Errors

Errors can be expressed in the following ways :-

1 **Absolute Error (Δa) :** The difference between the true value and the individual measured value of the quantity is called the absolute error of the measurement.

Suppose a physical quantity is measured *n* times and the measured values are $a_1, a_2, a_3 \ldots\ldots\ldots a_n$. The arithmetic mean ($a_m$) of these values is

1 $$a_m = \frac{a_1 + a_2 + a_3 + \ldots\ldots\ldots\ldots a_n}{n} = \frac{1}{n}\sum_{i=1}^{n} a_i \quad \ldots(1)$$

If the true value of the quantity is not given then mean value (a_m) *can be taken as the true value*. Then the absolute errors in the individual measured values are –

2
$$\begin{aligned} \Delta a_1 &= a_m - a_1 \\ \Delta a_2 &= a_m - a_2 \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ \Delta a_n &= a_m - a_n \end{aligned}$$

The arithmetic mean of all the absolute errors is defined as the final or mean absolute error $(\Delta a)_m$ or $\overline{\Delta a}$ of the value of the physical quantity a

3 $$(\Delta a)_m = \frac{|\Delta a_1| + |\Delta a_2| + \ldots\ldots\ldots + |\Delta a_n|}{n} = \frac{1}{n}\sum_{i=1}^{n}|\Delta a_i| \quad \ldots(2)$$

So if the measured value of a quantity be 'a' and the error in measurement be Δa, then the true value (a_t) can be written as $a_t = a \pm \Delta a$ ...(3)

Relative or Fractional Error : It is defined as the ratio of the mean absolute error ($(\Delta a)_m$ or $\overline{\Delta a}$) to the true value or the mean value (a_m or $\bar{a}$) of the quantity measured.

2 $$\text{Relative or fractional error} = \frac{\text{Mean absolute error}}{\text{Mean value}} = \frac{(\Delta a)_m}{a_m} \quad \text{or} \quad \frac{\overline{\Delta a}}{\bar{a}} \qquad \ldots(4)$$

When the relative error is expressed in percentage, it is known as percentage error,

5 $$\text{percentage error} = \text{relative error} \times 100\ \%$$

$$\text{or} \quad \text{percentage error} = \frac{\text{mean absolute error}}{\text{true value}} \times 100\% = \frac{\overline{\Delta a}}{a} \times 100\% \qquad \ldots\ldots\ldots(5)$$

Ex Measured length of a wire are 2.20 cm 2.10 cm 2.30 cm 2.05 cm 2.15 cm

Find (1) True / mean value
(2) Absolute Error in measurements
(3) mean Absolute Error
(4) fractional ERROR
(5) Percentage ERROR in measurement

$l_1 = 2.20$ cm
$l_2 = 2.10$ cm
$l_3 = 2.30$ cm
$l_4 = 2.05$ cm
$l_5 = 2.15$ cm

① True value

$$l_m = \frac{l_1 + l_2 + l_3 + l_4 + l_5}{5}$$

$$= \frac{2.20 + 2.10 + 2.30 + 2.05 + 2.15}{5}$$

$$\boxed{l_m = \frac{10.80}{5} = 2.16 \text{ cm}}$$

② Absolute Error in measurements

$\Delta l_1 = l_m - l_1 = 2.16 - 2.20 = -0.04$ cm

$\Delta l_2 = l_m - l_2 = 2.16 - 2.10 = +0.06$ cm

$\Delta l_3 = l_m - l_3 = 2.16 - 2.30 = -0.14$ cm

$\Delta l_4 = l_m - l_4 = 2.16 - 2.05 = +0.11$ cm

$\Delta l_5 = l_m - l_5 = 2.16 - 2.15 = +0.01$ cm

③ Absolute mean ERROR

$$\Delta l_m = \frac{|\Delta l_1| + |\Delta l_2| + |\Delta l_3| + |\Delta l_4| + |\Delta l_5|}{5}$$

$$= \frac{0.04 + 0.06 + 0.14 + 0.11 + 0.01}{5}$$

$$\boxed{\Delta l_m = \frac{0.36}{5} = 0.072 \text{ cm}}$$

④ Fractional / Relative Error {Dimensionless Quantity}

$$\boxed{\frac{\Delta l_m}{l_m} = \frac{0.072}{2.16} = 0.033}$$

⑤ Percentage ERROR

$$\frac{\Delta l_m}{l_m} \times 100 = 0.033 \times 100 = 3.3\% \quad \underline{\text{Ans}}$$

Final length $= l_m \pm \Delta l_m$

$$\boxed{l = (2.16 \pm 0.072) \text{ cm}}$$

PROPAGATION OF ERRORS IN MATHEMATICAL OPERATIONS :

(a) If $x = a + b$, then the maximum possible absolute error in measurements of x will be $\Delta x = \Delta a + \Delta b$

(b) If $x = a - b$, then the maximum possible absolute error in measurement of x will be $\Delta x = \Delta a + \Delta b$

(c) If $x = \frac{a}{b}$ then the maximum possible fractional error will be $\frac{\Delta x}{x} = \frac{\Delta a}{a} + \frac{\Delta b}{b}$

(d) If $x = a^n$ then the maximum possible fractional error will be $\frac{\Delta x}{x} = n\frac{\Delta a}{a}$

(e) If $x = \frac{a^n b^m}{c^p}$ then the maximum possible fractional error will be $\frac{\Delta x}{x} = n\frac{\Delta a}{a} + m\frac{\Delta b}{b} + p\frac{\Delta c}{c}$

(f) If $x = \log_e a$ then the maximum possible fractional error will be $\frac{\Delta x}{x} = \frac{1}{x}\frac{\Delta a}{a}$

ERROR In Add./Subtr.

(a) If $x = a + b$, then the maximum possible absolute error in measurements of x will be $\Delta x = \Delta a + \Delta b$

(b) If $x = a - b$, then the maximum possible absolute error in measurement of x will be $\Delta x = \Delta a + \Delta b$

$$\boxed{X = X_m \pm \Delta X_m}$$

If $X = a + b$

$$\boxed{X_m = a_m + b_m}$$
$$\boxed{\Delta X_m = \Delta a_m + \Delta b_m}$$

If $x = a - b$

$$\boxed{X_m = a_m - b_m}$$
$$\boxed{\Delta X_m = \Delta a_m + \Delta b_m}$$

If $x = a - b + c$

then $X_m = a_m - b_m + c_m$

$\Delta X_m = \Delta a_m + \Delta b_m + \Delta c_m$

Ex.-1 The inital and final temperatures of water as recorded by an observer are $(40.6 \pm 0.2)°C$ and $(78.3 \pm 0.3)°C$. Calculate the rise in temperature with proper error limits.

Ans

$\therefore$ rise in temperature = $(37.7 \pm 0.5)°C$

$\theta_i = (40.6 \pm 0.2)°C$

$\theta_f = (78.3 \pm 0.3)°C$

Rise in Temp $= \theta = \theta_f - \theta_i$

Find $\theta = \theta_m \pm \Delta\theta_m$

$\theta_m = (\theta_f)_m - (\theta_i)_m$

$= 78.3 - 40.6$

$\theta_m = 37.7$

$\Delta\theta_m = (\Delta\theta_f)_m + (\Delta\theta_i)_m$

$= 0.3 + 0.2$

$= 0.5$

$\theta = (37.7 \pm 0.5)°C$ Ans

Range of Rise in Tem

$37.2°C$ — $38.2°C$

Ex Two Resistance of wire are $R_1 = (20 \pm 0.5)\,\Omega$ $R_2 = (10 \pm 0.6)\,\Omega$ If they are Connected in series Find Final Resistance of combination in Error limits

$$[R_S = R_1 + R_2]$$

Find $R_S = (R_S)_m \pm (\Delta R_S)_m$

$$(R_S)_m = (R_1)_m + (R_2)_m$$

$$= 20 + 10$$

$$= 30\,\Omega$$

$$(\Delta R_S)_m = (\Delta R_1)_m + (\Delta R_2)_m$$

$$= 0.5 + 0.6$$

$$= 1.1\,\Omega$$

$$\boxed{R_S = (30 \pm 1.1)\,\Omega}$$ Ans

fractional ERROR $= \frac{1.1}{30}$

% ERROR $= \frac{1.1}{30} \times 100 = \frac{11}{3}\,\%$

Multi / Divide :→

(c) If $x = \frac{a}{b}$ then the maximum possible fractional error will be $\frac{\Delta x}{x} = \frac{\Delta a}{a} + \frac{\Delta b}{b}$

(d) If $x = a^n$ then the maximum possible fractional error will be $\boxed{\frac{\Delta x}{x} = n\frac{\Delta a}{a}}$

Given $x = ab$ or $x = \frac{a}{b}$

Find $x = x_m \pm \Delta x_m$

$x_m = (a_m b_m)$

$x_m = \frac{a_m}{b_m}$

$$\boxed{\frac{\Delta x_m}{x_m} = \frac{\Delta a_m}{a_m} + \frac{\Delta b_m}{b_m}} \qquad \boxed{\frac{\Delta x_m}{x_m} = \frac{\Delta a_m}{a_m} + \frac{\Delta b_m}{b_m}}$$

let $x = \frac{a}{bc}$

$x_m = \frac{a_m}{b_m c_m}$

$$\frac{\Delta x_m}{x_m} = \frac{\Delta a_m}{a_m} + \frac{\Delta b_m}{b_m} + \frac{\Delta c_m}{c_m}$$

let $x = a^n$

$x_m = a_m^n$

$$\frac{\Delta x_m}{x_m} = n\frac{\Delta a_m}{a_m}$$

let $x = a^n b^m$

$x_m = (a_m)^n (b_m)^m$

$$\frac{\Delta x_m}{x_m} = n\frac{\Delta a_m}{a_m} + m\frac{\Delta b_m}{b_m}$$

Ex. The length and breadth of a rectangle are (5.7 ± 0.1) cm and (3.4 ± 0.2) cm. Calculate area of the rectangle with error limits.

(Ans $\therefore$ Area = (19.38 ± 1.48) sq. cm)

$l = (5.7 \pm 0.1)\,cm \qquad b = (3.4 \pm 0.2)\,cm$

Area of Rectangle = length × breadth

$A = lb$

Find $A = A_m \pm \Delta A_m$

$A_m = l_m \cdot b_m$

$= 5.7 \times 3.4$

$A_m = 19.38\ cm^2$

$$\frac{\Delta A_m}{A_m} = \frac{\Delta l_m}{l_m} + \frac{\Delta b_m}{b_m}$$

$$= \frac{0.1}{5.7} + \frac{0.2}{3.4}$$

$$\frac{\Delta A_m}{A_m} = 0.08$$

$$\Delta A_m = 0.08 \times A_m$$

$$= 0.08 \times 19.38$$

$$= 1.48\ cm^2$$

$$\boxed{A = (19.38 \pm 1.48)\,cm^2}$$

Ans

Ex. A body travels uniformly a distance (13.8 ± 0.2) m in a time (4.0 ± 0.3) s. Calculate its velocity with error limits. What is the percentage error in velocity ?

Ans ∴ $v = (3.5 \pm 0.31)\ ms^{-1}$

Percentage error in velocity $= \frac{\Delta v}{v} \times 100 = \pm 0.0895 \times 100 = \pm 8.95\% = \pm 9\%$

$$\text{velocity} = \frac{\text{distance}}{\text{time}}$$

$$v = \frac{d}{t}$$

$$\boxed{V = V_m \pm \Delta V_m} \quad \text{—(1)}$$

$$V_m = \frac{d_m}{t_m} = \frac{13.8}{4.0} = 3.45\ m/s$$

$$\frac{\Delta V_m}{V_m} = \frac{\Delta d_m}{d_m} + \frac{\Delta t_m}{t_m}$$

$$= \frac{0.2}{13.8} + \frac{0.3}{4}$$

$$\frac{\Delta V_m}{V_m} = 0.089$$

$$\Delta V_m = 0.089 \times V_m$$

$$= 0.089 \times 3.45$$

$$\simeq 0.31$$

$$V = V_m \pm \Delta V_m$$

$$= (3.45 \pm 0.31)\ \frac{m}{s}$$

(e) If $\boxed{x = \frac{a^n b^m}{c^p}}$ then the maximum possible fractional error will be $\boxed{\frac{\Delta x}{x} = n\frac{\Delta a}{a} + m\frac{\Delta b}{b} + p\frac{\Delta c}{c}}$

(f) If $x = \log_e a$ then the maximum possible fractional error will be $\boxed{\frac{\Delta x}{x} = \frac{1}{x}\frac{\Delta a}{a}}$

$X = x_m \pm \Delta x_m \qquad x_m = \log_e a_m$

Ex In the measurement of a physical quantity $X = \frac{A^2 B}{C^{1/3} D^3}$, the percentage errors in the measurements of quantities A, B, C and D are 1%, 2%, 3% and 4% respectively. Find the percentage error in the measurement of X.

$X = \frac{A^2 B}{C^{1/3} D^3}$

$\frac{\Delta x}{x} = 2\frac{\Delta A}{A} + \frac{\Delta B}{B} + \frac{1}{3}\frac{\Delta C}{C} + 3\frac{\Delta D}{D}$

$\frac{\Delta x}{x}\times 100 = 2\left(\frac{\Delta A}{A}\times 100\right) + \left(\frac{\Delta B}{B}\times 100\right) + \frac{1}{3}\left(\frac{\Delta C}{C}\times 100\right) + 3\left(\frac{\Delta D}{D}\times 100\right)$

Given

$\frac{\Delta A}{A}\times 100 = 1\%$

$\frac{\Delta B}{B}\times 100 = 2\%$

$\frac{\Delta C}{C}\times 100 = 3\%$

$\frac{\Delta D}{D}\times 100 = 4\%$

$\left(\frac{\Delta x}{x}\times 100\right) = 2\times 1\% + 2\% + \frac{1}{3}\times 3\% + 3\times 4\%$

$= (2 + 2 + 1 + 12)\%$

$= 17\%$ Ans

$\frac{\Delta x}{x} = \frac{17}{100}$ Ans

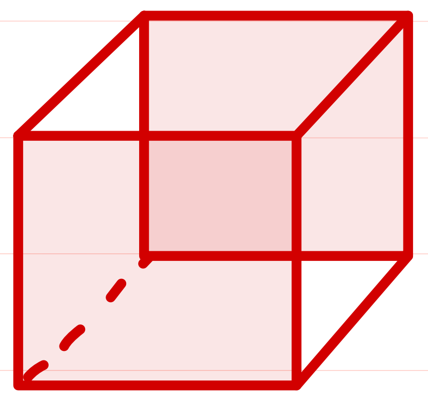
fractional Error

Ex. A thin copper wire of length L increase in length by 2% when heated from T_1 to T_2. If a copper cube having side 10 L is heated from T_1 to T_2 what will be the percentage change in

(i) area of one face of the cube and.

(ii) volume of the cube.

Given $\frac{\Delta l}{l} = \frac{2}{100}$



Area of one face $= (10L)^2$

$A = 100\,L^2$

$\frac{\Delta A}{A} = \left(2\frac{\Delta l}{L}\right)$

% ERROR

$\left(\frac{\Delta A}{A}\times 100\right) = 2\times 2$

$= 4\%$ Ans

H.W

Question For Practice

Q1. Two rods have lengths measured as (1.8 ± 0.2)m and (2.3 ± 0.1)m. Calculate their combined length with error limits.

Q2. The original length of wire is (153.7 ± 0.6) cm . It is stretched to (155.3 ± 0.2) cm. Calculate the elongation in the wire with error limits.

Q3. In an experiment, values of two resistances are measured to be $r_1 = (5.0 \pm 0.2)$ ohm and $r_2 = (10.0 \pm 0.1)$ ohm. Find the values of total resistance in series with limits of percentage error.

Q4. The radius of a sphere is measured to be (2.1 ± 0.5) cm. Calculate its surface area with absolute error limits.

Q5. A physical quantity x is calculated from the relation $x = a^3b^2\sqrt{cd}$. Calculate percentage error in x, if a, b, c and d are measured respectively with an error of 1%, 3%, 4% and 2%.

Ans. 1. (4.1 ± 0.3) m 2. (1.6 ± 0.8) cm 3. $R_s = 15$ ohm $\pm 2\%$

4. (55.4 ± 26.4) cm^2 5. $\pm 12\%$