Trigonometric Ratios and Identities

Lecture - 10 & 11

Application of Trigonometry in maximising & minimizing :→

Type I :→

$$-1 \le \sin x \le 1$$

$$-1 \le \cos x \le 1$$

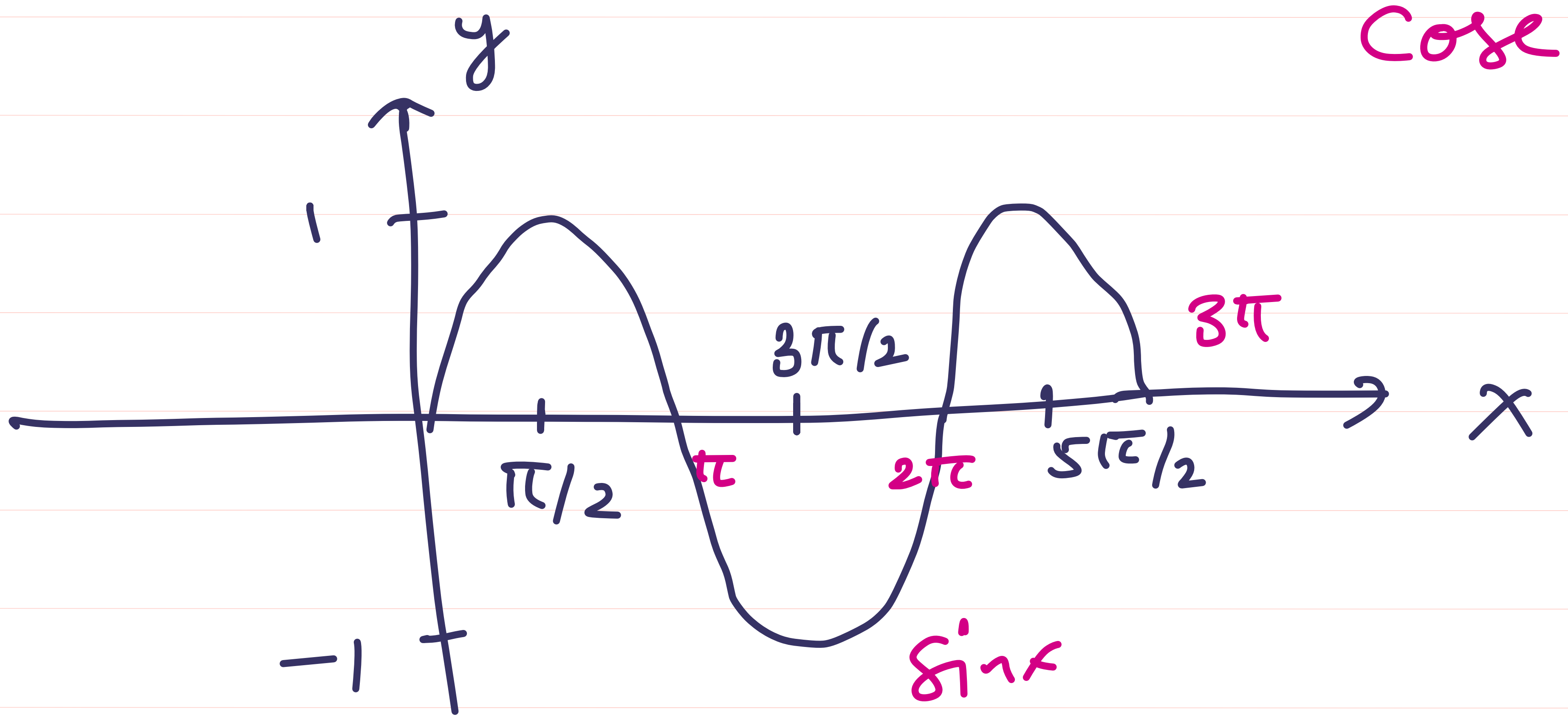
$$\tan x \in (-\infty, \infty)$$

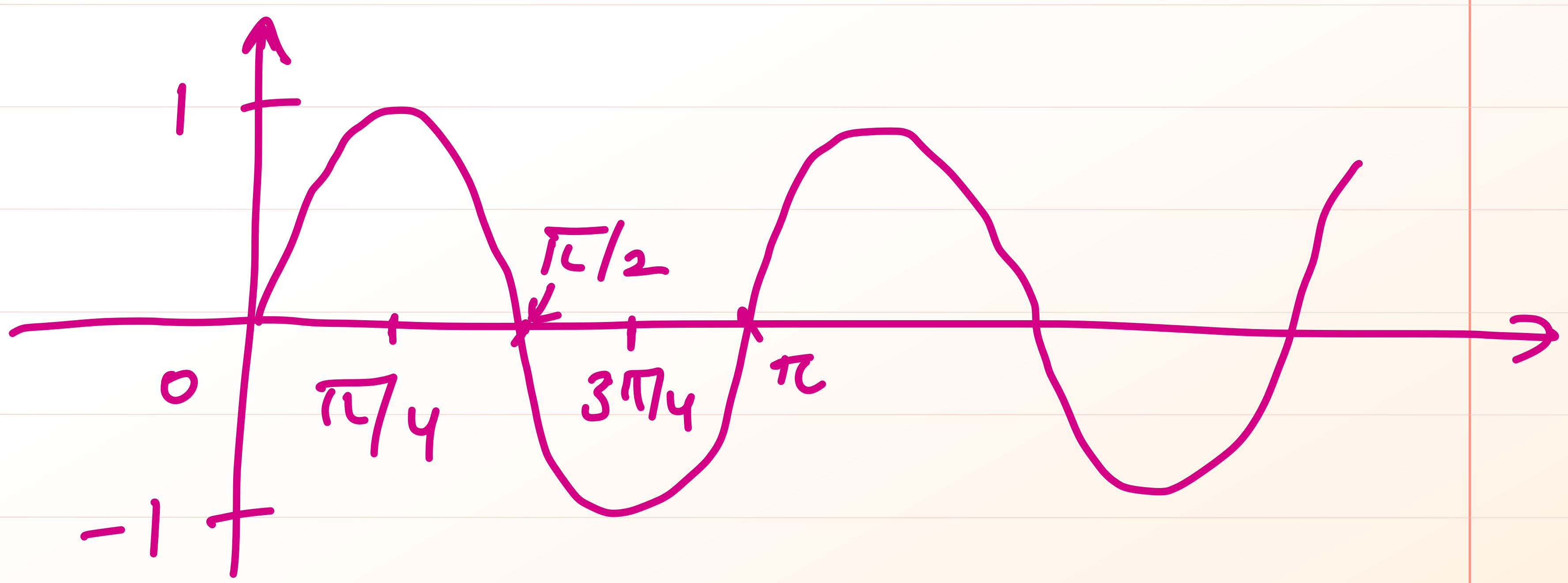
$$\cot x \in (-\infty, \infty)$$

$$\sec x \in (-\infty, -1] \cup [1, \infty)$$

$$\operatorname{cosec} x \in (-\infty, -1] \cup [1, \infty)$$

Range of $\sin x$ is $[-1, 1]$





sin 2x

① find range of $y = \cos^4 x - \sin^4 x$

$$= (\cos^2 x + \sin^2 x)(\cos^2 x - \sin^2 x)$$

$$= 1\,(\underbrace{\cos^2 x - \sin^2 x})$$

$$= \cos 2x$$

$$\boxed{y \in [-1, 1]}$$

(2) (a)

$$y = 4\tan x \cos x$$

$$y = 4\frac{\sin x}{\cos x}(\cos x)$$

$$y = 4\sin x$$

$$\cancel{y \in [-4,4]}$$

$$y \in (-4,4)$$

$$y = 4\sin x$$

$$y \in [-4,4]$$

③ ⓐ $y = \sin(\sqrt{x})$

$\boxed{y \in [-1, 1]}$

ⓑ $y = \sin^2 x$

$-1 \leq \sin x \leq 1$

$0 \leq \sin^2 x \leq 1$

$\boxed{y \in [0, 1]}$

ⓒ $y = \sin 3x$

$\boxed{y \in [-1, 1]}$

Q $y = \sin^2\left(\frac{15\pi}{8} - 4x\right) - \sin^2\left(\frac{17\pi}{8} - 4x\right)$

$\sin^2(A) - \sin^2 B$

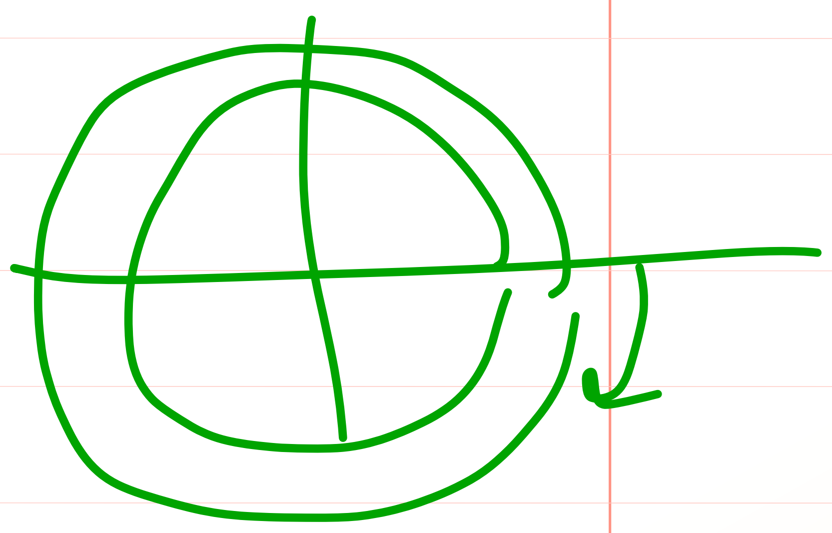
$= \sin(A+B)\sin(A-B)$

$y = \underbrace{\sin(4\pi - 8x)} \cdot \underbrace{\sin\left(-\frac{\pi}{4}\right)}$

$= +\underbrace{\sin(8x)}\ \frac{1}{\sqrt{2}}$

$=$

$\boxed{y \in \left[-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]}$



Type II when argument of sine & cosine are same :→

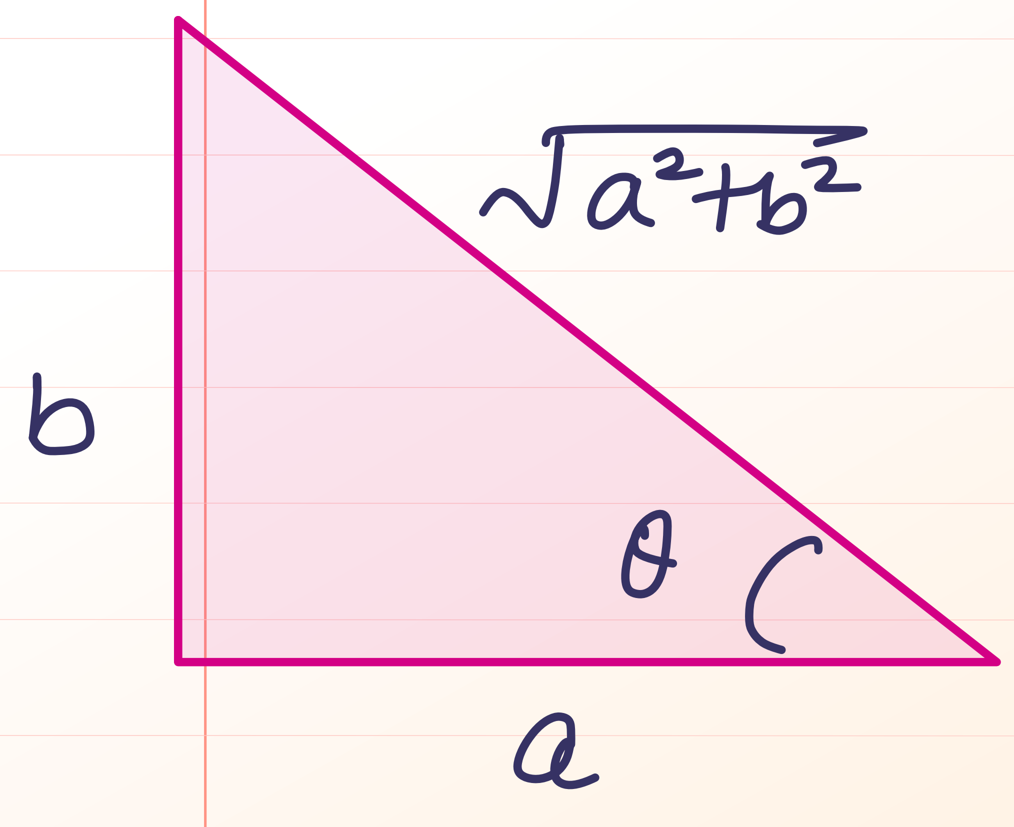
$a\sin \underline{x} + b\cos \underline{x}$

$$y = \left(\frac{a\sin x}{\sqrt{a^2+b^2}} + \frac{b\cos x}{\sqrt{a^2+b^2}}\right)\sqrt{a^2+b^2}$$

$$y = \sqrt{a^2+b^2}\left[\frac{a}{\sqrt{a^2+b^2}}\sin x + \frac{b}{\sqrt{a^2+b^2}}\cos x\right]$$

$$= \sqrt{a^2+b^2}\left[\cos\theta \cdot \sin x + \sin\theta \cos x\right]$$

$$= \sqrt{a^2+b^2}\left[\sin(x+\theta)\right]$$



$$\cos\theta = \frac{a}{\sqrt{a^2+b^2}}$$

$$\sin\theta = \frac{b}{\sqrt{a^2+b^2}}$$

$$-1 \le \sin(x+\theta) \le 1$$

$$-\sqrt{a^2+b^2} \le \underline{\sqrt{a^2+b^2}\,(\sin(x+\theta))} \le \sqrt{a^2+b^2}$$

$$\boxed{y \in \left[-\sqrt{a^2+b^2},\ +\sqrt{a^2+b^2}\right]}$$

find range

Q (1) $y = \sin x + \cos x$

$-\sqrt{1^2+1^2} \le \underbrace{\sin x + \cos x} \le \sqrt{1^2+1^2}$

$-\sqrt{2} \le y \le \sqrt{2}$

$\boxed{y \in [-\sqrt{2}, \sqrt{2}]}$

Q2 $y = 3\sin x + 4\cos x + 5$

$-\sqrt{3^2+4^2} \le 3\sin x + 4\cos x \le +\sqrt{3^2+4^2}$

$-5 \le \underbrace{3\sin x + 4\cos x \quad +5} \le +5$

$+5 \qquad\qquad\qquad\qquad +5$

$0 \le y \le 10$

$y \in [0, 10]$

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Q-3

$$y = \log_2\left(\frac{3\sin x - 4\cos x + 15}{10}\right)$$

$$-\sqrt{3^2+4^2} \le 3\sin x - 4\cos x \le \sqrt{3^2+4^2}$$

$$15-5 \le 3\sin x - 4\cos x + 15 \le 5+15$$

$$\frac{10}{10} \le \frac{3\sin x - 4\cos x + 15}{10} \le \frac{20}{10}$$

$$\underbrace{\log_2(1)} \le \log_2\left(\frac{3\sin x - 4\cos x + 15}{10}\right) \le \log_2 2$$

$$0 \le y \le 1$$

$$\boxed{y \in [0,1]}$$

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Q 4 If $b \le 3\sin^2 x + 6\cos^2 x \underbrace{-4\sin x \cos x} + 5 \le a$ find a & b.

$$y = \underline{3\sin^2 x} + \underline{6\cos^2 x} \underbrace{-4\sin x \cos x} + 5$$

$$= \underbrace{3\sin^2 x + 3\cos^2 x} + 3\cos^2 x - 2\sin 2x$$

$$= 3(\sin^2 x + \cos^2 x) + 3\cos^2 x - 2\sin 2x + 5$$

$$= 8 + \underline{3\cos^2 x} - \underline{2\sin 2x}$$

$$= 8 + 3\left(\frac{\cos 2x + 1}{2}\right) - 2\sin 2x$$

$$y = 8 + \frac{3}{2}\cos 2x + \frac{3}{2} - 2\sin 2x$$

$$\boxed{y = \frac{19}{2} + \frac{3}{2}\cos 2x - 2\sin 2x}$$

$\cos 2x = 2\cos^2 x - 1$

$\cos^2 x = \frac{\cos 2x + 1}{2}$

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$$\boxed{y = \frac{19}{2} + \frac{3}{2}\cos 2x - 2\sin 2x}$$

$$-\sqrt{\left(\frac{3}{2}\right)^2 + (2)^2} \leq \frac{3}{2}\cos 2x - 2\sin 2x \leq \sqrt{\left(\frac{3}{2}\right)^2 + (2)^2}$$

$$-\frac{5}{2} \leq \frac{3}{2}\cos 2x - 2\sin 2x \leq \frac{5}{2}$$

$$+\frac{19}{2} \qquad +\frac{19}{2} \qquad +\frac{19}{2}$$

$$7 \leq y \leq 12$$

$$y \in [7, 12]$$

Q 6 find range of $y = 5\sin\left(x+\frac{\pi}{6}\right) + 3\cos x$ Ans $[-7, 7]$

Q-7 find range of $y = \sin\left(x+\frac{\pi}{6}\right) + 3\cos\left(x-\frac{\pi}{3}\right)$ Ans $[-4, 4]$

Q-8 find maximum & minimum value of

$$y = \frac{17 + \boxed{(5\sin x + 12\cos x)}}{17 - (5\sin x + 12\cos x)} \rightarrow [-13, 13]$$

$$y_{max} = \frac{17+13}{17-13} = \frac{15}{2}$$

$$y_{min} = \frac{17-13}{17-(-13)} = \frac{2}{15}$$

Type-3 Argument of sine & cosine are different.

(1) $y = \underbrace{\cos 2x} + 3\underbrace{\sin x}$

$= \underbrace{1 - 2\sin^2 x} + 3\sin x$

$= \underbrace{-2\sin^2 x + 3\sin x} + 1$

$= -2\left[\sin^2 x - \frac{3}{2}\sin x\right] + 1$

$= -2\left[\sin^2 x - 2\cdot\sin x\cdot\left(\frac{3}{4}\right) + \left(\frac{3}{4}\right)^2\right] - \left(\frac{3}{4}\right)^2\underline{(-2)} + 1$

$= -2\left[\sin x - \frac{3}{4}\right]^2 + \frac{9}{8} + 1$

$y = -2\left[\sin x - \frac{3}{4}\right]^2 + \frac{17}{8}$

$$y = -2\left[\sin x - \frac{3}{4}\right]^2 + \frac{17}{8}$$

$$y = \frac{17}{8} - 2\left[\sin x - \frac{3}{4}\right]^2$$

when $\sin x = \frac{3}{4}$

$$y_{max} = \frac{17}{8} - 0 = \frac{17}{8}$$

when $\sin x = -1$

$$y_{min} = \frac{17}{8} - 2\left[-1 - \frac{3}{4}\right]^2$$

$$= \frac{17}{8} - 2\left[-\frac{7}{4}\right]^2$$

$$= \frac{17}{8} - \frac{49}{8} = -4$$

$$\boxed{y \in \left[-4, \frac{17}{8}\right]}$$

Q

$y = \sin^2 x - 20\cos x + 1$

$y = 1 - \cos^2 x - 20\cos x + 1$

$= -\cos^2 x - 20\cos x + 2$

$= -1\left[\cos^2 x + 20\cos x\right] + 2$

$= -1\left[\cos^2 x + 20\cos x + (10)^2\right] - (10)^2(-1) + 2$

$= -1\left[\cos x + 10\right]^2 + 102$

$y = 102 - (\underbrace{10 + \cos x})^2$

$\cos x = -1 \Rightarrow y_{max} = 102 - (10-1)^2 = 21$

$\cos = 1 \Rightarrow y_{min} = 102 - (11)^2 = -19$

Q $y = \cos^2 x - 4\cos x + 13$

Type - 4

$a^2+b^2 = (a+b)^2 - 2ab$

$a^2+b^2 = (a-b)^2 + 2ab$

Q $y = a^2\tan^2\theta + b^2\cot^2\theta$; $(a, b \geq 0)$

$= (a\tan\theta - b\cot\theta)^2 + 2ab\cot\theta\tan\theta$

$= (a\tan\theta - b\cot\theta)^2 + 2ab$

$y_{min} = 2ab$ when $a\tan\theta - b\cot\theta = 0$

$a\tan\theta = b\cot\theta$

$a\tan\theta = \dfrac{b}{\tan\theta}$

$\tan^2\theta = \dfrac{b}{a}$

$\boxed{\tan\theta = \sqrt{\dfrac{b}{a}}}$

2

$y = a^2 \sec^2\theta + b^2 \operatorname{cosec}^2\theta$

$= a^2(1+\tan^2\theta) + b^2(1+\cot^2\theta)$

$y = a^2+b^2 + \underbrace{a^2\tan^2\theta + b^2\cot^2\theta}$

$y = a^2+b^2 + (a\tan\theta - b\cot\theta)^2 + 2ab$

$y_{min} = a^2+b^2+2ab$ when $a\tan\theta - b\cot\theta = 0$

$\Rightarrow \tan\theta = \sqrt{\frac{b}{a}}$

(3) $y = 8\sec^2\theta + 18\cos^2\theta$

$$y = (2\sqrt{2}\sec\theta)^2 + (3\sqrt{2}\cos\theta)^2$$

$$= (2\sqrt{2}\sec\theta - 3\sqrt{2}\cos\theta)^2 + 2\cdot 2\sqrt{2}\cdot\sec\theta\cdot 3\sqrt{2}\cos\theta$$

$$= (2\sqrt{2}\sec\theta - 3\sqrt{2}\cos\theta)^2 + 24$$

$$\boxed{y_{min} = 24}$$

when $2\sqrt{2}\sec\theta - 3\sqrt{2}\cos\theta = 0$

$$\frac{2\sqrt{2}}{\cos\theta} = 3\sqrt{2}\cos\theta$$

$$\cos^2\theta = \frac{2\sqrt{2}}{3\sqrt{2}}$$

$$\boxed{\cos\theta = \sqrt{\frac{2}{3}}}$$

find range

(4) $y = 4\sin^2\theta + \operatorname{cosec}^2\theta$

(5) $y = \sin^2\theta + 4\operatorname{cosec}^2\theta$

(6) $y = 4\sin^2 x + 27\operatorname{cosec}^2 x$

(7) $y = 18\sec^2 x + 8\cos^2 x$

$\alpha+\beta = 60^\circ$

Note : $\rightarrow$

(i) If $\alpha, \beta \in \left(0, \frac{\pi}{2}\right)$ and $\alpha+\beta = \sigma$ (constant) then

(a) max. value of the expression $\cos\alpha\cos\beta$, $\cos\alpha+\cos\beta$, $\sin\alpha\cdot\sin\beta$ or $\sin\alpha+\sin\beta$ occurs when $\alpha=\beta=\sigma/2$

(b) Minimum value of $\sec\alpha+\sec\beta$, $\tan\alpha+\tan\beta$, $\operatorname{cosec}\alpha+\operatorname{cosec}\beta$ occurs when $\alpha=\beta=\sigma/2$

(ii) If A, B, C are angles of a triangle then maximum value of $\sin A+\sin B+\sin C$ and $\sin A\sin B\sin C$ occurs when $A=B=C=60^\circ$.

Q If $x^2+y^2=4$; $a^2+b^2=8$; find minimum & maximum value of $ax+by$.

$x^2+y^2=(2)^2$

$x=2\cos\theta$

$y=2\sin\theta$

$a^2+b^2=(2\sqrt{2})^2$

$a=2\sqrt{2}\cos\phi$

$b=2\sqrt{2}\sin\phi$

If $x^2+y^2=r^2$
then
$x=r\cos\theta$
$y=r\sin\theta$

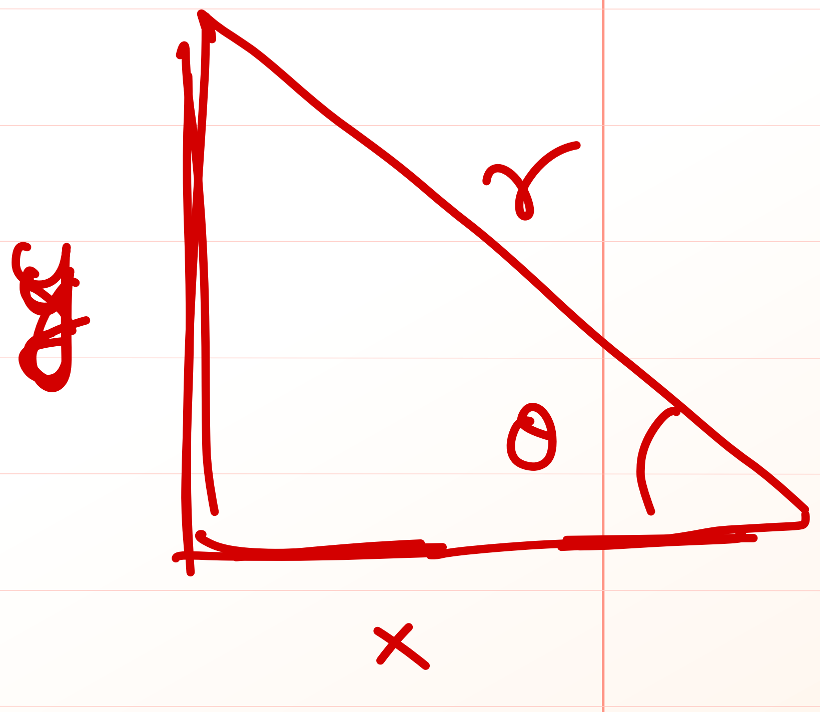
$ax+by=(2\sqrt{2}\cos\phi)(2\cos\theta)+(2\sqrt{2}\sin\phi)(2\sin\theta)$

$=4\sqrt{2}[\cos\theta\cos\phi+\sin\theta\sin\phi]$

$ax+by=4\sqrt{2}[\cos(\theta-\phi)]$

$ax+by\big|_{min}=-4\sqrt{2}$

$ax+by\big|_{max}=+4\sqrt{2}$



$\cos\theta=\frac{x}{r}$

$x=r\cos\theta$

$\sin\theta=\frac{y}{r}\Rightarrow y=r\sin\theta$

Q Prove that $\frac{\tan 3x}{\tan x}$ can not lies from $\frac{1}{3}$ to 3.

$$y = \frac{\tan 3x}{\tan x}$$

$$y = \frac{3\tan x - \tan^3 x}{(1 - 3\tan^2 x)(\tan x)}$$

$$y = \frac{3 - \tan^2 x}{1 - 3\tan^2 x}$$

$$y - 3y\tan^2 x = 3 - \tan^2 x$$

$$y - 3 = \tan^2 x\,(3y - 1)$$

$$\tan^2 x = \frac{y-3}{3y-1} \Rightarrow$$

$$\frac{y-3}{3y-1} = \tan^2 x$$

$$\frac{y-3}{3y-1} \geq 0$$

$$\boxed{y \in \left(-\infty, \frac{1}{3}\right) \cup [3, \infty)}$$

Continued product of sine & cosine Series :—

$$\prod_{r=1}^{n} \sin(r\theta) = \sin\theta \cdot \sin 2\theta \cdot \sin 3\theta \cdot \sin 4\theta \,----\, \sin n\theta$$

$$\sum_{r=1}^{n} \sin(r\theta) = \sin\theta + \sin 2\theta + \sin 3\theta + ---- + \sin n\theta$$