Dynamic Programming

Chapter 4, Sutton & Barto

DP concepts

- What is a backup?
  - how many states would be involved?
  - what do we use to determine the states involved in a backup?
- What is a sweep?
  - what determines the states involved in a sweep?

Relating DP concepts to incremental learning

- How do we handle exploration in dynamic programming?
  - what is exploration?
  - we don’t interact with the environment!
- Can dynamic programming work in non-deterministic domains? How do we modify the algorithms to account for stochastic outcomes? Where did we assume determinism?
- Computing state value in a sweep != visiting a state!

Policy Evaluation

- Given a policy π, compute the state-value function v_π

- Recall the **State-value function for policy** π

$$v_\pi(s) \doteq \mathbb{E}_\pi[G_t \mid S_t = s] = \mathbb{E}_\pi\left[\sum_{k=0}^{\infty} \gamma^k R_{t+k+1} \,\middle|\, S_t = s\right], \text{ for all } s \in \mathcal{S}$$

- and the Bellman equation for v_π

$$v_\pi(s) = \sum_a \pi(a|s) \sum_{s',r} p(s', r|s, a)\Big[r + \gamma v_\pi(s')\Big]$$

Iterative methods

- DP methods are called *iterative* because they produce a sequence of value function updates that eventually converge to v_π:

$$v_0 \rightarrow v_1 \rightarrow \cdots \rightarrow v_k \rightarrow v_{k+1} \rightarrow \cdots \rightarrow v_\pi$$

- Policy-evaluation methods perform what is called a **full backup** because they update the value of each state, based on **all possible** next states

- **Every iteration** backs up the value of **every single state**: a sweep consists of applying a **backup operation** to each state.

Basic idea of full policy-evaluation backup

- Turn the Bellman equation into an incremental update formula:

$$v_\pi(s) = \sum_a \pi(a|s) \sum_{s',r} p(s', r|s, a) \Big[r + \gamma v_\pi(s') \Big]$$

- How?

Expected update

Basic idea of full policy-evaluation backup

- Turn the Bellman equation into an incremental update formula:

$$v_\pi(s) = \sum_a \pi(a|s) \sum_{s',r} p(s', r|s, a) \Big[r + \gamma v_\pi(s') \Big]$$

- How?

$$v_{k+1}(s) = \sum_a \pi(a|s) \sum_{s',r} p(s', r|s, a) \Big[r + \gamma v_k(s') \Big] \qquad \forall s \in \mathcal{S}$$

Expected update

Iterative Policy Evaluation – One array version

Input π, the policy to be evaluated
Initialize an array $V(s) = 0$, for all $s \in \mathcal{S}^+$
Repeat
    $\Delta \leftarrow 0$
    For each $s \in \mathcal{S}$:
        $v \leftarrow V(s)$
        $V(s) \leftarrow \sum_a \pi(a|s) \sum_{s',r} p(s', r|s, a)\big[r + \gamma V(s')\big]$
        $\Delta \leftarrow \max(\Delta, |v - V(s)|)$
until $\Delta < \theta$ (a small positive number)
Output $V \approx v_\pi$

Iterative Policy Evaluation – One array version

$$v_{k+1}(s) = \sum_a \pi(a|s) \sum_{s',r} p(s', r|s, a) \Big[r + \gamma v_k(s') \Big] \qquad \forall s \in \mathcal{S}$$

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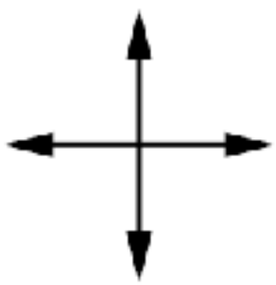
A Small Gridworld



actions

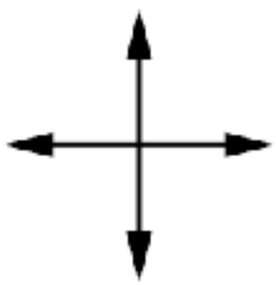
	1	2	3
4	5	6	7
8	9	10	11
12	13	14	

$R = -1$
on all transitions

$\gamma = 1$

- An undiscounted episodic task
- Nonterminal states: 1, 2, . . ., 14;
- One terminal state (shown twice as shaded squares)
- Actions that would take agent off the grid leave state unchanged
- Reward is –1 until the terminal state is reached

A Small Gridworld



actions

	1	2	3
4	5	6	7
8	9	10	11
12	13	14	

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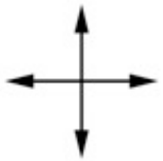
on all transitions

$\gamma = 1$

- What is p(6, -1 | 5, right) = ?
- p(10, r | 5, right) = ?
- p(13, -1 | 13, down) = ?

Iterative Policy Eval for the Small Gridworld



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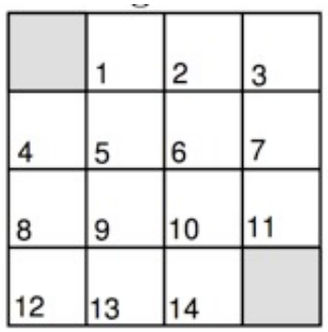

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Single array version

Input π, the policy to be evaluated
Initialize an array $V(s) = 0$, for all $s \in \mathcal{S}^+$
Repeat
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  For each $s \in \mathcal{S}$:
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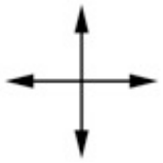
    $\Delta \leftarrow \max(\Delta, |v - V(s)|)$
until $\Delta < \theta$ (a small positive number)
Output $V \approx v_\pi$

$\pi =$ equiprobable random action choices

Iterative Policy Eval for the Small Gridworld



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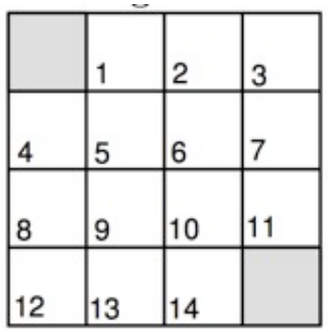

$R = -1$
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$\gamma = 1$

$k = 0$

v_k for the
Random Policy

0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0

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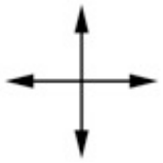

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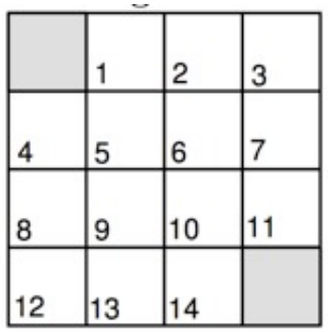

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$k = 0$

v_k for the Random Policy

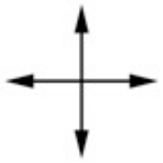
0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0

$k = 1$

0.0	-1.0	-1.0	-1.0
-1.0	-1.0	-1.0	-1.0
-1.0	-1.0	-1.0	-1.0
-1.0	-1.0	-1.0	0.0

$\pi =$ equiprobable random action choices

Iterative Policy Eval for the Small Gridworld



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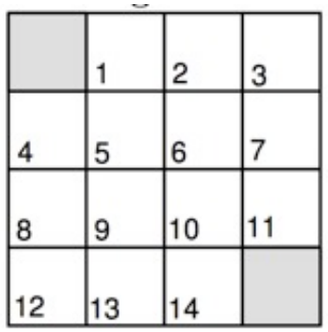

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v_k for the Random Policy

$k = 0$

0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0

$k = 1$

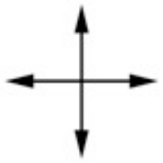
0.0	-1.0	-1.0	-1.0
-1.0	-1.0	-1.0	-1.0
-1.0	-1.0	-1.0	-1.0
-1.0	-1.0	-1.0	0.0

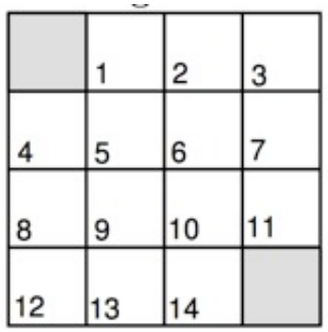
$k = 2$

0.0	-1.7	-2.0	-2.0
-1.7	-2.0	-2.0	-2.0
-2.0	-2.0	-2.0	-1.7
-2.0	-2.0	-1.7	0.0

$\pi =$ equiprobable random action choices

Iterative Policy Eval for the Small Gridworld





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0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0

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0.0	-1.0	-1.0	-1.0
-1.0	-1.0	-1.0	-1.0
-1.0	-1.0	-1.0	-1.0
-1.0	-1.0	-1.0	0.0

$k = 2$

0.0	-1.7	-2.0	-2.0
-1.7	-2.0	-2.0	-2.0
-2.0	-2.0	-2.0	-1.7
-2.0	-2.0	-1.7	0.0

$k = 3$

0.0	-2.4	-2.9	-3.0
-2.4	-2.9	-3.0	-2.9
-2.9	-3.0	-2.9	-2.4
-3.0	-2.9	-2.4	0.0

$k = 10$

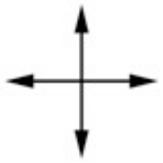
0.0	-6.1	-8.4	-9.0
-6.1	-7.7	-8.4	-8.4
-8.4	-8.4	-7.7	-6.1
-9.0	-8.4	-6.1	0.0

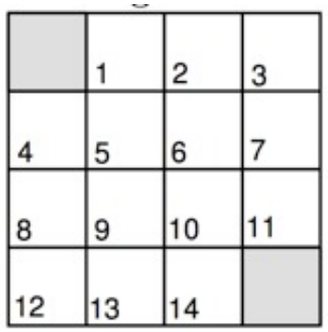
$k = \infty$

0.0	-14.	-20.	-22.
-14.	-18.	-20.	-20.
-20.	-20.	-18.	-14.
-22.	-20.	-14.	0.0

$\pi =$ equiprobable random action choices

Iterative Policy Eval for the Small Gridworld





$R = -1$
on all transitions

$\gamma = 1$

Single array version

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  For each $s \in \mathcal{S}$:
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    $\Delta \leftarrow \max(\Delta, |v - V(s)|)$
until $\Delta < \theta$ (a small positive number)
Output $V \approx v_\pi$

results using **2-array version**

v_k for the Random Policy

$k = 0$

0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0

$k = 1$

0.0	-1.0	-1.0	-1.0
-1.0	-1.0	-1.0	-1.0
-1.0	-1.0	-1.0	-1.0
-1.0	-1.0	-1.0	0.0

$k = 2$

0.0	-1.7	-2.0	-2.0
-1.7	-2.0	-2.0	-2.0
-2.0	-2.0	-2.0	-1.7
-2.0	-2.0	-1.7	0.0

$k = 3$

0.0	-2.4	-2.9	-3.0
-2.4	-2.9	-3.0	-2.9
-2.9	-3.0	-2.9	-2.4
-3.0	-2.9	-2.4	0.0

$k = 10$

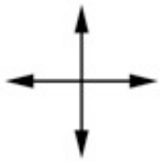
0.0	-6.1	-8.4	-9.0
-6.1	-7.7	-8.4	-8.4
-8.4	-8.4	-7.7	-6.1
-9.0	-8.4	-6.1	0.0

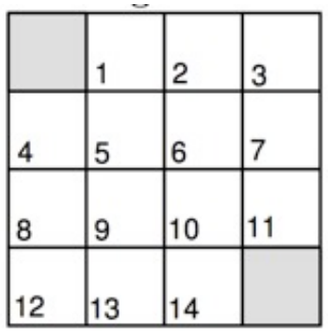
$k = \infty$

0.0	-14.	-20.	-22.
-14.	-18.	-20.	-20.
-20.	-20.	-18.	-14.
-22.	-20.	-14.	0.0

$\pi =$ equiprobable random action choices

Iterative Policy Eval for the Small Gridworld





$R = -1$
on all transitions

$\gamma = 1$

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Output $V \approx v_\pi$

results using **2-array version**

v_k for the Random Policy

$k = 0$

0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0

$k = 1$

0.0	-1.0	-1.0	-1.0
-1.0	-1.0	-1.0	-1.0
-1.0	-1.0	-1.0	-1.0
-1.0	-1.0	-1.0	0.0

$k = 2$

0.0	-1.7	-2.0	-2.0
-1.7	-2.0	-2.0	-2.0
-2.0	-2.0	-2.0	-1.7
-2.0	-2.0	-1.7	0.0

$k = 3$

0.0	-2.4	-2.9	-3.0
-2.4	-2.9	-3.0	-2.9
-2.9	-3.0	-2.9	-2.4
-3.0	-2.9	-2.4	0.0

$k = 10$

0.0	-6.1	-8.4	-9.0
-6.1	-7.7	-8.4	-8.4
-8.4	-8.4	-7.7	-6.1
-9.0	-8.4	-6.1	0.0

$k = \infty$

0.0	-14.	-20.	-22.
-14.	-18.	-20.	-20.
-20.	-20.	-18.	-14.
-22.	-20.	-14.	0.0

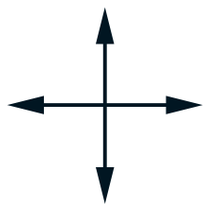
$\pi =$ equiprobable random action choices

- $v_{k=\infty}$ is v_π.
- What does v_π tell us?

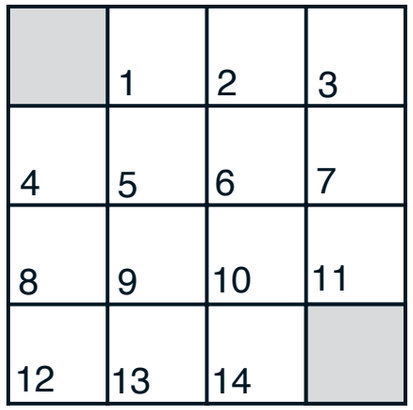
Implementing Iterative Policy Eval

- Two array version:
  - use one array for the old values, and another for the new values
- One array version:
  - Update in place
  - Each new value immediately overwrites the old value
  - Sometimes new values from the current **sweep** will be used instead of old values
- Both versions converge, and the one array version is often faster

Relating q and v



actions

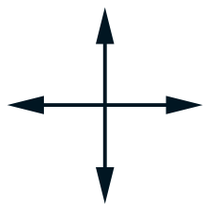


$R = -1$
on all transitions

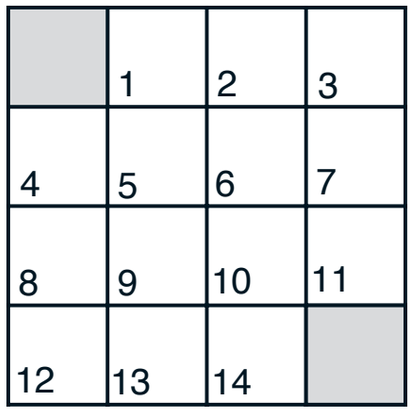
v_k for the
Random Policy

0.0	-14.	-20.	-22.
-14.	-18.	-20.	-20.
-20.	-20.	-18.	-14.
-22.	-20.	-14.	0.0

Relating q and v



actions



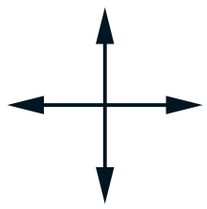
$R = -1$
on all transitions

v_k for the
Random Policy

0.0	-14.	-20.	-22.
-14.	-18.	-20.	-20.
-20.	-20.	-18.	-14.
-22.	-20.	-14.	0.0

- what is $q_\pi(11, \textit{down})$?

Relating q and v



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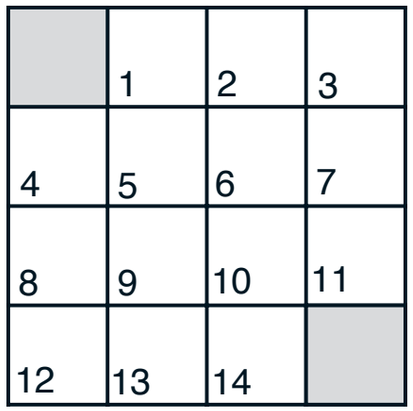

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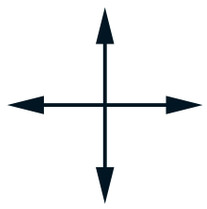
- what is $q_\pi(11, \mathit{down})$?
- what is $q_\pi(7, \mathit{down})$?

Relating q and v



actions

	1	2	3
4	5	6	7
8	9	10	11
12	13	14	

$R = -1$
on all transitions

v_k for the
Random Policy

0.0	-14.	-20.	-22.
-14.	-18.	-20.	-20.
-20.	-20.	-18.	-14.
-22.	-20.	-14.	0.0

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- what is $q_\pi(7, \textit{down})$?

$$q_\pi(s, a) = \mathbb{E}_\pi[R_{t+1} + \gamma v_\pi(S_{t+1}) | S_t = s, A_t = a]$$

Policy evaluation

- Given a policy, we can use iterative policy **evaluation** to tell us how good the policy is, in terms of expected discounted total future reward
  - v_π allows us to *predict*, in each state, the expected return

- RL is largely about finding near-optimal policies, maybe we can use the Bellman equation again to make a DP algorithm to find better policies…

Enough Prediction,
let's start towards Control!

Policy improvement

- We want to know how to change the policy to make it better
- Could ask: what if we did a **different** action in state s, and then followed π after that
  - Is this better or worse than π
  - Bettor or worse than $v\pi(s)$

Policy improvement theorem

- Given the value function for *any policy* π

$$q_\pi(s, a) \qquad \text{for all } s, a$$

- It can always be greedified to obtain a *better policy*:

$$\pi'(s) = \arg\max_a q_\pi(s, a) \qquad (\pi' \text{ is not unique})$$

- where better means:

$$q_{\pi'}(s, a) \geq q_\pi(s, a) \qquad \text{for all } s, a$$

- with equality only if both policies are optimal

Policy Improvement

Suppose we have computed v_π for a deterministic policy π.

For a given state s,
would it be better to do an action $a \neq \pi(s)$?

And, we can compute $q_\pi(s,a)$ from v_π by:

$$\begin{aligned} q_\pi(s,a) &= \mathbb{E}_\pi[R_{t+1} + \gamma v_\pi(S_{t+1}) \mid S_t = s, A_t = a] \\ &= \sum_{s',r} p(s', r|s, a)\Big[r + \gamma v_\pi(s')\Big]. \end{aligned}$$

Policy Improvement

Suppose we have computed v_π for a deterministic policy π.

For a given state s,
would it be better to do an action $a \neq \pi(s)$?

It is better to switch to action a for state s if and only if

$$q_\pi(s,a) > v_\pi(s)$$

And, we can compute $q_\pi(s,a)$ from v_π by:

$$\begin{aligned} q_\pi(s,a) &= \mathbb{E}_\pi[R_{t+1} + \gamma v_\pi(S_{t+1}) \mid S_t = s, A_t = a] \\ &= \sum_{s',r} p(s',r|s,a)\Big[r + \gamma v_\pi(s')\Big]. \end{aligned}$$

Policy Improvement Cont.

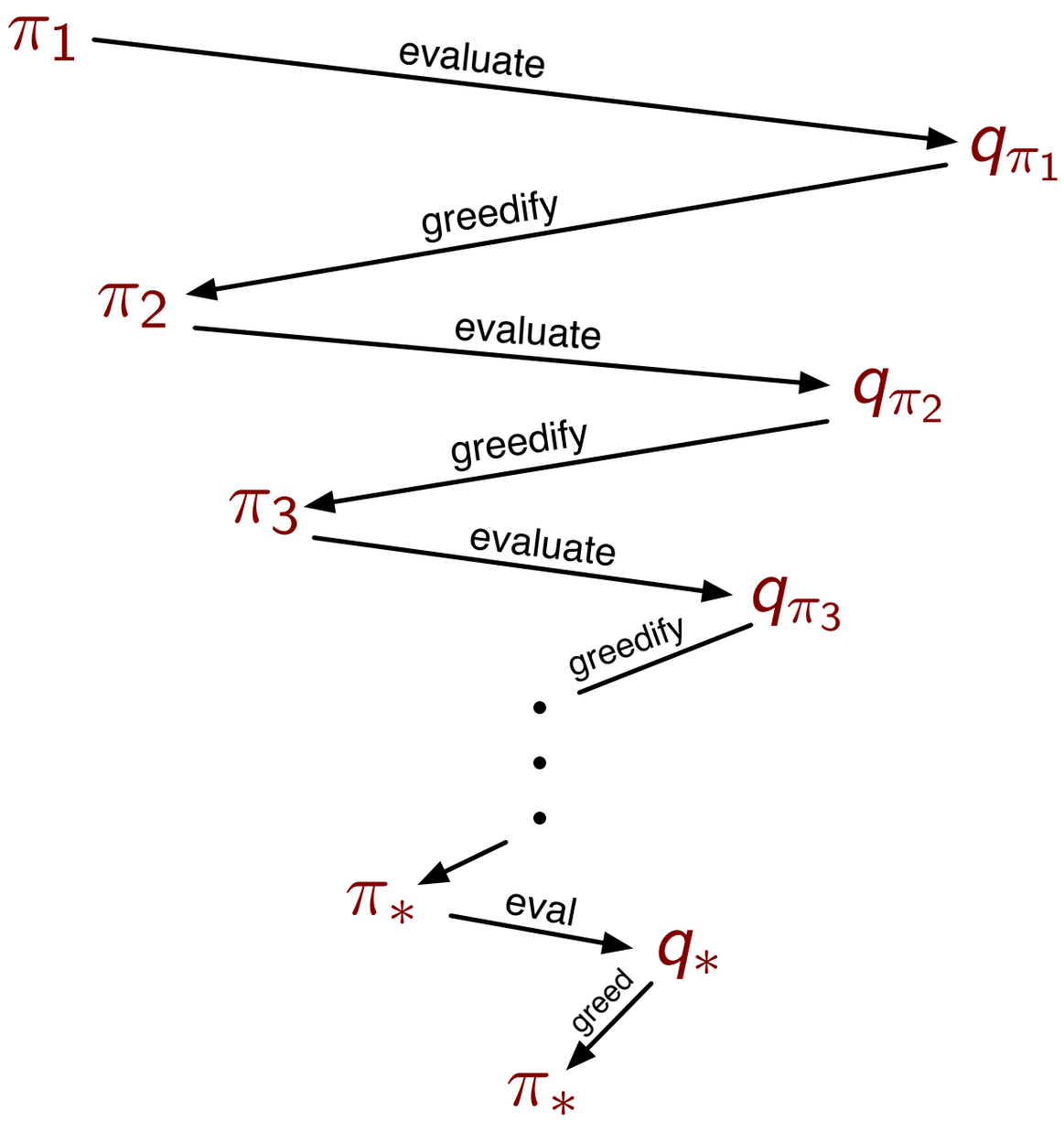
Do this for all states to get a new policy $\pi' \geq \pi$ that is **greedy** with respect to v_π :

$$\begin{aligned}
\pi'(s) &= \arg\max_a q_\pi(s,a) \\
&= \arg\max_a \mathbb{E}[R_{t+1} + \gamma v_\pi(S_{t+1}) \mid S_t = s, A_t = a] \\
&= \arg\max_a \sum_{s',r} p(s',r|s,a)\Big[r + \gamma v_\pi(s')\Big],
\end{aligned}$$

What if the policy is unchanged by this?

Then the policy must be optimal!

The dance of policy and value (Policy Iteration)



Any policy evaluates to a unique value function (soon we will see how to learn it)

which can be greedified to produce a better policy

That in turn evaluates to a value function

which can in turn be greedified…

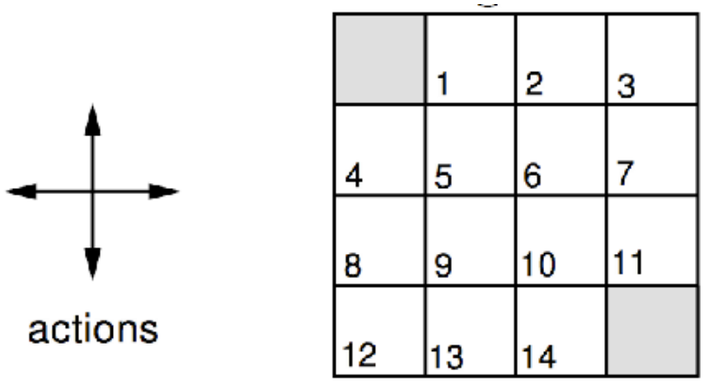
Each policy is *strictly better* than the previous, until *eventually both are optimal*

There are *no local optima*

The dance converges in a finite number of steps, usually very few

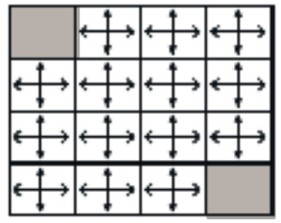
Iterative Policy Eval for the Small Gridworld

π = equiprobable random action choices



$R = -1$
on all transitions

$\gamma = 1$

	V_k for the Random Policy	Greedy Policy w.r.t. V_k
$k = 0$	0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0	 ← random policy
$k = 1$	0.0 -1.0 -1.0 -1.0 -1.0 -1.0 -1.0 -1.0 -1.0 -1.0 -1.0 -1.0 -1.0 -1.0 -1.0 0.0	
$k = 2$	0.0 -1.7 -2.0 -2.0 -1.7 -2.0 -2.0 -2.0 -2.0 -2.0 -2.0 -1.7 -2.0 -2.0 -1.7 0.0	
$k = 3$	0.0 -2.4 -2.9 -3.0 -2.4 -2.9 -3.0 -2.9 -2.9 -3.0 -2.9 -2.4 -3.0 -2.9 -2.4 0.0	
$k = 10$	0.0 -6.1 -8.4 -9.0 -6.1 -7.7 -8.4 -8.4 -8.4 -8.4 -7.7 -6.1 -9.0 -8.4 -6.1 0.0	
$k = \infty$	0.0 -14. -20. -22. -14. -18. -20. -20. -20. -20. -18. -14. -22. -20. -14. 0.0	

- An undiscounted episodic task
- Nonterminal states: 1, 2, . . ., 14;
- One terminal state (shown twice as shaded squares)
- Actions that would take agent off the grid leave state unchanged
- Reward is –1 until the terminal state is reached

Iterative Policy Eval for the Small Gridworld

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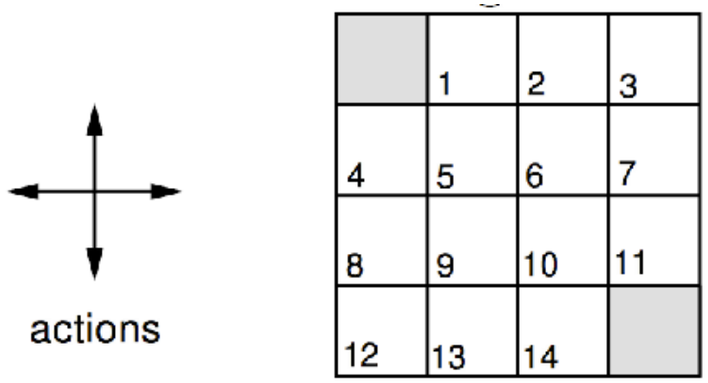

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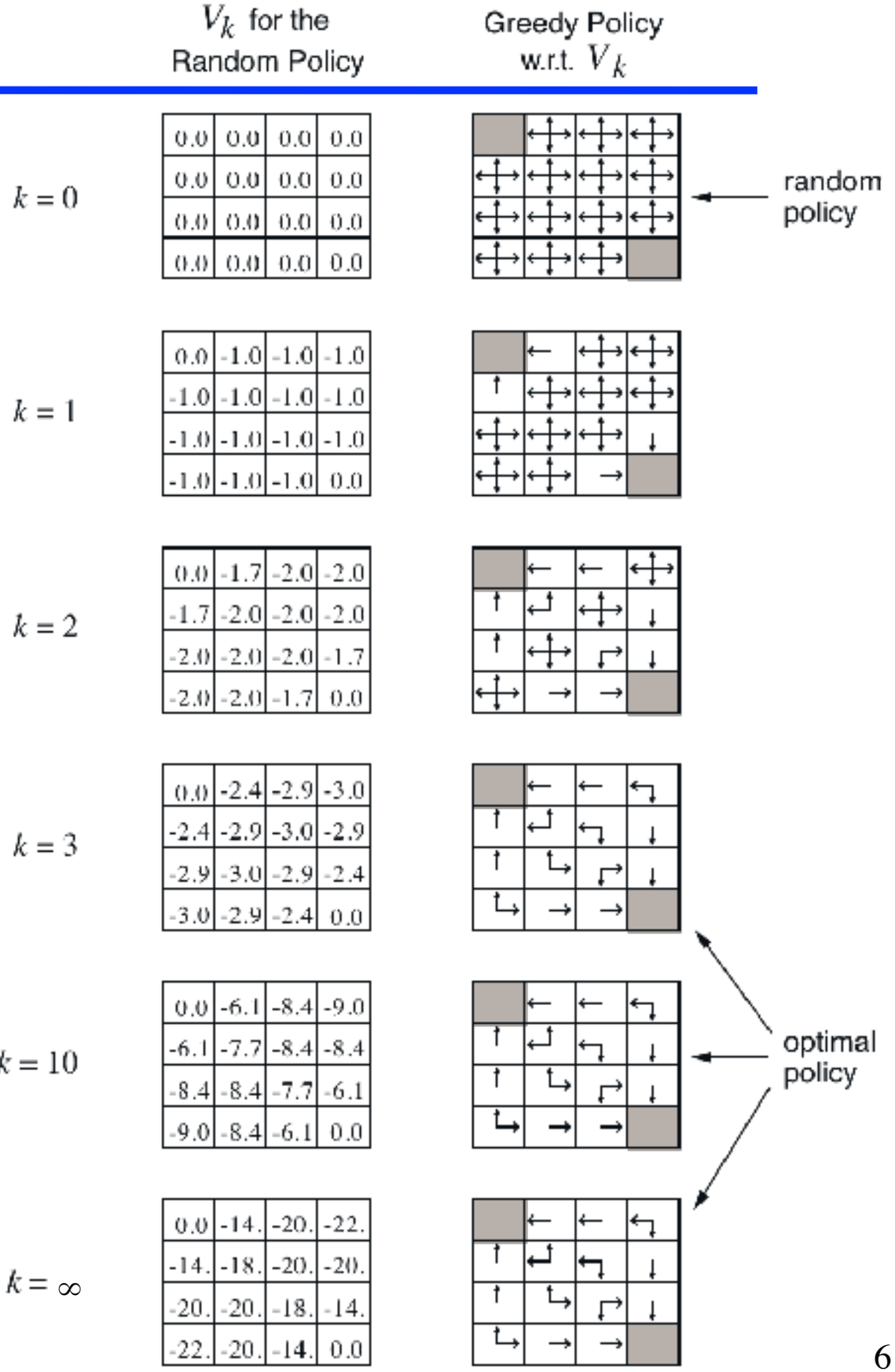
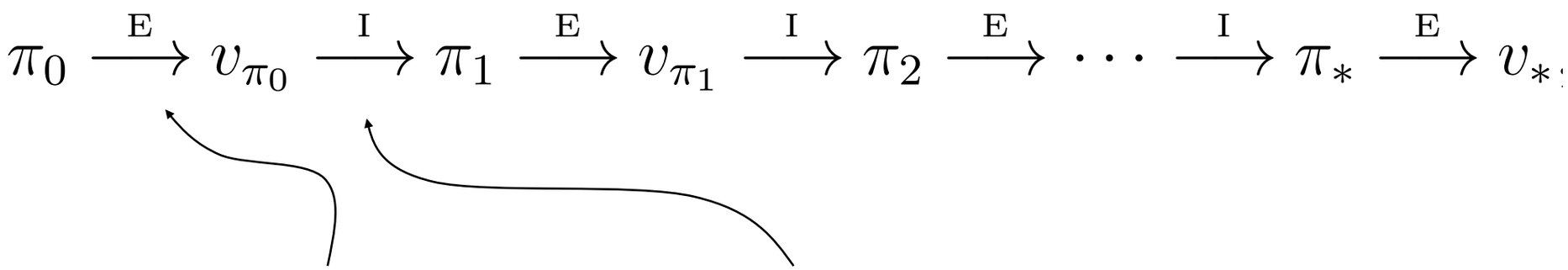


Policy Iteration

Once we improve π using v_π, we get a new policy π', then we can compute $v_{\pi'}$, then we can use $v_{\pi'}$ to get π'', and so on



Policy iteration: basic algorithmic structure

1. initialize V and π

2. run **policy evaluation** till near convergence (compute V_π)

3. greedify: produce π' from π, making π' equal to the action the maximizes V_π in **each state**

4. set $\pi = \pi$'

5. Goto 2

Policy Iteration – One array version (+ policy)

1. Initialization
   $V(s) \in \mathbb{R}$ and $\pi(s) \in \mathcal{A}(s)$ arbitrarily for all $s \in \mathcal{S}$

2. Policy Evaluation
   Repeat
      $\Delta \leftarrow 0$
      For each $s \in \mathcal{S}$:
         $v \leftarrow V(s)$
         $V(s) \leftarrow \sum_{s',r} p(s', r|s, \pi(s))\big[r + \gamma V(s')\big]$
         $\Delta \leftarrow \max(\Delta, |v - V(s)|)$
   until $\Delta < \theta$ (a small positive number)

3. Policy Improvement
   policy-stable $\leftarrow$ *true*
   For each $s \in \mathcal{S}$:
      $a \leftarrow \pi(s)$
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think of this part like argmax Q(s,a)

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Step 2 involves repeated sweeps over the states

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         $\Delta \leftarrow \max(\Delta, |v - V(s)|)$
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each time we call policy evaluation (step 2) we start with the value function of previous policy

>> speeds up convergence

Step 2 involves repeated sweeps over the states

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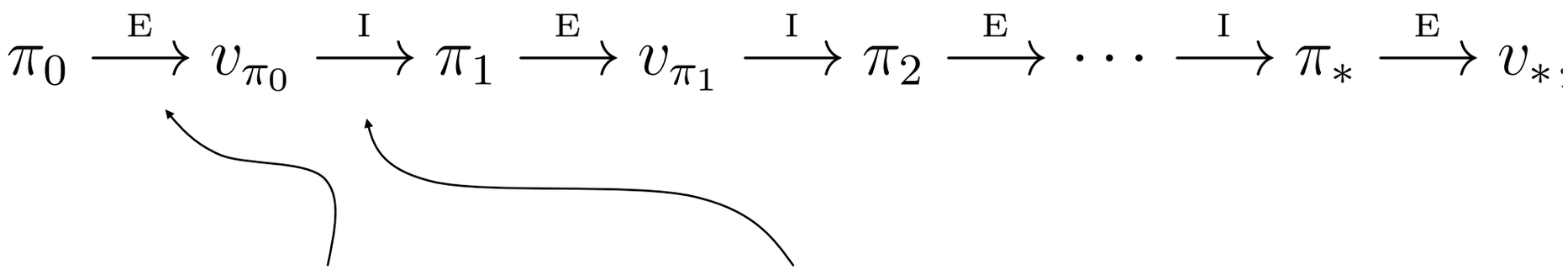
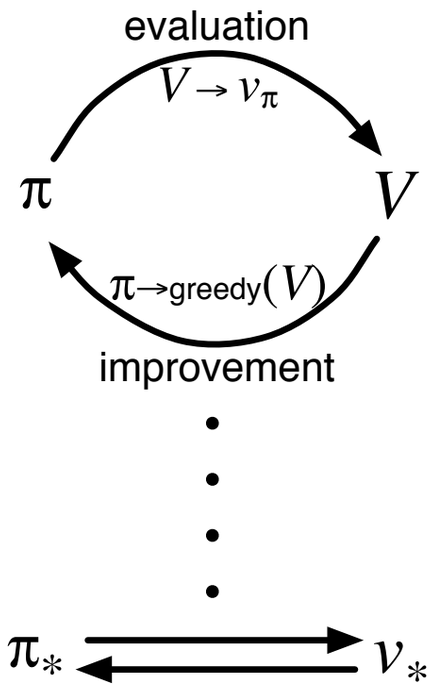


Policy Iteration



- In policy iteration:
  1. *policy evaluation* makes the value function consistent with the current policy
  2. *policy improvement* makes the policy greedy wrt to the current value function

- These two processes alternate, one then the other, each waiting for the other to complete

Modifying Policy iteration

- What if we only did a few iterations of **policy evaluation**—updating v ?
    - or just one?

- Recall the impact of greedifying the value function for the random policy in the gridworld

Value iteration

- Recall the Bellman optimality equation, or the Bellman equation for π*:

$$v_\star(s) = \max_a \sum_{s',r} p(s', r|s, a) \left[r + \gamma v_\star(s')\right]$$

- and the full value-iteration backup:

$$v_{k+1}(s) = \max_a \sum_{s',r} p(s', r|s, a) \Big[r + \gamma v_k(s')\Big] \qquad \forall s \in \mathcal{S}$$

Value iteration algorithm

Initialize array V arbitrarily (e.g., $V(s) = 0$ for all $s \in \mathcal{S}^+$)

Repeat
    $\Delta \leftarrow 0$
    For each $s \in \mathcal{S}$:
        $v \leftarrow V(s)$
        $V(s) \leftarrow \max_a \sum_{s',r} p(s', r|s, a)\big[r + \gamma V(s')\big]$
        $\Delta \leftarrow \max(\Delta, |v - V(s)|)$
until $\Delta < \theta$ (a small positive number)

Output a deterministic policy, π, such that
    $\pi(s) = \arg\max_a \sum_{s',r} p(s', r|s, a)\big[r + \gamma V(s')\big]$

Value iteration example: Gambler's problem

- Gambler can repeatedly bet money on a coin flip
- Gambler has capital, total money he has to bet
- He bet $0, $1, unto his total capital call it stake
- If coin lands on **heads** he gets his stake
- If coin lands tails he losses his stake (subtracted from his capital)
- Coin in unfair
  - Heads (gambler win) with probability p =.4
- Game ends when he wins, gets $100, or losses gets $0

Gambler’s problem

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- What are the states?

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  - terminal states 0 and 100; Let $V_0 = 0$
- What are the rewards?
  - +1 win, zero otherwise
- What are the actions:
  - a = [0:min(s,100-s)]

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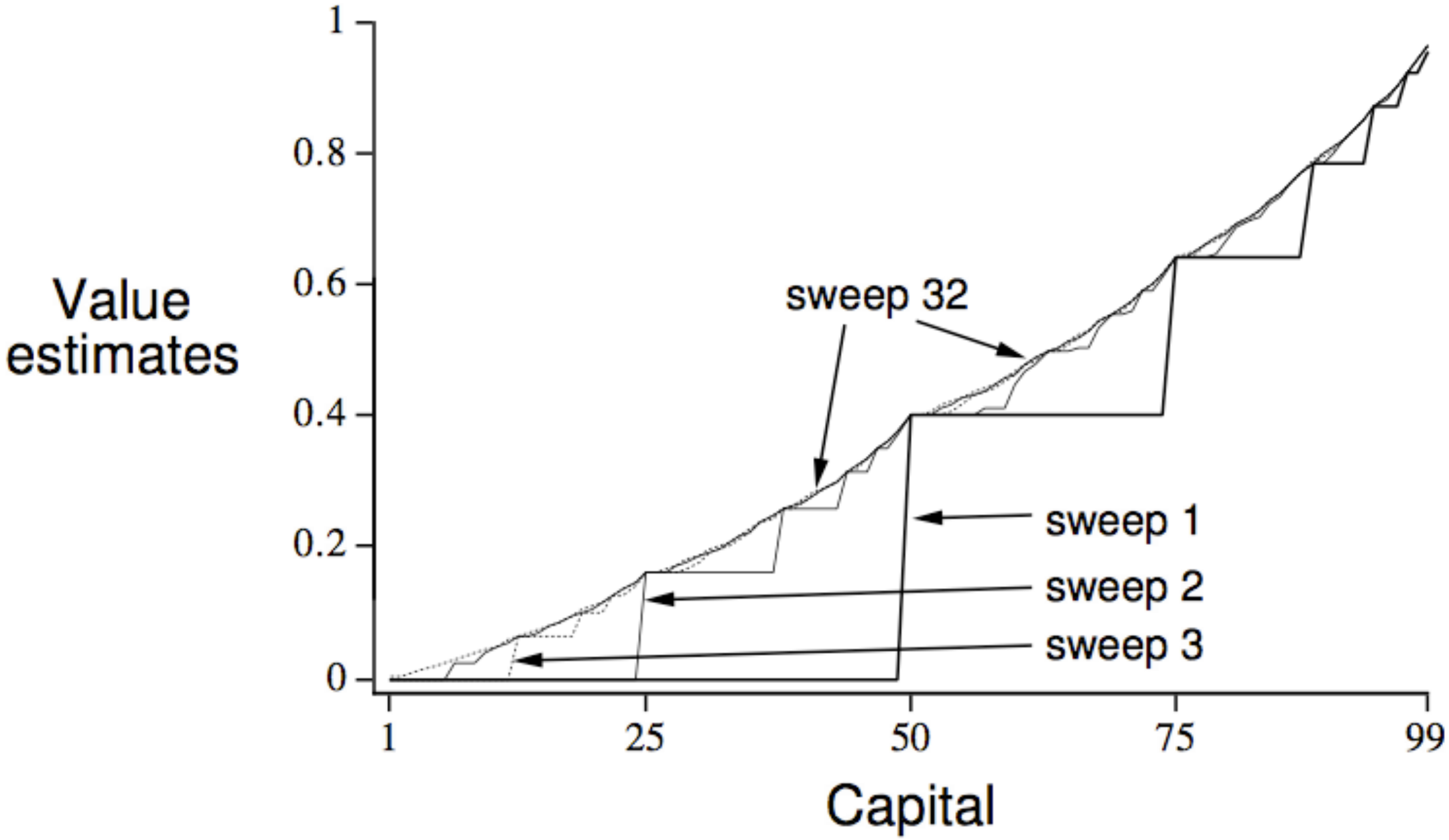
- Why is it important we know pr(heads) = 0.4?

Gambler's Problem Solution



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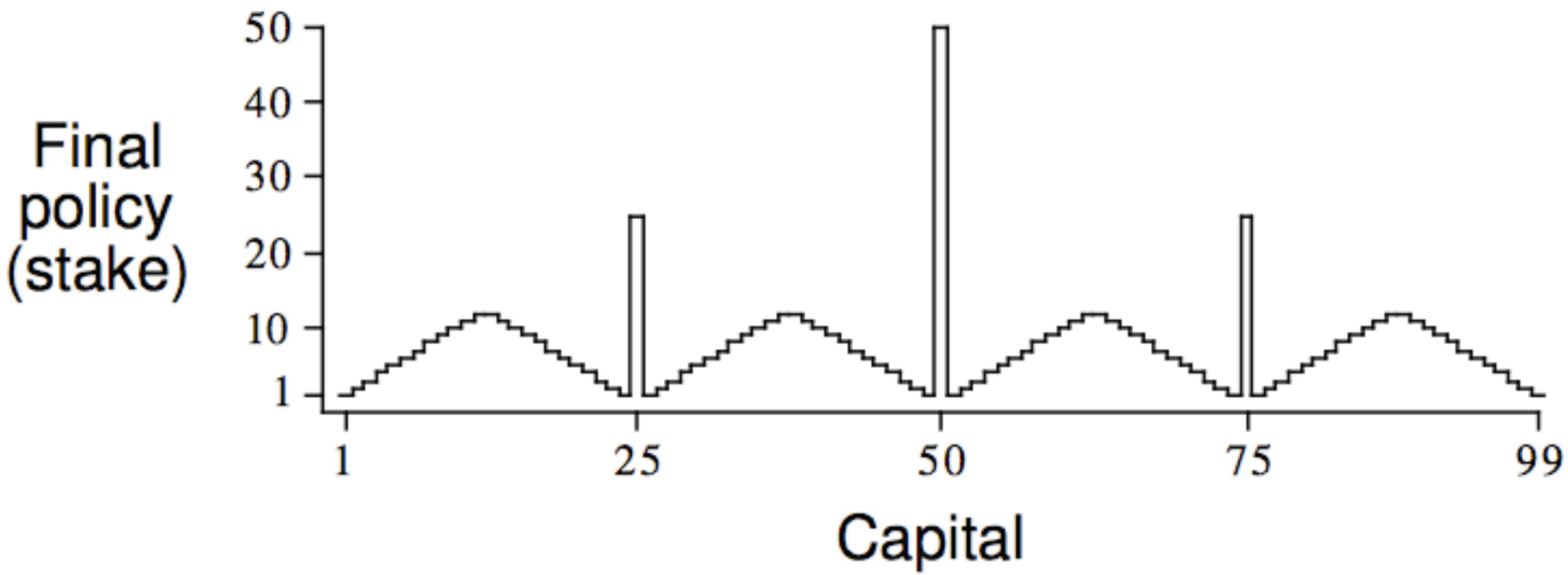

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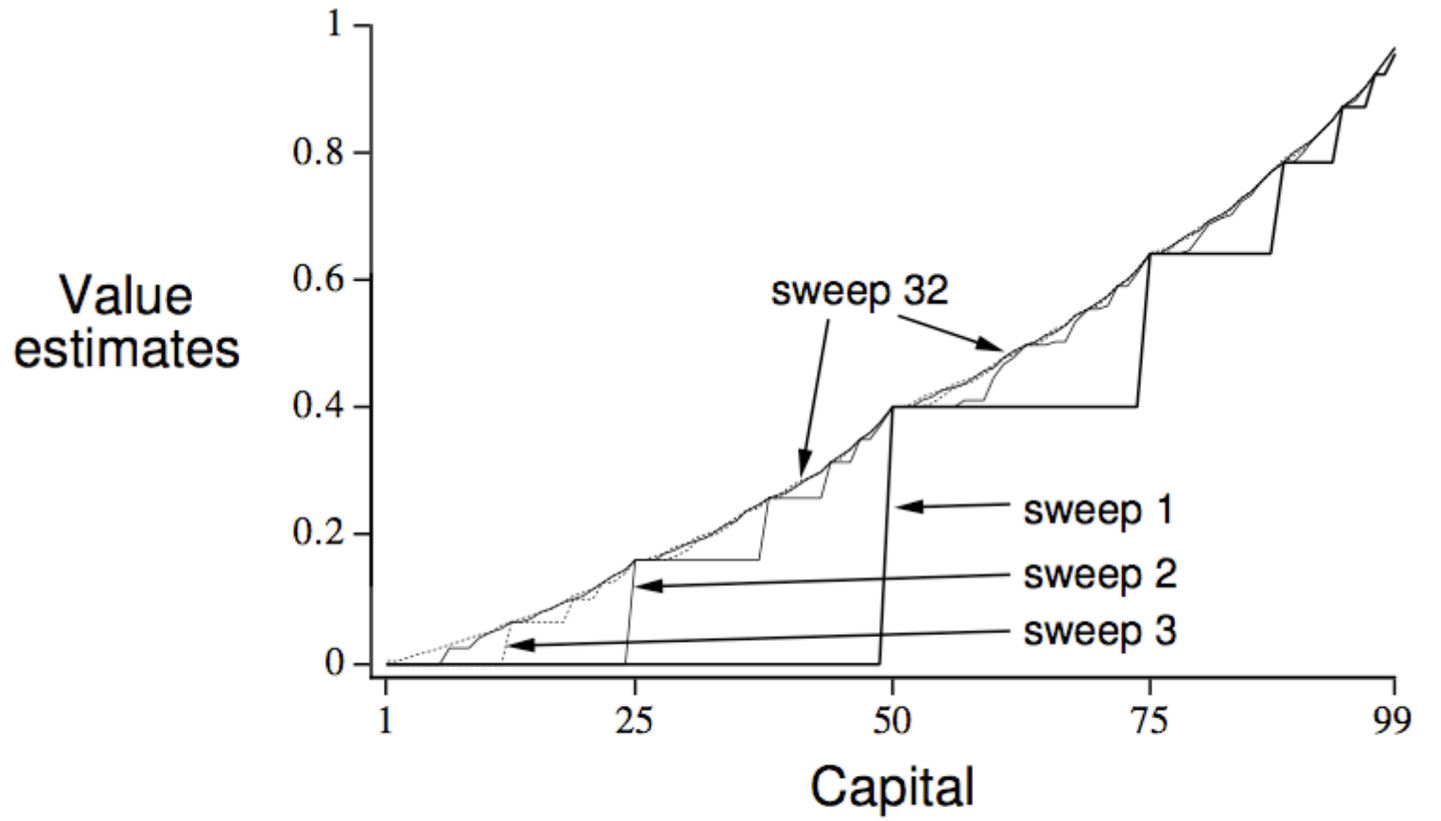
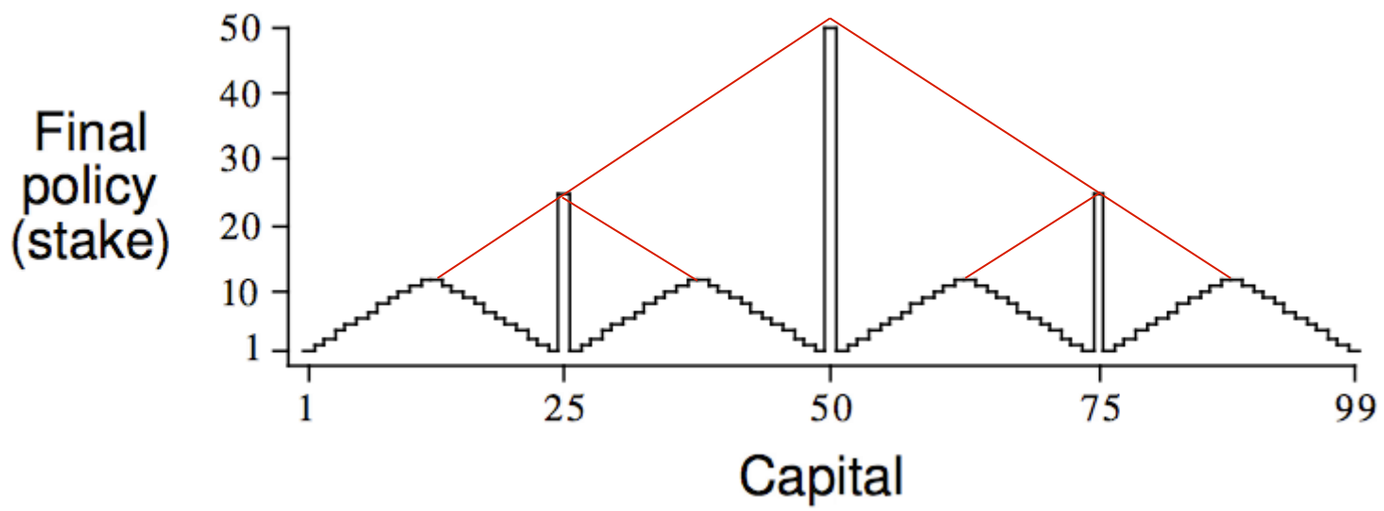




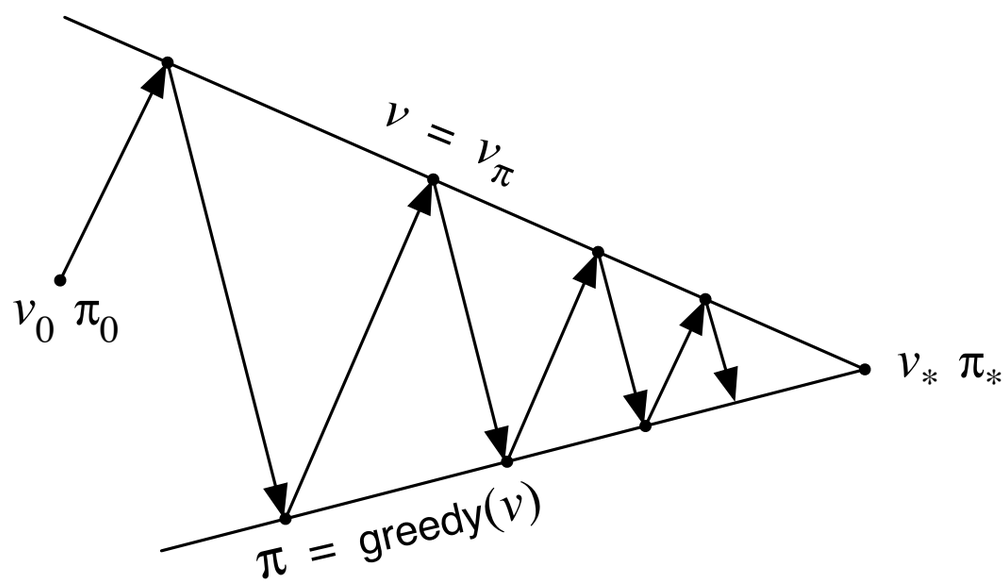
Generalized Policy Iteration

- In policy iteration consists of two simultaneous interacting processes: policy evaluation and policy improvement
  - Alternating, each completing before the other begins
- What other ways could this work?
  - a single iteration of policy iteration as in value iteration
  - only update the value or policy in some states
  - ??
- GPI: any interaction of policy evaluation and policy improvement independent of their granularity
  - As long long as both processes (PI and PE) continue to update all states, then this typically converges to optimal value func & policy
- Most RL methods can be described as GPI

GPI

- We have two interacting goals V —> v_π, and π=greedy(V)
- Progress towards one goal takes us away from the other, but the two processes together bring us closer to the goal of optimality
  - PI typically makes the value function incorrect wrt π'
  - PE (making V consistent with the policy), typically makes π no longer greedy wrt V

A geometric metaphor for convergence of GPI:



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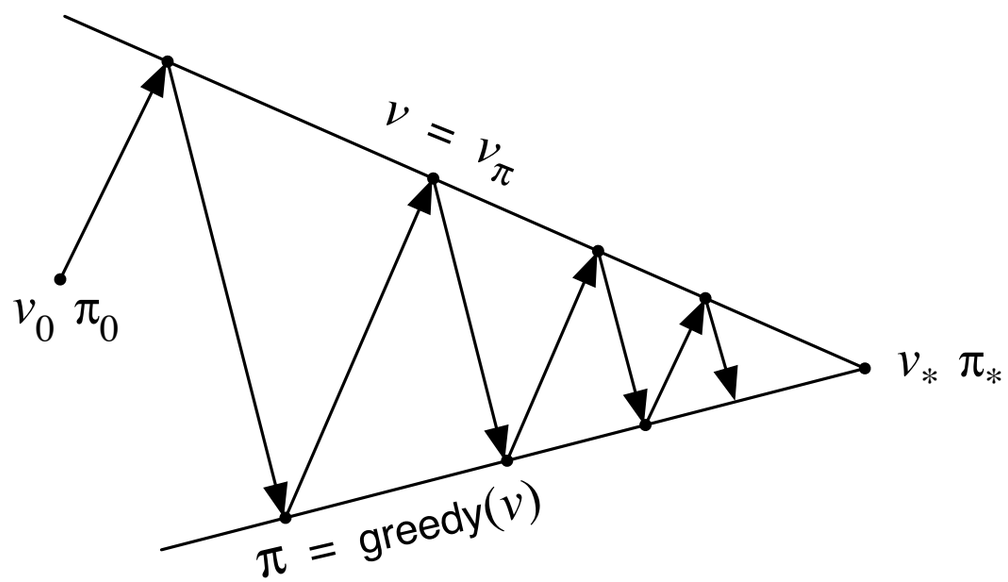

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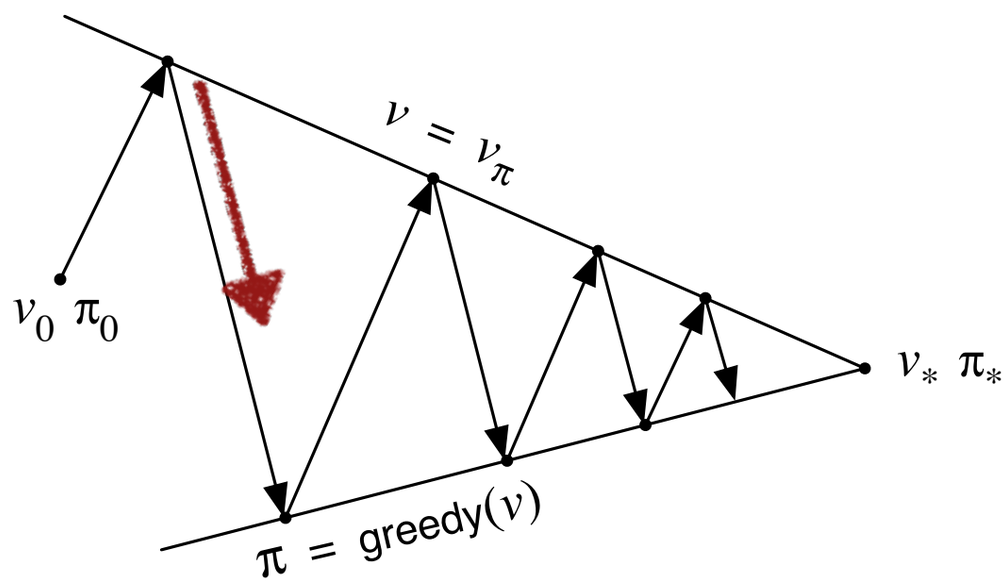

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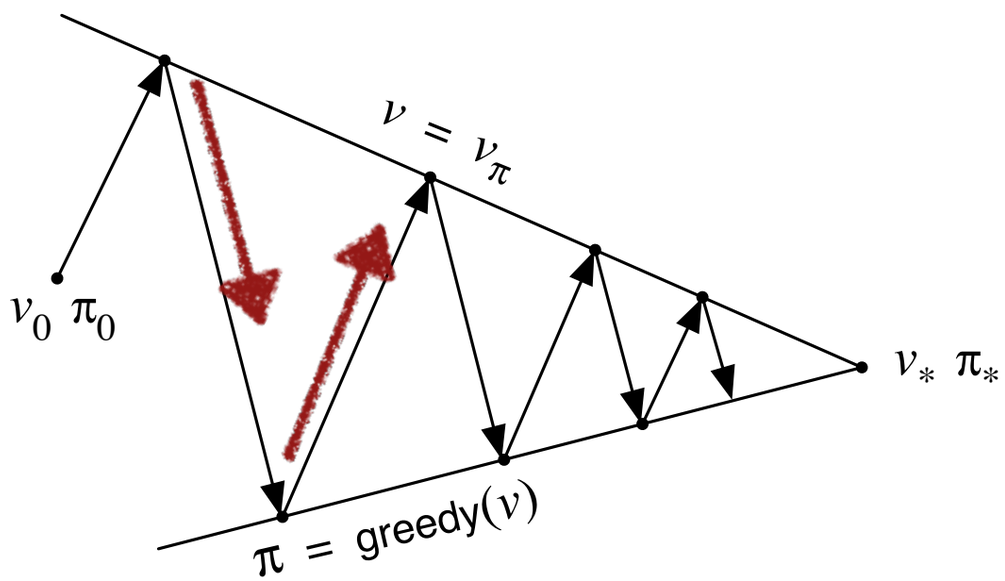

Asynchronous Dynamic Programming

- All the DP methods we talked about so far require systematic sweeps of the entire state set: **loop over s**
- Asynchronous DP are organized in terms of systematic sweeps. Instead it works like this:
  - Repeat until convergence criterion is met:
    - ➡ Pick **a state** and apply the appropriate backup
- Still need lots of computation, but does not get locked into hopelessly long sweeps
- Can you select states to backup intelligently?
  - yes: an agent's experience can act as a guide
- Distinct idea from GPI

Efficiency & Computation

- DP methods can find the optimal policy with time that is polynomial in the # states and actions, **worst case**
    - even though the # of policies is exponential in |*S*| and |*A*|
        - a policy search would require exponential, exhaustive search to get guarantees that we have for DP methods
- BUT, the number of states is often astronomical, e.g., often growing exponentially with the number of state variables (what Bellman called “the curse of dimensionality”).

DP in Practice

- DP methods typically converge must faster than the worst case
    - good initializations help!
- In practice, classical DP can be applied to problems with a few millions of states.
- Asynchronous DP can be applied to larger problems, and is appropriate for parallel computation.
    - however, relatively few states occur along the optimal solution path
- Not clear in practice if policy iteration or value iteration is to be preferred

Jack's Car Rental

- We have **two rental locations**
- **Everyday** we have stochastic car returns to each location
- Everyday we have stochastic rental requests at each location
- Cars are available for rent the **day after** they are returned

Jack's Car Rental

- Jack can move cars between the two locations overnight
  - max cars you can move is 5, at a cost of $2 per car
- **Jack wants to rent cars, but he also only wants to move cars only when required**
  - every car he rents he gets $10
- Let's say each rental location holds 20 cars
  - if more than 20 are returned they go to national rental center

Jack’s car rental s,a,r

Jack’s car rental s,a,r

- What is t, the timestep

Jack’s car rental s,a,r

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  - days

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- What is t, the timestep
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Jack’s car rental s,a,r

- What is t, the timestep
  - days
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  - $[s_1,s_2]$, s1 ϵ [0,20] and s2 ϵ [0,20] // at the end of the day

Jack's car rental s,a,r

- What is t, the timestep
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Jack's car rental s,a,r

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  - a ϵ {-5,-4,…,0,1,…,5} // car movement overnight

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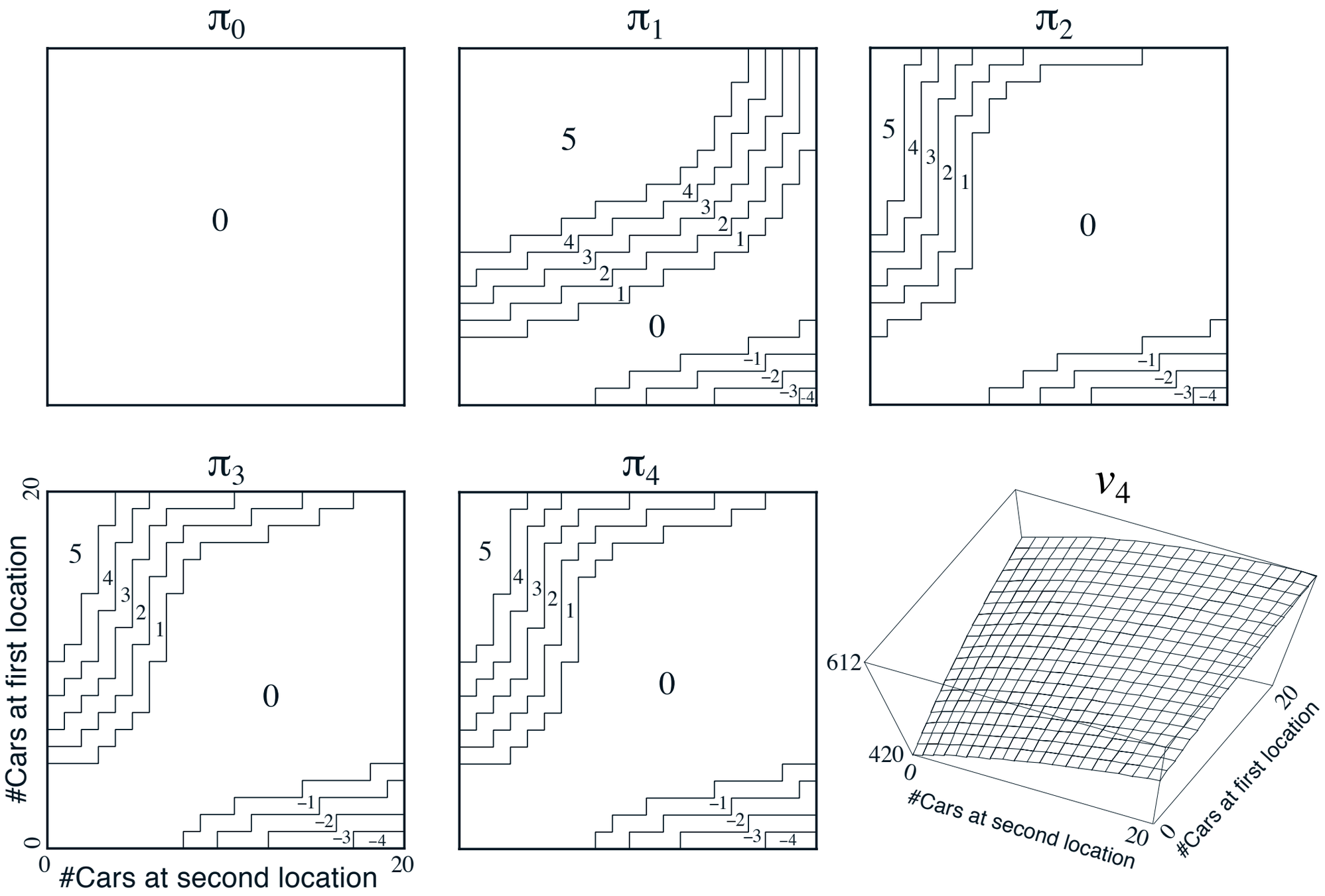
- What are the rewards?
  - R_t =10* satisfied_requests -2*moved
- Is this episodic or continuing?
  - gamma = 0.9

Jack's car rental …

- Cars returned and requested randomly (these are not actions)
  - Poisson distribution governs the process
- What is p(s',r | s,a)?
  - location 1: #out ~ Poisson(λ=3), #in ~ Poisson(λ=3)
  - location 2: #out ~ Poisson(λ=4), #in ~ Poisson(λ=2)

Jack's Car Rental: Policy iteration result



Jack's Car Rental is different

- The state is 2D
- The transition dynamics are stochastic and based on more complex distributions
- Nontrivial implementation, compared to Gamble's domain

Summary

- Policy evaluation: backups without a max
- Policy improvement: form a greedy policy, if only locally
- Policy iteration: alternate the above two processes
- Value iteration: backups with a max
- Full backups (to be contrasted later with sample backups)
- Generalized Policy Iteration (GPI)
- Asynchronous DP: a way to avoid exhaustive sweeps
- **Bootstrapping**: updating estimates based on other estimates
- Biggest limitation of DP is that it requires a *probability model* (as opposed to a generative or simulation model)