Lecture 5

Solving the New Keynesian DSGE Model

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EC417

This term

Part I: Shocking theory of the business cycle (weeks 1-6)

- Introduction to business cycles ✓
- Real Business Cycle (RBC) Model ✓
- New Keynesian DSGE Models $\Leftarrow$

Part II: Perspectives on business cycles and steady states (weeks 7-10)

- Persistent effects of recessions
- Aggregate shocks? Firm-heterogeneity and the business cycle
- Interesting steady states: firms, productivity, market power

New Keynesian DSGE lectures

- Lecture 1: Introduction to nominal rigidity, set up NK-DSGE ✓
- Lecture 2: Solve model with sticky prices, determinacy, analysis $\Leftarrow$
- Lecture 3: Unemployment in NK-DSGE, extensions, critiques

Nominal rigidities

New Keynesian DSGE add **nominal rigidities** to the RBC model

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New Keynesian DSGE add **nominal rigidities** to the RBC model

- **Price rigidity**: price-adjustments are less frequent than expected
- Wage (..): wage-adjustments are very infrequent, esp. downwards

Key conceptual difference: business cycle is **inefficient**

- Output and employment are lower (or higher) than optimal
- Model can allow for **involuntary unemployment**

Reference

Gali (2008) *Monetary Policy, Inflation, and the Business Cycle*, Chapter 3

Three-equation canonical model

- New Keynesian Philips Curve

$$\pi_t = \beta \mathbb{E}_t \left(\pi_{t+1} \right) + \kappa \widehat{y}_t$$

  - $\widehat{y}_t$ is the *output gap* vis a vis flexible prices, π_{t+1} is *inflation rate*
  - Comes from **firm optimization problem**

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- Dynamic IS Equation

$$\widehat{y_t} = -\frac{1}{\sigma}\left(i_t - \mathbb{E}_t(\pi_{t+1}) - \rho\right) + \mathbb{E}_t\left(\widehat{y_{t+1}}\right)$$

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- Monetary policy rule:

$$i_t = \rho + \phi_\pi \pi_t + \phi_y \widehat{y_t} + v_t$$

Previous lecture

Household:

$$\max_{C_t, N_t, B_t} \quad \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U(C_t, L_t),$$

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subject to

$$\int_0^1 P_{i,t} C_{i,t} di + Q_t B_t \leq B_{t-1} + W_t L_t + Profits_t$$

- B_t: one-period, riskless, bonds maturing in $t+1$
- Q_t: price of bond paying one unit of money at maturity

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- B_t: one-period, riskless, bonds maturing in $t+1$
- Q_t: price of bond paying one unit of money at maturity
- Consumption is an aggregate of individual goods i:

$$C_t = \left[\int_0^1 C_{i,t}^{1-1/\epsilon} di \right]^{\frac{\epsilon}{\epsilon - 1}}$$

Previous lecture

Optimal expenditure allocation

$$C_{i,t} = \left(\frac{P_{i,t}}{P_t}\right)^{-\epsilon} C_t$$

Euler equation (log-linearized):

$$c_t = \mathbb{E}_t\left(c_{t+1}\right) - \frac{1}{\sigma}\left(\underbrace{i_t}_{-\log Q_t} - E_t \underbrace{\left[\pi_{t+1}\right]}_{\log P_{t+1}/P_t} - \underbrace{\rho}_{-\log \beta}\right)$$

Static labor vs consumption (log-linearized):

$$w_t - p_t = \sigma c_t + \varphi l_t$$

Previous lecture: flexible price equilibrium

Firms have market power in product market: charge markup

$$P_t = \frac{\epsilon}{\epsilon - 1} MC_t$$

Wages are marked down because firms have product-market power:

$$\frac{W_t}{P_t} = \frac{\epsilon - 1}{\epsilon} \left(\underbrace{A_t^{\frac{1}{1-\alpha}} C_t^{-\frac{\alpha}{1-\alpha}} (1-\alpha)}_{MPL_t} \right)$$

This reduces labor supply and therefore equilibrium output:

$$Y_t = A_t^{\frac{\varphi+1}{\zeta}} \left(\frac{\epsilon - 1}{\epsilon} \right)^{\frac{1-\alpha}{\zeta}} (1-\alpha)^{\frac{1-\alpha}{\zeta}}$$

where $\zeta = \sigma(1-\alpha) + \alpha + \varphi$

This lecture

- Derive the three-equation linear NK-DSGE model
- Conditions for determinacy
- (Understanding and analyzing the model using Dynare)

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- **Derive the three-equation linear NK-DSGE model**
- Conditions for determinacy
- (Understanding and analyzing the model using Dynare)

Sorry

My kidnappers returning me after talking for two hours about the derivation of the three equation New Keynesian Dynamic Stochastic General Equilibrium Model using Calvo pricing

Source: Borui Zhu (MSc EME 2021)

Three-equation canonical model

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Calvo pricing

The Calvo Fairy

Calvo pricing

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- Firms change price with probability $(1 - \theta)$

Calvo pricing

Optimal price setting with nominal rigidity:

- At time t, set price P^*_{it} to maximize present value of dividends:

$$P^*_{it} = \arg\max_{P_{it}} \sum_{k=0}^{\infty} \theta^k \mathbb{E}_t Q_{t,t+k} \left(P_{i,t} Y_{i,t+k} - W_{t+k} L_{i,t+k} \right)$$

$$\text{s.t. } Y_{it+k} = \left(P_{it} / P_{t+k} \right)^{-\epsilon} C_{t+k} \text{ and } Y_{i,t+k} = A_{t+k} L_{it+k}^{1-\alpha}$$

where

$$P_t = \left[\int_0^1 P_{i,t}^{1-\epsilon} di \right]^{\frac{1}{1-\epsilon}}$$

Firms: sticky prices

- Firms will no longer be in a **symmetric equilibrium**
- At time t, set price P_{it}^* to maximize present value of dividends:

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- Define $\Psi_{t+k}\left(Y_{t+k|t}\right)$ as costs at $t+k$ for firm that set prices at t

$$\max \sum_{k=0}^{\infty} \theta^k \mathbb{E}_t Q_{t,t+k} \left(P_t^* Y_{t+k|t} - \Psi_{t+k}\left(Y_{t+k|t}\right)\right) \text{ s.t. } Y_{t+k|t} = \left(\frac{P_t^*}{P_{t+k}}\right)^{-\epsilon} C_{t+k}$$

Firms: sticky prices

First order condition:

$$\sum_{k=0}^{\infty} \theta^k \mathbb{E}_t Q_{t,t+k} \left(Y_{t+k|t} + P_t^* \frac{\partial Y_{t+k|t}}{\partial P_t^*} - \frac{\partial \Psi_{t+k}\left(Y_{t+k|t}\right)}{\partial Y_{t+k|t}} \frac{\partial Y_{t+k|t}}{\partial P_t^*} \right) = 0$$

where:

$$\frac{\partial Y_{t+k|t}}{\partial P_t^*} = -\left(\frac{\epsilon}{P_t^*}\right)\left(\frac{P_t^*}{P_{t+k}}\right)^{-\epsilon} C_{t+k} = -\epsilon\left(\frac{Y_{t+k|t}}{P_t^*}\right)$$

$$\frac{\partial \Psi_{t+k}\left(Y_{t+k|t}\right)}{\partial Y_{t+k|t}} = \psi_{t+k|t} \Rightarrow \textbf{nominal marginal cost}$$

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$$\frac{\partial \Psi_{t+k}\left(Y_{t+k|t}\right)}{\partial Y_{t+k|t}} = \psi_{t+k|t} \Rightarrow \textbf{nominal marginal cost}$$

such that:

$$\sum_{k=0}^{\infty} \theta^k \mathbb{E}_t Q_{t,t+k} \left[Y_{t+k|t}(1-\varepsilon) - \psi_{t+k|t}(-\epsilon)\left(\frac{Y_{t+k|t}}{P_t^*}\right)\right] = 0$$

$$\sum_{k=0}^{\infty} \theta^k \mathbb{E}_t Q_{t,t+k} Y_{t+k|t} \left[P_t^* - \underbrace{\left(\frac{\epsilon}{\epsilon-1}\right)\psi_{t+k|t}}_{\text{flex price m.u. x mar. costs}} \right] = 0$$

Firms: sticky prices log-linearized

Rewrite the first order condition in terms with well-defined steady state:

$$\sum_{k=0}^{\infty} \theta^k \mathbb{E}_t Q_{t,t+k} Y_{t+k|t} \left[P_t^* - \left(\frac{\epsilon}{\epsilon - 1} \right) \psi_{t+k|t} \right] = 0$$

$$\sum_{k=0}^{\infty} \theta^k \mathbb{E}_t Q_{t,t+k} Y_{t+k|t} \left[\frac{P_t^*}{P_{t-1}} - \left(\frac{\epsilon}{\epsilon - 1} \right) \underbrace{MC_{t+k|t}}_{\psi_{t+k|t}/P_{t+k}} \underbrace{\Pi_{t-1,t+k}}_{P_{t+k}/P_{t-1}} \right] = 0$$

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To log-linearize around the steady state, use:

- Zero inflation: $P_t^*/P_{t-1} = 1$ and $\Pi_{t-1,t+k} = 1$
- Symmetry: $Y_{t,t+k} = Y$, $MC_{t+k|t} = MC$, $P^* = P_{t+k}$
- No inflation, growth: same discounting for income and utility $Q_{t,t+k} = \beta^k$

Firms: sticky prices log-linearized

From the first-order condition of firms:

$$\sum_{k=0}^{\infty} \theta^k \mathbb{E}_t Q_{t,t+k} Y_{t+k|t} \left[\frac{P_t^*}{P_{t-1}} - \left(\frac{\epsilon}{\epsilon - 1} \right) \underbrace{MC_{t+k|t}}_{\psi_{t+k|t}/P_{t+k}} \underbrace{\Pi_{t-1,t+k}}_{P_{t+k}/P_{t-1}} \right] = 0$$

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Log-linearize around zero-inflation, symmetric steady state:

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Log-linearize around zero-inflation, symmetric steady state:

$$p_t^* - p_{t-1} \quad = \quad (1 - \beta\theta) \sum_{k=0}^{\infty} (\theta\beta)^k \; \mathbb{E}_t \big(\widehat{mc}_{t+k|t} + p_{t+k} - p_{t-1} \big)$$

Notation:

$$\widehat{x}_t \equiv x_t - x, \text{ and } x_t \equiv \log X_t$$

Price index

Advantage of log-linearization: straightforward expression for inflation

- How does price index develop?

$$
\begin{aligned}
P_t &= \left[\int_0^1 P_{i,t}^{1-\epsilon} di\right]^{\frac{1}{1-\epsilon}} \\
&= \left[\theta \underbrace{\int_0^1 P_{i,t-1}^{1-\epsilon} di}_{P_{t-1}^{1-\epsilon}} + (1-\theta) \underbrace{\int_0^1 (P_t^*)^{1-\epsilon} di}_{(P_t^*)^{1-\epsilon}}\right]^{\frac{1}{1-\epsilon}} \\
&= \left[\theta P_{t-1}^{1-\epsilon} + (1-\theta)(P_t^*)^{1-\epsilon}\right]^{\frac{1}{1-\epsilon}}
\end{aligned}
$$

- Note: continuum of firms, **law of large numbers** applies

Price index

- Index:

$$P_t = \left[\theta P_{t-1}^{1-\epsilon} + (1-\theta)(P_t^*)^{1-\epsilon}\right]^{\frac{1}{1-\epsilon}}$$

- **Define:** gross inflation rate is

$$\Pi_t \equiv \frac{P_t}{P_{t-1}} \Rightarrow \Pi_t^{1-\epsilon} = \theta + (1-\theta)\left(\frac{P_t^*}{P_{t-1}}\right)^{1-\epsilon}$$

- Log-linearized:

$$\pi_t = (1-\theta)(p_t^* - p_{t-1})$$

Remember: Overview

- New Keynesian Phillips Curve $\Leftarrow$

$$\pi_t = \beta \mathbb{E}_t (\pi_{t+1}) + \kappa \tilde{y}_t$$

- Dynamic IS Equation

$$\tilde{y}_t = -\frac{1}{\sigma} (i_t - \mathbb{E}_t(\pi_{t+1}) - r_t^n) + \mathbb{E}_t (\tilde{y}_{t+1})$$

- Monetary policy rule:

$$i_t = \rho + \phi_\pi \pi_t + \phi_y \tilde{y}_t + v_t$$

Remember: Overview

The three-equation system is the **equilibrium** system:

- All of the first order conditions hold
- All of the constraints hold
- In deriving the three-equation system, we will use all of them

Our results so far

Solution to price-setting problem (log-linearized)

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$$p_t^* - p_{t-1} \quad = \quad (1-\beta\theta)\sum_{k=0}^{\infty}(\theta\beta)^k \mathbb{E}_t\left(\widehat{mc}_{t+k|t} + p_{t+k} - p_{t-1}\right)$$

Inflation:

$$\pi_t = (1-\theta)(p_t^* - p_{t-1})$$

We want to get:

$$\pi_t = \beta\mathbb{E}_t\left(\pi_{t+1}\right) + \kappa\tilde{y}_t$$

New Keynesian Phillips Curve (NKPC)

Start from the log-linear pricing first order condition:

$$p_t^* - p_{t-1} = (1 - \beta\theta) \sum_{k=0}^{\infty} (\beta\theta)^k \mathbb{E}_t[\widehat{mc}_{t+k|t} + (p_{t+k} - p_{t-1})]$$

Steps:

1. Find an expression for marginal costs
2. Use pricing FOC to express inflation in terms of marginal costs
3. Express marginal costs in terms of output to obtain NKPC

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Insert the production function, demand function:

$$L_t = \int_0^1 \left(\frac{Y_{it}}{A_t}\right)^{\frac{1}{1-\alpha}} di = \left(\frac{Y_t}{A_t}\right)^{\frac{1}{1-\alpha}} \int_0^1 \left(\frac{P_{it}}{P_t}\right)^{\frac{-\varepsilon}{1-\alpha}} di$$

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In logs:

$$(1-\alpha) l_t = y_t - a_t + d_t$$

where price dispersion $d \equiv (1-\alpha)\int_0^1 \left(\frac{P_{it}}{P_t}\right)^{\frac{-\varepsilon}{1-\alpha}}$ is approximately 0
(see Gali Appendix 3.3 if you want)

New Keynesian Phillips Curve (NKPC)

1. Find an expression for marginal costs $\widehat{mc}_{t+k|t}$

Economy's average real marginal costs:

$$\begin{aligned} mc_t &= w_t - p_t - mpl_t \\ &= w_t - p_t - \underbrace{(a_t - \alpha l_t + \log(1-\alpha))}_{Y_t = A_t L_t^{1-\alpha} \text{ (follows previous approx.)}} \\ &= w_t - p_t - \frac{1}{1-\alpha}(a_t - \alpha y_t) - \log(1-\alpha) \end{aligned}$$

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Firm-level (or 'cohort' t-level) marginal costs:

$$\begin{aligned} mc_{t+k|t} &= w_{t+k} - p_{t+k} - mpl_{t+k|t} \\ &= w_{t+k} - p_{t+k} - \frac{1}{1-\alpha}\left(a_{t+k} - \alpha y_{t+k|t}\right) - \log(1-\alpha) \\ &= mc_{t+k} + \frac{\alpha}{1-\alpha}\left(y_{t+k|t} - y_{t+k}\right) \\ &= mc_{t+k} - \frac{\alpha\epsilon}{1-\alpha}\left(p_t^* - p_{t+k}\right) \end{aligned}$$

New Keynesian Phillips Curve (NKPC)

(2) Use pricing FOC to express inflation in terms of marginal costs

$$p_t^* - p_{t-1} = (1 - \beta\theta) \sum_{k=0}^{\infty} (\beta\theta)^k \mathbb{E}_t[\widehat{mc}_{t+k|t} - (p_{t+k} - p_{t-1})]$$

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&= (1-\beta\theta)\sum_{k=0}^{\infty}(\beta\theta)^k \mathbb{E}_t[\widehat{mc}_{t+k} - \frac{\alpha\epsilon\left(p_t^* - p_{t+k}\right)}{1-\alpha} + (p_{t+k} - p_{t-1})]
\end{aligned}
$$

Next move $p_t^* - p_{t-1}$ to LHS:

New Keynesian Phillips Curve (NKPC)

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$$
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&= (1-\beta\theta)\sum_{k=0}^{\infty}(\beta\theta)^k \mathbb{E}_t[\widehat{mc}_{t+k} - \frac{\alpha\epsilon\,(p_t^* - p_{t+k})}{1-\alpha} + (p_{t+k} - p_{t-1})]
\end{aligned}
$$

Next move $p_t^* - p_{t-1}$ to LHS:

$$
= (1-\beta\theta)\sum_{k=0}^{\infty}(\beta\theta)^k \mathbb{E}_t\left[\frac{1-\alpha}{1-\alpha+\alpha\epsilon}\widehat{mc}_{t+k} + (p_{t+k} - p_{t-1})\right]
$$

Next expand the final term, simplify

$$
= (1-\beta\theta)\frac{1-\alpha}{1-\alpha+\alpha\epsilon}\sum_{k=0}^{\infty}(\beta\theta)^k \mathbb{E}_t\widehat{mc}_{t+k} + \sum_{k=0}^{\infty}(\beta\theta)^k \mathbb{E}_t\left(\pi_{t+k}\right)
$$

Note: steps 3/4 contain a lot of simple/tedious algebra ⇒ try to derive

New Keynesian Phillips Curve (NKPC)

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$$p_t^* - p_{t-1} \quad = \quad (1-\beta\theta)\frac{1-\alpha}{1-\alpha+\alpha\epsilon}\sum_{k=0}^{\infty}(\beta\theta)^k \mathbb{E}_t \widehat{mc}_{t+k} + \sum_{k=0}^{\infty}(\beta\theta)^k \mathbb{E}_t\left(\pi_{t+k}\right)$$

Recursive formulation:

$$p_t^* - p_{t-1} = (1-\beta\theta)\frac{1-\alpha}{1-\alpha+\alpha\epsilon}\widehat{mc}_t + \pi_t + \beta\theta\mathbb{E}_t\left(p_{t+1}^* - p_t\right)$$

New Keynesian Phillips Curve (NKPC)

(2) Use pricing FOC to express inflation in terms of marginal costs

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Recursive formulation:

$$p_t^* - p_{t-1} = (1-\beta\theta)\frac{1-\alpha}{1-\alpha+\alpha\epsilon}\widehat{mc}_t + \pi_t + \beta\theta\mathbb{E}_t\left(p_{t+1}^* - p_t\right)$$

Recall: inflation is

$$\pi_t = (1-\theta)(p_t^* - p_{t-1})$$

Combined, isolating inflation on the LHS:

$$\pi_t = (1-\beta\theta)\frac{(1-\theta)}{\theta}\frac{(1-\alpha)}{1-\alpha+\alpha\epsilon}\widehat{mc}_t + \beta\mathbb{E}_t\left(\pi_{t+1}\right)$$

New Keynesian Phillips Curve (NKPC)

(3) Express marginal costs in terms of output to obtain NKPC

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New Keynesian Phillips Curve (NKPC)

(3) Express marginal costs in terms of output to obtain NKPC

Define y_t^n as the level of output under flexible prices. Then:

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where $\tilde{y}_t = y_t - y_t^n$ Such that:

$$\pi_t = \kappa \tilde{y}_t + \beta \mathbb{E}_t \left(\pi_{t+1}\right)$$

where the 'slope of the Phillips Curve' κ is:

$$\kappa = (1 - \beta\theta) \left(\frac{1 - \theta}{\theta}\right) \left(\frac{(1 - \alpha)}{1 - \alpha + \alpha\epsilon}\right) \left(\sigma + \frac{\varphi + \alpha}{1 - \alpha}\right)$$

Properties of the NKPC

1. Effect of output gap on inflation:
   - Decreases in price rigidity θ
   - Decreases in the demand elasticity ε
     $\Rightarrow$ More competition? Keep p^* closer to price level (smaller update)

Properties of the NKPC

1. Effect of output gap on inflation:
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     $\Rightarrow$ More competition? Keep p^* closer to price level (smaller update)

2. Inflation is forward looking:

$$\pi_t = \kappa \sum_{k=0}^{\infty} \beta^k \mathbb{E}_t(\tilde{y}_{t+k})$$

   - Does not depend on lagged values of inflation

Overview

- New Keynesian Phillips Curve ✓

$$\pi_t = \beta \mathbb{E}_t \left(\pi_{t+1} \right) + \kappa \tilde{y}_t$$

- Dynamic IS Equation

$$\tilde{y}_t = -\frac{1}{\sigma} \left(i_t - \mathbb{E}_t(\pi_{t+1}) - r_t^n \right) + \mathbb{E}_t \left(\tilde{y}_{t+1} \right)$$

- Monetary policy rule:

$$i_t = \rho + \phi_\pi \pi_t + \phi_y \tilde{y}_t + v_t$$

Dynamic IS equation

Start from the Euler equation (log-linearized):

$$c_t = -\frac{1}{\sigma}\left(i_t - \mathbb{E}_t\left[\pi_{t+1}\right] - \rho\right) + \mathbb{E}_t\left(c_{t+1}\right)$$

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From the clearance of the goods market we have:

$$C_{it} = Y_{it} \rightarrow Y_t = C_t$$

Hence:

$$y_t = -\frac{1}{\sigma}\left(i_t - \mathbb{E}_t\left[\pi_{t+1}\right] - \rho\right) + \mathbb{E}_t\left(y_{t+1}\right)$$

Dynamic IS equation

DIS relates deviations of output from **flexible price equilibrium** to i_t

From last lecture, recall that:

$$Y_t^n = A_t^{\frac{\varphi+1}{\zeta}} \left(\frac{\epsilon - 1}{\epsilon}\right)^{\frac{1-\alpha}{\zeta}} (1-\alpha)^{\frac{1-\alpha}{\zeta}}$$

where $\zeta = \sigma(1-\alpha) + \alpha + \varphi$. Log-linear:

$$y_t^n = \left(\frac{\varphi + 1}{\zeta}\right) a_t + \frac{1-\alpha}{\zeta} \log[(\epsilon - 1)(1-\alpha)/\epsilon]$$

Dynamic IS equation

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For the DIS curve, we need an expression for changes in natural output:

$$\mathbb{E}_t[y_{t+1}^n] - y_t^n = \left(\frac{\varphi+1}{\zeta}\right) \mathbb{E}_t[\Delta a_{t+1}]$$

Dynamic IS equation

Starting from:

$$y_t = -\frac{1}{\sigma}\left(i_t - \mathbb{E}_t\left[\pi_{t+1}\right] - \rho\right) + \mathbb{E}_t\left(y_{t+1}\right)$$

Subtract y_t^n from both sides:

$$\tilde{y}_t \quad = \quad -\frac{1}{\sigma}\left(i_t - \mathbb{E}_t\left[\pi_{t+1}\right] - \rho\right) + \mathbb{E}_t\left(y_{t+1} - y_t^n\right)$$

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&= -\frac{1}{\sigma}\left(i_t - \mathbb{E}_t\left[\pi_{t+1}\right] - r_t^n\right) + \mathbb{E}_t\left(\tilde{y}_{t+1}\right)
\end{aligned}$$

where r_t^n is the **natural interest rate**:

$$r_t^n \equiv \rho + \sigma\left(\frac{\varphi+1}{\zeta}\right)\mathbb{E}_t[\Delta a_{t+1}]$$

Note: this is just ρ in a model without productivity shocks

Dynamic IS equation

Hence the log-linearized curve gives us:

$$\tilde{y}_t = -\frac{1}{\sigma}\left(i_t - \mathbb{E}_t\left[\pi_{t+1}\right] - r_t^n\right) + \mathbb{E}_t\left(\tilde{y}_{t+1}\right)$$

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- This is the **dynamic IS equation**
- Output is suppressed if real interest rate is above natural rate

Dynamic IS equation

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- This is the **dynamic IS equation**
- Output is suppressed if real interest rate is above natural rate
- Iterating forward and assuming $\lim_{T\to\infty} \mathbb{E}_t(y_{t+T}) = 0$:

$$\tilde{y}_t = -\frac{1}{\sigma}\sum_{k=0}^{\infty}(\mathbb{E}_t r_{t+k} - r_{t+k}^n)$$

Overview

- New Keynesian Phillips Curve ✓

$$\pi_t = \beta \mathbb{E}_t(\pi_{t+1}) + \kappa \tilde{y}_t$$

- Dynamic IS Equation ✓

$$\tilde{y}_t = -\frac{1}{\sigma}\left(i_t - \mathbb{E}_t(\pi_{t+1}) - r_t^n\right) + \mathbb{E}_t(\tilde{y}_{t+1})$$

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$$i_t = \rho + \phi_\pi \pi_t + \phi_y \tilde{y}_t + v_t$$

Monetary policy rule

Close the model: assume simple monetary policy rule

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Monetary policy rule

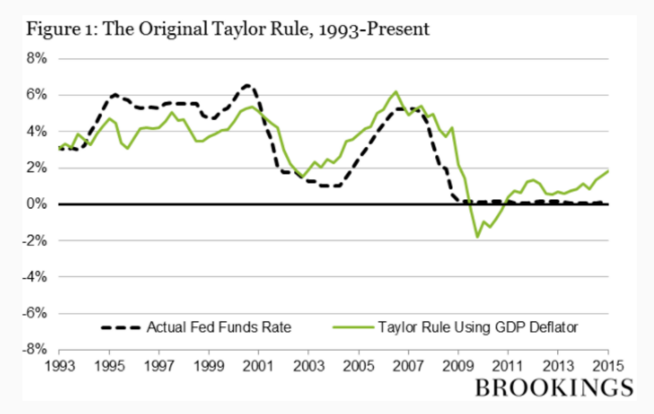
Close the model: assume simple monetary policy rule

$$i_t = \rho + \phi_\pi \pi_t + \phi_y \tilde{y}_t + v_t$$

- ϕ_π is weight on inflation. Taylor principle: $\phi_\pi > 1$ (see determinacy)
- ϕ_y is weight on output gap ('dual mandate')
- v_t is a **monetary policy shock**

Note: steady state nominal interest rate is ρ (consistent with 0 inflation)

Why the Taylor rule?



Source: Bernanke (2015)

$$i_t = 2 + \pi_t + 0.5\,(\pi_t - 2) + 0.5\,(\tilde{y}_t)$$

Why the Taylor rule?

It fits the data really well. Uses:

- It can serve as a **description**
- It can serve as a **benchmark**
- It can serve as a **prescription**

Note: zero-lower bound

Taylor rule prescribes **negative** nominal interest rates from 2009-2012

- Practical complication: there is a **nominal lower bound** on interest
- Any interest rate < 0: **cash** becomes the **dominant** asset
  - Some cost of storage etc.: **effective lower bound** $0 > ELB > -1\%$
- Also: potential adverse side effects of negative interest rates
  - Heider et al. (RFS, 2019): micro-data analysis of ECB negative deposit facility rate
  - Banks that rely on consumer deposits: relatively high **cost of capital**
  - These banks **cut** lending in response to negative rates

Note: monetary policy without money?

Straightforward to add money. E.g. demand for real balances:

$$\frac{M_t}{P_t} = \frac{Y_t}{i_t^{\eta}}$$

- intuition: transaction motive; opportunity cost of holding money

Note: monetary policy without money?

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- intuition: transaction motive; opportunity cost of holding money

Money growth:

$$\Delta m_t = \pi_t + \Delta y_t - \eta \Delta i_t$$

$\Rightarrow$ expansionary monetary policy (drop in i_t) requires money growth

This lecture

- Derive the three-equation linear NK-DSGE model
- **Conditions for determinacy**
- (Understanding and analyzing the model using Dynare)

Recall: Blanchard Kahn conditions

Linear rational expectations model:

$$\begin{bmatrix} \mathbb{E}_t Y_{t+1} \\ X_{t+1} \end{bmatrix} = \mathrm{A} \begin{bmatrix} Y_t \\ X_t \end{bmatrix} + \mathrm{B} Z_t$$

where:

- Y_{t+1}: $N_Y \times 1$ vector of endogenous non-predetermined (jump) vars
- X_{t+1}: $N_X \times 1$ vector of endogenous predetermined (state) variables
- Z_t: $N_Z \times 1$ vector of exogenous (incl. shock) variables

Recall: Blanchard Kahn conditions

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Key result lecture 2:

- Let N_λ denote number of eigenvalues of A with $\mid \lambda \mid > 1$
- Unique, saddle-point stable solution exists (determinacy) if $N_\lambda = N_Y$

Determinacy in the NK-DSGE model

To check whether our model can be solved:

1. Write the model in compact (matrix) form
2. Calculate the determinant and trace of the coefficient matrix
3. Check the Blanchard Kahn condition for determinacy

Determinacy in the NK-DSGE model

Linearized system:

$$\begin{bmatrix} \tilde{y}_t \\ \pi_t \end{bmatrix} = \Omega \begin{bmatrix} \sigma & 1 - \beta\phi_\pi \\ \sigma\kappa & \kappa + \beta(\sigma + \phi_y) \end{bmatrix} \begin{bmatrix} \mathbb{E}_t(\tilde{y}_{t+1}) \\ \mathbb{E}_t(\pi_{t+1}) \end{bmatrix} - \Omega \begin{bmatrix} 1 \\ \kappa \end{bmatrix} v_t$$

where $\Omega = \frac{1}{\sigma + \phi_y + \kappa\phi_\pi}$

- Insert monetary policy rule and NKPC into DIS
- Solve for output gap, insert into NKPC
- We'll abstract from productivity shocks: $r^n = \rho$

Extra slide: Determinacy in the NK-DSGE model

$$\begin{bmatrix} \mathbb{E}_t Y_{t+1} \\ X_{t+1} \end{bmatrix} = \mathrm{A} \begin{bmatrix} Y_t \\ X_t \end{bmatrix} + \mathrm{B} Z_t$$

$$\begin{bmatrix} Y_t \\ X_t \end{bmatrix} = \mathrm{A}^{-1} \begin{bmatrix} \mathbb{E}_t Y_{t+1} \\ X_{t+1} \end{bmatrix} - \mathrm{A}^{-1} \mathrm{B} Z_t$$

Our system is of the second form:

$$\begin{bmatrix} \tilde{y}_t \\ \pi_t \end{bmatrix} = \Omega \begin{bmatrix} \sigma & 1 - \beta\theta_\pi \\ \sigma\kappa & \kappa + \beta(\sigma + \phi_y) \end{bmatrix} \begin{bmatrix} \mathbb{E}_t(\tilde{y}_{t+1}) \\ \mathbb{E}_t(\pi_{t+1}) \end{bmatrix} - \Omega \begin{bmatrix} 1 \\ \kappa \end{bmatrix} v_t$$

where $\Omega = \frac{1}{\sigma + \phi_y + \kappa\phi_\pi}$

- Note that we've inverted the system, so for determinacy, need eigenvalues of $\tilde{A} = \mathrm{A}^{-1}$ to satisfy $\lambda_1 < 1$, $\lambda_2 < 1$.

Determinacy in the NK-DSGE model

$$\tilde{A} = \Omega \begin{bmatrix} \sigma & 1 - \beta\phi_y \\ \sigma\kappa & \kappa + \beta(\sigma + \phi_y) \end{bmatrix} \text{ where } \Omega = \frac{1}{\sigma + \phi_y + \kappa\phi_\pi}$$

- Note that both the trace and the determinant of $\tilde{A}$ are positive, which rules out that either of the eigenvalues is negative
- Now use the fact that that the eigenvalues must be the solution to:

$$x^2 - (\lambda_1 + \lambda_2)x + \lambda_1\lambda_2 = 0$$

- Use that the roots of the equation are only smaller than 1 if $\lambda_1\lambda_2 <$1 and $(\lambda_1 + \lambda_2) < \lambda_1\lambda_2 + 1$, and

$$\begin{aligned} \lambda_1\lambda_2 &= det(\tilde{A}) \\ &= \Omega^2 \left(\sigma\kappa + \sigma\beta[\sigma + \phi_y] - \sigma\kappa(1 - \beta\phi_\pi)\right) \\ &= \sigma\beta/(\sigma + \phi_y + \phi_\pi\kappa) < 1 \end{aligned}$$

Determinacy in the NK-DSGE model

$$\tilde{A} = \Omega \begin{bmatrix} \sigma & 1 - \beta\phi_y \\ \sigma\kappa & \kappa + \beta(\sigma + \phi_y) \end{bmatrix} \text{ where } \Omega = \frac{1}{\sigma + \phi_y + \kappa\phi_\pi}$$

- The second condition $(\lambda_1 + \lambda_2) < \lambda_1\lambda_2 + 1$:

$$\begin{aligned} 0 &< \lambda_1\lambda_2 + 1 - (\lambda_1 + \lambda_2) \\ &< det(\tilde{A}) + 1 - tr(\tilde{A}) \end{aligned}$$

where:

$$\begin{aligned} \text{det } (\tilde{A}) &= \Omega^2 \left(\sigma\kappa + \sigma\beta[\sigma + \phi_y] - \sigma\kappa(1 - \beta\phi_\pi)\right) \\ \text{tr } (\tilde{A}) &= \Omega \left(\sigma + \kappa + \beta[\sigma + \phi_y]\right) \end{aligned}$$

Determinacy in the NK-DSGE model

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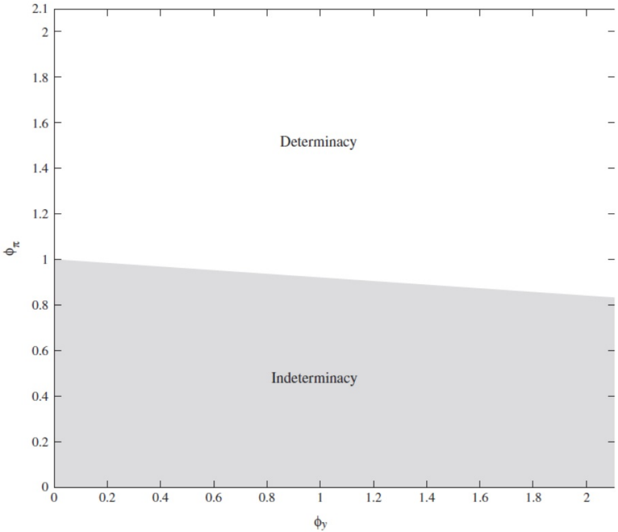
Insert this in the Blanchard Kahn condition:

$$0 < \text{ det } (\tilde{A}) + 1 - tr(\tilde{A})$$

Simplifying (try!):

$$0 < \kappa(\phi_\pi - 1) + (1 - \beta)\phi_y$$

Determinacy in the NK-DSGE model

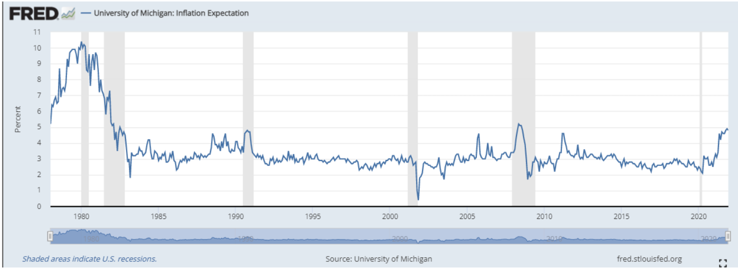


Combinations of ϕ_y and ϕ_π such that Blanchard-Kahn condition holds (Gali fig. 4.1)

Calibration: $\kappa = 0.11, \beta = 0.99$

Taylor principle, today

$$\tilde{y}_t = -\frac{1}{\sigma}\left(i_t - \mathbb{E}_t(\pi_{t+1}) - r_t^n\right) + \mathbb{E}_t\left(\tilde{y}_{t+1}\right)$$



U.S. inflation expectations from households

This lecture

- Derive the three-equation linear NK-DSGE model
- Conditions for determinacy
- **Understanding and analyzing the model using Dynare**
  $\rightarrow$ next time and problemsets