University of Cape Town

STA4016H

Portfolio Theory

Backtesting a Black-Litterman Portfolio

Author:
Chongo Nkalamo

Student Number:
NKLCHO001

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1 Derivations and Theory

1.1 The Mutual Fund Separation Theorem.

Applying Kuhn-Tacker methods to finding the Lagrangian of (1) we have

$$\begin{aligned} L &= \omega^T \mu - \frac{\alpha}{2}\omega^T \Sigma \omega - \lambda_\omega(\omega^T \mathbf{1} - \mathbf{1}) \\ &= \frac{\partial L}{\partial \omega} = \mu - \gamma\omega\Sigma - \lambda\mathbf{1} = 0 \\ &= \frac{\partial L}{\partial \lambda} = -(\omega^T \mathbf{1} - \mathbf{1}) = 0 \end{aligned} \tag{1}$$

Thus solving for ω from (1) which will be denoted by ω^* we have

$$\mu - \lambda\mathbf{1} = \gamma\omega\Sigma \implies \omega* = \frac{\mu - \lambda\mathbf{1}}{\gamma\Sigma} \tag{2}$$

From (2) we simply return the our constraint of a fully invested portfolio i.e

$$\omega^T \mathbf{1} = \mathbf{1} \tag{3}$$

Using (3) we can solve for the Lagrange multiplier λ and thus eliminate it from (2). We proceed as follows:

$$\begin{aligned} \omega^* &= \frac{1}{\gamma}\Sigma^{-1}\mu - \frac{1}{\gamma}\Sigma^{-1}\lambda\mathbf{1} \\ \frac{1}{\gamma}\Sigma^{-1}\lambda\mathbf{1} &= \frac{1}{\gamma}\Sigma^{-1}\mu - \omega* \\ \frac{1}{\gamma}\Sigma^{-1}\lambda\mathbf{1} &= \frac{1}{\gamma}\Sigma^{-1}\mu - \mathbf{1} \\ \mathbf{1}^T\Sigma^{-1}\mathbf{1}\lambda &= \mathbf{1}^T\Sigma^{-1}\mu - \gamma \\ \implies \lambda &= \frac{\mathbf{1}^T\Sigma^{-1}\mu}{\mathbf{1}^T\Sigma^{-1}\mathbf{1}} - \frac{\gamma}{\mathbf{1}^T\Sigma^{-1}\mathbf{1}} \end{aligned} \tag{4}$$

Therefore substituting the λ expression back into (2) we rewrite $\omega*$ as

$$
\begin{aligned}
\omega^* &= \frac{1}{\gamma}\Sigma^{-1}\Big(\mu - \frac{\mu \mathbf{1}^T\Sigma^{-1}\mathbf{1} - \gamma\mathbf{1}}{\mathbf{1}^T\Sigma^{-1}\mathbf{1}}\Big) \\
&= \frac{\mu\Sigma^{-1}}{\gamma} - \frac{\mu\mathbf{1}^T\Sigma^{-1}\mathbf{1}}{\gamma(\mathbf{1}^T\Sigma^{-1}\mathbf{1})} + \mathbf{1}\frac{\Sigma^{-1}\mathbf{1}}{\mathbf{1}^T\Sigma^{-1}\mathbf{1}} \\
&= \frac{\Sigma^{-1}\mathbf{1}}{\mathbf{1}^T\Sigma^{-1}\mathbf{1}}\Big(1 - \frac{1}{\gamma}\mathbf{1}^T\Sigma^{-1}\mu\Big) + \frac{\Sigma^{-1}\mu}{\gamma} \\
&= \frac{\Sigma^{-1}\mathbf{1}}{\mathbf{1}^T\Sigma^{-1}\mathbf{1}}\Big(1 - \frac{1}{\gamma}\mathbf{1}^T\Sigma^{-1}\mu\Big) + \Big(\frac{1}{\gamma}\mathbf{1}^T\Sigma^{-1}\mu\Big)\frac{\Sigma^{-1}\mu}{\mathbf{1}^T\Sigma^{-1}\mu}
\end{aligned} \tag{5}
$$

1.2 The Mutual Fund Separation Theorem in terms of an equilibrium view.

From (5), if we let $\mathbf{a}(\gamma) = \frac{1}{\gamma}\mathbf{1}^T\Sigma^{-1}$, then we can rewrite ω^* as

$$
\begin{aligned}
\omega^* &= \omega_G(1 - \mathbf{a}(\gamma) + \mathbf{a}(\gamma)\frac{\omega_R}{\omega_G}) \\
&= \omega_G(\mathbf{a}(\gamma)(\frac{\omega_R}{\omega_G} - 1) + 1) \\
&= \omega_G + \mathbf{a}(\gamma)\omega_R - \mathbf{a}(\gamma)\omega_G(\gamma) \\
&= \omega_G + \mathbf{a}(\gamma)(\omega_R - \omega_G)
\end{aligned} \tag{6}
$$

re-substituting the appropriate expressions we get

$$
\begin{aligned}
\omega^* &= \omega_G + \frac{1}{\gamma}\Big(\mathbf{1}^T\Sigma^{-1}\mu\Big)\Big(\frac{\Sigma^{-1}\mu}{\mathbf{1}^T\Sigma^{-1}\mu}\Big) - \frac{1}{\gamma}\mathbf{1}^T\Sigma^{-1}\mu\Big(\frac{\Sigma^{-1}\mathbf{1}}{\mathbf{1}^T\Sigma^{-1}\mathbf{1}}\Big) \\
&= \omega_G + \frac{1}{\gamma}\Sigma^{-1}\Big(\mu - \mathbf{1}\frac{\mathbf{1}^T\Sigma^{-1}\mu}{\mathbf{1}^T\Sigma^{-1}\mathbf{1}}\Big)
\end{aligned} \tag{7}
$$

Now, considering an investor with some long term investment horizon. This investor will pick some long-term equilibrium vector of returns for given assets, Π. Incorporating this long term equilibrium, we further adjust our above portfolio as

$$
\omega^* = \omega_G + \frac{1}{\gamma}\Sigma^{-1}((\mu + \Pi - \Pi) - \mathbf{1}\frac{\mathbf{1}^T\Sigma^{-1}(\mu + \Pi - \Pi)}{\mathbf{1}^T\Sigma^{-1}\mathbf{1}} \tag{8}
$$

Thus we can factorize the above into a risky term that holds absolute equilibrium views and another term that holds relative views to the equilibrium view $(E(\mathbf{R}) - \Pi)$.

$$\omega^* = \omega_G + \frac{1}{\gamma}\Sigma^{-1}\left(\Pi - \mathbf{1}\frac{\Pi^T\Sigma^{-1}\mathbf{1}}{\mathbf{1}^T\Sigma^{-1}\mathbf{1}}\right) + \frac{1}{\gamma}\Sigma^{-1}((\mu - \Pi) - \mathbf{1}\frac{(\mu - \Pi)^T\Sigma^{-1}\mathbf{1}}{\mathbf{1}^T\Sigma^{-1}\mathbf{1}} \tag{9}$$

Now factoring out the denominator $1^T\Sigma^{-1}1$ to find

$$\omega^* = \omega_G + \frac{1}{\gamma}\frac{\Sigma^{-1}}{\mathbf{1}^T\Sigma^{-1}\mathbf{1}}(\Pi\mathbf{1}^T\Sigma^{-1}\mathbf{1} - \mathbf{1}\Pi^T\Sigma^{-1}\mathbf{1}) + \frac{1}{\gamma}\frac{\Sigma^{-1}}{\mathbf{1}^T\Sigma^{-1}\mathbf{1}}((\mu - \Pi)\mathbf{1}^T\Sigma^{-1}\mathbf{1} - \mathbf{1}(\mu - \Pi)^T\Sigma^{-1}\mathbf{1}) \tag{10}$$

Finally factorizing out the inverse co-variance matrix, Σ^{-1}using matrix algebra.

$$\omega^* = \omega_G + \frac{\Sigma^{-1}}{\gamma}\left(\frac{(\Pi\mathbf{1}^T - \mathbf{1}\Pi^T)}{\mathbf{1}^T\Sigma\mathbf{1}}\right)\Sigma^{-1}\mathbf{1} + \frac{\Sigma^{-1}}{\gamma}\left(\frac{(\mu - \Pi)\mathbf{1}^T - \mathbf{1}(\mu - \Pi^T)}{\mathbf{1}^T\Sigma^{-1}\mathbf{1}}\right)\Sigma^{-1}\mathbf{1} \tag{11}$$

The weight can thus be decomposed into a three fund separation of the efficient frontier. Thus,

$$\omega^* = \omega_B + \omega_S + \omega_T$$

where,

$$\begin{aligned}
\omega_B &= \frac{\Sigma^{-1}\mathbf{1}}{\mathbf{1}^T\Sigma^{-1}\mathbf{1}} \\
\omega_S &= \frac{\Sigma^{-1}}{\gamma}\left(\frac{(\Pi\mathbf{1}^T - \mathbf{1}\Pi^T)}{\mathbf{1}^T\Sigma\mathbf{1}}\right)\Sigma^{-1}\mathbf{1} \\
\omega_T &= \frac{\Sigma^{-1}}{\gamma}\left(\frac{(\mu - \Pi)\mathbf{1}^T - \mathbf{1}(\mu - \Pi^T)}{\mathbf{1}^T\Sigma^{-1}\mathbf{1}}\right)\Sigma^{-1}\mathbf{1}
\end{aligned} \tag{12}$$

1.3 The Black-Litterman mean and covariance

1.3.1 Certain Views:

When there is certainty the error terms in the views are zero, $\mathbf{e} = 0$. We can then solve the following optimization

$$E(\mathbf{R})^* = \min\,(E(\mathbf{R}) - \Pi)^T(\tau\Sigma)^{-1}(E(\mathbf{R}) - \Pi) \tag{13}$$

subject to

$$\mathbf{P}E(\mathbf{R}) = \mathbf{V}$$

We solve the above optimization using Lagrangian methods as follows

$$L = (E(\mathbf{R}) - \Pi)^T(\tau\Sigma)^{-1}(E(\mathbf{R}) - \Pi) - \lambda(\mathbf{P}E(\mathbf{R}) - \mathbf{V}) \tag{14}$$

Differentiating (1) with respect to $E(\mathbf{R})$ and λ respectively we obtain

$$\frac{\partial L}{\partial E(\mathbf{R})} = 2E(\mathbf{R}) - \Pi)^T(\tau\Sigma)^{-1} - \lambda\mathbf{P} = 0 \tag{15}$$

$$\frac{\partial L}{\partial \lambda} = \mathbf{P}(E(\mathbf{R}) - \mathbf{V} = 0 \tag{16}$$

Solving for λ in (2) we obtain

$$\begin{aligned} \lambda &= 2E(\mathbf{R}) - \Pi)^T(\tau\Sigma)^{-1}\mathbf{P}^{-1} \\ \lambda^T &= 2(\mathbf{P}^T)^{-1}(\tau\Sigma)^{-1}(E(\mathbf{R}) - \Pi) \end{aligned} \tag{17}$$

The inversion of the matrix $\mathbf{P}$ comes about due to their being a generalized inverse of $\mathbf{P}^{-1}$.

From (3) we obtain $E(\mathbf{R}) = \mathbf{P}^{-1}\mathbf{V}$ by a similar argument.

Now, using the fact that $(AB)^{-1} = B^{-1}A^{-1}$ we have that

$$\lambda^T = 2(\mathbf{P}\tau\Sigma\mathbf{P}^T)^{-1}(\mathbf{V} - \mathbf{P}\Pi) \tag{18}$$

Solving for $E(\mathbf{R})$

$$\begin{aligned} E(\mathbf{R}) &= \frac{1}{2}(\tau\Sigma)\mathbf{P}^T\lambda^T + \Pi \\ &= \frac{1}{2}(\tau\Sigma)\mathbf{P}^T 2(\mathbf{P}\tau\Sigma\mathbf{P}^T)^{-1}(\mathbf{V} - \mathbf{P}\Pi) + \Pi \\ &= \Pi + \tau\Sigma\mathbf{P}^T(\mathbf{P}\tau\Sigma\mathbf{P}^T)^{-1}(\mathbf{V} - \mathbf{P}\Pi) \end{aligned} \tag{19}$$

1.3.2 Condition distribution of equilibrium returns $\Pi|E(\mathbf{R})$

We have that $P(E(\mathbf{R}) \propto e^{-\frac{1}{2}(\mathbf{P}E(\mathbf{R})-\mathbf{V}^T)\Omega^{-1}(\mathbf{P}E(\mathbf{R})-\mathbf{V}))}$ and $P(\Pi|E(\mathbf{R}) \propto e^{-\frac{1}{2}((\Pi-E(\mathbf{R})^T\tau\Sigma^{-1}(\Pi-E(\mathbf{R})))}$

Using Bayes theorem for this portfolio

$$P(E(\mathbf{R}|\Pi) = \frac{P(E(\Pi|\mathbf{R})P(E(\mathbf{R}))}{P(\Pi)} \tag{20}$$

We thus it follows that

$$P(E(\mathbf{R}|\Pi) = \frac{k}{P(\Pi)}e^{-\frac{1}{2}(\Pi-E(\mathbf{R})^T\tau\Sigma^{-1}(\Pi-E(\mathbf{R}))}e^{-\frac{1}{2}(\mathbf{P}E(\mathbf{R})-\mathbf{V}^T)\Omega^{-1}(\mathbf{P}E(\mathbf{R})-\mathbf{V})} \tag{21}$$

1.3.3 Derivation the mean of the conditional distribution

For this section, μ is taken to be a vector of returns and not a scalar. Consider the expansion of only exponent terms from (7):

$$\begin{aligned} &= (\Pi - \mu)^T(\tau\Sigma)^{-1}(\Pi - \mu) + (\mathbf{P}\mu - \mathbf{V})^T\Omega^{-1}(\mathbf{P}\mu - \mathbf{V}) \\ &= \Pi^T(\tau\Sigma)^{-1}\Pi - 2\mu^T(\tau\Sigma)^{-1}\Pi + \mu^T(\tau\Sigma)^{-1}\mu + \mu^T\mathbf{P}^T\Sigma^{-1}\mathbf{P}\mu - 2\mu^T\mathbf{P}^T\Omega^{-1}\mathbf{V} + \mathbf{V}^T\Omega^{-1}\mathbf{V} \end{aligned} \tag{22}$$

Rearranging the terms further, we obtain

$$= (\mathbf{V}^T\Omega^{-1}\mathbf{V} + \Pi^T(\tau\Sigma)^{-1}\Pi) - 2\mu^T((\tau\Sigma)^{-1}\Pi + \mathbf{P}^T\Omega^{-1}\mathbf{V}) + \mu^T((\tau\Sigma)^{-1} + \mathbf{P}^T\Omega^{-1}\mathbf{P})\mu \tag{23}$$

The above can be expressed as

$$\mathbf{A} - 2\mu^T\mathbf{C} + \mu^T\mathbf{H}\mu \tag{24}$$

where

$$\begin{aligned} \mathbf{A} &= \mathbf{V}^T\Omega^{-1}\mathbf{V} + \Pi^T(\tau\Sigma)^{-1}\Pi \\ \mathbf{H} &= (\tau\Sigma)^{-1} + \mathbf{P}^T\Omega^{-1}\mathbf{P}) \\ \mathbf{C} &= (\tau\Sigma)^{-1}\Pi + \mathbf{P}^T\Omega^{-1}\mathbf{V}) \end{aligned} \tag{25}$$

We can rewrite the above exponent term as

$$\begin{aligned} &= \mu^T\mathbf{H}\mu - 2\mu^T\mathbf{C} + \mathbf{A} \\ &\implies \mu^T\mathbf{H}^T\mu - 2\mathbf{C}^T\mu + \mathbf{A} \\ &\implies \mu^T\mathbf{H}^T\mathbf{H}^{-1}\mathbf{H}\mu - 2\mathbf{C}^T\mathbf{H}^{-1}\mathbf{H}\mu + \mathbf{A} \\ &\implies (\mathbf{H}\mu - \mathbf{C})^T\mathbf{H}^{-1}(\mathbf{H}\mu - \mathbf{C}) + \mathbf{A} - \mathbf{C}^T\mathbf{H}^{-1}\mathbf{C} \\ &\implies (\mu - \mathbf{H}^{-1}\mathbf{C})^T\mathbf{H}(\mu - \mathbf{H}^{-1}\mathbf{C}) + \mathbf{A} - \mathbf{C}^T\mathbf{H}^{-1}\mathbf{C} \end{aligned} \tag{26}$$

from which we obtain the mean, $E(\mathbf{R})^* = H^{-1}C$ and covariance $\Sigma^* = H$ as

$$\begin{aligned} E(\mathbf{R})^* &= ((\tau\Sigma)^{-1} + \mathbf{P}^T\Omega^{-1}\mathbf{P}))^{-1}((\tau\Sigma)^{-1}\Pi + \mathbf{P}^T\Omega^{-1}\mathbf{V})) \\ \Sigma^* &= (\tau\Sigma)^{-1} + \mathbf{P}^T\Omega^{-1}\mathbf{P}) \end{aligned} \tag{27}$$

1.4 Black-Litterman weights

The Black-Litterman mean was by given by equation (27). Construction of the Black-Litterman weights considering only updated expectation (27) is done the same way as outlined in equation (7) except we replace the vector μ with the expression in (27) ,$E(\mathbf{R})^*$ to obtain Black-Litterman weights

$$\omega_{BL} = \omega_G + \frac{1}{\gamma}\Sigma^{-1}\left(E(\mathbf{R})^* - \mathbf{1}\frac{\mathbf{1}^T\Sigma^{-1}E(\mathbf{R})^*}{\mathbf{1}^T\Sigma^{-1}\mathbf{1}}\right)$$

Table 1: Bench mark portfolio

Asset:	**ALBI**	**J500**	**J510**	**J520**	**J530**	**J540**	**J550**	**J560**	**J580**	**J590**
Weight:	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1

2 Backtesting the Black-Litterman Model.

2.1 Introduction.

Black-Litterman is a model framework that essentially tries to integrate the methods or approaches traditional fund managers and Quantitative managers have. This is done through the use of Bayesian methodology which effectively updates currently held opinions with data to form new opinions.(prescribed reading)Hence, the final forecast reflects the views/forecasts the judgmental managers have which are then added to the quantitative model the quantitative manager builds on the same data.

In this section, a portfolio of ten assets (by section name) , ALBI, J500, J510, J520, J530, J540, J550, J560, J580 and J590 will be built. We seek to find the optimal weights that will insure a good investment and ensure that our portfolio is fully invested. The weights that will be computed will not be the weights our mean-variance optimizer produces but rather Black-Litterman weights which will mix some views with a model. An appropriate benchmark portfolio will be selected in order to track and compare how well the Black-Litterman portfolio will perform. Other performance measures that will be calculated will be the Jensens alpha and systematic risk factor (or beta) of our resulting portfolio. All these calculations will be based off selecting our market portfolio as the All-Share-Index (**ALSI**). Furthermore, the model will be test using rolling-window method.

2.2 Model building.

A benchmark that will be used to track our portfolio performance will be an equally weighted portfolio of the ten assets we want in our portfolio. Using rolling-windows we evaluate how both our model and the benchmark perform and seek to find out if our model beats the selected benchmark. Hence our benchmark portfolio is as given in table 1.

We seek to find optimal weights for this selection of assets using a Black-Litterman approach. Hence, the weights we compute will be Black-Litterman weights or posterior weights that are obtained after taking in views on the assets and mixing with the data available. The Black-Litterman weights (as seen in section 1) take in as inputs a scaling factor τ,an n x n covariance matrix of returns, Σ, a k x n matrix $\mathbf{P}$ that is a matrix that contains our views,a k x k uncertainty covariance matrix Ω and a k x 1 matrix $\mathbf{V}$ that expresses the returns of our views. Finally, an n x 1 equilibrium view matrix Π, which indicates the equilibrium returns of the assets in our portfolio.

The covariance matrix of returns is the covariance variance of the individual asset returns in our portfolio. However, given that covariance is a statistical estimation problems, we can always run into miss-estimation due to random sampling error, inappropriate choice of an optimizer or function used to find the covariance and so on. To avoid incorportaing this estimation risk into our portfolio, we further shrink our Σ in order to take away any estimation error. By doing this we essentially eliminate any form of shortselling that would have been observed due to estimation risk that would not have actually taken place. Our new (or shrunk) covariance matrix is formed by taking the diagonal of the matrix computed below

$$\Sigma^* = ((1-\alpha) \times \Sigma) + (\alpha \times \mathbf{diag}(1))$$

As given in the literature, it is a common rule of thumb to set $\alpha = \frac{1}{2}$.

The scaling factor τ can be obtained through the use of basic statistics as outlined in Polovenko[2]. Polovenko suggests that given a covariance matrix from historical data,

$$\tau = \frac{1}{T}$$

which is the maximum likelihood estimator of τ and T is given to be the number of data points or the sample size.

The matrix representing the uncertainty of our views, Ω as outlined in Lee[1] can be estimated by

$$\Omega = diag(\mathbf{P}^T \Sigma \mathbf{P})$$

in this case, Σ is the shrunk covariance matrix. The equilibrium return matrix, Π was obtained through reverse engineering through the relationship

$$\Pi = \gamma \Sigma \omega_B$$

where

$$\gamma = \frac{E(R_P) - r_f}{\sigma_m^2}$$

Table 2: Black-Litterman portfolio

Asset:	**ALBI**	**J500**	**J510**	**J520**	**J530**	**J540**	**J550**	**J560**	**J580**	**J590**
Weight:	0.0887	0.1063	0.10059	0.0981	0.10019	0.0990	0.09827	0.09994	0.10158	0.10728

The Black-Litterman weights were calculated using in-sample data of 74 months and we obtain the portfolio in Table 2.

As can be seen from the table, all the weights obtained are positive and they sum to one ensuring that the portfolio is fully invested and there is no short-selling taking place.

2.3 Selection of parameters.

Two views will be considered apriori for our portfolio. These will be that Financial services (J580) will outperform basic materials (J510) by 3%. The second view is also a relative view in that we believe that Consumer goods industry (J530) will outperform the consumer services sector (J560) by 2%. Therefore, **P** will be

$$\mathbf{P} = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 \end{pmatrix}$$

hence given our beliefs **V** will be

$$\mathbf{V} = \begin{pmatrix} 0.03 \\ 0.02 \end{pmatrix}$$

As will be described shortly, windows of 74 months will be considered. If we choose to select τ based off the estimate given in the previous section, then T = 74 and thus we select $\tau = \frac{1}{74}$.

2.4 Testing the model and Portfolio performance.

Given our initial in-sample window Black-Litterman weights, we further test the performance of our model by using a rolling-window method to track the performance of our Black-Litterman porfolio relative to the benchmark for the entire period provided in the data. The procedure goes as follows, choose a window of size 74 months, compute the Black-Litterman weights for the first window (in-sample), use the returns in the 75th month to compute the return of the portfolio at the end of that month. Roll the window to compute the Black-Litterman weights for the second month to the seventy-fifth month and use the 76th month returns as out-of-sample data to compute the portfolio return and the end of the 76th month and so on. After implementing this, the portfolio produced a geometric return of 1.198298. For the same data over the same period, the benchmark geometric return is computed to be 1.158779 which indicates that our Black-Litterman portfolio out performed our selected benchmark by 3.952% compared to an ex-ante tracking error in the region of 3%-10%. Figure 1, shows a plot of the cumulative geometric returns and thus performance of both our Black-Litterman model and our benchmark.

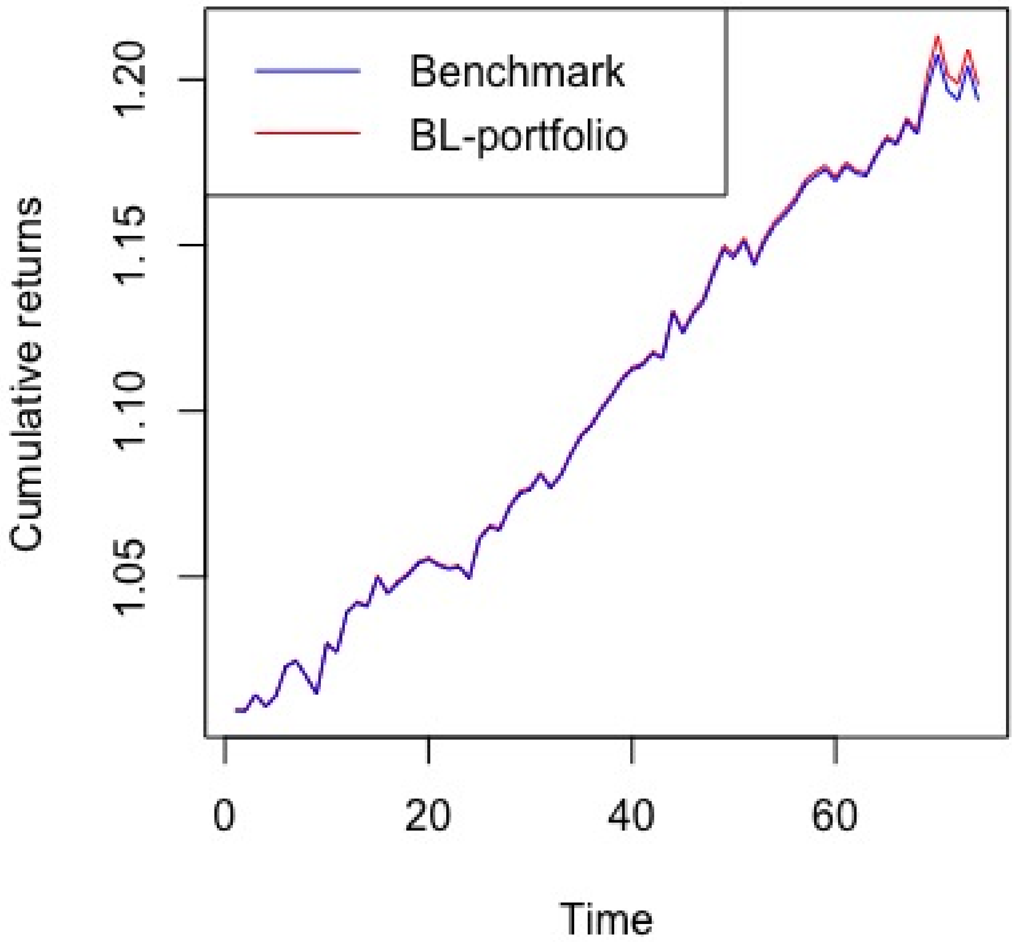

Figure 1: Black-Litterman portfolio relative to the Benchmark portfolio

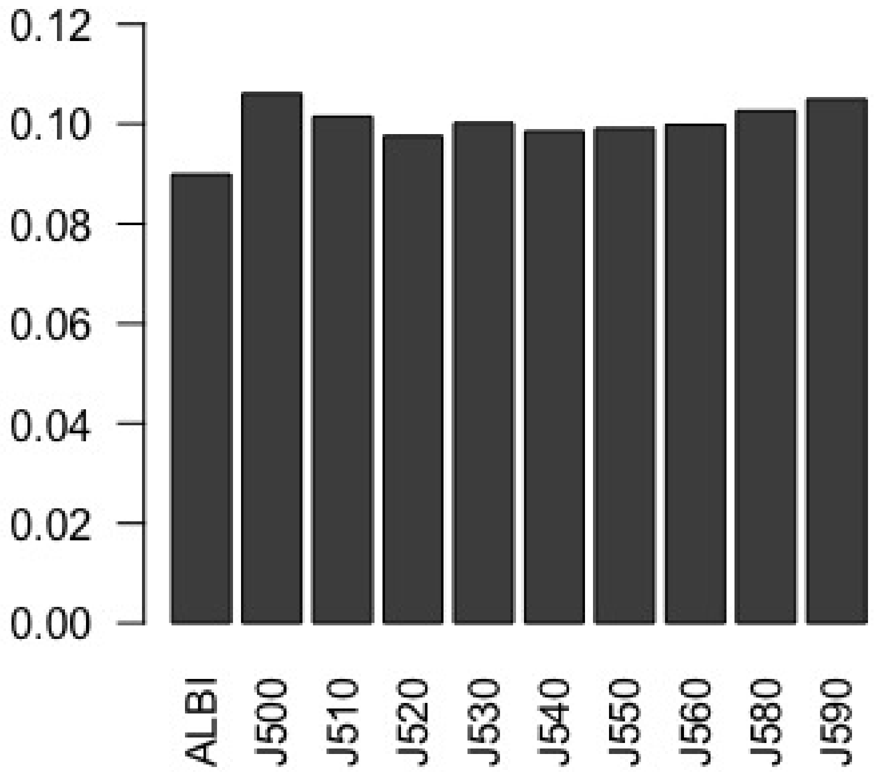

Figure 2: Proportion of weights across assets in the Portfolio.

As can be seen from Figure 1, for a period of about 65 months, our modelled portfolio performs quite similarly relative to the benchmark. After this period, we begin to see our portfolio start to rise above our targeted benchmark indicating an out performance. This trend still observed through to the end of our period. Figure 2 also shows the an indication of how the weights across each asset in our portfolio were distributed looked throughout the period on average.

In order to confirm the above claim that our modelled portfolio outperforms our benchmark, we seek to calculate further performance statistics. We compute the beta of the portfolio which is essentially the systematic risk of our portfolio relative to portfolios held on the market. Secondly, we compute the Jensen alpha,

$$\alpha_p = (E(R_P) - R_f) - \beta_p(E(R_M) - R_f)$$

where

•$E(R_P)$ is the expected return of the portfolio.
•R_f is the risk free return.
•$E(R_M)$ is the expected market return.
•β_p is the beta of the portfolio.

The Jensen alpha is a risk-adjusted measure that represents the average return of the portfolio above or below that of which is predicted by the capital asset pricing model(CAPM), given the portfolio's or investment's beta and the average market return. It is essentially a measure of how an asset manager performs relative to the market. For ex-post measures of the α_p, the manager is considered to have skill if this is significantly positive. Therefore, if $\alpha_p = 0$ then the managers views have not been able to beat the benchmark hence, managers are always in search for positive α_p. The Jensen alpha computed was $\alpha_p = 0.1371429$ which is positive and above 0 indicating that our modelled portfolio did outperform the selected benchmark and Figure 1 is indeed a testament to this. To further understand why this was the case, we note that the Jensen alpha is calculated using the portfolio beta or systematic risk. The systematic risk β_p is simply calculated as the co-variance between our portfolio returns and the market returns. Essentially, a β_p ¡ 1 indicates that the prices of the securities in the portfolio are less volatile compared to the market hence the portfolio is less risky with the actual securities/assets included in the portfolio than without them. The beta for our portfolio was calculated to be 2.469241e-05 which is between 0 and 1 thus this would mean that the portfolio will take on roughly(but not exactly) the same path (up and down) as the All-Share index or benchmark will take. The Sharpe ration obtained from this exercise was given to be 1.59 which means that the excess return of the portfolio for risk taken was 1.59.

All of the above can be once again seen in Figure 1 and hence the performance measures conform to the results obtained through the modelling exercise indicating that for the given views, benchmark and assumptions (Black-Litterman theory) chosen, the Black-Litterman portfolio constructed outperformed our selected benchmark.

References

[1] Lee, W. Theory and Methodology of Tactical Asset Allocation, Fabrozzi Associates, 2000
[2] Polovenko, Black-Litterman Model, Institute of Financial and Actuarial Mathematics at Vienna University of Technology, 2017.

3 Appendix

```
blacklitterman <- function(v, Sig, Pi, p, omeg, gam, tao){
  H <- solve(tao*Sig) + t(p) %*% solve(omeg) %*% p
  Bl.mean <- solve(H) %*% (solve(tao*Sig) %*% Pi + t(p) %*% solve(omeg) %*% v)
  I <- seq(1,1,length.out = nrow(H))
  I <- cbind(I)
  invHI <- solve(H)
  #global portfolio
  #wGlob <- as.numeric( 1 / (t(I) %*% solve(Sig) %*%  I)) * (solve(Sig) %*% I)
  wben <- c(rep(1/10,10))
  wben<- as.matrix(wben)
  #blacklitterman weights/portfolio
  wBL <- wben + ((1/gam) * solve(Sig)) %*% (Bl.mean - ( I %*% ((t(I) %*%
  solve(Sig) %*% Bl.mean) * (1 / (t(I) %*% solve(Sig) %*% I)))))
  #s <- sum(wBL) #sum of weights must be 1 to ensure fully invested
  out.bl <- list(posterior.returns = Bl.mean, blacklitterman.weights = wBL)
  return(wBL)
}

##############################################################################
library(nloptr)
library(quadprog)
load(file = "~/Documents/Portfolio theory/PT-TAA.RData")
Entities <- colnames(tsGRet)
Entities <- Entities[-c(grep('STEFI',Entities))]
Entities <- Entities[-c(grep('ALSI',Entities))]
#market.port <- tsGRet[,12]
ind <- 1:73
dat <- na.omit(tsGRet[,Entities])
dat.RF <- na.omit(tsGRet[,"STEFI"])
dat.mrkt <- na.omit(tsGRet[,"ALSI"])
test.sample <- dat[-ind,]
test.RF <- tsGRet[-ind,"STEFI"] #riskfree for test
test.mrkt <- tsGRet[-ind,"ALSI"] #all share index
#IS.weights <- matrix(NA,74,10)
rolling.window <- function(x){

  newdat <- dat[x,]
  rf <- dat.RF[x,]
  newdat <- na.omit(newdat)
  mean.returns <- colMeans(newdat, na.rm = TRUE)
  sd.returns <- colStdevs(newdat, na.rm = TRUE)
```

```
var.returns <- var(newdat, na.rm = TRUE)
risk.free.train <- colMeans(rf, na.rm = TRUE)

#annualize
mean.returns <- mean.returns*(12)
sd.returns <- sqrt(12)*sd.returns
var.returns <- var.returns*(12)
risk.free.train <- risk.free.train*(12)

one.vec <- rep(1,length(mean.returns))
init.wts <- one.vec / length(one.vec)
IS.weights <- matrix(NA,1,length(mean.returns))

sharpe <- function(x) {
  return(-(x %*% mean.returns - risk.free.train)
         / sqrt(x %*% var.returns %*% x))
}

#ensuring fully invested
constraint <- function(x) {
  return(x%*%one.vec - 1)
}

#make use of sqp to solve for tangency portfolio
soln <- slsqp(init.wts, fn = sharpe, gr = NULL, # target returns
              lower = rep(0,length(init.wts)), # no short-selling
              upper = rep(1,length(init.wts)), # no leverage
              heq = constraint, # fully invested constraint function
              control = list(xtol_rel = 1e-8)) # SQP
IS.weights <- soln$par
portfolio.return <- IS.weights %*% (mean.returns) #portfolio returns
portfolio.volatiliy <- IS.weights %*% var.returns %*% IS.weights #portfolio(market)
risk.free.train

risk.aversion <- (portfolio.return - risk.free.train) / (var(dat.mrkt[x,]))
risk.aversion <- as.numeric(risk.aversion)

portfolio.varcov <- var.returns  #we shrink to eliminate shortselling
alpha <- 1/2 #shrinkage parameters
portfolio.varcov.new <- (1+alpha)*portfolio.varcov + alpha*diag(1, 10, 10)
portfolio.varcov.new <- diag(diag(portfolio.varcov.new), 10, 10)

benchmark.weights <- c(rep(1/10,10))
benchmark.weights <- as.matrix(benchmark.weights)
```

```
  pi <- risk.aversion*(portfolio.varcov.new%*%benchmark.weights)

  tao <- 1
  p <- matrix(c(0,0,-1,0,0,0,0,0,1,0,0,0,0,0,1,0,0,-1,0,0),byrow = T, 2) #p matrix
  omega <- diag(p %*% (tao*var.returns)%*%t(p)) #omega
  omega <- as.numeric(omega)
  omega <- diag(omega)
  v <- matrix(c(0.05,0.02), 2, 1)
  #v <- p%*%mean.returns

  #return(out)
  bl <- blacklitterman(v,portfolio.varcov.new,pi,p,omega,risk.aversion,tao)
  return(bl)
}

compound.returns <- exp(test.sample)^(12/74)
test.RF <- exp(test.RF)^(12/74)

#rolling.window(1:74)
bl.weights <- matrix(NA,74,10)
post.returns <- c(rep(0,74))
#benchmark.weights2 <- matrix(NA,74,10)
#I <- seq(1,1,length.out = nrow(10))
#I <- cbind(I)
#weights.active <- bl.weights - benchmark.weights.matrix
for (i in 1:74) {
  bl.weights[i,] <- rolling.window(i:(i+73))
  post.returns[i] <- bl.weights[i,]%*%t(compound.returns[i,])
  #act.port[i] <-  weights.active[i,]%*%t(compound.returns[i,])

}

benchmark.weights <- c(rep(1/10,10))
benchmark.weights <- as.matrix(benchmark.weights)
bench.port <- t(benchmark.weights) %*% t(compound.returns)
bench.port <-  as.numeric(bench.port)

benchmark.weights.matrix <- matrix(c(rep(1/10,10)), 74,10)


#portfolio.returns.prod <- prod(post.returns);portfolio.returns.prod
plot(cumprod(post.returns), col ="red", type ="l", ylab = "Cumulative returns",
xlab = "Time")
#plot.ts(post.returns), time series plot.
```

```
lines(cumprod(bench.port), col ="blue")
#lines(cumprod(act.port), col="green")
legend("topleft", legend = c("Benchmark","BL-portfolio"), col = c("blue","red"),
lty = 1)
#beta of the portfolio
test.mrkt <- exp(test.mrkt)^(12/74) #standardize
test.mrkt<- as.numeric(test.mrkt)
test.mrkt <- test.mrkt[2:75]

bet <- cov(test.mrkt,post.returns)
a <- prod(test.mrkt)
b <-prod(post.returns)
c<-prod(test.RF)
#jenson alpha
jen.alpha <- (b - c) - bet*(a - c) #alpha is positive

#sharp ratio
port.variance <- var(post.returns*(12)) ;port.variance

S.R <- (a - c) / sqrt(port.variance)

T.E <- sqrt(sum((b-a)^2)/(74-1))
T.E

p.data <- data.frame(post.returns,bench.port, time = 1:74)
s<-ggplot(p.data, aes( x = time))
+ geom_line(aes(y =cumprod(post.returns)), col = "blue")
+ geom_line(aes(y = cumprod(bench.port)),
col = "red")
+ xlab("Time")+ylab("Cumulative Returns");s
s + theme(legend.position="topleft")
grid.arrange(s,d)

bl.weights1 <- colMeans(bl.weights)
df <- as.data.frame(bl.weights1)
rownames(df) <- c("ALBI","J500","J510","J520","J530","J540","J550","J560","J580"
,"J590")
df <- as.matrix(df)
df <-t(df)
barplot(df, ylim = c(0,0.12),las =2)
```

blacklitterman function

Chongo Nkalamo

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```r
blacklitterman <- function(v, Sig, Pi, p, omeg, gam, tao){
  H <- solve(tao*Sig) + t(p) %*% solve(omeg) %*% p
  Bl.mean <- solve(H) %*% (solve(tao*Sig) %*% Pi + t(p) %*% solve(omeg) %*% v)
  I <- seq(1,1,length.out = nrow(H))
  I <- cbind(I)
  invHI <- solve(H)
  #global portfolio
  wGlob <-
    as.numeric( 1 / (t(I) %*% solve(Sig) %*%  I)) * (solve(Sig) %*% I)
  #wben <- c(rep(1/10,10))
  #wben<- as.matrix(wben)
  #blacklitterman weights/portfolio
  wBL <- wGlob + ((1/gam) * solve(Sig)) %*% (Bl.mean - ( I %*% ((t(I) %*% solve(Sig) %*% Bl.mean) * (1
  #s <- sum(wBL) #sum of weights must be 1 to ensure fully invested
  out.bl <- list(posterior.returns = Bl.mean, blacklitterman.weights = wBL)
  return(out.bl)
}
```

```r
#test data

Pi <- matrix(c(0.25,0.1,0.05),3,1)
sig <- matrix(c(0.0900,0.024,-0.006,0.024,0.01,0.0003,-0.006,0.0003,0.0025),byrow = T,3)
v <- matrix(c(0.2,-0.05),2,1)
p <- matrix(c(1,-1,0,0,1,-1),byrow = T,2)
omeg <- matrix(c(0.3,0,0,0.55),byrow = T,2)
blacklitterman(v,sig,Pi,p,omeg,2,0.25)
```

```
## $posterior.returns
##            [,1]
## [1,] 0.25131706
## [2,] 0.10012729
## [3,] 0.04984403
##
## $blacklitterman.weights
##              I
## [1,]  2.029481
## [2,] -3.193920
## [3,]  2.164439
```