CONTENTS

FUNDAMENTAL OF MATHEMATICS

Recap of Early Classes

We have already studied numbers, plane geometry, algebraic formulae, linear equations and their applications in different way. This chapter is a bridge between these concepts and their advance application along with some other vital terms and their application. Modulus and Logarithm are entirely new concepts for the students and needed to be studied with due attention.

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FUNDAMENTALS OF MATHEMATICS

1.0 NUMBER SYSTEM

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1.1 Natural Numbers

Counting numbers are called natural numbers e.g.1, 2, 3, 4, 5.....
Set of natural numbers is represented by N. ***N* = {*1, 2, 3, 4, 5, ...*}**

(a) ***Prime Number –*** The natural numbers (except 1) which are divisible by 1 and itself only are called prime numbers. e.g. 2, 3, 5, 7, 11, 13...
In other words prime number is a natural number having only ***two natural factors*** 1 and itself.

NOTE

(i) Smallest Prime number is 2 (only even prime).
(ii) Smallest odd prime number is 3.
(iii) 1 is not a prime number.
(iv) Every prime number greater than 3 is of the form $6k \pm 1$ where $k \in N$ but converse needs not to be true.

Primality Test – To check whether any number 'n' is a prime number or not. Divide the number from 2 to integer part of $\sqrt{n}$ if it is divisible by any of the numbers it is not prime, else it is prime.
e.g. to check 101 is prime or not. $[\sqrt{101}] = 10$. 101 is not divisible by any of the number from 2 to 10. so it is a prime number. Also it is of the form $6k - 1$.

(b) ***Composite Number –*** The numbers except 1 and which are not prime, are called composite number. e.g. 4, 6, 8, 9, 10, 14...

In other words, composite number is a natural number having more than two natural factors.

NOTE

(i) Smallest composite number is 4.
(ii) Smallest odd composite number is 9.
(iii) 1 is not a composite number.
(iv) Composite number can be represented as exponent of primes. e.g. $100 = 2^2 \cdot 5^2$, $440 = 2^3 \cdot 5^1 \cdot 11^1$

(c) ***Co-prime Number or Relative Prime Numbers –*** Two natural numbers are said to be co-prime number whose H.C.F. is 1 e.g. (1, 3) , (3, 5), (25, 33)....

NOTE

(i) Two prime numbers are always coprime but converse is not necessarily true .
(ii) Two consecutive natural numbers are always co-prime.

(d) ***Twin Prime –*** Two prime numbers are said to be twin prime if there difference is 2. e.g. (3, 5), (5, 7), (11, 13)

1.2 Whole Number

All natural numbers including zero are called whole numbers e.g. 0, 1, 2, 3, 4, 5...
Set of whole numbers is represented by W. ***W* = {*0, 1, 2, 3, 4, 5, ...*}**

1.3 Integers

The numbers ....-3, -2, -1, 0, 1, 2, 3.... are called integers. Set of integers is represented by I or Z.
I or ***Z*** **= {*0, ± 1, ± 2, ± 3, ± 4, ± 5, ...*}**

(a) ***Positive Integers –*** Set of positive integers is represented by I^+ or Z^+.
I^+ or $Z^+ = \{1, 2, 3, 4, 5, ...\} = N$

(b) ***Negative Integers:*** Set of negative integers is represented by I^- or Z^-.
I^- or $Z^- = \{-1,-2,-3,-4,-5, ...\}$

(c) ***Non-negative Integers –*** Set of non-negative integers is $\{0, 1, 2, 3, 4, 5, ...\} = W$

(d) ***Non- positive Integers –*** Set of non-positive integers is $\{0, -1,-2,-3,-4,-5, ...\}$

(e) ***Even Integers –*** Integers which are divisible by 2 are called even integers. e.g. $0, \pm 2, \pm 4, \pm 6, \pm 8$...... It is generally represented by 2n. $n \in I$

(f) ***Odd Integers –*** Integers which are not divisible by 2 are called odd integers. e.g. $\pm 1, \pm 3, \pm 5, \pm 7$...... It is generally represented by $(2n -1)$ or $(2n +1)$. $n \in I$

1.4 Rational Number

The number which can be expressed in the form of p/q where p, q $\in$ I and $q \neq 0$ e.g. 2, $\frac{3}{2}, \frac{5}{1}, \frac{6}{4}$.

Set of rational numbers is represented by ***Q***.

In decimal notation, ***terminating numbers*** (3.25, 7.2934) or ***non-terminating but repeating numbers*** (3.2222..... = $3.\bar{2}$, 0.3333... = $0.\bar{3}$) are called rational numbers. As they can also be represented in the form of p/q.

1.5 Irrational Number

The number which can not be expressed in the form of p/q where p, q $\in$ I and $q \neq 0$ e.g. $\sqrt{2}$, $\sqrt{3}$, π, $\sqrt{10}$, e (Napier's constant) etc.

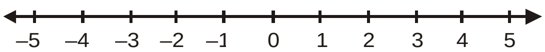
In other words, the numbers which are not rational is called irrational number. Set of irrational numbers is represented by $\boldsymbol{Q^c}$.

In decimal notation, ***non-terminating and non-repeating numbers*** are called irrational numbers. As they can not be represented in the form of p/q.

1.6 Real Number

The complete set of rational and irrational numbers is the set of real numbers and isdenotedby R.Thus $R = Q \cup Q^C$. e.g. $2, \sqrt{2}$, 5, 6, $\frac{9}{4}$, $\frac{6}{9}$, π, e etc. Set of real numbers is represented by ***R***.

Real Number Line – A line on which all the real numbers can be shown is called real number line.



All the real numbers follow the order property i.e. if there are two ***distinct*** real numbers a and b then either $a < b$ or $a > b$.

NOTE

(a) Integers are rational numbers, but converse need not be true.

(b) Negative of an irrational number is an irrational number.

(c) Sum or difference of a rational number and an irrational number is always an irrational number. e.g. $2 + \sqrt{3}$, $3 - \sqrt{5}$

(d) The product or quotient of a non zero rational number & an irrational number will always be an irrational number.

(e) If $a \in Q$ and $b \notin Q$, then ab = rational number, only if $a = 0$.

(f) Sum, difference, product and quotient of two irrational numbers need not be an irrational number (it may be a rational number also).

(g) There exists infinitely many rationals & irrational numbers between any two real numbers

1.7 Complex Number

A number of the form a + ib is called a complex number, where a, b $\in$ R and i = $\sqrt{-1}$. A Complex number is usually denoted by 'z' and a set of complex numbers is denoted by C

Note – It may be noted that $N \subset W \subset I \subset Q \subset R \subset C$.

- **Iota (i)**

$i^1 = i$	$i^5 = i$	$i^9 = i$	$\boldsymbol{i^{4n+1} = i}$	
$i^2 = -1$	$i^6 = -1$	$i^{10} = -1$	$\boldsymbol{i^{4n+2} = -1}$	(Here $n \in I$)
$i^3 = -i$	$i^7 = -i$	$i^{11} = -i$	$\boldsymbol{i^{4n+3} = -i}$	
$i^4 = 1$	$i^8 = 1$	$i^{12} = 1$	$\boldsymbol{i^{4n} = 1}$	

- ***Algebraic Operations of complex numbers*** Let $z_1 = a_1 + i\,b_1$ and $z_2 = a_2 + i\,b_2$

Addition $z_1 + z_2 = (a_1 + a_2) + i\,(b_1 + b_2)$

Subtraction $z_1 - z_2 = (a_1 - a_2) + i\,(b_1 - b_2)$

Multiplication $z_1 . z_2 = (a_1 . a_2 - b_1 . b_2) + i\,(a_1 b_2 + a_2 b_1)$

- **Conjugate of a Complex Number**

If $z = a + ib$, where $a, b \in R$, be a complex number then its conjugate complex number is represented by $\bar{z}$ and $\bar{z} = a - ib$. To find conjugate replace i by – i

- **Division** $\dfrac{z_1}{z_2} = \dfrac{a_1 + ib_1}{a_2 + ib_2}$

$$= \frac{a_1 + ib_1}{a_2 + ib_2} \cdot \frac{a_2 - ib_2}{a_2 - ib_2} \text{ (Multiplying and dividing by conjugate of denominator)}$$

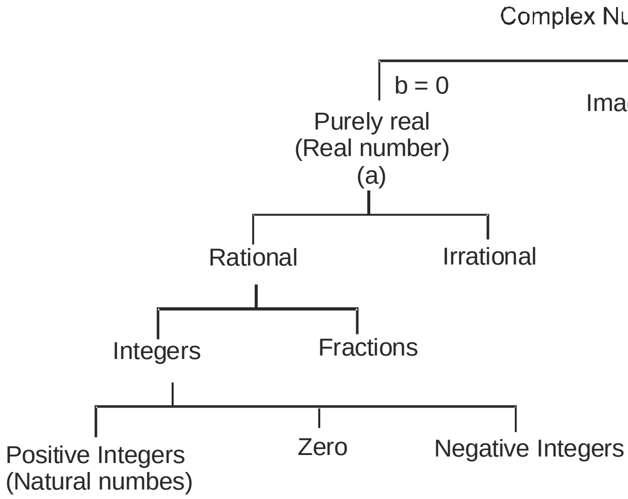
$$= \frac{(a_1 a_2 + b_1 b_2) + i(a_2 b_1 - a_1 b_2)}{a_2^2 + b_2^2}$$

- **Equality of two complex numbers**

$z_1 = z_2 \Leftrightarrow \mathrm{Re}(z_1) = \mathrm{Re}(z_2)$ and $\mathrm{Im}(z_1) = \mathrm{Im}(z_2)$

$z_1 = z_2 \Leftrightarrow a_1 = a_2$ and $b_1 = b_2$

- **Heirarchy Chart of Numbers**



1.8 Squares, Cubes & Square roots

Number	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	
Square	4	9	16	25	36	49	64	81	100	
Cube	8	27	64	125	216	343	512	729	1000	
Sq. Root	1.41	1.73	2	2.24	2.45	2.65	2.83	3	3.16	
Number	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**
Square	121	144	169	196	225	256	289	324	361	400
Cube	1331	1728	2197	2744	3375	4096	4913	5832	6859	8000

Illustrations

Illustration 1. If p, q, r are prime numbers such that $p^2 - q^2 = r$. Find all the possible ordered pairs. (p, q)

Solution. $(p + q)(p-q) = r$

since r is factorized into two integers, the smaller of them must be 1. i.e. $p - q = 1$, which is possible only for

$p = 3$ and $q = 2$.

$\therefore$ $r = 5$

$\therefore$ only one ordered pair.

Illustration 2. Prove that $x^4 + 4$ is prime only for one value of $x \in N$

Solution.

$x^4 + 4 = x^4 + 4x^2 + 4 - 4x^2$

$= (x^2 + 2)^2 - (2x)^2$

$= (x^2 - 2x + 2)(x^2 + 2x + 2)$

again smaller factor $x^2 - 2x + 2 = 1$

$\Rightarrow x = 1$

$x^4 + 4 = 5 \in$ prime

Illustration 3. Simplify $i^{100} + i^{50} + i^{48} + i^{46}$

Solution. $i^{4 \times 25} + i^{4.((7)+2)} + 2^{4 \times 12} + i^{4.(11)+2} = 1 + (-1) + 1 + -1 = 0$

Illustration 4. Express the form in form of a + ib

(i) $(-1 + 2i) + \left(\frac{1}{2} - i\right)$ (ii) $\left(\frac{1}{2} + \frac{i}{4}\right)\left(\frac{-2}{3} - \frac{i}{4}\right)$ (iii) $\left(\frac{1-i}{1+i}\right)$

Solution.

(i) $(-1 + 2i) + \left(\frac{1}{2} - i\right) = \left(-1 + \frac{1}{2}\right) + (2i - i) = \left(\frac{-2+1}{2}\right) + (2i - i) = \left(\frac{-1}{2}\right) + i$

(ii) $\left(\frac{1}{2} + \frac{i}{4}\right)\left(\frac{-2}{3} - \frac{i}{4}\right) = \left(-\frac{1}{3} + \frac{1}{16}\right) + i\left(-\frac{2}{12} - \frac{1}{8}\right)$

$\Rightarrow \left(\frac{-16+3}{48}\right) + i\left(-\frac{2}{12} - \frac{1}{8}\right) = -\frac{13}{48} - \frac{7i}{24}$

(iii) $\left(\frac{1-i}{1+i}\right) = \left(\frac{1-i}{1+i} \times \frac{1-i}{1-i}\right) = \frac{(1-i)^2}{1-i^2} = \frac{1+i^2-2i}{2} = \frac{1-1-2i}{2} = \frac{-2i}{2} = -i$

Note - (i) $i^2 = -1 \Rightarrow i = -\frac{1}{i}$ (ii) $\frac{1+i}{1-i} = i$

Illustration 5. Find the conjugate

(i) $(2 + 3i)(1 - i)$ (ii) $\frac{1}{i}$

Solution. (i) $z = (2 + 3i)(1 - i) = 2 - 2i + 3i - 3i^2 = 2 + i + 3 = 5 + i$

$\bar{z} = 5 - i$

(ii) $z = \frac{1}{i} \Rightarrow \bar{z} = \frac{1}{-i} = i$

***Illustration 6.** If $x = 2 - 3i$, then find the value of $x^2 - 4x + 10$

***Solution.** $x = 2 - 3i$

$x - 2 = 3i$

$\Rightarrow \quad x^2 - 4x + 4 = 9i^2$

$\Rightarrow \quad x^2 - 4x + 4 = -9$

$\Rightarrow \quad x^2 - 4x + 10 = -3$

2.0 ALGEBRAIC FORMULAE

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If a, b, c $\in$ ***C***

(i) $(a + b)^2 = a^2 + 2ab + b^2 = (a - b)^2 + 4ab$

(ii) $(a - b)^2 = a^2 - 2ab + b^2 = (a + b)^2 - 4ab$

(iii) $a^2 - b^2 = (a + b)(a - b)$

(iv) $(a + b)^3 = a^3 + b^3 + 3ab(a + b)$

(v) $(a - b)^3 = a^3 - b^3 - 3ab(a - b)$

(vi) $a^3 + b^3 = (a + b)^3 - 3ab(a + b) = (a + b)(a^2 + b^2 - ab)$

(vii) $a^3 - b^3 = (a - b)^3 + 3ab(a - b) = (a - b)(a^2 + b^2 + ab)$

(viii) $(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca = a^2 + b^2 + c^2 + 2abc\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)$

(ix) $a^2 + b^2 + c^2 - ab - bc - ca = 1/2\,[(a - b)^2 + (b - c)^2 + (c - a)^2]$

(x) $a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$

$= 1/2\,(a + b + c)[(a - b)^2 + (b - c)^2 + (c - a)^2]$

(xi) $a^4 - b^4 = (a + b)(a - b)(a^2 + b^2)$

(xii) $a^4 + a^2 + 1 = (a^2 + 1)^2 - a^2 = (1 + a + a^2)(1 - a + a^2)$

- **Cyclic Factors**

  If an expression remain same after replacing a by b, b by c & c by a, then it is called cyclic expression and its factors are called cyclic factors. e.g. $a(b - c) + b(c - a) + c(a - b)$

Illustrations

Illustration 7. If $x = \sqrt{3} + \sqrt{2}$, then find the value of $x + \frac{1}{x}$, $x^2 + \frac{1}{x^2}$, $x^3 + \frac{1}{x^3}$, $x^4 + \frac{1}{x^4}$

Solution. $x = \sqrt{3} + \sqrt{2}$, $\frac{1}{x} = \frac{1}{\sqrt{3} + \sqrt{2}} \times \frac{\sqrt{3} - \sqrt{2}}{\sqrt{3} - \sqrt{2}} = \sqrt{3} - \sqrt{2}$

(i) $x + \frac{1}{x} = \sqrt{3} + \sqrt{2} + \sqrt{3} - \sqrt{2} = 2\sqrt{3}$

(ii) $\left(x + \frac{1}{x}\right)^2 = x^2 + \frac{1}{x^2} + 2.x.\frac{1}{x} \quad \Rightarrow \quad (2\sqrt{3})^2 = x^2 + \frac{1}{x^2} + 2$

$\Rightarrow \quad 12 - 2 = x^2 + \frac{1}{x^2} \quad \Rightarrow \quad x^2 + \frac{1}{x^2} = 10$

(iii) $x^3 + \frac{1}{x^3}$

$$\Rightarrow \left(x+\frac{1}{x}\right)^3 = x^3 + \frac{1}{x^3} + 3.x.\frac{1}{x}\left(x+\frac{1}{x}\right)$$

$$\Rightarrow (2\sqrt{3})^3 = x^3 + \frac{1}{x^3} + 3 \times 2\sqrt{3} \Rightarrow 24\sqrt{3} = x^3 + \frac{1}{x^3} + 6\sqrt{3}$$

$$\Rightarrow 24\sqrt{3} - 6\sqrt{3} = x^3 + \frac{1}{x^3} \qquad \Rightarrow \sqrt{3}\,(24-6) = x^3 + \frac{1}{x^3}$$

$$\Rightarrow 18\sqrt{3} = x^3 + \frac{1}{x^3} \qquad \therefore\ x^3 + \frac{1}{x^3} = 18\sqrt{3}$$

(iv) $x^4 + \dfrac{1}{x^4}$

$$\Rightarrow \left(x^2+\frac{1}{x^2}\right)^2 = x^4 + \frac{1}{x^4} + 2.x^2.\frac{1}{x^2}$$

$$\Rightarrow (10)^2 = x^4 + \frac{1}{x^4} + 2 \qquad \Rightarrow 100 - 2 = x^4 + \frac{1}{x^4}$$

$$\Rightarrow 98 = x^4 + \frac{1}{x^4} \qquad \therefore\ x^4 + \frac{1}{x^4} = 98$$

***Illustration 8.** Suppose that a, b are two real numbers such that $a^2 + b^2 + 8a - 14b + 65 = 0$ find a and b

Solution. $a^2 + 8a + 16 + b^2 - 14b + 49 = 0$

which is possible only when

$a = -4$ and $b = 7$

Illustration 9. Simplify the expression $E = \dfrac{(a^2-b^2)^3+(b^2-c^2)^3+(c^2-a^2)^3}{(a-b)^3+(b-c)^3+(c-a)^3}$

Solution. Since, $(a^2-b^2) + (b^2-c^2) + (c^2-a^2) = 0$ and $(a-b) + (b-c) + (c-a) = 0$

$$E = \frac{(a^2-b^2)^3+(b^2-c^2)^3+(c^2-a^2)^3}{(a-b)^3+(b-c)^3+(c-a)^3} = \frac{3(a^2-b^2)(b^2-c^2)(c^2-a^2)}{3(a-b)(b-c)(c-a)} = (a+b)(b+c)(c+a)$$

***Illustration 10.** Solve the equation $a^3 + b^3 + 3ab = 1$ and find the relation between the real numbers a and b.

Solution. $a^3 + b^3 + 3ab = 1 \Rightarrow a^3 + b^3 + (-1)^3 = 3ab(-1) \Rightarrow a + b + (-1) = 0$ or $a = b = -1$

Illustration 11. Factorize (i) $x^4 + 5x^2 + 9$ (ii) $x^4 + 4$

Solution. (i) $(x^4 + 6x^2 + 9) - x^2 = (x^2+3)^2 - x^2 = (x^2 + 3 + x)(x^2 + 3 - x)$

(ii) $x^4 + 4 \Rightarrow (x^4 + 4x^2 + 4) - 4x^2 \Rightarrow (x^2+2)^2 - (2x)^2 \Rightarrow (x^2 + 2 - 2x)(x^2 + 2 + 2x)$

Illustration 12. Find the sum $\dfrac{1}{1+\sqrt{2}} + \dfrac{1}{\sqrt{2}+\sqrt{3}} + \dfrac{1}{\sqrt{3}+\sqrt{4}} + \ldots\ldots\ldots + $ upto 99 terms

Solution.

$$S = \frac{1}{\sqrt{2}+1} + \frac{1}{\sqrt{3}+\sqrt{2}} + \ldots\ldots\ldots + \frac{1}{\sqrt{100}+\sqrt{99}}$$

$$S = \frac{(\sqrt{2}-1)}{1} + \frac{(\sqrt{3}-\sqrt{2})}{1} + \frac{(\sqrt{4}-\sqrt{3})}{1} + \ldots\ldots\ldots + \frac{(\sqrt{100}-\sqrt{99})}{1}$$ (After rationalization of every term)

$S = \sqrt{100} - 1 \qquad \Rightarrow S = 10 - 1 = 9$

This method is called ***difference method***

3.0 DIVISIBILITY RULES

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Divisible by	**Remark**
2.	Last digit 0, 2, 4, 6, 8
3.	Sum of digits divisible by 3 (Remainder will be same when number is divided by 3 or sum of digits is divided by 3.)
4.	Last two digits divisible by 4 (Remainder will be same whether we divide the number or its last two digits)
5.	Last digit 0 or 5
6.	Divisible by 2 and 3 simultaneously.
8.	Last three digits is divisible by 8 (Remainder will be same whether we divide the number or its last three digits)
9.	Sum of digits divisible by 9. (Remainder will be same when number is divided by 9 or sum of digit is divided by 9)
10.	Last digits 0
11.	(Sum of digits at even places) – (Sum of digits at odd places) = divisible by 11

- **LCM and HCF**

(a) HCF is the highest common factor between any two or more numbers or algebraic expressions. When dealing only with numbers, it is also called "Greatest common divisor" (GCD).

(b) LCM is the lowest common multiple of two or more numbers or algebraic expressions.

(c) The product of HCF and LCM of two numbers (or expressions) is equal to the product of the numbers.

(d) LCM of $\left(\frac{a}{b}, \frac{p}{q}, \frac{l}{m}\right) = \frac{\text{L.C.M. of (a, p, l)}}{\text{H.C.F. of (b, q, m)}}$

Illustrations

Illustration 13. The smallest natural number of the form 123X43Y, which is exactly divisible by 6, is ________

Solution. For the number 123X43Y to be divisible by 6, it should be divisible by 2 and 3.

For divisibility by 2, Y should be even

$\therefore$ $Y \in \{0, 2, 4, 6, 8\}$ and for divisibility by 3,

sum of digits i.e. $1 + 2 + 3 + X + 4 + 3 + Y = X + Y + 13$,

should be divisible by 3. for number to be smallest $x = 0$ and $y = 2$, which satisfies all the conditions.

Illustration 14. Prove that $n \in N$

(i) $n^3 - n$ is divisible by 3

(ii) $n^5 - n$ is divisible by 5

Solution. (i) $n^3 - n = \underbrace{(n-1)\, n\, (n+1)}_{\text{3 consecutive integers}}$

so, atleast one is divisible by 3.

(ii) $n^5 - n = n(n-1)(n+1)(n^2+1)$

$= n(n-1)(n+1)[(n+2)(n-2)+5]$

$= \underbrace{(n-2)(n-1)\,n\,(n+1)(n+2)}_{\text{5 consecutive integers}} + \underbrace{5\{n(n-1)(n+1)\}}_{\text{divisible by 5}}$

BEGINNER'S BOX-1

TOPIC COVERED : NUMBER SYSTEM AND ALGEBRAIC EXPRESSION

1. Represent the following in fractional form ($\frac{p}{q}$, where p, q $\in$ I and q $\neq$ 0)

(i) $1.1\overline{4}$ (ii) $3.3\overline{79}$

****2.*** Which of the following is greater ?

(i) $\frac{7}{8}, \frac{6}{7}$ (ii) $\sqrt{13}-\sqrt{12}, \sqrt{14}-\sqrt{13}$ (iii) $\frac{9}{\sqrt{11}-\sqrt{2}}, \frac{6}{3\sqrt{3}}$

3. Remove the irrationality in the denominator

(i) $\sqrt{\frac{\sqrt{2}-1}{\sqrt{2}+1}}$ (ii) $\frac{1}{1+\sqrt{2}+\sqrt{3}}$

****4.*** Simplify and express the result in the form of a + bi :

(a) $-i\,(9+6i)\,(2-i)^{-1}$ (b) $\left(\frac{4i^3-i}{2i+1}\right)^2$

5. If $x-\frac{1}{x}=3$, then find the value of the expression $2\left(x^3-\frac{1}{x^3}\right)-3\left(x^2+\frac{1}{x^2}\right)-39$:

****6.*** If $x = 1+\sqrt{2}$ then find the value of the expression $x^4-x^3-2x^2-3x+1$

7. If $a^2 + b^2 + c^2 - ab - bc - ca \leq 0$, (where a, b, c are non-zero real number) then value of $\frac{a+b}{c}$ is :

****8.*** If $x = 4 + 2i$ then prove that value of the expression $x^3 - 7x^2 + 12x + 25$ is divisible by 1 and 5 (where $i=\sqrt{-1}$).

****9.*** $N = (3+1)(3^2 + 1)(3^4 + 1)(3^8 + 1) .....(3^{64} + 1)$. If N can be simplified as $\frac{(3^a-1)}{2}$ then find the value of a

4.0 POLYNOMIAL

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An expression of the form $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + ... + a_{n-1}x + a_n$, where n is a ***non negative integer*** and $a_0, a_1, a_2, ..., a_n$ are real numbers and $a_0 \neq 0$, is called a polynomial of degree n.

4.1 Remainder Theorem

Let P(x) be any polynomial of degree greater than or equal to one and 'a' be any real number. If P(x) is divided by (x – a), then the remainder is equal to P(a).

4.2 Factor Theorem

Let P(x) be a polynomial of degree greater than or equal to 1 and 'a' be a real number such that P(a) = 0, then (x – a) is a factor of P(x). Conversely, if (x – a) is a factor of P(x), then P(a) = 0.

Illustrations

Illustration 15. If a polynomial has remainder 3 and 5 when divided by x – 1 and x – 2 respectively, find the remainder when f(x) is divided by (x –1) (x – 2)

Solution. Clearly by remainder theorem. for p(x)

p(1) = 3 and p(2) = 5

now, when p(x) is divided by (x – 1) (x – 2), the remainder is at most linear.

Let remainder by ax + b and quotient q(x)

$\therefore$ $p(x) = (x-1)(x-2)\,q(x) + ax + b$

Putting x = 1

$p(1) = a + b \Rightarrow a + b = 3$

and putting x = 2

$p(2) = 2a + b \Rightarrow 2a + b = 5$

$\therefore$ a = 2 and b = 1 so, remainder is 2x + 1

Illustration 16. If $x^5 - 5qx + 4r$ is divisible by $(x-2)^2$. Find the value of q and r.

Solution. $p(x) = x^5 - 5q + 4r$ is divisible by (x –2)

$\therefore$ $p(2) = 0 \Rightarrow 32 - 10q + 4r = 0$

$\Rightarrow$ $5q = 16 + 2r$

$p(x) = x^5 - 16x - 2rx + 4r$

$= x(x^2 + 4)(x + 2)(x - 2) - 2r(x-2)$

$$= (x-2)\underbrace{\left[x(x+2)(x^2+4) - 2r\right]}_{Q(x)}$$

again $Q(x) = x(x + 2)(x^2 + 4) - 2r$ is divisible by x –2 and by factor theorem

$\Rightarrow$ $Q(2) = 0$

$\Rightarrow$ $2 \,.\, 4.\, 8 - 2r = 0 \Rightarrow r = 32$ and $q = 16$

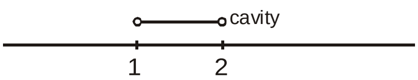
5.0 INTERVALS

AL

Intervals are basically subsets of R and are commonly used in solving inequalities or in finding domains. If there are two numbers a, b $\in$ R such that a < b, we can define three types of intervals as follows :

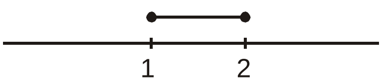
5.1 Open Interval – $(a, b) = \{x : a < x < b\}$ i.e. end points are not included.

Example: $1 < x < 2 \Rightarrow x \in (1, 2) \rightarrow$ paranthesis or $x \in\,]1, 2[$



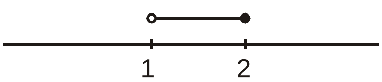
5.2 Closed Interval – $[a, b] = \{x : a \le x \le b\}$ i.e. end points are also included. This is possible only when both a and b are finite.

Example: $1 \le x \le 2 \Rightarrow x \in [1, 2] \rightarrow$ square bracket



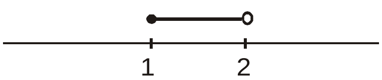
5.3 Semi Open Semi Closed Interval – $(a, b] = \{x : a < x \le b\}$

Example: $1 < x \le 2 \Rightarrow x \in (1, 2]$



5.4 Semi Closed Semi Open Interval – $[a, b) = \{x : a \le x < b\}$

Example: $1 \le x < 2 \Rightarrow x \in [1, 2)$



The infinite intervals are defined as follows :

(i) $(a, \infty) = \{x : x > a\}$ (ii) $[a, \infty) = \{x : x \geq a\}$

(iii) $(-\infty, b) = \{x : x < b\}$ (iv) $(-\infty, b] = \{x : x \leq b\}$

(v) $(-\infty, \infty) = \{x : x \in R\}$

Discrete set – If there are discrete points in a set then they are represented in curly bracket.

Example: $x = 2, 3, 4, -\sqrt{2}, -7 \Rightarrow x \in \{-7, -\sqrt{2}, 2, 3, 5\}$ →curly bracket

- **Some more examples**

$3 \leq x \leq 5$,	$x \in [3, 5]$		
$3 < x < 5$,	$x \in (3, 5)$	or	$]3, 5[$
$3 \leq x < 5$,	$x \in [3, 5)$	or	$[3, 5[$
$3 < x \leq 5$,	$x \in (3, 5]$	or	$]3, 5]$
$x \geq 3$,	$x \in [3, \infty)$	or	$[3, \infty[$
$x > 3$,	$x \in (3, \infty)$	or	$]3, \infty[$
$x \leq 3$,	$x \in (-\infty, 3]$	or	$]-\infty, 3]$
$x < 3$,	$x \in (-\infty, 3)$	or	$]-\infty, 3[$
$x \in R$,	$x \in (-\infty, \infty)$	or	$]-\infty, \infty[$

If there is ***no solution*** then, $x \in \phi$ (***Null set*** or ***Empty set*** or ***Void set***)

Subset (symbol $\subseteq$) and ***Proper subset (symbol $\subset$)***

e.g., $A = \{1, 2, 3, 4\}$

$B = \{1, 2, 3\}$

$C = \{1, 2, 3, 4\}$

$B \subseteq A \rightarrow$ True

$B \subset A \rightarrow$ True

$C \subseteq A \rightarrow$ True

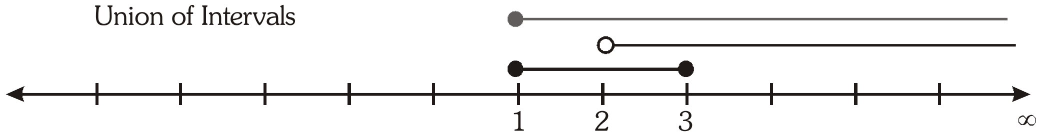
$C \subset A \rightarrow$ False

5.5 Union & Intersection

These are phenomenon of set theory, whenever there are numbers defined in one or more intervals and associated with the statement ***'OR'*** Union of the set of numbers gives the result. And if the sets of numbers are associated with the statement ***'AND'*** Intersection of the numbers gives the result.

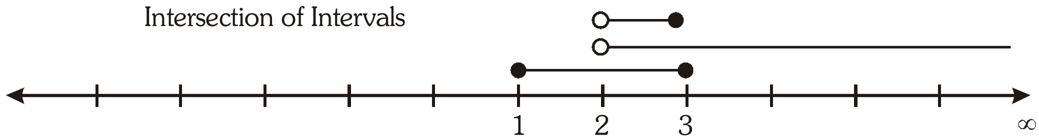
e.g. $1 \leq x \leq 3$ ***OR*** $2 < x < \infty \Rightarrow x \in [1,3]$ ***OR*** $x \in (2, \infty)$

Union of Intervals



$1 \leq x \leq 3$ ***AND*** $2 < x < \infty \Rightarrow x \in [1,3]$ ***AND*** $x \in (2, \infty)$

Intersection of Intervals



Illustrations

Illustration 17. ***True/False***

(i) $3 \in (3, 5) \rightarrow$ False

(ii) $-7 \in (-2, 9) \rightarrow$ False

(iii) $-2 \notin \{-1, -2, -3, -4\} \rightarrow$ False

(iv) $(2, 3) \subset [2, 3] \rightarrow$ True

(v) $(2, 3) \subseteq (2, 3) \rightarrow$ True

(vi) $-1 \in [-1, 3) \rightarrow$ True

6.0 VARIOUS TYPES OF FUNCTIONS

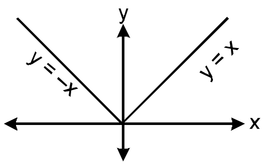
AL

6.1 Rational Function

A rational function is a function of the form, $y = f(x) = \dfrac{g(x)}{h(x)}$, where g (x) & h (x) are polynomial functions.

6.2 Absolute Value Function / Modulus Function

The symbol of modulus function is $f(x) = |x|$ and is defined as: $y = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$.



y = |x|

6.3 Greatest Integer Function or Step Up Function

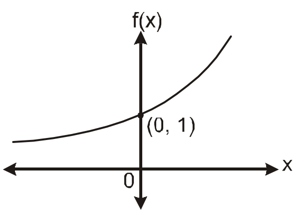
The function $y = f(x) = [x]$ is called the greatest integer function, where [x] equals to the greatest integer less than or equal to x. For example :

$[0.8] = 0$, $[1.5] = 1$, $[7.8] = 7$, $[-1.2] = -2$ etc.

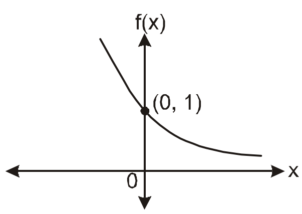
6.4 Exponential Function

A function $f(x) = a^x = e^{x \ln a}$ $(a > 0, a \neq 1, x \in R)$ is called an exponential function. Graph of exponential function can be as follows :

Case **- I**
For a > 1



Case **- II**
For 0 < a < 1



7.0 DEFINITION OF INDICES

AL

If 'a' is any non zero real or imaginary number and 'm' is a positive integer, then $a^m = a. a. a. ...a$ (m times). Here 'a' is called the base and m is the index, power or exponent.

- **Law of indices**

(i) $a^0 = 1$, $(a \neq 0)$

(ii) $a^{-m} = \frac{1}{a^m}$, $(a \neq 0)$

(iii) $a^{m+n} = a^m \cdot a^n$, where m and n are real numbers

(iv) $a^{m-n} = \frac{a^m}{a^n}$, where m and n are real numbers, $a \neq 0$

(v) $(a^m)^n = a^{mn}$

(vi) $a^{p/q} = \sqrt[q]{a^p}$

8.0 RATIO & PROPORTION

AL

8.1 Ratio

(i) If A and B be two quantities of the same kind, then their ratio is A : B; which may be denoted by the fraction $\frac{A}{B}$ (This may be an integer or fraction)

(ii) A ratio may be represented in a number of ways e.g. $\frac{a}{b} = \frac{ma}{mb} = \frac{na}{nb} = \ldots..$ where m, n,..... are non-zero numbers.

(iii) To compare two or more ratios, reduce them to common denominator.

8.2 Proportion

When two ratios are equal, then the four quantities composing them are said to be proportionals. If $\frac{a}{b} = \frac{c}{d}$, then it is written as a : b = c : d or a : b :: c : d

(i) 'a' and 'd' are known as extremes and 'b' and 'c' are known as means.

(ii) An important property of proportion : Product of extremes = product of means.

(iii) a : b = c : d,
$\Leftrightarrow$ b : a = d : c (Invertendo)

(iv) a : b = c : d,
$\Leftrightarrow$ a : c = b : d (Alternando)

(v) a : b = c : d,

$\Leftrightarrow \frac{a+b}{b} = \frac{c+d}{d}$ (Componendo)

(vi) a : b = c : d,

$\Leftrightarrow \frac{a-b}{b} = \frac{c-d}{d}$ (Dividendo)

(vii) a : b = c : d,

$\Leftrightarrow \frac{a+b}{a-b} = \frac{c+d}{c-d}$ (Componendo and Dividendo)

Illustrations

Illustration 18. Solve the equation $\dfrac{3x^4+x^2-2x-3}{3x^4-x^2+2x+3}=\dfrac{5x^4+2x^2-7x+3}{5x^4-2x^2+7x-3}$

Solution. $\dfrac{(3x^4)+(x^2-2x-3)}{(3x^4)-(x^2-2x-3)}=\dfrac{(5x^4)+(2x^2-7x+3)}{(5x^4)-(2x^2-7x+3)}$

$\Rightarrow \quad \dfrac{3x^4}{x^2-2x-3}=\dfrac{5x^4}{2x^2-7x+3}$ $\quad\quad \therefore$ ***x = 0 is a solution.***

$\dfrac{3x^4}{x^2-2x-3}=\dfrac{5x^4}{2x^2-7x+3}$ $\quad \Rightarrow \quad 6x^2-21x+9=5x^2-10x-15$

$\Rightarrow \quad 6x^2-5x^2-21x+10x+9+15=0$ $\quad \Rightarrow \quad x^2-11x+24=0$

$\Rightarrow \quad x^2-8x-3x+24=0$ $\quad \Rightarrow \quad x(x-8)-3(x-8)=0$

$\Rightarrow \quad (x-8)(x-3)=0$ $\quad \Rightarrow \quad x=8$ or $x=3$

Final Solution $\quad x = 0, 3, 8$

9.0 RATIONAL INEQUALITY

AL

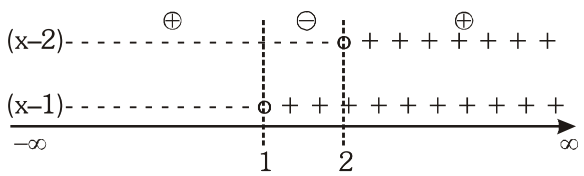
9.1 Method Of Interval

For solving rational inequalities of the following type :

$$\frac{(x-a_1)^{n_1}(x-a_2)^{n_2}.....(x-a_m)^{n_m}}{(x-b_1)^{p_1}(x-b_2)^{p_2}.....(x-b_m)^{p_m}}<0$$

(or >0 , ≥ 0, ≤ 0), where $a_1, a_2, ...a_m, b_1, b_2, ...b_m$ are real number and $n_1, n_2....n_m, p_1, p_2, ...p_m$ are natural number. We analyse change of sign at every zero of numerator and denaninator. On real number line

$(x - 1)(x - 2) > 0$



For the above inequality we can easily see $x = 1$ and w are critical points expression $(x - 1)$ and $(x - 2)$ chagnes sign at their critical point respectively and it divides the real number line in 3 intervals clearly solution set is $x \in (-\infty, 1) \cup (2, \infty)$.

Similary $\quad (x - 1)(x - 2) < 0 \Rightarrow x\in (1, 2)$

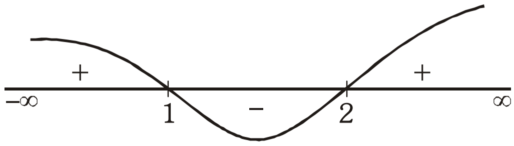
$(x - 1)(x - 2) \geq 0 \Rightarrow x\in (-\infty, 1] \cup [2, \infty)$

$(x - 1)(x - 2) \leq 0 \Rightarrow x\in [1, 2]$

9.2 Wavy Curve Method

The above analysis shwos a direct approach of solving rational inequality in following steps.

Locate critical point on real number line. Start a wave from extreme right critical point above the real number line which pass through all critical points making trough & crest above and below real number line as shown in figure. Trough shows tre and crest shows –ve.

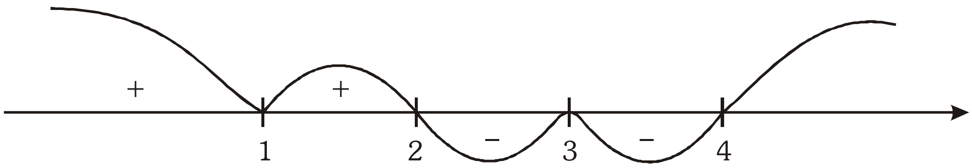


Clearly, $(x-1)(x-2) > 0 \Rightarrow x \in (-\infty, 1) \cup (2, \infty)$

$(x-1)(x-2) > 0 \Rightarrow x \in (1, 2)$

If rational inequality consisits of natural powers of linear factors, then for even power wave touches real number line and for odd power wave cuts real numbers line as shown in figures for given example.

$$\frac{(x-1)^2(x-2)^3}{(x-3)^4(x-4)^7} \le 0,\ x \ne 3,\ 4$$



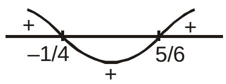
Hence solution set is $x \in \{1\} \cup [2, 3) \cup (3, 4)$

Note that $x = 1$ is in the soluton set which fulfills equality only

Illustrations

llustration 19. Solve for x :

(i) $\dfrac{6x-5}{4x+1} < 0\quad x \ne -\dfrac{1}{4}$



$\Rightarrow\quad x \in \left(-\dfrac{1}{4}, \dfrac{5}{6}\right)$

(ii) $\dfrac{2x-3}{3x-7} > 0\quad x \ne \dfrac{7}{3}$

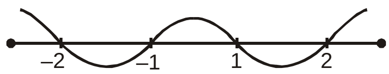
$\Rightarrow\quad \boldsymbol{x} \in \left(-\infty, \dfrac{3}{2}\right) \cup \left(\dfrac{7}{3}, \infty\right)$

(iii) $x^4 - 5x^2 + 4 < 0$

$\Rightarrow\quad (x^2)^2 - 4x^2 - x^2 + 4 < 0$

$\Rightarrow\quad (x^2 - 4)(x^2 - 1) < 0$

$\Rightarrow\quad (x-2)(x+2)(x-1)(x+1) < 0$



$\Rightarrow\quad x \in (-2, -1) \cup (1, 2)$

(iv) $\dfrac{3}{x-2} < 1\qquad x \ne 2$

$\Rightarrow\quad \dfrac{3-x+2}{x-2} < 0$

$\Rightarrow\quad \dfrac{5-x}{x-2} < 0$

$\Rightarrow\quad \dfrac{x-5}{x-2} > 0$

$\therefore\quad x \in (-\infty, 2) \cup (5, \infty)$

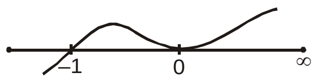
(v) $\dfrac{x^4+x^2+1}{x^2-4x-5} < 0$

$\Rightarrow \quad \dfrac{x^2+x^2+1}{x-5x+x-5} < 0$

$\Rightarrow \quad \dfrac{x^4+x^2+1}{(x+1)(x-5)} < 0$

$\Rightarrow \quad x \in (-1, 5)$

(vi) $\dfrac{x-1}{x+1} - x < 0$



$\dfrac{x-1-x^2-x}{x+1} < 0$

$\Rightarrow \quad \dfrac{-x^2-1}{x+1} < 0 \quad \Rightarrow \quad \dfrac{x^2+1}{x+1} > 0$

$\therefore \quad x \in (-1, \infty)$

(vii) $\dfrac{2(x-3)}{x(x-6)} - \dfrac{1}{x-1} \le 0$

$\Rightarrow \quad \dfrac{2(x-3)(x-1)-(x^2-6x)}{x(x-6)(x-1)} \le 0$

$\Rightarrow \quad \dfrac{2x^2-8x+6-x^2+6x}{x(x-6)(x-1)} \le 0 \quad \Rightarrow \quad \dfrac{x^2-2x+6}{x(x-6)(x-1)} \le 0$

$x \in (-\infty, 0) \cup (1, 6)$

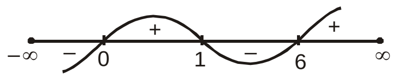


***Illustration 20.** Solve $\dfrac{x^2+6x-7}{x^2+1} \le 2$

Solution. $\dfrac{x^2+6x-7}{x^2+1} \le 2$

$\Rightarrow \quad x^2 + 6x - 7 \le 2x^2 + 2$

$\Rightarrow \quad x^2 - 6x + 9 \ge 0$

$\Rightarrow \quad (x-3)^2 \ge 0$

$\Rightarrow \quad x \in R$

BEGINNER'S BOX-2

TOPIC COVERED : POLYNOMIALS AND RATIONAL INEQUALITIES

1. If $x - a$ is a factor of $x^3 - a^2x + x + 2$, then find the value of 'a'

***2.** For any real numbers a, b, c find the smallest value of the expression $3a^2 + 27b^2 + 5c^2 - 18ab - 30c + 237$:

3. When a polynomial P(x) is divided by $(x - 2)$ and $(x - 3)$, remainders are 3 & 2 respectively. What is the remainder when the same polynomial is divided by $(x - 2)(x - 3)$?

***4.** The remainder when polynomial P(x) of degree 5 is divided by x + 1 and x – 1 is 1 and 2 respatively. Find the remainder when P(x) is divided by $x^2 - 1$.

***5.** If P(x) is a polynomial of degree 3 such that $P(i) = \frac{1}{i+1}\ \forall\ i = \{1,2,3,4\}$ Then find P(5).

***6.** If $P(x) = ax^7 + bx^5 + cx^3 + 3$ and P(7) = 2, P(–7) = ?

7. Solve $\frac{(x-5)}{(x^2+x+5)(x^2-4x-5)} > 0$

8. Solve $\frac{1}{x-1} > \frac{1}{x+1}$

9. Solve $\frac{2x}{x^2-9} \le \frac{1}{x+2}$

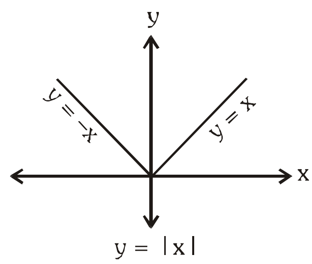
***10.** Solve $(x^2 + 3x + 1)(x^2 + 3x - 3) \ge 5$

10.0 ABSOLUTE VALUE FUNCTION / MODULUS FUNCTION

AL

The symbol of modulus function is $|x|$

and is defined as : $y = |x| = \begin{cases} x & \text{if } x \ge 0 \\ -x & \text{if } x < 0 \end{cases}$



Properties of Modulus :

For any a, b ∈ R

(a) $|a| \ge 0$

(b) $|a| = |-a|$

(c) $|ab| = |a||b|$

(d) $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$

(e) $|a + b| \le |a| + |b|$

(f) $|a| - |b| \le |a - b|$

(g) $||a| - |b|| = |a - b|$ iff $ab \ge 0$

10.1 Modulus Equation

Equation consisting of variable with in modulus.

Following points to be remembered.

$$|x| = a \Rightarrow \begin{cases} x = \pm a, & a > 0 \\ x = 0, & a = 0 \\ x \in \varnothing & a < 0 \end{cases}$$

It can also be seen graphically.

Illustrations

Illustration 21. If $||x-1| - 2| = 5$, then find x.

Solution. $|x - 1| - 2 = \pm 5$

$|x - 1| = 7, -3$

Case–I When $|x - 1| = 7 \Rightarrow x - 1 = \pm 7 \Rightarrow x = 8, -6$

Case–II When $|x - 1| = -3$ (reject)

Illustration 22. If $|x-1| + |x+1| = 2$, then find x.

Solution. ***Case–I*** If $x \le -1$

$-(x-1)-(x+1) = 2$

$\Rightarrow \quad -x+1-x-1 = 2$

$\Rightarrow \quad -2x = 2 \Rightarrow x = -1$ .... (i)

Case–II If $-1 < x < 1$

$-(x-1)+(x+1) = 2$

$\Rightarrow \quad -x+1+x+1 = 2$

$\Rightarrow \quad 2 = 2 \quad \Rightarrow -1 < x < 1$ .... (ii)

Case–III If $x \ge 1$

$x-1+x+1 = 2$

$\Rightarrow \quad x = 1$ .... (iii)

Thus from (i), (ii) and (iii) $-1 \le x \le 1$

****Illustration 23.*** Solve : $x|x+3| + 2|x+2| = 0$

Solution. ***Case–I*** $x < -3$

$-x(x+3) - 2(x+2) = 0$

$x^2 + 5x + 4 = 0 \Rightarrow x = -1, -4$

$\Rightarrow x = -4. \quad \because \quad x = -1$ (reject)

Case–II $-3 < x < -2$

$(x)(x+3) - 2x - 4 = 0$

$x^2 + x - 4 = 0$

$$\Rightarrow \quad x = \frac{-1+\sqrt{17}}{2}, \frac{-1-\sqrt{17}}{2}$$

$$\Rightarrow \quad x = \frac{-1-\sqrt{17}}{2} \qquad \because \qquad x = \frac{-1+\sqrt{17}}{2} \text{ (reject)}$$

Case–III $x > -2$

$x(x+3) + 2x + 4 = 0$

$x^2 + 5x + 4 = 0$

$\Rightarrow \quad x = -1, -4.$

$\Rightarrow \quad x = -1 \qquad \because \quad x = -4$ (reject)

Hence $x = -4, \dfrac{-1-\sqrt{17}}{2}, -1.$

Illustration 24. Solve the following equation

(i) $|x-3| = 4$

(ii) $||x-1| + 1| = 4$

(iii) $|x| - |x-2| = 2$

Solution. (i) $|x-3| = 4$

$\Rightarrow \quad (x-3) = \pm 4$

$\Rightarrow \quad x = 3 \pm 4$

$\Rightarrow \quad x = 7, -1$ (these values satisfy the original equation).

Final Solution $x \in \{7, -1\}$

(ii) $||x-1| + 1| = 4$

$\Rightarrow \quad |x-1| + 1 = \pm 4$

$\Rightarrow \quad |x-1| = \pm 4 - 1$

$\Rightarrow \quad |x-1| = 3 - 5^x$

$\Rightarrow \quad |x-1| = 3$

$\Rightarrow \quad (x-1) = \pm 3$

$\Rightarrow \quad x = 1 \pm 3$

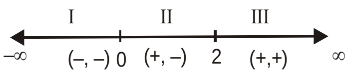
$\Rightarrow \quad x = 4, -2$ (these values satisfy the original equation).

(iii) $|x| - |x-2| - 2$

We know that $|x| = \begin{cases} x \ , & x \geq 0 \\ -x \ , & x < 0 \end{cases}$

$|x-2| = \begin{cases} (x-2) & x \geq 2 \\ -(x-2) & x < 2 \end{cases}$

Here $x = 0, 2$ are two critical points hence there are three intervals.



Case–1 when $-\infty < x < 0$

$|x| - |x-2| = 2$

$\Rightarrow \quad -x + x - 2 = 2$

$\Rightarrow \quad -2 = 2$ it is not possible

Hence $x \in [2, \infty)$

Case–2 when $0 \leq x < 2$

$|x| - |x-2| = 2$

$\Rightarrow \quad x - \{-(x-2)\} = 2$

$\Rightarrow \quad x + x - 2 = 2$

$\Rightarrow \quad 2x = 4$

$\Rightarrow \quad x = 2$ not in the taken interval $\therefore$ no solution $x \in \phi$

Case–3 when $2 \leq x < \infty$

$|x| - |x-2| = 2$

$\Rightarrow \quad x - (x-2) = 2$

$\Rightarrow \quad x - x + 2 = 2$

$\Rightarrow \quad 2 = 2$ it is an identity

Hence all the value in this interval. $\therefore x \in [2, \infty)$

Final solution $x \in [2, \infty)$

***Illustration 25.** Solve the equation

(i) $x^2 + 7|x| + 10 = 0$

(ii) $|3x-2| + x = 11$

(iii) $|x+1| + |x-2| + |x-5| = 2$

Solution. (i) ***Method 1***

$x^2 + 7|x| + 10 > 0 \ \forall \ x \in R$

Hence $x^2 + 7|x| + 10 = 0$

$x \in \phi$

Method 2

$x^2 = |x|^2$

$|x|^2 + 7|x| + 10 = 0$

$|x|^2 + 5|x| + 2|x| + 10 = 0$

$|x| = -2, -5$. (Absurd)

$\Rightarrow \quad x \in \phi$

(ii) $|3x-2| + x = 11$

Method–1 Here critical point is $x = 2/3$ so consider two cases:

Case–1 $-\infty < x < \frac{2}{3}$

$-(3x-2) + x = 11$

$x = -\frac{9}{2}$ permissible

Case–2 $\frac{2}{3} \le x < \infty$

$+3x - 2 + x = 11$

$\Rightarrow \quad x = \frac{13}{4}$

Final Solution $x \in \left\{-\frac{9}{2}, \frac{13}{4}\right\}$

Method–2

$|3x-2| + x = 11 \quad \Rightarrow \quad |3x-2| = 11 - x \quad \Rightarrow \quad (3x-2) = \pm(11-x)$

Taking +ve sign

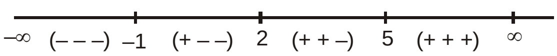
$3x - 2 = 11 - x \quad \Rightarrow \quad 3x + x = 13 \quad \Rightarrow \quad x = \frac{13}{4}$

Taking –ve sign

$(3x-2) = -11 + x \quad \Rightarrow \quad 3x - x = -11 + 2 \quad \Rightarrow \quad x = -\frac{9}{2}$

Final Solution $x \in \left\{-\frac{9}{2}, \frac{13}{4}\right\}$

(iii) $|x+1| + |x-2| + |x-5| = 2$ Critical point $\to -1, 2, 5$



Here –1, 2, 5 are three critical points hence four cases

Case–1 $-\infty < x < -1$

$-(x+1) - (x-2) - (x-5)$

$-3x + 6 = 2 \quad \Rightarrow -3x = 4 \Rightarrow x = -\frac{4}{3}$ (not in the taken interval hence not permissible)

Case–2 $1 \le x < 2$

$(x+1) - (x-2) - (x-5) = 2 \Rightarrow x = 6$ (not in the taken interval hence not permissible)

Case–3 $2 \le x < 5$

$(x+1) + (x-2) - (x-5) = 2$

$\Rightarrow \quad x + 1 + x - 2 - x + 5 = 2$

$\Rightarrow \quad 2x - x + 4 = 2$

$\Rightarrow \quad x = -2$ (not in the taken interval hence not permissible)

Case–4 $5 \le x < \infty$

$(x+1) + (x-2) + (x-5) = 2$

$\Rightarrow \quad x = \frac{8}{3}$ (not in the taken interval hence not permissible) Final solution $x \in \phi$

***Illustration 26.** Find the value of x, $|x-3| + 2|x+1| = 4$

Solution. Here critical point are 3, – 1

***Case*–I** if $x \geq 3$

$|x-3| + 2|x+1| = 4$

$\Rightarrow \quad (x-3) + 2(x+1) = 4$

$\Rightarrow \quad 3x - 1 = 4$

$$x = \frac{4+1}{3} = \frac{5}{3} = 1.666 \text{ (approximate)}$$

but here $x \geq 3$ Hence , there is no value of x in this interval

***Case*–II** if $-1 \leq x < 3$

$|x-3| + 2|x+1| = 4$

$\Rightarrow \quad -(x-3) + 2(x+1) = 4$

$\Rightarrow \quad -x + 3 + 2x + 2 = 4$

$\Rightarrow \quad x + 5 = 4$

$\Rightarrow \quad x = -5 + 4 = -1$

$\therefore \quad x = -1$

***Case*–III** if $x < -1$

$|x-3| + 2|x+1| = 4$

$\Rightarrow \quad -x + 3 - 2x - 2 = 4$

$\Rightarrow \quad -3x + 1 = 4$

$\Rightarrow \quad -3x = 3$

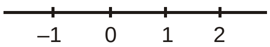
$\Rightarrow \quad x = -1$

but $x < -1$, Hence there is no value of x in this interval

Taking union of all the three cases final solution is $x \in \{-1\}$

***Illustration 27.** $|x+1| - |x| + 3|x-1| - 2|x-2| = x + 2$

Solution. Here, –1, 0, 1, 2 are four critical points hence five cases



***Case*–I** when $x \geq 2$

$|x+1| - |x| + 3|x-1| - 2|x-2| = x + 2$

$\Rightarrow \quad x + 1 - x + 3x - 3 - 2x + 4 = x + 2$

$\Rightarrow \quad x + 2 = x + 2$

Hence this is as identity so all the values of this interval will satisfy the equation

$\therefore \quad x \geq 2 \quad x \in [2, \infty)$

***Case*–II** when $1 \leq x < 2$

$|x+1| - |x| + 3|x-1| - 2|x-2| = x + 2$

$\Rightarrow \quad (x+1) - x + 3(x-1) + 2(x-2) = x + 2$

$\Rightarrow \quad x + 1 - x + 3x - 3 + 2x - 4 = x + 2$

$\Rightarrow \quad 5x - 6 = x + 2$

$\Rightarrow \quad 4x = 8$

$\Rightarrow \quad x = 2$

But $1 \leq x < 2$

Hence there is no value of x in this interval

***Case*–III** when $0 \leq x < 1$

$|x+1| - |x| + 3|x-1| - 2|x-2| = x + 2$

$\Rightarrow \quad (x+1) - x + (-3)(x-1) - (-2)(x-2) = x + 2$

$\Rightarrow \quad x + 1 - x - 3x + 3 + 2x - 4 = x + 2$

$\Rightarrow \quad -x = x + 2$

$\Rightarrow \quad -2x = 2$

$\Rightarrow \quad x = -1$ but $0 \leq x < 1$

hence, there is no value of x in this interval

Case–IV when $-1 \le x < 0$

$|x + 1| - |x| + 3|x - 1| - 2|x - 2| = x + 2$

$\Rightarrow$ $-(x + 1) + x - 3(x - 1) + 2(x - 2) = x + 2$

$\Rightarrow$ $x + 1 + x - 3x + 3 + 2x - 4 = x + 2$

$\Rightarrow$ $x = x + 2$

$\Rightarrow$ $0 = 2$

Hence there is no solution for x

Case–V when $x < -1$

$|x + 1| - |x| + 2|x - 1| - 2|x - 1| = x + 2$

$\Rightarrow$ $-(x + 1) + x - 3(x - 1) + 2(x - 2) = x + 2$

$\Rightarrow$ $-x - 1 + x - 3x + 2x - 4 = x + 2$

$\Rightarrow$ $-x - 2 = x + 2$

$\Rightarrow$ $-2x = 4$

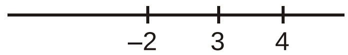
$\Rightarrow$ $x = -2$

Hence $x \in \{-2\} \cup [2, \infty)$

***Illustration 28.** Solve the equation $|x - 3| + |x + 2| - |x - 4| = 3$

Solution. $|x - 3| + |x + 2| - |x - 4| = 3$

$x = 3, -2, 4$ are three critical points hence four cases



when $x \ge 4$

$|x - 3| + |x + 2| - |x - 4| = 3$

$\Rightarrow$ $x - 3 + x + 2 - x + 4 = 3$

$\Rightarrow$ $x + 6 - 3 = 3$

$\Rightarrow$ $x + 3 = 3$

$\Rightarrow$ $x = 0$ (no solution)

when $3 \le x < 4$

$|x - 3| + |x + 2| - |x - 4| = 3$

$\Rightarrow$ $(x - 3) + (x + 2) - \{-(x - 4)\} = 3$

$\Rightarrow$ $x - 3 + x + 2 + x - 4 = 3$

$\Rightarrow$ $3x - 5 = 3$

$\Rightarrow$ $x = \frac{8}{3}$ (no solution)

when $-2 \le x < 3$

$|x - 3| + |x + 2| - |x - 4| = 3$

$\Rightarrow$ $-(x - 3) + (x + 2) - \{-(x - 4)\} = 3$

$\Rightarrow$ $-x - 3 + x + 2 + x - 4 = 3$

$\Rightarrow$ $x + 1 = 3$

$\Rightarrow$ $x = 2$

when $x < -2$

$|x - 3| + |x + 2| - |x - 4| = 3$

$\Rightarrow$ $-(x - 3) + \{-(x + 2)\} - \{-(x - 4)\} = 3$

$\Rightarrow$ $-x + 3 - x - 2 + x - 4 = 3$

$\Rightarrow$ $-x + 3 - 6 = 3$

$\Rightarrow$ $-x - 3 = 3$

$\Rightarrow$ $x = -6$

Hence, $x \in \{-6, 2\}$

***Illustration 29.** Solve for x : $2^{|x+1|} - 2^x = |2^x - 1| + 1$

Solution. Find critical points

$x + 1$ and $2^x - 1 = 0$

$\Rightarrow$ $x = -1$ and $x = 0$

so critical points are $x = 0$ and $x = -1$

Consider following cases :

$\boldsymbol{x \le -1}$ ...(i)

$2^{-(x+1)} - 2^x = -(2^x - 1) + 1$

$2^{-x-1} - 2^x = -2^x + 2$

$\Rightarrow$ $2^{-x-1} = 2$

$\Rightarrow$ $-x - 1 = 1$

$\Rightarrow$ $x = -2$

As $x = -2$ satisfies (i), one solution is $x = -2$

$\boldsymbol{-1 < x \le 0}$ ....(ii)

$2^{x+1} - 2^x = -(2^x - 1) + 1$

$\Rightarrow$ $2^{x+1} = 2$

$\Rightarrow$ $x + 1 = 1$

$\Rightarrow$ $x = 0$

As $x = 0$ satisfies (ii), second solution is $x = 0$

$\boldsymbol{x > 0}$ ...(iii)

$2^{x+1} - 2^x = (2^x - 1) + 1$

$\Rightarrow$ $2^{x+1} = 2^{x+1}$

$\Rightarrow$ identity in x, i.e. true for all $x \in R$

On combining $x \in R$ with (iii), we get :

$x > 0$

Now combining all cases, we have the final solution as :

$x \ge 0$ and $x = -2$

BEGINNER'S BOX-3

TOPIC COVERED : MODULUS EQUALITY

Solve the following equations

***1.** Solve : $|x + 3| = 2(5 - x)$

2. Solve : $x|x| + 7x - 8 = 0$

3. $|x| + 2 = 3$

4. $|x| - 2x + 5 = 0$

5. $x|x| = 4$

***6.** $\big||x - 1| - 2\big| = 1$

7. $|x|^2 - |x| + 4 = 2x^2 - 3|x| + 1$

8. $|x - 3| + 2|x + 1| = 4$

***9.** $\big||x - 1| - 2\big| = |x - 3|$

***10.** $|x - 1| + |x + 3| + |x - 5| = k$

find k if this equation has.

(i) only one solution

(ii) two solution

(iii) no solution

10.2 Modulus Inequality

Inequality that consist of variable in modulus.

Following point to be remembered :

$$|x| < a \Rightarrow \begin{cases} -a < x < a, & a > 0 \\ x \in \emptyset & a \le 0 \end{cases}$$

$$|x| \le a \Rightarrow \begin{cases} -a \le x \le a, & a > 0 \\ x = 0, & a = 0 \\ x \in \emptyset, & a < 0 \end{cases}$$

$$|x| > a \Rightarrow \begin{cases} x \in (-\infty, -a) U (a, \infty), & a > 0 \\ x \in R - \{0\}, & a = 0 \\ x \in R, & a < 0 \end{cases}$$

$$|x| \ge a \Rightarrow \begin{cases} x \in (-\infty, -a) U (a, \infty), & a > 0 \\ x \in R, & a \le 0 \end{cases}$$

It can also be seen graphically.

Illustrations

***Illustration 30.** Solve $\dfrac{x^2+x+1}{|x+1|} > 0$.

Solution. $\dfrac{x^2+x+1}{|x+1|} > 0$.

$\therefore$ $|x+1| > 0$

$\forall\ x \in R - \{-1\}$

$\therefore$ $x^2 + x + 1 > 0$

$\therefore$ $D = 1 - 4 = -3 < 0$

$\therefore$ $x^2 + x + 1 > 0\ \forall\ x \in R$

$\therefore$ $x \in (-\infty, -1) \cup (-1, \infty)$

***Illustration 31.** $\left|\dfrac{x^2-3x-1}{x^2+x+1}\right| < 3$.

Solution. $\dfrac{|x^2-3x-1|}{x^2+x+1} < 3$.

$\because$ in $x^2 + x + 1$

$D = 1 - 4 = -3 < 0$

$\therefore$ $x^2 + x + 1 > 0\ \forall\ x \in R$

$\therefore$ $|x^2 - 3x - 1| < 3(x^2 + x + 1)$

$\Rightarrow$ $(x^2 - 3x - 1)^2 - \{3(x^2 + x + 1)\}^2 < 0$

$\Rightarrow$ $(4x^2 + 2)(-2x^2 - 6x - 4) < 0$

$\Rightarrow$ $(2x^2 + 1)(x + 2)(x + 1) > 0$

$\Rightarrow$ $x \in (-\infty, -2) \cup (-1, \infty)$

BEGINNER'S BOX-4

TOPIC COVERED : MODULUS INEQUALITIES

Solve the following inequalities

1. $||x-1| + 2| \le 4$

***2.** $\left|\dfrac{2x-1}{x-1}\right| > 2$

3. $|x-3| + |x+4| \ge 12$

4. Solve for x, $\dfrac{|x-1|}{x+2} < 1 \quad x \in R$

***5.** $|x+1| + |x-1| = |2x|$

6. $|x^2-1| \le |2x-1|$

7. $|x^2 + x| - 5 < 0$

8. $x^2 - 7x + 12 < |x-4|$

***9.** Solve $\left|\dfrac{x^2-5x+4}{x^2-4}\right| \le 1$

***10.** If $|x-1| + |y-2| + (z-3)^2 \le 0$ then find the value of x + y + z (where x, y, z $\in$ R)

11.0 LOGARITHM

AL

- **Definition**

Every positive real number N can be expressed in exponential form as $a^x = N$ where 'a' is also a positive real number different than unity and is called the base and 'x' is called an exponent.

We can write the relation $a^x = N$ in logarithmic form as $\log_a N = x$. Hence $a^x = N \Leftrightarrow \log_a N = x$.

Hence logarithm of a number to some base is the exponent by which the base must be raised in order to get that number.

Limitations of logarithm – $\log_a N$ is defined only when

(i) $N > 0$ (ii) $a > 0$ (iii) $a \ne 1$

NOTE

(i) For a given value of N, $\log_a N$ will give us a unique value.

(ii) Logarithm of zero does not exist.

(iii) Logarithm of negative reals are not defined in the system of real numbers.

Illustrations

Illustration 32. The value of N, satisfying $\log_a[1 + \log_b\{1 + \log_c(1 + \log_p N)\}] = 0$ is -

(A) 4 (B) 3 (C) 2 (D) 1

Solution. $1 + \log_b\{1 + \log_c(1 + \log_p N)\} = a^0 = 1$

$\Rightarrow \log_b\{1 + \log_c(1 + \log_p N)\} = 0$

$\Rightarrow 1 + \log_c(1 + \log_p N) = 1$

$\Rightarrow \log_c(1 + \log_p N) = 0$

$\Rightarrow 1 + \log_p N = 1$

$\Rightarrow \log_p N = 0$

$\Rightarrow N = 1$ ***Ans. (D)***

****Illustration 33.*** If $\log_5 p = a$ and $\log_2 q = a$, then prove that $\dfrac{p^4q^4}{100} = 100^{2a-1}$

Solution. $\log_5 p = a \Rightarrow p = 5^a$

$\log_2 q = a \Rightarrow q = 2^a$

$$\Rightarrow \frac{p^4q^4}{100} = \frac{5^{4a}.2^{4a}}{100} = \frac{(10)^{4a}}{100} = \frac{(100)^{2a}}{100} = 100^{2a-1}$$

- **Basic Definition of Logarithm**

  Using the basic definition of logarithm we have 3 important deductions :

  (a) $\log_a 1 = 0$ i.e. logarithm of unity to any base is zero.

  (b) $\log_N N = 1$ i.e. logarithm of a number to the same base is 1.

  (c) $\log_{\frac{1}{N}} N = -1 = \log_N \frac{1}{N}$ i.e. logarithm of a number to the base as its reciprocal is –1.

  Note : $N = (a)^{\log_a N}$ e.g. $2^{\log_2 7} = 7$

BEGINNER'S BOX–5

TOPIC COVERED : DEFINITION OF LOGARITHM

1. Express the following in logarithmic form :

(a) $81 = 3^4$ (b) $0.001 = 10^{-3}$ (c) $2 = 128^{1/7}$

2. Express the following in exponential form :

(a) $\log_2 32 = 5$ (b) $\log_{\sqrt{2}} 4 = 4$ (c) $\log_{10} 0.01 = -2$

3. If $\log_4 m = 1.5$, then find the value of m.

***4.** If $\log_{2\sqrt{3}} 1728 = x$, then find x.

5 Find the value of the following :

(a) $\log_{\cot 22\frac{1}{2}^\circ}(\sec^2 x - \tan^2 x)$ ***(b)*** $\log_{1.4\bar{3}} \frac{43}{30}$ ***(c)*** $\left(\frac{1}{2}\right)^{\log_2 5}$

6. If $E = (\sin 10° + \cos 10°)^2 + (\cos 10° - \sin 10°)^2$, then find $\log_{0.5} E$

***7.** If $4^{\log_2 2x} = 36$, then find x.

***8.** Let $a = \left(\frac{1}{9}\right)^{-2\log_3 7}$ and $b = 2^{-\log_{\frac{1}{2}}(7)}$ then $a = (b)^k$ where k is equal to :

9. If $\log_3 5 = x$ and $\log_{25} 11 = y$ then the value of $\log_3\left(\frac{11}{3}\right)$ in terms of x and y is

- **The Principal Properties of Logarithms**

  If m,n are arbitrary positive numbers where $a > 0$, $a \neq 1$, then-

  (1) $\log_a mn = \log_a m + \log_a n$ (2) $\log_a \frac{m}{n} = \log_a m - \log_a n$

Illustrations

Illustration 34. Prove that $7\log\frac{16}{15} + 5\log\frac{25}{24} + 3\log\frac{81}{80} = \log 2$

Solution.

$$7\log\frac{16}{15} + 5\log\frac{25}{24} + 3\log\frac{81}{80}$$

$$= \log\left(\frac{16}{15}\right)^7 + \log\left(\frac{25}{24}\right)^5 + \log\left(\frac{81}{80}\right)^3 = \log\left(\left(\frac{16}{15}\right)^7 \times \left(\frac{25}{24}\right)^5 \times \left(\frac{81}{80}\right)^3\right)$$

$$= \log\left[\left(\frac{2^4}{3\times5}\right)^7 \times \left(\frac{5^2}{2^3\times3}\right)^5 \times \left(\frac{3^4}{2^4\times5}\right)^3\right] = \log\left[\frac{2^{28}}{3^7\times5^7} \times \frac{5^{10}}{2^{15}\times3^5} \times \frac{3^{12}}{2^{12}\times5^3}\right]$$

$$= \log\left[2^{28-15-12} \times 5^{10-7-3} \times 3^{12-7-5}\right] = \log(2^1 \times 5^0 \times 3^0) = \log 2$$

Illustration 35. If $a^2 + b^2 = 23ab$, then prove that $\log\frac{(a+b)}{5} = \frac{1}{2}(\log a + \log b)$.

Solution. $a^2 + b^2 = (a + b)^2 - 2ab = 23ab$

$\Rightarrow$ $(a + b)^2 = 25ab \Rightarrow a+b = 5\sqrt{ab}$ ....(i)

Using (i)

$$\text{L.H.S.} = \log\frac{(a+b)}{5} = \log\frac{5\sqrt{ab}}{5} = \frac{1}{2}\log ab = \frac{1}{2}(\log a + \log b) = \text{R.H.S.}$$

***Illustration 36.** If $\log_a x = p$ and $\log_b x^2 = q$, then $\log_x \sqrt{ab}$ is equal to (where $a, b, x \in R^+ - \{1\}$) -

(A) $\frac{1}{p} + \frac{1}{q}$ (B) $\frac{1}{2p} + \frac{1}{q}$ (C) $\frac{1}{p} + \frac{1}{2q}$ (D) $\frac{1}{2p} + \frac{1}{2q}$

Solution. $\log_a x = p \Rightarrow a^p = x \Rightarrow a = x^{1/p}$.
similarly $b^q = x^2 \Rightarrow b = x^{2/q}$

$$\text{Now, } \log_x \sqrt{ab} = \log_x \sqrt{x^{1/p}x^{2/q}} = \log_x x^{\left(\frac{1}{p}+\frac{2}{q}\right)\frac{1}{2}} = \frac{1}{2p} + \frac{1}{q}$$

- **Base Changing Theorem**

It can be stated as "quotient of the logarithm of two numbers is independent of their common base."

Symbolically, $\log_b m = \frac{\log_a m}{\log_a b}$ where $a > 0$, $a \neq 1$, $b > 0$, $b \neq 1$

NOTE

(i) $\log_b a . \log_a b = \frac{\log a}{\log b} . \frac{\log b}{\log a} = 1$; hence $\log_b a = \frac{1}{\log_a b}$

(ii) $a^{\log_b c} = c^{\log_b a}$

(iii) ***Base power formula -*** $\log_{a^k} m = \frac{1}{k}\log_a m$

(iv) The base of the logarithm can be any positive number other than 1, but in normal practice, only two bases are popular, these are 10 and $e(=2.718$ approx). Logarithms of numbers to the base 10 are named as 'common logarithm' and the logarithms of numbers to the base e are called Natural or Napierian logarithm. ***We will consider logx as $\log_e x$ or ℓnx.***

(v) Conversion of base e to base 10 & viceversa :

$$\log_e a = \frac{\log_{10} a}{\log_{10} e} = 2.303 \times \log_{10} a \; ; \qquad \log_{10} a = \frac{\log_e a}{\log_e 10} = \log_{10} e \times \log_e a = 0.434 \log_e a$$

Illustrations

***Illustration 37.** If a, b, c are distinct positive real numbers different from 1 such that
$(\log_b a . \log_c a - \log_a a) + (\log_a b . \log_c b - \log_b b) + (\log_a c . \log_b c - \log_c c) = 0$, then abc is equal to-

(A) 0 (B) e (C) 1 (D) none of these

Solution. $(\log_b a \log_c a - 1) + (\log_a b . \log_c b - 1) + (\log_a c \log_b c - 1) = 0$

$$\Rightarrow \frac{\log a}{\log b}.\frac{\log a}{\log c} + \frac{\log b}{\log a}.\frac{\log b}{\log c} + \frac{\log c}{\log a}.\frac{\log c}{\log b} = 3$$

$\Rightarrow$ $(\log a)^3 + (\log b)^3 + (\log c)^3 = 3\log a \log b \log c$

$\Rightarrow$ $(\log a + \log b + \log c) = 0$ [$\because$ If $a^3 + b^3 + c^3 - 3abc = 0$, then $a + b + c = 0$ if $a \neq b \neq c$]

$\Rightarrow$ $\log abc = \log 1 \Rightarrow abc = 1$

Illustration 38. Evaluate : $81^{1/\log_5 3}+27^{\log_9 36}+3^{4/\log_7 9}$

Solution. $81^{\log_3 5}+3^{3\log_9 36}+3^{4\log_9 7}$

$=3^{4\log_3 5}+3^{\log_3 (36)^{3/2}}+3^{\log_3 7^2}$

$= 625 + 216 + 49 = 890.$

BEGINNER'S BOX-6

TOPIC COVERED : PROPERTIES OF LOGARITHM

1. Show that $\frac{1}{2}\log 9+2\log 6+\frac{1}{4}\log 81-\log 12=3\log 3$

****2.*** If $\log_e x - \log_e y = a$, $\log_e y - \log_e z = b$ & $\log_e z - \log_e x = c$, then find the value of $\left(\frac{x}{y}\right)^{b-c}\times\left(\frac{y}{z}\right)^{c-a}\times\left(\frac{z}{x}\right)^{a-b}$

****3.*** Evaluate : $\frac{\log_3 135}{\log_{15} 3}-\frac{\log_3 5}{\log_{405} 3}$

4. Evaluate : $\log_9 27 - \log_{27} 9$

5. Evaluate : $2^{\log_3 5}-5^{\log_3 2}$

6. Evaluate : $\log_3 4 \,.\, \log_4 5 \,.\, \log_5 6 \,.\, \log_6 7 \,.\, \log_7 8 \,.\, \log_8 9$

****7.*** If $\log_a 3 = 2$ and $\log_b 8 = 3$, then $\log_a b$ is -

(A) $\log_3 2$ (B) $\log_2 3$ (C) $\log_3 4$ (D) $\log_4 3$

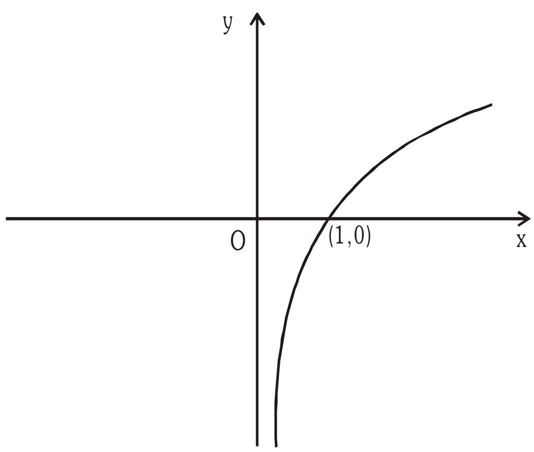
8. Let $S = \log_2\left(\sqrt{7}+\sqrt{5}\right)$, then find the value of $\log_2\left(\sqrt{7}-\sqrt{5}\right)$ in terms of S :

****9.*** If $\log_b 125 = c$ then $\log_b 25$ is what percent of the value of c, is $(b > 1)$

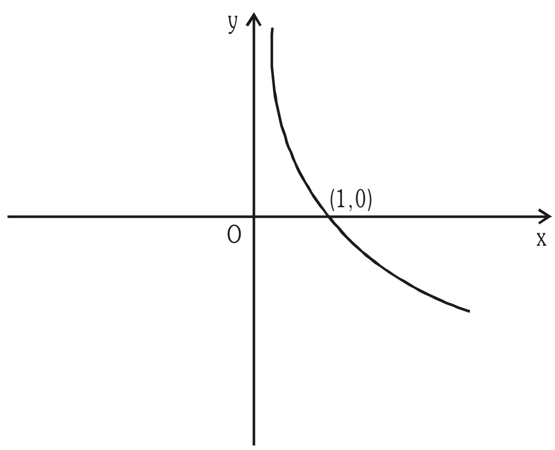
****10.*** Prove that the solution of the expression $\frac{1}{\log_4(18)}+\frac{1}{2\log_6(3)+\log_6(2)}+\frac{5}{\log_3(18)}$ is odd integer .

- **Graph of Logarithmic Functions**

Graph of $y = \log_a x$



When a > 1



When 0 < a < 1

- **Points to remember**

(i) If base of logarithm is greater than 1 then logarithm of greater number is greater. i.e. $\log_2 8 = 3$, $\log_2 4 = 2$ etc. and if base of logarithm is between 0 and 1 then logarithm of greater number is smaller. i.e. $\log_{1/2} 8 = -3$, $\log_{1/2} 4 = -2$ etc.

$$\log_a x < \log_a y \Leftrightarrow \begin{cases} x < y & \text{if } a > 1 \\ x > y & \text{if } 0 < a < 1 \end{cases}$$

(ii) It must be noted that whenever the number and the base are on the same side of unity then logarithm of that number to that base is positive, however if the number and the base are located on different side of unity then logarithm of that number to that base is negative.

e.g. $\log_{10} \sqrt[3]{10} = \frac{1}{3}$; $\log_{\sqrt{7}} 49 = 4$; $\log_{\frac{1}{2}} \left(\frac{1}{8}\right) = 3$; $\log_2 \left(\frac{1}{32}\right) = -5$; $\log_{10}(0.001) = -3$

(iii) $x + \frac{1}{x} \ge 2$ if x is positive real number and $x + \frac{1}{x} \le -2$ if x is negative real number

(iv) $n \ge 2$, $n \in N$

$\sqrt[n]{a} = a^{1/n} \Rightarrow n^{th}$ root of 'a' ('a' is a non negative number)

Some important values : $\log_{10} 2 = 0.3010$; $\log_{10} 3 = 0.4771$; $\ell n 2 = 0.693$, $\ell n 10 = 2.303$

- **Characteristic and Mantissa**

For any given number N, logarithm can be expressed as $\log_a N$ = Integer + Fraction

The integer part is called characteristic and the fractional part is called mantissa. When the value of log n is given, then to find digits of 'n' we use only the mantissa part. The characteristic is used only in determining the number of digits in the integral part (if $n \ge 1$) or the number of zeros after decimal & before first non-zero digit in the number (if $0 < n < 1$).

NOTE

(i) The mantissa part of logarithm of a number is always positive $(0 \le m < 1)$

(ii) If the characteristic of $\log_{10} N$ be n, then the number of digits in N is (n + 1)

(iii) If the characteristic of $\log_{10} N$ be (–n), then there exist (n – 1) zeros after decimal in N.

- **Antilogarithm**

The positive real number 'n' is called the antilogarithm of a number 'm' if log n = m

Thus, ***log n = m*** $\Leftrightarrow$ ***n = antilog m***

- **Logarithm Equation**

Any equation consisting of variable with logarithmic function.

Illustrations

Illustration 39. $\log_3 (x + 1) + \log_3 (x + 3) = 1$

Solution. $\log_3 (x + 1) + \log_3 (x + 3) = 1$ .....(i)

$\Rightarrow \log_3 (x + 1)(x + 3) = 1$

$\Rightarrow x^2 + 4x + 3 = 3$

$\Rightarrow x(x + 4) = 0$

$\Rightarrow x = 0, -4$

But x = – 4 does not satisfy the equation (i)

$\therefore x = 0$

Illustration 40. $\log_2 (3 - x) + \log_2 (1 - x) = 3$

Solution. $\log_2 (3 - x) + \log_2 (1 - x) = 3$ .....(i)

$\Rightarrow \log_2 (3 - x)(1 - x) = 3$

$\Rightarrow 3 - 4x + x^2 = 8$

$\Rightarrow x^2 - 4x - 5 = 0$

$\Rightarrow (x - 5)(x + 1) = 0$

$\Rightarrow x = 5, -1$

but x = 5 does note satisfy the equation (i)

$x = -1$

Illustration 41. $\log_2 \log_4 \log_5 x = 0$

Solution. $\log_2 \log_4 \log_5 x = 0$

$\Rightarrow \quad \log_4 \log_5 x = 1$

$\Rightarrow \quad \log_5 x = 4$

$\Rightarrow \quad x = 5^4 = 625$

Illustration 42. $\log_4 [2\log_3 [1 + \log_2(1 + 3\log_3 x)]] = \frac{1}{2}$

Solution. $\log_4 [2\log_3 [1 + \log_2 (1 + 3 \log_3 x)]] = \frac{1}{2}$

$\Rightarrow \quad 2\log_3 [1 + \log_2 (1 + 3 \log_3 x)] = 2$

$\Rightarrow \quad 1 + \log_2 (1 + 3 \log_3 x) = 3$

$\Rightarrow \quad \log_2 (1 + 3\log_3 x) = 2$

$\Rightarrow \quad 1 + 3 \log_3 x = 4$

$\Rightarrow \quad 3\log_3 x = 3$

$\Rightarrow \quad \log_3 x = 1$

$\Rightarrow \quad x = 3$

Illustration 43. Find the value of x , $\log_3[5 + 4 \log_3 (x - 1)] = 2$

Solution. $\log_3[5 + 4 \log_3 (x - 1)] = 2$

$\Rightarrow \quad 5 + 4 \log_3 (x - 1) = 9$

$\Rightarrow \quad 4 \log_3 (x - 1) = 4$

$\Rightarrow \quad \log_3 (x - 1) = 1$

$\Rightarrow \quad x - 1 = 3$

$\Rightarrow \quad x = 4$

***Illustration 44.** Find the value of x , $5^{2\log_5 x} - x - 6 = 0$

Solution. $5^{2\log_5 x} - x - 6 = 0$ ....(i)

$\Rightarrow \quad 5^{\log_5 x^2} - x - 6 = 0$

$\Rightarrow \quad x^2 - x - 6 = 0$

$\Rightarrow \quad (x - 3)(x + 2) = 0$

$\Rightarrow \quad x = 3, -2$

since $x = -2$ does not satisfy the equation (i)

Hence, $x = 3$

Illustration 45. Find the value of x, $5^{\log_5 x^2} - x - 6 = 0$

Solution. $5^{\log_5 x^2} - x - 6 = 0$

$\Rightarrow \quad x^2 - x - 6 = 0$

$\Rightarrow \quad (x - 3)(x + 2) = 0$

$\Rightarrow \quad x = 3, -2$

***Illustration 46.** Find the value of x, $\log_4 (x + 3) - \log_4(x - 1) = \log_4 8 - 2$.

Solution. $\log_4 (x + 3) - \log_4(x - 1) = \log_4 8 - 2$ ......(i)

$\Rightarrow \quad \log_4 \cdot \frac{x+3}{8(x-1)} = -2$

$$\Rightarrow \quad \frac{x+3}{8(x+1)} = \frac{1}{16}$$

$\Rightarrow \quad 2x + 6 = x + 1$

$\Rightarrow \quad x = -7$

since $x = -7$ does not satisfy the equation (i)

Hence, there is no value of x

$x \in \phi$

BEGINNER'S BOX-7

TOPIC COVERED : CHARACTERISTIC AND MANTISSA AND LOGARITHM EQUATION

1. Evaluate : $\log(0.06)^6$

***2.** Find number of digits in 18^{20}

***3.** Determine number of cyphers (zeros) between decimal & first significant digit in $\left(\frac{1}{6}\right)^{200}$

4. Find antilog of $\frac{5}{6}$ to the base 64.

***5.** Given that $\log 2 = 0.301$, find the number of digits before decimal in the solution to the equation $\log_5\left(\log_4\left(\log_3(\log_2 x)\right)\right) = 0$.

6. The value(s) of x satisfying the equation $\log x + \log(x-2) = \log(x^2 - 2x)$, is

***7.** The sum of all the solutions to the equation $7^{3x^2} \cdot 5^x = 11$, is

***8.** Let x and y are solutions of the system of equations

$$\begin{cases} \log_8(1) + \log_3(x+2) = \log_3(3-2y) \\ 2^{x+y} - 8^{3-y} = 0 \end{cases}$$

Then the value of $(y-x)$ is :

(A) -3 (B) 5 (C) 11 (D) None of these

***9.** The solution x of the equation $\log_4(3x+7) - \log_4(x-5) = 2$ would lie within which of the given ranges?

(A) $0 \le x \le 3$ (B) $3 \le x \le 6$ (C) $6 \le x \le 9$ (D) $9 \le x \le 12$

- **Logarithmic Inequality**

Logarithmic inequality : Inequality consisting of variable with logarithmic function.

Following points to be remembered.

(i) $\log_a x > p \Rightarrow \begin{cases} x > a^p, & a > 1 \\ 0 < x < a^p, & 0 < a < 1 \end{cases}$

(ii) $\log_a x < p \Rightarrow \begin{cases} 0 < x < a^p, & a > 1 \\ x > a^p, & 0 < a < 1 \end{cases}$

(iii) $\log_a x < \log_a y \Rightarrow \begin{cases} 0 < x < y, & a > 1 \\ x > y > 0, & 0 < a < 1 \end{cases}$

(iv) $\log_a x < \log_a y \Rightarrow \begin{cases} 0 < x < y, & a > 1 \\ x > y > 0, & 0 < a < 1 \end{cases}$

BEGINNER'S BOX-8

TOPIC COVERED : LOGARITHMIC INEQUATION

1. $\log_{\frac{1}{3}}\frac{2-3x}{x} \geq -1$

2. $\log_2 \log_4 \log_5 x > 0$

***3.** $\log_x\left(2x-\frac{3}{4}\right) > 2$

***4.** $\left(\frac{1}{3}\right)^{\frac{|x+2|}{2-|x|}} > 9$

***5.** $\left(\log_2\frac{x-1}{x+2}\right) > 0$

6. $\log_{\frac{1}{4}}(2-x) > \log_{\frac{1}{4}}\left(\frac{2}{x+1}\right)$

***7.** The equation $\log_2(\log_{1/2}(|x|-1)) > 0$ has
(A) No solution
(B) Infinite integral solution
(C) Two integral solution
(D) Infinite solutions.

8. The solution of the inequation
$\log_3(1-\log_{1/3}(x-1)) < 1$ is
(A) $x \in (3, 9)$
(B) $x \in (1, 9)$
(C) $x \in (1, 10)$
(D) None of these

***9.** Solve the inequation $\log_{|x|}(2-|x|) > 2$
(A) $x \in (-1, 1) - \{0\}$
(B) $x \in (-1, 1)$
(C) $x \in (1, \infty)$
(D) No solution

12.0 BASIC CONCEPTS OF GEOMETRY

AL

12.1 Basic theorems & results of triangles

(a) Two polygons are similar if (i) their corresponding angles are equal, (ii) the length of their corresponding sides are proportional. (Both conditions are independent & necessary)
In case of a triangle, any one of the conditions is sufficient, other satisfies automatically.

(b) ***Thales Theorem (Basic Proportionality Theorem) –*** In a triangle, a line drawn parallel to one side, to intersect the other sides in distinct points, divides the two sides in the same ratio.
Converse – If a line divides any two sides of a triangle in the same ratio then the line must be parallel to the third side.

(c) ***Similarity Theorem***

(i) ***AAA similarity*** – If in two triangles, corresponding angles are equal i.e. two triangles are equiangular, then the triangles are similar.

(ii) ***SSS similarity***– If the corresponding sides of two triangles are proportional, then they are similar.

(iii) ***SAS similarity –*** If in two triangles, one pair of corresponding sides are proportional and the included angles are equal then the two triangles are similar.

(iv) If two triangles are similar then

(1) They are equiangular

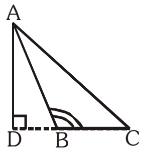
(2) The ratio of the corresponding (I) Sides (all), (II) Perimeters, (III) Medians, (IV) Angle bisector segments, (V) Altitudes are same (converse also true)

(3) The ratio of the areas is equal to the ratio of the squares of corresponding (I) Sides (all), (II) Perimeters, (III) Medians, (IV) Angle bisector segments, (V) Altitudes (converse also true)

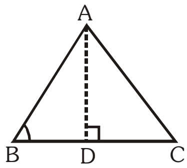
(d) *Pythagoras theorem*

(i) In a right triangle the square of hypotenuse is equal to the sum of square of the other two sides.

Converse – In a triangle if square of one side is equal to sum of the squares of the other two sides. then the angle opposite to the first side is a right angle.

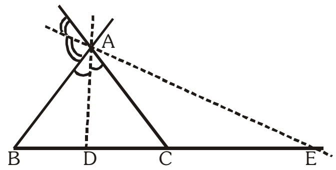
(ii) In obtuse Δ 

$AC^2 = AB^2 + BC^2 + 2BC \,.\, BD$

(iii) In Acute Δ 

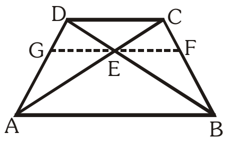
$AC^2 = AB^2 + BC^2 - 2BC \,.\, BD$

(e) The internal/external bisector of an angle of a triangle divides the opposite side internally/externally in the ratio of sides containing the angle (converse is also true) i.e. $\dfrac{AB}{AC} = \dfrac{BD}{DC} = \dfrac{BE}{CE}$



(f) The line joining the mid points of two sides of a triangle is parallel & half of the third side. (It's converse is also true)

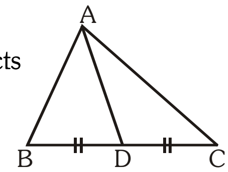
(g) (i) The diagonals of a trapezium divided each other proportionally. (converse is also true) i.e. $\dfrac{AE}{EC} = \dfrac{BE}{ED}$



(ii) Any line parallel to the parallel sides of a trapezium divides the non parallel sides proportionally i.e. $\dfrac{DG}{GA} = \dfrac{CF}{FB}$

(iii) If three or more parallel lines are intersected by two transversals, then intercepts made by them on transversals are proportional.

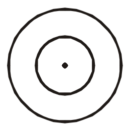
(h) In any triangle the sum of squares of any two sides is equal to twice the square of half of the third side together with twice the square of the median which bisects the third side. i.e. $AB^2 + AC^2 = 2\left(\dfrac{1}{2}BC\right)^2 + 2\,(AD)^2 = 2(AD^2 + BD^2)$



(i) In any triangle the three times the sum of squares of the sides of a triangle is equal to four times the sum of the square of the medians of the triangle.

(j) The altitudes, medians and angle bisectors of a triangle are concurrent among themselves.

12.2 Basic Theorems & Results of Circles



(a) *Concentric circles* – Circles having same centre.

(b) *Congruent circles –* Iff their radii are equal.

(c) *Congruent arcs –* Iff they have same degree measure at the centre.

Theorem 1

(i) If two arcs of a circle (or of congruent circles) are congruent, the corresponding chords are equal.

Converse – If two chords of a circle are equal then their corresponding arcs are congruent.

(ii) Equal chords of a circle (or of congruent circles) subtend equal angles at the centre.

Converse – If the angle subtended by two chords of a circle (or of congruent circles) at the centre are equal, the chords are equal.

Theorem 2

(i) The perpendicular from the centre of a circle to a chord bisects the chord.

Converse – The line joining the mid point of a chord to the centre of a circle is perpendicular to the chord.

(ii) Perpendicular bisectors of two chords of a circle intersect at its centre.

Theorem 3

(i) There is one and only one circle passing through three non collinear points.

(ii) If two circles intersects in two points, then the line joining the centres is perpendicular bisector of common chords.

Theorem 4

(i) Equal chords of a circle (or of congruent circles) are equidistant from the centre.

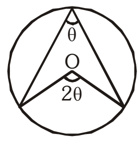
Converse – Chords of a circle (or of congruent circles) which are equidistant from the centre are equal.

(ii) If two equal chords are drawn from a point on the circle, then the centre of circle will lie on angle bisector of these two chords.

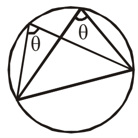
(iii) Of any two chords of a circle larger will be near to centre.

Theorem 5

(i) The degree measure of an arc or angle subtended by an arc at the centre is double the angle subtended by it at any point of alternate segment.

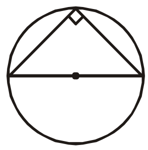


(ii) Angle in the same segment of a circle are equal.



(iii) The angle in a semi circle is right angle.

Converse – The arc of a circle subtending a right angle in alternate segment is semi circle.



Theorem 6

Any angle subtended by a minor arc in the alternate segment is acute and any angle subtended by a major arc in the alternate segment is obtuse.

Theorem 7

If a line segment joining two points subtends equal angles at two other points lying on the same side of the line segment, the four points are concyclic, i.e. lie on the same circle.

(d) Cyclic Quadrilaterals

A quadrilateral is called a cyclic quadrilateral if its all vertices lie on a circle.

Theorem 1

The sum of either pair of opposite angles of a cyclic quadrilateral is 180°

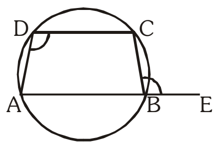
OR

The opposite angles of a cyclic quadrilateral are supplementary.

Converse – If the sum of any pair of opposite angle of a quadrilateral is 180°, then the quadrilateral is cyclic.

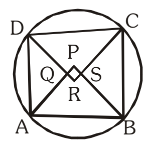
Theorem 2

If a side of a cyclic quadrilateral is produced, then the exterior angle is equal to the interior opposite angle. i.e. $\angle CBE = \angle ADC$

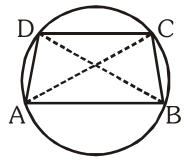


Theorem 3

The quadrilateral PQRS formed by angle bisectors of a cyclic quadrilateral is also cyclic.



Theorem 4



If two sides of a cyclic quadrilateral are parallel then the remaining two sides are equal and the diagonals are also equal. i.e. $AB \,||\, CD \Leftrightarrow AC = BD \;\&\; AD = BC$

OR

A cyclic trapezium is isosceles and its diagonals are equal.

Converse – If two non-parallel sides of a trapezium are equal, then it is cyclic.

OR

An isosceles trapezium is always cyclic.

Theorem 5

The bisectors of the angles formed by producing the opposite sides of a cyclic quadrilateral (provided that they are not parallel), intersect at right angle.

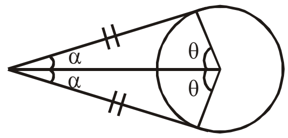
12.3 Tangents To A Circle

Theorem 1

A tangent to a circle is perpendicular to the radius through the point of contact.

Converse – A line drawn through the end point of a radius and perpendicular to it is a tangent to the circle.

Theorem 2



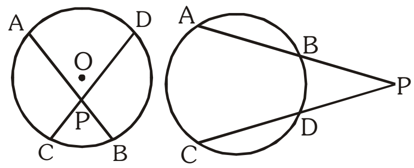
If two tangents are drawn to a circle from an external point, then :

(i) they are equal.

(ii) they subtend equal angles at the centre,

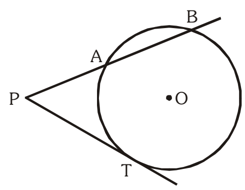
(iii) they are equally inclined to the segment, joining the centre to that point.

Theorem 3



If two chords of a circle intersect inside or outside the circle when produced, the rectangle formed by the two segments of one chord is equal in area to the rectangle formed by the two segments of the other chord. $PA \times PB = PC \times PD$

Theorem 4

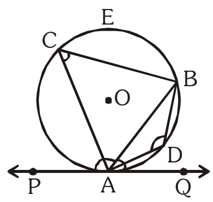


If PAB is a secant to a circle intersecting the circle at A and B and PT is tangent segment, then $PA \times PB = PT^2$

OR

Area of the rectangle formed by the two segments of a chord is equal to the area of the square of side equal to the length of the tangent from the point on the circle.

Theorem 5



If a chord is drawn through the point of contact of a tangent to a circle, then the angles which this chord makes with the given tangent are equal respectively to the angles formed in the corresponding alternate segments.

$\angle BAQ = \angle ACB$ and $\angle BAP = \angle ADB$

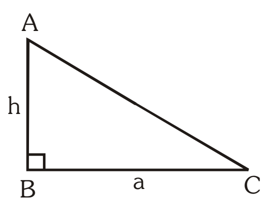
Converse

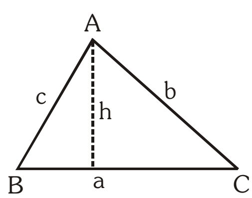
If a line is drawn through an end point of a chord of a circle so that the angle formed with the chord is equal to the angle subtended by the chord in the alternate segment, then the line is a tangent to the circle.

13.0 BASIC CONCEPT OF MENSURATION

AL

13.1 Triangle





(a) Sum of three angle is 180°

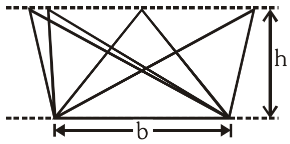
(b) Perimeter = Sum of three sides = $a + b + c = 2s$

Semi perimeter $s = (a + b + c)/2$

(c) Area= 1/2 (Base × Height)

$= \frac{1}{2}$ (Any side × Altitude over it) $= \Delta = \sqrt{s(s-a)(s-b)(s-c)}$

Note : Area of triangles formed between two same parallel lines and on the same base is same

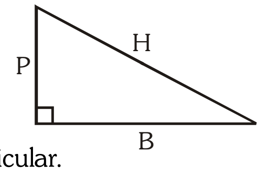


$$\text{Area} = \frac{1}{2}bh$$

(d) ***Right Angle Triangle*** – One angle 90° (Right angle)

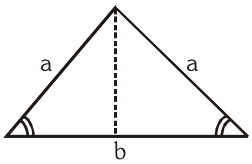
& Hypotenuse^2 = Perpendicular^2 + Base^2 (Pythagoras theorem)

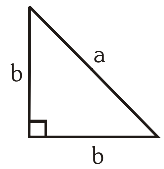
$$\text{Area} = \frac{1}{2}PB$$



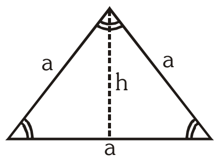
(e) ***Isosceles Triangle –*** Two sides equal hence two angle are equal.

Special case – Isosceles Right Triangle : Two sides equal and Base = Perpendicular.





(f) ***Equilateral Triangle –*** All three sides and angles (60°) are equal; $h = \left(\frac{\sqrt{3}}{2}\right)a$;

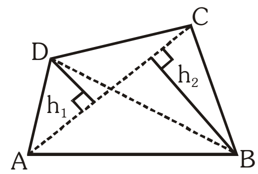


$$\text{Area} = \left(\frac{1}{2}\right)\text{base} \times \text{height} = \left(\frac{1}{2}\right)(a) \times \left(\frac{\sqrt{3}}{2}\right)a = \left(\frac{\sqrt{3}}{4}\right)a^2 = \frac{h^2}{\sqrt{3}}$$

13.2 Quadrilateral

(a) Sum of all angles is 360°

$$\text{Area} = \frac{1}{2}(AC)(h_1 + h_2) \text{ i.e. sum of areas of } \Delta\ ACD + \Delta ABC = \frac{1}{2}d_1d_2\sin\theta$$



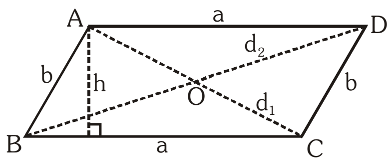
(b) ***Parallelogram***

(i) Opposite sides are parallel and equal.

(ii) Opposite angles are equal. ($\angle B = \angle D$ and $\angle A = \angle C$)

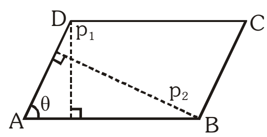
(iii) Diagonals bisects each other. AO = OC & BO = OD

(iv) Perimeter = 2(a + b) ;



(v) $\text{Area} = \frac{1}{2}(ah) + \frac{1}{2}(ah) = ah$ i.e. sum of areas of Δ ACD + ΔABC

also, $\text{Area} = \frac{p_1p_2}{\sin\theta}$



(c) ***Special cases of parallelogram***

(i) ***Rhombus –*** All sides are equal and opposite angles are equal.

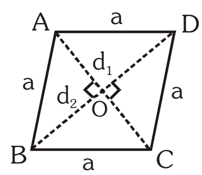
AB = BC = CD = DA = a

$\angle A = \angle C$ & $\angle B = \angle D$

Diagonals are not equal ($d_1 \neq d_2$) but bisects each other at 90°

AC ≠ BD but AC ⊥ BD

$$\text{Area} = \frac{1}{2}(d_1 \times d_2) \text{ i.e. sum of areas of } \Delta\ ACD + \Delta ABC$$

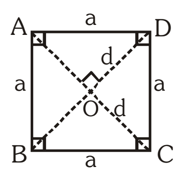


(ii) ***Square –*** All sides are equal and all angle are equal (90°)

Diagonals are equal and perpendicular bisectors of each other

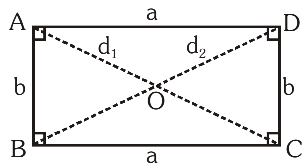
$$\text{Area} = a^2 = \frac{d^2}{2}$$

AC ⊥ BD & AO = OC, BO = OD



(iii) ***Rectangle –*** Opposite sides are equal and parallel, all angles are equal (90°) and diagonal are equal and bisects each other but not at 90°.

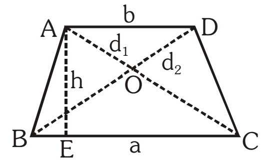
Area = a × b; Perimeter = 2(a + b)



(iv) ***Trapezium–*** Any two opposite sides are parallel but not equal. Diagonals cuts in same proportion. AD || BC ; AD ≠ BC ; $d_1 \neq d_2$

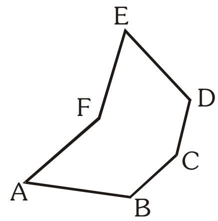
Area $= \left(\frac{1}{2}\right)(a + b)\,h$ i.e. sum of area of ΔABC + ΔACD

$\frac{AO}{OC} = \frac{OD}{OB}$ (∵ ΔBOC ~ ΔDOA)



13.3 Polygon

A plane figure enclosed by line segments (sides of polygon).

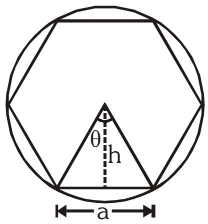


(a) ***n sides polygon have n sides –*** Triangle and quadrilaterals are polygon of three and four sides respectively. The polygons having 5 to 10 sides are called, PENTAGON, HEXAGON, HEPTAGON, OCTAGON, NANOGON and DECAGON respectively.

(b) Regular polygon – Polygon which has all equal sides and equal angles and can be inscribed in a circle whose center coincides with the center of polygon. Therefore the center is equidistant from all its vertices.

(i) A regular polygon can also circumscribe a circle.

(ii) A 'n' sided regular polygon can be divided into 'n' Isosceles Congruent Triangles with a common vertex i.e. centre of polygon.



(iii) Area $= n \times \left(\frac{1}{2}\right) \times a \times h$

(iv) Perimeter = na

(v) Each interior angle of polygon $= \left(\frac{n-2}{n}\right) \times 180°$

(vi) Angle subtended at the centre of inscribed/circumscribed circle by one side = 360°/n

(vii) Each exterior angle $= \left(\frac{360}{n}\right)^{\circ}$

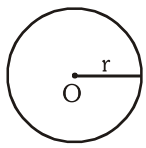
(viii) Sum of all interior angle = (n – 2) × 180°

(ix) Sum of all exterior angles = 360°

(x) ***Convex polygon –*** If any two consecutive vertices are joined then remaining all other vertices will lie on same side.

13.4 Circle

Area A = πr^2 ; Circumference (perimeter) = $2\pi r$

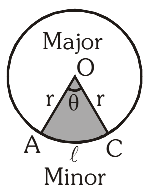


(a) ***Sector of a circle –*** Bounded by arc of circle (subtending angle 'θ' at center) and two radii. Circle is divided into minor (containing 'θ') and major sectors

(i) Arc length of sector : $\ell = \left(\frac{\theta°}{360°}\right) 2\pi r$

(ii) Area : A $= \left(\frac{\theta°}{360°}\right)\pi r^2 = \left(\frac{1}{2}\right)\ell r$

(iii) Perimeter of sector AOC = 2r + ℓ

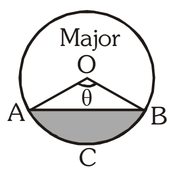


(b) ***Segment of a circle –*** Bounded by arc of the circle and the chord (determining the segment).

(i) Circle is divided into two segments minor segment and major segment.

(ii) When chord is diameter, sector coincides with segment.

(iii) Area (segment ACB) = Area of sector OACB - Area of ΔAOB



$$= \left(\frac{\theta^\circ}{360^\circ}\right) \times \pi r^2 - \left(\frac{1}{2}\right) \times \left(2r\sin\frac{\theta}{2}\right) \times \left(r\cos\frac{\theta}{2}\right)$$

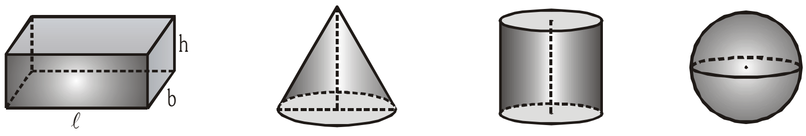
$$\text{Area} = \left(\frac{\theta^\circ}{360^\circ}\right)\pi r^2 - \left(\frac{1}{2}\right) r^2 \sin\theta$$

13.5 Solid

Require three dimension to describe

(a) ***Surfaces of solids –*** Plane areas bounding the solid e.g. six rectangle faces bounding a brick. Surface area is measured in square units.

(b) ***Volume of solids –*** Space occupied by a solid and is measured in cubic units.



Cuboid Cone Cylinder Sphere

13.5.1 Cuboid

Rectangular shaped solid also known as rectangular parallelopiped (e.g. match box, brick)

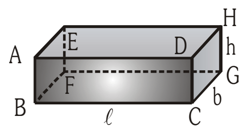
(a) Have six rectangular faces with opposite faces parallel and congruent.

(b) Have twelve edges (Edge - The line segment where two adjacent faces meets).

(c) Three adjacent faces meet at a point called vertex and cuboid have eight vertices

(d) ***Surface area :*** $A = 2[\ell \times b + b \times h + h \times \ell]$ square unit.

(e) ***Volume :*** $V = l \times b \times h$ cubic unit.

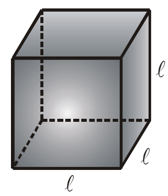


13.5.2 Cube

Special case of cuboid having all sides equal.

Area $= 6\ell^2$; Volume $= \ell^3$ Unit cube : Side $\ell = 1$

Volume is 1 cubic unit (From this cubic unit is derived)



13.5.3. Cylinder

Having a lateral (curved) surface and two congruent circular cross section.

(e.g. Jar, Circular Pillars, Drums, Pipes etc.)

(a) ***Axis –*** Line joining the centers of two circular cross section.

(b) ***Right circular cylinder –*** When axis is perpendicular to circular cross section.

(c) ***Generators –*** Lines parallel to axis and lying on the lateral surface.

(d) ***Base –*** With cylinder in vertical position, the lower circular end is base.

(e) ***Height (h) –*** Distance between two circular faces.

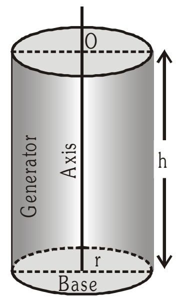
(f) ***Radius (r) –*** Radius of base or top circle.

(g) ***Total surface area –*** Base area + curved surface area

$= 2\pi r^2 + 2\pi rh = 2\pi r(h + r)$(including two circular ends).

Without circular ends (Hollow cylinder) $= 2\pi rh$

(h) ***Volume –*** $V = \pi r^2 h$



13.5.4. Cone

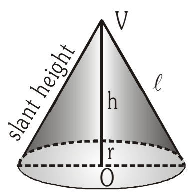
Have a curved surface with a vertex (V) and circular base radius : r and center O)

(a) ***Axis –*** Line joining vertex and center of base circle (VO)

(b) ***Height of cone (h) –*** Length of VO

(c) ***Slant height (Q) –*** Distance of vertex from any point of base circle

$\ell = \sqrt{r^2 + h^2}$



(d) ***Right circular cone –*** When axis is perpendicular to base.

(e) The cross section of a cone parallel to base is a circle and perpendicular to base is an isosceles triangle.

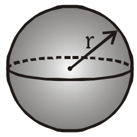
(f) ***Volume –*** $(1/3)\pi r^2 h$ (volume of a cone is 1/3rd of volume of a cylinder with same height and base radius).

(g) Curved surface Area : $\pi r\ell$

(h) Total surface Area : $\pi r\ell + \pi r^2 = \pi r(\ell + r)$

(i) A right circular cone can be generated by rotating a right angled triangle about its right angle forming side.

13.5.5. Sphere

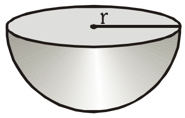


All point on its surface are equidistant from its center, the distance is called radius (r) and any line passing through center with end points on surface is called diameter.

(a) ***Volume –*** $(4/3)\pi r^3$

(b) ***Surface area –*** $4\pi r^2$

13.5.6. Hemisphere



A sphere is divided into two hemi spheres by a plane passing through center.

(a) ***Volume –*** $(2/3)\pi r^3$

(b) ***Curved surface area –*** $= 2\pi r^2$

(c) ***Total surface area –*** $= 2\pi r^2 + \pi r^2 = 3\pi r^2$

GOLDEN KEY POINTS

- (i) Any square of natural number can't have 2, 3, 7, 8 as unit digit
  (ii) All the squares of natural number are of $3k$ or $3k+1$ type
  (iii) All the squares of natural number are of $4k$ or $4k+1$ type
  (iv) All the squares of natural number are of $5k$, $5k+1$ or $5k+4$ type
  (v) A square can't have odd number of zeroes at the end.
  (vi) Cube of any natural number is of the form of $9k$, $9k+1$ or $9k+8$. $k \in N$.

m	n	m + n	m – n	m.n
Even	Even	Even	Even	Even
Even	Odd	Odd	Odd	Even
Odd	Odd	Even	Even	Odd

  $m, n \in I$

  From the above table we can notice that in reference to even and odd (m + n) and (m – n) are of same nature.

- Perfect square of an integer is of 4k or (4k+1) type where $k \in W$.

m	n	m + n	m – n	m.n	$\frac{m}{n}, n \neq 0$
Rational	Rational	Rational	Rational	Rational	Rational
Rational	Irrational	Irrational	Irrational	$\begin{cases} \text{Rational, if} & m = 0 \\ \text{Irrational, if} & m \neq 0 \end{cases}$	$\begin{cases} \text{Rational, if} & m = 0 \\ \text{Irrational, if} & m \neq 0 \end{cases}$
Irrational	Irrational	Real	Real	Real	Real

  From the above table we can conclude
  (i) Sum, difference, product and division of two rational number is always rational.
  (ii) Sum, difference, product and division of non-zero rational number and an irrational number is irrational.
  (iii) Sum, difference, product and division of two irrational number is a real number.

- Zero is neither positive integer nor negative integer. It is neutral integer.
- Two distinct prime numbers are always co-prime but converse need not be true.
- .Consecutive natural numbers are always co-prime numbers.
- Square of a real number is always non negative (i.e. $x^2 \geq 0$)
- Square root of a positive number is always positive e.g. $\sqrt{4} = 2$
- $\sqrt{x^2} = |x|$

SOME WORKED OUT ILLUSTRATIONS

Illustration 1. Show that $\log_4 18$ is an irrational number.

Solution. $\log_4 18 = \log_4(3^2 \times 2) = 2\log_4 3 + \log_4 2 = 2\dfrac{\log_2 3}{\log_2 4} + \dfrac{1}{\log_2 4} = \log_2 3 + \dfrac{1}{2}$

assume the contrary, that this number $\log_2 3$ is rational number.

$\Rightarrow \quad \log_2 3 = \dfrac{p}{q}$. Since $\log_2 3 > 0$ and $p, q, \in I$, p & q are coprimes,

$\Rightarrow \quad 3 = 2^{p/q} \;\Rightarrow\; 2^p = 3^q$

But this is not possible for any natural number p and q. The resulting contradiction completes the proof.

****Illustration 2.*** If in a right angled triangle, a and b are the lengths of sides and c is the length of hypotenuse and $c - b \neq 1$, $c + b \neq 1$, then show that

$\log_{c+b} a + \log_{c-b} a = 2\log_{c+b} a \,.\, \log_{c-b} a$.

Solution. We know that in a right angled triangle

$c^2 = a^2 + b^2$

$c^2 - b^2 = a^2$ .... (i)

$$\text{LHS} = \frac{1}{\log_a(c+b)} + \frac{1}{\log_a(c-b)} = \frac{\log_a(c-b) + \log_a(c+b)}{\log_a(c+b).\log_a(c-b)}$$

$$= \frac{\log_a(c^2 - b^2)}{\log_a(c+b).\log_a(c-b)} = \frac{\log_a a^2}{\log_a(c+b).\log_a(c-b)} \qquad \text{(using (i))}$$

$$= \frac{2}{\log_a(c+b).\log_a(c-b)} = 2\log_{(c+b)} a \,.\, \log_{(c-b)} a = \text{RHS}$$

****Illustration 3.*** Find the value of x, $|1 - \log_{1/5} x| + 2 = |3 - \log_{1/5} x|$

Solution. $|1 - \log_{1/5} x| + 2 = |3 - \log_{1/5} x|$

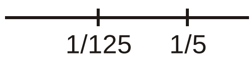
$\Rightarrow \quad 1 - \log_{1/5} x = 0$

$\Rightarrow \quad \log_{1/5} x = 1 \quad \Rightarrow x = \dfrac{1}{5}$

$3 = \log_{1/5} x = 0$

$\Rightarrow \quad \log_{1/5} x = 3 \quad \Rightarrow x = \dfrac{1}{125}$

Here creatical point $\dfrac{1}{5}, \dfrac{1}{125}$



when $x \geq \dfrac{1}{5}$

$|1 - \log_{1/5} x| + 2 = |3 - \log_{1/5} x|$

$\Rightarrow \quad 1 - \log_{1/5} x + 2 = 3 - \log_{1/5} x$

$\Rightarrow \quad 3 = 3$

it is an identity hence all the value in this interval will be satisfied

$x \in \left[\dfrac{1}{5}, \infty\right)$

when $\frac{1}{125} \le x < \frac{1}{5}$

$|1 - \log_{1/5} x| + 2 = |3 - \log_{1/5} x|$

$\Rightarrow$ $-(1 - \log_{1/5} x) + 2 = 3 - \log_{1/5} x$

$\Rightarrow$ $-1 + \log_{1/5} x + 2 = 3 - \log_{1/5} x$

$\Rightarrow$ $2 \log_{1/5} x = 2$

$\Rightarrow$ $x = \frac{1}{5}$ (no solution)

when $x < \frac{1}{125}$

$|1 - \log_{1/5} x| + 2 = |3 - \log_{1/5} x|$

$\Rightarrow$ $-1 + \log_{1/5} x + 2 = -3 + \log_{1/5} x$

$\Rightarrow$ $1 = -3$ not possible

Hence $x \in \left[\frac{1}{5}, \infty\right)$

Illustration 4. $\log(x+1)^{x^2} = 4 \log (x + 1)$

Solution. $\log(x+1)^{x^2} = 4 \log(x + 1)$ ...(i)

$\Rightarrow$ $x^2 \log (x + 1) - 4. \log (x + 1) = 0$

$\Rightarrow$ $\log (x + 1) [(x^2 - 4)] = 0$

$\log (x + 1) = 0$ or $x^2 - 4 = 0$

$\Rightarrow$ $x + 1 = 1$ $\Rightarrow$ $x^2 = 4$

$\Rightarrow$ $x = 0$ $\Rightarrow$ $x = \pm 2$

since $x = -2$ does not satisfy the equation (i)

Hence $x = 0, 2$

****Illustration 5.*** Find the value of x, $(x + 1)^{\log (x + 1)} = 100 (x + 1)$

Solution. Taking log in both the sides

$\Rightarrow$ $\log (x + 1)^{\log (x + 1)} = \log (100 (x + 1))$

$\Rightarrow$ $\log (x + 1) \log (x + 1) = \log 100 + \log (x + 1)$

$\Rightarrow$ $a^2 = 2 + a$ Let $a = \log (x + 1)$

$\Rightarrow$ $a^2 - a - 2 = 0$

$\Rightarrow$ $(a - 2) (a + 1) = 0$

$\Rightarrow$ $a = 2, -1$

$\Rightarrow$ $\log (x + 1) = 2, -1$

$\Rightarrow$ $x + 1 = 100, \frac{1}{10}$

$\Rightarrow$ $x = 99, -\frac{9}{10}$

Illustration 6. Find the value of x, $x^{1 + \log x} = 10x$

Solution. $x^{1 + \log x} = 10x$

Taking log in both sides

$\log x^{1 + \log x} = \log (10x)$

$\Rightarrow$ $(1 + \log x) \log x = \log 10 + \log x$

$\Rightarrow$ $(1 + \log x) \log x - (1 + \log x) = 0$

$\Rightarrow$ $(1 + \log x) (\log x - 1) = 0$

$\therefore \quad 1 + \log x = 0 \quad \text{or} \quad \log x - 1 = 0$

$\Rightarrow \quad \log x = -1 \quad \Rightarrow \log x = 1$

$\Rightarrow \quad x = \dfrac{1}{10}$

$\therefore \quad x = 10, \dfrac{1}{10}$

Illustration 7. Find the value of x, $x^{\log(x+1)} = x^2$

Solution. $x^{\log(x+1)} = x^2$

Taking log in both sides

$\Rightarrow \quad \log x^{\log(x+1)} = \log x^2$

$\Rightarrow \quad \log(x+1) \,.\, \log x = 2\log x$

$\Rightarrow \quad \log x\,[\log(x+1) - 2] = 0$

$\log x = 0 \quad \text{or} \log(x+1) = 2$

$\Rightarrow \quad x = 1 \quad \Rightarrow x + 1 = 100$

$\Rightarrow \quad x = 99$

$\therefore \quad x = 1, 99$

Illustration 8. Find the value of x, $2 \cdot x^{\log_4 3} + 3^{\log_4 x} = 27$

Solution. $2 \cdot x^{\log_4 3} + 3^{\log_4 x} = 27$

$\Rightarrow \quad 2\,.\,(x^{\log_x 3})^{\frac{1}{\log_x 4}} + 3^{\log_4 x} = 27$

$\Rightarrow \quad 2\,.\,3^{\log_4 x} + 3^{\log_4 x} = 27$

$\Rightarrow \quad 3^{\log_4 x} = 9 \quad \Rightarrow 3^{\log_4 x} = 3^2$

$\Rightarrow \quad \log_4 x = 2 \quad \Rightarrow x = 16$

****Illustration 9.*** Find the value of x, $\log_2(9 + 2^x) = 3$

Solution. $\log_2(9 + 2^x) = 3$

$\Rightarrow \quad 9 + 2^x = 8$

$\Rightarrow \quad 2^x = -1$

it is not true for any value of x

Hence, there is no solution of x

****Illustration 10.*** Find the value of x , $\log 4 + \left(1 + \dfrac{1}{2x}\right)\log 3 = \log\left(\sqrt[x]{3} + 27\right)$

Solution. $\log 4 + \left(1 + \dfrac{1}{2x}\right)\log 3 = \log\left(\sqrt[x]{3} + 27\right)$

$\Rightarrow \quad \log 4 + \log 3^{\left(1 + \frac{1}{2x}\right)} = \log(3^{1/x} + 27)$

$\Rightarrow \quad \log(4 \times 3 \times 3^{1/2x}) = \log(3^{1/x} + 27)$

$\Rightarrow \quad 12.3^{1/2x} = 3^{1/x} + 27$

Let $\quad 3^{1/2x} = a$

$\Rightarrow \quad 12a = a^2 + 27$

$\Rightarrow \quad a^2 - 12a + 27 = 0$

$\Rightarrow \quad (a-9)(a-3) = 0$

$\Rightarrow \quad a = 9, 3$

$\Rightarrow \quad 3^{1/2x} = 3^2, 3$

$$\Rightarrow \quad \frac{1}{2x} = 2, 1$$

$$x = \frac{1}{4}, \frac{1}{2}$$

Since for $\sqrt[x]{3}$ to be valid $x > 2$ ($x \in N$)

Hence there is no solution

Note– $\begin{bmatrix} \sqrt[n]{a} = a^{1/n} & n \geq 2,\ n \in N \\ a^{1/n} = y & \forall \quad n \in R \end{bmatrix}$

Illustration 11. Find the value of x, $\log_5 x + \log_{25} x = \log_{1/5} \sqrt{3}$

Solution. $\log_5 x + \log_{25} x = \log_{x/5} \sqrt{3}$

$$\Rightarrow \quad \log_5 x + \frac{1}{2}\log_5 x = -\frac{1}{2}\log_5 3$$

$$\Rightarrow \quad \frac{3}{2}\log_5 x = -\frac{1}{2}\log_5 3$$

$$\Rightarrow \quad \log_5 x^3 = \log_5 \frac{1}{3}$$

$$\Rightarrow \quad x^3 = \frac{1}{3}$$

$$\Rightarrow \quad x = \left(\frac{1}{3}\right)^{1/3}$$

$$\therefore \quad x \in \left\{\left(\frac{1}{3}\right)^{1/3}\right\}$$

****Illustration 12.*** Find the value of x, $\sqrt{\log_2 x} - \frac{1}{2} = \log_2 \sqrt{x}$

Solution. $\sqrt{\log_2 x} - \frac{1}{2} = \log_2 \sqrt{x}$

$$\Rightarrow \quad \sqrt{\log_2 x} - \frac{1}{2} = \frac{1}{2}\log_2 x$$

$\Rightarrow \quad 2\sqrt{a} - 1 = a$ Let $\log_2 x = a$

$\Rightarrow \quad a - 2\sqrt{a} + 1 = 0$

$\Rightarrow \quad (\sqrt{a} - 1)^2 = 0$

$\Rightarrow \quad a = 1$

$\Rightarrow \quad x = 2$

$\Rightarrow \quad \log_2 x = 1$

$\therefore \quad x \in \{2\}$

ANSWERS

BEGINNER'S BOX-1

1. ***(i)*** $\frac{103}{90}$ ***(ii)*** $\frac{1673}{495}$

2. ***(i)*** $\frac{7}{8}$ ***(ii)*** $\sqrt{13}-\sqrt{12}$ ***(iii)*** $\frac{9}{\sqrt{11}-\sqrt{2}}$

3. ***(i)*** $\left(\sqrt{2}-1\right)$ ***(ii)*** $\frac{2+\sqrt{2}-\sqrt{6}}{4}$

4. ***(i)*** $\frac{21-12i}{5}$ ***(ii)*** $3+4i$

5. 0 **6.** 2 **7.** 2 **9.** 128

BEGINNER'S BOX-2

1. –2 **2.** 192 **3.** 5–x **4.** $\frac{x+3}{2}$ **5.** $\frac{2}{15}$

6. 4 **7.** $(-1, \infty) - \{5\}$ **8.** $(-\infty, -1) \cup (1, \infty)$

9. $x \in (-\infty, -3)$ U$(-2, 3)$ **10.** $x \in (-\infty, -4]$ $[-2, -1]$ U $[1, \infty)$

BEGINNER'S BOX-3

1. $x \in \left\{\frac{7}{3}\right\}$ **2.** $x \in \{1\}$ **3.** $x \in \{-1, 1\}$

4. $x \in \{5\}$ **5.** $x \in \{2\}$ **6.** $x \in \{-2, 0, 2, 4\}$

7. $x \in \{-3, 3\}$ **8.** $x \in \{-1\}$ **9.** $x \geq 1$

10. (i) $k = 8$, (ii) $k > 8$ (iii) $k < 8$

BEGINNER'S BOX-4

1. $[-1, 3]$ **2.** $\left(\frac{3}{4}, 1\right) \cup (1, \infty)$ **3.** $\left(-\infty, -\frac{13}{2}\right] \cup \left[\frac{11}{2}, \infty\right)$

4. $(-\infty, -2) \cup (-\frac{1}{2}, \infty)$ **5.** $x \in (-\infty, -1] \cup [1, \infty)$ **6.** $[-\sqrt{3}-1, 0] \cup [\sqrt{3}-1, 2]$

7. $\left(-\left(\frac{1+\sqrt{21}}{2}\right), \left(\frac{\sqrt{21}-1}{2}\right)\right)$ **8.** $(2, 4)$ **9.** $\left[0, \frac{8}{5}\right] \cup \left[\frac{5}{2}, \infty\right)$

10. 6

BEGINNER'S BOX-5

1. (a) $\log_3 81 = 4$ (b) $\log_{10} 0.001 = -3$ (c) $\log_{128} 2 = \frac{1}{7}$

2. ***(a)*** $32 = 2^5$ (b) $4 = \sqrt{2}^4$ (c) $0.01 = 10^{-2}$

3. $m = 8$ **4.** $x = 6$ **5.** ***(a)*** 0, ***(b)*** 1, ***(c)*** $\frac{1}{5}$

6. –1 **7.** 3 **8.** 4 **9.** $2xy - 1$

BEGINNER'S BOX-6

2. 1 **3.** 3 **4.** $\frac{5}{6}$ **5.** 0 **6.** 2

7. C **8.** 1–S **9.** $66\frac{2}{3}$

BEGINNER'S BOX-7

1. – 7.3314 **2.** 26 **3.** 155 **4.** 32 **5.** 25

6. $x > 2$ **7.** $\frac{-\ell n5}{3\ell n7}$ **8.** (D) **9.** (C)

BEGINNER'S BOX-8

1. $x \in \left[\frac{1}{3}, \frac{2}{3}\right)$ **2.** $x > 625$ **3.** $x \in \left(\frac{3}{8}, \frac{1}{2}\right) \cup \left(1, \frac{3}{2}\right)$

4. (2, 6) **5.** $(-\infty, -2)$ **6.** $x \in (-1, 0) \cup (1, 2)$

7. (D) **8.** (D) **9.** (D)

EXERCISE – 1 MCQ (SINGLE CHOICE CORRECT)

1. If $x + \frac{1}{x} = 2$, then $x^3 + \frac{1}{x^3}$ is equal to

(A) 0 (B) 1 (C) 2 (D) 3

***2.** The number of real roots of the equation, $(x-1)^2 + (x-2)^2 + (x-3)^2 + (x-4)^2 + \ldots\ldots + (x-n)^2 = 0$, $(n > 1)$ is :

(A) 0 (B) 1 (C) 2 (D) 3

3. If p, q, r are real and distinct numbers, then the value of $\frac{(p-q)^3 + (q-r)^3 + (r-p)^3}{(p-q).(q-r).(r-p)}$ is

(A) 1 (B) p q r (C) 2 (D) 3

***4.** The polynomials $P(x) = kx^3 + 3x^2 - 3$ and $Q(x) = 2x^3 - 5x + k$, when divided by $(x-4)$ leave the same remainder. The value of k is

(A) 2 (B) 1 (C) 0 (D) –1

5. Solution of $|4x + 3| + |3x - 4| = 12$ is

(A) $x = -\frac{7}{3}, \frac{3}{7}$ (B) $x = -\frac{5}{2}, \frac{2}{5}$ (C) $x = -\frac{11}{7}, \frac{13}{7}$ (D) $x = -\frac{3}{7}, \frac{7}{5}$

6. The number of real roots of the equation $|x|^2 - 5|x| + 6 = 0$ is :

(A) 1 (B) 2 (C) 3 (D) 4

7. The value of $[\pi] - [-e]$ is, where [.] denotes greatest integer function

(A) 5 (B) 6 (C) 7 (D) 8

8. If $3^{2\log_3 x} - 2x - 3 = 0$, then the number of values of 'x' satisfying the equation is

(A) zero (B) 1 (C) 2 (D) more than 2

***9.** Number of real solutions of the equation $\sqrt{\log_{10}(-x)} = \log_{10}\sqrt{x^2}$ is :

(A) none (B) exactly 1 (C) exactly 2 (D) 4

***10.** Let $a, b \in R^+$ for which $60^a = 3$ and $60^b = 5$, then $12^{\frac{1-a-b}{2(1-b)}}$ is equal to

(A) 2 (B) 3 (C) 6 (D) 12

11. Number of cyphers after decimal before a significant figure comes in $\left(\frac{5}{3}\right)^{-100}$ is -

(A) 21 (B) 22 (C) 23 (D) none

12. $\dfrac{1}{\log_{\sqrt{bc}} abc} + \dfrac{1}{\log_{\sqrt{ca}} abc} + \dfrac{1}{\log_{\sqrt{ab}} abc}$ has the value equal to -

(A) 1/2 (B) 1 (C) 2 (D) 4

13. The value of $3^{\log_4 5} + 4^{\log_5 3} - 5^{\log_4 3} - 3^{\log_5 4}$ is -

(A) 0 (B) 1 (C) 2 (D) none of these

***14.** $\log_A B$, where $B = \dfrac{12}{3+\sqrt{5}+\sqrt{8}}$ and $A = \sqrt{1}+\sqrt{2}+\sqrt{5}-\sqrt{10}$ is -

(A) a negative integer (B) a prime integer

(C) a positive integer (D) an even–natural number

15. Number of integral solution of the equation, $4\log_{x/2}\left(\sqrt{x}\right) + 2\log_{4x}\left(x^2\right) = 3\log_{2x}\left(x^3\right)$. is :

(A) 0 (B) 1 (C) 2 (D) none of these

EXERCISE – 2 MCQ (ONE OR MORE CHOICE CORRECT)

***1.** If x, y, z are distinct real numbers, then the value of $\left(\frac{1}{x-y}\right)^2+\left(\frac{1}{y-z}\right)^2+\left(\frac{1}{z-x}\right)^2$ is

(A) $\left(\frac{1}{x-y}+\frac{1}{y-z}+\frac{1}{z-x}\right)^2$

(B) $\left(\frac{1}{x-y}+\frac{1}{y-z}+\frac{1}{z-x}\right)^2-2\sum\frac{1}{(x-y)\ (y-z)}$

(C) $\left(\frac{1}{x-y}+\frac{1}{y-z}+\frac{1}{z-x}\right)^2+2\sum\frac{1}{(x-y)\ (y-z)}$

(D) $\left(\frac{1}{x-y}+\frac{1}{y-z}+\frac{1}{z-x}\right)^2+\sum\frac{1}{(x-y)\ (y-z)}$

***2.** Let $t_n = \underbrace{11.........1}_{\text{n times}}$ then

(A) t_{102} is not prime (B) t_{951} is not prime (C) t_{540} is not prime (D) t_{91} is not prime

3. If 2576a456b is divisible by 15, then

(A) a may take the value 5
(B) b may take the value 0
(C) a may take the value 4
(D) a may take the value 6

4. If x & y are real numbers and $\frac{y}{x}=x$, then 'y' cannot take the value(s) :

(A) – 1 (B) 0 (C) 1 (D) 2

5. Which of the following when simplified, vanishes ?

(A) $\frac{1}{\log_3 2}+\frac{2}{\log_9 4}-\frac{3}{\log_{27} 8}$

(B) $\log_2\left(\frac{2}{3}\right)+\log_4\left(\frac{9}{4}\right)$

(C) $-\log_8 \log_4 \log_2 16$

(D) $\log_{10}\cot 1° + \log_{10}\cot 2° + \log_{10}\cot 3° + ... + \log_{10}\cot 89°$

***6.** If p, q ∈ N satisfy the equation $x^{\sqrt{x}}=\left(\sqrt{x}\right)^x$ then p & q are -

(A) relatively prime
(B) twin prime
(C) coprime
(D) if $\log_q p$ is defined then $\log_p q$ is not & vice versa

***7.** If $\log_p q + \log_q r + \log_r p$ vanishes where p, q and r are positive reals different than unity then the value of $(\log_p q)^3 + (\log_q r)^3 + (\log_r p)^3$ is -

(A) an odd prime (B) an even prime (C) an odd composite (D) an irrational number

Match the Column

Following question contains statements given in two columns, which have to be matched. The statements in ***Column-I*** are labelled as A, B, C and D while the statements in ***Column-II*** are labelled as p, q, r and s. Any given statement in ***Column-I*** can have correct matching with ***ONE OR MORE*** statement(s) in ***Column-II***.

***8.** Match the column for values of x which satisfy the equation in Column-I

	Column-I		Column-II
(A)	$\frac{\log_{10}(x-3)}{\log_{10}(x^2-21)} = \frac{1}{2}$	(p)	5
(B)	$x^{\log x+4} = 32$, where base of logarithm is 2	(q)	100
(C)	$5^{\log x} - 3^{\log x-1} = 3^{\log x+1} - 5^{\log x-1}$ where the base of logarithm is 10	(r)	2
(D)	$9^{1+\log x} - 3^{1+\log x} - 210 = 0$; where base of log is 3	(s)	$\frac{1}{32}$

Comprehension Based Questions

$\left(\frac{x^3-6x^2+11x-6}{(x^2-9x+20)^{1001}(x^2-x+30)}\right) \le 0$ has complete solution set $x \in (-\infty,1] \cup [a,b] \cup (c,d)$ then

9. value of a + b + c + d =

(A) 10 (B) 12 (C) 14 (D) 16

10. value of $\frac{b-a}{d-c}$

(A) 1 (B) 3 (C) 5 (D) 7

11. value of abc d = 15 k then k is

(A) 6 (B) 8 (C) 10 (D) none of these

EXERCISE - 3 SUBJECTIVE

1. What can be said about the numbers, a_1, a_2,......., a_n if it is known that,

$|a_1| + |a_2| + |a_3| + ... + |a_n| = 0$.

2. Solve the simultaneous equations

$|x + 2| + y = 5, \qquad x - |y| = 1$

3. Calculate : $7^{\log_3 5} + 3^{\log_5 7} - 5^{\log_3 7} - 7^{\log_5 3}$

4. If $4^A + 9^B = 10^C$, where $A = \log_{16}4$, $B = \log_3 9$ & $C = \log_x 83$, then find x.

***5.** If $\frac{\log a}{b-c} = \frac{\log b}{c-a} = \frac{\log c}{a-b}$, show that $a^a . b^b . c^c = 1$.

6. Which is greater

(a) $\log_2 3$ or $\log_{1/2} 5$ (b) $\log_7 11$ or $\log_8 5$

7. $\log_4 \log_3 \log_2 x = 0$

***8.** $\log_3\left(\log_9 x + \frac{1}{2} + 9^x\right) = 2x$.

9. $2\log_4 (4 - x) = 4 - \log_2 (-2 - x)$.

10. If $\log_{10} 2 = 0.3010$ and $\log_{10} 3 = 0.4771$, then find :

(a) the number of integers in 6^{15}

(b) the number of zeros immediately after the decimal in 3^{-100}

***11.** Compute the following :

(a) $\log_{1/3} \sqrt[4]{729.\sqrt[3]{9^{-1}.27^{-4/3}}}$ (b) $a^{\frac{\log_b(\log_b N)}{\log_b a}}$

12. Which is smaller ? 2 or $\left(\log_{e^{-1}} 2 + \log_2 e - 1\right)$.

EXERCISE – 4 RECAP OF AIEEE/JEE (MAIN)

1. Let $S = \{x \in R : x \geq 0$ and $2|\sqrt{x} - 3| + \sqrt{x}(\sqrt{x} - 6) + 6 = 0\}$. Then S : **[JEE-MAIN 2018]**

(A) contains exactly one element. (B) contains exactly two elements.

(C) contains exactly four elements. (D) is an empty set.

2. If 5, 5r, $5r^2$ are the lengths of the sides of a triangle, then r cannot be equal to : **[JEE-MAIN 2019]**

(A) $\frac{3}{2}$ (B) $\frac{3}{4}$ (C) $\frac{5}{4}$ (D) $\frac{7}{4}$

3. Consider the statement : "P(n): $n^2 - n + 41$ is prime." Then which one of the following is true? **[JEE-MAIN 2019]**

(A) P(5) is false but P(3) is true (B) Both P(3) and P(5) are false

(C) P(3) is false but P(5) is true (D) Both P(3) and P(5) are true

4. The sum of the solutions of the equation $\left|\sqrt{x} - 2\right| + \sqrt{x}\left(\sqrt{x} - 4\right) + 2 = 0, (x > 0)$ is equal to :**[JEE-MAIN 2019]**

(A) 4 (B) 9 (C) 10 (D) 12

5. The number of real roots of the equation $5 + |2^x - 1| = 2^x(2^x - 2)$ is : **[JEE-MAIN 2019]**

(A) 2 (B) 3 (C) 4 (D) 1

EXERCISE – 5 RECAP OF IIT-JEE/JEE (ADVANCED)

1. Number of solutions of $\log_4(x-1) = \log_2(x-3)$ is **[JEE 2001 (Screening)]**

(A) 3 (B) 1 (C) 2 (D) 0

***2.** Let (x_0, y_0) be the solution of the following equations **[JEE 2011]**

$(2x)^{\ln 2} = (3y)^{\ln 3}$ and $3^{\ln x} = 2^{\ln y}$

Then x_0 is

(A) $\frac{1}{6}$ (B) $\frac{1}{3}$ (C) $\frac{1}{2}$ (D) 6

***3.** The value of $6+\log_{\frac{3}{2}}\left(\frac{1}{3\sqrt{2}}\sqrt{4-\frac{1}{3\sqrt{2}}\sqrt{4-\frac{1}{3\sqrt{2}}\sqrt{4-\frac{1}{3\sqrt{2}}\ldots\ldots}}}\right)$ is **[JEE 2012]**

***4.** If $3^x = 4^{x-1}$, then $x =$ **[JEE 2013]**

(A) $\frac{2\log_3 2}{2\log_3 2-1}$ (B) $\frac{2}{2-\log_2 3}$ (C) $\frac{1}{1-\log_4 3}$ (D) $\frac{2\log_2 3}{2\log_2 3-1}$

5. The value of $\left((\log_2 9)^2\right)^{\frac{1}{\log_2(\log_2 9)}} \times \left(\sqrt{7}\right)^{\frac{1}{\log_4 7}}$ is —— **[JEE 2019]**

ANSWERS

EXERCISE-1

Que.	1	2	3	4	5	6	7	8	9	10
Ans.	C	A	D	B	C	D	B	B	C	A
Que.	11	12	13	14	15					
Ans.	B	B	A	C	C					

EXERCISE-2

Que.	1	2	3	4	5	6	7
Ans.	ABCD	ABCD	ABC	AB	ABCD	ACD	A

- ***Match the Column*** ***8.*** (A) $\rightarrow$ (p), (B) $\rightarrow$ (r,s), (C) $\rightarrow$ (q), (D) $\rightarrow$ (p)

- ***Comprehension Based Questions***

9. (C) ***10.*** (A) ***11.*** (B)

EXERCISE-3

1. $a_1 = a_2 = \ldots = a_n = 0$ ***2.*** $x = 2, y = 1$

3. 0 ***4.*** $x = 10$ ***6.*** (a) $\log_2 3$ (b) $\log_7 11$

7. $x = 8$ ***8.*** $x = \frac{1}{3}$ ***9.*** $x = -4$

10. (a) 12, (b) 47 ***11***. (a) –1, (b) $\log_b N$ ***12***. 2

EXERCISE-4

Que.	1	2	3	4	5
Ans.	A	D	D	C	D

EXERCISE-5

1. ***B*** ***2.*** ***C*** ***3.*** ***4*** ***4.*** ***ABC*** ***5.*** ***8***

TRIGONOMETRIC RATIOS & IDENTITIES

Recap of Early Classes

In earlier classes, we have studied the trigonometric ratios of acute angles as the ratio of the sides of a right angled triangle. We have also studied the trigonometric identities and application of trigonometric ratios in solving the problems related to heights and distances.

Index

TRIGONOMETRIC RATIOS & IDENTITIES

1.0 INTRODUCTION TO TRIGONOMETRY

SL AL

The word 'trigonometry' is derived from the Greek words 'trigon' and 'metron' and it means 'measuring the sides of a triangle'. The subject was originally developed to solve geometric problems involving triangles. It was studied by sea captains for navigation, surveyor to map out the new lands, by engineers and others. Currently, trigonometry is used in many areas such as the science of seismology, designing electric circuits, describing the state of an atom, predicting the heights of tides in the ocean, analysing a musical tone and in many other areas.

(a) ***Measurement of angles –*** Two systems of measurement of angles.

(i) ***English System –*** Here 1 right angle = 90° (degrees)

1° = 60' (minutes)

1' = 60" (seconds)

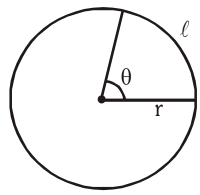
(ii) ***Circular system –*** Here an angle is measured in radians. One radian corresponds to the angle subtended by an arc of length 'r' at the centre of the circle of radius r. It is a constant quantity and does not depend upon the radius of the circle.

(b) Relation between the two systems : $\frac{D}{90} = \frac{R}{\pi/2}$

(c) If θ is the angle subtended at the centre of a circle of radius 'r',

by an arc of length 'ℓ' then $\frac{\ell}{r} = \theta$.

Note that here ℓ, r are in the same units and θ is always in radians.



Illustrations

Illustration 1. If the arcs of same length in two circles subtend angles of 60° and 75° at their centres. Find the ratio of their radii.

Solution. Let r_1 and r_2 be the radii of the given circles and let their arcs of same length s subtend angles of 60° and 75° at their centres.

$$\text{Now, } 60^\circ = \left(60 \times \frac{\pi}{180}\right)^c = \left(\frac{\pi}{3}\right)^c \text{ and } 75^\circ = \left(75 \times \frac{\pi}{180}\right)^c = \left(\frac{5\pi}{12}\right)^c$$

$$\therefore \quad \frac{\pi}{3} = \frac{s}{r_1} \text{ and } \frac{5\pi}{12} = \frac{s}{r_2}$$

$$\Rightarrow \quad \frac{\pi}{3} r_1 = s \text{ and } \frac{5\pi}{12} r_2 = s \quad \Rightarrow \quad \frac{\pi}{3} r_1 = \frac{5\pi}{12} r_2 \quad \Rightarrow \quad 4r_1 = 5r_2 \quad \Rightarrow \quad r_1 : r_2 = 5 : 4$$

Ans.

2.0 BASIC TRIGONOMETRIC IDENTITIES

SL AL

(1) $\sin\theta . \operatorname{cosec}\theta = 1$

(2) $\cos\theta . \sec\theta = 1$

(3) $\tan\theta . \cot\theta = 1$

(4) $\tan\theta = \frac{\sin\theta}{\cos\theta}$ & $\cot\theta = \frac{\cos\theta}{\sin\theta}$

(5) $\sin^2\theta + \cos^2\theta = 1$ or $\sin^2\theta = 1 - \cos^2\theta$ or $\cos^2\theta = 1 - \sin^2\theta$

(6) $\sec^2\theta - \tan^2\theta = 1$ or $\sec^2\theta = 1 + \tan^2\theta$ or $\tan^2\theta = \sec^2\theta - 1$

(7) $\sec\theta + \tan\theta = \dfrac{1}{\sec\theta - \tan\theta}$

(8) $\mathrm{cosec}^2\,\theta - \cot^2\theta = 1$ or $\mathrm{cosec}^2\theta = 1 + \cot^2\theta$ or $\cot^2\theta = \mathrm{cosec}^2\,\theta - 1$

(9) $\mathrm{cosec}\theta + \cot\theta = \dfrac{1}{\mathrm{cosec}\theta - \cot\theta}$

(10) Expressing trigonometrical ratio in terms of each other :

	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\cot\theta$	$\sec\theta$	$\mathrm{cosec}\,\theta$
$\sin\theta$	$\sin\theta$	$\sqrt{1-\cos^2\theta}$	$\dfrac{\tan\theta}{\sqrt{1+\tan^2\theta}}$	$\dfrac{1}{\sqrt{1+\cot^2\theta}}$	$\dfrac{\sqrt{\sec^2\theta-1}}{\sec\theta}$	$\dfrac{1}{\mathrm{cosec}\,\theta}$
$\cos\theta$	$\sqrt{1-\sin^2\theta}$	$\cos\theta$	$\dfrac{1}{\sqrt{1+\tan^2\theta}}$	$\dfrac{\cot\theta}{\sqrt{1+\cot^2\theta}}$	$\dfrac{1}{\sec\theta}$	$\dfrac{\sqrt{\mathrm{cosec}^2\theta-1}}{\mathrm{cosec}\,\theta}$
$\tan\theta$	$\dfrac{\sin\theta}{\sqrt{1-\sin^2\theta}}$	$\dfrac{\sqrt{1-\cos^2\theta}}{\cos\theta}$	$\tan\theta$	$\dfrac{1}{\cot\theta}$	$\sqrt{\sec^2\theta-1}$	$\dfrac{1}{\sqrt{\mathrm{cosec}^2\theta-1}}$
$\cot\theta$	$\dfrac{\sqrt{1-\sin^2\theta}}{\sin\theta}$	$\dfrac{\cos\theta}{\sqrt{1-\cos^2\theta}}$	$\dfrac{1}{\tan\theta}$	$\cot\theta$	$\dfrac{1}{\sqrt{\sec^2\theta-1}}$	$\sqrt{\mathrm{cosec}^2\theta-1}$
$\sec\theta$	$\dfrac{1}{\sqrt{1-\sin^2\theta}}$	$\dfrac{1}{\cos\theta}$	$\sqrt{1+\tan^2\theta}$	$\dfrac{\sqrt{1+\cot^2\theta}}{\cot\theta}$	$\sec\theta$	$\dfrac{\mathrm{cosec}\theta}{\sqrt{\mathrm{cosec}^2\theta-1}}$
$\mathrm{cosec}\,\theta$	$\dfrac{1}{\sin\theta}$	$\dfrac{1}{\sqrt{1-\cos^2\theta}}$	$\dfrac{\sqrt{1+\tan^2\theta}}{\tan\theta}$	$\sqrt{1+\cot^2\theta}$	$\dfrac{\sec\theta}{\sqrt{\sec^2\theta-1}}$	$\mathrm{cosec}\,\theta$

Illustrations

Illustration 2. If $\sin\theta + \sin^2\theta = 1$, then prove that $\cos^{12}\theta + 3\cos^{10}\theta + 3\cos^8\theta + \cos^6\theta - 1 = 0$

Solution. Given that $\sin\theta = 1 - \sin^2\theta = \cos^2\theta$

L.H.S. $= \cos^6\theta(\cos^2\theta + 1)^3 - 1 = \sin^3\theta(1 + \sin\theta)^3 - 1 = (\sin\theta + \sin^2\theta)^3 - 1 = 1 - 1 = 0$

Illustration 3. $2(\sin^6\theta + \cos^6\theta) - 3(\sin^4\theta + \cos^4\theta) + 1$ is equal to

(A) 0 (B) 1 (C) –2 (D) none of these

Solution. $2[(\sin^2\theta + \cos^2\theta)^3 - 3\sin^2\theta\cos^2\theta(\sin^2\theta + \cos^2\theta)] - 3[(\sin^2\theta + \cos^2\theta)^2 - 2\sin^2\theta\cos^2\theta] + 1$

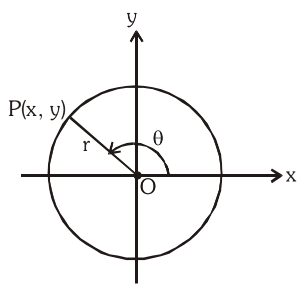
$= 2[1 - 3\sin^2\theta\cos^2\theta] - 3[1 - 2\sin^2\theta\cos^2\theta] + 1$

$= 2 - 6\sin^2\theta\cos^2\theta - 3 + 6\sin^2\theta\cos^2\theta + 1 = 0$ ***Ans.(A)***

3.0 DEFINITION OF T-RATIOS

SL AL

By using rectangular coordinates the definitions of trigonometric functions can be extended to angles of any size in the following way (see diagram). A point P is taken with coordinates (x, y). The radius vector OP has length r and the angle θ is taken as the directed angle measured anticlockwise from the x-axis. The three main trigonometric functions are then defined in terms of r and the coordinates x and y.



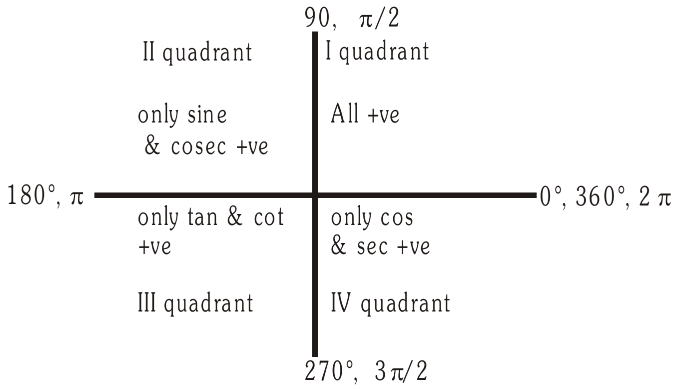
$\sin\theta = y/r$, $\cos\theta = x/r$, $\tan\theta = y/x$,

(The other function are reciprocals of these)

This can give negative values of the trigonometric functions.

4.0 SIGNS OF TRIGONOMETRIC FUNCTIONS IN DIFFERENT QUADRANTS

SL AL



5.0 TRIGONOMETRIC FUNCTIONS OF ALLIED ANGLES

SL AL

(a) $\sin(2n\pi + \theta) = \sin\theta$, $\cos(2n\pi + \theta) = \cos\theta$, where $n \in I$

(b)

$\sin(-\theta) = -\sin\theta$	$\cos(-\theta) = \cos\theta$	$\tan(-\theta) = -\tan\theta$
$\sin(90° - \theta) = \cos\theta$	$\cos(90° - \theta) = \sin\theta$	$\tan(90° - \theta) = \cot\theta$
$\sin(90° + \theta) = \cos\theta$	$\cos(90° + \theta) = -\sin\theta$	$\tan(90° + \theta) = -\cot\theta$
$\sin(180° - \theta) = \sin\theta$	$\cos(180° - \theta) = -\cos\theta$	$\tan(180° - \theta) = -\tan\theta$
$\sin(180° + \theta) = -\sin\theta$	$\cos(180° + \theta) = -\cos\theta$	$\tan(180° + \theta) = \tan\theta$
$\sin(270° - \theta) = -\cos\theta$	$\cos(270° - \theta) = -\sin\theta$	$\tan(270° - \theta) = \cot\theta$
$\sin(270° + \theta) = -\cos\theta$	$\cos(270° + \theta) = \sin\theta$	$\tan(270° + \theta) = -\cot\theta$
$\sin(360° - \theta) = -\sin\theta$	$\cos(360° - \theta) = \cos\theta$	$\tan(360° - \theta) = -\tan\theta$
$\sin(360° + \theta) = \sin\theta$	$\cos(360° + \theta) = \cos\theta$	$\tan(360° + \theta) = \tan\theta$

6.0 VALUES OF T-RATIOS OF SOME STANDARD ANGLES

SL AL

Angles / T-ratio	0°	30°	45°	60°	90°	180°	270°
	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	π	$3\pi/2$
$\sin\theta$	0	1/2	$1/\sqrt{2}$	$\sqrt{3}/2$	1	0	–1
$\cos\theta$	1	$\sqrt{3}/2$	$1/\sqrt{2}$	1/2	0	–1	0
$\tan\theta$	0	$1/\sqrt{3}$	1	$\sqrt{3}$	N.D.	0	N.D.
$\cot\theta$	N.D.	$\sqrt{3}$	1	$1/\sqrt{3}$	0	N.D.	0
$\sec\theta$	1	$2/\sqrt{3}$	$\sqrt{2}$	2	N.D.	–1	N.D.
$\text{cosec}\theta$	N.D.	2	$\sqrt{2}$	$2/\sqrt{3}$	1	N.D.	–1

N.D. → Not Defined

(a) $\sin n\pi = 0$; $\cos n\pi = (-1)^n$; $\tan n\pi = 0$ where $n \in I$

(b) $\sin(2n+1)\dfrac{\pi}{2} = (-1)^n$; $\cos(2n+1)\dfrac{\pi}{2} = 0$ where $n \in I$

Illustrations

Illustration 4. If $\sin\theta = -\frac{1}{2}$ and $\tan\theta = \frac{1}{\sqrt{3}}$ then θ is equal to -

(A) 30° (B) 150° (C) 210° (D) none of these

Solution. Let us first find out θ lying between 0 and 360°.

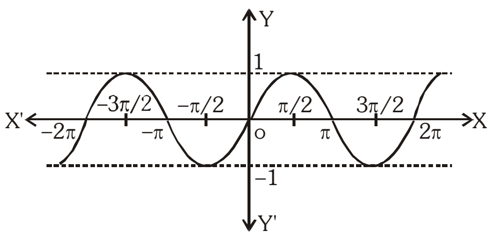
Since $\sin\theta = -\frac{1}{2} \Rightarrow \theta = 210°$ or $330°$ and $\tan\theta = \frac{1}{\sqrt{3}} \Rightarrow \theta = 30°$ or $210°$

Hence, $\theta = 210°$ or $\frac{7\pi}{6}$ is the value satisfying both. ***Ans. (C)***

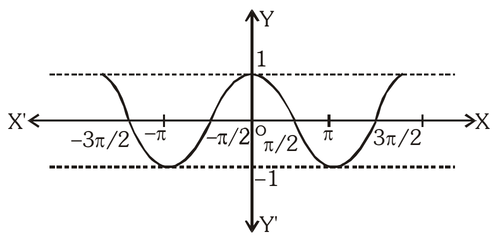
7.0 GRAPH OF TRIGONOMETRIC FUNCTIONS

SL AL

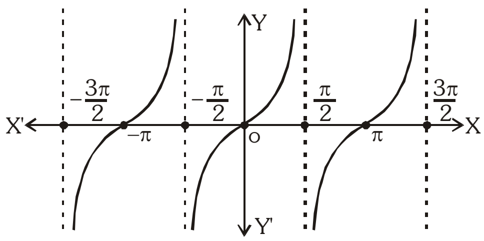
(i) ***y = sinx***



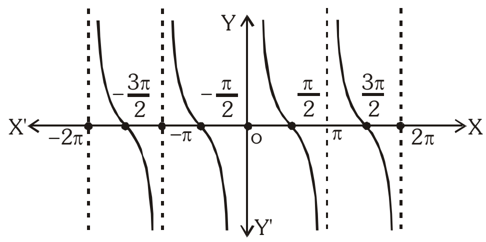
(ii) ***y = cosx***



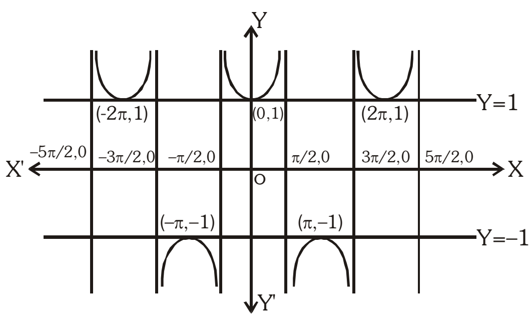
(iii) ***y = tanx***



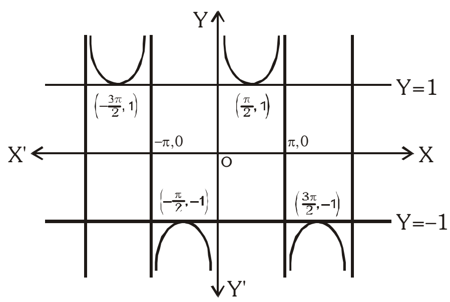
(iv) ***y = cotx***



(v) ***y = secx***



(vi) ***y = cosecx***



8.0 TRIGONOMETRIC RATIOS OF THE SUM & DIFFERENCE OF TWO ANGLES

SL AL

(i) $\sin(A + B) = \sin A\cos B + \cos A\sin B$.

(ii) $\sin(A - B) = \sin A\cos B - \cos A\sin B$.

(iii) $\cos(A + B) = \cos A\cos B - \sin A\sin B$

(iv) $\cos(A - B) = \cos A\cos B + \sin A\sin B$

(v) $\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$

(vi) $\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A\tan B}$

(vii) $\cot(A + B) = \frac{\cot B\cot A - 1}{\cot B + \cot A}$

(viii) $\cot(A - B) = \frac{\cot B\cot A + 1}{\cot B - \cot A}$

- ***Some more results***

  (i) $\sin^2 A - \sin^2 B = \sin(A + B).\sin(A - B) = \cos^2 B - \cos^2 A$.

  (ii) $\cos^2 A - \sin^2 B = \cos(A+B).\cos(A - B)$.

Illustrations

****Illustration 5.*** Prove that $\sqrt{3}$ cosec20° – sec20° = 4.

Solution.

$$\text{L.H.S.} = \frac{\sqrt{3}}{\sin 20^\circ} - \frac{1}{\cos 20^\circ} = \frac{\sqrt{3}\cos 20^\circ - \sin 20^\circ}{\sin 20^\circ.\cos 20^\circ}$$

$$= \frac{4\left(\frac{\sqrt{3}}{2}\cos 20^\circ - \frac{1}{2}\sin 20^\circ\right)}{2\sin 20^\circ \cos 20^\circ}$$

$$= \frac{4(\sin 60.\cos 20^\circ - \cos 60^\circ.\sin 20^\circ)}{\sin 40^\circ}$$

$$= 4.\frac{\sin(60^\circ - 20^\circ)}{\sin 40^\circ} = 4.\frac{\sin 40^\circ}{\sin 40^\circ} = 4 = \text{R.H.S.}$$

****Illustration 6.*** Prove that tan70° = cot70° + 2cot40° .

Solution.

$$\text{L.H.S.} = \tan 70^\circ = \tan(20^\circ + 50^\circ) = \frac{\tan 20^\circ + \tan 50^\circ}{1 - \tan 20^\circ \tan 50^\circ}$$

or $\tan 70^\circ - \tan 20^\circ \tan 50^\circ \tan 70^\circ = \tan 20^\circ + \tan 50^\circ$

or $\tan 70^\circ = \tan 70^\circ \tan 50^\circ \tan 20^\circ + \tan 20^\circ + \tan 50^\circ = 2\tan 50^\circ + \tan 20^\circ$

$= \cot 70^\circ + 2\cot 40^\circ = \text{R.H.S.}$

BEGINNER'S BOX–1

TOPIC COVERED : INTRODUCTION, BASIC IDENTITIES, DEFINITION OF T-RATIOS, TRIGONOMETRIC FUNCTIONS OF ALLIED ANGLES, VALUES OF T-RATIOS OF SOME STANDARD ANGLES, GRAPHS, TRIGONOMETRIC RATIOS OF THE SUM & DIFFERENCE OF TWO ANGLES

1. The radius of a circle is 30 cm. Find the length of an arc of this circle if the length of the chord of the arc is 30 cm.

2. If $\cot\theta = \frac{4}{3}$, then find the value of sinθ, cosθ and cosecθ in first quadrant.

3. If sinθ + cosecθ = 2, then find the value of $\sin^8\theta + \text{cosec}^8\theta$

4. If $\cos\theta = -\frac{1}{2}$ and $\pi < \theta < \frac{3\pi}{2}$, then find the value of $4\tan^2\theta - 3\text{cosec}^2\theta$.

5. Prove that cos570° sin510° + sin(–330°) cos(–390°) = 0

****6.*** Prove that $\tan\frac{11\pi}{3} - 2\sin\frac{9\pi}{3} - \frac{3}{4}\text{cosec}^2\frac{\pi}{4} + 4\cos^2\frac{17\pi}{6} = \frac{3 - 2\sqrt{3}}{2}$

7. If $\sin A = \frac{3}{5}$ and $\cos B = \frac{9}{41}$, $0 < A \& B < \frac{\pi}{2}$, then find the value of the following :

(a) sin(A + B) (b) sin(A – B) (c) cos(A + B) (d) cos(A – B)

8. The value of $\frac{\cos 4x + \cos 3x + \cos 2x}{\sin 4x + \sin 3x + \sin 2x}$ is

(A) –1 (B) tan 3x (C) cot 3x (D) cot x

***9.** If $\cos\theta + \sin\theta = \sqrt{2}\sin\theta$ then $\sin\theta - \cos\theta$, is

(A) $\sqrt{2}\sin\theta$ (B) $\sqrt{2}\cos\theta$ (C) 0 (D) none of these

10. If $\sin x + \sin^2 x = 1$ then the value of $\cos^2 x + \cos^4 x + \cot^4 x - \cot^2 x$ is

(A) 1 (B) 0 (C) 2 (D) None of these

***11.** Which of the following is correct ?

(A) cos 1° > cos 1 (B) sin 1° < sin 1 (C) sin 1° = sin 1 (D) cos 1° < cos 1

9.0 FORMULAE TO TRANSFORM THE PRODUCT INTO SUM OR DIFFERENCE

SL AL

(i) 2 sin A cos B = sin (A+ B) + sin (A – B).

(ii) 2 cos A sin B = sin (A + B) – sin (A – B).

(iii) 2 cos A cos B = cos (A + B) + cos (A – B)

(iv) 2 sin A sin B = cos (A – B) – cos (A + B)

Illustrations

Illustration 7. If sin2A = λ sin2B, then prove that $\frac{\tan(A+B)}{\tan(A-B)} = \frac{\lambda+1}{\lambda-1}$

Solution. Given sin2A = λ sin2B

$$\Rightarrow \quad \frac{\sin 2A}{\sin 2B} = \frac{\lambda}{1}$$

Applying componendo & dividendo,

$$\frac{\sin 2A + \sin 2B}{\sin 2B - \sin 2A} = \frac{\lambda+1}{1-\lambda}$$

$$\Rightarrow \quad \frac{2\sin\left(\frac{2A+2B}{2}\right)\cos\left(\frac{2A-2B}{2}\right)}{2\cos\left(\frac{2B+2A}{2}\right)\sin\left(\frac{2B-2A}{2}\right)} = \frac{\lambda+1}{1-\lambda}$$

$$\Rightarrow \quad \frac{\sin(A+B)\cos(A-B)}{\cos(A+B)\sin\{-(A-B)\}} = \frac{\lambda+1}{1-\lambda}$$

$$\Rightarrow \quad \frac{\sin(A+B)\cos(A-B)}{\cos(A+B)\times -\sin(A-B)} = \frac{\lambda+1}{-(\lambda-1)}$$

$$\Rightarrow \quad \frac{\sin(A+B)\cos(A-B)}{\cos(A+B)\sin(A-B)} = \frac{\lambda+1}{\lambda-1}$$

$$\Rightarrow \quad \tan(A+B)\cot(A-B) = \frac{\lambda+1}{\lambda-1}$$

$$\Rightarrow \quad \frac{\tan(A+B)}{\tan(A-B)} = \frac{\lambda+1}{\lambda-1}$$

10.0 FORMULAE TO TRANSFORM SUM OR DIFFERENCE INTO PRODUCT

SL AL

(i) $\sin C + \sin D = 2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$

(ii) $\sin C - \sin D = 2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$

(iii) $\cos C + \cos D = 2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$

(iv) $\cos C - \cos D = 2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{D-C}{2}\right)$

Illustrations

Illustration 8. $\dfrac{\sin 5\theta + \sin 2\theta - \sin\theta}{\cos 5\theta + 2\cos 3\theta + 2\cos^2\theta + \cos\theta}$ is equal to -

(A) $\tan\theta$ (B) $\cos\theta$ (C) $\cot\theta$ (D) none of these

Solution. L.H.S. $= \dfrac{2\sin 2\theta\cos 3\theta + \sin 2\theta}{2\cos 3\theta.\cos 2\theta + 2\cos 3\theta + 2\cos^2\theta} = \dfrac{\sin 2\theta[2\cos 3\theta + 1]}{2\left[\cos 3\theta(\cos 2\theta + 1) + (\cos^2\theta)\right]}$

$$= \frac{\sin 2\theta[2\cos 3\theta + 1]}{2\left[\cos 3\theta(2\cos^2\theta) + \cos^2\theta\right]} = \frac{\sin 2\theta(2\cos 3\theta + 1)}{2\cos^2\theta(2\cos 3\theta + 1)} = \tan\theta$$

Ans. (A)

*Illustration 9. Show that $\sin 12°.\sin 48°.\sin 54° = 1/8$

Solution. L.H.S. $= \frac{1}{2}[\cos 36° - \cos 60°]\sin 54° = \frac{1}{2}\left[\cos 36°\sin 54° - \frac{1}{2}\sin 54°\right]$

$$= \frac{1}{4}[2\cos 36°\sin 54° - \sin 54°] = \frac{1}{4}[\sin 90° + \sin 18° - \sin 54°]$$

$$= \frac{1}{4}[1 - (\sin 54° - \sin 18°)] = \frac{1}{4}[1 - 2\sin 18°\cos 36°]$$

$$= \frac{1}{4}\left[1 - \frac{2\sin 18°}{\cos 18°}\cos 18°\cos 36°\right] = \frac{1}{4}\left[1 - \frac{\sin 36°\cos 36°}{\cos 18°}\right]$$

$$= \frac{1}{4}\left[1 - \frac{2\sin 36°\cos 36°}{2\cos 18°}\right] = \frac{1}{4}\left[1 - \frac{\sin 72°}{2\sin 72°}\right] = \frac{1}{4}\left[1 - \frac{1}{2}\right] = \frac{1}{8} = \text{R.H.S.}$$

11.0 TRIGONOMETRIC RATIOS OF SUM OF MORE THAN TWO ANGLES

SL AL

(i) $\sin(A+B+C) = \sin A\cos B\cos C + \sin B\cos A\cos C + \sin C\cos A\cos B - \sin A\sin B\sin C$

$= \Sigma\sin A\cos B\cos C - \Pi\sin A$

$= \cos A\cos B\cos C\,[\tan A + \tan B + \tan C - \tan A\tan B\tan C]$

(ii) $\cos(A+B+C) = \cos A\cos B\cos C - \sin A\sin B\cos C - \sin A\cos B\sin C - \cos A\sin B\sin C$

$= \Pi\cos A - \Sigma\sin A\sin B\cos C$

$= \cos A\cos B\cos C\,[1 - \tan A\tan B - \tan B\tan C - \tan C\tan A]$

(iii) $\tan(A+B+C) = \dfrac{\tan A + \tan B + \tan C - \tan A\tan B\tan C}{1 - \tan A\tan B - \tan B\tan C - \tan C\tan A} = \dfrac{S_1 - S_3}{1 - S_2}$

12.0 TRIGONOMETRIC RATIOS OF MULTIPLE ANGLES

SL AL

(a) ***Trigonometrical ratios of an angle 2θ in terms of the angle θ***

(i) $\sin 2\theta = 2\sin\theta\cos\theta = \dfrac{2\tan\theta}{1+\tan^2\theta}$

(ii) $\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1$

$= 1 - 2\sin^2\theta = \dfrac{1-\tan^2\theta}{1+\tan^2\theta}$

(iii) $1 + \cos 2\theta = 2\cos^2\theta$

(iv) $1 - \cos 2\theta = 2\sin^2\theta$

(v) $\tan\theta = \dfrac{1-\cos 2\theta}{\sin 2\theta} = \dfrac{\sin 2\theta}{1+\cos 2\theta}$

(vi) $\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$

(b) ***Trigonometrical ratios of an angle 3θ in terms of the angle θ***

(i) $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$.

(ii) $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$.

(iii) $\tan 3\theta = \dfrac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta}$

Illustrations

Illustration 10. Prove that : $\dfrac{2\cos 2A+1}{2\cos 2A-1} = \tan(60°+A)\tan(60°-A)$

Solution. R.H.S. $= \tan(60° + A)\tan(60° - A)$

$$= \left(\frac{\tan 60° + \tan A}{1-\tan 60°\tan A}\right)\left(\frac{\tan 60° - \tan A}{1+\tan 60°\tan A}\right)$$

$$= \left(\frac{\sqrt{3}+\tan A}{1-\sqrt{3}\tan A}\right)\left(\frac{\sqrt{3}-\tan A}{1+\sqrt{3}\tan A}\right)$$

$$= \frac{3-\tan^2 A}{1-3\tan^2 A} = \frac{3-\dfrac{\sin^2 A}{\cos^2 A}}{1-3\dfrac{\sin^2 A}{\cos^2 A}} = \frac{3\cos^2 A - \sin^2 A}{\cos^2 A - 3\sin^2 A}$$

$$= \frac{2\cos^2 A + \cos^2 A - 2\sin^2 A + \sin^2 A}{2\cos^2 A - 2\sin^2 A - \sin^2 A - \cos^2 A}$$

$$= \frac{2(\cos^2 A - \sin^2 A) + \cos^2 A + \sin^2 A}{2(\cos^2 A - \sin^2 A) - (\sin^2 A + \cos^2 A)}$$

$$= \frac{2\cos 2A + 1}{2\cos 2A - 1} = \text{L.H.S.}$$

Illustration 11. Prove that : tanA + tan(60° + A) + tan(120° + A) = 3tan3A

Solution. L.H.S. = tanA + tan(60° + A) + tan(120° + A)

= tanA + tan(60° + A) + tan{180° –(60° – A)}

= tanA + tan(60° + A) – tan(60° – A) [∵ tan(180° – θ) = –tanθ]

$$= \tan A + \frac{\tan 60° + \tan A}{1 - \tan 60° \tan A} - \frac{\tan 60° - \tan A}{1 + \tan 60° \tan A}$$

$$= \tan A + \frac{\sqrt{3} + \tan A}{1 - \sqrt{3}\tan A} - \frac{\sqrt{3} - \tan A}{1 + \sqrt{3}\tan A}$$

$$= \tan A + \frac{\sqrt{3} + \tan A + 3\tan A + \sqrt{3}\tan^2 A - \sqrt{3} + \tan A + 3\tan A - \sqrt{3}\tan^2 A}{(1 - \sqrt{3}\tan A)(1 + \sqrt{3}\tan A)}$$

$$= \tan A + \frac{8\tan A}{1 - 3\tan^2 A} = \frac{\tan A - 3\tan^3 A + 8\tan A}{1 - 3\tan^2 A}$$

$$= \frac{9\tan A - 3\tan^3 A}{1 - 3\tan^2 A} = 3\left(\frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}\right) = 3\tan 3A = \text{R.H.S.}$$

BEGINNER'S BOX-2

TOPIC COVERED : TRANSFORM THE PRODUCT INTO SUM OR DIFFERENCE AND SUM OR DIFFERENCE INTO PRODUCT, TRIGONOMETRIC RATIOS OF SUM OF MORE THAN TWO ANGLES, TRIGONOMETRIC RATIOS OF MULTIPLE ANGLES.

1. Simplify $\frac{\sin 75° - \sin 15°}{\cos 75° + \cos 15°}$

2. Prove that (sin3A + sinA)sinA + (cos3A – cosA)cosA = 0

3. Find the value of cos20°cos40°cos60°cos80°

4. Prove that $\frac{\sin 8\theta \cos\theta - \sin 6\theta \cos 3\theta}{\cos 2\theta \cos\theta - \sin 3\theta \sin 4\theta} = \tan 2\theta$

5. Prove that :

(a) $\frac{\sin 2\theta}{1 + \cos 2\theta} = \tan\theta$ (b) $\frac{1 + \sin 2\theta + \cos 2\theta}{1 + \sin 2\theta - \cos 2\theta} = \cot\theta$ (c) $\frac{\sec 8\theta - 1}{\sec 4\theta - 1} = \frac{\tan 8\theta}{\tan 2\theta}$

****6.*** Prove that :

(a) cot θ cot (60° – θ) cot (60° + θ) = cot 3θ

(b) $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$

(c) $\sin 4\theta = 4\sin\theta\cos^3\theta - 4\cos\theta\sin^3\theta$

7. The value of $\frac{\tan 245° + \tan 335°}{\tan 205° - \tan 115°}$ is equal to

(A) cos 40° (B) sin 40° (C) –sin 50° (D) cos 50°

13.0 TRIGONOMETRIC RATIOS OF SUB MULTIPLE ANGLES

SL AL

Since the trigonometric relations are true for all values of angle θ, they will be true if instead of θ be substitute $\frac{\theta}{2}$

(i) $\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} = \dfrac{2\tan\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}$

(ii) $\cos\theta = \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} = 2\cos^2\frac{\theta}{2} - 1 = 1 - 2\sin^2\frac{\theta}{2} = \dfrac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}$

(iii) $1 + \cos\theta = 2\cos^2\frac{\theta}{2}$

(iv) $1 - \cos\theta = 2\sin^2\frac{\theta}{2}$

(v) $\tan\frac{\theta}{2} = \dfrac{1-\cos\theta}{\sin\theta} = \dfrac{\sin\theta}{1+\cos\theta}$

(vi) $\tan\theta = \dfrac{2\tan\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}}$

(vii) $\sin\frac{\theta}{2} = \pm\sqrt{\dfrac{1-\cos\theta}{2}}$

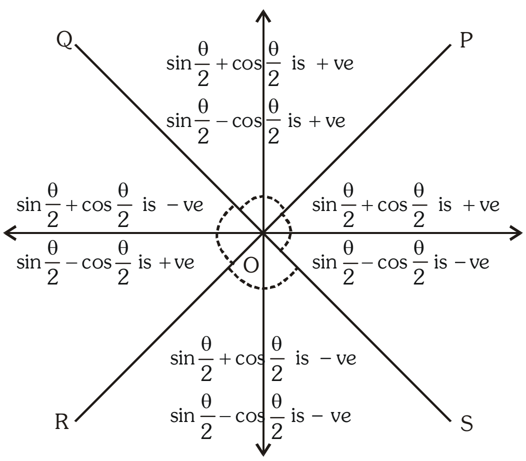
(viii) $\cos\frac{\theta}{2} = \pm\sqrt{\dfrac{1+\cos\theta}{2}}$

(ix) $\tan\frac{\theta}{2} = \pm\sqrt{\dfrac{1-\cos\theta}{1+\cos\theta}}$

(x) $2\sin\frac{\theta}{2} = \pm\sqrt{1+\sin\theta} \pm \sqrt{1-\sin\theta}$

(xi) $2\cos\frac{\theta}{2} = \pm\sqrt{1+\sin\theta} \mp \sqrt{1-\sin\theta}$

(xii) $\tan\frac{\theta}{2} = \dfrac{\pm\sqrt{1+\tan^2\theta}-1}{\tan\theta}$



Illustrations

Illustration 12. $\sin 67\frac{1}{2}^\circ + \cos 67\frac{1}{2}^\circ$ is equal to

(A) $\frac{1}{2}\sqrt{4+2\sqrt{2}}$ (B) $\frac{1}{2}\sqrt{4-2\sqrt{2}}$ (C) $\frac{1}{4}\left(\sqrt{4+2\sqrt{2}}\right)$ (D) $\frac{1}{4}\left(\sqrt{4-2\sqrt{2}}\right)$

Solution. $\sin 67\frac{1}{2}^\circ + \cos 67\frac{1}{2}^\circ = \sqrt{1+\sin 135^\circ} = \sqrt{1+\frac{1}{\sqrt{2}}}$ (using $\cos A + \sin A = \sqrt{1+\sin 2A}$)

$= \frac{1}{2}\sqrt{4+2\sqrt{2}}$

Ans.(A)

14.0 TRIGONOMETRIC RATIOS OF SOME STANDARD ANGLES

SL AL

(i) $\sin 18° = \sin\frac{\pi}{10} = \frac{\sqrt{5}-1}{4} = \cos 72° = \cos\frac{2\pi}{5}$

(ii) $\cos 36° = \cos\frac{\pi}{5} = \frac{\sqrt{5}+1}{4} = \sin 54° = \sin\frac{3\pi}{10}$

(iii) $\sin 72° = \sin\frac{2\pi}{5} = \frac{\sqrt{10+2\sqrt{5}}}{4} = \cos 18° = \cos\frac{\pi}{10}$

(iv) $\sin 36° = \sin\frac{\pi}{5} = \frac{\sqrt{10-2\sqrt{5}}}{4} = \cos 54° = \cos\frac{3\pi}{10}$

(v) $\sin 15° = \sin\frac{\pi}{12} = \frac{\sqrt{3}-1}{2\sqrt{2}} = \cos 75° = \cos\frac{5\pi}{12}$

(vi) $\cos 15° = \cos\frac{\pi}{12} = \frac{\sqrt{3}+1}{2\sqrt{2}} = \sin 75° = \sin\frac{5\pi}{12}$

(vii) $\tan 15° = \tan\frac{\pi}{12} = 2-\sqrt{3} = \frac{\sqrt{3}-1}{\sqrt{3}+1} = \cot 75° = \cot\frac{5\pi}{12}$

(viii) $\tan 75° = \tan\frac{5\pi}{12} = 2+\sqrt{3} = \frac{\sqrt{3}+1}{\sqrt{3}-1} = \cot 15° = \cot\frac{\pi}{12}$

(ix) $\tan(22.5°) = \tan\frac{\pi}{8} = \sqrt{2}-1 = \cot(67.5°) = \cot\frac{3\pi}{8}$

(x) $\tan(67.5°) = \tan\frac{3\pi}{8} = \sqrt{2}+1 = \cot(22.5°) = \cot\frac{\pi}{8}$

Illustrations

Illustration 13 Evaluate sin78° – sin66° – sin42° + sin6°.

Solution. The expression $= (\sin 78° - \sin 42°) - (\sin 66° - \sin 6°) = 2\cos(60°)\sin(18°) - 2\cos 36°.\sin 30°$

$$= \sin 18° - \cos 36° = \left(\frac{\sqrt{5}-1}{4}\right) - \left(\frac{\sqrt{5}+1}{4}\right) = -\frac{1}{2}$$

15.0 CONDITIONAL TRIGONOMETRIC IDENTITIES

SL AL

If A + B + C = 180°, then

(i) tan A + tan B + tan C = tan A tan B tan C

(ii) cot A cot B + cot B cot C + cot C cot A = 1

(iii) $\tan\frac{A}{2}\tan\frac{B}{2} + \tan\frac{B}{2}\tan\frac{C}{2} + \tan\frac{C}{2}\tan\frac{A}{2} = 1$

(iv) $\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2} = \cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}$

(v) sin 2A + sin 2B + sin 2C = 4 sinA sinB sinC

(vi) cos 2A + cos 2B + cos 2C =–1–4 cosA cosB cosC

(vii) sin A + sin B + sin C = 4 cos $\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$

(viii) cos A + cos B + cos C = 1 + 4 $\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

Illustrations

Illustration 14. In any triangle ABC, sin A – cos B = cos C, then angle B is

(A) $\pi/2$ (B) $\pi/3$ (C) $\pi/4$ (D) $\pi/6$

Solution. We have , sin A – cos B = cos C

sin A = cos B + cos C

$$\Rightarrow \quad 2\sin\frac{A}{2}\cos\frac{A}{2}=2\cos\left(\frac{B+C}{2}\right)\cos\left(\frac{B-C}{2}\right)$$

$$\Rightarrow \quad 2\sin\frac{A}{2}\cos\frac{A}{2}=2\cos\left(\frac{\pi-A}{2}\right)\cos\left(\frac{B-C}{2}\right) \qquad \because\ A+B+C=\pi$$

$$\Rightarrow \quad 2\sin\frac{A}{2}\cos\frac{A}{2}=2\sin\frac{A}{2}\cos\left(\frac{B-C}{2}\right)$$

$$\Rightarrow \quad \cos\frac{A}{2}=\cos\frac{B-C}{2} \quad \text{or} \quad A = B - C \ ; \ \text{But} \ A + B + C = \pi$$

Therefore 2B = $\pi \Rightarrow$ B = $\pi/2$ ***Ans.(A)***

****Illustration 15.*** If A + B + C = $\frac{3\pi}{2}$, then cos 2A + cos 2B + cos2C is equal to-

(A) 1 – 4cosA cosB cosC (B) 4 sinA sin B sinC

(C) 1 + 2cosA cosB cosC (D) 1 – 4 sinA sinB sinC

Solution. cos 2A + cos 2B + cos 2C = 2 cos (A + B) cos (A – B) + cos 2C

$= 2\cos\left(\frac{3\pi}{2}-C\right)\cos(A-B)+\cos 2C \quad \because\ A+B+C=\frac{3\pi}{2}$

$= -2\sin C\cos(A-B)+1-2\sin^2 C = 1-2\sin C\,[\cos(A-B)+\sin C\,)$

$= 1-2\sin C\,[\cos(A-B)+\sin\left(\frac{3\pi}{2}-(A+B)\right)]$

$= 1-2\sin C\,[\cos(A-B)-\cos(A+B)] = 1-4\sin A\sin B\sin C$ ***Ans.(D)***

BEGINNER'S BOX–3

TOPIC COVERED : TRIGONOMETRIC RATIOS OF SUB MULTIPLE ANGLES, STANDARD ANGLES, CONDITIONAL TRIGONOMETRIC IDENTITIES.

1. Find the value of

(a) $\sin\frac{\pi}{8}$ (b) $\cos\frac{\pi}{8}$ (c) $\tan\frac{\pi}{8}$

2. Find the value of

(a) $\sin\frac{\pi}{10}+\sin\frac{13\pi}{10}$ (b) $\cos^2 48^\circ-\sin^2 12^\circ$

3. If ABCD is a cyclic quadrilateral, then find the value of sinA + sinB – sinC – sinD

4. If A + B + C = $\frac{\pi}{2}$, then find the value of tanA tanB + tanBtanC + tanC tanA

5. Given that $5\cos^2\alpha - 2\sin\alpha - 2 = 0$, $\left(\frac{5\pi}{4} < \alpha < \frac{7\pi}{4}\right)$, the value of $\cot\frac{\alpha}{2}$ is

(A) 1 (B) –1 (C) 2 (D) none of these

6. If $\sin\alpha + \sin\beta = a$ and $\cos\alpha - \cos\beta = b$ then

(A) $\tan\left(\frac{\alpha-\beta}{2}\right) = -\frac{b}{a}$ (B) $\tan\left(\frac{\alpha-\beta}{2}\right) = \frac{b}{a}$

(C) $\cos(\alpha+\beta) = \frac{a^2+b^2-2}{2}$ (D) $\cos(\alpha+\beta) = \frac{2-a^2-b^2}{2}$

7. Prove that:

(a) $\tan 20° . \tan 40° . \tan 60° . \tan 80° = 3$ (b) $\tan 9° - \tan 27° - \tan 63° + \tan 81° = 4$.

(c) $(4\cos^2 9° - 3)(4\cos^2 27° - 3) = \tan 9°$. (d) $\cos^6\left(\frac{\pi}{16}\right) + \cos^6\left(\frac{3\pi}{16}\right) + \cos^6\left(\frac{5\pi}{16}\right) + \cos^6\left(\frac{7\pi}{16}\right) = \frac{5}{4}$

(e) $4\cos\frac{2\pi}{7}\cdot\cos\frac{\pi}{7} - 1 = 2\cos\frac{2\pi}{7}$. (f) $4\sin 27° = \left(5+\sqrt{5}\right)^{1/2} - \left(3-\sqrt{5}\right)^{1/2}$.

(g) $\tan 10° - \tan 50° + \tan 70° = \sqrt{3}$ (h) $\tan^2\frac{\pi}{16} + \tan^2\frac{3\pi}{16} + \tan^2\frac{5\pi}{16} + \tan^2\frac{7\pi}{16} = 28$

16.0 MAXIMUM & MINIMUM VALUES OF TRIGONOMETRIC EXPRESSIONS

AL

(i) $a\cos\theta + b\sin\theta$ will always lie in the interval $[-\sqrt{a^2+b^2}, \sqrt{a^2+b^2}]$ i.e. the maximum and minimum values are $\sqrt{a^2+b^2}$, $-\sqrt{a^2+b^2}$ respectively.

(ii) Minimum value of $a^2\tan^2\theta + b^2\cot^2\theta = 2ab$ where $a, b > 0$

(iii) $-\sqrt{a^2+b^2+2ab\cos(\alpha-\beta)} \le a\cos(\alpha+\theta) + b\cos(\beta+\theta) \le \sqrt{a^2+b^2+2ab\cos(\alpha-\beta)}$ where α and β are known angles.

(iv) If $\alpha, \beta, \in \left(0, \frac{\pi}{2}\right)$ and $\alpha + \beta = \sigma$ (constant) then

(i) Maximum value of the expression $\cos\alpha\cos\beta$, $\cos\alpha + \cos\beta$, $\sin\alpha\sin\beta$ or $\sin\alpha + \sin\beta$ occurs when $\alpha = \beta = \sigma/2$

(ii) Minimum value of $\sec\alpha + \sec\beta$, $\tan\alpha + \tan\beta$, $\operatorname{cosec}\alpha + \operatorname{cosec}\beta$ occurs when $\alpha = \beta = \sigma/2$

(v) If A, B, C are the angles of a triangle then maximum value of $\sin A + \sin B + \sin C$ and $\sin A \sin B \sin C$ occurs when $A = B = C = 60°$

(vi) In case a quadratic in $\sin\theta$ & $\cos\theta$ is given then the maximum or minimum values can be obtained by making perfect square.

Illustrations

Illustration 16. Prove that : $-4 \le 5\cos\theta + 3\cos\left(\theta+\frac{\pi}{3}\right) + 3 \le 10$, for all values of θ.

Solution. We have, $5\cos\theta + 3\cos\left(\theta+\frac{\pi}{3}\right) = 5\cos\theta + 3\cos\theta\cos\frac{\pi}{3} - 3\sin\theta\sin\frac{\pi}{3} = \frac{13}{2}\cos\theta - \frac{3\sqrt{3}}{2}\sin\theta$

Since, $-\sqrt{\left(\frac{13}{2}\right)^2+\left(-\frac{3\sqrt{3}}{2}\right)^2} \le \frac{13}{2}\cos\theta - \frac{3\sqrt{3}}{2}\sin\theta \le \sqrt{\left(\frac{13}{2}\right)^2+\left(-\frac{3\sqrt{3}}{2}\right)^2}$

$\Rightarrow \quad -7 \le \frac{13}{2}\cos\theta - \frac{3\sqrt{3}}{2}\sin\theta \le 7$

$\Rightarrow \quad -7 \le 5\cos\theta + 3\cos\left(\theta+\frac{\pi}{3}\right) \le 7$ for all θ.

$\Rightarrow \quad -7+3 \le 5\cos\theta + 3\cos\left(\theta+\frac{\pi}{3}\right)+3 \le 7+3$ for all θ.

$\Rightarrow \quad -4 \le 5\cos\theta + 3\cos\left(\theta+\frac{\pi}{3}\right)+3 \le 10$ for all θ.

***Illustration 17.** Find the maximum value of $1 + \sin\left(\frac{\pi}{4}+\theta\right) + 2\cos\left(\frac{\pi}{4}-\theta\right)$ -

(A) 1 (B) 2 (C) 3 (D) 4

Solution. We have $1 + \sin\left(\frac{\pi}{4}+\theta\right) + 2\cos\left(\frac{\pi}{4}-\theta\right)$

$= 1 + \frac{1}{\sqrt{2}}(\cos\theta + \sin\theta) + \sqrt{2}(\cos\theta + \sin\theta)$

$= 1 + \left(\frac{1}{\sqrt{2}}+\sqrt{2}\right)(\cos\theta + \sin\theta)$

$= 1 + \left(\frac{1}{\sqrt{2}}+\sqrt{2}\right) . \sqrt{2}\cos\left(\theta-\frac{\pi}{4}\right)$

$\therefore$ maximum value $= 1+\left(\frac{1}{\sqrt{2}}+\sqrt{2}\right) . \sqrt{2} = 4$ **Ans. (D)**

17.0 IMPORTANT RESULTS

AL

(i) $\sin\theta\sin(60^\circ-\theta)\sin(60^\circ+\theta) = \frac{1}{4}\sin 3\theta$

(ii) $\cos\theta.\cos(60^\circ-\theta)\cos(60^\circ+\theta) = \frac{1}{4}\cos 3\theta$

(iii) $\tan\theta\tan(60^\circ-\theta)\tan(60^\circ+\theta) = \tan 3\theta$

(iv) $\cot\theta\cot(60^\circ-\theta)\cot(60^\circ+\theta) = \cot 3\theta$

(v) (a) $\sin^2\theta + \sin^2(60^\circ+\theta) + \sin^2(60^\circ-\theta) = \frac{3}{2}$ (b) $\cos^2\theta + \cos^2(60^\circ+\theta) + \cos^2(60^\circ-\theta) = \frac{3}{2}$

(vi) (a) If tan A + tan B + tan C = tan A tan B tan C, then $A + B + C = n\pi, n \in I$

(b) If tan A tan B + tan B tan C + tan C tan A = 1, then $A + B + C = (2n + 1)\frac{\pi}{2}, n \in I$

(vii) $\cos\theta \cos 2\theta \cos 4\theta \ldots \cos(2^{n-1}\theta) = \dfrac{\sin(2^n\theta)}{2^n \sin\theta}$

(viii) (a) $\cot A - \tan A = 2\cot 2A$ (b) $\cot A + \tan A = 2\operatorname{cosec} 2A$

(ix) $\sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \ldots \sin(\alpha + \overline{n-1}\,\beta) = \dfrac{\sin\left\{\alpha + \left(\frac{n-1}{2}\right)\beta\right\}\sin\left(\frac{n\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)}$

(x) $\cos\alpha + \cos(\alpha+\beta) + \cos(\alpha + 2\beta) + \ldots + \cos(\alpha + \overline{n-1}\,\beta) = \dfrac{\cos\left\{\alpha + \left(\frac{n-1}{2}\right)\beta\right\}\sin\left(\frac{n\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)}$

Illustrations

***Illustration 18.** Prove that tanA + 2tan2A + 4tan4A + 8cot8A = cot A.

Solution. $8\cot 8A = \cot A - \tan A - 2\tan 2A - 4\tan 4A$

$= 2\cot 2A - 2\tan 2A - 4\tan 4A$ (using viii (a) in above results)

$= 4\cot 4A - 4\tan 4A$ (using viii (a) in above results)

$= 8\cot 8A.$

Aliter Method : L.H.S. $= \tan A + 2\tan 2A + 4\tan 4A + 8\left(\dfrac{1-\tan^2 4A}{2\tan 4A}\right)$

$= \tan A + 2\tan 2A + \left(\dfrac{4\tan^2 4A + 4 - 4\tan^2 4A}{\tan 4A}\right)$

$= \tan A + 2\tan 2A + 4\cot 4A = \tan A + 2\tan 2A + 4\left(\dfrac{1-\tan^2 2A}{2\tan 2A}\right)$

$= \tan A + \left[\dfrac{2\tan^2 2A + 2 - 2\tan^2 2A}{\tan 2A}\right] = \tan A + 2\cot 2A$

$= \tan A + 2\left(\dfrac{1-\tan^2 A}{2\tan A}\right) = \dfrac{\tan^2 A + 1 - \tan^2 A}{\tan A} = \cot A = \text{R.H.S.}$

Illustration 19. Evaluate $\sum_{r=1}^{n-1}\cos^2\left(\dfrac{r\pi}{n}\right)$; $n \geq 2$

Solution. Sum $= \dfrac{1}{2}\sum_{r=1}^{n-1}\left(1+\cos\dfrac{2r\pi}{n}\right) = \dfrac{1}{2}(n-1) + \dfrac{1}{2}\left\{\cos\dfrac{2\pi}{n} + \cos\dfrac{4\pi}{n} + \ldots\ldots + \cos\dfrac{(2n-2)\pi}{n}\right\}$

$$= \frac{1}{2}(n-1) + \frac{1}{2}\left\{\frac{\sin(n-1)\frac{2\pi}{2n}}{\sin\frac{2\pi}{n.2}}.\cos\left\{\frac{2\left(\frac{2\pi}{n}\right) + (n-2)\frac{2\pi}{n}}{2}\right\}\right\}$$

$$\left\{\text{Using}, \cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + \ldots\ldots + \cos(\alpha+(n-1)\beta) = \frac{\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}.\cos\left\{\frac{2\alpha+(n-1)\beta}{2}\right\}\right\}$$

$$= \frac{1}{2}(n-1) + \frac{1}{2}\left\{\frac{\sin\frac{(n-1)\pi}{n}.\cos\pi}{\sin\left(\frac{\pi}{n}\right)}\right\} = \frac{1}{2}(n-1) - \frac{1}{2} = \frac{n}{2} - 1$$

$$\therefore \quad \sum_{r=1}^{n-1}\cos^2\left(\frac{r\pi}{n}\right) = \frac{n-2}{2}$$ **Ans.**

***Illustration 20.** Prove that : $(1 + \sec 2\theta)(1 + \sec 2^2\theta)(1 + \sec 2^3\theta)\ldots\ldots(1 + \sec 2^n\theta) = \tan 2^n\theta.\cot\theta$.

Solution. L.H.S. $= \left(1+\frac{1}{\cos 2\theta}\right)\left(1+\frac{1}{\cos 2^2\theta}\right)\left(1+\frac{1}{\cos 2^3\theta}\right)\ldots\left(1+\frac{1}{\cos 2^n\theta}\right)$

$$= \left(\frac{1+\cos 2\theta}{\cos 2\theta}\right)\left(\frac{1+\cos 2^2\theta}{\cos 2^2\theta}\right)\left(\frac{1+\cos 2^3\theta}{\cos 2^3\theta}\right)\ldots\left(\frac{1+\cos 2^n\theta}{\cos 2^n\theta}\right)$$

$$= \frac{2\cos^2\theta.2\cos^2 2\theta.2\cos^2 2^2\theta\ldots.2\cos^2 2^{n-1}\theta}{\cos 2\theta.\cos 2^2\theta.\cos 2^3\theta\ldots.\cos 2^n\theta}$$

$$= \cos\theta(2\cos\theta)(2\cos 2\theta)(2\cos 2^2\theta)\ldots(2\cos 2^{n-1}\theta).\frac{1}{\cos 2^n\theta}$$

$$= \frac{\cos\theta}{\sin\theta}(2\sin\theta\cos\theta)(2\cos 2\theta)(2\cos 2^2\theta)\ldots(2\cos 2^{n-1}\theta).\frac{1}{\cos 2^n\theta}$$

$$= \frac{\cos\theta}{\sin\theta}(2\sin 2\theta.\cos 2\theta)(2\cos 2^2\theta)\ldots(2\cos 2^{n-1}\theta).\frac{1}{\cos 2^n\theta}$$

$$= \frac{\cos\theta}{\sin\theta}(2\sin 2^{n-1}\theta.\cos 2^{n-1}\theta).\frac{1}{\cos 2^n\theta}$$

$$= \frac{\cos\theta}{\sin\theta}.\sin 2^n\theta.\frac{1}{\cos 2^n\theta} = \tan 2^n\theta.\cot\theta = \text{R.H.S.}$$

BEGINNER'S BOX-4

TOPIC COVERED : MAXIMUM & MINIMUM VALUES OF TRIGONOMETRIC EXPRESSIONS AND IMPORTANT RESULTS

1. Find maximum and minimum value of $5\cos\theta + 3\sin\left(\theta+\frac{\pi}{6}\right)$ for all real values of θ.

2. Find the minimum value of $\cos\theta + \cos 2\theta$ for all real values of θ.

***3.** Find maximum and minimum value of $\cos^2\theta - 6\sin\theta\cos\theta + 3\sin^2\theta + 2$.

4. Evaluate $\sin\frac{\pi}{n} + \sin\frac{3\pi}{n} + \sin\frac{5\pi}{n} + \ldots\ldots\ldots$ to n terms

***5.** If $(2^n + 1)\theta = \pi$, then find the value of $2^n \cos\theta\cos2\theta\cos2^2\theta \ldots\ldots\ldots \cos2^{n-1}\theta$.

***6.** If $\frac{1}{\sin1^\circ\sin2^\circ} + \frac{1}{\sin2^\circ\sin3^\circ} \ldots\ldots + \frac{1}{\sin89^\circ\sin90^\circ} = \cot\theta^\circ\operatorname{cosec}\theta^\circ \quad \forall\ \theta \in (0,\ 90)$. Find θ

7. Which of the following functions have the maximum value unity ?

(A) $\sin^2 x - \cos^2 x$

(B) $\frac{\sin2x - \cos2x}{\sqrt{2}}$

(C) $-\frac{\sin2x - \cos2x}{\sqrt{2}}$

(D) $\sqrt{\frac{6}{5}}\left(\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{3}}\cos x\right)$

8. If $\alpha + \beta = c$ where $\alpha, \beta > 0$ each lying between 0 and $\pi/2$ and c is a constant, find the maximum or minimum value of –

(a) $\sin\alpha + \sin\beta$ (b) $\sin\alpha \sin\beta$ (c) $\tan\alpha + \tan\beta$

***9.** Find the maximum & minimum values of $27^{\cos2x}.81^{\sin2x}$.

GOLDEN KEY POINTS

- The quantity by which the cosine falls short of unity i.e. $1 - \cos\theta$, is called the versed sine θ of θ and also by which the sine falls short of unity i.e. $1 - \sin\theta$ is called the coversed sine of θ.

- If $x + y = 45^\circ$, then :

  (a) $(1 + \tan x)(1 + \tan y) = 2$ (b) $(\cot x - 1)(\cot y - 1) = 2$

SOME WORKED OUT ILLUSTRATIONS

Illustration 1. Prove that

$\tan\alpha + 2\tan 2\alpha + 2^2\tan^2\alpha + \ldots\ldots + 2^{n-1}\tan 2^{n-1}\alpha + 2^n\cot 2^n\alpha = \cot\alpha$

Solution We know $\tan\theta = \cot\theta - 2\cot 2\theta$ .....(i)

Putting $\theta = \alpha, 2\alpha, 2^2\alpha$, ...in (i), we get

$\tan\alpha = (\cot\alpha - 2\cot 2\alpha)$

$2(\tan 2\alpha) = 2(\cot 2\alpha - 2\cot 2^2\alpha)$

$2^2(\tan 2^2\alpha) = 2^2(\cot 2^2\alpha - 2\cot 2^3\alpha)$

....

$2^{n-1}(\tan 2^{n-1}\alpha) = 2^{n-1}(\cot 2^{n-1}\alpha - 2\cot 2^n\alpha)$

Adding,

$\tan\alpha + 2\tan 2\alpha + 2^2\tan^2\alpha + \ldots\ldots + 2^{n-1}\tan 2^{n-1}\alpha = \cot\alpha - 2^n\cot 2^n\alpha$

$\therefore \quad \tan\alpha + 2\tan 2\alpha + 2^2\tan^2\alpha + \ldots\ldots + 2^{n-1}\tan 2^{n-1}\alpha + 2^n\cot 2^n\alpha = \cot\alpha$

Illustration 2. If A,B,C and D are angles of a quadrilateral and $\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\sin\frac{D}{2} = \frac{1}{4}$, prove that $A = B = C = D = \pi/2$.

Solution

$$\left(2\sin\frac{A}{2}\sin\frac{B}{2}\right)\left(2\sin\frac{C}{2}\sin\frac{D}{2}\right) = 1$$

$$\Rightarrow \left\{\cos\left(\frac{A-B}{2}\right) - \cos\left(\frac{A+B}{2}\right)\right\}\left\{\cos\left(\frac{C-D}{2}\right) - \cos\left(\frac{C+D}{2}\right)\right\} = 1$$

Since, $A + B = 2\pi - (C + D)$, the above equation becomes,

$$\Rightarrow \left\{\cos\left(\frac{A-B}{2}\right) - \cos\left(\frac{A+B}{2}\right)\right\}\left\{\cos\left(\frac{C-D}{2}\right) + \cos\left(\frac{A+B}{2}\right)\right\} = 1$$

$$\Rightarrow \cos^2\left(\frac{A+B}{2}\right) - \cos\left(\frac{A+B}{2}\right)\left\{\cos\left(\frac{A-B}{2}\right) - \cos\left(\frac{C-D}{2}\right)\right\} + 1 - \cos\left(\frac{A-B}{2}\right)\cos\left(\frac{C-D}{2}\right) = 0$$

This is a quadratic equation in $\cos\left(\frac{A+B}{2}\right)$ which has real roots.

$$\Rightarrow \left\{\cos\left(\frac{A-B}{2}\right) - \cos\left(\frac{C-D}{2}\right)\right\}^2 - 4\left\{1 - \cos\left(\frac{A-B}{2}\right).\cos\left(\frac{C-D}{2}\right)\right\} \geq 0$$

$$\left(\cos\frac{A-B}{2} + \cos\frac{C-D}{2}\right)^2 \geq 4$$

$\Rightarrow \cos\frac{A-B}{2} + \cos\frac{C-D}{2} \geq 2$, Now both $\cos\frac{A-B}{2}$ and $\cos\frac{C-D}{2} \leq 1$

$$\Rightarrow \cos\frac{A-B}{2} = 1 \,\&\, \cos\frac{C-D}{2} = 1$$

$$\Rightarrow \frac{A-B}{2} = 0 = \frac{C-D}{2}$$

$\Rightarrow$ A = B, C = D.

Similarly A = C, B = D $\Rightarrow$ A = B = C = D = $\pi/2$

ANSWERS

BEGINNER'S BOX-1

1. 10π cm **2.** $\frac{3}{5}, \frac{4}{5}, \frac{5}{3}$ **3.** 2 **4.** 8

7. **(a)** $\frac{187}{205}$ **(b)** $\frac{-133}{205}$ **(c)** $\frac{-84}{205}$ **(d)** $\frac{156}{205}$

8. (C) **9.** (B) **10.** (C) **11.** (AB)

BEGINNER'S BOX-2

1. $\frac{1}{\sqrt{3}}$ **3.** $\frac{1}{16}$ **7.** (BD)

BEGINNER'S BOX-3

1. **(a)** $\sqrt{\frac{\sqrt{2}-1}{2\sqrt{2}}}$ **(b)** $\sqrt{\frac{\sqrt{2}+1}{2\sqrt{2}}}$ **(c)** $\sqrt{2}-1$

2. **(a)** $-\frac{1}{2}$ **(b)** $\frac{\sqrt{5}+1}{8}$

3. 0 **4.** 1 **5.** (B) **6.** (AD)

BEGINNER'S BOX-4

1. 7 & –7 **2.** $-\frac{9}{8}$ **3.** $4+\sqrt{10}$ & $4-\sqrt{10}$

4. 0 **5.** 1 **6.** 1 **7.** (ABCD)

8. (a) max. = 2sin c/2 (b) max. = $\sin^2 c/2$ (c) min. = 2tan c/2

9. (a) Minimum Value = 3^{-5} ; Maximum Value = 3^{5}

EXERCISE – 1 MCQ (SINGLE CHOICE CORRECT)

1. The expression $\dfrac{\tan\left(x-\dfrac{\pi}{2}\right).\cos\left(\dfrac{3\pi}{2}+x\right)-\sin^3\left(\dfrac{7\pi}{2}-x\right)}{\cos\left(x-\dfrac{\pi}{2}\right).\tan\left(\dfrac{3\pi}{2}+x\right)}$ simplifies to -

(A) $(1+\cos^2x)$ (B) $\sin^2x$ (C) $-(1+\cos^2x)$ (D) $\cos^2x$

***2.** Exact value of $\cos^2 73° + \cos^2 47° - \sin^2 43° + \sin^2 107°$ is equal to -
(A) 1/2 (B) 3/4 (C) 1 (D) None of these

3. The expression $\dfrac{\sin 22°\cos 8° + \cos 158°\cos 98°}{\sin 23°\cos 7° + \cos 157°\cos 97°}$ when simplified reduces to -
(A) 1 (B) – 1 (C) 2 (D) None of these

***4.** The two legs of right triangle are $\sin\theta + \sin\left(\dfrac{3\pi}{2}-\theta\right)$ and $\cos\theta - \cos\left(\dfrac{3\pi}{2}-\theta\right)$. The length of its hypotenuse is

(A) 1 (B) $\sqrt{2}$ (C) 2 (D) some function of θ

5. The expression $\dfrac{\sin(\alpha+\theta)-\sin(\alpha-\theta)}{\cos(\beta-\theta)-\cos(\beta+\theta)}$ is –
(A) independent of α (B) independent of β
(C) independent of θ (D) independent of α and β

6. The tangents of two acute angles are 3 and 2. The sine of twice their difference is -
(A) 7/24 (B) 7/48 (C) 7/50 (D) 7/25

7. If $\dfrac{\sin 2\alpha - \sin 3\alpha + \sin 4\alpha}{\cos 2\alpha - \cos 3\alpha + \cos 4\alpha} = \tan k\alpha$ is an identity then the value of k is equal to -
(A) 2 (B) 3 (C) 4 (D) 6

***8.** If $\cos(\theta+\phi) = m\cos(\theta-\phi)$, then $\tan\theta$ is equal to -

(A) $\left(\dfrac{1+m}{1-m}\right)\tan\phi$ (B) $\left(\dfrac{1-m}{1+m}\right)\tan\phi$ (C) $\left(\dfrac{1-m}{1+m}\right)\cot\phi$ (D) $\left(\dfrac{1+m}{1-m}\right)\cot\phi$

9. If $\sin\theta + \text{cosec}\,\theta = 2$, then the value of $\sin^8\theta + \text{cosec}^8\theta$ is equal to -
(A) 2 (B) 2^8 (C) 2^4 (D) None of these

10. If the expression $4\sin 5\alpha\cos 3\alpha\cos 2\alpha$ is expressed as the sum of three sines then two of them are $\sin 4\alpha$ and $\sin 10\alpha$. The third one is -
(A) $\sin 8\alpha$ (B) $\sin 6\alpha$ (C) $\sin 5\alpha$ (D) $\sin 12\alpha$

***11.** The expression, $3\left[\sin^4\left(\dfrac{3\pi}{2}-\alpha\right)+\sin^4(3\pi+\alpha)\right]-2\left[\sin^6\left(\dfrac{\pi}{2}+\alpha\right)+\sin^6(5\pi-\alpha)\right]$ when simplified is equal to -
(A) 0 (B) 1 (C) 3 (D) $\sin 4\alpha + \cos 6\alpha$

12. If $\cos\theta = \dfrac{1}{2}\left(a+\dfrac{1}{a}\right)$ then $\cos 3\theta$ in terms of 'a' =

(A) $\dfrac{1}{4}\left(a^3+\dfrac{1}{a^3}\right)$ (B) $4\left(a^3+\dfrac{1}{a^3}\right)$ (C) $\dfrac{1}{2}\left(a^3+\dfrac{1}{a^3}\right)$ (D) None of these

13. The product cot 123°. cot 133° . cot 137° . cot 147°, when simplified is equal to -
(A) – 1 (B) tan 37° (C) cot 33° (D) 1

***14.** Given sin B = $\frac{1}{5}$ sin (2A + B) then, tan (A + B) = k tan A, where k has the value equal to -
(A) 1 (B) 2 (C) 2/3 (D) 3/2

***15.** The value of the expression $\frac{1-4\sin 10^\circ \sin 70^\circ}{2\sin 10^\circ}$ is –
(A) 1/2 (B) 1 (C) 2 (D) None of these

16. Which of the following number (s) is / are rational ?
(A) sin15° (B) cos15° (C) sin15°cos15° (D) sin15°cos75°

17. If α and β are two positive acute angles satisfying $\alpha - \beta = 15°$ and $\sin\alpha = \cos 2\beta$ then the value of $\alpha + \beta$ is equal to-
(A) 35° (B) 55° (C) 65° (D) 85°

***18.** If $\alpha + \beta + \gamma = 2\pi$, then -
(A) $\tan\frac{\alpha}{2}+\tan\frac{\beta}{2}+\tan\frac{\gamma}{2}=\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\tan\frac{\gamma}{2}$ (B) $\tan\frac{\alpha}{2}\tan\frac{\beta}{2}+\tan\frac{\beta}{2}\tan\frac{\gamma}{2}+\tan\frac{\gamma}{2}\tan\frac{\alpha}{2}=1$
(C) $\tan\frac{\alpha}{2}+\tan\frac{\beta}{2}+\tan\frac{\gamma}{2}=-\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\tan\frac{\gamma}{2}$ (D) $\tan\frac{\alpha}{2}\tan\frac{\beta}{2}+\tan\frac{\beta}{2}\tan\frac{\gamma}{2}+\tan\frac{\gamma}{2}\tan\frac{\alpha}{2}=0$

19. The value of sin10° + sin20° + sin30° +......+ sin360° is -
(A) 1 (B) 0 (C) –1 (D) None of these

20. If A and C are two angles such that $A + C = \frac{3\pi}{4}$, then $(1+\cot A)(1+\cot C)$ equals -
(A) 1 (B) 2 (C) -1 (D) -2

***21.** $\log_{t_1}(4\sin 9^\circ \cos 9^\circ)$; where $t_1 = 4\sin 63^\circ \cos 63^\circ$, equals -
(A) $\frac{\sqrt{5}+1}{4}$ (B) $\frac{\sqrt{5}-1}{4}$ (C) 1 (D) None of these

22. If (a + b) tan($\theta - \phi$) = (a – b) tan($\theta + \phi$), then $\frac{\sin(2\theta)}{\sin(2\phi)}$ is equal to -
(A) ab (B) $\frac{a}{b}$ (C) $\frac{b}{a}$ (D) a^2b^2

EXERCISE – 2 MCQ (ONE OR MORE CHOICE CORRECT)

1. Let m = tan 3 and n = sec 6, then which of the following statement(s) does/do not hold good ?
(A) m & n both are positive (B) m & n both are negative
(C) m is positive and n is negative (D) m is negative and n is positive

***2.** If $\sqrt{\frac{1-\sin A}{1+\sin A}}+\frac{\sin A}{\cos A}=\frac{1}{\cos A}$, for all permissible values of A, then A belongs to -
(A) first quadrant (B) second quadrant (C) third quadrant (D) fourth quadrant

***3.** If $\pi < 2\theta < \frac{3\pi}{2}$, then $\sqrt{2+\sqrt{2+2\cos 4\theta}}$ equals -
(A) $-2\cos\theta$ (B) $-2\sin\theta$ (C) $2\cos\theta$ (D) $2\sin\theta$

4. If sec A = $\frac{17}{8}$ and cosecB = $\frac{5}{4}$ then sec (A + B) can have the value equal to -
(A) $\frac{85}{36}$ (B) $-\frac{85}{36}$ (C) $-\frac{85}{84}$ (D) $\frac{85}{84}$

***5.** Which of the following when simplified reduces to unity ?
(A) $\frac{1-2\sin^2\alpha}{2\cot\left(\frac{\pi}{4}+\alpha\right)\cos^2\left(\frac{\pi}{4}-\alpha\right)}$ (B) $\frac{\sin(\pi-\alpha)}{\sin\alpha-\cos\alpha\tan\frac{\alpha}{2}}+\cos(\pi-\alpha)$

(C) $\frac{1}{4\sin^2\alpha\cos^2\alpha}+\frac{(1-\tan^2\alpha)^2}{4\tan^2\alpha}$ (D) $\frac{1+\sin 2\alpha}{(\sin\alpha+\cos\alpha)^2}$

6. If $\frac{\sin 3\theta}{\sin\theta}=\frac{11}{25}$ then tan $\frac{\theta}{2}$ can have the value equal to -
(A) 2 (B) 1/2 (C) – 2 (D) – 1/2

7. If A + B – C = 3π, then sinA + sinB – sinC is equal to -
(A) $4\sin\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2}$ (B) $-4\sin\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2}$ (C) $4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$ (D) $-4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$

8. If $\tan^3\theta+\cot^3\theta=52$, then the value of $\tan^2\theta+\cot^2\theta$ is equal to -
(A) 14 (B) 15 (C) 16 (D) 17

***9.** The maximum value of $\log_{20}(3\sin x - 4\cos x + 15)$ -
(A) 1 (B) 2 (C) 3 (D) 4

10. If $\tan\frac{\theta}{2}=\text{cosec}\theta-\sin\theta$, then -
(A) $\sin^2\frac{\theta}{2}=2\sin^2 18°$ (B) $\cos 2\theta+2\cos\theta+1=0$
(C) $\sin^2\frac{\theta}{2}=4\sin^2 18°$ (D) $\cos 2\theta + 2\cos\theta - 1 = 0$

Match The Column

Following question contains statements given in two columns, which have to be matched. The statements in ***Column-I*** are labelled as A, B, C and D while the statements in ***Column-II*** are labelled as p, q, r and s. Any given statement in ***Column-I*** can have correct matching with ***ONE*** statement in ***Column-II***.

11. If maximum and minimum values of expression are λ and μ respectively then match the columns :

	Column-I		**Column-II**
(A)	$\sin^6\theta+\cos^6\theta$ for all θ	(p)	$\lambda + \mu = 2$
(B)	$\log_{\sqrt{5}}\left[\sqrt{2}(\sin\theta-\cos\theta)+3\right]$ for all θ	(q)	$\lambda + \mu = 6$
(C)	$\dfrac{7+6\tan\theta-\tan^2\theta}{(1+\tan^2\theta)}$ for all real values of $\theta \sim \dfrac{\pi}{2}$	(r)	$\lambda - \mu = 10$
(D)	$5\cos\theta + 3\cos(\theta + \dfrac{\pi}{3}) + 3$ for all real	(s)	$\lambda - \mu = 14$
	values of θ	(t)	$\lambda + \mu = \dfrac{5}{4}$

Comprehension Based Questions

Comprehension - 1

Continued product $\cos\alpha\cos2\alpha\cos2^2\alpha$ .... $\cos2^{n-1}\alpha$

$$= \begin{cases} \dfrac{\sin 2^n\alpha}{2^n\sin\alpha}, & \text{if } \alpha \neq n\pi \\ \dfrac{1}{2^n}, & \text{if } \alpha = \dfrac{\pi}{2^n+1} \quad \text{i.e. } 2^n\alpha = \pi - \alpha \\ -\dfrac{1}{2^n}, & \text{if } \alpha = \dfrac{\pi}{2^n-1} \quad \text{i.e. } 2^n\alpha = \pi + \alpha \end{cases}$$

Where, $n \in I$ (Integer)

On the basis of above information, answer the following questions :

***12.** The value of $\cos\dfrac{2\pi}{7}\cos\dfrac{4\pi}{7}\cos\dfrac{6\pi}{7}$ is -

(A) –1/2 (B) 1/2 (C) 1/4 (D) 1/8

***13.** If $\alpha = \dfrac{\pi}{15}$, then the value of $\prod_{r=1}^{7}\cos r\alpha$ is -

(A) $\dfrac{1}{128}$ (B) $-\dfrac{1}{128}$ (C) $\dfrac{1}{64}$ (D) $\dfrac{1}{32}$

***14.** The value of $\sin\left(\dfrac{\pi}{14}\right)\sin\left(\dfrac{3\pi}{14}\right)\sin\left(\dfrac{5\pi}{14}\right)\sin\left(\dfrac{7\pi}{14}\right)\sin\left(\dfrac{9\pi}{14}\right)\sin\left(\dfrac{11\pi}{14}\right)\sin\left(\dfrac{13\pi}{14}\right)$ is -

(A) 1 (B) $\dfrac{1}{8}$ (C) $\dfrac{1}{32}$ (D) $\dfrac{1}{64}$

EXERCISE - 3 SUBJECTIVE

***1.** If $\cos(y-z)+\cos(z-x)+\cos(x-y)=-\frac{3}{2}$, prove that $\cos x+\cos y+\cos z=0=\sin x+\sin y+\sin z$.

2. For all values of α, β, γ prove that :

$$\cos\alpha+\cos\beta+\cos\gamma+\cos(\alpha+\beta+\gamma)=4\cos\frac{\alpha+\beta}{2}.\cos\frac{\beta+\gamma}{2}.\cos\frac{\gamma+\alpha}{2}.$$

***3.** Prove that $\sin^4\frac{\pi}{16}+\sin^4\frac{3\pi}{16}+\sin^4\frac{5\pi}{16}+\sin^4\frac{7\pi}{16}=\frac{3}{2}$

***4.** Prove that : $\operatorname{cosec}\theta+\operatorname{cosec}2\theta+\operatorname{cosec}2^2\theta+\ldots\ldots+\operatorname{cosec}2^{n-1}\theta=\cot(\theta/2)-\cot 2^{n-1}\theta$

5. Let $\alpha=4\sin^2 10^\circ+4\sin^2 50^\circ\cos 20^\circ+\cos 80^\circ$ and $\beta=\cos^2\frac{\pi}{5}+\cos^2\frac{2\pi}{15}+\cos^2\frac{8\pi}{15}$. Find $(\alpha+\beta)$.

6. Let $x_1=\prod_{r=1}^{5}\cos\frac{r\pi}{11}$ and $x_2=\sum_{r=1}^{5}\cos\frac{r\pi}{11}$, then show that $x_1\cdot x_2=\frac{1}{64}\left(\operatorname{cosec}\frac{\pi}{22}-1\right)$, where Π denotes the continued product.

7. Find the smallest positive values of x & y satisfying, $x-y=\frac{\pi}{4}$, $\cot x+\cot y=2$

8. Prove that $\sin 6^\circ.\sin 42^\circ.\sin 66^\circ.\sin 78^\circ=\cos 6^\circ.\cos 42^\circ.\cos 66^\circ.\cos 78^\circ=\frac{1}{16}$

9. If $(1+\sin t)(1+\cos t)=\frac{5}{4}$. Find the value of $(1-\sin t)(1-\cos t)$.

10. Given that $3\sin x+4\cos x=5$ where $x\in(0,\pi/2)$. Find the value of $2\sin x+\cos x+4\tan x$.

EXERCISE – 4 RECAP OF AIEEE/JEE (MAIN)

1. If $y = \sec^2\theta + \cos^2\theta,\ \theta \neq 0$, then- **[AIEEE-2002]**

(A) $y = 0$ (B) $y \leq 2$ (C) $y \geq -2$ (D) $y > 2$.

2. The value of $\dfrac{1-\tan^2 15^\circ}{1+\tan^2 15^\circ} =$ **[AIEEE-2002]**

(A) 1 (B) $\sqrt{3}$ (C) $\dfrac{\sqrt{3}}{2}$ (D) 2

3. If α is a root of $25\cos^2\theta + 5\cos\theta - 12 = 0$, $\dfrac{\pi}{2} < \alpha < \pi$, then $\sin 2\alpha =$ **[AIEEE-2002]**

(A) $\dfrac{24}{25}$ (B) $-\dfrac{24}{25}$ (C) $\dfrac{13}{18}$ (D) $-\dfrac{13}{18}$

***4.** If $\sin(\alpha+\beta) = 1$, $\sin(\alpha-\beta) = \dfrac{1}{2}$, then $\tan(\alpha+2\beta)\tan(2\alpha+\beta) =$ **[AIEEE-2002]**

(A) 1 (B) –1 (C) zero (D) None of these

5. If $\tan\theta = -\dfrac{4}{3}$, then $\sin\theta$ is- **[AIEEE-2002]**

(A) $-\dfrac{4}{5}$ but not $\dfrac{4}{5}$ (B) $-\dfrac{4}{5}$ or $\dfrac{4}{5}$ (C) $\dfrac{4}{5}$ but not $-\dfrac{4}{5}$ (D) None of these

***6.** $\sec^2\theta = \dfrac{4xy}{(x+y)^2}$ is true if and only if - **[AIEEE-2003]**

(A) $x + y \neq 0$ (B) $x = y, x \neq 0$ (C) $x = y$ (D) $x \neq 0,\ y \neq 0$

***7.** If $u = \sqrt{a^2\cos^2\theta + b^2\sin^2\theta} + \sqrt{a^2\sin^2\theta + b^2\cos^2\theta}$ then the difference between the maximum and minimum values of u^2 is given by- **[AIEEE-2004]**

(A) $2(a^2 + b^2)$ (B) $2\sqrt{a^2+b^2}$ (C) $(a + b)^2$ (D) $(a - b)^2$

8. Let α, β be such that $\pi < \alpha - \beta < 3\pi$.
If $\sin\alpha + \sin\beta = -\dfrac{21}{65}$ and $\cos\alpha + \cos\beta = -\dfrac{27}{65}$, then the value of $\cos\dfrac{\alpha-\beta}{2}$ is- **[AIEEE-2004]**

(A) $-\dfrac{3}{\sqrt{130}}$ (B) $\dfrac{3}{\sqrt{130}}$ (C) $\dfrac{6}{65}0$ (D) $\dfrac{-6}{65}$

***9.** If $0 < x < \pi$, and $\cos x + \sin x = \dfrac{1}{2}$, then $\tan x$ is- **[AIEEE-2006]**

(A) $(4-\sqrt{7})/3$ (B) $-(4+\sqrt{7})/3$ (C) $(1+\sqrt{7})/4$ (D) $(1-\sqrt{7})/4$

10. Let $\cos(\alpha+\beta) = \dfrac{4}{5}$ and let $\sin(\alpha-\beta) = \dfrac{5}{13}$, where $0 \leq \alpha, \beta \leq \dfrac{\pi}{4}$. Then $\tan 2\alpha =$ **[AIEEE-2010]**

(A) $\dfrac{25}{16}$ (B) $\dfrac{56}{33}$ (C) $\dfrac{19}{12}$ (D) $\dfrac{20}{7}$

11. If A = $\sin^2x + \cos^4x$, then for all real x :- ***[JEE MAIN-2011]***

(A) $1 \le A \le 2$ (B) $\frac{3}{4} \le A \le \frac{13}{16}$ (C) $\frac{3}{4} \le A \le 1$ (D) $\frac{13}{16} \le A \le 1$

12. In a ΔPQR, if $3 \sin P + 4 \cos Q = 6$ and $4 \sin Q + 3 \cos P = 1$, then the angle R is equal to : ***[JEE MAIN-2012]***

(A) $\frac{3\pi}{4}$ (B) $\frac{5\pi}{6}$ (C) $\frac{\pi}{6}$ (D) $\frac{\pi}{4}$

13.** Let $f_k(x) = \frac{1}{k}(\sin^k x + \cos^k x)$ where $x \in \boldsymbol{R}$ and $K \geq 1$. Then $f_4(x) - f_6(x)$ equals: ***[JEE MAIN-2014,2019]

(A) $\frac{1}{4}$ (B) $\frac{1}{12}$ (C) $\frac{1}{6}$ (D) $\frac{1}{3}$

14. If $5(\tan^2 x - \cos^2 x) = 2\cos 2x + 9$, then the value of $\cos 4x$ is : ***[JEE MAIN-2017]***

(A) $\frac{2}{9}$ (B) $-\frac{7}{9}$ (C) $-\frac{3}{5}$ (D) $\frac{1}{3}$

15. For any $\theta \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$ the expression $3(\sin\theta - \cos\theta)^4 + 6(\sin\theta + \cos\theta)^2 + 4\sin^6\theta$ equals :

(A) $13 - 4\cos^6\theta$ (B) $13 - 4\cos^4\theta + 2\sin^2\theta\cos^2\theta$ ***[JEE MAIN-2019]***

(C) $13 - 4\cos^2\theta + 6\cos^4\theta$ (D) $13 - 4\cos^2\theta + 6\sin^2\theta\cos^2\theta$

16. The value of $\cos\frac{\pi}{2^2}\cdot\cos\frac{\pi}{2^3}\cdot\ldots\cdot\cos\frac{\pi}{2^{10}}\cdot\sin\frac{\pi}{2^{10}}$ is: ***[JEE MAIN-2019]***

(A) $\frac{1}{256}$ (B) $\frac{1}{2}$ (C) $\frac{1}{512}$ (D) $\frac{1}{1024}$

17. If $\cos(\alpha+\beta) = \frac{3}{5}, \sin(\alpha-\beta) = \frac{5}{13}$ and $0 < \alpha, \beta < \frac{\pi}{4}$, then $\tan(2\alpha)$ is equal to : ***[JEE MAIN-2019]***

(A) $\frac{21}{16}$ (B) $\frac{63}{52}$ (C) $\frac{33}{52}$ (D) $\frac{63}{16}$

18. Let $S = \{\theta \in [-2\pi, 2\pi] : 2\cos^2\theta + 3\sin\theta = 0\}$. Then the sum of the elements of S is ***[JEE MAIN-2019]***

(A) $\frac{13\pi}{6}$ (B) π (C) 2π (D) $\frac{5\pi}{3}$

19. The value of $\cos^2 10° - \cos 10° \cos 50° + \cos^2 50°$ is ***[JEE MAIN-2019]***

(A) $\frac{3}{2}(1+\cos 20°)$ (B) $\frac{3}{4}$ (C) $\frac{3}{4} + \cos 20°$ (D) $\frac{3}{2}$

20. The value of $\sin 10° \sin 30° \sin 50° \sin 70°$ is :- ***[JEE MAIN-2019]***

(A) $\frac{1}{36}$ (B) $\frac{1}{32}$ (C) $\frac{1}{18}$ (D) $\frac{1}{16}$

EXERCISE – 5 RECAP OF IIT-JEE/JEE (ADVANCED)

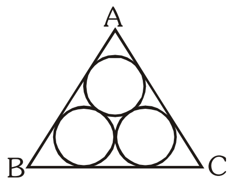
1.** If $\alpha+\beta=\frac{\pi}{2}$ and $\beta+\gamma=\alpha$ then $\tan\alpha$ equals - ***[JEE 2001 Screening]

(A) $2(\tan\beta+\tan\gamma)$ (B) $\tan\beta+\tan\gamma$ (C) $\tan\beta+2\tan\gamma$ (D) $2\tan\beta+\tan\gamma$

2. If θ and ϕ are acute angles satisfying $\sin\theta=\frac{1}{2}$, $\cos\phi=\frac{1}{3}$, then $\theta+\phi\in$ ***[JEE 2004 Screening]***

(A) $\left(\frac{\pi}{3},\frac{\pi}{2}\right]$ (B) $\left(\frac{\pi}{2},\frac{2\pi}{3}\right)$ (C) $\left(\frac{2\pi}{3},\frac{5\pi}{6}\right)$ (D) $\left(\frac{5\pi}{6},\pi\right)$

3.** In an equilateral triangle, 3 coins of radii 1 unit each are kept so that they touch each other and also the sides of the triangle. Area of the triangle is - ***[JEE 2005 Screening]



(A) $4+2\sqrt{3}$ (B) $6+4\sqrt{3}$ (C) $12+\frac{7\sqrt{3}}{4}$ (D) $3+\frac{7\sqrt{3}}{4}$

4.** Let $\theta\in\left(0,\frac{\pi}{4}\right)$ and $t_1=(\tan\theta)^{\tan\theta}$, $t_2=(\tan\theta)^{\cot\theta}$, $t_3=(\cot\theta)^{\tan\theta}$, $t_4=(\cot\theta)^{\cot\theta}$, then - ***[JEE 2006]

(A) $t_1>t_2>t_3>t_4$ (B) $t_4>t_3>t_1>t_2$ (C) $t_3>t_1>t_2>t_4$ (D) $t_2>t_3>t_1>t_4$

One or more than one is/are correct : [Q.5(a) & (b)]

5.(a) If $\frac{\sin^4 x}{2}+\frac{\cos^4 x}{3}=\frac{1}{5}$, then ***[JEE 2009]***

(A) $\tan^2 x=\frac{2}{3}$ (B) $\frac{\sin^8 x}{8}+\frac{\cos^8 x}{27}=\frac{1}{125}$ (C) $\tan^2 x=\frac{1}{3}$ (D) $\frac{\sin^8 x}{8}+\frac{\cos^8 x}{27}=\frac{2}{125}$

***(b)** For $0<\theta<\frac{\pi}{2}$, the solution(s) of $\sum_{m=1}^{6}\operatorname{cosec}\left(\theta+\frac{(m-1)\pi}{4}\right)\operatorname{cosec}\left(\theta+\frac{m\pi}{4}\right)=4\sqrt{2}$ is (are) -

(A) $\frac{\pi}{4}$ (B) $\frac{\pi}{6}$ (C) $\frac{\pi}{12}$ (D) $\frac{5\pi}{12}$

6. The maximum value of the expression $\frac{1}{\sin^2\theta+3\sin\theta\cos\theta+5\cos^2\theta}$ is ***[JEE 2010]***

7. Let $P = \{\theta : \sin\theta - \cos\theta = \sqrt{2}\cos\theta\}$ and $Q = \{\theta : \sin\theta + \cos\theta = \sqrt{2}\sin\theta\}$ be two sets. Then

(A) $P \subset Q$ and $Q - P \neq \varnothing$ (B) $Q \not\subset P$

(C) $P \not\subset Q$ (D) $P = Q$ ***[JEE 2011]***

8. The value of $\sum_{k=1}^{13} \dfrac{1}{\sin\left(\dfrac{\pi}{4} + \dfrac{(k-1)\pi}{6}\right)\sin\left(\dfrac{\pi}{4} + \dfrac{k\pi}{6}\right)}$ is equal to ***[JEE 2016]***

(A) $3-\sqrt{3}$ (B) $2(3-\sqrt{3})$ (C) $2(\sqrt{3}-1)$ (D) $2(2+\sqrt{3})$

9. Let a, b, c be three non-zero real numbers such that the equation

$$\sqrt{3}a\cos x + 2b\sin x = c, \quad x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

has two distinct real roots α and β with $\alpha + \beta = \dfrac{\pi}{3}$. Then the value of $\dfrac{b}{a}$ is ***[JEE 2018]***

ANSWER KEY

EXERCISE-1

Que.	1	2	3	4	5	6	7	8	9	10
Ans.	B	C	A	B	C	D	B	C	A	B
Que.	11	12	13	14	15	16	17	18	19	20
Ans.	B	C	D	D	B	C	C	A	B	B
Que.	21	22								
Ans.	D	B								

EXERCISE-2

Que.	1	2	3	4	5	6	7	8	9	10
Ans.	ABC	AD	D	ABCD	ABD	ABCD	D	A	A	AD

- ***Match the Column*** **11.** (A)$\rightarrow$(t), (B)$\rightarrow$(p), (C)$\rightarrow$(q,r), (D)$\rightarrow$(q,s)
- ***Comprehension Based Questions***

Comprehension-1 **12.** (D) **13.** (A) **14.** (D)

EXERCISE-3

5. 4

7. $x=\frac{5\pi}{12}, y=\frac{\pi}{6}$

9. $\frac{13}{4}-\sqrt{10}$

10. 5

EXERCISE-4

Que.	1	2	3	4	5	6	7	8	9	10
Ans.	D	C	B	A	B	B	D	A	B	B
Que.	11	12	13	14	15	16	17	18	19	20
Ans.	C	C	B	B	D	C	D	C	B	D

EXERCISE-5

1. (C) **2.** (B) **3.** (B) **4.** (B) **5.*(a)*** (AB) ***(b)*** (CD)

6. 2 **7.** (D) **8.** (C) **9.** 0.5

TRIGONOMETRIC EQUATION

Recap of Early Classes

In earlier chapter, we had an extensive idea about trigonometric ratios and their applications. In this chapter, we are going to explore about solving trigonometric equation and in equations using sum, difference and product formulas of trigonometric ratios.

Index

TRIGONOMETRIC EQUATION

1.0 TRIGONOMETRIC EQUATION

An equation involving one or more trigonometrical ratios of unknown angles is called a trigonometrical equation.

2.0 SOLUTION OF TRIGONOMETRIC EQUATION

A value of the unknown angle which satisfies the given equation is called a solution of the trigonometric equation.

(a) ***Principal solution –*** The solution of the trigonometric equation lying in the interval $[0, 2\pi)$.

(b) ***General solution –*** Since all the trigonometric functions are many one & periodic, hence there are infinite values of θ for which trigonometric functions have the same value. All such possible values of θ for which the given trigonometric function is satisfied is given by a general formula. Such a general formula is called general solution of trigonometric equation.

(c) ***Particular solution –*** The solution of the trigonometric equation lying in the given interval.

3.0 GENERAL SOLUTIONS OF SOME TRIGONOMETRIC EQUATIONS

SL AL

(a) If $\sin\theta = 0$, then $\theta = n\pi$, $n \in I$ (set of integers)

(b) If $\cos\theta = 0$, then $\theta = (2n+1)\frac{\pi}{2}$, $n \in I$

(c) If $\tan\theta = 0$, then $\theta = n\pi$, $n \in I$

(d) If $\sin\theta = \sin\alpha$, then $\theta = n\pi + (-1)^n\alpha$

where $\alpha \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$, $n \in I$

(e) If $\cos\theta = \cos\alpha$, then $\theta = 2n\pi \pm \alpha$, $n \in I$, $\alpha \in [0,\pi]$

(f) If $\tan\theta = \tan\alpha$, then $\theta = n\pi + \alpha$, $n \in I$, $\alpha \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$

(g) If $\sin\theta = 1$, then $\theta = 2n\pi + \frac{\pi}{2} = (4n+1)\frac{\pi}{2}$, $n \in I$

(h) If $\cos\theta = 1$ then $\theta = 2n\pi$, $n \in I$

(i) If $\sin^2\theta = \sin^2\alpha$ or $\cos^2\theta = \cos^2\alpha$ or $\tan^2\theta = \tan^2\alpha$,

then $\theta = n\pi \pm \alpha$, $n \in I$

(j) For $n \in I$, $\sin n\pi = 0$ and $\cos n\pi = (-1)^n$, $n \in I$

$\sin(n\pi + \theta) = (-1)^n \sin\theta$

$\cos(n\pi + \theta) = (-1)^n \cos\theta$

(k) $\cos n\pi = (-1)^n$, $n \in I$

If n is an odd integer, then $\sin\frac{n\pi}{2} = (-1)^{\frac{n-1}{2}}, \cos\frac{n\pi}{2} = 0,$

$$\sin\left(\frac{n\pi}{2} + \theta\right) = (-1)^{\frac{n-1}{2}}\cos\theta$$

$$\cos\left(\frac{n\pi}{2} + \theta\right) = (-1)^{\frac{n+1}{2}}\sin\theta$$

Illustrations

Illustration 1. Find the set of values of x for which $\frac{\tan 3x - \tan 2x}{1 + \tan 3x . \tan 2x} = 1$.

Solution We have, $\frac{\tan 3x - \tan 2x}{1 + \tan 3x . \tan 2x} = 1$

$\Rightarrow \quad \tan(3x - 2x) = 1 \Rightarrow \tan x = 1$

$\Rightarrow \quad \tan x = \tan\frac{\pi}{4}$

$\Rightarrow \quad x = n\pi + \frac{\pi}{4}, n \in I \qquad \{\text{using } \tan\theta = \tan\alpha \Leftrightarrow \theta = n\pi + \alpha)$

But for this value of x, tan 2x is not defined.

Hence the solution set for x is ϕ. ***Ans.***

4.0 DIFFERENT STRATEGIES FOR SOLVING TRIGONOMETRIC EQUATIONS

SL AL

4.1 Solving trigonometric equations by factorisation

e.g. $(2 \sin x - \cos x)(1 + \cos x) = \sin^2 x$

$\therefore \quad (2 \sin x - \cos x)(1 + \cos x) - (1 - \cos^2 x) = 0$

$\therefore \quad (1 + \cos x)(2 \sin x - \cos x - 1 + \cos x) = 0$

$\therefore \quad (1 + \cos x)(2 \sin x - 1) = 0$

$\Rightarrow \quad \cos x = -1 \text{ or } \sin x = \frac{1}{2}$

$\Rightarrow \quad \cos x = -1 = \cos\pi$

$\Rightarrow \quad x = 2n\pi + \pi = (2n + 1)\pi, n \in I$

or $\quad \sin x = \frac{1}{2} = \sin\frac{\pi}{6}$

$\Rightarrow \quad x = k\pi + (-1)^k \frac{\pi}{6}, k \in I$

Illustrations

Illustration 2. If $\frac{1}{6}\sin\theta$, $\cos\theta$ and $\tan\theta$ are in G.P. then the general solution for θ is -

(A) $2n\pi \pm \frac{\pi}{3}$ (B) $2n\pi \pm \frac{\pi}{6}$ (C) $n\pi \pm \frac{\pi}{3}$ (D) none of these

Solution Since, $\frac{1}{6}\sin\theta$, $\cos\theta$, $\tan\theta$ are in G.P.

$\Rightarrow \quad \cos^2\theta = \frac{1}{6}\sin\theta . \tan\theta$

$\Rightarrow \quad 6\cos^3\theta + \cos^2\theta - 1 = 0$

$\therefore \quad (2\cos\theta - 1)(3\cos^2\theta + 2\cos\theta + 1) = 0$

$\Rightarrow \quad \cos\theta = \frac{1}{2}$ (other values of $\cos\theta$ are imaginary)

$\Rightarrow \quad \cos\theta = \cos\frac{\pi}{3}$

$\Rightarrow \quad \theta = 2n\pi \pm \frac{\pi}{3}, n \in I.$ ***Ans. (A)***

4.2 Solving of trigonometric equation by reducing it to a quadratic equation

e.g. $6 - 10\cos x = 3\sin^2 x$

$\therefore$ $6 - 10\cos x = 3 - 3\cos^2 x$

$\Rightarrow$ $3\cos^2 x - 10\cos x + 3 = 0$

$\Rightarrow$ $(3\cos x - 1)(\cos x - 3) = 0$

$\Rightarrow$ $\cos x = \frac{1}{3}$ or $\cos x = 3$

Since $\cos x = 3$ is not possible as $-1 \le \cos x \le 1$

$\therefore$ $\cos x = \frac{1}{3} = \cos\left(\cos^{-1}\frac{1}{3}\right)$

$\Rightarrow$ $x = 2n\pi \pm \cos^{-1}\left(\frac{1}{3}\right),\ n \in I$

Illustrations

Illustration 3. Solve $\sin^2\theta - \cos\theta = \frac{1}{4}$ for θ and write the values of θ in the interval $0 \le \theta \le 2\pi$.

Solution The given equation can be written as

$$1 - \cos^2\theta - \cos\theta = \frac{1}{4}$$

$\Rightarrow$ $\cos^2\theta + \cos\theta - 3/4 = 0$

$\Rightarrow$ $4\cos^2\theta + 4\cos\theta - 3 = 0$

$\Rightarrow$ $(2\cos\theta - 1)(2\cos\theta + 3) = 0$

$\Rightarrow$ $\cos\theta = \frac{1}{2}, -\frac{3}{2}$

Since, $\cos\theta = -3/2$ is not possible as $-1 \le \cos\theta \le 1$

$\therefore$ $\cos\theta = \frac{1}{2}$

$\Rightarrow$ $\cos\theta = \cos\frac{\pi}{3}$

$\Rightarrow$ $\theta = 2n\pi \pm \frac{\pi}{3},\ n \in I$

For the given interval, $n = 0$ and $n = 1$.

$\Rightarrow$ $\theta = \frac{\pi}{3}, \frac{5\pi}{3}$ ***Ans.***

Illustration 4. Find the number of solutions of $\tan x + \sec x = 2\cos x$ in $[0, 2\pi]$.

Solution Here, $\tan x + \sec x = 2\cos x$

$\Rightarrow$ $\sin x + 1 = 2\cos^2 x$

$\Rightarrow$ $2\sin^2 x + \sin x - 1 = 0$

$\Rightarrow$ $\sin x = \frac{1}{2}, -1$

But $\sin x = -1 \Rightarrow x = \frac{3\pi}{2}$ for which $\tan x + \sec x = 2\cos x$ is not defined.

Thus $\sin x = \frac{1}{2} \Rightarrow x = \frac{\pi}{6}, \frac{5\pi}{6}$

$\Rightarrow$ number of solutions of $\tan x + \sec x = 2\cos x$ is 2. ***Ans.***

Illustration 5. Solve the equation $5\sin^2x - 7\sin x\cos x + 16\cos^2 x = 4$

Solution To solve this equation we use the fundamental formula of trigonometric identities,

$\sin^2x + \cos^2x = 1$

writing the equation in the form,

$5\sin^2x - 7\sin x \,.\, \cos x + 16\cos^2x = 4(\sin^2x + \cos^2x)$

$\Rightarrow$ $\sin^2x - 7\sin x\cos x + 12\cos^2 x = 0$

dividing by $\cos^2x$ on both side we get,

$\tan^2x - 7\tan x + 12 = 0$

Now it can be factorized as :

$(\tan x - 3)(\tan x - 4) = 0$

$\Rightarrow$ $\tan x = 3, 4$

$\Rightarrow$ $x = n\pi + \tan^{-1} 3$

or $x = n\pi + \tan^{-1} 4,\ n \in I.$ **Ans.**

Illustration 6. If $x \neq \frac{n\pi}{2}$, $n \in I$ and $(\cos x)^{\sin^2 x - 3\sin x + 2} = 1$, then find the general solutions of x.

Solution As $x \neq \frac{n\pi}{2}$ $\Rightarrow$ $\cos x \neq 0, 1, -1$

So, $(\cos x)^{\sin^2 x - 3\sin x + 2} = 1$

$\Rightarrow$ $\sin^2x - 3\sin x + 2 = 0$

$\therefore$ $(\sin x - 2)(\sin x - 1) = 0$

$\Rightarrow$ $\sin x = 1, 2$

where $\sin x = 2$ is not possible and $\sin x = 1$ which is also not possible as $x \neq \frac{n\pi}{2}$

$\therefore$ no general solution is possible. **Ans.**

***Illustration 7.** Solve the equation $\sin^4x + \cos^4 x = \frac{7}{2}\sin x \,.\, \cos x.$

Solution $\sin^4x + \cos^4x = \frac{7}{2}\sin x \,.\, \cos x$

$\Rightarrow$ $(\sin^2x + \cos^2x)^2 - 2\sin^2x\cos^2x = \frac{7}{2}\sin x \,.\, \cos x$

$\Rightarrow$ $1 - \frac{1}{2}(\sin 2x)^2 = \frac{7}{4}(\sin 2x)$

$\Rightarrow$ $2\sin^2 2x + 7\sin 2x - 4 = 0$

$\Rightarrow$ $(2\sin 2x - 1)(\sin 2x + 4) = 0$

$\Rightarrow$ $\sin 2x = \frac{1}{2}$ or $\sin 2x = -4$ (which is not possible)

$\Rightarrow$ $2x = n\pi + (-1)^n\frac{\pi}{6},\ n \in I$

i.e., $x = \frac{n\pi}{2} + (-1)^n\frac{\pi}{12},\ n \in I$ **Ans.**

BEGINNER'S BOX-1

TOPIC COVERED : SOLUTION OF $\sin\theta = \sin\alpha$, $\cos\theta = \cos\alpha$, $\tan\theta = \tan\alpha$ AND OTHER ELEMENTRY EQUATIONS.

1. Match the following

	Column-1		**column-2**
(a)	$\cos x = -\frac{1}{2}$	(p)	$x = \frac{7\pi}{3}$
(b)	$\sin x = \frac{\sqrt{3}}{2}$	(q)	$x = \frac{19\pi}{6}$
(c)	$\tan x = \frac{1}{\sqrt{3}}$	(r)	$x = \frac{8\pi}{3}$
(d)	$\cot x = -1$	(s)	$x = \frac{11\pi}{4}$

2. If $0 \le x \le 2\pi$, then find the number of solutions of the equation $\sin 2x = \cos 3x$.

3. The smallest positive root of the eqution $\tan x = x$ lies in

(A) $\left(0, \frac{\pi}{2}\right)$ (B) $\left(\frac{\pi}{2}, \pi\right)$ (C) $\left(\pi, \frac{3\pi}{2}\right)$ (D) $\left(\frac{3\pi}{2}, 2\pi\right)$

4. The number of real solution of the equation $\sin(e^x) = 2^x + 2^{-x}$ is

(A) 0 (B) 1 (C) 2 (D) infinite

5. If $\sin 2x = \sqrt{2}\cos x$ then which of the following is not correct, $(n \in z)$

(A) $x = n\pi + \frac{\pi}{2}$ (B) $x = 2n\pi + \frac{\pi}{4}$ (C) $x = 2n\pi - \frac{\pi}{4}$ (D) $x = n\pi - \frac{\pi}{2}(n \in I)$

6. Which of the following satisfies $\sin x + \sin 2x = 0$

(A) $\sin x = \frac{1}{2}$ (B) $\tan x = -1$ (C) $\cos x = -\frac{1}{2}$ (D) None of these

4.3 Solving trigonometric equations by introducing an auxilliary argument

Consider, $a \sin\theta + b\cos\theta = c$ ..... (i)

$$\therefore \quad \frac{a}{\sqrt{a^2+b^2}}\sin\theta + \frac{b}{\sqrt{a^2+b^2}}\cos\theta = \frac{c}{\sqrt{a^2+b^2}}$$

equation (i) has a solution only if $|c| \le \sqrt{a^2+b^2}$

let $\frac{a}{\sqrt{a^2+b^2}} = \cos\phi$, $\frac{b}{\sqrt{a^2+b^2}} = \sin\phi$ & $\phi = \tan^{-1}\frac{b}{a}$

by introducing this auxillary argument ϕ, equation (i) reduces to

$\sin(\theta+\phi) = \frac{c}{\sqrt{a^2+b^2}}$ Now this equation can be solved easily.

Illustrations

Illustration 8. Find the number of distinct solutions of secx + tanx = $\sqrt{3}$, where $0 \le x \le 3\pi$.

Solution Here, sec x + tanx = $\sqrt{3}$

$\Rightarrow \quad 1 + \sin x = \sqrt{3}\cos x$

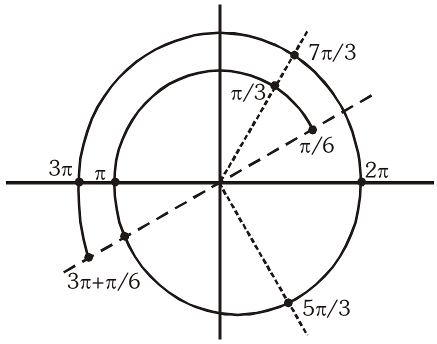
or $\quad \sqrt{3}\cos x - \sin x = 1$

dividing both sides by $\sqrt{a^2+b^2}$ i.e. $\sqrt{4} = 2$, we get

$$\Rightarrow \quad \frac{\sqrt{3}}{2}\cos x - \frac{1}{2}\sin x = \frac{1}{2}$$

$$\Rightarrow \quad \cos\frac{\pi}{6}\cos x - \sin\frac{\pi}{6}\sin x = \frac{1}{2}$$

$$\Rightarrow \quad \cos\left(x+\frac{\pi}{6}\right) = \frac{1}{2}$$



As $0 \le x \le 3\pi$

$$\frac{\pi}{6} \le x + \frac{\pi}{6} \le 3\pi + \frac{\pi}{6}$$

$$\Rightarrow \quad x + \frac{\pi}{6} = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}$$

$$\Rightarrow \quad x = \frac{\pi}{6}, \frac{3\pi}{2}, \frac{13\pi}{6}$$

But at $x = \frac{3\pi}{2}$, tanx and secx is not defined.

$\therefore$ Total number of solutions are 2. ***Ans.***

****Illustration 9.*** Prove that the equation kcosx – 3sinx = k + 1 possess a solution iff $k \in (-\infty, 4]$.

Solution Here, k cosx – 3sinx = k + 1, could be re-written as :

$$\frac{k}{\sqrt{k^2+9}}\cos x - \frac{3}{\sqrt{k^2+9}}\sin x = \frac{k+1}{\sqrt{k^2+9}}$$

or $\quad \cos(x+\phi) = \dfrac{k+1}{\sqrt{k^2+9}}$, where $\tan\phi = \dfrac{3}{k}$

which possess a solution only if $-1 \le \dfrac{k+1}{\sqrt{k^2+9}} \le 1$

i.e., $\quad \left|\dfrac{k+1}{\sqrt{k^2+9}}\right| \le 1$

i.e., $\quad (k+1)^2 \le k^2 + 9$

i.e., $\quad k^2 + 2k + 1 \le k^2 + 9$

or $\quad k \le 4$

$\Rightarrow$ The interval of k for which the equation (kcosx – 3sinx = k + 1) has a solution is $(-\infty, 4]$. ***Ans.***

BEGINNER'S BOX-2

TOPIC COVERED : SOLUTION USING FACTORIZATION, QUADRATIC REDUCTION & AUXILLIARY ARGUMENT

1. Find general solutions of the following equations :

(a) $\sin\theta = \frac{1}{2}$ (b) $\cos\left(\frac{3\theta}{2}\right) = 0$ (c) $\tan\left(\frac{3\theta}{4}\right) = 0$

(d) $\cos^2 2\theta = 1$ (e) $\sqrt{3}\sec 2\theta = 2$ (f) $\operatorname{cosec}\left(\frac{\theta}{2}\right) = -1$

2. Solve the following equations :

(a) $3\sin x + 2\cos^2 x = 0$ (b) $\sec^2 2\alpha = 1 - \tan 2\alpha$

(c) $7\cos^2\theta + 3\sin^2\theta = 4$ (d) $4\cos\theta - 3\sec\theta = \tan\theta$

3. Solve the equation : $2\sin^2\theta + \sin^2 2\theta = 2$ for $\theta \in (-\pi, \pi)$.

4. Solve the following equations :

(a) $\sin x + \sqrt{2} = \cos x$.

(b) $\operatorname{cosec}\theta = 1 + \cot\theta$.

5. If $\sin\theta = k$ for exactly one value of θ, $\theta \in \left[0, \frac{7\pi}{3}\right]$, then find sum of all values of 'k'.

6. Find number of all possible triplets (a_1, a_2, a_3) such that $a_1 + a_2 \cos 2x + a_3 \sin^2 x = 0$, for all x.

7. Find general solution of the equation $\left(\sqrt{3}-1\right)\sin\theta + \left(\sqrt{3}+1\right)\cos\theta = 2$.

8. Find number of solutions of the equation $\sin x = x^2 + x + 1$.

4.4 Solving trigonometric equations by transforming sum of trigonometric functions into product

e.g. $\cos 3x + \sin 2x - \sin 4x = 0$

$\cos 3x - 2\sin x \cos 3x = 0$

$\Rightarrow (\cos 3x)(1 - 2\sin x) = 0$

$\Rightarrow \cos 3x = 0$ or $\sin x = \frac{1}{2}$

$\Rightarrow \cos 3x = 0 = \cos\frac{\pi}{2}$ or $\sin x = \frac{1}{2} = \sin\frac{\pi}{6}$

$\Rightarrow 3x = 2n\pi \pm \frac{\pi}{2}$

or $x = m\pi + (-1)^m \frac{\pi}{6}$

$\Rightarrow x = \frac{2n\pi}{3} \pm \frac{\pi}{6}$

or $x = m\pi + (-1)^m \frac{\pi}{6}$; $(n, m \in I)$

Illustrations

Illustration 10. Solve : $\cos\theta + \cos3\theta + \cos5\theta + \cos7\theta = 0$

Solution We have $\cos\theta + \cos7\theta + \cos3\theta + \cos5\theta = 0$

$\Rightarrow \quad 2\cos4\theta\cos3\theta + 2\cos4\theta\cos\theta = 0$

$\Rightarrow \quad \cos4\theta(\cos3\theta + \cos\theta) = 0$

$\Rightarrow \quad \cos4\theta(2\cos2\theta\cos\theta) = 0$

$\Rightarrow$ Either $\cos\theta = 0 \Rightarrow \theta = (2n_1 + 1)\pi/2,\ n_1 \in I$

or $\quad \cos2\theta = 0 \Rightarrow \theta = (2n_2 + 1)\dfrac{\pi}{4},\ n_2 \in I$

or $\quad \cos4\theta = 0 \Rightarrow \theta = (2n_3 + 1)\dfrac{\pi}{8},\ n_3 \in I$ ***Ans.***

4.5 Solving trigonometric equations by transforming a product into sum

e.g. $\sin5x.\cos3x = \sin6x.\cos2x$

$\sin8x + \sin2x = \sin8x + \sin4x$

$\therefore \quad 2\sin2x\,.\cos2x - \sin2x = 0$

$\Rightarrow \quad \sin2x(2\cos2x - 1) = 0$

$\Rightarrow \quad \sin2x = 0$ or $\cos2x = \dfrac{1}{2}$

$\Rightarrow \quad \sin2x = 0 = \sin0$ or $\cos2x = \dfrac{1}{2} = \cos\dfrac{\pi}{3}$

$\Rightarrow \quad 2x = n\pi + (-1)^n \times 0,\ n \in I$ or $2x = 2m\pi \pm \dfrac{\pi}{3},\ m \in I$

$\Rightarrow \quad x = \dfrac{n\pi}{2},\ n \in I$ or $x = m\pi \pm \dfrac{\pi}{6},\ m \in I$

Illustrations

****Illustration 11.*** Solve : $\cos\theta\cos2\theta\cos3\theta = \dfrac{1}{4}$; where $0 \le \theta \le \pi$.

Solution $\dfrac{1}{2}(2\cos\theta\cos3\theta)\cos2\theta = \dfrac{1}{4}$

$\Rightarrow \quad (\cos2\theta + \cos4\theta)\cos2\theta = \dfrac{1}{2}$

$\Rightarrow \quad \dfrac{1}{2}[2\cos^2 2\theta + 2\cos4\theta\cos2\theta] = \dfrac{1}{2}$

$\Rightarrow \quad 1 + \cos4\theta + 2\cos4\theta\cos2\theta = 1$

$\therefore \quad \cos4\theta(1 + 2\cos2\theta) = 0$

$\cos4\theta = 0$ ......(1) or $(1 + 2\cos2\theta) = 0$ ......(2)

Now from the first equation : $\cos4\theta = 0 = \cos(\pi/2)$

$\therefore \quad 4\theta = \left(n + \dfrac{1}{2}\right)\pi$

$\Rightarrow \quad \theta = (2n+1)\dfrac{\pi}{8},\ n \in I$

for $n = 0,\ \theta = \frac{\pi}{8};\ n = 1,\ \theta = \frac{3\pi}{8};$

$n = 2,\ \theta = \frac{5\pi}{8}\ ;\ n = 3,\ \theta = \frac{7\pi}{8}\quad (\because\ 0 \le \theta \le \pi)$

and from the second equation :

$\cos 2\theta = -\frac{1}{2} = -\cos(\pi/3) = \cos(\pi-\pi/3) = \cos(2\pi/3)$

$\therefore\quad 2\theta = 2k\pi \pm 2\pi/3\qquad \therefore\ \theta = k\pi \pm \pi/3,\ k \in I$

again for $k = 0,\ \theta = \frac{\pi}{3};$

$k = 1,\ \theta = \frac{2\pi}{3}\qquad (\because\ 0 \le \theta \le \pi)$

$\therefore\quad \theta = \frac{\pi}{8}, \frac{\pi}{3}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{2\pi}{3}, \frac{7\pi}{8}$ **Ans.**

4.6 Solving equations by a change of variable

(i) Equations of the form P (sin x ± cos x, sin x. cos x) = 0 can be solved by the substitution $\cos x \pm \sin x = t \Rightarrow 1 \pm 2\sin x.\cos x = t^2$.

e.g. $\sin x + \cos x = 1 + \sin x.\cos x$.

put $\sin x + \cos x = t$

$\Rightarrow\quad \sin^2 x + \cos^2 x + 2\sin x.\cos x = t^2$

$\Rightarrow\quad 2\sin x\cos x = t^2 - 1\quad (\because \sin^2 x + \cos^2 x = 1)$

$\Rightarrow\quad \sin x.\cos x = \left(\frac{t^2-1}{2}\right)$

Substituting above result in given equation, we get :

$t = 1 + \frac{t^2-1}{2}$

$\Rightarrow\quad 2t = t^2 + 1 \Rightarrow t^2 - 2t + 1 = 0$

$\Rightarrow\quad (t-1)^2 = 0 \Rightarrow\quad t = 1$

$\Rightarrow\quad \sin x + \cos x = 1$

Dividing both sides by $\sqrt{1^2+1^2}$ i.e. $\sqrt{2}$, we get

$\Rightarrow\quad \frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x = \frac{1}{\sqrt{2}}$

$\Rightarrow\quad \cos x\cos\frac{\pi}{4} + \sin x.\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$\Rightarrow\quad \cos\left(x - \frac{\pi}{4}\right) = \cos\frac{\pi}{4}$

$\Rightarrow\quad x - \frac{\pi}{4} = 2n\pi \pm \frac{\pi}{4}$

$\Rightarrow\quad x = 2n\pi$ or $x = 2n\pi + \frac{\pi}{2} = (4n+1)\frac{\pi}{2},\ n \in I$

(ii) Equations of the form of asinx + bcosx + d = 0, where a, b & d are real numbers can be solved by changing sin x & cos x into their corresponding tangent of half the angle.

e.g. $3\cos x + 4\sin x = 5$

$$\Rightarrow \quad 3\left(\frac{1-\tan^2 x/2}{1+\tan^2 x/2}\right) + 4\left(\frac{2\tan x/2}{1+\tan^2 x/2}\right) = 5$$

$$\Rightarrow \quad \frac{3-3\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}} + \frac{8\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}} = 5$$

$$\Rightarrow \quad 3 - 3\tan^2\frac{x}{2} + 8\tan\frac{x}{2} = 5 + 5\tan^2\frac{x}{2}$$

$$\Rightarrow \quad 8\tan^2\frac{x}{2} - 8\tan\frac{x}{2} + 2 = 0$$

$$\Rightarrow \quad 4\tan^2\frac{x}{2} - 4\tan\frac{x}{2} + 1 = 0$$

$$\Rightarrow \quad \left(2\tan\frac{x}{2} - 1\right)^2 = 0$$

$$\Rightarrow \quad 2\tan\frac{x}{2} - 1 = 0$$

$$\Rightarrow \quad \tan\frac{x}{2} = \frac{1}{2} = \tan\left(\tan^{-1}\frac{1}{2}\right)$$

$$\Rightarrow \quad \frac{x}{2} = n\pi + \tan^{-1}\left(\frac{1}{2}\right),\ n \in I$$

$$\Rightarrow \quad x = 2n\pi + 2\tan^{-1}\frac{1}{2},\ n \in I$$

(iii) Many equations can be solved by introducing a new variable.

e.g. $\sin^4 2x + \cos^4 2x = \sin 2x.\cos 2x$

substituting $\sin 2x.\cos 2x = y$ $\because$ $(\sin^2 2x + \cos^2 2x)^2 = \sin^4 2x + \cos^4 2x + 2\sin^2 2x.\cos^2 2x$

$\Rightarrow$ $\sin^4 2x + \cos^4 2x = 1 - 2\sin^2 2x.\cos^2 2x$ substituting above result in given equation :

$1 - 2y^2 = y$

$$\Rightarrow \quad 2y^2 + y - 1 = 0 \quad \Rightarrow \quad 2(y+1)\left(y - \frac{1}{2}\right) = 0$$

$$\Rightarrow \quad y = -1 \quad \text{or} \quad y = \frac{1}{2}$$

$$\Rightarrow \quad \sin 2x.\cos 2x = -1 \quad \text{or} \quad \sin 2x.\cos 2x = \frac{1}{2}$$

$\Rightarrow$ $2\sin 2x.\cos 2x = -2$ or $2\sin 2x.\cos 2x = 1$

$\Rightarrow$ $\sin 4x = -2$ (which is not possible) or $2\sin 2x.\cos 2x = 1$

$$\Rightarrow \quad \sin 4x = 1 = \sin\frac{\pi}{2}$$

$$\Rightarrow \quad 4x = n\pi + (-1)^n\frac{\pi}{2},\ n \in I$$

$$\Rightarrow \quad x = \frac{n\pi}{4} + (-1)^n\frac{\pi}{8},\ n \in I$$

Illustrations

Illustration 12. Find the general solution of equation $\sin^4x + \cos^4x = \sin x \cos x$.

Solution Using half-angle formulae, we can represent given equation in the form :

$$\left(\frac{1-\cos 2x}{2}\right)^2 + \left(\frac{1+\cos 2x}{2}\right)^2 = \sin x \cos x$$

$\Rightarrow \quad (1 - \cos 2x)^2 + (1 + \cos 2x)^2 = 4\sin x \cos x$

$\Rightarrow \quad 2(1 + \cos^2 2x) = 2\sin 2x$

$\Rightarrow \quad 1 + 1 - \sin^2 2x = \sin 2x$

$\Rightarrow \quad \sin^2 2x + \sin 2x = 2$

$\Rightarrow \quad \sin 2x = 1$ or $\sin 2x = -2$ (which is not possible)

$\Rightarrow \quad 2x = 2n\pi + \dfrac{\pi}{2},\ n \in I$

$\Rightarrow \quad x = n\pi + \dfrac{\pi}{4},\ n \in I$ ***Ans.***

4.7 Solving trigonometric equations with the use of the boundness of the functions involved

e.g. $\sin x\left(\cos\dfrac{x}{4} - 2\sin x\right) + \left(1 + \sin\dfrac{x}{4} - 2\cos x\right).\cos x = 0$

$\therefore \quad \sin x \cos \dfrac{x}{4} + \cos x \sin \dfrac{x}{4} + \cos x = 2$

$\therefore \quad \sin\left(\dfrac{5x}{4}\right) + \cos x = 2$

$\Rightarrow \quad \sin\left(\dfrac{5x}{4}\right) = 1$

& $\quad \cos x = 1 \qquad$ (as $\sin\theta \le 1$ & $\cos\theta \le 1$)

Now consider

$\cos x = 1 \quad \Rightarrow \quad x = 2\pi, 4\pi, 6\pi, 8\pi \ ....$

and $\sin\dfrac{5x}{4} = 1 \quad \Rightarrow \quad x = \dfrac{2\pi}{5}, \dfrac{10\pi}{5}, \dfrac{18\pi}{5} \ ....$

Common solution to above APs will be the AP having

First term $= 2\pi$

Common difference $=$ LCM of 2π and $\dfrac{8\pi}{5} = \dfrac{40\pi}{5} = 8\pi$

$\therefore \quad$ General solution will be general term of this AP i.e. $2\pi + (8\pi)n,\ n \in I$

$\Rightarrow \quad x = 2(4n + 1)\pi,\ n \in I$

Illustrations

****Illustration 13.*** Solve the equation $(\sin x + \cos x)^{1+\sin 2x} = 2$, when $0 \le x \le \pi$.

Solution We know, $-\sqrt{a^2+b^2} \le a\sin\theta + b\cos\theta \le \sqrt{a^2+b^2}$ and $-1 \le \sin\theta \le 1$.

$\therefore \quad (\sin x + \cos x)$ admits the maximum value as $\sqrt{2}$

and $\quad (1 + \sin 2x)$ admits the maximum value as 2.

Also $\quad (\sqrt{2})^2 = 2$.

$\therefore$ the equation could hold only when, $\sin x + \cos x = \sqrt{2}$ and $1 + \sin 2x = 2$

Now, $\sin x + \cos x = \sqrt{2} \Rightarrow \cos\left(x - \frac{\pi}{4}\right) = 1$

$\Rightarrow x = 2n\pi + \pi/4,\ n \in I$ ...... (i)

and $1 + \sin 2x = 2 \Rightarrow \sin 2x = 1 = \sin\frac{\pi}{2}$

$\Rightarrow 2x = m\pi + (-1)^m\frac{\pi}{2},\ m \in I \Rightarrow x = \frac{m\pi}{2} + (-1)^m\frac{\pi}{4}$ ...... (ii)

The value of x in $[0, \pi]$ satisfying equations (i) and (ii) is $x = \frac{\pi}{4}$ (when $n = 0$ & $m = 0$) ***Ans.***

Note – $\sin x + \cos x = -\sqrt{2}$ and $1 + \sin 2x = 2$ also satisfies but as $x \geq 0$, this solution is not in domain.

Illustration 14. Solve for x and y : $2^{\frac{1}{\cos^2 x}}\sqrt{y^2 - y + 1/2} \leq 1$

Solution

$2^{\frac{1}{\cos^2 x}}\sqrt{y^2 - y + 1/2} \leq 1$ .... (i)

$2^{\frac{1}{\cos^2 x}}\sqrt{\left(y - \frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} \leq 1$

Minimum value of $2^{\frac{1}{\cos^2 x}} = 2$

Minimum value of $\sqrt{\left(y - \frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \frac{1}{2}$

$\Rightarrow$ Minimum value of $2^{\frac{1}{\cos^2 x}}\sqrt{y^2 - y + \frac{1}{2}}$ is 1

$\Rightarrow$ (i) is possible when $2^{\frac{1}{\cos^2 x}}\sqrt{\left(y - \frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = 1$

$\Rightarrow \cos^2 x = 1$ and $y = 1/2 \Rightarrow \cos x = \pm 1 \Rightarrow x = n\pi$, where $n \in I$.
Hence $x = n\pi,\ n \in I$ and $y = 1/2$. ***Ans.***

Illustration 15. The number of solution(s) of $2\cos^2\left(\frac{x}{2}\right)\sin^2 x = x^2 + \frac{1}{x^2}$, $0 \leq x \leq \pi/2$, is/are -

(A) 0 (B) 1 (C) infinite (D) none of these

Solution Let $y = 2\cos^2\left(\frac{x}{2}\right)\sin^2 x = x^2 + \frac{1}{x^2}$

$\Rightarrow y = (1 + \cos x)\sin^2 x$ and $y = x^2 + \frac{1}{x^2}$

when $y = (1 + \cos x)\sin^2 x = $ (a number < 2)(a number ≤ 1) $\Rightarrow y < 2$ .... (i)

and when $y = x^2 + \frac{1}{x^2} = \left(x - \frac{1}{x}\right)^2 + 2 \geq 2 \Rightarrow y \geq 2$ .... (ii)

No value of y can be obtained satisfying (i) and (ii), simultaneously

$\Rightarrow$ No real solution of the equation exists. ***Ans. (A)***

Note– If L.H.S. of the given trigonometric equation is always less than or equal to k and RHS is always greater than k, then no solution exists. If both the sides are equal to k for same value of θ, then solution exists and if they are equal for different values of θ, then solution does not exist.

BEGINNER'S BOX-3

TOPIC COVERED : TRANSFORMING PRODUCT INTO SUM, CHANGE OF VARIABLE, BOUNDEDNESS OF FUNCTIONS.

1. Solve $4\sin\theta\sin2\theta\sin4\theta = \sin3\theta$.

2. Solve for x : $\sin x + \sin3x + \sin5x = 0$.

3. If $x^2 - 4x + 5 - \sin y = 0$, $y \in [0, 2\pi)$, then -

(A) $x = 1, y = 0$ (B) $x = 1, y = \pi/2$ (C) $x = 2, y = 0$ (D) $x = 2, y = \pi/2$

4. If $\sin x + \cos x = \sqrt{y + \frac{1}{y}}$, $y > 0$, $x \in [0, \pi]$, then find the least positive value of x satisfying the given condition.

5. Find the number of solution of the equation $\sin5x\cos3x = \sin9x\cos7x$ in $\left[0, \frac{\pi}{4}\right]$.

6. Find number of real roots of the equation $\sec\theta + \operatorname{cosec}\theta = \sqrt{15}$ lying between 0 and π.

7. If $\cos 3x + \sin\left(2x - \frac{7\pi}{6}\right) = -2$ then x is equal to, $(k \in z)$

(A) $\frac{\pi}{3}(6k+1)$ (B) $\frac{\pi}{3}(6k-1)$ (C) $\frac{\pi}{3}(2k+1)$ (D) None of these

8. Find general solution of the equation $\sin^8 x + \cos^8 x = \frac{17}{32}$.

5.0 TRIGONOMETRIC INEQUALITIES

AL

There is no general rule to solve trigonometric inequations and the same rules of algebra are valid provided the domain and range of trigonometric functions should be kept in mind.

Illustrations

Illustration 16. Find the solution set of inequality $\sin x > 1/2$.

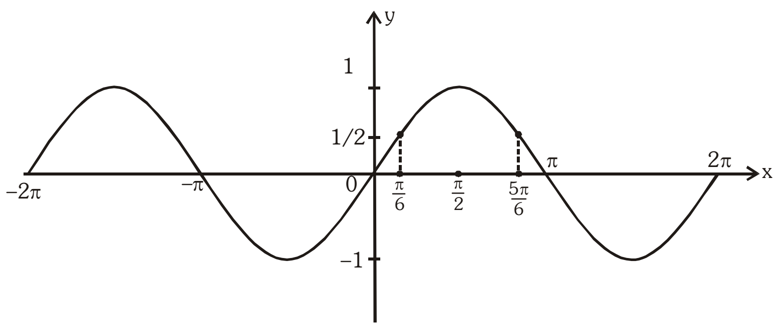
Solution When $\sin x = \frac{1}{2}$, the two values of x between 0 and 2π are $\pi/6$ and $5\pi/6$.

From the graph of $y = \sin x$, it is obvious that between 0 and 2π,

$\sin x > \frac{1}{2}$ for $\pi/6 < x < 5\pi/6$

Hence, $\sin x > 1/2$

$\Rightarrow \quad 2n\pi + \pi/6 < x < 2n\pi + 5\pi/6, n \in I$



Thus, the required solution set is $\bigcup\limits_{n\in I}\left(2n\pi + \frac{\pi}{6}, 2n\pi + \frac{5\pi}{6}\right)$ ***Ans.***

***Illustration 17.** Find the value of x in the interval $\left[-\frac{\pi}{2},\frac{3\pi}{2}\right]$ for which $\sqrt{2}\sin 2x+1\le 2\sin x+\sqrt{2}\cos x$

Solution We have, $\sqrt{2}\sin 2x+1\le 2\sin x+\sqrt{2}\cos x$

$\Rightarrow \quad 2\sqrt{2}\sin x\cos x-2\sin x-\sqrt{2}\cos x+1\le 0$

$\Rightarrow \quad 2\sin x(\sqrt{2}\cos x-1)-1(\sqrt{2}\cos x-1)\le 0$

$\Rightarrow \quad (2\sin x-1)(\sqrt{2}\cos x-1)\le 0$

$\Rightarrow \quad \left(\sin x-\frac{1}{2}\right)\left(\cos x-\frac{1}{\sqrt{2}}\right)\le 0$

Above inequality holds when :

Case-I – $\sin x-\frac{1}{2}\le 0$ and $\cos x-\frac{1}{\sqrt{2}}\ge 0$

$\Rightarrow \quad \sin x\le\frac{1}{2}$ and $\cos x\ge\frac{1}{\sqrt{2}}$

Now considering the given interval of x :

for $\sin x\le\frac{1}{2}$: $x\in\left[-\frac{\pi}{2},\frac{\pi}{6}\right]\cup\left[\frac{5\pi}{6},\frac{3\pi}{2}\right]$

and for $\cos x\ge\frac{1}{\sqrt{2}}$: $x\in\left[-\frac{\pi}{4},\frac{\pi}{4}\right]$

For both to simultaneously hold true : $x\in\left[-\frac{\pi}{4},\frac{\pi}{6}\right]$

Case-II – $\sin x-\frac{1}{2}\ge 0$ and $\cos x\le\frac{1}{\sqrt{2}}$

Again, for the given interval of x :

for $\sin x\ge\frac{1}{2}$: $x\in\left[\frac{\pi}{6},\frac{5\pi}{6}\right]$

and for $\cos x\le\frac{1}{\sqrt{2}}$: $x\in\left[-\frac{\pi}{2},-\frac{\pi}{4}\right]\cup\left[\frac{\pi}{4},\frac{3\pi}{2}\right]$

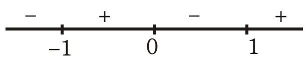
For both to simultaneously hold true : $x\in\left[\frac{\pi}{4},\frac{5\pi}{6}\right]$

$\therefore$ Given inequality holds for $x\in\left[-\frac{\pi}{4},\frac{\pi}{6}\right]\cup\left[\frac{\pi}{4},\frac{5\pi}{6}\right]$ ***Ans.***

Illustration 18. Find the values of α lying between 0 and π for which the inequality : $\tan\alpha>\tan^3\alpha$ is valid.

Solution We have : $\tan\alpha-\tan^3\alpha>0\Rightarrow\tan\alpha\ (1-\tan^2\alpha)>0$

$\Rightarrow \quad (\tan\alpha)(\tan\alpha+1)(\tan\alpha-1)<0$



So $\tan\alpha<-1,\ 0<\tan\alpha<1$

$\therefore$ Given inequality holds for $\alpha\in\left(0,\frac{\pi}{4}\right)\cup\left(\frac{\pi}{2},\frac{3\pi}{4}\right)$ ***Ans.***

BEGINNER'S BOX-4

TOPIC COVERED : TRIGONOMETRIC INEQUALITIES

1. Find the solution set of the inequality : $\cos x \geq -1/2$.

2. Find the values of x in the interval $[0, 2\pi]$ for which $4\sin^2 x - 8\sin x + 3 \leq 0$.

3. If $0 \leq x \leq 2x$ and $|\cos x| \leq \sin x$, then

(A) $x \in \left[0, \frac{\pi}{4}\right]$ (B) $x \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right]$ (C) $x \in \left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$ (D) None of these

4. The solution of $\log_{1/2} \sin\theta > \log_{1/2} \cos\theta$ in $[0, 2\pi]$ is

(A) $\left(0, \frac{\pi}{2}\right)$ (B) $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$ (C) $\left(0, \frac{\pi}{4}\right)$ (D) None of these

5. If $3(1 + \sin x) \geq 1 + \cos 2x$, $x \in [0, \pi]$ then the number of value of x is

(A) 0 (B) 1 (C) 2 (D) infinite

6. The number of values of x in $[0, 2\pi]$ satisfying $|\cos x - \sin x| \geq \sqrt{2}$ is

(A) 0 (B) 1 (C) 2 (D) 3

7. Find number of solutions of $\operatorname{cosec} x \leq 1$ in $[0, \pi]$.

****8.*** $\left|1 - \frac{|\sin x|}{1 + |\sin x|}\right| \geq \frac{2}{3}$, then set of all possible values of sin x is

(A) $\left(-\infty, -\frac{1}{2}\right] \cup \left[\frac{1}{2}, \infty\right)$ (B) $\left(-\frac{1}{2}, \frac{1}{2}\right)$ (C) $\left[-\frac{1}{2}, \frac{1}{2}\right]$ (D) None of these

GOLDEN KEY POINTS

- For equations of the type $\sin\theta = k$ or $\cos\theta = k$, one must check that $|k| \leq 1$.
- Avoid squaring the equations, if possible, because it may lead to extraneous solutions. Reject extra solutions if they do not satisfy the given equation.
- Do not cancel the common variable factor from the two sides of the equations which are in a product because we may loose some solutions.
- The answer should not contain such values of θ, which make any of the terms undefined or infinite.
  (i) Check that denominator is not zero at any stage while solving equations.
  (ii) If $\tan\theta$ or $\sec\theta$ is involved in the equations, θ should not be odd multiple of $\frac{\pi}{2}$.
  (iii) If $\cot\theta$ or $\operatorname{cosec}\theta$ is involved in the equation, θ should not be multiple of π or 0.

SOME WORKED OUT EXAMPLES

Illustration 1. Solve the following equation : $\tan^2\theta + \sec^2\theta + 3 = 2(\sqrt{2}\sec\theta + \tan\theta)$

Solution We have $\tan^2\theta + \sec^2\theta + 3 = 2\sqrt{2}\sec\theta + 2\tan\theta$

$\Rightarrow \quad \tan^2\theta - 2\tan\theta + \sec^2\theta - 2\sqrt{2}\sec\theta + 3 = 0$

$\Rightarrow \quad \tan^2\theta + 1 - 2\tan\theta + \sec^2\theta - 2\sqrt{2}\sec\theta + 2 = 0$

$\Rightarrow \quad (\tan\theta - 1)^2 + (\sec\theta - \sqrt{2})^2 = 0$

$\Rightarrow \quad \tan\theta = 1$ and $\sec\theta = \sqrt{2}$

As the periodicity of $\tan\theta$ and $\sec\theta$ are not same, we get

$$\theta = 2n\pi + \frac{\pi}{4}, n \in I$$

Ans.

Illustration 2. Find the solution set of equation $5^{(1 + \log_5 \cos x)} = 5/2$.

Solution Taking log to base 5 on both sides in given equation :

$(1 + \log_5 \cos x).\ \log_5 5 = \log_5(5/2)$

$\Rightarrow \quad \log_5 5 + \log_5 \cos x = \log_5 5 - \log_5 2$

$\Rightarrow \quad \log_5 \cos x = -\log_5 2$

$\Rightarrow \quad \cos x = 1/2$

$\Rightarrow \quad x = 2n\pi \pm \pi/3,\ n \in I$ ***Ans.***

Illustration 3. If the set of all values of x in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ satisfying $|4\sin x + \sqrt{2}| < \sqrt{6}$ is $\left(\frac{a\pi}{24}, \frac{b\pi}{24}\right)$ then find the value of $\left|\frac{a-b}{3}\right|$.

Solution $|4\sin x + \sqrt{2}| < \sqrt{6}$

$\Rightarrow \quad -\sqrt{6} < 4\sin x + \sqrt{2} < \sqrt{6}$

$\Rightarrow \quad -\sqrt{6} - \sqrt{2} < 4\sin x < \sqrt{6} - \sqrt{2}$

$\Rightarrow \quad \frac{-(\sqrt{6}+\sqrt{2})}{4} < \sin x < \frac{\sqrt{6}-\sqrt{2}}{4}$

$\Rightarrow \quad -\frac{5\pi}{12} < x < \frac{\pi}{12}$ for $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Comparing with $\frac{a\pi}{24} < x < \frac{b\pi}{24}$, we get, $a = -10$, $b = 2$

$\therefore \quad \left|\frac{a-b}{3}\right| = \left|\frac{-10-2}{3}\right| = 4$ ***Ans.***

****Illustration 4.*** Find the values of x in the interval $[0, 2\pi]$ which satisfy the inequality :

$3|2\sin x - 1| \geq 3 + 4\cos^2 x$.

Solution The given inequality can be written as :

$3|2\sin x - 1| \geq 3 + 4(1 - \sin^2 x)$

$\Rightarrow \quad 3|2\sin x - 1| \geq 7 - 4\sin^2 x$

Let $\sin x = t \Rightarrow 3|2t - 1| \geq 7 - 4t^2$

Case I – For $2t - 1 \geq 0$ i.e. $t \geq 1/2$ we have, $|2t - 1| = (2t - 1)$

$\Rightarrow$ $3(2t - 1) \geq 7 - 4t^2$ $\Rightarrow$ $6t - 3 \geq 7 - 4t^2$

$\Rightarrow$ $4t^2 + 6t - 10 \geq 0$ $\Rightarrow$ $2t^2 + 3t - 5 \geq 0$

$\Rightarrow$ $(t-1)(2t + 5) \geq 0$ $\Rightarrow$ $t \leq -\frac{5}{2}$ and $t \geq 1$

Now for $t \geq \frac{1}{2}$, we get $t \geq 1$ from above conditions i.e. $\sin x \geq 1$

The inequality holds true only for x satisfying the equation $\sin x = 1$ $\therefore$ $x = \frac{\pi}{2}$ (for $x \in [0,2\pi]$)

Case II – For $2t - 1 < 0$ $\Rightarrow$ $t < \frac{1}{2}$

we have, $|2t - 1| = -(2t - 1)$

$\Rightarrow$ $-3(2t - 1) \geq 7 - 4t^2$ $\Rightarrow$ $-6t + 3 \geq 7 - 4t^2$

$\Rightarrow$ $4t^2 - 6t - 4 \geq 0$ $\Rightarrow$ $2t^2 - 3t - 2 \geq 0$

$\Rightarrow$ $(t - 2)(2t + 1) \geq 0$ $\Rightarrow$ $t \leq -\frac{1}{2}$ and $t \geq 2$

Again, for $t < \frac{1}{2}$ we get $t \leq -\frac{1}{2}$ from above conditions

i.e. $\sin x \leq -\frac{1}{2}$ $\Rightarrow$ $\frac{7\pi}{6} \leq x \leq \frac{11}{6}\pi$ (for $x \in [0,2\pi]$)

Thus, $x \in \left[\frac{7\pi}{6}, \frac{11\pi}{6}\right] \cup \left\{\frac{\pi}{2}\right\}$ **Ans.**

Illustration 5. Find the values of θ, for which $\cos 3\theta + \sin 3\theta + (2\sin 2\theta - 3)(\sin\theta - \cos\theta)$ is always positive.

Solution Given expression can be written as :

$4\cos^3\theta - 3\cos\theta + 3\sin\theta - 4\sin^3\theta + (2\sin 2\theta - 3)(\sin\theta - \cos\theta)$

Applying given condition, we get

$\Rightarrow$ $-4(\sin^3\theta - \cos^3\theta) + 3(\sin\theta - \cos\theta) + (\sin\theta - \cos\theta)(2\sin 2\theta - 3) > 0$

$\Rightarrow$ $-4(\sin\theta - \cos\theta)(\sin^2\theta + \cos^2\theta + \sin\theta\cos\theta) + 3(\sin\theta - \cos\theta) + (\sin\theta - \cos\theta)(2\sin 2\theta - 3) > 0$

$\Rightarrow$ $-4(\sin\theta - \cos\theta)(1 + \sin\theta\cos\theta) + 3(\sin\theta - \cos\theta) + (\sin\theta - \cos\theta)(4\sin\theta\cos\theta - 3) > 0$

$\Rightarrow$ $(\sin\theta - \cos\theta)\{-4 - 4\sin\theta\cos\theta + 3 + 4\sin\theta\cos\theta - 3\} > 0$

$\Rightarrow$ $-4(\sin\theta - \cos\theta) > 0$

$\Rightarrow$ $-4\sqrt{2}\sin\left(\theta - \frac{\pi}{4}\right) > 0$ $\Rightarrow$ $\sin\left(\theta - \frac{\pi}{4}\right) < 0$

$\Rightarrow$ $2n\pi - \pi < \theta - \frac{\pi}{4} < 2n\pi,\ n \in I$

$\Rightarrow$ $2n\pi - \frac{3\pi}{4} < \theta < 2n\pi + \frac{\pi}{4}$

$\Rightarrow \theta \in \left(2n\pi - \frac{3\pi}{4}, 2n\pi + \frac{\pi}{4}\right),\ n \in I$ **Ans.**

ANSWERS

BEGINNER'S BOX-1

1. (a)-(r); (b)-(p,r); (c)-(q); (d)-(s) **2.** 6 **3.** (C) **4.** (A) **5.** (C) **6.** (C)

BEGINNER'S BOX-2

1. ***(a)*** $\theta = n\pi + (-1)^n \frac{\pi}{6}, n \in I$; ***(b)*** $\theta = (2n+1)\frac{\pi}{3}, n \in I$ ***(c)*** $\theta = \frac{4n\pi}{3}, n \in I$

(d) $\theta = \frac{n\pi}{2}, n \in I$; ***(e)*** $\theta = n\pi \pm \frac{\pi}{12}, n \in I$; ***(f)*** $\theta = 2n\pi + (-1)^{n+1}\pi$, $n \in I$

2. ***(a)*** $x = n\pi + (-1)^{n+1}\frac{\pi}{6}$, $n \in I$ ***(b)*** $\alpha = \frac{n\pi}{2}$ or $\alpha = \frac{k\pi}{2} + \frac{3\pi}{8}$, $n,k \in I$

(c) $\theta = n\pi \pm \frac{\pi}{3}, n \in I$ ***(d)*** $\theta = n\pi + (-1)^n\alpha$, where $\alpha = \sin^{-1}\left(\frac{\sqrt{17}-1}{8}\right)$

or $\sin^{-1}\left(\frac{-1-\sqrt{17}}{8}\right)$, $n \in I$

3. $\theta = \left\{-\frac{\pi}{4}, -\frac{3\pi}{4}, -\frac{\pi}{2}, \frac{\pi}{4}, \frac{3\pi}{4}, \frac{\pi}{2}\right\}$

4. ***(a)*** $x = 2n\pi - \frac{\pi}{4}, n \in I$; ***(b)*** $2m\pi + \frac{\pi}{2}, m \in I$

5. 0 **6.** Infinite **7.** $2n\pi \pm \frac{\pi}{4} + \frac{\pi}{12}$ **8.** (A)

BEGINNER'S BOX-3

1. $\theta = n\pi$ or $\theta = \frac{m\pi}{3} \pm \frac{\pi}{9}$; $n,m \in I$ **2.** $x = \frac{n\pi}{3}, n \in I$ and $k\pi \pm \frac{\pi}{3}, k \in I$

3. D **4.** $x = \frac{\pi}{4}$ **5.** 5 **6.** 4 **7.** (A) **8.** $\frac{n\pi}{2} \pm \frac{\pi}{8}$

BEGINNER'S BOX-4

1. $\bigcup_{n\in I}\left[2n\pi - \frac{2\pi}{3}, 2n\pi + \frac{2\pi}{3}\right]$ **2.** $\left[\frac{\pi}{6}, \frac{5\pi}{6}\right]$ **3.** (C) **4.** (C)

5. (A) **6.** (C) **7.** 1 **8.** (C)

EXERCISE – 1 MCQ (SINGLE CHOICE CORRECT)

1. The number of solutions of the equation $\frac{\sec x}{1-\cos x}=\frac{1}{1-\cos x}$ in $[0, 2\pi]$ is equal to -

(A) 3 (B) 2 (C) 1 (D) 0

2. The number of solutions of equation $2 + 7\tan^2\theta = 3.25 \sec^2\theta (0° < \theta < 360°)$ is -

(A) 2 (B) 4 (C) 6 (D) 8

3. The number of solutions of the equation $\tan^2 x - \sec^{10} x + 1 = 0$ in $(0, 10)$ is -

(A) 3 (B) 6 (C) 10 (D) 11

4. If $(\cos\theta + \cos 2\theta)^3 = \cos^3\theta + \cos^3 2\theta$, then the least positive value of θ is equal to -

(A) $\frac{\pi}{6}$ (B) $\frac{\pi}{4}$ (C) $\frac{\pi}{3}$ (D) $\frac{\pi}{2}$

5. The number of solution(s) of $\sin 2x + \cos 4x = 2$ in the interval $(0, 2\pi)$ is -

(A) 0 (B) 2 (C) 3 (D) 4

6. The complete solution of the equation $7\cos^2 x + \sin x \cos x - 3 = 0$ is given by -

(A) $n\pi+\frac{\pi}{2}; (n \in I)$ (B) $n\pi-\frac{\pi}{4}; (n \in I)$

(C) $n\pi+\tan^{-1}\frac{4}{3}; \ (n \in I)$ (D) $n\pi+\frac{3\pi}{4}, k\pi+\tan^{-1}\frac{4}{3}; (n,k \in I)$

7. If $\cos(\sin x) = 0$, then x lies in -

(A) $\left(\frac{\pi}{4},\frac{\pi}{2}\right)\cup\left(\frac{\pi}{2},\pi\right)$ (B) $\left(-\frac{\pi}{4},0\right)$ (C) $\left(\pi,\frac{3\pi}{2}\right)$ (D) null set

8. If $0 \le \alpha, \beta \le 90°$ and $\tan(\alpha+\beta) = 3$ and $\tan(\alpha-\beta) = 2$ then value of $\sin 2\alpha$ is -

(A) $-\frac{1}{\sqrt{2}}$ (B) $\frac{1}{\sqrt{2}}$ (C) $\frac{1}{2}$ (D) none of these

9. If tanA and tanB are the roots of $x^2 - 2x - 1 = 0$, then $\sin^2(A+B)$ is -

(A) 1 (B) $\frac{1}{\sqrt{2}}$ (C) $\frac{1}{2}$ (D) 0

***10.** If $\cos 2x - 3\cos x + 1 = \frac{\operatorname{cosec} x}{\cot x - \cot 2x}$, then which of the following is true ?

(A) $x=(2n+1)\frac{\pi}{2}, n \in I$ (B) $x = 2n\pi, n \in I$

(C) $x=2n\pi\pm\cos^{-1}\left(\frac{2}{5}\right), n \in I$ (D) no real x

11. If $x \in \left[-\frac{5\pi}{2}, \frac{5\pi}{2}\right]$, then the greatest positive solution of $1 + \sin^4 x = \cos^2 3x$ is -

(A) π (B) 2π (C) $\frac{5\pi}{2}$ (D) none of these

***12.** Number of values of 'x' in $(-2\pi, 2\pi)$ satisfying the equation $2^{\sin^2 x} + 4.2^{\cos^2 x} = 6$ is -

(A) 8 (B) 6 (C) 4 (D) 2

13. General solution for $|\sin x| = \cos x$ is -

(A) $2n\pi + \frac{\pi}{4}, n \in I$ (B) $2n\pi \pm \frac{\pi}{4}, n \in I$ (C) $n\pi + \frac{\pi}{4}, n \in I$ (D) none of these

14. The most general solution of $\tan\theta = -1, \cos\theta = \frac{1}{\sqrt{2}}$ is -

(A) $n\pi + \frac{7\pi}{4}, n \in I$ (B) $n\pi + (-1)^n \frac{7\pi}{4}, n \in I$ (C) $2n\pi + \frac{7\pi}{4}, n \in I$ (D) none of these

15. The solutions set of $(2\cos x - 1)(3 + 2\cos x) = 0$ in the interval $0 \le x \le 2\pi$ is :

(A) $\left\{\frac{\pi}{3}\right\}$ (B) $\left\{\frac{\pi}{3}, \frac{5\pi}{3}\right\}$ (C) $\left\{\frac{\pi}{3}, \frac{5\pi}{3}, \cos^{-1}\left(\frac{-3}{2}\right)\right\}$ (D) none of these

16. The number of solutions of $2\cos(x/2) = 3^x + 3^{-x}, x \in [0, 2\pi]$ is :

(A) 0 (B) 1 (C) 2 (D) infinite

17. If $\sin\theta + 7\cos\theta = 5$, then $\tan(\theta/2)$ is a root of the equation :

(A) $x^2 - 6x + 1 = 0$ (B) $6x^2 - x - 1 = 0$ (C) $6x^2 + x + 1 = 0$ (D) $x^2 - x + 6 = 0$

18. The most general solution of $\tan\theta = -1$ and $\cos\theta = \frac{1}{\sqrt{2}}$ is :

(A) $n\pi + \frac{7\pi}{4}, n \in I$ (B) $n\pi + (-1)^n \frac{7\pi}{4}, n \in I$ (C) $2n\pi + \frac{7\pi}{4}, n \in I$ (D) none of these

19. A triangle ABC is such that $\sin(2A + B) = \frac{1}{2}$. If A, B, C are in A.P., then the angle A, B, C are respectively :

(A) $\frac{5\pi}{12}, \frac{\pi}{4}, \frac{\pi}{3}$ (B) $\frac{\pi}{4}, \frac{\pi}{3}, \frac{5\pi}{12}$ (C) $\frac{\pi}{3}, \frac{\pi}{4}, \frac{5\pi}{12}$ (D) $\frac{\pi}{3}, \frac{5\pi}{12}, \frac{\pi}{4}$

20. The number of solutions of the equation $|\cot x| = \cot x + \frac{1}{\sin x} (0 \le x \le 2\pi)$ is :

(A) 0 (B) 1 (C) 2 (D) 3

21. The number of solutions of the equation $\sin^3 x \cos x + \sin^2 x \cos^2 x + \sin x \cos^3 x = 1$ in the interval $[0, 2\pi]$ is :

(A) 0 (B) 2 (C) 3 (D) infinite

22. The set of values of x for which $\frac{\tan 3x - \tan 2x}{1 + \tan 3x \tan 2x} = 1$, is :

(A) ϕ (B) $\left\{\frac{\pi}{4}\right\}$ (C) $\left\{n\pi + \frac{\pi}{4}, n \in I\right\}$ (D) $\left\{2n\pi + \frac{\pi}{4}, n \in I\right\}$

23. The value of a for which the equation $4\text{cosec}^2 (\pi (a + x)) + a^2 - 4a = 0$ has a real solution, is :

(A) $a = 1$ (B) $a = 2$ (C) $a = 10$ (D) none of these

24. If $\left(\frac{1 + \sin\theta - \cos\theta}{1 + \sin\theta + \cos\theta}\right)^2 = \lambda\left(\frac{1 - \cos\theta}{1 + \cos\theta}\right)$, then λ equals :

(A) –1 (B) 1 (C) 2 (D) –2

25. The number of solution(s) of the equation $\cos 2\theta = \left(\sqrt{2} + 1\right)\left(\cos\theta - \frac{1}{\sqrt{2}}\right)$, in the interval $\left(-\frac{\pi}{4}, \frac{3\pi}{4}\right)$, is -

(A) 4 (B) 1 (C) 2 (D) 3

EXERCISE – 2 MCQ (ONE OR MORE CHOICE CORRECT)

Select the correct alternatives (one or more than one correct answers)

1. The solution(s) of the equation cos2x sin6x = cos3x sin5x in the interval $[0, \pi]$ is/are -

(A) $\frac{\pi}{6}$ (B) $\frac{\pi}{2}$ (C) $\frac{2\pi}{3}$ (D) $\frac{5\pi}{6}$

2. The equation $4\sin^2 x - 2(\sqrt{3}+1)\sin x + \sqrt{3} = 0$ has -

(A) 2 solutions in $(0, \pi)$ (B) 4 solutions in $(0, 2\pi)$ (C) 2 solutions in $(-\pi, \pi)$ (D) 4 solutions in $(-\pi, \pi)$

3. If $\cos^2 2x + 2\cos^2 x = 1$, $x \in (-\pi, \pi)$, then x can take the values -

(A) $\pm\frac{\pi}{2}$ (B) $\pm\frac{\pi}{4}$ (C) $\pm\frac{3\pi}{4}$ (D) none of these

4. The solution(s) of the equation sin 7x + cos 2x = –2 is/are -

(A) $x = \frac{2k\pi}{7} + \frac{3\pi}{14}, k \in I$ (B) $x = n\pi + \frac{\pi}{4}, n \in I$ (C) $x = 2n\pi + \frac{\pi}{2}, n \in I$ (D) none of these

5. Set of values of x in $(-\pi, \pi)$ for which $|4\sin x - 1| < \sqrt{5}$ is given by -

(A) $\left(\frac{\pi}{10}, \frac{3\pi}{10}\right)$ (B) $\left(-\frac{\pi}{10}, \frac{3\pi}{10}\right)$ (C) $\left(\frac{\pi}{10}, -\frac{3\pi}{10}\right)$ (D) $\left(-\frac{\pi}{10}, -\frac{3\pi}{10}\right)$

6. If $\cos^2 x + \cos^2 2x + \cos^2 3x = 1$ then -

(A) $x = (2n + 1)\frac{\pi}{4}, n \in I$ (B) $x = (2n + 1)\frac{\pi}{2}, n \in I$

(C) $x = n\pi \pm \frac{\pi}{6}, n \in I$ (D) none of these

7. If $4\cos^2\theta + \sqrt{3} = 2(\sqrt{3} + 1)\cos\theta$, then θ is -

(A) $2n\pi \pm \frac{\pi}{3}, n \in I$ (B) $2n\pi \pm \frac{\pi}{4}, n \in I$ (C) $2n\pi \pm \frac{\pi}{6}, n \in I$ (D) none of these

***8.** If $(a + 2)\sin\alpha + (2a - 1)\cos\alpha = (2a + 1)$, then $\tan\alpha$ =

(A) 3/4 (B) 4/3 (C) $\frac{2a}{a^2+1}$ (D) $\frac{2a}{a^2-1}$

9. The value(s) of θ lying between 0 & 2π satisfying the equation : $r\sin\theta = \sqrt{3}$ & $r + 4\sin\theta = 2(\sqrt{3} + 1)$ is/are -

(A) $\frac{\pi}{6}$ (B) $\frac{\pi}{3}$ (C) $\frac{2\pi}{3}$ (D) $\frac{5\pi}{6}$

10. The solution(s) of $4\cos^2 x\sin x - 2\sin^2 x = 3\sin x$ is/are -

(A) $n\pi$; $n \in I$

(B) $n\pi + (-1)^n \frac{\pi}{10}$; $n \in I$

(C) $n\pi + (-1)^n\left(-\frac{3\pi}{10}\right)$; $n \in I$

(D) none of these

11. Using four values of θ satisfying the equation $8\cos^4\theta + 15\cos^2\theta - 2 = 0$ in the interval $(0,4\pi)$, an arithmetic progression is formed, then :

(A) The common difference of A.P. may be π.

(B) The common difference of A.P. may be 2π.

(C) Two such different A.P. can be formed.

(D) Four such different A.P. can be formed.

Match the Column

Following question contains statements given in two columns, which have to be matched. The statements in ***Column-I*** are labelled as A, B, C and D while the statements in ***Column-II*** are labelled as p, q, r and s. Any given statement in ***Column-I*** can have correct matching with ***ONE*** statement in ***Column-II***.

***12.** On the left, equation with interval is given and on the right number of solutions are given, match the column.

	Column-I		***Column-II***
(A)	$n\lvert\sin x\rvert = m\lvert\cos x\rvert$ in $[0, 2\pi]$ where $n > m$ and are positive integers	(p)	2
(B)	$\sum_{r=1}^{5} \cos rx = 5$ in $[0,2\pi]$	(q)	4
(C)	$2^{1+\lvert\cos x\rvert+\lvert\cos x\rvert^2 \ldots\ldots\infty} = 4$ in $(-\pi, \pi)$	(r)	3
(D)	$\tan\theta + \tan 2\theta + \tan 3\theta = \tan\theta\tan 2\theta\tan 3\theta$ in $(0, \pi)$	(s)	1

Comprehension Based Questions

Consider $\cos^n x - \sin^n x = 1$, where n is a natural number and $-\pi < x \le \pi$.

On the basis of above information, answer the following questions

13. When $n = 1$, the sum of the values of x satisfying the equation is

(A) $-\frac{\pi}{2}$ (B) 0 (C) $\frac{\pi}{4}$ (D) $\frac{\pi}{2}$

14. When n is an even natural number then the value of x is

(A) $\frac{\pi}{3}$ (B) $\frac{\pi}{2}$ (C) $\frac{2\pi}{3}$ (D) π

15. When n is an odd natural number other than 1, then the value of x is

(A) $-\pi$ (B) 0 (C) π (D) 3π

EXERCISE - 3 SUBJECTIVE

1. If $\sin A = \sin B$ & $\cos A = \cos B$, find the values of A in terms of B.

2. Solve the equation : $1 + 2\operatorname{cosec} x = -\dfrac{\sec^2 \frac{x}{2}}{2}$.

3. Solve the equation : $\dfrac{\sqrt{3}}{2}\sin x - \cos x = \cos^2 x$.

4. Solve the equation : $\cot x - 2\sin 2x = 1$.

***5.** If α & β satisfy the equation, $a\cos 2\theta + b\sin 2\theta = c$ then prove that : $\cos^2\alpha + \cos^2\beta = \dfrac{a^2+ac+b^2}{a^2+b^2}$.

6. Solve for x, $\sqrt{13-18\tan x} = 6\tan x - 3$, where $-2\pi < x < 2\pi$.

7. Find all the values of θ satisfying the equation : $\sin\theta + \sin 5\theta = \sin 3\theta$ such that $0 \le \theta \le \pi$.

8. Solve : $\cot\theta + \operatorname{cosec}\theta = \sqrt{3}$ for values of θ between $0°$ & $360°$.

9. Solve : $\sin 5x = \cos 2x$ for all values of x between $0°$ & $180°$.

10. Solve the equation : $(1 - \tan\theta)(1 + \sin 2\theta) = 1 + \tan\theta$.

11. Find the general solution of $\sec 4\theta - \sec 2\theta = 2$.

12. Solve the equation : $\cos 3x.\cos^3 x + \sin 3x.\sin^3 x = 0$.

***13.** Solve the inequality : $\sin 3x < \sin x$.

14. Solve the inequality : $\tan^2 x - (\sqrt{3} + 1)\tan x + \sqrt{3} < 0$.

EXERCISE – 4 RECAP OF AIEEE/JEE (MAIN)

1. The number of values of x in the interval $[0, 5\pi]$ satisfying the equation $3\sin^2 x - 7\sin x + 2 = 0$ is - ***[JEE 98]***

(A) 0 (B) 5 (C) 6 (D) 10

2. General solution of $\tan 5\theta = \cot 2\theta$ is- ***[AIEEE 2002]***

(A) $\theta = \frac{n\pi}{7} + \frac{\pi}{14}$ (B) $\theta = \frac{n\pi}{7} + \frac{\pi}{5}$ (C) $\theta = \frac{n\pi}{7} + \frac{\pi}{2}$ (D) $\theta = \frac{n\pi}{7} + \frac{\pi}{3}, n \in Z$

3. The number of values of x in the interval $[0,3\pi]$ satisfying the equation $2\sin^2 x + 5\sin x - 3 = 0$ is-

(A) 6 (B) 1 (C) 2 (D) 4 ***[AIEEE 2006]***

4. If $0 < x < \pi$, and $\cos x + \sin x = \frac{1}{2}$, then tan x is - ***[AIEEE 2006]***

(A) $\frac{(4-\sqrt{7})}{3}$ (B) $\frac{-(4+\sqrt{7})}{3}$ (C) $\frac{(1+\sqrt{7})}{4}$ (D) $\frac{(1-\sqrt{7})}{4}$

***5.** Let A and B denote the statements

A : $\cos\alpha + \cos\beta + \cos\gamma = 0$

B : $\sin\alpha + \sin\beta + \sin\gamma = 0$

If $\cos(\beta-\gamma) + \cos(\gamma-\alpha) + \cos(\alpha-\beta) = -\frac{3}{2}$, then :- ***[AIEEE 2009]***

(A) Both ***A*** and ***B*** are true (B) Both ***A*** and ***B*** are false

(C) ***A*** is true and ***B*** is false (D) ***A*** is false and ***B*** is true

6.** The possible values of $\theta \in (0, \pi)$ such that $\sin(\theta) + \sin(4\theta) + \sin(7\theta) = 0$ are: ***[AIEEE 2011]

(A) $\frac{2\pi}{9}, \frac{\pi}{4}, \frac{4\pi}{9}, \frac{\pi}{2}, \frac{3\pi}{4}, \frac{8\pi}{9}$ (B) $\frac{\pi}{4}, \frac{5\pi}{12}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{3\pi}{4}, \frac{8\pi}{9}$

(C) $\frac{2\pi}{9}, \frac{\pi}{4}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{3\pi}{4}, \frac{35\pi}{36}$ (D) $\frac{2\pi}{9}, \frac{\pi}{4}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{3\pi}{4}, \frac{8\pi}{9}$

7. If $0 \le x < \frac{\pi}{2}$, then the number of values of x for 2 which $\sin x - \sin 2x + \sin 3x = 0$, is **[JEE-MAIN(2019)]**

(A) 2 (B) 1 (C) 3 (D) 4

8. The sum of all values of $\theta \in \left(0, \frac{\pi}{2}\right)$ satisfying $\sin^2 2\theta + \cos^4 2\theta = \frac{3}{4}$ is **[JEE-MAIN(2019)]**

(A) $\frac{\pi}{2}$ (B) π (C) $\frac{3\pi}{8}$ (D) $\frac{5\pi}{4}$

9. If $\sin^4\alpha + 4\cos^4\beta + 2 = 4\sqrt{2}\sin\alpha\cos\beta$; $\alpha, \pi \in [0, \pi]$, then $\cos(\alpha+\beta) - \cos(\alpha-\beta)$ is equal to :

[JEE-MAIN(2019)]

(A) 0 (B) $-\sqrt{2}$ (C) -1 (D) $\sqrt{2}$

10. The maximum value of $3\cos\theta + 5\sin\left(\theta - \frac{\pi}{6}\right)$ for any real value of θ is : **[JEE-MAIN(2019)]**

(A) $\sqrt{19}$ (B) $\frac{\sqrt{79}}{2}$ (C) $\sqrt{31}$ (D) $\sqrt{34}$

11. All the pairs (x, y) that satisfy the inequality $2\sqrt{\sin^2 x - 2\sin x + 5} \cdot \frac{1}{4^{\sin^2 y}} \le 1$ also satisfy the eauation.

[JEE-MAIN(2019)]

(A) $\sin x = |\sin y|$ (B) $\sin x = 2\sin y$ (C) $2|\sin x| = 3\sin y$ (D) $2\sin x = \sin y$

12. The number of solutions of the equation $1 + \sin^4 x = \cos^2 3x,\ x \in \left[-\frac{5\pi}{2}, \frac{5\pi}{2}\right]$ is : **[JEE-MAIN(2019)]**

(A) 5 (B) 4 (C) 7 (D) 3

13. Let S be the set of all $\alpha \in R$ such that the equation, $\cos 2x + \alpha\sin x = 2\alpha - 7$ has a solution. Then S is equal to : **[JEE-MAIN(2019)]**

(A) [2, 6] (B) [3,7] (C) R (D) [1,4]

EXERCISE – 5 RECAP OF IIT-JEE/JEE (ADVANCED)

1. If $0 < \theta < 2\pi$, then the intervals of values of θ for which $2\sin^2\theta - 5\sin\theta + 2 > 0$, is **[JEE 2006, 3]**

(A) $\left(0, \frac{\pi}{6}\right) \cup \left(\frac{5\pi}{6}, 2\pi\right)$ (B) $\left(\frac{\pi}{8}, \frac{5\pi}{6}\right)$ (C) $\left(0, \frac{\pi}{8}\right) \cup \left(\frac{\pi}{6}, \frac{5\pi}{6}\right)$ (D) $\left(\frac{41\pi}{48}, \pi\right)$

2. The number of solutions of the pair of equations
$2\sin^2\theta - \cos 2\theta = 0$
$2\cos^2\theta - 3\sin\theta = 0$
in the interval $[0, 2\pi]$ is **[JEE 2007, 3]**
(A) zero (B) one (C) two (D) four

3. The number of values of θ in the interval $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$ such that $\theta \neq \frac{n\pi}{5}$ for $n = 0, \pm1, \pm2$ and $\tan\theta = \cot 5\theta$ as well as $\sin 2\theta = \cos 4\theta$, is **[JEE 2010, 3]**

4. The positive integer value of $n > 3$ satisfying the equation

$$\frac{1}{\sin\left(\frac{\pi}{n}\right)} = \frac{1}{\sin\left(\frac{2\pi}{n}\right)} + \frac{1}{\sin\left(\frac{3\pi}{n}\right)}$$ is **[JEE 2011, 4]**

5. Let $\theta, \varphi \in [0, 2\pi]$ be such that

$$2\cos\theta(1 - \sin\varphi) = \sin^2\theta\left(\tan\frac{\theta}{2} + \cot\frac{\theta}{2}\right)\cos\varphi - 1, \ \tan(2\pi - \theta) > 0 \ \text{ and } \ -1 < \sin\theta < -\frac{\sqrt{3}}{2}.$$

Then φ ***cannot*** satisfy- **[JEE 2012, 4]**

(A) $0 < \varphi < \frac{\pi}{2}$ (B) $\frac{\pi}{2} < \varphi < \frac{4\pi}{3}$ (C) $\frac{4\pi}{3} < \varphi < \frac{3\pi}{2}$ (D) $\frac{3\pi}{2} < \varphi < 2\pi$

6. For $x \in (0, \pi)$, the equation $\sin x + 2\sin 2x - \sin 3x = 3$ has **[JEE 2014]**
(A) infinitely many solutions (B) three solutions
(C) one solutions (D) no solutions

7. The number of distinct solutions of the equation $\frac{5}{4}\cos^2 2x + \cos^4 x + \sin^4 x + \cos^6 x + \sin^6 x = 2$ in the interval $[0, 2\pi]$ is ——— **[JEE 2015]**

8. Let $S = \left\{x \in (-\pi, \pi) : x \neq 0, \pm\frac{\pi}{2}\right\}$. The sum of all distinct solution of the equation

$\sqrt{3}\sec x + \operatorname{cosec} x + 2(\tan x - \cot x) = 0$ in the set S is equal to - **[JEE 2016]**

(A) $-\frac{7\pi}{9}$ (B) $-\frac{2\pi}{9}$ (C) 0 (D) $\frac{5\pi}{9}$

9. Let α and β be nonzero real numbers such that $2(\cos\beta - \cos\alpha) + \cos\alpha\cos\beta = 1$. Then which of the following is/are true ? ***[JEE 2017]***

(A) $\tan\left(\frac{\alpha}{2}\right) - \sqrt{3}\tan\left(\frac{\beta}{2}\right) = 0$

(B) $\sqrt{3}\tan\left(\frac{\alpha}{2}\right) - \tan\left(\frac{\beta}{2}\right) = 0$

(C) $\tan\left(\frac{\alpha}{2}\right) + \sqrt{3}\tan\left(\frac{\beta}{2}\right) = 0$

(D) $\sqrt{3}\tan\left(\frac{\alpha}{2}\right) + \tan\left(\frac{\beta}{2}\right) = 0$

ANSWER-KEY

EXERCISE-1

Que.	1	2	3	4	5	6	7	8	9	10
Ans.	D	B	A	B	A	D	D	B	C	D
Que.	11	12	13	14	15	16	17	18	19	20
Ans.	B	C	B	C	B	B	B	C	B	B
Que.	21	22	23	24	25					
Ans.	A	A	B	B	C					

EXERCISE-2

Que.	1	2	3	4	5	6	7	8	9	10
Ans.	A,B,D	B,D	A,B,C	C	B	A,B,C	A,C	B,D	A,B,C,D	A,B,C
Que.	11									
Ans.	A,D									

- ***Match the Column*** **12.** (A) $\rightarrow$ (q), (B) $\rightarrow$ (p), (C) $\rightarrow$ (q), (D) $\rightarrow$ (p)
- ***Comprehension Based Questions***

  Comprehension – 3 **13.** (A) **14.** (D) **15.** (B)

EXERCISE-3

1. $A = 2n\pi + B,\ n \in I$ **2.** $x = 2n\pi - \frac{\pi}{2},\ n \in I$ **3.** $x = 2n\pi \pm \pi$ or $2n\pi + \frac{\pi}{3},\ n \in I$

4. $x = \frac{\pi}{8} + \frac{K\pi}{2}$ or $x = \frac{3\pi}{4} + K\pi,\ K \in I$ **6.** $\alpha - 2\pi\ ;\ \alpha - \pi,\ \alpha,\ \alpha + \pi$, where $\alpha = \tan^{-1}\frac{2}{3}$

7. $0,\ \frac{\pi}{6},\ \frac{\pi}{3},\ \frac{2\pi}{3},\ \frac{5\pi}{6}\ \&\ \pi$ **8.** $\theta = 60°$

9. $\frac{90°}{7},\ 30°,\ \frac{450°}{7},\ \frac{810°}{7},\ 150°,\ \frac{1170°}{7}$ **10.** $n\pi$ or $\left(n\pi - \frac{\pi}{4}\right),\ n \in I$

11. $\theta = \frac{2n\pi}{5} \pm \frac{\pi}{10}$ or $2n\pi \pm \frac{\pi}{2},\ n \in I$ **12.** $(2n+1)\frac{\pi}{4},\ n \in I$

13. $x \in \left(2n\pi + \frac{\pi}{4}, 2n\pi + \frac{3\pi}{4}\right) \cup \left(2n\pi - \frac{\pi}{4}, 2n\pi\right) \cup \left(2n\pi + \pi,\ 2n\pi + \frac{5\pi}{4}\right),\ n \in I$

14. $n\pi + \frac{\pi}{4} < x < n\pi + \frac{\pi}{3},\ n \in I$

EXERCISE-4

Que.	1	2	3	4	5	6	7	8	9	10
Ans.	C	A	D	B	A	A	A	A	B	A
Que.	11	12	13							
Ans.	A	A	A							

EXERCISE-5

1. (A) **2.** (C) **3.** 3 **4.** 7 **5.** (A,C,D) **6.** (D) **7.** 8

8. (C) **9.** (A,C)

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Important Notes