ALLEN CAREER INSTITUTE
KOTA (RAJASTHAN)

CONTENTS

MATHEMATICAL TOOLS

Recap of Early Classes

Mathematics is the language of physics. It becomes easier to describe, understand and apply the physical principles, if one has a good knowledge of mathematics. In the present course we shall be constantly using the techniques of algebra, trigonometry and geometry as well as vector algebra, differential calculus and integral calculus. In this chapter we shall discuss the latter three topics.

Index

SECTION [A] : ELEMENTARY ALGEBRA

1.0 QUADRATIC EQUATION

SL AL

An algebraic equation of second order (highest power of the variable is equal to 2) is called a quadratic equation. The equation $ax^2 + bx + c = 0$....(i) is the general form of quadratic equation where $a \neq 0$. The general solution of above equation is

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

If values of x be x_1 and x_2 then $x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$ and $x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$

Here x_1 and x_2 are called roots of equation (i). We can easily see that

$$\text{sum of roots} = x_1 + x_2 = -\frac{b}{a} \text{ and product of roots} \Rightarrow x_1 x_2 = \frac{c}{a}$$

Illustrations

Illustration 1. Find roots of equation $2x^2 - x - 3 = 0$.

Solution Compare this equation with standard quadratic equation $ax^2 + bx + c = 0$, we have $a=2, b=-1, c=-3$.

$$\text{Now from } x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a};\ x = \frac{-(1) \pm \sqrt{(-1)^2 - 4(2)(-3)}}{2(2)}$$

$$x = \frac{1 \pm \sqrt{1+24}}{4} = \frac{1 \pm 5}{4} \Rightarrow x = \frac{6}{4} \Rightarrow x = \frac{-4}{4} \Rightarrow x = \frac{3}{2} \text{ or } x = -1$$

Illustration 2. In quadratic equation $ax^2 + bx + c = 0$, if discriminant $D = b^2 - 4ac$, then roots of quadratic equation are:

(a) real and distinct, if $D > 0$
(b) real and equal (repeated roots), if $D = 0$
(c) non-real (imaginary), if $D < 0$
(d) None of the above

Ans. ***(ABC)***

2.0 BINOMIAL EXPRESSION

SL AL

An algebraic expression containing two terms is called a binomial expression.

For example $(a+b)$, $(a+b)^3$, $(2x—3y)^{-1}$, $\left(x + \frac{1}{y}\right)$ etc. are binomial expressions.

2.1 Binomial Theorem

SL AL

$$(a+b)^n = a^n + na^{n-1}b^1 + \frac{n(n-1)}{2 \times 1}a^{n-2}b^2 + \ldots \quad (1+x)^n = 1 + nx + \frac{n(n-1)}{2 \times 1}x^2 + \ldots$$

2.2 Binomial Approximation

SL AL

If x is very small, then terms containing higher powers of x can be neglected so

$$(1+x)^n = 1 + nx$$

Illustrations

Illustration 3. The mass m of a body moving with a velocity v is given by $m=\frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}$ where m_0 = rest mass of body = 10 kg and c = speed of light = 3×10^8 m/s. Find the value of m at v = 3×10^7 m/s.

Solution

$$m=m_0\left(1-\frac{v^2}{c^2}\right)^{-1/2}=10\left[1-\left(\frac{3\times10^7}{3\times10^8}\right)^2\right]^{-1/2}=10\left[1-\frac{1}{100}\right]^{-1/2}$$

$$\approx 10\left[1-\left(-\frac{1}{2}\right)\left(\frac{1}{100}\right)\right]=10+\frac{10}{200}\approx 10.05\text{ kg}$$

3.0 LOGARITHM

SL AL

Common formulae

- $\log mn = \log m + \log n$
- $\log \frac{m}{n} = \log m - \log n$
- $\log m^n = n \log m$
- $\log_e m = 2.303 \log_{10} m$
- $\log_a b = \frac{\log_c b}{\log_c a}$
- $\log_{a^n} b = \frac{1}{n}\log_a b$

Illustrations

Illustration 4. If log 2 = 0.3010 and log 3 = 0.4771, find the value of $\log\sqrt{24}$.

Solution

$$\log\sqrt{24}=\log(24)^{1/2}=\frac{1}{2}\log(2^3\times3)$$

$$=\frac{1}{2}[3\log2+\log3]=\frac{1}{2}[3\times0.3010+0.4771]=0.69005$$

4.0 COMPONENDO AND DIVIDENDO RULE

SL AL

If $\frac{p}{q}=\frac{a}{b}$ then $\frac{p+q}{p-q}=\frac{a+b}{a-b}$

4.1 Arithmetic Progression (AP)

SL AL

General form : a, a + d, a +2d, ..., a + (n–1)d

Here a = first term, d = common difference

Sum of n terms $S_n = \frac{n}{2}[a+a+(n-1)d] = \frac{n}{2}[\text{I}^{\text{st}}\text{ term} + n^{\text{th}}\text{ term}]$

Illustrations

Illustration 5. Find sum of first n natural numbers.

Solution Let sum be S_n then $S_n = 1 + 2 + 3 + ... + n$; $S_n=\frac{n}{2}[1+n]=\left[\frac{n(n+1)}{2}\right]$

4.2 Geometrical Progression (GP)

SL AL

General form : a, ar, ar^2,..., ar^{n-1} Here a = first term, r = common ratio

Sum of n terms $S_n = \frac{a(1-r^n)}{1-r}$, if | r | < 1 then sum of infinite term $S_\infty = \frac{a}{1-r}$

Illustrations

Illustration 6. Find $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+....\text{upto } \infty$.

Solution Here, a = 1, r = $\frac{1}{2}$ So, $S_\infty = \frac{a}{1-r} = \frac{1}{1-\frac{1}{2}} = 2$

BEGINNER'S BOX-1

Sequences and Logarithm

1. Find the sum of the series 1 + 4 + 7 + 10 + ... to 40 terms.

2. Which term of the sequence 2, 1, $\frac{1}{2}, \frac{1}{4}$, ... is $\frac{1}{128}$

3.* Expand $(1 + x)^6$.

4. If log 3 = 0.48 and log 7 = 0.84, find the value of log $\sqrt{63}$.

5. Evaluate : $\log_{10} 8 + \log_{10} 25 + 2\log_{10} 3 - \log_{10} 18$

6. Evaluate : $5\log 2 + \frac{3}{2}\log 25 + \frac{1}{2}\log 49 - \log 28$

7. Find $\ell n\left[1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+......\right]$

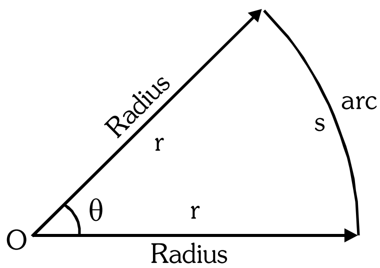
SECTION [B] : TRIGONOMETRY

5.0 ANGLE

SL AL

it is measure of change in direction.

$$\text{Angle } (\theta) = \frac{\text{Arc}(s)}{\text{Radius}(r)}$$



Angels measured in anticlockwise and clockwise direction are usually taken positive and negative respectively.

5.1 System of Measurement of an Angle

SL AL

Sexagesimal system

In this system, angle is measured in degrees.

In this system, 1 right angle = 90°, 1° = 60' (arc minutes), 1' = 60" (arc seconds)

Circular system

In this system, angle is measured in radian.

if arc = radius then $\theta = 1$ rad

Relation between degrees and radian

2π rad = 360°

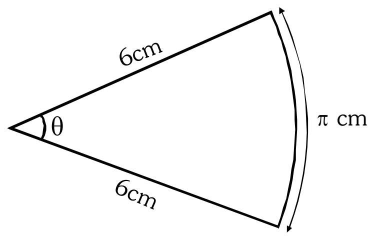
$$\pi \text{ rad} = 180° \Rightarrow 1 \text{ rad} = \frac{180°}{\pi} = 57.3°$$

To convert from degree to radian multiply by $\frac{\pi}{180°}$

To convert from radian to degree multiply by $\frac{180°}{\pi}$

Illustrations

Illustration 7. A circular arc of length π cm. Find angle subtended by it at the centre in radian and degree.



Solution $\theta = \frac{s}{r} = \frac{\pi \text{ cm}}{6 \text{ cm}} = \frac{\pi}{6} \text{rad} = 30°$

Illustration 8. When a clock shows 4 o'clock, how much angle do its minute and hour needles make?

(A) 120° (B) $\frac{\pi}{3}$ rad (C) $\frac{2\pi}{3}$ rad (D) 160°

Ans. ***(A,C)***

Solution From diagram angle $\theta = 4 \times 30° = 120° = \frac{2\pi}{3}$ rad

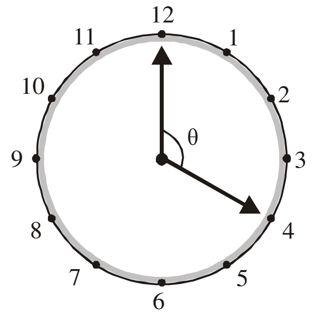


Illustration 9. The moon's distance from the earth is 360000 km and its diameter subtends an angle of 42' at the eye of the observer. The diameter of the moon in kilometers is :

(A) 4400 (B) 1000 (C) 3600 (D) 8800

Ans. ***(A)***

Solution Here angle is very small so diameter $\approx$ arc

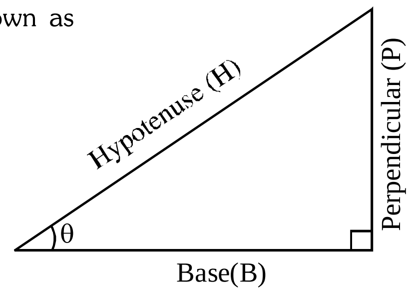
$$\theta = 42' = \left(42 \times \frac{1}{60}\right)^0 = 42 \times \frac{1}{60} \times \frac{\pi}{180} = \frac{7\pi}{1800} \text{ rad}$$

$$\text{Diameter} = R\theta = 360000 \times \frac{7}{1800} \times \frac{22}{7} = 4400 \text{ km}$$

6.0 TRIGONOMETRIC RATIOS (T-RATIOS)

SL AL

Following ratios of the sides of a right angled triangle are known as trigonometrical ratios.



$$\sin\theta = \frac{P}{H} \qquad \cos\theta = \frac{B}{H} \qquad \tan\theta = \frac{P}{B} \qquad \text{cosec}\theta = \frac{1}{\sin\theta} = \frac{H}{P}$$

$$\sec\theta = \frac{1}{\cos\theta} = \frac{H}{B} \qquad \cot\theta = \frac{1}{\tan\theta} = \frac{B}{P}$$

6.1 Trigonometric Identities

SL AL

In figure , $P^2 + B^2 = H^2$ Divide by H^2, $\left(\frac{P}{H}\right)^2 + \left(\frac{B}{H}\right)^2 = 1 \Rightarrow \sin^2\theta + \cos^2\theta = 1$

Divide by B^2, $\left(\frac{P}{B}\right)^2 + 1 = \left(\frac{H}{B}\right)^2 \Rightarrow 1 + \tan^2\theta = \sec^2\theta$

Divide by P^2, $1 + \left(\frac{B}{P}\right)^2 = \left(\frac{H}{P}\right)^2 \Rightarrow 1 + \cot^2\theta = \text{cosec}^2\theta$

Commonly Used Values of Trigonometric Functions

Angle(θ)	0°	30°	37°	45°	53°	60°	90°
sin θ	0	$\frac{1}{2}$	$\frac{3}{5}$	$\frac{1}{\sqrt{2}}$	$\frac{4}{5}$	$\frac{\sqrt{3}}{2}$	1
cos θ	1	$\frac{\sqrt{3}}{2}$	$\frac{4}{5}$	$\frac{1}{\sqrt{2}}$	$\frac{3}{5}$	$\frac{1}{2}$	0
tan θ	0	$\frac{1}{\sqrt{3}}$	$\frac{3}{4}$	1	$\frac{4}{3}$	$\sqrt{3}$	∞

6.2 Four Quadrants and ASTC Rule

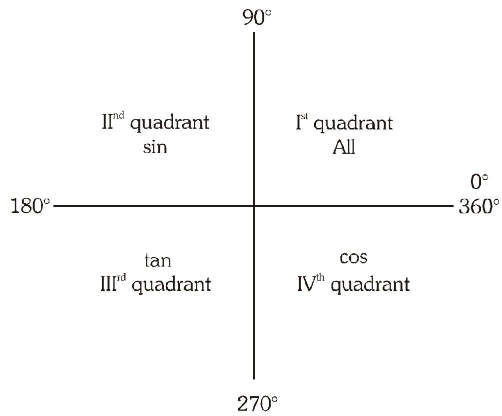
SL AL

In first quadrant, all trigonometric ratios are positive.

In second quadrant, only sinθ and cosecθ are positive.

In third quadrant, only tanθ and cotθ are positive.

In fourth quadrant, only cosθ and secθ are positive



6.3 Trigonometrical Ratios of General Angles (Reduction Formulae)

SL AL

(i) Trigonometric function of an angle $2n\pi + \theta$ where n=0, 1, 2, 3,.... will remain same.
$\sin(2n\pi + \theta) = \sin\theta$ $\cos(2n\pi+\theta) = \cos\theta$ $\tan(2n\pi+\theta)=\tan\theta$

(ii) Trigonometric function of an angle $\left(\frac{n\pi}{2} + \theta\right)$ will remain same if n is even and sign of trigonometric function will be according to value of that function in quadrant.

$\sin(\pi-\theta) = +\sin\theta$ $\cos(\pi-\theta) = -\cos\theta$ $\tan(\pi-\theta) = -\tan\theta$

$\sin(\pi+\theta) = -\sin\theta$ $\cos(\pi+\theta) = -\cos\theta$ $\tan(\pi+\theta) = +\tan\theta$

$\sin(2\pi-\theta) = -\sin\theta$ $\cos(2\pi-\theta) = +\cos\theta$ $\tan(2\pi-\theta) = -\tan\theta$

(iii) Trigonometric function of an angle $\left(\frac{n\pi}{2}+\theta\right)$ will be changed into co-function if n is odd and sign of trigonometric function will be according to value of that function in quadrant.

$$\sin\left(\frac{\pi}{2}+\theta\right)=+\cos\theta \qquad \cos\left(\frac{\pi}{2}+\theta\right)=-\sin\theta \qquad \tan\left(\frac{\pi}{2}+\theta\right)=-\cot\theta$$

$$\sin\left(\frac{\pi}{2}-\theta\right)=+\cos\theta \qquad \cos\left(\frac{\pi}{2}-\theta\right)=+\sin\theta \qquad \tan\left(\frac{\pi}{2}-\theta\right)=+\cot\theta$$

(iv) Trigonometric function of an angle $-\theta$ (negative angles)
$\sin(-\theta)=-\sin\theta$ $\cos(-\theta)=+\cos\theta$ $\tan(-\theta)=-\tan\theta$

Illustrations

Illustration 10. The two shorter sides of right angled triangle are 5 cm and 12 cm. Let θ denote the angle opposite to the 5 cm side. Find sinθ, cosθ and tanθ.

Solution

$$\sin\theta=\frac{P}{H}=\frac{5\text{cm}}{13\text{cm}}=\frac{5}{13}$$

$$\cos\theta=\frac{B}{H}=\frac{12\text{cm}}{13\text{cm}}=\frac{12}{13}$$

$$\tan\theta=\frac{P}{B}=\frac{5\text{cm}}{12\text{cm}}=\frac{5}{12}$$

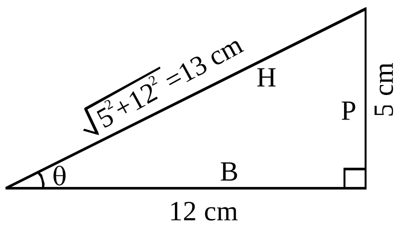
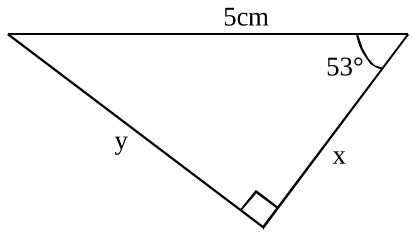


Illustration 11. Find x, y and perimeter of the triangle



Solution $\frac{y}{5}=\sin 53°=\frac{4}{5}\Rightarrow y=4$ cm and $\frac{x}{5}=\cos 53°=\frac{3}{5}\Rightarrow x=3$ cm

Perimeter of the triangle = x + y + 5 =3 + 4 + 5 = 12 cm

Illustration 12*. The values of $\sin\theta_1$, $\cos^2\theta_2$ and $\tan\theta_3$ are given as ½ , – ½ and 3 (not in order), for some angles θ_1, θ_2 and θ_3. Choose incorrect statement.

(A) The value of $\tan\theta_3$ could be –½
(B) The value of $\sin\theta_1$ can not be 3.
(C) The value of $\cos^2\theta_2$ can't be –½
(D) The value of $\cos^2\theta_2$ could be 3.

Ans. ***(D)***

Solution $-1\le\sin\theta_1\le1,\ 0\le\cos^2\theta_2\le1,\ -\infty<\tan\theta_3<\infty$

6.4 Addition/Subtraction Formulae for Trigonometrical Ratios

SL AL

- sin (A+B)=sinA cosB + cosA sinB
- sin(A–B)=sinA cosB–cosA sinB
- cos(A+B)=cosA cosB –sinA sinB
- cos(A–B)=cosA cosB + sinA sinB

Illustrations

Illustration 13*. By using above basic addition/ subtraction formulae, prove that :

(i) $\tan(A+B)=\frac{\tan A+\tan B}{1-\tan A\tan B}$

(ii) $\tan(A-B)=\frac{\tan A-\tan B}{1+\tan A\tan B}$

(iii) $\sin2\theta=2\sin\theta\cos\theta$

(iv) $\cos2\theta=\cos^2\theta-\sin^2\theta=1-2\sin^2\theta=2\cos^2\theta-1$

(v) $\tan2\theta=\frac{2\tan\theta}{1-\tan^2\theta}$

Solution (i) $\tan(A+B) = \frac{\sin(A+B)}{\cos(A+B)} = \frac{\sin A\cos B + \cos A\sin B}{\cos A\cos B - \sin A\sin B}$

$$= \frac{\cos A\cos B\left(\frac{\sin A}{\cos A} + \frac{\sin B}{\cos B}\right)}{\cos A\cos B\left(1 - \frac{\sin A\sin B}{\cos A\cos B}\right)} = \frac{\tan A + \tan B}{1 - \tan A\tan B}$$

(ii) $\tan(A-B) = \frac{\sin(A-B)}{\cos(A-B)} = \frac{\sin A\cos B - \cos A\sin B}{\cos A\cos B + \sin A\sin B}$

$$= \frac{\cos A\cos B\left[\frac{\sin A}{\cos A} - \frac{\sin B}{\cos B}\right]}{\cos A\cos B\left[1 + \frac{\sin A\sin B}{\cos A\cos B}\right]} = \frac{\tan A - \tan B}{1 + \tan A\tan B}$$

(iii) $\sin2\theta = \sin(\theta+\theta) = \sin\theta\cos\theta + \cos\theta\sin\theta = 2\sin\theta\cos\theta$

(iv) $\cos2\theta = \cos(\theta+\theta) = \cos\theta\cos\theta - \sin\theta\sin\theta = \cos^2\theta - \sin^2\theta = 1 - \sin^2\theta - \sin^2\theta = 1 - 2\sin^2\theta$
$= 1 - 2(1-\cos^2\theta) = 2\cos^2\theta - 1$

(v) $\tan2\theta = \tan(\theta+\theta) = \frac{\tan\theta + \tan\theta}{1 - \tan\theta\tan\theta} = \frac{2\tan\theta}{1 - \tan^2\theta}$

Illustration 14. Find the value of :
(i*) sin 74° (ii*) cos 106° (iii) sin 15° (iv) cos 75°

Solution (i) $\sin 74° = \sin(2\times 37°) = 2\sin 37\cos 37° = 2\left(\frac{3}{5}\right)\left(\frac{4}{5}\right) = \frac{24}{25}$

(ii) $\cos 106° = \cos(2\times53°) = \cos^2 53° \ \sin^2 53° = \left(\frac{3}{5}\right)^2 - \left(\frac{4}{5}\right)^2 = \frac{9-16}{25} = -\frac{7}{25}$

(iii) $\sin 15° = \sin(45°-30°) = \sin 45°\cos 30° - \cos 45°\sin 30° = \frac{1}{\sqrt{2}}\times\frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}}\times\frac{1}{2} = \frac{\sqrt{3}-1}{2\sqrt{2}}$

(iv) $\cos 75° = \cos(45°+30°) = \cos 45°\cos30° - \sin 45°\sin 30° = \frac{1}{\sqrt{2}}\times\frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}}\times\frac{1}{2} = \frac{\sqrt{3}-1}{2\sqrt{2}}$

6.5 Small Angle Approximation

SL AL

If θ is small (say < 5°) then $\sin\theta \approx \theta$, $\cos\theta \approx 1$ and $\tan\theta \approx \theta$

Note : here θ must be in radian.

Illustrations

Illustration 15. Find the approximate values of (i) sin 1° (ii) tan 2° (iii) cos 1°.

Solution (i) $\sin1° = \sin\left(1°\times\frac{\pi}{180°}\right) = \sin\frac{\pi}{180} \approx \frac{\pi}{180}$ (ii) $\tan2° = \tan\left(2°\times\frac{\pi}{180°}\right) = \tan\frac{\pi}{90} \approx \frac{\pi}{90}$

(iii) $\cos1° = \cos\left(1°\times\frac{\pi}{180°}\right) = \cos\frac{\pi}{180} = 1$

6.6 Maximum and Minimum Values of Some useful Trigonometric Functions

SL AL

- $-1 \le \sin\theta \le 1$
- $-1 \le \cos\theta \le 1$

$-\sqrt{a^2+b^2} \le a\cos\theta + b\sin\theta \le \sqrt{a^2+b^2}$

Illustrations

Illustration 16. Find maximum and minimum values of y :

(i) $y = 2\sin x$ (ii) $y = 4 - \cos x$ (iii) $y = 3\sin x + 4\cos x$

Solution (i) $y_{max} = 2(1) = 2$ and $y_{min} = 2(-1) = -2$ (ii) $y_{max} = 4 - (-1) = 4 + 1 = 5$ and $y_{min} = 4 - (1) = 3$

(iii) $y_{max} = \sqrt{3^2 + 4^2} = 5$ and $y_{min} = -\sqrt{3^2 + 4^2} = -5$

Illustration 17*. The position of a particle moving along x-axis varies with time t according to equation $x = \sqrt{3}\sin\omega t - \cos\omega t$ where ω is constants. Find the region in which the particle is confined.

Solution $\because x = \sqrt{3}\sin\omega t - \cos\omega t$

$\therefore x_{max} = \sqrt{\left(\sqrt{3}\right)^2 + (-1)^2} = 2$ and $x_{min} = \sqrt{\left(\sqrt{3}\right)^2 + (-1)^2} = -2$

Thus, the particle is confined in the region $-2 \le x \le 2$

SECTION [C] : COORDINATE GEOMETRY

To specify the position of a point in space, we use right handed rectangular axes coordinate system. This system consists of (i) origin (ii) axis or axes. If point is known to be on a given line or in a particular direction, only one coordinate is necessary to specify its position, if it is in a plane, two coordinates are required, if it is in space three coordinates are needed.

7.0 COORDINATE GEOMETRY

SL AL

7.1 Axis or Axes

SL AL

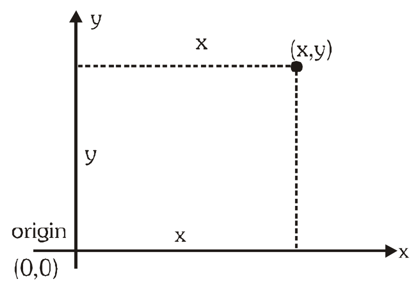
Origin is any fixed point which is convenient to you. All measurement are taken w.r.t. this fixed point. Any fixed direction passing through origin and convenient to you can be taken as an axis. If the position of a point or position of all the points under consideration always happen to be in a particular direction, then only one axis is required. This is generally called the x-axis. If the positions of all the points under consideration are always in a plane, two perpendicular axes are required. These are generally called x and y-axis. If the points are distributed in a space, three perpendicular axes are taken which are called x, y and z-axis.

7.2 Position of a point in xy plane

SL AL

The position of a point is specified by its distances from origin along (or parallel to) x and y-axis as shown in figure.

Here x-coordinate and y-coordinate is called abscissa and ordinate respectively.



7.3 Distance Formula

SL AL

The distance between two points (x_1, y_1) and (x_2, y_2) is given by $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Illustrations

Illustration 18. For point (2, 14) find abscissa and ordinates. Also find distance from y and x-axis.

Solution Abscissa = x-coordinate = 2 = distance from y-axis.

Ordinate = y-coordinate = 14 = distance from x-axis.

Illustration 19. Find value of a if distance between the points (–9 cm, a cm) and (3 cm, 3cm) is 13 cm.

Solution By using distance formula $d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2} \Rightarrow 13=\sqrt{[3-(-9)]^2+[3-a]^2}$

$\Rightarrow 13^2 = 12^2 + (3–a)^2 \Rightarrow (3–a)^2 = 13^2 – 12^2 = 5^2 \Rightarrow (3–a) = \pm 5 \Rightarrow a = –2$ cm or 8 cm

Illustration 20*. A dog wants to catch a cat. The dog follows the path whose equation is $y–x = 0$ while the cat follows the path whose equation is $x^2 + y^2 = 8$. The coordinates of possible points of catching the cat are :

(A) (2, –2) (B) (2, 2) (C) (–2, 2) (D) (–2, –2)

Ans. ***(B,D)***

Solution Let catching point be (x_1, y_1) then, $y_1–x_1=0$ and $x_1^2 + y_1^2 = 8$

Therefore, $2x_1^2 = 8 \Rightarrow x_1^2 = 4 \Rightarrow x_1 = \pm 2$

So possible points are (2, 2) and (–2, –2).

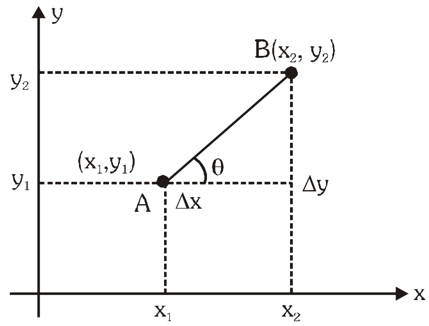
7.4 Slope of a Line

SL AL

The slope of a line joining two points $A(x_1, y_1)$ and $B(x_2, y_2)$ is denoted by m and is given by

$$m=\frac{\Delta y}{\Delta x}=\frac{y_2-y_1}{x_2-x_1}=\tan\theta \quad \text{[If both axes have identical scales]}$$

Here θ is the angle made by line with positive x-axis. Slope of a line is a quantitative measure of inclination.



Illustrations

Illustration 21. Distance between two points (8, –4) and (0, a) is 10. All the values are in the same unit of length. Find the positive value of a.

Ans. ***2***

Solution From distance formula $(8–0)^2 + (–4–a)^2 = 100 \Rightarrow (4+a)^2 = 36 \Rightarrow a = 2$

SECTION [D] : CALCULUS

Calculus is the study of how things change. In this we study the relationship between continuously varying functions.

8.0 DIFFERENTIAL CALCULUS

SL AL

The purpose of differential calculus to study the nature (i.e., increase or decrease) and the amount of variation in a quantity when another quantity (on which first quantity depends) varies independently.

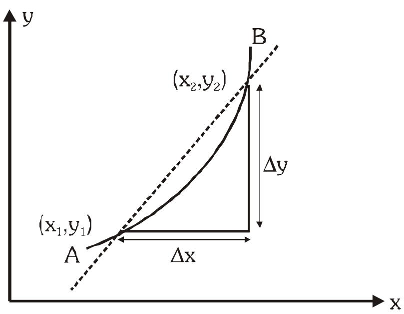
8.1 Average rate of change

SL AL

Let a function y = f(x) be plotted as shown in figure. Average rate of change in y w.r.t. x in interval $[x_1, x_2]$ is

$$\text{Average rate of change} = \frac{\text{change in y}}{\text{change in x}} = \frac{\Delta y}{\Delta x} = \frac{y_2-y_1}{x_2-x_1}$$

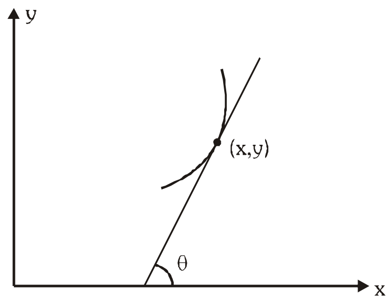
= slope of chord AB.



8.2 Instantaneous rate of change

SL AL

It is defined as the rate of change in y with x at a particular value of x. It is measured graphically by the slope of the tangent drawn to the y-x graph at the point (x,y) and algebraically by the first derivative of function y =f(x).



Instantaneous rate of change = $\frac{dy}{dx}$ = slope of tangent = $\tan\theta$

8.3 First Derivatives of Commonly used Functions

SL AL

- $y = \text{constant} \Rightarrow \frac{dy}{dx} = 0$
- $y = x^n \Rightarrow \frac{dy}{dx} = nx^{n-1}$
- $y = e^x \Rightarrow \frac{dy}{dx} = e^x$
- $y = \ln x \Rightarrow \frac{dy}{dx} = \frac{1}{x}$
- $y = \sin x \Rightarrow \frac{dy}{dx} = \cos x$
- $y = \cos x \Rightarrow \frac{dy}{dx} = -\sin x$
- $y = \tan x \Rightarrow \frac{dy}{dx} = \sec^2 x$
- $y = \cot x \Rightarrow \frac{dy}{dx} = -\operatorname{cosec}^2 x$

8.4 Method of Differentiation or Rules of Differentiation

SL AL

(i) Function multiplied by a constant i.e., $y = kf(x) \Rightarrow \frac{dy}{dx} = kf'(x)$

Illustrations

Illustration 22. Find derivatives of the following functions :

(i) $y = 2x^3$ (ii) $y = \frac{4}{x}$ (iii*) $y = 3e^x$ (iv) $y = 6 \ln x$ (v) $y = 5 \sin x$

Solution

(i) $y = 2x^3 \Rightarrow \frac{dy}{dx} = 2\left[3x^{3-1}\right] = 6x^2$

(ii) $y = \frac{4}{x} = 4x^{-1} \Rightarrow \frac{dy}{dx} = 4\left[(-1)x^{-1-1}\right] = -\frac{4}{x^2}$

(iii) $y = 3e^x \Rightarrow \frac{dy}{dx} = 3e^x$

(iv) $y = 6\ln x \Rightarrow \frac{dy}{dx} = 6\left(\frac{1}{x}\right) = \frac{6}{x}$

(v) $y = 5\sin x \Rightarrow \frac{dy}{dx} = 5(\cos x) = 5\cos x$

8.5 Sum or Subtraction of Two functions

SL AL

i.e., $y = f(x) \pm g(x) \Rightarrow \frac{dy}{dx} = f'(x) \pm g'(x)$

Illustrations

Illustration 23. Find differentiation of y w.r.t x.

(i) $y = x^2 - 6x$ (ii*) $y = x^5 + 2e^x$ (iii) $y = 4 \ln x + \cos x$

Solution (i) $\frac{dy}{dx} = 2x^{2-1} - 6(1) = 2x - 6$ (ii) $\frac{dy}{dx} = 5x^{5-1} + 2e^x = 5x^4 + 2e^x$

(iii) $\frac{dy}{dx} = 4\left(\frac{1}{x}\right) + (-\sin x) = \frac{4}{x} - \sin x$

8.6 Product of two functions : Product rule

SL AL

$$y = f(x) \cdot g(x) \Rightarrow \frac{dy}{dx} = f(x) \cdot g'(x) + f'(x) \cdot g(x)$$

Illustrations

Illustration 24. Find first derivative of y w.r.t. x. (i*) $y = x^2 \sin x$ (ii) $y = 4(e^x)\cos x$

Solution (i) $\frac{dy}{dx} = x^2(\cos x) + (2x)(\sin x) = x^2 \cos x + 2x \sin x$

(ii) $\frac{dy}{dx} = 4\left[(e^x)(\cos x) + (e^x)(-\sin x)\right] = 4e^x[\cos x - \sin x]$

8.7 Division of Two Functions : Quotient Rule

SL AL

$$y = \frac{f(x)}{g(x)} \Rightarrow \frac{dy}{dx} = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

Illustrations

Illustration 25. Find differentiation of y w.r.t. x. (i) $y = \frac{\sin x}{x}$ (ii*) $y = \frac{4x^3}{e^x}$

Solution (i) Here $f(x) = \sin x$, $g(x) = x$ So $f'(x) = \cos x$,, $g'(x) = 1$

Therefore $\frac{dy}{dx} = \frac{(\cos x)(x) - (\sin x)(1)}{x^2} = \frac{x \cos x - \sin x}{x^2}$

(ii) Here $f(x) = 4x^3$, $g(x) = e^x$ So $f'(x) = 12x^2, g'(x) = e^x$

Therefore, $\frac{dy}{dx} = \frac{12x^2(e^x) - 4x^3(e^x)}{(e^x)^2} = \frac{12x^2 - 4x^3}{e^x}$

8.8 Function of Functions : Chain rule

SL AL

Let f be a function of x, which in turn is a function of t. The first derivative of f w.r.t. t is equal to the product of $\frac{df}{dx}$ and $\frac{dx}{dt}$ Therefore $\frac{df}{dt} = \frac{df}{dx} \times \frac{dx}{dt}$

Illustrations

Illustration 26. Find first derivative of y w.r.t. x. :

(i) $y = e^{-x}$ (ii) $y = 4 \sin 3x$ (iii*) $y = 4e^{x^2-2x}$

Solution (i) $y = e^{-x} = e^{z}$ where $z = -x$ so $\frac{dy}{dx} = \frac{dy}{dz} \times \frac{dz}{dx} = (e^{z})(-1) = -e^{z} = -e^{-x}$

(ii) $y = 4 \sin 3x = 4 \sin z$ where $z = 3x$ so $\frac{dy}{dx} = \frac{dy}{dz} \times \frac{dz}{dx} = 4(\cos z)(3) = 12 \cos 3x$

(iii) $y = 4e^{x^2-2x} = 4e^{z}$ where $z = x^2 - 2x$ so $\frac{dy}{dx} = \frac{dy}{dz} \times \frac{dz}{dx} = 4(z)(2x-2) = (8x-8)e^{x^2-2x}$

Illustration 27. The position of a particle moving along x-axis varies with time t as $x=4t-t^2+1$. Find the time interval(s) during which the particle is moving along positive x-direction.

Solution If the particle moves along positive x-direction, its x-coordinate must increase with time t.

x-coordinate will increase with time t if $\frac{dx}{dt} > 0$.

$$\frac{dx}{dt} = 4 - 2t$$

$$\frac{dx}{dt} > 0 \Rightarrow 4 - 2t > 0 \Rightarrow t < 2$$

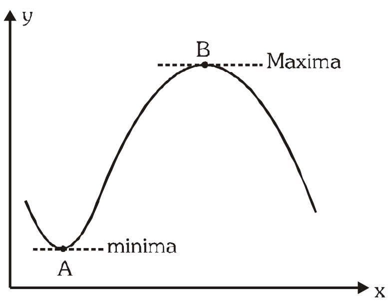
Hence, the particle moves in positive x-direction during time-interval $0 < t < 2$.

8.9 Maximum and Minimum value of a Function

SL AL

Higher order derivatives are used to find the maximum and minimum values of a function. At the points of maxima and minima, first derivative becomes zero.

At point 'A' (minima) : As we see in figure, in the neighborhood of A, slope increases so $\frac{d^2y}{dx^2} > 0$.



Condition for minima : $\boxed{\frac{dy}{dx} = 0 \text{ and } \frac{d^2y}{dx^2} > 0}$

At point 'B' (maxima) : As we see in figure, in the neighborhood of B, slope decreases so $\frac{d^2y}{dx^2} < 0$

Condition for maxima : $\boxed{\frac{dy}{dx} = 0 \text{ and } \frac{d^2y}{dx^2} < 0}$

Illustrations

Illustration 28. The minimum value of $y = 5x^2 - 2x + 1$ is :

(A) $\frac{1}{5}$ (B) $\frac{2}{5}$ (C) $\frac{4}{5}$ (D) $\frac{3}{5}$

Ans. ***(C)***

Solution For maximum/minimum value $\frac{dy}{dx} = 0 \Rightarrow 5(2x) - 2(1) + 0 = 0 \Rightarrow x = \frac{1}{5}$

Now at $x=\frac{1}{5}$, $\frac{d^2y}{dx^2} = 10$ which is positive so minima at $x = \frac{1}{5}$.

Therefore $y_{min} = 5\left(\frac{1}{5}\right)^2 - 2\left(\frac{1}{5}\right) + 1 = \frac{4}{5}$

Illustration 29*. The radius of a circular plate increases at the rate of 0.1 cm per second. At what rate does the area increase when the radius of plate is $\frac{5}{\pi}$ cm ?

(A) 1 cm^2/s (B) 0.1 cm^2/s (C) 0.5 cm^2/s (D) 2 cm^2/s

Ans. ***(A)***

Solution Area of disk, $A = \pi r^2$ (where r = radius of disk)

$$\frac{dA}{dt} = \pi\left(2r\frac{dr}{dt}\right) = 2\pi r\frac{dr}{dt} \text{ so } \frac{dA}{dt} = 2\pi \times \frac{5}{\pi} \times 0.1 = 1 \text{ cm}^2 / s$$

Illustration 30. A particle moves along the curve $12y = x^3$. Which coordinate changes at faster rate at $x=10$?

(A) x–coordinate (B) y–coordinate
(C) Both x and y–coordinate (D) Data insufficient

Ans. ***(B)***

Solution $12y = x^3 \Rightarrow 12dy = 3x^2dx \Rightarrow \frac{dy}{dt} = \left(\frac{x}{2}\right)^2\left(\frac{dx}{dt}\right)$

Therefore for $\left(\frac{x}{2}\right)^2 > 1$ or $x > 2$, y– coordinate changes at faster rate.

Illustration 31. If $y = 3x^4 - 8x^3 - 6x^2 + 24x$. Find values of x where function is maximum or minimum.

Solution To find values of x for maximum & minimum.

$$\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = 12x^3 - 24x^2 - 12x + 24$$
$$= 12(x^3 - 2x^2 - x + 2)$$
$$= 12(x-1)(x+1)(x-2) = 0$$

$$\left.\begin{array}{l} x = 1 \\ x = -1 \\ x = +2 \end{array}\right]$$

To check maxima / minima $\frac{d^2y}{dx^2} = (+) \Rightarrow$ minima or $(-) \Rightarrow$ maxima

at $x = -1$ $\quad \frac{d^2y}{dx^2} = 12(3+4-1) = (+)$ minima

at $x = 1$ $\quad \frac{d^2y}{dx^2} = 12(3x^2 - 4x - 1) = 12(3 - 4 - 1) = (-)$ maxima

at $x = +2$ $\quad \frac{d^2y}{dx^2} = 12(12 - 8 - 1) = (+)$ minima

Illustration 32*. If surface area of a cube is changing at a rate of 5 m^2/s, find the rate of change of body diagonal at the moment when side length is 1 m.

(A) 5 m/s (B) $5\sqrt{3}$ m/s (C) $\frac{5}{2}\sqrt{3}$ m/s (D) $\frac{5}{4\sqrt{3}}$ m/s

Ans. ***(D)***

Solution Surface area of cube $S=6a^2$ (where a = side of cube)

Body diagonal $\ell = \sqrt{3}a$. Therefore $S=2l^2$

Differentiating it w.r.t. time $\frac{dS}{dt} = 2(2\ell)\frac{d\ell}{dt} \Rightarrow \frac{d\ell}{dt} = \frac{1}{4(\sqrt{3}a)}\frac{dS}{dt} = \frac{5}{4\sqrt{3}}$ m/s

BEGINNER'S BOX-2

Differentiation

1. Find dy/dx for the following functions :

(i) $y = 1 + 3x - x^2$ (ii) $y = x^3 - 2x^2 - 3$ (iii) $y = \frac{3}{x}$ (iv) $y = \sqrt{2x}$

(v) $y = (ax + b)^2$

2. In each of the following evaluate y' when x = 2 :

(i) $y = 8x - x^3$ (ii) $y = ax^2 + bx + c$ (iii*) $y = (x + a)^3$ (iv) $y = \frac{1}{\sqrt{2x}}$

3. If $y = 3x^2 - 2x + 1$, find the value of x for which $y' = 0$.

4*. If $f(x) = x^3 - 3x^2$, find the values of x for which $f'(x) = 0$.

5. If $y = \sin x$, then find $\frac{dy}{dx}$ when $x = \frac{\pi}{2}$.

6. For the function $y = x\sqrt{x}$ find the value of x for which the rate of change of y with respect to x is 6.

7. Find f' (x) in each of the following :

(i) $y = x^3 (2x^2 - 1)$ (ii*) $y = (x - 1)(2x^2 + 3)$ (iii) $y = 2x^{1/2} + 3x^{2/3}$

8. In each of the following find y' for the given value of x:

(i) $y = 2\sqrt{x}(3x - 2)$, $x = 4$ (ii) $y = \frac{x+1}{\sqrt{x}}$, $x = \frac{1}{4}$

9. If $y = x^2 - x$ then find $\frac{dy}{dx}$ at $x = 0$.

10. For what values of x is the derivative of x^3 equal to the derivative of $x^2 + x$?

11. If $y = \tan x$ then at what value of x, $\frac{dy}{dx} = 1$.

9.0 INTEGRAL CALCULUS

SL AL

Integration is the reverse process of differentiation. By help of integration we can find a function whose derivative is known. Consider a function F(x) whose differentiation w.r.t. x is equal to f(x) then

$$\int f(x)\,dx = F(x) + c$$

here c is the constant of integration and this is called indefinite integration.

Few basic formulae of integration are :

$$\int x^n dx = \frac{x^{n+1}}{n+1} + c \qquad \int \frac{1}{x} dx = \ln x + c$$

$$\int \sin x dx = -\cos x + c \qquad \int \cos x dx = \sin x + c$$

$$\int \sec^2 x dx = \tan x + c \qquad \int e^x dx = e^x + c$$

$$\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c \qquad \int \frac{dx}{ax+b} = \frac{\ln(ax+b)}{a} + c$$

$$\int \sin(ax+b)\,dx = \frac{-\cos(ax+b)}{a}+c \qquad \int \cos(ax+b)\,dx = \frac{\sin(ax+b)}{a}+c$$

$$\int \sec^2(ax+b)\,dx = \frac{\tan(ax+b)}{a}+c \qquad \int e^{ax+b}dx = \frac{e^{ax+b}}{a}+c$$

Illustrations

Illustration 33. Integrate the following w.r.t. x :

(i) $4x^3$ (ii) $x-\frac{1}{x}$ (iii) $\frac{1}{2x+3}$ (iv) cos $(4x+3)$ (v*) $\cos^2 x$

Solution

(i) $\int 4x^3dx = 4\left(\frac{x^{3+1}}{3+1}\right)+c = \frac{4x^4}{4}+c = x^4+c$

(ii) $\int\left(x-\frac{1}{x}\right)dx = \int xdx - \int\frac{1}{x}dx = \frac{x^2}{2}-\ln x + c$

(iii) $\int\frac{dx}{2x+3} = \frac{\ln(2x+3)}{2}+c$

(iv) $\int\cos(4x+3)\,dx = \frac{\sin(4x+3)}{4}+c$

(v) $\int\cos^2 xdx = \int\frac{2\cos^2 x}{2}dx = \int\frac{(1+\cos 2x)}{2}dx = \frac{1}{2}\int dx + \frac{1}{2}\int \cos 2xdx$

$= \frac{x}{2}+\frac{1}{2}\frac{\sin 2x}{2}+c = \frac{x}{2}+\frac{1}{4}\sin 2x + c$

9.1 Definite Integration

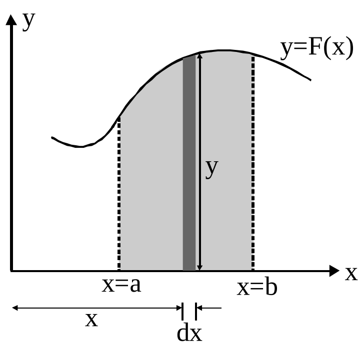
SL AL

When a function is integrated between a lower limit and an upper limit, it is called a definite integral. Consider a function F(x) whose differentiation w.r.t. x is equal to f(x), in an interval $a \le x \le b$ then

$$\int_a^b f(x)\,dx = F(b)-F(a)$$

9.2 Area Under a Curve and Definite Integration

SL AL



Area of small shown element = ydx = f(x) dx

If we sum up all areas between x=a and x= b then

$\int_a^b f(x)dx$ = shaded area between curve and x-axis.

Illustrations

Illustration 34. The integral $\int_1^5 x^2dx$ is equal to :

(A) $\frac{125}{3}$ (B) $\frac{124}{3}$ (C) $\frac{1}{3}$ (D) 45

Ans. ***(B)***

Solution $\int_1^5 x^2 dx = \left[\frac{x^3}{3}\right]_1^5 = \left[\frac{5^3}{3} - \frac{1^3}{3}\right] = \frac{125}{3} - \frac{1}{3} = \frac{124}{3}$

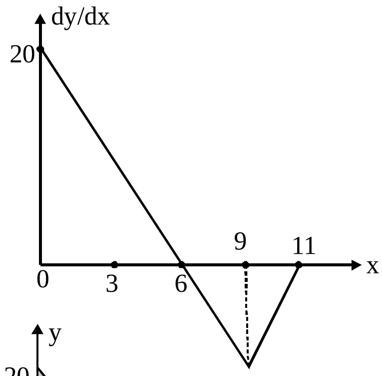
Illustration 35*. The following curve represents rate of change of a variable y w.r.t x. The change in the value of y when x changes from 0 to 11 is:

(A) 60

(B) 25

(C) 35

(D) 85



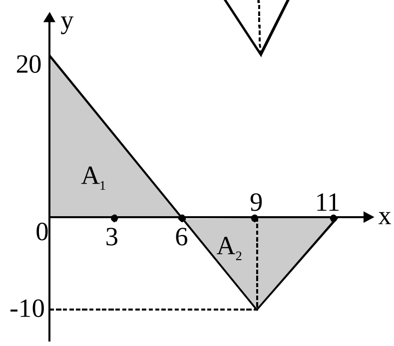
Solution As $dy = \left(\frac{dy}{dx}\right)dx$ So $\Delta y = \int dy = \int_0^{11}\left(\frac{dy}{dx}\right)dx$

Area under the curve

$$A_1 = \frac{1}{2} \times 6 \times 20 = 60$$

$$A_2 = -\frac{1}{2} \times (11-6)(10) = -25$$

$$\Delta y = A_1 + A_2 = 60 - 25 = 35$$



9.3 Average value of a continuous function in an interval

SL AL

Average value of a function y = f(x), over an interval $a \le x \le b$ is given by

$$y_{av} = \frac{\int_a^b y dx}{\int_a^b dx} = \frac{\int_a^b y dx}{b-a}$$

Illustrations

Illustration 36. Determine the average value of y = 2x +3 in the interval $0 \le x \le 1$.

(A) 1 (B) 5 (C) 3 (D) 4

Ans. ***(D)***

Solution $y_{av} = \frac{\int_0^1 y dx}{1-0} = \int_0^1 (2x+3)dx = \left[2\left(\frac{x^2}{2}\right) + 3x\right]_0^1 = 1^2 + 3(1) - 0^2 - 3(0) = 1 + 3 = 4$

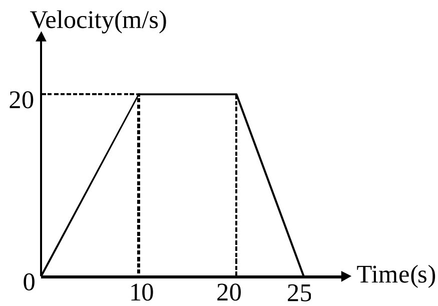
Illustration 37. The velocity-time graph of a car moving along a straight road is shown in figure. The average velocity of the car is first 25 seconds is :

(A) 20 m/s

(B) 14 m/s

(C) 10 m/s

(D) 17.5 m/s



Ans. ***(B)***

Solution Average velocity

$$= \frac{\int_0^{25} v dt}{25-0} = \frac{\text{Area of v-t graph between t=0 to t = 25 s}}{25} = \frac{1}{25}\left[\left(\frac{25+10}{2}\right)(20)\right] = 14 \text{ m/s}$$

BEGINNER'S BOX-3

Integration

1. Integrate the following expressions :

(i) $\int 5\,dx$ (ii) $\int (3x^2 - 5)\,dx$ (iii) $\int (x - 7)\,dx$ (iv) $\int (x + 1)(2 - x)\,dx$

2. Integrate the following expressions :

(i) $\int_0^1 (2 - x)\,dx$ (ii) $\int_{-2}^{2} (2x + x^2)\,dx$ (iii) $\int_{-3}^{-2} x(x + 1)^2\,dx$ (iv*) $\int_0^a (\sqrt{a} - \sqrt{x})^2\,dx$

(v) $\int_2^5 \left(x^2 + \frac{1}{x^2}\right) dx$

3*. Evaluate $\int e^{\sin\theta} \cos\theta\,d\theta$

4. Evaluate $\int \frac{2 + e^x}{e^x}\,dx$

5*. Evaluate $\int_1^4 \frac{e^{\sqrt{x}}\,dx}{\sqrt{x}}$

6. Evaluate $\int_0^1 (e^x + x^e)\,dx$

SECTION [E] : VECTORS

Precise description of laws of physics and physical phenomena requires expressing them in form of mathematical equations. In doing so we encounter several physical quantities, some of them have only magnitude and other have direction in addition to magnitude. Quantities of the former kind are referred as scalars and the latter as vectors and mathematical operations with vectors are collectively known as vector analysis.

10.0 VECTORS

SL AL

A vector has both magnitude and sense of direction, and follows triangle law of vector addition.
For example, displacement, velocity, and force are vectors.

Vector quantities are usually denoted by putting an arrow over the corresponding letter, as $\vec{A}$ or $\vec{a}$. Sometimes in print work (books) vector quantities are usually denoted by boldface letters as ***A*** or ***a***.

Magnitude of a vector $\vec{A}$ is a positive scalar and written as $|\vec{A}|$ or A.

10.1 Unit Vector

SL AL

It is mathematical way to express direction of a vector and defined by the ratio of a vector to its magnitude. When a unit vector is multiplied with a scalar magnitude, we get a vector of corresponding magnitude in the direction of the unit vector. A unit vector is usually represented by putting a sign (ˆ) known as cap, hat or caret over a letter assigned to the unit vector. This letter may be the same as used for the vector, or its lower case letter, or some other symbol. For example, if we assign lower case letter a to unit vector in the direction of vector $\vec{A}$, the unit vector denoted by $\hat{a}$ is expressed by the following equation.

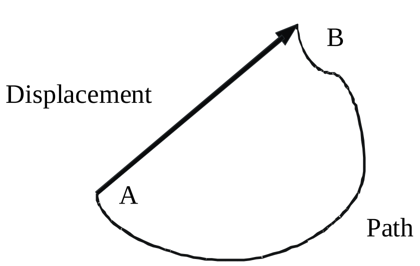
$$\vec{A} = A\hat{a}$$

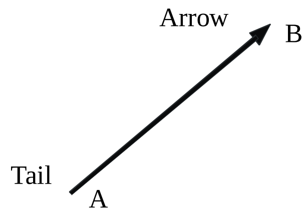
10.2 Geometrical Representation of Vectors

SL AL

Geometrically a vector is represented by a directed straight-line segment drawn to a scale. Starting point of the directed line segment is known as tail and the end-point as arrow, head, or tip. The orientation of the line and the arrow collectively show the direction and the length of the line drawn to a scale shows the magnitude.

For example let a particle moves from point A to B following a curvilinear path shown in the figure. It displacement vector is straight line AB directed form A to B. If straight-line distance between A and B is 25 m, the directed line segment has to be drawn to suitable scale. If we assume the scale 1.0 cm = 10 m, the geometrical length of the displacement vector AB must be 2.5 cm.





Geometrical representation of Displacement Vector

10.3 Addition of Vectors: The Triangle Law

SL AL

Use of geometry in solving problems involving vectors is of fundamental nature. The triangle law also uses principles of plane geometry. This law states:

The vectors to be added are drawn in such a manner that the tail of a vector coincides the tip of the preceding vector (in tip to tail fashion); their resultant is defined by the vector drawn from the tail of the first vector to the tip of the second vector. The two vectors to be added and their resultant are coplanar.

Consider vectors $\vec{A}$ and $\vec{B}$ shown in the figure-I. Using the triangle law, we obtain geometrical construction shown in the figure-II, where it is shown that two vectors and their sum $\vec{C} = \vec{A} + \vec{B}$ always make a closed triangle. If we change order of vector $\vec{A}$ and $\vec{B}$, it shown in figure-III that sum given by equation $\vec{C} = \vec{B} + \vec{A}$ remain unchanged. Therefore, vector addition is commutative.

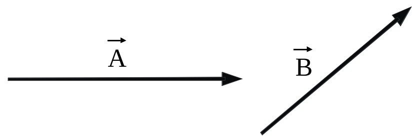


Fig. (i)

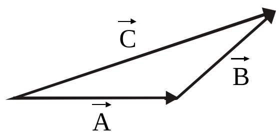


Fig. (ii)

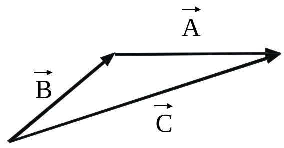
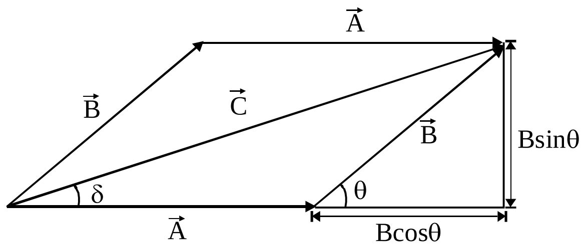


Fig. (iii)

Construction, which is combination of the figure-II and III, is in form of a parallelogram and is shown in figure-IV.



Geometry of the above figure suggests the following results.

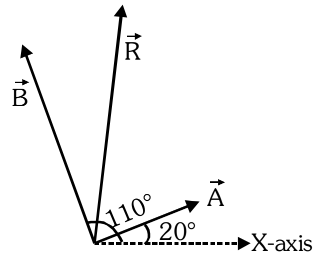
$$C = \sqrt{A^2 + B^2 + 2AB\cos\theta}\ ; \quad \delta = \tan^{-1}\left(\frac{B\sin\theta}{A + B\cos\theta}\right)$$

Illustrations

Illustration 38. A vector $\vec{A}$ and $\vec{B}$ make angles of 20° and 110° respectively with the X–axis. The magnitudes of these vectors are 5m and 12m respectively. Find their resultant vector.

Solution Angle between the $\vec{A}$ and $\vec{B}$ = 110°– 20° = 90°

$$R = \sqrt{A^2 + B^2 + 2AB\cos 90^\circ} = \sqrt{5^2 + 12^2} = 13\text{m}$$

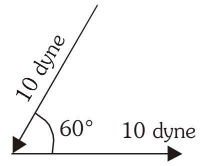


Let angle of $\vec{R}$ from $\vec{A}$ is α

$$\tan\alpha = \frac{B\sin\theta}{A+B\cos\theta} = \frac{12\sin 90^\circ}{5+12\cos 90^\circ} = \frac{12\times 1}{5+12\times 0} = \frac{12}{5}$$

or $\alpha = \tan^{-1}\left(\frac{12}{5}\right)$ with vector $\vec{A}$ or $(\alpha + 20°)$ with X–axis

Illustration 39. Two forces each numerically equal to 10 dynes are acting as shown in the figure, then find resultant of these two vectors.



Solution The angle θ between the two vectors is 120° and not 60°.

$$\therefore R = \sqrt{(10)^2 + (10)^2 + 2(10)(10)(\cos 120^\circ)}$$

$$= \sqrt{100+100-100} = 10 \text{ dyne}$$

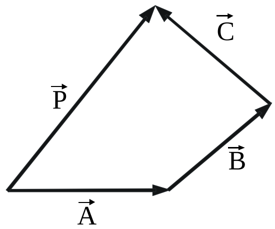
10.4 Addition of more than two Vectors

SL AL

The triangle law can be extended to define addition of more than two vectors. Accordingly, if vectors to be added are drawn in tip to tail fashion, resultant is defined by a vector drawn from the tail of the first vector to the tip of the last vector. This is also known as the polygon rule for vector addition.

Operation of addition of three vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$ and their resultant $\vec{P}$ are shown in figure.

$$\vec{A}+\vec{B}+\vec{C}=\vec{P}$$



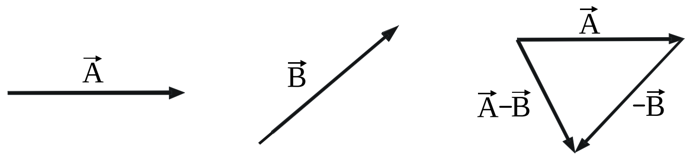
Here it is not necessary that three or more vectors and their resultant are coplanar. In fact, the vectors to be added and their resultant may be in different planes. However if all the vectors to be added are coplanar, their resultant must also be in the same plane containing the vectors.

10.5 Subtraction of Vectors

SL AL

A vector opposite in direction but equal in magnitude to another vector $\vec{A}$ is known as negative vector of $\vec{A}$. It is written as $-\vec{A}$. Addition of a vector and its negative vector results a vector of zero magnitude, which is known as a null vector. A null vector is denoted by arrowed zero $(\vec{0})$.

The idea of negative vector explains operation of subtraction as addition of negative vector. Accordingly to subtract a vector from another consider vectors $\vec{A}$ and $\vec{B}$ shown in the figure. To subtract $\vec{B}$ from $\vec{A}$, the negative vector $-\vec{B}$ is added to $\vec{A}$ according to the triangle law as shown in figure-II.



10.6 Multiplying by a Number

SL AL

Multiplication by a positive number changes magnitude of the vector but not the direction and multiplication by a negative number changes magnitude and reverses direction.

Thus multiplying a vector by a number n makes magnitude of the vector n times. $n\vec{A} = (nA)\hat{a}$

Here $\hat{a}$ denotes the unit vector in the direction of vector $\vec{A}$.

10.7 Resolution of a Vector into Components

SL AL

Following laws of vector addition, a vector can be represented as a sum of two (in two-dimensional space) or three (in three-dimensional space) vectors each along predetermined directions. These directions are called axes and parts of the original vector along these axes are called components of the vector.

10.8 Cartesian Components in Two Dimensions

SL AL

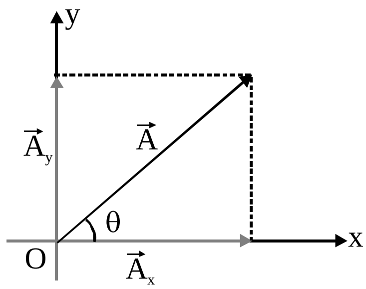
If a vector is resolved into its components along mutually perpendicular directions, the components are called Cartesian or rectangular components.

In figure is shown, a vector $\vec{A}$ resolved into its Cartesian components $\vec{A}_x$ and $\vec{A}_y$ along the x and y-axis. Magnitudes A_x and A_y of these components are given by the following equation.

$$A_x = A\cos\theta \quad \text{and} \quad A_y = A\sin\theta$$

$$\vec{A} = A_x\hat{i} + A_y\hat{j}$$

$$A = \sqrt{A_x^{\ 2} + A_y^{\ 2}}$$



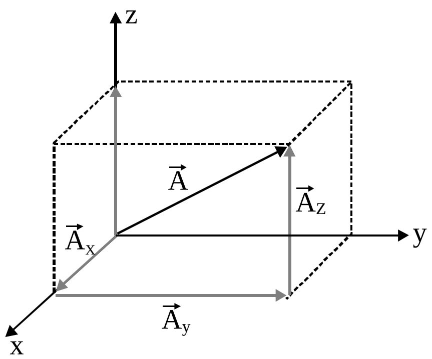
Here $\hat{i}$ and $\hat{j}$ are the unit vectors for x and y coordinates respectively.

Mathematical operations e.g. addition, subtraction, differentiation and integration can be performed independently on these components. This is why in most of the problems use of Cartesian components becomes desirable.

10.9 Cartesian components in three dimensions

SL AL

A vector $\vec{A}$ resolved into its three Cartesian components one along each of the directions x, y, and z-axis is shown in the figure.



$$\vec{A} = \vec{A}_x + \vec{A}_y + \vec{A}_z = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}\ ;\ A = \sqrt{A_x^2 + A_y^2 + A_z^2}$$

10.10 Equal Vectors

SL AL

Two vectors of equal magnitudes and same directions are known as equal vectors. Their x, y and z components in the same coordinates system must be equal.

If two vectors $\vec{a} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}$ and $\vec{b} = b_x\hat{i} + b_y\hat{j} + b_z\hat{k}$ are equal vectors, we have

$$\vec{a} = \vec{b} \ \Rightarrow\ a_x = b_x,\ a_y = b_y \text{ and } a_z = b_z$$

10.11 Parallel Vectors

SL AL

Two parallel vectors must have the same direction and may have unequal magnitudes. Their x, y and z components in the same coordinate system bear the same ratio.

Consider two vectors $\vec{a} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}$ and $\vec{b} = b_x\hat{i} + b_y\hat{j} + b_z\hat{k}$, if they are parallel, we have

$$\vec{a}\cdot\vec{b} \Rightarrow \frac{a_x}{b_x} = \frac{a_y}{b_y} = \frac{a_z}{b_z}$$

BEGINNER'S BOX-4

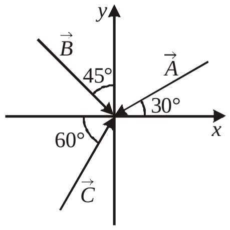
Addition of vectors and Resolution

1. The maximum value of magnitude of $(\vec{A}-\vec{B})$ is :

(A) A – B (B) A (C) A + B (D) $\sqrt{(A^2+B^2)}$

2. The resultant of two forces–5 newtons and 10 newtons can never be :
(A) 12 N (B) 5 N (C) 10 N (D) 4 N

3*. Vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$ are shown in figure. Find angle between.

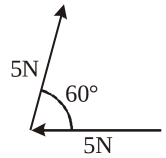


(i) $\vec{A}$ and $\vec{B}$

(ii) $\vec{A}$ and $\vec{C}$

(iii) $\vec{B}$ and $\vec{C}$

4. The forces, each numerically equal to 5N, are acting as shown in figure. Find the angle between forces ?



5. Two force $\vec{F}_1$ and $\vec{F}_2$ are acting at right angles to each other, find their resultant ?

6. The vector sum of the forces of 10 N and 6 N can be :
(A) 2 N (B) 8 N (C) 18 N (D) 20 N

7. The resultant of two forces has magnitude 20 N. One of the forces is of magnitude 20 $\sqrt{3}$ N and makes an angle of 30° with the resultant. Then, the other force must be of magnitude :

(A) 10 N (B) 10 $\sqrt{3}$ N (C) 20 N (D) 20 $\sqrt{3}$ N

8. The x and y components of a force are 2N and –3N. The force is :

(A) $2\hat{i}-3\hat{j}$ (B) $2\hat{i}+3\hat{j}$ (C) $-2\hat{i}-3\hat{j}$ (D) $3\hat{i}+2\hat{j}$

9. Write the unit vector in the direction of $\vec{A}=5\vec{i}+\vec{j}-2\vec{k}$.

10*. A truck traveling due north at 20 m/s turns east and travels at the same speed. What is the change in velocity :

(A) 40 m/s north east (B) 20 $\sqrt{2}$ m/s south east (C) 20 $\sqrt{2}$ m/s south west (D) 20 $\sqrt{2}$ m/s north west

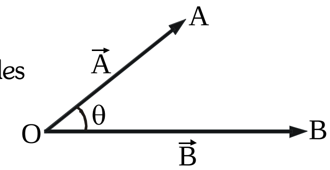
10.12 Product of Vectors

SL AL

In all physical situation, whose description involve product of two vectors, only two categories are observed. One category where product is also a vector involves multiplication of magnitudes of two vectors and sine of the angle between them, while the other category where product is a scalar involves multiplication of magnitudes of two vectors and cosine of the angle between them. Accordingly, we define two kinds of product operation. The former category is known as vector or cross product and the latter category as scalar or dot product.

10.13 Scalar or Dot Product of Two Vectors

SL AL

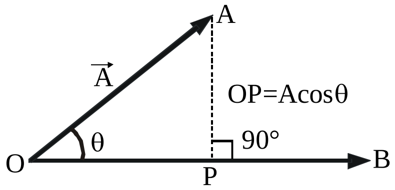


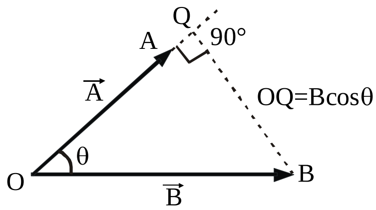
The scalar product of two vectors $\vec{A}$ and $\vec{B}$ equals to the product of their magnitudes and the cosine of the angle θ between them.

$$\vec{A}\cdot\vec{B} = AB\cos\theta = OA\cdot OB\cdot\cos\theta$$

The above equation can also be written in the following ways.

$$\vec{A}\cdot\vec{B} = (A\cos\theta)B = OP\cdot OB$$
$$\vec{A}\cdot\vec{B} = A(B\cos\theta) = OA\cdot OQ$$





Above two equations and figures, suggest a scalar product as product of magnitude of the one vector and magnitude of the component of another vector in the direction of the former vector.

Illustrations

Illustration 40. If $|\vec{A}+\vec{B}| = |\vec{A}-\vec{B}|$, then find the angle between $\vec{A}$ and $\vec{B}$.

Solution $\because |\vec{A}+\vec{B}| = |\vec{A}-\vec{B}| \quad \therefore \sqrt{A^2+B^2+2AB\cos\theta} = \sqrt{A^2+B^2-2AB\cos\theta}$

or $A^2 + B^2 + 2AB\cos\theta = A^2 + B^2 - 2AB\cos\theta$ or $\cos\theta = 0 \therefore \theta = 90°$

Illustration 41. If $\vec{A} = 4\hat{i}+n\hat{j}-2\hat{k}$ and $\vec{B} = 2\hat{i}+3\hat{j}+\hat{k}$, then find the value of n so that $\vec{A}\perp\vec{B}$.

Solution Dot product of two mutually perpendicular vectors is zero $\vec{A}\cdot\vec{B} = 0$

$\therefore (4\hat{i}+n\hat{j}-2\hat{k}).(2\hat{i}+3\hat{j}+\hat{k}) = 0 \Rightarrow (4\times2)+(n\times3)+(-2\times1) = 0 \Rightarrow 3n = -6 \quad \Rightarrow n = -2$

10.14 Vector or Cross Product of Two Vectors

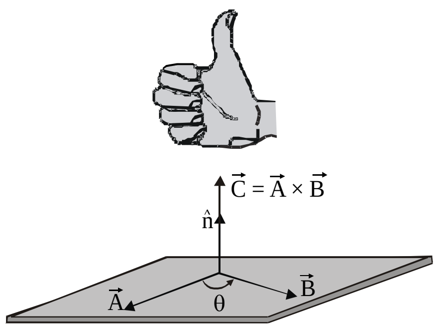
SL AL

The vector product $\vec{C}$ of two vectors $\vec{A}$ and $\vec{B}$ is defined as

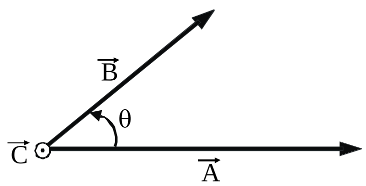
Its magnitude is the product of magnitudes of $\vec{A}$ and $\vec{B}$ and of the sine of angle θ between vectors $\vec{A}$ and $\vec{B}$.

Its direction is perpendicular to the plane containing vectors $\vec{A}$ and $\vec{B}$ and is decided by right hand rule by curling fingers in the direction from the first vector towards the second vector. In figure, where it is represented by $\hat{n}$.

$$\vec{C} = \vec{A}\times\vec{B} = (AB\sin\theta)\cdot\hat{n}$$



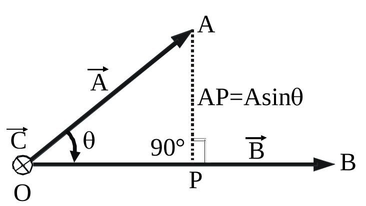
On paper vectors perpendicularly out and into the plane of paper are represented by encircled dot $\odot$ and encircled cross $\otimes$ signs respectively. Following this convention, cross product $\vec{C} = \vec{A}\times\vec{B}$ is shown in the figure.

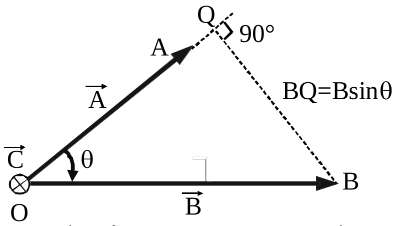


To have different symbols for scalar and vector products, symbols dot (·) and cross (×) respectively are written between the vectors undergoing these operations.

Cross product $\vec{C} = \vec{A}\times\vec{B}$, can also be written in the following ways.

$\vec{C} = \vec{A}\times\vec{B} = A(B\sin\theta)\hat{n}$ $\qquad$ $\vec{C} = \vec{A}\times\vec{B} = (A\sin\theta)B\hat{n}$





The above two equations and figures explain that the magnitude of vector or cross product is the product of magnitude of one vector and magnitude of the component of the other vector in the direction perpendicular to the first one.

Illustrations

Illustration 42. Find a unit vector perpendicular to both the vectors $(2\hat{i}+3\hat{j}+\hat{k})$ and $(\hat{i}-\hat{j}+2\hat{k})$.

Solution Let $\vec{A} = 2\hat{i}+3\hat{j}+\hat{k}$ and $\vec{B} = \hat{i}-\hat{j}+2\hat{k}$

unit vector perpendicular to both $\vec{A}$ and $\vec{B}$ is $\hat{n} = \dfrac{\vec{A}\times\vec{B}}{|\vec{A}\times\vec{B}|}$

$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 1 \\ 1 & -1 & 2 \end{vmatrix} == \hat{i}(6+1)-\hat{j}(4-1)+\hat{k}(-2-3) = 7\hat{i}-3\hat{j}-5\hat{k}$$

$$\therefore |\vec{A}\times\vec{B}| = \sqrt{7^2+(-3)^2+(-5)^2} = \sqrt{83}\ \text{unit} \qquad \therefore \hat{n} = \frac{1}{\sqrt{83}}(7\hat{i}-3\hat{j}-5k)$$

11.0 RATE OF CHANGE OF A VECTOR WITH TIME

SL AL

It is derivative of a vector function with respect to time. Cartesian components of a time dependent vector, if given as function of time as $\vec{r}(t) = x(t)\hat{i}+y(t)\hat{j}+z(t)\hat{k}$, the time rate of change can be calculated according to equation

$$\frac{d\vec{r}(t)}{dt} = \frac{dx(t)\hat{i}}{dt}+\frac{dy(t)\hat{j}}{dt}+\frac{dz(t)\hat{k}}{dt}$$

11.1 Methods of Differentiation of Vector Functions

SL AL

Methods of differentiation of scalar functions are also applicable to differentiation of vector functions.

1. $\dfrac{d}{dt}\left(\vec{F}\pm\vec{G}\right) = \dfrac{d\vec{F}}{dt}\pm\dfrac{d\vec{G}}{dt}$

2. $\dfrac{d}{dt}\left(\vec{F}\cdot\vec{G}\right) = \dfrac{d\vec{F}}{dt}\cdot\vec{G}+\vec{F}\cdot\dfrac{d\vec{G}}{dt}$

3. $\dfrac{d}{dt}\left(X\vec{F}\right) = \dfrac{dX}{dt}\vec{F}+X\dfrac{d\vec{F}}{dt}$ $\qquad$ Here X is a scalar function of time.

4. $\dfrac{d}{dt}\left(\vec{F}\times\vec{G}\right) = \dfrac{d\vec{F}}{dt}\times\vec{G}+\vec{F}\times\dfrac{d\vec{G}}{dt}$ Order of the vector functions $\vec{F}$ and $\vec{G}$ must be retained.

BEGINNER'S BOX-5

Product of Vectors

1. The angles which $\hat{i}+\hat{j}+\sqrt{2}\hat{k}$ makes with x-axis, y-axis and z-axis respectively are :

(A) 60°, 60° and 45° (B) 60°, 45° and 45° (C) 45°, 45° and 45° (D) 60°, 60° and 60°

2. The magnitude of scalar product of two vectors is 8 and that of vector product is $8\sqrt{3}$. The angle between them is:

(A) 30° (B) 60° (C) 90° (D) 150°

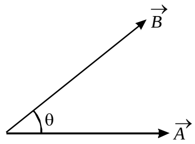
3*. A vector $\vec{A}$ points vertically downward and $\vec{B}$ points towards east, then the vector product $\vec{A}\times\vec{B}$ is :

(A) Along west (B) Along east (C) Zero (D) None of above

4. If $\vec{a}=2\vec{i}+3\vec{j}+4\vec{k}$ and $\vec{b}=4\vec{i}+3\vec{j}+2\vec{k}$, find the angle between $\vec{a}$ and $\vec{b}$.

5. If $\vec{A}=\hat{i}+\hat{j}+\hat{k}$ and $\vec{B}=2\hat{i}+\hat{j}$ find (a) $\vec{A}\cdot\vec{B}$ (b) $\vec{A}\times\vec{B}$

6. If $|\vec{A}|=4, |\vec{B}|=3$ and $\theta = 60°$ in the figure, find ***(a)*** $\vec{A}\cdot\vec{B}$ ***(b)*** $|\vec{A}\times\vec{B}|$

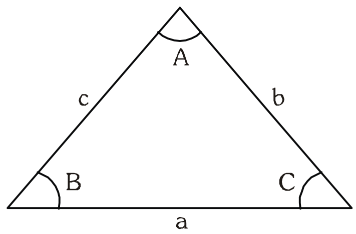


GOLDEN KEY POINTS

- Logarithms to the base 10 are known as common logarithms.
- If no base is given, the base is always taken as 10. i.e. $\log 5 = \log_{10} 5$; $\log 8 = \log_{10} 8$; $\log a = \log_{10} a$ and so on.
- ***Remember :***

$$\frac{\log_a m}{\log_a n} \neq \log_a m - \log_a n$$

- Sine Formula or Sine Rule



In ΔABC, $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

- Cosine Formula or Cosine Rule

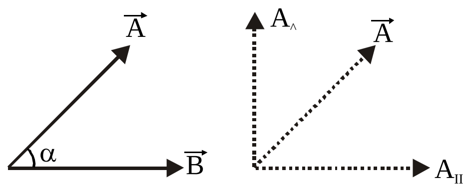
In ΔABC,

(i) $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$ or $a^2 = b^2 + c^2 - 2bc\cos A$

(ii) $\cos B = \dfrac{a^2 + c^2 - b^2}{2ac}$ or $b^2 = a^2 + c^2 - 2ac\cos B$

(iii) $\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$ or $c^2 = a^2 + b^2 - 2ab\cos C$

- Dot product of two vectors is commutative: $\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$.
- If two vectors are perpendicular, their dot product is zero. $\vec{A} \cdot \vec{B} = 0$, if $\vec{A} \perp \vec{B}$.
- Dot product of a vector by itself is known as self-product. $\vec{A} \cdot \vec{A} = A^2 \Rightarrow A = \sqrt{\vec{A} \cdot \vec{A}}$.
- The angle between the vectors $\theta = \cos^{-1}\left(\dfrac{\vec{A} \cdot \vec{B}}{AB}\right)$.
- (a) Component of $\vec{A}$ in direction of $\vec{B}$



$$\vec{A}_{\parallel} = \left(|\vec{A}|\cos\theta\right)\hat{B} = |\vec{A}|\left(\frac{\vec{A}.\vec{B}}{|\vec{A}||\vec{B}|}\right)\hat{B} = \left(\frac{\vec{A}.\vec{B}}{|\vec{B}|}\right)\hat{B} = (\vec{A}.\hat{B})\hat{B}.$$

(b) Component of $\vec{A}$ perpendicular to $\vec{B}$: $\vec{A}_{\perp} = \vec{A} - \vec{A}_{\parallel}$.

- Dot product of Cartesian unit vectors: $\hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = 1$.

$\hat{i} \cdot \hat{j} = \hat{j} \cdot \hat{k} = \hat{k} \cdot \hat{i} = 0$

- If $\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$ and $\vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$, their dot product is given by

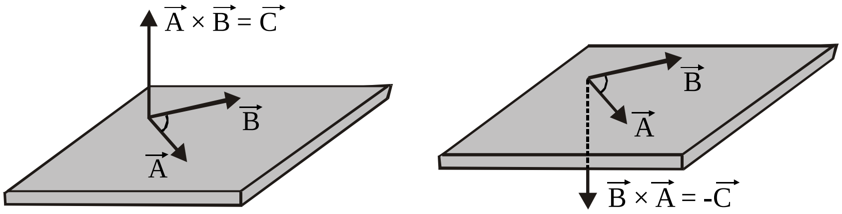
$$\vec{A} \cdot \vec{B} = A_xB_x + A_yB_y + A_zB_z$$

- Vector product of two vectors is always a vector perpendicular to the plane containing the two vectors, i.e., orthogonal (perpendicular) to both the vectors $\vec{A}$ and $\vec{B}$.

  Unit vector perpendicular to $\vec{A}$ and $\vec{B}$ is $\hat{n} = \pm\dfrac{\vec{A}\times\vec{B}}{|\vec{A}\times\vec{B}|}$.

- Vector product of two vectors is not commutative i.e. cross products $\vec{A}\times\vec{B}$ and $\vec{B}\times\vec{A}$ have equal magnitudes but opposite directions as shown in the figure.

$$\vec{A}\times\vec{B} = -\vec{B}\times\vec{A}$$



- The vector product is distributive when the order of the vectors is strictly maintained,

  i.e. $\vec{A}\times(\vec{B}+\vec{C}) = \vec{A}\times\vec{B}+\vec{A}\times\vec{C}$

- Angle θ between two vectors $\vec{A}$ and $\vec{B}$ is given by $\theta = \sin^{-1}\left[\dfrac{|\vec{A}\times\vec{B}|}{|\vec{A}||\vec{B}|}\right]$

- The self cross product, i.e., product of a vector by itself is a zero vector or a null vector.

$$\vec{A}\times\vec{A} = (AA\sin 0°)\hat{n} = \vec{0} = \hat{i}\times\hat{i} = \hat{j}\times\hat{j} = \hat{k}\times\hat{k}$$

- In case of orthogonal unit vectors $\hat{i}$, $\hat{j}$ and $\hat{k}$; according to right hand thumb rule

$$\hat{i}\times\hat{j} = \hat{k}, \hat{j}\times\hat{k} = \hat{i}, \hat{k}\times\hat{i} = \hat{j} \text{ and } \hat{j}\times\hat{i} = -\hat{k}, \hat{k}\times\hat{j} = -\hat{i}, \hat{i}\times\hat{k} = -\hat{j}$$

- If $\vec{A} = A_x\hat{i}+A_y\hat{j}+A_z\hat{k}$ and $\sqrt{2}$, their cross-products is given by

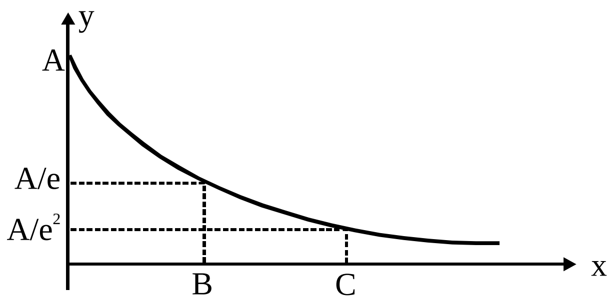
$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \hat{i}(A_yB_z - A_zB_y) - \hat{j}(A_xB_z - A_zB_x) + \hat{k}(A_xB_y - A_yB_x)$$

- If $\vec{A}$, $\vec{B}$ and $\vec{C}$ are coplanar, then $\vec{A}\cdot(\vec{B}\times\vec{C}) = 0$

SOME WORKED OUT ILLUSTRATIONS

Illustration 1*.

In the given figure, a function $y = 15e^{-x}$ is shown. What is the numerical value of expression A/(B+C)?



Ans. 5

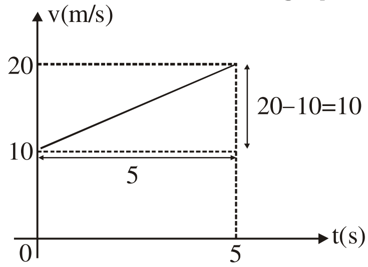
Solution

From graph A = 15 ; B =1 ; C =2 . Therefore [A/(B+C) = 15/3 = 5]

Illustration 2.

A car changes its velocity linearly from 10 m/s to 20 m/s in 5 seconds. Plot v-t graph and write velocity as a function of time.

Solution:

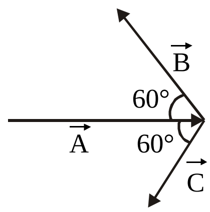


Slope $= \frac{20-10}{5-0} = 2 =$ m

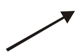
y-intercept = 10 = c $\Rightarrow$ v = 2t + 10

Illustration 3*.

Three coplanar vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$ have magnitudes 4, 3 and 2 respectively. If the angle between any two vectors is 120° then which of the following vector may be equal to



$\frac{3\vec{A}}{4} + \frac{\vec{B}}{3} + \frac{\vec{C}}{2}$:

(A)  (B)  (C)  (D) 

Ans. (B)

Solution

As $\left|\frac{\vec{B}}{3}\right| = \left|\frac{\vec{C}}{2}\right|$ so $\frac{\vec{B}}{3} + \frac{\vec{C}}{2} = -\frac{\vec{A}}{4}$ therefore $\frac{3\vec{A}}{4} + \frac{\vec{B}}{3} + \frac{\vec{C}}{2} = \frac{\vec{A}}{2}$

Illustration 4.

The magnitude of pairs of displacement vectors are given. Which pairs of displacement vectors cannot be added to give a resultant vector of magnitude 13 cm?

(A) 4 cm, 16 cm (B) 20 cm, 7 cm (C) 1 cm, 15 cm (D) 6 cm, 8 cm

Ans. (C)

Solution

Resultant of two vectors $\vec{A}$ and $\vec{B}$ must satisfy $A \sim B \le R \le A + B$

Illustration 5.

Three non zero vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$ satisfy the relation $\vec{A}\cdot\vec{B} = 0$ & $\vec{A}\cdot\vec{C} = 0$. Then $\vec{A}$ can be parallel to :

(A) $\vec{B}$ (B) $\vec{C}$ (C) $\vec{B}\cdot\vec{C}$ (D) $\vec{B}\times\vec{C}$

Ans. (D)

Solution

$\vec{A}\cdot\vec{B}=0 \Rightarrow \vec{A}\perp\vec{B}$ & $\vec{A}\cdot\vec{C}=0 \Rightarrow \vec{A}\perp\vec{C}$

But $\vec{B}\times\vec{C}$ is perpendicular to both $\vec{B}$ and $\vec{C}$ so $\vec{A}$ is parallel to $\vec{B}\times\vec{C}$.

Illustration 6.

α and β are the angle made by a vector from positive x & positive y-axes respectively. Which set of α and β is not possible :

(A) $45^0, 60^0$ (B) $30^0, 60^0$ (C) $60^0, 60^0$ (D) $30^0, 45^0$

Ans. (D)

Solution

α,β must satisfy $\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1$

Illustration 7.

Let $\vec{A}$, $\vec{B}$ and $\vec{C}$, be unit vectors. Suppose that $\vec{A}\cdot\vec{B}=\vec{A}\cdot\vec{C}=0$ and the angle between $\vec{B}$ and $\vec{C}$ is $\frac{\pi}{6}$ then :

(A) $\vec{A}=(\vec{B}\times\vec{C})$ (B) $\vec{A}=2(\vec{B}\times\vec{C})$ (C) $\vec{A}=2(\vec{C}\times\vec{B})$ (D) $|\vec{B}\times\vec{C}|=\frac{\sqrt{3}}{2}$

Ans. (BC)

Solution :

As $\vec{A}\perp\vec{B}$ and $\vec{A}\perp\vec{C}$ so $\vec{A}=\pm\frac{(\vec{B}\times\vec{C})}{|\vec{B}\times\vec{C}|}$ But $|\vec{B}\times\vec{C}|=BC\sin 30^\circ=\frac{1}{2}$

So $\vec{A}=\pm 2(\vec{B}\times\vec{C}) \Rightarrow \vec{A}=2(\vec{B}\times\vec{C})$ and $\vec{A}=-2(\vec{B}\times\vec{C})=2(\vec{C}\times\vec{B})$

Illustration 8*.

Angle between $\vec{a}$ and $\vec{b}$ is 60° than :

(A) The component of $\vec{a}-\vec{b}$ along $\vec{a}+\vec{b}$ will be $\frac{a^2-b^2}{\sqrt{a^2+b^2+ab}}$

(B) $\vec{a}\times\vec{b}$ is perpendicular to resultant of $(\vec{a}+2\vec{b})$ and $(\vec{a}-\vec{b})$

(C) The component of $\vec{a}-\vec{b}$ along $\vec{a}+\vec{b}$ will be $\frac{a^2-b^2}{\sqrt{a^2+b^2+2ab}}$

(D) The component of $\vec{a}+\vec{b}$ along $\vec{a}-\vec{b}$ will be $\frac{a^2-b^2}{\sqrt{a^2+b^2+\sqrt{3}ab}}$

Ans. (A,B)

Solution

For (A) : Required component $=\frac{(\vec{a}-\vec{b})\cdot(\vec{a}+\vec{b})}{|\vec{a}+\vec{b}|}=\frac{a^2-b^2}{\sqrt{a^2+b^2+2ab\cos 60^\circ}}=\frac{a^2-b^2}{\sqrt{a^2+b^2+ab}}$

For (B) : $\vec{a}+2\vec{b}+\vec{a}-\vec{b}=2\vec{a}+\vec{b}$ which lies in the plane of $\vec{a}$ and $\vec{b}$

$\Rightarrow$ resultant is perpendicular to $\vec{a}\times\vec{b}$

Illustration 9.

Which of the following sets of concurrent forces may be in equilibrium?

(A) $F_1=3N, F_2=5N, F_3=1N$ (B) $F_1=3N, F_2=5N, F_3=6N$

(C) $F_1=3N, F_2=5N, F_3=9N$ (D) $F_1=3N, F_2=5N, F_3=16N$

Ans. (B)

Solution

For equilibrium, net resultant force must be zero. These forces form a closed traingle such that

$F_1 \sim F_2 \le F_3 \le F_1+F_2 \Rightarrow 2N \le F_3 \le 8N$

Illustration 10.

Consider three vectors

$\vec{A} = 2\hat{i} + 3\hat{j} - 2\hat{k}$ $\quad \vec{B} = 5\hat{i} + n\hat{j} + \hat{k}$ $\quad \vec{C} = -\hat{i} + 2\hat{j} + 3\hat{k}$

If these three vectors are coplanar, then value of n will be :

(A) 0 (B) 12 (C) 16 (D) 18

Ans. (D)

Solution:

For coplanar vectors $\vec{A} \cdot (\vec{B} \times \vec{C}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} = 0$

$$\Rightarrow \begin{vmatrix} 2 & 3 & -2 \\ 5 & n & 1 \\ -1 & 2 & 3 \end{vmatrix} = 2(3n-2) - 3(15+1) - 2(10+n) = 0 \;\Rightarrow 4n - 72 = 0 \Rightarrow n = 18$$

Illustration 11 to 13.

Vector product of three vectors is given by $\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A}.\vec{C}) - \vec{C}(\vec{A}.\vec{B})$:

11*. The value of $\hat{i} \times (\hat{j} \times \hat{k})$ is :

(A) 0 (B) $\vec{0}$ (C) 1 (D) 3

12*. The plane of vector $\vec{A} \times (\vec{A} \times \vec{B})$ lies in the plane of :

(A) $\vec{A}$ (B) $\vec{B}$ (C) $\vec{A} \times \vec{B}$ (D) $\vec{A}$ and $\vec{B}$

13*. The value of $\hat{i} \times (\hat{i} \times \hat{j}) + \hat{j} \times (\hat{j} \times \hat{k}) + \hat{k} \times (\hat{k} \times \hat{i})$ is :

(A) $\hat{i} + \hat{j} + \hat{k}$ (B) $-\hat{i} - \hat{j} - \hat{k}$ (C) $\vec{0}$ (D) $-3\hat{i} - 3\hat{j} - 3\hat{k}$

Solution

11. Ans. (B)

$\hat{i} \times (\hat{j} \times \hat{k}) = \hat{j}(\hat{i} \cdot \hat{k}) - \hat{k}(\hat{i} \cdot \hat{j}) = \vec{0}$

12. Ans. (D)

$\vec{A} \times (\vec{A} \times \vec{B}) = \vec{A}(\vec{A} \cdot \vec{B}) - \vec{B}(\vec{A} \cdot \vec{A})$ $\Rightarrow$This vector lies in plane of $\vec{A}$ and $\vec{B}$

13. Ans. (B)

$\Sigma\hat{i} \times (\hat{i} \times \hat{j}) = \Sigma\hat{i}(\hat{i} \cdot \hat{j}) - \hat{j}(\hat{i} \cdot \hat{i}) = -\Sigma\hat{j} = -(\hat{i} + \hat{j} + \hat{k})$

Illustration 14.

If $\vec{A} = \hat{i} + 2\hat{j} + 3\hat{k}$, $\vec{B} = -\hat{i} + \hat{j} + 4\hat{k}$ and $\vec{C} = 3\hat{i} - 3\hat{j} - 12\hat{k}$, then find the angle between the vectors $(\vec{A} + \vec{B} + \vec{C})$ and $(\vec{A} \times \vec{B})$ in degrees.

Ans. 90

Solution

$$\vec{P} = \vec{A} + \vec{B} + \vec{C} = 3\hat{i} - 5\hat{k} \text{ and } \vec{Q} = \vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ -1 & 1 & 4 \end{vmatrix} = 5\hat{i} - 7\hat{j} + 3\hat{k}$$

Angle between $\vec{P}$ & $\vec{Q}$ is given by $\cos\theta = \dfrac{\vec{P} \cdot \vec{Q}}{PQ} = \dfrac{15-15}{PQ} = 0 \Rightarrow \theta = 90°$

Illustration 15.

$\vec{a}$ and $\vec{b}$ are unit vectors and angle between them is $\frac{\pi}{k}$. If $\vec{a}+2\vec{b}$ and $5\vec{a}-4\vec{b}$ are perpendicular to each other then find the integer value of k.

Ans. ***3***

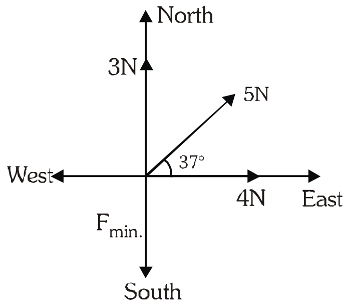
Solution :

$$(\vec{a}+2\vec{b}).(5\vec{a}-4\vec{b}) = 0 \Rightarrow 5a^2 + 10\vec{a}.\vec{b} - 8\vec{b}^2 - 4\vec{a}.\vec{b} \Rightarrow -3 + 6\,\vec{a}.\vec{b} = 0$$

$$\Rightarrow ab\cos\theta = \frac{3}{6} \Rightarrow \cos\theta = \frac{1}{2} = \theta = \frac{\pi}{3} \Rightarrow k = 3$$

Illustration 16.

For shown situation, what will be the magnitude of minimum force in newton that can be applied in any direction so that the resultant force is along east direction?



Ans. ***6***

Solution

Let force be F so resultant is in east direction

$$4\hat{i} + 3\hat{j} + \left(5\cos 37°\hat{i} + 5\sin 37°\hat{j}\right) + \vec{F} = k\hat{i}$$

$$\Rightarrow 4\hat{i} + 3\hat{j} + 4\hat{i} + 3\hat{j} + \vec{F} = k\hat{i} \Rightarrow 8\hat{i} + 6\hat{j} + \vec{F} = k\hat{i}$$

$$\Rightarrow \vec{F} = (k-8)\,\hat{i} - 6\hat{j}$$

$$\Rightarrow F = \sqrt{(k-8)^2 + (6)^2} \Rightarrow F_{min} = 6N$$

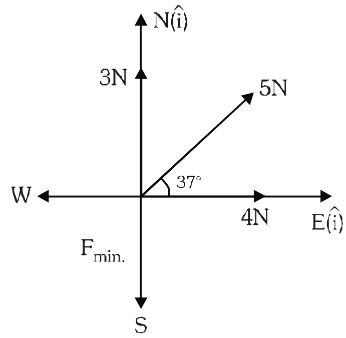


Illustration 17.

Column–I	***Column II***
(Operation of nonzero vectors $\vec{P}$ and $\vec{Q}$)	***(Possible angle between $\vec{P}$ and $\vec{Q}$)***
(A) $\mid\vec{P}\times\vec{Q}\mid = 0$	(p) 90°
(B) $\left\|\vec{P}\times\vec{Q}\right\| = \sqrt{3}\vec{P}.\vec{Q}$	(q) 180°
(C) $\vec{P}+\vec{Q}=\vec{R}$ and P+Q=R	(r) 60°
(D) $\left\|\vec{P}+\vec{Q}\right\| = \left\|\vec{P}-\vec{Q}\right\|$	(s) 0°
	(t) 30°

Ans. ***(A) q,s; (B) r ; (C) s ; (D) p***

Solution

For (A) $\mid\vec{P}\times\vec{Q}\mid = 0 \Rightarrow$ Angle between $\vec{P}$ and $\vec{Q}$ is 0^0 or 180^0

For (B) $|PQ\sin\theta| = \sqrt{3}\ PQ\cos\theta \Rightarrow |\sin\theta| = \sqrt{3}\ \cos\theta$

Here $\cos\theta$ must be positive so $\theta = 60°$

For (C) Here $P^2 + Q^2 + 2PQ\cos\theta = P^2 + Q^2 + 2PQ \Rightarrow \cos\theta = 1 \Rightarrow \theta = 0°$

For (D) Here $P^2 + Q^2 + 2PQ\cos\theta = P^2 + Q^2 - 2PQ\cos\theta \Rightarrow \cos\theta = 0, \Rightarrow \theta = 90°$

Illustration 18*.

The position of a particle moving in XY-plane varies with time t as $x = t$, $y = 3t - 5$.

(i) What is the path traced by the particle?

(ii) When does the particle cross-x-axis?

Solution

(i) $x = t$, $y = 3t - 5$ By eliminating t from above two equations $y = 3x - 5$

This is the equation of a straight line.

(ii) The particle crosses x-axis when $y = 0$. So $0 = 3t - 5 \Rightarrow t = \frac{5}{3}$

Illustration 19*.

Two particles A and B move along the straight lines $x+2y+3=0$ and $2x+y-3=0$ respectively. Their position vector, at the time of meeting will be :

(A) $3\hat{i}+3\hat{j}$ (B) $3\hat{i}-3\hat{j}$ (C) $\frac{\hat{i}}{3}-\frac{\hat{j}}{3}$ (D) Particles never meet

Ans. (B)

Solution

The particles meet at the point of intersection of lines.

By solving them $x = 3$, $y = -3$, So position vector of meeting point will be $3\hat{i}-3\hat{j}$

ANSWERS

BEGINNER'S BOX-1

1. 2380 **2.** n = 9 **3.** $1 + 6x + 15x^2 + 20x^3 + 15x^4 + 6x^5 + x^6$

4. 0.90 **5.** 2 **6.** 3 **7.** $\ell n2$

BEGINNER'S BOX-2

1. (i) $[3-2x]$ (ii) $[3x^2-4x]$ (iii) $\left[-\frac{3}{x^2}\right]$ (iv) $\left[\frac{1}{\sqrt{2x}}\right]$ (v) $[2a(ax+b)]$

2. (i) $[-4]$ (ii) $[4a+b]$ (iii) $[3(2+a)^2]$ (iv) $\left[-\frac{1}{8}\right]$

3. $\left[\frac{1}{3}\right]$

4. [(0, 2)]

5. [0]

6. [16]

7. (i) $[10x^4-3x^2]$ (ii) $[(6x^2-4x+3)]$ (iii) $[x^{-1/2}+2x^{-1/3}]$

8. (i) [17] (ii) [–3]

9. –1

10. $\left[1, -\frac{1}{3}\right]$

11. 0

BEGINNER'S BOX-3

1. (i)$[5x+C]$ (ii)$[x^3-5x+C]$ (iii)$\left[\frac{x^2}{2}-7x+C\right]$ (iv)$\left[-\frac{x^3}{3}+\frac{x^2}{2}+2x+C\right]$

2. (i) 3/2 (ii) 16/3 (iii) –73/12 (iv) 1/6 a^2 (v) [39.3]

3. $[e^{\sin\theta}+C]$ **4.** $[x-2e^{-x}+C]$ **5.** $2(e^2-e)$ **6.** $\frac{e^2}{e+1}$

BEGINNER'S BOX-4

1. (C) **2.** (D) **3.** [(i) 105°, (ii) 150°, (iii) 105°] **4.** [120°] **5.** $\left[\sqrt{F_1^2+F_2^2}\right]$

6. (B) **7.** (C) **8.** A **9.** $[\frac{5}{\sqrt{30}}\hat{i}+\frac{1}{\sqrt{30}}\hat{j}-\frac{2}{\sqrt{30}}\hat{k}]$ **10.** (B)

BEGINNER'S BOX-5

1. A **2.** (B) **3.** D **4.** $[q=\cos^{-1}\left(\frac{25}{29}\right)]$ **5.** [**A** 3, **B** $-\hat{i}+2\hat{j}-\hat{k}$]

6. [A 6, B $6\sqrt{3}$]

EXERCISE – 1 DIFFERENTIATION

1*. If y = ln (ln x), then dy/dx is equal to :

(A) $\frac{1}{x\ln x}$ (B) $\frac{1}{x}$ (C) $\frac{1}{\ln x}$ (D) e^x

2*. If x = a cos t, y = b sin t then $\frac{dy}{dx}$ equals :

(A) $-\frac{b}{a}\tan t$ (B) $-\frac{b}{a}\cot t$ (C) $-\frac{a}{b}\cot t$ (D) $-\frac{a}{b}\tan t$

3. $\frac{d}{dx}\ln(x^{10})$ equals :

(A) x^{-10} (B) 10 x (C) 10/x (D) $10x^9$

4. If f (x) = $x^2 - 6x + 8$, $2 \le x \le 4$, then the value of x for which f ′(x) vanishes is :

(A) 9/4 (B) 5/2 (C) 3 (D) 7/2

5. If f (x) = x – 5, then f ‘ (5) is equal to :

(A) 0 (B) 1 (C) 5 (D) ∞

6*. $\frac{d}{dx}(a^x)$ equals :

(A) $x\,a^{x-1}$ (B) $a^x/\log_a e$ (C) $a^x \log_a e$ (D) None of these

7*. If f (x) = $\frac{2x+c}{x-1}$ and f ‘ (0) = 0, then c equals :

(A) 0 (B) 1 (C) 2 (D) – 2

8. If y = sin x + cos 2x, then $\frac{d^2y}{dx^2}$ equals:

(A) sin x + 4 sin 2x (B) – sin x + 4 cos 2x (C) – (sin x + 4 cos 2 x) (D) None of these

9. Find points at which the tangent to the curve $y = x^3 - 3x^2 - 9x + 7$ is parallel to the x–axis

(A) (3,–20) and (–1, 12) (B) (3,20) and (1, 12) (C) (3,–10) and (1, 12) (D) None of these

10. ***Find all points of local maxima and minima and the corresponding maximum and minimum values of this functions if any.***

$y = x^3 - 6x^2 + 9x + 15.$

11. y = cos (sin x) Find $\frac{dy}{dx}$.

12. If $y = \frac{1+x^2+x^4}{1+x+x^2}$ and $\frac{dy}{dx} = ax + b$, then values of a & b are -

(A) a = 2 , b = 1 (B) a = –2 , b = 1 (C) a = 2 , b = –1 (D) a = –2 , b = –1

13. $\frac{d}{dx}(\log \tan x) =$

(A) $2\sec 2x$ (B) $2\operatorname{cosec} 2x$ (C) $\sec 2x$ (D) $\operatorname{cosec} 2x$

14. $\frac{d}{dx}\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)^2=$

(A) $1-\frac{1}{x^2}$ (B) $1+\frac{1}{x^2}$ (C) $1-\frac{1}{2x}$ (D) None of these

15. $\frac{d}{dx}\left(\frac{1}{x^4 \sec x}\right)=$

(A) $\frac{x\sin x+4\cos x}{x^5}$ (B) $\frac{-(x\sin x+4\cos x)}{x^5}$ (C) $\frac{4\cos x-x\sin x}{x^5}$ (D) None of these

16. $\frac{d}{dx}\left(x^2\sin\frac{1}{x}\right)=$

(A) $\cos\left(\frac{1}{x}\right)+2x\sin\left(\frac{1}{x}\right)$ (B) $2x\sin\left(\frac{1}{x}\right)-\cos\left(\frac{1}{x}\right)$ (C) $\cos\left(\frac{1}{x}\right)-2x\sin\left(\frac{1}{x}\right)$ (D) None of these

17. $f(x)=x^2-3x$, then the points at which $f(x)=f'(x)$ are

(A) 1, 3 (B) 1, – 3 (C) – 1, 3 (D) None of these

18. $\frac{d}{dx}\left(\frac{\cot^2 x-1}{\cot^2 x+1}\right)=$

(A) $-\sin 2x$ (B) $2\sin 2x$ (C) $2\cos 2x$ (D) $-2\sin 2x$

19. $\frac{d}{dx}(e^x\log\sin 2x)=$

(A) $e^x(\log\sin 2x+2\cot 2x)$ (B) $e^x(\log\cos 2x+2\cot 2x)$

(C) $e^x(\log\cos 2x+\cot 2x)$ (D) None of these

20. If $y=t^{4/3}-3t^{-2/3}$, then $dy/dt=$

(A) $\frac{2t^2+3}{3t^{5/3}}$ (B) $\frac{2t^2+3}{t^{5/3}}$ (C) $\frac{2(2t^2+3)}{t^{5/3}}$ (D) $\frac{2(2t^2+3)}{3t^{5/3}}$

EXERCISE – 2 INTEGRATION

Single Choice Correct

1. $\int \left(x+\frac{1}{x}\right)^3 dx =$

(A) $\frac{1}{4}\left(x+\frac{1}{x}\right)^4+c$
(B) $\frac{x^4}{4}+\frac{3x^2}{2}+3\log x-\frac{1}{2x^2}+c$
(C) $\frac{x^4}{4}+\frac{3x^2}{2}+3\log x+\frac{1}{x^2}+c$
(D) None of these

2. $\int \frac{x-1}{(x+1)^2} dx =$

(A) $\log(x+1)+\frac{2}{x+1}+c$
(B) $\log(x+1)-\frac{2}{x+1}+c$
(C) $\frac{2}{x+1}-\log(x+1)+c$
(D) None of these

3. $\int \left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)^2 dx =$

(A) $x+\cos x+c$
(B) $2\cos^2\frac{x}{2}+c$
(C) $\frac{1}{3}\left(\cos\frac{x}{2}-\frac{x}{2}\right)^3+c$
(D) $x-\cos x+c$

4. $\int \frac{x^2+x-6}{(x-2)(x-1)} dx =$

(A) $x+2\log(x-1)+c$
(B) $2x+2\log(x-1)+c$
(C) $x+4\log(1-x)+c$
(D) $x+4\log(x-1)+c$

5. $\int \frac{\sin 3x}{\sin x} dx =$

(A) $x+\sin 2x+c$
(B) $3x+\sin 2x+c$
(C) $3x+\sin^2 x+c$
(D) None of these

6. $\int a^x \, da =$

(A) $\frac{a^x}{\log_e a}+c$
(B) $a^x \log_e a+c$
(C) $\frac{a^{x+1}}{x+1}+c$
(D) $xa^{x-1}+c$

Subjective Type

7. Evaluate $\int (2x+1)^3 \, dx$

8. Evaluate $\int \cos(3z+4) \, dz$

9. Evaluate $\int_{-4}^{-1} \frac{\pi}{2} \, d\theta$

10. Evaluate $\int_{\pi}^{2\pi} \theta \, d\theta$

11. Evaluate $\int_{0}^{\pi} \cos x dx$

12*. Evaluate $\int_{0}^{1} \frac{dx}{3x+2}$

Direction (Q 13 to 17) Use a definite integral to find the area of the region between the given curve and the x-axis on the interval [0, b]:

13. $y = 3x^2$

14. $y = 2x$

15. $y = \frac{x}{2} + 1$

16. Use a definite integral to find the area of the region between the given curve and the x-axis on the interval $[0, \pi]$:

$y = \sin x$.

17. ***Evaluate the following integrals.***

(i) $\int_0^1 (4x^3 + 3x^2 - 2x + 1)dx$ ***(ii)*** $\int_4^5 e^x dx$ ***(iii)*** $\int_3^2 \frac{(x^2-1)}{(x-1)}dx$

EXERCISE – 3 **VECTORS**

Single Choice Correct

1. Two vectors $\vec{a}$ and $\vec{b}$ inclined at an angle θ w.r.t. each other have a resultant $\vec{c}$ which makes an angle β with $\vec{a}$. If the directions of $\vec{a}$ and $\vec{b}$ are interchanged, then the resultant will have the same :
(A) Magnitude (B) Direction
(C) Magnitude as well as direction (D) Neither magnitude nor direction

2. Given : $\vec{C} = \vec{A} + \vec{B}$. Also, the magnitude of $\vec{A}$, $\vec{B}$ and $\vec{C}$ are 12, 5 and 13 units respectively. The angle between $\vec{A}$ and $\vec{B}$ is :
(A) 0° (B) $\frac{\pi}{4}$ (C) $\frac{\pi}{2}$ (D) π

3. If $\vec{P} + \vec{Q} = \vec{P} - \vec{Q}$ and θ is the angle between $\vec{P}$ and $\vec{Q}$, then :
(A) $\theta = 0°$ (B) $\theta = 90°$ (C) P = 0 (D) Q = 0

4*. The sum and difference of two perpendicular vectors of equal lengths are :
(A) Of equal lengths and have an acute angle between them
(B) Of equal length and have an obtuse angle between them
(C) Also perpendicular to each other and are of different lengths
(D) Also perpendicular to each other and are of equal lengths

5. If the angle between two forces increases, the magnitude of their resultant :
(A) Decreases (B) Increases
(C) Remains unchanged (D) First decreases and then increases

6*. A car is moving on a straight road due north with a uniform speed of 50 km h^{-1} when it turns left through 90°. If the speed remains unchanged after turning, the change in the velocity of the car in the turning process is :
(A) Zero (B) $50\sqrt{2}$ km h^{-1} S-W direction
(C) $50\sqrt{2}$ km h^{-1} N-W direction (D) 50 km h^{-1} due west

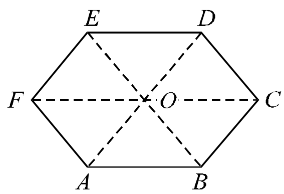
7*. If $|\vec{A} + \vec{B}| = |\vec{A}| = |\vec{B}|$, then the angle between $\vec{A}$ & $\vec{B}$ and is :
(A) 0° (B) 60° (C) 90° (D) 120°

8. The vector sum of 6 coplanar forces, each of magnitude F, when each force is making an angle of $\pi/3$ with the preceding it, is :
(A) F (B) 6F (C) 3F/2 (D) zero

9. The following sets of three vectors act on a body, whose resultant can not be zero :
(A) 10, 10, 10 (B) 10, 10, 20 (C) 10, 20, 20 (D) 10, 20, 40

10*. Which of the following sets of displacements might be capable of returning a car to its starting point ?
(A) 4, 6, 8 and 15 km (B) 10, 30, 50 and 120 km
(C) 5, 10, 30 and 50 km (D) 40, 50, 75 and 200 km

11*. ABCDEF is a regular hexagon. What is the value of $\vec{AB} + \vec{AC} + \vec{AD} + \vec{AE} + \vec{AF}$?

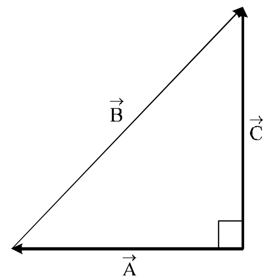


(A) 0 (B) $2\vec{AO}$
(C) $4\vec{AO}$ (D) $6\vec{AO}$

12. A set of vectors taken in a given order gives a closed polygon. Then the resultant of these vectors is a :
(A) Scalar quantity (B) Pseudo vector (C) Unit vector (D) Null vector

13. The vectors $\vec{A}$ and $\vec{B}$ lie in a plane. Another vector $\vec{C}$ lies outside this plane. The resultant $\vec{A}+\vec{B}+\vec{C}$ of these three vectors :
(A) Can be zero (B) Cannot be zero
(C) Lies in the plane of $\vec{A}$ and $\vec{B}$ (D) Lies in the plane of $\vec{A}$ and $\vec{A}+\vec{B}$

14*. In the adjoining vector diagram, what is the angle between $\vec{A}$ and $\vec{B}$?



(Given : C = $\frac{B}{2}$) :
(A) 30° (B) 60°
(C) 120° (D) 150°

15. The vector sum of two force P and Q is minimum when the angle θ between their positive directions, is :
(A) $\frac{\pi}{4}$ (B) $\frac{\pi}{3}$ (C) $\frac{\pi}{2}$ (D) π

16. The vector sum of two vectors $\vec{A}$ and $\vec{B}$ is maximum, then the angle θ between two vectors is :
(A) 0° (B) 30° (C) 45° (D) 60°

17. Given : $\vec{A}=2\hat{i}+3\hat{j}$ and $\vec{B}=5\hat{i}-6\hat{j}$. The magnitude of $\vec{A}+\vec{B}$ is :
(A) 4 units (B) 10 units (C) $\sqrt{58}$ units (D) $\sqrt{61}$ units

18. The modulus of $2\hat{i}-\hat{j}+\hat{k}$ is :
(A) $\sqrt{6}$ (B) 2 (C) 4 (D) 6

19*. A particle has position vector $(3\hat{i}-\hat{j}+2\hat{k})$ metre at time t = 0. It moves with constant velocity $(-\hat{i}+\hat{j}-3\hat{k})$ ms^{-1}. The position vector (in m) of the particle after 3 second is :
(A) $2\hat{j}-7\hat{k}$ (B) $2\hat{i}+\hat{k}$ (C) $\hat{j}$ (D) $3\hat{k}$

20. The vector joining the points A (1, 1, –1) and B (2, –3, 4) and pointing from A to B is :
(A) $-\hat{i}+4\hat{j}-5\hat{k}$ (B) $\hat{i}+4\hat{j}+5\hat{k}$ (C) $\hat{i}-4\hat{j}+5\hat{k}$ (D) $-\hat{i}-4\hat{j}-5\hat{k}$

21. A hall has the dimensions 10 m × 12 m × 14 m. A fly starting at one corner ends up at a diametrically opposite corner. The magnitude of its displacement is nearly :
(A) 16 m (B) 17 m (C) 18 m (D) 21 m

22. Given : $\vec{A}=2\hat{i}-\hat{j}+2\hat{k}$ and $\vec{B}=-\hat{i}-\hat{j}+\hat{k}$. The unit vector of $\vec{A}-\vec{B}$ is :
(A) $\frac{3\hat{i}+\hat{k}}{\sqrt{10}}$ (B) $\frac{3\hat{i}}{\sqrt{10}}$ (C) $\frac{\hat{k}}{\sqrt{10}}$ (D) $\frac{-3\hat{i}-\hat{k}}{\sqrt{10}}$

23. If $\vec{P}=2\hat{i}-3\hat{j}+\hat{k}$ and $\vec{Q}=3\hat{i}+2\hat{j}$, then $\vec{P}\cdot\vec{Q}$ is :
(A) zero (B) 6 (C) 12 (D) 15

24. A force $\vec{F} = 3\hat{i} - 2\hat{j} + 4\hat{k}$ displaces a body from a point A (8, – 2, – 3) to the point B (–2, 0, 6). The work done is:

(A) 1 unit (B) 2 units (C) 3 units (D) 4 units

25. Three non zero vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$ satisfy the relation $\vec{A}\cdot\vec{B} = 0$ and $\vec{A}\cdot\vec{C} = 0$. Then $\vec{A}$ can be parallel to :

(A) $\vec{B}$ (B) $\vec{C}$ (C) $\vec{B}\cdot\vec{C}$ (D) $\vec{B}\times\vec{C}$

26. A vector is not changed if :

(A) It is displaced parallel to itself
(B) It is rotated through an arbitrary angle
(C) It is cross-multiplied by a unit vector
(D) It is multiplied by an arbitrary scalar

27. When two vector $\vec{a}$ and $\vec{b}$ are added, the magnitude of the resultant vector is always :

(A) Greater than (a + b)
(B) Less than or equal to (a + b)
(C) Less than (a + b)
(D) Equal to (a + b)

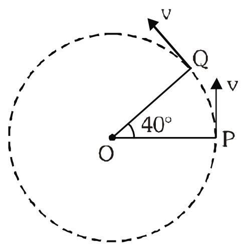
28*. Given : $\vec{a} + \vec{b} + \vec{c} = 0$. Out of the three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ two are equal in magnitude. The magnitude of the third vector is $\sqrt{2}$ times that of either of the two having equal magnitude. The angles between the vectors are:

(A) 90°, 135°, 135° (B) 30°, 60°, 90° (C) 45°, 45°, 90° (D) 45°, 60°, 90°

29*. What is the angle between two vector forces of equal magnitude such that the resultant is one-third as much as either of the original forces ?

(A) $\cos^{-1}\left(-\frac{17}{18}\right)$ (B) $\cos^{-1}\left(\frac{1}{3}\right)$ (C) 45° (D) $2\cos^{-1}\left(\frac{1}{12}\right)$

30*. A particle is moving in a circle or radius r centred at O with constant speed v. The change in velocity in moving from P to Q ($\angle$POQ = 40°) is:



(A) 2v cos 40°

(B) 2v sin 40°

(C) 2v cos 20°

(D) 2v sin 20°

EXERCISE – 4 VECTORS

One or More Choice Correct

1. Which of the following is a true statement ?

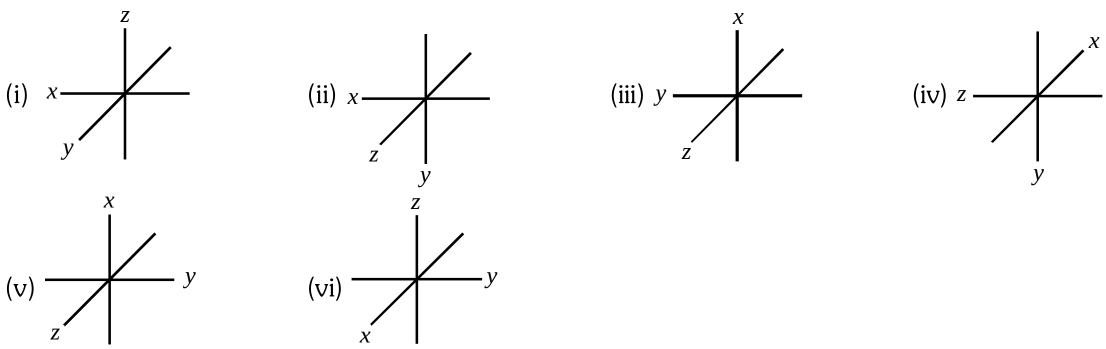
(A) A vector cannot be divided by another vector

(B) Angular displacement can either be a scalar or a vector

(C) Since addition of vectors is commutative therefore vector subtraction is also commutative

(D) The resultant of two equal forces of magnitude F acting at a point is F if the angle between the two forces is 120°

2*. Which of the arrangement of axes in figure. can be labelled "right-handed coordinate system " ? As usual, each axis label indicates the positive side of the axis :

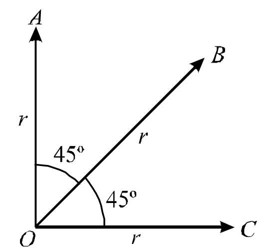


(A) (i), (ii) (B) (iii), (iv) (C) (vi) (D) (v)

Subjective Type

3. A force of 30 N is inclined at an angle θ to the horizontal. If its vertical component is 18 N, find the horizontal component and the value of θ.

4. Find the resultant of the three vectors $\overline{OA}$, $\overline{OB}$, $\overline{OC}$ each of magnitude r as shown in figure ?



5*. The x and y components of vector $\vec{A}$ are 4 m and 6 m respectively. The x, y components of vector $\vec{A}+\vec{B}$ are 10 m and 9 m respectively. Find the length of $\vec{B}$ and angle that $\vec{B}$ makes with the x-axis.

6*. If $\vec{a} = x_1\hat{i} + y_1\hat{j}$ and $\vec{B} = x_2\hat{i} + y_2\hat{j}$. Find the condition that would make $\vec{a}$ and $\vec{b}$ parallel to each other.

7*. A particle whose speed is 50 m/s moves along the line from A (2, 1) to B (9, 25). Find its velocity vector in the form of $a\hat{i} + b\hat{j}$.

8. A man walks 40 m North, then 30 m East and then 40 m South. Find the displacement from the starting point?

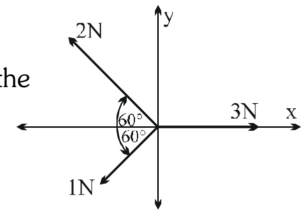
9. Find the magnitude of $3\hat{i} + 2\hat{j} + \hat{k}$?

10. If $0.5\hat{i} + 0.8\hat{j} + C\hat{k}$ is a unit vector. Find the value of C.

11*. If $\vec{A} = 3\hat{i} + 4\hat{j}$ and $\vec{B} = \hat{i} + \hat{j} + 2\hat{k}$ then find out unit vector along $\vec{A} + \vec{B}$.

12. The rectangular components of a vector are (2, 2). The corresponding rectangular components of another vector are (1, $\sqrt{3}$). Find the angle between the two vectors.

13*. Three forces of 3N, 2N and 1N act on a particle as shown in the figure. Calculate the
(A) net force along the x-axis.
(B) net force along the y-axis.



14*. Four forces of magnitudes P, 2P, 3P and 4P act along the four sides of a square ABCD in cyclic order. Use the vector method to find the resultant force.

15*. Two force of $\vec{F_1}$ = 500 N due east and $\vec{F_2}$ = 250 N due north. Find $\vec{F_2} - \vec{F_1}$?

16*. The resultant of two vectors of magnitudes 2A and $\sqrt{2}$ A acting at an angle θ is $\sqrt{10}$ A . Find the value of θ ?

17. Two vectors acting in the opposite directions have a resultant of 10 units. If they act at right angles to each other, then the resultant is 50 units. Calculate the magnitude of two vectors.

18. $\vec{a} + \vec{b} = \vec{c}$ and $\vec{a} + 2\vec{b} = \vec{R}$. It is given that $\vec{R}$ is perpendicular to $\vec{a}$. Find $\dfrac{|\vec{b}|}{|\vec{c}|}$

19. Let the resultant of three forces of magnitude 5 N, 12 N & 13 N acting on a body be zero. If sin 23° = $\dfrac{5}{13}$, find the angle between the 5 N force & 13 N force.

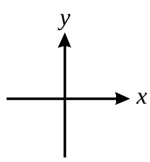
20. $\vec{A} = \hat{i} + \hat{j} + \hat{k}$

$\vec{B} = x\hat{i} + \hat{j} - 2\hat{k}$

If $\vec{A}$ & $\vec{B}$ are perpendicular find x.

21. If $\vec{A} = 3\hat{i} + 4\hat{j}$ then find $\hat{A}$.

22*. What are the x and the y components of a 25 m displacement at an angle of 210° with the x-axis (anti clockwise)?



23. One of the rectangular components of a velocity of 60 km h^{-1} is 30 km h^{-1}. Find other rectangular component?

24. A vector of magnitude 30 and direction eastwards is added with another vector of magnitude 40 and direction northwards. Find the magnitude and direction of resultant with the east.

ANSWERS

EXERCISE-1

1. (A) **2.** (B) **3.** (C) **4.** (C) **5.** (B) **6.** (B) **7.** (D)
8. (C) **9.** (A)
10. local max. at $x = 1$, local max. value $= 19$; local min. at $x = 3$, local min. value $= 15$
11. (B). **12.** (C) **13.** (B) **14.** (A) **15.** (A) **16.** (B) **17.** (D) **18.** (D)
19. (A) **20.** (D)

EXERCISE-2

1. (B) **2.** (A) **3.** (A) **4.** (D) **5.** (A) **6.** (C)

7. $\frac{(2x+1)^4}{8} + (C)$ **8.** $\frac{1}{3}\sin(3z+4) + (C)$ **9.** $\frac{3\pi}{2}$ **10.** $\frac{3\pi^2}{2}$ **11.** 0

12. $\frac{1}{3}\ln\frac{5}{2} = \ln\left(\frac{5}{2}\right)^{1/3}$ **13.** b^3 **14.** Area $= b^2$ **15.** $\frac{b(4+b)}{4}$

16. 2 **17.** ***(i)*** (2), ***(ii)*** $e^4(e-1)$, ***(iii)*** $-\frac{7}{2}$

EXERCISE-3

1. (A) **2.** (C) **3.** (D) **4.** (D) **5.** (A) **6.** (B) **7.** (D) **8.** (D)
9. (D) **10.** (A) **11.** (D) **12.** (D) **13.** (B) **14.** (D) **15.** (D) **16.** (A)
17. (C) **18.** (A) **19.** (A) **20.** (C) **21.** (D) **22.** (A) **23.** (A) **24.** (B)
25. (D) **26.** (A) **27.** (B) **28.** (A) **29.** (A) **30.** (D)

EXERCISE-4

- ***One or More Choice correct*** **1.** (A),(B),(D) **2.** (A),(B),(C)
- ***Subjective Type*** **3.** [24 N; 37° approx] **4.** [$r(1+\sqrt{2})$]

5. [$3\sqrt{5}$, $\tan^{-1}\frac{1}{2}$] **6.** [$\frac{x_1}{x_2} = \frac{y_1}{y_2}$]

7. [$2(7\hat{i}+24\hat{j})$] **8.** [30 m East] **9.** [$\sqrt{14}$]

10. [$\sqrt{11/10}$] **11.** [$\frac{4\hat{i}+5\hat{j}+2\hat{k}}{\sqrt{45}}$] **12.** [15°]

13. (a) $\frac{3}{2}$N, (b) $\frac{\sqrt{3}}{2}$N

14. [$2\sqrt{2}$P at 45° to 4P force]

15. [$250\sqrt{5}$ N, $\tan^{-1}(2)$ W of N]

16. [45°] **17.** [P = 40; Q = 30] **18.** [1]

19. [113°] **20.** [x = 1] **21.** [$\frac{3\hat{i}+4\hat{j}}{5}$]

22. [– 25 cos 30° and – 25 sin 30°] **23.** [$30\sqrt{3}$ km h^{-1}]

24. [50, 53° with East]

* * * * *

RECTILINEAR MOTION

Recap of Early Classes

Motion is change in position of an object with time. How does the position change with time ? In this chapter, we shall learn how to describe motion. For this we develop the concept of velocity and acceleration. We shall confine overselves to the study of motion of objects along a straight line, also known as rectilinear motion. Here we will study ways to describe motion without going into the causes of motion. For the case of rectilinear motion with uniform velocity, uniform acceleration and variable acceleration, a set of simple equations and some methods will be discussed. After going through this topic you will be able to analyze all the cases of rectilinear motion of particles.

Index

RECTILINEAR MOTION

1.0 KINEMATICS

SL AL

In kinematics we study how a body moves without knowing why it moves. All particles of a rigid body in translation motion move in identical fashion hence any of the particles of a rigid body in translation motion can be used to represent translation motion of the body. This is why, while analyzing its translation motion, a rigid body is considered a particle and kinematics of translation motion as particle kinematics.

Particle kinematics deals with nature of motion i.e. how fast and on what path an object moves and relates the position, velocity, acceleration, and time without any reference to mass, force and energy. In other words, it is study of geometry of motion.

1.1 Types of Translation Motion

SL AL

A body in translation motion can move on either a straight–line path or curvilinear path.

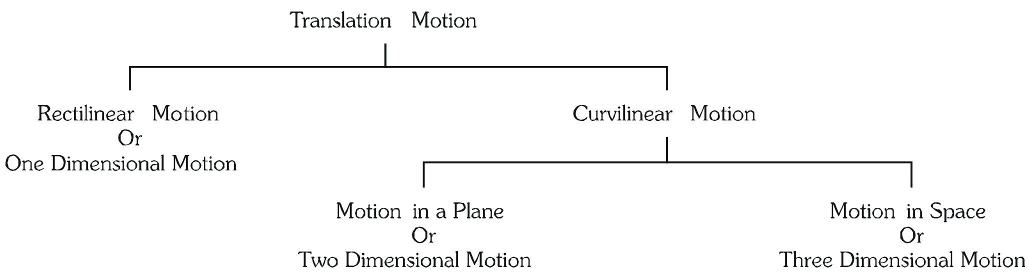
Rectilinear Motion

Translation motion on straight–line path is known as rectilinear translation. It is also known as one–dimensional motion. A car running on a straight road, train running on a straight track and a ball thrown vertically upwards or dropped from a height etc are very common Illustrations of rectilinear translation.

Curvilinear Motion

Translation motion of a body on curvilinear path is known as curvilinear translation. If the trajectory is in a plane, the motion is known as two–dimensional motion. A ball thrown at some angle with the horizontal describes a curvilinear trajectory in a vertical plane; a stone tied to a string when whirled describes a circular path and an insect crawling on the floor or on a wall are Illustrations of two–dimensional motion.

If path is not in a plane and requires a region of space or volume, the motion is known as three–dimensional motion or motion is space. An insect flying randomly in a room, motion of a football in soccer game over considerable duration of time etc are common Illustrations of three–dimensional motion.



Reference Frame

Motion of a body can only be observed if it changes its position with respect to some other body. Therefore, for a motion to be observed there must be a body, which is changing its position with respect to other body and a person who is observing motion. The person observing motion is known as observer. The observer for the purpose of investigation must have its own clock to measure time and a point in the space attached with the other body as origin and a set of coordinate axes. These two things the time measured by the clock and the coordinate system are collectively known as reference frame.

In this way, motion of the moving body is expressed in terms of its position coordinates changing with time.

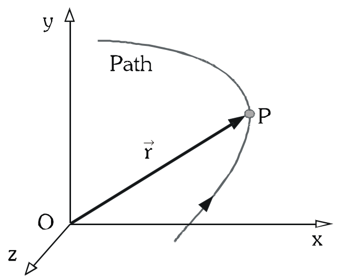
1.2 Position Vector, Velocity and Acceleration Vector

SL AL

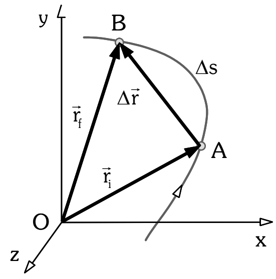
For analyzing translation motion, we assume the moving body as a particle and represent it as mathematical point. Consider a particle P moving on a curvilinear path.

Position–Vector–It describes position of a particle relative to other particle and is a vector from the later towards the first. To study motion of a particle we have to assume a reference frame fixed with some other body. The vector drawn from the origin of the coordinate system representing the reference frame to the location of the particle P is known as position vector of the particle P.

Consider a particle P moving in space traces a path shown in the figure. Its position continuously changes with time and so does the position vector. At an instant of time, its position vector $\vec{r}$ is shown in the following figure.



Position Vector



Displacement Vector & Distance Travelled

Displacement and distance travelled

Displacement is measure of change in place i.e. position of particle. It is defined by a vector from the initial position to the final position. Let the particle moves from point A to B on the curvilinear path. The vector $\overrightarrow{AB} = \Delta\vec{r}$ is displacement.

Distance travelled is length of the path traversed. We can say it "path length". Here in the figure length of the curve Δs from A to B is the distance travelled.

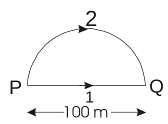
Distance travelled between two places is greater than the magnitude of displacement vector wherever particle changes its direction during its motion. In unidirectional motion, both of them are equal. Displacement is vector while distance travelled is scalar.

Illustrations

Illustration 1. Ram takes path 1 (straight line) to go from P to Q and Shyam takes path 2 (semicircle).

(a) Find the distance travelled by Ram and Shyam?

(b) Find the displacement of Ram and Shyam?



Solution (a) Distance travelled by Ram = 100 m

Distance travelled by Shyam = π(50 m) = 50π m

(b) Displacement of Ram = 100 m

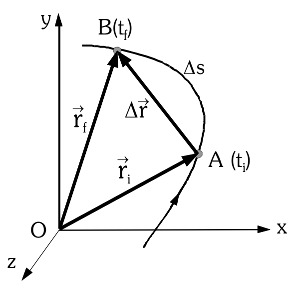
Displacement of Shyam = 100 m

Average Velocity and Average Speed

SL AL

Average velocity of a particle in a time interval is that constant velocity with which particle would have covered the same displacement in the same time interval as it covers in its actual motion. It is defined as the ratio of displacement to the concerned time interval.

If the particle moves from point A to point B in time interval t_i to t_f, the average velocity $\vec{v}_{av}$ in this time interval is given by the following equation.



$$\vec{v}_{av} = \frac{\Delta\vec{r}}{\Delta t} = \frac{\vec{r}_f - \vec{r}_i}{t_f - t_i}$$

Similar to average velocity, average speed in a time interval is that constant speed with which particle would travel the same distance on the same path in the same time interval as it travels in its actual motion. It is defined as the ratio of distance travelled to the concerned time interval.

If in moving from point A to B, the particle travels path length i.e. distance Δs in time interval t_i to t_f, its average speed c_{av} is given by the following equation.

$$c_{av} = \frac{\Delta s}{\Delta t} = \frac{\text{Path Length}}{t_f - t_i}$$

Average speed in a time interval is greater than the magnitude of average velocity vector wherever particle changes its direction during its motion. In unidirectional motion, both of them are equal. Average velocity is vector while average speed is scalar.

Illustrations

Illustration 2. In the Illustration 1, if Ram takes 4 seconds and Shyam takes 5 seconds to go from P to Q, find
(a) Average speed of Ram and Shyam?
(b) Average velocity of Ram and Shyam?

Solution (a) Average speed of Ram $= \frac{100}{4}$ m/s = 25 m/s

Average speed of Shyam $= \frac{50\pi}{5}$ m/s $= 10\pi$ m/s

(b) Average velocity of Ram $= \frac{100}{4}$ m/s = 25 m/s

Average velocity of Shyam $= \frac{100}{5}$ m/s = 20 m/s

Illustration 3. A particle travels half of total distance with speed v_1 and next half with speed v_2 along a straight line. Find out the average speed of the particle?

Solution Let total distance travelled by the particle be 2s.

Time taken to travel first half $= \frac{s}{v_1}$

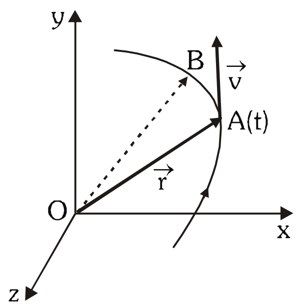
Time taken to travel next half $= \frac{s}{v_2}$

$$\text{Average speed} = \frac{\text{Total distance covered}}{\text{Total time taken}} = \frac{2s}{\frac{s}{v_1} + \frac{s}{v_2}} = \frac{2v_1 v_2}{v_1 + v_2}$$

Instantaneous Velocity and speed

SL AL

If we assume the time interval Δt to be infinitesimally small i.e. $\Delta t \to 0$, the point B approaches A making the chord AB to coincide with the tangent at A. Now we can express the instantaneous velocity $\vec{v}$ by the following equations.



$$\vec{v} = \lim_{\Delta t \to 0} \frac{\Delta \vec{r}}{\Delta t} = \frac{d\vec{r}}{dt}$$

The instantaneous velocity equals to the rate of change in its position vector $\vec{r}$ with time. Its direction is along the tangent to the path. Instantaneous speed is defined as the time rate of distance travelled.

$$c = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \frac{ds}{dt}$$

You can easily conceive that when $\Delta t \to 0$, not only the chord AB but also the arc AB both approach to coincide with each other and with the tangent. Therefore $ds = |d\vec{r}|$. Now we can say that speed equals to magnitude of instantaneous velocity.

Instantaneous speed tells us how fast a particle moves at an instant and instantaneous velocity tells us in what direction and with what speed a particle moves at an instant of time.

Acceleration

SL AL

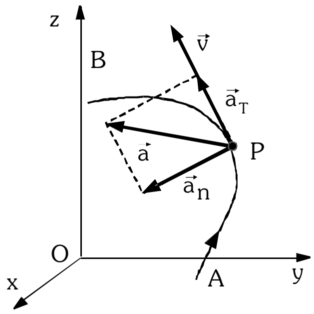
Instantaneous acceleration $\vec{a}$ is measure of how fast velocity of a body changes i.e. how fast direction of motion and speed change with time.

At an instant, it equals to the rate of change in velocity vector $\vec{v}$ with time.

$$\vec{a} = \frac{d\vec{v}}{dt}$$

A vector quantity changes, when its magnitude or direction or both change. Accordingly, acceleration vector may have two components, one responsible to change only speed and the other responsible to change only direction of motion.

Component of acceleration responsible to change speed must be in the direction of motion. It is known as tangential component of acceleration $\vec{a}_T$. The component responsible to change direction of motion must be perpendicular to the direction of motion. It is known as normal component of acceleration $\vec{a}_n$. Acceleration vector $\vec{a}$ of a particle moving on a curvilinear path and its tangential and Normal components are shown in the figure.



Illustrations

Illustration 4. Position of a particle as a function of time is given as $x = 5t^2 + 4t + 3$. Find the velocity and acceleration of the particle at $t = 2$ s?

Solution

Velocity; $v = \dfrac{dx}{dt} = 10t + 4$

At $t = 2$ s, $\quad v = 10(2) + 4 \Rightarrow v = 24$ m/s

Acceleration; $\quad a = \dfrac{d^2x}{dt^2} = 10$

Acceleration is constant, so at $t = 2$ s

$a = 10$ m/s^2

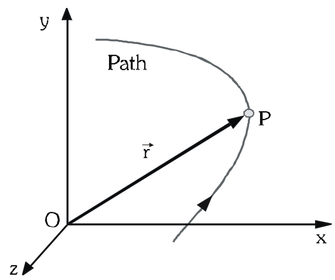
2.0 CURVILINEAR TRANSLATION IN CARTESIAN COORDINATE SYSTEM

Superposition of three rectilinear Motions

SL AL

Consider a particle moving on a three dimensional curvilinear path AB. At an instant of time t it is at point P (x, y, z) moving with velocity $\vec{v}$ and acceleration $\vec{a}$. Its position vector is defined by equations

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$



Differentiating it with respect to time, we get velocity vector.

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k} = v_x\hat{i} + v_y\hat{j} + v_z\hat{k}$$

Here $v_x = dx/dt$, $v_y = dy/dt$ and $v_z = dz/dt$ are the components of velocity vectors in the x, y and z– directions respectively.

Now the acceleration can be obtained by differentiating velocity vector $\vec{v}$ with respect to time.

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{dv_x}{dt}\hat{i} + \frac{dv_y}{dt}\hat{j} + \frac{dv_z}{dt}\hat{k} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}$$

Acceleration vector can also be obtained by differentiating position vector twice with respect to time.

$$\vec{a} = \frac{d^2\vec{r}}{dt^2} = \frac{d^2x}{dt^2}\hat{i} + \frac{d^2x}{dt^2}\hat{j} + \frac{d^2x}{dt^2}\hat{k} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}$$

In the above two equations, $a_x = d^2x/dt^2 = dv_x/dt$, $a_y = d^2y/dt^2 = dv_y/dt$ and $a_z = d^2z/dt^2 = dv_z/dt$ are the components of acceleration vectors in the x, y and z– directions respectively.
In the above equations, we can analyze each of the components x, y and z of motion as three individual rectilinear motions each along one of the axes x, y and z.

Along the x–axis $\quad v_x = \frac{dx}{dt}$ and $a_x = \frac{dv_x}{dt}$

Along the y–axis $\quad v_y = \frac{dy}{dt}$ and $a_y = \frac{dv_y}{dt}$

Along the z–axis $\quad v_z = \frac{dz}{dt}$ and $a_z = \frac{dv_z}{dt}$

A curvilinear motion can be analyzed as superposition of three simultaneous rectilinear motions each along one of the coordinate axes.

Illustrations

Illustration 5*. Position vector $\vec{r}$ of a particle varies with time t according to the law

$\vec{r} = \left(\frac{1}{2}t^2\right)\hat{i} - \left(\frac{4}{3}t^{1.5}\right)\hat{j} + (2t)\hat{k}$, where r is in meters and t is in seconds.

(a) Find suitable expression for its velocity and acceleration as function of time.

(b) Find magnitude of its displacement and distance travelled in the time interval t = 0 to 4 s.

Solution

(a) Velocity $\vec{v}$ is defined as the first derivative of position vector with respect to time.

$$\vec{v} = \frac{d\vec{r}}{dt} = t\hat{i} - 2\sqrt{t}\hat{j} + 2\hat{k} \text{ m/s}$$

Acceleration $\vec{a}$ is defined as the first derivative of velocity vector with respect to time.

$$\vec{a} = \frac{d\vec{v}}{dt} = \hat{i} - \frac{1}{\sqrt{t}}\hat{j} \text{ m/s}^2$$

(b) Displacement $\Delta\vec{r}$ is defined as the change in place of position vector.

$$\Delta\vec{r} = 8\hat{i} - \frac{32}{3}\hat{j} + 8\hat{k} \text{ m}$$

Magnitude of displacement $\Delta r = \sqrt{8^2 + \left(\frac{32}{3}\right)^2 + 8^2} = 15.55$ m

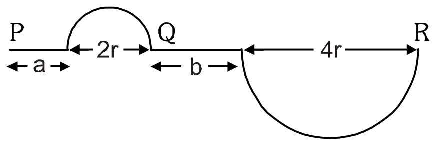
Distance Δs is defined as the path length and can be calculated by integrating speed over the concerned time interval.

$$\Delta s = \int_0^4 v dt = \int_0^4 \sqrt{t^2 + 4t + 4} dt = \int_0^4 (t+2) dt = 16 \text{ m}$$

BEGINNER'S BOX-1

Displacement, Velocity and Acceleration

1. A car starts from P and follows the path as shown in figure. Finally car stops at R. Distance travelled and displacement of the car if a = 7 m, b = 8 m and $r = \frac{11}{\pi}$ m? [Take $\pi = \frac{22}{7}$]



(A) 48m, 36m (B) 48m, 42m (C) 48m, 32m (D) 48m, 40m

2. A body covers first $\frac{1}{3}$ part of its journey with a velocity of 2 m/s, next $\frac{1}{3}$ part with a velocity of 3 m/s and rest of the journey with a velocity 6m/s. The average velocity of the body will be

(A) 3 m/s (B) $\frac{11}{3}$ m/s (C) $\frac{8}{3}$ m/s (D) $\frac{4}{3}$ m/s

3. A particle moves in straight line in same direction for 20 seconds with velocity 3 m/s and then moves with velocity 4 m/s for another 20 sec and finally moves with velocity 5 m/s for next 20 seconds. What is the average velocity of the particle?
(A) 3 m/s (B) 4 m/s (C) 5 m/s (D) Zero

4. An athlete completes one round of a circular track of radius R in 40 sec. What will be his displacement at the end of 2 min. 20 sec
(A) Zero (B)2R (C) $2\pi R$ (D) $7\pi R$

5. The displacement x of a particle along a straight line at time t is given by $x = a_0 + a_1t + a_2t^2$. The acceleration of the particle is
(A) a_0 (B) a_1 (C) $2a_2$ (D) a_2

6*. The position of a particle moving in the xy-plane at any time t is given by $x = (3t^2 - 6t)$ metres, $y = (t^2 - 2t)$ metres. Select the correct statement about the moving particle from the following
(A) The acceleration of the particle is zero at t = 0 second
(B) The velocity of the particle is zero at t = 0 second
(C) The velocity of the particle is zero at t = 1 second
(D) The velocity and acceleration of the particle are never zero

7. The acceleration 'a' in m/s^2 of a particle is given by $a = 3t^2 + 2t + 2$ where t is the time. If the particle starts out with a velocity u = 2 m/s at t = 0, then the velocity at the end of 2 second is
(A) 12 m/s (B)18 m/s (C) 27 m/s (D) 36 m/s

8. The displacement of a particle starting from rest (at t = 0) is given by $s = 6t^2 - t^3$. The time in seconds at which the particle will attain zero velocity again, is
(A) 2 (B)4 (C) 6 (D) 8

9*. The relation $3t = \sqrt{3x} + 6$ describes the position of a particle in one direction where x is in metres and t in sec. The displacement, when velocity is zero, is
(A) 24 metres (B) 12 metres (C) 5 metres (D) Zero

3.0 RECTILINEAR MOTION

SL AL

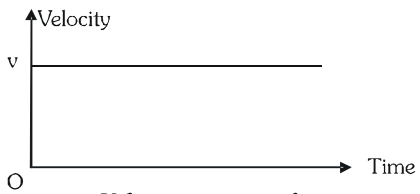
Curvilinear motion can be conceived as superposition of three rectilinear motions each along one of the Cartesian axes. Therefore, we first study rectilinear motion in detail.
We can classify rectilinear motion problems in following categories according to given information.

- Rectilinear Motion
  - Uniform Velocity Motion
  - Accelerated Motion
    - Uniform Acceleration Motion
    - Variable Acceleration Motion
      - I. Acceleration as function of time.
      - II. Acceleration as function of position.
      - III. Acceleration as function of velocity.

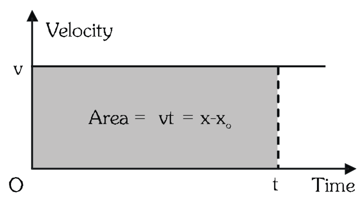
3.1 Uniform Velocity Motion

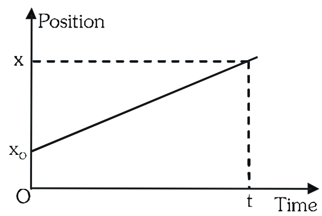
SL AL

In uniform velocity motion, a body moves with constant speed on a straight–line path without change in direction.
If a body starting from position $x = x_o$ at the instant $t = 0$, moves with uniform velocity v in the positive x–direction, its equation of motion at any time t is $x = x_o + vt$
Velocity–time (v–t) graph for this motion is shown in the following figure.
As we know that, the area between v–t graph and the time axes equals to change in position i.e displacement, the position–time relationship or position at any instant can be obtained.



Velocity-time graph



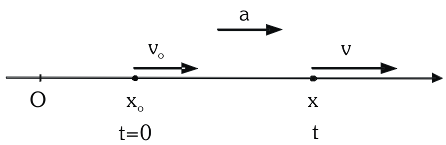


Position-time graph

3.2 Uniform Acceleration Motion

SL AL

Motion in which acceleration remains constant in magnitude as well as direction is called uniform acceleration motion. In the motion diagram, is shown a particle moving in positive x–direction with uniform acceleration a. It passes the position x_o, moving with velocity v_o at the instant $t = 0$ and acquires velocity v at a latter instant t.



$$dv = adt \Rightarrow \int_{v_o}^{v} dv = a\int_{0}^{t} dt \Rightarrow v - v_o = at$$

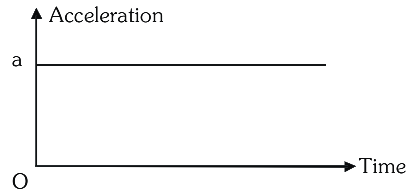
$$v = v_o + at \qquad \text{...(i)}$$

Now from the above equation, we have $dx = vdt \Rightarrow \int_{x_o}^{x} dx = \int_{0}^{t} (v_o + at)dt \Rightarrow x - x_o = v_o t + \frac{1}{2}at^2$

$$x = x_0 + v_0 t + \frac{1}{2}at^2 \qquad \text{...(ii)}$$

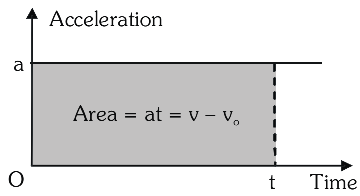
Eliminating time t, from the above two equations, we have $v^2 = v_o^2 + 2a(x - x_o)$ ...(iii)

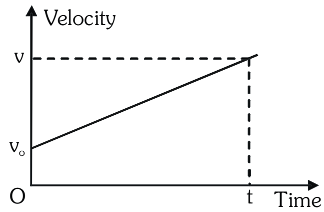
Equations (i), (ii) and (iii) are known as the first, second and third equations of motion for uniformly accelerated bodies. Acceleration–time (a–t) graph for this motion is shown in the following figure.



Acceleration-time graph

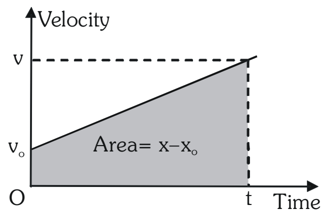
As we know that, the area between a–t graph and the time axes equals to change in velocity, velocity–time relation or velocity at any instant can be obtained.



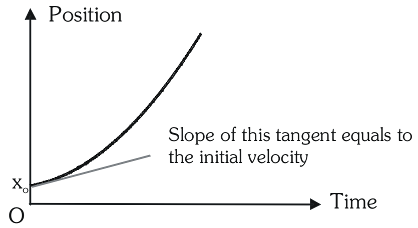


Velocity-time graph

The area between v–t graph and the time axes equals to change in position. Therefore, position–time relation or position at any instant can be obtained.



$$\text{Area} = x - x_o = \left(\frac{v_o + v}{2}\right)t$$



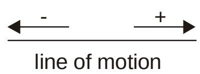
Position-time graph

3.3 Directions of Vectors in Straight Line Motion

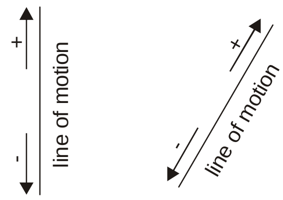
SL AL

In straight line motion, all the vectors (position, displacement, velocity & acceleration) will have only one component (along the line of motion) and there will be only two possible directions for each vector.

For example, if a particle is moving in a horizontal line (x–axis), the two directions are right and left. Any vector directed towards right can be represented by a positive number and towards left can be represented by a negative number.



For vertical or inclined motion, upward direction can be taken +ve and downward as -ve



For objects moving vertically near the surface of the earth, the only force acting on the particle is its weight (mg) i.e. the gravitational pull of the earth. Hence acceleration for this type of motion will always be $a = -g$ i.e. $a = -9.8 \text{ m/s}^2$ (–ve sign, because the force and acceleration are directed downwards, If we select upward direction as positive).

NOTE :

(a) If acceleration is in same direction as velocity, then speed of the particle increases.

(b) If acceleration is in opposite direction to the velocity then speed decreases i.e. the particle slows down. This situation is known as *retardation*.

Illustrations

Illustration 6. A particle moving with uniform acceleration passes the point x = 2 m with velocity 20 m/s at the instant t = 0. Some time latter it is observed at the point x = 32 m moving with velocity 10 m/s.

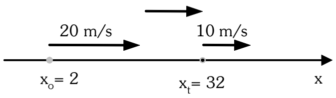
(a) What is its acceleration?

(b) Find its position and velocity at the instant t = 8 s.

(c) What is the distance travelled during the interval t = 0 to 8 s?

Solution In the adjoining figure the given and required information shown are not to a scale. As motion diagram is a schematic representation only.

(a) Using the third equation of uniform acceleration motion, we have



$$v_t^2 = v_o^2 + 2a(x_t - x_o) \rightarrow a = \frac{v_t^2 - v_o^2}{2(x_t - x_o)} = \frac{10^2 - 20^2}{2(32-2)} = -5 \text{ m/s}^2$$

(b) Using second equation of uniform acceleration motion, we have

$$x_t = x_o + v_o t + \tfrac{1}{2}at^2 \rightarrow x_8 = 2 + 20 \times 8 + \tfrac{1}{2}(-5)8^2 = 2 \text{ m}$$

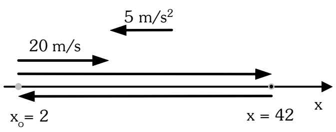
Using the first equation of uniform acceleration motion, we have

$$v_t = v_o + at \rightarrow v_8 = 20 + (-5) \times 8 = -20 \text{ m/s}$$

(c) Where the particle returns, its velocity must be zero. Using the third equation of uniform acceleration motion, we have

$$v^2 = v_o^2 + 2a(x - x_o) \rightarrow x = x_o + \frac{v^2 - v_o^2}{2a} = 2 + \frac{0 - 20^2}{2(-5)} = 42 \text{ m}$$

This location is shown in the adjoining modified motion diagram.



The distance-travelled Δs is $\quad \Delta s = |x - x_o| + |x_o - x| = 80 \text{ m}$

Illustration 7*. A ball is dropped from the top of a building. The ball takes 0.50 s to fall past the 3 m length of a window, which is some distance below the top of the building.

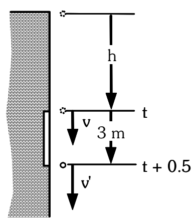
(a) How fast was the ball going as it passed the top of the window?

(b) How far is the top of the window from the point at which the ball was dropped?

Assume acceleration g in free fall due to gravity be 10 m/s^2 downwards.

Solution The ball is dropped, so it start falling from the top of the building with zero initial velocity (v_o = 0). The motion diagram is shown with the given information in the adjoining figure.

Using the first equation of the constant acceleration motion, we have



$$v_t = v_o + at \rightarrow \quad v = 0 + 10t = 10t \qquad \text{...(i)}$$

$$v' = 0 + 10(t + 0.5) = 10t + 5 \qquad \text{...(ii)}$$

Using values of v and v' in following equation, we have

$$x - x_o = \left(\frac{v_o + v}{2}\right)t \rightarrow \text{window height} \left(\frac{v + v'}{2}\right) \times 0.5 \Rightarrow t = 0.35\text{s}$$

(a) From equation (i), we have $v = 10t = 3.5$ m/s

(b) From following equation, we have

$$x - x_o = \left(\frac{v_o + v}{2}\right)t \rightarrow h = \left(\frac{0 + v}{2}\right)t = 61.25 \text{ cm}$$

BEGINNER'S BOX-2

Uniform Acceleration Motion

1*. A stone is dropped into a well in which the level of water is h below the top of the well. If v is velocity of sound, the time T after which the splash is heard is given by :

(A) T = 2h/v (B) $T=\sqrt{\frac{2h}{g}}+\frac{h}{v}$ (C) $T=\sqrt{\frac{2h}{g}}+\frac{h}{2v}$ (D) $T=\sqrt{\frac{h}{2g}}+\frac{2h}{v}$

2. A stone is thrown vertically upward with an initial velocity u from the top of a tower, reaches the ground with a velocity 3u. The height of the tower is :

(A) $\frac{3u^2}{g}$ (B) $\frac{4u^2}{g}$ (C) $\frac{6u^2}{g}$ (D) $\frac{9u^2}{g}$

3*. A particle moving with a constant acceleration from *A* to *B* in the straight line *AB* has velocities *u* and *v* at *A* and *B* respectively. If *C* is the mid-point of *AB* then the velocity of particle while passing *C* will be :

(A) $\sqrt{\frac{v^2+u^2}{2}}$ (B) $\frac{v+u}{2}$ (C) $\frac{v-u}{2}$ (D) $\frac{\left(\frac{1}{v}+\frac{1}{u}\right)}{2}$

4. A ball thrown up in vacuum returns after 12 sec. Its position after five seconds will be same as after :
(A) 7 sec (B) 3 sec (C) 4 sec (D) 3.5 sec

5. A car travelling at 72 km/h decelerates uniformly at 2 m/s^2. Calculate (a) the distance it goes before it stops, (b) the time it takes to stop, and (c) the distance it travels during the first and third seconds.

6. A ball is dropped from a tower. In the last second of its motion it travels a distance of 15m. Find the height of the tower. [take g = 10 m/sec^2]

4.0 VARIABLE ACCELERATION MOTION

SL AL

More often, problems in rectilinear motion involve acceleration that is not constant. In these cases acceleration is expressed as a function of one or more of the variables t, x and v. Let us consider three common cases.

4.1 Acceleration given as function of time

SL AL

If acceleration is a given function of time say a = f(t), from equation a = dv/dt we have

$$dv = f(t)dt \Rightarrow \int dv = \int f(t)dt$$

The above equation expresses v as function of time, say v = g(t). Now substituting g(t) for v in equation v = dx/dt, we have

$$dx = g(t)\,dt \Rightarrow \int dx = \int g(t)dt$$

The above equation yield position as function of time.

Illustrations

Illustration 8. The acceleration of a particle moving along the x-direction is given by equation a = (3–2t) m/s^2.
At the instants :
t = 0 and t = 6 s, it occupies the same position.
(a) Find the initial velocity v_o.
(b) What will be the velocity at t = 2 s?

Solution By substituting the given equation in equation $a = dv/dt$, we have

$$dv = (3-2t)dt \Rightarrow \int_{v_o}^{v} dv = \int_0^t (3-2t)dt \Rightarrow v = v_o + 3t - t^2 \qquad \ldots(i)$$

By substituting eq. (i) in equation v = dx/dt , we have

$$dx = (v_o + 3t - t^2)dt \Rightarrow \int_{x_o}^{x} dx = \int_0^t (v_o + 3t - t^2)dt \Rightarrow x = x_o + v_o t + \tfrac{3}{2}t^2 - \tfrac{1}{3}t^3 \quad \ldots(ii)$$

(a) Applying the given condition that the particle occupies the same x coordinate at the instants t = 0 and t = 6 s in eq. (ii), we have

$$x_o = x_6 \Rightarrow x_o = x_o + 6v_o + 54 - 72 \Rightarrow v_o = 3 \text{ m/s}$$

(b) Using v_o in eq. (i), we have $v = 3 + 3t - t^2 \Rightarrow v_2 = 5$ m/s

4.2 Acceleration as function of position

SL AL

If acceleration is a given function of position say a = f(x), we have to use equation a = vdv/dx. Rearranging term in this equation we have vdv = adx. Now substituting f(x) for a, we have

$$vdv = f(x)dx \Rightarrow \int vdv = \int f(x)dx$$

The above equation provides us with velocity as function of position. Let relation obtained in this way is v = g(x). Now substituting g(x) for v in equation v = dx/dt, we have

$$dt = \frac{dx}{g(x)} \Rightarrow \int dt = \int \frac{dx}{g(x)}$$

The above equation yields the desired relation between x and t.

Illustrations

Illustration 9*. Acceleration of a particle moving along the x-axis is defined by the law $a = -4x$, where a is in m/s^2 and x is in meters. At the instant t = 0, the particle passes the origin with a velocity of 2 m/s moving in the positive x-direction.

(a) Find its velocity v as function of its position coordinates.

(b) Find its position x as function of time t.

(c) Find the maximum distance it can go away from the origin.

Solution (a) By substituting given expression in the equation a = v dv/dx and rearranging, we have

$$vdv = -4xdx \Rightarrow \int_2^v vdv = -4\int_0^x xdx \Rightarrow v = \pm 2\sqrt{1-x^2} \rightarrow v = 2\sqrt{1-x^2}$$

Since the particle passes the origin with positive velocity of 2 m/s, so the minus sign in the eq. (i) has been dropped.

(b) By substituting above obtained expression of velocity in the equation v = dx/dt and rearranging, we have

$$\frac{dx}{\sqrt{1-x^2}} = 2dt \Rightarrow \int_0^x \frac{dx}{\sqrt{1-x^2}} = 2\int_0^t dt \Rightarrow \sin^{-1}(x) = 2t \rightarrow x = \sin 2t$$

(c) The maximum distance it can go away from the origin is 1m because maximum magnitude of sine function is unity.

4.3 Acceleration as function of velocity

SL AL

If acceleration is given as function of velocity say a=f(v), by using equation a = dv/dt we can obtain velocity as function of time.

$$dt = \frac{dv}{f(v)} \Rightarrow \int dt = \int \frac{dv}{f(v)}$$

Now using equation v = dx/dt we can obtain position as function of time

In another way if we use equation a = vdv/dx, we obtain velocity as function of position.

$$dx = \frac{vdv}{f(v)} \Rightarrow \int dx = \int \frac{vdv}{f(v)}$$

Now using equation v = dx/dt we can obtain position as function of time

Illustrations

Illustration 10*. Acceleration of particle moving along the x-axis varies according to the law a = –2v , where a is in m/s^2 and v is in m/s. At the instant t = 0, the particle passes the origin with a velocity of 2 m/s moving in the positive x-direction.

(a) Find its velocity v as function of time t.
(b) Find its position x as function of time t.
(c) Find its velocity v as function of its position coordinates.
(d) Find the maximum distance it can go away from the origin.
(e) Will it reach the above-mentioned maximum distance?

Solution

(a) By substituting the given relation in equation $a = dv/dt$, we have

$$\frac{dv}{v} = -2dt \Rightarrow \int_2^v \frac{dv}{v} = -2\int_0^t dt \rightarrow v = 2e^{-2t} \qquad ...(i)$$

(b) By substituting the above equation in v = dx/dt, we have

$$dx = 2e^{-2t}dt \Rightarrow \int_0^x dx = 2\int_0^t e^{-2t}dt \rightarrow x = 1 - e^{-2t} \qquad ...(ii)$$

(c) Substituting given expression a in the equation $a = v\,dv/dx$ and rearranging, we have

$$dv = -2dx \Rightarrow \int_2^v dv = -2\int_0^x dx \rightarrow v = 2(1-x) \qquad ...(iii)$$

(d) Eq. (iii) suggests that it will stop at x = 1 m. Therefore, the maximum distance away from the origin it can go is 1 m.

(e) Eq. (ii) suggests that to cover 1 m it will take time whose value tends to infinity. Therefore, it can never cover this distance.

BEGINNER'S BOX-3

Motion with variable acceleration

1. The acceleration of a particle is increasing linearly with time t as bt . The particle starts from the origin with an initial velocity v_0 The distance travelled by the particle in time t will be

(A) $v_0t + \frac{1}{3}bt^2$ (B) $v_0t + \frac{1}{3}bt^3$ (C) $v_0t + \frac{1}{6}bt^3$ (d) $v_0t + \frac{1}{2}bt^3$

2. The acceleration of a moving body can be found from

(A) Area under velocity-time graph (B) Area under distance-time graph
(C) Slope of the velocity-time graph (D) Slope of distance-time graph

3. The velocity of a body depends on time according to the equation v = 20 + 0.1t^2. The body is undergoing

(A) Uniform acceleration (B) Uniform retardation
(C) Non-uniform acceleration (D) Zero acceleration

4. The displacement of a particle is given by $y = a + bt + ct^2 - dt^4$. The initial velocity and acceleration are respectively

(A) $b, -4d$ (B) $-b, 2c$ (C) $b, 2c$ (D) $2c, -4d$

5. A particle moves along a straight line such that its displacement at any time t is given by

$S = t^3 - 6t^2 + 3t + 4$ metres

The velocity when the acceleration is zero is

(A) $3\ ms^{-1}$ (B) $-12\ ms^{-1}$ (C) $42\ ms^{-1}$ (D) $-9\ ms^{-1}$

6. The position x of a particle varies with time t as $x = at^2 - bt^3$. The acceleration of the particle will be zero at time t equal to

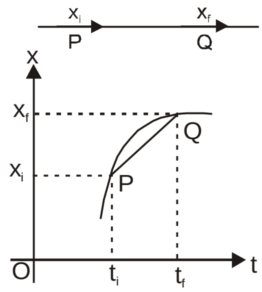
(A) $\frac{a}{b}$ (B) $\frac{2a}{3b}$ (C) $\frac{a}{3b}$ (D) Zero

5.0 GRAPHICAL ANALYSIS OF SOME QUANTITIES

SL AL

5.1 Average Velocity

SL AL

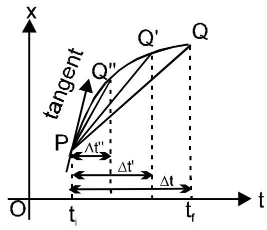


If a particle passes a point P (x_i) at time $t = t_i$ and reaches Q (x_f) at a later time instant $t = t_f$, its average velocity in the interval PQ is $V_{av} = \frac{\Delta x}{\Delta t} = \frac{x_f - x_i}{t_f - t_i}$

This expression suggests that the average velocity is equal to the slope of the line (chord) joining the points corresponding to P and Q on the x–t graph.

5.2 Instantaneous Velocity

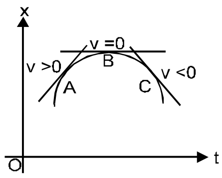
SL AL



Consider the motion of the particle between the two points P and Q on the x–t graph shown. As the point Q is brought closer and closer to the point P, the time interval between PQ (Δt, $\Delta t'$, $\Delta t''$,......) get progressively smaller. The average velocity for each time interval is the slope of the appropriate dotted line (PQ, PQ′, PQ″......). As the point Q approaches P, the time interval approaches zero, but at the same time the slope of the dotted line approaches that of the tangent to the curve at the point P.

As $\Delta t \to 0$, $V_{av}\ (=\Delta x/\Delta t) \to V_{inst.}$

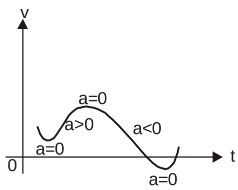
Geometrically, as $\Delta t \to 0$, chord PQ → tangent at P.



Hence the instantaneous velocity at P is the slope of the tangent at P in the x – t graph. When the slope of the x – t graph is positive, v is positive (as at the point A in figure). At C, v is negative because the tangent has negative slope. The instantaneous velocity at point B (turning point) is zero as the slope is zero.

5.3 Instantaneous Acceleration

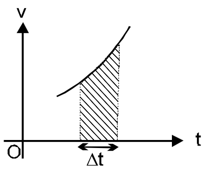
SL AL



The derivative of velocity with respect to time is the slope of the tangent in velocity time (v–t) graph.

5.4 Displacement from v - t graph [CBSE + JEE]

SL AL

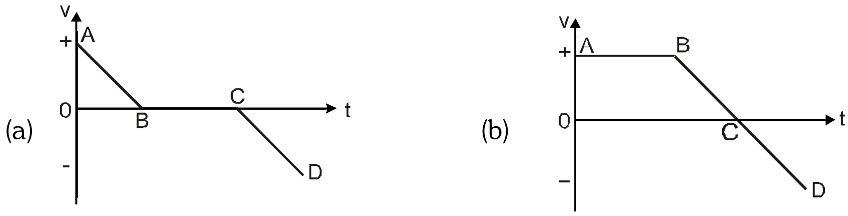


Displacement = Δx = area under v–t graph.

Since a negative velocity causes a negative displacement, areas below the time axis are taken negative. In similar way, we can see that $\Delta v = a\,\Delta t$ leads to the conclusion that ***area under a – t graph gives the change in velocity Δv during that interval.***

Illustrations

Illustration 11. Describe the motion shown by the following velocity-time graphs.



Solution

(a) ***During interval AB:*** velocity is +ve so the particle is moving in +ve direction, but it is slowing down as acceleration (slope of v-t curve) is negative. ***During interval BC:*** particle remains at rest as velocity is zero. Acceleration is also zero. ***During interval CD:*** velocity is -ve so the particle is moving in -ve direction and is speeding up as acceleration is also negative.

(b) ***During interval AB:*** particle is moving in +ve direction with constant velocity and acceleration is zero. ***During interval BC:*** particle is moving in +ve direction as velocity is +ve, but it slows down until it comes to rest as acceleration is negative. ***During interval CD:*** velocity is -ve so the particle is moving in -ve direction and is speeding up as acceleration is also negative.

6.0 ANALYSIS OF SOME MORE GRAPHS

SL AL

6.1 Position vs Time graph

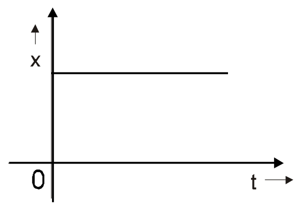
SL AL

Zero Velocity

As position of particle is fix at all the time, so the body is at rest.

Slope; $\frac{dv}{dt} = \tan\theta = \tan 0^\circ = 0$

Velocity of particle is zero

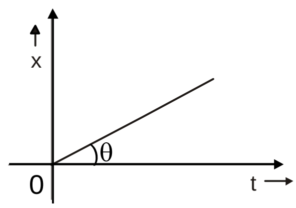


Uniform Velocity

Here $\tan\theta$ is constant $\qquad \tan\theta = \frac{dv}{dt}$

$\therefore \quad \frac{dv}{dt}$ is constant.

$\therefore \quad$ velocity of particle is constant.



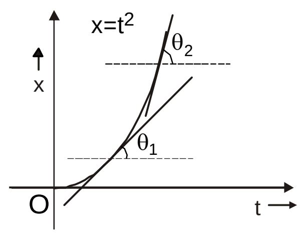
Non uniform velocity (increasing with time)

In this case;

As time is increasing, θ is also increasing.

$\therefore \quad \frac{dv}{dt} = \tan\theta$ is also increasing

Hence, velocity of particle is increasing.



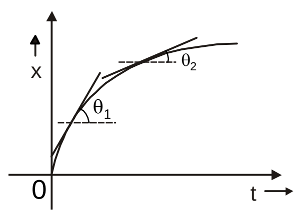
Non uniform velocity (decreasing with time)

In this case;

As time increases, θ decreases.

$\therefore \quad \frac{dv}{dt} = \tan\theta$ also decreases.

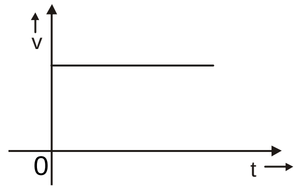
Hence, velocity of particle is decreasing.



6.2 Velocity vs time graph

SL AL

Zero acceleration



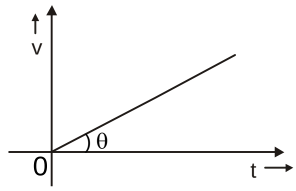
Velocity is constant.

$\tan\theta = 0$

$\therefore \quad \dfrac{dv}{dt} = 0$

Hence, acceleration is zero.

Uniform acceleration

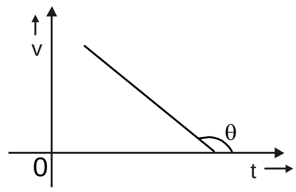


$\tan\theta$ is constant.

$\therefore \quad \dfrac{dv}{dt} = \text{constant}$

Hence, it shows constant acceleration.

Uniform retardation



Since $\theta > 90^\circ$

$\therefore \quad \tan\theta$ is constant and negative.

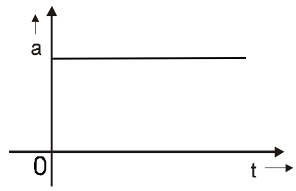
$\therefore \quad \dfrac{dv}{dt} = \text{negative constant}$

Hence, it shows constant retardation.

6.3 Acceleration vs time graph

SL AL

Constant acceleration

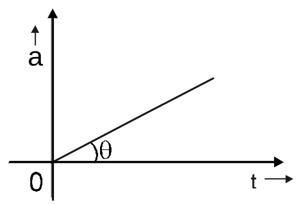


$\tan\theta = 0$

$\therefore \quad \dfrac{dv}{dt} = 0$

Hence, acceleration is constant.

Uniformly increasing acceleration



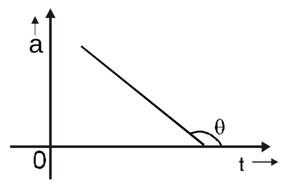
θ is constant.

$0^\circ < \theta < 90^\circ \Rightarrow \tan\theta > 0$

$\therefore \quad \dfrac{da}{dt} = \tan\theta = \text{constant} > 0$

Hence, acceleration is uniformly increasing with time.

Uniformly decreasing acceleration



Since $\theta > 90^\circ$

$\therefore \quad \tan\theta$ is constant and negative.

$\therefore \quad \dfrac{da}{dt} = \text{negative constant}$

Hence, acceleration is uniformly decreasing with time

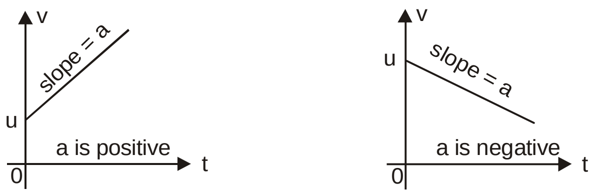
7.0 GRAPHS IN UNIFORMLY ACCELERATED MOTION

SL AL

x is a quadratic polynomial in terms of t. Hence x – t graph is a parabola.

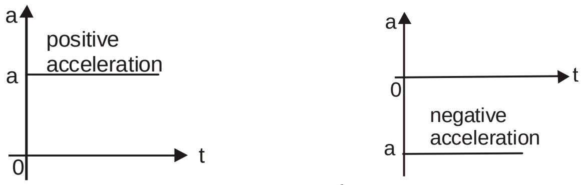
x-t graph

v is a linear polynomial in terms of t. Hence v–t graph is a straight line of slope a.



v-t graph

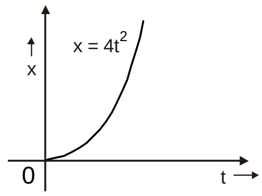
a–t graph is a horizontal line because a is constant.



a-t graph

Illustrations

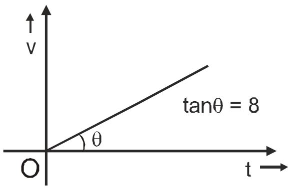
Illustration 12*. The displacement vs time graph of a particle moving along a straight line is shown in the figure. Draw velocity vs time and acceleration vs time graph.

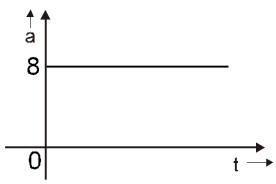


Solution $x = 4t^2 \quad \Rightarrow \quad v = \dfrac{dx}{dt} = 8t$

Hence, velocity-time graph is a straight line having slope i.e. $\tan\theta = 8$.

$a = \dfrac{dv}{dt} = 8$. Hence, acceleration is constant throughout and is equal to 8.

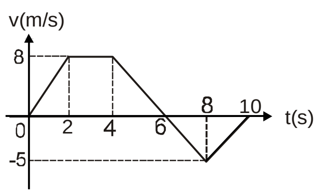




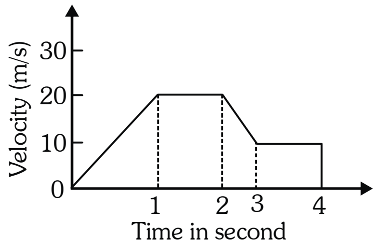
BEGINNER'S BOX–4

Graphical Analysis

1. For a particle moving along x-axis, velocity-time graph is as shown in figure. Find the distance travelled and displacement of the particle?



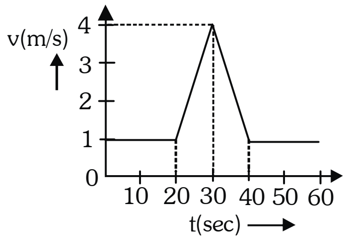
2. The variation of velocity of a particle with time moving along a straight line is illustrated in the following figure. The distance travelled by the particle in four seconds is



(A) 60 m (B) 55 m

(C) 25 m (D) 30 m

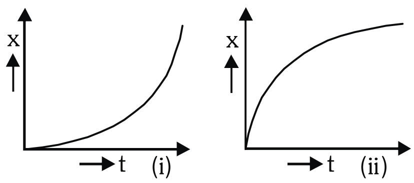
3*. Velocity-time (v-t) graph for a moving object is shown in the figure. Total displacement of the object during the time interval when there is non-zero acceleration and retardation is



(A) 60 m (B) 50 m

(C) 30 m (D) 40 m

4*. Figures (i) and (ii) below show the displacement-time graphs of two particles moving along the x-axis (curves are parabolic).We can say that



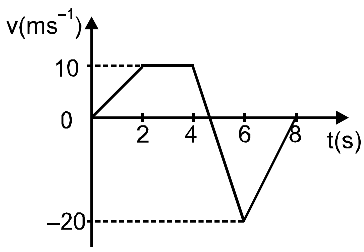
(A) Both the particles are having a uniformly acceleration

(B) Both the particles are having a non uniformly acceleration

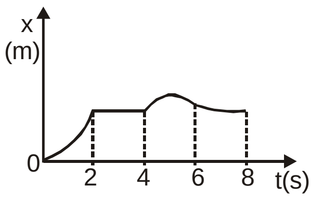
(C) Particle (i) is speeding up.

(D) Particle (ii) is slowing down.

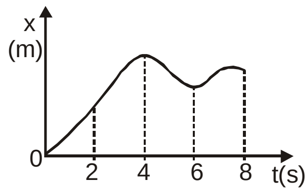
5*. The figure shows a velocity-time graph of a particle moving along a straight line, The correct displacement-time graph of the particle is shown as :



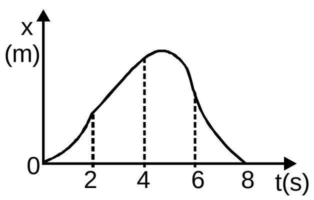
(A)



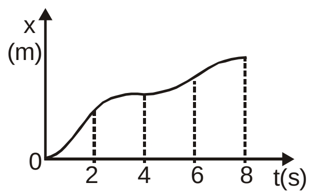
(B)



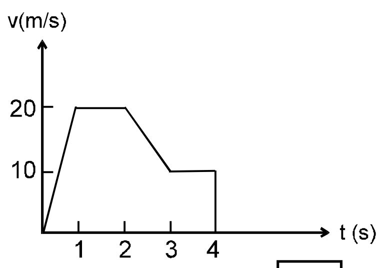
(C)



(D)



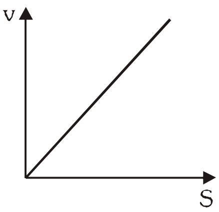
6. The variation of velocity of a particle moving along a straight line is shown in the figure. The distance travelled by the particle in 4 s is :

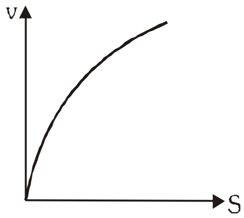


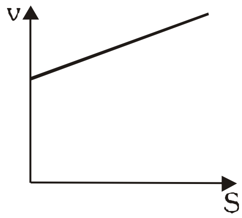
(A) 25 m (B) 30 m

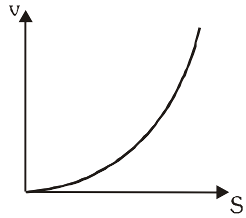
(C) 55 m (D) 60 m

7*. A particle starts from rest and moves along a straight line with constant acceleration. The variation of velocity v with displacement S is :

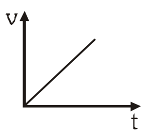
(A) 

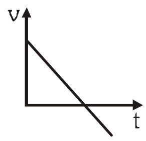
(B) 

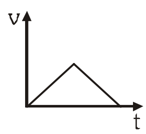
(C) 

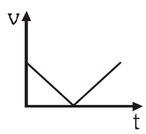
(D) 

8. Which of the following graph correctly represents velocity-time relationship for a particle released from rest to fall freely under gravity?

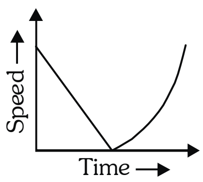
(A) 

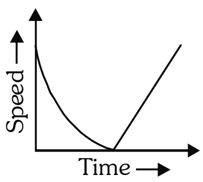
(B) 

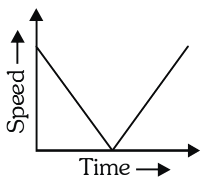
(C) 

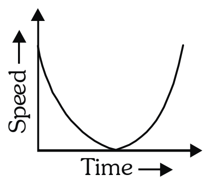
(D) 

9. A ball is thrown vertically upwards. Which of the following plots represents the speed-time graph of the ball during its height if the air resistance is not ignored. (Air resistance force is uniform and acts opposite to velocity).

(A) 

(B) 

(C) 

(D) 

GOLDEN KEY POINTS

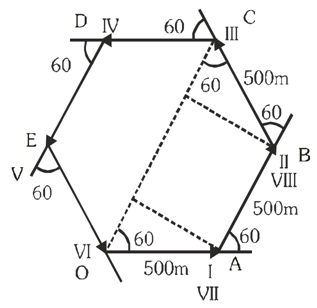
- ***Reaction Time :*** When a situation demands our immediate action. It takes some time before we really respond. Reaction time is the time a person takes to observe, think and act.
- If acceleration is in opposite direction to the velocity then speed decreases i.e. the particle slows down. This situation is known as *retardation*.
- For uniformly accelerated motion ($a \neq 0$), x–t graph is a parabola (opening upwards if $a > 0$ and opening downwards if $a < 0$). The slope of tangent at any point of the parabola gives the velocity at that instant.
- For uniformly accelerated motion ($a \neq 0$), v–t graph is a straight line whose slope gives the acceleration of the particle.
- In general, the slope of tangent in x–t graph is velocity and the slope of tangent in v–t graph is the acceleration.
- The area under a–t graph gives the change in velocity.
- The area between the v–t graph gives the distance travelled by the particle, if we take all areas as positive.
- Area under v–t graph gives displacement, if areas below the t–axis are taken negative.

SOME WORKED OUT ILLUSTRATIONS

Illustration 1.

On an open ground a motorist follows a track that turns to his left by an angle of 60° after every 500 m. Starting from a given turn, specify the displacement of the motorist at the third, sixth and eighth turn. Compare the magnitude of displacement with the total path length covered by the motorist in each case.

Solution



At III turn

Displacement $= \overrightarrow{OA} + \overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{OC} = 500 \cos 60° + 500 + 500 \cos 60°$

$= 500 \times \frac{1}{2} + 500 + 500 \times \frac{1}{2} = 1000$ m from O to C

Distance $= 500 + 500 + 500 = 1500$ m. So $\frac{\text{Displacement}}{\text{Distance}} = \frac{1000}{1500} = \frac{2}{3}$

At VI turn : $\because$ initial and final positions are same so displacement

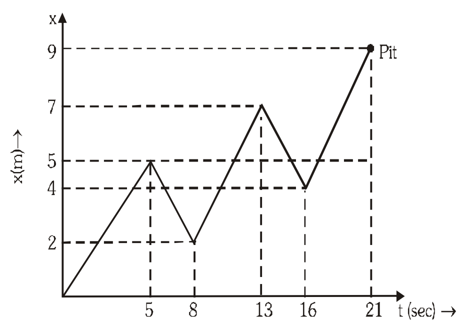
$= 0$ and distance $= 500 \times 6 = 3000$ m $\therefore \frac{\text{Displacement}}{\text{Distance}} = \frac{0}{3000} = 0$

At VIII turn : Displacement $= 2(500)\cos\left(\frac{60°}{2}\right) = 1000 \times \cos 30° = 1000 \times \frac{\sqrt{3}}{2} = 500\sqrt{3}$ m

Distance $= 500 \times 8 = 4000$ m $\quad \therefore \frac{\text{Displacement}}{\text{Distance}} = \frac{500\sqrt{3}}{4000} = \frac{\sqrt{3}}{8}$

Illustration 2*.

A drunkard walking in a narrow lane takes 5 steps forward and 3 steps backward, followed again by 5 steps forward and 3 steps backward, and so on. Each step is 1m long and requires 1s. Plot the x–t graph of his motion. Determine graphically or otherwise how long the drunkard takes to fall in a pit 9m away from the start.



Solution

from x–t graph time taken = 21 s

OR

(5m – 3m) + (5m – 3m) + 5m = 9m $\Rightarrow$ total steps = 21 $\Rightarrow$ time = 21 s

Illustration 3.

A man walks on a straight road from his home to a market 2.5 km away with a speed of 5 km/h. On reaching the market he instantly turns and walks back with a speed of 7.5 km/h. What is the

(a) magnitude of average velocity and

(b) average speed of the man, over the interval of time (i) 0 to 30 min. (ii) 0 to 50 min (iii) 0 to 40 min.

Solution

Time taken by man to go from his home to market, $t_1 = \dfrac{\text{distance}}{\text{speed}} = \dfrac{2.5}{5} = \dfrac{1}{2}$ h

Time taken by man to go from market to his home, $t_2 = \dfrac{2.5}{7.5} = \dfrac{1}{3}$ h

$\therefore$ Total time taken $= t_1 + t_2 = \dfrac{1}{2} + \dfrac{1}{3} = \dfrac{5}{6}$ h = 50 min.

(i) 0 to 30 min

Average velocity $= \dfrac{\text{displacement}}{\text{time interval}} = \dfrac{2.5}{\frac{30}{60}} = 5$ km/h towards market

Average speed $= \dfrac{\text{distance}}{\text{time interval}} = \dfrac{2.5}{\frac{30}{60}} = 5$ km/h

(ii) 0 to 50 min

Total displacement = zero so average velocity = 0

So, average speed $= \dfrac{5}{50/60} = 6$ km/h

Total distance travelled = 2.5 + 2.5 = 5 km.

(iii) 0 to 40 min

Distance covered in 30 min (from home to market) = 2.5 km.

Distance covered in 10 min (from market to home) with speed 7.5 km/h = $7.5 \times \dfrac{10}{60} = 1.25$ km

So, displacement = 2.5 – 1.25 = 1.25 km (towards market)

Distance travelled = 2.5 + 1.25 = 3.75 km

Average velocity $= \dfrac{1.25}{\frac{40}{60}} = 1.875$ km/h. (towards market)

Average speed $= \dfrac{3.75}{\frac{40}{60}} = 5.625$ km/h.

Note : Moving body with uniform speed may have variable velocity. e.g. in uniform circular motion speed is constant but velocity is non–uniform.

Illustration 4.

A driver takes 0.20 s to apply the brakes after he sees a need for it. This is called the reaction time of the driver. If he is driving a car at a speed of 54 km/h and the brakes cause a deceleration of $6.0 m/s^2$, find the distance travelled by the car after he sees the need to put the brakes on ?

Solution

Distance covered by the car during the application of brakes by driver

$$s_1 = ut = \left(54 \times \frac{5}{18}\right)(0.2) = 15 \times 0.2 = 3.0 \text{ m}$$

After applying the brakes; $v = 0$, $u = 15$ m/s, $a = 6$ m/s^2 $s_2 = ?$

Using $v^2 = u^2 - 2as \Rightarrow 0 = (15)^2 - 2 \times 6 \times s_2 \Rightarrow 12\, s_2 = 225 \Rightarrow s_2 = \frac{225}{12} = 18.75\text{m}$

Distance travelled by the car after driver sees the need for it $s = s_1 + s_2 = 3 + 18.75 = 21.75$ m.

Illustration 5*.

A passenger is standing d distance away from a bus. The bus begins to move with constant acceleration a. To catch the bus, the passenger runs at a constant speed u towards the bus. What must be the minimum speed of the passenger so that he may catch the bus?

Solution

Let the passenger catch the bus after time t.

The distance travelled by the bus, $s_1 = 0 + \frac{1}{2}at^2$ ....(i)

and the distance travelled by the passenger $s_2 = ut + 0$ ....(ii)

Now the passenger will catch the bus if $d + s_1 = s_2$ ....(iii)

$$\Rightarrow d + \frac{1}{2}at^2 = ut \Rightarrow \frac{1}{2}at^2 - ut + d = 0 \Rightarrow t = \frac{[u \pm \sqrt{u^2 - 2ad}]}{a}$$

So the passenger will catch the bus if t is real, i.e., $u^2 \geq 2\,ad \Rightarrow u \geq \sqrt{2\ ad}$

So the minimum speed of passenger for catching the bus is $\sqrt{2ad}$.

Illustration 6.

If a body travels half its total path in the last second of its fall from rest, find : (a) The time and (b) height of its fall. Explain the physically unacceptable solution of the quadratic time equation. ($g = 9.8$ m/s^2)

Solution

If the body falls a height h in time t, then

$$h = \frac{1}{2}gt^2 \text{ [} u = 0 \text{ as the body starts from rest]} \quad \text{... (i)}$$

Now, as the distance covered in $(t - 1)$ second is $h' = \frac{1}{2}g(t-1)^2$ ... (ii)

So from Equations (i) and (ii) distance travelled in the last second.

$$h - h' = \frac{1}{2}gt^2 - \frac{1}{2}g(t-1)^2 \text{ i.e., } h - h' = \frac{1}{2}g(2t-1)$$

But according to given problem as $(h - h') = \frac{h}{2}$

$$\Rightarrow \left(\frac{1}{2}\right)h = \left(\frac{1}{2}\right)g(2t-1) \text{ or } \left(\frac{1}{2}\right)gt^2 = g(2t-1) \text{ [as from equation (i) } h = \left(\frac{1}{2}\right)gt^2 \text{]}$$

$$\Rightarrow t^2 - 4t + 2 = 0 \text{ or } t = [4 \pm \sqrt{(4^2 - 4 \times 2)}]/2 \quad \Rightarrow t = 2 \pm \sqrt{2} \Rightarrow t = 0.59 \text{ s or } 3.41 \text{ s}$$

0.59 s is physically unacceptable as it gives the total time t taken by the body to reach ground lesser than one sec while according to the given problem time of motion must be greater than 1s.

so $t = 3.41s$ and $h = \frac{1}{2} \times (9.8) \times (3.41)^2 = 57$ m

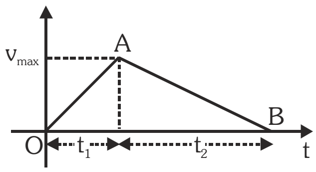
Illustration 7.

A car accelerates from rest at a constant rate α for some time, after which it decelerates at a constant rate β, to come to rest. If the total time elapsed is t evaluate (a) the maximum velocity attained and (b) the total distance travelled.

Solution

(a) Let the car accelerates for time t_1 and decelerates for time t_2 then $t = t_1 + t_2$....(i)

and corresponding velocity–time graph will be as shown in. fig.



From the graph α = slope of line OA $= \dfrac{v_{max}}{t_1} \Rightarrow t_1 = \dfrac{v_{max}}{\alpha}$

and $\beta = -$ slope of line AB $= \dfrac{v_{max}}{t_2} \Rightarrow t_2 = \dfrac{v_{max}}{\beta}$

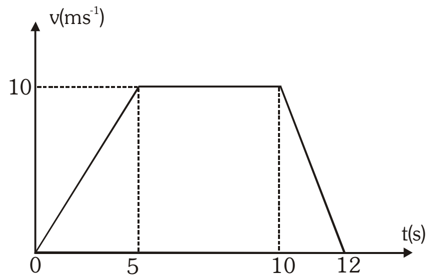
$$\Rightarrow \frac{v_{max}}{\alpha} + \frac{v_{max}}{\beta} = t \Rightarrow v_{max}\left(\frac{\alpha+\beta}{\alpha\beta}\right) = t \Rightarrow v_{max} = \frac{\alpha\beta t}{\alpha+\beta}$$

(b) Total distance = area under v–t graph $= \dfrac{1}{2} \times t \times v_{max} = \dfrac{1}{2} \times t \times \dfrac{\alpha\beta t}{\alpha+\beta} = \dfrac{1}{2}\left(\dfrac{\alpha\beta t^2}{\alpha+\beta}\right)$

Note: This problem can also be solved by using equations of motion ($v = u + at$, etc.).

Illustration 8*.

Draw displacement time and acceleration – time graph for the given velocity–time graph.



Solution

For $0 \le t \le 5$ $v \propto t$ $\Rightarrow s \propto t^2$ and a_1 = constant $\dfrac{10}{5} = 2$ ms^{-2}

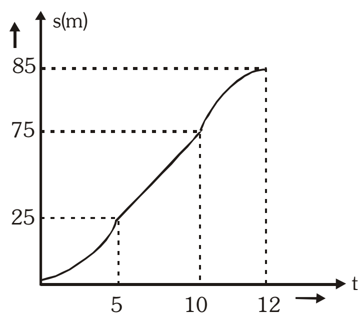
for whole interval s_1 = Area under the curve $= \dfrac{1}{2} \times 5 \times 10 = 25$ m

For $5 \le t \le 10$, $v = 10$ms$^{-1} \Rightarrow a = 0$

for whole interval s_2 = area under the curve $= 5 \times 10 = 50$ m

For $10 \le t \le 12$ v linearly decreases with time $\Rightarrow a_3 = -\dfrac{10}{2} = -5$ ms^{-2}

for whole interval s_3 = Area under the curve $= \dfrac{1}{2} \times 2 \times 10 = 10$ m



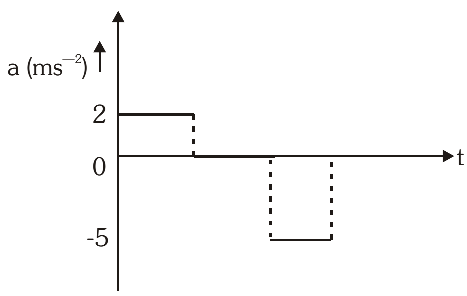
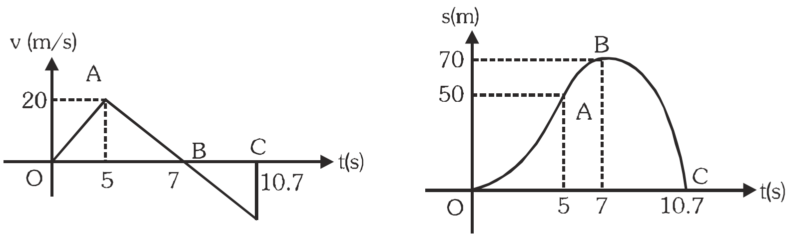


Illustration 9.

A rocket is fired upwards vertically with a net acceleration of 4 m/s^2 and initial velocity zero. After 5 seconds its fuel is finished and it decelerates with g. At the highest point its velocity becomes zero. Then it accelerates downwards with acceleration g and return back to ground. Plot velocity–time and displacement–time graphs for the complete journey. Take g = 10 m/s^2.

Solution



In the graphs, $v_A = at_{OA} = (4)(5) = 20$ m/s $\quad v_B = 0 = v_A - gt_{AB}$

$\therefore t_{AB} = \dfrac{v_A}{g} = \dfrac{20}{10} = 2s \qquad \therefore t_{OAB} = (5+2)s = 7s$

Now, s_{OAB} = area under v–t graph between 0 to 7 s = $\dfrac{1}{2}(7)(20) = 70$ m

Now, $s_{OAB} = s_{BC} = \dfrac{1}{2}gt^2_{BC} \qquad \therefore 70 = \dfrac{1}{2}(10)\,t^2_{BC}$

$\therefore t_{BC} = \sqrt{14} = 3.7$ s $\qquad \therefore t_{OABC} = 7+3.7 = 10.7s$

Also s_{OA} = area under v–t graph between OA = $\dfrac{1}{2}(5)(20) = 50$ m

Illustration 10*.

At the height of 500m, a particle A is thrown up with v = 75 ms^{-1} and particle B is released from rest. Draw, acceleration –time, velocity–time, speed–time and displacement–time graph of each particle.

Solution

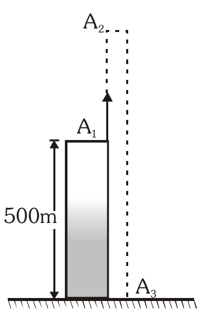
For particle A :

Time of flight

$-500 = +75\,t - \dfrac{1}{2} \times 10t^2$

$\Rightarrow t^2 - 15t - 100 = 0$

$\Rightarrow t = 20$ s



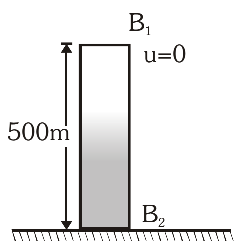
Time taken for A_1A_2

$v = 0 = 75 - 10t \Rightarrow t = 7.5$ s

Velocity at A_3, $v = 75 - 10 \times 20 = -125$ ms^{-1}

Height $A_2A_1 = 75 \times 7.5 - \dfrac{1}{2}(10)(7.5)^2 = 281.25$ m

For Particle B



Time of flight

$500 = \dfrac{1}{2}(10)t^2$

$\Rightarrow t = 10$ s

Velocity at B_2

$v = 0 - (10)(10) = -100$ ms^{-1}

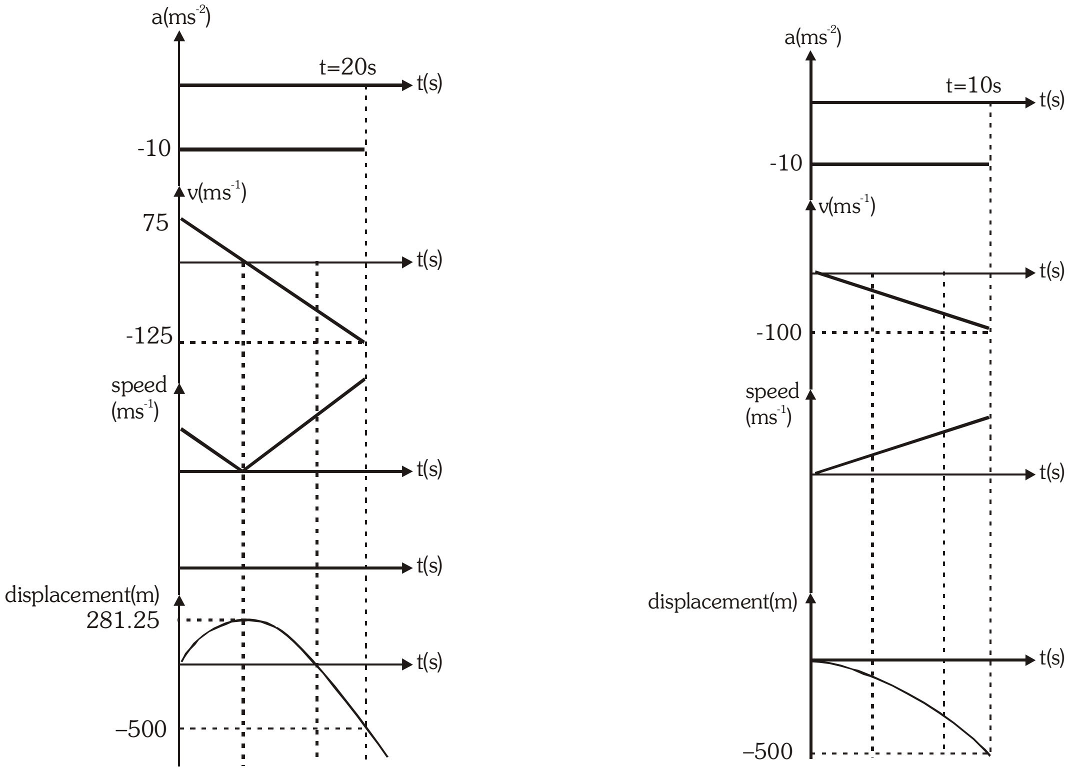


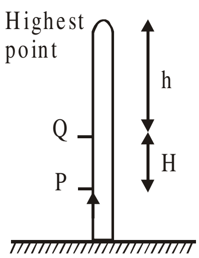
Illustration 11.

A particle is thrown vertically upwards from the surface of the earth. Let T_P be the time taken by the particle to travel from a point P above the earth to its highest point and back to the point P. Similarly, let T_Q be the time taken by the particle to travel from another point Q above the earth to its highest point and back to the same point Q. If the distance between the points P and Q is H, the expression for acceleration due to gravity in terms of T_P, T_Q and H, is :

(A) $\dfrac{6H}{T_P^2 + T_Q^2}$ (B) $\dfrac{8H}{T_P^2 - T_Q^2}$ (C) $\dfrac{2H}{T_P^2 + T_Q^2}$ (D) $\dfrac{H}{T_P^2 - T_Q^2}$

Ans. (B)

Solution



Time taken from point P to point P $\quad T_P = 2\sqrt{\dfrac{2(h+H)}{g}}$

Time taken from point Q to point Q $\quad T_Q = 2\sqrt{\dfrac{2h}{g}}$

$$\Rightarrow T_P^2 = \frac{8(h+H)}{g} \text{ and } T_Q^2 = \frac{8h}{g} \Rightarrow T_P^2 = T_Q^2 + \frac{8H}{g} \Rightarrow g = \frac{8H}{T_P^2 - T_Q^2}$$

Illustration 12.

Some informations are given for a body moving in a straight line. The body starts its motion at t=0.

Information I : The velocity of a body at the end of 4s is 16 m/s

Information II : The velocity of a body at the end of 12s is 48 m/s

Information III : The velocity of a body at the end of 22s is 88 m/s

The body is certainly moving with :

(A) Uniform velocity (B) Uniform speed

(C) Uniform acceleration (D) Data insufficient for generalization

Ans. (D)

Solution

Here average acceleration $= \dfrac{16-0}{4-0} = \dfrac{48-16}{12-4} = \dfrac{88-48}{22-12} = 4$

But we can't say certainly that body have uniform acceleration.

Illustration 13*.

A large number of particles are moving each with speed v having directions of motion randomly distributed. What is the average relative velocity between any two particles averaged over all the pairs?

(A) v (B) $(\pi/4)$v (C) $(4/\pi)$v (D) Zero

Ans. (C)

Solution

Relative velocity , $v_r = |\vec{v}_1 - \vec{v}_2|$ where $v_1 = v_2 = v$

If angle between them be θ, then $v_r = \sqrt{v^2 + v^2 - 2v^2\cos\theta} = \sqrt{2v^2(1-\cos\theta)} = 2v\sin\left(\dfrac{\theta}{2}\right)$

Hence, average relative velocity $\bar{v}_r = \dfrac{\int_0^{2\pi} 2v\sin\frac{\theta}{2}\,d\theta}{\int_0^{2\pi} d\theta} = \dfrac{4v}{\pi}$

ANSWERS

BEGINNER'S BOX-1

1. A	**2.** A	**3.** B	**4.** B	**5.** C	**6.** C	**7.** B	**8.** B	**9.** D

BEGINNER'S BOX-2

1. B　**2.** B　**3.** A　**4.** A

5. (a) 100 m; (b) 10 s; (c) 19 m, 15 m　**6.** 20 m

BEGINNER'S BOX-3

1. C　**2.** C　**3.** C　**4.** C　**5.** D

6. C

BEGINNER'S BOX-4

1. 42 m , 22 m　**2.** B　**3.** B　**4.** ACD

5. C　**6.** C　**7.** B　**8.** A　**9.** C

EXERCISE – 1 MCQ (SINGLE CHOICE CORRECT)

1. A particle is moving in x–y–plane at 2 m/s along x–axis, 2 seconds later, its velocity is 4 m/s in a direction making 60° with positive x–axis. Its average acceleration for this period of motion is:

(A) $\sqrt{5}$ m/s², along y–axis
(B) $\sqrt{3}$ m/s², along y–axis
(C) $\sqrt{5}$ m/s², along at 60° with positive x–axis
(D) 3m/s², at 60° with positive x–axis.

2. The coordinates of a moving particle at time t are given by $x = ct^2$ and $y = bt^2$. The speed of the particle is given by (c and b are constants) :

(A) $2t(c+b)$
(B) $2t\sqrt{c^2-b^2}$
(C) $t\sqrt{c^2+b^2}$
(D) $2t\sqrt{c^2+b^2}$

3. A balloon is going upwards with a velocity 12 m s^{-1}. It release a packet when it is at a height of 65 m from the ground. How much time the packet will take to reach the ground if g = 10 m s^{-2} ?

(A) 1 s
(B) 2 s
(C) 4 s
(D) 5 s

4. A particle initially at rest moves along x-axis. It is subjected to an acceleration which varies with time according to the equation : $a = 2t + 5$. Its velocity after 2 second will be

(A) 9 m s^{-1}
(B) 12 m s^{-1}
(C) 14 m s^{-1}
(D) 18 m s^{-1}

5. A, B, C and D are points in a vertical line such that AB = BC = CD. If a body falls from rest from A, then the times of descend through AB, BC and CD are in the ratio :

(A) $1 : \sqrt{2} : \sqrt{3}$
(B) $\sqrt{2} : \sqrt{3} : 1$
(C) $\sqrt{3} : 1 : \sqrt{2}$
(D) $1 : (\sqrt{2}-1) : (\sqrt{3}-\sqrt{2})$

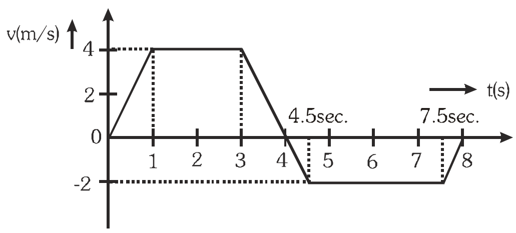
6. A particle is projected vertically upward and it reaches the maximum height H in T seconds. The height of the particle at any time t (t > T) will be :

(A) $H - g(t-T)^2$
(B) $g(t-T)^2$
(C) $H - \frac{1}{2}g(t-T)^2$
(D) $\frac{g}{2}(t-T)^2$

7*. With what speed should a body be thrown upwards so that the distances traversed in 5th second and 6th second are equal ?

(A) 58.4 m/s
(B) 49 m/s
(C) $\sqrt{98}$ m/s
(D) 98 m/s

8. A particle is projected vertically upwards from a point A on the ground. It takes t_1 time to reach a point B but it still continues to move up. If it takes further t_2 time to reach the ground from point B then height of point B from the ground is :

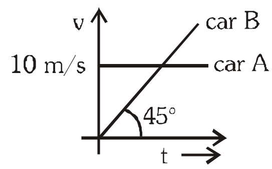


(A) $\frac{1}{2}g(t_1+t_2)^2$
(B) gt_1t_2
(C) $\frac{1}{8}g(t_1+t_2)^2$
(D) $\frac{1}{2}gt_1t_2$

9. The velocity – time graph of a linear motion is shown in figure. The displacement & distance after 8 sec. is :

(A) 5 m, 19m (B) 16 m, 22m (C) 8 m,19m (D) 6 m, 5m

10*. Initially car A is 10.5 m ahead of car B. Both start moving at time t = 0 in the same direction along a straight line. The velocity time graph of two cars is shown in figure. The time when the car B will catch the car A, will be :

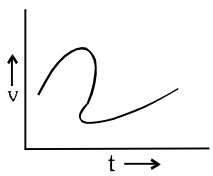


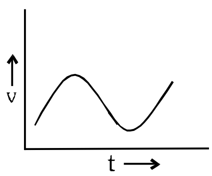
(A) t = 21 sec (B) t = $2\sqrt{5}$ sec (C) 20 sec (D) None of these

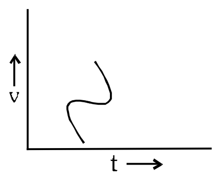
11. The displacement x of a body varies with time *t* as $x = \frac{2}{3}t^2 - 16t + 2$ In what time that body comes to rest ? (x is measured in metre and *t* in second).

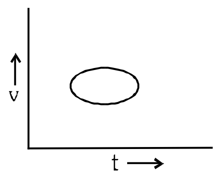
(A) 6 s (B) 12 s (C) 18 s (D) 20 s

12. Which of the following velocity–time graph shows a realistic situation for a body in motion :

(A) 

(B) 

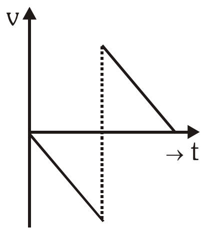
(C) 

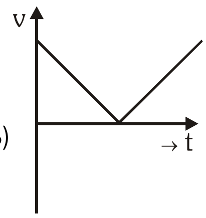
(D) 

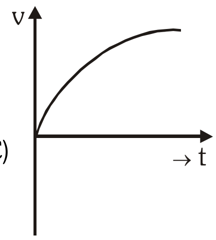
13. Two balls are dropped to the ground from different heights. One ball is dropped 2 s after the other but they both strike the ground at the same time, 5 s after the first was dropped. The difference in the heights at which they were dropped is (Given : g = 10 m s^{-2})

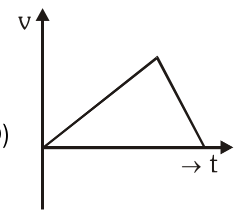
(A) 10 m (B) 20 m (C) 40 m (D) 80 m

14. The velocity – time graph of a body falling from rest under gravity and rebounding from a solid surface is best represented by which of the following graphs?

(A) 

(B) 

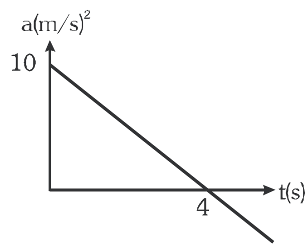
(C) 

(D) 

15. The acceleration of a particle increase linearly with time *t* as '6t'. If the initial velocity of the particle is zero and the particle starts from the origin, then distance travelled by the particle in time *t* will be

(A) t (B) t^2 (C) t^3 (D) t^4

16. The acceleration–time graph of a particle moving along a straight line is as shown in figure. At what time the particle acquires its initial velocity?



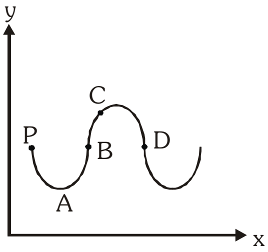
(A) 12 sec

(B) 5 sec

(C) 8 sec

(D) 16 sec

17*. A man moves in x-y plane along the path shown. At what point is his average velocity vector is in the same direction as his instaneous velocity vector. The man starts from point P.



(A) A
(B) B
(C) C
(D) D

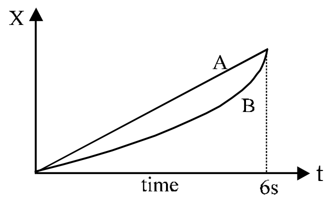
18*. The relation between time t and distance x for a particle moving in straight line is $t = \alpha x^2 + \beta x$ where α and β are constants. The retardation is :

(A) $2\alpha v^3$ (B) $2\beta v^2$ (C) $2\alpha\beta v^2$ (D) $2\beta^2 v^3$

19*. A person drops a stone from a building of height 20 m. At the same instant the front end of a truck passes below the building moving with constant acceleration of 1 m/s^2 and velocity of 2 m/s at that instant. Length of the truck if the stone just misses to hit its rear part is :

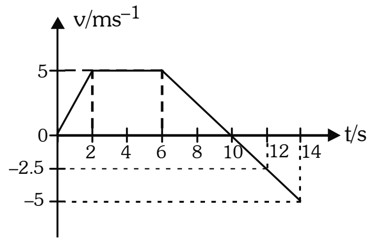
(A) 6 m (B) 4 m (C) 5 m (D) 2 m

20. In the diagram shown, the displacement (x) of particles is given as a function of time. The particle A is moving under constant velocity of 9 m/s. The particle B is moving under variable acceleration. From time t = 0 s. to t = 6 s., the average velocity of the particle B will be equal to :



(A) 2.5 m/s (B) 4 m/s
(C) 9 m/s (D) None of these

21. The variation of velocity of a particle moving along a straight line is shown in figure. The distance travelled by the particle in 12s is

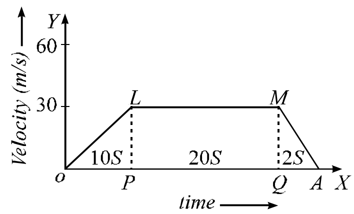


(A) 37.5 m (B) 32.5 m
(C) 35.0 m (D) none of these

22. A particle starts from the origin and moves along the X-axis such that the velocity at any instant is given by $(4t^3 - 2t)$, where t is in second and velocity is in ms^{-1}. What is the acceleration of the particle when it is 2 m from the origin?

(A) 10 m s^{-2} (B) 12 m s^{-1} (C) 22 m s^{-2} (D) 28 m s^{-2}

23*. The following figure shows the velocity-time graph of a train. The total distance travelled by the train is :



(A) 780 m (B) 1200 m
(C) 660 m (D) 1500 m

24. A particle moves along a straight line OX. At a time t (in seconds) the distance X (in metres) of the particle from O is given by $X = 40 + 12t - t^3$. How long would the particle travel before coming to rest for a moment?

(A) 24 m (B) 40 m (C) 56 m (D) 16 m

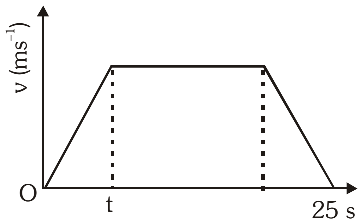
25*. A particle is thrown upwards from ground. It experiences a constant resistance force which can produce retardation 2 m/s^2. The ratio of time of ascent to the time of descent is : [g = 10 m/s^2]

(A) 1 : 1 (B) $\sqrt{\frac{2}{3}}$ (C) $\frac{2}{3}$ (D) $\sqrt{\frac{3}{2}}$

26*. Drops of water fall from the roof of a building 9m high at regular intervals of time, the first drop reaching the ground at the same instant fourth drop starts to fall. What are the distances of the second and third drops from the roof ?

(A) 6 m and 2 m (B) 6 m and 3 m (C) 4 m and 1 m (D) 4 m and 2 m

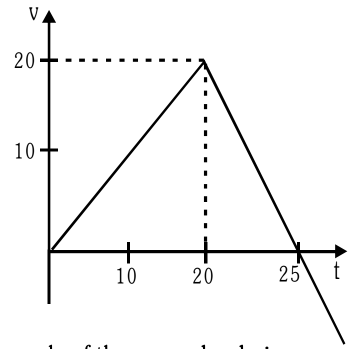
27. The velocity – time graph of the particle moving along a straight line is shown. The rate of acceleration and deceleration is constant and it is equal to 5 ms^{-2}. If the average velocity during the motion is 20 ms^{-1}, then the value of t is



(A) 3 sec (B) 5 sec

(C) 10 sec (D) 12 sec

28*. The figure shows the v–t graph of a particle moving in straight line. Find the time (from starting) when particle returns to the starting point.



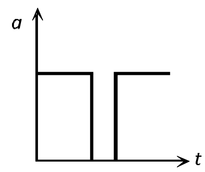
(A) 30 sec

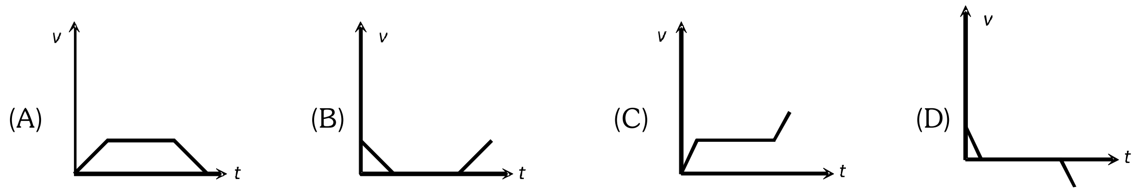
(B) 34.5 sec

(C) 36.2 sec

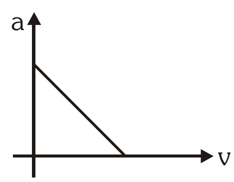
(D) 35.4 sec

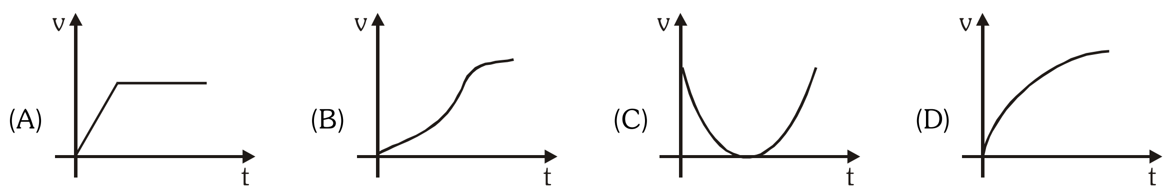
29. Acceleration-time graph of a body is shown. The corresponding velocity-time graph of the same body is





30. Acceleration versus velocity graph of a particle moving in a straight line starting from rest is as shown in figure. The corresponding velocity – time graph would be :





31. A 150 m long train is moving with a uniform velocity of 45 km/h. The time taken by the train to cross a bridge of length 850 meters is

(A) 56 sec (B) 68 sec (C) 80 sec (D) 92 sec

32. A particle starts from rest with uniform acceleration a. Its velocity after n seconds is v. The displacement of the body in the last two seconds is :

(A) $\frac{2v(n-1)}{n}$ (B) $\frac{v(n-1)}{n}$ (C) $\frac{v(n+1)}{n}$ (D) $\frac{2v(2n+1)}{n}$

33. A stone A is dropped from rest from a height h above the ground. A second stone B is simultaneously thrown vertically up from a point on the ground with velocity v. The line of motion of both the stones is same. The value of v which would enable the stone B to meet the stone A midway (at mid point) between their initial positions is:

(A) 2 gh (B) $2\sqrt{gh}$ (C) $\sqrt{gh}$ (D) $\sqrt{2gh}$

EXERCISE – 2 **MCQ (ONE OR MORE CHOICE CORRECT)**

1. A ball is dropped from the top of a building. The ball takes 0.5 s to fall the 3m length of a window some distance from the top of the building. If the magnitude of velocities of the ball at the top and at the bottom of the window are v_T and v_B respectively, then (take g = 9.8 m/s^2) :

(A) $v_T + v_B = 12\ ms^{-1}$ (B) $v_B - v_T = 4.9\ ms^{-1}$ (C) $v_B v_T = 1\ ms^{-1}$ (D) $\frac{v_B}{v_T} = 1\ ms^{-1}$

2*. A particle moves in the xy plane and at time t is at the point $(t^2, t^3 - 2t)$. Then :

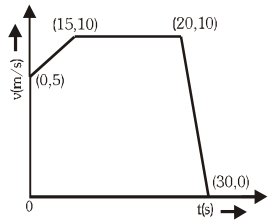
(A) At $t = 2/3$ s, directions of velocity and acceleration are perpendicular

(B) At $t = 0$, directions of velocity and acceleration are perpendicular

(C) At $t = \sqrt{\frac{2}{3}}$ s, particle is moving parallel to x–axis

(D) Acceleration of the particle when it is at point (4, 4) is $2\hat{i} + 12\hat{j}$

3. The figure shows the velocity time graph of a particle which moves along a straight line starting with velocity at 5 m/sec and coming to rest at $t = 30s$. Then :



(A) Distance travelled by the particle is 212.5 m

(B) Distance covered by the particle when it moves with constant velocity is 100 m

(C) Velocity of the particle at $t = 25s$ is 5 m/sec

(D) Velocity of the particle at $t = 9s$ is 8 m/sec .

4. An object may have :

(A) Varying speed without having varying acceleration

(B) Varying velocity without having varying speed

(C) Non–zero acceleration without having varying velocity

(D) Non–zero acceleration without having varying speed .

5. A particle moving along a straight line with uniform acceleration has velocities 7m/s at A and 17m/s at C. B is the mid point of AC. Then :

(A) The velocity at B is 12m/s

(B) The average velocity between A and B is 10m/s

(C) The ratio of the time to go from A to B to that from B to C is 3 : 2

(D) The average velocity between B and C is 15m/s

6. A particle moves along the X–axis as $x = u(t-2) + a(t-2)^2$:

(A) The initial velocity of the particle is u

(B) The acceleration of the particle is a

(C) The acceleration of the particle is 2a

(D) At t = 2s particle is at the origin.

7. Which of the following statements are true for a moving particle ?

(A) If its speed changes, its velocity must change and it must have some acceleration

(B) If its velocity changes, its speeds must change and it must have some acceleration

(C) If its velocity changes, its speed may or may not change, and it must have some acceleration

(D) If its speed changes but direction of motion does not change, its velocity may remain constant

8. If velocity of the particle is given by $v = \sqrt{x}$, where x denotes the position of the particle and initially particle was at x = 4m, then which of the following are correct.

(A) At t = 2 s, the position of the particle is at x = 9m

(B) Particle acceleration at t = 2 s is 1 m/s^2

(C) Particle acceleration is 1/2 m/s^2 through out the motion

(D) Particle will never go in negative direction from its starting position

9. The displacement x of a particle depends on time t as : $x = \alpha t^2 - \beta t^3$

(A) The particle will return to its starting point after time α/β.

(B) The particle will come to rest after time $2\alpha/3\beta$.

(C) The initial velocity of the particle was zero but its initial acceleration was not zero.

(D) No net force will act on the particle at $t = \alpha/3\beta$.

10. A bullet is fired vertically upwards. After 10 second, it returns to the point of firing. Which of the following statements are correct ? (Take g = 10 ms^{-2})

(A) The net displacement of the bullet in 10 s is zero

(B) The total distance traveled by the bullet in 10 s is 250 m

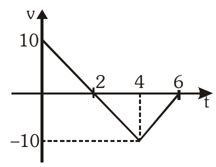
(C) The rate of change of velocity with time is constant throughout the motion of the bullet

(D) The bullet is fired at an initial velocity of 50 ms^{-1} directed vertically upwards

Match the column

11. For the velocity–time graph shown in figure, in a time interval from t = 0 to t = 6 s, match the following:

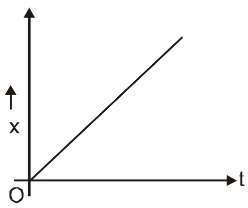
Column I	***Column II***
(A) Change in velocity	(p) – 5/3 SI unit
(B) Average acceleration	(q) – 20 SI unit
(C) Total displacement	(r) – 10 SI unit
(D) Acceleration at t=3s	(s) – 5 SI unit

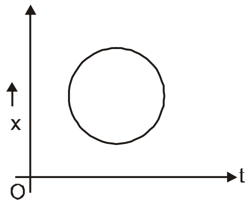


12. Drops of water fall at regular intervals from the roof of a building of height 16 m. The first drop strikes the ground at the same moment as the fifth drop detaches itself from the roof.

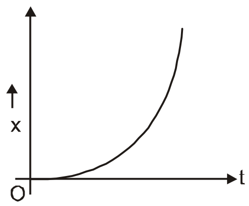
Column I	***Column II***
(A) Distance between 1st and 2nd drops	(p) 1 m
(B) Distance between 2nd and 3rd drops	(q) 7 m
(C) Distance between 3rd and 4th drops	(r) 3 m
(D) Distance between 4th and 5th drops	(s) 5 m

13. Match the displacement-time graph in column I with the type of motion in column II:

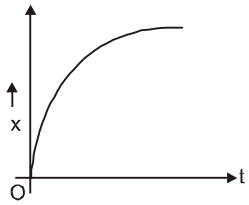
Column-I Displacement Time Graph	***Column-II*** Description of Motion
(A) 	(p) Constant acceleration

(B) 

(q) Constant retardation

(C) 

(r) Uniform circular motion

(D) 

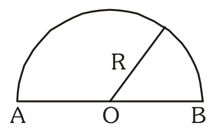
(s) Not possible

Comprehension Based Questions

Comprehension-1

Distance is a scalar quantity. Displacement is a vector quantity. The magnitude of displacement is always less than or equal to distance. For a moving body displacement can be zero but distance cannot be zero. Same concept is applicable regarding velocity and speed. Acceleration is the rate of change of velocity. If acceleration is constant, then equations of kinematics are applicable for one dimensional motion. Motion under the gravity in which air resistance is considered, then the value of acceleration depends on the density of medium. Each motion is measured with respect of frame of reference.

14. A particle moves from A to B. Then the ratio of distance to displacement is :



(A) $\frac{\pi}{2}$ (B) $\frac{2}{\pi}$

(C) $\frac{\pi}{4}$ (D) 1 : 1

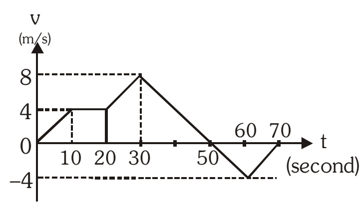
15. A person is going 40m north, 30 m east and then $30\sqrt{2}$ m southwest. The net displacement will be :

(A) 10 m towards east (B) 10 m towards west

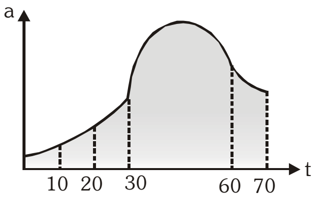
(C) 10 m towards south (D) 10 m towards north

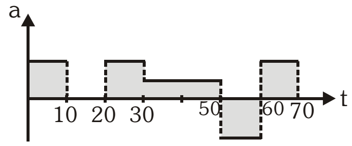
Comprehension-2

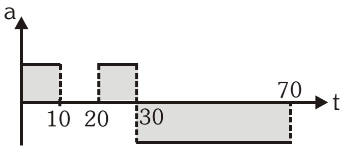
A car is moving on a straight road. The velocity of the car varies with time as shown in the figure. Initially (at t = 0), the car was at x = 0, where, x is the position of the car at any time 't'.

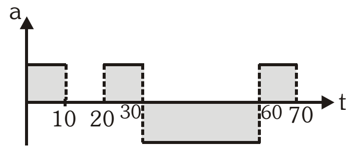


16. The variation of acceleration (a) with time (t) will be best represented by :

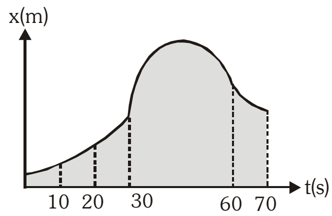
(A) 

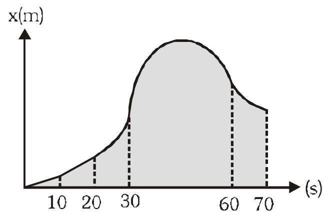
(B) 

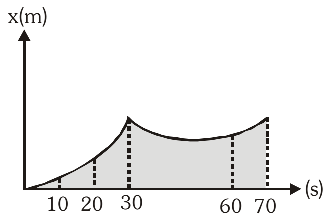
(C) 

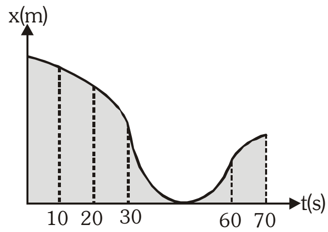
(D) 

17. The displacement time graph will be best represented by :

(A) 

(B) 

(C) 

(D) 

18. The maximum displacement from the starting position will be :

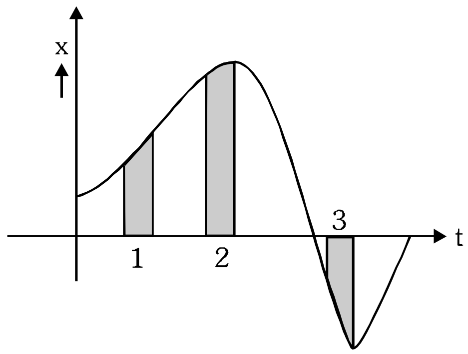
(A) 200 m (B) 250 m (C) 160 m (D) 165 m

19. Average speed from t = 0 to t = 70 s will be :

(A) $\frac{16}{7}$ m/s (B) $\frac{24}{7}$ m/s (C) $\frac{20}{7}$ m/s (D) zero

EXERCISE – 3 **SUBJECTIVE LEVEL-I**

1. For shown situation in which interval is the average speed greatest ? (Given each interval is of equal duration)



2. A body moving with uniform acceleration (in the direction of velocity), covers a distance of 20 m in the 7th second and 24 m in the 9th second. How much shall it cover in 15th second ?

3. A lift accelerates downwards from rest at rate of 2 m/s^2, starting 100 m above the ground. After 3 sec, an object falls out of the lift. Which will reach the ground first ? What is the time interval between their striking the ground

4. A ball is thrown vertically upwards with a velocity of 20 ms^{-1} from the top of a tower. The height of the tower is 25 m from the ground.

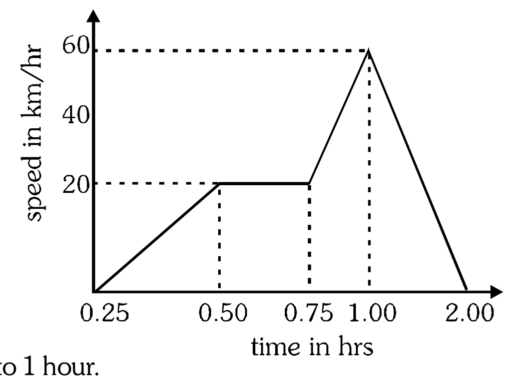
(i) How high will the ball rise ?

(ii) How long will it be before the ball hits the ground ? (Take g = 10 ms^{-2})

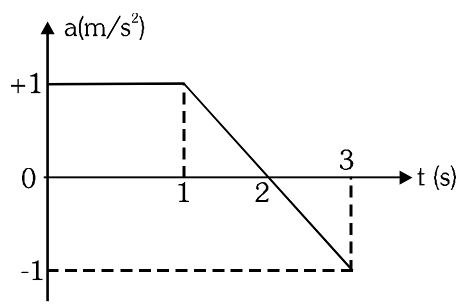
5. A balloon is going upwards with a constant velocity 15 m/s. When the balloon is at 50 m height, a stone is dropped outside from the balloon. How long will stone take to reach at the ground ? (take g = 10 m/s^2)

6. A train moves from one station to another in two hours time. Its speed during the motion is shown in the graph. Calculate :

(i) Maximum acceleration during the journey .

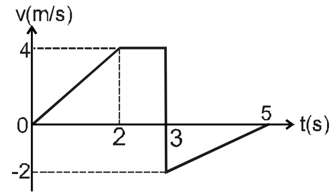


(ii) Distance covered during the time interval from 0.75 hour to 1 hour.



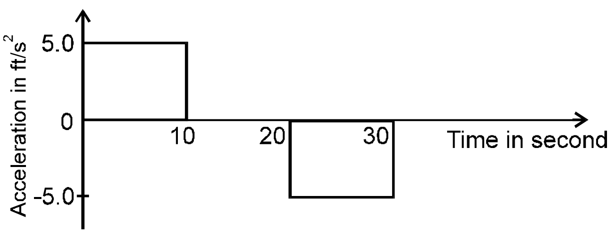
7. A particle starts motion from rest and moves along a straight line. Its acceleration–time graph is shown. Find out speed of particle at t = 2s and at t = 3s.

8. Two cars travelling towards each other on a straight road at velocity 10 m/s and 12 m/s respectively. When they are 150 metre apart, both drivers apply their brakes and each car decelerates at 2 m/s^2 until it stops. How far apart will they be when they have both come to a stop ?



9. For a particle moving along x-axis, velocity-time graph is as shown in figure. Find the distance travelled and displacement of the particle? Also find the average velocity of the particle?

10. The acceleration of a cart started at t = 0, varies with time as shown in figure. Find the distance travelled in 30 seconds and draw the position-time graph.



EXERCISE – 4 RECAP OF AIEEE/JEE (MAIN)

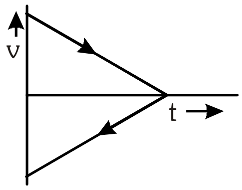
1. From a tower of height H, a particle is thrown vertically upwards with a speed u. The time taken by the particle, to hit the ground, is n times that taken by it to reach the highest point of its path. The relation between H, u and n is : ***[JEE(Main) - 2014]***

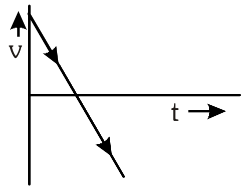
(A) $2g\ H = n^2u^2$ (B) $g\ H = (n-2)^2\ u^2$ (C) $2g\ H = nu^2(n-2)$ (D) $g\ H = (n-2)u^2$

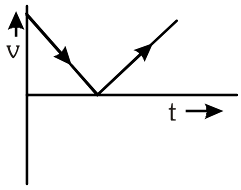
2. A body of mass $m = 10^{-2}$ kg is moving in a medium and experiences a frictional force $F = -kv^2$. Its initial speed is $v_0 = 10\ ms^{-1}$. If, after 10 s, its energy is $\frac{1}{8}mv_0^2$, the value of k will be : ***[JEE(Main) - 2017]***

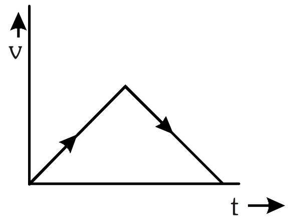
(A) 10^{-3} kg s^{-1} (B) 10^{-4} kg m^{-1} (C) 10^{-1} kg m^{-1} s^{-1} (D) 10^{-3} kg m^{-1}

3. A body is thrown vertically upwards. Which one of the following graphs correctly represent the velocity vs time? ***[JEE(Main) - 2017]***

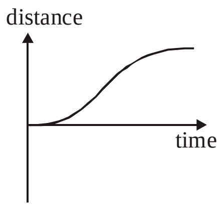
(A) 

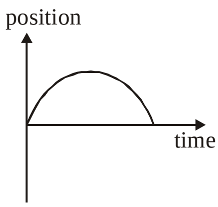
(B) 

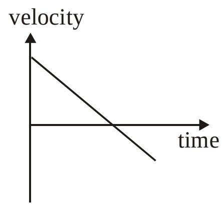
(C) 

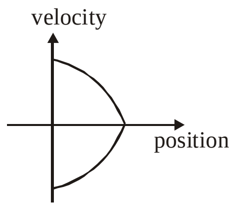
(D) 

4. All the graphs below are intended to represent the same motion. One of them does it incorrectly. Pick it up. ***[JEE(Main) - 2018]***

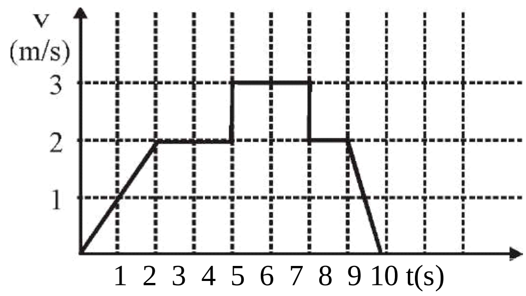
(A) 

(B) 

(C) 

(D) 

5. A particle starts from the origin at time t = 0 and moves along the positive x–axis. The graph of velocity with respect to time is shown in figure. What is the position of the particle at time t = 5s ? ***[JEE(Main) - 2019]***



(A) 6 m (B) 9 m (C) 3 m (D) 10 m

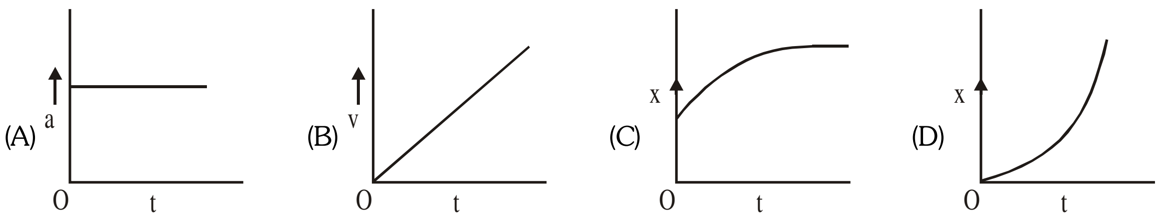
6. A particle moves from the point $(2.0\hat{i}+4.0\hat{j})$ at t = 0, with an initial velocity $(5.0\hat{i}+4.0\hat{j})\ ms^{-1}$. It is acted upon by a constant force which produces a constant acceleration $(4.0\hat{i}+4.0\hat{j})\ ms^{-2}$. What is the distance of the particle from the origin at time 2 s ? ***[JEE(Main) - 2019]***

(A) $20\sqrt{2}$ m (B) $10\sqrt{2}$ m (C) 5 m (D) 15 m

7. A person standing on an open ground hears the sound of a jet aeroplane, coming from north at an angle 60° with ground level. But he finds the aeroplane right vertically above his position. If u is the speed of sound, speed of the plane is : ***[JEE(Main) - 2019]***

(A) $\frac{2v}{\sqrt{3}}$ (B) v (C) $\frac{v}{2}$ (D) $\frac{\sqrt{3}}{2}v$

8. A particle starts from origin O from rest and moves with a uniform acceleration along the positive x-axis. Identify all figures that correctly represent the motion qualitatively.
(a = acceleration, v = velocity, x = displacement, t = time) ***[JEE(Main) - 2019]***



(A) (A), (B), (C) (B) (A) (C) (A), (B), (D) (D) (B), (C)

9. The position of a particle as a function of time t, is given by $x(t) = at + bt^2 - ct^3$ where a, b and c are constants. When the particle attains zero acceleration, then its velocity will be : ***[JEE(Main) - 2019]***

(A) $a + \frac{b^2}{4c}$ (B) $a + \frac{b^2}{c}$ (C) $a + \frac{b^2}{2c}$ (D) $a + \frac{b^2}{3c}$

10. A particle is moving with speed $v = b\sqrt{x}$ along positive x-axis. Calculate the speed of the particle at time $t = \tau$(assume that the particle is at origin at t = 0). ***[JEE(Main) - 2019]***

(A) $\frac{b^2\tau}{4}$ (B) $\frac{b^2\tau}{2}$ (C) $b^2\tau$ (D) $\frac{b^2\tau}{\sqrt{2}}$

11. The position co–ordinates of a particle moving in a 3–D coordinate system is given by

$x = a\cos\omega t$

$y = a\sin\omega t$ and $z = a\omega t$

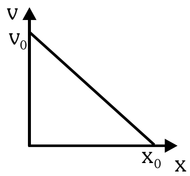
The speed of the particle is : ***[JEE(Main)-2019]***

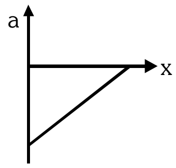
(A) $a\omega$ (B) $\sqrt{3}\ a\omega$ (C) $\sqrt{2}\ a\omega$ (D) $2a\omega$

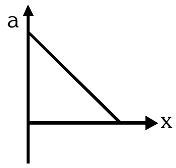
EXERCISE – 5 **RECAP OF IIT-JEE/JEE (ADVANCED)**

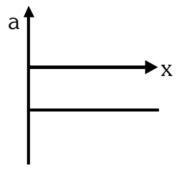
Single Choice Correct

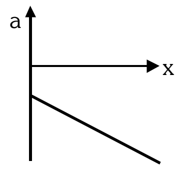
1. The given graph shows the variation of velocity with displacement. Which one of the graph given below correctly represents the variation of acceleration with displacement : ***[IIT-JEE - 2005]***



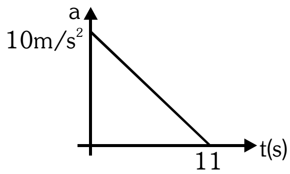
(A) 

(B) 

(C) 

(D) 

2. A particle starts from rest. Its acceleration (a) versus time (t) is as shown in the figure. The maximum speed of the particle will be : ***[IIT-JEE - 2004]***

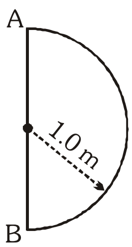


(A) 110 m/s (B) 55 m/s (C) 550 m/s (D) 660 m/s

3. A small block slides without friction down an inclined plane starting from rest. Let s_n be the distance travelled from t = n – 1 to t = n. Then $\frac{s_n}{s_{n+1}}$ is : ***[IIT-JEE - 2004]***

(A) $\frac{2n-1}{2n}$ (B) $\frac{2n+1}{2n-1}$ (C) $\frac{2n-1}{2n+1}$ (D) $\frac{2n}{2n+1}$

4. In 1.0 s, a particle goes from point A to point B, moving in a semicircle (see figure). The magnitude of the average velocity is– ***[IIT-JEE 1999]***



(A) 3.14 m/s (B) 2.0 m/s (C) 1.0 m/s (D) zero

5. A person measures the depth of a well by measuring the time interval between dropping a stone and receiving the sound of impact with the bottom of the well. The error in his measurement of time is $\delta T = 0.01$ seconds and he measures the depth of the well to be $L = 20$ meters. Take the acceleration due to gravity $g = 10\ ms^{-2}$ and the velocity of sound is $300\ ms^{-1}$. Then the fractional error in the measurement, $\frac{\delta L}{L}$, is closest to

[JEE(Adv.) - 2017]

(A) 0.2% (B) 5% (C) 3% (D) 1%

6. Consider an expanding sphere of instantaneous radius R whose total mass remains constant. The expansion is such that the *instantaneous* density ρ remains uniform throughout the volume. The rate of fractional change in density $\left(\frac{1}{\rho}\frac{d\rho}{dt}\right)$ is constant. The velocity v of any point on the surface of the expanding sphere is proportional to

[JEE(Adv.) - 2017]

(A) R^3 (B) $\frac{1}{R}$ (C) R (D) $R^{2/3}$

One or More Choice Correct

7. The position vector $\vec{r}$ of a particle of mass m is given by the following equation $\vec{r}(t) = \alpha t^3\hat{i} + \beta t^2\hat{j}$, where $\alpha = 10/3\ ms^{-3}$, $\beta = 5\ ms^{-2}$ and $m = 0.1$ kg. At $t = 1$ s, which of the following statement (s) is (are) true about the particle ? ***[JEE(Adv.) 2016]***

(A) The velocity $\vec{v}$ is given by $\vec{v} = (10\hat{i} + 10\hat{j})\,ms^{-1}$

(B) The angular momentum $\vec{L}$ with respect to the origin is given by $\vec{L} = -5(5/3)\hat{k}$ N ms

(C) The force $\vec{F}$ is given by $\vec{F} = (\hat{i} + 2\hat{j})N$

(D) The torque $\vec{\tau}$ with respect to the origin is given by $\vec{\tau} = -(20/3)\hat{k}$ N m

ANSWERS

EXERCISE-1

Que.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Ans.	B	D	D	C	D	C	B	D	A	A	B	B	D	A	C
Que.	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
Ans.	C	C	A	A	C	A	C	A	D	B	C	B	C	C	D
Que.	31	32	33												
Ans.	C	A	C												

EXERCISE-2

Que.	1	2	3	4	5	6	7	8	9	10
Ans.	A,B	A,B,C,D	A,C,D	A,B,D	B,C,D	C,D	A,C	A,C,D	A,B,C,D	A,B,C,D

- ***Match the Column***

  11. A–r, B–p, C–r D–s
  12. A–q, B–s, C–r D–p
  13. A-r; B-s; C-p; D-q

- ***Comprehension Based Questions***

  Comprehension 1 : ***14.*** (A) ***15.*** (D)
  Comprehension 2 : ***16.*** (D) ***17.*** (B) ***18.*** (A) ***19.*** (B)

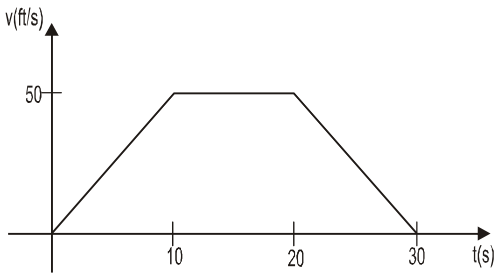
EXERCISE-3

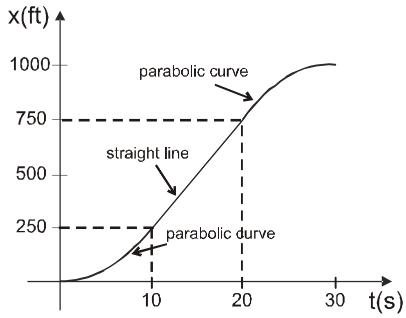
1. 3 ***2***. 36 m ***3***. object, 3.3 s ***4***. (i) 20 m (ii) 5s ***5***. 5s

6. (i) 160 km/hr^2 (ii) 10 km ***7.*** 1.5 m/s, 1 m/s ***8***. 89m

9. distance travelled = 10 m; displacement = 6 m; average velocity = 1.2 m/s

10. 1000 ft.,





EXERCISE-4

Que.	1	2	3	4	5	6	7	8	9	10	11
Ans.	C	B	B	A	B	A	C	C	D	B	C

EXERCISE-5

- ***Single Choice correct*** ***1.*** (A) ***2.*** (B) ***3.*** (C) ***4.*** (B) ***5.*** (D) ***6.*** (C)
- ***One or More Choice correct*** ***7.*** (ABD)

* * * * *

PROJECTILE MOTION

Recap of Early Classes

Motion is common to everything in the universe. We walk, run and ride a bicycle. Even when we are sleeping, air moves into and out of our lungs and blood flows in arteries and veins. We see leaves falling from trees and water flowing down a dam. automobiles and planes carry people from one place to the other. The earth rotates once every twenty-four hours and revolves round the sun once in a year. The sun itself is in motion in the Milky Way, which is again moving within its local group of galaxies.

The motion of a projectile may be thought of as the result of two separate, simultaneously occurring components of motions. One component is along a horizontal direction without any acceleration and the other along the vertical direction with constant acceleration due to the force of gravity. It was Galileo who first stated this independency of the horizontal and the vertical components of projectile motion in his Dialogue on the great world systems.

Index

PROJECTILE MOTION

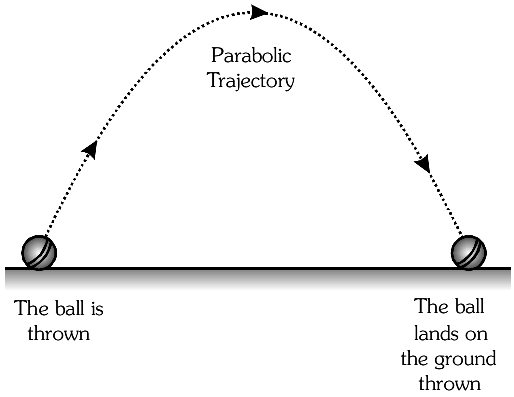
1.0 PROJECTILE MOTION

SL AL

An object projected by an external force when continues to move by its own inertia is known as projectile and its motion as projectile motion.

A football kicked by a player, an arrow shot by an archer, water sprinkling out a water–fountain, an athlete in long jump or high jump, a bullet or an artillery shell fired from a gun are some Illustrations of projectile motion.

In simplest case when a projectile does reach great heights above the ground as well as does not cover a very large distance on the ground, acceleration due to gravity can be assumed uniform throughout its motion. Moreover, such a projectile does not spend much time in air not permitting the wind and air resistance to gather appreciable effects. Therefore, while analyzing them, we can assume gravity to be uniform and neglect effects of wind as well as air resistance. Under these circumstances when an object is thrown in a direction other than the vertical, its trajectory assumes shape of a parabola. In the figure, a ball thrown to follow a parabolic trajectory is shown as an Illustration of projectile motion.



At present, we study projectiles moving on parabolic trajectories and by the term projectile motion; we usually refer to this kind of motion.

For a projectile to move on parabolic trajectory, the following conditions must be fulfilled.

- Acceleration vector must be uniform.
- Velocity vector never coincides with line of acceleration vector.

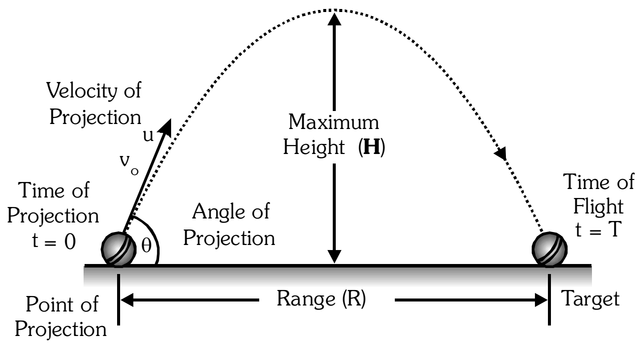
1.1 Analyzing Projectile motion

SL AL

Since parabola is a plane curve, projectile motion on parabolic trajectory becomes an Illustration of a two–dimensional motion. It can be conceived as superposition of two simultaneous rectilinear motions in two mutually perpendicular directions, which can be analyzed separately as two Cartesian components of the projectile motion.

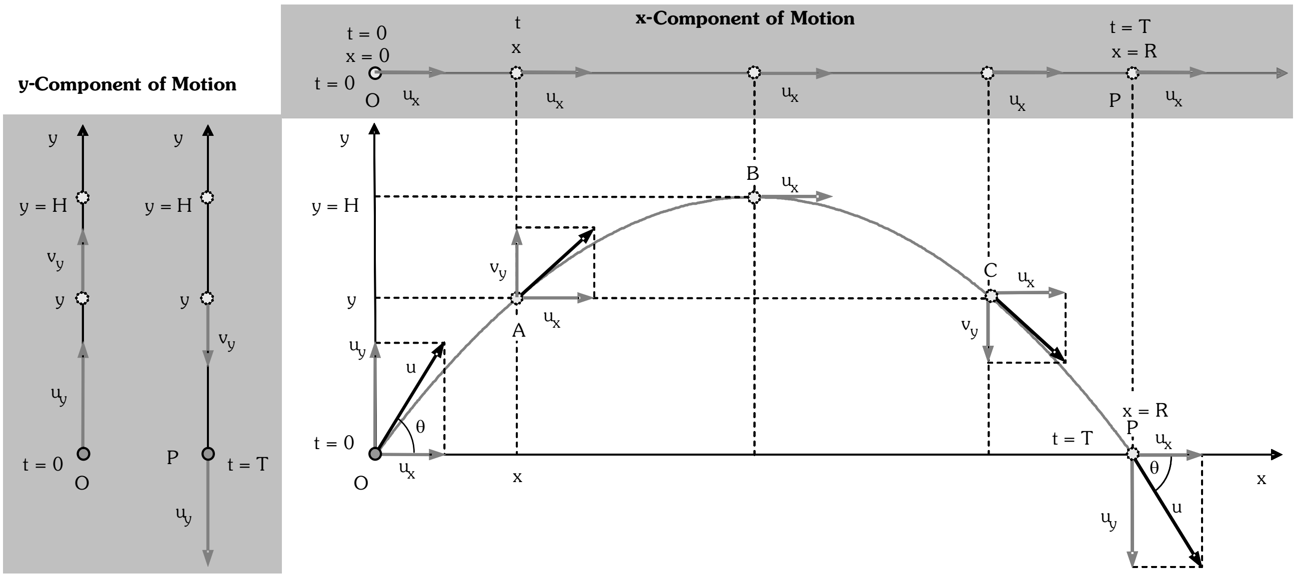
Projectile Motion near the Horizontal or Flat Ground using Cartesian components

Consider motion of a ball thrown from ground as shown in the figure. The point from where it is projected is known as point of projection, the point where it falls on the ground is known as point of landing or target. The distance between these two points is known as horizontal range or range, the height from the ground of the highest point it reaches during flight is known as maximum height and the duration for which it remain in the air is known as air time or time of flight. The velocity with which it is thrown is known as velocity of projection and angle which velocity of projection makes with the horizontal is known as angle of projection.



A careful observation of this motion reveals that when a ball is thrown its vertical component of velocity decreases in its upward motion, vanishes at the highest point and thereafter increases in its downward motion due to gravity similar to motion of a ball thrown vertically upwards. At the same time, the ball continues to move uniformly in horizontal direction due to inertia. The actual projectile motion on its parabolic trajectory is superposition of these two simultaneous rectilinear motions.

In the following figure, the above ideas are shown representing the vertical by y-axis and the horizontal by x-axis.



Projectile motion resolved into its two Cartesian components.
Projectile motion as superposition of two rectilinear motions one in vertical and other in horizontal direction.

Vertical or y-component of motion

Component of initial velocity in the vertical direction is u_y. Since forces other than gravitational pull of the earth are negligible, vertical component of acceleration a_y of the ball is g vertically downwards. This component of motion is described by the following three equations. Here v_y denotes y-component of velocity, y denotes position coordinate y at any instant t.

$v_y = u_y - gt$ ...(i) $\qquad y = u_y t - \frac{1}{2}gt^2$ ...(ii) $\qquad v_y^2 = u_y^2 - 2gy$ ...(iii)

Horizontal or x-component of motion

Since effects of wind and air resistance are assumed negligible as compared to effect of gravity, the horizontal component of acceleration of the ball becomes zero and the ball moves with uniform horizontal component of velocity u_x. This component of motion is described by the following equation.

$$x = u_x t \qquad \text{...(iv)}$$

Equation of trajectory

Equation of the trajectory is relation between the x and the y coordinates of the ball without involvement of time t. To eliminate t, we substitute its expression from equation (iv) into equation (ii).

$$y = x\tan\theta - \frac{g}{2u^2\cos^2\theta}x^2 \qquad \text{...(v)}$$

Every projectile motion can be analyzed using the above five equations. In a special case of interest, if the projectile lands the ground again, its time of flight, the maximum height reached and horizontal range are obtained using the above equations.

Time of Flight

At the highest point of trajectory when $t = \frac{1}{2}T$, the vertical component of velocity becomes zero. At the instant t =T, the ball strikes the ground with vertical component of velocity $v_y = -u_y$. By substituting either of these conditions in equation (i), we obtain the time of flight.

$$T = \frac{2u_y}{a_y} = \frac{2u_y}{g}$$

Maximum Height

At the highest point of trajectory where y = H, the vertical component of velocity becomes zero. By substituting this information in equation (iii), we obtain the maximum height.

$$H = \frac{u_y^2}{2g}$$

Horizontal Range

The horizontal range or simply the range of the projectile motion of the ball is distance traveled on the ground in its whole time of flight.

$$R = u_x T = \frac{2u_x u_y}{g} = \frac{u^2 \sin 2\theta}{g}$$

Maximum Range

It is the maximum distance traveled by a projectile in the horizontal direction for a certain velocity of projection. The above expression of range makes obvious that to obtain maximum range the ball must be projected at angle $\theta = 45°$.

Substituting this condition in the expression of range, we obtain the maximum range R_m.

$$R_m = \frac{u^2}{g}$$

Trajectory Equation

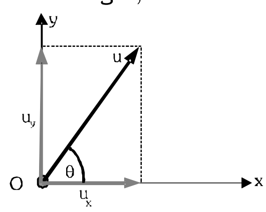
If range is known in advance, the equation of trajectory can be written in an alternate form involving horizontal range.

$$y = x\tan\theta\left(1 - \frac{x}{R}\right)$$

Illustrations

Illustration 1. A ball is thrown with 25 m/s at an angle 53° above the horizontal. Find its time of flight, maximum height and range.

Solution In the adjoining figure velocity of projection u = 25 m/s, angle of projection θ = 53°, the horizontal and vertical components u_x and u_y of velocity of projection are shown. From these information we have



$$u_x = u\cos 53° = 15 \text{ m/s and } u_y = u\sin 53° = 20 \text{ m/s}$$

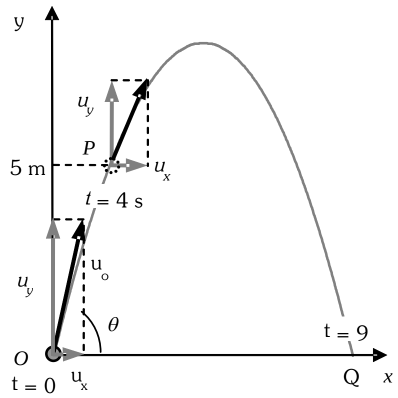
Using equations for time of flight T, maximum height H and range R, we have

$$T = \frac{2u_y}{g} = \frac{2\times 20}{10} = 4 \text{ s}; \quad H = \frac{u_y^2}{2g} = \frac{20^2}{2\times 10} = 20 \text{ m}$$

$$R = \frac{2u_x u_y}{g} = \frac{2\times 15\times 20}{10} = 60 \text{ m}$$

Illustration 2*. A ball 4 s after the instant it was thrown from the ground passes through a point P, and strikes the ground after 5 s from the instant it passes through the point P. Assuming acceleration due to gravity to be 9.8 m/s^2 find height of the point P above the ground.

Solution The ball projected with velocity $\vec{u} = u_x\hat{i} + u_y\hat{j}$ from O reaches the point P with velocity $\vec{v} = u_x\hat{i} + v_y\hat{j}$ and hits the ground at point Q at the instant T = 4 + 5 = 9 s as shown in the adjoining motion diagram.



From equation of time of flight, we have its initial y-component of velocity u_y $\quad T = \frac{2u_y}{g} \rightarrow u_y = \frac{1}{2}gT$

Substituting above in eq. (ii) and rearranging terms, we have the height y of the point P.

$$y = u_y t - \tfrac{1}{2}gt^2 \rightarrow y = \tfrac{1}{2}gt(T-t) = \tfrac{1}{2}\times 9.8\times 4(9-4) = 98\text{m}$$

BEGINNER'S BOX–1

Ground to Ground Projectile

1. The velocity of projection of a projectile is $(6\hat{i} + 8\hat{j})$ ms^{-1}. The horizontal range of the projectile is :

(A) 4.9 m (B) 9.6 m (C) 19.6 m (D) 14 m

2. A projectile is fired with velocity u making angle θ with the horizontal. What is the change in velocity (from initial) when it is at the highest point :

(A) u cosθ (B) u (C) u sinθ (D) (u cosθ – u)

3. A projectile can have the same range R for two angles of projection. If t_1 and t_2 be the times of flight in two cases, then the product of two times of flight is :

(A) $t_1t_2 \propto R^2$ (B) $t_1t_2 \propto R$ (C) $t_1t_2 \propto \frac{1}{R}$ (D) $t_1t_2 \propto \frac{1}{R^2}$

4. In case of projectile motion if two projectiles A and B are projected with same speed at angles 15° and 75° respectively to the horizontal then :

(A) $H_A > H_B$ (B) $H_A < H_B$ (C) $T_A > T_B$ (D) $T_A < T_B$

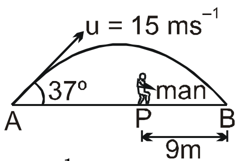
5. Four bodies P, Q, R and S are projected with equal speed having angles of projection 15°, 30°, 45° and 60° with the horizontal respectively. The body having shortest range is :

(A) P (B) Q (C) R (D) S

6*. A projectile is fired with a speed u at an angle θ with the horizontal. Its speed when its direction of motion makes an angle 'α' with the horizontal is :

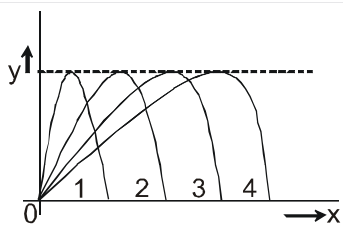
(A) u secθ cosα (B) u secθ sinα (C) u cosθ secα (D) u sin θ secα

7*. A ball is hit by a batsman at an angle of 37° as shown in figure. The man standing at P should run at what minimum velocity so that he catches the ball before it strikes the ground. Assume that height of man is negligible in comparison to maximum height of projectile.



(A) 3 ms^{-1} (B) 5 ms^{-1} (C) 9 ms^{-1} (D) 12 ms^{-1}

8. Figure shows four paths for a kicked football. Ignoring the effects of air on the flight, rank the paths according to initial horizontal velocity component, highest first :



(A) 1, 2, 3, 4 (B) 2, 3, 4, 1

(C) 3, 4, 1, 2 (D) 4, 3, 2, 1

9. A projectile is given an initial velocity of $(\hat{i} + 2\hat{j})$ m/sec. The cartesian equation of its path is : (g = 10 m/s^2)

(A) $y = 2x - 5x^2$ (B) $y = x - 5x^2$ (C) $4y = 2x - 5x^2$ (D) $y = 2x - 25x^2$

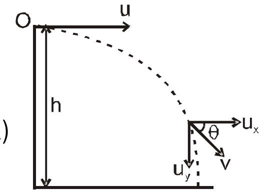
10*. The equation of projectile is $y = 16x - \frac{5x^2}{4}$. The horizontal range is :

(A) 16 m (B) 8 m (C) 3.2 m (D) 12.8 m

2.0 PROJECTILE THROWN PARALLEL TO THE HORIZONTAL FROM SOME HEIGHT

SL AL

Consider a projectile thrown from point O at some height h from the ground with a velocity u. Now we shall study the characteristics of projectile motion by resolving the motion along horizontal and vertical directions.



	Horizontal direction	*Vertical direction*
(i)	Initial velocity $u_x = u$	Initial velocity $u_y = 0$
(ii)	Acceleration $a_x = 0$	Acceleration $a_y = g$ (downward)

2.1 Time of flight

This is equal to the time taken by the projectile to return to ground. From equation of motion

$$S = ut + \frac{1}{2}at^2 \text{ , along vertical direction, we get}$$

$$-h = u_y t + \frac{1}{2}(-g)t^2$$

$$\Rightarrow \quad h = \frac{1}{2}gt^2 \quad \Rightarrow \boldsymbol{t} = \sqrt{\frac{2h}{g}}$$

2.2 Horizontal range

Distance covered by the projectile along the horizontal direction between the point of projection to the point on the ground.

$$R = u_x \cdot t$$

$$R = u\sqrt{\frac{2h}{g}}$$

2.3 Velocity at a general point P(x, y)

$$v = \sqrt{u_x^2 + u_y^2}$$

Here horizontal velocity of the projectile after time t

$$v_x = u$$

velocity of projectile in vertical direction after time t

$$v_y = 0 + (-g)t = -gt = gt \text{ (downward)}$$

$$\therefore \quad v = \sqrt{u^2 + g^2t^2}$$

and $\tan\theta = v_y/v_x$

2.4 Velocity with which the projectile hits the ground :

$$V_x = u$$

$$V_y^2 = 0^2 - 2g(-h)$$

$$V_y = \sqrt{2gh}$$

$$V = \sqrt{V_x^2 + V_y^2} \Rightarrow \boldsymbol{V} = \sqrt{u^2 + 2gh}$$

2.5 Trajectory equation

The path traced by projectile is called the trajectory.
After time t,

$$x = ut \quad ....(1)$$

$$y = -\frac{1}{2}gt^2 \quad ....(2)$$

From equation (1)

$$t = \frac{x}{t}$$

Put value of t in equation (2)

$$y = -\frac{1}{2} g \cdot \frac{x^2}{u^2}$$

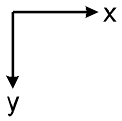
This is trajectory equation of the particle projected horizontally from some height.

Illustrations

Illustration 9*. A projectile is fired horizontally with a speed of 98 ms^{-1} from the top of a hill 490 m high. Find (i) the time taken to reach the ground (ii) the distance of the target from the hill and (iii) the velocity with which the projectile hits the ground. (take $g = 9.8\ m/s^2$)

Solution

(i) The projectile is fired from the top O of a hill with speed $u = 98\ ms^{-1}$ along the horizontal as shown as OX. It reaches the target P at vertical depth OA, in the coordinate system as shown, OA = y = 490 m

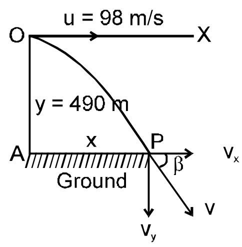


As, $y = 1/2gt^2$

$\therefore\quad 490 = 1/2 \times 9.8\, t^2 \quad$ or $\quad t = \sqrt{100} = 10$ s.

(ii) Distance of the target from the hill is given by,

AP = x = Horizontal velocity × time = 98 × 10 = 980 m.



(iii) The horizontal and vertical components of velocity v of the projectile at point P are

$$v_x = u = 98\ ms^{-1}$$

$$v_y = u_y + gt = 0 + 9.8 \times 10 = 98\ ms^{-1}$$

$$V = \sqrt{v_x^2 + v_y^2} = \sqrt{98^2 + 98^2} = 98\sqrt{2}\ ms^{-1}$$

Now if the resultant velocity v makes an angle β with the horizontal, then

$$\tan\beta = \frac{v_y}{v_x} = \frac{98}{98} = 1 \qquad \therefore \beta = 45°$$

Illustration 10. A motorcycle stunt rider rides off the edge of a cliff. Just at the edge his velocity is horizontal, with magnitude 9.0 m/s. Find the motorcycle's position, distance from the edge of the cliff and velocity after 0.5 s.

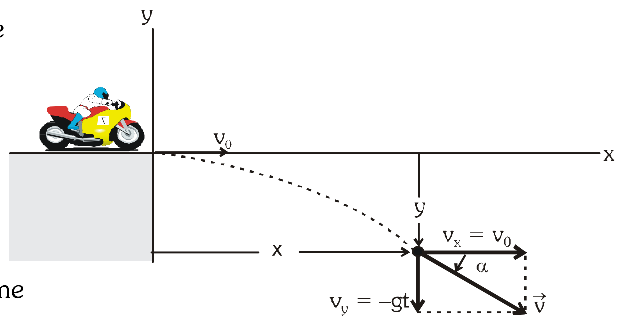
Solution

At t = 0.50 s, the x and y-coordinates are

$$x = v_{0x}t = (9.0\ m/s)\ (0.50\ s) = 4.5\ m$$

$$y = -\frac{1}{2}gt^2 = -\frac{1}{2}(10\ m/s^2)\ (0.50\ s)^2$$

$$= -\frac{5}{4}\ m$$



The negative value of y shows that this time

The motorcycle is below its starting point.

The motorcycle's distance from the origin at this time

$$r = \sqrt{x^2 + y^2} = \sqrt{\left(\frac{9}{2}\right)^2 + \left(\frac{5}{4}\right)^2} = \frac{\sqrt{349}}{4}\ m.$$

The components of velocity at this time are

$v_x = v_{0x} = 9.0$ m/s

$v_y = -gt = (-10\ m/s^2)\ (0.50\ s) = -5$ m/s.

The speed (magnitude of the velocity) at this time is

$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{(9.0\ m/s)^2 + (-5\ m/s)^2} = \sqrt{106}\ m/s$$

Illustration 11*. An object is thrown between two tall buildings. 180 m froms each other. The object is thrown horizontally from a window 55 m above ground from one building through a window 10.9 m above ground in the order building. Find out the speed of projection. (use $g = 9.8\ m/s^2$)

Solution

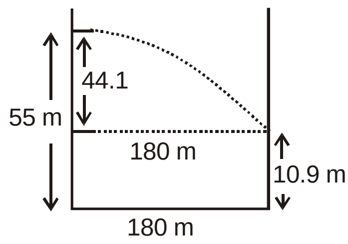
$t = \sqrt{\frac{2h}{g}} = \sqrt{\frac{2 \times 44.1}{9.8}} = \sqrt{9}$

$t = 3$ sec.

$R = uT$

$\frac{180}{3} = u$

$u = 60$ m/s



3.0 PROJECTION FROM A TOWER

SL AL

***Case (i)* –** Horizontal projection

$u_x = u$;

$u_y = 0$;

$a_y = -g$

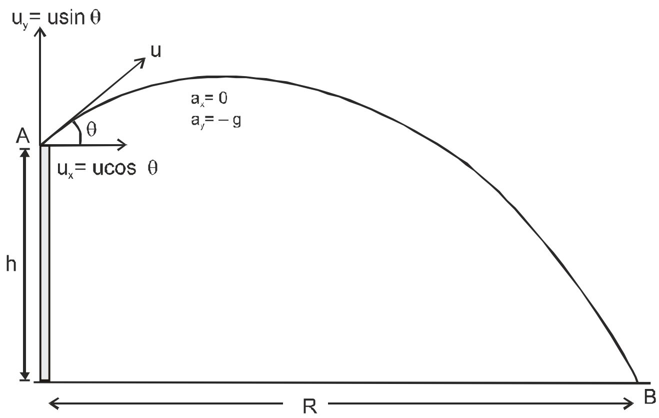
This is same as previous section (section 4)

***Case (ii)* –** Projection at an angle θ above horizontal

$u_x = u\cos\theta$;

$u_y = u\sin\theta$;

$a_y = -g$



Equation of motion between A & B (in Y direction)

$S_y = -h$, $u_y = u\sin\theta$, $a_y = -g$, $t = T$

$S_y = u_y t + \frac{1}{2} a_y t^2 \Rightarrow -h = u\sin\theta\ t - \frac{1}{2} gt^2$

solving this equation we will get time of flight, T and range,

$R = u_x T = u\cos\theta\ T$

Also, $v_y^2 = u_y^2 + 2a_y S_y = u^2 \sin^2\theta + 2gh$

$v_x = u\cos\theta$

$v_B = \sqrt{v_y^2 + v_x^2}$

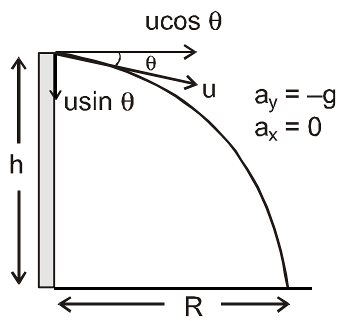
$\boldsymbol{v_B = \sqrt{u^2 + 2gh}}$

***Case (iii)* –** Projection at an angle θ below horizontal

$u_x = u\cos\theta$;

$u_y = -u\sin\theta$;

$a_y = -g$



$S_y = u_y t + \frac{1}{2} a_y t^2$

$S_y = -h$, $u_y = -u\sin\theta$, $t = T$, $a_y = -g$

$\Rightarrow -h = -u\sin\theta\ T - \frac{1}{2} gT^2$

$\Rightarrow h = u\sin\theta + \frac{1}{2} gt^2$.

solving this equation we will get time of flight, T.

and range, $R = u_x T = u \cos\theta\, T$

$$v_x = u \cos \theta$$

$$v_y^2 = u_y^2 + 2a_y S_y$$

$$= u^2 \sin^2\theta + 2(-g)(-h)$$

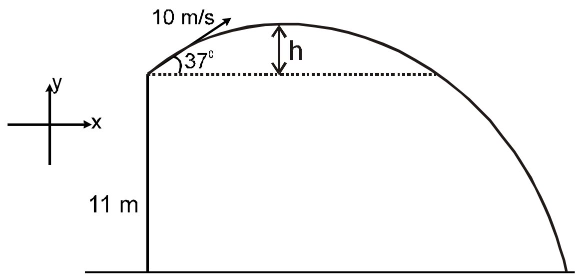
$$v_y^2 = u^2 \sin^2\theta + 2gh$$

$$v_B = \sqrt{v_x^2 + v_y^2} = \sqrt{u^2 + 2gh}$$

Note – Objects thrown from same height in different directions with same initial speed will strike the ground with the same final speed. But the time of flight will be different.

Illustrations

Illustration 12. From the top of a 11 m high tower a stone is projected with speed 10 m/s, at an angle of 37° as shown in figure. Find



(a) speed after 2s (b) time of flight. (c) horizontal range.

(d) the maximum height attained by the particle. (e) speed just before striking the ground.

Solution : (a) Initial velocity in horizontal direction $= 10 \cos 37° = 8$ m/s

Initial velocity in vertical direction $= 10 \sin 37° = 6$ m/s

Speed after 2 seconds

$$\vec{v} = v_x \hat{i} + v_y \hat{j} = 8\hat{i} + (u_y + a_y t)\hat{j} = 8\hat{i} + (6 - 10 \times 2)\hat{j} = 8\hat{i} - 14\hat{j}$$

(b) $S_y = u_y t + \frac{1}{2} a_y t^2 \Rightarrow -11 = 6 \times t + \frac{1}{2} \times (-10)\, t^2$

$5t^2 - 6t - 11 = 0 \Rightarrow (t+1)(5t - 11) = 0 \Rightarrow t = \frac{11}{5}$ sec.

(c) Range $= 8 \times \frac{11}{5} = \frac{88}{5}$ m

(d) Maximum height above the level of projection,

$$h = \frac{u_y^2}{2g} = \frac{6^2}{2 \times 10} = 1.8 \text{ m}$$

$\therefore$ maximum height above ground $= 11 + 1.8 = 12.8$ m

(e) $v = \sqrt{u^2 + 2gh} = \sqrt{100 + 2 \times 10 \times 11}$

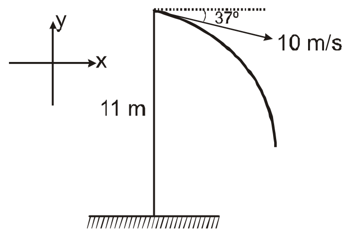
$\Rightarrow v = 8\sqrt{5}$ m/s

Illustration 13. From the top of a 11 m high tower a stone is projected with speed 10 m/s, at an angle of 37° as shown in figure. Find

(a) time of flight.

(b) horizontal range.

(c) speed just before striking the ground.



Solution $u_x = 10\cos 37° = 8$ m/s , $u_y = -10 \sin 37° = -6$ m/s

(a) $S_y = u_y t + \frac{1}{2} a_y t^2 \Rightarrow -11 = -6 \times t + \frac{1}{2} \times (-10)\, t^2$

$5t^2 + 6t - 11 = 0 \Rightarrow (t-1)(5t+11) = 0$

$\Rightarrow t = 1$ sec

(b) Range $= 8 \times 1 = 8$ m

(c) $v = \sqrt{u^2 + 2gh} = \sqrt{100 + 2 \times 10 \times 11}$

$v = \sqrt{320}$ m/s $= 8\sqrt{5}$ m/s

BEGINNER'S BOX–2

Projection From Height

1. An aeroplane is flying horizontally with a velocity of 600 km/h at a height of 1960 m. When it is vertically at a point A on the ground, a bomb is released from it. The bomb strikes the ground at point B. The distance AB is

(A) 1200 m (B) 0.33 km (C) 3.33 km (D) 33 km

2. A ball is rolled off the edge of a horizontal table at a speed of 4 m/second. It hits the ground after 0.4 second. Which statement given below is true

(A) It hits the ground at a horizontal distance 1.6 m from the edge of the table

(B) The speed with which it hits the ground is 4.0 m/second

(C) Height of the table is 0.8 m

(D) It hits the ground at an angle of 60° to the horizontal

3. An aeroplane flying 490 m above ground level at 100 m/s, releases a block. How far on ground will it strike

(A) 0.1 km (B) 1 km (C) 2 km (D) None

4. A body is thrown horizontally from the top of a tower of height 5 m. It touches the ground at a distance of 10 m from the foot of the tower. The initial velocity of the body is ($g = 10\ ms^{-2}$)

(A) 2.5 ms^{-1} (B) 5 ms^{-1} (C) 10 ms^{-1} (D) 20 ms^{-1}

5. An aeroplane moving horizontally with a speed of 720 km/h drops a food pocket, while flying at a height of 396.9 m. the time taken by a food pocket to reach the ground and its horizontal range is (Take g = 9.8 m/sec^2)

(A) 3 sec and 2000 m (B) 5 sec and 500 m (C) 8 sec and 1500 m (D) 9 sec and 1800 m

6. A particle (A) is dropped from a height and another particle (B) is thrown in horizontal direction with speed of 5 m/sec from the same height. The correct statement is

(A) Both particles will reach at ground simultaneously

(B) Both particles will reach at ground with same speed

(C) Particle (A) will reach at ground first with respect to particle (B)

(D) Particle (B) will reach at ground first with respect to particle (A)

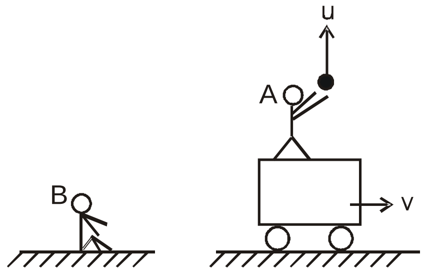
7. A body is projected horizontally from top of a tower with initial velocity 18m/s. It hits the ground at an angle of 45°. What is vertical component of its velocity just before hitting the ground?

8. A man standing on the roof of a house of height h throws one particle vertically downwards and another particle horizontally with same velocity u. find the ratio of their speeds when they reach ground.

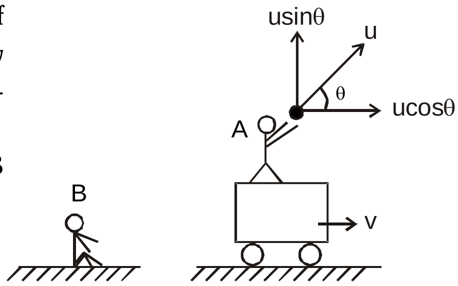
4.0 PROJECTION FROM A MOVING PLATFORM

AL

Case (1) – When a ball is thrown upward from a truck moving with uniform speed, then observer A standing in the truck, will see the ball moving in straight vertical line (upward & downward).
The observer B sitting on road, will see the ball move in a parabolic path. The horizontal speed of the ball is equal to the speed of the truck.

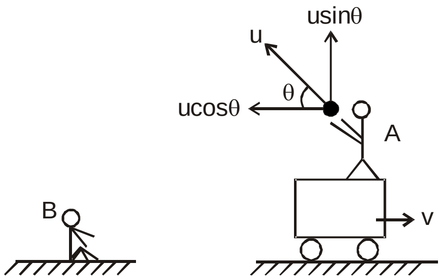


Case (2) – When a ball is thrown at some angle 'θ' in the direction of motion of the truck, horizontal & vertical component of ball's velocity w.r.t. observer A standing on the truck, is ucosθ, and usinθ respectively.
Horizontal & vertical component of ball's velocity w.r.t. observer B sitting on the ground, is u_x = ucosθ + v and u_y=usinθ respectively.



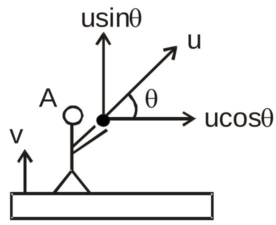
Case (3) – When a ball is thrown at some angle 'θ' in the opposite direction of motion of the truck, horizontal & vertical component of ball's velocity w.r.t. observer A standing on the truck, is ucosθ, and usinθ respectively.
Horizontal & vertical component of ball's velocity w.r.t. observer B sitting on the ground, is

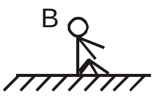
u_x = ucosθ – v and u_y=usinθ respectively.



Case (4) – When a ball is thrown at some angle 'θ' from a platform moving with speed v upwards, horizontal & vertical component of ball's velocity w.r.t. observer A standing on the moving platform, is ucosθ and usinθ respectively.
Horizontal & vertical component of ball's velocity w.r.t. observer B sitting on the ground, is

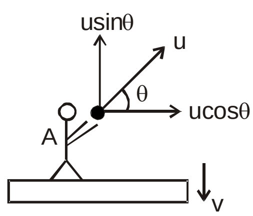
u_x = ucosθ and u_y = usinθ + v respectively.

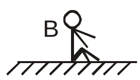




Case (5) – When a ball is thrown at some angle 'θ' from a platform moving with speed v downwards, horizontal & vertical component of ball's velocity w.r.t. observer A standing on the moving platform, is ucosθ and usinθ respectively.
Horizontal & vertical component of ball's velocity w.r.t. observer B sitting on the ground, is

u_x = ucosθ and u_y=usinθ – v respectively.





Illustrations

Illustration 14*. A boy standing on a long railroad car throws a ball straight upwards. The car is moving on the horizontal road with an acceleration of 1 m/s^2 and the projection speed in the vertical direction is 9.8 m/s. How far behind the boy will the ball fall on the car ?

Solution Let the initial velocity of car be 'u'.

time of flight,

$$t = \frac{2u_y}{g} = 2$$

where u_y = component of velocity in vertical direction

distance travelled by car

$$x_c = u \times 2 + \frac{1}{2} \times 1 \times 2^2 = 2u + 2$$

distance travelled by ball

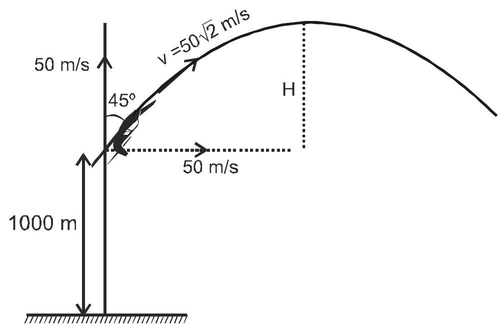
$x_b = u \times 2$

$x_c - x_b = 2u + 2 - 2u = 2m$ ***Ans.***

Illustration 15*. A fighter plane moving upward at an angle of 45° with vertical speed of $50\sqrt{2}$ m/s drop a bomb. Find :

(a) time of flight

(b) maximum height of the bomb above ground



Solution (a) $y = u_y t + \frac{1}{2}a_y t^2$

$$-1000 = 50t - \frac{1}{2} \times 10 \times t^2$$

$t^2 - 10t - 200 = 0$

$(t - 20)(t + 10) = 0$

$t = 20$ sec

(b) $$H = \frac{u_y^2}{2g} = \frac{50^2}{2g} = \frac{50 \times 50}{20} = 125 \text{ m}$$

Hence maximum hight above ground H = 1000 + 125 = 1125 m

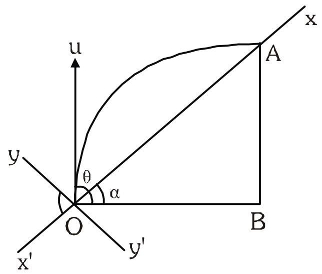
5.0 PROJECTILE MOTION ON AN INCLINED PLANE

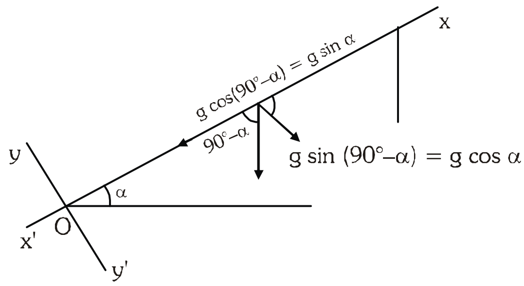
AL

5.1 Up the Plane

AL

A projectile is projected up the inclined plane from the point O with an initial velocity u at an angle θ with horizontal. The angle of inclination of the plane with horizontal is α [Fig.].





$u_x = u\cos(\theta - \alpha)$ and $a_x = -g\sin\alpha$

$u_y = u\sin(\theta - \alpha)$ and $a_y = -g\cos\alpha$

Time of flight : During motion from point O to A, the displacement along y-axis is zero.

$\therefore$ $s_y = 0$ at $t = T$

$\because$ $$s_y = u_y t + \frac{1}{2}a_y t^2$$

or $$0 = u\sin(\theta - \alpha)T - \frac{1}{2}g\cos\alpha\, T^2$$

$\therefore$ $$T = \frac{2u\sin(\theta - \alpha)}{g\cos\alpha}$$

Range : As shown in Fig, OA is the range of projectile.
Horizontal component of initial velocity $u_H = u\cos\theta$

$\therefore \quad OB = u_H T \qquad\qquad (\text{as } a_H = 0)$

$$= \frac{(u\cos\theta)2u\sin(\theta-\alpha)}{g\cos\alpha} = \frac{2u^2\sin(\theta-\alpha)\cos\theta}{g\cos\alpha}$$

$$\therefore \quad R = OA = \frac{OB}{\cos\alpha} = \frac{2u^2\sin(\theta-\alpha)\cos\theta}{g\cos^2\alpha}$$

Using, $2\sin A\cos B = \sin(A+B) + \sin(A-B)$
Range can also be written as,

$$R = \frac{u^2}{g\cos^2\alpha}\left[\sin(2\theta-\alpha)-\sin\alpha\right]$$

This range will be maximum when

$$2\theta-\alpha = \frac{\pi}{2} \quad \text{or} \quad \theta = \frac{\pi}{4}+\frac{\alpha}{2} \quad \text{and} \quad R_{max} = \frac{u^2}{g\cos^2\alpha}[1-\sin\alpha]$$

we see that for $\alpha = 0$, range will be maximum for $\theta = \frac{\pi}{4}$ or 45°.

$$R_{max} = \frac{u^2}{g\cos^2 0^\circ}(1-\sin 0^\circ)$$

$$R_{max} = \frac{u^2}{g}$$

Alternative method :

For range, $\qquad s_x = R,\ t = T$

$$s_x = u_x t + \frac{1}{2}a_x t^2$$

or $$R = u\cos(\theta-\alpha)T - \frac{1}{2}g\sin\alpha\ T^2$$

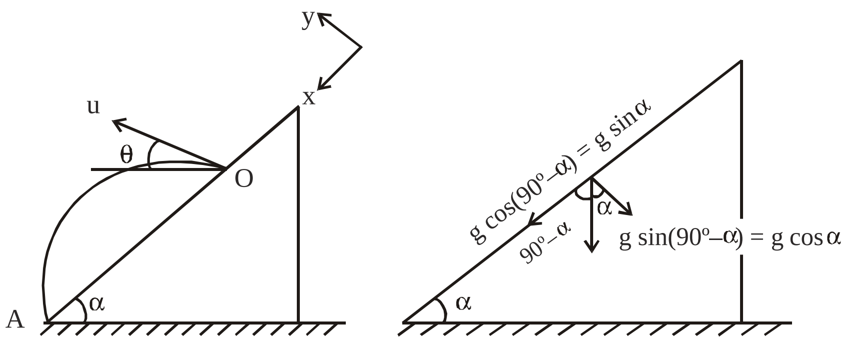
Substituting the value of $T = \frac{2u\sin(\theta-\alpha)}{g\cos\alpha}$, in above equation for R.

$$\therefore \quad R = \frac{u^2}{g\cos^2\alpha}\left[\sin(2\theta-\alpha)-\sin\alpha\right]$$

5.2 Down the Plane

AL

A projectile is projected down the plane from the point O with an initial velocity u at an angle θ with horizontal [Fig.]. The angle of inclination of plane with horizontal is α.



Therefore, $u_x = u\cos(\theta + \alpha)$, $a_x = g\sin\alpha$

$u_y = u\sin(\theta + \alpha)$, $a_y = -g\cos\alpha$

Proceeding in the similar manner, we get the following results :

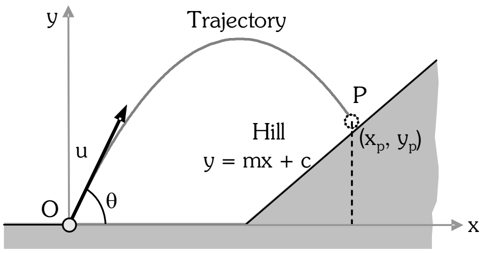
$$T = \frac{2u\sin(\theta+\alpha)}{g\cos\alpha}$$

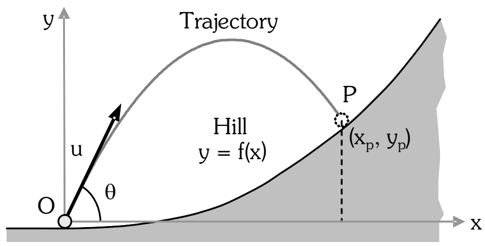
$$R = \frac{u^2}{g\cos^2\alpha}\left[\sin(2\theta+\alpha)+\sin\alpha\right]$$

5.3 Analysis of projectile on an incline plane using Equation of trajectory

AL

Sometimes the hill may be away from the point of projection or the hill may not have uniform slope as shown in the following two figures.





In these cases, the shape of the hill can be expressed by a suitable equation of the form $y = mx + c$ for uniform slope hill or $y = f(x)$ for nonuniform slope hill. The target P where the projectile hits the hill is the intersection of trajectory of the projectile and the hill. Therefore, coordinates (x_p, y_p) of the target can be obtained by simultaneously solving equation of the hill and equation of trajectory of the projectile.

Time of flight

Since a projectile move with uniform horizontal component of the velocity (u_x), its time of flight T can be calculated from the following equation.

$$T = \frac{x_p}{u_x} = \frac{x_p}{u\cos\theta}$$

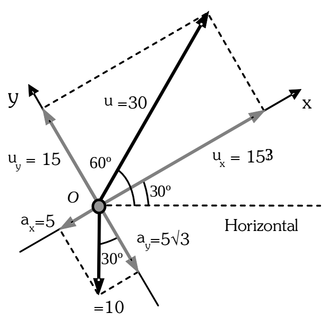
Illustrations

Illustration 16*. A particle is projected with a velocity of 30 m/s at an angle 60° above the horizontal on a slope of inclination 30°. Find its range, time of flight and angle of hit.

Solution The coordinate system, projection velocity and its component, and acceleration due to gravity and its component are shown in the adjoining figure.

Substituting corresponding values in following equation,

we get the time of flight.

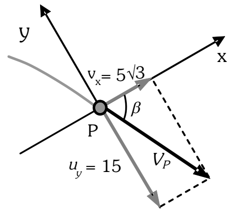


$$T = \frac{2u_y}{a_y} \rightarrow T = \frac{2\times 15}{5\sqrt{3}} = 2\sqrt{3}\ \text{s}$$

Substituting value of time of flight in following equation, we get the range R.

$$R = u_xT - \tfrac{1}{2}a_xT^2 \Rightarrow R = 15\sqrt{3}\times 2\sqrt{3} - \tfrac{1}{2}\times 5\times(2\sqrt{3})^2 = 60\text{m}$$

In the adjoining figure, components of velocity $\vec{v}_P$ when the projectile hits the slope at point P are shown. The angle β which velocity vector makes with the x-axis is known as angle of hit. The projectile hits the slope with such a velocity $\vec{v}_P$, whose y-component is equal in magnitude to that of velocity of projection. The x-component of velocity v_x is calculated by substituting value of time of flight in following equation.



$$v_x = u_x - a_x t \rightarrow v_x = 15\sqrt{3} - 5 \times 2\sqrt{3} = 5\sqrt{3}$$

$$\beta = \tan^{-1}\left(\frac{v_y}{v_x}\right) \rightarrow \beta = 60°$$

BEGINNER'S BOX-3

Projectile from Moving platform and Inclined Plane

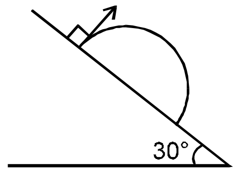
1. A plane surface is inclined making an angle θ with the horizontal. From the bottom of this inclined plane, a bullet is fired with velocity v. The maximum possible range of the bullet on the inclined plane is

(A) $\frac{v^2}{g}$ (B) $\frac{v^2}{g(1+\sin\theta)}$ (C) $\frac{v^2}{g(1-\sin\theta)}$ (D) $\frac{v^2}{g(1+\cos\theta)}$

2. A ball is projected horizontal with a speed v from the top of a plane inclined at an angle 45° with the horizontal. How far from the point of projection with the ball strike the plane?

(A) $\frac{v^2}{g}$ (B) $\sqrt{2}\,\frac{v^2}{g}$ (C) $\frac{2v^2}{g}$ (D) $\sqrt{2}\left[\frac{2v^2}{g}\right]$

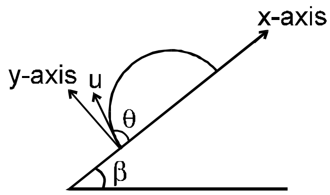
3*. A ball is projected from point A with a velocity 10 m/s perpendicular to the inclined plane as shown in figure. Range of the ball on the inclined plane is :



(A) $\frac{40}{3}$ m (B) $\frac{20}{13}$ m

(C) $\frac{13}{20}$ m (D) $\frac{13}{40}$ m

4. A particle is projected at an angle θ with an inclined plane making an angle β with the horizontal as shown in figure, speed of the particle is u, after time t find :



(a) x component of acceleration ?
(b) y component of acceleration ?
(c) x component of velocity ?
(d) y component of velocity ?
(e) x component of displacement ?
(f) y component of displacement ?
(g) y component of velocity when particle is at maximum distance from the incline plane ?

5. A person is standing on a trolley which moves horizontally with uniform velocity $5\hat{i}$ m/s. At t=0, person throws a ball with velocity $(5\cos\theta\hat{i} + 5\sin\theta\hat{j})$ m/s wrt himself. He always sees ball overhead. Then

(A) $\theta = \frac{\pi}{2}$ (B) $\theta = \frac{\pi}{4}$ (C) $\theta = \pi$ (D) $\theta = \frac{\pi}{3}$

6. A person is standing on a trolley which moves horizontally with uniform velocity $2\hat{i}$ m/s. At t=0, person throws a ball with velocity $(2\hat{j})$ m/s wrt trolley. Displacement of ball when it returns to initial horizontal level is

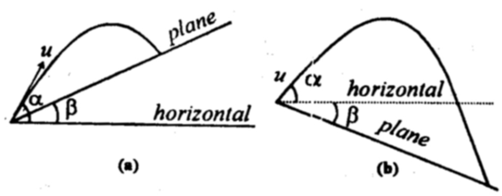
(A) 0.6m (B) 0.2m (C) 0m (D) 0.8m

7. A shot is fired at an angle 'θ' to the horizontal up a hill of inclination $\sin^{-1}\left(\frac{4}{5}\right)$ to the horizontal. If the shot strikes the hill horizontally, then the value of 'θ' is

(A) $\sin^{-1}\left(\frac{3}{\sqrt{73}}\right)$ (B) $\cos^{-1}\left(\frac{3}{\sqrt{73}}\right)$ (C) $\tan^{-1}\left(\frac{3}{\sqrt{73}}\right)$ (D) $\cot^{-1}\left(\frac{3}{\sqrt{73}}\right)$

8*. For a given velocity of projection from a point on the inclined plane, the maximum range down the plane is three times the maximum range up the incline. Then find the angle of inclination of the inclined plane.

9. A particle of projected at an angle 'α' to the horizon, up and down is a plane, inclined at an angle β to the horizontal.



If the ratio of time flights be 1 : 2, then the ratio $\frac{\tan\alpha}{\tan\beta}$ is equal to

(A) $\frac{2}{1}$ (B) $\frac{3}{1}$ (C) $\frac{4}{1}$ (D) $\frac{5}{3}$

GOLDEN KEY POINTS

- For maximum range $\theta = 45°$

$$R_{max} = \frac{u^2}{g}; \quad H_{max} = \frac{R_{max}}{2}$$

- We get the same range for two angle of projections α and $(90 - \alpha)$ but in both cases, maximum heights attained by the particles are different.

This is because, $R = \frac{u^2 \sin 2\theta}{g}$, and $\sin 2\,(90 - \alpha) = \sin 180 - 2\alpha = \sin 2\alpha$

- If $\quad R = H$

i.e. $\quad \frac{u^2 \sin 2\theta}{g} = \frac{u^2 \sin^2 \theta}{2g} \Rightarrow \tan \theta = 4$

- ***Standard results for projectile motion on an inclined plane***

	Up the Incline	Down the Incline
Range	$\frac{2u^2 \sin\alpha \cos(\alpha+\beta)}{g \cos^2\beta}$	$\frac{2u^2 \sin\alpha \cos(\alpha-\beta)}{g \cos^2\beta}$
Time of flight	$\frac{2u \sin\alpha}{g \cos\beta}$	$\frac{2u \sin\alpha}{g \cos\beta}$
Angle of projection for maximum range	$\frac{\pi}{4} - \frac{\beta}{2}$	$\frac{\pi}{4} + \frac{\beta}{2}$
Maximum Range	$\frac{u^2}{g(1+\sin\beta)}$	$\frac{u^2}{g(1-\sin\beta)}$

Here α is the angle of projection with the incline and β is the angle of incline.

- For a given speed, the direction which gives the maximum range of the projectile on an incline, bisects the angle between the incline and the vertical, for upward or downward projection.

SOME WORKED OUT ILLUSTRATIONS

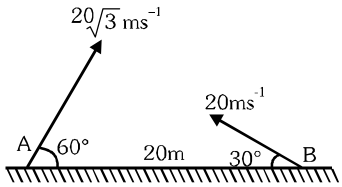


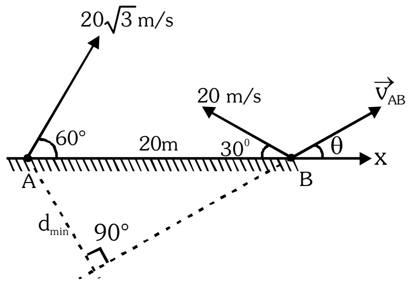
Illustration 1*.

In the figure shown, the two projectile are fired simultaneously. Find the minimum distance between them during their flight.

Solution

Taking origin at A and x axis along AB

Velocity of A w.r.t. B



$$= 20\sqrt{3}\left(\cos 60\hat{i} + \sin 60\hat{j}\right) - 20\left(\cos 150\hat{i} + \sin 150\hat{j}\right)$$

$$= 20\sqrt{3}\left(\frac{1}{2}\hat{i} + \frac{\sqrt{3}}{2}\hat{j}\right) - 20\left(-\frac{\sqrt{3}}{2}\hat{i} + \frac{1}{2}\hat{j}\right) = 20\sqrt{3}\hat{i} + 20\hat{j}$$

$$\tan\theta = \frac{20}{20\sqrt{3}} = \frac{1}{\sqrt{3}} \Rightarrow \theta = 30^0 \text{ so } \frac{d_{min.}}{20} = \sin\theta = \sin 30^\circ = \frac{1}{2} \Rightarrow d_{min.} = 10\text{m}$$

Illustration 2.

A ball is thrown from the ground to clear a wall 3 m high at a distance of 6 m and falls 18 m away from the wall, the angle of projection of ball is

(A) $\tan^{-1}\left(\frac{3}{2}\right)$ (B) $\tan^{-1}\left(\frac{2}{3}\right)$ (C) $\tan^{-1}\left(\frac{1}{2}\right)$ (D) $\tan^{-1}\left(\frac{3}{4}\right)$

Ans. (B)

Solution

From equation of trajectory $y = x\tan\theta\left[1 - \frac{x}{R}\right] \Rightarrow 3 = 6\tan\theta\left[1 - \frac{1}{4}\right] \Rightarrow \tan\theta = \frac{2}{3}$

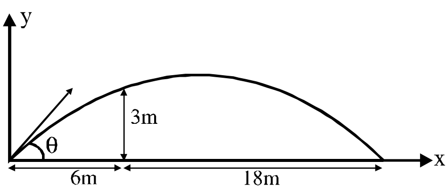
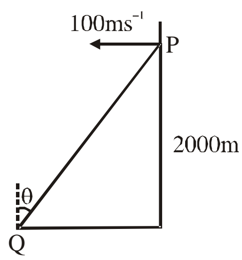


Illustration 3.



An aeroplane is travelling horizontally at a height of 2000 m from the ground. The aeroplane, when at a point P, drops a bomb to hit a stationary target Q on the ground. In order that the bomb hits the target, what angle θ must the line PQ make with the vertical ? [$g = 10ms^{-2}$]

(A) 15° (B) 30°
(C) 90° (D) 45°

Ans. (D)

Solution

Let t be the time taken by bomb to hit the target.

$h = 2000 = \frac{1}{2}gt^2 \Rightarrow t = 20$ sec

$R = ut = (100)(20) = 2000$ m

$\because \tan\theta = \frac{R}{h} = \frac{2000}{2000} = 1 \Rightarrow \theta = 45^\circ$

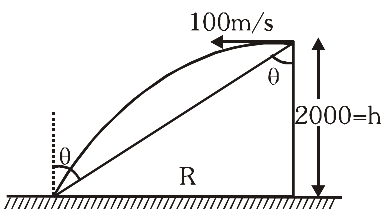
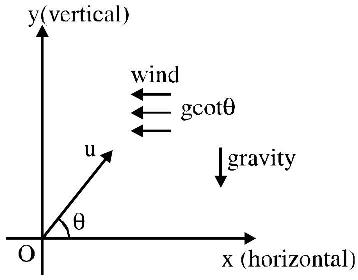


Illustration 4*.



A ball is projected as shown in figure. The ball will return to point :

(A) O (B) left to point O

(C) right to point O (D) none of these

Ans. (A)

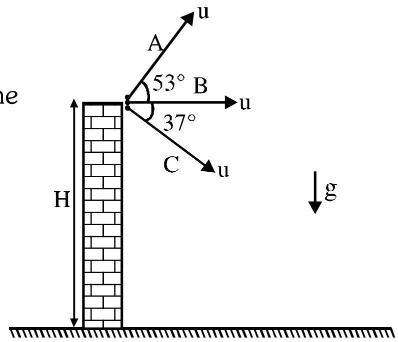
Solution

Here $\frac{a_x}{a_y} = \frac{g\cot\theta}{g} = \frac{1}{\tan\theta} = \frac{u_x}{u_y}$

$\Rightarrow$ Initial velocity & acceleration are opposite to each other.

$\Rightarrow$ Ball will return to point O.

Illustration 5*.



Three point particles A, B and C are projected from same point with same speed at t=0 as shown in figure.

For this situation select correct statement(s).

(A) All of them reach the ground at same time.

(B) All of them reach the ground at different time.

(C) All of them reach the ground with same speed.

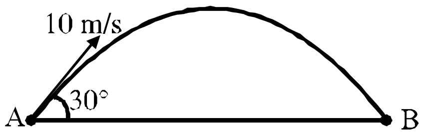
(D) All of them have same horizontal displacement when they reach the ground.

Ans. (B, C)

Solution

Vertical component of initial velocities are different $\Rightarrow$ reach the ground at different time.

Illustration 6.



As shown in the figure there is a particle of mass $\sqrt{3}$ kg, is projected with speed 10 m/s at an angle 30° with horizontal (take g = 10 m/s^2) then match the following

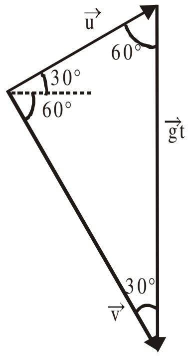
Column I		***Column II***	
(A)	Average velocity (in m/s) during half of the time of flight, is	(p)	$\frac{1}{2}$
(B)	The time (in sec) after which the angle between velocity vector and initial velocity vector becomes $\pi/2$, is	(q)	$\frac{5}{2}\sqrt{13}$
(C)	Horizontal range (in m), is	(r)	$5\sqrt{3}$
(D)	Change in linear momentum (in N-s) when particle is at highest point, is	(s)	At an angle of $\tan^{-1}\left(\frac{1}{2\sqrt{3}}\right)$ from horizontal
		(t)	2

Ans. (A) (q,s); (B) (t); (C) (r); (D) (r)

Solution :

For (A) : $v_{av} = \sqrt{(v_{avx})^2 + (v_{avy})^2} = \sqrt{(10\cos 30°)^2 + \left(\frac{10\sin 30° + 0}{2}\right)^2} = \sqrt{75 + \frac{25}{4}} = \frac{5}{2}\sqrt{13}$ m/s

Angle with horizontal $\theta = \tan^{-1}\left(\frac{v_{avy}}{v_{avx}}\right) = \tan^{-1}\left(\frac{5/2}{5\sqrt{3}}\right) = \tan^{-1}\left(\frac{1}{2\sqrt{3}}\right)$



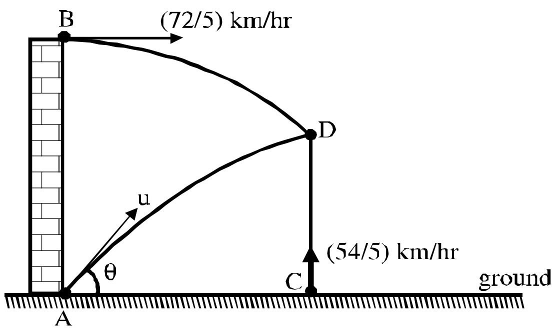
For (B) : By using $\vec{v}=\vec{u}+\vec{a}t$ We have $\frac{u}{gt}=\sin 30° \Rightarrow t=\frac{10}{(10)(1/2)}=2$

For (C) : Horizontal range(R) $=\frac{u^2\sin 2\theta}{g}=\frac{100\times\sqrt{3}/2}{10}=5\sqrt{3}$ m

For (D) : Change in linear momentum $= mu_y = \sqrt{3}\times 10\sin 30° = 5\sqrt{3}$ N-s

Illustration 7*.

In the given figure points A and C are on the horizontal ground & A and B are in same vertical plane. Simultaneously bullets are fired from A, B and C and they collide at D. The bullet at B is fired horizontally with speed of $\frac{72}{5}$ km/hr and the bullet at C is projected vertically upward at velocity of $\frac{54}{5}$ km/hr. Find velocity of the bullet projected from A in m/s.



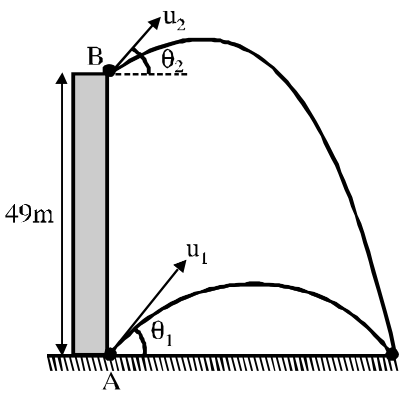
Ans. 5

Solution

For collision $u=\sqrt{u_B^2+u_C^2}=\sqrt{\left(\frac{72}{5}\times\frac{5}{18}\right)^2+\left(\frac{54}{5}\times\frac{5}{18}\right)^2}=\sqrt{4^2+3^2}=5$ m/s

Illustration 8.

Two stones A and B are projected simultaneously as shown in figure. It has been observed that both the stones reach the ground at the same place after 7 sec of their projection. Determine difference in their vertical components of initial velocities in m/s. ($g = 9.8$ m/s^2)



Ans. 7

Solution :

In time of flight i.e. 7 s, the vertical displacement of A is zero and that of B is 49 m so for relative motion of B w.r.t. A

$(u_2\sin\theta_2 - u_1\sin\theta_1)\times 7 = 49 \Rightarrow u_2\sin\theta_2 - u_1\sin\theta_1 = 7$ m/s

ANSWERS

BEGINNER'S BOX-1

1. (B) **2.** (C) **3.** (B) **4.** (B),(D) **5.** (A) **6.** (C) **7.** (B) **8.** (D)

9. (A) **10.** (D)

BEGINNER'S BOX-2

1. (C) **2.** AC **3.** (B) **4.** (C) **5.** (D) **6.** (A) **7.** 18m/s **8.** 1:1

BEGINNER'S BOX-3

1. (B)

2. (D)

3. (A)

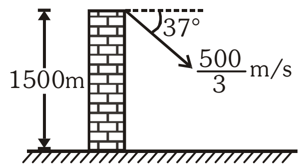
4. (a) $-g\sin\beta$, (b) $-g\cos\beta$, (c) $u\cos\theta - g\sin\beta \times t$, (d) $u\sin\theta - g\cos\beta \times t$,

(e) $u\cos\theta \times t - \frac{1}{2}g\sin\beta \times t^2$, (f) $u\sin\theta \times t - \frac{1}{2}g\cos\beta \times t^2$, (g) zero.

5. (A)
6. (D)
7. (B)
8. 30°
9. (B)

EXERCISE – 1 MCQ (SINGLE CHOICE CORRECT)

1. A particle is projected from a tower as shown in figure, then the distance from the foot of the tower where it will strike the ground will be :
(take $g = 10\ m/s^2$)



(A) 4000/3 m (B) 5000/3 m
(C) 2000 m (D) 3000 m

2. Particle is dropped from the height of 20m on horizontal ground. There is wind blowing due to which horizontal acceleration of the particle becomes 6 ms^{-2}. Find the horizontal displacement of the particle till it reaches ground.
(A) 6m (B) 10 m (C) 12m (D) 24 m

3. A projectile is projected at an angle ($\alpha > 45°$) with an initial velocity u. The time t, at which its magnitude of horizontal velocity will equal the magnitude of vertical velocity is :

(A) $t = \frac{u}{g}(\cos\alpha - \sin\alpha)$ (B) $t = \frac{u}{2g}(\cos\alpha + \sin\alpha)$ (C) $t = \frac{u}{g}(\sin\alpha - \cos\alpha)$ (D) $t = \frac{u}{g}(\sin^2\alpha - \cos^2\alpha)$.

4. A particle is projected from a horizontal plane (x–z plane) such that its velocity vector at time t is given by $\vec{v} = a\hat{i} + (b - ct)\hat{j}$. Its range on the horizontal plane is given by :

(A) $\frac{ba}{c}$ (B) $\frac{2ba}{c}$ (C) $\frac{3ba}{c}$ (D) None

5. A particle is dropped from a height h . Another particle which was initially at a horizontal distance 'd' from the first, is simultaneously projected with a horizontal velocity 'u' and the two particles just collide on the ground . The three quantities h. d and u are related as :

(A) $d^2 = \frac{u^2h}{2g}$ (B) $d^2 = \frac{2u^2h}{g}$ (C) $d = h$ (D) $gd^2 = u^2h$

6. A person can throw a stone to a maximum horizontal distance of h. The greatest height to which he can throw the stone is
(A) h (B) h/2 (C) 2h (D) 3h

7. The point from where a ball is projected is taken as the origin of the coordinate axis. The x and y components of its displacement are given by $x = 6t$ and $y = 8t - 5t^2$. What is the velocity of projection?
(A) 6 m/s (B) 8 m/s (C) 10 m/s (D) 14 m/s

8. A particle is projected with a velocity v, so that its range on a horizontal plane is twice the greatest height attained. If g is acceleration due to gravity, then its range is

(A) $\frac{4v^2}{5g}$ (B) $\frac{4g}{5v^2}$ (C) $\frac{4v^3}{5g^2}$ (D) $\frac{4v}{5g^2}$

9. A body is thrown horizontally with a velocity $\sqrt{2gh}$ from the top of a tower of height h. It strikes the level ground through the foot of the tower at a distance x from the tower. The value of x is :
(A) h (B) h/2 (C) 2h (D) 2h/3

10. A ball is thrown upwards at an angle of 60° to the horizontal. It falls on the ground at a distance of 90 m. If the ball is thrown with the same initial velocity at an angle 30°, it will fall on the ground at a distance of
(A) 120 m (B) 90 m (C) 60 m (D) 30 m

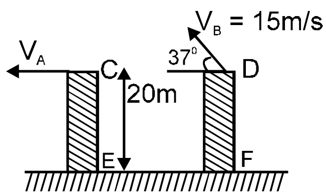
11. A ball is projected from a certain point on the surface of a planet at a certain angle with the horizontal surface. The horizontal and vertical displacements x and y vary with time t in second as : $x = 10\sqrt{3}\,t$; $y = 10t - t^2$. The maximum height attained by the ball is :

(A) 100m (B) 75m (C) 50 m (D) 25m

12. The horizontal range of a projectile is R and the maximum height attained by it is H. A strong wind now begins to blow in the direction of motion of the projectile, giving it a constant horizontal acceleration = g/2. Under the same conditions of projection. Find the horizontal range of the projectile.

(A) R +H (B) R + 2H (C) R (D) R +H/2

13. CE and DF are two walls of equal height (20 meter) from which two particles A and B of same mass are projected as shown in the figure. A is projected horizontally towards left while B is projected at an angle 37^0 (with horizontal towards left) with velocity 15 m/sec. If A always sees B to be moving perpendicular to EF, then the range of A on ground is :



(A) 24 m (B) 30 m

(C) 26 m (D) 28 m

14. Two projectiles are thrown simultaneously in the same plane from the same point, their velocities are v_1 and v_2 at angles θ_1 and θ_2 respectively from the horizontal.

If $v_1\cos\theta_1 = v_2\cos\theta_2$, then choose the incorrect statement :

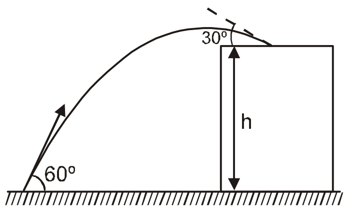
(A) one particle will remain exactly below or above the other particle

(B) the trajectory of one with respect to other will be a vertical straight line

(C) both will have the same range

(D) none of these

15*. A stone projected at an angle of 60° from the ground level strikes at an angle of 30° on the roof of a building of height 'h= 30m '. Find the speed of projection(in m/s) of the stone.



(A) 30 (B) 40

(C) 50 (D) 60

16. A ball is thrown eastward across level ground. A wind blows horizontally to the east, and assume that the effect of wind is to provide a constant force to the east, equal in magnitude to the weight of the ball. The angle θ (with respect to horizontal) at which the ball should be projected so that it travels maximum horizontal distance is

(A) 45° (B) 37° (C) 53° (D) 67.5°

17*. A particle at a height 'h' from the ground is projected with an angle 30° from the horizontal, it strikes the ground making angle 45° with horizontal. It is again projected from the same point at height h with the same speed but with an angle of 60° with horizontal. Find the angle it makes with the horizontal when it strikes the ground :

(A) $\tan^{-1}$ (D) (B) $\tan^{-1}$ (5) (C) $\tan^{-1}\left(\sqrt{5}\right)$ (D) $\tan^{-1}\left(\sqrt{3}\right)$

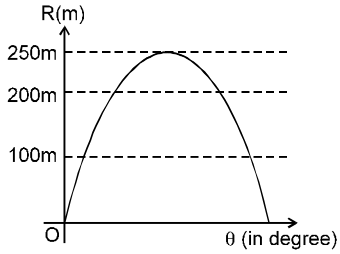
18. An airplane moving horizontally with a speed of 180 kmph drops a food packet while flying at a height of 490 m. The horizontal range of the food packet is ($g = 9.8\ m/s^2$)

(A) 180 m (B) 980 m (C) 500 m (D) 670 m

19. A projectile is thrown with a speed u, at an angle θ to an inclined plane of inclination β. The angle θ at which the projectile is thrown such that it strikes the inclined plane normally is

(A) $\cot^{-1}(2\tan\beta)$ (B) $\cot^{-1}(\tan\beta)$ (C) $\tan^{-1}\dfrac{(\cot\beta)}{2}$ (D) None of these

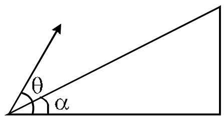
20*. From the ground level, a ball is to be shot with a certain speed. Graph shows the range R it will have versus the launch angle θ. The least speed the ball will have during its flight if θ is chosen such that the flight time is half of its maximum possible value, is equal to (take $g = 10\ m/s^2$)



(A) 250 m/s (B) $50\sqrt{3}$ m/s

(C) 50 m/s (D) $25\sqrt{3}$ m/s

21. A projectile is fired at an angle θ with the horizontal. Find the condition under which it lands perpendicular on an inclined plane inclination α as shown in figure.



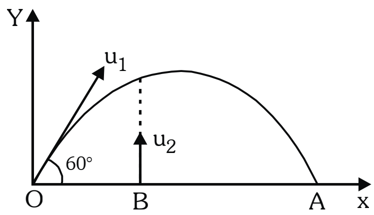
(A) $\sin\alpha = \cos(\theta - \alpha)$ (B) $\cos\alpha = \sin(\theta - \alpha)$

(C) $\tan\theta = \cot(\theta - \alpha)$ (D) $\cot(\theta - \alpha) = 2\tan\alpha$

22. A body is projected horizontally from the top of a tower with initial velocity 18 m/s. It hits the ground at angle 45°. What is the vertical component of velocity when it strikes the ground?

(A) 9 m/s (B) $9\sqrt{2}$ m/s (C) 18 m/s (D) $18\sqrt{2}$ m/s

23. As shown in figure, a body is projected with velocity u_1 from the point O. At the same time another body is projected vertically upwards with the velocity u_2 from the point B. The value of u_1/u_2 so that both the bodies collide is



(A) $\dfrac{2}{\sqrt{3}}$ (B) $\dfrac{1}{\sqrt{3}}$

(C) $\sqrt{3}$ (D) $\sqrt{2}$

24. It was calculated that a shell when fired from a gun with a certain velocity and at an angle of elevation $\dfrac{5\pi}{36}$ rad should strike a given target in the same horizontal plane. In actual practice, it was found that a hill just prevented the trajectory. At what angle of elevation should the gun be fired to hit the target.

(A) $\dfrac{5\pi}{36}$ rad (B) $\dfrac{11\pi}{36}$ rad (C) $\dfrac{7\pi}{36}$ rad (D) $\dfrac{13\pi}{36}$ rad.

25. A particle moves along the parabolic path $y = ax^2$ in such a way that the x component of the velocity remains constant, say c. The acceleration of the particle is :

(A) $ac\hat{k}$ (B) $2ac^2\hat{j}$ (C) $ac^2\hat{j}$ (D) $a^2c\hat{j}$

EXERCISE – 2 **MCQ (ONE OR MORE CHOICE CORRECT)**

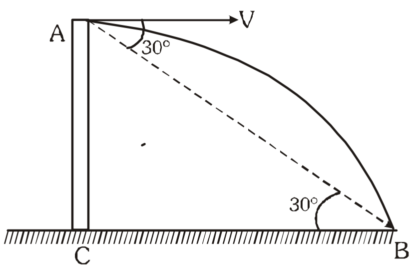
1. A particle is projected from a point P with a velocity v at an angle θ with horizontal. At a certain point Q it moves at right angle to its initial direction. Then :
(A) Velocity of particle at Q is $v\sin\theta$
(B) Velocity of particle at Q is $v\cot\theta$
(C) Time of flight from P to Q is $(v/g)\operatorname{cosec}\theta$
(D) Time of flight from P to Q is $(v/g)\sec\theta$

2. Two particles A & B projected along different directions from the same point P on the ground with the same velocity of 70 m/s in the same vertical plane . They hit the ground at the same point Q such that PQ = 480 m. Then : $(g = 9.8 m/s^2)$
(A) Ratio of their times of flight is 4 : 5
(B) Ratio of their maximum heights is 9 : 16
(C) Ratio of their minimum speeds during flights is 4 : 3
(D) The bisector of the angle between their directions of projection makes 45^0 with horizontal

3. Two particles P & Q are projected simultaneously from a point O on a level ground in the same vertical plane with the same speed in directions making angle of 30^0 and 60^0 respectively with the horizontal.
(A) Both reach the ground simultaneously
(B) P reaches the ground earlier than Q
(C) Both strike the same point on the level ground
(D) The maximum height attained by Q is thrice that attained by P

4. One stone is projected horizontally from a 20 m high cliff with an initial speed of $10\ ms^{-1}$. A second stone is simultaneously dropped from that cliff. Which of the following is true?
(A) Both strike the ground with the same speed.
(B) The ball with initial speed $10\ ms^{-1}$ reaches the ground first.
(C) Both the balls hit the ground at the same time.
(D) Both strike the ground with different speed

5. An object is thrown horizontally from a point 'A' from a tower and hits the ground 3s later at B. The line from 'A' to 'B' makes an angle of 30° with the horizontal. The initial velocity of the object is : (take $g = 10\ m/s^2$)



(A) $15\sqrt{3}$ m/s
(B) 15 m/s
(C) $10\sqrt{3}$ m/s
(D) $25/\sqrt{3}$ m/s

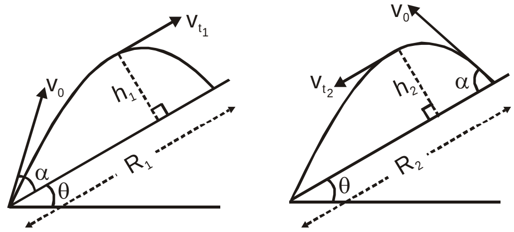
6. Two particles are projected from the same point with the same speed, at different angles θ_1 and θ_2 to the horizontal. They have the same horizontal range. Their time of flight are t_1 and t_2 respectively. Then :
(A) $\theta_1 + \theta_2 = 90°$
(B) $\frac{t_1}{t_2} = \tan\theta_1$
(C) $\frac{t_1}{t_2} = \tan\theta_2$
(D) $\frac{t_1}{\sin\theta_1} = \frac{t_2}{\sin\theta_2}$

7. Choose the correct alternative (s) :
(A) If the greatest height to which a man can throw a stone is h, then the greatest horizontal distance upto which he can throw the stone is 2h.
(B) The angle of projection for a projectile motion whose range R is n times the maximum height is $\tan^{-1}(4/n)$
(C) The time of flight T and the horizontal range R of a projectile are connected by the equation $gT^2 = 2R\tan\theta$ where θ is the angle of projection.
(D) A ball is thrown vertically up. Another ball is thrown at an angle θ with the vertical. Both of them remain in air for the same period of time. Then the ratio of maximum heights attained by the two balls 1 : 1.

8. A particle is launched from the origin with an initial velocity $\vec{u} = (3\hat{i})\,ms^{-1}$ under the influence of a constant acceleration $\vec{a} = -\left(\hat{i} + \frac{1}{2}\hat{j}\right) ms^{-2}$. Its velocity $\vec{v}$ and position vector $\vec{r}$ when it reaches its maximum x-coordinate are

(A) $\vec{v} = \left(-1.5\hat{j}\right) ms^{-1}$ (B) $\vec{v} = -2\hat{j}$ (C) $\vec{r} = (3\hat{i} - 2\hat{j})\,m$ (D) $\vec{r} = (4.5\hat{i} - 2.25\hat{j})\,m$

9. Two balls are thrown from an inclined plane at angle of projection α with the plane, one up the inclined and other down the inclined as shown in figure. Then :



(A) $h_1 = h_2 = \dfrac{v_0^2 \sin^2 \alpha}{2g\cos\theta}$

(B) $T_1 = T_2 \dfrac{2v_0 \sin\alpha}{g\cos\theta}$

(C) $R_2 - R_1 = g\sin\theta \cdot T^2$ (where T represents time of flight)

(D) $v_{t_2} = v_{t_1}$

10. At what angle should a body be projected with a velocity 24 ms^{-1} just to pass over the obstacle 14 m high at a distance of 24 m. [Take g = 10 ms^{-2}]

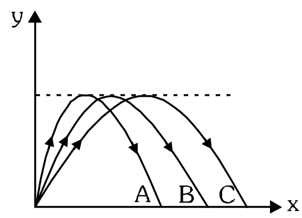
(A) $\tan\theta = 3.8$ (B) $\tan\theta = \sqrt{2}$ (C) $\tan\theta = 3.2$ (D) $\tan\theta = 2$

Match the column

11. A particle is projected from level ground. Assuming projection point as origin, x-axis along horizontal and y-axis along vertically upwards. If particle moves in x-y plane and its path is given by $y = ax - bx^2$ where a, b are positive constants. Then match the physical quantities given in column-I with the values given in column-II. (g in column II is acceleration due to gravity) :

Column I	**Column II**
(A) Horizontal component of velocity	(p) $\frac{a}{b}$
(B) Time of flight	(q) $\frac{a^2}{4b}$
(C) Maximum height	(r) $\sqrt{\frac{g}{2b}}$
(D) Horizontal range	(s) $\sqrt{\frac{2a^2}{bg}}$

12. The trajectories of the motion of three particles are shown in figure. Match the entries of column I with the entries of column II.



	Column-I		**Column-II**
(A)	Time of flight is least for	(p)	A
(B)	Vertical component of the velocity is greatest for	(q)	B
(C)	Horizontal component of the velocity is greatest for	(r)	C
(D)	Launch speed is least for	(s)	No appropriate match given

13. A body is projected with a velocity of 60 m s^{-1} at 30° to horizontal.

	Column-I		Column-II
(A)	Initial velocity vector	(p)	$60\sqrt{3}\,\hat{i}+40\hat{j}$
(B)	Velocity after 3s	(q)	$30\sqrt{3}\,\hat{i}+10\hat{j}$
(C)	Displacement after 2s	(r)	$30\sqrt{3}\,\hat{i}+30\hat{j}$
(D)	Velocity after 2s	(s)	$30\sqrt{3}\,\hat{i}$

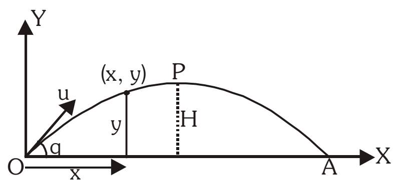
14. A particle projected with a velocity u at an angle θ above the horizontal and R is the horizontal range of this projectile.

	Column-I		Column-II
(A)	Time of flight	(p)	$y = x\tan\theta\left(1-\frac{x}{R}\right)$
(B)	Equation of trajectory	(q)	$\sqrt{(u\cos\theta)^2+(u\sin\theta-gf)^2}$
(C)	Speed at anytime	(r)	$\dfrac{2u\sin\theta}{g}$
(D)	Speed at highest point	(s)	$u\cos\theta$

Comprehension based Questions

Comprehension-1

The trajectory of a projectile in a vertical plane is $y=\sqrt{3}\,x-2x^2$. [$g = 10\ m/s^2$]. x and y are in metres.



15. Angle of projection θ is :

(A) 30° (B) 60° (C) 45° (D) $\sqrt{3}$ rad

16. Maximum height H is :

(A) $\frac{8}{3}$ m (B) $\frac{3}{8}$ m (C) $\sqrt{3}$ m (D) $\frac{2}{\sqrt{3}}$ m

17. Range OA is :

(A) $\frac{\sqrt{3}}{2}$ (B) $\frac{\sqrt{3}}{4}$ (C) $\sqrt{3}$ (D) $\frac{3}{8}$

18. Time of flight of the projectile is :

(A) $\sqrt{\frac{3}{10}}$s (B) $\sqrt{\frac{10}{3}}$s (C) 1s (D) 2s

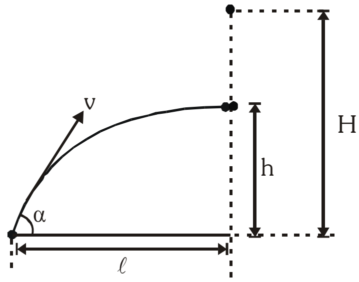
EXERCISE – 3 SUBJECTIVE

1. A particle is projected with a speed v and an angle θ to the horizontal. After a time t, the magnitude of the instantaneous velocity is equal to the magnitude of the average velocity from 0 to t. Find t.

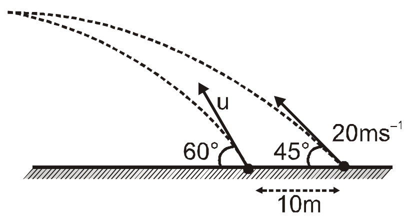
2. A projectile is thrown with speed u making angle θ with horizontal at t =0. It just crosses the two points at equal height at time t=1 s and t = 3sec respectively. Calculate maximum height attained by it. ($g=10m/s^2$)

3. A Bomber flying upward at an angle of 53° with the vertical releases a bomb at an altitude of 800 m . The bomb strikes the ground 20s after its release . Find :[Given sin 53° = 0.8 ; g = 10 ms^{-2}]
(i) The velocity of the bomber at the time of release of the bomb .
(ii) The maximum height attained by the bomb .
(iii) The horizontal distance travelled by the bomb before it strikes the ground
(iv) The velocity (magnitude & direction) of the bomb just when it strikes the ground .

4. Two particles are projected from the two towers simultaneously, as shown in the figure. What should be value of 'd' for their collision.



5. A particle leaves the origin with an initial velocity $\vec{v} = (3\hat{i})m/s$ and a constant acceleration $a = (-1.0\hat{i} - 0.50\hat{j})m/s^2$. When the particle reaches its maximum x coordinate, what are
(a) its velocity and (b) its position vector ?



6. A particle is moving in x-y plane. At time t = 0, particle is at (1m, 2m) and has velocity $(4\hat{i}+6\hat{j})m/s$. At t = 4s, particle reaches at (6m, 4m) and has velocity $(2\hat{i}+10\hat{j})m/s$. In the given time interval, find
(a) average velocity
(b) average acceleration and
(c) from the given data, can you find average speed ?

7. A particle of mass 1 kg has a velocity of 2 m/s. A constant force of 2N acts on the particle for 1s in a direction perpendicular to its initial velocity. Find the velocity and displacement of the particle at the end of 1s.

8. At time t = 0, a particle is at (2m, 4m). It starts moving towards positive x-axis with constant acceleration 2 m/s^2 (initial velocity = 0). After 2s an acceleration of 4 m/s^2 starts acting on the particle in negative y-direction also. Find after next 2s:
(a) velocity and
(b) coordinates of particle

9. A body is projected up such that its position vector varies with times as $\vec{r} = (3t\hat{i} + (4t - 5t^2)\hat{j})$. Here, t is in seconds. Find the time and x-coordinate of particle when its y-coordinate is zero.

10. At what angle should a projectile be thrown such that the horizontal range of the projectile will be equal to half of its maximum value ?

EXERCISE – 4 RECAP OF AIEEE/JEE (MAIN)

1*. A particle is moving with velocity $\vec{v} = K(y\hat{i} + x\hat{j})$, where K is a constant. The general equation for its path is: **[JEE(Main) 2010]**

(A) $y = x^2$ + constant (B) $y^2 = x$ + constant (C) xy = constant (D) $y^2 = x^2$ + constant

2*. A water fountain on the ground sprinkles water all around it. If the speed of water coming out of the fountain is v, the total area around the fountain that gets wet is : **[JEE(Main) 2011]**

(A) $\pi\frac{v^4}{g^2}$ (B) $\frac{\pi}{2}\frac{v^4}{g^2}$ (C) $\pi\frac{v^2}{g^2}$ (D) $\pi\frac{v^4}{g}$

3. A boy can throw a stone up to a maximum height of 10 m. The maximum horizontal distance that the boy can throw the same stone up to will be : **[JEE(Main) 2012]**

(A) $20\sqrt{2}$ m (B) 10 m (C) $10\sqrt{2}$ m (D) 20 m

4. A projectile is given an initial velocity of $(\hat{i} + 2\hat{j})$ m/s, where $\hat{i}$ is along the ground and $\hat{j}$ is along the vertical. If g = 10 m/s², the equation of its trajectory is : **[JEE(Main) 2013]**

(A) $y = 2x - 5x^2$ (B) $4y = 2x - 5x^2$ (C) $4y = 2x - 25x^2$ (D) $y = x - 5x^2$

5*. From a tower of height H, a particle is thrown vertically upwards with a speed u. The time taken by the particle, to hit the ground, is n times that taken by it to reach the highest point of its path.
The relation between H, u and n is : **[JEE(Main) 2013]**

(A) $2gH = n^2u^2$ (B) $gH = (n-2)^2u^2$ (C) $2gH = nu^2(n-2)$ (D) $gH = (n-2)u^2$

6. A particle is moving with a velocity $\vec{v} = K(y\hat{i} + x\hat{j})$ where K is a constant. The general equation for its path is **[JEE(Main) 2019]**

(A) xy = constant (B) $y^2 = x^2$ + constant (C) $y = x^2$ + constant (D) $y^2 = x$ + constant

7. A particle moves from the point $(2.0\hat{i} + 4.0\hat{j})$ at t = 0, with an initial velocity $(5.0\hat{i} + 4.0\hat{j})\ ms^{-1}$. It is acted upon by a constant force which produces a constant acceleration $(4.0\hat{i} + 4.0\hat{j})\ ms^{-2}$. What is the distance of the particle from the origin at time 2 s ? **[JEE(Main) 2019]**

(A) $20\sqrt{2}$ m (B) $10\sqrt{2}$ m (C) 5 m (D) 15 m

8. Two guns A and B can fire bullets at speeds 1 km/s and 2 km/s respectively. From a point on a horizontal ground, they are fired in all possible directions. The ratio of maximum areas covered by the bullets fired by the two guns, on the ground is : **[JEE(Main) 2019]**

(A) 1 : 2 (B) 1 : 4 (C) 1 : 8 (D) 1 : 16

9. The trajectory of a projectile near the surface of the earth is given as $y = 2x - 9x^2$. If it were launched at an angle θ_0 with speed v_0 then ($g = 10\ ms^{-2}$) : **[JEE(Main) 2019]**

(A) $\theta_0 = \cos^{-1}\left(\frac{1}{\sqrt{5}}\right)$ and $v_0 = \frac{5}{3}ms^{-1}$ (B) $\theta_0 = \sin^{-1}\left(\frac{1}{\sqrt{5}}\right)$ and $v_0 = \frac{5}{3}ms^{-1}$

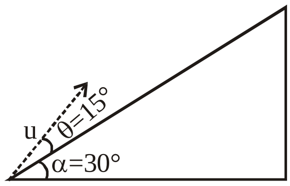
(C) $\theta_0 = \sin^{-1}\left(\frac{2}{\sqrt{5}}\right)$ and $v_0 = \frac{3}{5}ms^{-1}$ (D) $\theta_0 = \cos^{-1}\left(\frac{2}{\sqrt{5}}\right)$ and $v_0 = \frac{3}{5}ms^{-1}$

10. The position vector of a particle changes with time according to the relation $\vec{r}(t) = 15t^2\hat{i} + (4 - 20t^2)\hat{j}$. What is the magnitude of the acceleration at t = 1 ? (g = 10 ms^{-2}) : ***[JEE(Main) 2019]***

(A) 40 (B) 100 (C) 25 (D) 50

11. A plane is inclined at an angle $\alpha = 30°$ with a respect to the horizontal. A particle is projected with a speed $u = 2\ ms^{-1}$ from the base of the plane, making an angle $\theta = 15°$ with respect to the plane as shown in the figure. The distance from the base, at which the particle hits the plane is close to : (Take $g = 10\ ms^{-2}$)

[JEE(Main) 2019]



(A) 14 cm (B) 20 cm (C) 18 cm (D) 26 cm

12. A shell is fired from a fixed artillery gun with an initial speed u such that it hits the target on the ground at a distance R from it. If t_1 and t_2 are the values of the time taken by it to hit the target in two possible ways, the product t_1t_2 is : ***[JEE(Main) 2019]***

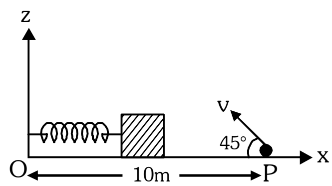
(A) R/g (B) R/4g (C) 2R/g (D) R/2g

13. Two particles are projected from the same point with the same speed u such that they have the same range R, but different maximum heights, h_1 and h_2. Which of the following is correct ? ***[JEE(Main) 2019]***

(A) $R^2 = 2\ h_1h_2$ (B) $R^2 = 16h_1h_2$ (C) $R^2 = 4\ h_1h_2$ (D) $R^2 = h_1h_2$

EXERCISE – 5 RECAP OF IIT-JEE/JEE (ADVANCED)

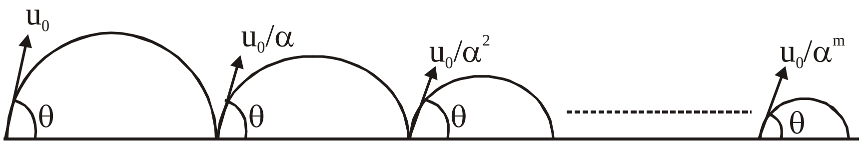
1. A small block is connected to one end of a massless spring of un-stretched length 4.9 m. The other end of the spring (see the figure) is fixed. The system lies on a horizontal frictionless surface. The block is stretched by 0.2 m and released from rest at t = 0. It then executes simple harmonic motion with angular frequency $\omega = \pi/3$ rad/s. Simultaneously at t = 0, a small pebble is projected with speed v from point P at an angle of 45° as shown in the figure. Point P is at a horizontal distance of 10 m from O. If the pebble hits the block at t = 1s, the value of v is (take g = 10 m/s^2) ***[JEE 2012]***



(A) $\sqrt{50}\,m/s$ (B) $\sqrt{51}\,m/s$ (C) $\sqrt{52}\,m/s$ (D) $\sqrt{53}\,m/s$

2. A ball is projected from the ground at an angle of 45° with the horizontal surface. It reaches a maximum height of 120 m and returns to the ground. Upon hitting the ground for the first time, it loses half of its kinetic energy. Immediately after the bounce, the velocity of the ball makes an angle of 30° with the horizontal surface. The maximum height it reaches after the bounce, in metres, is......... ***[JEE 2018]***

3. A ball is thrown from ground at an angle θ with horizontal and with an initial speed u_0. For the resulting projectile motion, the magnitude of average velocity of the ball up to the point when it hits the ground for the first time is V_1. After hitting the ground, ball rebounds at the same angle θ but with a reduced speed of u_0/α. Its motion continues for a long time as shown in figure. If the magnitude of average velocity of the ball for entire duration of motion is 0.8 V_1, the value of α is______ ***[JEE 2019]***



ANSWERS

EXERCISE-1

Que.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Ans.	A	C	C	B	B	B	C	A	C	B	D	B	A	C	A
Que.	16	17	18	19	20	21	22	23	24	25					
Ans.	D	C	C	C	D	D	C	A	D	B					

EXERCISE-2

Que.	1	2	3	4	5	6	7	8	9	10
Ans.	B,C	B,C,D	B,C,D	C,D	A	A,B,D	A,B,C,D	A,D	A,B,C	A

- ***Match the Column*** **11.** (A) r (B) s (C) q (D) p **12.** (A) p; (B) q; (C) r; (D) s
  13. (A) s; (B) s; (C) r; (D) p **14.** (A) r; (B) s; (C) p; (D) q
- ***Comprehension Based Questions:***
  Comprehension 1: **15** (B) **16** (B) **17.** (A) **18.** (A)

EXERCISE-3

1. $\frac{4}{3}\frac{v\sin\theta}{g}=t$

2. 20 m

3. (i) 100 m/s (ii) 980 m (iii) 1600 m (iv) $(80\hat{i}-140\hat{j})$

4. 20 m

5. (a) $(-1.5\hat{j})m/s$ (b) $(4.5\hat{i}-2.25\hat{j})m$

6. (a) $(1.25\hat{i}+0.5\hat{j})m/s$; (b) $(-0.5\hat{i}+\hat{j})m/s^2$; (c) No

7. $-2\sqrt{2}\,m/s$ at 45° with initial velocity.; $\sqrt{5}\,m$ at $\tan^{-1}\left(\frac{1}{2}\right)$ from initial velocity.

8. (a) $(8\hat{i}-8\hat{j})m/s$ (b) (18 m, – 4m)

9. Time = 0, 0.8 sec.; x-coordinate = 0, 2.4 m

10. 15°, 75°

EXERCISE-4

Que.	1	2	3	4	5	6	7	8	9	10	11	12	13
Ans.	D	A	D	A	C	B	A	D	A	D	B	C	B

EXERCISE-5

1. A **2.** 30.00 **3.** 4.00

RELATIVE MOTION

Recap of Early Classes

You must be familiar with the experience of travelling in a train and being overtaken by another train moving in the same direction as you are. While that train must be travelling faster than you to be able to pass you, it does seem slower to you than it would be to someone standing on the ground and watching both the trains. In case both the trains have the same velocity with respect to the ground, then to you the other train would seem to be not moving at all. To understand such observations, we now introduce the concept of relative motion.

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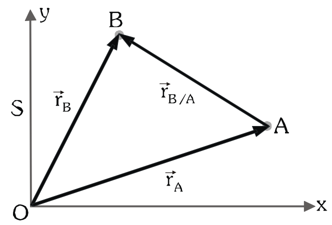
RELATIVE MOTION

Motion of a body can only be observed, when it changes its position with respect to some other body. In this sense, motion is a relative concept. To analyze motion of a body say A, therefore we have to fix our reference frame to some other body say B. The result obtained is motion of body A relative to body B.

1.0 RELATIVE POSITION, RELATIVE VELOCITY AND RELATIVE ACCELERATION

SL AL



Let two bodies represented by particles A and B at positions defined by position vectors $\vec{r}_A$ and $\vec{r}_B$, moving with velocities $\vec{v}_A$ and $\vec{v}_B$ and accelerations $\vec{a}_A$ and $\vec{a}_B$ with respect to a reference frame S. For analyzing motion of terrestrial bodies the reference frame S is fixed with the ground.

The vectors $\vec{r}_{B/A}$ denotes position vector of B relative to A. Following triangle law of vector addition, we have

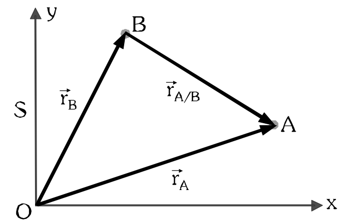
$$\vec{r}_B = \vec{r}_A + \vec{r}_{B/A} \qquad ...(i)$$

First derivatives of $\vec{r}_A$ and $\vec{r}_B$ with respect to time equals to velocity of particle A and velocity of particle B relative to frame S and first derivative of $\vec{r}_{B/A}$ with respect to time defines velocity of B relative to A.

$$\vec{v}_B = \vec{v}_A + \vec{v}_{B/A} \qquad ...(ii)$$

Second derivatives of $\vec{r}_A$ and $\vec{r}_B$ with respect to time equals to acceleration of particle A and acceleration of particle B relative to frame S and second derivative of $\vec{r}_{B/A}$ with respect to time defines acceleration of B relative to A.

$$\vec{a}_B = \vec{a}_A + \vec{a}_{B/A} \qquad ...(iii)$$



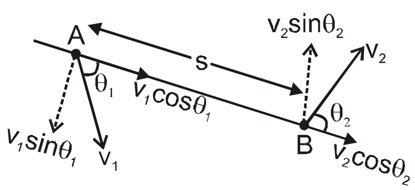
In similar fashion motion of particle A relative to particle B can be analyzed with the help of adjoining figure. You can observe in the figure that position vector of A relative to B is directed from B to A and therefore

$$\vec{r}_{B/A} = -\vec{r}_{A/B}\ ,\ \vec{v}_{B/A} = -\vec{v}_{A/B} \text{ and } \vec{a}_{B/A} = -\vec{a}_{A/B}\ .$$

The above equations elucidate that how a body A appears moving to another body B is opposite to how body B appears moving to body A.

2.0 VELOCITY OF APPROACH / SEPARATION

SL AL



It is the component of relative velocity of one particle w.r.t. another, along the line joining them.

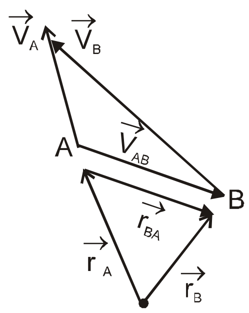
$v_2 \cos \theta_2 - v_1 \cos \theta_1 = \dfrac{ds}{dt}$ rate of increase/decrease in separation distance

If the separation is decreasing, we say it is velocity of approach and if separation is increasing, then we say it is velocity of separation.

In one dimension, since relative velocity is along the line joining A and B, hence velocity of approach / separation is simply equal to magnitude of relative velocity of A w.r.t. B

2.1 Condition to collide or to reach at the same point

SL AL



When the relative velocity of one particle w.r.t. to other particle is directed towards each other then they will collide. (If there is a zero relative acceleration).

Two particles collide if $\vec{r}_{BA}$ and $\vec{v}_{AB}$ have same direction. For the same direction of these two vectors unit vectors in the direction of $\vec{r}_{BA}$ and $\vec{v}_{AB}$ must be equal.

$$\frac{\vec{r}_B - \vec{r}_A}{|\vec{r}_B - \vec{r}_A|} = -\frac{\vec{v}_B - \vec{v}_A}{|\vec{v}_B - \vec{v}_A|}$$

2.2 Minimum / Maximum distance between two particles

AL

If the separation between two particles decreases and after some time it starts increasing then the separation between them will be minimum at the instant, velocity of approach changes to velocity of separation. (at this instant $v_{app} = 0$)

Mathematically separation between two particles A and B

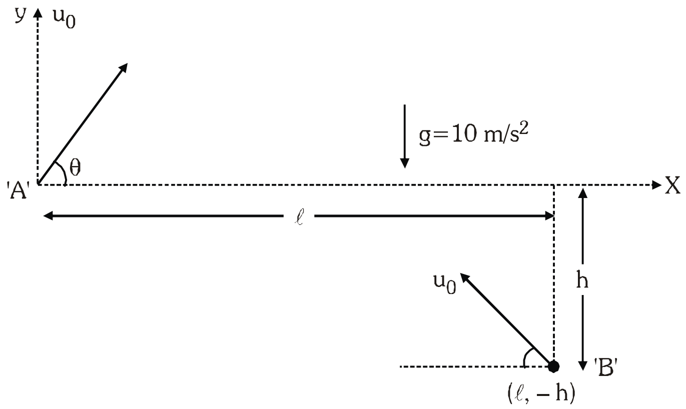
S_{AB} is minimum when $\frac{dS_{AB}}{dt} = 0$

Similarly for maximum separation $v_{sep} = 0$.

If the initial position of two particles are $\vec{r}_1$ and $\vec{r}_2$ and their velocities are $\vec{v}_1$ and $\vec{v}_2$ then shortest distance between the particles, $d_{shortest} = \frac{|\vec{r}_{12} \times \vec{v}_{12}|}{|\vec{v}_{12}|}$ and time after which this situation will occur, $t = -\frac{\vec{r}_{12} \cdot \vec{v}_{12}}{|\vec{v}_{12}|^2}$

Illustrations

Illustration 1*. Two particles 'A' and 'B' are projected in the vertical plane with same intial velocity u_0 from point (0, 0) and (ℓ, - h) towards each other as shown in figure at t = 0.



(I) The path of particle 'A' with respect to particle 'B' will be :

(A) parabola
(B) straight line parallel to x - axis.
(C) straight line parallel to y-axis
(D) none of these.

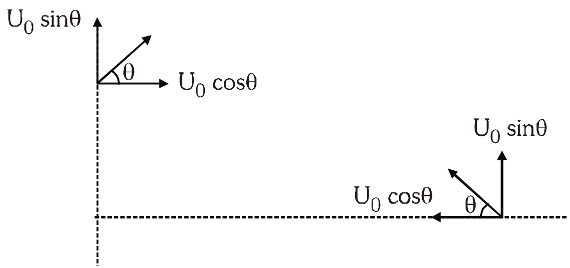
(II) Minimum distance between particle A and B during motion will be :

(A) ℓ
(B) h
(C) $\sqrt{\ell^2 + b^2}$
(D) $\ell + h$

(III) The time when separation between A and B is minimum is :

(A) $\frac{x}{u_0 \cos\ \theta}$
(B) $\sqrt{\frac{2h}{g}}$
(C) $\frac{\ell}{2u_0 \cos\ \theta}$
(D) $\frac{2\ell}{u_0 \cos\ \theta}$

Solution The path of a projectile as observed by other projectile is a straight line.



$\vec{V}_A = u_0 \cos\theta\hat{i} + (u_0 \sin\theta - gt)\hat{j}.V_{AB} = (2u_0 \cos\ \theta)\hat{i}$

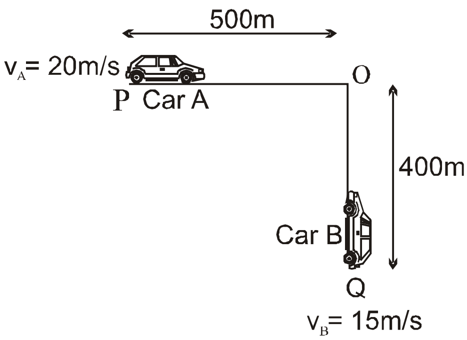
$\vec{V}_B = -u_0 \cos\theta\hat{i} + (u_0 \sin\theta - gt)\ \hat{j}$.

$a_{BA} = g - g = 0$

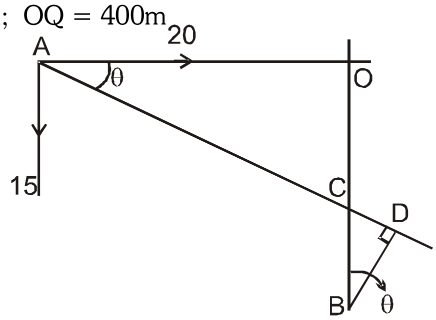
The vertical component $u_0\sin\theta$ will get cancelled.
The relative velocity will only be horizontal which is equal to $2u_0\cos\theta$.
Hence B will travel horizontally towards left w.r.t. A with constant speed $2u_0\cos\theta$ and minimum distance will be h.

Time to attain this separation will obviously be $\dfrac{S_{rel}}{V_{rel}} = \dfrac{\ell}{2u_0 \cos\ \theta}$

Illustration 2*. Two roads intersect at right angles. Car A is situated at P which is 500m from the intersection O on one of the roads. Car B is situated at Q which is 400m from the intersection on the other road. They start out at the same time and travel towards the intersection at 20m/s and 15m/s respectively.What is the minimum distance between them ?
How long do they take to reach it.



Solution First we find out the velocity of car B relative to A
As can be seen from figure,
$v_A = 20$ m/s, $V_B = 15$ m/s, OP = 500 m; OQ = 400m



$\tan\theta = \dfrac{15}{20} = \dfrac{3}{4}$;

$\cos\theta = \dfrac{4}{5}$;

$\sin\theta = \dfrac{3}{5}$

$OC = OA\tan\theta = 500 \times \dfrac{3}{4} = 375$ m

$BC = OB - OC = 400 - 375 = 25$ m

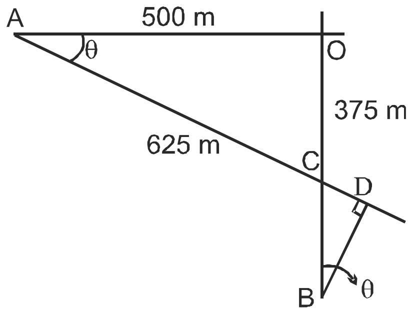
$BD = BC(\cos\theta) = 25 \times \dfrac{4}{5} = 20$ m

shortest distance = 20 m

$AD = AC + CD = 625 + 15 = 640$

$|\vec{v}_{AB}| = 25$ m/s

$t = \dfrac{640}{25} = 25.6$ sec.



BEGINNER'S BOX–1

Relative Motion

1. Two particles are moving with velocities v_1 and v_2. Their relative velocity is the maximum, when the angle between their velocities is :

(A) zero (B) $\pi/4$ (C) $\pi/2$ (D) π

2. A car A is going north east at 80 kmh^{-1} and another car B is going south east with a velocity of 60 kmh^{-1}. The velocity of A relative to B makes an angle with the north equal to

(A) $\tan^{-1}\left(\dfrac{2}{7}\right)$ (B) $\tan^{-1}\left(\dfrac{7}{2}\right)$ (C) $\tan^{-1}(7)$ (D) $\tan^{-1}\left(\dfrac{1}{7}\right)$

3*. A coin is released inside a lift at a height of 2 m from the floor of the lift. The height of the lift is 10 m. The lift is moving with an acceleration of 11 m/s^2 downwards. The time after which the coin will strike with the lift is :

(A) 4 s (B) 2 s (C) $\dfrac{4}{\sqrt{21}}$ s (D) $\dfrac{2}{\sqrt{11}}$ s

4. A ship is travelling due east at 10 km/h. A ship heading 30° east of north is always due north from the first ship. The speed of the second ship in km/h is -

(A) $20\sqrt{2}$ (B) $20\sqrt{3/2}$ (C) 20 (D) $20/\sqrt{2}$

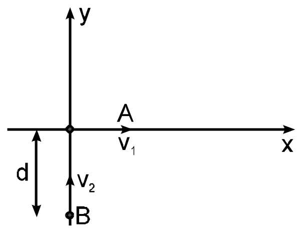
5. Three ships A, B & C are in motion. The motion of A as seen by B is with speed v towards north – east. The motion of B as seen by C is with speed v towards the north – west. Then as seen by A, C will be moving towards

(A) north (B) south (C) east (D) west

6*. Three stones A, B and C are simultaneously projected from same point with same speed. A is thrown upwards, B is thrown horizontally and C is thrown downwards from a building. When the distance between stone A and C becomes 10 m, then distance between A and B will be :

(A) 10 m (B) 5 m (C) $5\sqrt{2}$ m (D) $10\sqrt{2}$ m

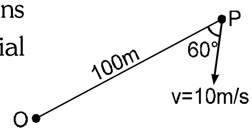
7*. Two particles A and B move with velocities v_1 and v_2 respectively along the x & y axis. The initial separation between them is 'd' as shown in the fig. Find the least distance between them during their motion.



(A) $\dfrac{d.v_1^2}{v_1^2+v_2^2}$ (B) $\dfrac{d.v_2^2}{v_1^2+v_2^2}$

(C) $\dfrac{d.v_1}{\sqrt{v_1^2+v_2^2}}$ (D) $\dfrac{d.v_2}{\sqrt{v_1^2+v_2^2}}$

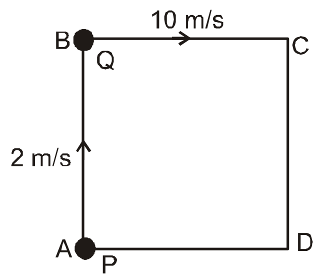
8*. P is a point moving with constant speed 10 m/s such that its velocity vector always maintains an angle 60° with line OP as shown in figure (O is a fixed point in space). The initial distance between O and P is 100 m. After what time shall P reach O.



(A) 10 sec. (B) 15 sec.

(C) 20 sec. (D) $20\sqrt{3}$ sec

9. Two men P & Q are standing at corners A & B of square ABCD of side 8 m. They start moving along the track with constant speed 2 m/s and 10 m/s respectively. The time when they will meet for the first time, is equal to :



(A) 2 sec (B) 3 sec

(C) 1 sec (D) 6 sec

10. An express train is moving with a velocity v_1. Its driver finds another train is moving on the same track in the same direction with velocity v_2. To escape collision, driver applies a retardation a on the train. The minimum time of escaping collision will be

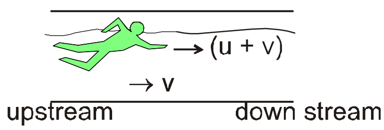
(A) $t=\dfrac{v_1-v_2}{a}$ (B) $t=\dfrac{v_1^2-v_2^2}{2}$ (C) None (D) Both

3.0 RELATIVE MOTION IN RIVER FLOW

SL AL

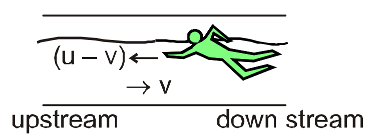
If a man can swim relative to water with velocity $\vec{v}_{mR}$ and water is flowing relative to ground with velocity $\vec{v}_R$, velocity of man relative to ground $\vec{v}_m$ will be given by :

$$\vec{v}_{mR} = \vec{v}_m - \vec{v}_R \quad \text{or} \quad \vec{v}_m = \vec{v}_{mR} + \vec{v}_R$$



So, if he is swimming is in the direction of flow of water,

$$v_m = v_{mR} + v_R \qquad (v_{mR} = u,\ v_R = v)$$



and if the swimming is opposite to the flow of water,

$$v_m = v_{mR} - v_R$$

3.1 Swimming / Rowing across the river

SL AL

The angle θ at which the man should swim so that the time taken to cross the river be minimum

$\vec{v}_m = \vec{v}_{mR} + \vec{v}_R$

$$= v_{mR}(\cos\theta\,\hat{i} + \sin\theta\,\hat{j}) + v_R\,\hat{i} = (v_{mR}\cos\theta + v_R)\,\hat{i} + v_{mR}\sin\theta\,\hat{j}$$

$$\Rightarrow \quad \frac{x\hat{i}}{t} + \frac{d}{t}\hat{j} = \vec{v}_m \qquad \Rightarrow \frac{d}{t} = v_{mR}\sin\theta \quad \Rightarrow t = \frac{d}{v_{mR}\sin\theta}$$

t minimum at $\sin\theta$ maximum. $\Rightarrow \theta = 90°$

$t_{minimum} = d/v_{mR}$. So the man should try to swim perpendicular to the river flow to minimize the time in each case.

The angle θ at which the man should swim so that the length of path be minimum

For minimum length of the path, drift x should be minimum.

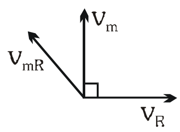
$$x = (v_{mR}\cos\theta + v_R)\,t = \frac{d}{v_{mR}\sin\theta}(v_{mR}\cos\theta + v_R)$$

$$x = d\left[\cot\theta + \frac{v_R}{v_{mR}}\operatorname{cosec}\theta\right]$$

Case-I : $v_{mR} > v_R$ or the river flow is mush less then the effort of the man.

In such case the minimum possible drift will be zero. So the man should swim at the angle.

$\cos\theta = -v_r/v_{mR}$



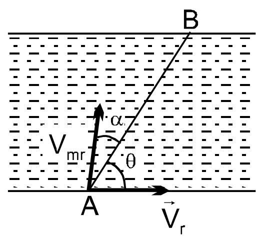
Case-II : $v_{mR} < v_R$ or the river flow is greater than velocity of man's effort. In such case one thing should be carefully noted that the velocity of boat is less than the river flow velocity. In such a case, boat cannot reach the point directly opposite to its starting point. i.e. drift can never be zero.

$$\therefore \quad x = \frac{d}{v_{mR}\sin\theta}(v_R + v_{mR}\cos\theta) \therefore \qquad \text{for x to be minimum } \frac{dx}{d\theta} = 0$$

$$\therefore \quad \frac{dx}{d\theta} = d\left[\frac{v_{mR}\sin\theta\;(-v_{mR}\sin\theta) - v_{mR}\cos\theta(v_R + v_{mR}\cos\theta)}{(v_{mR}\sin\theta)^2}\right]$$

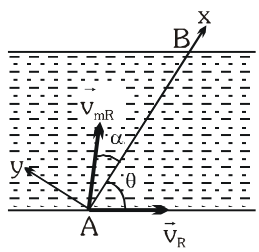
$$\therefore \quad \cos\theta = \frac{-v_{mR}}{v_R}$$

Thus, to minimize the drift, boat starts at an angle θ from the river flow.



Swimming in a desired direction : to find the direction in which to swim so that the actual velocity is along line AB.

We assume AB to be the reference line the resultant of v_{mR} and v_R is along line AB. Thus the components of v_{mR} and v_R in a direction perpendicular to line AB should cancel each other.



$\vec{v}_m = \vec{v}_{mR} + \vec{v}_R$

$$\vec{v}_m = [v_{mR}\cos\alpha\,\hat{i} + v_{mR}\sin\alpha\,\hat{j}] + [v_R\cos\theta\hat{i} - v_R\sin\theta\hat{j}]$$

$$v_{mR}\sin\alpha - v_R\sin\theta = 0 \Rightarrow v_{mR}\sin\alpha = v_R\sin\theta$$

Illustrations

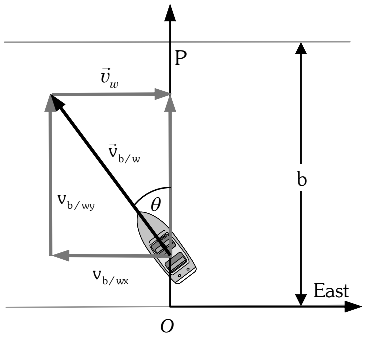
Illustration 3*. A boat can be rowed at 5 m/s on still water. It is used to cross a 200 m wide river from south bank to the north bank. The river current has uniform velocity of 3 m/s due east.

(a) In which direction must it be steered to cross the river perpendicular to current?

(b) How long will it take to cross the river in a direction perpendicular to the river flow?

(c) In which direction must the boat be steered to cross the river in minimum time? How far will it drift?

Solution (a) Velocity of a boat on still water is its capacity to move on water surface and equals to its velocity relative to water.



$\vec{v}_{b/w}$ = Velocity of boat relative to water = Velocity of boat on still water

On flowing water, the water carries the boat along with it. Thus velocity $\vec{v}_b$ of the boat relative to the ground equals to vector sum of $\vec{v}_{b/w}$ and $\vec{v}_w$. The boat crosses the river with the velocity $\vec{v}_b$.

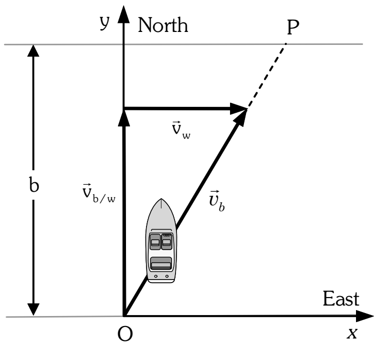
$$\vec{v}_b = \vec{v}_{b/w} + \vec{v}_w$$

(b) To cross the river perpendicular to current the boat must be steered in a direction so that one of the components of its velocity ($\vec{v}_{b/w}$) relative to water becomes equal and opposite to water flow velocity $\vec{v}_w$ to neutralize its effect. It is possible only when velocity of boat relative to water is grater than water flow velocity. In the adjoining figure it is shown that the boat starts from the point O and moves along the line OP (y-axis) due north relative to ground with velocity $\vec{v}_b$. To achieve this it is steered at an angle θ with the y-axis.

$$v_{b/w}\sin\theta = v_w \rightarrow 5\sin\theta = 3 \Rightarrow \theta = 37°$$

(c) The boat will cover river width b with velocity

$v_b = v_{b/wy} = v_{b/w}\sin 37° = 4$ m/s in time t, which is given by $t = b/v_b \rightarrow t = 50s$



(d) To cross the river in minimum time, the component perpendicular to current of its velocity relative to ground must be kept to maximum value. It is achieved by steering the boat always perpendicular to current as shown in the adjoining figure. The boat starts from O at the south bank and reaches point P on the north bank. Time t taken by the boat is given by

$$t = b/v_{b/w} \rightarrow t = 40s$$

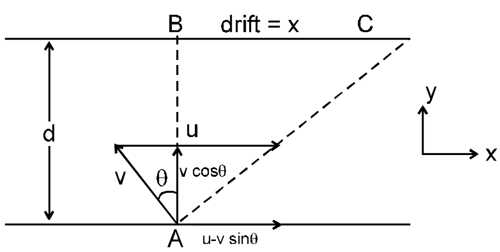
Drift is the displacement along the river current measured from the starting point. Thus, it is given by the following equation. We denote it by x_d.

$$x_d = v_{bx}t$$

Substituting $v_{bx} = v_w = 3$ m/s, from the figure, we have $x_d = 120$ m

Illustration 4*. A boat moves relative to water with a velocity which is n times less than the river flow velocity. At what angle to the stream direction must the boat move to minimize drifting?

Solution In this problem, one thing should be carefully meted that the velocity of boat is less than the river flow velocity. In such a case, boat cannot reach the point directly opposite to its starting point. i.e. drift can never be zero. Thus, to minimize the drift, boat starts at an angle θ from the normal direction up stream as shown.



Now, again if we find the components of velocity of boat along and perpendicular to the flow, these are,
velocity along the river, $v_x = u - v \sin \theta$.
and velocity perpendicular to the river, $v_y = v \cos \theta$.

time taken to cross the river is $t = \dfrac{d}{v_y} = \dfrac{d}{v\cos\theta}$.

In this time, drift $x = (v_x)t = (u - v \sin \theta)\dfrac{d}{v\cos\theta}$

or $x = \dfrac{ud}{v}\sec\theta - d\tan\theta$

The drift x is minimum, when $\dfrac{dx}{d\theta} = 0$,

or $\left(\dfrac{ud}{v}\right)(\sec\theta \,.\, \tan\theta) - d\sec^2\theta = 0$

or $\dfrac{u}{v}\sin\theta = 1$

or $\sin\theta = \dfrac{v}{u} = \dfrac{1}{n}$ $\left(\text{as } v = \dfrac{u}{n}\right)$

so, for minimum drift, the boat must move at an angle $\theta = \sin^{-1}\left(\dfrac{v}{u}\right)$ from normal direction or an angle $\dfrac{\pi}{2} + \sin^{-1}\left(\dfrac{v}{u}\right)$ from stream direction.

4.0 WIND AIRPLANE PROBLEMS

SL AL

This is very similar to boat river flow problems the only difference is that boat is replaced by a plane and river is replaced by wind.
Thus, velocity of aeroplane with respect to wind

$$\vec{v}_{aw} = \vec{v}_a - \vec{v}_w \quad \text{or} \quad \vec{v}_a = \vec{v}_{aw} + \vec{v}_w$$

where, $\vec{v}_a$ = velocity of aeroplane with respect to ground

and, $\vec{v}_w$ = velocity of wind with respect to ground

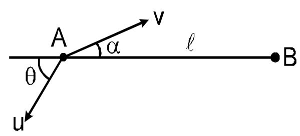
In general resultant velocity for object moving in any medium

$$\underbrace{\vec{v}_{object/ground}}_{\text{Resultant velocity}} = \underbrace{\vec{v}_{object/medium}}_{\substack{\text{Velocity of object}\\ \text{relative to medium}}} + \underbrace{\vec{v}_{medium/ground}}_{\text{Velocity of medium}}$$

Illustrations

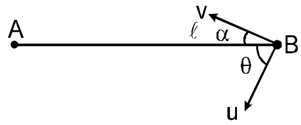
Illustration 5*. An aeroplane flies along a straight path A to B and returns back again. The distance between A and B is ℓ and the aeroplane maintains the constant speed v. There is a steady wind with a speed u at an angle θ with line AB. Determine the expression for the total time of the trip.

Solution A to B :



Velocity of plane along AB = $v\cos\alpha - u\cos\theta$,
and for no-drift from line
AB : $v \sin\alpha = u\sin\theta \Rightarrow \sin\alpha = \dfrac{u\sin\theta}{v}$

time taken from A to B : $t_{AB} = \dfrac{\ell}{v\cos\alpha - u\cos\theta}$



B to A :

velocity of plane along BA = vcosα + u cosθ

and for no drift from line AB : vsinα = usinθ

$$\Rightarrow \quad \sin\alpha = \frac{u\sin\theta}{v}$$

time taken from B to A : $t_{BA} = \dfrac{\ell}{v\cos\alpha + u\cos\theta}$

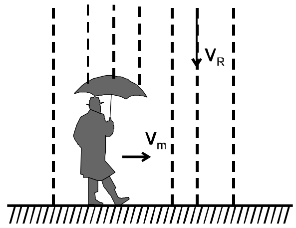
total time taken = $t_{AB} + t_{BA}$

$$= \frac{\ell}{v\cos\alpha - u\cos\theta} + \frac{\ell}{v\cos\alpha + u\cos\theta} = \frac{2v\ell\cos\alpha}{v^2\cos^2\alpha - u^2\cos^2\theta} = \frac{2v\ell\sqrt{1-\dfrac{u^2\sin^2\theta}{v^2}}}{v^2-u^2}.$$

5.0 RAIN PROBLEMS

SL AL

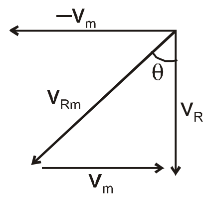
If rain is falling vertically with a velocity $\vec{v}_R$ and an observer is moving horizontally with velocity $\vec{v}_m$, the velocity of rain relative to observer will be :



$$\vec{v}_{Rm} = \vec{v}_R - \vec{v}_m$$

or $\quad v_{Rm} = \sqrt{v_R^2 + v_m^2}$

and direction $\theta = \tan^{-1}\left(\dfrac{v_m}{v_R}\right)$ with the vertical as shown in figure.

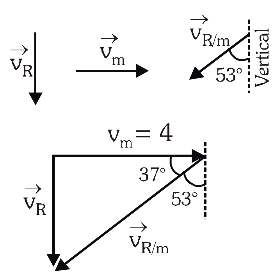


Illustrations

Illustration 6. A man when standstill observes the rain falling vertically and when he walks at 4 km/h he has to hold his umbrella at an angle of 53° from the vertical. Find velocity of the raindrops.

Solution Assigning usual symbols $\vec{v}_m$, $\vec{v}_r$ and $\vec{v}_{r/m}$ to velocity of man, velocity of rain and velocity of rain relative to man, we can express their relationship by the following eq.

$$\vec{v}_r = \vec{v}_m + \vec{v}_{r/m}$$



The above equation suggests that a standstill man observes velocity $\vec{v}_r$ of rain relative to the ground and while he is moving with velocity $\vec{v}_m$, he observes velocity of rain relative to himself $\vec{v}_{r/m}$. It is a common intuitive fact that umbrella must be held against $\vec{v}_{r/m}$ for optimum protection from rain. According to these facts, directions of the velocity vectors are shown in the adjoining figure.

The addition of velocity vectors is represented according to the above equation is also represented. From the figure we have

$$v_r = v_m \tan 37° = 3 \text{ km/h}$$

Illustration 7. A man wearing a hat of extended length 12 cm is running in rain falling vertically downwards with speed 10 m/s. The maximum speed with which man can run, so that rain drops do not fall on his face (the length of his face below the extended part of the hat is 16 cm) will be :

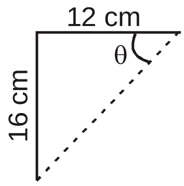
Solution $V_{R/G(x)} = 0$, $V_{R/G(y)} = 10$ m/s

Let, velocity of man = v

$$\tan\theta = \frac{16}{12} = \frac{4}{3}$$

then, $v_{R/man} = v$ (opposite to man)

For the required condition :



$$\tan\theta = \frac{V_{R/M(y)}}{V_{R/M(x)}} = \frac{10}{v} = \frac{4}{3}$$

$$\Rightarrow V = \frac{10 \times 3}{4} = 7.5 \ \textbf{Ans.}$$

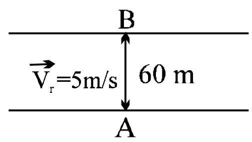
BEGINNER'S BOX–2

Applications of Relative Motion

1. Raindrops are falling vertically with a velocity of 10 m/s. To a cyclist moving on a straight road the raindrops appear to be coming with a velocity of 20 m/s. The velocity of cyclist is :

(A) 10 m/s (B) $10\sqrt{3}$ m/s (C) 20 m/s (D) $20\sqrt{3}$ m/s

2. A man is crossing a river flowing with velocity of 5 m/s. He reaches a point directly across at a distance of 60 m in 5 sec. His velocity in still water should be



(A) 12 m/s (B) 13 m/s

(C) 5 m/s (D) 10 m/s

3*. A person standing on the escalator takes time t_1 to reach the top of a tower when the escalator is moving. He takes time t_2 to reach the top of the tower when the escalator is standing. How long will he take if he walks up a moving escalator?

(A) $t_2 - t_1$ (B) $t_1 + t_2$ (C) $t_1t_2/(t_1 - t_2)$ (D) $t_1t_2/(t_1 + t_2)$

4*. A battalion of soldiers is ordered to swim across a river 500 ft wide. At what minimum rate should they swim perpendicular to river flow in order to avoid being washed away by the waterfall 300 ft downstream. The speed of current being 3 m.p.h. :

(A) 6 m.p.h. (B) 5 m.p.h. (C) 4 m.p.h. (D) 2 m.p.h.

5. A car with a vertical wind shield moves along in a rain storm at the speed of 40 km/hr. The rain drops fall vertically with a terminal speed of 20 m/s. The angle with the vertical at which the rain drop strike the wind shield is -

(A) $\tan^{-1}(5/9)$ (B) $\tan^{-1}(9/5)$ (C) $\tan^{-1}(3/2)$ (D) $\tan^{-1}(3)$

6. An airplane is flying with velocity $50\sqrt{2}$ km/hour in north-east direction. Wind is blowing at 25 km/hr from north to south. What is the resultant displacement of airplane in 2 hours ?

7*. An airplane pilot sets a compass course due west and maintains an air speed of 240 km. hr^{-1}. After flying for $\frac{1}{2}$ hr, he finds himself over a town that is 150 km west and 40 km south of his starting point.

(a) Find the wind velocity, in magnitude and direction.

(b) If the wind velocity were 120 km. hr^{-1} due south, in what direction should the pilot set his course in order to travel due west ? Take the same air speed of 240 km. hr^{-1}.

8. A bus is going southwards at 5m/s. To a man sitting in bus a car appears to move towards west at $2\sqrt{6}$ m/s. what is the actual speed of car?

(A) 4m/s (B) 3m/s (C)7m/s (D) None of these

9. A girl is walking on a road with velocity of 8kph. Suddenly rain starts falling at 10kph in vertically downward direction. The velocity of rain w.r.t to girl is

(A) $\sqrt{7}$ kph (B) $\sqrt{13}$ kph (C) $\sqrt{6}$ kph (D) $\sqrt{164}$ kph

10. A boat of sent across (perpendicular) a river with a velocity 8kph. If the resultant velocity of boat is 10kph, the river is flowing with velocity

(A) 6 kph (b) $\sqrt{13}$ kph (c) $\sqrt{6}$ kph (d) $\sqrt{109}$ kph

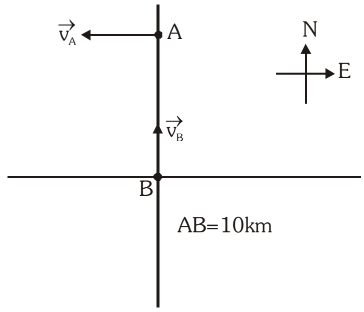
SOME WORKED OUT ILLUSTRATIONS

Illustration 1*.

Two ships A and B are 10 km apart on a line running south to north. Ship A farther north is streaming west at 20 km/h and ship B is streaming north at 20 km/h. What is their distance of closest approach and how long do they take to reach it?

Solution

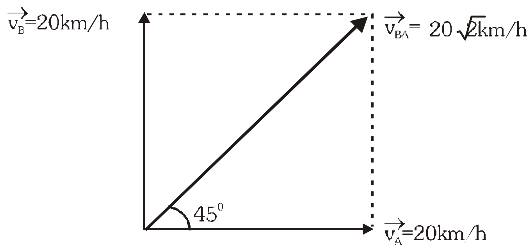
Ships A and B are moving with same speed 20 km/h in the directions shown in figure. It is a two dimensional, two body problem with zero acceleration. Let us find $\vec{v}_{BA}$



$$\vec{v}_{BA} = \vec{v}_B - \vec{v}_A$$

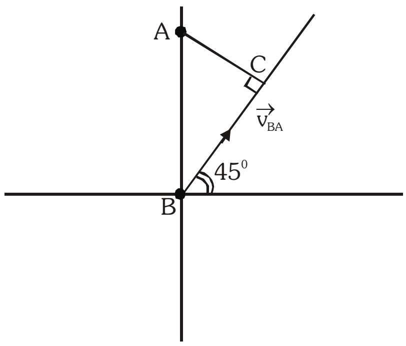
Here, $|\vec{v}_{BA}| = \sqrt{(20)^2 + (20)^2} = 20\sqrt{2}$ km/h

i.e., $\vec{v}_{BA}$ is $20\sqrt{2}$ km/h at an angle of 45° from east towards north. Thus, the given problem can be simplified as :



A is at rest and B is moving with $\vec{v}_{BA}$ in the direction shown in figure.

Therefore, the minimum distance between the two is



$$s_{min} = AC = AB \sin 45^0 = 10\left(\frac{1}{\sqrt{2}}\right) \text{ km} = 5\sqrt{2} \text{ km}$$

and the desired time is $t = \dfrac{BC}{|\vec{v}_{BA}|} = \dfrac{5\sqrt{2}}{20\sqrt{2}} = \dfrac{1}{4}$ h = 15 min

Illustration 2.

A projectile is thrown with speed u into air from a point on the horizontal ground at an angle θ with horizontal. If the air exerts a constant horizontal resistive force on the projectile then select correct alternative(s).

(A) At the farthest point, the velocity is horiozntal.
(B) The time for ascent equals the time for descent.
(C) The path of the projectile may be parabolic
(D) The path of the projectile may be a straight line.

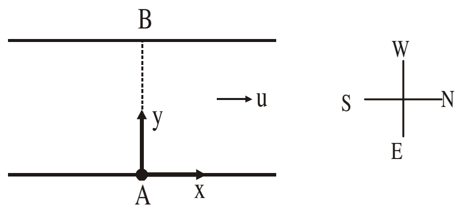
Ans. (C,D)

Solution

Here total acceleration $a = \sqrt{g^2 + a_x^2}$ = constant, so path may be parabolic or straight line.

Illustration 3 to 5.

A river of width 'd' with straight parallel banks flows due North with speed u. A boat, whose speed is v relative to water, starts from A and crosses the river. If the boat is steered due West and u varies with y as $u = \dfrac{y(d-y)v}{d^2}$ then answer the following questions.



3*. The time taken by boat to cross the river is

(A) $\frac{d}{\sqrt{2}v}$ (B) $\frac{d}{v}$ (C) $\frac{d}{2v}$ (D) $\frac{2d}{v}$

***Ans.* (B)**

Solution

Time taken $= \frac{d}{v_y} = \frac{d}{v}$

4*. Absolute velocity of boat when it reaches the opposite bank is

(A) $\frac{4}{3}$v, towards East (B) v, towards West (C) $\frac{4}{3}$v, towards West (D) v, towards East

***Ans.* (B)**

Solution

At y=d, u =0 so absolute velocity of boat = v towards West.

5*. Equation of trajectory of the boat is

(A) $y = \frac{x^2}{2d}$ (B) $x = \frac{y^2}{2d}$ (C) $y = \frac{x^2}{2d} - \frac{x^3}{3d^2}$ (D) $x = \frac{y^2}{2d} - \frac{y^3}{3d^2}$

***Ans.* (D)**

Solution

For boat (w.r.t. ground) $v_y = v, v_x = u = \frac{y(d-y)}{d^2}v \Rightarrow \frac{dy}{dt} = v$ and $\frac{dx}{dt} = \frac{y(d-y)}{d^2}v$

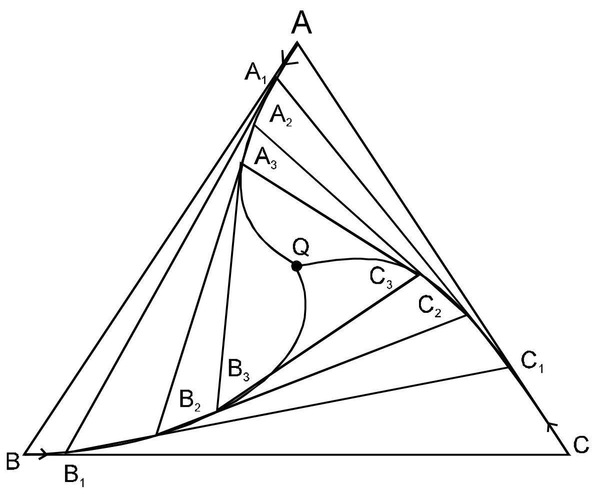
$$\Rightarrow \frac{dx}{dy} = \frac{y(d-y)}{d^2} \Rightarrow \int_0^x dx = \int_0^y \frac{(yd-y^2)}{d^2}dy \Rightarrow x = \frac{y^2}{2d} - \frac{y^3}{3d^2}$$

Illustration 6*.

Three particles A, B and C are situated at the vertices of an equilateral triangle ABC of side a at t = 0. Each of the particles moves with constant speed v. A always has its velocity along AB, B along BC and C along CA. At what time will the particle meet each other?

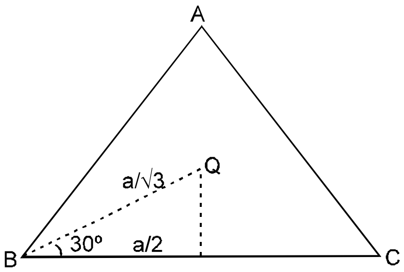
Solution

The motion of the particles is roughly sketched in figure. By symmetry they will meet at the centroid O of the triangle. At any instant the particles will form an equilateral triangle ABC with the same



Centroid O. All the particles will meet at the centre. Concentrate on the motion of any one particle, say B. At any instant its velocity makes angle 30° with BO.

The component of this velocity along BO is v cos 30°. This component is the rate of decrease of the distance BO. Initially.

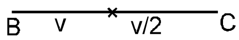


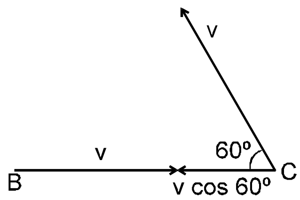
$$BO = \frac{a/2}{\cos 30^\circ} = \frac{a}{\sqrt{3}} = \text{displacement of each particle.}$$

Therefore, the time taken for BO to become zero

$$= \frac{d/\sqrt{3}}{v\cos 30^\circ} = \frac{2d}{\sqrt{3}v \times \sqrt{3}} = \frac{2d}{3v}.$$

Alternative : Velocity of B is v along BC. The velocity of C is along CA. Its component along BC is v cos 60° = v/2. Thus, the separation BC decreases at the rate of approach velocity.

∴ 



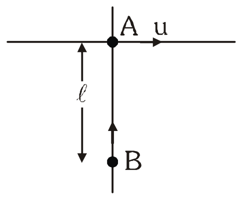
$\therefore$ approach velocity $= v + \dfrac{v}{2} = \dfrac{3v}{2}$

Since, the rate of approach is constant, the time taken in reducing the separation BC from a to zero is

$$t = \frac{a}{\frac{3v}{2}} = \frac{2a}{3v}$$

Illustration 7*.

'A' moves with constant velocity u along the 'x' axis. B always has velocity towards A. After how much time will B meet A if B moves with constant speed v. What distance will be travelled by A and B.

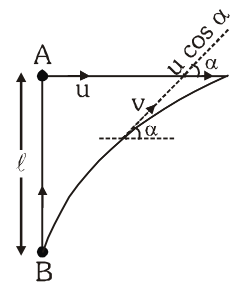


Solution

Let at any instant the velocity of B makes an angle α with that of x axis and the time to collide is T.

$$v_{app.} = v - u\cos\alpha$$

$$\ell = \int_0^T v_{app.}\,dt = \int_0^T (v - u\cos\alpha)\,dt \qquad ....(1)$$



Now equating the displacement of A and B along x-direction we get

$$uT = \int v\cos\alpha\,dt \qquad ....(2)$$

Now from (1) and (2) we get

$$\ell = vT - \int_0^T u\cos\alpha\,dt = vT - \frac{u}{v}\int_0^T v\cos\alpha\,dt = vT - \frac{u}{v}\cdot uT$$

$$\Rightarrow \qquad T = \frac{\ell v}{v^2 - u^2}$$

Now distance travelled by A $= \dfrac{uv\ell}{v^2 - u^2}$ and by B $= \dfrac{v^2\ell}{v^2 - u^2}$

ANSWERS

BEGINNER'S BOX-1

1. (D) ***2.*** (D) ***3.*** (A) ***4.*** (C) ***5.*** (B) ***6.*** (C) ***7.*** (C) ***8.*** (C)
9. (B) ***10.*** (A)

BEGINNER'S BOX-2

1. (B) ***2.*** (B) ***3.*** (D) ***4.*** (B) ***5.*** (A) ***6.*** $50\sqrt{5}$ km

7. (a) 100 km/hr, 37° W of S (b) 30° N of W ***8.*** (C) ***9.*** (D) ***10.*** (A)

EXERCISE – 1 MCQ (SINGLE CHOICE CORRECT)

1. Rain is falling vertically downwards with a velocity of 3 kmph. A man walks in the rain with a velocity of 4 kmph. The rain drops will appear to be falling on the man with a velocity of

(A) 1 kmph (B) 3 kmph (C) 4 kmph (D) 5 kmph

2. A car A is going north east at 80 kmh–1 and another car B is going south east with a velocity of 60 kmh^{-1}. The velocity of A relative to B makes an angle with the north equal to

(A) $\tan^{-1}\left(\frac{2}{7}\right)$ (B) $\tan^{-1}\left(\frac{7}{2}\right)$ (C) $\tan^{-1}(7)$ (D) $\tan^{-1}\left(\frac{1}{7}\right)$

3. A boat is moving with a velocity $3\hat{i}+4\hat{j}$ with respect to the ground. The water in the river is flowing with a velocity $-3\hat{i}-4\hat{j}$ with respect to the ground. The velocity of the boat relative to the water is

(A) $6\hat{i}+8\hat{j}$ (B) $8\hat{i}+6\hat{j}$ (C) $6\hat{i}+6\hat{j}$ (D) none of these

4 A boy can swim in still water at 1 m/s. He swims across a river flowing at 0.6 m/s which is 336 m wide. If he travels in shortest possible time, then what time he takes to cross the river.

(A) 250 s (B) 420 s (C) 340 s (D) 336 s

5. A man can swim in still water with a speed of 2 m/s. If he wants to cross a river of water current speed $\sqrt{3}$ m/s along shortest possible path, then in which direction should he swim?

(A) at an angle 120° to the water current (B) at an angle 150° to the water current

(C) at an angle 90° to the water current (D) none of these

6. A man wishes to cross a river in a boat. If he crosses the river in minimum time he takes 10 minutes with a drift of 120 m. If he crosses the river taking shortest route, he takes 12.5 minutes, find velocity of the boat with respect to water

(A) 20 m/min (B) 12 m/min (C) 10 m/min (D) 8 m/min

7*. For four particles A, B, C & D, the velocities of one with respect to other are given as $\vec{V}_{DC}$ is 20 m/s towards north, $\vec{V}_{BC}$ is 20 m/s towards east and $\vec{V}_{BA}$ is 20 m/s towards south. Then $\vec{V}_{DA}$ is

(A) 20 m/s towards north (B) 20 m/s towards south

(C) 20 m/s towards east (D) 20 m/s towards west

8. Two particles P and Q are moving with velocities of $(\hat{i}+\hat{j})$ and $(-\hat{i}+2\hat{j})$ respectively. At time t = 0, P is at origin and Q is at a point with position vector $(2\hat{i}+\hat{j})$. Then the shortest distance between P & Q is :-

(A) $\frac{2\sqrt{5}}{5}$ (B) $\frac{4\sqrt{5}}{5}$ (C) $\sqrt{5}$ (D) $\frac{3\sqrt{5}}{5}$

9. A, B & C are three objects each moving with constant velocity. A's speed is 10 m/s in a direction $\overline{PQ}$. The velocity of B relative to A is 6 m/s at an angle of, $\cos^{-1}(15/24)$ to PQ. The velocity of C relative to B is 12 m/s in a direction $\overline{QP}$. Then the magnitude of the velocity of C is :-

(A) 5 m/s (B) $2\sqrt{10}$ m/s (C) 3 m/s (D) 4 m/s

10. Two boats A and B are moving along perpendicular paths in a still lake at night. Boat A move with a speed of 3 m/s and boat B moves with a speed of 4 m/s in the direction such that they collide after sometime. At t = 0, the boats are 300 m apart. The ratio of distance travelled by boat A to the distance travelled by boat B at the instant of collision is :

(A) 1 (B) 1/2 (C) 3/4 (D) 4/3

11 A body is thrown up in a lift with a velocity u relative to the lift and the time of flight is found to be 't'. The acceleration with which the lift is moving up is :

(A) $\frac{u-gt}{t}$ (B) $\frac{2u-gt}{t}$ (C) $\frac{u+gt}{t}$ (D) $\frac{2u+gt}{t}$

12. A boat can go across a lake and return in time T_0 at a speed V. On a rough day there is uniform current at speed u to help the onward journey and impede the return journey. If the time taken to go across and return on the rough day be T, then T/T_0 =

(A) $1-\frac{u^2}{V^2}$ (B) $\frac{1}{1-\frac{u^2}{V^2}}$ (C) $1+\frac{u^2}{V^2}$ (D) $\frac{1}{1+\frac{u^2}{V^2}}$

13. To cross the river in shortest distance, a swimmer should swim making angle θ with the upstream. What is the ratio of the time taken to swim across in the shortest time to that in swimming across over shortest distance. [Assume speed of swimmer in still water is greater than the speed of river flow]

(A) $\cos\theta$ (B) $\sin\theta$ (C) $\tan\theta$ (D) $\cot\theta$

14. A boat which has a speed of 5km per hour in still water crosses a river of width 1km along the shortest possible path in fifteen minutes. The velocity of the river water in km per hour is :-

(A) 1 (B) 2 (C) 3 (D) $\sqrt{41}$

15. Rain is falling vertically with a speed of 20 ms^{-1} relative to air. A person is running in the rain with a velocity of 5 ms^{-1} and a wind is also blowing with a speed of 15 ms^{-1} (both towards east). Find the angle with the vertical at which the person should hold his umbrella so that he may not get drenched.

(A) $\tan^{-1}\sqrt{2}$ (B) $\tan^{-1}\left(\frac{1}{2}\right)$ (C) $\tan^{-1}(2)$ (D) 45°

16. The velocity of a boat in still water is η times less than the velocity of flow of the river ($\eta > 1$). The angle with the stream direction at which the boat must move to minimise drifting is

(A) $\sin^{-1}\left(\frac{1}{\eta}\right)$ (B) $\cot^{-1}\left(\frac{1}{\eta}\right)$ (C) $\frac{\pi}{2}+\sin^{-1}\left(\frac{1}{\eta}\right)$ (D) $\frac{\pi}{2}+\cot^{-1}\left(\frac{1}{\eta}\right)$

17. A pilot in a plane wants to go 500 km towards north. To reach straight to his desired position the pilot has to drive his plane 53° west of north in presence of wind, which is blowing in due east. The time taken by pilot to reach his destination is 10 hr. The velocity of wind is [take tan 37° = 3/4]

(A) 200/3 km/hr (B) 100/3 km/hr (C) 200 km/hr (D) 150 km/hr

18. A swimmer crosses a river with minimum possible time 10 second. And when he reaches the other end starts swimming in the direction towards the point from where he started swimming. Keeping the direction fixed the swimmer crosses the river in 15 sec. The ratio of speed of swimmer with respect to water and the speed of river flow is (Assume constant speed of river & swimmer) :

(A) $\frac{3}{2}$ (B) $\frac{9}{4}$ (C) $\frac{2}{\sqrt{5}}$ (D) $\frac{\sqrt{5}}{2}$

19. Two men A and B are running with velocities $5\hat{i}$ m/s and $-5\hat{i}$ m/s respectively. The rain is falling vertically downward with the velocity $-5\sqrt{3}\,\hat{j}$ m/s. In order to be safe from rain in which direction they should keep their umbrellas ?

(A) Both should keep the umbrella 60° from the positive x-axis.
(B) Both should keep the umbrella 120° from the positive x-axis.
(C) Man A should keep the umbrella at 60° and B at 120° from positive x-axis.
(D) Man A should keep the umbrella at 120° and B at 60° from the positive x-axis.

20. A car leaves station X for station Y every 10 min. The distance between X and Y is 60 km. The car travels at speed 60 kmh^{-1}. A man drives a car from Y towards X at speed 60 kmh^{-1}. If he starts at the moment when first car leaves station X. The number of cars he would meet on route is :

(A) 5 (B) 7 (C) 10 (D) 20

21. A jet airplane travelling from east to west at a speed of 500 km h^{-1} ejected out gases of combustion at a speed of 1500 km h^{-1} with respect to the jet plane. What is the velocity of the gases with respect to an observer on the ground?

(A) 1000 km h^{-1} in the direction west to east (B) 1000 km h^{-1} in the direction east to west
(C) 2000 km h^{-1} in the direction west to east (D) 2000 km h^{-1} in the direction east to west

22. A police jeep is chasing with, velocity of 45 km/h a thief in another jeep moving with velocity 153 km/h. Police fires a bullet with muzzle velocity of 180 m/s. The velocity it will strike the car of the thief is

(A) 150 m/s (B) 27 m/s (C) 450 m/s (D) 250 m/s

23. A man is moving due east with a speed 1 km/hr and rain is falling vertically with a speed $\sqrt{3}$ km/hr. At what angle from vertical the man has to hold his umbrella to keep the rain away. Also find the speed of rain drops w.r.t. man.

(A) $\theta = 30°$, 2 km hr^{-1} (B) $\theta = 45°$, 3 km hr^{-1} (C) $\theta = 60°$, 5 km hr^{-1} (D) None of these

24. A boat is moving with velocity of $3\hat{i}+4\hat{j}$ in river and water is moving with a velocity of $-3\hat{i}-4\hat{j}$ with respect to ground. Relative velocity of boat with respect to water is :

(A) $-6\hat{i}-8\hat{j}$ (B) $6\hat{i}+8\hat{j}$ (C) $8\hat{i}$ (D) $6\hat{i}$

25. The distance between two particles is decreasing at the rate of 6 m/sec. If these particles travel with same speeds and in the same direction, then the separation increases at the rate of 4 m/sec. The particles have speeds as

(A) 5 m/sec ; 1 m/sec (B) 4 m/sec ; 1 m/sec (C) 4 m/sec ; 2 m/sec (D) 5 m/sec ; 2 m/sec

EXERCISE – 2 MCQ (ONE OR MORE CHOICE CORRECT)

1. A ball is thrown vertically upward (relative to the train) in a compartment of a moving train.
(A) The ball will maintain the same horizontal velocity as that of the person (or the compartment) at the time of throwing.
(B) If the train is accelerating then the horizontal velocity of the ball will be different from that of the train velocity, at the time of throwing.
(C) If the ball appears to be moving backward to the person sitting in the compartment it means that speed of the train is increasing.
(D) If the ball appears to be moving ahead of the person sitting in the compartment it means the train's motion is retarding.

2. Two boats A and B having same speed relative to river are moving in a river. Boat A moves normal to the river current as observed by an observer moving with velocity of river current. Boat B moves normal to the river as observed by the observer on the ground.
(A) To a ground observer boat B moves faster than A
(B) To a ground observer boat A moves faster than B
(C) To the given moving observer boat B moves faster than A
(D) To the given moving observer boat A moves faster than B

3. A river is flowing towards East with a velocity of 5 ms^{-1}. The boat velocity is 10 ms^{-1}. The boat crosses the river by shortest path. Hence,
(A) The direction of boat's velocity is 30° west of north.
(B) The direction of boat's velocity is north-west.
(C) Resultant velocity is $5\sqrt{3}\ ms^{-1}$.
(D) resultant velocity of boat is $5\sqrt{2}\ ms^{-1}$.

4. A stationary person observes that rain is falling vertically down at 30 kmh^{-1}. A cyclist is moving up on an inclined plane making an angle 30° with horizontal at 10 kmh^{-1}. In which direction should the cyclist hold his umbrella to prevent himself from the rain ?
(A) At an angle $\tan^{-1}\left(\frac{3\sqrt{3}}{5}\right)$ with inclined plane
(B) At an angle $\tan^{-1}\left(\frac{3\sqrt{3}}{5}\right)$ with horizontal
(C) At angle $\tan^{-1}\left(\frac{\sqrt{3}}{7}\right)$ with inclined plane
(D) At an angle $\tan^{-1}\left(\frac{\sqrt{3}}{7}\right)$ with vertical

5. Ship A is located 4 km north and 3 km east of ship B. Ship A has a velocity of 20 kmh^{-1} towards the south and ship B is moving at 40 kmh^{-1} in a direction 37° north of east. Take x-and y-axes along east and north directions, respectively.
(A) Velocity of A relative to B is $-32\hat{i} - 44\hat{j}$.
(B) Position of A relative to B as a function of time is given by $\vec{r}_{AB} = (3-32t)\hat{i} + (4-44t)\hat{j}$ where t = 0 when the ships are in position described above.
(C) Velocity of B relative to A is $-32\hat{i} - 44\hat{j}$
(D) At some moment A will be west of B.

6. Consider a shell that has a muzzle velocity of 45 ms^{-1} fired from the tail gun of an airplane moving horizontally with a velocity of 215 ms^{-1}. The tail gun can be directed at any angle with the vertical in the plane of motion of the airplane. The shell is fired when the plane is above point A on ground, and the plane is above point B on ground when the shell hits the ground. (Assume for simplicity that the Earth is flat)
(A) Shell may hit the ground at point A.
(B) Shell may hit the ground at point B.
(C) Shell may hit a point on earth which is behind point A.
(D) Shell may hit a point on earth which is ahead of point B.

7. A boat is traveling due east at 12 ms^{-1}. A flag on the boat flaps at 53°N of W. Another flag on the shore flaps due north.

(A) Speed of wind with respect to ground is 16 ms^{-1} (B) Speed of wind with respect to ground is 20 ms^{-1}

(C) Speed of wind with respect to boat is 20 ms^{-1} (D) Speed of wind with respect to boat is 16 ms^{-1}

8. A cubical box dimension L = 5/4 m starts moving with an acceleration $\vec{a} = 0.5\,ms^{-2}\hat{i}$ from the state of rest. At the same time, a stone is thrown from the origin with velocity $\vec{v} = v_1\hat{i} + v_2\hat{j} - v_3\hat{k}$ with respect to earth. Acceleration due to gravity $\vec{g} = 10\,ms^{-2}(-\hat{j})$. The stone just touches the roof of box and finally falls at the diagonally opposite point. then

(A) $v_1 = \frac{3}{2}$ (B) $v_2 = 5$ (C) $v_3 = \frac{5}{4}$ (D) $v_3 = \frac{5}{2}$

9. A river of width 500m flows with velocity $5\hat{i}$ m/s. Velocity of boat 1 and boat 2 in still water are $(2\hat{i} + 2\hat{j})$ m/s and $(-2\hat{i} + 2\hat{j})$ m/s respectively. Choose the correct option(s):

(A) velocity of boat 1 relative to boat 2 is $(4\hat{i})$ m/s

(B) time taken by boat 1 to cross the river is 250secs

(C) time taken by boat 2 to cross the river is 250secs

(D) velocity of boat 1 relative to ground makes angle $\cot^{-1}\left(\frac{7}{2}\right)$ with bank

10. A and B are two point on a same vertical line. A is 20m above ground while B is 40m above ground. Two small balls are released from rest, one from A and B each at t=0. Neglect air resistance. All collisions are perfectly inelastic. Choose the correct option(s):

(A) acceleration of A relative to B is zero

(B) acceleration of A relative to B is 9.8ms^{-2}

(C) acceleration of A relative to B is zero in $0 \text{ sec} \le t \le 2 \text{ sec}$

(D) acceleration of A relative to B is 9.8ms^{-2} in $2 \text{ sec} \le t \le 2\sqrt{2} \text{ sec}$

Match the column

11. Two particles A and B moving in x-y plane are at origin at t = 0 sec. The initial velocity vectors of A and B are $\bar{u}_A = 8\hat{i}$ m/s and $\bar{u}_B = 8\hat{j}$ m/s. The acceleration of A and B are constant and are $\bar{a}_A = -2\hat{i}$ m/s^2 and $\bar{a}_B = -2\hat{j}$ m/s^2. Column I gives certain statements regarding particle A and B. Column II gives corresponding results. Match the statements in column I with corresponding results in Column II.

	Columns I	***Column II***
(A)	The time (in seconds) at which velocity of A relative to B is zero	(p) $16\sqrt{2}$
(B)	The distance (in metres) betwen A and B when their relative velocity is zero.	(q) $8\sqrt{2}$
(C)	The time (in seconds) after t = 0 sec. at which A and B are at same position	(r) 8
(D)	The magnitude of relative velocity of A w.r. to and B at the instant when they are at same position.	(s) 4
		(t) 6 seconds

Comprehension Based Questions

Comprehension-1.

When a particle is undergoing motion, the displacement of the particle has a magnitude that is equal to or smaller than the total distance travelled by the particle. In many cases the displacement of the particle may actually be zero, while the distance travelled by it is non–zero. Both these quantities, however depend on the frame of reference in which motion of the particle is being observed. Consider a particle which is projected in the earth's gravitational field, close to its surface, with a speed of $100\sqrt{2}$ m/s, at an angle of 45° with the horizontal in the eastward direction. Ignore air resistance and assume that the acceleration due to gravity is 10 m/s^2.

12. The motion of the particle is observed in two different frames : one in the ground frame (A) and another frame (B), in which the horizontal component of the displacement is always zero. Two observers located in these frames will agree on :-
(A) The total distance travelled by the particle
(B) The horizontal range of the particle
(C) The maximum height risen by the particle
(D) None of the above

13. "A third observer (C) close to the surface of the earth reports that particle is initially travelling at a speed of 100 √2 m/s making on angle of 45° with the horizontal, but its horizontal motion is northward". The third observer is moving in :
(A) The south–west direction with a speed of 100 √2 m/s
(B) The south–east direction with a speed of 100 √2 m/s
(C) The north–west direction with a speed of 100 √2 m/s
(D) The north–east direction with a speed of 100 √2 m/s

14. There exists a frame (D) in which the distance travelled by the particle is a minimum. This minimum distance is equal to :
(A) 2 km (B) 1 km (C) 0 km (D) 500 m

Comprehension-2.

Raindrops are falling with a velocity $10\sqrt{2}$ m/s making an angle of 45° with the vertical. The drops appear to be falling vertically to a man running with constant horizontal velocity. The velocity of rain drops change such that the rain drops now appear to be falling vertically with $\sqrt{3}$ times the velocity it appeared earlier to the same person running with same velocity.

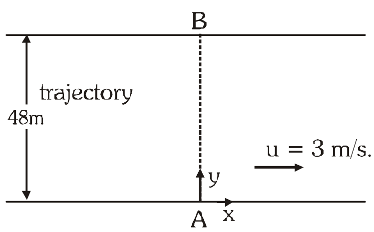
15. The magnitude of velocity of man with respect to ground is :
(A) $10\sqrt{2}$ m/s (B) $10\sqrt{3}$ m/s (C) 20 m/s (D) 10 m/s

16. After the velocity of rain drops change, the magnitude of velocity of raindrops with respect to ground is:
(A) 20 m/s (B) $20\sqrt{3}$ (C) $20\sqrt{2}$ m/s (D) $10\sqrt{3}$ m/s

17. The angle (in degrees) between the initial and the final velocity vectors of the raindrops with respect to the ground is :
(A) 8 (B) 15 (C) 22.5 (D) 37

Comprehension- 3.

A man starts swimming at time t = 0 from point A on the ground and he wants to reach the point B directly opposite to the point A. His velocity in still water is $5\frac{\text{m}}{\text{sec.}}$ and width of river is 48 m. River flow velocity 'u' = 3m/s.



18. Direction (with line AB) in which he should make stroke is :
(A) 37° (B) 75° (C) 60° (D) 45°

19. The time taken by man to cross the river
(A) 8 sec. (B) 12 sec.. (C) 24 sec. (D) 20 sec.

20. Trajectory of path.
(A) $x = 0$ (B) $y = 0$ (C) $x = y$ (D) $x = \frac{3}{4}y$

Comprehension- 4.

Ram and Shyam are walking on two perpendicular tracks with speed 3 ms^{-1} and 4 ms^{-1}, respectively. At a certain moment (say $t = 0$ s), Ram and Shyam are at 20 m and 40 m away from the intersection of tracks, respectively, and moving towards the intersection of the tracks.

21. During the motion the magnitude of velocity of ram with respect to Shyam is
(A) 1 ms^{-1} (B) 4 ms^{-1} (C) 5 ms^{-1} (D) 7 ms^{-1}

22. Shortest distance between them subsequently is
(A) 18 m (B) 15 m (C) 25 m (D) 8 m

23. The time t when they are at shortest distance from each other subsequently, is-
(A) 8.8 s (B) 12 s (C) 15 s (D) 44 s

EXERCISE – 3 SUBJECTIVE

1. An aeroplane is flying vertically upwards with a uniform speed of 500 m/s. When it is at a height of 1000 m above the ground a shot is fired at it with a speed of 700 m/s from a point directly below it. What should be the uniform acceleration of the aeroplane now so that it may escape from being hit?
($g = 10$ m/s^2)

2*. Men are running along a road at 15 km/h behind one another at equal intervals of 20 m. Cyclists are riding in the same direction at 25 km/h at equal intervals of 30 m. At what speed in km/h an observer travelling along the road in opposite direction so that whenever he meets a runner he also meets a cyclist?

3. Two trains A and B 100 m and 60 m long are moving in opposite directions on the parallel tracks. The speed of shorter train is 3 times that of the longer one. If the train take 4 seconds to cross each other then find the velocities of the trains ?

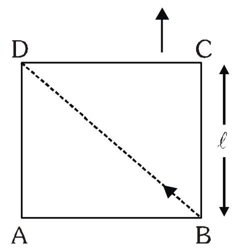
4*. Hailstones falling vertically with a speed of 10 m/s, hit the wind screen (wind screen makes an angle 30° with the horizontal) of a moving car and rebound elastically. Find the velocity of the car if the driver finds the hailstones rebound vertically after striking.

5. Two motor cars start from A simultaneously & reach B after 2 hour . The first car travelled half the distance at a speed of $v_1 = 30$ km hr^{-1} & the other half at a speed of $v_2 = 60$ km hr^{-1} . The second car covered the entire distance with a constant acceleration . At what instant of time, were the speeds of both the vehicles same ? Will one of them overtake the other enroute ?

6. A man crosses a river in a boat. If he crosses the river in minimum time he takes 10 minutes with a horizontal drift 120 m. If he crosses the river taking shortest path in 12.5 minutes then find width of the river, velocity of the boat w.r.t. water and speed of flow of river.

7. Two cars A and B are racing along straight line. Car A is leading, such that their relative velocity is directly proportional to the distance between the two cars. When the lead of car A is $l_1 = 10$ m, its running 10 m/s faster than car B. Determine the time car A will take to increase its lead to $l_2 = 20$ m from car B.

8. In the figure the top view of a compartment of a train is shown. A man is sitting at a corner 'B' of the compartment. The man throws a ball (with respect to himself) along the surface of the floor towards the corner 'D' of the compartment of the train. The ball hits the corner 'A' of the compartment, then find the time at which it hits A after the ball is thrown. Assume no other collision during motion and floor is smooth. The length of the compartment is given as 'ℓ' and the train is moving with constant acceleration 'a' in the direction shown in the figure.



9. A swimmer jumps from a bridge over a canal and swims 1 km upstream. After that first km, he passes a floating cork. He continues swimming for half an hour and then turns around and swims back to the bridge. The swimmer and the cork reach the bridge at the same time. The swimmer has been swimming at a constant speed. How fast does the water in the canal flow ?

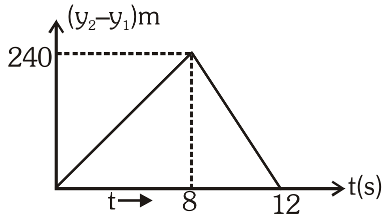
10. A platform is moving upwards with a constant acceleration of 2 ms^{-2}. At time $t = 0$, a boy standing on the platform throws a ball upwards with a relative speed of 8 ms^{-1}. At this instant, platform was at the height of 4m from the ground and was moving with a speed of 2 ms^{-1}. take $g = 10$ ms^{-2}. Find
(a) When and where the ball strikes the platform
(b) The maximum height attained by the ball from the ground.
(c) The maximum distance of the ball from the platform.

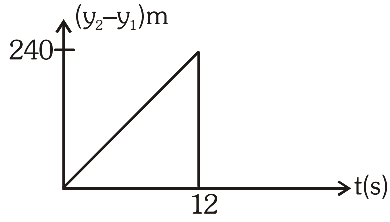
EXERCISE – 4 RECAP OF AIEEE/JEE (MAIN)

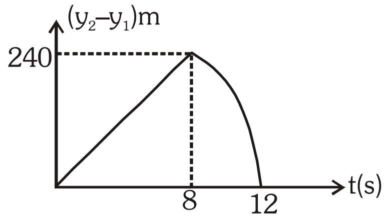
1*. Two stones are thrown up simultaneously from the edge of a cliff 240 m high with initial speed of 10 m/s and 40 m/s respectively. Which of the following graph best represents the time variation of relative position of the second stone with respect to the first ?

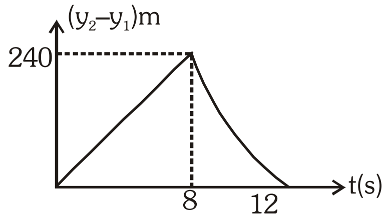
(Assume stones do not rebound after hitting the ground and neglect air resistance, take $g = 10\ m/s^2$)

(The figure are schematic and not drawn to scale) ***[JEE(Main)-2015]***

(A) 

(B) 

(C) 

(D) 

2. In a car race on straight road, car A takes a time t less than car B at the finish and passes finishing point with a speed 'v' more than that of car B. Both the cars start from rest and travel with constant acceleration a_1 and a_2 respectively. Then 'v' is equal to : ***[JEE(Main)-2019]***

(A) $\frac{a_1 + a_2}{2}t$ (B) $\sqrt{2a_1a_2}\,t$ (C) $\frac{2a_1a_2}{a_1 + a_2}t$ (D) $\sqrt{a_1a_2}\,t$

3. A passenger train of length 60m travels at a speed of 80 km/hr. Another freight train of length 120 m travels at a speed of 30 km/hr. The ratio of times taken by the passenger train to completely cross the freight train when : (i) they are moving in the same direction, and (ii) in the opposite directions is : ***[JEE(Main)-2019]***

(A) $\frac{5}{2}$ (B) $\frac{25}{11}$ (C) $\frac{3}{2}$ (D) $\frac{11}{5}$

4. A person standing on an open ground hears the sound of a jet aeroplane, coming from north at an angle 60° with ground level. But he finds the aeroplane right vertically above his position. If u is the speed of sound, speed of the plane is : ***[JEE(Main)-2019]***

(A) $\frac{2v}{\sqrt{3}}$ (B) v (C) $\frac{v}{2}$ (D) $\frac{\sqrt{3}}{2}v$

5. Ship A is sailing towards north-east with velocity $\vec{v} = 30\hat{i} + 50\hat{j}$ km/hr where $\hat{i}$ points east and $\hat{j}$, north. Ship B is at a distance of 80 km east and 150 km north of Ship A and is sailing towards west at 10 km/hr. A will be at minimum distance from B in : ***[JEE(Main)-2019]***

(A) 4.2 hrs. (B) 2.2 hrs. (C) 3.2 hrs. (D) 2.6 hrs.

6. The stream of a river is flowing with a speed of 2km/h. A swimmer can swim at a speed of 4km/h. What should be the direction of the swimmer with respect to the flow of the river to cross the river straight ? ***[JEE(Main)-2019]***

(A) 60° (B) 150° (C) 90° (D) 120°

EXERCISE – 5 **RECAP OF IIT-JEE/JEE (ADVANCED)**

Single Choice Correct

1*. ***STATEMENT -1 :*** For an observer looking out through the window of a fast moving train, the nearby objects appear to move in the opposite direction to the train, while the distant objects appear to be stationary.

and

STATEMENT -2 : If the observer and the object are moving at velocities $\vec{V}_1$ and $\vec{V}_2$ respectively with reference to a laboratory frame, the velocity of the object with respect to the observer is $\vec{V}_2 - \vec{V}_1$. ***[JEE-2008]***

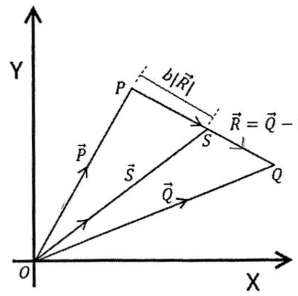
(A) STATEMENT-1 is True, STATEMENT-2 is True; STATEMENT-2 ***is*** a correct explanation for STATEMENT-1

(B) STATEMENT-1 is True, STATEMENT-2 is True; STATEMENT-2 ***is NOT*** a correct explanation for STATEMENT-1

(C) STATEMENT-1 is True, STATEMENT -2 is False

(D) STATEMENT-1 is False, STATEMENT -2 is True.

2. Three vectors $\vec{P}$, $\vec{Q}$ and $\vec{R}$ are shown in the figure. Let S be any point on the vector $\vec{R}$. The distance between the points P and S is $b|\vec{R}|$. The general relation among vector $\vec{P}$, $\vec{Q}$ and $\vec{S}$ is ***[JEE(Adv.) - 2017]***



(A) $\vec{S} = (1-b)\vec{P} + b^2\vec{Q}$

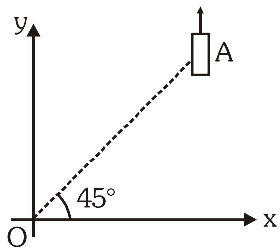
(B) $\vec{S} = (b-1)\vec{P} + b\vec{Q}$

(C) $\vec{S} = (1-b)\vec{P} + b\vec{Q}$

(D) $\vec{S} = (1-b^2)\vec{P} + b\vec{Q}$

Subjective Type

3*. On a frictionless horizontal surface, assumed to be the x-y plane, a small trolley A is moving along a straight line parallel to the y-axis (see figure) with a constant velocity of $(\sqrt{3}-1)$ m/s. At a particular instant when the line OA makes an angle of 45° with the x-axis, a ball is thrown along the surface from the origin O. Its velocity makes an angle ϕ with the x-axis and it hits the trolley. ***[IIT-JEE 2002]***



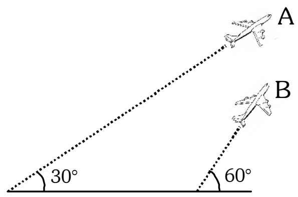
(a) The motion of the ball is observed from the frame of the trolley. Calculate the angle ϕ made by the velocity vector of the ball with the x-axis in this frame.

(b) Find the speed of the ball with respect to the surface, if $\phi = 4\theta/3$.

4. A train is moving along a straight line with a constant acceleration 'α'. A boy standing in the train throws a ball forward with a speed of 10 m/s, at an angle of 60° to the horizontal. The body has to move forward by 1.15 m inside the train to catch the ball back at the initial height. The acceleration of the train, in m/s^2 is

[JEE-2011]

5. Airplanes A and B are flying with constant velocity in the same vertical plane at angles 30° and 60° with respect to the horizontal respectively as shown in figure. The speed of A is $100\sqrt{3}$ ms^{-1}. At time t = 0s, an observer in A finds B at a distance of 500 m. This observer sees B moving with a constant velocity perpendicular to the line of motion of A. If at t = t_0. A just escapes being hit by B, t_0 in seconds is : ***[JEE(Adv.) 2014]***



ANSWERS

EXERCISE-1

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
D	D	A	D	C	A	D	B	A	C	B	B	B	C	B
16	**17**	**18**	**19**	**20**	**21**	**22**	**23**	**24**	**25**					
C	A	C	D	B	A	A	A	B	A					

EXERCISE-2

Que.	1	2	3	4	5	6	7	8	9	10
Ans.	A,C,D	B	A,C	A,D	A,B	B,D	A,C	A,B,C	ABCD	CD

- ***Match the column :*** **11.** A–s ; B–p ; C–r; D–q

- ***Comprehension Based Questions***

Comprehension 1 : **12.** (C) **13.** (B) **14.** (C)

Comprehension 2 : **15.** (D) **16.** (A) **17.** (B)

Comprehension 3 : **18.** (A) **19.** (B) **20.** (A)

Comprehension 4 : **21.** (C) **22.** (D) **23.** (A)

EXERCISE-3

1. $a > 10\ m/s^2$ **2.** 5 **3.** $v_A = 10\ ms^{-1}$, $v_B = 30\ ms^{-1}$

4. $10\sqrt{3}$ m/s **5.** 0.75 hr, 1.5 hr after the moment of departure,No overtaking

6. 200m, $v_{boat} = 1.20$ km/hr, $v_{water} = 0.72$ km/hr, **7.** $(\log_e 2)$ sec **8.** $\sqrt{\frac{2\ell}{a}}$

9. 1 km/h **10.** (a) $\frac{4}{3}$ s ; (b) 9 m; (c) $\frac{8}{3}$ m

EXERCISE-4

Que.	1	2	3	4	5	6
Ans.	C	A	D	C	D	D

EXERCISE-5

1. (B) **2.** (C) **3.** (a) 45° (b) 2 m/s **4.** 5 **5.** (5)

* * * * *

Important Notes