CONTENTS

STRAIGHT LINE

Recap of Early Classes

In the previous classes, we have studied plane geometry related to lines, triangles, circles etc. In plane geometry location of figure is not important. From this chapter onwards we will study co-ordinate geometry in which location is given due importance.

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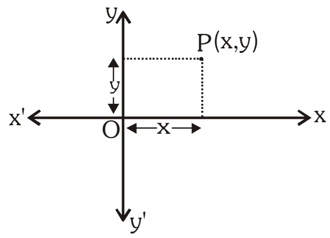
STRAIGHT LINE

1.0 INTRODUCTION OF COORDINATE GEOMETRY

SL AL

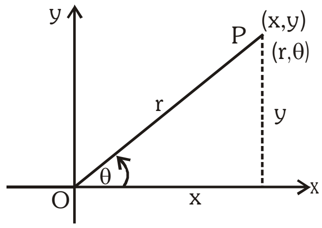
Coordinate geometry is the combination of algebra and geometry. A systematic study of geometry by the use of algebra was first carried out by celebrated French philosopher and mathematician René Descartes. The resulting combination of analysis and geometry is referred as ***analytical geometry***.

1.1 Cartesian co-ordinates system



In two dimensional coordinate system, two lines are used; the lines are at right angles, forming a rectangular coordinate system. The horizontal axis is the x-axis and the vertical axis is y-axis. The point of intersection O is the origin of the coordinate system. Distances along the x-axis to the right of the origin are taken as positive, distances to the left as negative. Distances along the y-axisabove the origin are positive; distances below are negative. The position of a point anywhere in the plane can be specified by two numbers, the coordinates of the point, written as (x, y). The x-coordinate (or abscissa) is the distance of the point from the y-axis in a direction parallel to the x-axis (i.e. horizontally). The y-coordinate (or ordinate) is the distance from the x-axis in a direction parallel to the y-axis (vertically). The origin O is the point (0, 0).

1.2 Polar co-ordinates system



A coordinate system in which the position of a point is determined by the length of a line segment from a fixed origin together with the angle that the line segment makes with a fixed line. The origin is called the pole and the line segment is the radius vector (r).

The angle θ between the polar axis and the radius vector is called the vectorial angle. By convention, positive values of θ are measured in an anticlockwise sense, negative values in clockwise sense. The coordinates of the point are then specified as (r, θ).

If (x,y) are cartesian co-ordinates of a point P, then: $x = r\cos\theta$, $y = r\sin\theta$ and $r = \sqrt{x^2 + y^2}$, $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

2.0 DISTANCE FORMULA AND ITS APPLICATIONS

SL AL

If $A(x_1,y_1)$ and $B(x_2,y_2)$ are two points, then $AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

NOTE

(i) Three given points A,B and C are collinear, when sum of any two distances out of AB,BC, CA is equal to the remaining third otherwise the points will be the vertices of a triangle.

(ii) Let A,B,C & D be the four given points in a plane. Then the quadrilateral will be :

(a) Square if $AB = BC = CD = DA$ & $AC = BD$; $AC \perp BD$

(b) Rhombus if $AB = BC = CD = DA$ and $AC \neq BD$; $AC \perp BD$

(c) Parallelogram if $AB = DC$, $BC = AD$; $AC \neq BD$; $AC \not\perp BD$

(d) Rectangle if $AB = CD$, $BC = DA$, $AC = BD$; $AC \not\perp BD$

Illustrations

Illustration 1. The number of points on x-axis which are at a distance $c(c < 3)$ from the point (2, 3) is

(A) 2 (B) 1 (C) infinite (D) no point

Solution. Let a point on x-axis is $(x_1, 0)$ then its distance from the point (2, 3)

$= \sqrt{(x_1 - 2)^2 + 9} = c$ or $(x_1 - 2)^2 = c^2 - 9$

$\therefore$ $x_1 - 2 = \pm\sqrt{c^2 - 9}$ since $c < 3 \Rightarrow c^2 - 9 < 0$

$\therefore$ x_1 will be imaginary. **Ans. (D)**

Illustration 2. The distance between the point $P(a\cos\alpha, a\sin\alpha)$ and $Q(a\cos\beta, a\sin\beta)$ is -

(A) $\left|4a\sin\frac{\alpha-\beta}{2}\right|$ (B) $\left|2a\sin\frac{\alpha+\beta}{2}\right|$ (C) $\left|2a\sin\frac{\alpha-\beta}{2}\right|$ (D) $\left|2a\cos\frac{\alpha-\beta}{2}\right|$

Solution. $d^2 = (a\cos\alpha - a\cos\beta)^2 + (a\sin\alpha - a\sin\beta)^2 = a^2(\cos\alpha-\cos\beta)^2 + a^2(\sin\alpha - \sin\beta)^2$

$$= a^2\left\{2\sin\frac{\alpha+\beta}{2}\sin\frac{\beta-\alpha}{2}\right\}^2 + a^2\left\{2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}\right\}^2$$

$$= 4a^2\sin^2\frac{\alpha-\beta}{2}\left\{\sin^2\frac{\alpha+\beta}{2}+\cos^2\frac{\alpha+\beta}{2}\right\} = 4a^2\sin^2\frac{\alpha-\beta}{2} \Rightarrow d = \left|2a\sin\frac{\alpha-\beta}{2}\right|$$ ***Ans. (C)***

3.0 SECTION FORMULA

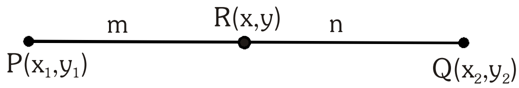
SL AL

The co-ordinates of a point dividing a line joining the points $P(x_1,y_1)$ and $Q(x_2,y_2)$ in the ratio m:n is given by :

3.1 Internal division

P - R - Q $\Rightarrow$ R divides line segment PQ, internally.

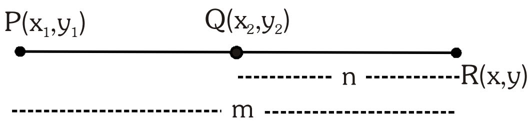
$$(x, y) \equiv \left(\frac{mx_2+nx_1}{m+n}, \frac{my_2+ny_1}{m+n}\right)$$



3.2 External division

R - P - Q or P - Q - R $\Rightarrow$ R divides line segment PQ, externally.

$$(x, y) \equiv \left(\frac{mx_2-nx_1}{m-n}, \frac{my_2-ny_1}{m-n}\right)$$



$\frac{\mathbf{(PR)}}{\mathbf{(QR)}} < 1$ $\Rightarrow$ R lies on the left of P & $\frac{\mathbf{(PR)}}{\mathbf{(QR)}} > 1$ $\Rightarrow$ R lies on the right of Q

3.3 Harmonic conjugate

If P divides AB internally in the ratio m : n & Q divides AB externally in the ratio m : n then P & Q are said to be harmonic conjugate of each other w.r.t. AB.

Mathematically ; $\frac{2}{AB} = \frac{1}{AP} + \frac{1}{AQ}$ i.e. AP, AB & AQ are in ***H.P.***

Illustrations

Illustration 3. Determine the ratio in which $y - x + 2 = 0$ divides the line joining (3, –1) and (8, 9).

Solution. Suppose the line $y - x + 2 = 0$ divides the line segment joining A(3, –1) and B(8, 9) in the ratio $\lambda : 1$ at a point P, then the co-ordinates of the point P are $\left(\frac{8\lambda+3}{\lambda+1}, \frac{9\lambda-1}{\lambda+1}\right)$

But P lies on $y - x + 2 = 0$ therefore $\left(\frac{9\lambda-1}{\lambda+1}\right) - \left(\frac{8\lambda+3}{\lambda+1}\right) + 2 = 0$

$\Rightarrow$ $9\lambda - 1 - 8\lambda - 3 + 2\lambda + 2 = 0$

$\Rightarrow$ $3\lambda - 2 = 0$ or $\lambda = \frac{2}{3}$

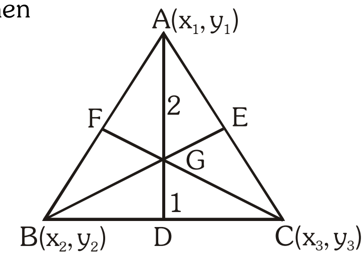
So, the required ratio is $\frac{2}{3} : 1$, i.e., 2 : 3 (internally) since here λ is positive.

4.0 CO-ORDINATES OF SOME PARTICULAR POINTS

AL

Let $A(x_1,y_1)$, $B(x_2,y_2)$ and $C(x_3,y_3)$ are vertices of any triangle ABC, then

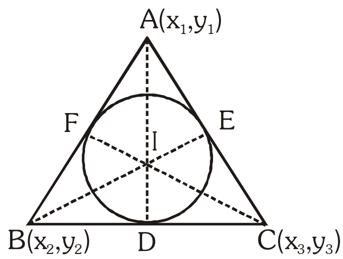
4.1 Centroid



The centroid is the point of intersection of the medians (line joining the mid point of sides and opposite vertices). Centroid divides each median in the ratio of 2 : 1.

Co-ordinates of centroid $G\left(\dfrac{x_1+x_2+x_3}{3}, \dfrac{y_1+y_2+y_3}{3}\right)$

4.2 Incentre



The incentre is the point of intersection of internal bisectors of the angles of a triangle. Also it is a centre of the circle touching all the sides of a triangle.

Co-ordinates of incentre $I\left(\dfrac{ax_1+bx_2+cx_3}{a+b+c}, \dfrac{ay_1+by_2+cy_3}{a+b+c}\right)$

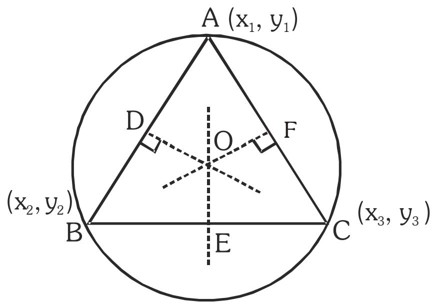
where a, b, c are the sides of triangle ABC.

NOTE

(i) Angle bisector divides the opposite sides in the ratio of remaining sides. *e.g.* $\dfrac{BD}{DC}=\dfrac{AB}{AC}=\dfrac{c}{b}$

(ii) Incentre divides the angle bisectors in the ratio $(b+c):a$, $(c+a):b$, $(a+b):c$.

4.3 Circumcentre



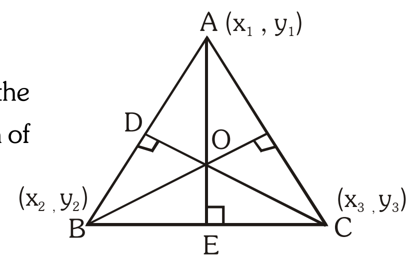
It is the point of intersection of perpendicular bisectors of the sides of a triangle. If O is the circumcentre of any triangle ABC, then $OA^2 = OB^2 = OC^2$. Also it is a centre of a circle touching all the vertices of a triangle.

NOTE

(i) If the triangle is right angled, then its circumcentre is the mid point of hypotenuse.

(ii) Co-ordinates of circumcentre $\left(\dfrac{x_1\sin 2A+x_2\sin 2B+x_3\sin 2C}{\sin 2A+\sin 2B+\sin 2C}, \dfrac{y_1\sin 2A+y_2\sin 2B+y_3\sin 2C}{\sin 2A+\sin 2B+\sin 2C}\right)$

4.4 Orthocentre



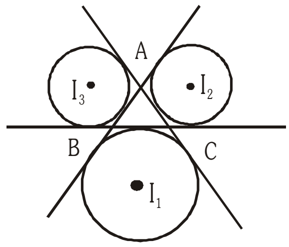
It is the point of intersection of perpendiculars drawn from vertices on the opposite sides of a triangle and it can be obtained by solving the equation of any two altitudes.

NOTE

(i) If a triangle is right angled, then orthocentre is the point where right angle is formed.

(ii) Co-ordinates of orthocentre $\left(\dfrac{x_1\tan A+x_2\tan B+x_3\tan C}{\tan A+\tan B+\tan C}, \dfrac{y_1\tan A+y_2\tan B+y_3\tan C}{\tan A+\tan B+\tan C}\right)$

4.5 Ex-centres



The centre of a circle which touches side BC and the extended portions of sides AB and AC is called the ex-centre of ΔABC with respect to the vertex A. It is denoted by I_1 and its coordinates are

$$I_1\left(\frac{-ax_1+bx_2+cx_3}{-a+b+c},\frac{-ay_1+by_2+cy_3}{-a+b+c}\right)$$

Similarly ex-centres of ΔABC with respect to vertices B and C are denoted by I_2 and I_3 respectively , and

$$I_2\left(\frac{ax_1-bx_2+cx_3}{a-b+c},\frac{ay_1-by_2+cy_3}{a-b+c}\right),\quad I_3\left(\frac{ax_1+bx_2-cx_3}{a+b-c},\frac{ay_1+by_2-cy_3}{a+b-c}\right)$$

Illustrations

Illustration 4. If $\left(\frac{5}{3},3\right)$ is the centroid of a triangle and its two vertices are (0, 1) and (2, 3), then find its third vertex, circumcentre, circumradius & orthocentre.

Solution. Let the third vertex of triangle be (x, y), then

$$\frac{5}{3}=\frac{x+0+2}{3}\Rightarrow x=3 \text{ and } 3=\frac{y+1+3}{3}\Rightarrow y=5$$. So third vertex is (3, 5).

Now three vertices are A(0, 1), B(2, 3) and C(3, 5)

Let circumcentre be P(h, k),

then AP = BP = CP = R (circumradius)

$\Rightarrow$ $AP^2 = BP^2 = CP^2 = R^2$

$h^2+(k-1)^2=(h-2)^2+(k-3)^2=(h-3)^2+(k-5)^2=R^2$ .... (i)

from the first two equations, we have

$h+k=3$ .... (ii)

from the first and third equation, we obtain

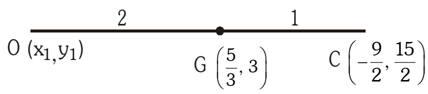
$6h+8k=33$ .... (iii)

On solving, (ii) & (iii), we get

$$h=-\frac{9}{2},\quad k=\frac{15}{2}$$

Substituting these values in (i), we have

$$R=\frac{5}{2}\sqrt{10}$$



Let $O(x_1, y_1)$ be the orthocentre,

then $$\frac{x_1+2\left(-\frac{9}{2}\right)}{3}=\frac{5}{3}$$

$$\Rightarrow\quad x_1=14,\ \frac{y_1+2\left(\frac{15}{2}\right)}{3}=3$$

$\Rightarrow$ $y_1=-6$. Hence orthocentre of the triangle is (14, –6).

Illustration 5. The vertices of a triangle are A(0, –6), B(–6, 0) and C(1,1) respectively, then coordinates of the ex-centre opposite to vertex A is :

(A) $\left(\frac{-3}{2},\frac{-3}{2}\right)$ (B) $\left(-4,\frac{3}{2}\right)$ (C) $\left(\frac{-3}{2},\frac{3}{2}\right)$ (D) (–4, 6)

Solution. $a = BC = \sqrt{(-6-1)^2+(0-1)^2} = \sqrt{50} = 5\sqrt{2}$

$b = CA = \sqrt{(1-0)^2+(1+6)^2} = \sqrt{50} = 5\sqrt{2}$

$c = AB = \sqrt{(0+6)^2+(-6-0)^2} = \sqrt{72} = 6\sqrt{2}$

Coordinates of ex-centre opposite to vertex A will be :

$$x = \frac{-ax_1+bx_2+cx_3}{-a+b+c} = \frac{-5\sqrt{2}.0+5\sqrt{2}(-6)+6\sqrt{2}(1)}{-5\sqrt{2}+5\sqrt{2}+6\sqrt{2}} = \frac{-24\sqrt{2}}{6\sqrt{2}} = -4$$

$$y = \frac{-ay_1+by_2+cy_3}{-a+b+c} = \frac{-5\sqrt{2}(-6)+5\sqrt{2}.0+6\sqrt{2}(1)}{-5\sqrt{2}+5\sqrt{2}+6\sqrt{2}} = \frac{36\sqrt{2}}{6\sqrt{2}} = 6$$

Hence coordinates of ex-centre is (–4, 6) **Ans. (D)**

BEGINNER'S BOX-1

TOPIC COVERED : BASICS OF COORDINATE GEOMETRY

1. Find the distance between the points P(–3, 2) and Q(2, –1).

2. If the distance between the points P(–3, 5) and Q(–x, – 2) is $\sqrt{58}$, then find the value(s) of x.

3. A line segment is of the length 15 units and one end is at the point (3, 2), if the abscissa of the other end is 15, then find possible ordinates.

4. Find the co-ordinates of the point dividing the join of A(1, – 2) and B(4, 7) :
(a) Internally in the ratio 1 : 2 (b) Externally in the ratio of 2 : 1

5. In what ratio is the line joining A(8, 9) and B(– 7, 4) is divided by
(a) the point (2, 7) (b) the x-axis (c) the y-axis.

6. The coordinates of the vertices of a triangle are (0, 1), (2, 3) and (3, 5) :
(a) Find centroid of the triangle.
(b) Find circumcentre & the circumradius.
(c)Find Orthocentre of the triangle.

7. Find the distances between the following pairs of points
(a) $\left(t_1^2,\ 2t_1\right)$ and $\left(t_2^2,\ 2t_2\right)$ if t_1 and t_2 are the roots of $x^2 - 2\sqrt{3}x + 2 = 0$.
(b) (a cos θ, a sin θ) and (a cos φ, a sin φ)

8. The length of a line segment AB is 10 units. If the coordinates of one extremity are (2, – 3) and the abscissa of the other extremity is 10 then the sum of all possible values of the ordinate of the other extremity is
(A) 3 (B) – 4 (C) 12 (D) – 6

***9.** If P(1, 2), Q(4, 6), R(5, 7) & S(a, b) are the vertices of a parallelogram PQRS, then :
(A) a = 2, b = 4 (B) a = 3, b = 4 (C) a = 2, b = 3 (D) a = 3, b = 5

10. If A and B are the points (–3, 4) and (2, 1), then the co–ordinates of the point C on AB produced such that AC = 2BC are :
(A) (2, 4) (B) (3, 7) (C) (7, –2) (D) $\left(-\frac{1}{2},\frac{5}{2}\right)$

5.0 AREA OF TRIANGLE

SL AL

Let $A(x_1,y_1)$, $B(x_2,y_2)$ and $C(x_3,y_3)$ are vertices of a triangle, then

$$\text{Area of } \Delta ABC = \left|\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}\right| = \frac{1}{2}\,|[x_1(y_2-y_3) + x_2(y_3-y_1) + x_3(y_1-y_2)]|$$

To remember the above formula, take the help of the following method :

$$= \frac{1}{2}\begin{bmatrix} x_1 & & x_2 & & x_3 & & x_1 \\ & \times & & \times & & \times & \\ y_1 & & y_2 & & y_3 & & y_1 \end{bmatrix} = \left|\frac{1}{2}[(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + (x_3y_1 - x_1y_3)]\right|$$

Remarks

(i) If the area of triangle joining three points is zero, then the points are collinear.

(ii) ***Area of Equilateral triangle :*** If length of altitude of any equilateral triangle is P, then its area $= \frac{P^2}{\sqrt{3}}$. If 'a' be the length of side of equilateral triangle, then its area $= \left(\frac{a^2\sqrt{3}}{4}\right)$.

(iii) Area of quadrilateral with given vertices $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$, $D(x_4, y_4)$

$$\text{Area of quad. ABCD} = \frac{1}{2}\left|\begin{bmatrix} x_1 & & x_2 & & x_3 & & x_4 & & x_1 \\ & \times & & \times & & \times & & \times & \\ y_1 & & y_2 & & y_3 & & y_4 & & y_1 \end{bmatrix}\right|$$

Note – Area of a polygon can be obtained by dividing the polygon into disjoined triangles and then adding their areas.

Illustrations

Illustration 6. If the vertices of a triangle are (1, 2), (4, –6) and (3, 5) then its area is

(A) $\frac{25}{2}$ sq. units (B) 12 sq. units (C) 5 sq. units (D) 25 sq. units

Solution. $\Delta = \frac{1}{2}[1(-6-5) + 4(5-2) + 3(2+6)] = \frac{1}{2}[-11+12+24] = \frac{25}{2}$ square units ***Ans. (A)***

Illustration 7. The point A divides the join of the points (–5, 1) and (3, 5) in the ratio k:1 and coordinates of points B and C are (1, 5) and (7, –2) respectively. If the area of ΔABC be 2 units, then k equals-

(A) 7, 9 (B) 6, 7 (C) 7, $\frac{31}{9}$ (D) 9, $\frac{31}{9}$

Solution. $A \equiv \left(\frac{3k-5}{k+1}, \frac{5k+1}{k+1}\right)$

Area of $\Delta ABC = 2$ units

$$\Rightarrow \frac{1}{2}\left[\frac{3k-5}{k+1}(5+2) + 1\left(-2 - \frac{5k+1}{k+1}\right) + 7\left(\frac{5k+1}{k+1} - 5\right)\right] = \pm 2$$

$$\Rightarrow \quad 14k - 66 = \pm 4(k+1) \quad \Rightarrow k = 7 \text{ or } \frac{31}{9}$$

Ans. (C)

6.0 CONDITIONS FOR COLLINEARITY OF THREE GIVEN POINTS

SL AL

Three given points A (x_1, y_1), B (x_2, y_2), C (x_3, y_3) are collinear if any one of the following conditions are satisfied.

(a) Area of triangle ABC is zero i.e. $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$

(b) Slope of AB = slope of BC = slope of AC. i.e. $\dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{y_3 - y_2}{x_3 - x_2} = \dfrac{y_3 - y_1}{x_3 - x_1}$

(c) Find the equation of line passing through 2 given points, if the third point satisfies the given equation of the line, then three points are collinear.

7.0 LOCUS

SL AL

The locus of a moving point is the path traced out by that point under one or more geometrical conditions.

(a) ***Equation of Locus***

The equation to a locus is the relation which exists between the coordinates of any point on the path, and which holds for no other point except those lying on the path.

(b) ***Procedure for finding the equation of the locus of a point***

(i) If we are finding the equation of the locus of a point P, assign coordinates (h, k) to P.

(ii) Express the given condition as equations in terms of the known quantities to facilitate calculations. We sometimes include some unknown quantities known as parameters.

(iii) Eliminate the parameters, so that the eliminant contains only h, k and known quantities.

(iv) Replace h by x, and k by y, in the eliminant. The resulting equation would be the equation of the locus of P.

Illustrations

****Illustration 8.*** The ends of the rod of length ℓ moves on two mutually perpendicular lines, the locus of the point on the rod which divides it in the ratio $m_1 : m_2$, is

(A) $m_1^2x^2 + m_2^2y^2 = \dfrac{\ell^2}{(m_1+m_2)^2}$ (B) $(m_2x)^2 + (m_1y)^2 = \left(\dfrac{m_1m_2\ell}{m_1+m_2}\right)^2$

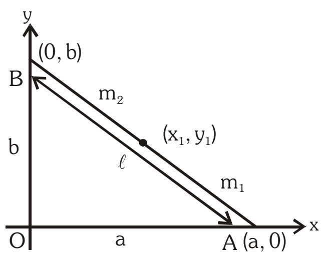
(C) $(m_1x)^2 + (m_2y)^2 = \left(\dfrac{m_1m_2\ell}{m_1+m_2}\right)^2$ (D) none of these

Solution. Let (x_1, y_1) be the point that divide the rod $AB = \ell$, in the ratio $m_1 : m_2$, and $OA = a$, $OB = b$ say

$\therefore \quad a^2 + b^2 = \ell^2$ ..... (i)

Now $x_1 = \left(\dfrac{m_2a}{m_1+m_2}\right) \Rightarrow a = \left(\dfrac{m_1+m_2}{m_2}\right)x_1$

$y_1 = \left(\dfrac{m_1b}{m_1+m_2}\right) \Rightarrow b = \left(\dfrac{m_1+m_2}{m_1}\right)y_1$



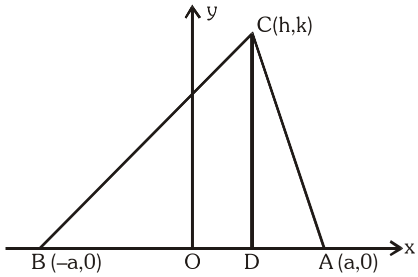
putting these values in (i) $\dfrac{(m_1+m_2)^2}{m_2^2}x_1^2 + \dfrac{(m_1+m_2)^2}{m_1^2}y_1^2 = \ell^2$

$\therefore$ Locus of (x_1, y_1) is $m_1^2x^2 + m_2^2y^2 = \left(\dfrac{m_1m_2\ell}{m_1+m_2}\right)^2$ ***Ans. (C)***

***Illustration 9.** A(a,0) and B(–a, 0) are two fixed points of ΔABC. If its vertex C moves in such a way that cotA + cotB = λ, where λ is a constant, then the locus of the point C is -

(A) $y\lambda = 2a$ (B) $y = \lambda a$ (C) $ya = 2\lambda$ (D) none of these

Solution. Given that coordinates of two fixed points A and B are (a, 0) and (–a, 0) respectively. Let variable point C is (h, k). From the adjoining figure



$$\cot A = \frac{DA}{CD} = \frac{a-h}{k}$$

$$\cot B = \frac{BD}{CD} = \frac{a+h}{k}$$

But cotA + cotB = λ, so we have

$$\frac{a-h}{k} + \frac{a+h}{k} = \lambda \Rightarrow \frac{2a}{k} = \lambda$$

Hence locus of C is $y\lambda = 2a$ ***Ans. (A)***

BEGINNER'S BOX-2

TOPIC COVERED : CONDITION OF COLLINEARITY AND LOCUS

1. Find the area of the triangle whose vertices are A(1,1), B(7, – 3) and C(12, 2)

2. Find the area of the quadrilateral whose vertices are A(1,1) B(7, – 3), C(12,2) and D(7, 21)

3. Prove that the points A(a, b + c), B(b, c + a) and C(c, a + b) are collinear (By determinant method)

4. Prove that the points (–1, –1), (2, 3) and (8, 11) are collinear.

5. Find the value of x so that the points (x, –1), (2, 1) and (4, 5) are collinear.

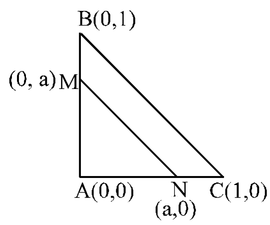
6. Find the locus of a variable point which is at a distance of 2 units from the y-axis.

7. Find the locus of a point which is equidistant from both the axes.

8. Find the locus of a point whose co-ordinates are given by $x = at^2$, $y = 2at$, where 't' is a parameter.

9. If the points (x, y) be equidistant from the points (6, – 1) and (2, 3), prove that x – y = 3.

***10.** ΔABC lies in the plane with A = (0, 0), B = (0, 1) and C = (1, 0). Points M and N are chosen on AB and AC, respectively, such that MN is parallel to BC and MN divides the area of ΔABC in half. Find the coordinates of M.



***11.** A point P (x, y) moves so that the sum of the distances from P to the coordinate axes is equal to the distance from P to the point A(1, 1). The equation of the locus of P in the first quadrant is

(A) (x + 1) (y + 1) = 1 (B) (x + 1) (y + 1) = 2 (C) (x – 1) (y – 1) = 1 (D) (x – 1) (y – 1) = 2

***12.** Let A(2, –3) and B(–2, 1) be vertices of a Δ ABC. If the centroid of Δ ABC moves on the line 2x + 3y = 1, then the locus of the vertex C is

(A) 2x + 3y = 9 (B) 2x – 3y = 7 (C) 3x + 2y = 5 (D) 3x – 2y = 3

8.0 STRAIGHT LINE

SL AL

Introduction – A relation between x and y which is satisfied by co-ordinates of every point lying on a line is called equation of the straight line. Here, remember that every one degree equation in variable x and y always represents a straight line i.e. $ax + by + c = 0$; a & b $\neq 0$ simultaneously.

(a) Equation of a line parallel to x-axis at a distance 'a' is $y = a$ or $y = -a$

(b) Equation of x-axis is $y = 0$

(c) Equation of a line parallel to y-axis at a distance 'b' is $x = b$ or $x = -b$

(d) Equation of y-axis is $x = 0$

Illustrations

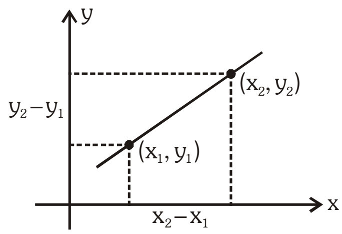
Illustration 10. Prove that every first degree equation in x, y represents a straight line.

Solution. Let $ax + by + c = 0$ be a first degree equation in x, y
where a, b, c are constants.
Let $P(x_1, y_1)$ & $Q(x_2, y_2)$ be any two points on the curve represented by $ax + by + c = 0$. Then $ax_1 + by_1 + c = 0$ and $ax_2 + by_2 + c = 0$
Let R be any point on the line segment joining P & Q
Suppose R divides PQ in the ratio $\lambda : 1$. Then, the coordinates of R are $\left(\frac{\lambda x_2 + x_1}{\lambda + 1}, \frac{\lambda y_2 + y_1}{\lambda + 1}\right)$
We have $a\left(\frac{\lambda x_2 + x_1}{\lambda + 1}\right) + b\left(\frac{\lambda y_2 + y_1}{\lambda + 1}\right) + c = \lambda\, 0 + 0 = 0$
$\therefore$ $R\left(\frac{\lambda x_2 + x_1}{\lambda + 1}, \frac{\lambda y_2 + y_1}{\lambda + 1}\right)$ lies on the curve represented by $ax + by + c = 0$. Thus every point on the line segment joining P & Q lies on $ax + by + c = 0$.
Hence $ax + by + c = 0$ represents a straight line.

9.0 SLOPE OF LINE

SL AL

If a given line makes an angle $\theta(0° \leq \theta < 180°, \theta \neq 90°)$ with the positive direction of x-axis, then slope of this line will be $\tan\theta$ and is usually denoted by the letter m i.e. $m = \tan\theta$. If $A(x_1, y_1)$ and $B(x_2, y_2)$ & $x_1 \neq x_2$ then slope of line AB $= \frac{y_2 - y_1}{x_2 - x_1}$



Remark

(i) If $\theta = 90°$, m does not exist and line is parallel to y-axis.

(ii) If $\theta = 0°$, $m = 0$ and the line is parallel to x-axis.

(iii) Let m_1 and m_2 be slopes of two given lines (none of them is parallel to y-axis)
 (a) If lines are parallel, $m_1 = m_2$ and vice-versa.
 (b) If lines are perpendicular, $m_1m_2 = -1$ and vice-versa

10.0 STANDARD FORMS OF EQUATIONS OF A STRAIGHT LINE

SL AL

10.1 Slope Intercept form – Let m be the slope of a line and c its intercept on y-axis. Then the equation of this straight line is written as : $y = mx + c$
If the line passes through origin, its equation is written as $y = mx$

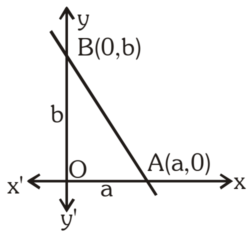
10.2 Point Slope form – If m be the slope of a line and it passes through a point (x_1,y_1), then its equation is written as : $y - y_1 = m(x - x_1)$

10.3 Two point form – Equation of a line passing through two points (x_1,y_1) and (x_2,y_2) is written as :

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) \text{ or } \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

10.4 Intercept form – If a and b are the intercepts made by a line on the axes of x and y, its equation is written

as : $\frac{x}{a} + \frac{y}{b} = 1$



(i) Length of intercept of line between the coordinate axes $= \sqrt{a^2 + b^2}$

(ii) Area of triangle AOB $= \frac{1}{2}$OA.OB $= \left|\frac{1}{2}ab\right|$

Illustrations

Illustration 11. The equation of the lines which passes through the point (3, 4) and the sum of its intercepts on the axes is 14 is -

(A) $4x - 3y = 24$, $x - y = 7$ (B) $4x + 3y = 24$, $x + y = 7$
(C) $4x + 3y + 24 = 0$, $x + y + 7 = 0$ (D) $4x - 3y + 24 = 0$, $x - y + 7 = 0$

Solution. Let the equation of the line be $\frac{x}{a} + \frac{y}{b} = 1$ .....(i)

This passes through (3 , 4) , therefore $\frac{3}{a} + \frac{4}{b} = 1$ .....(ii)

It is given that $a + b = 14 \Rightarrow b = 14 - a$. Putting $b = 14 - a$ in (ii) , we get

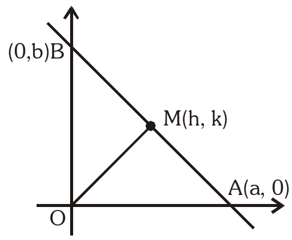
$\frac{3}{a} + \frac{4}{14-a} = 1 \Rightarrow a^2 - 13a + 42 = 0 \Rightarrow (a - 7)(a - 6) = 0 \Rightarrow a = 7 , 6$

For $a = 7$, $b = 14 - 7 = 7$ and for $a = 6$, $b = 14 - 6 = 8$

Putting the values of a and b in (i) , we get the equations of the lines

$\frac{x}{7} + \frac{y}{7} = 1$ and $\frac{x}{6} + \frac{y}{8} = 1$ or $x + y = 7$ and $4x + 3y = 24$ ***Ans. (B)***

****Illustration 12.*** Two points A and B move on the positive direction of x-axis and y-axis respectively, such that OA + OB = K. Show that the locus of the foot of the perpendicular from the origin O on the line AB is $(x + y)(x^2 + y^2) = Kxy$.



Solution. Let the equation of AB be $\frac{x}{a} + \frac{y}{b} = 1$ .... (i)

given, $a + b = K$ .... (ii)

now, $m_{AB} \times m_{OM} = -1 \Rightarrow ah = bk$ .... (iii)

from (ii) and (iii),

$$a = \frac{kK}{h+k} \text{ and } b = \frac{hK}{h+k}$$

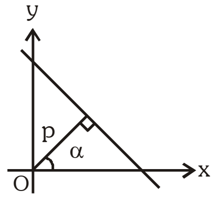
$\therefore$ from (i) $\frac{x(h+k)}{k.K} + \frac{y(h+k)}{h.K} = 1$

as it passes through (h, k)

$\frac{h(h+k)}{k.K} + \frac{k(h+k)}{h.K} = 1 \Rightarrow (h + k)(h^2 + k^2) = Khk$

$\therefore$ locus of (h, k) is $(x + y)(x^2 + y^2) = Kxy$.

10.5 Normal form – If p is the length of perpendicular on a line from the origin, and α the angle which this perpendicular makes with positive x-axis, then the equation of this line is written as : $x\cos\alpha + y\sin\alpha = p$ (p is always positive) where $0 \le \alpha < 2\pi$.

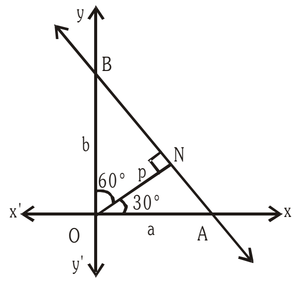


Illustrations

Illustration 13. Find the equation of the straight line on which the perpendicular from origin makes an angle 30° with positive x-axis and which forms a triangle of area $\left(\frac{50}{\sqrt{3}}\right)$ sq. units with the co-ordinates axes.

Solution. $\angle NOA = 30°$

Let ON = p > 0, OA = a, OB = b



In ΔONA, $\cos 30° = \frac{ON}{OA} = \frac{p}{a} \Rightarrow \frac{\sqrt{3}}{2} = \frac{p}{a}$

or $\quad a = \frac{2p}{\sqrt{3}}$

and in ΔONB, $\cos 60° = \frac{ON}{OB} = \frac{p}{b} \Rightarrow \frac{1}{2} = \frac{p}{b}$

or $\quad b = 2p$

$\because$ Area of $\Delta OAB = \frac{1}{2}ab = \frac{1}{2}\left(\frac{2p}{\sqrt{3}}\right)(2p) = \frac{2p^2}{\sqrt{3}}$

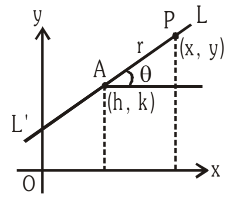
$\therefore \quad \frac{2p^2}{\sqrt{3}} = \frac{50}{\sqrt{3}} \Rightarrow p^2 = 25$

or $\quad p = 5$

$\therefore$ Using $x\cos\alpha + y\sin\alpha = p$, the equation of the line AB is $x\cos 30° + y\sin 30° = 5$

or $\quad x\sqrt{3} + y = 10$

10.6 Parametric form – To find the equation of a straight line which passes through a given point A(h, k) and makes a given angle θ with the positive direction of the x-axis. P(x, y) is any point on the line LAL'.



Let AP = r, then $x - h = r\cos\theta$, $y - k = r\sin\theta$ & $\frac{x-h}{\cos\theta} = \frac{y-k}{\sin\theta} = r$ is the equation of the straight line LAL'.

Any point P on the line will be of the form $(h + r\cos\theta, k + r\sin\theta)$, where $|r|$ gives the distance of the point P from the fixed point (h, k).

Illustrations

Illustration 14. Equation of a line which passes through point A(2, 3) and makes an angle of 45° with x axis. If this line meet the line $x + y + 1 = 0$ at point P then distance AP is -

(A) $2\sqrt{3}$ (B) $3\sqrt{2}$ (C) $5\sqrt{2}$ (D) $2\sqrt{5}$

Solution. Here $x_1 = 2$, $y_1 = 3$ and $\theta = 45°$ hence $\frac{x-2}{\cos 45°} = \frac{y-3}{\sin 45°} = r$

from first two parts $\Rightarrow x - 2 = y - 3 \Rightarrow x - y + 1 = 0$

Co-ordinate of point P on this line is $\left(2 + \frac{r}{\sqrt{2}}, 3 + \frac{r}{\sqrt{2}}\right)$.

If this point is on line $x + y + 1 = 0$ then

$\left(2 + \frac{r}{\sqrt{2}}\right) + \left(3 + \frac{r}{\sqrt{2}}\right) + 1 = 0 \Rightarrow r = -3\sqrt{2}$; $|r| = 3\sqrt{2}$ ***Ans. (B)***

Illustration 15. A straight line through P(–2, –3) cuts the pair of straight lines $x^2 + 3y^2 + 4xy - 8x - 6y - 9 = 0$ in Q and R. Find the equation of the line if PQ. PR = 20.

Solution. Let line be $\frac{x+2}{\cos\theta} = \frac{y+3}{\sin\theta} = r$

$\Rightarrow$ $x = r\cos\theta - 2,\ y = r\sin\theta - 3$ .... (i)

Now, $x^2 + 3y^2 + 4xy - 8x - 6y - 9 = 0$ .... (ii)

Taking intersection of (i) with (ii) and considering terms of r^2 and constant (as we need PQ . PR = $r_1 . r_2$ = product of the roots)

$r^2(\cos^2\theta + 3\sin^2\theta + 4\sin\theta\cos\theta) + (\text{some terms})r + 80 = 0$

$\therefore$ $r_1.r_2 = PQ.\ PR = \frac{80}{\cos^2\theta + 4\sin\theta\cos\theta + 3\sin^2\theta}$

$\therefore$ $\cos^2\theta + 4\sin\theta\cos\theta + 3\sin^2\theta = 4$ $(\because\ PQ . PR = 20)$

$\therefore$ $\sin^2\theta - 4\sin\theta\cos\theta + 3\cos^2\theta = 0$

$\Rightarrow$ $(\sin\theta - \cos\theta)(\sin\theta - 3\cos\theta) = 0$

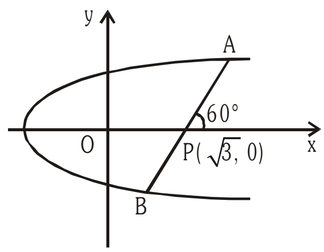
$\therefore$ $\tan\theta = 1,\ \tan\theta = 3$

hence equation of the line is $y + 3 = 1(x + 2) \Rightarrow x - y = 1$

and $y + 3 = 3(x + 2) \Rightarrow 3x - y + 3 = 0$.

****Illustration 16.*** If the line $y - \sqrt{3}x + 3 = 0$ cuts the parabola $y^2 = x + 2$ at A and B, then find the value of PA.PB {where $P \equiv (\sqrt{3}, 0)$}

Solution. Slope of line $y - \sqrt{3}x + 3 = 0$ is $\sqrt{3}$



If line makes an angle θ with x-axis, then $\tan\theta = \sqrt{3}$

$\therefore$ $\theta = 60°$

$\frac{x-\sqrt{3}}{\cos 60°} = \frac{y-0}{\sin 60°} = r \Rightarrow \left(\sqrt{3} + \frac{r}{2}, \frac{r\sqrt{3}}{2}\right)$

be a point on the parabola $y^2 = x + 2$

then $\frac{3}{4}r^2 = \sqrt{3} + \frac{r}{2} + 2$

$\Rightarrow$ $3r^2 - 2r - 4(2 + \sqrt{3}) = 0$

$\therefore$ $PA.PB = r_1r_2 = \left|\frac{-4\left(2+\sqrt{3}\right)}{3}\right| = \frac{4\left(2+\sqrt{3}\right)}{3}$

10.7 General form – We know that a first degree equation in x and y, ax +by + c = 0 always represents a straight line. This form is known as general form of straight line.

(i) Slope of this line $= \frac{-a}{b} = -\frac{\text{coeff. of } x}{\text{coeff. of } y}$

(ii) Intercept by this line on x-axis $= -\frac{c}{a}$ and intercept by this line on y-axis $= -\frac{c}{b}$

(iii) To change the general form of a line to normal form, first take c to right hand side and make it positive, then divide the whole equation by $\sqrt{\mathbf{a^2 + b^2}}$.

11.0 ANGLE BETWEEN TWO LINES

SL AL

(a) If θ be the angle between two lines : $y = m_1x + c$ and $y = m_2x + c_2$, then $\tan\theta = \pm\left(\frac{m_1 - m_2}{1 + m_1m_2}\right)$

NOTE

(i) There are two angles formed between two lines but usually the acute angle is taken as the angle between the lines. So we shall find θ from the above formula only by taking positive value of $\tan\theta$.

(ii) Let m_1, m_2, m_3 are the slopes of three lines $L_1 = 0$; $L_2 = 0$; $L_3 = 0$ where $m_1 > m_2 > m_3$ then the interior angles of the Δ ABC found by these formulas are given by,

$$\tan A = \frac{m_1 - m_2}{1 + m_1 m_2}\ ;\ \tan B = \frac{m_2 - m_3}{1 + m_2 m_3}\ \ \&\ \ \tan C = \frac{m_3 - m_1}{1 + m_3 m_1}$$

(b) If equation of lines are $a_1x+b_1y+c_1=0$ and $a_2x+b_2y+c_2=0$, then these line are -

(i) Parallel $\Leftrightarrow$ $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$

(ii) Perpendicular $\Leftrightarrow$ $a_1a_2+b_1b_2=0$

(iii) Coincident $\Leftrightarrow$ $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

Illustrations

Illustration 17. If $x + 4y - 5 = 0$ and $4x + ky + 7 = 0$ are two perpendicular lines then k is -
(A) 3 (B) 4 (C) –1 (D) –4

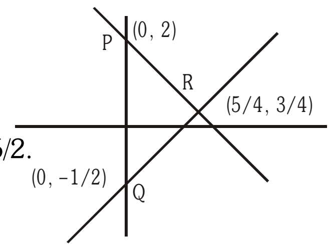
Solution. $m_1 = -\frac{1}{4}$ $\quad m_2 = -\frac{4}{k}$

Two lines are perpendicular if $m_1 m_2 = -1$

$\Rightarrow \left(-\frac{1}{4}\right)\times\left(-\frac{4}{k}\right) = -1 \Rightarrow k = -1$ ***Ans. (C)***

****Illustration 18.*** A line L passes through the points (1, 1) and (0, 2) and another line M which is perpendicular to L passes through the point (0, –1/2).The area of the triangle formed by these lines with y-axis is
(A) 25/8 (B) 25/16 (C) 25/4 (D) 25/32

Solution. Equation of the line L is $y - 1 = \frac{-1}{1}(x - 1) \Rightarrow y = -x + 2$



Equation of the line M is $y = x - 1/2$.

If these lines meet y-axis at P (0, –1/2) and Q (0, 2) then PQ = 5/2.

Also x-coordinate of their point of intersection R = 5/4

$\therefore$ area of the $\Delta PQR = \frac{1}{2}\left(\frac{5}{2}\times\frac{5}{4}\right) = 25/16.$ ***Ans. (B)***

12.0 EQUATION OF LINES PARALLEL AND PERPENDICULAR TO A GIVEN LINE

SL AL

(a) Equation of line parallel to line $ax + by + c = 0$

$ax + by + \lambda = 0$

(b) Equation of line perpendicular to line $ax + by + c = 0$

$bx - ay + k = 0$

Here λ, k, are parameters and their values are obtained with the help of additional information given in the problem.

13.0 STRAIGHT LINE MAKING A GIVEN ANGLE WITH A LINE.

AL

Equations of lines passing through a point (x_1,y_1) and making an angle α, with the line $y = mx+c$ is written as :

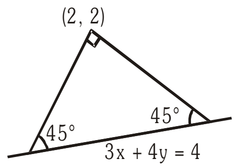
$$y - y_1 = \frac{m \pm \tan\alpha}{1 \mp m\tan\alpha}(x - x_1)$$

Illustrations

Illustration 19. Find the equation to the sides of an isosceles right-angled triangle, the equation of whose hypotenuse is $3x + 4y = 4$ and the opposite vertex is the point (2, 2).

Solution. The problem can be restated as :

Find the equations of the straight lines passing through the given point (2, 2) and making equal angles of 45° with the given straight line $3x + 4y - 4 = 0$. Slope of the line $3x + 4y - 4 = 0$ is $m_1 = -3/4$.



$\Rightarrow \quad \tan 45° = \pm \dfrac{m - m_1}{1 + m_1 m}$,

i.e., $1 = \pm \dfrac{m + 3/4}{1 - \frac{3}{4}m}$

$m_A = \dfrac{1}{7}$, and $m_B = -7$

Hence the required equations of the two lines are

$y - 2 = m_A(x - 2)$ and $y - 2 = m_B(x - 2)$

$\Rightarrow \quad 7y - x - 12 = 0$ and $7x + y = 16$ ***Ans.***

BEGINNER'S BOX-3

TOPIC COVERED : VARIOUS FORMS OF STRAIGHT LINES, ANGLE BETWEEN TWO LINES

1. Reduce the line $2x - 3y + 5 = 0$,
(a) In slope- intercept form and hence find slope & Y-intercept
(b) In intercept form and hence find intercepts on the axes.
(c) In normal form and hence find perpendicular distance from the origin and angle made by the perpendicular with the positive x-axis.

2. Find distance of point A (2, 3) measured parallel to the line $x - y = 5$ from the line $2x + y + 6 = 0$.

****3.*** A triangle ABC is formed by three lines $x + y + 2 = 0$, $x - 2y + 5 = 0$ and $7x + y - 10 = 0$. P is a point inside the triangle ABC such that areas of the triangles PAB, PBC and PCA are equal. If the co-ordinates of the point P are (a, b) and the area of the triangle ABC is δ, then find $(a + b + \delta)$.

4. The line through point (m, – 9) and (7, m) has slope *m*. The *y*-intercept of this line, is
(A) – 18 (B) – 6 (C) 6 (D) 18

5. A line passes through (2, 2) and cuts a triangle of area 9 square units from the first quadrant. The sum of all possible values for the slope of such a line, is
(A) – 2.5 (B) – 2 (C) – 1.5 (D) – 1

****6.*** The equations of L_1 and L_2 are $y = mx$ and $y = nx$, respectively. Suppose L_1 makes twice as large of an angle with the horizontal (measured counterclockwise from the positive x-axis) as does L_2 and that L_1 has 4 times the slope of L_2. If L_1 is not horizontal, then the value of the product (mn) equals

(A) $\dfrac{\sqrt{2}}{2}$ (B) $-\dfrac{\sqrt{2}}{2}$ (C) 2 (D) –2

***7.** The extremities of the base of an isosceles triangle ABC are the points A (2, 0) and B (0, 1). If the equation of the side AC is x = 2 then the slope of the side BC is

(A) $\frac{3}{4}$ (B) $\frac{4}{3}$ (C) $\frac{3}{2}$ (D) $\sqrt{3}$

8. A line with gradient 2 intersects a line with gradient 6 at the point (40, 30). The distance between x-intercepts of these lines, is

(A) 6 (B) 8 (C) 10 (D) 12

9. The equations to the straight lines which join the origin and the points of trisection of the portion of the line $x + 3y - 12 = 0$ intercepted between the axes of co-ordinates, is

(A) $y = \frac{2}{3}x$ (B) $y = \frac{x}{6}$ (C) $y = \frac{x}{3}$ (D) $y = \frac{4}{3}x$

10. The equations to the straight lines each of which passses through the point (3, 2) and intersects the x-axis and y-axis in A, B respectively such that OA – OB = 2, can be

(A) $3x + 3y = 7$ (B) $x - y = 1$ (C) $2x + 3y = 12$ (D) $3x - y = 1$

11. Find the angle between the lines $3x + y - 7 = 0$ and $x + 2y - 9 = 0$.

12. Find the line passing through the point (2, 3) and perpendicular to the straight line $4x - 3y = 10$.

13. Find the equation of the line which has positive y-intercept 4 units and is parallel to the line $2x - 3y - 7 = 0$. Also find the point where it cuts the x-axis.

***14.** A variable line passing through the origin intersects two given straight lines $2x + y = 4$ and $x + 3y = 6$ at R and S respectively. A point P is taken on this variable line. Find the equation to the locus of the point P if

(a) OP is the arithmetic mean of OR and OS.
(b) OP is the geometric mean of OR and OS.
(c) OP is an harmonic mean of OR and OS.

14.0 LENGTH OF PERPENDICULAR FROM A POINT ON A LINE

SL AL

Length of perpendicular from a point (x_1,y_1) on the line $ax + by + c = 0$ is $\left|\frac{ax_1 + by_1 + c}{\sqrt{a^2+b^2}}\right|$

In particular, the length of the perpendicular from the origin on the line $ax + by + c = 0$ is $P = \frac{|c|}{\sqrt{a^2+b^2}}$

Illustrations

****Illustration 20.*** If the algebraic sum of perpendiculars from n given points on a variable straight line is zero then prove that the variable straight line passes through a fixed point.

Solution. Let n given points be (x_i, y_i) where i = 1, 2.... n and the variable straight line is $ax + by + c = 0$.

Given that $\sum_{i=1}^{n}\left(\frac{ax_i + by_i + c}{\sqrt{a^2+b^2}}\right) = 0$

$\Rightarrow \quad a\Sigma x_i + b\Sigma y_i + cn = 0$

$\Rightarrow \quad a\frac{\Sigma x_i}{n} + b\frac{\Sigma y_i}{n} + c = 0$.

Hence the variable straight line always passes through the fixed point $\left(\frac{\Sigma x_i}{n}, \frac{\Sigma y_i}{n}\right)$.

Ans.

Illustration 21. Prove that no line can be drawn through the point (4, –5) so that its distance from (–2, 3) will be equal to 12.

Solution. Suppose, if possible.

Equation of line through (4, –5) with slope of m is

$$y + 5 = m(x - 4)$$

$\Rightarrow \quad mx - y - 4m - 5 = 0$

Then $\dfrac{|m(-2) - 3 - 4m - 5|}{\sqrt{m^2 + 1}} = 12$

$\Rightarrow \quad |-6m - 8| = 12\sqrt{(m^2 + 1)}$

On squaring, $(6m + 8)^2 = 144(m^2 + 1)$

$\Rightarrow \quad 4(3m + 4)^2 = 144(m^2 + 1)$

$\Rightarrow \quad (3m + 4)^2 = 36(m^2 + 1)$

$\Rightarrow \quad 27m^2 - 24m + 20 = 0 \qquad \text{.... (i)}$

Since the discriminant of (i) is $(-24)^2 - 4.27.20 = -1584$ which is negative, there is no real value of m. Hence no such line is possible.

15.0 DISTANCE BETWEEN TWO PARALLEL LINES

SL AL

(a) The distance between two parallel lines $ax + by + c_1 = 0$ and $ax + by + c_2 = 0$ is $= \dfrac{|c_1 - c_2|}{\sqrt{a^2 + b^2}}$

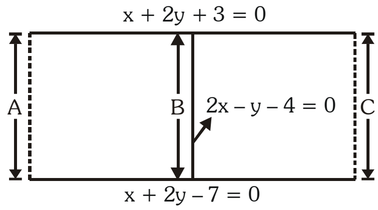
(***Note –*** The coefficients of x & y in both equations should be same)

(b) The area of the parallelogram $= \dfrac{p_1 p_2}{\sin\theta}$, where p_1 & p_2 are distances between two pairs of opposite sides & θ is the angle between any two adjacent sides . Note that area of the parallelogram bounded by the lines $y = m_1x + c_1$, $y = m_1x + c_2$ and $y = m_2x + d_1$, $y = m_2x + d_2$ is given by $\left|\dfrac{(c_1 - c_2)(d_1 - d_2)}{m_1 - m_2}\right|$.

Illustrations

Illustration 22. Three lines $x + 2y + 3 = 0$, $x + 2y - 7 = 0$ and $2x - y - 4 = 0$ form 3 sides of two squares. Find the equation of remaining sides of these squares.

Solution. Distance between the two parallel lines is $\dfrac{|7+3|}{\sqrt{5}} = 2\sqrt{5}$.



The equations of sides A and C are of the form $2x - y + k = 0$.

Since distance between sides A and B

= distance between sides

B and C $\dfrac{|k - (-4)|}{\sqrt{5}} = 2\sqrt{5}$

$\Rightarrow \quad \dfrac{k+4}{\sqrt{5}} = \pm 2\sqrt{5} \Rightarrow k = 6, -14.$

Hence the fourth sides of the two squares are (i) $2x - y + 6 = 0$ (ii) $2x - y - 14 = 0$.***Ans.***

16.0 POSITION OF TWO POINTS WITH RESPECT TO A GIVEN LINE

AL

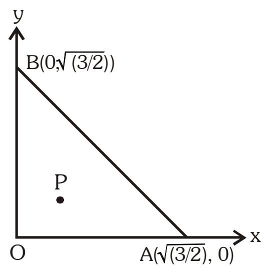
Let the given line be $ax + by + c = 0$ and $P(x_1, y_1)$, $Q(x_2, y_2)$ be two points. If the expressions $ax_1 + by_1 + c$ and $ax_2 + by_2 + c$ have the same signs, then both the points P and Q lie on the same side of the line $ax + by + c = 0$. If the quantities $ax_1 + by_1 + c$ and $ax_2 + by_2 + c$ have opposite signs, then they lie on the opposite sides of the line.

Illustrations

***Illustration 23.** Let P $(\sin\theta, \cos\theta)$ $(0 \le \theta \le 2\pi)$ be a point and let OAB be a triangle with vertices (0, 0), $\left(\sqrt{\frac{3}{2}}, 0\right)$ and $\left(0, \sqrt{\frac{3}{2}}\right)$. Find θ if P lies inside the ΔOAB.

Solution. Equations of lines along OA, OB and AB are $y = 0$, $x = 0$ and $x + y = \sqrt{\frac{3}{2}}$ respectively. Now P and B will lie on the same side of $y = 0$ if $\cos\theta > 0$. Similarly P and A will lie on the same side of $x = 0$ if $\sin\theta > 0$ and P and O will lie on the same side of $x + y = \sqrt{\frac{3}{2}}$ if $\sin\theta + \cos\theta < \sqrt{\frac{3}{2}}$.

Hence P will lie inside the ΔABC, if $\sin\theta > 0$, $\cos\theta > 0$ and $\sin\theta + \cos\theta < \sqrt{\frac{3}{2}}$.



Now $\sin\theta + \cos\theta < \sqrt{\frac{3}{2}}$

$\Rightarrow \quad \sin(\theta + \frac{\pi}{4}) < \frac{\sqrt{3}}{2}$

i.e. $0 < \theta + \frac{\pi}{4} < \pi/3$

or $\frac{2\pi}{3} < \theta + \frac{\pi}{4} < \pi$

Since $\sin\theta > 0$ and $\cos\theta > 0$,

so $0 < \theta < \frac{\pi}{12}$ or $\frac{5\pi}{12} < \theta < \frac{\pi}{2}$.

17.0 CONCURRENCY OF LINES

AL

(a) Three lines $a_1x + b_1y + c_1 = 0$; $a_2x + b_2y + c_2 = 0$ and $a_3x + b_3y + c_3 = 0$ are concurrent,

if $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_2 & c_3 \end{vmatrix} = 0$

(b) To test the concurrency of three lines, first find out the point of intersection of any two of the three lines. If this point lies on the remaining line (i.e. coordinates of the point satisfy the equation of the line) then the three lines are concurrent otherwise not concurrent.

Illustrations

***Illustration 24.** If the lines ax + by + p = 0, xcosα + ysinα – p = 0 (p ≠ 0) and xsinα – ycosα = 0 are concurrent and the first two lines include an angle $\frac{\pi}{4}$, then $a^2 + b^2$ is equal to -

(A) 1 (B) 2 (C) 4 (D) p^2

Solution. Since the given lines are concurrent,

$$\begin{vmatrix} a & b & p \\ \cos\alpha & \sin\alpha & -p \\ \sin\alpha & -\cos\alpha & 0 \end{vmatrix} = 0$$

$\Rightarrow \quad a\cos\alpha + b\sin\alpha + 1 = 0$ ..... (i)

As ax + by + p = 0 and $x\cos\alpha + y\sin\alpha - p = 0$ include an angle $\frac{\pi}{4}$.

$$\pm\tan\frac{\pi}{4} = \frac{-\frac{a}{b} + \frac{\cos\alpha}{\sin\alpha}}{1 + \frac{a}{b}\frac{\cos\alpha}{\sin\alpha}}$$

$\Rightarrow \quad$ –a sinα + bcosα = ± (bsinα + acosα)

$\Rightarrow \quad$ –a sinα + bcosα = ±1 [from (i)] ..... (ii)

Squaring and adding (i) & (ii), we get

$a^2 + b^2 = 2$. ***Ans. (B)***

BEGINNER'S BOX-4

TOPIC COVERED : LENGTH OF PERPENDICULAR, SHORTEST DISTANCE BETWEEN TWO PARALLEL LINES, CONCURRENCY OF LINES, POSITION OF TWO POINTS W.R.T. A LINE

1. Classify the following pairs of lines as coincident, parallel or intersecting :
(a) x + 2y – 3 = 0 & –3x – 6y + 9 = 0
(b) x + 2y + 1 = 0 & 2x + 4y + 3 = 0
(c) 3x – 2y + 5 = 0 & 2x + y – 5 = 0

2. Find the distances between the following pair of parallel lines :
(a) 3x + 4y = 13, 3x + 4y = 3
(b) 3x – 4y + 9 = 0, 6x – 8y – 15 = 0

3. Find the points on the x-axis such that their perpendicular distance from the line $\frac{x}{a} + \frac{y}{b} = 1$ is 'a', a, b > 0.

***4.** Show that the area of the parallelogram formed by the lines
2x – 3y + a = 0, 3x – 2y – a = 0, 2x – 3y + 3a = 0 and 3x – 2y – 2a = 0 is $\frac{2a^2}{5}$ square units.

5. Examine the positions of the points (3, 4) and (2, – 6) w.r.t. 3x – 4y = 8

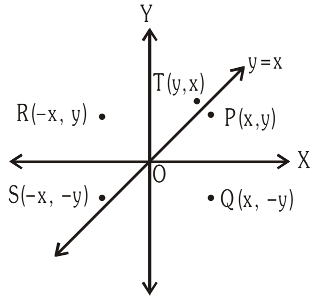
***6.** If (2, 9), (– 2, 1) and (1, –3) are the vertices of a triangle, then prove that the origin lies inside the triangle.

7. Find the equation of the line joining the point (2, – 9) and the point of intersection of lines 2x + 5y – 8 = 0 and 3x – 4y – 35 = 0.

8. Find the value of λ, if the lines 3x – 4y – 13 = 0, 8x – 11 y – 33 = 0 and 2x – 3y + λ = 0 are concurrent.

18.0 REFLECTION OF A POINT

AL



Let P(x, y) be any point, then its image with respect to

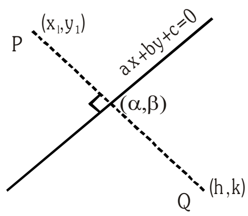
(a) x-axis is Q(x, –y)

(b) y-axis is R(–x, y)

(c) origin is S(–x,–y)

(d) line y = x is T(y, x)

(e) Reflection of a point about any arbitrary line : The image (h,k) of a point $P(x_1, y_1)$ about the line ax + by + c = 0 is given by following formula.



$$\frac{h-x_1}{a}=\frac{k-y_1}{b}=-2\frac{(ax_1+by_1+c)}{a^2+b^2}$$

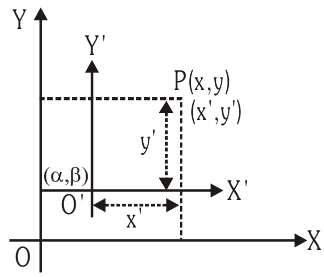
and the foot of perpendicular (α,β) from a point (x_1, y_1) on the line ax + by + c = 0 is given by following formula.

$$\frac{\alpha-x_1}{a}=\frac{\beta-y_1}{b}=-\frac{ax_1+by_1+c}{a^2+b^2}$$

19.0 TRANSFORMATION OF AXES

AL

19.1 Shifting of origin without rotation of axes



Let P (x, y) with respect to axes OX and OY.

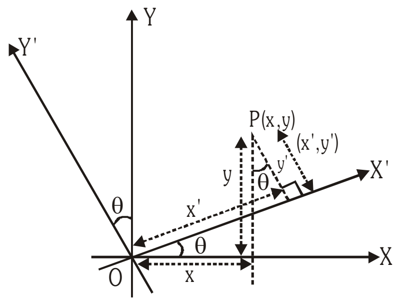
Let O' (α, β) is new origin with respect to axes OX and OY and let P (x', y') with respect to axes O'X' and O'Y' , where OX and O'X' are parallel and OY and O'Y' are parallel.

Then x = x' + α, y = y' + β

or x' = x – α, y' = y – β

Thus if origin is shifted to point (α, β) without rotation of axes, then new equation of curve can be obtained by putting x + α in place of x and y + β in place of y.

19.2 Rotation of axes without shifting the origin



Let O be the origin. Let P (x, y) with respect to axes OX and OY and let P (x', y') with respect to axes OX' and OY' where ∠X'OX = ∠YOY' = θ, where θ is measured in anticlockwise direction.

then x = x' cos θ – y' sin θ

y = x' sin θ + y' cos θ

and x' = x cos θ + y sin θ

y' = –x sin θ + y cos θ

The above relation between (x, y) and (x', y') can be easily obtained with the help of following table

Old / New	x ↓	y ↓
x' →	cos θ	sin θ
y' →	–sin θ	cos θ

Illustrations

Illustration 25. Through what angle should the axes be rotated so that the equation $9x^2 - 2\sqrt{3}\,xy + 7y^2 = 10$ may be changed to $3x^2 + 5y^2 = 5$?

Solution. Let angle be θ then replacing (x, y) by $(x\cos\theta - y\sin\theta,\ x\sin\theta + y\cos\theta)$

then $9x^2 - 2\sqrt{3}xy + 7y^2 = 10$ becomes

$9(x\cos\theta - y\sin\theta)^2 - 2\sqrt{3}(x\cos\theta - y\sin\theta)(x\sin\theta + y\cos\theta) + 7(x\sin\theta + y\cos\theta)^2 = 10$

$\Rightarrow\quad x^2(9\cos^2\theta - 2\sqrt{3}\sin\theta\cos\theta + 7\sin^2\theta) + 2xy(-9\sin\theta\cos\theta - \sqrt{3}\cos2\theta + 7\sin\theta\cos\theta)$

$+ y^2(9\cos^2\theta + 2\sqrt{3}\sin\theta\cos\theta + 7\cos^2\theta) = 10$

On comparing with $3x^2 + 5y^2 = 5$ (coefficient of xy = 0)

We get $\quad -9\sin\theta\cos\theta - \sqrt{3}\cos2\theta + 7\sin\theta\cos\theta = 0$

or $\quad \sin2\theta = -\sqrt{3}\cos2\theta$

or $\quad \tan2\theta = -\sqrt{3} = \tan(180° - 60°)$

or $\quad 2\theta = 120° \qquad \therefore \qquad \theta = 60°$

20.0 EQUATION OF BISECTORS OF ANGLES BETWEEN TWO LINES

AL

If equation of two intersecting lines are $a_1x+b_1y+c_1=0$ and $a_2x + b_2y+c_2=0$, then equation of bisectors of the angles between these lines are written as :

$$\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}} = \pm\frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}} \qquad ...(i)$$

20.1 Equation of bisector of angle containing origin

If the equation of the lines are written with constant terms c_1 and c_2 positive, then the equation of the bisectors of the angle containing the origin is obtained by taking positive sign in (i)

20.2 Equation of bisector of acute/obtuse angles

To find the equation of the bisector of the acute or obtuse angle :

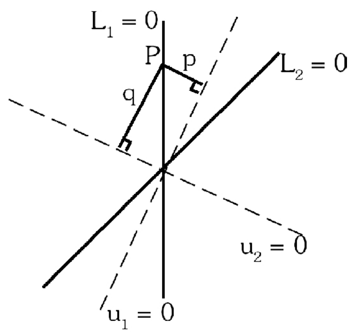
(i) let ϕ be the angle between one of the two bisectors and one of two given lines. Then if $\tan\phi < 1$ i.e. $\phi < 45°$ i.e. $2\phi < 90°$, the angle bisector will be bisector of acute angle.

(ii) See whether the constant terms c_1 and c_2 in the two equation are +ve or not. If not then multiply both sides of given equation by –1 to make the constant terms positive.

If sign of $a_1a_2 + b_1b_2$	For obtuse angle bisector	For acute angle bisector
+	use + sign in eq. (1)	use − sign in eq. (1)
−	use − sign in eq. (1)	use + sign in eq. (1)

i.e. if $a_1a_2 + b_1b_2 > 0$, then the bisector corresponding to + sign gives obtuse angle bisector

$$\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}} = \frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}}$$



(iii) Another way of identifying an acute and obtuse angle bisector is as follows :

Let $L_1 = 0$ & $L_2 = 0$ are the given lines & $u_1 = 0$ and $u_2 = 0$ are the bisectors between $L_1 = 0$ & $L_2 = 0$. Take a point P on any one of the lines $L_1 = 0$ or $L_2 = 0$ and drop perpendicular on $u_1 = 0$ & $u_2 = 0$ as shown . If,

$|p| < |q| \Rightarrow u_1$ is the acute angle bisector .

$|p| > |q| \Rightarrow u_1$ is the obtuse angle bisector .

$|p| = |q| \Rightarrow$ the lines L_1 & L_2 are perpendicular .

Note – Equation of straight lines passing through $P(x_1, y_1)$ & equally inclined with the lines $a_1x + b_1y + c_1 = 0$ & $a_2x + b_2y + c_2 = 0$ are those which are parallel to the bisectors between these two lines & passing through the point P.

Illustrations

***Illustration 26.** For the straight lines $4x + 3y - 6 = 0$ and $5x + 12y + 9 = 0$, find the equation of the
(i) bisector of the obtuse angle between them. (ii) bisector of the acute angle between them.
(iii) bisector of the angle which contains origin. (iv) bisector of the angle which contains (1, 2).

Solution. Equations of bisectors of the angles between the given lines are

$$\frac{4x+3y-6}{\sqrt{4^2+3^2}} = \pm\frac{5x+12y+9}{\sqrt{5^2+12^2}} \Rightarrow 9x - 7y - 41 = 0 \text{ and } 7x + 9y - 3 = 0$$

If θ is the acute angle between the line $4x + 3y - 6 = 0$ and the bisector

$9x - 7y - 41 = 0$, then $\tan\theta = \left|\dfrac{-\frac{4}{3}-\frac{9}{7}}{1+\left(\frac{-4}{3}\right)\frac{9}{7}}\right| = \dfrac{11}{3} > 1$

Hence
(i) bisector of the obtuse angle is $9x - 7y - 41 = 0$
(ii) bisector of the acute angle is $7x + 9y - 3 = 0$
(iii) bisector of the angle which contains origin

$$\frac{-4x-3y+6}{\sqrt{(-4)^2+(-3)^2}} = \frac{5x+12y+9}{\sqrt{5^2+12^2}} \Rightarrow 7x+9y-3=0$$

(iv) $L_1(1, 2) = 4 \times 1 + 3\times 2 - 6 = 4 > 0$
$L_2(1, 2) = 5 \times 1 + 12 \times 2 + 9 = 38 > 0$

+ve sign will give the required bisector, $\dfrac{4x+3y-6}{5} = +\dfrac{5x+12y+9}{13}$

$\Rightarrow$ $9x - 7y - 41 = 0$.

Alternative

Making c_1 and c_2 positive in the given equation, we get $-4x - 3y + 6 = 0$ and $5x + 12y + 9 = 0$

Since $a_1a_2 + b_1b_2 = -20 - 36 = -56 < 0$, so the origin will lie in the acute angle.
Hence bisector of the acute angle is given by

$$\frac{-4x-3y+6}{\sqrt{4^2+3^2}} = \frac{5x+12y+9}{\sqrt{5^2+12^2}} \Rightarrow 7x+9y-3=0$$

Similarly bisector of obtuse angle is $9x - 7y - 41 = 0$.

***Illustration 27.** A ray of light is sent along the line $x - 2y - 3 = 0$. Upon reaching the line mirror $3x - 2y - 5 = 0$, the ray is reflected from it. Find the equation of the line containing the reflected ray.

Solution. Let Q be the point of intersection of the incident ray and the line mirror, then

$x_1 - 2y_1 - 3 = 0$ & $3x_1 - 2y_1 - 5 = 0$

on solving these equations, we get

$x_1 = 1$ & $y_1 = -1$

Since $P(-1, -2)$ be a point lies on the incident ray, so we can find the image of the point P on the reflected ray about the line mirror (by property of reflection).
Let $P'(h, k)$ be the image of point P about line mirror, then

$$\frac{h+1}{3} = \frac{k+2}{-2} = \frac{-2(-3+4-5)}{13}$$

$\Rightarrow \quad h = \frac{11}{13}$ and $k = \frac{-42}{13}$.

So, $\quad P'\left(\frac{11}{13}, \frac{-42}{13}\right)$

Then equation of reflected ray will be

$$(y + 1) = \frac{\left(\frac{-42}{13}+1\right)(x-1)}{\left(\frac{11}{13}-1\right)}$$

$\Rightarrow \quad 2y - 29x + 31 = 0$ is the required equation of reflected ray.

21.0 FAMILY OF LINES

AL

If equation of two lines be $P \equiv a_1x + b_1y + c_1 = 0$ and $Q \equiv a_2x + b_2y + c_2 = 0$, then the equation of the lines passing through the point of intersection of these lines is :
$P + \lambda Q = 0$ or $a_1x + b_1y + c_1 + \lambda\,(a_2x + b_2y + c_2) = 0$. The value of λ is obtained with the help of the additional informations given in the problem.

Illustrations

***Illustration 28.** Prove that each member of the family of straight lines $(3\sin\theta + 4\cos\theta)x + (2\sin\theta - 7\cos\theta)y + (\sin\theta + 2\cos\theta) = 0$ (θ is a parameter) passes through a fixed point.

Solution. The given family of straight lines can be rewritten as
$(3x + 2y + 1)\sin\theta + (4x - 7y + 2)\cos\theta = 0$
or, $(4x - 7y + 2) + \tan\theta(3x + 2y + 1) = 0$ which is of the form $L_1 + \lambda L_2 = 0$
Hence each member of it will pass through a fixed point which is the intersection of

$4x - 7y + 2 = 0$ and $3x + 2y + 1 = 0$ i.e. $\left(\frac{-11}{29}, \frac{2}{29}\right)$.

BEGINNER'S BOX-5

TOPIC COVERED : TRONSFORMATION OF AXIS, ANGLE BISECTOR AND FAMILY OF LINES

***1.** The point (4, 1) undergoes the following transformations, then the match the correct alternatives :

	Column-I		***Column-II***
(A)	Reflection about x-axis is	(p)	(4, –1)
(B)	Reflection about y-axis is	(q)	(–4, –1)
(C)	Reflection about origin is	(r)	$\left(-\frac{12}{25}, -\frac{59}{25}\right)$
(D)	Reflection about the line $y = x$ is	(s)	(–4, 1)
(E)	Reflection about the line $4x + 3y - 5 = 0$ is	(t)	(1, 4)

2. On what point must the origin be shifted, if the coordinates of a point (4, 5) become (– 3, 9).

3. If the axes be turned through an angle $\tan^{-1} 2$ (in anticlockwise direction), what does the equation $4xy - 3x^2 = a^2$ become ?

4. Find the equations of bisectors of the angle between the lines $4x + 3y = 7$ and $24x + 7y - 31 = 0$. Also find which of them is (a) the bisector of the angle containing origin (b) the bisector of the acute angle.

5. Find the equations of the line which pass through the point of intersection of the lines $4x - 3y = 1$ and $2x - 5y + 3 = 0$ and is equally inclined to the coordinate axes.

6. Find the equation of the line through the point of intersection of the lines $3x - 4y + 1 = 0$ & $5x + y - 1 = 0$ and perpendicular to the line $2x - 3y = 10$.

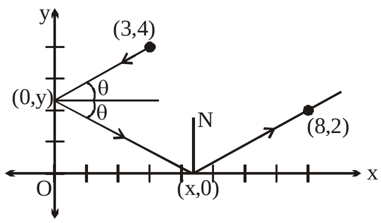
7. The sides of a triangle ABC lie on the lines $3x + 4y = 0$; $4x + 3y = 0$ and $x = 3$. Let (h, k) be the centre of the circle inscribed in ΔABC. The value of (h + k) equals
(A) 0 (B) 1/4 (C) – 1/4 (D) 1/2

***8.** A ray of light passing through the point A(1, 2) is reflected at a point B on the x–axis and then passes through (5, 3). Then the equation of AB is :
(A) $5x + 4y = 13$ (B) $5x - 4y = -3$
(C) $4x + 5y = 14$ (D) $4x - 5y = -6$

***9.** In a triangle ABC, side AB has the equation $2x + 3y = 29$ and the side AC has the equation, $x + 2y = 16$. If the mid–point of BC is (5, 6) then the equation of BC is :
(A) $x - y = -1$ (B) $5x - 2y = 13$ (C) $x + y = 11$ (D) $3x - 4y = -9$

***10.** A ray of light is sent along the line $x - 2y - 3 = 0$. Upon reaching the line $3x - 2y - 5 = 0$, the ray is reflected from it. If the equation of the line containing the reflected ray is $ax - 2y = b$, then find the value of (a + b).

***11.** Suppose that a ray of light leaves the point (3, 4), reflects off the y-axis towards the x-axis, reflects off the x-axis, and finally arrives at the point (8, 2). The value of x, is



(A) $x = 4\frac{1}{2}$ (B) $x = 4\frac{1}{3}$

(C) $x = 4\frac{2}{3}$ (D) $x = 5\frac{1}{3}$

22.0 PAIR OF STRAIGHT LINES

AL

22.1 Homogeneous equation of second degree

Let us consider the homogeneous equation of 2nd degree as

$ax^2 + 2hxy + by^2 = 0$ ...(i)

which represents pair of straight lines passing through the origin.

Now, we divide by x^2, we get

$$a + 2h\left(\frac{y}{x}\right) + b\left(\frac{y}{x}\right)^2 = 0$$

$\frac{y}{x} = m$ (say)

then $a + 2hm + bm^2 = 0$ ...(ii)

if m_1 & m_2 are the roots of equation (ii), then $m_1 + m_2 = -\frac{2h}{b}$, $m_1m_2 = \frac{a}{b}$

and also, $\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1m_2}\right| = \left|\frac{\sqrt{(m_1+m_2)^2 - 4m_1m_2}}{1+m_1m_2}\right| = \left|\frac{\sqrt{\frac{4h^2}{b^2} - \frac{4a}{b}}}{1+\frac{a}{b}}\right| = \pm\,\mathbf{\frac{2\sqrt{h^2 - ab}}{a + b}}$

These line will be –

(i) Real and different, if $h^2 - ab > 0$

(ii) Real and coincident, if $h^2 - ab = 0$

(iii) Imaginary, if $h^2 - ab < 0$

(iv) The condition that these lines are :

(1) At right angles to each other is $a + b = 0$. i.e. coefficient of x^2 + coefficient of $y^2 = 0$.

(2) Coincident is $h^2 = ab$.

(3) Equally inclined to the axes of x is $h = 0$. i.e. coefficient of $xy = 0$.

Homogeneous equation of 2nd degree $ax^2 + 2hxy + by^2 = 0$ always represent a pair of straight lines whose equations are

$$y = \left(\frac{-h \pm \sqrt{h^2 - ab}}{b}\right) x \equiv y = m_1 x \ \& \ y = m_2 x \quad \text{and} \quad m_1 + m_2 = -\frac{2h}{b} \ ; m_1 m_2 = \frac{a}{b}$$

These straight lines passes through the origin.

Note – A homogeneous equation of degree n represents n straight lines passing through ***origin***.

22.2 The combined equation of angle bisectors

The combined equation of angle bisectors between the lines represented by homogeneous equation of 2nd degree is given by $\dfrac{\mathbf{x^2 - y^2}}{\mathbf{a - b}} = \dfrac{\mathbf{xy}}{\mathbf{h}}$, $a \neq b$, $h \neq 0$.

NOTE (i) If $a = b$, the bisectors are $x^2 - y^2 = 0$ i.e. $x - y = 0$, $x + y = 0$

(ii) If $h = 0$, the bisectors are $xy = 0$ i.e. $x = 0$, $y = 0$.

(iii) The two bisectors are always at right angles, since we have coefficient of x^2 + coefficient of $y^2 = 0$

22.3 General Equation and Non-homogeneous Equation of Second Degree

(i) The general equation of second degree $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of straight lines, if $\Delta = abc + 2fgh - af^2 - bg^2 - ch^2 = 0$ i.e. $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$

(ii) If θ be the angle between the lines, then $\tan\theta = \pm\dfrac{2\sqrt{h^2 - ab}}{a + b}$

Obviously these lines are

(1) Parallel, if $\Delta = 0$, $h^2 = ab$ or if $h^2 = ab$ and $bg^2 = af^2$

(2) Perpendicular, if $a + b = 0$ i.e. coeff. of x^2 + coeff. of $y^2 = 0$.

Illustrations

Illustration 29. If $\lambda x^2 - 10xy + 12y^2 + 5x - 16y - 3 = 0$ represents a pair of straight lines, then λ is equal to -

(A) 4 (B) 3 (C) 2 (D) 1

Solution. Here $a = \lambda$, $b = 12$, $c = -3$, $f = -8$, $g = 5/2$, $h = -5$

Using condition $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$, we have

$\lambda(12)(-3) + 2(-8)(5/2)(-5) - \lambda(64) - 12(25/4) + 3(25) = 0$

$\Rightarrow \quad -36\lambda + 200 - 64\lambda - 75 + 75 = 0 \quad \Rightarrow \quad 100\lambda = 200$

$\therefore \quad \lambda = 2$ ***Ans. (C)***

***Illustration 30.** Show that the two straight lines $x^2(\tan^2\theta+\cos^2\theta)-2xy\tan\theta+y^2\sin^2\theta=0$ represented by the equation are such that the difference of their slopes is 2.

Solution. The given equation is $x^2(\tan^2\theta+\cos^2\theta)-2xy\tan\theta+y^2\sin^2\theta=0$ .... (i)

and general equation of second degree $ax^2+2hxy+by^2=0$ .... (ii)

Comparing (i) and (ii), we get $a=\tan^2\theta+\cos^2\theta$

$h=-\tan\theta$

and $b=\sin^2\theta$

Let separate lines of (ii) are $y=m_1x$ and $y=m_2x$

where $m_1=\tan\theta_1$ and $m_2=\tan\theta_2$

therefore, $m_1+m_2=-\dfrac{2h}{b}=\dfrac{2\tan\theta}{\sin^2\theta}$

and $m_1.m_2=\dfrac{\tan^2\theta+\cos^2\theta}{\sin^2\theta}$

$\therefore\quad m_1-m_2=\sqrt{(m_1+m_2)^2-4m_1m_2}$

$\Rightarrow\quad \tan\theta_1-\tan\theta_2=\sqrt{\dfrac{4\tan^2\theta}{\sin^4\theta}-\dfrac{4(\tan^2\theta+\cos^2\theta)}{\sin^2\theta}}$

$=\dfrac{2}{\sin^2\theta}\sqrt{\tan^2\theta-\sin^2\theta(\tan^2\theta+\cos^2\theta)}$

$=\dfrac{2\sin\theta}{\sin^2\theta}\sqrt{(\sec^2\theta-\tan^2\theta-\cos^2\theta)}$

$=\dfrac{2\sin\theta}{\sin^2\theta}\sqrt{(1-\cos^2\theta)}=\dfrac{2}{\sin\theta}\sin\theta=2$ ***Ans.***

23.0 HOMOGENIZATION

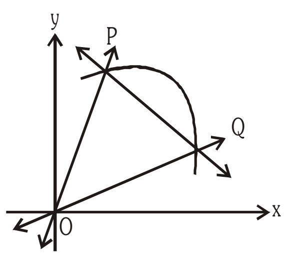
AL

(a) Let the equation of curve be –

$ax^2+2hxy+by^2+2gx+2fy+c=0$ ...(i)

and straight line be

$lx+my+n=0$ ...(ii)



Now joint equation of line OP and OQ joining the origin and points of intersection P and Q can be obtained by making the equation (i) homogenous with the help of equation of the line. Thus required equation is given by

$$ax^2+2hxy+by^2+2(gx+fy)\left(\frac{\ell x+my}{-n}\right)+c\left(\frac{\ell x+my}{-n}\right)^2=0$$

$\Rightarrow\quad (an^2+2gln+cl^2)x^2+2(hn^2+gmn+fln+clm)xy+(bn^2+2fmn+cm^2)y^2=0$ ...(iii)

All points which satisfy (i) and (ii) simultaneously, will satisfy (iii)

(b) Any second degree curve through the four points of intersection of $f(x, y)=0$ & $xy=0$ is given by $f(x, y)+\lambda\,xy=0$ where $f(x, y)=0$ is also a second degree curve.

Illustrations

***Illustration 31.** The chord $\sqrt{6}y = \sqrt{8}px + \sqrt{2}$ of the curve $py^2 + 1 = 4x$ subtends a right angle at origin then find the value of p.

Solution. $\sqrt{3}y - 2px = 1$ is the given chord. Homogenizing the equation of the curve, we get,

$py^2 - 4x(\sqrt{3}\,y - 2px) + (\sqrt{3}\,y - 2px)^2 = 0$

$\Rightarrow$ $(4p^2 + 8p)x^2 + (p + 3)y^2 - 4\sqrt{3}\,xy - 4\sqrt{3}\,pxy = 0$

Now, angle at origin is 90°

$\therefore$ coefficient of x^2 + coefficient of $y^2 = 0$

$\therefore$ $4p^2 + 8p + p + 3 = 0 \Rightarrow 4p^2 + 9p + 3 = 0$

$\therefore$ $p = \dfrac{-9 \pm \sqrt{81-48}}{8} = \dfrac{-9 \pm \sqrt{33}}{8}$.

BEGINNER'S BOX-6

TOPIC COVERED : PAIR OF STRAIGHT LINES, HOMOGENIZATION

1. Prove that the equation $x^2 - 5xy + 4y^2 = 0$ represents two lines passing through the origin. Also find their equations.

2. If the equation $6x^2 - 11xy - 10y^2 - 19y + c = 0$ represents a pair of lines, find their equations. Also find the angle between the two lines.

***3.** Find the angle subtended at the origin by the intercept made on the curve $x^2 - y^2 - xy + 3x - 6y + 18 = 0$ by the line $2x - y = 3$.

***4.** Find the equation of the lines joining the origin to the points of intersection of the curve $2x^2 + 3xy - 4x + 1 = 0$ and the line $3x + y = 1$.

5. Let $S = \{(x, y) \mid x^2 + 2xy + y^2 - 3x - 3y + 2 = 0\}$, then S
(A) consists of two coincident lines.
(B) consists of two parallel lines which are not coincident.
(C) consists of two intersecting lines.
(D) is a parabola.

***6.** If the straight lines joining the origin and the points of intersection of the curve
$5x^2 + 12xy - 6y^2 + 4x - 2y + 3 = 0$ and $x + ky - 1 = 0$
are equally inclined to the co-ordinate axes then the value of k :
(A) is equal to 1 (B) is equal to -1
(C) is equal to 2 (D) does not exist in the set of real numbers .

***7.** The angles between the straight lines joining the origin to the points common to
$7x^2 + 8y^2 - 4xy + 2x - 4y - 8 = 0$ and $3x - y = 2$ is :
(A) $\tan^{-1}\sqrt{2}$ (B) $\dfrac{\pi}{3}$ (C) $\dfrac{\pi}{4}$ (D) $\dfrac{\pi}{2}$

***8.** A pair of perpendicular straight lines is drawn through the origin forming with the line $2x + 3y = 6$ an isosceles triangle right angled at the origin . The equation to the line pair is :
(A) $5x^2 - 24xy - 5y^2 = 0$ (B) $5x^2 - 26xy - 5y^2 = 0$
(C) $5x^2 + 24xy - 5y^2 = 0$ (D) $5x^2 + 26xy - 5y^2 = 0$

***9.** Through a point A on the x-axis a straight line is drawn parallel to y-axis so as to meet the pair of straight lines $ax^2 + 2hxy + by^2 = 0$ in B and C. If AB = BC then
(A) $h^2 = 4ab$ (B) $8h^2 = 9ab$ (C) $9h^2 = 8ab$ (D) $4h^2 = ab$

GOLDEN KEY POINTS

- Equidistant collinear points have their x co-ordinates (or y-co-ordinates) in A.P.
- If the triangle is equilateral, then centroid, incentre, orthocentre, circumcentre coincide.
- In a triangle orthocentre, centroid and circumcentre are always collinear and centroid divides the line joining orthocentre and circumcentre in the ratio 2 : 1
- In an isosceles triangle centroid, orthocentre, incentre & circumcentre lie on the same line.
- Image point of Orthocentre through any side of triangle lies on circumcircle.
- Pair of straight lines perpendicular to the lines $ax^2 + 2hxy + by^2 = 0$ and through origin are given by $bx^2 - 2hxy + ay^2 = 0$.
- The product of the perpendiculars drawn from the point (x_1, y_1) on the lines $ax^2 + 2hxy + by^2 = 0$ is

  $$\left|\frac{ax_1^2 + 2hx_1y_1 + by_1^2}{\sqrt{(a-b)^2 + 4h^2}}\right|$$

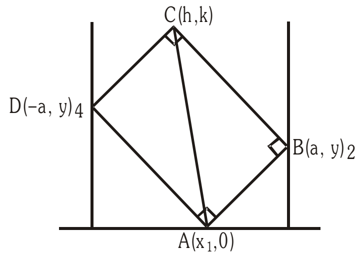
- The product of the perpendiculars drawn from the origin to the lines

  $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ is $\left|\dfrac{c}{\sqrt{(a-b)^2 + 4h^2}}\right|$

SOME WORKED OUT ILLUSTRATIONS

***Illustration 1.** ABCD is a variable rectangle having its sides parallel to fixed directions (say m). The vertices B and D lie on x = a and x = –a and A lies on the line y = 0. Find the locus of C.

Solution. Let A be $(x_1, 0)$, B(a, y_2) and D be $(-a, y_4)$. We are given AB and AD have fixed directions and hence their slopes are constants. i.e. m & m_1 (say)



$$\therefore \quad \frac{y_2}{a-x_1} = m \text{ and } \frac{y_4}{-a-x_1} = m_1 .$$

Further, $mm_1 = -1$. Since ABCD is a rectangle.

$$\frac{y_2}{a-x_1} = m \text{ and } \frac{y_4}{-a-x_1} = -\frac{1}{m}$$

The mid point of BD is $\left(0, \frac{y_2+y_4}{2}\right)$ and mid point of AC is $\left(\frac{x_1+h}{2}, \frac{k}{2}\right)$,

where C is taken to be (h, k). This gives $h = -x_1$

and $k = y_2 + y_4$. So C is $(-x_1, y_2 + y_4)$.

Also, $\frac{y_2}{a-x_1} = m$ and $\frac{y_4}{a+x_1} = \frac{1}{m}$

gives $m(k - y_2) = a + x_1 = m(k - m(a - x_1)) = a + x_1$

$\Rightarrow \quad mk - m^2(a - x_1) = a + x_1$

$\Rightarrow \quad m^2(a + h) - mk + a - h = 0$

$\Rightarrow \quad (m^2 - 1)h - mk = -(m^2 + 1)a$

$\Rightarrow \quad (1 - m^2)h + mk = (m^2 + 1)a$

$\Rightarrow \quad (1 - m^2)x + my = (m^2 + 1)a$

The locus of C is $(1 - m^2)x + my = (m^2 + 1)a$.

***Illustration 2.** Prove that the co-ordinates of the vertices of an equilateral triangle can not all be rational.

Solution. Let $A(x_1\ y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ be the vertices of a triangle ABC. If possible let $x_1, y_1, x_2, y_2, x_3, y_3$ be all rational.

Now area of $\Delta ABC = \frac{1}{2}\,|x_1(y_2-y_3) + x_2(y_3-y_1) + x_3(y_1-y_2)|$ = Rational ... (i)

Since ΔABC is equilateral

$$\therefore \quad \text{Area of } \Delta ABC = \frac{\sqrt{3}}{4}(\text{side})^2 = \frac{\sqrt{3}}{4}(AB)^2$$

$$= \frac{\sqrt{3}}{4}\{(x_1-x_2)^2 + (y_1-y_2)^2\} = \text{Irrational} \qquad \text{... (ii)}$$

From (i) and (ii),

Rational = Irrational

which is contradiction.

Hence $x_1, y_1, x_2, y_2, x_3, y_3$ cannot all be rational.

***Illustration 3.** A variable line is drawn through O, to cut two fixed straight lines L_1 and L_2 in A_1 and A_2, respectively. A point A is taken on the variable line such that $\frac{m+n}{OA} = \frac{m}{OA_1} + \frac{n}{OA_2}$.

Show that the locus of A is a straight line passing through the point of intersection of L_1 and L_2 where O is being the origin.

Solution. Let the variable line passing through the origin is $\frac{x}{\cos\theta} = \frac{y}{\sin\theta} = r_i$ ...... (i)

Let the equation of the line L_1 is $p_1x + q_1y = 1$ ...... (ii)

Equation of the line L_2 is $p_2x + q_2y = 1$ ...... (iii)

the variable line intersects the line (ii) at A_1 and (iii) at A_2.

Let $OA_1 = r_1$.

Then $A_1 = (r_1\cos\theta, r_1\sin\theta) \Rightarrow A_1$ lies on L_1

$\Rightarrow \quad r_1 = OA_1 = \dfrac{1}{p_1\cos\theta + q_1\sin\theta}$

Similarly, $r_2 = OA_2 = \dfrac{1}{p_2\cos\theta + q_2\sin\theta}$

Let OA = r

Let co-ordinate of A are (h, k) $\Rightarrow$ (h, k) $\equiv$ ($r\cos\theta$, $r\sin\theta$)

Now $\frac{m+n}{r} = \frac{m}{OA_1} + \frac{n}{OA_2} \Rightarrow \frac{m+n}{r} = \frac{m}{r_1} + \frac{n}{r_2}$

$\Rightarrow \quad m + n = m(p_1r\cos\theta + q_1r\sin\theta) + n(p_2r\cos\theta + q_2r\sin\theta)$

$\Rightarrow \quad (p_1h + q_1k - 1) + \frac{n}{m}(p_2h + q_2k - 1) = 0$

Therefore, locus of A is $(p_1x+q_1y - 1) + \frac{n}{m}(p_2x + q_2y - 1) = 0$

$\Rightarrow \quad L_1 + \lambda L_2 = 0$ where $\lambda = \frac{n}{m}$.

This is the equation of the line passing through the intersection of L_1 and L_2.

***Illustration 4.** If the straight line $3x + 4y + 5 - k(x + y + 3) = 0$ is parallel to y-axis, then the value of k is -

(A) 1 (B) 2 (C) 3 (D) 4

Solution. A straight line is parallel to y-axis, if its y - coefficient is zero, i.e. $4 - k = 0$ i.e. $k = 4$.***Ans. (D)***

***Illustration 5.** If pairs of straight lines $x^2 - 2pxy - y^2 = 0$ and $x^2 - 2qxy - y^2 = 0$ be such that each pair bisects the angle between the other pair, prove that $pq = -1$.

Solution. According to the question, the equation of the bisectors of the angle between the lines

$x^2 - 2pxy - y^2 = 0$ ... (i)

is $x^2 - 2qxy - y^2 = 0$ ... (ii)

$\therefore$ The equation of bisectors of the angle between the lines (i) is $\frac{x^2 - y^2}{1-(-1)} = \frac{xy}{-p}$

$\Rightarrow \quad -px^2 - 2xy + py^2 = 0$

Since (ii) and (iii) are identical, comparing (ii) and (iii), we get $\frac{1}{-p} = \frac{-2q}{-2} = \frac{-1}{p}$

$\Rightarrow \quad pq = -1$

ANSWERS

BEGINNER'S BOX-1

1. PQ = $\sqrt{34}$; **2.** x = 6 or x = 0 **3.** 11, –7

4. ***(a)*** (2,1) ; ***(b)*** (7,16) ; **5.** ***(a)*** 2 : 3 (internally) ; ***(b)*** 9 : 4 (externally) ; ***(c)*** 8 : 7 (internally)

6. ***(a)*** $\left(\frac{5}{3},3\right)$; ***(b)*** $\left(-\frac{9}{2},\frac{15}{2}\right)$, $\frac{5\sqrt{10}}{2}$, ***(c)*** (14, – 6)

7. (i) 8; (b) $2a\sin\left|\frac{\theta-\phi}{2}\right|$ **8.** (D) **9.** (C) **10.** (C)

BEGINNER'S BOX-2

1. 25 square units; **2.** 132 square units ; **5.** 1 **6.** x = ±2 ; **7.** y = ± x ;

8. $y^2 = 4ax$ **10.** $\left(0,\frac{1}{\sqrt{2}}\right)$ **11.** (B) **12.** (A)

BEGINNER'S BOX-3

1. ***(a)*** $y=\frac{2x}{3}+\frac{5}{3},\ \frac{2}{3},\ \frac{5}{3}$; ***(b)*** $\frac{x}{(-5/2)}+\frac{y}{(5/3)}=1,\ -\frac{5}{2},\ \frac{5}{3}$;

(c) $-\frac{2x}{\sqrt{13}}+\frac{3y}{\sqrt{13}}=\frac{5}{\sqrt{13}},\ \frac{5}{\sqrt{13}},\ \alpha=-\tan^{-1}\left(\frac{3}{2}\right)$; **2.** $13\sqrt{2}/3$ units

3. 15 **4.** (A) **5.** (A) **6.** (C) **7.**(A)

8. (C) **9.** (AB) **10.** (BC)

11. θ = 135° or 45° **12.** 3x + 4y = 18 ; **13.** 2x – 3y + 12 = 0, (–6, 0)

14. (a) $2x^2+7xy+3y^2-8x-9y=0$; (b) $2x^2+7xy+3y^2-24=0$; (c) 8x + 9y – 24 = 0]

BEGINNER'S BOX-4

1. ***(a)*** Coincident, ***(b)*** Parallel, ***(c)*** Intersecting **2.** ***(a)*** 2; ***(b)*** 33/10 ;

3. $\left(\frac{a}{b}\left(b\pm\sqrt{a^2+b^2}\right),0\right)$ **5.** opposite sides of the line; **7.** – y + x = 11; **8.** λ = – 7

BEGINNER'S BOX-5

1. (A) → (p), (B) → (s), (C)→ (q), (D) → (t), (E) → (r), **2.** (7, – 4) **3.** $x^2-4y^2=a^2$

4. x – 2y + 1 = 0 & 2x + y – 3 = 0; ***(a)*** x – 2y + 1 = 0; ***(b)*** 2x + y – 3 = 0

5. x + y = 2, x = y; **6.** 69x + 46y – 25 = 0 **7.** (A) **8.** (A)

9. (C) **10.** 60 **11.** (B)

BEGINNER'S BOX-6

1. x – y = 0 & x – 4y = 0; **2.** 2x – 5y – 2 = 0 & 3x + 2y + 3 = 0 ; $\pm\tan^{-1}\left(\frac{19}{4}\right)$

3. $\theta=\pm\tan^{-1}\frac{4}{7}$; **4.** $x^2-y^2-5xy=0$;

5. ***(B)*** **6.**(B) **7.**(D) **8.**(A) **9.**(B)

EXERCISE – 1 MCQ (SINGLE CHOICE CORRECT)

1. If (3, –4) and (–6, 5) are the extremities of a diagonal of a parallelogram and (2, 1) is its third vertex, then its fourth vertex is -
(A) (–1, 0) (B) (–1, 1) (C) (0, –1) (D) (–5, 0)

2. The ratio in which the line joining the points (3, –4) and (–5, 6) is divided by x-axis -
(A) 2 : 3 (B) 6 : 4 (C) 3 : 2 (D) none of these

3. The circumcentre of the triangle with vertices (0, 0), (3, 0) and (0, 4) is -
(A) (1, 1) (B) (2, 3/2) (C) (3/2, 2) (D) none of these

4. The mid points of the sides of a triangle are (5, 0), (5, 12) and (0, 12), then orthocentre of this triangle is -
(A) (0, 0) (B) (0, 24) (C) (10, 0) (D) $\left(\frac{13}{3}, 8\right)$

***5.** Area of a triangle whose vertices are $(a\cos\theta, b\sin\theta)$, $(-a\sin\theta, b\cos\theta)$ and $(-a\cos\theta, -b\sin\theta)$ is -
(A) $a\,b\sin\theta\cos\theta$ (B) $a\cos\theta\sin\theta$ (C) $\frac{1}{2}ab$ (D) ab

***6.** If $A(\cos\alpha, \sin\alpha)$, $B(\sin\alpha, -\cos\alpha)$, C (1,2) are the vertices of a ΔABC, then as α varies, the locus of its centroid is -
(A) $x^2 + y^2 - 2x - 4y + 3 = 0$ (B) $x^2 + y^2 - 2x - 4y + 1 = 0$
(C) $3(x^2 + y^2) - 2x - 4y + 1 = 0$ (D) none of these

7. The points with the co-ordinates (2a, 3a), (3b, 2b) & (c, c) are collinear-
(A) for no value of a, b, c (B) for all values of a, b, c
(C) if a, $\frac{c}{5}$, b are in H.P. (D) if a, $\frac{2}{5}$c, b are in H.P.

8. A stick of length 10 units rests against the floor and a wall of a room. If the stick begins to slide on the floor then the locus of its middle point is -
(A) $x^2 + y^2 = 2.5$ (B) $x^2 + y^2 = 25$ (C) $x^2 + y^2 = 100$ (D) none

9. The equation of the line cutting an intercept of 3 units on negative y-axis and inclined at an angle $\tan^{-1}\frac{3}{5}$ to the x-axis is -
(A) $5y - 3x + 15 = 0$ (B) $5y - 3x = 15$ (C) $3y - 5x + 15 = 0$ (D) none of these

***10.** The equation of a straight line which passes through the point (–3, 5) such that the portion of it between the axes is divided by the point in the ratio 5 : 3, internally (reckoning from x-axis) will be -
(A) $x + y - 2 = 0$ (B) $2x + y + 1 = 0$ (C) $x + 2y - 7 = 0$ (D) $x - y + 8 = 0$

11.** The points $\left(0, \frac{8}{3}\right)$, (1, 3) and (82, 30) are vertices of- ***[IIT-JEE 1986]
(A) an obtuse angled triangle (B) an acute angled triangle
(C) a right angled triangle (D) None of these

12. The straight lines $x + y = 0$, $3x + y - 4 = 0$, $x + 3y - 4 = 0$ form a triangle which is- ***[IIT-JEE 1983]***
(A) isosceles (B) equilateral (C) right angled (D) none of these

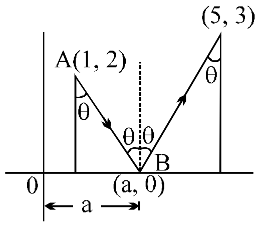
13. A ray of light passing through the point $A(1, 2)$ is reflected at a point B on the x−axis and then passes through $(5, 3)$. Then the equation of AB is :

(A) $5x + 4y = 13$

(B) $5x - 4y = -3$

(C) $4x + 5y = 14$

(D) $4x - 5y = -6$



14. Points A & B are in the first quadrant ; point 'O' is the origin. If the slope of OA is 1, slope of OB is 7 and OA = OB, then the slope of AB is -

(A) –1/5 (B) –1/4 (C) –1/3 (D) –1/2

15. On the portion of the straight line, $x + 2y = 4$ intercepted between the axes, a square is constructed on the side of the line away from the origin. Then the point of intersection of its diagonals has co-ordinates

(A) (2, 3) (B) (3, 2) (C) (3, 3) (D) (2, 2)

16. The equation of the line passing through the point (c, d) and parallel to the line $ax + by + c = 0$ is -

(A) $a(x + c) + b(y + d) = 0$ (B) $a(x + c) - b(y + d) = 0$

(C) $a(x - c) + b(y - d) = 0$ (D) none of these

17. The position of the point (8,–9) with respect to the lines $2x + 3y - 4 = 0$ and $6x + 9y + 8 = 0$ is -

(A) point lies on the same side of the lines (B) point lies on one of the lines

(C) point lies on the different sides of the line (D) point lies between the lines

18. Distance between the two lines represented by the line pair, $x^2 - 4xy + 4y^2 + x - 2y - 6 = 0$ is :

(A) $\frac{1}{\sqrt{5}}$ (B) $\sqrt{5}$ (C) $2\sqrt{5}$ (D) none

***19.** If the point (a, 2) lies between the lines $x - y - 1 = 0$ and $2(x - y) - 5 = 0$, then the set of values of a is -

(A) $(-\infty, 3) \cup (9/2, \infty)$ (B) $(3, 9/2)$ (C) $(-\infty, 3)$ (D) $(9/2, \infty)$

20. If P = (1,0) ; Q = (–1,0) & R = (2,0) are three given points, then the locus of the points S satisfying the relation, $SQ^2 + SR^2 = 2\,SP^2$ is -

(A) A straight line parallel to x-axis (B) A circle passing through the origin

(C) A circle with the centre at the origin (D) A straight line parallel to y-axis

21. The area of triangle formed by the lines $x + y - 3 = 0$, $x - 3y + 9 = 0$ and $3x - 2y + 1 = 0$ is -

(A) $\frac{16}{7}$ sq. units (B) $\frac{10}{7}$ sq. units (C) 4 sq. units (D) 9 sq. units

22. Given the family of lines, $a(3x + 4y + 6) + b(x + y + 2) = 0$. The line of the family situated at the greatest distance from the point P (2, 3) has equation :

(A) $4x + 3y + 8 = 0$ (B) $5x + 3y + 10 = 0$ (C) $15x + 8y + 30 = 0$ (D) none

23.** If the sum of the distances of a point from two perpendicular lines in a plane is 1, then its locus is- ***[JEE 1992]

(A) square (B) circle (C) straight line (D) two intersecting lines

24. Distance of the point (2, 5) from the line $3x + y + 4 = 0$ measured parallel to the line $3x - 4y + 8 = 0$ is -

(A) 15/2 (B) 9/2 (C) 5 (D) none

25. The co-ordinates of the point of reflection of the origin (0, 0) in the line $4x - 2y - 5 = 0$ is -

(A) (1, –2) (B) (2, –1) (C) $\left(\frac{4}{5}, -\frac{2}{5}\right)$ (D) (2, 5)

***26.** If the axes are rotated through an angle of 30° in the anti-clockwise direction, the coordinates of point $(4,-2\sqrt{3})$ with respect to new axes are-

(A) $(2, \sqrt{3})$ (B) $(\sqrt{3}, -5)$ (C) (2, 3) (D) $(\sqrt{3}, 2)$

***27.** If one diagonal of a square is along the line $x = 2y$ and one of its vertex is (3, 0), then its sides through this vertex are given by the equations -

(A) $y - 3x + 9 = 0,\ x - 3y - 3 = 0$
(B) $y - 3x + 9 = 0,\ x - 3y - 3 = 0$
(C) $y + 3x - 9 = 0,\ x + 3y - 3 = 0$
(D) $y - 3x + 9 = 0,\ x + 3y - 3 = 0$

28. The line $(p + 2q)x + (p - 3q)y = p - q$ for different values of p and q passes through a fixed point whose co-ordinates are -

(A) $\left(\frac{3}{2}, \frac{5}{2}\right)$ (B) $\left(\frac{2}{5}, \frac{2}{5}\right)$ (C) $\left(\frac{3}{5}, \frac{3}{5}\right)$ (D) $\left(\frac{2}{5}, \frac{3}{5}\right)$

***29.** The equation $2x^2 + 4xy - py^2 + 4x + qy + 1 = 0$ will represent two mutually perpendicular straight lines, if

(A) p=1 and q = 2 or 6
(B) p = – 2 and q = – 2 or 8
(C) p = 2 and q = 0 or 8
(D) p = 2 and q = 0 or 6

***30.** Equation of the pair of straight lines through origin and perpendicular to the pair of straight lines $5x^2 - 7xy - 3y^2 = 0$ is -

(A) $3x^2 - 7xy - 5y^2 = 0$
(B) $3x^2 + 7xy + 5y^2 = 0$
(C) $3x^2 - 7xy + 5y^2 = 0$
(D) $3x^2 + 7xy - 5y^2 = 0$

EXERCISE – 2 MCQ (ONE OR MORE CHOICE CORRECT)

***1.** Coordinates of a point which is at 3 units distance from the point (1, –3) on the line 2x + 3y + 7 = 0 is/are -

(A) $\left(1+\frac{9}{\sqrt{13}}, 3-\frac{6}{\sqrt{13}}\right)$ (B) $\left(1-\frac{9}{\sqrt{13}}, -3+\frac{6}{\sqrt{13}}\right)$

(C) $\left(1+\frac{9}{\sqrt{13}}, -3-\frac{6}{\sqrt{13}}\right)$ (D) $\left(1-\frac{9}{\sqrt{13}}, 3-\frac{6}{\sqrt{13}}\right)$

2. The angle between the lines $y - x + 5 = 0$ and $\sqrt{3}\,x - y + 7 = 0$ is/are -

(A) 15° (B) 60° (C) 165° (D) 75°

***3.** If line $y - x + 2 = 0$ is shifted parallel to itself towards the x-axis by a perpendicular distance of $3\sqrt{2}$ units, then the equation of the new line is may be -

(A) $y = x + 4$ (B) $y = x + 1$ (C) $y = x - (2 + 3\sqrt{2})$ (D) $y = x - 8$

4. Three lines $px + qy + r = 0$, $qx + ry + p = 0$ and $rx + py + q = 0$ are concurrent if - ***[JEE 1985]***

(A) $p + q + r = 0$ (B) $p^2 + q^2 + r^2 = pr + qr + pq$

(C) $p^3 + q^3 + r^3 = 3pqr$ (D) None of these

5.** All points lying inside the triangle formed by the points (1, 3), (5, 0) and (–1, 2) satisfy - ***[JEE 1986]

(A) $3x + 2y \geq 0$ (B) $2x + y - 13 \geq 0$ (C) $2x - 3y - 12 \leq 0$ (D) $-2x + y \geq 0$

6. The diagonals of a square are along the pair of lines whose equation is $2x^2 - 3xy - 2y^2 = 0$. If (2, 1) is a vertex of the square, then the vertex of the square adjacent to it may be -

(A) (1, 4) (B) (–1, –4) (C) (–1, 2) (D) (1, –2)

7. The line PQ whose equation is $x - y = 2$ cuts the x axis at P and Q is (4,2). The line PQ is rotated about P through 45° in the anticlockwise direction. The equation of the line PQ in the new position is -

(A) $y = -\sqrt{2}$ (B) $y = 2$ (C) $x = 2$ (D) $x = -2$

***8.** If one vertex of an equilateral triangle of side 'a' lies at the origin and the other lies on the line $x - \sqrt{3}y = 0$, then the co-ordinates of the third vertex are -

(A) (0, a) (B) $\left(\frac{\sqrt{3}a}{2}, -\frac{a}{2}\right)$ (C) (0, –a) (D) $\left(-\frac{\sqrt{3}a}{2}, \frac{a}{2}\right)$

9. If the equation $ax^2 - 6xy + y^2 + bx + cx + d = 0$ represents a pair of lines whose slopes are m and m^2, then value(s) of a is/are -

(A) a = –8 (B) a = 8 (C) a = 27 (D) a = –27

10. If the vertices P, Q, R of a triangle PQR are rational points, which of the following points of the triangle PQR is/are always rational point (s) ? ***[JEE 1998]***

(A) centriod (B) incentre (C) circumcentre (D) orthocentre

Match the column

Following questions contains statements given in two columns, which have to be matched. The statements in ***Column-I*** are labelled as A, B, C and D while the statements in ***Column-II*** are labelled as p, q, r and s. Any given statement in ***Column-I*** can have correct matching with ***ONE*** statement in ***Column-II***.

****11.***

	Column-I		***Column-II***
(A)	If $3a - 2b + 5c = 0$, then family of straight lines $ax + by + c = 0$ are always concurrent at a point whose co-ordinates is (α, β), then the values of $5(\alpha - \beta)$	(p)	$3\sqrt{2}$
(B)	Number of integral values of b for which the origin and the point (1, 1) lie on the same side of the straight line $a^2x + aby + 1 = 0$ for all $a \in R - \{0\}$ is	(q)	5
(C)	Vetices of a right angled triangle lie on a circle and extrimites of whose hypotenuse are (6, 0) and (0, 6), then radius of circle is	(r)	12
(D)	If the slope of one of the lines represented by $ax^2 - 6xy + y^2 = 0$ is square of the other, then a is	(s)	3
		(t)	8

Comprehension Based Questions

For points $P \equiv (x_1, y_1)$ and $Q \equiv (x_2, y_2)$ of the coordinate plane, a new distance d(P, Q) is defined by $d(P, Q) = |x_1 - x_2| + |y_1 - y_2|$

Let $O \equiv (0, 0)$, $A \equiv (1, 2)$, $B \equiv (2, 3)$ and $C \equiv (4, 3)$ are four fixed points on x – y plane.

On the basis of above information, answer the following questions

12. Let R(x, y), such that R is equidistant from the points O and A with respect to new distance and if $0 \le x < 1$ and $0 \le y < 2$, then R lies on a line segment whose equation is -

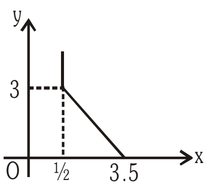
(A) $x + y = 3$ (B) $x + 2y = 3$ (C) $2x + y = 3$ (D) $2x + 2y = 3$

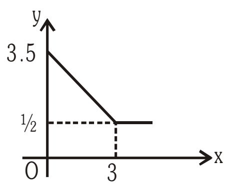
13. Let S(x, y), such that S is equidistant from points O and B with respect to new distance and if $x \ge 2$ and $0 \le y < 3$, then locus of S is -

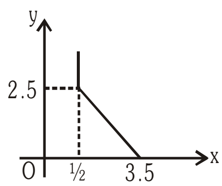
(A) a line segment (B) a line

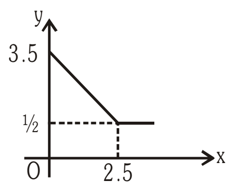
(C) a vertical ray (D) a horizontal ray

14. Let T(x, y), such that T is equidistant from point O and C with respect to new distance and if T lies in first quadrant, then T consists of the union of a line segment of finite length and an infinite ray whose labelled diagram is -

(A) 

(B) 

(C) 

(D) 

EXERCISE – 3 SUBJECTIVE

1. The area of a triangle is 5. Two of its vertices are (2, 1) & (3, – 2). The third vertex lies on $y = x + 3$. Find the third vertex.

2. Two vertices of a triangle are (4, – 3) & (– 2, 5). If the orthocentre of the triangle is at (1, 2), find the coordinates of the third vertex.

***3.** The line $3x + 2y = 24$ meets the y-axis at A & the x-axis at B. The perpendicular bisector of AB meets the line through (0,–1) parallel to x-axis at C. Find the area of the triangle ABC.

***4.** A line is such that its segment between the straight lines $5x – y – 4 = 0$ and $3x + 4y – 4 = 0$ is bisected at the point (1, 5). Obtain the equation.

5. A straight line L is perpendicular to the line $5x – y = 1$. The area of the triangle formed by the line L & the coordinate axes is 5. Find the equation of the line.

6. The vertices of a triangle OBC are O(0, 0), B (–3, –1), C (–1, –3). Find the equation of the line parallel to BC & intersecting the sides OB & OC, whose perpendicular distance from the point (0, 0) is half.

7. If the straight line drawn through the point $P\left(\sqrt{3},2\right)$ & making an angle $\frac{\pi}{6}$ with the x-axis, meets the line $\sqrt{3}x - 4y + 8 = 0$ at Q. Find the length PQ.

8. The points (1, 3) & (5, 1) are two opposite vertices of a rectangle. The other two vertices lie on the line $y = 2x + c$. Find c & the remaining vertices.

9.** Two sides of a rhombus ABCD are parallel to the lines $y = x + 2$ and $y = 7x + 3$. If the diagonals of the rhombus intersect at the point (1, 2) and the vertex A is on the y-axis. find possible co-ordinates of A. ***[IIT-JEE 1985]

10. Find the direction in which a straight line may be drawn through the point (2, 1) so that its point of intersection with the line $4y – 4x + 4 + 3\sqrt{2} + 3\sqrt{10} = 0$ is at a distance of 3 unit from (2, 1).

11. Find the equation of the line which bisects the obtuse angle between the lines $x – 2y + 4 = 0$ and $4x – 3y + 2 = 0$. ***[IIT-JEE 1978]***

***12.** A line through A (–5, –4) meets the line $x + 3y + 2 = 0$, $2x + y + 4 = 0$ and $x – y – 5 = 0$ at the points B, C & D respectively, if $\left(\frac{15}{AB}\right)^2 + \left(\frac{10}{AC}\right)^2 = \left(\frac{6}{AD}\right)^2$. Find the equation of the line.

***13.** Show that all the chords of the curve $3x^2 + 3y^2 – 2x + 4y = 0$ which subtend a right angle at the origin are concurrent. Also find the point of concurrency.

EXERCISE - 4 RECAP OF AIEEE/JEE (MAIN)

1. The angle between the straight lines $x^2 + 4xy + y^2 = 0$ is- ***[AIEEE 2002]***
(A) 30° (B) 45° (C) 60° (D) 90°

2.** The distance between a pair of parallel lines $9x^2 - 24xy + 16y^2 - 12x + 16y - 12 = 0$. ***[AIEEE 2002]
(A) 5 (B) 8 (C) 8/5 (D) 5/8

3.** A square of sides a lies above the x-axis and has one vertex at the origin. The side passing through the origin makes an angle α $(0 < \alpha < \pi/4)$ with the positive direction of x-axis.The equation of its diagonal not passing through the origin is- ***[AIEEE 2003]
(A) $y(\cos\alpha + \sin\alpha) + x(\cos\alpha - \sin\alpha) = a$
(B) $y(\cos\alpha - \sin\alpha) - x(\sin\alpha - \cos\alpha) = a$
(C) $y(\cos\alpha + \sin\alpha) + x(\sin\alpha - \cos\alpha) = a$
(D) $y(\cos\alpha + \sin\alpha) + x(\sin\alpha + \cos\alpha) = a$

4.** If the pair of straight lines $x^2 - 2pxy - y^2 = 0$ and $x^2 - 2qxy - y^2 = 0$ be such that each pair bisects the angle between the other pair, then- ***[AIEEE 2003]
(A) $pq = -1$ (B) $p = q$ (C) $p = -q$ (D) $pq = 1$

5.** Locus of centroid of the triangle whose vertices are (a cos t, a sin t), (b sin t, – b cos t) and (1,0), where t is a parameter, is- ***[AIEEE 2003]
(A) $(3x + 1)^2 + (3y)^2 = a^2 - b^2$
(B) $(3x - 1)^2 + (3y)^2 = a^2 - b^2$
(C) $(3x - 1)^2 + (3y)^2 = a^2 + b^2$
(D) $(3x + 1)^2 + (3y)^2 = a^2 + b^2$

6.** If the equation of the locus of a point equidistant from the points (a_1,b_1) and (a_2,b_2) is $(a_1-a_2)x + (b_1-b_2)y + c = 0$, then the value of 'c' is- ***[AIEEE 2003]
(A) $\sqrt{a_1^2 + b_1^2 - a_2^2 - b_2^2}$
(B) $\frac{1}{2}(a_2^2 + b_2^2 - a_1^2 - b_1^2)$
(C) $a_1^2 - a_2^2 + b_1^2 - b_2^2$
(D) $\frac{1}{2}(a_1^2 + a_2^2 + b_1^2 + b_1^2)$

7. The equation of the straight line passing through the point (4,3) and making intercepts on the coordinate axes whose sum is –1 is- ***[AIEEE 2004]***
(A) $\frac{x}{2}+\frac{y}{3}=-1$ and $\frac{x}{-2}+\frac{y}{1}=-1$
(B) $\frac{x}{2}-\frac{y}{3}=-1$ and $\frac{x}{-2}+\frac{y}{1}=-1$
(C) $\frac{x}{2}+\frac{y}{3}=1$ and $\frac{x}{2}+\frac{y}{1}=1$
(D) $\frac{x}{2}-\frac{y}{3}=1$ and $\frac{x}{-2}+\frac{y}{1}=1$

8. If the sum of the slopes of the lines given by $x^2 - 2cxy - 7y^2 = 0$ is four times their product, then c has the value- ***[AIEEE 2004]***
(A) 1 (B) –1 (C) 2 (D) –2

9. If one of the lines given by $6x^2 - xy + 4cy^2 = 0$ is $3x + 4y = 0$, then c equals- ***[AIEEE 2004]***
(A) 1 (B) –1 (C) 3 (D) –3

***10.** The line parallel to the x-axis and passing through the intersection of the lines $ax + 2by + 3b = 0$ and $bx - 2ay - 3a = 0$ is, (where $(a,b) \neq (0,0)$) **[AIEEE 2005]**

(A) below the x-axis at a distance of $\frac{3}{2}$ from it
(B) below the x-axis at a distance of $\frac{2}{3}$ from it
(C) above the x-axis at a distance of $\frac{3}{2}$ from it
(D) above the x-axis at a distance of $\frac{2}{3}$ from it

***11.** If non-zero numbers a,b,c are in H.P., then the straight line $\frac{x}{a}+\frac{y}{b}+\frac{1}{c}=0$ always passes through a fixed point that point is- **[AIEEE 2005]**

(A) $(-1,2)$ (B) $(-1,-2)$ (C) $(1,-2)$ (D) $\left(1,-\frac{1}{2}\right)$

12. A straight line passing through the point A(3,4) is such that its intercept between the axes is bisected at A. Then its equation is- **[AIEEE 2006]**

(A) $3x - 4y + 7 = 0$ (B) $4x + 3y = 24$ (C) $3x + 4y = 25$ (D) $x + y = 7$

13. If (a,a^2) falls inside the angle made by the lines $y = \frac{x}{2}$, $x > 0$ and $y = 3x$, $x > 0$, then a belongs to- **[AIEEE 2006]**

(A) $(3,\infty)$ (B) $\left(\frac{1}{2},3\right)$ (C) $\left(-3,-\frac{1}{2}\right)$ (D) $\left(0,\frac{1}{2}\right)$

14. Let P(–1,0) Q(0,0) and R($3, 3\sqrt{3}$) be three points. The equation of the bisector of the angle PQR is- **[AIEEE 2007], [IIT Scr. 2002]**

(A) $\sqrt{3}x + y = 0$ (B) $x+\frac{\sqrt{3}}{2}y=0$ (C) $\frac{\sqrt{3}}{2}x+y=0$ (D) $x + \sqrt{3}y = 0$

***15.** If one of the lines of $my^2 + (1 - m^2)xy - mx^2 = 0$ is a bisector of the angle between the lines $xy=0$, then m is- **[AIEEE 2007]**

(A) $-\frac{1}{2}$ (B) -2 (C) 1 (D) 2

16. The perpendicular bisector of the line segment joining P (1, 4) and Q (k, 3) has y–intercept –4. Then a possible value of k is- **[AIEEE 2008]**

(A) 1 (B) 2 (C) –2 (D) –4

***17.** The lines $p(p^2 + 1)x - y + q = 0$ and $(p^2 + 1)^2x + (p^2 + 1)y + 2q = 0$ are **[AIEEE 2009]**
Perpendicular to a common line for :

(A) Exactly two values of p
(B) More than two values of p
(C) No value of p
(D) Exactly one value of p

18. The line L given by $\frac{x}{5}+\frac{y}{b}=1$ passes through the point (13, 32). The line K is parallel to L and has the equation $\frac{x}{c}+\frac{y}{3}=1$. Then the distance between L and K is : **[AIEEE-2010]**

(A) $\frac{23}{\sqrt{15}}$ (B) $\sqrt{17}$ (C) $\frac{17}{\sqrt{15}}$ (D) $\frac{23}{\sqrt{17}}$

19.** The lines $L_1 : y - x = 0$ and $L_2 : 2x + y = 0$ intersect the line $L_3 : y + 2 = 0$ at P and Q respectively. The bisector of the acute angle between L_1 and L_2 intersects L_3 at R. ***[AIEEE 2011]

Statement-1– The ratio PR : RQ equals $2\sqrt{2} : \sqrt{5}$

Statement-2– In any triangle, bisector of an angle divides the triangle into two similar triangles.

(A) Statement-1 is true, Statement-2 is false.

(B) Statement-1 is false, Statement-2 is true

(C) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1

(D) Statement-1 is true, Statement-2 is true; Statement-2 is ***not*** a correct explanation for Statement-1.

20.** The lines $x + y = |a|$ and $ax - y = 1$ intersect each other in the first quadrant. Then the set of all possible values of a is the interval : ***[AIEEE 2011]

(A) $(-1, 1]$ (B) $(0, \infty)$ (C) $[1, \infty)$ (D) $(-1, \infty)$

21. A line is drawn through the point (1, 2) to meet the coordinate axes at P and Q such that it forms a triangle OPQ, where O is the origin. If the area of the triangle OPQ is least, then the slope of the line PQ is : ***[AIEEE 2012]***

(A) $-\frac{1}{2}$ (B) $-\frac{1}{4}$ (C) –4 (D) –2

22. If the line $2x + y = k$ passes through the point which divides the line segment joining the points (1, 1) and (2, 4) in the ratio 3 : 2, then k equals : ***[AIEEE 2012]***

(A) $\frac{11}{5}$ (B) $\frac{29}{5}$ (C) 5 (D) 6

23. A ray of light along $x + \sqrt{3}y = \sqrt{3}$ gets reflected upon reaching x-axis, the equation of the reflected ray is : ***[JEE(Main)-2013]***

(A) $y = x + \sqrt{3}$ (B) $\sqrt{3}y = x - \sqrt{3}$ (C) $y = \sqrt{3}x - \sqrt{3}$ (D) $\sqrt{3}y = x - 1$

24. The x-coordinate of the incentre of the triangle that has the coordinates of mid points of its sides as (0, 1)(1, 1) and (1, 0) is : ***[JEE(Main)-2013]***

(A) $2 + \sqrt{2}$ (B) $2 - \sqrt{2}$ (C) $1 + \sqrt{2}$ (D) $1 - \sqrt{2}$

25.** Let PS be the median of the triangle with vertices P(2, 2), Q(6 – 1) and R(7, 3). The equation of the line passing through (1 – 1) and parallel to PS is : ***[JEE(Main)-2014]

(A) $4x + 7y + 3 = 0$ (B) $2x - 9y - 11 = 0$ (C) $4x - 7y - 11 = 0$ (D) $2x + 9y + 7 = 0$

26.** Let a, b, c and d be non-zero numbers. If the point of intersection of the lines $4ax + 2ay + c = 0$ and $5bx + 2by + d = 0$ lies in the fourth quadrant and is equidistant from the two axes then: ***[JEE(Main)-2014]

(A) $3bc - 2ad = 0$ (B) $3bc + 2ad = 0$ (C) $2bc - 3ad = 0$ (D) $2bc + 3ad = 0$

27. The number of points, having both co-ordinates as integers, that lie in the interior of the triangle with vertices (0, 0), (0, 41) and (41, 0) is : ***[JEE(Main)-2015]***

(A) 820 (B) 780 (C) 901 (D) 861

28. Locus of the image of the point (2, 3) in the line $(2x - 3y + 4) + k(x - 2y + 3) = 0$, $k \in R$, is a

(A) circle of radius $\sqrt{2}$ (B) circle of radius $\sqrt{3}$ **[JEE(Main)-2015]**

(C) straight line parallel to x-axis (D) straight line parallel to y-axis

29. Two sides of a rhombus are along the lines, $x - y + 1 = 0$ and $7x - y - 5 = 0$. If its diagonals intersect at $(-1, -2)$, then which one of the following is a vertex of this rhombus ? **[JEE(Main)-2016]**

(A) $\left(-\frac{10}{3}, -\frac{7}{3}\right)$ (B) $(-3, -9)$ (C) $(-3, -8)$ (D) $\left(\frac{1}{3}, -\frac{8}{3}\right)$

30. Let k be an integer such that triangle with vertices $(k, -3k)$, $(5, k)$ and $(-k, 2)$ has area 28 sq. units. Then the orthocentre of this triangle is at the point: **[JEE(Main)-2017]**

(A) $\left(2, \frac{1}{2}\right)$ (B) $\left(2, -\frac{1}{2}\right)$ (C) $\left(1, \frac{3}{4}\right)$ (D) $\left(1, -\frac{3}{4}\right)$

31. A straight line through a fixed point (2, 3) intersects the coordinate axes at distinct points P and Q. If O is the origin and the rectangle OPRQ is completed, then the locus of R is : **[JEE(Main)-2018]**

(A) $2x + 3y = xy$ (B) $3x + 2y = xy$ (C) $3x + 2y = 6xy$ (D) $3x + 2y = 6$

32. Let the orthocentre and centroid of a triangle be A(–3, 5) and B(3, 3) respectively. If C is the circumcentre of this triangle, then the radius of the circle having line segment AC as diameter, is: **[JEE(Main)-2018]**

(A) $2\sqrt{10}$ (B) $3\sqrt{\frac{5}{2}}$ (C) $\frac{3\sqrt{5}}{2}$ (D) $\sqrt{10}$

33. If in a parallelogram ABDC, the coordinates of A, B and C are respectively (1, 2), (3, 4) and (2, 5), then the equation of the diagonal AD is: **[JEE(Main)-2019]**

(A) $5x + 3y - 11 = 0$ (B) $3x - 5y + 7 = 0$ (C) $3x + 5y - 13 = 0$ (D) $5x - 3y + 1 = 0$

34. Let the equations of two sides of a triangle be $3x - 2y + 6 = 0$ and $4x + 5y - 20 = 0$. If the orthocentre of this triangle is at (1, 1), then the equation of its third side is : **[JEE(Main)-2019]**

(A) $122y - 26x - 1675 = 0$ (B) $26x + 61y + 1675 = 0$

(C) $122y + 26x + 1675 = 0$ (D) $26x - 122y - 1675 = 0$

35. Let S be the set of all triangles in the xy–plane, each having one vertex at the origin and the other two vertices lie on coordinate axes with integral coordinates. If each triangle in S has area 50 sq. units, then the number of elements in the set S is : **[JEE(Main)-2019]**

(A) 9 (B) 18 (C) 32 (D) 36

36. Consider the set of all lines $px + qy + r = 0$ such that $3p + 2q + 4r = 0$. Which one of the following statements is true ? **[JEE(Main)-2019]**

(A) The lines are all parallel.

(B) Each line passes through the origin.

(C) The lines are not concurrent The lines are concurrent at the point

(D) $\left(\frac{3}{4}, \frac{1}{2}\right)$

37. Two vertices of a triangle are (0, 2) and (4, 3). If its orthocentre is at the origin, then its third vertex lies in which quadrant ? ***[JEE(Main)-2019]***

(A) Fourth (B) Second (C) Third (D) First

38. Two sides of a parallelogram are along the lines, $x + y = 3$ and $x - y + 3 = 0$. If its diagonals intersect at (2,4), then one of its vertex is : ***[JEE(Main)-2019]***

(A) (2, 6) (B) (2, 1) (C) (3, 5) (D) (3, 6)

39. If the line $3x + 4y - 24 = 0$ intersects the x–axis at the point A and the y–axis at the point B, then the incentre of the triangle OAB, where O is the origin, is ***[JEE(Main)-2019]***

(A) (3, 4) (B) (2, 2) (C) (4, 4) (D) (4, 3)

40. A point P moves on the line $2x - 3y + 4 = 0$. If Q(l, 4) and R(3, –2) are fixed points, then the locus of the centroid of ΔPQR is a line : ***[JEE(Main)-2019]***

(A) parallel to x–axis (B) with slope $\frac{2}{3}$ (C) with slope $\frac{3}{2}$ (D) parallel to y–axis

41. The straight line $x + 2y = 1$ meets the coordinate axes at A and B. A circle is drawn through A, B and the origin. Then the sum of perpendicular distances from A and B on the tangent to the circle at the origin is : ***[JEE(Main)-2019]***

(A) $\frac{\sqrt{5}}{4}$ (B) $\frac{\sqrt{5}}{2}$ (C) $2\sqrt{5}$ (D) $4\sqrt{5}$

42. If a straight line passing thourgh the point P(–3, 4) is such that its intercepted portion between the coordinate axes is bisected at P, then its equation is : ***[JEE(Main)-2019]***

(A) $x - y + 7 = 0$ (B) $3x - 4y + 25 = 0$ (C) $4x + 3y = 0$ (D) $4x - 3y + 24 = 0$

43. If the straight line, $2x - 3y + 17 = 0$ is perpendicular to the line passing through the points (7, 17) and (15, β), then β equals : ***[JEE(Main)-2019]***

(A) –5 (B) $-\frac{35}{3}$ (C) $\frac{35}{3}$ (D) 5

44. A point on the straight line, $3x + 5y = 15$ which is equidistant from the coordinate axes will lie only in ***[JEE(Main)-2019]***

(A) 1^{st} and 2^{nd} quadrants (B) 4^{th} quadrant

(C) 1^{st}, 2^{nd} and 4^{th} quadrant (D) 1^{st} quadrant

45. Let O(0, 0) and A(0, 1) be two fixed points. Then the locus of a point P such that the perimeter of ΔAOP is 4, is : ***[JEE(Main)-2019]***

(A) $8x^2 - 9y^2 + 9y = 18$ (B) $9x^2 + 8y^2 - 8y = 16$ (C) $8x^2 + 9y^2 - 9y = 18$ (D) $9x^2 - 8y^2 + 8y = 16$

46. Suppose that the points (h,k), (1,2) and (–3,4) lie on the line L_1. If a line L_2 passing through the points (h,k) and (4,3) is perpendicular to L_1, then $\frac{k}{h}$ equals : ***[JEE(Main)-2019]***

(A) 3 (B) $-\frac{1}{7}$ (C) $\frac{1}{3}$ (D) 0

47. Slope of a line passing through P(2, 3) and intersecting the line, $x + y = 7$ at a distance of 4 units from P, is **[JEE(Main)-2019]**

(A) $\frac{\sqrt{5}-1}{\sqrt{5}+1}$ (B) $\frac{1-\sqrt{5}}{1+\sqrt{5}}$ (C) $\frac{1-\sqrt{7}}{1+\sqrt{7}}$ (D) $\frac{\sqrt{7}-1}{\sqrt{7}+1}$

48. If the two lines $x + (a-1)y = 1$ and $2x + a^2y = 1 (a \in R - \{0, 1\})$ are perpendicular, then the distance of their point of intersection from the origin is :- **[JEE(Main)-2019]**

(A) $\frac{2}{5}$ (B) $\frac{2}{\sqrt{5}}$ (C) $\frac{\sqrt{2}}{5}$ (D) $\sqrt{\frac{2}{5}}$

49. Let A(3, 0, –1), B (2, 10, 6) and C(1, 2, 1) be the vertices of a triangle and M be the midpoint of AC. If G divides BM in the ratio, 2 : 1, then $\cos(\angle GOA)$ (O being the origin) is equal to : **[JEE(Main)-2019]**

(A) $\frac{1}{\sqrt{30}}$ (B) $\frac{1}{6\sqrt{10}}$ (C) $\frac{1}{\sqrt{15}}$ (D) $\frac{1}{2\sqrt{15}}$

50. Lines are drawn parallel to the line $4x - 3y + 2 = 0$, at a distance $\frac{3}{5}$ from the origin. Then which one of the following points lies on any of these lines ? **[JEE(Main)-2019]**

(A) $\left(-\frac{1}{4},\frac{2}{3}\right)$ (B) $\left(\frac{1}{4},\frac{1}{3}\right)$ (C) $\left(-\frac{1}{4},-\frac{2}{3}\right)$ (D) $\left(\frac{1}{4},-\frac{1}{3}\right)$

51. The equation $y = \sin x \sin(x + 2) - \sin^2(x + 1)$ represents a straight line lying in : **[JEE(Main)-2019]**

(A) second and third quadrants only
(B) third and fourth quadrants only
(C) first, third and fourth quadrants
(D) first, second and fourth quadrants

52. A straight line L at a distance of 4 units from the origin makes positive intercepts on the coordinate axes and the perpendicular from the origin to this line makes an angle of 60° with the line $x + y = 0$. Then an equation of the line L is : **[JEE(Main)-2019]**

(A) $\left(\sqrt{3}+1\right)x+\left(\sqrt{3}-1\right)y=8\sqrt{2}$
(B) $\left(\sqrt{3}-1\right)x+\left(\sqrt{3}+1\right)y=8\sqrt{2}$
(C) $\sqrt{3}x+y=8$
(D) $x+\sqrt{3}y=8$

53. A triangle has a vertex at (1, 2) and the mid points of the two sides through it are (–1, 1) and (2,3). Then the centroid of this triangle is : **[JEE(Main)-2019]**

(A) $\left(\frac{1}{3},1\right)$ (B) $\left(\frac{1}{3},2\right)$ (C) $\left(1,\frac{7}{3}\right)$ (D) $\left(\frac{1}{3},\frac{5}{3}\right)$

EXERCISE – 5 RECAP OF IIT-JEE/JEE (ADVANCED)

***1.** (a) Let O(0, 0), P (3, 4), Q(6, 0) be the vertices of the triangle OPQ. The point R inside the triangle OPQ is such that the triangles OPR, PQR, OQR are of equal area. The coordinates of R are

(A) $(4/3,\ 3)$ (B) $(3,\ 2/3)$ (C) $(3,\ 4/3)$ (D) $(4/3,\ 2/3)$

(b) Lines $L_1 : y - x = 0$ and $L_2 : 2x + y = 0$ intersect the line $L_3 : y + 2 = 0$ at P and Q, respectively. The bisector of the acute angle between L_1 and L_2 intersects L_3 at R.

Statement-1 : The ratio PR : RQ equals $2\sqrt{2} : \sqrt{5}$

because

Statement-2 : In any triangle, bisector of an angle divides the triangle into two similar triangles.

(A) Statement-1 is true, statement-2 is true; statement-2 is a correct explanation for statement-1.

(B) Statement-1 is true, statement-2 is true; statement-2 is NOT a correct explanation for statement-1.

(C) Statement-1 is true, statement-2 is false.

(D) Statement-1 is false, statement-2 is true. ***[JEE 2007, 3+3]***

***2.** Consider the lines given by

$L_1 = x + 3y - 5 = 0$

$L_2 = 3x - ky - 1 = 0$

$L_3 = 5x + 2y - 12 = 0$

Match the statements / Expression in ***Column-I*** with the statements / Expressions in ***Column-II*** and indicate your answer by darkening the appropriate bubbles in the 4 × 4 matrix given in OMR. ***[JEE 2008, 6]***

Column-I	***Column-II***
(A) L_1, L_2, L_3 are concurrent, if	(P) $k = -9$
(B) One of L_1, L_2, L_3 is parallel to at least one of the other two, if	(Q) $k = -\frac{6}{5}$
(C) L_1, L_2, L_3 form a triangle, if	(R) $k = \frac{5}{6}$
(D) L_1, L_2, L_3 do not form a triangle, if	(S) $k = 5$

3. Let P, Q, R and S be the points on the plane with position vectors $-2\hat{i} - \hat{j}, 4\hat{i}, 3\hat{i} + 3\hat{j}$ and $-3\hat{i} + 2\hat{j}$ respectively. The quadrilateral PQRS must be a ***[JEE 2010, 3]***

(A) parallelogram, which is neither a rhombus nor a rectangle

(B) square

(C) rectangle, but not a square

(D) rhombus, but not a square

4.** A straight line L through the point (3, –2) is inclined at an angle 60° to the line $\sqrt{3}x + y = 1$. If L also intersect the x-axis, then the equation of L is ***[JEE 2011, 3 (–1)]

(A) $y + \sqrt{3}x + 2 - 3\sqrt{3} = 0$ (B) $y - \sqrt{3}x + 2 + 3\sqrt{3} = 0$

(C) $\sqrt{3}y - x + 3 + 2\sqrt{3} = 0$ (D) $\sqrt{3}y + x - 3 + 2\sqrt{3} = 0$

***5.** For $a > b > c > 0$, the distance between (1, 1) and the point of intersection of the lines $ax + by + c = 0$ and $bx + ay + c = 0$ is less than $2\sqrt{2}$. Then **[JEE(Advanced) 2013, 2M]**

(A) $a + b - c > 0$ (B) $a - b + c < 0$ (C) $a - b + c > 0$ (D) $a + b - c < 0$

***6.** For a point P in the plane, let d_1 (P) and d_2(P) be the distances of the point P from the lines $x - y = 0$ and $x + y = 0$ respectively. The area of the region R consisting of all points P lying in the first quadrant of the plane and satisfying $2 \leq d_1(P) + d_2(P) \leq 4$, is **[JEE(Advanced) 2014]**

7. Let $a,\lambda,m \in \mathbb{R}$. Consider the system of linear equations

$ax + 2y = \lambda$

$3x - 2y = \mu$

Which of the following statement(s) is(are) correct ? **[JEE 2016]**

(A) If $a = -3$, then the system has infinitely many solutions for all values of λ and μ

(B) If $a \neq -3$, then the system has a unique solution for all values of λ and μ

(C) If $\lambda + \mu = 0$, then the system has infinitely many solutions for $a = -3$

(D) If $\lambda + \mu \neq 0$, then the system has no solution for $a = -3$

ANSWER KEY

EXERCISE-1

Que.	1	2	3	4	5	6	7	8	9	10
Ans.	D	A	C	A	D	C	D	B	A	D
Que.	11	12	13	14	15	16	17	18	19	20
Ans.	D	A	A	D	C	C	A	B	B	D
Que.	21	22	23	24	25	26	27	28	29	30
Ans.	B	A	A	C	B	B	D	D	C	A

EXERCISE-2

Que.	1	2	3	4	5	6	7	8	9	10
Ans.	BC	AC	AD	ABC	AC	CD	C	ABCD	BD	ACD

- ***Match the Column*** **11.** (A)→(q); (B)→(s); (C)→(p); (D)→(t)
- ***Comprehension Based Questions*** **12.** D **13.** D **14.** A

EXERCISE-3

1. $\left(\frac{7}{2},\frac{13}{2}\right)$ or $\left(-\frac{3}{2},\frac{3}{2}\right)$ **2.** (33, 26) **3.** 91 sq. units

4. $83x - 35y + 92 = 0$ **5.** $x + 5y + 5\sqrt{2} = 0$ or $x + 5y - 5\sqrt{2} = 0$

6. $2x + 2y + \sqrt{2} = 0$ **7.** 6 units **8.** C = – 4 ; B (2, 0) ; D (4, 4)

9. $\left(0,\frac{5}{2}\right),(0,0)$ **10.** 171°, 99°

11. $(4+\sqrt{5})x-(2\sqrt{5}+3)y+(4\sqrt{5}+2)=0$ **12.** $2x + 3y + 22 = 0$

13. $\left(\frac{1}{3},-\frac{2}{3}\right)$

EXERCISE-4

Que.	1	2	3	4	5	6	7	8	9	10
Ans.	C	C	A	A	C	B	D	C	D	A
Que.	11	12	13	14	15	16	17	18	19	20
Ans.	C	B	B	A	C	D	D	D	A	C
Que.	21	22	23	24	25	26	27	28	29	30
Ans.	D	D	B	B	D	A	B	A	D	A
Que.	31	32	33	34	35	36	37	38	39	40
Ans.	B	B	D	D	D	D	B	D	B	B
Que.	41	42	43	44	45	46	47	48	49	50
Ans.	B	D	D	A	B	C	C	D	C	A
Que.	51	52	53							
Ans.	B	C	B							

EXERCISE-5

1. (a) C; (b) C **2.** (A) S; (B) P,Q; (C) R ; (D) P,Q,S **3.** (A) **4.** (B) **5.** (A) **6.** 6

7. (BCD)

CIRCLE

Recap of Early Classes

The circle has been known since before the begining of recorded history. Natural circles would have been observed such as moon, sun etc. Till previous class we have studied circle as simple shape of euclidean geometry and have also studied about chords, diameter, circumference area etc. Now these same terms along with some other important terms will be studied under co-ordinate geometry in this chapter.

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CIRCLE

1.0 DEFINITION

SL AL

A circle is the locus of a point which moves in a plane in such a way that its distance from a fixed point (in the same given plane) remains constant. The fixed point is called the centre of the circle and the constant distance is called the radius of the circle.

Equation of a circle

The curve traced by the moving point is called its circumference i.e. the equation of any circle is satisfied by co-ordinates of all points on its circumference.

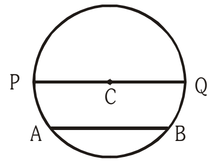
or

The equation of the circle means the equation of its circumference.

or

It is the set of all points lying on the circumference of the circle.

Chord and diameter - the line joining any two points on the circumference is called a chord. If any chord passing through its centre is called its diameter.



AB = chord, PQ = diameter

C = centre

2.0 STANDARD EQUATIONS OF THE CIRCLE

SL AL

2.1 Central Form

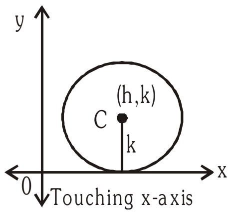
If (h, k) is the centre and r is the radius of the circle then its equation is $(x-h)^2 + (y-k)^2 = r^2$

Special Cases

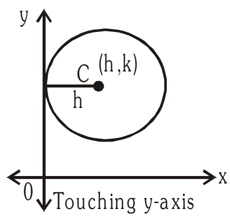
(i) If centre is origin (0,0) and radius is 'r' then equation of circle is $x^2 + y^2 = r^2$ and this is called the standard form.

(ii) If radius of circle is zero then equation of circle is $(x - h)^2 + (y - k)^2 = 0$. Such circle is called zero circle or point circle.

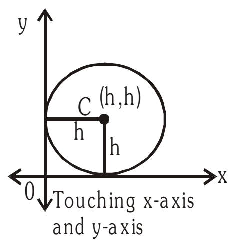
(iii) When circle touches x-axis then equation of the circle is $(x-h)^2 + (y-k)^2 = k^2$.



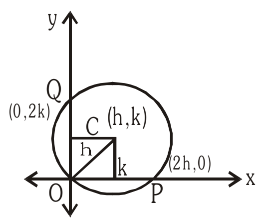
(iv) When circle touches y-axis then equation of circle is $(x-h)^2 + (y-k)^2 = h^2$.



(v) When circle touches both the axes (x-axis and y-axis) then equation of circle $(x-h)^2 + (y-h)^2 = h^2$.



(vi) When circle passes through the origin and centre of the circle is (h,k) then radius $\sqrt{h^2 + k^2} = r$ and intercept cut on x-axis OP =2h, and intercept cut on y-axis is OQ = 2k and equation of circle is $(x - h)^2 + (y-k)^2 = h^2 + k^2$ or $x^2 + y^2 - 2hx - 2ky = 0$



Note – Centre of the circle may exist in any quadrant hence for general cases use ± sign before h & k.

2.2 General equation of circle

$x^2 + y^2 + 2gx + 2fy + c = 0$. where g,f,c are constants and centre is $(-g,-f)$

i.e. $\left(-\dfrac{\text{coefficient of } x}{2}, -\dfrac{\text{coefficient of } y}{2}\right)$ and radius $r = \sqrt{g^2 + f^2 - c}$

NOTE

(i) If $(g^2 + f^2 - c) > 0$, then r is real and positive and the circle is a real circle.

(ii) If $(g^2 + f^2 - c) = 0$, then radius $r = 0$ and circle is a point circle.

(iii) If $(g^2 + f^2 - c) < 0$, then r is imaginary then circle is also an imaginary circle with real centre.

(iv) $x^2 + y^2 + 2gx + 2fy + c = 0$, has three constants and to get the equation of the circle at least three conditions should be known $\Rightarrow$ A unique circle passes through three non collinear points.

(v) The general second degree in x and y, $ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0$ represents a circle if.
- coefficient of x^2 = coefficient of y^2 or $a = b \neq 0$
- coefficient of $xy = 0$ or $h = 0$
- $(g^2 + f^2 - c) \geq 0$ (for a real circle)

2.3 Intercepts cut by the circle on axes

The intercepts cut by the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ on :

(i) x-axis $= 2\sqrt{g^2 - c}$ (ii) y-axis $= 2\sqrt{f^2 - c}$

NOTE

(i) If the circle cuts the x-axis at two distinct point, then $g^2 - c > 0$

(ii) If the cirlce cuts the y-axis at two distinct point, then $f^2 - c > 0$

(iii) If circle touches x-axis then $g^2 = c$.

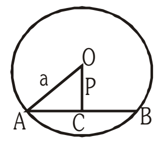
(iv) If circle touches y-axis then $f^2 = c$.

(v) Circle lies completely above or below the x-axis then $g^2 < c$.

(vi) Circle lies completely to the right or left to the y-axis, then $f^2 < c$.

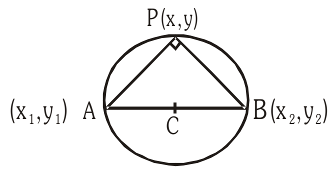
(vii) Intercept cut by a line on the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ or

length of chord of the circle $= 2\sqrt{a^2 - P^2}$ where a is the radius and P is the length of perpendicular from the centre to the chord.



2.4 Equation of circle in diameter form

If $A(x_1,y_1)$ and $B(x_2,y_2)$ are the end points of the diameter of the circle and $P(x,y)$ is the point other then A and B on the circle then from geometry we know that $\angle APB = 90°$.



$\Rightarrow$ (Slope of PA) $\times$ (Slope of PB) $= -1$

$$\Rightarrow \quad \therefore \left(\frac{y - y_1}{x - x_1}\right)\left(\frac{y - y_2}{x - x_2}\right) = -1$$

$$\Rightarrow \quad (x-x_1)(x-x_2) + (y-y_1)(y-y_2) = 0$$

Note – This will be the circle of least radius passing through (x_1, y_1) and (x_2, y_2)

2.5 Equation of circle in parametric forms

(i) The parametric equation of the circle $x^2+y^2 = r^2$ are $x = r\cos\theta$, $y = r\sin\theta$; $\theta \in [0, 2\pi)$ and $(r\cos\theta, r\sin\theta)$ are called the parametric co-ordinates.

(ii) The parametric equation of the circle $(x - h)^2 + (y - k)^2 = r^2$ is $x = h + r\cos\theta$, $y = k + r\sin\theta$ where θ is parameter.

(iii) The parametric equation of the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ are $x = -g + \sqrt{\mathbf{g^2 + f^2 - c}}\cos\theta$, $y = -f + \sqrt{\mathbf{g^2 + f^2 - c}}\sin\theta$ where θ is parameter.

Note– Equation of a straight line joining two point α & β on the circle $x^2 + y^2 = a^2$ is

$$x\cos\frac{\alpha+\beta}{2} + y\sin\frac{\alpha+\beta}{2} = a\cos\frac{\alpha-\beta}{2}.$$

Illustrations

Illustration 1. Find the centre and the radius of the circles

(a) $3x^2 + 3y^2 - 8x - 10y + 3 = 0$

(b) $x^2 + y^2 + 2x\sin\theta + 2y\cos\theta - 8 = 0$

(c) $2x^2 + \lambda xy + 2y^2 + (\lambda - 4)x + 6y - 5 = 0$, for some λ.

Solution. (a) We rewrite the given equation as

$$x^2 + y^2 - \frac{8}{3}x - \frac{10}{3}y + 1 = 0 \Rightarrow g = -\frac{4}{3},\ f = -\frac{5}{3},\ c = 1$$

Hence the centre is $\left(\frac{4}{3}, \frac{5}{3}\right)$ and the radius is $\sqrt{\frac{16}{9} + \frac{25}{9} - 1} = \sqrt{\frac{32}{9}} = \frac{4\sqrt{2}}{3}$ units

(b) $x^2 + y^2 + 2x\sin\theta + 2y\cos\theta - 8 = 0.$

Centre of this circle is $(-\sin\theta, -\cos\theta)$

Radius $= \sqrt{\sin^2\theta + \cos^2\theta + 8} = \sqrt{1+8} = 3$ units

(c) $2x^2 + \lambda xy + 2y^2 + (\lambda - 4)x + 6y - 5 = 0$

We rewrite the equation as

$$x^2 + \frac{\lambda}{2}xy + y^2 + \left(\frac{\lambda - 4}{2}\right)x + 3y - \frac{5}{2} = 0 \qquad \text{........ (i)}$$

Since, there is no term of xy in the equation of circle $\Rightarrow \frac{\lambda}{2} = 0 \Rightarrow \lambda = 0$

So, equation (i) reduces to $x^2 + y^2 - 2x + 3y - \frac{5}{2} = 0$

$\therefore$ centre is $\left(1, -\frac{3}{2}\right)$ Radius $= \sqrt{1 + \frac{9}{4} + \frac{5}{2}} = \frac{\sqrt{23}}{2}$ units.

Illustration 2. If the lines $3x - 4y + 4 = 0$ and $6x - 8y - 7 = 0$ are tangents to a circle, then the radius of the circle is -

(A) 3/2 (B) 3/4 (C) 1/10 (D) 1/20

Solution. The diameter of the circle is perpendicular distance between the parallel lines (tangents) $3x - 4y + 4 = 0$ and $3x - 4y - \frac{7}{2} = 0$ and so it is equal to $\frac{4 + 7/2}{\sqrt{9+16}} = \frac{3}{2}$.

Hence radius is $\frac{3}{4}$. ***Ans. (B)***

Illustration 3. If $y = 2x + m$ is a diameter to the circle $x^2 + y^2 + 3x + 4y - 1 = 0$, then find m

Solution. Centre of circle = $(-3/2, -2)$. This lies on diameter $y = 2x + m$

$\Rightarrow -2 = (-3/2) \times 2 + m \Rightarrow m = 1$

****Illustration 4.*** The equation of a circle which passes through the point $(1, -2)$ and $(4, -3)$ and whose centre lies on the line $3x + 4y = 7$ is

(A) $15(x^2 + y^2) - 94x + 18y - 55 = 0$ (B) $15(x^2 + y^2) - 94x + 18y + 55 = 0$

(C) $15(x^2 + y^2) + 94x - 18y + 55 = 0$ (D) none of these

Solution. Let the circle be $x^2 + y^2 + 2gx + 2fy + c = 0$ ..... (i)

Hence, substituting the points, $(1, -2)$ and $(4, -3)$ in equation (i)

$5 + 2g - 4f + c = 0$ ..... (ii)

$25 + 8g - 6f + c = 0$ ..... (iii)

centre $(-g, -f)$ lies on line $3x + 4y = 7$

Hence $-3g - 4f = 7$

solving for g, f,c, we get

Here $g = \frac{-47}{15}$, $f = \frac{9}{15}$, $c = \frac{55}{15}$

Hence the equation is $15(x^2 + y^2) - 94x + 18y + 55 = 0$ ***Ans. (B)***

***Illustration 5.** A circle has radius equal to 3 units and its centre lies on the line $y = x - 1$. Find the equation of the circle if it passes through (7, 3).

Solution. Let the centre of the circle be (α, β). It lies on the line $y = x - 1$

$\Rightarrow \beta = \alpha - 1$. Hence the centre is $(\alpha, \alpha - 1)$.

$\Rightarrow$ The equation of the circle is $(x - \alpha)^2 + (y - \alpha + 1)^2 = 9$

It passes through (7, 3) $\Rightarrow (7 - \alpha)^2 + (4 - \alpha)^2 = 9$

$\Rightarrow 2\alpha^2 - 22\alpha + 56 = 0 \Rightarrow \alpha^2 - 11\alpha + 28 = 0$

$\Rightarrow (\alpha - 4)(\alpha - 7) = 0 \Rightarrow \alpha = 4, 7$

Hence the required equations are

$x^2 + y^2 - 8x - 6y + 16 = 0$ and $x^2 + y^2 - 14x - 12y + 76 = 0$. **Ans.**

BEGINNER'S BOX-1

TOPIC COVERED : VARIOUS FORMS OF CIRCLE

1. Find the centre and radius of the circle $2x^2 + 2y^2 = 3x - 5y + 7$

2. Find the equation of the circle whose centre is the point of intersection of the lines $2x - 3y + 4 = 0$ & $3x + 4y - 5 = 0$ and passes through the origin.

***3.** Find the parametric form of the equation of the circle $x^2 + y^2 + px + py = 0$

4. Find the equation of the circle the end points of whose diameter are the centres of the circles $x^2 + y^2 + 16x - 14y = 1$ & $x^2 + y^2 - 4x + 10y = 2$

5. Find the coordinates of the centre and the radius of the circles whose equations are
(a) $3x^2 + 3y^2 - 5x - 6y + 4 = 0$ (b) $4x^2 + 4y^2 - 16x - 12y + 21 = 0$.

6. Find the equation of the circle which goes through the origin and cuts off intercepts equal to h and k from the positive parts of the axes.

***7.** Find the equation of the circle which touches the axis of x and passes through the two points (1, –2) and (3, – 4).

8. Find the equation of the circle which touches the axis of :
(a) x at a distance + 3 from the origin and intercepts a distance 6 on the axis of y.
(b) y at a distance – 3 from the origin and intercepts a length 8 on the axis of x.
(c) x, pass through the point (1, 1) and have line $x + y = 3$ as diameter.

***9.** Centres of the three circles

$x^2 + y^2 - 4x - 6y - 14 = 0$

$x^2 + y^2 + 2x + 4y - 5 = 0$

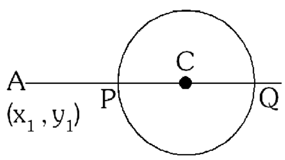
and $x^2 + y^2 - 10x - 16y + 7 = 0$

(A) are the vertices of a right triangle (B) the vertices of an isosceles triangle which is not regular
(C) vertices of a regular triangle (D) are collinear

3.0 POSITION OF A POINT W.R.T CIRCLE

AL

(a) Let the circle is $x^2 + y^2 + 2gx + 2fy + c = 0$ and the point is (x_1, y_1) then -
Point (x_1, y_1) lies out side the circle or on the circle or inside the circle according as
$\Rightarrow x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c >, =, < 0$ or $S_1 >, =, < 0$



(b) The greatest & the least distance of a point A from a circle with centre C & radius r is $AC + r$ & $|AC - r|$ respectively.

4.0 POWER OF A POINT W.R.T. CIRCLE

AL

Theorem – The power of point $P(x_1, y_1)$ w.r.t. the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is S_1

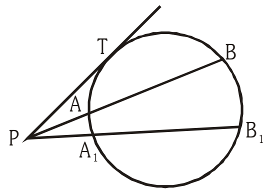
where $S_1 = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$

Note – If P outside, inside or on the circle then power of point is positive, negative or zero respectively.

If from a point $P(x_1, y_1)$, inside or outside the circle, a secant be drawn intersecting the circle in two points A & B, then PA . PB = constant. The product PA . PB is called power of point $P(x_1, y_1)$ w.r.t. the circle

$S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$, i.e. for number of secants

$PA.PB = PA_1 . PB_1 = PA_2 . PB_2 = \ldots\ldots = PT^2 = S_1$



Illustrations

Illustration 6. If P(2, 8) is an interior point of a circle $x^2 + y^2 - 2x + 4y - p = 0$ which neither touches nor intersects the axes, then set for p is -

(A) $p < -1$ (B) $p < -4$ (C) $p > 96$ (D) ϕ

Solution. For internal point p(2, 8), $4 + 64 - 4 + 32 - p < 0 \Rightarrow p > 96$

and x intercept $= 2\sqrt{1+p}$ therefore $1 + p < 0$

$\Rightarrow$ $p < -1$ and y intercept $= 2\sqrt{4+p}$ $\Rightarrow$ $p < -4$ ***Ans. (D)***

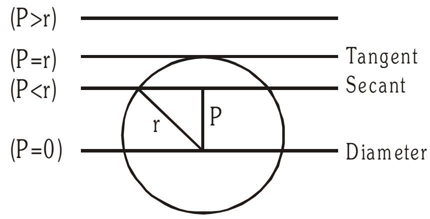
5.0 TANGENT LINE OF CIRCLE

AL

When a straight line meet a circle on two coincident points then it is called the tangent of the circle.

5.1 Condition of Tangency

The line L = 0 touches the circle S = 0 if P the length of the perpendicular from the centre to that line and radius of the circle r are equal i.e. P = r.



Illustrations

****Illustration 7.*** Find the range of parameter 'a' for which the variable line $y = 2x + a$ lies between the circles $x^2 + y^2 - 2x - 2y + 1 = 0$ and $x^2 + y^2 - 16x - 2y + 61 = 0$ without intersecting or touching either circle.

Solution. The given circles are $C_1 : (x-1)^2 + (y-1)^2 = 1$ and $C_2 : (x-8)^2 + (y-1)^2 = 4$

The line $y - 2x - a = 0$ will lie between these circle if centre of the circles lie on opposite sides of the line, i.e. $(1 - 2 - a)(1 - 16 - a) < 0$

$\Rightarrow$ $a \in (-15, -1)$

Line wouldn't touch or intersect the circles if,

$$\frac{|1-2-a|}{\sqrt{5}} > 1, \quad \frac{|1-16-a|}{\sqrt{5}} > 2$$

$\Rightarrow$ $|1 + a| > \sqrt{5}$, $|15 + a| > 2\sqrt{5}$

$\Rightarrow$ $a > \sqrt{5} - 1$

or $a < -\sqrt{5} - 1$, $a > 2\sqrt{5} - 15$

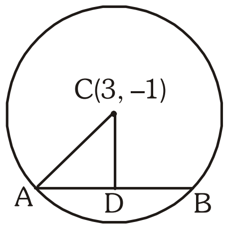
or $a < -2\sqrt{5} - 15$

Hence common values of 'a' are $(2\sqrt{5} - 15, -\sqrt{5} - 1)$.

Examples 8. The equation of a circle whose centre is (3, –1) and which cuts off a chord of length 6 on the line $2x - 5y + 18 = 0$

(A) $(x-3)^2 + (y+1)^2 = 38$ (B) $(x+3)^2 + (y-1)^2 = 38$

(C) $(x-3)^2 + (y+1)^2 = \sqrt{38}$ (D) none of these



Solution. Let AB(= 6) be the chord intercepted by the line $2x - 5y + 18 = 0$ from the circle and let CD be the perpendicular drawn from centre (3, –1) to the chord AB.

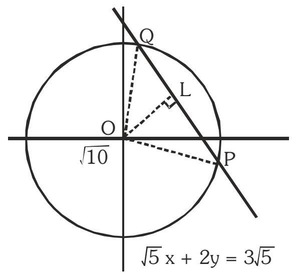
i.e., AD = 3, $CD = \dfrac{2.3 - 5(-1) + 18}{\sqrt{2^2 + 5^2}} = \sqrt{29}$

Therefore, $CA^2 = 3^2 + (\sqrt{29})^2 = 38$

Hence required equation is $(x-3)^2 + (y+1)^2 = 38$ **Ans. (A)**

Examples 9. The area of the triangle formed by line joining the origin to the points of intersection(s) of the line $x\sqrt{5} + 2y = 3\sqrt{5}$ and circle $x^2 + y^2 = 10$ is -

(A) 3 (B) 4 (C) 5 (D) 6



Solution. Length of perpendicular from origin to the line $x\sqrt{5} + 2y = 3\sqrt{5}$ is

$$OL = \frac{3\sqrt{5}}{\sqrt{(\sqrt{5})^2 + 2^2}} = \frac{3\sqrt{5}}{\sqrt{9}} = \sqrt{5}$$

Radius of the given circle = $\sqrt{10}$ = OQ = OP

$PQ = 2QL = 2\sqrt{OQ^2 - OL^2} = 2\sqrt{10-5} = 2\sqrt{5}$

Thus area of $\Delta OPQ = \dfrac{1}{2} \times PQ \times OL = \dfrac{1}{2} \times 2\sqrt{5} \times \sqrt{5} = 5$ **Ans. (C)**

5.2 Equation of the tangent (T = 0)

(a) Tangent at the point (x_1, y_1) on the circle $x^2 + y^2 = a^2$ is $xx_1 + yy_1 = a^2$.

(b) (1) The tangent at the point (acos t, asin t) on the circle $x^2 + y^2 = a^2$ is $x\cos t + y\sin t = a$

(2) The point of intersection of the tangents at the points $P(\alpha)$ and $Q(\beta)$ is $\left(\dfrac{a\cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}, \dfrac{a\sin\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}\right)$.

(c) The equation of tangent at the point (x_1, y_1) on the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$

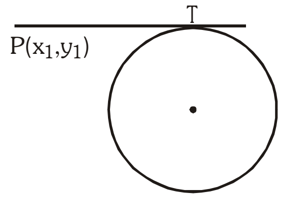
(d) If line $y = mx + c$ is a straight line touching the circle $x^2 + y^2 = a^2$, then $c = \pm a\sqrt{1+m^2}$ and contact points are $\left(\mp\dfrac{am}{\sqrt{1+m^2}}, \pm\dfrac{a}{\sqrt{1+m^2}}\right)$ or $\left(\mp\dfrac{a^2m}{c}, \pm\dfrac{a^2}{c}\right)$ and equation of tangent is $y = mx \pm a\sqrt{1+m^2}$.

(e) The equation of tangent with slope m of the circle $(x-h)^2 + (y-k)^2 = a^2$ is

$(y-k) = m(x-h) \pm a\sqrt{1+m^2}$

Note To get the equation of tangent at the point (x_1, y_1) on any second degree curve we replace xx_1 in place of x^2, yy_1 in place of y^2, $\dfrac{x+x_1}{2}$ in place of x, $\dfrac{y+y_1}{2}$ in place of y, $\dfrac{xy_1+yx_1}{2}$ in place of xy and c in place of c.

5.3 Length of tangent ($\sqrt{S_1}$)



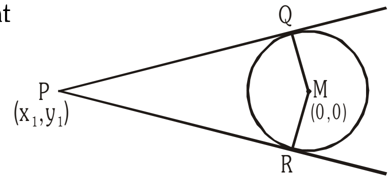
The length of tangent drawn from point (x_1,y_1) out side the circle
$S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ is,

$PT = \sqrt{S_1} = \sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c}$

Note : When we use this formula the coefficient of x^2 and y^2 must be 1.

5.4 Equation of Pair of tangents ($SS_1 = T^2$)

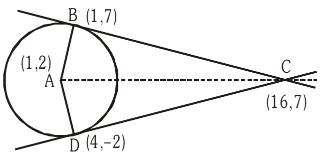
Let the equation of circle $S \equiv x^2 + y^2 = a^2$ and $P(x_1,y_1)$ is any point outside the circle. From the point we can draw two real and distinct tangent PQ & PR and combine equation of pair of tangents is -
$(x^2 + y^2 - a^2)(x_1^2 + y_1^2 - a^2) = (xx_1 + yy_1 - a^2)^2$ or
$SS_1 = T^2$.



Illustrations

Examples 10. Let A be the centre of the circle $x^2 + y^2 - 2x - 4y - 20 = 0$ and B(1, 7) and D(4, –2) are points on the circle then, if tangents be drawn at B and D, which meet at C, then area of quadrilateral ABCD is -

(A) 150 (B) 75
(C) 75/2 (D) none of these



Solution. Here centre A(1, 2) and Tangent at (1, 7) is
$x.1 + y.7 - 1(x + 1) - 2(y + 7) - 20 = 0$ or $y = 7$ ..... (i)
Tangent at D(4, –2) is $3x - 4y - 20 = 0$ ..... (ii)
Solving (i) and (ii), C is (16, 7)
Area ABCD = AB × BC = 5 × 15 = 75 units. ***Ans. (B)***

6.0 NORMAL OF CIRCLE

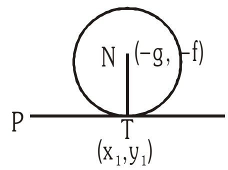
AL

Normal at a point is the straight line which is perpendicular to the tangent at the point of contact.

Note – Normal at point of the circle passes through the centre of the circle.

(a) Equation of normal at point (x_1,y_1) of circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is

$$y - y_1 = \left(\frac{y_1 + f}{x_1 + g}\right)(x - x_1)$$



(b) The equation of normal on any point (x_1,y_1) of circle $x^2 + y^2 = a^2$ is $\frac{y}{x} = \frac{y_1}{x_1}$.

(c) If $x^2 + y^2 = a^2$ is the equation of the circle then at any point 't' of this circle (a cos t, a sint), the equation of normal is xsint – ycost = 0.

Illustrations

Illustration 11. Find the equation of the normal to the circle $x^2 + y^2 - 5x + 2y - 48 = 0$ at the point (5, 6).

Solution. Since normal to the circle always passes through the centre so equation of the normal will be the line passing through (5, 6) & $\left(\frac{5}{2}, -1\right)$

i.e. $y + 1 = \frac{7}{5/2}\left(x - \frac{5}{2}\right)$

$\Rightarrow$ $5y + 5 = 14x - 35$

$\Rightarrow$ $14x - 5y - 40 = 0$ ***Ans.***

***Illustration 12.** If the straight line ax + by = 2; a, b $\neq$ 0 touches the circle $x^2 + y^2 - 2x = 3$ and is normal to the circle $x^2 + y^2 - 4y = 6$, then the values of a and b are respectively

(A) 1, –1 (B) 1, 2 (C) $-\frac{4}{3}, 1$ (D) 2, 1

Solution. Given $x^2 + y^2 - 2x = 3$

$\therefore$ centre is (1, 0) and radius is 2

Given $x^2 + y^2 - 4y = 6$

$\therefore$ centre is (0, 2) and radius is $\sqrt{10}$. Since line ax + by = 2 touches the first circle

$$\therefore \quad \frac{|a(1) + b(0) - 2|}{\sqrt{a^2 + b^2}} = 2$$

or $|(a-2)| = [2\sqrt{a^2 + b^2}]$ ..... (i)

Also the given line is normal to the second circle. Hence it will pass through the centre of the second circle.

$\therefore$ a(0) + b(2) = 2 or 2b = 2 or b = 1

Putting this value in equation (i) we get $|a - 2| = 2\sqrt{a^2 + 1^2}$

or $(a-2)^2 = 4(a^2 + 1)$

or $a^2 + 4 - 4a = 4a^2 + 4$ or $3a^2 + 4a = 0$

or $a(3a + 4) = 0$ or $a = 0, -\frac{4}{3}$ $(a \neq 0)$

$\therefore$ values of a and b are $\left(-\frac{4}{3}, 1\right)$. ***Ans. (C)***

***Illustration 13.** Find the equation of a circle having the lines $x^2 + 2xy + 3x + 6y = 0$ as its normal and having size just sufficient to contain the circle $x(x-4) + y(y-3) = 0$.

Solution. Pair of normals are $(x + 2y)(x + 3) = 0$

$\therefore$ Normals are x + 2y = 0, x + 3 = 0.

Point of intersection of normals is the centre of required circle i.e. $C_1(-3, 3/2)$ and centre of given circle is $C_2(2, 3/2)$ and radius $r_2 = \sqrt{4 + \frac{9}{4}} = \frac{5}{2}$

Let r_1 be the radius of required circle

$$\Rightarrow \quad r_1 = C_1C_2 + r_2 = \sqrt{(-3-2)^2 + \left(\frac{3}{2} - \frac{3}{2}\right)^2} + \frac{5}{2} = \frac{15}{2}$$

Hence equation of required circle is $x^2 + y^2 + 6x - 3y - 45 = 0$

BEGINNER'S BOX-2

TOPIC COVERED : POINT AND LINE WITH RESPECT TO A CIRCLE

1. Find the position of the points (1, 2) & (6, 0) w.r.t. the circle $x^2 + y^2 - 4x + 2y - 11 = 0$

***2.** Find the greatest and least distance of a point P(7, 3) from circle $x^2 + y^2 - 8x - 6y + 16 = 0$. Also find the power of point P w.r.t. circle.

3. Find the equation of tangent to the circle $x^2 + y^2 - 2ax = 0$ at the point $(a(1 + \cos\alpha), a\sin\alpha)$.

4. Find the equations of tangents to the circle $x^2+y^2-6x+4y-12=0$ which are parallel to the line $4x - 3y + 6 = 0$

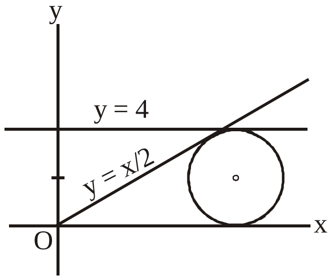
5. Find the equation of the tangents to the circle $x^2 + y^2 = 4$ which are perpendicular to the line $12x - 5y + 9 = 0$. Also find the points of contact.

6. Find the value of 'c' if the line $y = c$ is a tangent to the circle $x^2 + y^2 - 2x + 2y - 2 = 0$ at the point (1, 1)

7. Find the equation of the normal to the circle $x^2 + y^2 = 2x$, which is parallel to the line $x + 2y = 3$.

8. If the points $(\lambda, -\lambda)$ lies inside the circle $x^2 + y^2 - 4x + 2y - 8 = 0$, then find the range of λ.

***9.** The x-coordinate of the center of the circle in the first quadrant (see figure) tangent to the lines $y = \frac{1}{2}x$, $y = 4$ and the x-axis is



(A) $4 + 2\sqrt{5}$ (B) $4 + \frac{8\sqrt{5}}{5}$

(C) $2 + \frac{6\sqrt{5}}{5}$ (D) $8 + 2\sqrt{5}$

***10.** Consider 3 non collinear points A, B, C with coordinates (0, 6), (5, 5) and (–1, 1) respectively. Equation of a line tangent to the circle circumscribing the triangle ABC and passing through the origin is

(A) $2x - 3y = 0$ (B) $3x + 2y = 0$ (C) $3x - 2y = 0$ (D) $2x + 3y = 0$

***11.** (a) Find the shortest distance from the point M (-7, 2) to the circle $x^2 + y^2 - 10x - 14y - 151 = 0$.

(b) Find the co-ordinate of the point on the circle $x^2 + y^2 - 12x - 4y + 30 = 0$, which is farthest from the origin.

***12.** A variable circle C has the equation

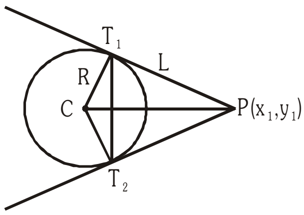
$x^2 + y^2 - 2(t^2 - 3t + 1)x - 2(t^2 + 2t)y + t = 0$, where t is a parameter.

If the power of point P(a,b) w.r.t. the circle C is constant then the ordered pair (a, b) is

(A) $\left(\frac{1}{10}, -\frac{1}{10}\right)$ (B) $\left(-\frac{1}{10}, \frac{1}{10}\right)$ (C) $\left(\frac{1}{10}, \frac{1}{10}\right)$ (D) $\left(-\frac{1}{10}, -\frac{1}{10}\right)$

7.0 CHORD OF CONTACT (T = 0)

AL



A line joining the two points of contacts of two tangents drawn from a point out side the circle, is called chord of contact of that point.

If two tangents PT_1 & PT_2 are drawn from the point $P(x_1, y_1)$ to the circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$, then the equation of the chord of contact T_1T_2 is :

$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$ (i.e. T = 0 same as equation of tangent).

Remember –

(a) Length of chord of contact $T_1T_2 = \frac{2LR}{\sqrt{R^2 + L^2}}$.

(b) Area of the triangle formed by the pair of the tangents & its chord of contact $= \frac{RL^3}{R^2 + L^2}$, where R is the radius of the circle & L is the length of the tangent from (x_1, y_1) on $S = 0$.

(c) Angle between the pair of tangents from $P(x_1, y_1) = \tan^{-1}\left(\frac{2RL}{L^2 - R^2}\right)$

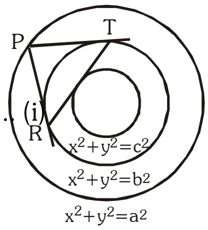
(d) Equation of the circle circumscribing the triangle PT_1T_2 or quadrilateral CT_1PT_2 is :
$(x - x_1)(x + g) + (y - y_1)(y + f) = 0$.

(e) The joint equation of a pair of tangents drawn from the point $A(x_1, y_1)$ to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is : $SS_1 = T^2$.
Where $S \equiv x^2 + y^2 + 2gx + 2fy + c$; $S_1 \equiv x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$
$T \equiv xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c$.

Illustrations

***Illustration 14.** The chord of contact of tangents drawn from a point on the circle $x^2 + y^2 = a^2$ to the circle $x^2 + y^2 = b^2$ touches the circle $x^2 + y^2 = c^2$. Show that a, b, c are in GP.

Solution. Let $P(a\cos\theta, a\sin\theta)$ be a point on the circle $x^2 + y^2 = a^2$.



Then equation of chord of contact of tangents drawn from

$P(a\cos\theta, a\sin\theta)$ to the circle $x^2 + y^2 = b^2$ is $ax\cos\theta + ay\sin\theta = b^2$ ..... (i)

This touches the circle $x^2 + y^2 = c^2$ ..... (ii)

$\therefore$ Length of perpendicular from (0, 0) to (i) = radius of (ii)

$$\therefore \quad \frac{|0+0-b^2|}{\sqrt{(a^2\cos^2\theta + a^2\sin^2\theta)}} = c$$

or $b^2 = ac \Rightarrow$ a, b, c are in GP.

8.0 EQUATION OF THE CHORD WITH A GIVEN MIDDLE POINT ($T = S_1$)

AL

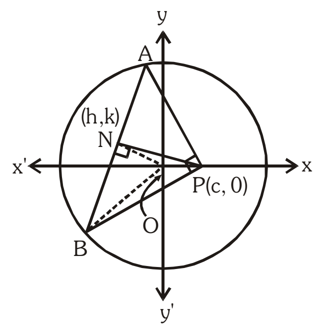
The equation of the chord of the circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ in terms of its mid point $M(x_1, y_1)$ is $y - y_1 = -\dfrac{x_1+g}{y_1+f}(x - x_1)$. This on simplification can be put in the form $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$ which is designated by $T = S_1$.

Note that – The shortest chord of a circle passing through a point 'M' inside the circle, is one chord whose middle point is M.

Illustrations

***Illustration 15.** Find the locus of middle points of chords of the circle $x^2 + y^2 = a^2$, which subtend right angle at the point (c, 0).

Solution. Let N(h, k) be the middle point of any chord AB,



which subtend a right angle at P(c, 0).

Since $\angle APB = 90°$

$\therefore$ NA = NB = NP

(Since distance of the vertices from middle point of the hypotenuse are equal)

or $(NA)^2 = (NB)^2 = (h - c)^2 + (k - 0)^2$ ..... (i)

But also $\angle BNO = 90°$

$\therefore$ $(OB)^2 = (ON)^2 + (NB)^2$

$\Rightarrow$ $-(NB)^2 = (ON)^2 - (OB)^2$

$\Rightarrow$ $-[(h - c)^2 + (k - 0)^2] = (h^2 + k^2) - a^2$

or $2(h^2 + k^2) - 2ch + c^2 - a^2 = 0$

$\therefore$ Locus of N(h, k) is $2(x^2 + y^2) - 2cx + c^2 - a^2 = 0$ ***Ans.***

***Illustration 16.** Let a circle be given by $2x(x - a) + y(2y - b) = 0$ $(a \neq 0, b \neq 0)$

Find the condition on a and b if two chords, each bisected by the x-axis, can be drawn to the circle from (a, b/2).

Solution. The given circle is $2x(x - a) + y(2y - b) = 0$

or $x^2 + y^2 - ax - by/2 = 0$

Let AB be the chord which is bisected by x-axis at a point M. Let its co-ordinates be M(h, 0).

and $S \equiv x^2 + y^2 - ax - by/2 = 0$

$\therefore$ Equation of chord AB is $T = S_1$

$$hx + 0 - \frac{a}{2}(x+h) - \frac{b}{4}(y+0) = h^2 + 0 - ah - 0$$

Since its passes through (a, b/2) we have $ah - \frac{a}{2}(a + h) - \frac{b^2}{8}$

$$= \quad h^2 - ah \quad \Rightarrow \quad h^2 - \frac{3ah}{2} + \frac{a^2}{2} + \frac{b^2}{8} = 0$$

Now there are two chords bisected by the x-axis, so there must be two distinct real roots of h.

$\therefore$ $B^2 - 4AC > 0$

$$\Rightarrow \quad \left(\frac{-3a}{2}\right)^2 - 4.1.\left(\frac{a^2}{2} + \frac{b^2}{8}\right) > 0 \quad \Rightarrow \quad a^2 > 2b^2.$$ ***Ans.***

9.0 DIRECTOR CIRCLE

AL

The locus of point of intersection of two perpendicular tangents to a circle is called director circle. Let P(h,k) is the point of intersection of two tangents drawn on the circle $x^2 + y^2 = a^2$. Then the equation of the pair of tangents is $SS_1 = T^2$

i.e. $(x^2 + y^2 - a^2)(h^2 + k^2 - a^2) = (hx + ky - a^2)^2$

As lines are perpendicular to each other then, coefficient of x^2 + coefficient of $y^2 = 0$

$\Rightarrow$ $[(h^2 + k^2 - a^2) - h^2] + [(h^2 + k^2 - a^2) - k^2] = 0$

$\Rightarrow$ $h^2 + k^2 = 2a^2$

$\therefore$ locus of (h,k) is $x^2 + y^2 = 2a^2$ which is the equation of the director circle.

$\therefore$ director circle is a concentric circle whose radius is $\sqrt{2}$ times the radius of the circle.

Note – The director circle of $x^2 + y^2 + 2gx + 2fy + c = 0$ is $x^2 + y^2 + 2gx + 2fy + 2c - g^2 - f^2 = 0$

Illustrations

****Illustration 17.*** Let P be any moving point on the circle $x^2 + y^2 - 2x = 1$, from this point chord of contact is drawn w.r.t. the circle $x^2 + y^2 - 2x = 0$. Find the locus of the circumcentre of the triangle CAB, C being centre of the circle and A, B are the points of contact.

Solution. The two circles are

$(x - 1)^2 + y^2 = 1$ .... (i)

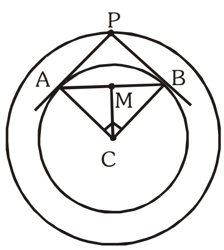
$(x - 1)^2 + y^2 = 2$ .... (ii)

So the second circle is the director circle of the first. So $\angle APB = \pi/2$

Also $\angle ACB = \pi/2$

Now circumcentre of the right angled triangle CAB would lie on the mid point of AB

So let the point be $M \equiv (h, k)$



Now, $CM = CB\sin 45° = \frac{1}{\sqrt{2}}$

So, $(h - 1)^2 + k^2 = \left(\frac{1}{\sqrt{2}}\right)^2$

So, locus of M is $(x - 1)^2 + y^2 = \frac{1}{2}$.

BEGINNER'S BOX-3

TOPIC COVERED : CHORD OF CIRCLE AND DIRECTOR CIRCLE

1. Find the equation of the chord of contact of the point (1, 2) with respect to the circle $x^2 + y^2 + 2x + 3y + 1 = 0$

***2.** Tangents are drawn from the point P(4, 6) to the circle $x^2 + y^2 = 25$. Find the area of the triangle formed by them and their chord of contact.

3. Find the equation of the chord of $x^2 + y^2 - 6x + 10 - a = 0$ which is bisected at (–2, 4).

***4.** Find the locus of mid point of chord of $x^2 + y^2 + 2gx + 2fy + c = 0$ that pass through the origin.

5. Find the equation of the director circle of the circle $(x - h)^2 + (y - k)^2 = a^2$.

***6.** If the angle between the tangents drawn to $x^2 + y^2 + 4x + 8y + c = 0$ from (0, 0) is $\frac{\pi}{2}$, then find value of 'c'

7. If two tangents are drawn from a point on the circle $x^2 + y^2 = 50$ to the circle $x^2 + y^2 = 25$, then find the angle between the tangents

***8.** Show that the line $3x - 4y - c = 0$ will meet the circle having centre at (2, 4) and the raidus 5 in real and distinct points if $-35 < c < 15$.

9. From (3, 4) chords are drawn to the circle $x^2 + y^2 - 4x = 0$. The locus of the mid points of the chords is :

(A) $x^2 + y^2 - 5x - 4y + 6 = 0$ (B) $x^2 + y^2 + 5x - 4y + 6 = 0$

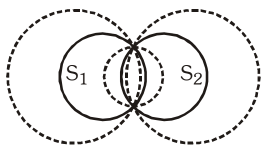
(C) $x^2 + y^2 - 5x + 4y + 6 = 0$ (D) $x^2 + y^2 - 5x - 4y - 6 = 0$

***10.** Chord AB of the circle $x^2 + y^2 = 100$ passes through the point (7, 1) and subtends an angle of 60° at the circumference of the circle. If m_1 and m_2 are the slopes of two such chords then the value of m_1m_2, is

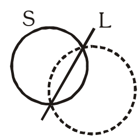
(A) – 1 (B) 1 (C) 7/12 (D) – 3

10.0 FAMILY OF CIRCLES

AL

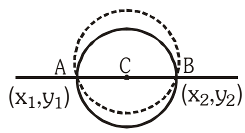


(a) The equation of the family of circles passing through the points of intersection of two circles $S_1 = 0$ & $S_2 = 0$ is : $S_1 + KS_2 = 0$ $(K \neq -1)$.

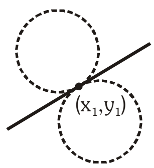


(b) The equation of the family of circles passing through the point of intersection of a circle S = 0 & a line L = 0 is given by S + KL = 0.

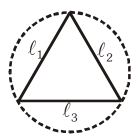
(c) The equation of a family of circles passing through two given points (x_1, y_1) & (x_2, y_2) can be written in the form :



$$(x-x_1)(x-x_2) + (y-y_1)(y-y_2) + K\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$ where K is a parameter.

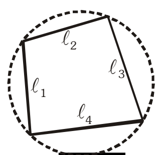


(d) The equation of a family of circles touching a fixed line $y - y_1 = m(x - x_1)$ at the fixed point (x_1, y_1) is $(x - x_1)^2 + (y - y_1)^2 + K[y - y_1 - m(x - x_1)] = 0$, where K is a parameter.



(e) Family of circles circumscribing a triangle whose sides are given by $L_1 = 0$; $L_2 = 0$ & $L_3 = 0$ is given by ; $L_1L_2 + \lambda L_2L_3 + \mu L_3L_1 = 0$ provided coefficient of xy = 0 & coefficient of x^2 = coefficient of y^2.

(f) Equation of circle circumscribing a quadrilateral whose sides in order are represented by the lines $L_1 = 0$, $L_2 = 0$, $L_3 = 0$ & $L_4 = 0$ is $L_1L_3 + \lambda L_2L_4 = 0$ provided coefficient of x^2 = coefficient of y^2 and coefficient of xy = 0.



Illustrations

***Illustration 18.** The equation of the circle through the points of intersection of $x^2 + y^2 - 1 = 0$, $x^2 + y^2 - 2x - 4y + 1 = 0$ and touching the line $x + 2y = 0$, is -

(A) $x^2 + y^2 + x + 2y = 0$ (B) $x^2 + y^2 - x + 20 = 0$

(C) $x^2 + y^2 - x - 2y = 0$ (D) $2(x^2 + y^2) - x - 2y = 0$

Solution. Family of circles is $x^2 + y^2 - 2x - 4y + 1 + \lambda(x^2 + y^2 - 1) = 0$

$(1 + \lambda) x^2 + (1 + \lambda) y^2 - 2x - 4y + (1 - \lambda) = 0$

$$x^2 + y^2 - \frac{2}{1+\lambda}x - \frac{4}{1+\lambda}y + \frac{1-\lambda}{1+\lambda} = 0$$

Centre is $\left(\frac{1}{1+\lambda}, \frac{2}{1+\lambda}\right)$ and radius $= \sqrt{\left(\frac{1}{1+\lambda}\right)^2 + \left(\frac{2}{1+\lambda}\right)^2 - \frac{1-\lambda}{1+\lambda}} = \frac{\sqrt{4+\lambda^2}}{|1+\lambda|}$.

Since it touches the line $x + 2y = 0$, hence

Radius = Perpendicular distance from centre to the line.

i.e., $\left|\frac{\frac{1}{1+\lambda} + 2\frac{2}{1+\lambda}}{\sqrt{1^2+2^2}}\right| = \frac{\sqrt{4+\lambda^2}}{|1+\lambda|} \Rightarrow \sqrt{5} = \sqrt{4+\lambda^2} \quad \Rightarrow \lambda = \pm 1$

$\lambda = -1$ cannot be possible in case of circle. So $\lambda = 1$.

Thus, we get the equation of circle. **Ans. (C)**

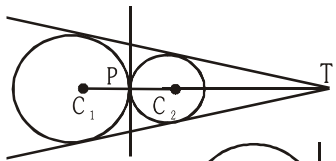
11.0 DIRECT AND TRANSVERSE COMMON TANGENTS

AL

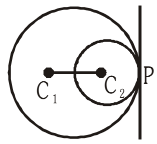
Let two circles having centre C_1 and C_2 and radii, r_1 and r_2 and C_1C_2 is the distance between their centres then :

(a) Both circles will touch

(i) Externally if $C_1C_2 = r_1 + r_2$ i.e. the distance between their centres is equal to sum of their radii and point P & T divides C_1C_2 in the ratio $r_1 : r_2$ (internally & externally respectively). In this case there are ***three common tangents.***

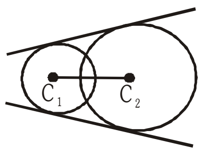


(ii) Internally if $C_1C_2 = |r_1 - r_2|$ i.e. the distance between their centres is equal to difference between their radii and point P divides C_1C_2 in the ratio $r_1 : r_2$ ***externally*** and in this case there will be only ***one common tangent.***



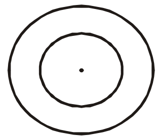
(b) The circles will intersect

when $|r_1 - r_2| < C_1C_2 < r_1 + r_2$ in this case there are ***two common tangents.***

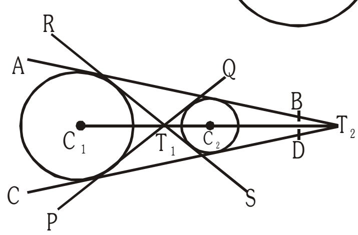


(c) The circles will not intersect

(i) One circle will lie inside the other circle if $C_1C_2 < |r_1 - r_2|$ In this case there will be no common tangent.



(ii) When circle are apart from each other then $C_1C_2 > r_1 + r_2$ and in this case there will be ***four common tangents.*** Lines PQ and RS are called ***transverse*** or ***indirect*** or ***internal*** common tangents and these lines meet line C_1C_2 on T_1 and T_1 divides the line C_1C_2 in the ratio $r_1 : r_2$ internally and lines AB & CD are called ***direct*** or ***external*** common tangents. These lines meet C_1C_2 produced on T_2. Thus T_2 divides C_1C_2 externally in the ratio $r_1 : r_2$.



Note – Length of direct common tangent $= \sqrt{(C_1C_2)^2 - (r_1 - r_2)^2}$

Length of transverse common tangent $= \sqrt{(C_1C_2)^2 - (r_1 + r_2)^2}$

Illustrations

Illustration 19. Prove that the circles $x^2 + y^2 + 2ax + c^2 = 0$ and $x^2 + y^2 + 2by + c^2 = 0$ touch each other, if $\frac{1}{a^2}+\frac{1}{b^2}=\frac{1}{c^2}$.

Solution. Given circles are $x^2 + y^2 + 2ax + c^2 = 0$ ... (i)

and $x^2 + y^2 + 2by + c^2 = 0$ ... (ii)

Let C_1 and C_2 be the centres of circles (i) and (ii), respectively and r_1 and r_2 be their radii, then

$C_1 = (-a, 0)$, $C_2 = (0, -b)$, $r_1 = \sqrt{a^2-c^2}$, $r_2 = \sqrt{b^2-c^2}$

Here we find the two circles touch each other internally or externally.

For touch, $|C_1C_2| = |r_1 \pm r_2|$

or $\sqrt{(a^2+b^2)} = \left|\sqrt{(a^2-c^2)} \pm \sqrt{(b^2-c^2)}\right|$

On squaring $a^2 + b^2 = a^2 - c^2 + b^2 - c^2 \pm 2\sqrt{(a^2-c^2)}\sqrt{(b^2-c^2)}$

or $c^2 = \pm\sqrt{a^2b^2 - c^2(a^2+b^2)+c^4}$

Again squaring, $c^4 = a^2b^2 - c^2(a^2 + b^2) + c^4$

or $c^2(a^2 + b^2) = a^2b^2$ or $\frac{1}{a^2}+\frac{1}{b^2}=\frac{1}{c^2}$

BEGINNER'S BOX-4

TOPIC COVERED : FAMILY OF CIRCLE AND COMMON TANGENTS OF CIRCLE

***1.** Find the equation of the circle passing through the points of intersection of the circle $x^2 + y^2 - 6x + 2y + 4 = 0$ & $x^2 + y^2 + 2x - 4y - 6 = 0$ and with its centre on the line $y = x$.

2. Find the equation of the circle through the points of intersection of the circles $x^2 + y^2 + 2x + 3y - 7 = 0$ and $x^2 + y^2 + 3x - 2y - 1 = 0$ and passing through the point (1, 2).

***3.** Two circles with radius 5 touches at the point (1, 2). If the equation of common tangent is $4x + 3y = 10$ and one of the circle is $x^2 + y^2 + 6x + 2y - 15 = 0$. Find the equation of other circle.

4. Find the number of common tangents to the circles $x^2 + y^2 = 1$ and $x^2 + y^2 - 2x - 6y + 6 = 0$.

***5.** Consider a family of circles which are passing through M (1, 1) and are tangent to x-axis. If (h, k) is the centre of circle, then

(A) $k \geq \frac{1}{2}$ (B) $-\frac{1}{2} \leq k \leq \frac{1}{2}$ (C) $k \leq \frac{1}{2}$ (D) $0 < k < \frac{1}{2}$

6. Find the equation of the circle passing through the points of intersection of the circles $x^2 + y^2 - 2x - 4y - 4 = 0$ and $x^2 + y^2 - 10x - 12y + 40 = 0$ and whose radius is 4.

***7.** Find the equation to the circle which passes through the points (1, – 2) and (4, – 3) and which has its centre on the straight line $3x + 4y = 7$.

***8.** Sum of the abscissa and ordinate of the centre of the circle touching the line $3x + y + 2 = 0$ at the point (–1, 1) and passing through the point (3, 5) is

(A) 2 (B) 3 (C) 4 (D) 5

9. A circle touches the bisector of the first and third quadrant at the origin and passes through the point (2, 0). The equation of the circle is

(A) $x^2 + y^2 - 2x - 2y = 0$ (B) $x^2 + y^2 - 2x + 2y = 0$

(C) $x^2 + y^2 + 2x + 2y = 0$ (D) none of these

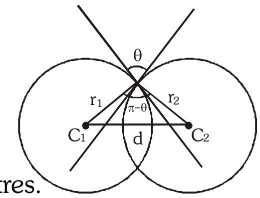
***10.** If the line $x \cos\theta + y \sin\theta = 2$ is the equation of a transverse common tangent to the circles $x^2 + y^2 = 4$ and $x^2 + y^2 - 6\sqrt{3}x - 6y + 20 = 0$, then the value of θ is :

(A) $5\pi/6$ (B) $2\pi/3$ (C) $\pi/3$ (D) $\pi/6$

12.0 THE ANGLE OF INTERSECTION OF TWO CIRCLES

AL

Definition – The angle between the tangents of two circles at the point of intersection of the two circles is called angle of intersection of two circles. If two circles are $S_1 \equiv x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$
$S_2 \equiv x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$ and θ is the acute angle between them

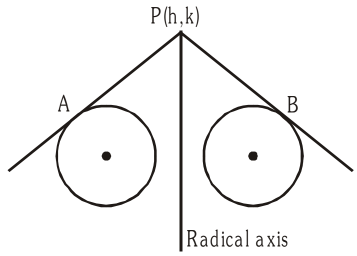


then $\cos\theta = \left|\left(\dfrac{r_1^2 + r_2^2 - d^2}{2r_1r_2}\right)\right|$ or $\cos\theta = \left|\dfrac{2g_1g_2 + 2f_1f_2 - c_1 - c_2}{2\sqrt{g_1^2 + f_1^2 - c_1}\ \sqrt{g_2^2 + f_2^2 - c_2}}\right|$

Here r_1 and r_2 are the radii of the circles and d is the distance between their centres.
If the angle of intersection of the two circles is a right angle then such circles are called "***Orthogonal circles***" and conditions for the circles to be orthogonal is - $2g_1g_2 + 2f_1f_2 = c_1 + c_2$

13.0 RADICAL AXIS OF THE TWO CIRCLES ($S_1 - S_2 = 0$)

AL



(a) Definition – The locus of a point, which moves in such a way that the length of tangents drawn from it to the circles are equal and is called the radical axis. If two circles are -
$S_1 \equiv x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$
$S_2 \equiv x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$
Let P(h,k) is a point and PA,PB are length of two tangents on the circles from point P, Then from definition -

$$\sqrt{h^2 + k^2 + 2g_1h + 2f_1k + c_1} = \sqrt{h^2 + k^2 + 2g_2h + 2f_2k + c_2} \quad \text{or} \quad 2(g_1-g_2)h + 2(f_1-f_2)k + c_1 - c_2 = 0$$

$\therefore$ locus of (h,k)
$2x(g_1-g_2) + 2y(f_1-f_2)k + c_1 - c_2 = 0$
$\boldsymbol{S_1 - S_2 = 0}$

which is the equation of radical axis.

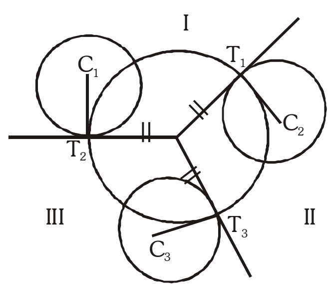
NOTE

(i) To get the equation of the radical axis first of all make the coefficient of x^2 and $y^2 = 1$
(ii) If circles touch each other then radical axis is the common tangent to both the circles.
(iii) When the two circles intersect on real points then common chord is the radical axis of the two circles.
(iv) The radical axis of the two circles is perpendicular to the line joining the centre of two circles but not always pass through mid point of it.
(v) Radical axis (if exist) bisects common tangent to two circles.
(vi) The radical axes of three circles (taking two at a time) meet at a point.
(vii) If circles are concentric then the radical axis does not always exist but if circles are not concentric then radical axis always exists.
(viii) If two circles are orthogonal to the third circle then radical axis of both circle passes through the centre of the third circle.
(ix) A system of circle, every pair of which have the same radical axis, is called a coaxial system of circles.

(b) Radical centre :

The radical centre of three circles is the point from which length of tangents on three circles are equal i.e. the point of intersection of radical axis of the circles is the radical centre of the circles.
To get the radical axis of three circles $S_1 = 0, S_2 = 0, S_3 = 0$ we have to solve any two
$S_1 - S_2 = 0, S_2 - S_3 = 0, S_3 - S_1 = 0$

NOTE



(i) The circle with centre as radical centre and radius equal to the length of tangent from radical centre to any of the circle, will cut the three circles orthogonally.
(ii) If three circles are drawn on three sides of a triangle taking them as diameter then its orthocenter will be its radical centre.
(iii) Locus of the centre of a variable circle orthogonal to two fixed circles is the radical axis between the two fixed circles.
(iv) If two circles are orthogonal, then the polar of a point 'P' on first circle w.r.t. the second circle passes through the point Q which is the other end of the diameter through P. Hence locus of a point which moves such that its polars w.r.t. the circles $S_1 = 0$, $S_2 = 0$ & $S_3 = 0$ are concurrent is a circle which is orthogonal to all the three circles.

Illustrations

***Illustration 20.** A and B are two fixed points and P moves such that PA = nPB where $n \neq 1$. Show that locus of P is a circle and for different values of n all the circles have a common radical axis.

Solution. Let $A \equiv (a, 0)$, $B \equiv (-a, 0)$ and $P(h, k)$

so PA = nPB

$\Rightarrow \quad (h-a)^2 + k^2 = n^2[(h+a)^2 + k^2]$

$\Rightarrow \quad (1-n^2)h^2 + (1-n^2)k^2 - 2ah(1+n^2) + (1-n^2)a^2 = 0$

$$\Rightarrow \quad h^2 + k^2 - 2ah\left(\frac{1+n^2}{1-n^2}\right) + a^2 = 0$$

Hence locus of P is

$$x^2 + y^2 - 2ax\left(\frac{1+n^2}{1-n^2}\right) + a^2 = 0$$, which is a circle of different values of n.

Let n_1 and n_2 are two different values of n so their radical axis is x = 0 i.e. y-axis. Hence for different values of n the circles have a common radical axis.

Illustration 21. Find the equation of the circle through the points of intersection of the circles $x^2 + y^2 - 4x - 6y - 12 = 0$ and $x^2 + y^2 + 6x + 4y - 12 = 0$ and cutting the circle $x^2 + y^2 - 2x - 4 = 0$ orthogonally.

Solution. The equation of the circle through the intersection of the given circles is

$$x^2 + y^2 - 4x - 6y - 12 + \lambda(-10x - 10y) = 0 \qquad \text{.... (i)}$$

where $(-10x - 10y = 0)$ is the equation of radical axis for the circle

$$x^2 + y^2 - 4x - 6y - 12 = 0 \text{ and } x^2 + y^2 + 6x + 4y - 12 = 0.$$

Equation (i) can be re-arranged as

$$x^2 + y^2 - x(10\lambda + 4) - y(10\lambda + 6) - 12 = 0$$

It cuts the circle $x^2 + y^2 - 2x - 4 = 0$ orthogonally.

Hence $2gg_1 + 2ff_1 = c + c_1$

$\Rightarrow \quad 2(5\lambda + 2)(1) + 2(5\lambda + 3)(0) = -12 - 4 \Rightarrow \lambda = -2$

Hence the required circle is

$$x^2 + y^2 - 4x - 6y - 12 - 2(-10x - 10y) = 0$$

i.e., $x^2 + y^2 + 16x + 14y - 12 = 0$

***Illustration 22.** Find the radical centre of circles $x^2 + y^2 + 3x + 2y + 1 = 0$, $x^2 + y^2 - x + 6y + 5 = 0$ and $x^2 + y^2 + 5x - 8y + 15 = 0$. Also find the equation of the circle cutting them orthogonally.

Solution. Given circles are $S_1 \equiv x^2 + y^2 + 3x + 2y + 1 = 0$

$S_2 \equiv x^2 + y^2 - x + 6y + 5 = 0$

$S_3 \equiv x^2 + y^2 + 5x - 8y + 15 = 0$

Equations of two radical axes are $S_1 - S_2 \equiv 4x - 4y - 4 = 0$ or $x - y - 1 = 0$

and $S_2 - S_3 \equiv -6x + 14y - 10 = 0$ or $3x - 7y + 5 = 0$

Solving them the radical centre is (3, 2). Also, if r is the length of the tangent drawn from the radical centre (3, 2) to any one of the given circles, say S_1, we have

$$r = \sqrt{S_1} = \sqrt{3^2 + 2^2 + 3.3 + 2.2 + 1} = \sqrt{27}$$

Hence (3, 2) is the centre and $\sqrt{27}$ is the radius of the circle intersecting them orthogonally.

$\therefore \quad$ Its equation is $\quad (x-3)^2 + (y-2)^2 = r^2 = 27 \Rightarrow x^2 + y^2 - 6x - 4y - 14 = 0$

Alternative Method

Let $x^2 + y^2 + 2gx + 2fy + c = 0$ be the equation of the circle cutting the given circles orthogonally.

$$\therefore \quad 2g\left(\frac{3}{2}\right) + 2f(1) = c + 1 \quad \text{or} \quad 3g + 2f = c + 1 \qquad \text{.... (i)}$$

$$2g\left(-\frac{1}{2}\right) + 2f(3) = c + 5 \quad \text{or} \quad -g + 6f = c + 5 \qquad \text{.... (ii)}$$

$$\text{and} \quad 2g\left(\frac{5}{2}\right) + 2f(-4) = c + 15 \quad \text{or} \quad 5g - 8f = c + 15 \qquad \text{.... (iii)}$$

Solving (i), (ii) and (iii) we get g = –3, f = –2 and c = –14

$\therefore \quad$ equation of required circle is $x^2 + y^2 - 6x - 4y - 14 = 0$ ***Ans.***

BEGINNER'S BOX-5

TOPIC COVERED : ANGLE BETWEEN TWO CIRCLE, RADICAL AXIS AND CENTRE

1. Find the angle of intersection of two circles

$S : x^2 + y^2 - 4x + 6y + 11 = 0$ & $S' : x^2 + y^2 - 2x + 8y + 13 = 0$

2. Find the equation of the radical axis of the circle $x^2 + y^2 - 3x - 4y + 5 = 0$ and

$3x^2 + 3y^2 - 7x - 8y + 11 = 0$

***3.** Find the radical centre of three circles described on the three sides $4x - 7y + 10 = 0$, $x + y - 5 = 0$ and $7x + 4y - 15 = 0$ of a triangle as diameters.

4. When the circles $x^2 + y^2 + 4x + 6y + 3 = 0$ and $2(x^2 + y^2) + 6x + 4y + c = 0$ intersect orthogonally, then find the value of c is

5. Write the condition so that circles $x^2 + y^2 + 2ax + c = 0$ and $x^2 + y^2 + 2by + c = 0$ touch externally.

6. If the circles $x^2 + y^2 + 2ax + cy + a = 0$ and $x^2 + y^2 - 3ax + dy - 1 = 0$ intersect in two distinct points P and Q then the line $5x + by - a = 0$ passes through P and Q for

(A) exactly one value of a
(B) no value of a
(C) infinitely many values of a
(D) exactly two values of a

7. If the circle $C_1 : x^2 + y^2 = 16$ intersects another circle C_2 of radius 5 in such a manner that the common chord is of maximum length and has a slope equal to 3/4, then the co-ordinates of the centre of C_2 are :

(A) $\left(\pm\frac{9}{5}, \pm\frac{12}{5}\right)$
(B) $\left(\pm\frac{9}{5}, \mp\frac{12}{5}\right)$
(C) $\left(\pm\frac{12}{5}, \pm\frac{9}{5}\right)$
(D) $\left(\pm\frac{12}{5}, \mp\frac{9}{5}\right)$

8. Find the radical centre of the following set of circles

$x^2 + y^2 - 3x - 6y + 14 = 0$; $x^2 + y^2 - x - 4y + 8 = 0$; $x^2 + y^2 + 2x - 6y + 9 = 0$

GOLDEN KEY POINTS

- If the circle $S_1 = 0$, bisects the circumference of the circle $S_2 = 0$, then their common chord will be the diameter of the circle $S_2 = 0$.
- The radius of the director circle of a given circle is $\sqrt{2}$ times the radius of the given circle.
- The locus of the middle point of a chord of a circle subtend a right angle at a given point will be a circle.
- The length of side of an equilateral triangle inscribed in the circle $x^2 + y^2 = a^2$ is $\sqrt{3}a$
- If the lengths of tangents from the points A and B to a circle are ℓ_1 and ℓ_2 respectively, then if the points A and B are conjugate to each other, then $(AB)^2 = \ell_1^2 + \ell_2^2$.
- Length of transverse common tangent is less than the length of direct common tangent.

SOME WORKED OUT ILLUSTRATIONS

***Illustration 1.** Find the equation of a circle which passes through the point (2, 0) and whose centre is the limit of the point of intersection of the lines $3x + 5y = 1$ and $(2 + c)x + 5c^2y = 1$ as $c \to 1$.

Solution. Solving the equations $(2 + c)x + 5c^2y = 1$ and $3x + 5y = 1$

then $(2 + c)x + 5c^2\left(\frac{1-3x}{5}\right) = 1$

or $(2 + c)x + c^2(1 - 3x) = 1$

$\therefore \quad x = \frac{1-c^2}{2+c-3c^2}$

or $x = \frac{(1+c)(1-c)}{(3c+2)(1-c)} = \frac{1+c}{3c+2}$

$\therefore \quad x = \lim_{c\to 1}\frac{1+c}{3c+2}$

or $x = \frac{2}{5}$

$\therefore \quad y = \frac{1-3x}{5} = \frac{1-\frac{6}{5}}{5} = -\frac{1}{25}$

Therefore the centre of the required circle is $\left(\frac{2}{5}, \frac{-1}{25}\right)$ but circle passes through (2, 0)

$\therefore$ Radius of the required circle $= \sqrt{\left(\frac{2}{5}-2\right)^2 + \left(-\frac{1}{25}-0\right)^2} = \sqrt{\frac{64}{25}+\frac{1}{625}} = \sqrt{\frac{1601}{625}}$

Hence the required equation of the circle is $\left(x-\frac{2}{5}\right)^2 + \left(y+\frac{1}{25}\right)^2 = \frac{1601}{625}$

or $25x^2 + 25y^2 - 20x + 2y - 60 = 0$ ***Ans.***

***Illustration 2.** Two straight lines rotate about two fixed points. If they start from their position of coincidence such that one rotates at the rate double that of the other. Prove that the locus of their point of intersection is a circle.

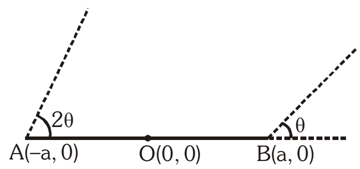
Solution. Let $A \equiv (-a, 0)$ and $B \equiv (a, 0)$ be two fixed points.

Let one line which rotates about B an angle θ with the x-axis at any time t and at that time the second line which rotates about A make an angle 2θ with x-axis.

Now equation of line through B and A are respectively

$y - 0 = \tan\theta(x - a)$ ...... (i)

and $y - 0 = \tan 2\theta(x + a)$ ...... (ii)



From (ii), $y = \frac{2\tan\theta}{1-\tan^2\theta}(x+a)$

$= \left\{\frac{\frac{2y}{(x-a)}}{1-\frac{y^2}{(x-a)^2}}\right\}(x+a)$ (from (i))

$\Rightarrow \quad y = \frac{2y(x-a)(x+a)}{(x-a)^2 - y^2}$

$\Rightarrow \quad (x - a)^2 - y^2 = 2(x^2 - a^2)$

or $x^2 + y^2 + 2ax - 3a^2 = 0$ which is the required locus.

Illustration 3. If the circle $x^2 + y^2 + 6x - 2y + k = 0$ bisects the circumference of the circle $x^2 + y^2 + 2x - 6y - 15 = 0$, then k =

(A) 21 (B) –21 (C) 23 (D) –23

Solution. Common chord $4x + 4y + K + 15 = 0$

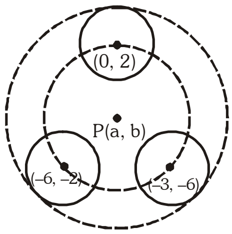
Pass through (–1, 3) $\Rightarrow K = -23$ **Ans. (D)**

***Illustration 4.** Find the equation of the circle of minimum radius which contains the three circles.

$S_1 \equiv x^2 + y^2 - 4y - 5 = 0$

$S_2 \equiv x^2 + y^2 + 12x + 4y + 31 = 0$

$S_3 \equiv x^2 + y^2 + 6x + 12y + 36 = 0$



Solution. For S_1, centre = (0, 2) and radius = 3

For S_2, centre = (–6, –2) and radius = 3

For S_3, centre = (–3, –6) and radius = 3

let P(a, b) be the centre of the circle passing through the centres of the three given circles, then

$$(a-0)^2 + (b-2)^2 = (a+6)^2 + (b+2)^2$$

$$\Rightarrow \quad (a+6)^2 - a^2 = (b-2)^2 - (b+2)^2$$

$$(2a+6)6 = 2b(-4)$$

$$b = \frac{2\times 6(a+3)}{-8} = -\frac{3}{2}(a+3)$$

again $(a-0)^2 + (b-2)^2 = (a+3)^2 + (b+6)^2$

$$\Rightarrow \quad (a+3)^2 - a^2 = (b-2)^2 - (b+6)^2$$

$$(2a+3)3 = (2b+4)(-8)$$

$$(2a+3)3 = -16\left[-\frac{3}{2}(a+3)+2\right]$$

$$6a + 9 = -8(-3a-5)$$

$$6a + 9 = 24a + 40 \Rightarrow 18a = -31$$

$$a = -\frac{31}{18},\ b = -\frac{23}{12}$$

radius of the required circle $= 3+\sqrt{\left(-\frac{31}{18}\right)^2+\left(-\frac{23}{12}-2\right)^2} = 3+\frac{5}{36}\sqrt{949}$

$\therefore$ equation of the required circle is $\left(x+\frac{31}{18}\right)^2+\left(y+\frac{23}{12}\right)^2=\left(3+\frac{5}{36}\sqrt{949}\right)^2$

***Illustration 5.** Find the equation of the image of the circle $x^2 + y^2 + 16x - 24y + 183 = 0$ by the line mirror $4x + 7y + 13 = 0$.

Solution. Centre of given circle = (–8, 12), radius = 5

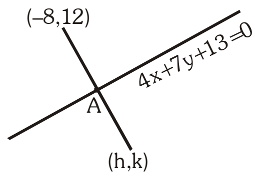
the given line is $4x + 7y + 13 = 0$

let the centre of required circle is (h, k)

since radius will not change. so radius of required circle is 5.

Now (h, k) is the reflection of centre (–8, 12) in the line $4x + 7y + 13 = 0$

Co-ordinates of A $= \left(\frac{-8+h}{2}, \frac{12+k}{2}\right)$



$$\Rightarrow \quad \frac{4(-8+h)}{2}+\frac{7(12+k)}{2}+13=0$$

$$-32 + 4h + 84 + 7k + 26 = 0$$

$$4h + 7k + 78 = 0 \quad ....(i)$$

Also $\frac{k-12}{h+8}=\frac{7}{4}$

$$4k - 48 = 7h + 56$$

$$4k = 7h + 104 \quad ....(ii)$$

solving (i) & (ii)

h = –16, k = –2

$\therefore$ required circle is $(x+16)^2 + (y+2)^2 = 5^2$

***Illustration 6.** The circle $x^2 + y^2 - 6x - 10y + k = 0$ does not touch or intersect the coordinate axes and the point (1, 4) is inside the circle. Find the range of the value of k.

Solution. Since (1, 4) lies inside the circle

$\Rightarrow \quad S_1 < 0$

$\Rightarrow \quad (1)^2 + (4)^2 - 6(1) - 10(4) + k < 0$

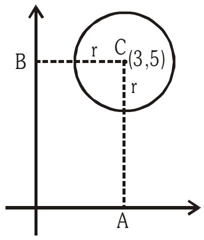
$\Rightarrow \quad k < 29$

Also centre of given circle is (3, 5) and circle does not touch or intersect the coordinate axes

$\Rightarrow \quad r < CA \quad \& \quad r < CB$

$CA = 5$

$CB = 3$



$\Rightarrow \quad r < 5 \quad \& \quad r < 3$

$\Rightarrow \quad r < 3 \quad \text{or} \quad r^2 < 9$

$r^2 = 9 + 25 - k$

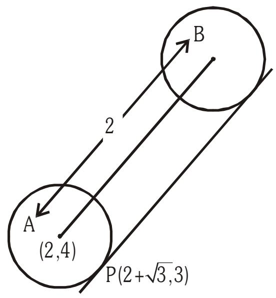
$r^2 = 34 - k \qquad \Rightarrow \qquad 34 - k < 9$

$k > 25$

$\Rightarrow \quad k \in (25, 29)$

***Illustration 7.** The circle $x^2 + y^2 - 4x - 8y + 16 = 0$ rolls up the tangent to it at $(2 + \sqrt{3}, 3)$ by 2 units, find the equation of the circle in the new position.

Solution. Given circle is $x^2 + y^2 - 4x - 8y + 16 = 0$



let $P \equiv (2 + \sqrt{3}, 3)$

Equation of tangent to the circle at $P(2 + \sqrt{3}, 3)$ will be

$$(2 + \sqrt{3})x + 3y - 2(x + 2 + \sqrt{3}) - 4(y + 3) + 16 = 0$$

or $\quad \sqrt{3}x - y - 2\sqrt{3} = 0$

slope $= \sqrt{3} \quad \Rightarrow \quad \tan\theta = \sqrt{3}$

$\theta = 60°$

line AB is parallel to the tangent at P

$\Rightarrow \quad$ coordinates of point B $= (2 + 2\cos 60°, 4 + 2\sin 60°)$

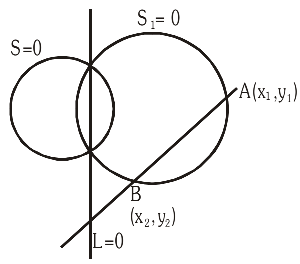
thus $B = (3, 4 + \sqrt{3})$

radius of circle $= \sqrt{2^2 + 4^2 - 16} = 2$

$\therefore \quad$ equation of required circle is $(x - 3)^2 + (y - 4 - \sqrt{3})^2 = 2^2$

***Illustration 8.** A fixed circle is cut by a family of circles all of which, pass through two given points $A(x_1, y_1)$ and $B(x_2, y_2)$. Prove that the chord of intersection of the fixed circle with any circle of the family passes through a fixed point.

Solution. Let $S = 0$ be the equation of fixed circle



let $S_1 = 0$ be the equation of any circle through A and B which intersect $S = 0$ in two points.

$L \equiv S - S_1 = 0$ is the equation of the chord of intersection of $S = 0$ and $S_1 = 0$

let $L_1 = 0$ be the equation of line AB

let S_2 be the equation of the circle whose diametrical ends are $A(x_1, y_1)$ & $B(x_2, y_2)$

then $S_1 \equiv S_2 - \lambda L_1 = 0$

$\Rightarrow \quad L \equiv S - (S_2 - \lambda L_1) = 0 \quad \text{or} \quad L \equiv (S - S_2) + \lambda L_1 = 0$

or $\quad L \equiv L' + \lambda L_1 = 0 \qquad$ ....(i)

(i) Implies each chord of intersection passes through the fixed point, which is the point of intersection of lines $L' = 0$ & $L_1 = 0$. Hence proved.

***Illustration 9.** Let L_1 be a straight line through the origin and L_2 be the straight line $x + y = 1$. If the intercepts made by the circle $x^2 + y^2 - x + 3y = 0$ on L_1 & L_2 are equal , then which of the following equations can represent L_1?

(A) $x + y = 0$ (B) $x - y = 0$ (C) $x + 7y = 0$ (D) $x - 7y = 0$

Solution. Let L_1 be $y = mx$

lines L_1 & L_2 will be at equal distances from centre of the circle centre of the circle is $\left(\frac{1}{2}, -\frac{3}{2}\right)$

$$\Rightarrow \quad \frac{\left|\frac{1}{2}m + \frac{3}{2}\right|}{\sqrt{1+m^2}} = \frac{\left|\frac{1}{2} - \frac{3}{2} - 1\right|}{\sqrt{2}}$$

$$\Rightarrow \quad \frac{(m+3)^2}{(1+m^2)} = 8$$

$\Rightarrow \quad 7m^2 - 6m - 1 = 0$

$\Rightarrow \quad (m - 1)(7m + 1) = 0$

$$\Rightarrow \quad m = 1, m = -\frac{1}{7}$$

$\Rightarrow \quad y = x, 7y + x = 0$ ***Ans.(B,C)***

ANSWERS

BEGINNER'S BOX-1

1. Centre $\left(\frac{3}{4}, -\frac{5}{4}\right)$, Radius $\frac{3\sqrt{10}}{4}$

2. $17(x^2 + y^2) + 2x - 44y = 0$

3. $x = \frac{p}{2}(-1+\sqrt{2}\cos\theta)$; $y = \frac{p}{2}(-1+\sqrt{2}\sin\theta)$

4. $x^2 + y^2 + 6x - 2y - 51 = 0$

5. (a) $\left(\frac{5}{6}, 1\right)$; $\frac{1}{6}\sqrt{13}$; (b) $\left(2, \frac{3}{2}\right)$, 1

6. $x^2 + y^2 - hx - ky = 0$

7. $x^2 + y^2 - 6x + 4y + 9 = 0$, or $x^2 + y^2 + 10x + 20y + 25 = 0$

8. (a) $x^2 + y^2 - 6x \pm 6\sqrt{2}\,y + 9 = 0$]; (b) $x^2 + y^2 \pm 10x + 6y + 9 = 0$
(c) $x^2 + y^2 + 4x - 10y + 4 = 0$; $x^2 + y^2 - 4x - 2y + 4 = 0$.

9. (D)

BEGINNER'S BOX-2

1. (1, 2) lie inside the circle and the point (6, 0) lies outside the circle

2. min = 0, max = 6, power = 0

3. $x\cos\alpha + y\sin\alpha = a(1 + \cos\alpha)$

4. $4x - 3y + 7 = 0$ & $4x - 3y - 43 = 0$

5. $5x + 12y = \pm 26$; $\left(\mp\frac{10}{13}, \mp\frac{24}{13}\right)$

6. 1

7. $x + 2y = 1$

8. [Ans. $\lambda \in (-1, 4)$]

9. (A)

10. (D)

11. (a) 2; (b) (9, 3)

12. (B)

BEGINNER'S BOX-3

1. $4x + 7y + 10 = 0$

2. $\frac{405\sqrt{3}}{52}$ sq. units

3. $5x - 4y + 26 = 0$

4. $x^2 + y^2 + gx + fy = 0$

5. $(x - h)^2 + (y - k)^2 = 2a^2$

6. 10

7. angle between the tangents = 90°

9. (A)

10. (A)

BEGINNER'S BOX-4

1. $x^2 + y^2 - \frac{10x}{7} - \frac{10y}{7} - \frac{12}{7} = 0$

2. $x^2 + y^2 + 4x - 7y + 5 = 0$

3. $(x - 5)^2 + (y - 5)^2 = 25$

4. 4

5. (A)

6. $2x^2 + 2y^2 - 18x - 22y + 69 = 0$ and $x^2 + y^2 - 2y - 15 = 0$

7. $15x^2 + 15y^2 - 94x + 18y + 55 = 0$

8. (C)

9. (B)

10. (D)

BEGINNER'S BOX-5

1. 135°

2. $x + 2y = 2$

3. (1, 2)

4. 18

5. $a^{-2} + b^{-2} = c^{-1}$

6. (B)

7. (B)

8. (1, 2)

EXERCISE – 1 MCQ (SINGLE CHOICE CORRECT)

1. The lines $2x - 3y = 5$ and $3x - 4y = 7$ are diameters of a circle of area 154 sq. units. The equation of the circle is -
(A) $x^2 + y^2 - 2x - 2y = 47$
(B) $x^2 + y^2 - 2x - 2y = 62$
(C) $x^2 + y^2 - 2x + 2y = 47$
(D) $x^2 + y^2 - 2x + 2y = 62$

2. If a be the radius of a circle which touches x-axis at the origin, then its equation is -
(A) $x^2 + y^2 + ax = 0$
(B) $x^2 + y^2 \pm 2ya = 0$
(C) $x^2 + y^2 \pm 2xa = 0$
(D) $x^2 + y^2 + ya = 0$

3. The equation of the circle which touches the axis of y at the origin and passes through (3,4) is -
(A) $4(x^2 + y^2) - 25x = 0$
(B) $3(x^2 + y^2) - 25x = 0$
(C) $2(x^2 + y^2) - 3x = 0$
(D) $4(x^2 + y^2) - 25x + 10 = 0$

4. The equation of the circle passing through (3,6) and whose centre is (2,–1) is -
(A) $x^2 + y^2 - 4x + 2y = 45$
(B) $x^2 + y^2 - 4x - 2y + 45 = 0$
(C) $x^2 + y^2 + 4x - 2y = 45$
(D) $x^2 + y^2 - 4x + 2y + 45 = 0$

***5.** The equation to the circle whose radius is 4 and which touches the negative x-axis at a distance 3 units from the origin is -
(A) $x^2 + y^2 - 6x + 8y - 9 = 0$
(B) $x^2 + y^2 \pm 6x - 8y + 9 = 0$
(C) $x^2 + y^2 + 6x \pm 8y + 9 = 0$
(D) $x^2 + y^2 \pm 6x - 8y - 9 = 0$

6. The equation of a circle which passes through the three points (3,0) (1,–6),(4,–1) is -
(A) $2x^2 + 2y^2 + 5x - 11y + 3 = 0$
(B) $x^2 + y^2 - 5x + 11y - 3 = 0$
(C) $x^2 + y^2 + 5x - 11y + 3 = 0$
(D) $2x^2 + 2y^2 - 5x + 11y - 3 = 0$

7. $y = \sqrt{3}x + c_1$ & $y = \sqrt{3}x + c_2$ are two parallel tangents of a circle of radius 2 units, then $|c_1 - c_2|$ is equal to -
(A) 8
(B) 4
(C) 2
(D) 1

***8.** Number of different circles that can be drawn touching 3 lines, no two of which are parallel and they are neither coincident nor concurrent, are -
(A) 1
(B) 2
(C) 3
(D) 4

9. B and C are fixed points having co-ordinates (3, 0) and (–3, 0) respectively. If the vertical angle BAC is 90°, then the locus of the centroid of the ΔABC has the equation -
(A) $x^2 + y^2 = 1$
(B) $x^2 + y^2 = 2$
(C) $9(x^2 + y^2) = 1$
(D) $9(x^2 + y^2) = 4$

***10.** If a circle of constant radius 3k passes through the origin 'O' and meets co-ordinate axes at A and B then the locus of the centroid of the triangle OAB is -
(A) $x^2 + y^2 = (2k)^2$
(B) $x^2 + y^2 = (3k)^2$
(C) $x^2 + y^2 = (4k)^2$
(D) $x^2 + y^2 = (6k)^2$

***11.** The area of an equilateral triangle inscribed in the circle $x^2 + y^2 - 2x = 0$ is :
(A) $\frac{3\sqrt{3}}{2}$
(B) $\frac{3\sqrt{3}}{4}$
(C) $\frac{3\sqrt{3}}{8}$
(D) None of these

12. The length of intercept on y-axis, by a circle whose diameter is the line joining the points (–4,3) and (12, –1) is -
(A) $3\sqrt{2}$
(B) $\sqrt{13}$
(C) $4\sqrt{13}$
(D) None of these

13. The gradient of the tangent line at the point ($a\cos\alpha$, $a\sin\alpha$) to the circle $x^2 + y^2 = a^2$, is -
(A) $\tan(\pi - \alpha)$ (B) $\tan\alpha$ (C) $\cot\alpha$ (D) $-\cot\alpha$

14. $\ell x + my + n = 0$ is a tangent line to the circle $x^2 + y^2 = r^2$, if -
(A) $\ell^2 + m^2 = n^2 r^2$ (B) $\ell^2 + m^2 = n^2 + r^2$ (C) $n^2 = r^2(\ell^2 + m^2)$ (D) None of these

15. Line $3x + 4y = 25$ touches the circle $x^2 + y^2 = 25$ at the point -
(A) (4, 3) (B) (3, 4) (C) (–3, –4) (D) None of these

16. The equations of the tangents drawn from the point (0,1) to the circle $x^2 + y^2 - 2x + 4y = 0$ are -
(A) $2x - y + 1 = 0$, $x + 2y - 2 = 0$ (B) $2x - y - 1 = 0$, $x + 2y - 2 = 0$
(C) $2x - y + 1 = 0$, $x + 2y + 2 = 0$ (D) $2x - y - 1 = 0$, $x + 2y + 2 = 0$

***17.** The greatest distance of the point P(10,7) from the circle $x^2 + y^2 - 4x - 2y - 20 = 0$ is -
(A) 5 (B) 15 (C) 10 (D) None of these

18. The centre of the smallest circle touching the circles $x^2 + y^2 - 2y - 3 = 0$ and $x^2 + y^2 - 8x - 18y + 93 = 0$ is :
(A) (3 , 2) (B) (4 , 4) (C) (2 , 7) (D) (2 , 5)

***19.** The parametric coordinates of any point on the circle $x^2 + y^2 - 4x - 4y = 0$ are-
(A) $(-2 + 2\cos\alpha, -2 + 2\sin\alpha)$ (B) $(2 + 2\cos\alpha, 2 + 2\sin\alpha)$
(C) $(2 + 2\sqrt{2}\cos\alpha, 2 + 2\sqrt{2}\sin\alpha)$ (D) $(-2 + 2\sqrt{2}\cos\alpha, -2 + 2\sqrt{2}\sin\alpha)$

20. The length of the tangent drawn from the point (2,3) to the circles $2(x^2 + y^2) - 7x + 9y - 11 = 0$ -
(A) 18 (B) 14 (C) $\sqrt{14}$ (D) $\sqrt{28}$

21. A pair of tangents are drawn from the origin to the circle $x^2 + y^2 + 20(x + y) + 20 = 0$. The equation of the pair of tangents is -
(A) $x^2 + y^2 + 5xy = 0$ (B) $x^2 + y^2 + 10xy = 0$
(C) $2x^2 + 2y^2 + 5xy = 0$ (D) $2x^2 + 2y^2 - 5xy = 0$

***22.** Tangents are drawn from (4, 4) to the circle $x^2 + y^2 - 2x - 2y - 7 = 0$ to meet the circle at A and B. The length of the chord AB is -
(A) $2\sqrt{3}$ (B) $3\sqrt{2}$ (C) $2\sqrt{6}$ (D) $6\sqrt{2}$

23. The angle between the two tangents from the origin to the circle $(x-7)^2 + (y + 1)^2 = 25$ equals -
(A) $\frac{\pi}{2}$ (B) $\frac{\pi}{3}$ (C) $\frac{\pi}{4}$ (D) None of these

***24.** Pair of tangents are drawn from every point on the line $3x + 4y = 12$ on the circle $x^2 + y^2 = 4$. Their variable chord of contact always passes through a fixed point whose co-ordinates are -
(A) $\left(\frac{4}{3}, \frac{3}{4}\right)$ (B) $\left(\frac{3}{4}, \frac{3}{4}\right)$ (C) (1, 1) (D) $\left(1, \frac{4}{3}\right)$

***25.** The locus of the mid-points of the chords of the circle $x^2 + y^2 - 2x - 4y - 11 = 0$ which subtend 60° at the centre is -
(A) $x^2 + y^2 - 4x - 2y - 7 = 0$ (B) $x^2 + y^2 + 4x + 2y - 7 = 0$
(C) $x^2 + y^2 - 2x - 4y - 7 = 0$ (D) $x^2 + y^2 + 2x + 4y + 7 = 0$

26. The locus of the centres of the circles such that the point (2,3) is the mid point of the chord $5x + 2y = 16$ is -

(A) $2x - 5y + 11 = 0$ (B) $2x + 5y - 11 = 0$

(C) $2x + 5y + 11 = 0$ (D) None of these

***27.** The locus of the centre of a circle which touches externally the circle, $x^2 + y^2 - 6x - 6y + 14 = 0$ and also touches the y-axis is given by the equation -

(A) $x^2 - 6x - 10y + 14 = 0$ (B) $x^2 - 10x - 6y + 14 = 0$

(C) $y^2 - 6x - 10y + 14 = 0$ (D) $y^2 - 10x - 6y + 14 = 0$

28. The equation of the circle having the lines $y^2 - 2y + 4x - 2xy = 0$ as its normals & passing through the point (2,1) is -

(A) $x^2 + y^2 - 2x - 4y + 3 = 0$ (B) $x^2 + y^2 - 2x + 4y - 5 = 0$

(C) $x^2 + y^2 + 2x + 4y - 13 = 0$ (D) None of these

***29.** A circle is drawn touching the x-axis and centre at the point which is the reflection of (a, b) in the line $y - x = 0$. The equation of the circle is -

(A) $x^2 + y^2 - 2bx - 2ay + a^2 = 0$ (B) $x^2 + y^2 - 2bx - 2ay + b^2 = 0$

(C) $x^2 + y^2 - 2ax - 2by + b^2 = 0$ (D) $x^2 + y^2 - 2ax - 2by + a^2 = 0$

***30.** The length of the common chord of circles $x^2 + y^2 - 6x - 16 = 0$ and $x^2 + y^2 - 8y - 9 = 0$ is -

(A) $10\sqrt{3}$ (B) $5\sqrt{3}$ (C) $5\sqrt{3}/2$ (D) None of these

EXERCISE – 2 MCQ (ONE OR MORE CHOICE CORRECT)

1. Equation $\frac{x-x_1}{\cos\theta}=\frac{y-y_1}{\sin\theta}=r$, may represents -
(A) Equation of straight line, if θ is constant and r is variable.
(B) Equation of a circle, if r is constant & θ is variable.
(C) A straight line passing through a fixed point & having a known slope.
(D) A circle with a known centre and given radius.

***2.** If r represent the distance of a point from origin & θ is the angle made by line joining origin to that point from line x-axis, then $r = |\cos\theta|$ represents -
(A) two circles of radii $\frac{1}{2}$ each. (B) two circles centred at $\left(\frac{1}{2},0\right)$ & $\left(-\frac{1}{2},0\right)$
(C) two circles touching each other at the origin. (D) pair of straight line

***3.** For the equation $x^2 + y^2 + 2\lambda x + 4 = 0$ which of the following can be true -
(A) It represents a real circle for all $\lambda \in R$.
(B) It represents a real circle for $|\lambda| > 2$.
(C) The radical axis of any two circles of the family is the y-axis.
(D) The radical axis of any two circles of the family is the x-axis.

4. If y = c is a tangent to the circle $x^2 + y^2 - 2x + 2y - 2 = 0$, then the value of c can be -
(A) 1 (B) 3 (C) –1 (D) –3

***5.** Three equal circles each of radius r touch one another. The radius of the circle touching all the three given circles internally is -
(A) $(2+\sqrt{3})r$ (B) $\frac{(2+\sqrt{3})}{\sqrt{3}}r$ (C) $\frac{(2-\sqrt{3})}{\sqrt{3}}r$ (D) $(2-\sqrt{3})r$

***6.** If $a^2 + b^2 = 1$, $m^2 + n^2 = 1$, then which of the following is true for all values of m, n, a, b -
(A) $|am + bn| \le 1$ (B) $|am - bn| \ge 1$ (C) $|am + bn| \ge 1$ (D) $|am - bn| \le 1$

7. $x^2 + y^2 + 6x = 0$ and $x^2 + y^2 - 2x = 0$ are two circles, then -
(A) They touch each other externally
(B) They touch each other internally
(C) Area of triangle formed by their common tangents is $3\sqrt{3}$ sq. units.
(D) Their common tangents do not form any triangle.

***8.** Slope of tangent to the circle $(x - r)^2 + y^2 = r^2$ at the point (x, y) lying on the circle is -
(A) $\frac{x}{y-r}$ (B) $\frac{r-x}{y}$ (C) $\frac{y^2-x^2}{2xy}$ (D) $\frac{y^2+x^2}{2xy}$

9. The circle passing through the distinct points (1,t) , (t,1) & (t,t) for all values of 't', passes through the point -
(A) (–1, –1) (B) (–1, 1) (C) (1, –1) (D) (1,1)

***10.** The centre(s) of the circle(s) passing through the points (0, 0), (1, 0) and touching the circle $x^2 + y^2 = 9$ is/are -
(A) $\left(\frac{3}{2},\frac{1}{2}\right)$ (B) $\left(\frac{1}{2},\frac{3}{2}\right)$ (C) $\left(\frac{1}{2},2^{1/2}\right)$ (D) $\left(\frac{1}{2},-2^{1/2}\right)$

11. The equation(s) of the tangent at the point (0, 0) on the circle, making intercepts of length 2a and 2b units on the co-ordinate x & y axes respectively, is (are) -
(A) ax + by = 0 (B) ax – by = 0 (C) x = y (D) None of these

***12.** The tangents drawn from the origin to the circle $x^2 + y^2 - 2rx - 2hy + h^2 = 0$ are perpendicular if -
(A) h = r (B) h = –r (C) $r^2 + h^2 = 1$ (D) $r^2 + h^2 = 2$

Match the column

Following question contains statements given in two columns, which have to be matched. The statements in ***Column-I*** are labelled as A, B, C and D while the statements in ***Column-II*** are labelled as p, q, r and s. Any given statement in ***Column-I*** can have correct matching with ***ONE OR MORE*** statement(s) in ***Column-II***.

****13.***

	Column-I		Column-II
(A)	If point of intersection and number of common tangents of two circles $x^2 + y^2 - 2x - 6y + 9 = 0$ and $x^2 + y^2 + 6x - 2y + 1 = 0$ are λ and μ respectively, then	(p)	$\mu - \lambda = 3$
(B)	If point of intersection and number of tangents of two circles $x^2 + y^2 - 6x = 0$ and $x^2 + y^2 + 2x = 0$ are λ and μ respectively, then	(q)	$\mu + \lambda = 5$
(C)	If the straight line $y = mx\ \forall\ m \in I$ touches or lies outside the circle $x^2 + y^2 - 20y + 90 = 0$ and the maximum and minimum values of $\lvert m\rvert$ are μ & λ respectively then	(r)	$\mu - \lambda = 4$
(D)	If two circle $x^2 + y^2 + px + py - 7 = 0$ and $x^2 + y^2 - 10x + 2py + 1 = 0$ cut orthogonally and the value of p are λ & μ respectively then	(s)	$\mu + \lambda = 4$

Comprehension Based Questions

P is a variable point of the line L = 0. Tangents are drawn to the circle $x^2 + y^2 = 4$ from P to touch it at Q and R. The parallelogram PQSR is completed.

On the basis of above information, answer the following questions

14. If $L \equiv 2x + y - 6 = 0$, then the locus of circumcetre of ΔPQR is -

(A) $2x - y = 4$ (B) $2x + y = 3$ (C) $x - 2y = 4$ (D) $x + 2y = 3$

15. If $P \equiv (6, 8)$, then the area of ΔQRS is -

(A) $\frac{(6)^{3/2}}{25}$ sq.units (B) $\frac{(24)^{3/2}}{25}$ sq.units (C) $\frac{48\sqrt{6}}{25}$ sq.units (D) $\frac{192\sqrt{6}}{25}$ sq.units

16. If $P \equiv (3, 4)$, then coordinate of S is -

(A) $\left(-\frac{46}{25}, -\frac{63}{25}\right)$ (B) $\left(-\frac{51}{25}, -\frac{68}{25}\right)$ (C) $\left(-\frac{46}{25}, -\frac{68}{25}\right)$ (D) $\left(-\frac{68}{25}, -\frac{51}{25}\right)$

EXERCISE – 3 SUBJECTIVE

1. Find the equations of the circles which have the radius $\sqrt{13}$ & which touch the line $2x - 3y + 1 = 0$ at $(1, 1)$.

2. (x_1, y_1) & (x_2, y_2) are the ends of a diameter of a circle such that x_1 & x_2 are the roots of $ax^2 + bx + c = 0$ & y_1 & y_2 are roots of $py^2 + qy + r = 0$. Find the equation of the circle, its centre & radius.

***3.** If the lines $a_1x + b_1y + c_1 = 0$ & $a_2x + b_2y + c_2 = 0$ cut the coordinate axes in concyclic points. Prove that $a_1a_2 = b_1b_2$.

***4.** Let A be the centre of the circle $x^2 + y^2 - 2x - 4y - 20 = 0$. Suppose that the tangents at the points $B(1, 7)$ & $D(4, -2)$ on the circle meet at the point C. Find the area of the quadrilateral ABCD.

5. Determine the nature of the quadrilateral formed by four lines $3x + 4y - 5 = 0$; $4x - 3y - 5 = 0$; $3x + 4y + 5 = 0$ and $4x - 3y + 5 = 0$. Find the equation of the circle inscribed and circumscribing this quadrilateral.

***6.** A circle is drawn with its centre on the line $x + y = 2$ to touch the line $4x - 3y + 4 = 0$ and pass through the point $(0, 1)$. Find its equation.

***7.** Obtain the equations of the straight lines passing through the point $A(2, 0)$ & making $45°$ angle with the tangent at A to the circle $(x + 2)^2 + (y - 3)^2 = 25$. Find the equations of the circles each of radius 3 whose centres are on these straight lines at a distance of $5\sqrt{2}$ from A.

***8.** Suppose the equation of the circle which touches both the coordinates axes and passes through the point with abscissa -2 and ordinate 1 has the equation $x^2 + y^2 + Ax + By + C = 0$, find all the possible ordered triplet (A, B, C).

9. Find the equation of the circle which passes through the point $(1, 1)$ & which touches the circle $x^2 + y^2 + 4x - 6y - 3 = 0$ at the point $(2, 3)$ on it.

***10.** A circle $S = 0$ is drawn with its centre at $(-1, 1)$ so as to touch the circle $x^2 + y^2 - 4x + 6y - 3 = 0$ externally. Find the intercept made by the circle $S = 0$ on the coordinates axes.

EXERCISE - 4 RECAP OF AIEEE/JEE (MAIN)

1. The square of the length of tangent from (3, –4) on the circle $x^2 + y^2 - 4x - 6y + 3 = 0$. ***[AIEEE-2002]***

(A) 20 (B) 30 (C) 40 (D) 50

2. Radical axis of the circles $x^2 + y^2 + 6x - 2y - 9 = 0$ and $x^2 + y^2 - 2x + 9y - 11 = 0$ is- ***[AIEEE-2002]***

(A) $8x - 11y + 2 = 0$ (B) $8x + 11y + 2 = 0$ (C) $8x + 11y - 2 = 0$ (D) $8x - 11y - 2 = 0$

***3.** If the two circles $(x - 1)^2 + (y - 3)^2 = r^2$ and $x^2 + y^2 - 8x + 2y + 8 = 0$ intersect in two distinct points, then-

(A) $r > 2$ (B) $2 < r < 8$ (C) $r < 2$ (D) $r = 2$ ***[AIEEE-2003]***

4. The lines $2x - 3y = 5$ and $3x - 4y = 7$ are diameters of a circle having area as 154 sq. units. Then the equation of the circle is- ***[AIEEE-2003]***

(A) $x^2 + y^2 - 2x + 2y = 62$ (B) $x^2 + y^2 + 2x - 2y = 62$

(C) $x^2 + y^2 + 2x - 2y = 47$ (D) $x^2 + y^2 - 2x + 2y = 47$

5.** If a circle passes through the point (a, b) and cuts the circle $x^2 + y^2 = 4$ orthogonally, then the locus of its centre is- ***[AIEEE-2004]

(A) $2ax + 2by + (a^2 + b^2 + 4) = 0$ (B) $2ax + 2by - (a^2 + b^2 + 4) = 0$

(C) $2ax - 2by + (a^2 + b^2 + 4) = 0$ (D) $2ax - 2by - (a^2 + b^2 + 4) = 0$

6. A variable circle passes through the fixed point A(p, q) and touches x-axis. The locus of the other end of the diameter through A is- ***[AIEEE-2004]***

(A) $(x - p)^2 = 4qy$ (B) $(x - q)^2 = 4py$ (C) $(y - p)^2 = 4qx$ (D) $(y - q)^2 = 4px$

7. If the lines $2x + 3y + 1 = 0$ and $3x - y - 4 = 0$ lie along diameters of a circle of circumference 10π, then the equation of the circle is- ***[AIEEE-2004]***

(A) $x^2 + y^2 - 2x + 2y - 23 = 0$ (B) $x^2 + y^2 - 2x - 2y - 23 = 0$

(C) $x^2 + y^2 + 2x + 2y - 23 = 0$ (D) $x^2 + y^2 + 2x - 2y - 23 = 0$

8. The intercept on the line $y = x$ by the circle $x^2 + y^2 - 2x = 0$ is AB. Equation of the circle on AB as a diameter is- ***[AIEEE-2004]***

(A) $x^2 + y^2 - x - y = 0$ (B) $x^2 + y^2 - x + y = 0$ (C) $x^2 + y^2 + x + y = 0$ (D) $x^2 + y^2 + x - y = 0$

9.** If the circles $x^2 + y^2 + 2ax + cy + a = 0$ and $x^2 + y^2 - 3ax + dy - 1 = 0$ intersect in two distinct point P and Q then the line $5x + by - a = 0$ passes through P and Q for- ***[AIEEE-2005]

(A) exactly one value of a (B) no value of a

(C) infinitely many values of a (D) exactly two values of a

10. A circle touches the x-axis and also touches the circle with centre at (0, 3) and radius 2. The locus of the centre of the circle is- ***[AIEEE-2005]***

(A) an ellipse (B) a circle (C) a hyperbola (D) a parabola

11. If a circle passes through the point (a, b) and cuts the circle $x^2 + y^2 = p^2$ orthogonally, then the equation of the locus of its centre is- ***[AIEEE-2005]***

(A) $x^2 + y^2 - 3ax - 4by + (a^2 + b^2 - p^2) = 0$ (B) $2ax + 2by - (a^2 - b^2 + p^2) = 0$

(C) $x^2 + y^2 - 2ax - 3by + (a^2 - b^2 - p^2) = 0$ (D) $2ax + 2by - (a^2 + b^2 + p^2) = 0$

12.** If the pair of lines $ax^2 + 2(a + b)xy + by^2 = 0$ lie along diameters of a circle and divide the circle into four sectors such that the area of one of the sectors is thrice the area of another sector then- ***[AIEEE-2005]

(A) $3a^2 - 10ab + 3b^2 = 0$
(B) $3a^2 - 2ab + 3b^2 = 0$
(C) $3a^2 + 10ab + 3b^2 = 0$
(D) $3a^2 + 2ab + 3b^2 = 0$

13. If the lines $3x - 4y - 7 = 0$ and $2x - 3y - 5 = 0$ are two diameters of a circle of area 49π square units, the equation of the circle is- ***[AIEEE-2006]***

(A) $x^2 + y^2 + 2x - 2y - 62 = 0$
(B) $x^2 + y^2 - 2x + 2y - 62 = 0$
(C) $x^2 + y^2 - 2x + 2y - 47 = 0$
(D) $x^2 + y^2 + 2x - 2y - 47 = 0$

14.** Let C be the circle with centre (0, 0) and radius 3 units. The equation of the locus of the mid points of the chords of the circle C that subtend an angle of $\frac{2\pi}{3}$ at its centre is - ***[AIEEE-2006, IIT-1996]

(A) $x^2 + y^2 = 1$
(B) $x^2 + y^2 = \frac{27}{4}$
(C) $x^2 + y^2 = \frac{9}{4}$
(D) $x^2 + y^2 = \frac{3}{2}$

15.** Consider a family of circles which are passing through the point (–1, 1) and are tangent to x-axis. If (h, k) are the co-ordinates of the centre of the circles, then the set of values of k is given by the interval- ***AIEEE-2007]

(A) $0 < k < 1/2$
(B) $k \geq 1/2$
(C) $-1/2 \leq k \leq 1/2$
(D) $k \leq 1/2$

16. The point diametrically opposite to the point (1, 0) on the circle $x^2 + y^2 + 2x + 4y - 3 = 0$ is- ***[AIEEE-2008]***

(A) (3, –4)
(B) (–3, 4)
(C) (–3, –4)
(D) (3, 4)

17.** Three distinct points A, B and C are given in the 2–dimensional coordinate plane such that the ratio of the distance of any one of them from the point (1, 0) to the distance from the point (–1, 0) is equal to $\frac{1}{3}$. Then the circumcentre of the triangle ABC is at the point :- ***[AIEEE-2009]

(A) $\left(\frac{5}{2}, 0\right)$
(B) $\left(\frac{5}{3}, 0\right)$
(C) (0, 0)
(D) $\left(\frac{5}{4}, 0\right)$

18.** If P and Q are the points of intersection of the circles $x^2 + y^2 + 3x + 7y + 2p - 5 = 0$ and $x^2 + y^2 + 2x + 2y - p^2 = 0$, then there is a circle passing through P, Q and (1, 1) for :- ***[AIEEE-2009]

(A) All except two values of p
(B) Exactly one value of p
(C) All values of p
(D) All except one value of p

19.** For a regular polygon, let r and R be the radii of the inscribed and the circumscribed circles. A false statement among the following is :- ***[AIEEE-2010]

(A) There is a regular polygon with $\frac{r}{R} = \frac{1}{2}$
(B) There is a regular polygon with $\frac{r}{R} = \frac{1}{\sqrt{2}}$
(C) There is a regular polygon with $\frac{r}{R} = \frac{2}{3}$
(D) There is a regular polygon with $\frac{r}{R} = \frac{\sqrt{3}}{2}$

20. The circle $x^2 + y^2 = 4x + 8y + 5$ intersects the line $3x - 4y = m$ at two distinct points if :- ***[AIEEE-2010]***

(A) $-85 < m < -35$
(B) $-35 < m < 15$
(C) $15 < m < 65$
(D) $35 < m < 85$

21. The two circles $x^2 + y^2 = ax$ and $x^2 + y^2 = c^2$ $(c > 0)$ touch each other if :- **[AIEEE-2011]**

(A) $a = 2c$ (B) $|a| = 2c$ (C) $2|a| = c$ (D) $|a| = c$

22. The equation of the circle passing through the points (1, 0) and (0, 1) and having the smallest radius is:

(A) $x^2 + y^2 + x + y - 2 = 0$ (B) $x^2 + y^2 - 2x - 2y + 1 = 0$ **[AIEEE-2011]**

(C) $x^2 + y^2 - x - y = 0$ (D) $x^2 + y^2 + 2x + 2y - 7 = 0$

23. The length of the diameter of the circle which touches the x-axis at the point (1, 0) and passes through the point (2, 3) is : **[AIEEE-2012]**

(A) 5/3 (B) 10/3 (C) 3/5 (D) 6/5

24. The circle passing through (1, – 2) and touching the axis of x at (3, 0) also passes through the point : **[JEE(Main)-2013]**

(A) (–5, 2) (B) (2, –5) (C) (5, –2) (D) (–2, 5)

25. Let C be the circle with centre at (1, 1) and radius = 1. If T is the circle centred at (0, y), passing through origin and touching the circle C externally, then the radius of T is equal to : **[JEE(Main)-2014]**

(A) $\frac{1}{2}$ (B) $\frac{1}{4}$ (C) $\frac{\sqrt{3}}{\sqrt{2}}$ (D) $\frac{\sqrt{3}}{2}$

***26.** The number of common tangents to the circles $x^2 + y^2 - 4x - 6y - 12 = 0$ and $x^2 + y^2 + 6x + 18y + 26 = 0$, is: **[JEE(Main)-2015]**

(A) 1 (B) 2 (C) 3 (D) 4

27. If the circles $x^2 + y^2 - 16x - 20y + 164 = r^2$ and $(x - 4)^2 + (y - 7)^2 = 36$ intersect at two distinct points, then **[JEE(Main)-2019]**

(A) $0 < r < 1$ (B) $1 < r < 11$ (C) $r > 11$ (D) $r = 11$

28. Three circles of radii a, b, c$(a < b < c)$ touch each other externally. If they have x–axis as a common tangent, then : **[JEE(Main)-2019]**

(A) $\frac{1}{\sqrt{a}} = \frac{1}{\sqrt{b}} + \frac{1}{\sqrt{c}}$ (B) a, b, c are in A. P.

(C) $\sqrt{a}, \sqrt{b}, \sqrt{c}$ are in A. P. (D) $\frac{1}{\sqrt{b}} = \frac{1}{\sqrt{a}} + \frac{1}{\sqrt{c}}$

29. If the area of an equilateral triangle inscribed in the circle, $x^2 + y^2 + 10x + 12y + c = 0$ is $27\sqrt{3}$ sq. units then c is equal to : **[JEE(Main)-2019]**

(A) 20 (B) 25 (C) 13 (D) –25

30. If a circle C passing through the point (4,0) touches the circle $x^2 + y^2 + 4x - 6y = 12$ externally at the point (1, –1), then the radius of C is: **[JEE(Main)-2019]**

(A) $\sqrt{57}$ (B) 4 (C) $2\sqrt{5}$ (D) 5

31. A circle cuts a chord of length 4a on the x–axis and passes through a point on the y–axis, distant 2b from the origin. Then the locus of the centre of this circle, is : **[JEE(Main)-2019]**

(A) A hyperbola (B) A parabola (C) A straight line (D) An ellipse

32. The area (in sq. units) of the region bounded by the curve $x^2 = 4y$ and the straight line $x = 4y - 2$: **[JEE(Main)-2019]**

(A) $\frac{5}{4}$ (B) $\frac{9}{8}$ (C) $\frac{3}{4}$ (D) $\frac{7}{8}$

33. A square is inscribed inthe circle $x^2 + y^2 - 6x + 8y - 103 = 0$ with its sides parallel to the corrdinate axes. Then the distance of the vertex of this square which is nearest to the origin is : **[JEE(Main)-2019]**

(A) 13 (B) $\sqrt{137}$ (C) 6 (D) $\sqrt{41}$

34. Two circles with equal radii are intersecting at the points (0, 1) and (0, –1). The tangent at the point (0, 1) to one of the circles passes through the centre of the other circle. Then the distance between the centres of these circles is : **[JEE(Main)-2019]**

(A) 1 (B) $\sqrt{2}$ (C) $2\sqrt{2}$ (D) 2

35. If a circle of radius R passes through the origin O and intersects the coordinate axes at A and B, then the locus of the foot of perpendicular from O on AB is : **[JEE(Main)-2019]**

(A) $(x^2 + y^2)^2 = 4Rx^2y^2$ (B) $(x^2 + y^2)(x + y) = R^2xy$

(C) $(x^2 + y^2)^3 = 4R^2x^2y^2$ (D) $(x^2 + y^2)^2 = 4R^2x^2y^2$

36. If a variable line, $3x + 4y - \lambda = 0$ is such that the two circles $x^2 + y^2 - 2x - 2y + 1 = 0$ and $x^2 + y^2 - 18x - 2y + 78 = 0$ are on its opposite sides, then the set of all values of X is the interval : **[JEE(Main)-2019]**

(A) [12, 21] (B) (2, 17) (C) (23, 31) (D) [13, 23]

37. Let C_1 and C_2 be the centres of the circles $x^2 + y^2 - 2x - 2y - 2 = 0$ and $x^2 + y^2 - 6x - 6y + 14 = 0$ respectively. If P and Q are the points of intersection of these circles, then the area (in sq. units) of the quadrilateral PC_1QC_2 is : **[JEE(Main)-2019]**

(A) 8 (B) 6 (C) 9 (D) 4

38. The sum of the squares of the lengths of the chords intercepted on the circle, $x^2 + y^2 = 16$, by the lines, $x + y = n$, $n \in N$, where N is the set of all natural numbers, is : **[JEE(Main)-2019]**

(A) 320 (B) 160 (C) 105 (D) 210

39. The tangent and the normal lines at the point $\left(\sqrt{3},1\right)$ to the circle $x^2 + y^2 = 4$ and the x-axis form a triangle. The area of this triangle (in square units) is : **[JEE(Main)-2019]**

(A) $\frac{1}{3}$ (B) $\frac{4}{\sqrt{3}}$ (C) $\frac{1}{\sqrt{3}}$ (D) $\frac{2}{\sqrt{3}}$

40. If a tangent to the circle $x^2 + y^2 = 1$ intersects the coordinate axes at distinct points P and Q, then the locus of the mid-point of PQ is **[JEE(Main)-2019]**

(A) $x^2 + y^2 - 2xy = 0$ (B) $x^2 + y^2 - 16x^2y^2 = 0$ (C) $x^2 + y^2 - 4x^2y^2 = 0$ (D) $x^2 + y^2 - 2x^2y^2 = 0$

41. The common tangent to the circles $x^2 + y^2 = 4$ and $x^2 + y^2 + 6x + 8y - 24 = 0$ also passes through the point **[JEE(Main)-2019]**

(A) (–4, 6) (B) (6, –2) (C) (–6, 4) (D) (4, –2)

42. A rectangle is inscribed in a circle with a diameter lying along the line $3y = x + 7$. If the two adjacent vertices of the rectangle are (–8, 5) and (6, 5), then the area of the rectangle (in sq. units) is :- **[JEE(Main)-2019]**

(A) 72 (B) 84 (C) 98 (D) 56

43. If the circles $x^2 + y^2 + 5Kx + 2y + K = 0$ and $2(x^2 + y^2) + 2Kx + 3y - 1 = 0$, $(K \in R)$, intersect at the points P and Q, then the line $4x + 5y - K = 0$ passes through P and Q for : ***[JEE(Main)-2019]***

(A) exactly two values of K
(B) exactly one value of K
(C) no value of K.
(D) infinitely many values of K

44. The line $x = y$ touches a circle at the point $(1, 1)$. If the circle also passes through the point $(1, -3)$, then its radius is : ***[JEE(Main)-2019]***

(A) $3\sqrt{2}$
(B) 3
(C) $2\sqrt{2}$
(D) 2

45. The locus of the centres of the circles, which touch the circle, $x^2 + y^2 = 1$ externally, also touch the y-axis and lie in the first quadrant, is : ***[JEE(Main)-2019]***

(A) $y = \sqrt{1+4x}$, $x \geq 0$
(B) $x = \sqrt{1+4y}$, $y \geq 0$
(C) $x = \sqrt{1+2y}$, $y \geq 0$
(D) $y = \sqrt{1+2x}$, $x \geq 0$

46. If the angle of intersection at a point where the two circles with radii 5 cm and 12 cm intersect is 90°, then the length (in cm) of their common chord is : ***[JEE(Main)-2019]***

(A) $\frac{60}{13}$
(B) $\frac{120}{13}$
(C) $\frac{13}{2}$
(D) $\frac{13}{5}$

47. A circle touching the x-axis at (3, 0) and making an intercept of length 8 on the y-axis passes through the point ***[JEE(Main)-2019]***

(A) (3, 10)
(B) (2,3)
(C) (1,5)
(D) (3,5)

EXERCISE – 5 RECAP OF IIT-JEE/JEE (ADVANCED)

1.** Let ABCD be a quadrilateral with area 18, with side AB parallel to the side CD and AB = 2CD. Let AD be perpendicular to AB and CD. If a circle is drawn inside the quadrilateral ABCD touching all the sides, then its radius is ***[JEE 2007, 3]

(A) 3 (B) 2 (C) 3/2 (D) 1

2. Tangents are drawn from the point (17, 7) to the circle $x^2 + y^2 = 169$. ***[JEE 2007, 3]***

Statement-1– The tangents are mutually perpendicular.

because

Statement-2– The locus of the points from which mutually perpendicular tangents can be drawn to the given circle is $x^2 + y^2 = 338$.

(A) Statement-1 is true, statement-2 is true; statement-2 is correct explanation for statement-1.

(B) Statement-1 is true, statement-2 is true; statement-2 is NOT a correct explanation for statement-1.

(C) Statement-1 is true, statement-2 is false.

(D) Statement-1 is false, statement-2 is true.

3. Consider the two curves $C_1 : y^2 = 4x$; $C_2 : x^2 + y^2 - 6x + 1 = 0$. Then, ***[JEE 2008, 3]***

(A) C_1 and C_2 touch each other only at one point

(B) C_1 and C_2 touch each other exactly at two points

(C) C_1 and C_2 intersect (but do not touch) at exactly two points

(D) C_1 and C_2 neither intersect nor touch each other

4. Consider, $L_1 : 2x + 3y + p - 3 = 0$; $L_2 : 2x + 3y + p + 3 = 0$, where p is a real number, ***[JEE 2008, 3]*** and $C : x^2 + y^2 + 6x - 10y + 30 = 0$.

Statement-1– If line L_1 is a chord of circle C, then line L_2 is not always a diameter of circle C.

and

Statement-2– If line L_1 is a diameter of circle C, then line L_2 is not a chord of circle C.

(A) Statement-1 is True, Statement-2 is True; statement-2 is a correct explanation for statement-1

(B) Statement-1 is True, Statement-2 is True; statement-2 is ***NOT*** a correct explanation for statement-1

(C) Statement-1 is True, Statement-2 is False

(D) Statement-1 is False, Statement-2 is True

****Comprehension (Q. 5 - 7)***

A circle C of radius 1 is inscribed in an equilateral triangle PQR. The points of contact of C with the sides PQ, QR, RP are D, E, F respectively. The line PQ is given by the equation $\sqrt{3}x + y - 6 = 0$ and the point D is $\left(\frac{3\sqrt{3}}{2}, \frac{3}{2}\right)$. Further, it is given that the origin and the centre of C are on the same side of the line PQ.

5. The equation of circle C is

(A) $(x - 2\sqrt{3})^2 + (y - 1)^2 = 1$ (B) $(x - 2\sqrt{3})^2 + (y + \frac{1}{2})^2 = 1$

(C) $(x - \sqrt{3})^2 + (y + 1)^2 = 1$ (D) $(x - \sqrt{3})^2 + (y - 1)^2 = 1$

6. Points E and F are given by

(A) $\left(\frac{\sqrt{3}}{2},\ \frac{3}{2}\right),\ (\sqrt{3},0)$ (B) $\left(\frac{\sqrt{3}}{2},\ \frac{1}{2}\right),\ (\sqrt{3},\ 0)$ (C) $\left(\frac{\sqrt{3}}{2},\frac{3}{2}\right),\left(\frac{\sqrt{3}}{2},\frac{1}{2}\right)$ (D) $\left(\frac{3}{2},\ \frac{\sqrt{3}}{2}\right),\ \left(\frac{\sqrt{3}}{2},\ \frac{1}{2}\right)$

7. Equations of the sides RP, RQ are **[JEE 2008, 4+4+4]**

(A) $y = \frac{2}{\sqrt{3}}x + 1,\ y = -\frac{2}{\sqrt{3}}x - 1$ (B) $y = \frac{1}{\sqrt{3}}x,\ y = 0$

(C) $y = \frac{\sqrt{3}}{2}x + 1,\ y = -\frac{\sqrt{3}}{2}x - 1$ (D) $y = \sqrt{3}\,x,\ y = 0$

8. Tangents drawn from the point P(l, 8) to the circle $x^2 + y^2 - 6x - 4y - 11 = 0$ touch the circle at the points A and B. The equation of the circumcircle of the triangle PAB is **[JEE 2009, 3]**

(A) $x^2 + y^2 + 4x - 6y + 19 = 0$ (B) $x^2 + y^2 - 4x - 10y + 19 = 0$

(C) $x^2 + y^2 - 2x + 6y - 29 = 0$ (D) $x^2 + y^2 - 6x - 4y + 19 = 0$

***9.** The centres of two circles C_1 and C_2 each of unit radius are at a distance of 6 units from each other. Let P be the mid point of the line segment joining the centres of C_1 and C_2 and C be a circle touching circles C_1 and C_2 externally. If a common tangent to C_1 and C passing through P is also a common tangent to C_2 and C, then the radius of the circle C is **[JEE 2009, 4]**

10. Two parallel chords of a circle of radius 2 are at a distance $\sqrt{3}+1$ apart. If the chords subtend at the center, angles of $\frac{\pi}{k}$ and $\frac{2\pi}{k}$, where k > 0, then the value of [k] is **[JEE 10, 3]**

[***Note –*** [k] denotes the largest integer less than or equal to k]

11. The circle passing through the point (–1,0) and touching the y-axis at (0, 2) also passes through the point -

(A) $\left(-\frac{3}{2},0\right)$ (B) $\left(-\frac{5}{2},2\right)$ (C) $\left(-\frac{3}{2},\frac{5}{2}\right)$ (D) (–4,0) **[JEE 2011, 3]**

***12.** The straight line 2x – 3y = 1 divides the circular region $x^2 + y^2 \le 6$ into two parts. If

$$S = \left\{\left(2, \frac{3}{4}\right), \left(\frac{5}{2}, \frac{3}{4}\right), \left(\frac{1}{4}, -\frac{1}{4}\right), \left(\frac{1}{8}, \frac{1}{4}\right)\right\},$$

then the number of point(s) in S lying inside the smaller part is **[JEE 2011, 4]**

13. The locus of the mid-point of the chord of contact of tangents drawn from points lying on the straight line 4x – 5y = 20 to the circle $x^2 + y^2 = 9$ is- **[JEE 2012, 3, –1]**

(A) $20(x^2 + y^2) - 36x + 45y = 0$ (B) $20(x^2 + y^2) + 36x - 45y = 0$

(C) $36(x^2 + y^2) - 20x + 45y = 0$ (D) $36(x^2 + y^2) + 20x - 45y = 0$

****Paragraph for Question 14 and 15***

A tangent PT is drawn to the circle $x^2 + y^2 = 4$ at the point $P(\sqrt{3}, 1)$. A straight line L, perpendicular to PT is a tangent to the circle $(x - 3)^2 + y^2 = 1$.

14. A common tangent of the two circles is **[JEE 2012, 3, –1]**

(A) $x = 4$ (B) $y = 2$ (C) $x+\sqrt{3}y=4$ (D) $x+2\sqrt{2}y=6$

15. A possible equation of L is **[JEE 2012, 3, –1]**

(A) $x-\sqrt{3}y=1$ (B) $x+\sqrt{3}y=1$ (C) $x-\sqrt{3}y=-1$ (D) $x+\sqrt{3}y=5$

16. Circle(s) touching x-axis at a distance 3 from the origin and having an intercept of length $2\sqrt{7}$ or y-axis is (are) **[JEE(Adv) 2013, 3,]**

(A) $x^2 + y^2 - 6x + 8y + 9 = 0$ (B) $x^2 + y^2 - 6x + 7y + 9 = 0$

(C) $x^2 + y^2 - 6x - 8y + 9 = 0$ (D) $x^2 + y^2 - 6x - 7y + 9 = 0$

***17.** A circle S passes through the point (0, 1) and is orthogonal to the circles $(x-1)^2 + y^2 = 16$ and $x^2 + y^2 = 1$. Then **[JEE(Advanced) 2014]**

(A) radius of S is 8 (B) radius of S is 7 (C) centre of S is (–7, 1) (D) centre of S is (–8, 1)

18. Let RS be the diameter of the circle $x^2 + y^2 = 1$, where S is the point (1,0). Let P be a variable point (other than R and S) on the circle and tangents to the circle at S and P meet at the point Q. The normal to the circle at P intersects a line drawn through Q parallel to RS at point E. then the locus of E passes through the point(s)- **[JEE 2016]**

(A) $\left(\frac{1}{3},\frac{1}{\sqrt{3}}\right)$ (B) $\left(\frac{1}{4},\frac{1}{2}\right)$ (C) $\left(\frac{1}{3},-\frac{1}{\sqrt{3}}\right)$ (D) $\left(\frac{1}{4},-\frac{1}{2}\right)$

19. For how many values of p, the circle $x^2 + y^2 + 2x + 4y - p = 0$ and the coordinate axes have exactly three common points ? **[JEE 2017]**

20. Let T be the line passing through the points P(–2, 7) and Q(2, –5). Let F_1 be the set of all pairs of circles (S_1, S_2) such that T is tangents to S_1 at P and tangent to S_2 at Q, and also such that S_1 and S_2 touch each other at a point, say, M. Let E_1 be the set representing the locus of M as the pair (S_1, S_2) varies in F_1. Let the set of all straight line segments joining a pair of distinct points of E_1 and passing through the point R(1, 1) be F_2. Let E_2 be the set of the mid-points of the line segments in the set F_2. Then, which of the following statement(s) is (are) TRUE ? **[JEE 2018]**

(A) The point (–2, 7) lies in E_1 (B) The point $\left(\frac{4}{5},\frac{7}{5}\right)$ does ***NOT*** lie in E_2

(C) The point $\left(\frac{1}{2},1\right)$ lies in E_2 (D) The point $\left(0,\frac{3}{2}\right)$ does ***NOT*** lie in E_1

Paragraph **"X"**

Let S be the circle in the xy-plane defined by the equation $x^2 + y^2 = 4$.

(*There are two question based on Paragraph* "X", *the question given below is one of them*)

21. Let E_1E_2 and F_1F_2 be the chord of S passing through the point $P_0(1, 1)$ and parallel to the x-axis and the y-axis, respectively. Let G_1G_2 be the chord of S passing through P_0 and having slop –1. Let the tangents to S at E_1 and E_2 meet at E_3, the tangents of S at F_1 and F_2 meet at F_3, and the tangents to S at G_1 and G_2 meet at G_3. Then, the points E_3, F_3 and G_3 lie on the curve **[JEE 2018]**

(A) $x + y = 4$ (B) $(x-4)^2 + (y-4)^2 = 16$

(C) $(x-4)(y-4) = 4$ (D) $xy = 4$

PARAGRAPH "X"

Let S be the circle in the xy-plane defined by the equation $x^2 + y^2 = 4$

(*There are two questions based on Paragraph "X", the question given below is one of them*)

22. Let P be a point on the circle S with both coordinates being positive. Let the tangent to S at P intersect the coordinate axes at the points M and N. Then, the mid-point of the line segment MN must lie on the curve - **[JEE 2018]**

(A) $(x + y)^2 = 3xy$ (B) $x^{2/3} + y^{2/3} = 2^{4/3}$ (C) $x^2 + y^2 = 2xy$ (D) $x^2 + y^2 = x^2y^2$

23. A line $y = mx + 1$ intersects the circle $(x - 3)^2 + (y + 2)^2 = 25$ at the points P and Q. If the midpoint of the line segment PQ has x-coordinate $-\frac{3}{5}$, then which one of the following options is correct ? **[JEE 2019]**

(A) $6 \le m < 8$ (B) $2 \le m < 4$ (C) $4 \le m < 6$ (D) $-3 \le m < -1$

24. Let the point B be the reflection of the point A(2, 3) with respect to the line $8x - 6y - 23 = 0$. Let Γ_A and Γ_B be circles of radii 2 and 1 with centres A and B respectively. Let T be a common tangent to the circles Γ_A and Γ_B such that both the circles are on the same side of T. If C is the point of intersection of T and the line passing through A and B, then the length of the line segment AC is _____ **[JEE 2019]**

25. ***Answer the following by appropriately matching the lists based on the information given in the paragraph***

Let the circles $C_1 : x^2 + y^2 = 9$ and $C_2 : (x-3)^2 + (y-4)^2 = 16$, intersect at the points X and Y. Suppose that another circle $C_3 : (x-h)^2 + (y-k)^2 = r^2$ satisfies the following conditions :

(i) centre of C_3 is collinear with the centres of C_1 and C_2

(ii) C_1 and C_2 both lie inside C_3, and

(iii) C_3 touches C_1 at M and C_2 at N.

Let the line through X and Y intersect C_3 at Z and W, and let a common tangent of C_1 and C_3 be a tangent to the parabola $x^2 = 8\alpha y$.

There are some expression given in the List-I whose values are given in List-II below:

List-I	List-II
(I) $2h + k$	(P) 6
(II) $\dfrac{\text{Length of ZW}}{\text{Length of XY}}$	(Q) $\sqrt{6}$
(III) $\dfrac{\text{Area of triangle MZN}}{\text{Area of triangle ZMW}}$	(R) $\frac{5}{4}$
(IV) α	(S) $\frac{21}{5}$
	(T) $2\sqrt{6}$
	(U) $\frac{10}{3}$

Which of the following is the only INCORRECT combination ? **[JEE 2019]**

Options

(A) (IV), (S) (B) (IV), (U) (C) (III), (R) (D) (I), (P)

26. *Answer the following by appropriately matching the lists based on the information given in the paragraph*

Let the circles $C_1 : x^2 + y^2 = 9$ and $C_2 : (x-3)^2 + (y-4)^2 = 16$, intersect at the points X and Y. Suppose that another circle $C_3 : (x-h)^2 + (y-k)^2 = r^2$ satisfies the following conditions :

(i) centre of C_3 is collinear with the centres of C_1 and C_2

(ii) C_1 and C_2 both lie inside C_3, and

(iii) C_3 touches C_1 at M and C_2 at N.

Let the line through X and Y intersect C_3 at Z and W, and let a common tangent of C_1 and C_3 be a tangent to the parabola $x^2 = 8\alpha y$.

There are some expression given in the List-I whose values are given in List-II below:

List-I	***List-II***
(I) $2h + k$	(P) 6
(II) $\dfrac{\text{Length of ZW}}{\text{Length of XY}}$	(Q) $\sqrt{6}$
(III) $\dfrac{\text{Area of triangle MZN}}{\text{Area of triangle ZMW}}$	(R) $\dfrac{5}{4}$
(IV) α	(S) $\dfrac{21}{5}$
	(T) $2\sqrt{6}$
	(U) $\dfrac{10}{3}$

Which of the following is the only CORRECT combination ? **[JEE 2019]**

Options

(A) (II), (T) (B) (I), (S) (C) (I), (U) (D) (II), (Q)

ANSWERS

EXERCISE-1

Que.	1	2	3	4	5	6	7	8	9	10
Ans.	C	B	B	D	C	D	A	D	A	A
Que.	11	12	13	14	15	16	17	18	19	20
Ans.	B	C	D	C	B	A	B	C	C	C
Que.	21	22	23	24	25	26	27	28	29	30
Ans.	C	B	A	D	C	A	D	A	B	B

EXERCISE-2

Que.	1	2	3	4	5	6	7	8	9	10
Ans.	ABCD	ABC	BC	AD	B	AD	AC	BC	D	CD
Que.	11	12								
Ans.	AB	AB								

- ***Match the Column*** **13.** (A) $\rightarrow$ (r, s) ; (B) $\rightarrow$ (s); (C) $\rightarrow$ (p); (D) $\rightarrow$ (q)
- ***Comprehension Based Questions*** **14.** B **15.** D **16.** B

EXERCISE-3

1. $x^2 + y^2 - 6x + 4y = 0$ or $x^2 + y^2 + 2x - 8y + 4 = 0$ **2.** $x^2 + y^2 + \left(\frac{b}{a}\right)x + \left(\frac{q}{p}\right)y + \left(\frac{c}{a} + \frac{r}{p}\right) = 0$

4. 75 sq.units **5.** square of side 2; $x^2 + y^2 = 1$; $x^2 + y^2 = 2$

6. $x^2 + y^2 - 2x - 2y + 1 = 0$ or $x^2 + y^2 - 42x + 38y - 39 = 0$

7. $x - 7y = 2$, $7x + y = 14$; $(x - 1)^2 + (y - 7)^2 = 3^2$;
$(x - 3)^2 + (y + 7)^2 = 3^2$; $(x - 9)^2 + (y - 1)^2 = 3^2$; $(x + 5)^2 + (y + 1)^2 = 3^2$

8. $x^2 + y^2 + 10x - 10y + 25 = 0$ or $x^2 + y^2 + 2x - 2y + 1 = 0$, (10, –10, 25) (2, –2, 1)

9. $x^2 + y^2 + x - 6y + 3 = 0$

10. zero, zero

EXERCISE-4

Que.	1	2	3	4	5	6	7	8	9	10
Ans.	C	A	B	D	B	A	A	A	B	D
Que.	11	12	13	14	15	16	17	18	19	20
Ans.	D	D	C	C	B	C	D	D	C	B
Que.	21	22	23	24	25	26	27	28	29	30
Ans.	D	C	B	C	B	C	B	A	B	D
Que.	31	32	33	34	35	36	37	38	39	40
Ans.	B	B	D	D	C	A	D	D	D	C
Que.	41	42	43	44	45	46	47			
Ans.	B	B	C	C	D	B	A			

EXERCISE-5

1. (B)	***2.*** (A)	***3.*** (B)	***4.*** (C)	***5.*** (D)	***6.*** (A)	***7.*** (D)	***8.*** (B)
9. (8)	***10.*** (3)	***11.*** (D)	***12.*** (2)	***13.*** (A)	***14.*** (D)	***15.*** (A)	
16. (AC)	***17.*** (BC)	***18.*** (AC)	***19.*** (2)	***20.*** (D)	***21.*** (A)	***22.*** (D)	***23.*** (2)
24. (10.00)	***25.*** (1)	***26.*** (4)					

CONIC SECTION

PARABOLA

Recap of Early Classes

In the previous Classes, we have studied various forms of the equations of a line and equation of circles. In this chapter we will studied about parabola, ellipse and hyperbola. The names parabola and hyperbola are given by Apollonius. These curves are in fact, known as conic sections or more commonly conics because they can be obtained as intersections of a plane with a double napped right circular cone. These curves have a very wide range of applications in fields such as planetary motion, design of telescopes and antennas, reflectors in flashlights and automobile headlights, etc. Now, in the subsequent sections we will see how the intersection of a plane with a double napped right circular cone results in different types of curves.

Index

PARABOLA

1.0 CONIC SECTIONS

A conic section, or conic is the locus of a point which moves in a plane so that its distance from a fixed point is in a constant ratio to its perpendicular distance from a fixed straight line and a fixed point does not lie on a fixed line.

(a) The fixed point is called the ***focus***.

(b) The fixed straight line is called the ***directrix***.

(c) The constant ratio is called the ***eccentricity*** denoted by e.

(d) The line passing through the focus & perpendicular to the directrix is called the ***axis***.

(e) A point of intersection of a conic with its axis is called a ***vertex***.

2.0 GENERAL EQUATION OF A CONIC : FOCAL DIRECTRIX PROPERTY

The general equation of a conic with focus (p, q) & directrix $lx + my + n = 0$ is :

$(l^2 + m^2)\,[(x - p)^2 + (y - q)^2] = e^2\,(lx + my + n)^2 \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

The conic represents –

a parabola	*an ellipse*	*a hyperbola*	*a rectangular hyperbola*
$e = 1\,;\, D \neq 0$ $h^2 = ab$	$0 < e < 1\,;\, D \neq 0$ $h^2 < ab$	$D \neq 0\,;\, e > 1\,;$ $h^2 > ab$	$e > 1\,;\, D \neq 0$ $h^2 > ab\,;\, a + b = 0$

3.0 PARABOLA

A parabola is the locus of a point which moves in a plane, such that its distance from a fixed point (focus) is equal to its perpendicular distance from a fixed straight line (directrix).

Standard equation of a parabola is $\boldsymbol{y^2 = 4ax}$. For this parabola :

(i) Vertex is $\boldsymbol{(0, 0)}$ (ii) Focus is $\boldsymbol{(a, 0)}$ (iii) Axis is $\boldsymbol{y = 0}$ (iv) Directrix is $\boldsymbol{x + a = 0}$

(a) ***Focal distance***

The distance of a point on the parabola from the focus is called the ***focal distance of the point.***

(b) ***Focal chord***

A chord of the parabola, which passes through the focus is called a ***focal chord***.

(c) ***Double ordinate***

A chord of the parabola perpendicular to the axis of the symmetry is called a ***double ordinate***.

(d) ***Latus rectum***

A double ordinate passing through the focus or a focal chord perpendicular to the axis of parabola is called the ***latus rectum***. For $y^2 = 4ax$.

(i) Length of the latus rectum = $\boldsymbol{4a}$.

(ii) Length of the semi latus rectum = $\boldsymbol{2a}$.

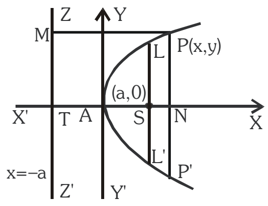
(iii) Ends of the latus rectum are $\boldsymbol{L(a, 2a)}$ & $\boldsymbol{L'(a, -2a)}$

4.0 PARAMETRIC REPRESENTATION

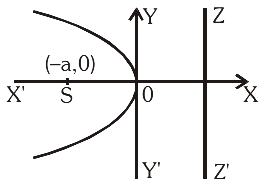
The simplest & the best form of representing the co-ordinates of a point on the parabola is $\boldsymbol{(at^2, 2at)}$. The equation $\boldsymbol{x = at^2}$ & $\boldsymbol{y = 2at}$ together represents the parabola $\boldsymbol{y^2 = 4ax}$, t being the parameter.

5.0 TYPE OF PARABOLA

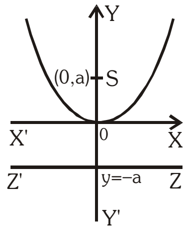
Four standard forms of the parabola are $\boldsymbol{y^2 = 4ax}$; $\boldsymbol{y^2 = -4ax}$; $\boldsymbol{x^2 = 4ay}$; $\boldsymbol{x^2 = -4ay}$



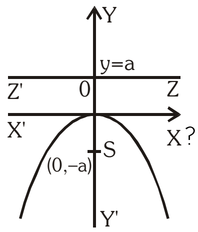
$\boldsymbol{y^2 = 4ax}$



$\boldsymbol{y^2 = -4ax}$



$\boldsymbol{x^2 = 4ay}$



$\boldsymbol{x^2 = -4ay}$

Parabola	**Vertex**	**Focus**	**Axis**	**Directrix**	Length of Latus rectum	Ends of Latus rectum	Parametric equation	Focal length
$y^2 = 4ax$	(0,0)	(a,0)	y=0	x=–a	4a	(a, ±2a)	$(at^2,2at)$	x + a
$y^2 = -4ax$	(0,0)	(–a,0)	y=0	x=a	4a	(–a, ±2a)	$(-at^2,2at)$	x–a
$x^2 = +4ay$	(0,0)	(0,a)	x=0	y=–a	4a	(±2a, a)	$(2at,at^2)$	y+a
$x^2 = -4ay$	(0,0)	(0,–a)	x=0	y=a	4a	(±2a, –a)	$(2at, -at^2)$	y–a
$(y-k)^2 = 4a(x-h)$	(h,k)	(h+a,k)	y=k	x+a–h =0	4a	(h+a, k±2a)	$(h+at^2,k+2at)$	x–h+a
$(x-p)^2 = 4b(y-q)$	(p,q)	(p,b+q)	x=p	y+b–q=0	4b	(p±2a,q+a)	$(p+2at,q+at^2)$	y–q+b

Illustrations

Illustration 1. Find the vertex, axis, directrix, focus, latus rectum and the tangent at vertex for the parabola $9y^2 - 16x - 12y - 57 = 0$.

Solution The given equation can be rewritten as $\left(y - \frac{2}{3}\right)^2 = \frac{16}{9}\left(x + \frac{61}{16}\right)$ which is of the form $Y^2 = 4AX$.

Hence the vertex is $\left(-\frac{61}{16}, \frac{2}{3}\right)$

The axis is $y - \frac{2}{3} = 0 \Rightarrow y = \frac{2}{3}$

The directrix is $X + A = 0 \Rightarrow x + \frac{61}{16} + \frac{4}{9} = 0 \Rightarrow x = -\frac{613}{144}$

The focus is $X = A$ and $Y = 0 \Rightarrow x + \frac{61}{16} = \frac{4}{9}$ and $y - \frac{2}{3} = 0$

$\Rightarrow$ focus $= \left(-\frac{485}{144}, \frac{2}{3}\right)$

Length of the latus rectum $= 4A = \frac{16}{9}$

The tangent at the vertex is $X = 0 \Rightarrow x = -\frac{61}{16}$.

Ans.

Illustration 2. The length of latus rectum of a parabola, whose focus is (2, 3) and directrix is the line $x - 4y + 3 = 0$ is -

(A) $\frac{7}{\sqrt{17}}$ (B) $\frac{14}{\sqrt{21}}$ (C) $\frac{7}{\sqrt{21}}$ (D) $\frac{14}{\sqrt{17}}$

Solution The length of latus rectum = 2 × perp. from focus to the directrix

$$= 2 \times \left|\frac{2 - 4(3) + 3}{\sqrt{(1)^2 + (4)^2}}\right| = \frac{14}{\sqrt{17}}$$

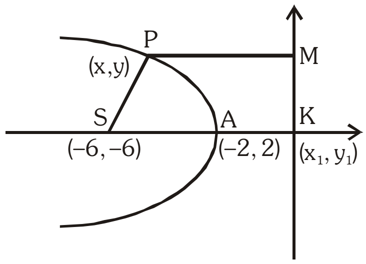
Ans. (D)

Illustration 3. Find the equation of the parabola whose focus is (–6, –6) and vertex (–2, 2).

Solution Let S(–6, –6) be the focus and A(–2, 2) is vertex of the parabola. On SA take a point $K(x_1, y_1)$ such that SA = AK. Draw KM perpendicular on SK. Then KM is the directrix of the parabola.

Since A bisects SK, $\left(\frac{-6 + x_1}{2}, \frac{-6 + y_1}{2}\right) = (-2, 2)$

$\Rightarrow$ $-6 + x_1 = -4$ and $-6 + y_1 = 4$ or $(x_1, y_1) = (2, 10)$



Hence the equation of the directrix KM is

$y - 10 = m(x - 2)$ .... (i)

Also gradient of SK = $\frac{10 - (-6)}{2 - (-6)} = \frac{16}{8} = 2; \Rightarrow m = \frac{-1}{2}$

$y - 10 = \frac{-1}{2}(x - 2)$ (from (i))

$\Rightarrow$ $x + 2y - 22 = 0$ is the directrix

Next, let PM be a perpendicular on the directrix KM from any point P(x, y) on the parabola. From SP = PM, the equation of the parabola is $\sqrt{\left\{(x+6)^2 + (y+6)^2\right\}} = \frac{|x + 2y - 22|}{\sqrt{(1^2 + 2^2)}}$

or $5(x^2 + y^2 + 12x + 12y + 72) = (x + 2y - 22)^2$

or $4x^2 + y^2 - 4xy + 104x + 148y - 124 = 0$

or $(2x - y)^2 + 104x + 148y - 124 = 0.$

Ans.

Illustration 4. The extreme points of the latus rectum of a parabola are (7, 5) and (7, 3). Find the equation of the parabola.

Solution Focus of the parabola is the mid-point of the latus rectum.

$\Rightarrow$ S is (7, 4). Also axis of the parabola is perpendicular to the latus rectum and passes through the focus. Its equation is

$$y - 4 = \frac{0}{5 - 3}(x - 7) \Rightarrow y = 4$$

Length of the latus rectum = (5 – 3) = 2

Hence the vertex of the parabola is at a distance 2/4 = 0.5 from the focus. We have two parabolas, one concave rightwards and the other concave leftwards.

The vertex of the first parabola is (6.5, 4) and its equation is $(y - 4)^2 = 2(x - 6.5)$ and it meets the x-axis at (14.5, 0). The equation of the second parabola is $(y - 4)^2 = -2(x - 7.5)$. It meets the x-axis at (–0.5, 0). **Ans.**

GOLDEN KEY POINTS

- Perpendicular distance from focus on directrix = half the latus rectum.
- Vertex is middle point of the focus & the point of intersection of directrix & axis.
- Two parabolas are said to be equal if they have the same latus rectum.

BEGINNER'S BOX-1

1. Name the conic represented by the equation $\sqrt{ax}+\sqrt{by}=1$, where a, b $\in$ R, a, b, > 0.

2. Find the vertex, axis, focus, directrix, latus rectum of the parabola $4y^2 + 12x - 20y + 67 = 0$.

3. Find the equation of the parabola whose focus is (1, –1) and whose vertex is (2, 1). Also find its axis and latus rectum.

4. Find the equation of the parabola whose latus rectum is 4 units, axis is the line $3x + 4y = 4$ and the tangent at the vertex is the line $4x - 3y + 7 = 0$.

6.0 POSITION OF A POINT RELATIVE TO A PARABOLA

The point (x_1, y_1) lies outside, on or inside the parabola $y^2 = 4ax$ according as the expression $y_1^2 - 4ax_1$ is positive, zero or negative.

Illustrations

Illustration 5. Find the value of α for which the point $(\alpha - 1, \alpha)$ lies inside the parabola $y^2 = 4x$.

Solution

$\because$ Point $(\alpha - 1, \alpha)$ lies inside the parabola $y^2 = 4x$

$\therefore$ $y_1^2 - 4x_1 < 0$

$\Rightarrow$ $\alpha^2 - 4(\alpha - 1) < 0$

$\Rightarrow$ $\alpha^2 - 4\alpha + 4 < 0$

$(\alpha - 2)^2 < 0 \Rightarrow \alpha \in \phi$ ***Ans.***

7.0 CHORD JOINING TWO POINTS

The equation of a chord of the parabola $y^2 = 4ax$ joining its two points $P(t_1)$ and $Q(t_2)$ is $y(t_1 + t_2) = 2x + 2at_1t_2$

Note –

(i) If PQ is focal chord then $t_1t_2 = -1$.

(ii) Extremities of focal chord can be taken as $(at^2, 2at)$ & $\left(\frac{a}{t^2}, \frac{-2a}{t}\right)$

Illustrations

Illustration 6. Through the vertex O of a parabola $y^2 = 4x$ chords OP and OQ are drawn at right angles to one another. Show that for all position of P, PQ cuts the axis of the parabola at a fixed point.

Solution The given parabola is $y^2 = 4x$ .... (i)

Let $P \equiv (t_1^2, 2t_1)$, $Q \equiv (t_2^2, 2t_2)$

Slope of OP $= \frac{2t_1}{t_1^2} = \frac{2}{t_1}$ and slope of OQ $= \frac{2}{t_2}$

Since OP $\perp$ OQ, $\frac{4}{t_1t_2} = -1$ or $t_1t_2 = -4$ .... (ii)

The equation of PQ is $y(t_1 + t_2) = 2(x + t_1t_2)$

$\Rightarrow y\left(t_1 - \frac{4}{t_1}\right) = 2(x-4)$ [from (ii)]

$\Rightarrow 2(x-4) - y\left(t_1 - \frac{4}{t_1}\right) = 0 \quad \Rightarrow \quad L_1 + \lambda L_2 = 0$

$\therefore$ variable line PQ passes through a fixed point which is point of intersection of $L_1 = 0$ & $L_2 = 0$ i.e. (4, 0) ***Ans.***

8.0 LINE & A PARABOLA

(a) The line $y = mx + c$ meets the parabola $y^2 = 4ax$ in two points real, coincident or imaginary according as $a >=< cm \Rightarrow$ condition of tangency is, $\mathbf{c = \frac{a}{m}}$.

Note – Line $y = mx + c$ will be tangent to parabola $x^2 = 4ay$ if $\mathbf{c = -am^2}$.

(b) Length of the chord intercepted by the parabola $y^2 = 4ax$ on the line $y = mx + c$ is : $\left(\frac{4}{m^2}\right)\sqrt{a(1+m^2)(a-mc)}$.

Note – Length of the focal chord making an angle α with the x - axis is $4a\,\text{cosec}^2\,\alpha$.

Illustrations

Illustration 7. If the line $y = 3x + \lambda$ intersect the parabola $y^2 = 4x$ at two distinct points then set of values of λ is -

(A) $(3, \infty)$ (B) $(-\infty, 1/3)$ (C) $(1/3, 3)$ (D) none of these

Solution Putting value of y from the line in the parabola -

$(3x + \lambda)^2 = 4x$

$\Rightarrow \quad 9x^2 + (6\lambda - 4)x + \lambda^2 = 0$

$\because \quad$ line cuts the parabola at two distinct points

$\therefore \quad D > 0$

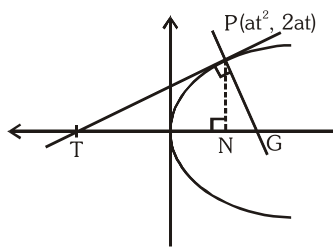
$\Rightarrow \quad 4(3\lambda - 2)^2 - 4.9\lambda^2 > 0$

$\Rightarrow \quad 9\lambda^2 - 12\lambda + 4 - 9\lambda^2 > 0$

$\Rightarrow \quad \lambda < 1/3$

Hence, $\lambda \in (-\infty, 1/3)$ ***Ans.(B)***

9.0 LENGTH OF SUBTANGENT & SUBNORMAL



PT and PG are the tangent and normal respectively at the point P to the parabola $y^2 = 4ax$. Then

TN = length of subtangent = twice the abscissa of the point P (Subtangent is always bisected by the vertex)

NG = length of subnormal which is constant for all points on the parabola & equal to its semilatus rectum (2a).

10.0 TANGENT TO THE PARABOLA $y^2 = 4ax$

(a) ***Point form***

Equation of tangent to the given parabola at its point (x_1, y_1) is

$\boldsymbol{yy_1 = 2a(x + x_1)}$

(b) ***Slope form***

Equation of tangent to the given parabola whose slope is 'm', is

$$y = mx + \frac{a}{m}, (m \neq 0)$$

Point of contact is $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$

(c) ***Parametric form***

Equation of tangent to the given parabola at its point P(t), is $\boldsymbol{ty = x + at^2}$

Note : Point of intersection of the tangents at the point $\boldsymbol{t_1}$ **&** $\boldsymbol{t_2}$ ***is*** $\boldsymbol{[at_1t_2, a(t_1 + t_2)]}$.

Illustrations

Illustration 8. A tangent to the parabola $y^2 = 8x$ makes an angle of 45° with the straight line $y = 3x + 5$. Find its equation and its point of contact.

Solution Let the slope of the tangent be m

$$\therefore \quad \tan 45° = \left|\frac{3-m}{1+3m}\right| \quad \Rightarrow \quad 1+3m = \pm(3-m)$$

$$\therefore \quad m = -2 \quad \text{or} \quad \frac{1}{2}$$

As we know that equation of tangent of slope m to the parabola $y^2 = 4ax$ is $y = mx + \frac{a}{m}$ and point of contact is $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$

for $m = -2$, equation of tangent is $y = -2x - 1$ and point of contact is $\left(\frac{1}{2}, -2\right)$

for $m = \frac{1}{2}$, equation of tangent is $y = \frac{1}{2}x + 4$ and point of contact is $(8, 8)$ **Ans.**

Illustration 9. Find the equation of the tangents to the parabola $y^2 = 9x$ which go through the point (4, 10).

Solution Equation of tangent to parabola $y^2 = 9x$ is

$$y = mx + \frac{9}{4m}$$

Since it passes through (4, 10)

$$\therefore \quad 10 = 4m + \frac{9}{4m} \quad \Rightarrow \quad 16m^2 - 40m + 9 = 0$$

$$m = \frac{1}{4}, \frac{9}{4}$$

$$\therefore \quad \text{Equation of tangent's are} \quad y = \frac{x}{4} + 9 \quad \& \quad y = \frac{9}{4}x + 1$$ **Ans.**

Illustration 10. Find the locus of the point P from which tangents are drawn to the parabola $y^2 = 4ax$ having slopes m_1 and m_2 such that -

(i) $m_1^2 + m_2^2 = \lambda$ (constant) (ii) $\theta_1 - \theta_2 = \theta_0$ (constant)

where θ_1 and θ_2 are the inclinations of the tangents from positive x-axis.

Solution Equation of tangent to $y^2 = 4ax$ is $y = mx + a/m$

Let it passes through P(h, k)

$\therefore \quad m^2h - mk + a = 0$

(i) $m_1^2 + m_2^2 = \lambda$

$(m_1 + m_2)^2 - 2m_1m_2 = \lambda$

$\frac{k^2}{h^2} - 2.\frac{a}{h} = \lambda$

$\therefore$ locus of P(h, k) is $y^2 - 2ax = \lambda x^2$

(ii) $\theta_1 - \theta_2 = \theta_0$

$\tan(\theta_1 - \theta_2) = \tan\theta_0$

$\frac{m_1 - m_2}{1 + m_1m_2} = \tan\theta_0$

$(m_1 + m_2)^2 - 4m_1m_2 = \tan^2\theta_0(1 + m_1m_2)^2$

$\frac{k^2}{h^2} - \frac{4a}{h} = \tan^2\theta_0\left(1 + \frac{a}{h}\right)^2$

$k^2 - 4ah = (h + a)^2 \tan^2\theta_0$

$\therefore$ locus of P(h, k) is $y^2 - 4ax = (x + a)^2\tan^2\theta_0$ **Ans.**

GOLDEN KEY POINTS

- The focal chord of parabola $y^2 = 4ax$ makes an angle α with x-axis is of length $4a\ \text{cosec}^2\alpha$.

BEGINNER'S BOX-2

1. Find the value of 'a' for which the point $(a^2 - 1, a)$ lies inside the parabola $y^2 = 8x$.

2. The focal distance of a point on the parabola $(x - 1)^2 = 16(y - 4)$ is 8. Find the co-ordinates.

3. Find the condition that the straight line $ax + by + c = 0$ touches the parabola $y^2 = 4kx$.

4. Find the length of the chord of the parabola $y^2 = 8x$, whose equation is $x + y = 1$.

5. Find the equation of the tangent to the parabola $y^2 = 12x$, which passes through the point (2, 5). Find also the co-ordinates of their points of contact.

11.0 NORMAL TO THE PARABOLA $y^2 = 4ax$

(a) Point form

Equation of normal to the given parabola at its point (x_1, y_1) is

$$y - y_1 = -\frac{y_1}{2a}(x - x_1)$$

(b) Slope form

Equation of normal to the given parabola whose slope is 'm', is

$y = mx - 2am - am^3$

foot of the normal is $(am^2, -2am)$

(c) Parametric form

Equation of normal to the given parabola at its point P(t), is

$y + tx = 2at + at^3$

Note :

(i) Point of intersection of normals at t_1 & t_2 is $(a(t_1^2 + t_2^2 + t_1t_2 + 2), -at_1t_2(t_1 + t_2))$.

(ii) If the normal to the parabola $y^2 = 4ax$ at the point t_1, meets the parabola again at the point t_2, then $t_2 = -\left(t_1 + \frac{2}{t_1}\right)$.

(iii) If the normals to the parabola $y^2 = 4ax$ at the points t_1 & t_2 intersect again on the parabola at the point 't_3' then $t_1t_2 = 2$; $t_3 = -(t_1 + t_2)$ and the line joining t_1 & t_2 passes through a fixed point (–2a, 0).

(iv) If normal drawn to a parabola passes through a point P(h,k) then $k = mh - 2am - am^3$, i.e. $am^3 + m(2a - h) + k = 0$.

This gives $m_1 + m_2 + m_3 = 0$; $m_1m_2 + m_2m_3 + m_3m_1 = \frac{2a - h}{a}$; $m_1m_2m_3 = \frac{-k}{a}$

where m_1, m_2, & m_3 are the slopes of the three concurrent normals :

- Algebraic sum of slopes of the three concurrent normals is zero.
- Algebraic sum of ordinates of the three co-normal points on the parabola is zero.
- Centroid of the Δ formed by three co-normal points lies on the axis of parabola (x-axis).

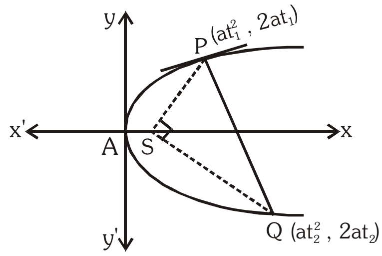
Illustrations

Illustration 11. Prove that the normal chord to a parabola $y^2 = 4ax$ at the point whose ordinate is equal to abscissa subtends a right angle at the focus.

Solution Let the normal at $P\left(at_1^2, 2at_1\right)$ meet the curve at $Q\left(at_2^2, 2at_2\right)$

$\therefore$ PQ is a normal chord.

and $t_2 = -t_1 - \dfrac{2}{t_1}$ ....(i)



By given condition $2at_1 = at_1^2$

$\therefore$ $t_1 = 2$ from equation (i), $t_2 = -3$

then P(4a, 4a) and Q(9a, –6a)

but focus S(a, 0)

$\therefore$ Slope of SP $= \dfrac{4a-0}{4a-a} = \dfrac{4a}{3a} = \dfrac{4}{3}$

and Slope of SQ $= \dfrac{-6a-0}{9a-a} = \dfrac{-6a}{8a} = -\dfrac{3}{4}$

$\therefore$ Slope of SP $\times$ Slope of SQ $= \dfrac{4}{3} \times -\dfrac{3}{4} = -1$

$\therefore$ $\angle PSQ = \pi/2$

i.e. PQ subtends a right angle at the focus S.

Illustration 12. If two normals drawn from any point to the parabola $y^2 = 4ax$ make angle α and β with the axis such that $\tan\alpha \,.\, \tan\beta = 2$, then find the locus of this point.

Solution Let the point is (h, k). The equation of any normal to the parabola $y^2 = 4ax$ is

$y = mx - 2am - am^3$

passes through (h, k)

$k = mh - 2am - am^3$

$am^3 + m(2a-h) + k = 0$ ....(i)

m_1, m_2, m_3 are roots of the equation, then $m_1 . m_2 . m_3 = -\dfrac{k}{a}$

but $m_1m_2 = 2$, $m_3 = -\dfrac{k}{2a}$

m_3 is root of (i) $\therefore$ $a\left(-\dfrac{k}{2a}\right)^3 - \dfrac{k}{2a}(2a-h) + k = 0 \Rightarrow k^2 = 4ah$

Thus locus is $y^2 = 4ax$. ***Ans.***

Illustration 13. Three normals are drawn from the point (14, 7) to the curve $y^2 - 16x - 8y = 0$. Find the coordinates of the feet of the normals.

Solution The given parabola is $y^2 - 16x - 8y = 0$ ..... (i)

Let the co-ordinates of the feet of the normal from (14, 7) be $P(\alpha, \beta)$. Now the equation of the tangent at $P(\alpha, \beta)$ to parabola (i) is

$y\beta - 8(x + \alpha) - 4(y + \beta) = 0$

or $(\beta - 4)y = 8x + 8a + 4\beta$ .... (ii)

Its slope $= \dfrac{8}{\beta - 4}$

Equation of the normal to parabola (i) at (α, β) is $y - \beta = \dfrac{4-\beta}{8}(x - \alpha)$

It passes through (14, 7)

$\Rightarrow$ $7 - \beta = \dfrac{4-\beta}{8}(14 - \alpha) \Rightarrow \alpha = \dfrac{6\beta}{\beta - 4}$ .... (iii)

Also (α, β) lies on parabola (i) i.e. $\beta^2 - 16\alpha - 8\beta = 0$ .... (iv)

Putting the value of α from (iii) in (iv), we get $\beta^2 - \dfrac{96\beta}{\beta - 4} - 8\beta = 0$

$\Rightarrow$ $\beta^2(\beta - 4) - 96\beta - 8\beta(\beta - 4) = 0 \Rightarrow \beta(\beta^2 - 4\beta - 96 - 8\beta + 32) = 0$

$\Rightarrow$ $\beta(\beta^2 - 12\beta - 64) = 0 \Rightarrow \beta(\beta - 16)(\beta + 4) = 0$

$\Rightarrow$ $\beta = 0, 16, -4$

from (iii), $\alpha = 0$ when $\beta = 0$; $\alpha = 8$, when $\beta = 16$; $\alpha = 3$ when $\beta = -4$

Hence the feet of the normals are (0, 0), (8, 16) and (3, –4) ***Ans.***

BEGINNER'S BOX-3

1. If three distinct and real normals can be drawn to $y^2 = 8x$ from the point (a, 0), then -

(A) $a > 2$ (B) $a \in (2, 4)$ (C) $a > 4$ (D) none of these

2. Find the number of distinct normal that can be drawn from (–2, 1) to the parabola $y^2 - 4x - 2y - 3 = 0$.

3. If $2x + y + k = 0$ is a normal to the parabola $y^2 = -16x$, then find the value of k.

4. Three normals are drawn from the point (7, 14) to the parabola $x^2 - 8x - 16y = 0$. Find the co-ordinates of the feet of the normals.

Illustrations

Illustration 14. If the equation $m^2(x + 1) + m(y - 2) + 1 = 0$ represents a family of lines, where 'm' is parameter then find the equation of the curve to which these lines will always be tangents.

Solution $m^2(x + 1) + m(y - 2) + 1 = 0$

The equation of the curve to which above lines will always be tangents can be obtained by equating its discriminant to zero.

$\therefore \quad (y - 2)^2 - 4(x + 1) = 0$

$y^2 - 4y + 4 - 4x - 4 = 0$

$y^2 = 4(x + y)$ ***Ans.***

12.0 PAIR OF TANGENTS

The equation of the pair of tangents which can be drawn from any point $P(x_1, y_1)$ outside the parabola to the parabola $y^2 = 4ax$ is given by : $SS_1 = T^2$ where :

$S \equiv y^2 - 4ax$; $S_1 \equiv y_1^2 - 4ax_1$; $T \equiv yy_1 - 2a(x + x_1)$.

13.0 DIRECTOR CIRCLE

Locus of the point of intersection of the perpendicular tangents to the parabola $y^2 = 4ax$ is called the ***director circle.*** It's equation is $x + a = 0$ which is parabola's own directrix.

Illustrations

Illustration 15. The angle between the tangents drawn from a point (–a, 2a) to $y^2 = 4ax$ is -

(A) $\pi/4$ (B) $\pi/2$ (C) $\pi/3$ (D) $\pi/6$

Solution The given point (–a, 2a) lies on the directrix $x = -a$ of the parabola $y^2 = 4ax$. Thus, the tangents are at right angle. ***Ans. (B)***

Illustration 16. The circle drawn with variable chord $x + ay - 5 = 0$ (a being a parameter) of the parabola $y^2 = 20x$ as diameter will always touch the line -

(A) $x + 5 = 0$ (B) $y + 5 = 0$ (C) $x + y + 5 = 0$ (D) $x - y + 5 = 0$

Solution Clearly $x + ay - 5 = 0$ will always pass through the focus of $y^2 = 20x$ i.e. (5, 0). Thus the drawn circle will always touch the directrix of the parabola i.e. the line $x + 5 = 0$. ***Ans.(A)***

14.0 CHORD OF CONTACT

Equation of the chord of contact of tangents drawn from a point $P(x_1, y_1)$ is $yy_1 = 2a(x + x_1)$

Note – The area of the triangle formed by the tangents from the point (x_1, y_1) & the chord of contact is

$\dfrac{\left(y_1^2 - 4ax_1\right)^{3/2}}{2a}$ i.e. $\dfrac{(S_1)^{3/2}}{2a}$, also note that the chord of contact exists only if the point P is not inside.

Illustrations

Illustration 17. If the line $x - y - 1 = 0$ intersect the parabola $y^2 = 8x$ at P & Q, then find the point of intersection of tangents at P & Q

Solution Let (h, k) be point of intersection of tangents then chord of contact is

$$yk = 4(x + h)$$

$$4x - yk + 4h = 0 \qquad \ldots\ldots\ (i)$$

But given line is

$$x - y - 1 = 0 \qquad \ldots\ldots\ (ii)$$

Comparing (i) and (ii)

$$\therefore \quad \frac{4}{1} = \frac{-k}{-1} = \frac{4h}{-1} \qquad \Rightarrow \qquad h = -1,\ k = 4$$

$\therefore$ point $\equiv (-1, 4)$

Ans.

Illustration 18. Find the locus of point whose chord of contact w.r.t. to the parabola $y^2 = 4bx$ is the tangent of the parabola $y^2 = 4ax$.

Solution Equation of tangent to $y^2 = 4ax$ is $y = mx + \frac{a}{m}$ .... (i)

Let it is chord of contact for parabola $y^2 = 4bx$ w.r.t. the point P(h, k)

$\therefore$ Equation of chord of contact is $yk = 2b(x + h)$

$$y = \frac{2b}{k}x + \frac{2bh}{k} \qquad \ldots\ldots\ldots\ (ii)$$

From (i) & (ii)

$$m = \frac{2b}{k},\ \frac{a}{m} = \frac{2bh}{k} \Rightarrow a = \frac{4b^2h}{k^2}$$

locus of P is $y^2 = \frac{4b^2}{a}x$. ***Ans.***

15.0 CHORD WITH A GIVEN MIDDLE POINT

Equation of the chord of the parabola $y^2 = 4ax$ whose middle point is (x_1, y_1) is $y - y_1 = \frac{2a}{y_1}(x - x_1)$.

This reduced to $T = S_1$, where $T \equiv yy_1 - 2a(x + x_1)$ & $S_1 \equiv y_1^2 - 4ax_1$.

Illustrations

Illustration 19. Find the locus of middle point of the chord of the parabola $y^2 = 4ax$ which pass through a given (p, q).

Solution Let P(h, k) be the mid point of chord of the parabola $y^2 = 4ax$,

so equation of chord is $yk - 2a(x + h) = k^2 - 4ah$.

Since it passes through (p, q)

$\therefore$ $qk - 2a(p + h) = k^2 - 4ah$

$\therefore$ Required locus is $y^2 - 2ax - qy + 2ap = 0$.

Illustration 20. Find the locus of the middle point of a chord of a parabola $y^2 = 4ax$ which subtends a right angle at the vertex.

Solution The equation of the chord of the parabola whose middle point is (α, β) is

$y\beta - 2a(x + \alpha) = \beta^2 - 4a\alpha$

$\Rightarrow$ $y\beta - 2ax = \beta^2 - 2a\alpha$

or $\dfrac{y\beta - 2ax}{\beta^2 - 2a\alpha} = 1$ .... (i)

Now, the equation of the pair of the lines OP and OQ joining the origin O i.e. the vertex to the points of intersection P and Q of the chord with the parabola $y^2 = 4ax$ is obtained by making the equation homogeneous by means of (i). Thus the equation of lines OP and OQ is

$$y^2 = \frac{4ax(y\beta - 2ax)}{\beta^2 - 2a\alpha}$$

$\Rightarrow$ $y^2(\beta^2 - 2a\alpha) - 4a\beta xy + 8a^2x^2 = 0$

If the lines OP and OQ are at right angles, then the coefficient of x^2 + the coefficient of $y^2 = 0$

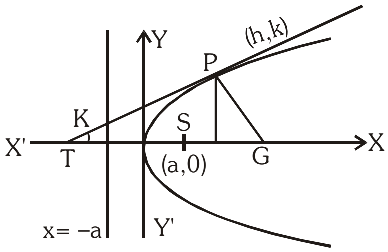
Therefore, $\beta^2 - 2a\alpha + 8a^2 = 0 \Rightarrow \beta^2 = 2a(\alpha - 4a)$

Hence the locus of (α, β) is $y^2 = 2a(x - 4a)$

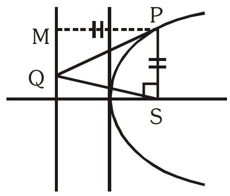
16.0 DIAMETER

The locus of the middle points of a system of parallel chords of a Parabola is called a DIAMETER. Equation to the diameter of a parabola is $y = 2a/m$, where m = slope of parallel chords.

17.0 PROPERTIES OF PARABOLA



(a) If the tangent & normal at any point 'P' of the parabola intersect the axis at T & G then ST = SG = SP where 'S' is the focus. In other words the tangent and the normal at a point P on the parabola are the bisectors of the angle between the focal radius SP & the perpendicular from P on the directrix. From this we conclude that all rays emanating from S will become parallel to the axis of the parabola after reflection.

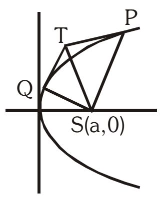


(b) The portion of a tangent to a parabola cut off between the directrix & the curve subtends a right angle at the ***focus***.

(c) The tangents at the extremities of a focal chord intersect at right angles on the ***directrix,*** and a circle on any focal chord as diameter touches the directrix. Also a circle on any focal radii of a point P $(at^2, 2at)$ as diameter touches the tangent at the vertex and intercepts a chord of length $a\sqrt{1 + t^2}$ on a normal at the point P.

(d) Any tangent to a parabola & the perpendicular on it from the focus meet on the tangent at the vertex.

(e) Semi latus rectum of the parabola $y^2 = 4ax$, is the harmonic mean between segments of any focal chord of the parabola is ; $\mathbf{2a = \dfrac{2bc}{b+c}}$ ***i.e.*** $\mathbf{\dfrac{1}{b} + \dfrac{1}{c} = \dfrac{1}{a}}$.

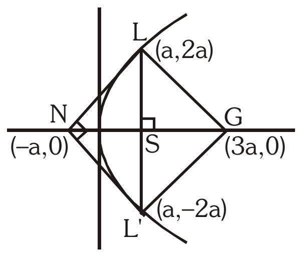


(f) If the tangents at P and Q meet in T, then :

(i) TP and TQ subtend equal angles at the focus S.

(ii) $ST^2 = SP \cdot SQ$ &

(iii) The triangles SPT and STQ are similar.



(g) Tangents and Normals at the extremities of the latus rectum of a parabola $y^2 = 4ax$ constitute a square, their points of intersection being (–a, 0) & (3a, 0).

NOTE

(i) The two tangents at the extremities of focal chord meet on the foot of the directrix.

(ii) Figure LNL'G is square of side $2\sqrt{2}a$

(h) The circle circumscribing the triangle formed by any three tangents to a parabola passes through the focus.

GOLDEN KEY POINTS

- If a family of straight lines can be represented by an equation $\lambda^2 P + \lambda Q + R = 0$ where λ is a parameter and P, Q, R are linear functions of x and y then the family of lines will be tangent to the curve $Q^2 = 4PR$.

BEGINNER'S BOX-4

1. Find the angle between the tangents drawn from the origin to the parabola, $y^2 = 4a(x - a)$.

2. Find the equation of the chord of contacts of tangents drawn from a point (2, 1) to the parabola $x^2 = 2y$.

3. Find the co-ordinates of the middle point of the chord of the parabola $y^2 = 16x$, the equation of which is $2x - 3y + 8 = 0$

4. Find the locus of the mid-point of the chords of the parabola $y^2 = 4ax$ such that tangent at the extremities of the chords are perpendicular.

5. Let P be the point (1, 0) and Q a point on the parabola $y^2 = 8x$, then find the locus of the mid point of PQ.

SOME WORKED OUT ILLUSTRATIONS

Illustration 1. The common tangent of the parabola $y^2 = 8ax$ and the circle $x^2 + y^2 = 2a^2$ is -

(A) $y = x + a$ (B) $x + y + a = 0$ (C) $x + y + 2a = 0$ (D) $y = x + 2a$

Solution Any tangent to parabola is $y = mx + \dfrac{2a}{m}$

Solving with the circle $x^2 + (mx + \dfrac{2a}{m})^2 = 2a^2 \Rightarrow x^2(1 + m^2) + 4ax + \dfrac{4a^2}{m^2} - 2a^2 = 0$

$B^2 - 4AC = 0$ gives $m = \pm 1$

Tangent $y = \pm x \pm 2a$ ***Ans. (C,D)***

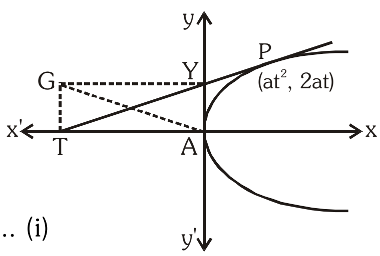
Illustration 2. If the tangent to the parabola $y^2 = 4ax$ meets the axis in T and tangent at the vertex A in Y and the rectangle TAYG is completed, show that the locus of G is $y^2 + ax = 0$.

Solution Let $P(at^2, 2at)$ be any point on the parabola $y^2 = 4ax$.

Then tangent at $P(at^2, 2at)$ is $ty = x + at^2$

Since tangent meet the axis of parabola in T and tangent at the vertex in Y.

$\therefore$ Co-ordinates of T and Y are $(-at^2, 0)$ and $(0, at)$ respectively.



Let co-ordinates of G be (x_1, y_1).

Since TAYG is rectangle.

$\therefore$ Mid-points of diagonals TY and GA is same

$\Rightarrow \dfrac{x_1 + 0}{2} = \dfrac{-at^2 + 0}{2} \Rightarrow x_1 = -at^2$ .... (i)

and $\dfrac{y_1 + 0}{2} = \dfrac{0 + at}{2} \Rightarrow y_1 = at$ .... (ii)

Eliminating t from (i) and (ii) then we get $x_1 = -a\left(\dfrac{y_1}{a}\right)^2$

or $y_1^2 = -ax_1$ or $y_1^2 + ax_1 = 0$

$\therefore$ The locus of $G(x_1, y_1)$ is $y^2 + ax = 0$

Illustration 3. If $P(-3, 2)$ is one end of the focal chord PQ of the parabola $y^2 + 4x + 4y = 0$, then the slope of the normal at Q is -

(A) $-1/2$ (B) 2 (C) $1/2$ (D) -2

Solution The equation of the tangent at $(-3, 2)$ to the parabola $y^2 + 4x + 4y = 0$ is

$2y + 2(x - 3) + 2(y + 2) = 0$

or $2x + 4y - 2 = 0 \Rightarrow x + 2y - 1 = 0$

Since the tangent at one end of the focal chord is parallel to the normal at the other end, the slope of the normal at the other end of the focal chord is $-\dfrac{1}{2}$. ***Ans.(A)***

Illustration 4. Prove that the two parabolas $y^2 = 4ax$ and $y^2 = 4c(x - b)$ cannot have common normal, other than the axis unless $b/(a - c) > 2$.

Solution Given parabolas $y^2 = 4ax$ and $y^2 = 4c(x - b)$ have common normals. Then equation of normals in terms of slopes are $y = mx - 2am - am^3$ and $y = m(x - b) - 2cm - cm^3$ respectively then normals must be identical, compare the co-efficients

$$1 = \frac{2am + am^3}{mb + 2cm + cm^3}$$

$\Rightarrow \quad m[(c-a)m^2 + (b+2c-2a)] = 0,\ m \neq 0$ $\quad (\because$ other than axis)

and $\quad m^2 = \dfrac{2a-2c-b}{c-a},\ m = \pm\sqrt{\dfrac{2(a-c)-b}{c-a}}$

or $\quad m = \pm\sqrt{\left(-2-\dfrac{b}{c-a}\right)}$

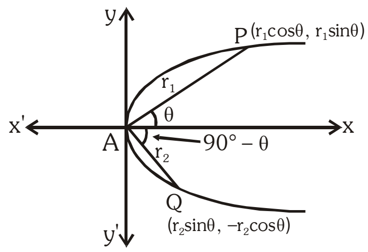
$\therefore \quad -2-\dfrac{b}{c-a} > 0$

or $\quad -2+\dfrac{b}{a-c} > 0 \Rightarrow \dfrac{b}{a-c} > 2$

Illustration 5. If r_1, r_2 be the length of the perpendicular chords of the parabola $y^2 = 4ax$ drawn through the vertex, then show that $(r_1r_2)^{4/3} = 16a^2\left(r_1^{2/3}+r_2^{2/3}\right)$.

Solution Since chord are perpendicular, therefore if one makes an angle θ then the other will make an angle $(90^\circ - \theta)$ with x-axis

Let $\quad AP = r_1$ and $AQ = r_2$



If $\quad \angle PAX = \theta$

then $\quad \angle QAX = 90^\circ - \theta$

$\therefore$ Co-ordinates of P and Q are $(r_1\cos\theta,\ r_1\sin\theta)$

and $\quad (r_2\sin\theta, -r_2\cos\theta)$ respectively.

Since P and Q lies on $y^2 = 4ax$

$\therefore \quad r_1^2\sin^2\theta = 4ar_1\cos\theta$ and $r_2^2\cos^2\theta = 4ar_2\sin\theta$

$\Rightarrow \quad r_1 = \dfrac{4a\cos\theta}{\sin^2\theta}$ and $r_2 = \dfrac{4a\sin\theta}{\cos^2\theta}$

$$\therefore \quad (r_1r_2)^{4/3} = \left(\frac{4a\cos\theta}{\sin^2\theta}\cdot\frac{4a\sin\theta}{\cos^2\theta}\right)^{4/3} = \left(\frac{16a^2}{\sin\theta\cos\theta}\right)^{4/3} \qquad \text{....... (i)}$$

$$\text{and} \quad 16a^2.\left(r_1^{2/3}+r_2^{2/3}\right) = 16a^2\left\{\left(\frac{4a\cos\theta}{\sin^2\theta}\right)^{2/3} + \left(\frac{4a\sin\theta}{\cos^2\theta}\right)^{2/3}\right\}$$

$$= 16a^2.(4a)^{2/3}\left\{\frac{(\cos\theta)^{2/3}}{(\sin\theta)^{4/3}} + \frac{(\sin\theta)^{2/3}}{(\cos\theta)^{4/3}}\right\} = 16a^2.(4a)^{2/3}\left\{\frac{\cos^2\theta+\sin^2\theta}{(\sin\theta)^{4/3}(\cos\theta)^{4/3}}\right\}$$

$$= \frac{16a^2.(4a)^{2/3}}{(\sin\theta\cos\theta)^{4/3}} = \left(\frac{16a^2}{\cos\theta\cos\theta}\right)^{4/3} = (r_1r_2)^{4/3} \qquad \{\text{from (i)}\}$$

Illustration 6. The area of the triangle formed by three points on a parabola is twice the area of the triangle formed by the tangents at these points.

Solution Let the three points on the parabola be

$(at_1^2, 2at_1), (at_2^2, 2at_2)$ and $(at_3^2, 2at_3)$

The area of the triangle formed by these points

$\Delta_1 = \dfrac{1}{2}[at_1^2(2at_2 - 2at_3) + at_2^2(2at_3 - 2at_1) + at_2^2(2a_1 - 2at_2)]$

$= -a^2(t_2-t_3)(t_3-t_1)(t_1-t_2)$.

The points of intersection of the tangents at these points are

$(at_2t_3, a(t_2 + t_3))$, $(at_3t_1, a(t_3 + t_1))$ and $(at_1t_2, a(t_1 + t_2))$

The area of the triangle formed by these three points

$$\Delta_2 = \frac{1}{2}\left\{at_2t_3(at_3 - at_2) + at_3t_1(at_1 - at_3) + at_1t_2(at_2 - at_1)\right\}$$

$$= \frac{1}{2}a^2(t_2 - t_3)(t_3 - t_1)(t_1 - t_2)$$

Hence $\Delta_1 = 2\Delta_2$

Illustration 7. Prove that the orthocentre of any triangle formed by three tangents to a parabola lies on the directrix.

Solution Let the equations of the three tangents be

$t_1y = x + at_1^2$ ......(i)

$t_2y = x + at_2^2$ ......(ii)

and $t_3y = x + at_3^2$ .....(iii)

The point of intersection of (ii) and (iii) is found, by solving them, to be $(at_2t_3, a(t_2 + t_3))$

The equation of the straight line through this point & perpendicular to (i) is

$y - a(t_2 + t_3) = -t_1(x - at_2t_3)$

i.e. $y + t_1x = a(t_2 + t_3 + t_1t_2t_3)$ .....(iv)

Similarly, the equation of the straight line through the point of intersection of (iii) and (i) & perpendicular to (ii) is

$y + t_2x = a(t_3 + t_1 + t_1t_2t_3)$ .....(v)

and the equation of the straight line through the point of intersection of (i) and (ii) & perpendicular to (iii) is

$y + t_1x = a(t_1 + t_2 + t_1t_2t_3)$ .....(vi)

The point which is common to the straight lines (iv), (v) and (vi)

i.e. the orthocentre of the triangle, is easily seen to be the point whose coordinates are

$x = -a$, $y = a(t_1 + t_2 + t_3 + t_1t_2t_3)$

and this point lies on the directrix.

ANSWERS

BEGINNER'S BOX-1

1. Parabola **2.** Vertex : $\left(-\frac{7}{2}, \frac{5}{2}\right)$, Axis : $y = \frac{5}{2}$, Focus : $\left(-\frac{17}{4}, \frac{5}{2}\right)$, Directrix : $x = -\frac{11}{4}$; LR = 3

3. $4x^2 + y^2 - 4xy + 8x + 46y - 71 = 0$; Axis : $2x - y = 3$; $LR = 4\sqrt{5}$ unit

4. $(3x + 4y - 4)^2 = 20(4x - 3y + 7)$

BEGINNER'S BOX-2

1. $\left(-\infty, -\sqrt{\frac{8}{7}}\right) \cup \left(\sqrt{\frac{8}{7}}, \infty\right)$ **2.** (–7, 8), (9, 8) **3.** $kb^2 = ac$ **4.** $8\sqrt{3}$

5. $x - y + 3 = 0$, (3, 6); $3x - 2y + 4 = 0$, $\left(\frac{4}{3}, 4\right)$

BEGINNER'S BOX-3

1. C **2.** 1 **3.** 48 **4.** (0, 0), (–4, 3) and (16, 8)

BEGINNER'S BOX-4

1. $\pi/2$ **2.** $2x = y + 1$ **3.** (14, 12)

4. $y^2 = 2a(x - a)$ **5.** $y^2 - 4x + 2 = 0$

EXERCISE – 1 MCQ (SINGLE CHOICE CORRECT)

1. Latus rectum of the parabola whose focus is (3, 4) and whose tangent at vertex has the equation $x + y = 7+5\sqrt{2}$ is -

(A) 5 (B) 10 (C) 20 (D) 15

2. Directrix of a parabola is x + y = 2. If it's focus is origin, then latus rectum of the parabola is equal to -

(A) $\sqrt{2}$ units (B) 2 units (C) $2\sqrt{2}$ units (D) 4 units

3. Which one of the following equations represents parametrically, parabolic profile ?

(A) $x = 3 \cos t$; $y = 4 \sin t$ (B) $x^2 - 2 = -\cos t$; $y = 4 \cos^2 \frac{t}{2}$

(C) $\sqrt{x} = \tan t$; $\sqrt{y} = \sec t$ (D) $x = \sqrt{1-\sin t}$; $y = \sin\frac{t}{2}+\cos\frac{t}{2}$

4. If $(t^2, 2t)$ is one end of a focal chord of the parabola $y^2 = 4x$ then the length of the focal chord will be-

(A) $\left(t+\frac{1}{t}\right)^2$ (B) $\left(t+\frac{1}{t}\right)\sqrt{\left(t^2+\frac{1}{t^2}\right)}$ (C) $\left(t-\frac{1}{t}\right)\sqrt{\left(t^2+\frac{1}{t^2}\right)}$ (D) none

5. The point of intersection of the curves whose parametric equations are $x = t^2+1$, $y = 2t$ and $x = 2s$, $y = 2/s$ is given by -

(A) (4, 1) (B) (2, 2) (C) (–2, 4) (D) (1, 2)

6. If M is the foot of the perpendicular from a point P of a parabola $y^2= 4ax$ to its directrix and SPM is an equilateral triangle, where S is the focus, then SP is equal to -

(A) a (B) 2a (C) 3a (D) 4a

7. The tangents to the parabola $x = y^2 + c$ from origin are perpendicular then c is equal to -

(A) $\frac{1}{2}$ (B) 1 (C) 2 (D) $\frac{1}{4}$

8. The locus of a point such that two tangents drawn from it to the parabola $y^2 = 4ax$ are such that the slope of one is double the other is -

(A) $y^2 = \frac{9}{2}ax$ (B) $y^2 = \frac{9}{4}ax$ (C) $y^2 = 9ax$ (D) $x^2 = 4ay$

9. The equation of the circle drawn with the focus of the parabola $(x - 1)^2 - 8y = 0$ as its centre and touching the parabola at its vertex is :

(A) $x^2 + y^2 - 4y = 0$ (B) $x^2 + y^2 - 4y + 1 = 0$

(C) $x^2 + y^2 - 2x - 4y = 0$ (D) $x^2 + y^2 - 2x - 4y + 1 = 0$

10. Length of the normal chord of the parabola, $y^2 = 4x$, which makes an angle of $\frac{\pi}{4}$ with the axis of x is-

(A) 8 (B) $8\sqrt{2}$ (C) 4 (D) $4\sqrt{2}$

11. Tangents are drawn from the point (–1, 2) on the parabola $y^2 = 4x$. The length , these tangents will intercept on the line $x = 2$:

(A) 6 (B) $6\sqrt{2}$ (C) $2\sqrt{6}$ (D) none of these

12. Locus of the point of intersection of the perpendiculars tangent of the curve $y^2 + 4y - 6x - 2 = 0$ is :

(A) $2x - 1 = 0$ (B) $2x + 3 = 0$ (C) $2y + 3 = 0$ (D) $2x + 5 = 0$

13. Tangents are drawn from the points on the line $x - y + 3 = 0$ to parabola $y^2 = 8x$. Then the variable chords of contact pass through a fixed point whose coordinates are-

(A) (3, 2) (B) (2, 4) (C) (3, 4) (D) (4, 1)

14. The line $4x - 7y + 10 = 0$ intersects the parabola, $y^2 = 4x$ at the points A & B. The co-ordinates of the point of intersection of the tangents drawn at the points A & B are :

(A) $\left(\frac{7}{2}, \frac{5}{2}\right)$ (B) $\left(-\frac{5}{2}, \frac{7}{2}\right)$ (C) $\left(\frac{5}{2}, \frac{7}{2}\right)$ (D) $\left(-\frac{7}{2}, \frac{5}{2}\right)$

15. From the point (4, 6) a pair of tangent lines are drawn to the parabola, $y^2 = 8x$. The area of the triangle formed by these pair of tangent lines & the chord of contact of the point (4, 6) is

(A) 2 (B) 4 (C) 8 (D) none

EXERCISE – 2 **MCQ (ONE OR MORE CHOICE CORRECT)**

1. The straight line joining any point P on the parabola $y^2 = 4ax$ to the vertex and perpendicular from the focus to the tangent at P, intersect at R, then the equation of the locus of R is -
(A) $x^2 + 2y^2 - ax = 0$ (B) $2x^2 + y^2 - 2ax = 0$ (C) $2x^2 + 2y^2 - ay = 0$ (D) $2x^2 + y^2 - 2ay = 0$

2. Let A be the vertex and L the length of the latus rectum of parabola, $y^2 - 2y - 4x - 7 = 0$. The equation of the parabola with point A as vertex, 2L as the length of the latus rectum and the axis at right angles to that of the given curve is -
(A) $x^2 + 4x + 8y - 4 = 0$ (B) $x^2 + 4x - 8y + 12 = 0$
(C) $x^2 + 4x + 8y + 12 = 0$ (D) $x^2 + 8x - 4y + 8 = 0$

3. The parametric coordinates of any point on the parabola $y^2 = 4ax$ can be -
(A) $(at^2, 2at)$ (B) $(at^2, -2at)$ (C) $(a\sin^2 t, 2a\sin t)$ (D) $(a\sin t, 2a\cos t)$

4. The length of the chord of the parabola $y^2 = x$ which is bisected at the point (2, 1) is-
(A) $5\sqrt{2}$ (B) $4\sqrt{5}$ (C) $4\sqrt{50}$ (D) $2\sqrt{5}$

5. If the tangents and normals at the extremities of a focal chord of a parabola intersect at (x_1, y_1) and (x_2, y_2) respectively, then -
(A) $x_1 = x_2$ (B) $x_1 = y_2$ (C) $y_1 = y_2$ (D) $x_2 = y_1$

6. Locus of the intersection of the tangents at the ends of the normal chords of the parabola $y^2 = 4ax$ is -
(A) $(2a + x)y^2 + 4a^3 = 0$ (B) $(x + 2a)y^2 + 4a^2 = 0$ (C) $(y + 2a)x^2 + 4a^3 = 0$ (D) none

7. The equation of a straight line passing through the point (3, 6) and cutting the curve $y = \sqrt{x}$ orthogonally is -
(A) $4x + y - 18 = 0$ (B) $x + y - 9 = 0$ (C) $4x - y - 6 = 0$ (D) none

8. AB, AC are tangents to a parabola $y^2 = 4ax$. p_1 p_2 and p_3 are the lengths of the perpendiculars from A, B and C respectively on any tangent to the curve, then p_2, p_1, p_3 are in-
(A) A.P. (B) G.P. (C) H.P. (D) none of these

9. Two parabolas have the same focus. If their directrices are the x-axis & the y-axis respectively, then the slope of their common chord is -
(A) 1 (B) –1 (C) 4/3 (D) 3/4

10. If the distance between a tangent to the parabola $y^2 = 4x$ and a parallel normal to the same parabola is $2\sqrt{2}$, then possible values of gradient of either of them are -
(A) –1 (B) +1 (C) $-\sqrt{\sqrt{5}-2}$ (D) $+\sqrt{\sqrt{5}-2}$

11. If PQ is a chord of parabola $x^2 = 4y$ which subtends right angle at vertex. Then locus of centroid of triangle PSQ (S is focus) is a parabola whose -
(A) vertex is (0, 3) (B) length of LR is 4/3
(C) axis is x = 0 (D) tangent at the vertex is x = 3

12. The normals to the parabola $y^2 = 4ax$ from the point (5a, 2a) are -

(A) $y = -3x + 33a$ (B) $x = -3y + 3a$ (C) $y = x - 3a$ (D) $y = -2x + 12a$

13. The equation of the lines joining the vertex of the parabola $y^2 = 6x$ to the points on it whose abscissa is 24, is -

(A) $2y + x + 1 = 0$ (B) $2y - x + 1 = 0$ (C) $x + 2y = 0$ (D) $x - 2y = 0$

14. The equation of the tangent to the parabola $y^2 = 9x$ which passes through the point (4, 10) is -

(A) $x + 4y + 1 = 0$ (B) $x - 4y + 36 = 0$ (C) $9x - 4y + 4 = 0$ (D) $9x + 4y + 4 = 0$

15. Consider the equation of a parabola $y^2 = 4ax$, $(a < 0)$ which of the following is false -

(A) tangent at the vertex is $x = 0$ (B) directrix of the parabola is $x = 0$

(C) vertex of the parabola is at the origin (D) focus of the parabola is at $(-a, 0)$

Match the Column

Following questions contains statements given in two columns, which have to be matched. The statements in ***Column-I*** are labelled as A, B, C and D while the statements in ***Column-II*** are labelled as p, q, r and s. Any given statement in ***Column-I*** can have correct matching with ***ONE*** statement in ***Column-II***.

16.

	Column-I		Column-II
(A)	Area of a triangle formed by the tangents drawn from a point $(-2, 2)$ to the parabola $y^2 = 4(x + y)$ and their corresponding chord of contact is	(p)	8
(B)	Length of the latus rectum of the conic $25\{(x-2)^2 + (y-3)^2\} = (3x + 4y - 6)^2$ is	(q)	$4\sqrt{3}$
(C)	If focal distance of a point on the parabola $y = x^2 - 4$ is 25/4 and points are of the form $(\pm\sqrt{a}, b)$ then value of a + b is	(r)	4
(D)	Length of side of an equilateral triangle inscribed in a parabola $y^2 - 2x - 2y - 3 = 0$ whose one angular point is vertex of the parabola, is	(s)	24/5

Comprehension Based Questions

If a source of light is placed at the fixed point of a parabola and if the parabola is a reflecting surface, then the ray will bounce back in a line parallel to the axis of the parabola.

On the basis of above information, answer the following questions

17. A ray of light is coming along the line $y = 2$ from the positive direction of x-axis and strikes a concave mirror whose intersection with the xy-plane is a parabola $y^2 = 8x$, then the equation of the reflected ray is -

(A) $2x + 5y = 4$ (B) $3x + 2y = 6$ (C) $4x + 3y = 8$ (D) $5x + 4y = 10$

18. A ray of light moving parallel to the x-axis gets reflected from a parabolic mirror whose equation is $y^2 + 10y - 4x + 17 = 0$ After reflection, the ray must pass through the point -

(A) $(-2, -5)$ (B) $(-1, -5)$ (C) $(-3, -5)$ (D) $(-4, -5)$

19. Two ray of light coming along the lines $y = 1$ and $y = -2$ from the positive direction of x-axis and strikes a concave mirror whose intersection with the xy-plane is a parabola $y^2 = x$ at A and B respectively. The reflected rays pass through a fixed point C, then the area of the triangle ABC is -

(A) $\frac{21}{8}$ sq. unit (B) $\frac{19}{2}$ sq. unit (C) $\frac{17}{2}$ sq. unit (D) $\frac{15}{2}$ sq. unit

EXERCISE - 3 SUBJECTIVE

1. Find the equation of parabola, whose focus is (–3, 0) and directrix is $x + 5 = 0$.

2. Find the vertex, axis, focus, directrix, latus rectum of the parabola $x^2 + 2y - 3x + 5 = 0$

3. Find the equation of the parabola whose focus is (1, –1) and whose vertex is (2, 1). Also find its axis and latus rectum.

4. Find the locus of the middle points of all chords of the parabola $y^2 = 4ax$ which are drawn through the vertex.

5. Find the length of the side of an equilateral triangle inscribed in the parabola, $y^2 = 4x$ so that one of its angular point is at the vertex.

6. Find the set of values of α in the interval $[\pi/2, 3\pi/2]$, for which the point $(\sin\alpha, \cos\alpha)$ does not lie outside the parabola $2y^2 + x - 2 = 0$.

7. Find the length of the focal chord of the parabola $y^2 = 4ax$ whose distance from the vertex is p.

8. If 'm' varies then find the range of c for which the line $y = mx + c$ touches the parabola $y^2 = 8(x + 2)$.

9. Find the equations of the tangents to the parabola $y^2 = 16x$, which are parallel & perpendicular respectively to the line $2x - y + 5 = 0$. Find also the coordinates of their points of contact.

10. Find the equations of the tangents of the parabola $y^2 = 12x$, which passes through the point (2, 5).

EXERCISE – 4 RECAP OF AIEEE/JEE (MAIN)

1. The length of the latus rectum of the parabola $x^2 - 4x - 8y + 12 = 0$ is- ***[AIEEE-2002]***

(A) 4 (B) 6 (C) 8 (D) 10

2. The equation of tangents to the parabola $y^2 = 4ax$ at the ends of its latus rectum is- ***[AIEEE-2002]***

(A) $x - y + a = 0$ (B) $x + y + a = 0$ (C) $x + y - a = 0$ (D) both (A) and (B)

3. The normal at the point $(bt_1^2, 2bt_1)$ on a parabola meets the parabola again in the point $(bt_2^2, 2bt_2)$, then- ***[AIEEE-2003]***

(A) $t_2 = t_1 + \frac{2}{t_1}$ (B) $t_2 = -t_1 - \frac{2}{t_1}$ (C) $t_2 = -t_1 + \frac{2}{t_1}$ (D) $t_2 = t_1 - \frac{2}{t_1}$

4. If $a \neq 0$ and the line $2bx + 3cy + 4d = 0$ passes through the points of intersection of the parabolas $y^2 = 4ax$ and $x^2 = 4ay$, then- ***[AIEEE-2004]***

(A) $d^2 + (2b + 3c)^2 = 0$ (B) $d^2 + (3b + 2c)^2 = 0$ (C) $d^2 + (2b - 3c)^2 = 0$ (D) $d^2 + (3b - 2c)^2 = 0$

5. The locus of the vertices of the family of parabolas $y = \frac{a^3x^2}{3} + \frac{a^2x}{2} - 2a$ is- ***[AIEEE-2006]***

(A) $xy = \frac{3}{4}$ (B) $xy = \frac{35}{16}$ (C) $xy = \frac{64}{105}$ (D) $xy = \frac{105}{64}$

6. The equation of a tangent to the parabola $y^2 = 8x$ is $y = x + 2$. The point on this line from which the other tangent to the parabola is perpendicular to the given tangents is- ***[AIEEE-2007]***

(A) $(-1, 1)$ (B) $(0, 2)$ (C) $(2, 4)$ (D) $(-2, 0)$

7. A parabola has the origin as its focus and the line $x = 2$ as the directrix. Then the vertex of the parabola is at -

(A) $(0, 2)$ (B) $(1, 0)$ (C) $(0, 1)$ (D) $(2, 0)$ ***[AIEEE-2008]***

8. If two tangents drawn from a point P to the parabola $y^2 = 4x$ are at right angles then the locus of P is :- ***[AIEEE-2010]***

(A) $x = 1$ (B) $2x + 1 = 0$ (C) $x = -1$ (D) $2x - 1 = 0$

9. Given : A circle, $2x^2 + 2y^2 = 5$ and a parabola, $y^2 = 4\sqrt{5}\,x$. ***[JEE Main-2013]***

Statement–I – An equation of a common tangent to these curves is $y = x + \sqrt{5}$.

Statement–II – If the line, $y = mx + \frac{\sqrt{5}}{m}$ $(m \neq 0)$ is their common tangent, then m satisfies $m^4 - 3m^2 + 2 = 0$.

(A) Statement-I is true, Statement-II is true; statement-II is a ***correct*** explanation for Statement-I.

(B) Statement-I is true, Statement-II is true; statement-II is ***not*** a correct explanation for Statement-I.

(C) Statement-I is true, Statement-II is false.

(D) Statement-I is false, Statement-II is true.

10. The slope of the line touching both the parabolas $y^2 = 4x$ and $x^2 = -32y$ is : ***[JEE Main-2014]***

(A) $\frac{1}{8}$ (B) $\frac{2}{3}$ (C) $\frac{1}{2}$ (D) $\frac{3}{2}$

11. Let O be the vertex and Q be any point on the parabola, $x^2 = 8y$. If the point P divides the line segment OQ internally in the ratio 1 : 3, then the locus of P is :- **[JEE Main-2015]**

(A) $y^2 = 2x$ (B) $x^2 = 2y$ (C) $x^2 = y$ (D) $y^2 = x$

12. Let P be the point on the parabola, $y^2 = 8x$ which is at a minimum distance from the cente C of the circle, $x^2 + (y + 6)^2 = 1$. Then the equation of the circle, passing through C and having its centre at P is:

(A) $x^2 + y^2 - 4x + 9y + 18 = 0$ (B) $x^2 + y^2 - 4x + 8y + 12 = 0$ **[JEE Main-2016]**

(C) $x^2 + y^2 - x + 4y - 12 = 0$ (D) $x^2 + y^2 - \frac{x}{4} + 2y - 24 = 0$

13. Tangent and normal are drawn at P(16, 16) on the parabola $y^2 = 16x$, which intersect the axis of the parabola at A and B, respectively. If C is the centre of the circle through the points P, A and B and $\angle CPB = \theta$, then a value of $\tan\theta$ is- **[JEE Main-2018]**

(A) 2 (B) 3 (C) $\frac{4}{3}$ (D) $\frac{1}{2}$

14. If the tangent at (1, 7) to the curve $x^2 = y - 6$ touches the circle $x^2 + y^2 + 16x + 12y + c = 0$ then the value of c is : **[JEE Main-2018]**

(A) 185 (B) 85 (C) 95 (D) 195

15. Let A(4, –4) and B(9, 6) be points on the parabola, $y^2 + 4x$. Let C be chosen on the arc AOB of the parabola, where O is the origin, such that the area of ΔACB is maximum. Then, the area (in sq. units) of ΔACB, is:

(A) $31\frac{3}{4}$ (B) 32 (C) $30\frac{1}{2}$ (D) $31\frac{1}{4}$ **[JEE Main-2019]**

16. Equation of a common tangent to the circle, $x^2 + y^2 - 6x = 0$ and the parabola, $y^2 = 4x$, is : **[JEE Main-2019]**

(A) $2\sqrt{3}\,y = 12x + 1$ (B) $2\sqrt{3}\,y = -x - 12$ (C) $\sqrt{3}\,y = x + 3$ (D) $\sqrt{3}\,y = 3x + 1$

17. Axis of a parabola lies along x–axis. If its vertex and focus are at distances 2 and 4 respectively from the origin, on the positive x–axis then which of the following points does not lie on it ? **[JEE Main-2019]**

(A) (4, –4) (B) $(5,\ 2\sqrt{6})$ (C) (8, 6) (D) $6, 4\sqrt{2}$

18. The length of the chord of the parabola $x^2 = 4y$ having equation $x - \sqrt{2}y + 4\sqrt{2} = 0$ is : **[JEE Main-2019]**

(A) $2\sqrt{11}$ (B) $3\sqrt{2}$ (C) $6\sqrt{3}$ (D) $8\sqrt{2}$

19. If the parabolas $y^2 = 4b(x - c)$ and $y^2 = 8ax$ have a common normal, then which one of the following is a valid choice for the ordered triad (a, b, c) **[JEE Main-2019]**

(A) (1, 1, 0) (B) $\left(\frac{1}{2},\ 2, 3\right)$ (C) $\left(\frac{1}{2},\ 2, 0\right)$ (D) (1, 1, 3)

20. If the area of the triangle whose one vertex is at the vertex of the parabola, $y^2 + 4(x - a^2) = 0$ and the other two vertices are the points of intersection of the parabola and y–axis, is 250 sq. units, then a value of 'a' is :

(A) $5\sqrt{5}$ (B) $(10)^{2/3}$ (C) $5(2^{1/3})$ (D) 5 **[JEE Main-2019]**

21. The equation of a tangent to the parabola, $x^2 = 8y$, which makes an angle θ with the positive direction of x–axis, is : ***[JEE Main-2019]***

(A) $x = y \cot\theta + 2\tan\theta$ (B) $x = y\cot\theta - 2\tan\theta$

(C) $y = x\tan\theta - 2\cot\theta$ (D) $y = x\tan\theta + 2\cot\theta$

22. The maximum area (in sq. units) of a rectangle having its base on the x-axis and its other two vertices on the parabola, $y = 12 - x^2$ such that the rectangle lies inside the parabola, is : ***[JEE Main-2019]***

(A) $20\sqrt{2}$ (B) $18\sqrt{3}$ (C) 32 (D) 36

23. Let P(4, –4) and Q(9, 6) be two points on the parabola, $y^2 = 4x$ and let X be any point on the arc POQ of this parabola, where O is the vertex of this parabola, such that the area of ΔPXQ is maximum. Then this maximum area (in sq. units) is : ***[JEE Main-2019]***

(A) $\frac{125}{4}$ (B) $\frac{125}{2}$ (C) $\frac{625}{4}$ (D) $\frac{75}{2}$

24. The tangent to the parabola $y^2 = 4x$ at the point where it intersects the circle $x^2 + y^2 = 5$ in the first quadrant, passes through the point : ***[JEE Main-2019]***

(A) $\left(-\frac{1}{3},\frac{4}{3}\right)$ (B) $\left(-\frac{1}{4},\frac{1}{2}\right)$ (C) $\left(\frac{3}{4},\frac{7}{4}\right)$ (D) $\left(\frac{1}{4},\frac{3}{4}\right)$

25. If one end of a focal chord of the parabola, $y^2 = 16x$ is at (1, 4), then the length of this focal chord is ***[JEE Main-2019]***

(A) 25 (B) 24 (C) 20 (D) 22

26. If the line $ax + y = c$, touches both the curves $x^2 + y^2 = 1$ and $y^2 = 4\sqrt{2}\,x$, then $|\,c\,|$ is equal to : ***[JEE Main-2019]***

(A) 1/2 (B) 2 (3) $\sqrt{2}$ (D) $\frac{1}{\sqrt{2}}$

27. The tangents to the curve $y = (x - 2)^2 - 1$ at its points of intersection with the line $x - y = 3$, intersect at the point : ***[JEE Main-2019]***

(A) $\left(-\frac{5}{2},-1\right)$ (B) $\left(-\frac{5}{2},1\right)$ (C) $\left(\frac{5}{2},-1\right)$ (D) $\left(\frac{5}{2},1\right)$

EXERCISE – 5 RECAP OF IIT-JEE/JEE (ADVANCED)

1. ***(a)*** If the line $x - 1 = 0$ is the directrix of the parabola $y^2 - kx + 8 = 0$, then one of the values of 'k' is:

(A) 1/8 (B) 8 (C) 4 (D) 1/4

(b) If $x + y = k$ is normal to $y^2 = 12x$, then 'k' is - ***[JEE-2000]***

(A) 3 (B) 9 (C) – 9 (D) – 3

2. ***(a)*** The equation of the common tangent touching the circle $(x-3)^2 + y^2 = 9$ and the parabola $y^2 = 4x$ above the x-axis is -

(A) $\sqrt{3}y = 3x+1$ (B) $\sqrt{3}y = -(x+3)$ (C) $\sqrt{3}y = x+3$ (D) $\sqrt{3}y = -(3x+1)$

(b) The equation of the directrix of the parabola $y^2 + 4y + 4x + 2 = 0$ is - ***[JEE-2001]***

(A) $x = -1$ (B) $x = 1$ (C) $x = -3/2$ (D) $x = 3/2$

3. The locus of the mid-point of the line segment joining the focus to a moving point on the parabola $y^2 = 4ax$ is another parabola with directrix ***[JEE-2002]***

(A) $x = -a$ (B) $x = -\frac{a}{2}$ (C) $x = 0$ (D) $x = \frac{a}{2}$

4. The equation of the common tangent to the curves $y^2 = 8x$ and $xy = -1$ is - ***[JEE-2002]***

(A) $3y = 9x + 2$ (B) $y = 2x + 1$ (C) $2y = x + 8$ (D) $y = x + 2$

5. If a focal chord of the parabola $y^2 = 16x$ is a tangent to the circle $(x - 6)^2 + y^2 = 2$. then the set of possible values of the slope of this chord, are - ***[JEE-2003]***

(A) $\{-1, 1\}$ (B) $\{-2, 2\}$ (C) $\left\{-2, \frac{1}{2}\right\}$ (D) $\left\{2, -\frac{1}{2}\right\}$

6. Normals with slopes m_1, m_2, m_3 are drawn from the point P to the parabola $y^2 = 4x$. If locus of P with $m_1 m_2 = \alpha$ is a part of the parabola itself, find α. ***[JEE-2004]***

7. Two tangents are drawn from point (1, 4) to the parabola $y^2 = 4x$. Angles between tangents is ***[JEE 2004]***

(A) $\pi/6$ (B) $\pi/4$ (C) $\pi/3$ (D) $\pi/2$

8. At any point P on the parabola $y^2 - 2y - 4x + 5 = 0$, a tangent is drawn which meets the directrix at Q. Find the locus of point R which divides QP externally in the ratio $\frac{1}{2} : 1$. ***[JEE 2004]***

9. Tangent to the curve $y = x^2 + 6$ at point P (1, 7) touches the circle $x^2 + y^2 + 16x + 12y + c = 0$ at a point Q. Then coordinate of Q is - ***[JEE 2005]***

(A) (–6, 11) (B) (6, –11) (C) (–6, –7) (D) (–6, –11)

10. The axis of a parabola is along the line $y = x$ and the distance of its vertex from origin is $\sqrt{2}$ and that of origin from its focus is $2\sqrt{2}$. If vertex and focus both lie in the first quadrant, then the equation of the parabola is - ***[JEE 2006]***

(A) $(x + y)^2 = (x - y - 2)$ (B) $(x - y)^2 = (x + y - 2)$

(C) $(x - y)^2 = 4(x + y - 2)$ (D) $(x - y)^2 = 8(x + y - 2)$

11. The equations of the common tangents to the parabola $y = x^2$ and $y = -x^2 + 4x - 4$ is/are- **[JEE 2006]**

(A) $y = 4(x - 1)$ (B) $y = 0$ (C) $y = -4(x - 1)$ (D) $y = -30x - 50$

12. ***Match the following*** **[JEE 2006]**

Normals are drawn at points P, Q and R lying on the parabola $y^2 = 4x$ which intersect at (3, 0). Then

(i) Area of ΔPQR	(A) 2
(ii) Radius of circumcircle of ΔPQR	(B) 5/2
(iii) Centroid of ΔPQR	(C) (5/2, 0)
(iv) Circumcentre of ΔPQR	(D) (2/3, 0)

13 to 15 are based on this paragraph **[JEE 2006]**

Let ABCD be a square of side length 2 units. C_2 is the circle through vertices A, B, C, D and C_1 is the circle touching all the sides of the square ABCD. L is a line through A.

13. If P is a point on C_1 and Q in another point on C_2, then $\dfrac{PA^2 + PB^2 + PC^2 + PD^2}{QA^2 + QB^2 + QC^2 + QD^2}$ is equal to -

(A) 0.75 (B) 1.25 (C) 1 (D) 0.5

14. A circle touches the line L and circle C_1 externally such that both the circles are on the same side of the line, then the locus of centre of the circle is -

(A) ellipse (B) hyperbola (C) parabola (D) pair of straight line

15. A line M through A is drawn parallel to BD. Point S moves such that its distances from the line BD and the vertex A are equal. If locus of S cuts M at T_2 and T_3 and AC at T_1 then area of $\Delta T_1T_2T_3$ is

(A) 1/2 sq. units (B) 2/3 sq. units (C) 1 sq. units (D) 2 sq. units

16 to 18 are based on this paragraph

Consider the circle $x^2 + y^2 = 9$ and the parabola $y^2 = 8x$. They intersect at P and Q in the first and the fourth quadrants, respectively. Tangents to the circle at P and Q intersect the x-axis at R and tangents to the parabola at P and Q intersect the x-axis at S.

16. The ratio of the areas of the triangle PQS and PQR is :- **[JEE 2007]**

(A) $1 : \sqrt{2}$ (B) 1 : 2 (C) 1 : 4 (D) 1 : 8

17. The radius of the circumcircle of the triangle PRS is :- **[JEE 2007]**

(A) 2 (B) $3\sqrt{3}$ (C) $3\sqrt{2}$ (D) $2\sqrt{3}$

18. The radius of the incircle of the triangle PQR is :- **[JEE 2007]**

(A) 4 (B) 3 (C) $\frac{8}{3}$ (D) 2

Assertion and Reason

19. ***Statement-1 –*** The curve $y = \dfrac{-x^2}{2} + x + 1$ is symmetric with respect to the line $x = 1$ because

Statement-2 – A parabola is symmetric about its axis. **[JEE 2007]**

(A) Statement-1 is True, Statement-2 is True ; Statement-2 is a correct explanation for Statement-1.

(B) Statement-1 is True, Statement-2 is True ; Statement-2 is NOT a correct explanation for Statement-1.

(C) Statement-1 is True, Statement-2 is False.

(D) Statement-1 is False, Statement-2 is True.

20. Consider the two curves $C_1 : y^2 = 4x$; $C_2 : x^2 + y^2 - 6x + 1 = 0$. Then **[JEE 2008]**

(A) C_1 and C_2 touch each other only at one point

(B) C_1 and C_2 touch each other exactly at two points

(C) C_1 and C_2 intersect (but do not touch) at exactly two points

(D) C_1 and C_2 neither intersect nor touch each other

21. The tangent PT and the normal PN to the parabola $y^2 = 4ax$ at a point P on it meet its axis at points T and N, respectively. The locus of the centroid of the triangle PTN is a parabola whose **[JEE 2009]**

(A) vertex is $\left(\frac{2a}{3}, 0\right)$ (B) directrix is $x = 0$ (C) latus rectum is $\frac{2a}{3}$ (D) focus is (a, 0)

22. Let A and B be two distinct points on the parabola $y^2 = 4x$. If the axis of the parabola touches a circle of radius r having AB as its diameter, then the slope of the line joining A and B can be - **[JEE 2010]**

(A) $-1/r$ (B) $1/r$ (C) $2/r$ (D) $-2/r$

23. Consider the parabola $y^2 = 8x$. Let Δ_1 be the area of the triangle formed by the end points of its latus rectum and the point $P\left(\frac{1}{2}, 2\right)$ on the parabola, and Δ_2 be the area of the triangle formed by drawing tangents at P and at the end points of the latus rectum. Then $\frac{\Delta_1}{\Delta_2}$ is **[JEE 2011]**

24. Let (x,y) be any point on the parabola $y^2 = 4x$. Let P be the point that divides the line segment from (0,0) to (x,y) in the ratio 1 : 3. Then the locus of P is - **[JEE 2011]**

(A) $x^2 = y$ (B) $y^2 = 2x$ (C) $y^2 = x$ (D) $x^2 = 2y$

25. Let L be a normal to the parabola $y^2 = 4x$. If L passes through the point (9,6), then L is given by - **[JEE 2011]**

(A) $y - x + 3 = 0$ (B) $y + 3x - 33 = 0$ (C) $y + x - 15 = 0$ (D) $y - 2x + 12 = 0$

26. Let S be the focus of the parabola $y^2 = 8x$ & let PQ be the common chord of the circle $x^2 + y^2 - 2x - 4y = 0$ and the given parabola. The area of the triangle PQS is **[JEE 2012]**

Paragraph for Question 27 and 28

Let PQ be a focal chord of the parabolas $y^2 = 4ax$. The tangents to the parabola at P and Q meet at a point lying on the line $y = 2x + a$, $a > 0$.

27. If chord PQ subtends an angle θ at the vertex of $y^2 = 4ax$, then $\tan\theta =$ **[JEE Adv 2013]**

(A) $\frac{2}{3}\sqrt{7}$ (B) $\frac{-2}{3}\sqrt{7}$ (C) $\frac{2}{3}\sqrt{5}$ (D) $\frac{-2}{3}\sqrt{5}$

28. Length of chord PQ is **[JEE Adv 2013]**

(A) 7a (B) 5a (C) 2a (D) 3a

29. A line L : $y = mx + 3$ meets y-axis at E(0,3) and the arc of the parabola $y^2 = 16x$, $0 < y < 6$ at the point $F(x_0, y_0)$. The tangent to the parabola at $F(x_0, y_0)$ intersects the y-axis at $G(0, y_1)$. The slope m of the line L is chosen such that the area of the triangle EFG has a local maximum.

Match List-I with List-II and select the correct answer using the code given below the lists.

	List-I		**List-II**
P.	m =	1.	$\frac{1}{2}$
Q.	Maximum area of ΔEFG is	2.	4
R.	y_0 =	3.	2
S.	y_1 =	4.	1

Codes :

	P	Q	R	S		P	Q	R	S
(A)	4	1	2	3	(B)	3	4	1	2
(C)	1	3	2	4	(D)	1	3	4	2

[JEE Adv 2014]

30. The common tangents to the circle $x^2 + y^2 = Z$ and the parabola $y^2 = 8x$ touch circle at the points P, Q and the parabola at the points R, S. Then the area of quadrilateral PQRS is **[JEE Adv 2014]**

(A) 3 (B) 6 (C) 9 (D) 15

Paragraph For Questions 31 and 32

Let a, r, s, t be nonzero real numbers. Let $P(at^2, 2at)$, Q, $R(ar^2, 2ar)$ and $S(as^2, 2as)$ be distinct points on the parabola $y^2 = 4ax$. Suppose that PQ is the focal chord and lines QR and PK are parallel, where K is the point (2a, 0).

31. The value of r is

(A) $-\frac{1}{t}$ (B) $\frac{t^2+1}{t}$ (C) $\frac{1}{t}$ (D) $\frac{t^2-1}{t}$

32. If st = 1, then the tangent at P and the normal at S to the parabola meet at a point whose ordinate is

(A) $\frac{(t^2+1)^2}{2t^3}$ (B) $\frac{a(t^2+1)^2}{2t^3}$ (C) $\frac{a(t^2+1)^2}{t^3}$ (D) $\frac{a(t^2+2)^2}{t^3}$

33. If the normals of the parabola $y^2 = 4x$ drawn at the end points of its latus rectum are tangents to the circle $(x-3)^2 + (y+2)^2 = r^2$, then the value of r^2 is **[JEE 2015]**

34. Let the curve C be the mirror image of the parabola $y^2 = 4x$ with respect to the line $x + y + 4 = 0$. If A and B are the points of intersection of C with the line $y = -5$, then the distance between A and B is **[JEE 2015]**

35. Let P and Q be distinct points on the parabola $y^2 = 2x$ such that a circle with PQ as diameter passes through the vertex O of the parabola. If P lies in the first quadrant and the area of the triangle ΔOPQ is $3\sqrt{2}$, then which of the following is(are) the coordinates of P ? **[JEE 2015]**

(A) $\left(4, 2\sqrt{2}\right)$ (B) $\left(9, 3\sqrt{2}\right)$ (C) $\left(\frac{1}{4}, -\frac{1}{\sqrt{2}}\right)$ (D) $\left(1, \sqrt{2}\right)$

36. The circle $C_1: x^2 + y^2 = 3$, with centre at O, intersects the parabola $x^2 = 2y$ at the point P in the first quadrant. Let the tangent to the circle C_1 at P touches other two circles C_2 and C_3 at R_2 and R_3, respectively. Suppose C_2 and C_3 have equal radii $2\sqrt{3}$ and centres Q_2 and Q_3, respectively. If Q_2 and Q_3 lie on the y-axis, then

(A) $Q_2Q_3 = 12$ (B) $R_2R_3 = 4\sqrt{6}$ **[JEE 2016]**

(C) area of the triangle OR_2R_3 is $6\sqrt{2}$ (D) area of the triangle PQ_2Q_3 is $4\sqrt{2}$

37. Let P be the point on the parabola $y^2 = 4x$ which is at the shortest distance from the center S of the circle $x^2 + y^2 - 4x - 16y + 64 = 0$. Let Q be the point on the circle dividing the line segment SP internally. Then- **[JEE 2016]**

(A) $SP = 2\sqrt{5}$

(B) $SQ : QP = \left(\sqrt{5}+1\right) : 2$

(C) the x-intercept of the normal to the parabola at P is 6

(D) the slope of the tangent to the circle at Q is $\frac{1}{2}$

38. If a chord, which is not a tangent, of the parabola $y^2 = 16x$ has the equation $2x + y = p$, and midpoint (h, k), then which of the following is(are) possible value(s) of p, h and k ? **[JEE 2017]**

(A) $p = 5, h = 4, k = -3$
(B) $p = -1, h = 1, k = -3$
(C) $p = -2, h = 2, k = -4$
(D) $p = 2, h = 3, k = -4$

39. Consider two straight lines, each of which is tangent to both the circle $x^2 + y^2 = \frac{1}{2}$ and the parabola $y^2 = 4x$. Let these lines intersect at the point Q. Consider the ellipse whose center is at the origin O(0, 0) and whose semi-major axis is OQ. If the length of the minor axis of this ellipse is $\sqrt{2}$, then the which of the following statement(s) is (are) TRUE ? **[JEE 2018]**

(A) For the ellipse, the eccentricity is $\frac{1}{\sqrt{2}}$ and the length of the latus rectum is 1

(B) For the ellipse, the eccentricity is $\frac{1}{2}$ and the length of the latus rectum is $\frac{1}{2}$

(C) The area of the region bounded by the ellipse between the lines $x = \frac{1}{\sqrt{2}}$ and $x = 1$ is $\frac{1}{4\sqrt{2}}(\pi - 2)$

(D) The area of the region bounded by the ellipse between the lines $x = \frac{1}{\sqrt{2}}$ and $x = 1$ is $\frac{1}{16}(\pi - 2)$

ANSWER KEY

EXERCISE-1

Que.	1	2	3	4	5	6	7	8	9	10
Ans.	C	C	B	A	B	D	D	A	D	B
Que.	11	12	13	14	15					
Ans.	B	D	C	C	A					

EXERCISE-2

Que.	1	2	3	4	5	6	7	8	9	10
Ans.	B	AB	AB	D	C	A	A	B	AB	ABCD
Que.	11	12	13	14	15					
Ans.	ABC	CD	CD	BC	BD					

- ***Match the Column*** **16.** (A) → (r); (B) → (s); (C) → (p); (D) → (q)
- ***Comprehension Based Questions*** **17.** C **18.** B **19.** A

EXERCISE-3

1. $y^2 = 4(x + 4)$

2. Vertex $\equiv \left(\frac{3}{2}, \frac{-11}{8}\right)$, focus $\equiv \left(\frac{3}{2}, \frac{-15}{8}\right)$, axis : $x = \frac{3}{2}$, directrix : $y = -\frac{7}{8}$, latus rectum $= 2$

3. $(2x - y - 3)^2 = -20(x + 2y - 4)$, axis : $2x - y - 3 = 0$. latus rectum $= 4\sqrt{5}$.

4. $y^2 = 2ax$ **5.** $8\sqrt{3}$

6. $\alpha \in [\pi/2, 5\pi/6] \cup [\pi, 3\pi/2]$ **7.** $\frac{4a^3}{p^2}$ **8.** $(-\infty, -4] \cup [4, \infty)$

9. $2x - y + 2 = 0$, $(1, 4)$; $x + 2y + 16 = 0$, $(16, -16)$

10. $3x - 2y + 4 = 0$; $x - y + 3 = 0$

EXERCISE-4

Que.	1	2	3	4	5	6	7	8	9	10
Ans	C	D	B	A	D	D	B	C	B	C
Que.	11	12	13	14	15	16	17	18	19	20
Ans	B	B	A	C	D	C	C	C	A	D
Que.	21	22	23	24	25	26	27			
Ans	A	C	A	C	A	C	C			

EXERCISE-5

1. (a) C (b) B **2.** (a) C (b) D **3.** (C) **4.** (D) **5.** (A) **6.** 2 **7.** (C)
8. $(x + 1)(y - 1)^2 + 4 = 0$ **9.** (C) **10.** (D) **11.** (A, B) **12.** (i) A, (ii) B, (iii)D, (iv)C
13. (A) **14.** (C) **15.** (C) **16.** (C) **17.** (B) **18.** (D) **19.** (A)
20. (B) **21.** (A,D) **22.** (C, D) **23.** (2) **24.** (C) **25.** (A,B,D) **26.** (4)
27. (D) **28.** (B) **29.** (A) **30.** (D) **31.** (D) **32.** (B) **33.** (2)
34. (4) **35.** (A,D) **36.** (ABC) **37.** (A,C,D) **38.** (D) **39.** (A,C)

ELLIPSE

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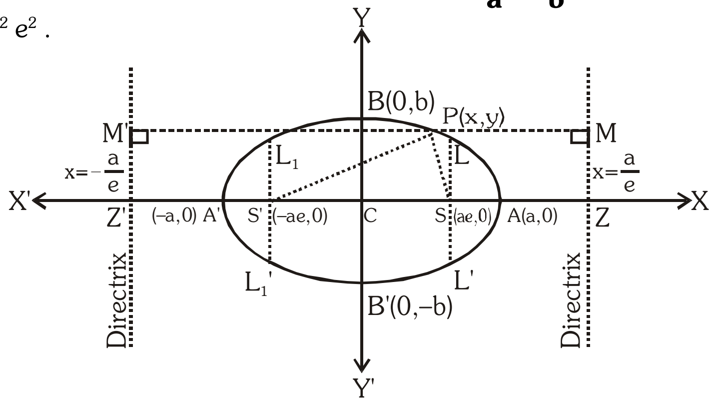
ELLIPSE

1.0 STANDARD EQUATION & DEFINITION

Standard equation of an ellipse referred to its principal axes along the co-ordinate axes is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. where $a > b$ & $b^2 = a^2(1-e^2) \Rightarrow a^2 - b^2 = a^2e^2$.

where e = eccentricity $(0 < e < 1)$.

FOCI – $S \equiv (ae, 0)$ & $S' \equiv (-ae, 0)$.



(a) ***Equation of directrices***

$$x = \frac{a}{e} \;\;\&\;\; x = -\frac{a}{e}.$$

(b) ***Vertices***

$A' \equiv (-a, 0)$ & $A \equiv (a, 0)$.

(c) ***Major axis –*** The line segment A' A in which the foci S' & S lie is of length 2a & is called the ***major axis*** (a > b) of the ellipse. Point of intersection of major axis with directrix is called ***the foot of the directrix (z)*** $\left(\pm\frac{a}{e}, 0\right)$.

(d) ***Minor Axis –*** The y-axis intersects the ellipse in the points $B' \equiv (0,-b)$ & $B \equiv (0, b)$. The line segment B'B of length 2b (b < a) is called the ***Minor Axis*** of the ellipse.

(e) ***Principal Axes –*** The major & minor axis together are called ***Principal Axes*** of the ellipse.

(f) ***Centre :*** The point which bisects every chord of the conic drawn through it is called the ***centre*** of the conic. $C \equiv (0,0)$ the origin is the centre of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

(g) ***Diameter –*** A chord of the conic which passes through the centre is called a ***diameter*** of the conic.

(h) ***Focal Chord –*** A chord which passes through a focus is called a ***focal chord.***

(i) ***Double Ordinate –*** A chord perpendicular to the major axis is called a ***double ordinate.***

(j) ***Latus Rectum –*** The focal chord perpendicular to the major axis is called the ***latus rectum***.

(i) Length of latus rectum (LL') $= \frac{2b^2}{a} = \frac{(\text{minor axis})^2}{\text{major axis}} = 2a(1-e^2)$

(ii) Equation of latus rectum : $x = \pm ae$.

(iii) Ends of the latus rectum are $L\left(ae, \frac{b^2}{a}\right), L'\left(ae, -\frac{b^2}{a}\right), L_1\left(-ae, \frac{b^2}{a}\right)$ *and* $L_1'\left(-ae, -\frac{b^2}{a}\right)$.

(k) ***Focal radii –*** $SP = a - ex$ & $S'P = a + ex \quad \Rightarrow \quad SP + S'P = 2a =$ ***Major axis.***

(l) ***Eccentricity –*** $e = \sqrt{1-\frac{b^2}{a^2}}$

NOTE

(i) The sum of the focal distances of any point on the ellipse is equal to the major Axis. Hence distance of focus from the extremity of a minor axis is equal to semi major axis. ***i.e BS = CA.***

(ii) If the equation of the ellipse is given as $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ & nothing is mentioned, then the rule is to assume that a > b.

Illustrations

Illustration 1. If LR of an ellipse is half of its minor axis, then its eccentricity is -

(A) $\frac{3}{2}$ (B) $\frac{2}{3}$ (C) $\frac{\sqrt{3}}{2}$ (D) $\frac{\sqrt{2}}{3}$

Solution As given $\frac{2b^2}{a} = b$ $\Rightarrow$ $2b = a$ $\Rightarrow$ $4b^2 = a^2$

$\Rightarrow$ $4a^2(1 - e^2) = a^2$ $\Rightarrow$ $1 - e^2 = 1/4$ $\therefore$ $e = \sqrt{3}/2$ **Ans. (C)**

Illustration 2. Find the equation of the ellipse whose foci are (2, 3), (–2, 3) and whose semi minor axis is of length $\sqrt{5}$.

Solution Here S is (2, 3) & S' is (–2, 3) and $b = \sqrt{5}$ $\Rightarrow$ $SS' = 4 = 2ae \Rightarrow ae = 2$

but $b^2 = a^2(1 - e^2)$ $\Rightarrow$ $5 = a^2 - 4$ $\Rightarrow$ $a = 3$.

Hence the equation to major axis is $y = 3$

Centre of ellipse is midpoint of SS' i.e. (0, 3)

$\therefore$ Equation to ellipse is $\frac{x^2}{a^2} + \frac{(y-3)^2}{b^2} = 1$ or $\frac{x^2}{9} + \frac{(y-3)^2}{5} = 1$ **Ans.**

Illustration 3. Find the equation of the ellipse having centre at (1, 2), one focus at (6, 2) and passing through the point (4, 6).

Solution With centre at (1, 2), the equation of the ellipse is $\frac{(x-1)^2}{a^2} + \frac{(y-2)^2}{b^2} = 1$. It passes through the point (4, 6)

$\Rightarrow$ $\frac{9}{a^2} + \frac{16}{b^2} = 1$ .... (i)

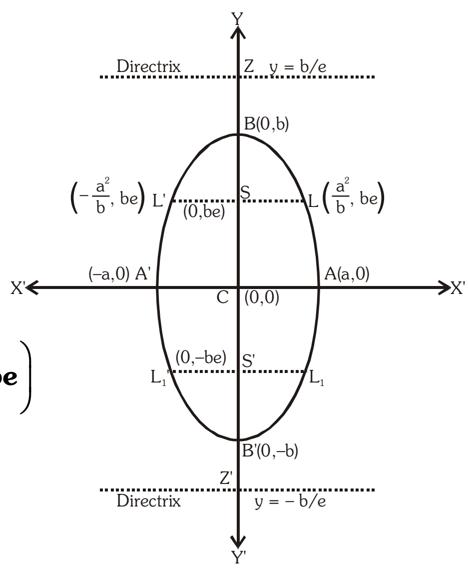
Distance between the focus and the centre $= (6 - 1) = 5 = ae$

$\Rightarrow$ $b^2 = a^2 - a^2e^2 = a^2 - 25$ .... (ii)

Solving for a^2 and b^2 from the equations (i) and (ii), we get $a^2 = 45$ and $b^2 = 20$.

Hence the equation of the ellipse is $\frac{(x-1)^2}{45} + \frac{(y-2)^2}{20} = 1$ **Ans.**

2.0 ANOTHER FORM OF ELLIPSE – $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, (a <b)



(a) ***AA' = Minor axis = 2a***

(b) ***BB' = Major axis = 2b***

(c) $a^2 = b^2(1 - e^2)$

(d) ***Latus rectum*** $LL' = L_1L_1' = \frac{2a^2}{b}$, ***equation*** $y = \pm be$

(e) ***Ends of the latus rectum are :***

$$L\left(\frac{a^2}{b}, be\right), L'\left(-\frac{a^2}{b}, be\right), L_1\left(\frac{a^2}{b}, -be\right), L_1'\left(-\frac{a^2}{b}, -be\right)$$

(f) ***Equation of directrix*** $y = \pm b/e$

(g) ***Eccentricity :*** $e = \sqrt{1 - \frac{a^2}{b^2}}$

Illustrations

Illustration 4. The equation of the ellipse with respect to coordinate axes whose minor axis is equal to the distance between its foci and whose LR = 10, will be-

(A) $2x^2 + y^2 = 100$ (B) $x^2 + 2y^2 = 100$ (C) $2x^2 + 3y^2 = 80$ (D) none of these

Solution When $a > b$

As given $2b = 2ae \Rightarrow b = ae$ ...... (i)

Also $\frac{2b^2}{a} = 10 \Rightarrow b^2 = 5a$ ...... (ii)

Now since $b^2 = a^2 - a^2e^2 \Rightarrow b^2 = a^2 - b^2$ [From (i)]

$\Rightarrow 2b^2 = a^2$ ...... (iii)

(ii), (iii) $\Rightarrow a^2 = 100, b^2 = 50$

Hence equation of the ellipse will be $\frac{x^2}{100} + \frac{y^2}{50} = 1 \Rightarrow x^2 + 2y^2 = 100$

Similarly when $a < b$ then required ellipse is $2x^2 + y^2 = 100$ **Ans. (A, B)**

BEGINNER'S BOX-1

1. If LR of an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $(a < b)$ is half of its major axis, then find its eccentricity.

2. Find the equation of the ellipse whose foci are (4, 6) & (16, 6) and whose semi-minor axis is 4.

3. Find the eccentricity, foci and the length of the latus-rectum of the ellipse $x^2 + 4y^2 + 8y - 2x + 1 = 0$.

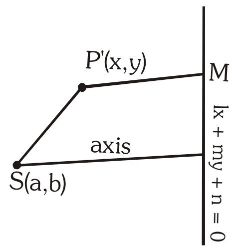
4. The foci of an ellipse are $(0, \pm 2)$ and its eccentricity is $\frac{1}{\sqrt{2}}$. Find its equation

5. Find the centre, the length of the axes, eccentricity and the foci of ellipse $12x^2 + 4y^2 + 24x - 16y + 25 = 0$

3.0 GENERAL EQUATION OF AN ELLIPSE

Let (a, b) be the focus S, and $lx + my + n = 0$ is the equation of directrix.

Let P(x, y) be any point on the ellipse. Then by definition.



$\Rightarrow$ ***SP = e PM*** (e is the eccentricity) $\Rightarrow (x-a)^2 + (y-b)^2 = e^2\frac{(lx+my+n)^2}{(l^2+m^2)}$

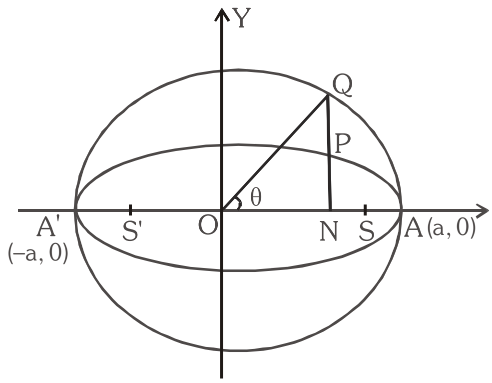
$\Rightarrow (l^2 + m^2)\{(x-a)^2 + (y-b)^2\} = e^2\{lx + my + n\}^2$

4.0 POSITION OF A POINT W.R.T. AN ELLIPSE

The point $P(x_1, y_1)$ lies ***outside***, ***inside*** or ***on*** the ellipse according as ; $\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1 > < or = 0$.

5.0 AUXILIARY CIRCLE/ECCENTRIC ANGLE

A circle described on major axis as diameter is called the ***auxiliary circle.*** Let Q be a point on the auxiliary circle $x^2 + y^2 = a^2$ such that QP produced is perpendicular to the x-axis then P & Q are called as the CORRESPONDING POINTS on the ellipse & the auxiliary circle respectively.'θ' is called the ***ECCENTRIC ANGLE*** of the point P on the ellipse $(0 \le \theta < 2\pi)$.



Note that $= \dfrac{\text{Semi minor axis}}{\text{Semi major axis}} = \dfrac{l(PN)}{l(QN)} = \dfrac{b}{a}$

Hence "If from each point of a circle perpendiculars are drawn upon a fixed diameter then the locus of the points dividing these perpendiculars in a given ratio is an ellipse of which the given circle is the auxiliary circle".

6.0 PARAMETRIC REPRESENTATION

The equations $x = a\cos\theta$ & $y = b\sin\theta$ together represent the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

where θ is a parameter (eccentric angle).

Note that if $P(\theta) \equiv (a\cos\theta, b\sin\theta)$ is on the ellipse then ; $Q(\theta) \equiv (a\cos\theta, a\sin\theta)$ is on the auxiliary circle.

7.0 LINE AND AN ELLIPSE

The line $y = mx + c$ meets the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ in two points real, coincident or imaginary according as c^2 is $<$ $=$ or $>$ $a^2m^2 + b^2$.

Hence $y = mx + c$ is tangent to the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ if $c^2 = a^2m^2 + b^2$.

The equation to the chord of the ellipse joining two points with eccentric angles α & β is given by

$$\frac{x}{a}\cos\frac{\alpha+\beta}{2} + \frac{y}{b}\sin\frac{\alpha+\beta}{2} = \cos\frac{\alpha-\beta}{2}.$$

Illustrations

Illustration 5. For what value of λ does the line $y = x + \lambda$ touches the ellipse $9x^2 + 16y^2 = 144$.

Solution $\because$ Equation of ellipse is $9x^2 + 16y^2 = 144$ or $\dfrac{x^2}{16} + \dfrac{y^2}{9} = 1$

Comparing this with $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ then we get $a^2 = 16$ and $b^2 = 9$

and comparing the line $y = x + \lambda$ with $y = mx + c$ $\quad\therefore\quad m = 1$ and $c = \lambda$

If the line $y = x + \lambda$ touches the ellipse $9x^2 + 16y^2 = 144$, then $c^2 = a^2m^2 + b^2$

$\Rightarrow \quad \lambda^2 = 16 \times 1^2 + 9 \quad \Rightarrow \quad \lambda^2 = 25 \quad\quad \therefore \quad \lambda = \pm 5$ ***Ans.***

Illustration 6. If α, β are eccentric angles of end points of a focal chord of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, then $\tan\alpha/2.\tan\beta/2$ is equal to -

(A) $\dfrac{e-1}{e+1}$ (B) $\dfrac{1-e}{1+e}$ (C) $\dfrac{e+1}{e-1}$ (D) $\dfrac{e-1}{e+1}$

Solution Equation of line joining points 'α' and 'β' is $\dfrac{x}{a}\cos\dfrac{\alpha+\beta}{2} + \dfrac{y}{b}\sin\dfrac{\alpha+\beta}{2} = \cos\dfrac{\alpha-\beta}{2}$

If it is a focal chord, then it passes through focus (ae, 0), so $e\cos\dfrac{\alpha+\beta}{2} = \cos\dfrac{\alpha-\beta}{2}$

$$\Rightarrow \quad \frac{\cos\frac{\alpha-\beta}{2}}{\cos\frac{\alpha+\beta}{2}} = \frac{e}{1} \quad \Rightarrow \quad \frac{\cos\frac{\alpha-\beta}{2} - \cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2} + \cos\frac{\alpha+\beta}{2}} = \frac{e-1}{e+1}$$

$\Rightarrow \quad \dfrac{2\sin\alpha/2\ \ \sin\beta/2}{2\cos\alpha/2\ \ \cos\beta/2} = \dfrac{e-1}{e+1} \quad \Rightarrow \quad \tan\dfrac{\alpha}{2}\tan\dfrac{\beta}{2} = \dfrac{e-1}{e+1}$

using (–ae, 0) , we get $\tan\dfrac{\alpha}{2}\tan\dfrac{\beta}{2} = \dfrac{e+1}{e-1}$ ***Ans. (A,C)***

8.0 TANGENT TO THE ELLIPSE $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$:

(a) ***Point form –*** Equation of tangent to the given ellipse at its point (x_1, y_1) is $\dfrac{xx_1}{a^2}+\dfrac{yy_1}{b^2}=1$

Note : For general ellipse replace x^2 by (xx_1), y^2 by (yy_1), $2x$ by $(x + x_1)$, $2y$ by $(y + y_1)$, $2xy$ by $(xy_1 + yx_1)$ & c by (c).

(b) ***Slope form –*** Equation of tangent to the given ellipse whose slope is 'm', is $y = mx \pm \sqrt{a^2m^2+b^2}$

Point of contact are $\left(\dfrac{\mp a^2m}{\sqrt{a^2m^2+b^2}}, \dfrac{\pm b^2}{\sqrt{a^2m^2+b^2}}\right)$

Note that there are two tangents to the ellipse having the same m, i.e. there are two tangents parallel to any given direction.

(c) ***Parametric form –*** Equation of tangent to the given ellipse at its point $(a\cos\theta, b\sin\theta)$, is

$$\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1$$

NOTE

(i) The eccentric angles of point of contact of two parallel tangents differ by π.

(ii) Point of intersection of the tangents at the point α & β is $\left(a\dfrac{\cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}, b\dfrac{\sin\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}\right)$

Illustrations

Illustration 7. Find the equations of the tangents to the ellipse $3x^2 + 4y^2 = 12$ which are perpendicular to the line $y + 2x = 4$.

Solution Let m be the slope of the tangent, since the tangent is perpendicular to the line $y + 2x = 4$.

$\therefore \quad mx - 2 = -1 \Rightarrow m = \dfrac{1}{2}$

Since $3x^2 + 4y^2 = 12$ or $\dfrac{x^2}{4}+\dfrac{y^2}{3}=1$

Comparing this with $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$

$\therefore \quad a^2 = 4$ and $b^2 = 3$

So the equation of the tangent are $y = \dfrac{1}{2}x \pm \sqrt{4\times\dfrac{1}{4}+3}$

$\Rightarrow \quad y = \dfrac{1}{2}x \pm 2$ or $x - 2y \pm 4 = 0$. ***Ans.***

Illustration 8. The tangent at a point P on an ellipse intersects the major axis in T and N is the foot of the perpendicular from P to the same axis. Show that the circle drawn on NT as diameter intersects the auxiliary circle orthogonally.

Solution Let the equation of the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Let P(acosθ, bsinθ) be a point on the ellipse. The equation of the tangent at P is $\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1$. It meets the major axis at T ≡ (a secθ, 0).

The coordinates of N are (a cosθ, 0). The equation of the circle with NT as its diameter is $(x-a\sec\theta)(x-a\cos\theta)+y^2=0$.

$\Rightarrow \quad x^2+y^2-ax(\sec\theta+\cos\theta)+a^2=0$

It cuts the auxiliary circle $x^2+y^2-a^2=0$ orthogonally if

$2g\,.\,0+2f\,.\,0=a^2-a^2=0$, which is true. ***Ans.***

BEGINNER'S BOX-2

1. Find the position of the point (4, 3) relative to the ellipse $2x^2+9y^2=113$.

2. A tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, (a > b) having slope –1 intersects the axis of x & y in point A & B respectively. If O is the origin then find the area of triangle OAB.

3. Find the condition for the line x cosθ + ysinθ = P to be a tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

4. Find the equation of the tangents to the ellipse $9x^2+16y^2=144$ which are parallel to the line x + 3y + k = 0.

5. Find the equation of the tangent to the ellipse $7x^2+8y^2=100$ at the point (2, –3).

9.0 NORMAL TO THE ELLIPSE $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

(a) ***Point form –*** Equation of the normal to the given ellipse at (x_1, y_1) is $\frac{a^2x}{x_1}-\frac{b^2y}{y_1}=a^2-b^2=a^2e^2$.

(b) ***Slope form –*** Equation of a normal to the given ellipse whose slope is 'm' is $y=mx\mp\frac{(a^2-b^2)m}{\sqrt{a^2+b^2m^2}}$.

(c) ***Parametric form –*** Equation of the normal to the given ellipse at the point ***(acosθ, bsinθ)*** is ***axsecθ – bycosecθ = (a² – b²)***.

Illustrations

Illustration 9. Find the condition that the line ℓx + my = n may be a normal to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Solution Equation of normal to the given ellipse at (a cos θ, b sin θ) is $\frac{ax}{\cos\theta}-\frac{by}{\sin\theta}=a^2-b^2$ ...(i)

If the line ℓx + my = n is also normal to the ellipse then there must be a value of θ for which line (i) and line ℓx + my = n are identical. For that value of θ we have

$$\frac{\ell}{\left(\frac{a}{\cos\theta}\right)}=\frac{m}{-\left(\frac{b}{\sin\theta}\right)}=\frac{n}{(a^2-b^2)} \quad \text{or} \quad \cos\theta=\frac{an}{\ell(a^2-b^2)} \qquad \text{...... (iii)}$$

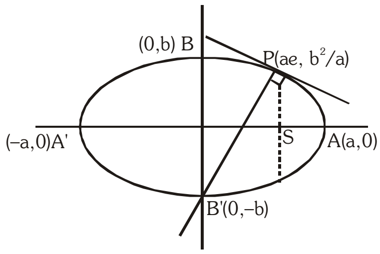
and $$\sin\theta=\frac{-bn}{m(a^2-b^2)} \qquad \text{...... (iv)}$$

Squaring and adding (iii) and (iv), we get $1=\frac{n^2}{(a^2-b^2)^2}\left(\frac{a^2}{\ell^2}+\frac{b^2}{m^2}\right)$ which is the required condition.

Illustration 10. If the normal at an end of a latus-rectum of an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ passes through one extremity of the minor axis, show that the eccentricity of the ellipse is given by $e=\sqrt{\frac{\sqrt{5}-1}{2}}$

Solution The co-ordinates of an end of the latus-rectum are (ae, b^2/a).

The equation of normal at P(ae, b^2/a) is



$\frac{a^2x}{ae}-\frac{b^2(y)}{b^2/a}=a^2-b^2$ or $\frac{ax}{e}-ay=a^2-b^2$

It passes through one extremity of the minor axis whose co-ordinates are (0, –b)

$\therefore$ $0+ab=a^2-b^2$ $\Rightarrow$ $(a^2b^2)=(a^2-b^2)^2$

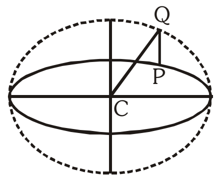
$\Rightarrow$ $a^2.a^2(1-e^2)=(a^2e^2)^2$ $\Rightarrow$ $1-e^2=e^4$

$\Rightarrow$ $e^4+e^2-1=0$ $\Rightarrow$ $(e^2)^2+e^2-1=0$

$\therefore$ $e^2=\frac{-1\pm\sqrt{1+4}}{2}$ $\Rightarrow$ $e=\sqrt{\frac{\sqrt{5}-1}{2}}$ (taking positive sign) **Ans.**

Illustration 11. P and Q are corresponding points on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and the auxiliary circles respectively. The normal at P to the ellipse meets CQ in R, where C is the centre of the ellipse. Prove that CR = a + b

Solution Let $P\equiv(a\cos\theta, b\sin\theta)$



$\therefore$ $Q\equiv(a\cos\theta, a\sin\theta)$

Equation of normal at P is

$(a\sec\theta)x-(b\,\mathrm{cosec}\,\theta)y=a^2-b^2$ .... (i)

equation of CQ is $y=\tan\theta\,.\,x$ .... (ii)

Solving equation (i) & (ii), we get $(a-b)x=(a^2-b^2)\cos\theta$

$x=(a+b)\cos\theta$, & $y=(a+b)\sin\theta$

$\therefore$ $R\equiv((a+b)\cos\theta, (a+b)\sin\theta)$

$\therefore$ CR = a + b **Ans.**

10.0 CHORD OF CONTACT

If PA and PB be the tangents from point $P(x_1,y_1)$ to the ellipse $\mathbf{\frac{x^2}{a^2}+\frac{y^2}{b^2}=1}$.

The equation of the chord of contact AB is $\mathbf{\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1}$ ***or T = 0 (at x_1,y_1)***.

Illustrations

Illustration 12. If tangents to the parabola $y^2=4ax$ intersect the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at A and B, the find the locus of point of intersection of tangents at A and B.

Solution Let $P\equiv(h, k)$ be the point of intersection of tangents at A & B

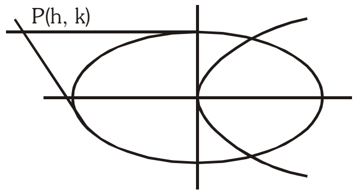
$\therefore$ Equation of chord of contact AB is $\frac{xh}{a^2} + \frac{yk}{b^2} = 1$ .... (i)

which touches the parabola.

Equation of tangent to parabola $y^2 = 4ax$ is $y = mx + \frac{a}{m}$

$\Rightarrow \quad mx - y = -\frac{a}{m}$ .... (ii)

Equation (i) & (ii) as must be same



$$\therefore \quad \frac{m}{\left(\frac{h}{a^2}\right)} = \frac{-1}{\left(\frac{k}{b^2}\right)} = \frac{-\frac{a}{m}}{1} \Rightarrow \quad m = -\frac{h\,b^2}{k\,a^2} \ \& \ m = \frac{ak}{b^2}$$

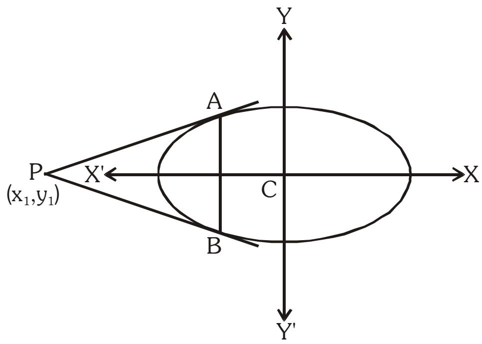
$$\therefore \quad -\frac{hb^2}{ka^2} = \frac{ak}{b^2} \quad \Rightarrow \quad \text{locus of P is } y^2 = -\frac{b^4}{a^3}.x$$ ***Ans.***

11.0 PAIR OF TANGENTS

If $P(x_1,y_1)$ be any point lies outside the ellipse $\frac{\mathbf{x^2}}{\mathbf{a^2}} + \frac{\mathbf{y^2}}{\mathbf{b^2}} = \mathbf{1}$,

and a pair of tangents PA, PB can be drawn to it from P.

Then the equation of pair of tangents of PA and PB is $SS_1 = T^2$



where $\mathbf{S_1} = \frac{\mathbf{x_1^2}}{\mathbf{a^2}} + \frac{\mathbf{y_1^2}}{\mathbf{b^2}} - \mathbf{1}$, $\mathbf{T} = \frac{\mathbf{xx_1}}{\mathbf{a^2}} + \frac{\mathbf{yy_1}}{\mathbf{b^2}} - \mathbf{1}$

i.e. $$\left(\frac{\mathbf{x^2}}{\mathbf{a^2}} + \frac{\mathbf{y^2}}{\mathbf{b^2}} - \mathbf{1}\right)\left(\frac{\mathbf{x_1^2}}{\mathbf{a^2}} + \frac{\mathbf{y_1^2}}{\mathbf{b^2}} - \mathbf{1}\right) = \left(\frac{\mathbf{xx_1}}{\mathbf{a^2}} + \frac{\mathbf{yy_1}}{\mathbf{b^2}} - \mathbf{1}\right)^2$$

12.0 DIRECTOR CIRCLE

Locus of the point of intersection of the tangents which meet at right angles is called the ***Director Circle.*** The equation to this locus is $\boldsymbol{x^2 + y^2 = a^2 + b^2}$ i.e. a circle whose centre is the centre of the ellipse & whose radius is the length of the line joining the ends of the major & minor axis.

Illustrations

Illustration 13. A tangent to the ellipse $x^2 + 4y^2 = 4$ meets the ellipse $x^2 + 2y^2 = 6$ at P and Q. Prove that the tangents at P and Q of the ellipse $x^2 + 2y^2 = 6$ are at right angles.

Solution Given ellipse are $\frac{x^2}{4} + \frac{y^2}{1} = 1$ .... (i)

and, $\frac{x^2}{6} + \frac{y^2}{3} = 1$ .... (ii)

any tangent to (i) is $\frac{x\cos\theta}{2} + \frac{y\sin\theta}{1} = 1$ .... (iii)

It cuts (ii) at P and Q, and suppose tangent at P and Q meet at (h, k) Then equation of chord of contact of (h, k) with respect to ellipse (ii) is $\frac{hx}{6} + \frac{ky}{3} = 1$ .... (iv) comparing (iii) and (iv), we get $\frac{\cos\theta}{h/3} = \frac{\sin\theta}{k/3} = 1 \Rightarrow \cos\theta = \frac{h}{3}$ and $\sin\theta = \frac{k}{3} \Rightarrow h^2 + k^2 = 9$

locus of the point (h, k) is $x^2 + y^2 = 9 \Rightarrow x^2 + y^2 = 6 + 3 = a^2 + b^2$

i.e. director circle of second ellipse. Hence the tangents are at right angles.

13.0 EQUATION OF CHORD WITH MID POINT (x_1, y_1)

The equation of the chord of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, whose mid-point be (x_1, y_1) is $T = S_1$

where $T = \frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1$, $S_1 = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1$, i.e. $\left(\frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1\right) = \left(\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1\right)$

Illustrations

Illustration 14. Find the locus of the mid-point of focal chords of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

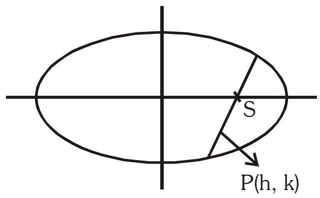
Solution Let $P \equiv (h, k)$ be the mid-point

$\therefore$ Equation of chord whose mid-point is given $\frac{xh}{a^2} + \frac{yk}{b^2} - 1 = \frac{h^2}{a^2} + \frac{k^2}{b^2} - 1$

since it is a focal chord,

$\therefore$ It passes through focus, either (ae, 0) or (–ae, 0)

If it passes through (ae, 0)



$\therefore$ locus is $\frac{ex}{a} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$

If it passes through (–ae, 0)

$\therefore$ locus is $-\frac{ex}{a} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ ***Ans.***

BEGINNER'S BOX-3

1. Find the equation of the normal to the ellipse $9x^2 + 16y^2 = 288$ at the point (4, 3)

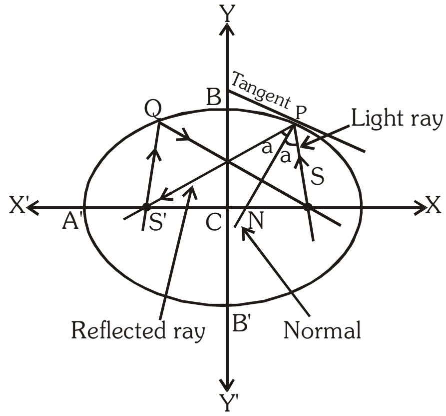
2. Let P be a variable point on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with foci F_1 and F_2. If A is the area of the triangle PF_1F_2, then find maximum value of A.

3. Show that for all real values of 't' the line $2tx + y\sqrt{1-t^2} = 1$ touches a fixed ellipse. Find the eccentricity of the ellipse.

4. Find the equation of chord of contact to the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$ at the point (1, 3).

5. Find the equation of chord of the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$ whose mid point be (–1, 1).

14.0 PROPERTIES OF ELLIPSE



Referring to an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

(a) If P be any point on the ellipse with S & S' as its foci then $\ell(SP)+\ell(S'P)=2a$.

(b) The tangent & normal at a point P on the ellipse bisect the external & internal angles between the focal distances of P. This refers to the well known reflection property of the ellipse which states that rays from one focus are reflected through other focus & vice versa .

(c) ***The product of the length's of the perpendicular segments from the foci on any tangent to the*** ellipse is b^2 and the feet of these perpendiculars lie on its auxiliary circle and the tangents at these feet to the auxiliary circle meet on the ordinate of P and that the locus of their point of intersection is a similar ellipse as that of the original one.

(d) The portion of the tangent to an ellipse between the point of contact & the directrix subtends a ***right angle*** at the corresponding focus.

(e) If the normal at any point P on the ellipse with centre C meet the major & minor axes in G & g respectively, & if CF be perpendicular upon this normal, then

(i) $PF \cdot PG = b^2$ *(ii)* $PF \cdot Pg = a^2$

(iii) $PG \cdot Pg = SP \cdot S'P$ *(iv)* $CG \cdot CT = CS^2$

(v) locus of the mid point of Gg is another ellipse having the same eccentricity as that of the original ellipse.

[where S and S' are the focii of the ellipse and T is the point where tangent at P meet the major axis]

(f) Atmost four normals & two tangents can be drawn from any point to an ellipse.

(g) The circle on any focal distance as diameter touches the auxiliary circle.

(h) Perpendiculars from the centre upon all chords which join the ends of any perpendicular diameters of the ellipse are of constant length.

(i) If the tangent at the point P of a standard ellipse meets the axes in T and t and CY is the perpendicular on it from the centre then,

(a) $Tt \cdot PY = a^2 - b^2$ ***and*** *(b)* ***least value of*** Tt ***is*** $a + b$.

BEGINNER'S BOX-4

1. A man running round a racecourse note that the sum of the distance of two flag-posts from him is always 20 meters and distance between the flag-posts is 16 meters. Find the area of the path be encloses in square meters

2. If chord of contact of the tangent drawn from the point (α, β) to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ touches the circle $x^2 + y^2 = k^2$, then find the locus of the point (α, β).

3. A point moves so that the sum of the squares of its distances from two intersecting straight lines is constant. Prove that its locus is an ellipse.

SOME WORKED OUT ILLUSTRATIONS

Illustration 1. Find the condition on 'a' and 'b' for which two distinct chords of the ellipse $\frac{x^2}{2a^2}+\frac{y^2}{2b^2}=1$ passing through (a, –b) are bisected by the line x + y = b.

Solution Let (t, b – t) be a point on the line x + y = b.

Then equation of chord whose mid point (t, b – t) is

$$\frac{tx}{2a^2}+\frac{y(b-t)}{2b^2}-1=\frac{t^2}{2a^2}+\frac{(b-t)^2}{2b^2}-1 \qquad \text{.......... (i)}$$

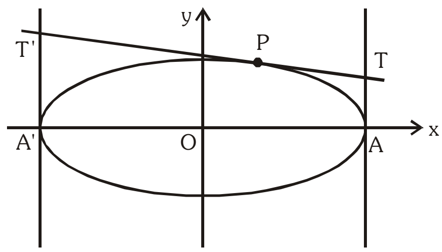
(a, –b) lies on (i) then $\frac{ta}{2a^2}-\frac{b(b-t)}{2b^2}=\frac{t^2}{2a^2}+\frac{(b-t)^2}{2b^2}$ $\Rightarrow$ $t^2(a^2+b^2)-ab(3a+b)t+2a^2b^2=0$

Since t is real $B^2-4AC\geq 0 \Rightarrow$ $a^2b^2(3a+b)^2-4(a^2+b^2)2a^2b^2\geq 0$

$\Rightarrow$ $a^2+6ab-7b^2\geq 0$ $\Rightarrow$ $a^2+6ab\geq 7b^2$, which is the required condition.

Illustration 2. Any tangent to an ellipse is cut by the tangents at the ends of the major axis in T and T '. Prove that circle on TT' as diameter passes through foci.

Solution Let ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$



and let P(acosϕ, bsinϕ) be any point on this ellipse

$\therefore$ Equation of tangent at P(acosϕ, bsinϕ) is

$$\frac{x}{a}\cos\phi+\frac{y}{b}\sin\phi=1 \qquad \text{.....(i)}$$

The two tangents drawn at the ends of the major axis are x = a and x = –a

Solving (i) and x = a we get $T=\left\{a,\frac{b(1-\cos\phi)}{\sin\phi}\right\}\equiv\left\{a,b\tan\left(\frac{\phi}{2}\right)\right\}$

and solving (i) and x = – a we get $T'=\left\{-a,\frac{b(1+\cos\phi)}{\sin\phi}\right\}\equiv\left\{-a,b\cot\left(\frac{\phi}{2}\right)\right\}$

Equation of circle on TT' as diameter is $(x-a)(x+a)+(y-b\tan(\phi/2))(y-b\cot(\phi/2))=0$

or $x^2+y^2-by(\tan(\phi/2)+\cot(\phi/2))-a^2+b^2=0$ .... (ii)

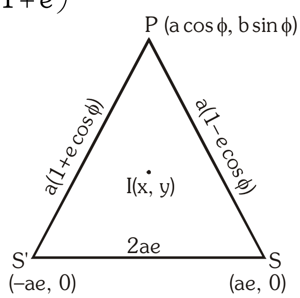
Now put x = ± ae and y = 0 in LHS of (ii), we get

$a^2e^2+0-0-a^2+b^2=a^2-b^2-a^2+b^2=0=$ RHS

Hence foci lie on this circle

Illustration 3. A variable point P on an ellipse of eccentricity e, is joined to its foci S, S'. Prove that the locus of the incentre of the triangle PSS' is an ellipse whose eccentricity is $\sqrt{\left(\frac{2e}{1+e}\right)}$.

Solution Let the given ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$



Let the co-ordinates of P are (a cosϕ, b sinϕ)

By hypothesis

$b^2=a^2(1-e^2)$ and S(ae, 0), S'(–ae, 0)

$\therefore$ SP = focal distance of the point P = a – ae cos ϕ

and S'P = a + ae cos ϕ

Also SS' = 2ae

If (x, y) be the incentre of the ΔPSS' then

$$\therefore \quad x = \frac{(2ae)a\cos\phi + a(1-e\cos\phi)(-ae) + a(1+e\cos\phi)ae}{2ae + a(1-e\cos\phi) + a(1+e\cos\phi)}$$

$$x = ae\cos\phi \qquad \ldots (i)$$

$$y = \frac{2ae(b\sin\phi) + a(1+e\cos\phi).0 + a(1-e\cos\phi).0}{2ae + a(1-e\cos\phi) + a(1+e\cos\phi)}$$

$$\Rightarrow \quad y = \frac{eb\sin\phi}{(e+1)} \qquad \ldots (ii)$$

Eliminating ϕ from equations (i) and (ii),

we get $\dfrac{x^2}{a^2e^2} + \dfrac{y^2}{\left[\dfrac{be}{e+1}\right]^2} = 1$

which represents an ellipse.

Let e_1 be its eccentricity.

$$\therefore \quad \frac{b^2e^2}{(e+1)^2} = a^2e^2(1-e_1^2)$$

$$\Rightarrow \quad e_1^2 = 1 - \frac{b^2}{a^2(e+1)^2}$$

$$= 1 - \frac{1-e^2}{(e+1)^2} = 1 - \frac{1-e}{1+e} = \frac{2e}{1+e}$$

$$\Rightarrow \quad e_1 = \sqrt{\left(\frac{2e}{1+e}\right)}$$

ANSWERS

BEGINNER'S BOX-1

1. $e=\frac{1}{\sqrt{2}}$ **2.** $\frac{(x-10)^2}{52}+\frac{(y-6)^2}{16}=1$ **3.** $e=\frac{\sqrt{3}}{2}$; foci $=(1\pm\sqrt{3},-1)$; LR $=1$

4. $\frac{x^2}{4}+\frac{y^2}{8}=1$

5. $C\equiv(-1,2)$, length of major axis $=2b=\sqrt{3}$, length of minor axis $=2a=1$; $e=\sqrt{\frac{2}{3}}$; $f\left(-1,\ 2\pm\frac{1}{\sqrt{2}}\right)$

BEGINNER'S BOX-2

1. On the ellipse **2.** $\frac{1}{2}(a^2+b^2)$ **3.** $P^2=a^2\cos^2\theta+b^2\sin^2\theta$

4. $3y+x\pm\sqrt{97}=0$ **5.** $7x-12y=50$

BEGINNER'S BOX-3

1. $4x-3y=7$ **2.** abe **3.** $\frac{\sqrt{3}}{2}$

4. $\frac{x}{16}+\frac{y}{3}=1$ **5.** $-9x+16y=25$

BEGINNER'S BOX-4

1. 60π **2.** $\frac{x^2}{a^4}+\frac{y^2}{b^4}=\frac{1}{k^2}$

EXERCISE – 1 MCQ (SINGLE CHOICE CORRECT)

1. If distance between the directrices be thrice the distance between the foci, then eccentricity of ellipse is -

(A) $\frac{1}{2}$ (B) $\frac{2}{3}$ (C) $\frac{1}{\sqrt{3}}$ (D) $\frac{4}{5}$

2. If the eccentricity of an ellipse be 5/8 and the distance between its foci be 10, then its latus rectum is -

(A) $\frac{39}{4}$ (B) 12 (C) 15 (D) $\frac{37}{2}$

3. The curve represented by $x = 3(\cos t + \sin t)$, $y = 4(\cos t - \sin t)$, is -

(A) ellipse (B) parabola (C) hyperbola (D) circle

4. If the distance of a point on the ellipse $\frac{x^2}{6}+\frac{y^2}{2}=1$ from the centre is 2, then the eccentric angle is-

(A) $\pi/3$ (B) $\pi/4$ (C) $\pi/6$ (D) $\pi/2$

5. An ellipse having foci at (3, 3) and (–4, 4) and passing through the origin has eccentricity equal to-

(A) $\frac{3}{7}$ (B) $\frac{2}{7}$ (C) $\frac{5}{7}$ (D) $\frac{3}{5}$

6. A tangent having slope of $-\frac{4}{3}$ to the ellipse $\frac{x^2}{18}+\frac{y^2}{32}=1$ intersects the major & minor axes in points A & B respectively. If C is the centre of the ellipse then the area of the triangle ABC is :

(A) 12 sq. units (B) 24 sq. units (C) 36 sq. units (D) 48 sq. units

7. The equation to the locus of the middle point of the portion of the tangent to the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$ included between the co-ordinate axes is the curve-

(A) $9x^2 + 16y^2 = 4x^2y^2$ (B) $16x^2 + 9y^2 = 4x^2y^2$

(C) $3x^2 + 4y^2 = 4x^2y^2$ (D) $9x^2 + 16y^2 = x^2y^2$

8. An ellipse is drawn with major and minor axes of lengths 10 and 8 respectively. Using one focus as centre, a circle is drawn that is tangent to the ellipse, with no part of the circle being outside the ellipse. The radius of the circle is-

(A) $\sqrt{3}$ (B) 2 (C) $2\sqrt{2}$ (D) $\sqrt{5}$

9. Which of the following is the common tangent to the ellipses $\frac{x^2}{a^2+b^2}+\frac{y^2}{b^2}=1$ & $\frac{x^2}{a^2}+\frac{y^2}{a^2+b^2}=1$?

(A) $ay = bx + \sqrt{a^4-a^2b^2+b^4}$ (B) $by = ax - \sqrt{a^4+a^2b^2+b^4}$

(C) $ay = bx - \sqrt{a^4+a^2b^2+b^4}$ (D) $by = ax - \sqrt{a^4-a^2b^2+b^4}$

10. Angle between the tangents drawn from point (4, 5) to the ellipse $\frac{x^2}{16}+\frac{y^2}{25}=1$ is -

(A) $\frac{\pi}{3}$ (B) $\frac{5\pi}{6}$ (C) $\frac{\pi}{4}$ (D) $\frac{\pi}{2}$

11. The point of intersection of the tangents at the point P on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, and its corresponding point Q on the auxiliary circle meet on the line -

(A) $x = a/e$ (B) $x = 0$ (C) $y = 0$ (D) none

12. The equation of the normal to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the positive end of latus rectum is -

(A) $x + ey + e^2a = 0$ (B) $x - ey - e^3a = 0$ (C) $x - ey - e^2a = 0$ (D) none of these

13. The eccentric angle of the point where the line, $5x - 3y = 8\sqrt{2}$ is a normal to the ellipse $\frac{x^2}{25}+\frac{y^2}{9}=1$ is -

(A) $\frac{3\pi}{4}$ (B) $\frac{\pi}{4}$ (C) $\frac{\pi}{6}$ (D) $\tan^{-1}2$

14. The equation of the chord of the ellipse $2x^2 + 5y^2 = 20$ which is bisected at the point (2, 1) is -

(A) $4x + 5y + 13 = 0$ (B) $4x + 5y = 13$ (C) $5x + 4y + 13 = 0$ (D) $4x + 5y = 13$

EXERCISE – 2 MCQ (ONE OR MORE CHOICE CORRECT)

1. $x-2y+4=0$ is a common tangent to $y^2=4x$ & $\frac{x^2}{4}+\frac{y^2}{b^2}=1$. Then the value of b and the other common tangent are given by -

(A) $b=\sqrt{3}\ ;\ x+2y+4=0$
(B) $b=3\ ;\ x+2y+4=0$
(C) $b=\sqrt{3}\ ;\ x+2y-4=0$
(D) $b=\sqrt{3}\ ;\ x-2y-4=0$

2. The tangent at any point P on a standard ellipse with foci as S & S' meets the tangents at the vertices A & A' in the points V & V', then -

(A) $l(AV).l(A'V')=b^2$
(B) $l(AV).l(A'V')=a^2$
(C) $\angle V'SV=90°$
(D) V'S' VS is a cyclic quadrilateral

3. The area of the rectangle formed by the perpendiculars from the centre of the standard ellipse to the tangent and normal at its point whose eccentric angle is $\pi/4$ is -

(A) $\frac{(a^2-b^2)ab}{a^2+b^2}$
(B) $\frac{(a^2+b^2)ab}{a^2-b^2}$
(C) $\frac{(a^2-b^2)}{ab(a^2+b^2)}$
(D) $\frac{(a^2+b^2)}{(a^2-b^2)ab}$

4. A circle has the same centre as an ellipse & passes through the foci F_1 & F_2 of the ellipse, such that the two curves intersect in 4 points. Let 'P' be any one of their point of intersection. If the major axis of the ellipse is 17 & the area of the triangle PF_1F_2 is 30, then the distance between the foci is -

(A) 11
(B) 12
(C) 13
(D) none

5. Point 'O' is the centre of the ellipse with major axis AB and minor axis CD. Point F is one focus of the ellipse. If OF = 6 and the diameter of the inscribed circle of triangle OCF is 2, then the product (AB)(CD) is equal to-

(A) 65
(B) 52
(C) 78
(D) none

6. If the chord through the points whose eccentric angles are θ & ϕ on the ellipse, $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ passes through the focus, then the value of $\tan(\theta/2)\tan(\phi/2)$ is -

(A) $\frac{e+1}{e-1}$
(B) $\frac{e-1}{e+1}$
(C) $\frac{1+e}{1-e}$
(D) $\frac{1-e}{1+e}$

7. If latus rectum of an ellipse $\frac{x^2}{16}+\frac{y}{b^2}=1$ $\{0< b < 4\}$, subtend angle 2θ at farthest vertex such that $\text{cosec}\theta=\sqrt{5}$, then -

(A) $e=\frac{1}{2}$
(B) no such ellipse exist
(C) $b=2\sqrt{3}$
(D) area of Δ formed by LR and nearest vertex is 6 sq. units

8. If $x-2y+k=0$ is a common tangent to $y^2=4x$ & $\frac{x^2}{a^2}+\frac{y^2}{3}=1$ ($a>\sqrt{3}$), then the value of a, k and other common tangent are given by -

(A) a = 2
(B) a = –2
(C) x + 2y + 4 = 0
(D) k = 4

9. All ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ $(0 < b < a)$ has fixed major axis. Tangent at any end point of latus rectum meet at a fixed point which can be -

(A) (a, a) (B) (0, a) (C) (0, –a) (D) (0, 0)

10. Eccentric angle of a point on the ellipse $x^2 + 3y^2 = 6$ at a distance $\sqrt{3}$ units from the centre of the ellipse is

(A) $\frac{5\pi}{3}$ (B) $\frac{\pi}{3}$ (C) $\frac{3\pi}{4}$ (D) $\frac{2\pi}{3}$

Match the column

Following question contains statements given in two columns, which have to be matched. The statements in ***Column-I*** are labelled as A, B, C and D while the statements in ***Column-II*** are labelled as p, q, r and s. Any given statement in ***Column-I*** can have correct matching with ***ONE OR MORE*** statement(s) in ***Column-II***.

11.

	Column - I		Column - II
(A)	The minimum and maximum distance of a point (2, 6) from the ellipse are $9x^2 + 8y^2 - 36x - 16y - 28 = 0$	(p)	0
(B)	The minimum and maximum distance of a point $\left(\frac{9}{5},\frac{12}{5}\right)$ from the ellipse $4(3x + 4y)^2 + 9(4x - 3y)^2 = 900$ are	(q)	2
(C)	If $E : 2x^2 + y^2 = 2$ and director circle of E is C_1, director circle of C_1 is C_2 director circle of C_2 is C_3 and so on. If r_1, r_2, r_3 .... are the radii of C_1, C_2, C_3 ... respectively then G.M. of r_1^2, r_2^2, r_3^2 is	(r)	6
(D)	Minimum area of the triangle formed by any tangent to the ellipse $x^2 + 4y^2 = 16$ with coordinate axes is	(s)	8

Comprehension Based Questions

An ellipse whose distance between foci S and S' is 4 units is inscribed in the triangle ABC touching the sides AB, AC and BC at P, Q and R. If centre of ellipse is at origin and major axis along x-axis, SP + S'P = 6.

On the basis of above information, answer the following questions

12. If $\angle BAC = 90°$, then locus of point A is -

(A) $x^2 + y^2 = 12$ (B) $x^2 + y^2 = 4$ (C) $x^2 + y^2 = 14$ (D) none of these

13. If chord PQ subtends 90° angle at centre of ellipse, then locus of A is -

(A) $25x^2 + 81y^2 = 620$ (B) $25x^2 + 81y^2 = 630$ (C) $9x^2 + 16y^2 = 25$ (D) none of these

14. If difference of eccentric angles of points P and Q is 60°, then locus of A is -

(A) $16x^2 + 9y^2 = 144$ (B) $16x^2 + 45y^2 = 576$ (C) $5x^2 + 9y^2 = 60$ (D) $5x^2 + 9y^2 = 15$

EXERCISE - 3 SUBJECTIVE

1. Find the equation to the ellipse, whose focus is the point (–1, 1), whose directrix is the straight line $x - y + 3 = 0$ and whose eccentricity is $\frac{1}{2}$.

2. Find the latus rectum, the eccentricity and the coordinates of the foci, of the ellipse

(a) $x^2 + 3y^2 = a^2, a > 0$ (b) $5x^2 + 4y^2 = 1$

3. An ellipse passes through the points (– 3 , 1) & (2, –2) & its principal axis are along the coordinate axes in order. Find its equation.

4. Find the latus rectum, eccentricity, coordinates of the foci, coordinates of the vertices, the length of the axes and the centre of the ellipse $4x^2 + 9y^2 - 8x - 36y + 4 = 0$.

5. Find the set of value(s) of α for which the point $\left(7-\frac{5}{4}\alpha, \alpha\right)$ lies inside the ellipse $\frac{x^2}{25}+\frac{y^2}{16}=1$.

6. Find the equation of tangents to the ellipse $\frac{x^2}{50}+\frac{y^2}{32}=1$ which passes through a point (15, –4).

7. If tangent drawn at a point $(t^2, 2t)$ on the parabola $y^2 = 4x$ is same as the normal drawn at a point $\left(\sqrt{5}\cos\phi, 2\sin\phi\right)$ on the ellipse $4x^2 + 5y^2 = 20$, then find the values of t & ϕ.

8. Find the locus of the point the chord of contact of the tangent drawn from which to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ touches the circle $x^2 + y^2 = c^2$, where $c < b < a$.

9. A tangent to the ellipse $x^2 + 4y^2 = 4$ meets the ellipse $x^2 + 2y^2 = 6$ at P & Q. Prove that the tangents at P & Q of the ellipse $x^2 + 2y^2 = 6$ are at right angles.

EXERCISE - 4 RECAP OF AIEEE/JEE (MAIN)

1. If distance between the foci of an ellipse is equal to its minor axis, then eccentricity of the ellipse is- ***[AIEEE-2002]***

(A) $e = \frac{1}{\sqrt{2}}$ (B) $e = \frac{1}{\sqrt{3}}$ (C) $e = \frac{1}{\sqrt{4}}$ (D) $e = \frac{1}{\sqrt{6}}$

2. The equation of an ellipse, whose major axis = 8 and eccentricity = 1/2 is- (a > b) ***[AIEEE-2002]***

(A) $3x^2 + 4y^2 = 12$ (B) $3x^2 + 4y^2 = 48$ (C) $4x^2 + 3y^2 = 48$ (D) $3x^2 + 9y^2 = 12$

3. The eccentricity of an ellipse, with its centre at the origin, is 1/2. If one of the directirices is x = 4, then the equation of the ellipse is- ***[AIEEE-2004]***

(A) $3x^2 + 4y^2 = 1$ (B) $3x^2 + 4y^2 = 12$ (C) $4x^2 + 3y^2 = 12$ (D) $4x^2 + 3y^2 = 1$

4. An ellipse has OB as semi minor axis, F and F' its focii and the angle FBF' is a right angle. Then the eccentricity of the ellipse is- ***[AIEEE-2005, IIT-1997]***

(A) $\frac{1}{\sqrt{2}}$ (B) $\frac{1}{2}$ (C) $\frac{1}{4}$ (D) $\frac{1}{\sqrt{3}}$

5. In an ellipse, the distance between its foci is 6 and minor axis is 8. Then its eccentricity is- ***[AIEEE-2006]***

(A) $\frac{1}{2}$ (B) $\frac{4}{5}$ (C) $\frac{1}{\sqrt{5}}$ (D) $\frac{3}{5}$

6. A focus of an ellipse is at the origin. The directrix is the line x = 4 and the eccentricity is 1/2. Then the length of the semi-major axis is- ***[AIEEE-2008]***

(A) 8/3 (B) 2/3 (C) 4/3 (D) 5/3

7. The ellipse $x^2 + 4y^2 = 4$ is inscribed in a rectangle aligned with the coordinate axes, which in turn is inscribed in another ellipse that passes through the point (4, 0). Then the equation of the ellipse is : -***[AIEEE-2009]***

(A) $4x^2 + 48y^2 = 48$ (B) $4x^2 + 64y^2 = 48$

(C) $x^2 + 16y^2 = 16$ (D) $x^2 + 12y^2 = 16$

8. Equation of the ellipse whose axes are the axes of coordinates and which passes through the point (–3, 1) and has eccentricity $\sqrt{2/5}$ is :- ***[AIEEE-2011]***

(A) $3x^2 + 5y^2 - 15 = 0$ (B) $5x^2 + 3y^2 - 32 = 0$

(C) $3x^2 + 5y^2 - 32 = 0$ (D) $5x^2 + 3y^2 - 48 = 0$

9. An ellipse is drawn by taking a diameter of the circle $(x - 1)^2 + y^2 = 1$ as its semi-minor axis and a diameter of the circle $x^2 + (y - 2)^2 = 4$ as its semi-major axis. If the centre of the ellipse is at the origin and its axes are the coordinate axes, then the equation of the ellipse is : ***[AIEEE-2012]***

(A) $x^2 + 4y^2 = 16$ (B) $4x^2 + y^2 = 4$ (C) $x^2 + 4y^2 = 8$ (D) $4x^2 + y^2 = 8$

10. ***Statement–1 :*** An equation of a common tangent to the parabola $y^2 = 16\sqrt{3}\,x$ and the ellipse $2x^2 + y^2 = 4$ is $y = 2x + 2\sqrt{3}$.

Statement–2 : If the line $y = mx + \frac{4\sqrt{3}}{m}$, $(m \neq 0)$ is a common tangent to the parabola $y^2 = 16\sqrt{3}\,x$ and the ellipse $2x^2 + y^2 = 4$, then m satisfies $m^4 + 2m^2 = 24$. ***[AIEEE-2012]***

(A) Statement–1 is true, Statement–2 is false.

(B) Statement–1 is false, Statement–2 is true.

(C) Statement–1 is true, Statement–2 is true ; Statement–2 is a correct explanation for Statement–1.

(D) Statement–1 is true, Statement–2 is true ; Statement–2 is not a correct explanation for Statement–1.

11. The equation of the circle passing through the foci of the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$ and having centre at (0, 3) is : ***[JEE (Main)-2013]***

(A) $x^2 + y^2 - 6y - 7 = 0$ (B) $x^2 + y^2 - 6y + 7 = 0$ (C) $x^2 + y^2 - 6y - 5 = 0$ (D) $x^2 + y^2 - 6y + 5 = 0$

12. The locus of the foot of perpendicular drawn from the centre of the ellipse $x^2 + 3y^2 = 6$ on any tangent to it is: ***[JEE (Main)-2014]***

(A) $(x^2 + y^2)^2 = 6x^2 + 2y^2$ (B) $(x^2 + y^2)^2 = 6x^2 - 2y^2$

(C) $(x^2 - y^2)^2 = 6x^2 + 2y^2$ (D) $(x^2 - y^2)^2 = 6x^2 - 2y^2$

13. The area (in sq. units) of the quadrilateral formed by the tangents at the end points of the latera recta to the ellipse $\frac{x^2}{9}+\frac{y^2}{5}=1$ is : ***[JEE (Main)-2015]***

(A) $\frac{27}{2}$ (B) 27 (C) $\frac{27}{4}$ (D) 18

14. The eccentricity of an ellipse whose centre is at the origin is $\frac{1}{2}$. If one of its directices is $x = -4$, then the equation of the normal to it at $\left(1,\frac{3}{2}\right)$ is :- ***[JEE (Main)-2017]***

(A) $x + 2y = 4$ (B) $2y - x = 2$ (C) $4x - 2y = 1$ (D) $4x + 2y = 7$

15. If the curves $y^2 = 6x$, $9x^2 + by^2 = 16$ intersect each other at right angles, then the value of b is :

(A) $\frac{7}{2}$ (B) 4 (C) $\frac{9}{2}$ (D) 6 ***[JEE (Main)-2018]***

16. Let the length of the latus rectum of an ellipse with its major axis is along x–axis and centre at the origin, be 8. If the distance between the foci of this ellipse is equal to the length of its minor axis, then which one of the following points lies on it ? ***[JEE (Main)-2019]***

(A) $(4\sqrt{3}, 2\sqrt{3})$ (B) $(4\sqrt{3}, 2\sqrt{2})$ (C) $(4\sqrt{2}, 2\sqrt{2})$ (D) $(4\sqrt{2}, 2\sqrt{3})$

17. If tangents are drawn to the ellipse $x^2 + 2y^2 = 2$ at all points on the ellipse other than its four vertices then the mid points of the tangents intercepted betwen the coordinate axes lie on the curve : ***[JEE (Main)-2019]***

(A) $\frac{x^2}{2}+\frac{y^2}{4}=1$ (B) $\frac{x^2}{4}+\frac{y^2}{2}=1$ (C) $\frac{1}{2x^2}+\frac{1}{4y^2}=1$ (D) $\frac{1}{4x^2}+\frac{1}{2y^2}=1$

18. Let S and S' be the foci of the ellipse and B be any one of the extremities of its minor axis. If AS'BS is a right angled triangle with right angle at B and area (AS'BS) = 8 sq. units, then the length of a latus rectum of the ellipse is : ***[JEE (Main)-2019]***

(A) $2\sqrt{2}$ (B) 2 (C) 4 (D) $4\sqrt{2}$

19. If the tangents on the ellipse $4x^2 + y^2 = 8$ at the points (1, 2) and (a, b) are perpendicular to each other, then a^2 is equal to : ***[JEE (Main)-2019]***

(A) $\frac{64}{17}$ (B) $\frac{2}{17}$ (C) $\frac{128}{17}$ (D) $\frac{4}{17}$

20. In an ellipse, with centre at the origin, if the difference of the lengths of major axis and minor axis is 10 and one of the foci is at $\left(0, 5\sqrt{3}\right)$, then the length of its latus rectum is: ***[JEE (Main)-2019]***

(A) 10 (B) 8 (C) 5 (D) 6

21. If the tangent to the parabola $y^2 = x$ at a point (α, β), $(\beta > 0)$ is also a tangent to the ellipse, $x^2 + 2y^2 = 1$, then α is equal to : ***[JEE (Main)-2019]***

(A) $2\sqrt{2}+1$ (B) $\sqrt{2}-1$ (C) $\sqrt{2}+1$ (D) $2\sqrt{2}-1$

22 If the line $x - 2y = 12$ is tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the point $\left(3,\frac{-9}{2}\right)$, then the length of the latus recturm of the ellipse is : ***[JEE (Main)-2019]***

(A) 9 (B) $8\sqrt{3}$ (C) $12\sqrt{2}$ (D) 5

23. The tangent and normal to the ellipse $3x^2 + 5y^2 = 32$ at the point P(2, 2) meet the x-axis at Q and R, respectively. Then the area (in sq. units) of the triangle PQR is : ***[JEE (Main)-2019]***

(A) $\frac{14}{3}$ (B) $\frac{16}{3}$ (C) $\frac{68}{15}$ (D) $\frac{34}{15}$

24. If the normal to the ellipse $3x^2 + 4y^2 = 12$ at a point P on it is parallel to the line, $2x + y = 4$ and the tangent to the ellipse at P passes through Q(4, 4) then PQ is equal to : ***[JEE (Main)-2019]***

(A) $\frac{\sqrt{221}}{2}$ (B) $\frac{\sqrt{157}}{2}$ (C) $\frac{\sqrt{61}}{2}$ (D) $\frac{5\sqrt{5}}{2}$

25. An ellipse, with foci at (0, 2) and (0, –2) and minor axis of length 4, passes through which of the following points ? ***[JEE (Main)-2019]***

(A) $(1, 2\sqrt{2})$ (B) $(2, \sqrt{2})$ (C) $(2, 2\sqrt{2})$ (D) $(\sqrt{2}, 2)$

EXERCISE – 5 RECAP OF IIT-JEE/JEE (ADVANCED)

1. Let ABC be an equilateral triangle inscribed in the circle $x^2 + y^2 = a^2$. Suppose perpendiculars from A, B, C to the major axis of the ellipse, $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, (a > b) meet the ellipse respectively at P,Q,R so that P, Q,R lie on the same side of the major axis as A, B,C respectively . Prove that the normals to the ellipse drawn at the points P, Q and R are concurrent. **[JEE 2000]**

2. Let C_1 and C_2 be two circles with C_2 lying inside C_1. A circle C lying inside C_1 touches C_1 internally and C_2 externally. Identify the locus of the centre of C. **[JEE 2001]**

3. Prove that, in an ellipse, the perpendicular from a focus upon any tangent and the line joining the centre of the ellipse to the point of contact meet on the corresponding directrix. **[JEE 2002]**

4. Tangent is drawn to ellipse $\frac{x^2}{27}+y^2=1$ at $(3\sqrt{3}\cos\theta, \sin\theta)$ (where $\theta \in (0, \pi/2)$). Then the value of θ, such that sum of intercepts on axes made by this tangent is least is - **[JEE 2003]**

(A) $\frac{\pi}{3}$ (B) $\frac{\pi}{6}$ (C) $\frac{\pi}{8}$ (D) $\frac{\pi}{4}$

5. The area of the quadrilateral formed by the tangents at the end points of the latus rectum of the ellipse $\frac{x^2}{9}+\frac{y^2}{5}=1$, is - **[JEE 2003]**

(A) 27/4 sq. units (B) 9 sq. units (C) 27/2 sq. units (D) 27 sq. units

6. Find a point on the curve $x^2 + 2y^2 = 6$ whose distance from the line $x + y = 7$, is as small as possible. **[JEE 2003]**

7. Locus of the mid points of the segments which are tangents to the ellipse $\frac{1}{2}x^2+y^2=1$ and which are intercepted between the coordinate axes is - **[JEE 2004]**

(A) $\frac{1}{2}x^2+\frac{1}{4}y^2=1$ (B) $\frac{1}{4}x^2+\frac{1}{2}y^2=1$ (C) $\frac{1}{3x^2}+\frac{1}{4y^2}=1$ (D) $\frac{1}{2x^2}+\frac{1}{4y^2}=1$

8. The minimum area of triangle formed by tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and coordinate axes - **[JEE 2005]**

(A) ab (B) $\frac{a^2+b^2}{2}$ (C) $\frac{(a+b)^2}{2}$ (D) $\frac{a^2+ab+b^2}{3}$

9. Find the equation of the common tangent in 1st quadrant to the circle $x^2 + y^2 = 16$ and the ellipse $\frac{x^2}{25}+\frac{y^2}{4}=1$. Also find the length of the intercept of the tangent between the coordinate axes. **[JEE 2005]**

10. Let $P(x_1, y_1)$ and $Q(x_2, y_2)$, $y_1 < 0$, $y_2 < 0$, be the end points of the latus rectum of the ellipse $x^2+4y^2=4$. The equations of parabolas with latus rectum PQ are - **[JEE 2008]**

(A) $x^2 + 2\sqrt{3}\,y = 3 + \sqrt{3}$ (B) $x^2 - 2\sqrt{3}\,y = 3 + \sqrt{3}$

(C) $x^2 + 2\sqrt{3}\,y = 3 - \sqrt{3}$ (D) $x^2 - 2\sqrt{3}\,y = 3 - \sqrt{3}$

11. The line passing through the extremity A of the major axis and extremity B of the minor axis of the ellipse $x^2 + 9y^2 = 9$ meets its auxiliary circle at the point M. Then the area of the triangle with vertices at A, M and the origin O is :- ***[JEE 2009]***

(A) $\frac{31}{10}$ (B) $\frac{29}{10}$ (C) $\frac{21}{10}$ (D) $\frac{27}{10}$

12. The normal at a point P on the ellipse $x^2 + 4y^2 = 16$ meets the x-axis at Q. If M is the mid point of the line segment PQ, then the locus of M intersects the latus rectums of the given ellipse at the points ***[JEE 2009]***

(A) $\left(\pm\frac{3\sqrt{5}}{2},\pm\frac{2}{7}\right)$ (B) $\left(\pm\frac{3\sqrt{5}}{2},\pm\frac{\sqrt{19}}{4}\right)$ (C) $\left(\pm2\sqrt{3},\pm\frac{1}{7}\right)$ (D) $\left(\pm2\sqrt{3},\pm\frac{4\sqrt{3}}{7}\right)$

Paragraph for Question 13 to 15

Tangents are drawn from the point P(3, 4) to the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$ touching the ellipse at points A and B.

[JEE 10]

13. The coordinates of A and B are

(A) (3, 0) and (0, 2)

(B) $\left(-\frac{8}{5},\frac{2\sqrt{161}}{15}\right)$ and $\left(-\frac{9}{5},\frac{8}{5}\right)$

(C) $\left(-\frac{8}{5},\frac{2\sqrt{161}}{15}\right)$ and (0, 2)

(D) (3, 0) and $\left(-\frac{9}{5},\frac{8}{5}\right)$

14. The orthocenter of the triangle PAB is

(A) $\left(5,\frac{8}{7}\right)$ (B) $\left(\frac{7}{5},\frac{25}{8}\right)$ (C) $\left(\frac{11}{5},\frac{8}{5}\right)$ (D) $\left(\frac{8}{25},\frac{7}{5}\right)$

15. The equation of the locus of the point whose distances from the point P and the line AB are equal, is

(A) $9x^2 + y^2 - 6xy - 54x - 62y + 241 = 0$

(B) $x^2 + 9y^2 + 6xy - 54x + 62y - 241 = 0$

(C) $9x^2 + 9y^2 - 6xy - 54x - 62y - 241 = 0$

(D) $x^2 + y^2 - 2xy + 27x + 31y - 120 = 0$

16. The ellipse $E_1 : \frac{x^2}{9}+\frac{x^2}{4}=1$ is inscribed in a rectangle R whose sides are parallel to the coordinate axes. Another ellipse E_2 passing through the point (0,4) circumscribes the rectangle R. The eccentricity of the ellipse E_2 is - ***[JEE 2012]***

(A) $\frac{\sqrt{2}}{2}$ (B) $\frac{\sqrt{3}}{2}$ (C) $\frac{1}{2}$ (D) $\frac{3}{4}$

17. A vertical line passing through the point (h,0) intersects the ellipse $\frac{x^2}{4}+\frac{y^2}{3}=1$ at the points P and Q. Let the tangents to the ellipse at P and Q meet at the point R. If $\Delta(h)$ = area of the triangle PQR, $\Delta_1 = \max\limits_{1/2\le h\le1}\Delta(h)$ and $\Delta_2 = \min\limits_{1/2\le h\le1}\Delta(h)$, then $\frac{8}{\sqrt{5}}\Delta_1 - 8\Delta_2 =$ ***[IIT-JEE 2013]***

18. If the normal from the point P(h, 1) on the ellipse $\frac{x^2}{6}+\frac{y^2}{3}=1$ is perpendicular to the line x + y = 8, then the value of h is ? **[IIT-JEE]**

19. Let E_1 and E_2 be two ellipses whose centers are at the origin. The major axes of E_1 and E_2 lie along the x-axis and the y-axis, respectively. Let S be the circle $x^2 + (y-1)^2 = 2$. The straight line x + y = 3 touches the curves S, E_1 and E_2 at P,Q and R, respectively. Suppose that $PQ = PR = \frac{2\sqrt{2}}{3}$. If e_1 and e_2 are the eccentricities of E_1 and E_2, respectively, then the correct expression(s) is(are) **[JEE 2015]**

(A) $e_1^2+e_2^2=\frac{43}{40}$ (B) $e_1+e_2=\frac{\sqrt{7}}{2\sqrt{10}}$ (C) $\left|e_1^2-e_2^2\right|=\frac{5}{8}$ (D) $e_1e_2=\frac{\sqrt{3}}{4}$

20. Suppose that the foci of the ellipse $\frac{x^2}{9}+\frac{y^2}{5}=1$ are $(f_1,0)$ and $(f_2,0)$ where $f_1>0$ and $f_2<0$. Let P_1 and P_2 be two parabolas with a common vertex at (0, 0) and with foci at $(f_1,0)$ and $(2f_2,0)$, respectively. Let T_1 be a tangent to P_1 which passes through $(2f_2,0)$ and T_2 be a tangent to P_2 which passes through $(f_1,0)$. If m_1 is the slope of T_1 and m_2 is the slope of T_2, then the value of $\left(\frac{1}{m_1^2}+m_2^2\right)$ is **[JEE 2015]**

Paragraph for Question 21 & 22

Let $F_1(x_1, 0)$ and $F_2(x_2, 0)$ for $x_1 < 0$ and $x_2 > 0$, be the foci of the ellipse $\frac{x^2}{9}+\frac{y^2}{8}=1$. Suppose a parabola having vertex at the origin and focus at F_2 intersects the ellipse at point M in the first quadrant and at point N in the fourth quadrant. **[JEE 2016]**

21. The orthocentre of the triangle F_1MN is-

(A) $\left(-\frac{9}{10},0\right)$ (B) $\left(\frac{2}{3},0\right)$ (C) $\left(\frac{9}{10},0\right)$ (D) $\left(\frac{2}{3},\sqrt{6}\right)$

22. If the tangents to the ellipse at M and N meet at R and the normal to the parabola at M meets the x-axis at Q, then the ratio of area of the triangle MQR to area of the quadrilateral MF_1NF_2 is-

(A) 3 : 4 (B) 4 : 5 (C) 5 : 8 (D) 2 : 3

23. Define the collections $\{E_1, E_2, E_3, ...\}$ of ellipses and $\{R_1, R_2, R_3, ...\}$ of rectangles as follows :

$E_1 : \frac{x^2}{9}+\frac{y^2}{4}=1$;

R_1 : rectangle of largest area, with sides parallel to the axes, inscribed in E_1 ;

E_n : ellipse $\frac{x^2}{a_n^2}+\frac{y^2}{b_n^2}=1$ of largest area inscribed in R_{n-1}, n > 1 ;

R_n : rectangle of largest area, with sides parallel to the axes, inscribed in E_n, n > 1.

Then which of the following options is/are correct ? **[JEE 2019]**

(A) The eccentricities of E_{18} and E_{19} are NOT equal

(B) The distance of a focus from the centre in E_9 is $\frac{\sqrt{5}}{32}$

(C) The length of latus rectum of E_9 is $\frac{1}{6}$

(D) $\sum_{n=1}^{N}(\text{area of } R_n) < 24$, for each positive integer N

ANSWER KEY

EXERCISE-1

Que.	1	2	3	4	5	6	7	8	9	10
Ans.	C	A	A	B	C	B	A	B	B	D
Que.	11	12	13	14						
Ans.	C	B	B	B						

EXERCISE-2

Que.	1	2	3	4	5	6	7	8	9	10
Ans.	A	ACD	A	C	A	AB	ACD	ABCD	BC	ABD

- ***Match the Column*** **11.** (A) $\to$ (q,s) ; (B) $\to$ (p,r); (C) $\to$ (r); (D) $\to$ (s)
- ***Comprehension Based Questions*** **12.** C **13.** B **14.** C

EXERCISE-3

1. $7x^2 + 2xy + 7y^2 + 10x - 10y + 7 = 0$ **2.** (a) $\frac{2a}{3}; \frac{1}{3}\sqrt{6}; \left(\pm\frac{a}{3}\sqrt{6}, 0\right)$ (b) $\frac{4}{5}; \frac{1}{5}\sqrt{5}; \left(0, \pm\frac{1}{10}\sqrt{5}\right)$

3. $3x^2 + 5y^2 = 32$ **4.** $\frac{8}{3}, \frac{\sqrt{5}}{3}$; $\left(1\pm\sqrt{5}, 2\right); \left(-2,2\right)$ and (4, 2); 6 and 4; (1, 2)

5. $\left(\frac{12}{5}, \frac{16}{5}\right)$ **6.** $4x + 5y = 40,\ 4x - 35y = 200$

7. $\phi = \pi - \tan^{-1}2,\ \ t = -\frac{1}{\sqrt{5}};\ \ \phi = \pi + \tan^{-1}2,\ \ t = \frac{1}{\sqrt{5}};\ \ \phi = \frac{\pi}{2}, \frac{3\pi}{2}\ \ t = 0$ **8.** $\frac{x^2}{a^4} + \frac{y^2}{b^4} = \frac{1}{c^2}$

EXERCISE-4

Que.	1	2	3	4	5	6	7	8	9	10
Ans	A	B	B	A	D	A	D	C	A	C
Que.	11	12	13	14	15	16	17	18	19	20
Ans	A	A	B	C	C	B	C	C	B	C
Que.	21	22	23	24	25					
Ans	C	A	C	D	D					

EXERCISE-5

2. Locus is an ellipse with foci as the centres of the circles C_1 and C_2. **4.** B **5.** D

6. (2,1) **7.** D **8.** A **9.** $y = -\frac{2}{\sqrt{3}}x + 4\sqrt{\frac{7}{3}}; \frac{14}{\sqrt{3}}$ **10.** B,C

11. D **12.** C **13.** D **14.** C **15.** A **16.** C **17.** 9

18. 2 **19.** A, B **20.** 4 **21.** A **22.** C **23.** C, D

CONIC SECTION

HYPERBOLA

Index

HYPERBOLA

The ***Hyperbola*** is a conic whose eccentricity is greater than unity. ***(e > 1)***.

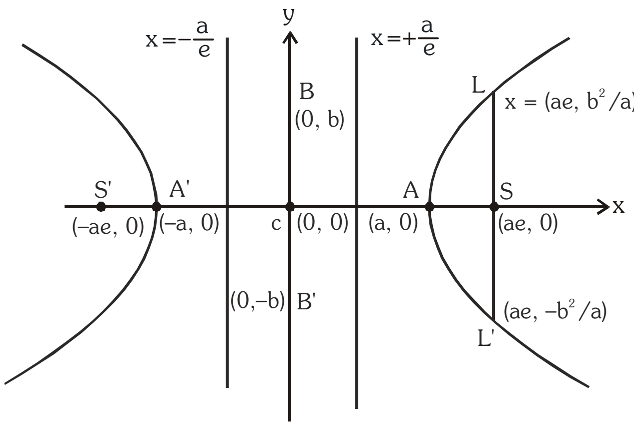
1.0 STANDARD EQUATION & DEFINITION(S)

Standard equation of the hyperbola is

$\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$, where $b^2 = a^2(e^2 - 1)$

or $a^2e^2 = a^2 + b^2$ i.e. $e^2 = 1 + \dfrac{b^2}{a^2}$

$= 1 + \left(\dfrac{\text{Conjugate Axis}}{\text{Transverse Axis}}\right)^2$



(a) ***Foci – S ≡ (ae, 0)*** & ***S′ ≡ (– ae, 0).***

(b) ***Equations of directrices –*** $x = \dfrac{a}{e}$ & $x = -\dfrac{a}{e}$.

(c) ***Vertices – A ≡ (a , 0)*** & ***A′ ≡ (– a , 0).***

(d) ***Latus rectum –***

(i) Equation – $x = \pm ae$

(ii) Length $\dfrac{2b^2}{a} = \dfrac{(\text{Conjugate Axis})^2}{(\text{Transverse Axis})} = 2a(e^2 - 1) = 2e$ (distance from focus to directrix)

(iii) Ends – $\left(ae, \dfrac{b^2}{a}\right), \left(ae, \dfrac{-b^2}{a}\right)$; $\left(-ae, \dfrac{b^2}{a}\right), \left(-ae, \dfrac{-b^2}{a}\right)$

(e) ***(i) Transverse Axis –*** The line segment A′A of length 2a in which the foci S′ & S both lie is called the ***Transverse Axis of the Hyperbola.***

(ii) Conjugate Axis – The line segment B′B between the two points B′ ≡ (0, – b) & B ≡ (0, b) is called as the ***Conjugate Axis of the Hyperbola.***

The Transverse Axis & the Conjugate Axis of the hyperbola are together called the ***Principal axes of the hyperbola.***

(f) ***Focal Property –*** The difference of the focal distances of any point on the hyperbola is constant and equal to transverse axis i.e. $\left| |PS| - |PS'| \right| = 2a$. The distance SS′ = focal length.

(g) ***Focal distance –*** Distance of any point P(x, y) on Hyperbola from foci ***PS = ex – a*** & ***PS′ = ex + a.***

Illustrations

Illustration 1. Find the equation of the hyperbola whose directrix is 2x + y = 1, focus (1, 2) and eccentricity $\sqrt{3}$.

Solution Let P(x, y) be any point on the hyperbola and PM is perpendicular from P on the directrix.

Then by definition SP = e PM

$\Rightarrow \quad (SP)^2 = e^2(PM)^2 \quad \Rightarrow \quad (x-1)^2 + (y-2)^2 = 3\left\{\dfrac{2x+y-1}{\sqrt{4+1}}\right\}^2$

$\Rightarrow \quad 5(x^2 + y^2 - 2x - 4y + 5) = 3(4x^2 + y^2 + 1 + 4xy - 2y - 4x)$

$\Rightarrow \quad 7x^2 - 2y^2 + 12xy - 2x + 14y - 22 = 0$

which is the required hyperbola.

Illustration 2. The eccentricity of the hyperbola $4x^2 - 9y^2 - 8x = 32$ is -

(A) $\frac{\sqrt{5}}{3}$ (B) $\frac{\sqrt{13}}{3}$ (C) $\frac{\sqrt{13}}{2}$ (D) $\frac{3}{2}$

Solution $4x^2 - 9y^2 - 8x = 32 \Rightarrow 4(x-1)^2 - 9y^2 = 36 \Rightarrow \frac{(x-1)^2}{9} - \frac{y^2}{4} = 1$

Here $a^2 = 9, b^2 = 4$

$\therefore$ eccentricity $e = \sqrt{1+\frac{b^2}{a^2}} = \sqrt{1+\frac{4}{9}} = \frac{\sqrt{13}}{3}$ **Ans.(B)**

Illustration 3. If foci of a hyperbola are foci of the ellipse $\frac{x^2}{25}+\frac{y^2}{9}=1$. If the eccentricity of the hyperbola be 2, then its equation is -

(A) $\frac{x^2}{4}-\frac{y^2}{12}=1$ (B) $\frac{x^2}{12}-\frac{y^2}{4}=1$ (C) $\frac{x^2}{12}+\frac{y^2}{4}=1$ (D) none of these

Solution For ellipse $e = \frac{4}{5}$, so foci $= (\pm 4, 0)$

For hyperbola $e = 2$, so $a = \frac{ae}{e} = \frac{4}{2} = 2$, $b = 2\sqrt{4-1} = 2\sqrt{3}$

Hence equation of the hyperbola is $\frac{x^2}{4}-\frac{y^2}{12}=1$ **Ans.(A)**

Illustration 4. Find the coordinates of foci, the eccentricity and latus-rectum, equations of directrices for the hyperbola $9x^2 - 16y^2 - 72x + 96y - 144 = 0$.

Solution Equation can be rewritten as $\frac{(x-4)^2}{4^2} - \frac{(y-3)^2}{3^2} = 1$ so $a = 4, b = 3$

$b^2 = a^2(e^2 - 1)$ given $e = \frac{5}{4}$

Foci : $X = \pm ae, Y = 0$ gives the foci as $(9, 3), (-1, 3)$

Centre : $X = 0, Y = 0$ i.e. $(4, 3)$

Directrices : $X = \pm \frac{a}{e}$ i.e. $x - 4 = \pm \frac{16}{5}$ $\therefore$ directrices are $5x - 36 = 0; 5x - 4 = 0$

Latus-rectum $= \frac{2b^2}{a} = 2.\frac{9}{4} = \frac{9}{2}$

2.0 CONJUGATE HYPERBOLA

Two hyperbolas such that transverse & conjugate axes of one hyperbola are respectively the conjugate & the transverse axes of the other are called ***Conjugate Hyperbolas*** of each other.

eg. $\mathbf{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}$ & $\mathbf{-\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1}$ are conjugate hyperbolas of each other .

Illustrations

Illustration 5. The eccentricity of the conjugate hyperbola to the hyperbola $x^2 - 3y^2 = 1$ is -

(A) 2 (B) $2/\sqrt{3}$ (C) 4 (D) 4/3

Solution Equation of the conjugate hyperbola to the hyperbola $x^2 - 3y^2 = 1$ is

$-x^2 + 3y^2 = 1 \quad \Rightarrow \quad -\frac{x^2}{1} + \frac{y^2}{1/3} = 1$

Here $a^2 = 1, b^2 = 1/3$

$\therefore$ eccentricity $e = \sqrt{1 + a^2/b^2} = \sqrt{1+3} = 2$ ***Ans. (A)***

GOLDEN KEY POINTS

- If e_1 & e_2 are the eccentricities of the hyperbola & its conjugate then $e_1^{-2} + e_2^{-2} = 1$.
- The foci of a ***hyperbola*** and its ***conjugate*** are ***concyclic and form the vertices of a square.***
- Two hyperbolas are said to be ***similar*** if they have the ***same eccentricity***.

BEGINNER'S BOX-1

1. Find the eccentricity of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ which passes through (4, 0) & $(3\sqrt{2}, 2)$

2. Find the equation to the hyperbola, whose eccentricity is $\frac{5}{4}$, focus is (a, 0) and whose directrix is $4x - 3y = a$.

3. In the hyperbola $4x^2 - 9y^2 = 36$, find length of the axes, the co-ordinates of the foci, the eccentricity, and the latus rectum.

4. Find the equation to the hyperbola, the distance between whose foci is 16 and whose eccentricity is $\sqrt{2}$.

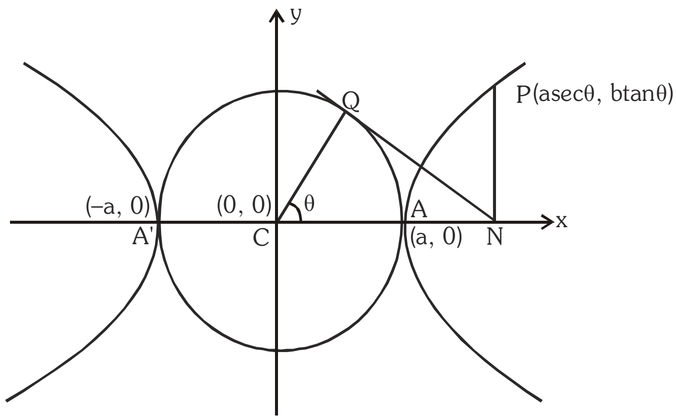
5. Find eccentricity of conjugate hyperbola of hyperbola $4x^2 - 16y^2 = 64$, also find area of quadrilateral formed by foci of hyperbola & its conjugate hyperbola

3.0 RECTANGULAR OR EQUILATERAL HYPERBOLA

The particular kind of hyperbola in which the lengths of the transverse & conjugate axis are equal is called an ***Equilateral Hyperbola***. Note that the eccentricity of the rectangular hyperbola is $\sqrt{2}$ and ***the length of it's latus rectum is equal to it's transverse or conjugate axis.***

4.0 AUXILIARY CIRCLE

A circle drawn with centre C & T.A. as a diameter is called the ***Auxiliary Circle*** of the hyperbola. Equation of the auxiliary circle is $x^2 + y^2 = a^2$.



Note from the figure that P & Q are called the "***Corresponding Points***" on the hyperbola & the auxiliary circle. 'θ' is called the ***eccentric angle*** of the point 'P' on the hyperbola. $(0 \le \theta < 2\pi)$.

Parametric Equation

The equations $x = a \sec\theta$ & $y = b\tan\theta$ together represents the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ where θ is a parameter. The parametric equations ; $x = a\cos h\,\phi$, $y = b\sin h\,\phi$ also represents the same hyperbola.

General Note

Since the fundamental equation to the hyperbola only differs from that to the ellipse in having $-b^2$ instead of b^2 it will be found that many propositions for the hyperbola are derived from those for the ellipse by simply changing the sign of b^2 .

5.0 POSITION OF A POINT 'P' w.r.t. A HYPERBOLA

The quantity $\frac{x_1^{\ 2}}{a^2} - \frac{y_1^{\ 2}}{b^2} = 1$ is ***positive , zero*** or ***negative*** according as the point (x_1, y_1) lies within , upon or outside the curve.

6.0 LINE AND A HYPERBOLA

The straight line $y = mx + c$ is a secant , a tangent or passes outside the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ according as – $c^2 > = < a^2m^2 - b^2$.

Equation of a chord of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ joining its two points $P(\alpha)$ & $Q(\beta)$ is

$$\frac{x}{a}\cos\frac{\alpha-\beta}{2} - \frac{y}{b}\sin\frac{\alpha+\beta}{2} = \cos\frac{\alpha+\beta}{2}$$

Illustrations

Illustration 6. Show that the line $x\cos\alpha + y\sin\alpha = p$ touches the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ if $a^2\cos^2\alpha - b^2\sin^2\alpha = p^2$.

Solution The given line is $x\cos\alpha + y\sin\alpha = p \Rightarrow y\sin\alpha = -x\cos\alpha + p$

$\Rightarrow \quad y = -x\cot\alpha + p\,\text{cosec}\,\alpha$

Comparing this line with $y = mx + c$

$m = -\cot\alpha,\ c = p\,\text{cosec}\,\alpha$

Since the given line touches the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ then

$c^2 = a^2m^2 - b^2 \quad \Rightarrow \quad p^2\,\text{cosec}^2\alpha = a^2\cot^2\alpha - b^2$ or $p^2 = a^2\cos^2\alpha - b^2\sin^2\alpha$

Illustration 7. If $(a\sec\theta,\ b\tan\theta)$ and $(a\sec\phi,\ b\tan\phi)$ are the ends of a focal chord of $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, then $\tan\frac{\theta}{2}\tan\frac{\phi}{2}$ equal to -

(A) $\frac{e-1}{e+1}$ (B) $\frac{1-e}{1+e}$ (C) $\frac{1+e}{1-e}$ (D) $\frac{e+1}{e-1}$

Solution Equation of chord connecting the points $(a\sec\theta,\ b\tan\theta)$ and $(a\sec\phi,\ b\tan\phi)$ is

$$\frac{x}{a}\cos\left(\frac{\theta-\phi}{2}\right) - \frac{y}{b}\sin\left(\frac{\theta+\phi}{2}\right) = \cos\left(\frac{\theta+\phi}{2}\right) \qquad \text{........ (i)}$$

If it passes through (ae, 0); we have, $e\cos\left(\frac{\theta-\phi}{2}\right) = \cos\left(\frac{\theta+\phi}{2}\right)$

$$\Rightarrow \quad e = \frac{\cos\left(\frac{\theta+\phi}{2}\right)}{\cos\left(\frac{\theta-\phi}{2}\right)} = \frac{1-\tan\frac{\theta}{2}.\tan\frac{\phi}{2}}{1+\tan\frac{\theta}{2}.\tan\frac{\phi}{2}} \Rightarrow \tan\frac{\theta}{2}.\tan\frac{\phi}{2} = \frac{1-e}{1+e}$$

Similarly if (i) passes through (–ae, 0), $\tan\frac{\theta}{2}.\tan\frac{\phi}{2} = \frac{1+e}{1-e}$ ***Ans. (B, C)***

7.0 TANGENT TO THE HYPERBOLA $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

(a) ***Point form –*** Equation of the tangent to the given hyperbola at the point (x_1, y_1) is $\frac{x x_1}{a^2} - \frac{y y_1}{b^2} = 1$.

Note – In general two tangents can be drawn from an external point $(x_1 y_1)$ to the hyperbola and they are $y - y_1 = m_1 (x - x_1)$ & $y - y_1 = m_2 (x - x_1)$, where m_1 & m_2 are roots of the equation $(x_1^2 - a^2) m^2 - 2 x_1 y_1 m + y_1^2 + b^2 = 0$. If $D < 0$, then ***no tangent*** can be drawn from (x_1, y_1) to ***the hyperbola.***

(b) ***Slope form –*** The equation of tangents of slope m to the given hyperbola is $y = mx \pm \sqrt{a^2m^2 - b^2}$.

Point of contact are $\left(\mp\frac{a^2m}{\sqrt{a^2m^2 - b^2}}, \frac{\mp b^2}{\sqrt{a^2m^2 - b^2}}\right)$

Note that there ***are two parallel tangents having the same slope m.***

(c) ***Parametric form –*** Equation of the tangent to the given hyperbola at the point $(a \sec\theta, b\tan\theta)$ is $\frac{x \sec\theta}{a} - \frac{y \tan\theta}{b} = 1$.

Note – Point of intersection of the tangents at θ_1 & θ_2 is $x = a\frac{\cos\left(\frac{\theta_1 - \theta_2}{2}\right)}{\cos\left(\frac{\theta_1 + \theta_2}{2}\right)}$, $y = b \tan\left(\frac{\theta_1 + \theta_2}{2}\right)$

Illustrations

Illustration 8. Find the equation of the tangent to the hyperbola $x^2 - 4y^2 = 36$ which is perpendicular to the line $x - y + 4 = 0$.

Solution Let m be the slope of the tangent. Since the tangent is perpendicular to the line $x - y = 0$

$\therefore \quad m \times 1 = -1 \Rightarrow m = -1$

Since $x^2 - 4y^2 = 36$ or $\frac{x^2}{36} - \frac{y^2}{9} = 1$

Comparing this with $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$\therefore \quad a^2 = 36$ and $b^2 = 9$

So the equation of tangents are $y = (-1)x \pm \sqrt{36 \times (-1)^2 - 9}$

$y = -x \pm \sqrt{27} \quad \Rightarrow \quad x + y \pm 3\sqrt{3} = 0$ ***Ans.***

Illustration 9. The locus of the point of intersection of two tangents of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ if the product of their slopes is c^2, will be -

(A) $y^2 - b^2 = c^2(x^2 + a^2)$
(B) $y^2 + b^2 = c^2(x^2 - a^2)$
(C) $y^2 + a^2 = c^2(x^2 - b^2)$
(D) $y^2 - a^2 = c^2(x^2 + b^2)$

Solution Equation of any tangent of the hyperbola with slope m is $y = mx \pm \sqrt{a^2m^2 - b^2}$

If it passes through (x_1, y_1) then

$(y_1 - mx_1)^2 = a^2m^2 - b^2 \quad \Rightarrow \quad (x_1^2 - a^2) m^2 - 2x_1y_1m + (y_1^2 + b^2) = 0$

If $m = m_1, m_2$ then as given $m_1m_2 = c^2 \quad \Rightarrow \frac{y_1^2 + b^2}{x_1^2 - a^2} = c^2$

Hence required locus will be : $y^2 + b^2 = c^2(x^2 - a^2)$ ***Ans.(B)***

Illustration 10. A common tangent to $9x^2 - 16y^2 = 144$ and $x^2 + y^2 = 9$ is -

(A) $y = 3\sqrt{\frac{2}{7}}x - \frac{15}{\sqrt{7}}$ (B) $y = 3\sqrt{\frac{2}{7}}x + \frac{15}{\sqrt{7}}$

(C) $y = -3\sqrt{\frac{2}{7}}x + \frac{15}{\sqrt{7}}$ (D) $y = -3\sqrt{\frac{2}{7}}x - \frac{15}{\sqrt{7}}$

Solution $\frac{x^2}{16} - \frac{y^2}{9} = 1$, $x^2 + y^2 = 9$

Equation of tangent $y = mx + \sqrt{16m^2 - 9}$ (for hyperbola)

Equation of tangent $y = m'x + 3\sqrt{1 + m'^2}$ (circle)

For common tangent $m = m'$ and $3\sqrt{1+m'^2} = \sqrt{16m^2 - 9}$

or $9 + 9m^2 = 16m^2 - 9$

or $7m^2 = 18 \Rightarrow m = \pm 3\sqrt{\frac{2}{7}}$

$\therefore$ required equation is $y = \pm 3\sqrt{\frac{2}{7}}\, x \pm 3\sqrt{1 + \frac{18}{7}}$

or $y = \pm 3\sqrt{\frac{2}{7}}x \pm \frac{15}{\sqrt{7}}$ ***Ans. (A,B,C,D)***

BEGINNER'S BOX-2

1. Find the condition for the line $\ell x + my + n = 0$ to touch the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

2. If the line $y = 5x + 1$ touch the hyperbola $\frac{x^2}{4} - \frac{y^2}{b^2} = 1$ $\{b > 4\}$, then -

(A) $b^2 = \frac{1}{5}$ (B) $b^2 = 99$ (C) $b^2 = 4$ (D) $b^2 = 100$

3. Find the equation of the tangent to the hyperbola $4x^2 - 9y^2 = 1$, which is parallel to the line $4y = 5x + 7$.

4. Find the equation of the tangent to the hyperbola $16x^2 - 9y^2 = 144$ at $\left(5, \frac{16}{3}\right)$.

5. Find the common tangent to the hyperbola $\frac{x^2}{16} - \frac{y^2}{9} = 1$ and an ellipse $\frac{x^2}{4} + \frac{y^2}{3} = 1$.

8.0 NORMAL TO THE HYPERBOLA $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

(a) ***Point form –*** The equation of the normal to the given hyperbola at the point P (x_1, y_1) on it is $\frac{a^2 x}{x_1} + \frac{b^2 y}{y_1} = a^2 + b^2 = a^2 e^2$.

(b) ***Slope form –*** The equation of normal of slope m to the given hyperbola is $y = mx \mp \frac{m(a^2 + b^2)}{\sqrt{(a^2 - m^2 b^2)}}$

foot of normal are $\left(\pm \frac{a^2}{\sqrt{(a^2 - m^2 b^2)}}, \mp \frac{mb^2}{\sqrt{(a^2 - m^2 b^2)}}\right)$

(c) ***Parametric form –*** The equation of the normal at the point ***P (a secθ , b tanθ)*** to the given hyperbola is $\frac{ax}{\sec\theta} + \frac{by}{\tan\theta} = a^2 + b^2 = a^2 e^2$.

Illustrations

Illustration 11. Line x cos α + y sin α = p is a normal to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, if -

(A) $a^2\sec^2\alpha - b^2\operatorname{cosec}^2\alpha = \frac{(a^2+b^2)^2}{p^2}$ (C) $a^2\sec^2\alpha + b^2\operatorname{cosec}^2\alpha = \frac{(a^2+b^2)^2}{p^2}$

(C) $a^2\cos^2\alpha - b^2\sin^2\alpha = \frac{(a^2+b^2)^2}{p^2}$ (D) $a^2\cos^2\alpha + b^2\sin^2\alpha = \frac{(a^2+b^2)^2}{p^2}$

Solution Equation of a normal to the hyperbola is $ax\cos\theta + by\cot\theta = a^2+b^2$

comparing it with the given line equation

$$\frac{a\cos\theta}{\cos\alpha}=\frac{b\cot\theta}{\sin\alpha}=\frac{a^2+b^2}{p} \quad\Rightarrow\quad \sec\theta=\frac{ap}{\cos\alpha(a^2+b^2)},\ \tan\theta=\frac{bp}{\sin\alpha(a^2+b^2)}$$

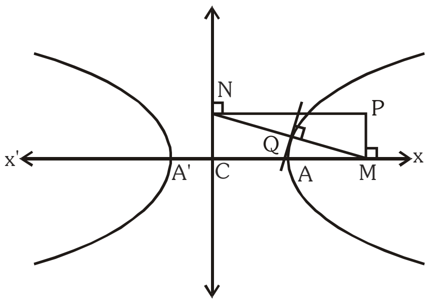
Eliminating θ, we get

$$\frac{a^2p^2}{\cos^2\alpha(a^2+b^2)^2}-\frac{b^2p^2}{\sin^2\alpha(a^2+b^2)^2}=1 \quad\Rightarrow\quad a^2\sec^2\alpha-b^2\operatorname{cosec}^2\alpha=\frac{(a^2+b^2)^2}{p^2}$$

Ans.(A)

Illustration 12. The normal to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ meets the axes in M and N, and lines MP and NP are drawn at right angles to the axes. Prove that the locus of P is hyperbola $(a^2x^2 - b^2y^2) = (a^2+b^2)^2$.

Solution Equation of normal at any point Q is $ax\cos\theta + by\cot\theta = a^2+b^2$



$$\therefore\quad M\equiv\left(\frac{a^2+b^2}{a}\sec\theta,\ 0\right),\ N\equiv\left(0,\ \frac{a^2+b^2}{b}\tan\theta\right)$$

$\therefore$ Let $P\equiv(h,k)$

$$\Rightarrow\quad h=\frac{a^2+b^2}{a}\sec\theta,\quad k=\frac{a^2+b^2}{b}\tan\theta$$

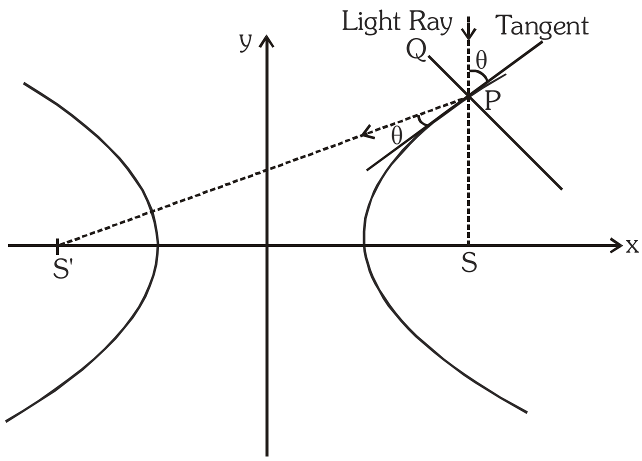
$$\Rightarrow\quad \frac{a^2h^2}{(a^2+b^2)}-\frac{b^2k^2}{(a^2+b^2)^2}=\sec^2\theta-\tan^2\theta=1$$

$\therefore$ locus of P is $(a^2x^2 - b^2y^2) = (a^2+b^2)$.

9.0 PROPERTIES ON TANGENT AND NORMAL

(a) Locus of the feet of the perpendicular drawn from focus of the hyperbola $\mathbf{\frac{x^2}{a^2}-\frac{y^2}{b^2}=1}$ upon any tangent is its auxiliary circle i.e. $\boldsymbol{x^2+y^2=a^2}$ & the product of lengths to these perpendiculars is $\boldsymbol{b^2}$ ***(semi Conjugate Axis)²***

(b) The portion of the tangent between the ***point of contact*** & ***the directrix*** subtends a ***right angle*** at the corresponding ***focus***.



(c) The tangent & normal at any point of a hyperbola ***bisect*** the angle between the ***focal radii***. This spells the ***reflection property of the hyperbola*** as "***An incoming light ray***" aimed towards one ***focus is reflected from the outer surface of the hyperbola towards the other focus.*** It follows that if an ellipse and a hyperbola have the same foci, they cut at ***right angles*** at any of their ***common point.***

Note that the ellipse $\mathbf{\frac{x^2}{a^2}+\frac{y^2}{b^2}=1}$ & the hyperbola $\mathbf{\frac{x^2}{a^2-k^2}-\frac{y^2}{k^2-b^2}}=1$ $(a > k > b > 0)$ are confocal and therefore orthogonal.

(d) The foci of the hyperbola and the points P and Q in which any tangent meets the tangents at the vertices are concyclic with PQ as diameter of the circle.

10.0 DIRECTOR CIRCLE

The locus of the intersection of tangents which are at ***right angles*** is known as the ***Director Circle*** of the hyperbola. The equation to the ***director circle is*** : $x^2 + y^2 = a^2 - b^2$.

If $b^2 < a^2$, this circle is ***real ;*** if $b^2 = a^2$ the ***radius*** of the ***circle is zero*** & it reduces to a ***point circle at the origin.*** In this case the ***centre is the only point*** from which the ***tangents at right angles*** can be drawn to the ***curve***.

If $b^2 > a^2$, the ***radius*** of the ***circle is imaginary***, so that there is ***no such circle*** & so ***no tangents at right angle*** can be drawn to the curve.

Note – Equations of chord of contact, chord with a given middle point, pair of tangents from an external point are to be interpreted in the similar way as in ellipse.

11.0 ASYMPTOTES

Definition – If the length of the perpendicular let fall from a point on a hyperbola to a straight line tends to zero as the point on the hyperbola moves to infinity along the hyperbola, then the straight line is called the ***Asymptote of the Hyperbola***.

To find the asymptote of the hyperbola –

Let $y = mx + c$ is the ***asymptote*** of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

Solving these two we get the quadratic as $(b^2 - a^2m^2)\, x^2 - 2a^2 mcx - a^2 (b^2 + c^2) = 0$ ....(1)

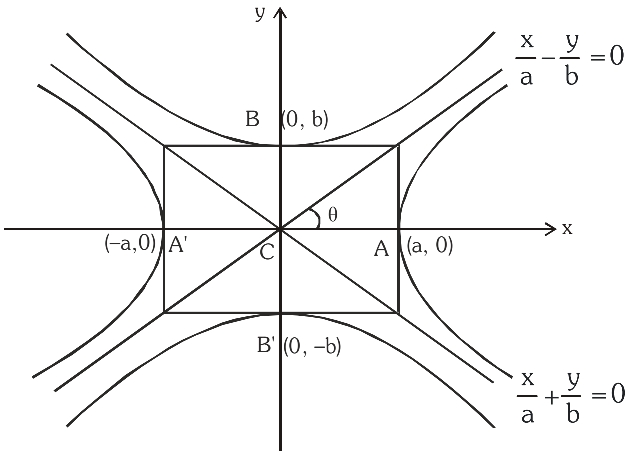
In order that $y = mx + c$ ***be an asymptote***, both ***roots*** of equation (1) must approach infinity, the conditions for which are : ***coefficient of*** $x^2 = 0$ & ***coefficient of*** $x = 0$.

$\Rightarrow$ $b^2 - a^2m^2 = 0$ or $m = \pm\frac{b}{a}$ &

$a^2 mc = 0 \Rightarrow c = 0$.

$\therefore$ Equations of asymptote are $\frac{x}{a} + \frac{y}{b} = 0$

and $\frac{x}{a} - \frac{y}{b} = 0$.



combined equation to the asymptotes $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$.

Particular Case :

When $b = a$ the asymptotes of the rectangular hyperbola.

$x^2 - y^2 = a^2$ are $y = \pm x$ which are at ***right angles***.

Note :

(i) ***Equilateral hyperbola $\Leftrightarrow$ rectangular hyperbola.***

(ii) If a ***hyperbola is equilateral*** then the ***conjugate hyperbola is also equilateral***.

(iii) A ***hyperbola*** and its ***conjugate*** have the ***same asymptote.***

(iv) The equation of the ***pair of asymptotes*** differ the ***hyperbola*** & ***the conjugate hyperbola*** by the ***same constant*** only.

(v) The asymptotes pass through the ***centre of the hyperbola*** & the bisectors of the angles between the asymptotes are the axes of the hyperbola.

(vi) The asymptotes of a hyperbola are the diagonals of the rectangle formed by the lines drawn through the extremities of each axis parallel to the other axis.

(vii) Asymptotes are the tangent to the hyperbola from the centre.

(viii) A simple method to find the co–ordinates of the centre of the hyperbola expressed as a general equation of degree 2 should be remembered as : Let $f(x, y) = 0$ represents a hyperbola.

Find $\frac{\partial f}{\partial x}$ & $\frac{\partial f}{\partial y}$. Then the point of intersection of $\frac{\partial f}{\partial x} = 0$ & $\frac{\partial f}{\partial y} = 0$ gives the ***centre*** of the ***hyperbola***.

Illustrations

Illustration 13. Find the asymptotes of the hyperbola $2x^2 + 5xy + 2y^2 + 4x + 5y = 0$. Find also the general equation of all the hyperbolas having the same set of asymptotes.

Solution Let $2x^2 + 5xy + 2y^2 + 4x + 5y + \lambda = 0$ be asymptotes. This will represent two straight line so

$$4\lambda + 25 - \frac{25}{2} - 8 - \frac{25}{4}\lambda = 0$$

$\Rightarrow$ $\lambda = 2$

$\Rightarrow$ $2x^2 + 5xy + 2y^2 + 4x + 5y + 2 = 0$ are asymptotes

$\Rightarrow$ $(2x + y + 2) = 0$ and $(x + 2y + 1) = 0$ are asymptotes

and $2x^2 + 5xy + 2y^2 + 4x + 5y + c = 0$ is general equation of hyperbola.

Illustration 14. Find the hyperbola whose asymptotes are $2x - y = 3$ and $3x + y - 7 = 0$ and which passes through the point (1, 1).

Solution The equation of the hyperbola differs from the equation of the asymptotes by a constant

$\Rightarrow$ The equation of the hyperbola with asymptotes $3x + y - 7 = 0$ and $2x - y = 3$ is $(3x + y - 7)(2x - y - 3) + k = 0$

It passes through (1, 1)

$\Rightarrow$ $k = -6$.

Hence the equation of the hyperbola is $(2x - y - 3)(3x + y - 7) = 6$.

BEGINNER'S BOX-3

1. Find the equation of normal to the hyperbola $\frac{x^2}{25} - \frac{y^2}{16} = 1$ at (5, 0).

2. Find the equation of normal to the hyperbola $\frac{x^2}{16} - \frac{y^2}{9} = 1$ at the point $\left(6, \frac{3}{2}\sqrt{5}\right)$.

3. Find the condition for the line $\ell x + my + n = 0$ is normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

4. Find the equation to the chords of the hyperbola $x^2 - y^2 = 9$ which is bisected at (5, –3)

5. Find the locus of the mid points of the chords of the circle $x^2 + y^2 = 16$ which are tangents to the hyperbola $9x^2 - 16y^2 = 144$.

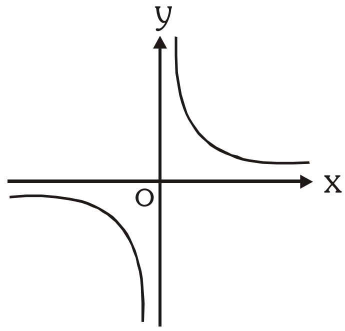
12.0 PROPERTIES OF ASYMPTOTES

(a) The tangent at any point P on a hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ with ***centre C***, meets the ***asymptotes in Q*** and ***R*** and cuts off a Δ ***CQR*** of ***constant area*** equal to ***ab*** from the asymptotes & the portion of the tangent intercepted between the asymptote is bisected at the point of contact . This implies that locus of the centre of the circle circumscribing the Δ ***CQR*** in case of a rectangular hyperbola is the ***hyperbola*** itself.

(b) If the angle between the asymptote of a hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is 2θ then the eccentricity of the ***hyperbola*** is $\sec\theta$.

13.0 RECTANGULAR HYPERBOLA

Rectangular hyperbola referred to its asymptotes as axis of coordinates.



(a) Equation is $xy = c^2$ with parametric representation

$x = ct, \quad y = c/t, \quad t \in R - \{0\}$.

(b) Equation of a chord joining the points $P(t_1)$ & $Q(t_2)$ is

$x + t_1 t_2 y = c(t_1 + t_2)$ with slope, $m = \dfrac{-1}{t_1 t_2}$

(c) Equation of the tangent at $P(x_1, y_1)$ is $\dfrac{x}{x_1} + \dfrac{y}{y_1} = 2$

& at P (t) is $\dfrac{x}{t} + ty = 2c$.

(d) Equation of normal is $y - \dfrac{c}{t} = t^2(x - ct)$

(e) Chord with a given middle point as (h, k) is $kx + hy = 2hk$.

NOTE

For the hyperbola, $xy = c^2$

(i) Vertices : (c, c) & $(-c, -c)$.

(ii) Foci : $(\sqrt{2}c, \sqrt{2}c)$ & $(-\sqrt{2}c, -\sqrt{2}c)$

(iii) Directrices : $x + y = \pm\sqrt{2}c$

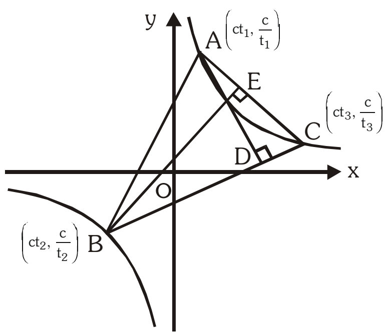
(iv) Latus rectum : $\ell = 2\sqrt{2}c = T.A = C.A$

Illustrations

Illustration 15. A triangle has its vertices on a rectangular hyperbola. Prove that the orthocentre of the triangle also lies on the same hyperbola.

Solution Let t_1, t_2 and t_3 are the vertices of the triangle ABC, described on the rectangular hyperbola $xy = c^2$.

$\therefore$ co-ordinates of A, B and C are $\left(ct_1, \dfrac{c}{t_1}\right), \left(ct_2, \dfrac{c}{t_2}\right)$ and $\left(ct_3, \dfrac{c}{t_3}\right)$ respectively



Now slope of BC is $\dfrac{\dfrac{c}{t_3} - \dfrac{c}{t_2}}{ct_3 - ct_2} = -\dfrac{1}{t_2 t_3}$

$\therefore$ Slope of AD is $t_2 t_3$

Equation of altitude AD is $y - \dfrac{c}{t_1} = t_2 t_3 (x - ct_1)$

or $t_1 y - c = x t_1 t_2 t_3 - c t_1^2 t_2 t_3$ .... (i)

Similarly equation of altitude BE is

$t_2 y - c = x t_1 t_2 t_3 - c t_1 t_2^2 t_3$ .... (ii)

Solving (i) and (ii), we get the orthocentre $\left(-\dfrac{c}{t_1 t_2 t_3}, -ct_1 t_2 t_3\right)$ which lies on $xy = c^2$.

GOLDEN KEY POINTS

- If from any point on the asymptote a straight line be drawn perpendicular to the transverse axis, the product of the segments of this line, intercepted between the point & the curve is always equal to the square of the semi conjugate axis.
- Perpendicular from the foci on either asymptote meet it in the same points as the corresponding directrix & the common points of intersection lie on the auxiliary circle.

BEGINNER'S BOX–4

1. If equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a rectangular hyperbola then write required conditions.

2. Find the equation of tangent at the point (1, 2) to the rectangular hyperbola $xy = 2$.

3. Prove that the locus of point, tangents from where to hyperbola $x^2 - y^2 = a^2$ inclined at an angle α & β with x-axis such that $\tan\alpha \tan\beta = 2$ is also a hyperbola. Find the eccentricity of this hyperbola.

SOME WORKED OUT ILLUSTRATIONS

Illustration 1. Chords of the circle $x^2 + y^2 = a^2$ touch the hyperbola $x^2/a^2 - y^2/b^2 = 1$. Prove that locus of their middle point is the curve $(x^2 + y^2)^2 = a^2x^2 - b^2y^2$.

Solution Let (h, k) be the mid-point of the chord of the circle $x^2 + y^2 = a^2$, so that its equation by $T = S_1$ is $hx + ky = h^2 + k^2$

or $\quad y = -\dfrac{h}{k}x + \dfrac{h^2+k^2}{k}$ i.e. of the form $y = mx + c$

It will touch the hyperbola if $c^2 = a^2m^2 - b^2$

$\therefore \quad \left(\dfrac{h^2+k^2}{k}\right)^2 = a^2\left(-\dfrac{h}{k}\right)^2 - b^2 \quad$ or $\quad (h^2 + k^2)^2 = a^2h^2 - b^2k^2$

Generalising, the locus of mid-point (h, k) is $(x^2 + y^2)^2 = a^2x^2 - b^2y^2$

Illustration 2. C is the centre of the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$. The tangent at any point P on this hyperbola meets the straight lines $bx - ay = 0$ and $bx + ay = 0$ in the points Q and R respectively. Show that $CQ\,.\,CR = a^2 + b^2$.

Solution P is $(a\sec\theta, b\tan\theta)$

Tangent at P is $\dfrac{x\sec\theta}{a} - \dfrac{y\tan\theta}{b} = 1$

It meets $bx - ay = 0$ $\quad$ i.e. $\dfrac{x}{a} = \dfrac{y}{b}$ in Q

$\therefore \quad$ Q is $\left(\dfrac{a}{\sec\theta - \tan\theta}, \dfrac{b}{\sec\theta - \tan\theta}\right)$

It meets $bx + ay = 0$ i.e. $\dfrac{x}{a} = -\dfrac{y}{b}$ in R.

$\therefore \quad$ R is $\left(\dfrac{a}{\sec\theta + \tan\theta}, \dfrac{-b}{\sec\theta + \tan\theta}\right)$

$\therefore \quad CQ.CR = \dfrac{\sqrt{(a^2+b^2)}}{\sec\theta - \tan\theta}.\dfrac{\sqrt{(a^2+b^2)}}{\sec\theta + \tan\theta} = a^2 + b^2 \qquad (\because \quad \sec^2\theta - \tan^2\theta = 1)$ ***Ans.***

Illustration 3. A circle of variable radius cuts the rectangular hyperbola $x^2 - y^2 = 9a^2$ in points P, Q, R and S. Determine the equation of the locus of the centroid of triangle PQR.

Solution Let the circle be $(x - h)^2 + (y - k)^2 = r^2$ where r is variable. Its intersection with $x^2 - y^2 = 9a^2$ is obtained by putting $y^2 = x^2 - 9a^2$.

$x^2 + x^2 - 9a^2 - 2hx + h^2 + k^2 - r^2 = 2k\sqrt{(x^2 - 9a^2)}$

or $\quad [2x^2 - 2hx + (h^2 + k^2 - r^2 - 9a^2)]^2 = 4k^2(x^2 - 9a^2)$

or $\quad 4x^4 - 8hx^3 + \ldots.. = 0$

$\therefore \quad$ Above gives the abscissas of the four points of intersection.

$\therefore \quad \Sigma x_1 = \dfrac{8h}{4} = 2h$

$\quad x_1 + x_2 + x_3 + x_4 = 2h$

Similarly $\quad y_1 + y_2 + y_3 + y_4 = 2k.$

Now if (α, β) be the centroid of ΔPQR, then $3\alpha = x_1 + x_2 + x_3$, $3\beta = y_1 + y_2 + y_3$

$\therefore \quad x_4 = 2h - 3\alpha,\ y_4 = 2k - 3\beta$

But (x_4, y_4) lies on $x^2 - y^2 = 9a^2$

$\therefore \quad (2h - 3\alpha)^2 + (2k - 3\beta)^2 = 9a^2$

Hence the locus of centroid (α, β) is $(2h - 3x)^2 + (2k - 3y)^2 = 9a^2$

or $\quad \left(x - \dfrac{2h}{3}\right)^2 + \left(y - \dfrac{2k}{3}\right)^2 = a^2$

Illustration 4. If a circle cuts a rectangular hyperbola $xy = c^2$ in A, B, C, D and the parameters of these four points be t_1, t_2, t_3 and t_4 respectively, then prove that :

(a) $t_1t_2t_3t_4 = 1$

(b) The centre of mean position of the four points bisects the distance between the centres of the two curves.

Solution (a) Let the equation of the hyperbola referred to rectangular asymptotes as axes be $xy = c^2$ or its parametric equation be

$x = ct,\ y = c/t$ .... (i)

and that of the circle be

$x^2 + y^2 + 2gx + 2fy + k = 0$ .... (ii)

Solving (i) and (ii), we get

$$c^2t^2 + \frac{c^2}{t^2} + 2gct + 2f\frac{c}{t} + k = 0$$

or $c^2t^4 + 2gct^3 + kt^2 + 2fct + c^2 = 0$ .... (iii)

Above equation being of fourth degree in t gives us the four parameters t_1, t_2, t_3, t_4 of the points of intersection.

$\therefore$ $t_1 + t_2 + t_3 + t_4 = -\dfrac{2gc}{c^2} = -\dfrac{2g}{c}$ .... (iv)

$t_1t_2t_3 + t_1t_2t_4 + t_3t_4t_1 + t_3t_4t_2$

$= -\dfrac{2fc}{c^2} = -\dfrac{2f}{c}$ .... (v)

$t_1t_2t_3t_4 = \dfrac{c^2}{c^2} = 1$. It proves (a) .... (vi)

Dividing (v) by (vi), we get

$$\frac{1}{t_1} + \frac{1}{t_2} + \frac{1}{t_3} + \frac{1}{t_4} = -\frac{2f}{c}$$ .... (vii)

(b) The centre of mean position of the four points of intersection is

$$\left[\frac{c}{4}(t_1 + t_2 + t_3 + t_4), \frac{c}{4}\left(\frac{1}{t_1} + \frac{1}{t_2} + \frac{1}{t_3} + \frac{1}{t_4}\right)\right] = \left[\frac{c}{4}\left(-\frac{2g}{c}\right), \frac{c}{4}\left(-\frac{2f}{c}\right)\right],$$ by (iv) and (vii)

$= (-g/2, -f/2)$

Above is clearly the mid-point of (0, 0) and $(-g, -f)$ i.e. the join of the centres of the two curves.

ANSWERS

BEGINNER'S BOX-1

1. $\sqrt{3}$ **2.** $7y^2 + 24xy - 24ax - 6ay + 15a^2 = 0$

3. 6, 4; $(\pm\sqrt{13}, 0)$; $\sqrt{13}/3$; 8/3 **4.** $x^2 - y^2 = 32$ **5.** $\sqrt{5}$ & 40 sq. units

BEGINNER'S BOX-2

1. $n^2 = a^2\ell^2 - b^2m^2$ **2.** B

3. $24y = 30x \pm \sqrt{161}$ **4.** $5x - 3y = 9$ **5.** $y = \pm x \pm \sqrt{7}$

BEGINNER'S BOX-3

1. $y = 0$; **2.** $8\sqrt{5}x + 18y = 75\sqrt{5}$ **3.** $\dfrac{a^2}{\ell^2} - \dfrac{b^2}{m^2} = \dfrac{(a^2+b^2)^2}{n^2}$

4. $5x + 3y = 16$ **5.** $(x^2 + y^2)^2 = 16x^2 - 9y^2$

BEGINNER'S BOX-4

1. $\Delta \neq 0$, $h^2 > ab$, $a + b = 0$ **2.** $2x + y = 4$ **3.** $e = \sqrt{3}$

EXERCISE – 1 | MCQ (SINGLE CHOICE CORRECT)

1. The eccentricity of the hyperbola $4x^2 - 9y^2 - 8x = 32$ is -

(A) $\frac{\sqrt{5}}{3}$ (B) $\frac{\sqrt{13}}{3}$ (C) $\frac{4}{3}$ (D) $\frac{3}{2}$

2. The locus of the point of intersection of the lines $\sqrt{3}x - y - 4\sqrt{3}k = 0$ and $\sqrt{3}kx + ky - 4\sqrt{3} = 0$ for different values of k is -

(A) ellipse (B) parabola (C) circle (D) hyperbola

3. If the latus rectum of an hyperbola be 8 and eccentricity be $\frac{3}{\sqrt{5}}$ then the equation of the hyperbola can be -

(A) $4x^2 - 5y^2 = 100$ (B) $5x^2 - 4y^2 = 100$ (C) $4x^2 + 5y^2 = 100$ (D) $5x^2 + 4y^2 = 100$

4. If the centre, vertex and focus of a hyperbola be (0,0), (4, 0) and (6,0) respectively, then the equation of the hyperbola is –

(A) $4x^2 - 5y^2 = 8$ (B) $4x^2 - 5y^2 = 80$ (C) $5x^2 - 4y^2 = 80$ (D) $5x^2 - 4y^2 = 8$

5. The equation of the hyperbola whose foci are (6,5), (–4, 5) and eccentricity 5/4 is–

(A) $\frac{(x-1)^2}{16} - \frac{(y-5)^2}{9} = 1$ (B) $\frac{x^2}{16} - \frac{y^2}{9} = 1$

(C) $\frac{(x-1)^2}{16} - \frac{(y-5)^2}{9} = -1$ (D) $\frac{(x-1)^2}{4} - \frac{(y-5)^2}{9} = 1$

6. The vertices of a hyperbola are at (0, 0) and (10,0) and one of its foci is at (18,0). The possible equation of the hyperbola is –

(A) $\frac{x^2}{25} - \frac{y^2}{144} = 1$ (B) $\frac{(x-5)^2}{25} - \frac{y^2}{144} = 1$ (C) $\frac{x^2}{25} - \frac{(y-5)^2}{144} = 1$ (D) $\frac{(x-5)^2}{25} - \frac{(y-5)^2}{144} = 1$

7. The length of the transverse axis of a hyperbola is 7 and it passes through the point (5, –2). The equation of the hyperbola is –

(A) $\frac{4}{49}x^2 - \frac{196}{51}y^2 = 1$ (B) $\frac{49}{4}x^2 - \frac{51}{196}y^2 = 1$ (C) $\frac{4}{49}x^2 - \frac{51}{196}y^2 = 1$ (D) none of these

8. AB is a double ordinate of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ such that ΔAOB (where 'O' is the origin) is an equilateral triangle, then the eccentricity e of the hyperbola satisfies -

(A) $e > \sqrt{3}$ (B) $1 < e < \frac{2}{\sqrt{3}}$ (C) $e = \frac{2}{\sqrt{3}}$ (D) $e > \frac{2}{\sqrt{3}}$

9. The equation of the tangent lines to the hyperbola $x^2 - 2y^2 = 18$ which are perpendicular to the line $y = x$ are -

(A) $y = x \pm 3$ (B) $y = -x \pm 3$ (C) $2x + 3y + 4 = 0$ (D) none of these

10. The equations to the common tangents to the two hyperbolas $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ and $\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$ are -

(A) $y = \pm x \pm \sqrt{b^2-a^2}$ (B) $y = \pm x \pm (a^2-b^2)$

(C) $y = \pm x \pm \sqrt{a^2-b^2}$ (D) $y = \pm x \pm \sqrt{a^2+b^2}$

11. Locus of the feet of the perpendiculars drawn from either foci on a variable tangent to the hyperbola $16y^2 - 9x^2 = 1$ is -

(A) $x^2 + y^2 = 9$ (B) $x^2 + y^2 = 1/9$ (C) $x^2 + y^2 = 7/144$ (D) $x^2 + y^2 = 1/16$

12. The equation of the common tangent to the parabola $y^2 = 8x$ and the hyperbola $3x^2 - y^2 = 3$ is -

(A) $2x \pm y + 1 = 0$ (B) $x \pm y + 1 = 0$ (C) $x \pm 2y + 1 = 0$ (D) $x \pm y + 2 = 0$

13. Equation of the chord of the hyperbola $25x^2 - 16y^2 = 400$ which is bisected at the point (6, 2) is -

(A) $16x - 75y = 418$ (B) $75x - 16y = 418$ (C) $25x - 4y = 400$ (D) none of these

14. The asymptotes of the hyperbola $xy - 3x - 2y = 0$ are-

(A) $x - 2 = 0$ and $y - 3 = 0$ (B) $x - 3 = 0$ and $y - 2 = 0$

(C) $x + 2 = 0$ and $y + 3 = 0$ (D) $x + 3 = 0$ and $y + 2 = 0$

15. If the product of the perpendicular distances from any point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ of eccentricity $e = \sqrt{3}$ on its asymptotes is equal to 6, then the length of the transverse axis of the hyperbola is -

(A) 3 (B) 6 (C) 8 (D) 12

16. Area of triangle formed by tangent to the hyperbola $xy = 16$ at (16, 1) and co-ordinate axes equals -

(A) 8 (B) 16 (C) 32 (D) 64

17. Locus of the middle points of the parallel chords with gradient m of the rectangular hyperbola $xy = c^2$ is -

(A) $y + mx = 0$ (B) $y - mx = 0$ (C) $my - x = 0$ (D) $my + x = 0$

EXERCISE – 2 MCQ (ONE OR MORE CHOICE CORRECT)

1. Variable circles are drawn touching two fixed circles externally, then locus of centre of variable circle is -
(A) parabola (B) ellipse (C) hyperbola (D) circle

2. The locus of the mid points of the chords passing through a fixed point (α, β) of the hyperbola, $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is -
(A) a circle with centre $\left(\frac{\alpha}{2},\frac{\beta}{2}\right)$ (B) an ellipse with centre $\left(\frac{\alpha}{2},\frac{\beta}{2}\right)$
(C) a hyperbola with centre $\left(\frac{\alpha}{2},\frac{\beta}{2}\right)$ (D) straight line through $\left(\frac{\alpha}{2},\frac{\beta}{2}\right)$

3. The locus of the foot of the perpendicular from the centre of the hyperbola $xy = c^2$ on a variable tangent is :
(A) $(x^2 - y^2)^2 = 4c^2 xy$ (B) $(x^2 + y^2)^2 = 2c^2 xy$ (C) $(x^2 - y^2) = 4c^2xy$ (D) $(x^2 + y^2)^2 = 4c^2 xy$

4. The equation to the chord joining two points (x_1, y_1) and (x_2, y_2) on the rectangular hyperbola $xy = c^2$ is -
(A) $\frac{x}{x_1+x_2}+\frac{y}{y_1+y_2}=1$ (B) $\frac{x}{x_1-x_2}+\frac{y}{y_1-y_2}=1$
(C) $\frac{x}{y_1-y_2}+\frac{y}{x_1-x_2}=1$ (D) $\frac{x}{y_1-y_2}+\frac{y}{x_1-x_2}=1$

5. The equation $9x^2 - 16y^2 - 18x + 32y - 151 = 0$ represent a hyperbola -
(A) The length of the transverse axes is 4 (B) Length of latus rectum is 9
(C) Equation of directrix is $x = \frac{21}{5}$ and $x = -\frac{11}{5}$ (D) none of these

6. From the points of the circle $x^2 + y^2 = a^2$, tangents are drawn to the hyperbola $x^2 - y^2 = a^2$; then the locus of the middle points of the chords of contact is -
(A) $(x^2 - y^2)^2 = a^2 (x^2 + y^2)$ (B) $(x^2 - y^2)^2 = 2a^2 (x^2 + y^2)$
(C) $(x^2 + y^2)^2 = a^2 (x^2 - y^2)$ (D) $2(x^2 - y^2)^2 = 3a^2 (x^2 + y^2)$

7. The tangent to the hyperbola, $x^2 - 3y^2 = 3$ at the point $(\sqrt{3}, 0)$ when associated with two asymptotes constitutes -
(A) isosceles triangle (B) an equilateral triangle
(C) a triangles whose area is $\sqrt{3}$ sq. units (D) a right isosceles triangle.

8. The asymptote of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ form with any tangent to the hyperbola a triangle whose area is $a^2 \tan\lambda$ in magnitude then its eccentricity is -
(A) $\sec\lambda$ (B) $\text{cosec}\,\lambda$ (C) $\sec^2\lambda$ (D) $\text{cosec}^2\lambda$

9. If θ is the angle between the asymptotes of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ with eccentricity e, then $\sec\frac{\theta}{2}$ can be -
(A) e (B) $e/2$ (C) $e/3$ (D) $\frac{e}{\sqrt{e^2-1}}$

10. If (5, 12) and (24, 7) are the focii of a conic passing through the origin, then the eccentricity of conic is -
(A) $\sqrt{386}/12$ (B) $\sqrt{386}/13$ (C) $\sqrt{386}/25$ (D) $\sqrt{386}/38$

11. The point of contact of line $5x + 12y = 9$ and hyperbola $x^2 - 9y^2 = 9$ will lie on
(A) $4x + 15y = 0$ (B) $7x + 12y = 19$ (C) $4x + 15y + 1 = 0$ (D) $7x - 12y = 19$

12. Equation $(2 + \lambda)x^2 - 2\lambda xy + (\lambda - 1)y^2 - 4x - 2 = 0$ represents a hyperbola if -
(A) $\lambda = 4$ (B) $\lambda = 1$ (C) $\lambda = 4/3$ (D) $\lambda = -1$

13. If the normal at point P to the rectangular hyperbola $x^2 - y^2 = 4$ meets the transverse and conjugate axes at A and B respectively and C is the centre of the hyperbola, then -
(A) PA = PC (B) PA = PB (C) PB = PC (D) AB = 2PC

14. Consider the hyperbola $3x^2 - y^2 - 24x + 4y - 4 = 0$ -
(A) its centre is (4, 2) (B) its centre is (2, 4)
(C) length of latus rectum = 24 (D) length of latus rectum = 12

Match the Column

Following question contains statements given in two columns, which have to be matched. The statements in ***Column-I*** are labelled as A, B, C and D while the statements in ***Column-II*** are labelled as p, q, r and s. Any given statement in ***Column-I*** can have correct matching with ***ONE*** statement in ***Column-II***.

15. Consider the hyperbola $9x^2 - 16y^2 - 36x + 96y + 36 = 0$.

	Column - I		Column - II
(A)	If directrices of the hyperbola are $y = k_1$ & $y = k_2$ then $k_1 + k_2$ is equal to	(p)	16
(B)	If foci of hyperbola are (a, b) & (a, c) then a + b + c is equal to	(q)	10
(C)	Product of the perpendiculars drawn from the foci upon its any tangent is	(r)	6
(D)	Distance between foci of the hyperbola is	(s)	8

EXERCISE – 3 SUBJECTIVE

1. The hyperbola $x^2/a^2 - y^2/b^2 = 1$ (a,b > 0) passes through the point of intersection of the lines, $7x+13y-87=0$ & $5x - 8y + 7 = 0$ and the latus rectum is $32\sqrt{2}/5$. Find 'a' & 'b'.

2. Find the eccentricity of the hyperbola whose latus rectum is half its transverse axis.

3. Find the centre, the foci, the directrices, the length of the latus rectum, the length & the equations of the axes $16x^2 - 9y^2 + 32x + 36y - 164 = 0$.

4. For the hyperbola $x^2/100 - y^2/25 = 1$, prove that

(a) eccentricity $= \sqrt{5}/2$

(b) SA . S'A = 25, where S & S' are the foci & A is the vertex.

5. Find the eccentricity of the conic represented by $x^2 - y^2 - 4x + 4y + 16 = 0$.

6. For what value of λ does the line $y = 3x + \lambda$ touch the hyperbola $9x^2 - 5y^2 = 45$.

7. Find the equation of the tangent to the hyperbola $x^2 - 4y^2 = 36$ which is perpendicular to the line $x - y + 4 = 0$.

8. Tangents are drawn to the hyperbola $3x^2 - 2y^2 = 25$ from the point (0, 5/2). Find their equations.

9. The variable chords of the hyperbola $x^2/a^2 - y^2/b^2 = 1$ (b > a) whose equation is $x \cos\alpha + y \sin\alpha = p$ subtends a right angle at the centre. Prove that it always touches a circle.

10. Find the asymptotes of the hyperbola $2x^2 - 3xy - 2y^2 + 3x - y + 8 = 0$. Also find the equation to the conjugate hyperbola & the equation of the principal axes of the curve.

EXERCISE - 4 RECAP OF AIEEE/JEE (MAIN)

1. The latus rectum of the hyperbola $16x^2 - 9y^2 = 144$ is- **[AIEEE-2002]**

(A) 16/3 (B) 32/3 (C) 8/3 (D) 4/3

2. The foci of the ellipse $\frac{x^2}{16} + \frac{y^2}{b^2} = 1$ and the hyperbola $\frac{x^2}{144} - \frac{y^2}{81} = \frac{1}{25}$ coincide. Then the value of b^2 is- **[AIEEE-2003]**

(A) 9 (B) 1 (C) 5 (D) 7

3. The locus of a point $P(\alpha, \beta)$ moving under the condition that the line $y = \alpha x + \beta$ is a tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is- **[AIEEE-2005]**

(A) a hyperbola (B) a parabola (C) a circle (D) an ellipse

4. For the hyperbola $\frac{x^2}{\cos^2\alpha} - \frac{y^2}{\sin^2\alpha} = 1$, which of the following remains constant when α varies ? **[AIEEE-2007, IIT-2003]**

(A) Abscissae of vertices (B) Abscissae of foci (C) Eccentricity (D) Directrix

5. The equation of the hyperbola whose foci are (–2,0) and (2, 0) and eccentricity is 2 is given by : **[AIEEE-2011]**

(A) $-3x^2 + y^2 = 3$ (B) $x^2 - 3y^2 = 3$ (C) $3x^2 - y^2 = 3$ (D) $-x^2 + 3y^2 = 3$

6. The eccentricity of the hyperbola whose length of the latus rectum is equal to 8 and the length of its conjugate axis is equal to half of the distance between its foci, is: **[JEE (Main)-2016]**

(A) $\sqrt{3}$ (B) $\frac{4}{3}$ (C) $\frac{4}{\sqrt{3}}$ (D) $\frac{2}{\sqrt{3}}$

7. A hyperbola passes through the point $P(\sqrt{2},\sqrt{3})$ and has foci at (± 2, 0). Then the tangent to this hyperbola at P also passes through the point : **[JEE (Main)-2017]**

(A) $(-\sqrt{2},-\sqrt{3})$ (B) $(3\sqrt{2},2\sqrt{3})$ (C) $(2\sqrt{2},3\sqrt{3})$ (D) $(\sqrt{3},\sqrt{2})$

8. Tangents are drawn to the hyperbola $4x^2 - y^2 = 36$ at the point P and Q. If these tangents intersect at the point T(0, 3) then the area (in sq. units) of ΔPTQ is - **[JEE (Main)-2018]**

(A) $54\sqrt{3}$ (B) $60\sqrt{3}$ (C) $36\sqrt{5}$ (D) $45\sqrt{5}$

9. A hyperbola has its centre at the origin, passes through the point (4, 2) and has transverse axis of length 4 along the x–axis. Then the eccentricity of the hyperbola is : **[JEE (Main)-2019]**

(A) $\frac{2}{\sqrt{3}}$ (B) $\frac{3}{2}$ (C) $\sqrt{3}$ (D) 2

10. Let $0 < \theta < \frac{\pi}{2}$. If the eccentricity of the hyperbola $\frac{x^2}{\cos^2\theta} - \frac{y^2}{\sin^2\theta} = 1$ is greater than 2, then the length of its latus rectum lies in the interval : **[JEE (Main)-2019]**

(A) (2, 3] (B) $(3, \infty)$ (C) (3/2, 2] (D) (1, 3/2]

11. Let $S=\left\{(x, y) \in R^2 : \frac{y^2}{1+r}-\frac{x^2}{1-r}=1\right\}$, where $r \pm 1$. Then S represents : **[JEE (Main)-2019]**

(A) A hyperbola whose eccentricity is $\frac{2}{\sqrt{r+1}}$, where $0 < r < 1$.

(B) An ellipse whose eccentricity is $\frac{1}{\sqrt{r+1}}$, where $r > 1$

(C) A hyperbola whose eccentricity is $\frac{2}{\sqrt{r+1}}$, when $0 < r < 1$.

(D) An ellipse whose eccentricity is $\sqrt{\frac{2}{r+1}}$, when $r > 1$

12. If a hyperbola has length of its conjugate axis equal to 5 and the distance between its foci is 13, then the eccentricity of the hyperbola is : **[JEE (Main)-2019]**

(A) 2 (B) $\frac{13}{6}$ (C) $\frac{13}{8}$ (D) $\frac{13}{12}$

13. If the vertices of a hyperbola be at (–2, 0) and (2, 0) and one of its foci be at (–3, 0), then which one of the following points does not lie on this hyperbola? **[JEE (Main)-2019]**

(A) $(4, \sqrt{15})$ (B) $(-6, 2\sqrt{10})$ (C) $(6, 5\sqrt{2})$ (D) $(2\sqrt{6}, 5)$

14. Equation of a common tangent to the parabola $y^2 = 4x$ and the hyperbole $xy = 2$ is : **[JEE (Main)-2019]**

(A) $x + 2y + 4 = 0$ (B) $x - 2y + 4 = 0$ (C) $x + y + 1 = 0$ (D) $4x + 2y + 1 = 0$

15. If the eccentricity of the standard hyperbola passing through the point (4,6) is 2, then the equation of the tangent to the hyperbola at (4,6) is- **[JEE (Main)-2019]**

(A) $2x - y - 2 = 0$ (B) $3x - 2y = 0$ (C) $2x - 3y + 10 = 0$ (D) $x - 2y + 8 = 0$

16. If the line $y = mx + 7\sqrt{3}$ is normal to the hyperbola $\frac{x^2}{24}-\frac{y^2}{18}=1$, then a value of m is **[JEE (Main)-2019]**

(A) $\frac{\sqrt{5}}{2}$ (B) $\frac{3}{\sqrt{5}}$ (C) $\frac{2}{\sqrt{5}}$ (D) $\frac{\sqrt{15}}{2}$

17. If a directrix of a hyperbola centred at the origin and passing through the point $(4, -2\sqrt{3})$ is $5x = 4\sqrt{5}$ and its eccentricity is e, then : **[JEE (Main)-2019]**

(A) $4e^4 - 24e^2 + 35 = 0$ (B) $4e^4 + 8e^2 - 35 = 0$
(C) $4e^4 - 12e^2 - 27 = 0$ (D) $4e^4 - 24e^2 + 27 = 0$

18. If $5x + 9 = 0$ is the directrix of the hyperbola $16x^2 - 9y^2 = 144$, then its corresponding focus is : **[JEE (Main)-2019]**

(A) $\left(-\frac{5}{3}, 0\right)$ (B) (5, 0) (C) (–5, 0) (D) $\left(\frac{5}{3}, 0\right)$

19. Let P be the point of intersection of the common tangents to the parabola $y^2 = 12x$ and the hyperbola $8x^2 - y^2 = 8$. If S and S' denote the foci of the hyperbola where S lies on the positive x-axis then P divides SS' in a ratio: **[JEE (Main)-2019]**

(A) 5:4 (B) 14:13 (C) 2:1 (D) 13:11

20. The equation of a common tangent to the curves, $y^2 = 16x$ and $xy = -4$ is : **[JEE (Main)-2019]**

(1) $x + y + 4 = 0$ (B) $x - 2y + 16 = 0$ (C) $2x - y + 2 = 0$ (D) $x - y + 4 = 0$

EXERCISE – 5 RECAP OF IIT-JEE/JEE (ADVANCED)

1. The equation of the common tangent to the curve $y^2 = 8x$ and $xy = -1$ is - ***[JEE 2002 Screening]***

(A) $3y = 9x + 2$ (B) $y = 2x + 1$ (C) $2y = x + 8$ (D) $y = x + 2$

2. For hyperbola $\frac{x^2}{\cos^2\alpha} - \frac{y^2}{\sin^2\alpha} = 1$ which of the following remains constant with change in α -

[JEE 2003 Screening]

(A) abscissae of vertices (B) abscissae of foci (C) eccentricity (D) directrix

3. The point of contact of the line $2x + \sqrt{6}\ y = 2$ and the hyperbola $x^2 - 2y^2 = 4$ is - ***[JEE 2004 Screening]***

(A) $(4\ ,\ -\sqrt{6})$ (B) $(\sqrt{6},\ 1)$ (C) $\left(\frac{1}{2}, \frac{1}{\sqrt{6}}\right)$ (D) $\left(\frac{1}{6}, \frac{3}{2}\right)$

4. Tangents are drawn from any point on the hyperbola $\frac{x^2}{9} - \frac{y^2}{4} = 1$ to the circle $x^2 + y^2 = 9$. Find the locus of mid-point of the chord of contact. ***[JEE 2005 Mains 4M out of 60]***

5. If a hyperbola passes through the focus of the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$ and its transverse and conjugate axis coincides with the major and minor axis of the ellipse and product of their eccentricities is 1, then - ***[JEE 2006, 5M)]***

(A) equation of hyperbola $\frac{x^2}{9} - \frac{y^2}{16} = 1$ (B) equation of hyperbola $\frac{x^2}{9} - \frac{y^2}{25} = 1$

(C) focus of hyperbola (5, 0) (D) focus of hyperbola $(5\sqrt{3}, 3)$

6. A hyperbola, having the transverse axis of length $2\sin\theta$, is confocal with the ellipse $3x^2 + 4y^2 = 12$. Then its equation is - ***[JEE 2007, 3M]***

(A) $x^2\text{cosec}^2\theta - y^2\sec^2\theta = 1$ (B) $x^2\sec^2\theta - y^2\text{cosec}^2\theta = 1$

(C) $x^2\sin^2\theta - y^2\cos^2\theta = 1$ (D) $x^2\cos^2\theta - y^2\sin^2\theta = 1$

7. Match the column - ***[2007, 6M]***

Column I	***Column II***
(A) Two intersecting circles	(p) have a common tangent
(B) Two mutually external circles	(q) have a common normal
(C) Two circles, one strictly inside the other	(r) do not have a common tangent
(D) Two branches of a hyperbola	(s) do not have a common normal

8. Let a and b be non-zero real numbers. Then, the equation $(ax^2 + by^2 + c)\ (x^2 - 5xy + 6y^2) = 0$ represents -

(A) four straight lines, when $c = 0$ and a, b are of the same sign

(B) two straight lines and a circle, when $a = b$, and c is of sign opposite to that of a

(C) two straight lines and a hyperbola, when a and b are of the same sign and c is of sign opposite to that of a

(D) a circle and an ellipse, when a and b are of the same sign and c is of sign opposite to that of a

[JEE 2008, 3M, –1M]

9. Consider a branch of the hyperbola $x^2 - 2y^2 - 2\sqrt{2}x - 4\sqrt{2}y - 6 = 0$ with vertex at the point A. Let B be one of the end points of its latus rectum. If C is the focus of the hyperbola nearest to the point A, then the area of the triangle ABC is - ***[JEE 2008, 3M, –1M]***

(A) $1 - \sqrt{\frac{2}{3}}$ (B) $\sqrt{\frac{3}{2}} - 1$ (C) $1 + \sqrt{\frac{2}{3}}$ (D) $\sqrt{\frac{3}{2}} + 1$

10. An ellipse intersects the hyperbola $2x^2 - 2y^2 = 1$ orthogonally. The eccentricity of the ellipse is reciprocal of that of the hyperbola. If the axes of the ellipse are along the coordinate axes, then :- ***[JEE 2009, 4M, –1M]***

(A) equation of ellipse is $x^2 + 2y^2 = 2$ (B) the foci of ellipse are $(\pm 1, 0)$

(C) equation of ellipse is $x^2 + 2y^2 = 4$ (D) the foci of ellipse are $\left(\pm\sqrt{2}, 0\right)$

Paragraph for Question 11 and 12 ***[JEE 10, (3M each), –1M]***

The circle $x^2 + y^2 - 8x = 0$ and hyperbola $\frac{x^2}{9} - \frac{y^2}{4} = 1$ intersect at the points A and B.

11. Equation of a common tangent with positive slope to the circle as well as to the hyperbola is -

(A) $2x - \sqrt{5}y - 20 = 0$ (B) $2x - \sqrt{5}y + 4 = 0$ (C) $3x - 4y + 8 = 0$ (D) $4x - 3y + 4 = 0$

12. Equation of the circle with AB as its diameter is -

(A) $x^2 + y^2 - 12x + 24 = 0$ (B) $x^2 + y^2 + 12x + 24 = 0$

(C) $x^2 + y^2 + 24x - 12 = 0$ (D) $x^2 + y^2 - 24x - 12 = 0$

13. The line $2x + y = 1$ is tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. If this line passes through the point of intersection of the nearest directrix and the x-axis, then the eccentricity of the hyperbola is ***[JEE 10, 3M]***

14. Let the eccentricity of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ be reciprocal to that of the ellipse $x^2 + 4y^2 = 4$. If the hyperbola passes through a focus of the ellipse, then - ***[JEE 2011, 4M]***

(A) the equation of the hyperbola is $\frac{x^2}{3} - \frac{y^2}{2} = 1$ (B) a focus of the hyperbola is (2,0)

(C) the eccentricity of the hyperbola is $\sqrt{\frac{5}{3}}$ (D) the equation of the hyperbola is $x^2 - 3y^2 = 3$

15. Let P(6, 3) be a point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. If the normal at the point P intersects the x-axis at (9, 0), then the eccentricity of the hyperbola is - ***[JEE 2011, 3M]***

(A) $\sqrt{\frac{5}{2}}$ (B) $\sqrt{\frac{3}{2}}$ (C) $\sqrt{2}$ (D) $\sqrt{3}$

16. Tangents are drawn to the hyperbola $\frac{x^2}{9} - \frac{y^2}{4} = 1$, parallel to the straight line $2x - y = 1$. The points of contact of the tangents on the hyperbola are ***[JEE 2012, 4M]***

(A) $\left(\frac{9}{2\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ (B) $\left(-\frac{9}{2\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$ (C) $\left(3\sqrt{3}, -2\sqrt{2}\right)$ (D) $\left(-3\sqrt{3}, 2\sqrt{2}\right)$

17. Consider the hyperbola $H : x^2 - y^2 = 1$ and a circle S with center $N(x_2, 0)$. Suppose that H and S touch each other at a point $P(x_1, y_1)$ with $x_1 > 1$ and $y_1 > 0$. The common tangent to H and S at P intersects the x-axis at point M. If (l, m) is the centroid of the triangle ΔPMN, then the correct expression(s) is(are)

(A) $\frac{dl}{dx_1} = 1 - \frac{1}{3x_1^2}$ for $x_1 > 1$

(B) $\frac{dm}{dx_1} = \frac{x_1}{3\left(\sqrt{x_1^2 - 1}\right)}$ for $x_1 > 1$ **[JEE 2015]**

(C) $\frac{dl}{dx_1} = 1 + \frac{1}{3x_1^2}$ for $x_1 > 1$

(D) $\frac{dm}{dy_1} = \frac{1}{3}$ for $y_1 > 1$

18. If $2x - y + 1 = 0$ is tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{16} = 1$, then which of the following CANNOT be sides of a right angled triangle ? **[JEE 2017]**

(A) 2a, 4, 1 (B) 2a, 8, 1 (C) a, 4, 1 (D) a, 4, 2

* Column 1, 2 and 3 contain conics, equation of tangents to the conics and points of contact, respectively. **[JEE 2017]**

Column 1	***Column 2***	***Column 3***
(I) $x^2 + y^2 = a^2$	(i) $my = m^2x + a$	(P) $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$
(II) $x^2 + a^2y^2 = a^2$	(ii) $y = mx + a\sqrt{m^2+1}$	(Q) $\left(\frac{-ma}{\sqrt{m^2+1}}, \frac{a}{\sqrt{m^2+1}}\right)$
(III) $y^2 = 4ax$	(iii) $y = mx + \sqrt{a^2m^2-1}$	(R) $\left(\frac{-a^2m}{\sqrt{a^2m^2+1}}, \frac{1}{\sqrt{a^2m^2+1}}\right)$
(IV) $x^2 - a^2y^2 = a^2$	(iv) $y = mx + \sqrt{a^2m^2+1}$	(S) $\left(\frac{-a^2m}{\sqrt{a^2m^2-1}}, \frac{-1}{\sqrt{a^2m^2-1}}\right)$

19. The tangent to a suitable conic (Column 1) at $\left(\sqrt{3}, \frac{1}{2}\right)$ is found to be $\sqrt{3}x + 2y = 4$, then which of the following options is the only ***CORRECT*** combination ?

(A) (II) (iii) (R) (B) (IV) (iv) (S) (C) (IV) (iii) (S) (D) (II) (iv) (R)

20. If a tangent to a suitable conic (Column 1) is found to be $y = x + 8$ and its point of contact is (8,16), then which of the following options is the only ***CORRECT*** combination ?

(A) (III) (i) (P) (B) (III) (ii) (Q) (C) (II) (iv) (R) (D) (I) (ii) (Q)

21. For $a = \sqrt{2}$, if a tangent is drawn to a suitable conic (Column 1) at the point of contact (–1,1), then which of the following options is the only ***CORRECT*** combination for obtaining its equation ?

(A) (II) (ii) (Q) (B) (III) (i) (P) (C) (I) (i) (P) (D) (I) (ii) (Q)

22. Let H : $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, where a > b > 0, be a hyperbola in the xy-plane whose conjugate axis LM subtends an angle of 60° at one of its vertices N. Let the area of the triangle LMN be $4\sqrt{3}$.

LIST-I	**LIST-II**
P. The length of the conjugate axis of H is	**1.** 8
Q. The eccentricity of H is	**2.** $\frac{4}{\sqrt{3}}$
R. The distance between the foci of H is	**3.** $\frac{2}{\sqrt{3}}$
S. The length of the latus rectum of H is	**4.** 4

The correct option is : ***[JEE 2018]***

(A) P $\rightarrow$ 4; Q $\rightarrow$ 2, R $\rightarrow$ 1; S $\rightarrow$ 3

(B) P $\rightarrow$ 4; Q $\rightarrow$ 3; R $\rightarrow$ 1; S $\rightarrow$ 2

(C) P $\rightarrow$ 4; Q $\rightarrow$ 1, R $\rightarrow$ 3; S $\rightarrow$ 2

(D) P $\rightarrow$ 3; Q $\rightarrow$ 4; R $\rightarrow$ 2; S $\rightarrow$ 1

ANSWER KEY

EXERCISE-1

Que.	1	2	3	4	5	6	7	8	9	10
Ans.	B	D	A	C	A	B	C	D	B	C
Que.	11	12	13	14	15	16	17			
Ans.	D	A	B	A	B	C	A			

EXERCISE-2

Que.	1	2	3	4	5	6	7	8	9	10
Ans.	C	C	D	A	C	A	ABC	A	AD	AD
Que.	11	12	13	14						
Ans.	AB	BD	ABCD	AC						

- ***Match the Column*** **15.** (A) $\rightarrow$ (r) ; (B) $\rightarrow$ (s); (C) $\rightarrow$ (p); (D) $\rightarrow$ (q)

EXERCISE-3

1. $a^2 = 25/2$; $b^2 = 16$ **2.** $\sqrt{\dfrac{3}{2}}$

3. $(-1, 2)$; $(4, 2)$ & $(-6, 2)$; $5x - 4 = 0$ & $5x + 14 = 0$; $\dfrac{32}{3}$; 6; 8; $y - 2 = 0$; $x + 1 = 0$

5. $\sqrt{2}$ **6.** $\lambda = \pm 6$ **7.** $x + y \pm 3\sqrt{3} = 0$ **8.** $3x + 2y - 5 = 0$; $3x - 2y + 5 = 0$

10. $x - 2y + 1 = 0$; $2x + y + 1 = 0$; $2x^2 - 3xy - 2y^2 + 3x - y - 6 = 0$; $3x - y + 2 = 0$; $x + 3y = 0$

EXERCISE-4

Que.	1	2	3	4	5	6	7	8	9	10
Ans.	B	D	A	B	C	D	C	D	A	B
Que.	11	12	13	14	15	16	17	18	19	20
Ans.	D	D	C	A	A	C	A	C	A	D

EXERCISE-5

1. D **2.** B **3.** A **4.** $\dfrac{x^2}{9} - \dfrac{y^2}{4} = \left(\dfrac{x^2 + y^2}{9}\right)^2$ **5.** A,C **6.** A

7. (A) $\rightarrow$ (p, q) ; (B) $\rightarrow$ (p, q) ; (C) $\rightarrow$ (q, r) ; (D) $\rightarrow$ (q, r) **8.** B **9.** B **10.** A, B

11. B **12.** A **13.** 2 **14.** B,D **15.** B **16.** A,B

17. A,B,D **18.** B,C,D **19.** D **20.** A **21.** D **22.** B

SOLUTION OF TRIANGLE

Recap of Early Classes

Some times we feel difficulty while solving problems of triangle, quadilatrel, polygons with the help of plane geometry. By the help of elementry laws of properties of triangle the problem solving becomes easier.

Index

SOLUTIONS OF TRIANGLE

The process of calculating the sides and angles of triangle using given information is called solution of triangle. In a ΔABC, the angles are denoted by capital letters A, B and C and the length of the sides opposite these angle are denoted by small letter a, b and c respectively.

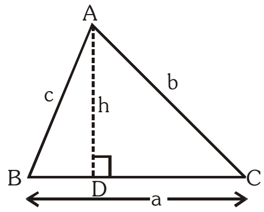
1.0 SINE FORMULAE

SL AL

In any triangle ABC

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = \lambda = \frac{abc}{2\Delta} = 2R$$

where R is circumradius and Δ is area of triangle.



Illustrations

Illustration 1. Angles of a triangle are in 4 : 1 : 1 ratio. The ratio between its greatest side and perimeter is

(A) $\frac{3}{2+\sqrt{3}}$ (B) $\frac{\sqrt{3}}{2+\sqrt{3}}$ (C) $\frac{\sqrt{3}}{2-\sqrt{3}}$ (D) $\frac{1}{2+\sqrt{3}}$

Solution. Angles are in ratio 4 : 1 : 1.

$\Rightarrow$ angles are 120°, 30°, 30°.

If sides opposite to these angles are a, b, c respectively, then a will be the greatest side.

Now from sine formula $\frac{a}{\sin 120^\circ} = \frac{b}{\sin 30^\circ} = \frac{c}{\sin 30^\circ}$

$$\Rightarrow \quad \frac{a}{\sqrt{3}/2} = \frac{b}{1/2} = \frac{c}{1/2}$$

$$\Rightarrow \quad \frac{a}{\sqrt{3}} = \frac{b}{1} = \frac{c}{1} = k \text{ (say)}$$

then $a = \sqrt{3}k$, perimeter $= (2+\sqrt{3})k$

$\therefore$ required ratio $= \frac{\sqrt{3}k}{(2+\sqrt{3})k} = \frac{\sqrt{3}}{2+\sqrt{3}}$ **Ans. (B)**

Illustration 2. In triangle ABC, if b = 3, c = 4 and $\angle B = \pi/3$, then number of such triangles is -

(A) 1 (B) 2 (C) 0 (D) infinite

Solution. Using sine formulae $\frac{\sin B}{b} = \frac{\sin C}{c}$

$$\Rightarrow \quad \frac{\sin \pi/3}{3} = \frac{\sin C}{4}$$

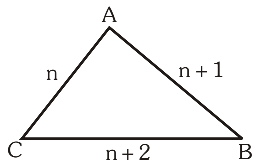
$$\Rightarrow \quad \frac{\sqrt{3}}{6} = \frac{\sin C}{4}$$

$\Rightarrow \quad \sin C = \frac{2}{\sqrt{3}} > 1$ which is not possible.

Hence there exist no triangle with given elements. **Ans. (C)**

Illustration 3. The sides of a triangle are three consecutive natural numbers and its largest angle is twice the smallest one. Determine the sides of the triangle.

Solution. Let the sides be n, n + 1, n + 2 cms.



i.e. AC = n, AB = n + 1, BC = n + 2

Smallest angle is B and largest one is A.

Here, $\angle A = 2\angle B$

Also, $\angle A + \angle B + \angle C = 180°$

$\Rightarrow \quad 3\angle B + \angle C = 180° \Rightarrow \angle C = 180° - 3\angle B$

We have, sine law as,

$$\frac{\sin A}{n+2} = \frac{\sin B}{n} = \frac{\sin C}{n+1} \quad \Rightarrow \frac{\sin 2B}{n+2} = \frac{\sin B}{n} = \frac{\sin(180-3B)}{n+1}$$

$$\Rightarrow \quad \underset{(i)}{\frac{\sin 2B}{n+2}} = \underset{(ii)}{\frac{\sin B}{n}} = \underset{(iii)}{\frac{\sin 3B}{n+1}}$$

from (i) and (ii);

$$\frac{2\sin B\cos B}{n+2} = \frac{\sin B}{n} \Rightarrow \cos B = \frac{n+2}{2n} \qquad \text{..... (iv)}$$

and from (ii) and (iii);

$$\frac{\sin B}{n} = \frac{3\sin B - 4\sin^3 B}{n+1} \Rightarrow \frac{\sin B}{n} = \frac{\sin B(3-4\sin^2 B)}{n+1}$$

$$\Rightarrow \quad \frac{n+1}{n} = 3 - 4(1-\cos^2 B) \qquad \text{..... (v)}$$

from (iv) and (v), we get

$$\frac{n+1}{n} = -1 + 4\left(\frac{n+2}{2n}\right)^2 \Rightarrow \frac{n+1}{n} + 1 = \left(\frac{n^2+4n+4}{n^2}\right)$$

$$\Rightarrow \quad \frac{2n+1}{n} = \frac{n^2+4n+4}{n^2} \Rightarrow 2n^2 + n = n^2 + 4n + 4$$

$\Rightarrow \quad n^2 - 3n - 4 = 0 \quad \Rightarrow (n-4)(n+1) = 0$

$n = 4$ or -1

where $n \neq -1$

$\therefore \quad n = 4$. Hence the sides are 4, 5, 6 ***Ans.***

2.0 COSINE FORMULAE

SL AL

(a) $\cos A = \dfrac{b^2+c^2-a^2}{2bc}$ ***(b)*** $\cos B = \dfrac{c^2+a^2-b^2}{2ca}$

(c) $\cos C = \dfrac{a^2+b^2-c^2}{2ab}$ or $a^2 = b^2 + c^2 - 2bc\cos A$

Illustrations

Illustration 4. In a triangle ABC, if B = 30° and $c = \sqrt{3}\,b$, then A can be equal to -

(A) 45° (B) 60° (C) 90° (D) 120°

Solution. We have $\cos B = \dfrac{c^2+a^2-b^2}{2ca} \Rightarrow \dfrac{\sqrt{3}}{2} = \dfrac{3b^2+a^2-b^2}{2\times\sqrt{3}b\times a}$

$\Rightarrow$ $a^2 - 3ab + 2b^2 = 0 \Rightarrow (a-2b)(a-b) = 0$

$\Rightarrow$ Either $a = b \Rightarrow A = 30°$

or $a = 2b \Rightarrow a^2 = 4b^2 = b^2 + c^2 \Rightarrow A = 90°$. **Ans. (C)**

Illustration 5. In a triangle ABC, $(a^2 - b^2 - c^2) \tan A + (a^2 - b^2 + c^2) \tan B$ is equal to -

(A) $(a^2 + b^2 - c^2) \tan C$ (B) $(a^2 + b^2 + c^2) \tan C$

(C) $(b^2 + c^2 - a^2) \tan C$ (D) none of these

Solution. Using cosine law :

The given expression is equal to $-2\,bc \cos A \tan A + 2\,ac \cos B \tan B$

$$= 2abc\left(-\frac{\sin A}{a} + \frac{\sin B}{b}\right) = 0$$ **Ans. (D)**

Illustration 6. If in a triangle ABC, $\dfrac{2\cos A}{a} + \dfrac{\cos B}{b} + \dfrac{2\cos C}{c} = \dfrac{a}{bc} + \dfrac{b}{ac}$, find the $\angle A$ =

(A) 90° (B) 60° (C) 30° (D) None of these

Solution. We have $\dfrac{2\cos A}{a} + \dfrac{\cos B}{b} + \dfrac{2\cos C}{c} = \dfrac{a}{bc} + \dfrac{b}{ac}$

Multiplying both sides of abc, we get

$\Rightarrow$ $2bc \cos A + ac \cos B + 2ab \cos C = a^2 + b^2$

$\Rightarrow$ $\left(b^2+c^2-a^2\right) + \dfrac{(a^2+c^2-b^2)}{2} + (a^2+b^2-c^2) = a^2+b^2$

$\Rightarrow$ $c^2 + a^2 - b^2 = 2a^2 - 2b^2$

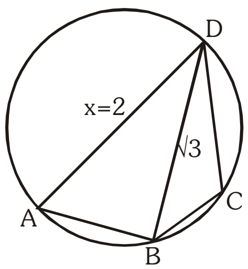
$\Rightarrow$ $b^2 + c^2 = a^2$

$\therefore$ ΔABC is right angled at A.

$\Rightarrow$ $\angle A = 90°$ **Ans. (A)**

****Illustration 7.*** A cyclic quadrilateral ABCD of area $\dfrac{3\sqrt{3}}{4}$ is inscribed in unit circle. If one of its side AB = 1, and the diagonal $BD = \sqrt{3}$, find lengths of the other sides.

Solution. $AB = 1$, $BD = \sqrt{3}$, $OA = OB = OD = 1$



The given circle of radius 1 is also circumcircle of Δ ABD

$\Rightarrow$ $R = 1$ for ΔABD

$\Rightarrow$ $\dfrac{a}{\sin A} = 2R \Rightarrow A = 60°$

and hence $C = 120°$

Also by cosine rule on ΔABD, $\left(\sqrt{3}\right)^2 = 1^2 + x^2 - 2x\cos 60°$

$\Rightarrow$ $x = 2$

Now, area ABCD = $\Delta ABD + \Delta BCD$

$\Rightarrow \quad \frac{3\sqrt{3}}{4} = \frac{1}{2}(1.2.\sin 60^\circ) + \frac{1}{2}(c.d.\sin 120^\circ)$

$\Rightarrow \quad cd = 1$, or $c^2d^2 = 1$

Also by cosine rule on triangle BCD we have

$$\left(\sqrt{3}\right)^2 = c^2 + d^2 - 2cd\cos 120^\circ = c^2 + d^2 + cd$$

$\Rightarrow \quad c^2 + d^2 = 2$ or $cd = 1$

$\Rightarrow \quad c^2$ and d^2 are the roots of $t^2 - 2t + 1 = 0$

$\therefore \quad c^2 = d^2 = 1 \therefore BC = 1 = CD$ and $AD = x = 2$.

3.0 PROJECTION FORMULAE

AL

(a) $b \cos C + c \cos B = a$ ***(b)*** $c \cos A + a \cos C = b$ ***(c)*** $a \cos B + b \cos A = c$

Illustrations

Illustration 8. In a ΔABC, $c\cos^2\frac{A}{2} + a\cos^2\frac{C}{2} = \frac{3b}{2}$, then show a, b, c are in A.P.

Solution. Here, $\frac{c}{2}(1+\cos A) + \frac{a}{2}(1+\cos C) = \frac{3b}{2}$

$\Rightarrow \quad a + c + (c \cos A + a \cos C) = 3b$

$\Rightarrow \quad a + c + b = 3b$ {using projection formula}

$\Rightarrow \quad a + c = 2b$

which shows a, b, c are in A.P.

BEGINNER'S BOX-1

TOPIC COVERED : SINE FORMULAE, COSINE FORMULAE, PROJECTION FORMULAE

1. If in a ΔABC, $\angle A = \frac{\pi}{6}$ and $b : c = 2 : \sqrt{3}$, find $\angle B$.

2. Show that, in any ΔABC : $a \sin(B - C) + b \sin(C - A) + c \sin(A - B) = 0$.

3. If in a ΔABC, $\frac{\sin A}{\sin C} = \frac{\sin(A-B)}{\sin(B-C)}$, show that a^2, b^2, c^2 are in A.P.

4. If in a ΔABC, $\angle A = 3\angle B$, then prove that $\sin B = \frac{1}{2}\sqrt{\frac{3b-a}{b}}$

5. If $a : b : c = 4 : 5 : 6$, then show that $\angle C = 2\angle A$.

6. In any ΔABC, prove that

(a) $\frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c} = \frac{a^2+b^2+c^2}{2abc}$

(b) $\frac{b^2}{a}\cos A + \frac{c^2}{b}\cos B + \frac{a^2}{c}\cos C = \frac{a^4+b^4+c^4}{2abc}$.

7. In a ΔABC, if $\angle A = \frac{\pi}{4}$, $\angle B = \frac{5\pi}{12}$, show that $a + c\sqrt{2} = 2b$.

8. In a ΔABC, prove that :

(a) $b(a\cos C - c\cos A) = a^2 - c^2$

(b) $2\left(b\cos^2\frac{C}{2} + c\cos^2\frac{B}{2}\right) = a + b + c$

4.0 NAPIER'S ANALOGY (TANGENT RULE)

AL

(a) $\tan\left(\frac{B-C}{2}\right) = \frac{b-c}{b+c}\cot\frac{A}{2}$ ***(b)*** $\tan\left(\frac{C-A}{2}\right) = \frac{c-a}{c+a}\cot\frac{B}{2}$ ***(c)*** $\tan\left(\frac{A-B}{2}\right) = \frac{a-b}{a+b}\cot\frac{C}{2}$

Illustrations

Illustration 9. In a ΔABC, the tangent of half the difference of two angles is one-third the tangent of half the sum of the angles. Determine the ratio of the sides opposite to the angles.

Solution. Here, $\tan\left(\frac{A-B}{2}\right) = \frac{1}{3}\tan\left(\frac{A+B}{2}\right)$ .... (i)

using Napier's analogy, $\tan\left(\frac{A-B}{2}\right) = \frac{a-b}{a+b}.\cot\left(\frac{C}{2}\right)$ .... (ii)

from (i) & (ii) ;

$$\frac{1}{3}\tan\left(\frac{A+B}{2}\right) = \frac{a-b}{a+b}.\cot\left(\frac{C}{2}\right)$$

$$\Rightarrow \quad \frac{1}{3}\cot\left(\frac{C}{2}\right) = \frac{a-b}{a+b}.\cot\left(\frac{C}{2}\right)$$

{as $A + B + C = \pi$

$$\therefore \quad \tan\left(\frac{B+C}{+}\right) = \tan\left(\frac{\pi}{2} - \frac{C}{2}\right) = \cot\frac{C}{2}\}$$

$$\Rightarrow \quad \frac{a-b}{a+b} = \frac{1}{3} \quad \text{or} \quad 3a - 3b = a + b$$

$$2a = 4b \quad \text{or} \quad \frac{a}{b} = \frac{2}{1} \Rightarrow \frac{b}{a} = \frac{1}{2}$$

Thus the ratio of the sides opposite to the angles is $b : a = 1 : 2$. ***Ans.***

5.0 HALF ANGLE FORMULAE

AL

$s = \frac{a+b+c}{2}$ = semi-perimeter of triangle.

(a) (i) $\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$ (ii) $\sin\frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{ca}}$ (iii) $\sin\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}}$

(b) (i) $\cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$ (ii) $\cos\frac{B}{2} = \sqrt{\frac{s(s-b)}{ca}}$ (iii) $\cos\frac{C}{2} = \sqrt{\frac{s(s-c)}{ab}}$

(c) (i) $\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$ (ii) $\tan\frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{s(s-b)}}$ (iii) $\tan\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$

$$= \frac{\Delta}{s(s-a)} = \frac{\Delta}{s(s-b)} = \frac{\Delta}{s(s-c)}$$

5.1 Area of Triangle

$$\Delta = \sqrt{s(s-a)(s-b)(s-c)} = \frac{1}{2}bc\sin A = \frac{1}{2}ca\sin B$$

$$= \frac{1}{2}ab\sin C = \frac{1}{2}ap_1 = \frac{1}{2}bp_2 = \frac{1}{2}cp_3,$$

where p_1, p_2, p_3 are altitudes from vertices A,B,C respectively.

Illustrations

Illustration 10. If in a triangle ABC, CD is the angle bisector of the angle ACB, then CD is equal to -

(A) $\frac{a+b}{2ab}\cos\frac{C}{2}$ (B) $\frac{2ab}{a+b}\sin\frac{C}{2}$ (C) $\frac{2ab}{a+b}\cos\frac{C}{2}$ (D) $\frac{b\sin\angle DAC}{\sin(B+C/2)}$

Solution. $\Delta CAB = \Delta CAD + \Delta CDB$

$$\Rightarrow \quad \frac{1}{2}ab\sin C = \frac{1}{2}b.CD.\sin\left(\frac{C}{2}\right) + \frac{1}{2}a.CD\sin\left(\frac{C}{2}\right)$$

$$\Rightarrow \quad CD(a+b)\sin\left(\frac{C}{2}\right) = ab\left(2\sin\left(\frac{C}{2}\right)\cos\left(\frac{C}{2}\right)\right)$$

So $\quad CD = \frac{2ab\cos(C/2)}{(a+b)}$

and in ΔCAD, $\frac{CD}{\sin\angle DAC} = \frac{b}{\sin\angle CDA}$ (by sine rule)

$$\Rightarrow \quad CD = \frac{b\sin\angle DAC}{\sin(B+C/2)}$$

Ans. (C,D)

Illustration 11. If Δ is the area and 2s the sum of the sides of a triangle, then show $\Delta \le \frac{s^2}{3\sqrt{3}}$.

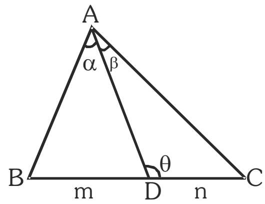
Solution. We have, $2s = a + b + c$, $\Delta^2 = s(s-a)(s-b)(s-c)$

Now, A.M. $\ge$ G.M.

$$\frac{(s-a)+(s-b)+(s-c)}{3} \ge \{(s-a)(s-b)(s-c)\}^{1/3}$$

or $\frac{3s-2s}{3} \ge \left(\frac{\Delta^2}{s}\right)^{1/3}$ or $\frac{s}{3} \ge \left(\frac{\Delta^2}{s}\right)^{1/3}$ or $\frac{\Delta^2}{s} \le \frac{s^3}{27} \Rightarrow \Delta \le \frac{s^2}{3\sqrt{3}}$ **Ans.**

6.0 m-n THEOREM



AL

(i) $(m + n)\cot\theta = m\cot\alpha - n\cot\beta$

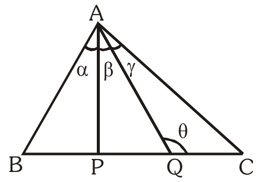
(ii) $(m + n)\cot\theta = n\cot B - m\cot C$. (BD : DC = m : n)

Illustrations

***Illustration 12.** The base of a Δ is divided into three equal parts. If t_1, t_2, t_3 be the tangents of the angles subtended by these parts at the opposite vertex, prove that : $\left(\frac{1}{t_1}+\frac{1}{t_2}\right)\left(\frac{1}{t_2}+\frac{1}{t_3}\right)=4\left(1+\frac{1}{t_2^2}\right)$

Solution. Let the points P and Q divide the side BC in three equal parts :

Such that BP = PQ = QC = x



Also let,

$\angle BAP = \alpha,\ \angle PAQ = \beta,\ \angle QAC = \gamma$

and $\angle AQC = \theta$

From question, $\tan\alpha = t_1$, $\tan\beta = t_2$, $\tan\gamma = t_3$.

Applying

m : n rule in triangle ABC we get,

$$(2x + x)\cot\theta = 2x\cot(\alpha+\beta) - x\cot\gamma \qquad \text{.... (i)}$$

from ΔAPC, we get

$$(x + x)\cot\theta = x\cot\beta - x\cot\gamma \qquad \text{.... (ii)}$$

dividing (i) and (ii), we get

$$\frac{3}{2}=\frac{2\cot(\alpha+\beta)-\cot\gamma}{\cot\beta-\cot\gamma}$$

or $$3\cot\beta-\cot\gamma=\frac{4(\cot\alpha.\cot\beta-1)}{\cot\beta+\cot\alpha}$$

or $$3\cot^2\beta-\cot\beta\cot\gamma+3\cot\alpha.\cot\beta-\cot\alpha.\cot\gamma=4\cot\alpha.\cot\beta-4$$

or $$4 + 4\cot^2\beta = \cot^2\beta + \cot\alpha\ .\ \cot\beta + \cot\beta\ .\ \cot\gamma + \cot\gamma\ .\ \cot\alpha$$

or $$4(1 + \cot^2\beta) = (\cot\beta + \cot\alpha)(\cot\beta + \cot\gamma)$$

or $$4\left(1+\frac{1}{t_2^2}\right)=\left(\frac{1}{t_1}+\frac{1}{t_2}\right)\left(\frac{1}{t_2}+\frac{1}{t_3}\right)$$

BEGINNER'S BOX-2

TOPIC COVERED : NAPIER'S ANALOGY (TANGENT RULE), HALF ANGLE FORMULAE, m-n THEOREM

1. In any ΔABC, prove that $\frac{b-c}{b+c}=\frac{\tan\left(\frac{B-C}{2}\right)}{\tan\left(\frac{B+C}{2}\right)}$

2. If ΔABC is right angled at C, prove that : (a) $\tan\frac{A}{2}=\sqrt{\frac{c-b}{c+b}}$ (b) $\sin(A-B)=\frac{a^2-b^2}{a^2+b^2}$

3. If in a ΔABC, two sides are a = 3, b = 5 and $\cos(A-B) = \frac{7}{25}$, find $\tan\frac{C}{2}$.

4. Given a = 6, b = 8, c = 10. Find

(a) sinA (b) tanA (c) $\sin\frac{A}{2}$ (d) $\cos\frac{A}{2}$ (e) $\tan\frac{A}{2}$

5. Prove that in any ΔABC, $(abcs)\ \sin\frac{A}{2}.\sin\frac{B}{2}.\sin\frac{C}{2} = \Delta^2$.

6. Show that if $\left(\tan\frac{A}{2}+\tan\frac{C}{2}\right) = \frac{2}{3}\cot\frac{B}{2}$, then a, b, c are in A.P.

7. The median AD of a ΔABC is perpendicular to AB, prove that tanA + 2tanB = 0

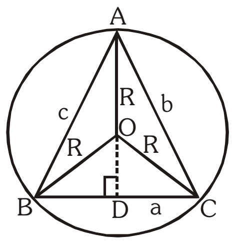
8. In a triangle ABC if $\Delta = a^2 - (b-c)^2$ then the value of $\tan\frac{A}{2}$ is :

(A) –1 (B) 0 (C) $\frac{1}{4}$ (D) $\frac{1}{2}$

7.0 RADIUS OF THE CIRCUMCIRCLE 'R'

AL

Circumcentre is the point of intersection of perpendicular bisectors of the sides and distance between circumcentre & vertex of triangle is called circumradius 'R'.

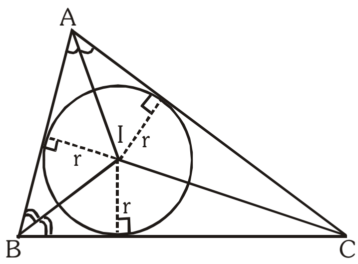


$$R = \frac{a}{2\sin A} = \frac{b}{2\sin B} = \frac{c}{2\sin C} = \frac{abc}{4\Delta}.$$

8.0 RADIUS OF THE INCIRCLE 'r'

AL

Point of intersection of internal angle bisectors is incentre and perpendicular distance of incentre from any side is called inradius 'r'.



$$r = \frac{\Delta}{s} = (s-a)\tan\frac{A}{2} = (s-b)\tan\frac{B}{2} = (s-c)\tan\frac{C}{2} = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.$$

$$= a\frac{\sin\frac{B}{2}\sin\frac{C}{2}}{\cos\frac{A}{2}} = b\frac{\sin\frac{A}{2}\sin\frac{C}{2}}{\cos\frac{B}{2}} = c\frac{\sin\frac{B}{2}\sin\frac{A}{2}}{\cos\frac{C}{2}}$$

Illustrations

Illustration 13. In a triangle ABC, if a : b : c = 4 : 5 : 6, then ratio between its circumradius and inradius is-

(A) $\frac{16}{7}$ (B) $\frac{16}{9}$ (C) $\frac{7}{16}$ (D) $\frac{11}{7}$

Solution. $\dfrac{R}{r}=\dfrac{abc}{4\Delta}\Big/\dfrac{\Delta}{s}=\dfrac{(abc)s}{4\Delta^2} \Rightarrow \dfrac{R}{r}=\dfrac{abc}{4(s-a)(s-b)(s-c)}$ ....(i)

$\because$ a : b : c = 4 : 5 : 6

$\Rightarrow \dfrac{a}{4}=\dfrac{b}{5}=\dfrac{c}{6}$ = k (say)

$\Rightarrow$ a = 4k, b = 5k, c = 6k

$\therefore \quad s=\dfrac{a+b+c}{2}=\dfrac{15k}{2},\ s-a=\dfrac{7k}{2},\ s-b=\dfrac{5k}{2},\ s-c=\dfrac{3k}{2}$

using (i) in these values $\dfrac{R}{r}=\dfrac{(4k)(5k)(6k)}{4\left(\dfrac{7k}{2}\right)\left(\dfrac{5k}{2}\right)\left(\dfrac{3k}{2}\right)}=\dfrac{16}{7}$ **Ans. (A)**

Illustration 14. If A, B, C are the angles of a triangle, prove that : cosA + cosB + cosC = $1+\dfrac{r}{R}$.

Solution. cosA + cosB + cosC = $2\cos\left(\dfrac{A+B}{2}\right).\cos\left(\dfrac{A-B}{2}\right)+\cos C$

$=\ 2\sin\dfrac{C}{2}.\cos\left(\dfrac{A-B}{2}\right)+1-2\sin^2\dfrac{C}{2}\ =\ 1+2\sin\dfrac{C}{2}\left[\cos\left(\dfrac{A-B}{2}\right)-\sin\left(\dfrac{C}{2}\right)\right]$

$=\ 1+2\sin\dfrac{C}{2}\left[\cos\left(\dfrac{A-B}{2}\right)-\cos\left(\dfrac{A+B}{2}\right)\right] \quad \left\{\because \dfrac{C}{2}=90^\circ-\left(\dfrac{A+B}{2}\right)\right\}$

$=\ 1+2\sin\dfrac{C}{2}.2\sin\dfrac{A}{2}.\sin\dfrac{B}{2}\ =\ 1+4\sin\dfrac{A}{2}.\sin\dfrac{B}{2}.\sin\dfrac{C}{2}$

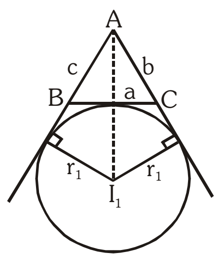
$=\ 1+\dfrac{r}{R}$ {as, r = 4R sin A/2 . sinB/2 . sinC/2}

$\Rightarrow$ cosA + cosB + cosC = $1+\dfrac{r}{R}$. Hence proved.

9.0 RADII OF THE EX-CIRCLES

AL

Point of intersection of two external angles and one internal angle bisectors is excentre and perpendicular distance of excentre from any side is called exradius. If r_1 is the radius of escribed circle opposite to $\angle A$ of ΔABC and so on, then -



(a) $r_1=\dfrac{\Delta}{s-a}=s\tan\dfrac{A}{2}=4R\sin\dfrac{A}{2}\cos\dfrac{B}{2}\cos\dfrac{C}{2}=\dfrac{a\cos\dfrac{B}{2}\cos\dfrac{C}{2}}{\cos\dfrac{A}{2}}$

(b) $r_2=\dfrac{\Delta}{s-b}=s\tan\dfrac{B}{2}=4R\cos\dfrac{A}{2}\sin\dfrac{B}{2}\cos\dfrac{C}{2}=\dfrac{b\cos\dfrac{A}{2}\cos\dfrac{C}{2}}{\cos\dfrac{B}{2}}$

(c) $r_3=\dfrac{\Delta}{s-c}=s\tan\dfrac{C}{2}=4R\cos\dfrac{A}{2}\cos\dfrac{B}{2}\sin\dfrac{C}{2}=\dfrac{c\cos\dfrac{A}{2}\cos\dfrac{B}{2}}{\cos\dfrac{C}{2}}$

I_1, I_2 and I_3 are taken as ex-centre opposite to vertex A, B, C repsectively.

Illustrations

Illustration 15. Value of the expression $\frac{b-c}{r_1}+\frac{c-a}{r_2}+\frac{a-b}{r_3}$ is equal to -

(A) 1 (B) 2 (C) 3 (D) 0

Solution. $\frac{(b-c)}{r_1}+\frac{(c-a)}{r_2}+\frac{(a-b)}{r_3}$

$\Rightarrow \quad (b-c)\left(\frac{s-a}{\Delta}\right)+(c-a)\left(\frac{s-b}{\Delta}\right)+(a-b).\left(\frac{s-c}{\Delta}\right)$

$\Rightarrow \quad \frac{(s-a)(b-c)+(s-b)(c-a)+(s-c)(a-b)}{\Delta}$

$=\frac{s(b-c+c-a+a-b)-[ab-ac+bc-ba+ac-bc]}{\Delta}=\frac{0}{\Delta}=0$

Thus, $\frac{b-c}{r_1}+\frac{c-a}{r_2}+\frac{a-b}{r_3}=0$ **Ans. (D)**

Illustration 16. If $r_1 = r_2 + r_3 + r$, prove that the triangle is right angled.

Solution. We have, $r_1 - r = r_2 + r_3$

$\Rightarrow \quad \frac{\Delta}{s-a}-\frac{\Delta}{s}=\frac{\Delta}{s-b}+\frac{\Delta}{s-c} \Rightarrow \frac{s-s+a}{s(s-a)}=\frac{s-c+s-b}{(s-b)(s-c)}$

$\Rightarrow \quad \frac{a}{s(s-a)}=\frac{2s-(b+c)}{(s-b)(s-c)}$ {as, $2s = a + b + c$}

$\Rightarrow \quad \frac{a}{s(s-a)}=\frac{a}{(s-b)(s-c)} \Rightarrow s^2-(b+c)s+bc=s^2-as$

$\Rightarrow \quad s(-a+b+c)=bc \Rightarrow \frac{(b+c-a)(a+b+c)}{2}=bc$

$\Rightarrow \quad (b+c)^2-(a)^2=2bc \Rightarrow b^2+c^2+2bc-a^2=2bc$

$\Rightarrow \quad b^2+c^2=a^2$

$\therefore \quad \angle A = 90°.$ **Ans.**

BEGINNER'S BOX-3

TOPIC COVERED : RADIUS OF THE CIRCUMCIRCLE 'R', RADIUS OF THE INCIRCLE 'R', RADII OF THE EX-CIRCLES

1. If in ΔABC, $a = 3$, $b = 4$ and $c = 5$, find

(a) Δ (b) R (c) r

2. In a ΔABC, show that : $\frac{a^2-b^2}{c}=2R\sin(A-B)$

3. In a ΔABC, show that : $r\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}=\frac{\Delta}{4R}$

4. In a ΔABC, show that : $a+b+c=\dfrac{abc}{2Rr}$

5. Let Δ & Δ' denote the areas of a Δ and that of its incircle. Prove that $\Delta : \Delta' = \left(\cot\dfrac{A}{2}.\cot\dfrac{B}{2}.\cot\dfrac{C}{2}\right) : \pi$

6. In an equilateral ΔABC, $R = 2$, find

(a) r (b) r_1 (c) a

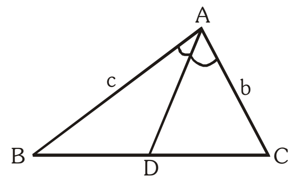
7. In a ΔABC, show that $r_1r_2 + r_2r_3 + r_3r_1 = s^2$, (notations has usual meaning)

8. In a ΔABC, show that $\sqrt{rr_1r_2r_3} = \Delta$

10.0 ANGLE BISECTORS & MEDIANS

AL

An angle bisector divides the base in the ratio of corresponding sides.



$$\frac{BD}{CD}=\frac{c}{b} \quad \Rightarrow \quad BD=\frac{ac}{b+c} \quad \& \quad CD=\frac{ab}{b+c}$$

If m_a and β_a are the lengths of a median and an angle bisector from the angle A then,

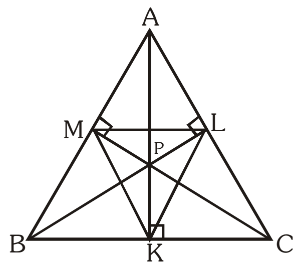
$$m_a=\frac{1}{2}\sqrt{2b^2+2c^2-a^2} \text{ and } \beta_a=\frac{2bc\cos\frac{A}{2}}{b+c}$$

Note that $m_a^2+m_b^2+m_c^2=\dfrac{3}{4}(a^2+b^2+c^2)$

11.0 ORTHOCENTRE & PEDAL TRIANGLE

AL

(a) Point of intersection of altitudes is orthocentre & the triangle KLM which is formed by joining the feet of the altitudes is called the pedal triangle.



(b) The distances of the orthocentre from the angular points A,B,C of the ΔABC are 2R cosA, 2R cosB, & 2R cosC.

(c) The distance of orthocentre from sides BC, CA, AB of the ΔABC are 2R cosB cosC, 2R cosC cosA and 2R cosA cosB.

12.0 THE DISTANCES BETWEEN THE SPECIAL POINTS

AL

(a) The distance between circumcentre and orthocentre is $= R\sqrt{1-8\cos A\cos B\cos C}$

(b) The distance between circumcentre and incentre is $=\sqrt{R^2-2Rr}$

(c) The distance between incentre and orthocentre is $=\sqrt{2r^2-4R^2\cos A\cos B\cos C}$

(d) The distances between circumcentre & excentres are

$$OI_1=R\sqrt{1+8\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}=\sqrt{R^2+2Rr_1} \text{ \& so on.}$$

Illustrations

Illustration 17. Prove that the distance between the circumcentre and the orthocentre of a triangle ABC is $R\sqrt{1-8\cos A\cos B\cos C}$.

Solution. Let O and P be the circumcentre and the orthocentre respectively. If OF is the perpendicular to AB, we have $\angle OAF = 90° - \angle AOF = 90° - C$. Also $\angle PAL = 90° - C$.

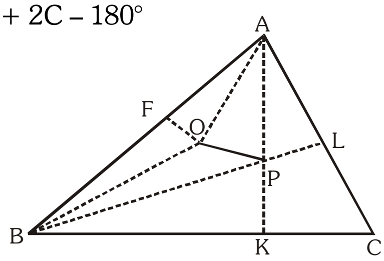
Hence, $\angle OAP = A - \angle OAF - \angle PAL = A - 2(90° - C) = A + 2C - 180°$

$= A + 2C - (A + B + C) = C - B$.

Also $OA = R$ and $PA = 2R\cos A$.

Now in ΔAOP,

$$OP^2 = OA^2 + PA^2 - 2OA.\ PA\cos OAP$$
$$= R^2 + 4R^2\cos^2 A - 4R^2\cos A\cos(C - B)$$
$$= R^2 + 4R^2\cos A[\cos A - \cos(C - B)]$$
$$= R^2 - 4R^2\cos A[\cos(B + C) + \cos(C - B)] = R^2 - 8R^2\cos A\cos B\cos C.$$



Hence $OP = R\sqrt{1-8\cos A\cos B\cos C}$. **Ans.**

BEGINNER'S BOX-4

TOPIC COVERED : ANGLE BISECTORS & MEDIANS, ORTHOCENTRE & PEDAL TRIANGLE, THE DISTANCES BETWEEN THE SPECIAL POINTS

1. If x, y, z are the distance of the vertices of ΔABC respectively from the orthocentre, then prove that $\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=\frac{abc}{xyz}$

2. If p_1, p_2, p_3 are respectively the perpendiculars from the vertices of a triangle to the opposite sides, prove that

(a) $p_1p_2p_3 = \frac{a^2b^2c^2}{8R^3}$ (b) $\Delta = \sqrt{\frac{1}{2}Rp_1p_2p_3}$

3. In a ΔABC, AD is altitude and H is the orthocentre prove that AH : DH = (tanB + tanC) : tanA

4. In a ΔABC, the lengths of the bisectors of the angle A, B and C are x, y, z respectively. Show that

$\frac{1}{x}\cos\frac{A}{2}+\frac{1}{y}\cos\frac{B}{2}+\frac{1}{z}\cos\frac{C}{2}=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$. Also show that $\frac{a}{b+c}=\sqrt{1-\frac{x^2}{bc}}$

5. Show that in an equilateral triangle, circumcentre, orthocentre and incentre overlap each other.

6. If the incentre and circumcentre of a triangle are equidistant from the side BC, show that cosB + cosC = 1.

7. In ΔABC show that length of bisector of angle A is

$$\frac{abc}{2R(b+c)}\operatorname{cosec}\frac{A}{2}$$

13.0 SOLUTION OF TRIANGLES

AL

The three sides a,b,c and the three angles A,B,C are called the elements of the triangle ABC. When any three of these six elements (except all the three angles) of a triangle are given, the triangle is known completely; that is the other three elements can be expressed in terms of the given elements and can be evaluated. This process is called the solution of triangles.

- If the three sides a,b,c are given, angle A is obtained from $\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$

  or $\cos A = \frac{b^2+c^2-a^2}{2bc}$. B and C can be obtained in the similar way.

- If two sides b and c and the included angle A are given, then $\tan\frac{B-C}{2} = \frac{b-c}{b+c}\cot\frac{A}{2}$ gives $\frac{B-C}{2}$. Also

  $\frac{B+C}{2} = 90° - \frac{A}{2}$, so that B and C can be evaluated. The third side is given by $a = b\,\frac{\sin A}{\sin B}$

  or $a^2 = b^2 + c^2 - 2bc\cos A$.

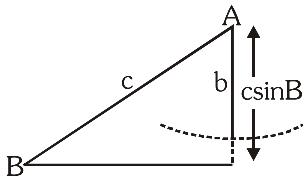
- If two sides b and c and an angle opposite the one of them (say B) are given then

  $$\sin C = \frac{c}{b}\sin B,\ \ A = 180° - (B+C)\ \text{ and }\ a = \frac{b\sin A}{\sin B}$$ given the remaining elements.

Case I

$b < c\sin B$.

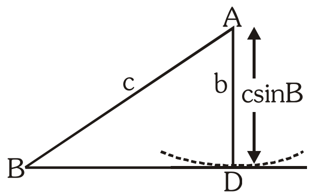
We draw the side c and angle B. Now it is obvious from the figure that there is no triangle possible.



Case II

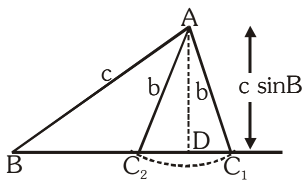
$b = c\sin B$ and B is an acute angle, there is only one triangle possible.

and it is right-angled at C.



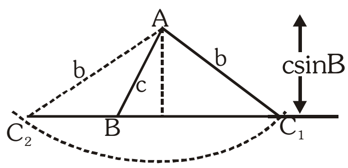
Case III

$b > c\sin B$, $b < c$ and B is an acute angle, then there are two triangles possible for two values of angle C.



Case IV

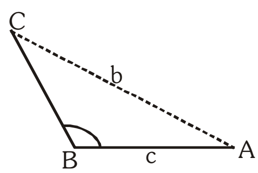
$b > c\sin B$, $c < b$ and B is an acute angle, then there is only one triangle.



Case V

$b > c\sin B$, $c > b$ and B is an obtuse angle.

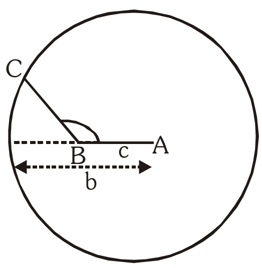
For any choice of point C, b will be greater than c which is a contradication as $c > b$ (given). So there is no triangle possible.



Case VI

$b > c\sin B$, $c < b$ and B is an obtuse angle.

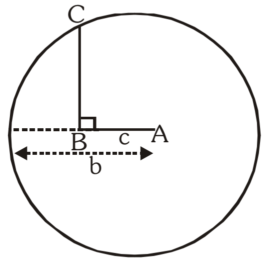
We can see that the circle with A as centre and b as radius will cut the line only in one point. So only one triangle is possible.



Case VII

b > c and B = 90°.

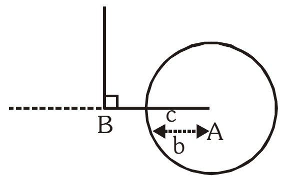
Again the circle with A as centre and b as radius will cut the line only in one point. So only one triangle is possible.



Case VIII

$b \le c$ and B = 90°.

The circle with A as centre and b as radius will not cut the line in any point. So no triangle is possible.



This is, sometimes, called an ambiguous case.

Alternative Method –

By applying cosine rule, we have $\cos B = \dfrac{a^2 + c^2 - b^2}{2ac}$

$\Rightarrow$ $a^2 - (2c\cos B)a + (c^2 - b^2) = 0 \Rightarrow a = c\cos B \pm \sqrt{(c\cos B)^2 - (c^2 - b^2)}$

$\Rightarrow$ $a = c\cos B \pm \sqrt{b^2 - (c\sin B)^2}$

This equation leads to following cases :

Case-I – If $b < c\sin B$, no such triangle is possible.

Case-II – Let $b = c\sin B$. There are further following case :

(a) B is an obtuse angle $\Rightarrow$ cosB is negative. There exists no such triangle.

(b) B is an acute angle $\Rightarrow$ cosB is positive. There exists only one such triangle.

Case-III – Let $b > c\sin B$. There are further following cases :

(a) B is an acute angle $\Rightarrow$ cosB is positive. In this case triangle will exist if and only if $c\cos B > \sqrt{b^2 - (c\sin B)^2}$ or $c > b \Rightarrow$ Two such triangle is possible. If $c < b$, only one such triangle is possible.

(b) B is an obtuse angle $\Rightarrow$ cosB is negative. In this case triangle will exist if and only if $\sqrt{b^2 - (c\sin B)^2} > |c\cos B| \Rightarrow b > c$. So in this case only one such triangle is possible. If $b<c$ there exists no such triangle.

This is called an ambiguous case.

- If one side a and angles B and C are given, then $A = 180° - (B + C)$, and $b = \dfrac{a\sin B}{\sin A}, c = \dfrac{a\sin C}{\sin A}$.
- If the three angles A,B,C are given, we can only find the ratios of the sides a,b,c by using sine rule (since there are infinite similar triangles possible).

Illustrations

Illustration 18. In the ambiguous case of the solution of triangles, prove that the circumcircles of the two triangles are of same size.

Solution. Let us say b,c and angle B are given in the ambiguous case. Both the triangles will have b and its opposite angle as B. so $\dfrac{b}{\sin B} = 2R$ will be given for both the triangles. So their circumradii and therefore their sizes will be same.

***Illustration 19.** If a,b and A are given in a triangle and c_1, c_2 are the possible values of the third side, prove that $c_1^2 + c_2^2 - 2c_1c_2 \cos 2A = 4a^2\cos^2 A$.

Solution. $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$

$\Rightarrow \quad c^2 - 2bc\cos A + b^2 - a^2 = 0.$

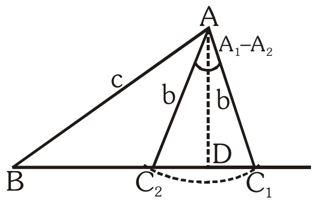
$c_1 + c_2 = 2b\cos A$ and $c_1c_2 = b^2 - a^2$.

$\Rightarrow \quad c_1^2 + c_2^2 - 2c_1c_2\cos 2A = (c_1 + c_2)^2 - 2c_1c_2(1 + \cos 2A)$

$= \quad 4b^2\cos^2 A - 2(b^2 - a^2)2\cos^2 A = 4a^2\cos^2 A.$

Illustration 20. If b,c,B are given and b < c, prove that $\cos\left(\dfrac{A_1 - A_2}{2}\right) = \dfrac{c\sin B}{b}$.

Solution. $\angle C_2AC_1$ is bisected by AD.



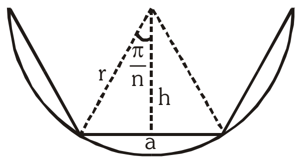
$\Rightarrow \quad$ In ΔAC_2D, $\cos\left(\dfrac{A_1 - A_2}{2}\right) = \dfrac{AD}{AC_2} = \dfrac{c\sin B}{b}$

Hence proved.

14.0 REGULAR POLYGON

AL

A regular polygon has all its sides equal. It may be inscribed or circumscribed.

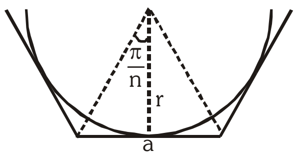


(a) *Inscribed in circle of radius r*

(i) $a = 2h\tan\dfrac{\pi}{n} = 2r\sin\dfrac{\pi}{n}$

(ii) Perimeter (P) and area (A) of a regular polygon of n sides inscribed in a circle of radius r are given by $P = 2nr\sin\dfrac{\pi}{n}$ and $A = \dfrac{1}{2}nr^2\sin\dfrac{2\pi}{n}$

(b) *Circumscribed about a circle of radius r*



(i) $a = 2r\tan\dfrac{\pi}{n}$

(ii) Perimeter (P) and area (A) of a regular polygon of n sides circumscribed about a given circle of radius r is given by $P = 2nr\tan\dfrac{\pi}{n}$ and $A = nr^2\tan\dfrac{\pi}{n}$

BEGINNER'S BOX-5

TOPIC COVERED : SOLUTION OF TRIANGLES, REGULAR POLYGON

1. If b,c,B are given and b<c, prove that $\sin\left(\dfrac{A_1 - A_2}{2}\right) = \dfrac{a_1 - a_2}{2b}$

2. In a ΔABC, b,c,B (c > b) are gives. If the third side has two values a_1 and a_2 such that

$a_1 = 3a_2$, show that $\sin B = \sqrt{\frac{4b^2 - c^2}{3c^2}}$.

3. If the perimeter of a circle and a regular polygon of n sides are equal, then prove that $\frac{\text{area of the circle}}{\text{area of polygon}} = \frac{\tan\frac{\pi}{n}}{\frac{\pi}{n}}$.

4. The ratio of the area of n-sided regular polygon, circumscribed about a circle, to the area of the regular polygon of equal number of sides inscribed in the circle is 4 : 3. Find the value of n.

5. In ΔABC if tan A : tan B : tan C = 1 : 2 : 3 then find $\angle A$

6. If in ΔABC, $\angle B = 45°$, $\angle C = 60$, $a = 2(\sqrt{3}+1)$ then find the area of ΔABC

7. In ΔABC, a, b, A are given then which of following gives us two such triangles.
(a) a < b sin A, (b) a = b sin A, (c) a > b sin A and a < b
(d) a > b sin A and a > b

8. Prove that area of qundrilateral ABCD = $\frac{1}{2}d_1d_2\sin\alpha$, (where d_1, d_2 are length of digonals AC, BD respectively and α is angle between them)

GOLDEN KEY POINTS

- (i) If a cos B = b cos A, then the triangle is isosceles.
  (ii) If a cos A = b cos B, then the triangle is isosceles or right angled.

- In right angle triangle
  (i) $a^2 + b^2 + c^2 = 8R^2$ (ii) $\cos^2 A + \cos^2 B + \cos^2 C = 1$

- In equilateral triangle
  (i) R = 2r (ii) $r_1 = r_2 = r_3 = \frac{3R}{2}$ (iii) $r : R : r_1 = 1 : 2 : 3$
  (iv) area $= \frac{\sqrt{3}a^2}{4}$ (v) $R = \frac{a}{\sqrt{3}}$

- (i) The circumcentre lies (1) inside an acute angled triangle (2) outside an obtuse angled triangle & (3) mid point of the hypotenuse of right angled triangle.
  (ii) The orthocentre of right angled triangle is the vertex at the right angle.
  (iii) The orthocentre, centroid & circumcentre are collinear & centroid divides the line segment joining orthocentre & circumcentre internally in the ratio 2 : 1 except in case of equilateral triangle. In equilateral triangle, all these centres coincide

- Area of a cyclic quadrilateral = $\sqrt{(s-a)(s-b)(s-c)(s-d)}$

  where a, b, c, d are lengths of the sides of quadrilateral and $s = \frac{a+b+c+d}{2}$.

SOME WORKED OUT ILLUSTRATIONS

Illustration 1. For a ΔABC, it is given that $\cos A + \cos B + \cos C = 3/2$. Prove that the triangle is equilateral.

Solution. If a, b, c are the sides of the ΔABC, then $\cos A + \cos B + \cos C = 3/2$

$$\Rightarrow \quad \frac{b^2+c^2-a^2}{2bc} + \frac{a^2+c^2-b^2}{2ac} + \frac{a^2+b^2-c^2}{2ab} = \frac{3}{2}$$

$$\Rightarrow \quad ab^2 + ac^2 - a^3 + bc^2 + ba^2 - b^3 + ca^2 + cb^2 - c^3 = 3abc$$

$$\Rightarrow \quad ab^2 + ac^2 + bc^2 + ba^2 + ca^2 + cb^2 - 6abc = a^3 + b^3 + c^3 - 3abc$$

$$\Rightarrow \quad a(b-c)^2 + b(c-a)^2 + c(a-b)^2 = \frac{(a+b+c)}{2}\left\{(a-b)^2 + (b-c)^2 + (c-a)^2\right\}$$

$$\Rightarrow \quad (a+b-c)(a-b)^2 + (b+c-a)(b-c)^2 + (c+a-b)(c-a)^2 = 0 \qquad \text{.... (i)}$$

as we know $a + b > c$, $b + c > a$, $c + a > b$

$\therefore$ each term on the left side of equation (i) has positive coefficient multiplied by perfect square, each must be separately zero.

$\Rightarrow \quad a = b = c.$

Hence Δ is equilateral. ***Ans.***

Illustration 2. In a triangle ABC, if $\cos A + 2\cos B + \cos C = 2$. Prove that the sides of the triangle are in A.P.

Solution. $\cos A + 2\cos B + \cos C = 2$ or $\cos A + \cos C = 2(1 - \cos B)$

$$\Rightarrow \quad 2\cos\left(\frac{A+C}{2}\right).\cos\left(\frac{A-C}{2}\right) = 4\sin^2 B/2$$

$$\Rightarrow \quad \cos\left(\frac{A-C}{2}\right) = 2\sin\frac{B}{2} \qquad \left\{\text{as } \cos\left(\frac{A+C}{2}\right) = \cos\left(\frac{\pi}{2} - \frac{B}{2}\right) = \sin\frac{B}{2}\right\}$$

$$\Rightarrow \quad \cos\left(\frac{A-C}{2}\right) = 2\cos\left(\frac{A+C}{2}\right)$$

$$\Rightarrow \quad \cos\frac{A}{2}.\cos\frac{C}{2} + \sin\frac{A}{2}.\sin\frac{C}{2} = 2\cos\frac{A}{2}.\cos\frac{C}{2} - 2\sin\frac{A}{2}.\sin\frac{C}{2}$$

$$\Rightarrow \quad \cot\frac{A}{2}.\cot\frac{C}{2} = 3 \qquad \Rightarrow \quad \sqrt{\frac{s(s-a)}{(s-b)(s-c)}}.\sqrt{\frac{s(s-c)}{(s-a)(s-b)}} = 3$$

$$\Rightarrow \quad \frac{s}{(s-b)} = 3 \Rightarrow s = 3s - 3b \qquad \Rightarrow \quad 2s = 3b$$

$\Rightarrow \quad a + c = 2b,$ $\quad \therefore$ a, b, c are in A.P. ***Ans.***

ANSWERS

BEGINNER'S BOX-1

1. 90°

BEGINNER'S BOX-2

3. $\frac{1}{3}$ **4.** ***(a)*** $\frac{3}{5}$ ***(b)*** $\frac{3}{4}$ ***(c)*** $\frac{1}{\sqrt{10}}$ ***(d)*** $\frac{3}{\sqrt{10}}$ ***(e)*** $\frac{1}{3}$ **8.** (C)

BEGINNER'S BOX-3

1. ***(a)*** 6 ***(b)*** $\frac{5}{2}$ ***(c)*** 1 **6.** ***(a)*** 1 ***(b)*** 3 ***(c)*** $2\sqrt{3}$

BEGINNER'S BOX-5

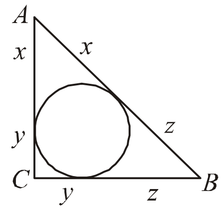
4. 6 **5.** $\frac{\pi}{4}$ **6.** $6+2\sqrt{3}$ **7.** (C)

EXERCISE – 1 MCQ (SINGLE CHOICE CORRECT)

1. The angle A of the triangle ABC, in which (a + b + c) (b + c –a) = 3bc is

(A) 30° (B) 45° (C) 60° (D) 120°

2. In a triangle ABC, Let $\angle C = \frac{\pi}{2}$, if r is the inradius and R is the circumradius of the triangle, then 2 (r + R) is equal to



(A) a + b (B) b + c

(C) c + a (D) a + b + c

3. In a triangle ABC, if $\frac{2\cos A}{a}+\frac{\cos B}{b}+\frac{2\cos C}{c}=\frac{a}{bc}+\frac{b}{ca}$, then the value of the angle A is

(A) 30° (B) 45° (C) 60° (D) 90°

4. If A = 45^0, B =75^0 then a + c$\sqrt{2}$ is equal to

(A) 2b (B) 3b (C) $\sqrt{2}$b (D) b

5. In a $\Delta ABC\left(\frac{a^2}{\sin A}+\frac{b^2}{\sin B}+\frac{c^2}{\sin C}\right).\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$ simplifies to -

(A) 2Δ (B) Δ (C) $\frac{\Delta}{2}$ (D) $\frac{\Delta}{4}$

(where Δ is the area of triangle)

6. In a ΔABC if b + c = 3a then $\cot\frac{B}{2}.\cot\frac{C}{2}$ has the value equal to -

(A) 4 (B) 3 (C) 2 (D) 1

7. If $\frac{a}{\sin A}$ = K, then the area of ΔABC in terms of K and sines of the angles is -

(A) $\frac{K^2}{4}$sinAsinBsinC (B) $\frac{K^2}{2}$sinAsinBsinC (C) 2K^{2}sinAsinBsin(A+B) (D) none

8. In a ΔABC, a semicircle is inscribed, whose diameter lies on the side c. Then the radius of the semicircle is (Where Δ is the area of the triangle ABC)

(A) $\frac{2\Delta}{a+b}$ (B) $\frac{2\Delta}{a+b-c}$ (C) $\frac{2\Delta}{s}$ (D) $\frac{c}{2}$

9. In triangle ABC where A, B, C are acute, the distances of the orthocentre from the sides are in the proportion

(A) cos A : cos B : cos C (B) sin A : sin B : sin C

(C) sec A : sec B : sec C (D) tan A : tan B : tan C

10. In a ΔABC, the value of $\frac{a\cos A+b\cos B+c\cos C}{a+b+c}$ is equal to -

(A) $\frac{r}{R}$ (B) $\frac{R}{2r}$ (C) $\frac{R}{r}$ (D) $\frac{2r}{R}$

11. If in a triangle ABC, $b + c = 4a$. Then $\cot\frac{B}{2}\cot\frac{C}{2}$ is equal to

(A) $\frac{5}{3}$ (B) $\frac{3}{5}$ (C) $\frac{5}{8}$ (D) None of these

12. With usual notation in a ΔABC $\left(\frac{1}{r_1}+\frac{1}{r_2}\right)\left(\frac{1}{r_2}+\frac{1}{r_3}\right)\left(\frac{1}{r_3}+\frac{1}{r_1}\right)=\frac{K\ R^3}{a^2\ b^2\ c^2}$ then K has value equal to -

(A) 1 (B) 16 (C) 64 (D) 128

13. In a triangle ABC, $\frac{r_1+r_2}{1+\cos C}$ is equal to -

(A) $2ab/c\Delta$ (B) $(a + b)/c\Delta$ (C) $abc/2\Delta$ (D) abc/Δ^2

14. With usual notations in a triangle ABC, if $r_1 = 2r_2 = 2r_3$ then -

(A) $4a = 3b$ (B) $3a = 2b$ (C) $4b = 3a$ (D) $2a = 3b$

15. If r_1, r_2, and r_3 be the radii of excircles of the triangle ABC, then $\frac{\sum r_1}{\sqrt{\sum r_1 r_2}}$ is equal to -

(A) $\sum\cot\frac{A}{2}$ (B) $\sum\cot\frac{A}{2}\cot\frac{B}{2}$ (C) $\sum\tan\frac{A}{2}$ (D) $\prod\tan\frac{A}{2}$

16. In a triangle ABC, then $2ac\sin\frac{1}{2}(A - B + C)$ is

(A) $a^2 + b^2 - c^2$ (B) $c^2 + a^2 - b^2$ (D) $b^2 - c^2 - a^2$ (D) $c^2 - a^2 - b^2$

17. If in a ΔABC, $\Delta = a^2 - (b - c)^2$, then tan A is equal to :

(A) 15/16 (B) 8/15 (C) 8/17 (D) 1/2

18. The line $\frac{x}{6}+\frac{y}{8}=1$ cuts the co-ordinate axis at A & B. If O is origin, then $\prod\sin\frac{A}{2}$ for the triangle OAB is -

(A) 5/6 (B) 1/10 (C) 5/4 (D) none of above

19. In a triangle ABC, CD is the bisector of the angle C. If $\cos\frac{C}{2}$ has the value $\frac{1}{3}$ and $\ell(CD) = 6$, then $\left(\frac{1}{a}+\frac{1}{b}\right)$ has the value equal to -

(A) $\frac{1}{9}$ (B) $\frac{1}{12}$ (C) $\frac{1}{6}$ (D) none

EXERCISE – 2 MCQ (ONE OR MORE CHOICE CORRECT)

Select the correct alternatives (one or more than one correct answers)

1. If in a triangle ABC p, q, r are the altitudes from the vertices A, B, C to the opposite sides, then which of the following does not hold good ?

(A) $(\Sigma p)\left(\Sigma\frac{1}{p}\right) = (\Sigma a)\left(\Sigma\frac{1}{a}\right)$ (B) $(\Sigma p)\ (\Sigma a) = \left(\Sigma\frac{1}{p}\right)\left(\Sigma\frac{1}{a}\right)$

(C) $(\Sigma p)\ (\Sigma pq)\ (\Pi a) = (\Sigma a)\ (\Sigma ab)\ (\Pi p)$ (D) $\left(\Sigma\frac{1}{p}\right)\Pi\left(\frac{1}{p}+\frac{1}{q}-\frac{1}{r}\right)\Pi a^2 = 16R^2$

2. If 'O' is the circum centre of the ΔABC and R_1, R_2 and R_3 are the radii of the circumcircles of triangles OBC, OCA and OAB respectively then $\frac{a}{R_1}+\frac{b}{R_2}+\frac{c}{R_3}$ has the value equal to -

(A) $\frac{abc}{2R^3}$ (B) $\frac{R^3}{abc}$ (C) $\frac{4\Delta}{R^2}$ (D) $\frac{abc}{R^3}$

3. In a triangle ABC, $(r_1 - r)\ (r_2 - r)\ (r_3 - r)$ is equal to -

(A) $4Rr^2$ (B) $\frac{4abc.\Delta}{(a+b+c)^2}$

(C) $16R^3(\cos A + \cos B + \cos C - 1)$ (D) $r^3 \operatorname{cosec}\frac{A}{2}\operatorname{cosec}\frac{B}{2}\operatorname{cosec}\frac{C}{2}$

4. Two rays emanate from the point A and form an angle of 43° with one another. Lines L_1, L_2 and L_3 (no two of which are parallel) each form an isosceles triangle with the original rays. The largest angle of the triangle formed by lines L_1, L_2 and L_3 is -

(A) 127° (B) 129° (C) 133° (D) 137°

5. In ΔABC, BC = 5, CA = 4, AB = 3 and D, E are points on BC such that BD = DE = EC, $\angle CAE = \theta$ then

(A) $AE^2 = \frac{73}{3}$ (B) $AE^2 = \frac{73}{9}$ (C) $\tan\theta = \frac{3}{8}$ (D) $\cos\theta = \frac{8}{\sqrt{73}}$

6. If a, b, A are given in a triangle and c_1 and c_2 are two possible values of third side such that $c_1^2 + c_1c_2 + c_2^2 = a^2$, then A is equal to -

(A) 30° (B) 60° (C) 90° (D) 120°

7. If A, B, C are angles of a triangle which of the following will not imply it is equilateral -

(A) $\tan A + \tan B + \tan C = 3\sqrt{3}$ (B) $\cot A + \cot B + \cot C = \sqrt{3}$

(C) $a + b + c = 2R$ (D) $a^2 + b^2 + c^2 = 9R^2$

8. In a ΔABC, $\frac{s}{R}$ is equal to -

(A) $\sin A + \sin B + \sin C$ (B) $4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$ (C) $4\sin A\sin B\sin C$ (D) $\frac{\Delta s}{abc}$

9. If cosA + cosB + 2cosC = 2 then the sides of the ΔABC are in-

(A) A.P. (B) G.P. (C) H.P. (D) none

10. If x, y and z are the distances of incentre from the vertices of the triangle ABC respectively then $\frac{abc}{xyz}$ is equal to

(A) $\prod \tan\frac{A}{2}$ (B) $\sum \cot\frac{A}{2}$ (C) $\sum \tan\frac{A}{2}$ (D) $\prod \cot\frac{A}{2}$

Match the Column

Following question contains statements given in two columns, which have to be matched. The statements in ***Column-I*** are labelled as A, B, C and D while the statements in ***Column-II*** are labelled as p, q, r and s. Any given statement in ***Column-I*** can have correct matching with ***ONE*** statement in ***Column-II***.

11. If p_1, p_2, p_3 are altitudes of a triangle ABC from the vertices A, B, C respectively and Δ is the area of the triangle and s is semi perimeter of the triangle, then match the columns

	Column-I		***Column-I***
(A)	If $\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}=\frac{1}{2}$ then the least value of $p_1p_2p_3$ is	(p)	$\frac{1}{R}$
(B)	The value of $\frac{\cos A}{p_1}+\frac{\cos B}{p_2}+\frac{\cos C}{p_3}$ is	(q)	216
(C)	The minimum value of $\frac{b^2p_1}{c}+\frac{c^2p_2}{a}+\frac{a^2p_3}{b}$ is	(r)	6Δ
(D)	The value of $p_1^{-2} + p_2^{-2} + p_3^{-2}$ is	(s)	$\frac{\Sigma a^2}{4\Delta^2}$

Comprehension Based Questions

In ΔABC, suppose AB = 5 cm, AC = 7 cm, $\angle ABC = \frac{\pi}{3}$.

12. The area of Δ ABC is :

(A) 10 cm^2 (B) $10\sqrt{3}$cm^2 (C) 20 cm^2 (D) $20\sqrt{3}$cm^2

13. The distance of the orthocentre of ΔABC from the vertex B is :

(A) $\frac{14}{\sqrt{3}}$cm (B) 14 cm (C) $\frac{7}{\sqrt{3}}$cm (D) 7 cm

14. The distance between of the incentre and circumcentre of ΔABC is

(A) $\sqrt{\frac{7}{3}}$cm (B) $\sqrt{\frac{64}{3}}$cm (C) 5 cm (D) $\sqrt{\frac{112}{3}}$cm

EXERCISE - 3 SUBJECTIVE

1. Prove that : $4R \sin A \sin B \sin C = a\cos A + b\cos B + c\cos C$.

2. Prove that : $a\cos B\cos C + b\cos C\cos A + c\cos A\cos B = \frac{\Delta}{R}$

3. Prove that : $\frac{abc}{s}\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} = \Delta$

4. Prove that : $\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}=\frac{1}{r}$

5. Prove that : $r_1 + r_2 + r_3 - r = 4R$

6. Prove that : $\frac{b-c}{r_1}+\frac{c-a}{r_2}+\frac{a-b}{r_3}=0$

***7.** Prove that : $\frac{r_1}{bc}+\frac{r_2}{ca}+\frac{r_3}{ab}=\frac{1}{r}-\frac{1}{2R}$

8. Prove that : $\frac{\tan\frac{A}{2}}{(a-b)(a-c)}+\frac{\tan\frac{B}{2}}{(b-a)(b-c)}+\frac{\tan\frac{C}{2}}{(c-a)(c-b)}=\frac{1}{\Delta}$

9. In a ΔABC, prove that :

(i) $a\sin(B-C) + b\sin(C-A) + c\sin(A-B) = 0$

(ii) $\frac{a^2\sin(B-C)}{\sin A}+\frac{b^2\sin(C-A)}{\sin B}+\frac{c^2\sin(A-B)}{\sin C}=0$

EXERCISE – 4 RECAP OF AIEEE/JEE (MAIN)

1. Which of the following pieces of data does NOT uniquely determine an acute-angled triangle ABC(R being the radius of the circumcircle)? **[JEE-2002, 3]**

(A) a, sinA, sinB (B) a, b, c (C) a, sinB, R (D) a, sinA, R

2. The ratio of the sides of a triangle ABC is $1 : \sqrt{3} : 2$. The ratio A : B : C is **[JEE-2004]**

(A) 3 : 5 : 2 (B) $1 : \sqrt{3} : 2$ (C) 3 : 2 : 1 (D) 1 : 2 : 3

3. In ΔABC, a, b, c are the lengths of its sides and A, B, C are the angles of triangle ABC. The correct relation is **[JEE-2005]**

(A) $(b-c)\sin\left(\frac{B-C}{2}\right) = a\cos\left(\frac{A}{2}\right)$ (B) $(b-c)\cos\left(\frac{A}{2}\right) = a\sin\left(\frac{B-C}{2}\right)$

(C) $(b+c)\sin\left(\frac{B+C}{2}\right) = a\cos\left(\frac{A}{2}\right)$ (D) $(b-c)\cos\left(\frac{A}{2}\right) = 2a\sin\left(\frac{B+C}{2}\right)$

4. Circles with radii 3, 4 and 5 touch each other externally if P is the point of intersection of tangents to these circles at their points of contact. Find the distance of P from the points of contact. **[JEE 2002, 5]**

(A) $\sqrt{5}$ (B) $2\sqrt{5}$ (C) 5 (D) 4

5. Given an isosceles triangle, whose one angle is 120° and radius of its incircle is $\sqrt{3}$. Then the area of triangle in sq. units is

(A) $7 + 12\sqrt{3}$ (B) $12 - 7\sqrt{3}$ (C) $12 + 7\sqrt{3}$ (D) 4π

6. If in a triangle PQR, sin P, sin Q, sin R are in A.P., then - **[JEE-1998]**

(A) the altitudes are in A.P. (B) the altitudes are in H.P.

(C) the medians are in G.P. (D) the medians are in A.P.

7. If the angles of elevation of the top of a tower from three collinear points A, B and C, on a line leading to the foot of the tower, are 30°, 45° and 60° respectively, then the ratio, AB : BC, is : **[JEE-2015]**

(A) $\sqrt{3} : 1$ (B) $\sqrt{3} : \sqrt{2}$ (C) $1 : \sqrt{3}$ (D) 2 : 3

8. With the usual notation, in ΔABC, if $\angle A + \angle B = 120°$, $a = \sqrt{3} + 1$ and $b = \sqrt{3} - 1$, then the ratio $\angle A : \angle B$, is : **[JEE-2019]**

(A) 7 : 1 (B) 5 : 3 (C) 9 : 7 (D) 3 : 1

9. Given $\frac{b+c}{11} = \frac{c+a}{12} = \frac{a+b}{13}$ for a ΔABC with usual notation. If $\frac{\cos A}{\alpha} = \frac{\cos B}{\beta} = \frac{\cos C}{\gamma}$, the ordered triad (a, p, y) has a value : **[JEE-2019]**

(A) (3, 4, 5) (B) (19, 7, 25) (C) (7, 19, 25) (D) (5, 12, 13)

10. In a triangle, the sum of lengths of two sides is x and the product of the lengths of the same two sides is y. If $x^2 - c^2 = y$, where c is the length of the third side of the triangle, then the circumradius of the triangle is :

(A) $\frac{y}{\sqrt{3}}$ (B) $\frac{c}{\sqrt{3}}$ (C) $\frac{c}{3}$ (D) $\frac{3}{2}y$ **[JEE-2019]**

11. If the lengths of the sides of a triangle are in A.P. and the greatest angle is double the smallest, then a ratio of lengths of the sides of this triangle is : ***[JEE-2019]***

(A) 5 : 9 : 13 (B) 5 : 6 : 7 (C) 4 : 5 : 6 (D) 3 : 4 : 5

12. The angles A, B and C of a triangle ABC are in A.P. and a : b = 1 : $\sqrt{3}$. If c = 4 cm, then the area (in sq. cm) of this triangle is : ***[JEE-2019]***

(A) $4\sqrt{3}$ (B) $\frac{2}{\sqrt{3}}$ (C) $2\sqrt{3}$ (D) $\frac{4}{\sqrt{3}}$

EXERCISE – 5 RECAP OF IIT-JEE/JEE (ADVANCED)

1. Internal bisector of $\angle A$ of a triangle ABC meets side BC at D. A line drawn through D perpendicular to AD intersects the side AC at E and the side AB at F. If a, b, c represent sides of ΔABC then ***[JEE 2006, 5]***

(A) AE is HM of b and c (B) $AD = \frac{2bc}{b+c}\cos\frac{A}{2}$ (C) $EF = \frac{4bc}{b+c}\sin\frac{A}{2}$ (D) The ΔAEF is isosceles

2. Let ABC and ABC′ be two non-congruent triangles with sides AB = 4, AC = AC′ = $2\sqrt{2}$ and angle B = 30°. The absolute value of the difference between the areas of these triangles is ***[JEE 2009, 5]***

3. If the angle A,B and C of a triangle are in an arithmetic progression and if a,b and c denote the length of the sides opposite to A,B and C respectively, then the value of the expression $\frac{a}{c}\sin 2C + \frac{c}{a}\sin 2A$, is

(A) $\frac{1}{2}$ (B) $\frac{\sqrt{3}}{2}$ (C) 1 (D) $\sqrt{3}$ ***[JEE 2010]***

4. Consider a triangle ABC and let a,b and c denote the length of the sides opposite to vertices A,B and C respectively. Suppose a = 6, b = 10 and the area of the triangle is $15\sqrt{3}$. If $\angle ACB$ is obtuse and if r denotes the radius of the incircle of the triangle, then r^2 is equal to ***[JEE 2010]***

5. Let ABC be a triangle such that $\angle ACB = \frac{\pi}{6}$ and let a,b and c denote the lengths of the sides opposite to A,B and C respectively. The value(s) of x for which $a = x^2 + x + 1$, $b = x^2 - 1$ and $c = 2x + 1$ is/are ***[JEE 2010]***

(A) $-(2+\sqrt{3})$ (B) $1+\sqrt{3}$ (C) $2+\sqrt{3}$ (D) $4\sqrt{3}$

6. Let PQR be a triangle of area Δ with a = 2, b = $\frac{7}{2}$ and $c = \frac{5}{2}$, where a, b and c are the lengths of the sides of the triangle opposite to the angles at P, Q and R respectively. Then $\frac{2\sin P - \sin 2P}{2\sin P + \sin 2P}$ equals ***[JEE 2012, 3]***

(A) $\frac{3}{4\Delta}$ (B) $\frac{45}{4\Delta}$ (C) $\left(\frac{3}{4\Delta}\right)^2$ (D) $\left(\frac{45}{4\Delta}\right)^2$

7.** In a ΔPQR, P is the largest angle and $\cos P = \frac{1}{3}$. Further the incircle of the triangle touches the sides PQ, QR and RQ and RP at N, L and M respectively such that the lengths of PN, QL and RM are consecutive even integers. Then possible lengths of side(s) of triangle is /are : ***[JEE 2013, ADV.]

(A) 16 (B) 18 (C) 24 (D) 22

8.** In a triangle the sum of two sides is x and the product of the same two sides is y. If $x^2 - c^2 = y$, where c is the third side of the triangle, then the ratio of the in-radius to the circum-radius of the triangle is ***[JEE 2014]

(A) $\frac{3y}{2x(x+c)}$ (B) $\frac{3y}{2c(x+c)}$ (C) $\frac{3y}{4x(x+c)}$ (D) $\frac{3y}{4c(x+c)}$

9. In a triangle XYZ, let x, y, z be the lengths of sides opposite to the angles X, Y, Z, respectively, ***[JEE 2016]***

and 2s = x + y + z. If $\frac{s-x}{4}=\frac{s-y}{3}=\frac{s-z}{2}$ and area of incircle of the triangle XYZ is $\frac{8\pi}{3}$, then

(A) area of the triangle XYZ is $6\sqrt{6}$

(B) the radius of circumcircle of the triangle XYZ is $\frac{35}{6}\sqrt{6}$

(C) $\sin\frac{X}{2}\sin\frac{Y}{2}\sin\frac{Z}{2}=\frac{4}{35}$

(D) $\sin^2\left(\frac{X+Y}{2}\right)=\frac{3}{5}$

10. The sides of the right angled triangle are in arithmetic progression. If the triangle has area 24, then what is the length of its smallest side ? ***[JEE 2017]***

11. In a triangle PQR, let $\angle PQR = 30°$ and the sides PQ and QR have lengths $10\sqrt{3}$ and 10, respectively. Then, which of the following statement(s) is (are) TRUE ? ***[JEE 2018]***

(A) $\angle QPR = 45°$

(B) The area of the triangle PQR is $25\sqrt{3}$ and $\angle QRP = 120°$

(C) The radius of the incircle of the triangle PQR is $10\sqrt{3}-15$

(D) The area of the circumcircle of the triangle PQR is 100π.

12. In a non-right-angled triangle ΔPQR, let p, q, r denote the lengths of the sides opposite to the angles at P, Q, R respectively. The median from R meets the side PQ at S, the perpendicular from P meets the side QR at E, and RS and PE intersect at O. If $p=\sqrt{3}$, $q = 1$, and the radius of the circumcircle of the ΔPQR equals 1, then which of the following options is/are correct ? ***[JEE 2019]***

(1) Area of $\Delta SOE=\frac{\sqrt{3}}{12}$

(2) Radius of incircle of $\Delta PQR=\frac{\sqrt{3}}{2}(2-\sqrt{3})$

(3) Length of $RS=\frac{\sqrt{7}}{2}$

(4) Length of $OE=\frac{1}{6}$

ANSWER KEY

EXERCISE-1

Que	1	2	3	4	5	6	7	8	9	10
Ans.	C	A	D	A	B	C	B	A	C	A
Que	11	12	13	14	15	16	17	18	19	
Ans.	A	C	C	C	C	B	B	B	A	

EXERCISE-2

Que	1	2	3	4	5	6	7	8	9	10
Ans.	B	C,D	A,B,D	B	BCD	B	C	A,B	A	B,D

- ***Match the Column*** 11. (A) $\rightarrow$ (q), (B) $\rightarrow$ (p), (C) $\rightarrow$ (r), (D) $\rightarrow$ (s)
- ***Comprehension Based Questions*** ***12.*** B ***13.*** C ***14.*** A

EXERCISE-4

1. (D) ***2.*** (D) ***3.*** (B) ***4.*** (A) ***5.*** (C) ***6.*** (B) ***7.*** (A) ***8.*** (A)
9. (C) ***10.*** (B) ***11.*** (C) ***12.*** (C)

EXERCISE-5

1. A,B,C,D ***2.*** (4) ***3.*** (D) ***4.*** (3) ***5.*** (B) ***6.*** C
7. B, D ***8.*** B ***9.*** (ACD) ***10.*** (6) ***11.*** (BCD)
12. (2,3,4)

Important Notes

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