Principle of Homogeneity of dimension

Consider a simple equation,

$$A + B = C.$$

If this is an equation of physics, i.e. if A, B and C are physical quantities, then this equation says that one physical quantity A, when added to another physical quantity B, gives a third physical quantity C. This equation will have no meaning in physics if the nature (i.e. the dimensions) of the quantities on the left-hand side of the equation is not the same as the nature of the quantity on the right-hand side. For example, if A is a length, B must also be a length and the result of addition of A and B must express a length. In other words, the dimensions of both sides of a physical equation must be identical. This is called the *principle of homogeneity of dimensions*.

According to this
"Each and Every term in Physical equation must have same Dimensions"

NOTE: Same Dimensions Can be added or Subtracted while Diff. Dimensions Can be Divide & multiply

$[A] = [B] = [C]$

If $AB = C$

then $[AB] = [C]$

Here $[A] \neq [B]$

USES OF DIMENSIONAL ANALYSIS

There are four important uses of dimensional equations:

1. Checking the correctness of an equation.
2. Derivation of the relationship between the physical quantities involved in any phenomenon.
3. Finding the dimensions of constants or variables in an equation.
4. Conversion of units from one system to another.

1.checking the correctness of an equation

To check the dimensional correctness of a given physical relation :

If in a given relation, the terms on both the sides have the same dimensions, then the relation is dimensionally correct. This is known as the *principle of homogeneity of dimensions*.

Ex-) Check the accuracy of the relation $T = 2\pi\sqrt{\frac{L}{g}}$ for a simple pendulum using dimensional analysis.

$T = 2\pi\sqrt{\frac{L}{g}}$

L = length of pendulum

g = Acc. due to gravity

T = time period

$[T] = [M^0 L^0 T^1]$

$\left[\sqrt{\frac{L}{g}}\right] = \left[\frac{L^1}{L^1} \times T^2\right]^{1/2}$

$\left[\sqrt{\frac{L}{g}}\right] = [M^0 L^0 T^1]$

Hence R.H.S = L.H.S

This is Dimensionally correct.

NOTE Numerical values does not have any Dimensions or UNIT.

Ex Show that following Eq. are Dimensionally correct or not

(1) $v = u + at$

v = final velocity

u = Initial velocity

a = Acceleration

t = time

According to P.D.H

$$[v] = [u] = [at]$$

$$[v] = [M^0 L^1 T^{-1}]$$

$$[u] = [M^0 L^1 T^{-1}]$$

$$[at] = [M^0 L^1 T^{-2} . T^1]$$

$$= [M^0 L^1 T^{-1}]$$

Hence $[v] = [u] = [at]$

we can say that this is Dimensionally correct

(ii) $S = ut + \frac{1}{2}at^2$ $S =$ displacement

$[S] = [M^0 L^1 T^0]$

$[ut] = [M^0 L^1 T^{-1} \times T^1] = [M^0 L^1 T^0]$

$[\frac{1}{2}at^2] = [M^0 L^1 T^{-2} . T^2]$

$= [M^0 L^1 T^0]$

Hence

$[S] = [ut] = [at^2]$

Given Relⁿ is Dimensionally correct

(iii)

$S_{n^{th}} = u + \frac{1}{2} a (2n-1)$ Disp. in n^{th} second.

$[S_{n^{th}}] = [M^0 L^1 T^0]$

$[u] = [M^0 L^1 T^{-1}]$

$[a] = [M^0 L^1 T^{-2}]$

Here $[S_{n^{th}}] \neq [u] \neq [a]$

this is Dimensionally incorrect by physically Right Eq.

2. To derive relationship between different physical quantities:

Ex-1 The time period (t) of a simple pendulum may depend upon m the mass of the bob, l the length of the string and g the acceleration due to gravity. Find the dependence of t on m, l and g.

$$t \propto m^a l^b g^c$$

P.O.H

Dim. of R.H.S = Dim. of L.H.S

$$[t] = [M^0 L^0 T^1]$$

$$[m^a l^b g^c] = [M^a L^b (LT^{-2})^c]$$

$$= [M^a L^{b+c} T^{-2c}]$$

$$[M^0 L^0 T^1] = [M^a L^{b+c} T^{-2c}]$$

$a = 0$ $\quad b + c = 0$ $\quad -2c = 1$

$\boxed{a = 0}$ $\quad \boxed{b = \frac{1}{2}}$ $\quad \boxed{c = -\frac{1}{2}}$

Put

$$t \propto m^0 l^{1/2} g^{-1/2}$$

$$t \propto \sqrt{\frac{l}{g}} \Rightarrow t = k\sqrt{\frac{l}{g}}$$

$$k = 2\pi$$

Ex-2 Find relationship between speed of sound in a medium (v), the elastic constant (E) and the density of the medium (ρ).

Elastic Constant [E] = $\frac{\text{Stress}}{\text{Strain}}$

Stress = $\frac{\text{Force}}{\text{Area}}$

$[\text{Stress}] = \left[\frac{M^1L^1T^{-2}}{L^2}\right] = [M^1L^{-1}T^{-2}]$

Strain ⇒ Dimension less quantity

$[E] = [M^1L^{-1}T^{-2}]$

$[v] = [M^0L^1T^{-1}]$

$[\rho] = [M^1L^{-3}T^0]$

$v \propto E^a \rho^b$

$[M^0L^1T^{-1}] = [M^1L^{-1}T^{-2}]^a[M^1L^{-3}T^0]^b$

$[M^0L^1T^{-1}] = [M^{a+b}L^{-a-3b}T^{-2a}]$

$a+b=0$ — ①

$-a-3b=1$ — ②

$-1 = -2a \Rightarrow a = \frac{1}{2}$ — ③

From ① & ③

$\frac{1}{2}+b=0$

$b=-\frac{1}{2}$

$v \propto E^{1/2}\rho^{-1/2}$

$$\boxed{v \propto \sqrt{E/\rho}}$$

$v = K\sqrt{\frac{E}{\rho}}$

In SI Unit K=1

$v = \sqrt{E/\rho}$

Ex-3 Centripetal force (F) on a body of mass (m) moving with uniform speed (v) in a circle of radius (r) depends upon m, v and r. Derive a formula for the centripetal force using theory of dimensions.

$F \propto m^a v^b r^c$

$[M^1 L^1 T^{-2}] = [M^1]^a [L T^{-1}]^b [L^1]^c$

$[M^1 L^1 T^{-2}] = [M^a L^{b+c} T^{-b}]$

$a = 1$ — (1)

$b + c = 1$ — (2)

$-b = -2$

$b = 2$ — (3)

$c = -1$

$F \propto m^1 v^2 r^{-1}$

$F \propto \frac{mv^2}{r}$

$F = K \frac{mv^2}{r}$

$K = 1$

$\boxed{F = \frac{mv^2}{r}}$

3. Finding the dimensions of constants and variables in an equation:

Ex-1 The distance x travelled by a body varies with time t as

$$x = at + bt^2, \quad \text{where } a \text{ and } b \text{ are constants.}$$

Find the dimensions of a and b.

$[x] = [at] \Rightarrow [a] = \left[\frac{x}{t}\right] = [M^0 L^1 T^{-1}]$

$[x] = [bt^2] \Rightarrow [b] = \left[\frac{x}{t^2}\right] = [M^0 L^1 T^{-2}]$

Ex-2 The pressure P, volume V and temperature T of a gas are related as

$$\left(P+\frac{a}{V^2}\right)(V-b) = cT$$

where a, b, and c are constants. Find the dimensions of $\frac{a}{b}$.

$\left(P+\frac{a}{V^2}\right)$ $\quad [P] = \left[\frac{a}{V^2}\right]$

$\therefore [a] = [PV^2]$

$[V-b]$ $\quad \therefore [V] = [b]$

$\left[\frac{a}{b}\right] = \left[\frac{PV^2}{V}\right] = [PV]$

$= \left[\frac{M^1L^1T^{-2}}{L^2} \times L^3\right]$

$\left[\frac{a}{b}\right] = [M^1L^2T^{-2}]$

$\underline{\underline{\text{Ans}}}$

$[c] = \left[\frac{PV}{T}\right]$

$[c] = [M^1L^2T^{-2}K^{-1}]$

NOTE

1. Trigonometric function (sin, cos, tan, cot etc) are dimensionless. The arguments of these functions are also dimensionless
2. Exponential functions are dimensionless. Their exponents are also dimensionless

$\sin(\theta)$ θ = Dimensionless

$e^{(2x+3)} \rightarrow$ Dimensionless

$\cos(2t+3) \rightarrow$ Dimensionless

Ex-1 When a plane wave travels in a medium, the displacement y of a particle located at x at time t is given by

$$y = a\sin(bt + cx)$$

where a, b and c are constants. Find the dimensions of $\frac{b}{c}$.

$[a] = [y] = [M^0 L^1 T^0]$

$[bt + cx] = [M^0 L^0 T^0]$

$[b] \propto \left[\frac{1}{t}\right] = [M^0 L^0 T^{-1}]$

$[c] \propto \frac{1}{[x]} = [M^0 L^{-1} T^0]$

$\left[\frac{b}{c}\right] = \left[\frac{M^0 L^0 T^{-1}}{M^0 L^{-1} T^0}\right]$

$= [M^0 L^1 T^{-1}]$

Ans

Ex-2 In the expression

$$P = \frac{a^2}{b} e^{-ax}$$

P is pressure, x is a distance and a and b are constants. Find the dimensional formula for b.

$[a] \propto \frac{1}{[x]}$

$$\boxed{[a] = [M^0 L^{-1} T^0]}$$

$[P] = \left[\frac{a^2}{b}\right]$

$b = \left(\frac{a^2}{P}\right)$

$= \frac{[M^0 L^{-2} T^0]}{[M^1 L^{-1} T^{-2}]}$

$$\boxed{[b] = [M^{-1} L^{-1} T^2]}$$