sum_{i'=1} T_{i'}^j \left[c_{i'}^j d_{ni'}^j \right]^{-\theta^j}}$$

Model-4 Solve Equilibrium



Model-4 Matlab code for Solving $\{w_i\}$,$\{P_i\}$,$\{X_i\}$