Function of Functions : Chain rule

Let f be a function of x, which in turn is a function of t. The first derivative of f w.r.t. t is equal to the product of

$\frac{df}{dx}$ and $\frac{dx}{dt}$ Therefore $\frac{df}{dt} = \frac{df}{dx} \times \frac{dx}{dt}$

$y = f(x)$ —① $x = f(t)$ —②

$\frac{dy}{dt} = ??$

$\frac{dy}{dx} = f'(x)$

$\frac{dx}{dt} = f'(t)$

$\rightarrow \frac{dy}{\cancel{dx}} \cdot \frac{\cancel{dx}}{dt} = f'(x) \cdot f'(t)$

$$\boxed{\frac{dy}{dt} = f'(x) \cdot f'(t)}$$

Illustration 26. ,27

Ex $y = \sin(x^2)$ let $t = x^2$

Find $\frac{dy}{dx} = ??$

$y = \sin(t)$ $t = x^2$

$\frac{dy}{dt} = \cos(t)$ $\frac{dt}{dx} = 2x^{2-1} = 2x$

$\frac{dy}{\cancel{dt}} \times \frac{\cancel{dt}}{dx} = \cos(t) \cdot 2x$

$$\boxed{\frac{dy}{dx} = 2x\cos(x^2)}$$

Method

$\frac{dy}{dx} = \cos(x^2) \cdot 2x$ Ans

Ex $y = \sin(e^x)$

$\frac{dy}{dx} = \cos(e^x) \cdot e^x$ Ans

Ex $y = [\cos(x)]^3$

$\frac{dy}{dx} = 3[\cos(x)]^{3-1} \cdot (-\sin x)$

$= -3\cos^2 x \sin x$ Ans

Ex $y = 2at \qquad x = at^2 \qquad \{a = \text{constant}\}$

Find $\dfrac{dy}{dx}$

$$\frac{dy}{dt} = 2a \qquad \frac{dx}{dt} = a(2t^{2-1}) = 2at$$

$$\frac{dy}{\cancel{dt}} \times \frac{\cancel{dt}}{dx} = \frac{\cancel{2a}}{\cancel{2a}t}$$

$$\boxed{\frac{dy}{dx} = \frac{1}{t}}$$ Ans

Ex $y = \log(e^x)$ Find $\dfrac{dy}{dx} = ??$

$$\frac{dy}{dx} = \frac{1}{e^x} \cdot e^x = 1$$ Ans

Ex $y = e^{\sin x}$ $\quad \dfrac{dy}{dx} = ??$

$$\frac{dy}{dx} = e^{\sin(x)} \cdot \cos(x)$$

Ex $y = \sin^2 x$ $\quad \frac{dy}{dx} = ??$

(i) $\sin(2x)$ (ii) $\cos(2x)$ (iii) $\cos^2 x$ (iv) none

Ex $y = (\sin x)^2$

$$\frac{dy}{dx} = 2(\sin x)^{2-1} \cdot \cos x$$

$$= 2 \sin x \cos x$$

$$= \sin(2x)$$

Ex $y = (\sin x)^5$ $\quad \frac{dy}{dx} = ??$

$$\frac{dy}{dx} = 5 \sin^4 x \cdot \cos x$$

Ex $y = e^{x^5}$ $\quad \frac{dy}{dx} = ??$

$$\frac{dy}{dx} = e^{x^5} \cdot 5x^4 \quad \text{Ans}$$

$y = e^{x^5}$

let $t = x^5$

$y = e^t$ $\quad t = x^5$

$\frac{dy}{dt} = e^t$ $\quad \frac{dt}{dx} = 5x^4$

$$\frac{dy}{\cancel{dt}} \frac{\cancel{dt}}{dx} = e^t \cdot 5x^4$$

$$\frac{dy}{dx} = e^{x^5} \cdot 5x^4$$

Ex $y = \sin\left(\sqrt{\sin x + \cos x}\right)$

let $t = \sqrt{\sin x + \cos x}$, $z = \sin x + \cos x$

$y = \sin(t)$	$t = \sqrt{z}$	$z = \sin x + \cos x$
$\frac{dy}{dt} = \cos t$ — (1)	$\frac{dt}{dz} = \frac{1}{2} z^{\frac{1}{2}-1} = \frac{1}{2\sqrt{z}}$ — (2)	$\frac{dz}{dx} = \cos x - \sin x$ — (3)

(1) × (2) × (3)

$$\frac{dy}{\cancel{dt}} \cdot \frac{\cancel{dt}}{\cancel{dz}} \cdot \frac{\cancel{dz}}{dx} = \cos t \cdot \frac{1}{2\sqrt{z}} \cdot \left[\cos(x) - \sin x\right]$$

$$\frac{dy}{dx} = \cos\left(\sqrt{\cos x + \sin x}\right) \frac{(\cos x - \sin x)}{2\sqrt{\cos x + \sin x}}$$

$$\frac{dy}{dx} = \cos\left(\sqrt{\sin x + \cos x}\right) \frac{1}{2}(\sin x + \cos x)^{\frac{1}{2}-1} \cdot (\cos x - \sin x)$$

Ex $y = \underline{\sin}(e^{x^2})$ $\quad \frac{dy}{dx} = ??$

$$\frac{dy}{dx} = \cos(e^{x^2}) \cdot e^{x^2} \cdot 2x$$

$\underline{\underline{Ans}}$

$y = e^{\underline{x}}$

$$\frac{dy}{dx} = e^x \cdot \{1 \cdot x^{1-1}\}$$

$$= e^x \cdot 1$$

$y = \sin(x)$

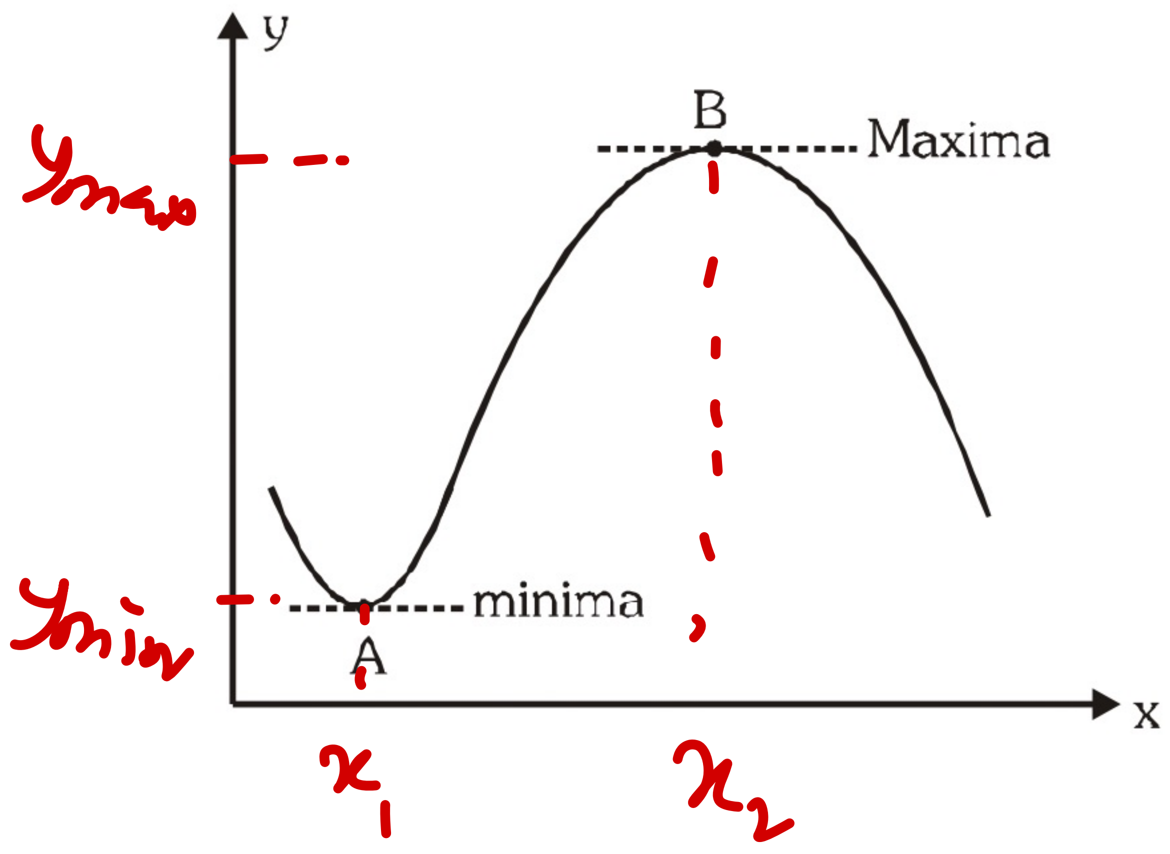
$$= \cos(x) \cdot 1 \cdot x^{1-1} = \underline{\cos x}$$

Maximum and Minimum value of a Function

SL AL

Higher order derivatives are used to find the maximum and minimum values of a function. At the points of maxima and minima, first derivative becomes zero.

At point 'A' (minima) : As we see in figure, in the neighborhood of A, slope increases so $\frac{d^2y}{dx^2} > 0$.



Condition for minima : $\boxed{\frac{dy}{dx} = 0 \text{ and } \frac{d^2y}{dx^2} > 0}$

At point 'B' (maxima) : As we see in figure, in the neighborhood of B, slope decreases so $\frac{d^2y}{dx^2} < 0$

Condition for maxima : $\boxed{\frac{dy}{dx} = 0 \text{ and } \frac{d^2y}{dx^2} < 0}$

Illustration 28. 29,30,31,32

$y = f(x)$

Step -(1) Find $\frac{dy}{dx}$

Step -(2) Put $\frac{dy}{dx} = 0$ And find values of x say x_1, x_2 - - - -

Step (3) do further diff. of $\frac{dy}{dx}$ $\frac{d^2y}{dx^2}$

Step (4) Put values of x in Step (3)

If $x = x_1$ $\left(\frac{d^2y}{dx^2}\right)_{x=x_1} > 0$ y is min. at x_1

$\left(\frac{d^2y}{dx^2}\right)_{x=x_1} < 0$ y is max. at x_1

Ex 1 The minimum value of $y = 5x^2 - 2x + 1$ is :

$y = 5x^2 - 2x + 1$

Step - 1 $\frac{dy}{dx} = 10x - 2$

Step - 2 $\frac{dy}{dx} = 0$

$10x - 2 = 0$

$x = \frac{1}{5}$

Step - 3 $\frac{d^2y}{dx^2} = 10 \times 1 - 0$

$\frac{d^2y}{dx^2} = 10$

Step (4) Put $x = \frac{1}{5}$ in step 3

$\frac{d^2y}{dx^2} = 10 > 0$

at $x = \frac{1}{5}$ y is minimum

$y_{min} = 5\left(\frac{1}{5}\right)^2 - 2\left(\frac{1}{5}\right) + 1$

$= \frac{1}{5} - \frac{2}{5} + 1$

$= -\frac{1}{5} + 1$

$= 4/5$ Ans

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Ex Find minimum & maximum values of $x^4 - 8x^2 + 5$

$y = x^4 - 8x^2 + 5$

(1) $\frac{dy}{dx} = 4x^3 - 16x + 0$

(2) $\frac{dy}{dx} = 0$

$4x^3 - 16x = 0$

$4x(x^2 - 4) = 0$

$x = 0 \quad | \quad x^2 - 4 = 0$

$x^2 = 4$

$x = \pm 2$

$x_1 = 0 \quad x_2 = -2 \quad x_3 = +2$

(3) $\frac{d^2y}{dx^2} = 4\{3x^2\} - 16 = 12x^2 - 16$

(4) $x_1 = 0 \quad \frac{d^2y}{dx^2}(x_1 = 0) = 12(0)^2 - 16 = -16 < 0$

Hence y at $x = 0$ is maximum

$y_{max} = 0^4 - 8.0^2 + 5 \Rightarrow \boxed{y_{max} = 5}$

$x_2 = 2 \quad \frac{d^2y}{dx^2}(x = 2) = 12(2)^2 - 16 = 32 > 0$

$x_3 = -2 \quad \frac{d^2y}{dx^2}(x = -2) = 12(-2)^2 - 16 = 32 > 0$

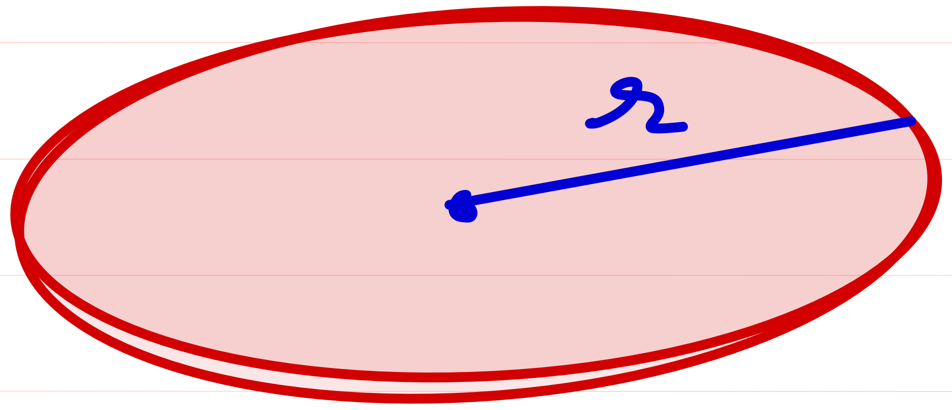
Hence y at $-2, +2$ is minimum

$y_{min} = (2)^4 - 8(2)^2 + 5 = -11$ Ans

$\boxed{y_{min} = -11}$

Ex 2 The radius of a circular plate increases at the rate of 0.1 cm per second. At what rate does the area increase when the radius of plate is $\frac{5}{\pi}$ cm ?

(A) 1 cm^2/s (B) 0.1 cm^2/s (C) 0.5 cm^2/s (D) 2 cm^2/s



$\frac{dr}{dt} = 0.1$ cm/sec

$\frac{dA}{dt} = ?$

$A = \pi r^2$

$\frac{dA}{dr} = \pi\{2r\}$

divide both sides by dt

$\left(\frac{dA}{dt}\right) = 2\pi r \frac{dr}{dt}$

$\frac{dA}{dt} = 2\cancel{\pi}\left(\frac{5}{\cancel{\pi}}\text{ cm}\right) \cdot \left(0.1\text{ cm/sec}\right)$

$= 10 \times 0.1 \times \frac{cm^2}{sec}$

$= 1 \frac{cm^2}{sec}$

Ex-3

1 Given $s = t^2 + 5t + 3$, find $\frac{ds}{dt}$.

2 If $s = ut + \frac{1}{2}at^2$, where u and a are constants. Obtain the value of $\frac{ds}{dt}$.

3 The area of a blot of ink is growing such that after t seconds, its area is given by $A = (3t^2+7)\ cm^2$. Calculate the rate of increase of area at t= 5 second.

4 The area of a circle is given by $A = \pi r^2$, where r is the radius. Calculate the rate of increase of area w.r.t. radius.

(1) $s = t^2 + 5t + 3$

$\frac{ds}{dt} = 2t + 5 + 0$

(2) $S = ut + \frac{1}{2}at^2$

$\frac{dS}{dt} = u \times 1 + \frac{a}{2}(2t)$

$= u + at$

(3) $A = (3t^2 + 7)\ cm^2$

$\frac{dA}{dt} = (6t + 0)\ cm^2/sec$

$= 6 \times 5\ cm^2/sec$

$= 30\ cm^2/sec$

(4) $A = \pi r^2$

$\frac{dA}{dr} = 2\pi r$ Ans

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Ex Find rate of change of velocity of particle is given $v = t^2 - 2t + 3$ at $t = 3$ sec

$$v = t^2 - 2t + 3$$

$$\frac{dv}{dt} = 2t - 2$$

$$(t=3) \quad \frac{dv}{dt} = 2 \times 3 - 2 = 4 \ (m/sec^2) \text{ Ans}$$

Ex Find volume rate of spherical object of volume $V = \alpha r^3$ (where α is constant)

at $r = 2$ If $\frac{dr}{dt} = 3$

$$\frac{dV}{dt} = 3\alpha r^2 \cdot \frac{dr}{dt}$$

$$\frac{dV}{dt} = 3\alpha \times (2)^2 \cdot 3$$

$$= 9\alpha 4$$

$$= 36\alpha$$

Find $\frac{dV}{dt} = ??$

$$\frac{dV}{dr} = \alpha(3r^2)$$

$$\left(\frac{dV}{dt}\right) = 3\alpha r^2 \left(\frac{dr}{dt}\right)$$

H.W

Race # 4 → complete

Module-1 BB # 2 (complete)

Illustration 27 → 32