Lecture 9

The Textbook Heterogeneous Agent Model: Aiyagari-Bewley-Huggett

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Macroeconomics EC417

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Recall: Plan for remaining lectures

1. [DONE] Consumption-saving problem with idiosyncratic labor income risk in partial equilibrium

2. [DONE] Numerical dynamic programming a.k.a. numerical solution of Bellman equations

3. Textbook heterogeneous agent model: Aiyagari-Bewley-Huggett

   - income fluctuation problem, embedded in general equilibrium

   - give some motivation for using this model

4. Textbook heterogeneous firm model: Hopenhayn & Rogerson

   - firm growth, entry, exit, and firm size distribution in GE

The Textbook Heterogeneous Agent Model
Aiyagari-Bewley-Huggett

From income fluctuation problem to agg capital supply

1. [DONE] Individuals are subject to exogenous income shocks. These shocks are not fully insurable because of the lack of a complete set of Arrow-Debreu contingent claims

2. [DONE] There is only a risk-free asset (i.e., and asset with non-state contingent rate of return) in which the individual can save/borrow, and that the individual faces a borrowing (liquidity) constraint

3. [DONE] A continuum of such agents subject to different shocks will give rise to a wealth distribution

4. Integrating wealth holdings across all agents will give rise to an aggregate supply of capital

Aggregate Capital Supply

- For a given interest rate r, we can compute stationary distribution $g(a, y; r)$. Since g is a density, it satisfies:

$$g(a, y) \geq 0, \qquad \sum_j \int_{\underline{a}}^{\infty} g(a, y_j; r) da = 1$$

- Note: dist will typically have mass points e.g. at $\underline{a}$ so we should really treat dist as measure and write $\sum_j \int_{\underline{a}}^{\infty} G(da, y_j; r) = 1$ etc
  - my notation will simply ignore this
  - numerical g is not a fn. anyway (vector or via simulation)

- Compute aggregate savings in stationary distribution:

$$A(r) = \sum_j \int_{\underline{a}}^{\infty} a g(a, y_j; r) da$$

- When $r = -1$ (discrete time), no-one saves so $A(-1) = \underline{a}$
- When $r = \beta^{-1} - 1$ (or $r = \rho$), assets explode: $A(r) \to \infty$

Precautionary Savings

- Intuition for why savings diverge when $1 + r = \beta^{-1}$ (equivalently $r = \rho$): Precautionary savings
- Households have three motives for saving in this model:
  1. Inter-temporal motive: difference between $1 + r$ and β
  2. Smoothing motive: concavity of utility function
  3. Precautionary motive: presence of occasionally binding borrowing constraint + income risk
- Precautionary motive leads agents to continue to save even when inter-temporal motive is shutdown, i.e. $r = \rho$. For total assets to remain bounded, we require $r < \rho$
  - proof uses the super-martingale convergence theorem, see e.g. Ljungqvist-Sargent textbook

Shape of Aggregate Savings Function, $A(r)$

- See graphs on whiteboard
- How would you compute these graphs on a computer?
- $A(r)$ is continuous if no discontinuity in underlying consumption-savings problem when varying r
  - claim based on results in Stokey-Lucas-Prescott
- If IES≥ 1, then $A(r)$ is strictly increasing (Achdou et al, 2017). But this is not a necessary condition. In general $A(r)$ need not be strictly increasing but in most applications it is.
  - e.g. with CRRA utility $\frac{c^{1-\sigma}}{1-\sigma}$, IES $= 1/\sigma$ so IES≥ 1 means $\sigma \leq 1$ so log utility or less concave
  - what's the intuition for the condition IES≥ 1?

Stationary Equilibrium Interest Rate

- Stationary equilibrium interest rate r determined by equating demand and supply in the market for assets in the ergodic distribution of households
- Since $A(r) \in [\underline{a}, \infty)$ and continuous, an equilibrium will exist if the demand for assets is either constant or decreasing in the interest rate.
- Different GE HA models: different assumptions about how to interpret assets and how they are supplied:
  1. Huggett model: private IOUs in zero net supply
  2. Bewley model: money or bonds in positive net supply
  3. Aiyagari model: capital in positive net supply
- Compare rep agent model: $A(r)$ perfectly elastic at $r = \rho$

Stationary equilibrium: some general remarks

- Conceptually = steady state: "if you start there you stay there"
    - difference to before: now looking for entire distribution such that this is true!
- Importantly: aggregates constant (like st. st. in growth model)...
- but rich dynamics at individual level
    - individuals "churning around" in stationary distribution
- Typically, no analytic solutions for stationary equilibrium
- $\Rightarrow$ solve for stationary equilibrium numerically
    - challenge: have to find stationary wealth distribution
    - much easier than time-varying equilibrium because prices (e.g. w^*, r^*) are just scalars

Huggett Model: Assets in Zero Net Supply

- Equilibrium interest rate determined by market clearing condition
$$A(r) = 0$$
- Important that households are allowed to borrow, i.e. $\underline{a} < 0$
- Compute by iterating on interest rate until convergence or using a one-dimensional equation solver

Huggett Model: Definition of Equilibrium

A stationary Recursive Competitive Equilibrium (RCE) is

1. Value and policy functions: $V(a, y), c(a, y), s(a, y)$
2. Distribution of households: $g(a, y)$
3. Interest rate: r

such that

1. Given r, the function $V(a, y)$ solves the household problem, i.e. satisfies the Bellman eqn:

   $$V(a, y_j) = \max_{c, a' \geq \underline{a}} u(c) + \beta \sum_{j'} p_{jj'} V(a', y_{j'}) \quad \text{s.t.} \quad a' = (1 + r)a + y - c$$

   The implied policy functions are $c(a, y)$ and $a'(a, y) = (1 + r)a + y - c(a, y)$.

2. Given the saving policy function $a'(a, y)$ and transition probabilities $p_{jj'}$, the distribution $g(a, y)$ is the corresponding stationary distribution

3. Given the distribution $g(a, y)$, the market for asset clears:

   $$\sum_j \int_{\underline{a}}^{\infty} a g(a, y_j) da = 0$$

Bewley Model: Assets in Positive Supply

- Government issues bonds B, finances interest payments and govt spending G by collecting taxes according to tax function $\tau(a, y)$
- Total tax revenues are

$$T(r) = \sum_j \int_a \tau\left(a, y_j\right) g\left(a, y_j; r\right) da$$

- Government budget constraint: $G + rB = T(r)$
- Market clearing condition

$$A(r) = B$$

- Computation with exogenous B: As in Huggett economy, determine $G(r) = T(r) - rB$ as residual, provided $G(r) \geq 0$
- Computation with exogenous G: Solve $A(r) = \frac{T(r)-G}{r}$ and determined equilibrium B endogenously

Aiyagari Model: Add Production Side

- Representative firm with CRS production technology

$$Y = K^{\alpha} L^{1-\alpha}$$

- Firm rents capital from households at rate r and hires efficiency units of labor at wage rate w:

$$r + \delta = \alpha \left(\frac{K}{L} \right)^{\alpha - 1}$$

$$w = (1 - \alpha) \left(\frac{K}{L} \right)^{\alpha}$$

- Note that this implies a one-to-one mapping between w and r

$$w(r) = (1 - \alpha) \left(\frac{\alpha}{r + \delta} \right)^{\frac{\alpha}{1-\alpha}}$$

- HH's supply "efficiency units of labor" y_j, budget constraint is

$$c + a' = w y_j + (1 + r) a$$

Market Clearing

- Labor market clearing: exogenous labor supply

$$
\begin{aligned}
L &= \sum_j \int_a y_j g\left(a, y_j; r\right) da \\
&= \sum_j y_j \pi_j
\end{aligned}
$$

where π_j := stationary dist of income process $= \int_a g\left(a, y_j; r\right) da$

- Capital market clearing

$$
\begin{aligned}
A(r) &= K(r) \\
&= L\left(\frac{\alpha}{r+\delta}\right)^{\frac{1}{1-\alpha}}
\end{aligned}
$$

Aiyagari Model: Definition of Equilibrium

A stationary Recursive Competitive Equilibrium (RCE) is

1. Value and policy functions: $V(a,y), c(a,y), s(a,y)$
2. Factor Demands: K, L
3. Distribution of households: $g(a,y)$
4. Prices: r,w

such that

1. Given r, w, the function $V(a,y)$ solves the hh problem, i.e. satisfies the Bellman eqn:

$$V(a,y_j) = \max_{c,a' \geq \underline{a}} u(c) + \beta \sum_{j'} p_{jj'} V(a', y_{j'}) \quad \text{s.t.} \quad a' = (1+r)a + wy - c$$

   The implied policy functions are $c(a,y)$ and $a'(a,y) = (1+r)a + wy - c(a,y)$.

2. Given r, w, the factor demands K, L solve the firm FOC

3. Given the saving policy function $a'(a,y)$ and transition probabilities $p_{jj'}$, the distribution $g(a,y)$ is the corresponding stationary distribution

4. Given the distribution $g(a,y)$, the markets for capital and labor clear:

$$\sum_j \int_{\underline{a}}^{\infty} a g(a, y_j) da = K \qquad \sum_j y_j \pi_j = L$$

Main Graph of Aiyagari (1994)

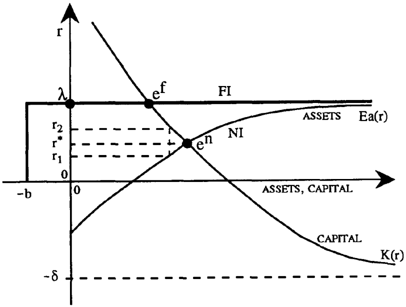


FIGURE IIb
Steady-State Determination

- Aiyagari's λ is our ρ, and his E$a(r)$ curve is our $A(r)$
- To get E$a(r)$, feed r & $w(r) = (1-\alpha)\left(\frac{\alpha}{r+\delta}\right)^{\frac{\alpha}{1-\alpha}}$ into hh problem

Aside: If assets = capital, how can they be < 0?

- Capital $K \geq 0$. If $\underline{a} < 0$, hh's can have negative assets $a < 0$
- At same time, agg assets = capital, $A(r) = K$. Or indexing individual hh's by $i \in [0, 1]$ (alternative notation = useful below)

$$\int_0^1 a_i di = K$$

- Question: So how does this make sense? If assets = capital, how can assets be negative?
- Answer: key is that assets = capital only in aggregate. There can still be borrowing and lending among different households.
- Easiest way to operationalize this:
  - households hold two assets capital $k_i \geq 0$ and bonds $b_i \gtrless 0$
  - household wealth $a_i = k_i + b_i \gtrless 0$
  - capital & bond market clearing (bonds in zero net supply)

$$K = \int_0^1 k_i di, \quad 0 = \int_0^1 b_i di \quad \Rightarrow \quad \int_0^1 a_i di = K$$

Computation of Equilibrium

- Any non-linear equation solver can be used to solve: $A(r) = K(r)$

- Often useful to iterate on $\kappa := \frac{K}{L}$. Using $r = \alpha\kappa^{\alpha-1} - \delta$:

$$\kappa = \frac{A\left(\alpha\kappa^{\alpha-1} - \delta\right)}{L}$$

- Suggests updating rule

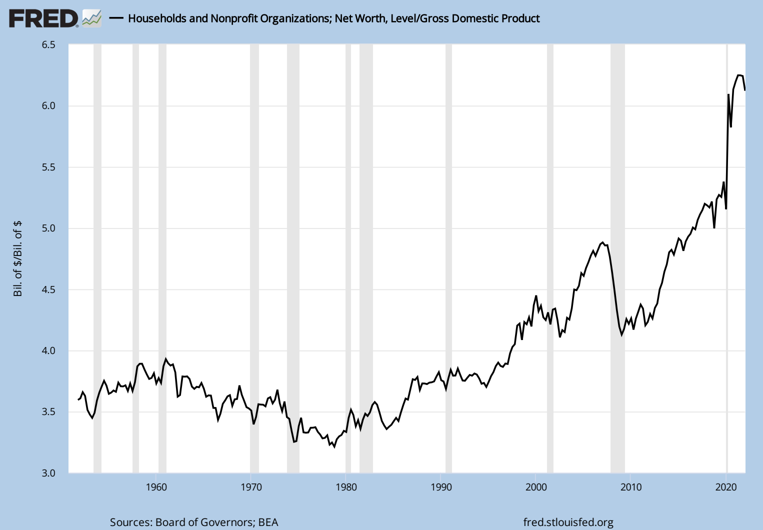
$$\kappa_{\ell+1} = \omega\frac{A\left(\alpha\kappa_\ell^{\alpha-1} - \delta\right)}{L} + (1-\omega)\,\kappa_\ell$$

where $\omega \in [0, 1]$ is dampening parameter

Aiyagari model: some aspects of the calibration

- **Discount rate:** choose discount rate β so as to match aggregate or average wealth-income ratio
  - option 1 = macro target: agg wealth/GDP from national accounts, e.g. for U.S. $\approx 3-5$ depending on time period and whether include residential capital
  - option 2 = micro target: e.g. average wealth from households survey, say SCF (next slides)
- **Labor income process:** estimate from micro data on individual income dynamics
- **Borrowing constraint:** calibrate the borrowing constraint in order to match, say, the fraction of agents with negative net worth which is around 10% in the U.S. economy.

Aggregate Wealth-GDP Ratio for U.S. Economy



Average Wealth (SCF 2016)

Wealth Definition		All	Exclude top 1%
Total	Mean	11.2	7.8
	Median	1.6	1.7
Non-housing	Mean	8.5	5.3
	Median	0.5	0.5
Financial	Mean	5.3	4.0
	Median	0.3	0.3
Mean earnings		\$61,600	\$54,900
Mean income		\$102,200	\$ 84,300

- Wealth numbers expressed as ratios to mean household earnings
- Average wealth $= 11.2 \times \$61,600 = \$689,920$!
- Average wealth/average income $= \$689,920/\$102,200 = 6.75$
  - surprisingly hard to square with agg numbers on previous slide

Low Wealth Households (SCF 2016)

Wealth Definition		
Total	$<= 0$	11%
	$<= \$2,000$	17%
Non-housing	$<= 0$	15%
	$<= \$2,000$	22%
Financial	$<= 0$	11%
	$<= \$2,000$	31%

Comparing Model and Data

Baseline Model Wealth Statistics – `egp_AR1_IID_tax.m`

Discount factor	0.945	0.95	**0.955**	0.96	0.97
Var log gross labor inc	0.982	0.982	**0.982**	0.982	0.982
Gini gross labor inc	0.505	0.505	**0.505**	0.505	0.505
Var log net labor inc	0.982	0.982	**0.982**	0.982	0.982
Gini net labor inc	0.505	0.505	**0.505**	0.505	0.505
Var log consumption	0.987	0.980	**0.966**	0.941	0.833
Gini consumption	0.497	0.493	**0.486**	0.476	0.443
Mean wealth	1.414	2.067	**3.053**	4.599	12.003
Median wealth	0.017	0.130	**0.379**	0.930	4.935
Gini wealth	0.858	0.831	**0.799**	0.762	0.662
P90-P50 wealth	220	44	**23**	14	7
P99-P50 wealth	1217	209	**94**	50	17
Frac wealth$<= 0$	47%	30%	**25%**	20%	6%
Frac wealth$<=$ 5% $E[y]$	53%	46%	**39%**	26%	11%
Top 10% wealth share	75%	70%	**64%**	59%	47%
Top 1% wealth share	22%	19%	**17%**	14%	9%
Top 0.1% wealth share	4%	3%	**3%**	2%	1%

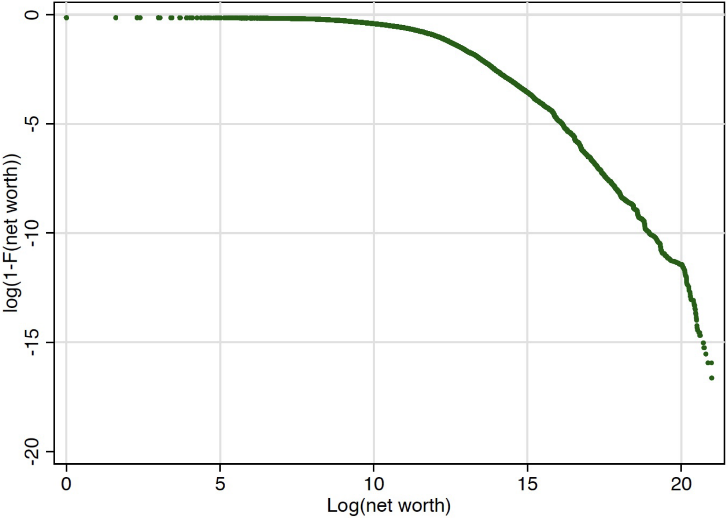
Wealth Statistics in SCF 2016

Full Distribution			
	Total	Non-Housing	Financial
Mean	11.19	8.48	5.34
Median	1.58	0.49	0.31
P90	19.3	12.6	9.6
P99	168.0	141.4	87.0
P99.9	700.5	634.6	358.9
P90-P50 Ratio	12	26	31
P99-P50 Ratio	106	288	284
Top 10% Share	77%	84%	81%
Top 1% Share	39%	45%	40%
Top 0.1% Share	15%	18%	14%
Gini	0.86	0.91	0.89
Frac <= 0	11%	15%	11%
Frac <=$2000	17%	22%	31%
Frac <= 5% Av Earns	18%	23%	34%

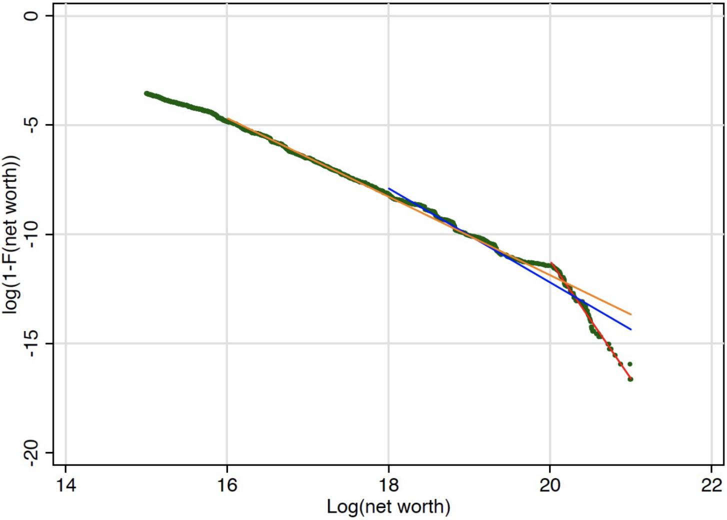
Wealth Statistics in SCF 2016: Exclude Top 1%

Excluding top 1%			
	Total	Non-Housing	Financial
Mean	7.79	5.32	3.95
Median	1.71	0.53	0.33
P90	19.8	12.6	9.7
P99	106.0	85.1	56.9
P99.9	179.1	163.1	133.5
P90-P50 Ratio	12	24	30
P99-P50 Ratio	62	160	174
Top 10% Share	65%	74%	73%
Top 1% Share	18%	23%	23%
Top 0.1% Share	2%	3%	4%
Gini	0.79	0.86	0.84
Frac <=0	12%	15%	11%
Frac <= $2000	17%	22%	31%
Frac <= 5% Av Earns	18%	23%	33%

Pareto Tail SCF 2013



Pareto Tail SCF 2013



Model Wealth Statistics with Transitory Shocks

Transitory shock size	0.05	0.1	**0.2**	0.25	**0.2**
Discount factor	0.955	0.955	**0.955**	0.955	**0.9**
Var log gross labor inc	0.985	0.992	**1.021**	1.043	**1.021**
Gini gross labor inc	0.509	0.513	**0.521**	0.526	**0.521**
Var log net labor inc	0.985	0.992	**1.021**	1.043	**1.021**
Gini net labor inc	0.509	0.513	**0.521**	0.526	**0.521**
Var log consumption	0.964	0.963	**0.959**	0.957	**1.001**
Gini consumption	0.486	0.486	**0.486**	0.486	**0.513**
Mean wealth	3.071	3.105	**3.212**	3.285	**0.178**
Median wealth	0.399	0.430	0.543	0.617	**0.026**
Gini wealth	0.795	0.790	**0.776**	0.767	**0.814**
P90-P50 wealth	22	21	**17**	15	**16**
P99-P50 wealth	90	83	**66**	59	**90**
Frac wealth<=0	6%	5%	**4%**	3%	**24%**
Frac wealth<= 5% $E[y]$	34%	29%	**20%**	16%	**59%**
Top 10% wealth share	64%	64%	**62%**	61%	**69%**
Top 1% wealth share	16%	16%	**16%**	16%	**25%**
Top 0.1% wealth share	3%	3%	**3%**	3%	**6%**

Modifications I: Discount Factor Heterogeneity

Discount factor spread	±5%	±6%	**±6.5%**	±7%	**±6.5%**
Mean discount factor	0.9	0.9	**0.9**	0.9	**0.9**
Switching probability	0	0	**0**	0	$\frac{1}{40}$
Var log consumption	0.992	0.977	**0.963**	0.946	**0.987**
Gini consumption	0.505	0.497	**0.491**	0.483	**0.507**
Mean wealth	1.152	2.400	**3.678**	6.073	**1.184**
Median wealth	0.046	0.064	**0.083**	0.120	**0.060**
Gini wealth	0.883	0.866	**0.850**	0.826	**0.864**
P90-P50 wealth	57	103	**133**	162	**48**
P99-P50 wealth	423	555	**594**	582	**306**
Frac wealth<=0	25%	24%	**24%**	24%	**22%**
Frac wealth<=5% $E[y]$	51%	48%	**46%**	44%	**48%**
Top 10% wealth share	82%	77%	**73%**	67%	**78%**
Top 1% wealth share	28%	22%	**19%**	16%	**25%**
Top 0.1% wealth share	5%	4%	**3%**	3%	**5%**

Modifications II: Bequests

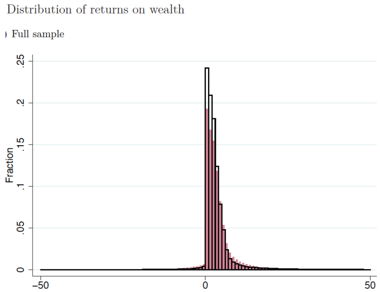
Warm-glow B	0	0.07	0.07	0.07	0.08
Luxury parameter ζ	0	0.01	4	6	4
Discount factor	0.955	0.955	0.955	0.955	0.955
Var log consumption	0.959	0.973	0.964	0.962	0.971
Gini consumption	0.486	0.488	0.485	0.484	0.485
Mean wealth	3.212	3.245	3.421	3.483	4.400
Median wealth	0.543	0.147	0.188	0.197	0.203
Gini wealth	0.776	0.900	0.887	0.884	0.893
P90-P50 wealth	17	37	33	33	39
P99-P50 wealth	66	421	332	318	398
Frac wealth<=0	4%	9%	8%	8%	8%
Frac wealth<= 5% $E[y]$	20%	33%	30%	30%	29%
Top 10% wealth share	62%	86%	83%	83%	84%
Top 1% wealth share	16%	36%	34%	33%	34%
Top 0.1% wealth share	3%	9%	8%	8%	8%

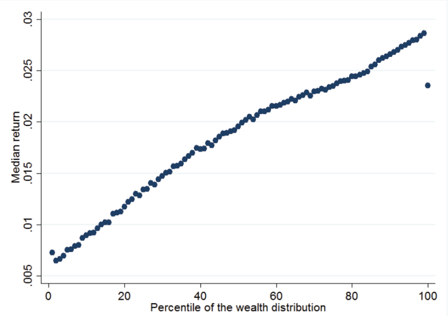
1. Wealth inequality at the top

- Standard model does not generate enough wealth at the top: e.g. top 1% wealth share in model = 15%, in data = 40%
  - Heterogeneity in discount factors: patients households (richer) save more (Krusell and Smith, 1998)
  - Non-homothetic preferences: rich save more, e.g. to bequeath (Atkinson, De Nardi, Straub)
  - High but transitory income realization "awesome state": rich save more for precautionary reasons (Castaneda & al, 2003)
  - Heterogeneous rates of return (Benhabib & al, 2014)
  - Entrepreneurs with projects yielding higher, but stochastic, rate of return than r (Quadrini, 2000)
- See survey "Skewed Wealth Distribution: Theory and Empirics" by Benhabib-Bisin
- Current work: empirical evidence for these ingredients?

Example: Returns to Wealth – Fagereng et al (2019)

- Using Norwegian administrative data (Norway has wealth tax), document massive heterogeneity in returns to wealth
  - range of over 500 basis points between 10th and 90th pctile
  - returns positively correlated with wealth





- Note: figures are from working paper version