Trigonometric Ratios and Identities

Lecture - 12

Continued product of sine & cosine Series :—

$$\prod_{r=1}^{n} \sin(r\theta) = \sin\theta \cdot \sin 2\theta \cdot \sin 3\theta \cdot \sin 4\theta \text{ ----- } \sin n\theta$$

$$\sum_{r=1}^{n} \sin(r\theta) = \sin\theta + \sin 2\theta + \sin 3\theta + \text{ ---- } + \sin n\theta$$

Q

$$P = \frac{2\sin\theta}{2\sin\theta}\cos\theta \cdot \cos 2\theta \cdot \cos 2^2\theta \cdot \cos 2^3\theta \cdot \ldots\ldots \cos 2^{n-1}\theta$$

$$= \frac{2\sin 2\theta}{2(2)\sin\theta}\cos 2\theta \cdot \cos 2^2\theta \cdot \cos 2^3\theta \cdot \ldots\ldots \cos 2^{n-1}\theta$$

$$= \frac{2\sin 4\theta}{2(2^2)\sin\theta}\cos 2^2\theta \cdot \cos 2^3\theta \cdot \ldots\ldots \cos 2^{n-1}\theta$$

$$= \frac{\sin(2^3\theta)}{2^3\sin\theta} \cdot \cos 2^3\theta \cdot \ldots\ldots \cos 2^{n-1}\theta.$$

$$= \frac{\sin(2^n\theta)}{2^n\sin\theta}$$

$$\boxed{\begin{aligned} &2^{n-1}\cdot 2 \\ &= 2^{n-1}\cdot 2^1 \\ &= 2^n \end{aligned}}$$

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Q

$$P = \sin\left(\frac{\pi}{14}\right)\sin\left(\frac{3\pi}{14}\right)\sin\left(\frac{5\pi}{14}\right)\underbrace{\sin\left(\frac{7\pi}{14}\right)}\sin\left(\frac{9\pi}{14}\right)\sin\left(\frac{11\pi}{14}\right)\sin\left(\frac{13\pi}{14}\right)$$

$$= \sin\left(\frac{\pi}{14}\right)\quad \sin\left(\frac{3\pi}{14}\right)\quad \sin\left(\frac{5\pi}{14}\right)\sin\left(\frac{\pi}{2}\right)\sin\left(\pi-\frac{5\pi}{14}\right)\sin\left(\pi-\frac{3\pi}{14}\right)\sin\left(\pi-\frac{\pi}{14}\right)$$

$$= \sin^2\frac{\pi}{14}\cdot\sin^2\left(\frac{3\pi}{14}\right)\sin^2\left(\frac{5\pi}{14}\right)$$

$$= \left(\sin\frac{\pi}{14}\cdot\sin\frac{3\pi}{14}\cdot\sin\frac{5\pi}{14}\right)^2$$

$$= \left[\sin\left(\frac{7\pi-6\pi}{14}\right)\sin\left(\frac{7\pi-4\pi}{14}\right)\cdot\sin\left(\frac{7\pi-2\pi}{14}\right)\right]^2$$

$$= \left[\sin\left(\frac{\pi}{2}-\frac{6\pi}{14}\right)\cdot\sin\left(\frac{\pi}{2}-\frac{4\pi}{14}\right)\cdot\sin\left(\frac{\pi}{2}-\frac{2\pi}{14}\right)\right]^2$$

$$= \left[\left(\cos\frac{3\pi}{7}\cdot\cos\frac{2\pi}{7}\cdot\cos\frac{\pi}{7}\cdot\frac{2\sin\left(\frac{\pi}{7}\right)}{2\sin(\pi/7)}\right]^2 =$$

$$= \left[\left(\cos\frac{3\pi}{7}\cdot\cos\frac{2\pi}{7}\cdot\cos\frac{\pi}{7}\cdot\frac{2\sin(\pi/7)}{2\sin(\pi/7)}\right)\right]^2$$

$$= \left[\cos\frac{3\pi}{7}\cdot\frac{\boxed{\cos\frac{2\pi}{7}\cdot\left(2\sin 2\pi/7\right)}}{2(2)\sin\pi/7}\right]^2$$

$$= \left[\underline{\cos\left(\frac{3\pi}{7}\right)}\cdot\frac{\sin\left(\frac{4\pi}{7}\right)}{4\sin(\pi/7)}\right]^2$$

$$\cos\frac{3\pi}{7} = \cos\left(\pi-\frac{4\pi}{7}\right) = -\cos\frac{4\pi}{7}$$

$$= \left[\frac{-\cos(4\pi/7)\cdot(2)\sin\frac{4\pi}{7}}{(2)\,4\sin(\pi/7)}\right]^2 = \left(\frac{-\sin\frac{8\pi}{7}}{8\sin\pi/7}\right)^2$$

$$= \left[\frac{-\sin\left(\pi+\frac{\pi}{7}\right)}{8\sin(\pi/7)}\right]^2 = \left[\frac{+\sin\pi/7}{8\sin(\pi/7)}\right]^2 = \boxed{\frac{1}{64}}$$

Ans

Q $\quad P = \sin\left(\frac{\pi}{16}\right)\cdot\sin\left(\frac{3\pi}{16}\right)\cdot\sin\left(\frac{5\pi}{16}\right)\cdot\sin\left(\frac{7\pi}{16}\right)$ $\quad \frac{8\pi}{16}$

$$= \underbrace{\sin\left(\frac{\pi}{16}\right)}\ \sin\left(\frac{3\pi}{16}\right)\ \sin\left(\frac{8\pi-3\pi}{16}\right)\underbrace{\sin\left(\frac{8\pi-\pi}{16}\right)}$$

$$= \underbrace{\sin\left(\frac{\pi}{16}\right)}\ \sin\frac{3\pi}{16}\ \sin\left(\frac{\pi}{2}-\frac{3\pi}{16}\right)\underbrace{\sin\left(\frac{\pi}{2}-\frac{\pi}{16}\right)}$$

$$= \frac{2}{2}\underbrace{\sin\frac{\pi}{16}}\cdot\frac{1}{2}\left(2\sin\frac{3\pi}{16}\cdot\cos\frac{3\pi}{16}\right)\underbrace{\cos\frac{\pi}{16}}$$

$$= \frac{1}{4}\left[\sin\left(\frac{\pi}{8}\right)\cdot\sin\left(\frac{3\pi}{8}\right)\right]$$

$$= \frac{1}{4}\cdot\left[\sin\left(\frac{\pi}{8}\right)\sin\left(\frac{4\pi-\pi}{8}\right)\right] = \frac{1}{4}\left[\frac{2}{2}\sin\frac{\pi}{8}\cdot\cos\frac{\pi}{8}\right]$$

$$= \frac{1}{8}\left[\sin\frac{\pi}{4}\right] = \frac{1}{8\sqrt{2}}$$ Ans

Summation of Trigonometric Series :→

Type I Sum of sine / cosine of n angles which are in AP. (i.e. successive argument of sine or cosine have the same difference.)

Q

$$2\sin\left(\frac{\beta}{2}\right) S = 2\sin\frac{\beta}{2}\left[\sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \sin(\alpha+3\beta) + \text{-----} + \sin(\alpha+(n-1)\beta)\right]$$

$$= \underbrace{2\sin\frac{\beta}{2}\sin\alpha} + \underbrace{2\sin\frac{\beta}{2}\sin(\alpha+\beta)} + \underbrace{2\sin\frac{\beta}{2}\cdot\sin(\alpha+2\beta)} + \text{----} + \underbrace{2\sin\frac{\beta}{2}\sin(\alpha+(n-1)\beta)}$$

$$= \left[\cos\left(\alpha-\frac{\beta}{2}\right) - \cos\left(\alpha+\frac{\beta}{2}\right)\right] + \left[\cos\left(\alpha+\beta-\frac{\beta}{2}\right) - \cos\left(\alpha+\beta+\frac{\beta}{2}\right)\right]$$

$$+ \left[\cos\left(\alpha+2\beta-\frac{\beta}{2}\right) - \cos\left(\alpha+2\beta+\frac{\beta}{2}\right)\right] + \text{----}$$

$$+ \left[\cos\left(\alpha+\underline{(n-1)\beta}-\frac{\beta}{2}\right) - \cos\left(\alpha+(n-1)\beta+\frac{\beta}{2}\right)\right]$$

$$\left(2\sin\frac{\beta}{2}\right)S = \cos\left(\alpha-\frac{\beta}{2}\right) - \cancel{\cos\left(\alpha+\frac{\beta}{2}\right)} + \cancel{\cos\left(\alpha+\frac{\beta}{2}\right)} - \cancel{\cos\left(\alpha+\frac{3\beta}{2}\right)}$$

$$+ \cancel{\cos\left(\alpha+\frac{3\beta}{2}\right)} - \cancel{\cos\left(\alpha+\frac{5\beta}{2}\right)} + - - - -$$

$$+ \cancel{\cos\left[\alpha+\left(n-\frac{3}{2}\right)\beta\right]} - \cos\left(\alpha+\left(n-\frac{1}{2}\right)\beta\right)$$

$$\left(2\sin\frac{\beta}{2}\right)\cdot S = \cos\underbrace{\left(\alpha-\frac{\beta}{2}\right)} - \cos\underbrace{\left(\alpha+\left(n-\frac{1}{2}\right)\beta\right)}$$

$$\alpha-\frac{\beta}{2}+\alpha+n\beta-\frac{\beta}{2}$$

$$\frac{2\alpha+n\beta-\beta}{2}$$

$$\cancel{\alpha}+n\beta-\cancel{\frac{\beta}{2}}-\cancel{\alpha}+\cancel{\frac{\beta}{2}}$$

$$\left(\cancel{2}\sin\frac{\beta}{2}\right)S = \cancel{2}\sin\left(\alpha+\frac{\beta(n-1)}{2}\right)\sin\left(\frac{n\beta}{2}\right)$$

$$S = \frac{\sin\left(\frac{n\beta}{2}\right)\cdot\sin\left(\alpha+\frac{(n-1)\beta}{2}\right)}{\sin\frac{\beta}{2}}$$

$$S = \cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + \text{- - - -} + \cos(\alpha + (n-1)\beta)$$

$$S = \frac{\sin\left(\frac{n\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)} \cdot \cos\left(\alpha + (n-1)\frac{\beta}{2}\right)$$

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Q $S = \sin\theta + \sin 2\theta + \sin 3\theta + \cdots + \sin(n-1)\theta + \sin n\theta$

$\alpha = \theta$

$\alpha + \beta = 2\theta \Rightarrow \theta + \beta = 2\theta \Rightarrow \beta = \theta$

$$S = \frac{\sin\left(\frac{n\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)} \cdot \sin\left(\alpha + (n-1)\frac{\beta}{2}\right)$$

$$= \frac{\sin\left(\frac{n\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)} \sin\left(\theta + (n-1)\frac{\theta}{2}\right)$$

$$S = \frac{\sin\left(\frac{n\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)} \cdot \sin\left((n+1)\frac{\theta}{2}\right)$$

Given $\theta = 2\pi/n$

$$S = \frac{\boxed{\sin\left(\frac{n}{2}\cdot\frac{2\pi}{n}\right)} \cdot \sin\left((n+1)\frac{2\pi}{2n}\right)}{\sin\left(\frac{2\pi}{2n}\right)}$$

$S = 0$