HARMONIC PROGRESSION - I

1. Find the fourth term in the following series : $2, 2\frac{1}{2}, 3\frac{1}{3}, \ldots$

2. Find the fourth term in the following series : $2, 2\frac{1}{2}, 3, \ldots$

Sol^n:

(1.) $2, \frac{5}{2}, \frac{10}{3} \ldots \longrightarrow$ H.P.

$\Rightarrow \frac{1}{2}, \frac{2}{5}, \frac{3}{10} \longrightarrow$ A.P.

Common difference $= \frac{2}{5} - \frac{1}{2} = \frac{4-5}{10}$

$= -\frac{1}{10}$

$T_4 = \frac{1}{2} + 3 \times \left(-\frac{1}{10}\right)$ $\quad T_4 = \frac{2}{10} = \frac{1}{5}$

Fourth term of H.P. will be 5.

(2.) $2, 2.5, 3 \ldots \rightarrow$ A.P.

So, $T_4 = 3.5 = 3\frac{1}{2}$

3. If the p^{th}, q^{th}, r^{th} terms of a H.P. be a, b, c respectively, prove that $(q-r)bc+(r-p)ca+(p-q)ab=0$.

Sol^n:

$$T_p = a \quad , \quad T_q = b \quad \& \quad T_r = c$$

$$\frac{1}{a} = a_1 + (p-1)d_1$$

$$\frac{1}{b} = a_1 + (q-1)d_1 \qquad \frac{1}{c} = a_1 + (r-1)d_1$$

$$\frac{1}{a} - \frac{1}{b} = (p-q)d_1$$

$$\frac{b-a}{ab} = (p-q)d_1 \Rightarrow (p-q)ab = \frac{(b-a)}{d_1}$$

Similarly, $$(q-r)bc = \frac{(c-b)}{d_1}$$

$$(r-p)ac = \frac{(a-c)}{d_1}$$

$$(q-r)bc + (r-p)ac + (p-q)ab = \frac{\cancel{b-a+c-b+a-c}}{d_1} \to 0$$

4. If the m^{th} term of a H.P. be equal to n, and the n^{th} term be equal to m, prove that the $(m + n)^{th}$ term is equal to $\frac{mn}{m+n}$.

Sol^n: Given, $T_m = n$ and $T_n = m$

$$\frac{1}{T_m} = \frac{1}{n} = a + (m-1)d \quad \text{———} \ (1)$$

$$\frac{1}{T_n} = \frac{1}{m} = a + (n-1)d \quad \text{———} \ (2)$$

$$(1) - (2) \Rightarrow \frac{\cancel{m-n}}{mn} = \cancel{(m-n)}d \Rightarrow \boxed{d = \frac{1}{mn}}$$

$$\boxed{a = \frac{1}{mn}}$$

$$\frac{1}{T_{m+n}} = \frac{1}{mn} + (m+n-1) \times \frac{1}{mn}$$

$$\frac{1}{T_{m+n}} = \frac{(m+n)}{mn} \Rightarrow \boxed{T_{m+n} = \frac{mn}{m+n}}$$

5. If a, b, c be in H.P., prove that

(A) $\frac{1}{b-a}+\frac{1}{b-c}=\frac{2}{b}$ (B) $\frac{b+a}{b-a}+\frac{b+c}{b-c}=2$ (C) $\left(\frac{1}{a}+\frac{1}{b}-\frac{1}{c}\right)\left(\frac{1}{b}+\frac{1}{c}-\frac{1}{a}\right)=\frac{4}{ac}-\frac{3}{b^2}$

Solⁿ: Given, a, b, c in H.P. then

$\frac{1}{a}, \frac{1}{b}, \frac{1}{c} \rightarrow$ A.P. Also, $b=\frac{2ac}{a+c}$

(A) $$\frac{1}{\frac{2ac}{(a+c)}-a}+\frac{1}{\frac{2ac}{a+c}-c}$$

$$\Rightarrow \frac{a+c}{ac-a^2}+\frac{(a+c)}{ac-c^2}$$

$$\Rightarrow \frac{(a+c)}{(c-a)}\left(\frac{1}{a}-\frac{1}{c}\right) \Rightarrow \frac{a+c}{ac}=\frac{2}{b}.$$

(B) $$\frac{b+a}{b-a}+\frac{b+c}{b-c}$$

$$\Rightarrow \frac{\frac{2ac}{a+c}+a}{\frac{2ac}{a+c}-a}+\frac{\frac{2ac}{a+c}+c}{\frac{2ac}{a+c}-c}$$

$$\frac{3ac+a^2}{ac-a^2} + \frac{3ac+c^2}{ac-c^2}$$

$$\Rightarrow \quad \frac{3c+a}{c-a} + \frac{3a+c}{a-c} \Rightarrow$$

$$\Rightarrow \quad \frac{3c+a-3a-c}{(c-a)} \Rightarrow \frac{2\cancel{(c-a)}}{\cancel{(c-a)}} \Rightarrow \boxed{2}$$

(C) $\left(\frac{1}{a}+\frac{1}{b}-\frac{1}{c}\right)\left(\frac{1}{b}+\frac{1}{c}-\frac{1}{a}\right)=\frac{4}{ac}-\frac{3}{b^2}$

$$\Rightarrow \left(\frac{1}{b}+\left(\frac{1}{a}-\frac{1}{c}\right)\right)\left(\frac{1}{b}-\left(\frac{1}{a}-\frac{1}{c}\right)\right)$$

$$\Rightarrow \quad \frac{1}{b^2} - \left(\frac{1}{a}-\frac{1}{c}\right)^2$$

$$\Rightarrow \quad \frac{1}{b^2} - \left(\frac{a-c}{ac}\right)^2$$

$$\Rightarrow \quad \frac{1}{b^2} - \left[\frac{a^2+c^2-2ac}{a^2c^2}\right]$$

$$\Rightarrow \quad \frac{1}{b^2} - \left[\frac{a^2+c^2-2ac+2ac-2ac}{a^2c^2}\right]$$

$$\frac{1}{b^2} - \frac{4(a+c)^2}{4a^2c^2} + \frac{4ac}{a^2c^2}$$

$$\Rightarrow \frac{1}{b^2} - \frac{4}{b^2} + \frac{4}{ac} \Rightarrow \frac{4}{ac} - \frac{3}{b^2}$$

6. If $\frac{1}{b-a} + \frac{1}{b-c} = \frac{1}{a} + \frac{1}{c}$, then prove that a, b, c are in H.P. unless b = a + c.

Solⁿ:

$$\frac{1}{(b-a)} - \frac{1}{c} + \frac{1}{(b-c)} - \frac{1}{a} = 0$$

$$\Rightarrow \frac{c-(b-a)}{c\cdot(b-a)} + \frac{a-(b-c)}{(b-c)\cdot a} = 0$$

$$(a+c-b)\left(\frac{1}{c\cdot(b-a)} + \frac{1}{a(b-c)}\right) = 0$$

$a+c=b$

$$(ab-ac+bc-ac) = 0$$

$$ab+bc = 2ac$$

$$\frac{2}{b} = \frac{1}{a} + \frac{1}{c} \rightarrow a, b, c \text{ in } \underline{H.P.}$$

7. If a, b, c, d are in H.P., then show that ab + bc + cd = 3ad.

$\Rightarrow$ To prove that

$$ab + bc + cd + da = 4ad$$

$$(a+c)b + (a+c)d$$

$$\Rightarrow (a+c)(b+d)$$

$$a, b, c \to \text{H.P.} \Rightarrow b = \frac{2ac}{(a+c)} \Rightarrow (a+c) = \frac{2ac}{b}$$

$$b, c, d \to \text{H.P.} \Rightarrow c = \frac{2bd}{(b+d)} \Rightarrow (b+d) = \frac{2bd}{c}$$

$$\Rightarrow \frac{2a\cancel{c}}{\cancel{b}} \times \frac{2\cancel{b}d}{\cancel{c}} = 4ad \quad \text{H.P.}$$

8. (A) Solve the equation $6x^3 - 11x^2 + 6x - 1 = 0$ if its roots are in harmonic progression.

(B) If the roots of $10x^3 - cx^2 - 54x - 27 = 0$ are in harmonic progression, then find c and all the roots.

Solⁿ:

8 A.

$$6x^3 - 11x^2 + 6x - 1 = 0 \quad \to \alpha, \beta, \gamma$$

$$\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma} \to A.P.$$

Eqⁿ having roots $\frac{1}{\alpha} \to \alpha'$, $\frac{1}{\beta} \to \beta'$ & $\frac{1}{\gamma} \to \gamma'$ is

$$-x^3 - 11x + 6x^2 + 6 = 0$$

$$x^3 - 6x^2 + 11x - 6 = 0 \quad \to \alpha', \beta', \gamma'$$

$$2\beta' = \alpha' + \gamma'$$

$$\alpha' + \beta' + \gamma' = 6 \qquad \beta' = 2.$$

$$\alpha' + \gamma' = 4 \qquad \alpha' \cdot \beta' \cdot \gamma' = 6$$

$$\alpha'\gamma' = 3$$

$$\alpha' = 1, \quad \beta' = 2, \quad \gamma' = 3$$

Roots of original Eqⁿ are $1, \frac{1}{2}, \frac{1}{3}$.

(B) If the roots of $10x^3 - cx^2 - 54x - 27 = 0$ are in harmonic progression, then find c and all the roots.

Solⁿ:

$$10x^3 - cx^2 - 54x - 27 = 0 \quad \rightarrow \alpha, \beta, \gamma$$

$$\alpha, \beta, \gamma \rightarrow H.P. \qquad \beta = \frac{2\alpha\gamma}{\alpha+\gamma}$$

$$\alpha\beta + \beta\gamma + \gamma\alpha = -\frac{54}{10}$$

$$(\alpha+\gamma)\cdot\beta + \alpha\gamma = -\frac{27}{5}$$

$$\Rightarrow \cancel{(\alpha+\gamma)}\cdot\frac{2\alpha\gamma}{\cancel{\alpha+\gamma}} + \alpha\gamma = -\frac{27}{5}$$

$$\boxed{\alpha\gamma = -\frac{9}{5}}$$

$$\alpha\cdot\beta\cdot\gamma = \frac{27}{10}$$

$$\Rightarrow \frac{\alpha\cdot\gamma\cdot 2\alpha\gamma}{(\alpha+\gamma)} = \frac{27}{10}$$

$$\frac{\cancel{81}^{3}}{\cancel{25}_{5}} \times \frac{2}{(\alpha+\gamma)} = \frac{\cancel{27}}{\cancel{10}\,2} \qquad \boxed{\alpha+\gamma = \frac{12}{5}}$$

$$\alpha = -\frac{3}{5} \qquad \gamma = 3$$

$$\beta = \frac{2\times -9/\cancel{5}}{12/\cancel{5}} \qquad \boxed{\beta = -\frac{3}{2}} \qquad \frac{c}{10} = \alpha+\beta+\gamma \qquad \boxed{c = 9}$$

9. If a, b, c are in G.P. and a – b, c – a and b – c are in H.P. then prove that a + 4b + c is equal to 0.

$\Rightarrow$ Given $b^2 = ac$

$$(c-a) = \frac{2(a-b)(b-c)}{(a-\cancel{b}+\cancel{b}-c)}$$

$$\Rightarrow \quad -(a-c)(a-c) = 2(a-b)(b-c)$$

$$2(ab-ac-b^2+bc) + (a-c)^2 = 0$$

$$\Rightarrow \quad 2ab - \underline{2ac} - \underline{2b^2} + 2bc + a^2 + c^2 - 2ac = 0$$

$(\because ac = b^2)$

$$\Rightarrow \quad 2ab + 2bc + a^2 + c^2 = 6b^2$$

$$a^2 + c^2 + b^2 + 2ab + 2bc + 2ca = 6b^2 + b^2 + 2ac$$

$$\Rightarrow \quad (a+b+c)^2 = 9b^2$$

$$\Rightarrow \quad (a+b+c)^2 - (3b)^2 = 0$$

$$\Rightarrow \quad (a+b+c-3b)(a+b+c+3b) = 0$$

$$(a+c-2b)(a+4b+c) = 0$$

$$\Rightarrow \quad \boxed{a+4b+c=0}$$

10. (A) If $\frac{1}{a(b+c)}, \frac{1}{b(c+a)}, \frac{1}{c(a+b)}$ be in H.P. then a, b, c are also in H.P.

(B) If b + c, c + a, a + b are in H.P. then prove that $\frac{a}{b+c}, \frac{b}{c+a}, \frac{c}{a+b}$ are in A.P.

(C) If a, b, c be in H.P. prove that $\frac{a}{b+c-a}, \frac{b}{c+a-b}, \frac{c}{a+b-c}$ are in H.P.

(A.)

$a(b+c),\ b(c+a),\ c(a+b)$ in A.P.

$$2b(c+a) = ab+ac+ac+bc$$

$$\Rightarrow \quad 2bc+2ab = ab+2ac+bc$$

$$ab+bc = 2ac$$

$$\frac{2}{b} = \frac{1}{a}+\frac{1}{c}$$

a, b, c in H.P.

(B)

$\frac{1}{(b+c)}, \frac{1}{(c+a)}, \frac{1}{(a+b)}$ in A.P.

$\frac{(a+b+c)}{(b+c)}, \frac{(a+b+c)}{(c+a)}, \frac{(a+b+c)}{(a+b)}$ in A.P.

$\Rightarrow \frac{a}{(b+c)}+1, \frac{b}{(c+a)}+1, \frac{c}{(a+b)}+1$ in A.P.

$\Rightarrow \frac{a}{(b+c)}, \frac{b}{(c+a)}, \frac{c}{(a+b)}$ in A.P.

(C) If a, b, c be in H.P. prove that $\frac{a}{b+c-a}, \frac{b}{c+a-b}, \frac{c}{a+b-c}$ are in H.P.

Sol^n:

$$\frac{1}{a}, \frac{1}{b}, \frac{1}{c} \text{ in A.P.}$$

$$\frac{b+c-a}{a}$$

$$\frac{(a+b+c)}{a}, \frac{(a+b+c)}{b}, \frac{(a+b+c)}{c} \text{ in A.P.}$$

$$\frac{(a+b+c)}{a}-2, \frac{(a+b+c)}{b}-2, \frac{(a+b+c)}{c}-2 \text{ in A.P.}$$

$$\Rightarrow \frac{b+c-a}{a}, \frac{a+c-b}{b}, \frac{a+b-c}{c} \text{ in A.P.}$$

$$\Rightarrow \frac{a}{b+c-a}, \frac{b}{a+c-b}, \frac{c}{a+b-c} \text{ in H.P.}$$

11. Let $a_1, a_2, ..., a_{10}$ be in A.P. and $h_1, h_2, ..., h_{10}$ be in H.P. If $a_1 = h_1 = 2$ and $a_{10} = h_{10} = 3$ then $a_4 h_7$ is

(A) 2 (B) 3 (C) 5 (D) 6

Sol^n:

$$a_{10} = a_1 + 9d$$

$$3 = 2 + 9d \qquad \boxed{d = 1/9}$$

$$a_4 = a_1 + 3d = 2 + 3 \times \frac{1}{9} = \frac{7}{3}$$

$$\frac{1}{h_{10}} = \frac{1}{h_1} + 9d'$$

$$\frac{1}{3} = \frac{1}{2} + 9d' \Rightarrow \frac{2-3}{6} = 9d'$$

$$d' = -\frac{1}{54}$$

$$\frac{1}{h_7} = \frac{1}{h_1} + 6d' \Rightarrow \frac{1}{h_7} = \frac{1}{2} - \frac{6}{54}$$

$$h_7 = \frac{18}{7}$$

$$a_4 \cdot h_7 = \frac{7}{3} \times \frac{18}{7} = 6$$

12. If x > 1, y > 1, z > 1 are in G.P., then $\frac{1}{1+\ell nx}, \frac{1}{1+\ell ny}, \frac{1}{1+\ell nz}$ are in

(A) A.P. (B) H.P. (C) G.P. (D) None of these

Sol^n:

$x, y, z \longrightarrow$ in G.P.

$\ln x, \ln y, \ln z$ in A.P.

$1+\ln x, 1+\ln y, 1+\ln z$ in A.P.

$\frac{1}{1+\ln x}, \frac{1}{1+\ln y}, \frac{1}{1+\ln z}$ in H.P.

13. If a, b, c, d are in H. P., then ab + bc + cd is equal to

(A) 3ad (B) (a + b) (c + d) (C) 3ac (D) (a + c)(b + d)

$$\Rightarrow \quad ab+bc+cd+da-da$$

$$(a+c)b + (a+c)d - ad$$

$$\Rightarrow \quad (a+c)(b+d) - ad$$

$a, b, c \rightarrow$ H.P. $\Rightarrow$ $b = \dfrac{2ac}{(a+c)} \Rightarrow (a+c) = \dfrac{2ac}{b}$

$b, c, d \rightarrow$ H.P. $\Rightarrow$ $c = \dfrac{2bd}{(b+d)} \Rightarrow (b+d) = \dfrac{2bd}{c}$

$$\Rightarrow \quad \frac{2a\cancel{c}}{\cancel{b}} \times \frac{2\cancel{b}d}{\cancel{c}} \Rightarrow 4ad - ad = \underline{\underline{3ad}}.$$

1. The value of n for which $\dfrac{a^{n+1}+b^{n+1}}{a^n+b^n}$ is the harmonic mean of a and b, is equal to

(A) –1 (B) 0 (C) 1/2 (D) 1

<u>Sol^n:</u>

$$\frac{a^{n+1}+b^{n+1}}{a^n+b^n} = \frac{2ab}{a+b}$$

$$(a+b)(a^{n+1}+b^{n+1}) = 2ab(a^n+b^n)$$

$$\Rightarrow \quad a^{n+2} + a \times b^{n+1} + b \cdot a^{n+1} + b^{n+2} = 2 a^{n+1} \cdot b + 2ab^{n+1}$$

$$a^{n+2} - a^{n+1} \cdot b + b^{n+2} - b^{n+1} \cdot a = 0$$

$$a^{n+1}(a-b) - b^{n+1}(a-b) = 0$$

$$(a-b)\left(a^{n+1} - b^{n+1}\right) = 0$$

$$a^{n+1} = b^{n+1} \Rightarrow \left(\frac{a}{b}\right)^{n+1} = 1$$

$$\left(\frac{a}{b}\right)^{n+1} = \left(\frac{a}{b}\right)^{0} \Rightarrow n+1 = 0$$

$$\boxed{n = -1}$$

2. The harmonic mean of the roots of the equation $(5+\sqrt{2})x^2 - (4+\sqrt{5})x + 8 + 2\sqrt{5} = 0$ is

(A) 2 (B) 4 (C) 6 (D) 8

Sol^n:

$$\frac{2\alpha\beta}{\alpha+\beta} = \frac{\dfrac{2 \times (8+2\sqrt{5})}{\cancel{(5+\sqrt{2})}}}{\dfrac{(4+\sqrt{5})}{\cancel{(5+\sqrt{2})}}} = 2 \times 2 = 4$$

3. If 2(y – a) is the H.M. between y – x and y – z, then x – a, y – a, z – a are in

(A) A.P. (B) G.P. (C) H.P. (D) none of these

2

Sol^n:

$y-x,\ 2(y-a),\ y-z$ in H.P.

$\frac{1}{(y-x)},\ \frac{1}{2(y-a)},\ \frac{1}{(y-z)}$ in A.P.

$\Rightarrow \frac{1}{2(y-a)} - \frac{1}{(y-x)} = \frac{1}{(y-z)} - \frac{1}{2(y-a)}$

$\Rightarrow \frac{y-x-2y+2a}{\cancel{2(y-a)}(y-x)} = \frac{2y-2a-y+z}{(y-z)\cdot\cancel{2(y-a)}}$

$\Rightarrow \frac{x+y-2a}{(x-y)} = \frac{(y+z-2a)}{(y-z)}$

$\Rightarrow \frac{(x-a)+(y-a)}{(x-a)-(y-a)} = \frac{(y-a)+(z-a)}{(y-a)-(z-a)}$

Applying Componendo & dividendo

$\Rightarrow \frac{2(x-a)}{2(y-a)} = \frac{2(y-a)}{2(z-a)}$

$\Rightarrow (y-a)^2 = (x-a)(z-a)$ in G.P.

4. If a, b, c are in H.P., then $a^2(b-c)^2$, $\frac{b^2}{4}(c-a)^2$, $c^2(a-b)^2$ are in

(A) H.P. (B) G.P. (C) A.P. (D) All of the above ✓

Soln:

$$\frac{1}{b} - \frac{1}{a} = \frac{1}{c} - \frac{1}{b}$$

$$\Rightarrow \frac{a-b}{ab} = \frac{b-c}{bc}$$

$$\Rightarrow \frac{a-b}{a} = \frac{b-c}{c}$$

$$a(b-c) = c(a-b)$$

T_1 & T_3 are Same

$$T_2 = \left(\frac{b(c-a)}{2}\right)^2$$

$$\frac{a-b}{a} = \frac{b-c}{c}$$

$$ac - bc = ab - ac$$

$$2ac = b(a+c+a-a)$$

$$2ac = b(2a+(c-a))$$

$$2a(c-b) = b(c-a)$$

$$a(c-b) = \frac{b(c-a)}{2}$$

$$a^2(b-c)^2 = \frac{b^2}{4}(c-a)^2$$

$$T_1 = T_2 = T_3 \qquad \text{So,} \quad \boxed{D}$$

5. If m is a root of the equation $(1 - ab)x^2 - (a^2 + b^2)x - (1 + ab) = 0$ and m harmonic means are inserted between a and b, then the difference between the last and the first of the means equals

(A) b – a (B) ab(b – a) (C) a(b – a) (D) ab(a – b)

Sol^n:

$$(1-ab)m^2 - (a^2+b^2)m - (1+ab) = 0$$

$$(m^2-1) - (m^2+1)ab = (a^2+b^2)m$$

$$\Rightarrow \quad (a^2+b^2)m + ab(m^2+1) = m^2-1 \quad -(1)$$

$$\frac{1}{a}, \frac{1}{H_1} \text{------} \frac{1}{H_m}, \frac{1}{b} \rightarrow \text{A.P.}$$

$$\frac{1}{H_1} = \frac{1}{a} + \frac{(a-b)}{ab(m+1)} = \frac{b(m+1)+a-b}{ab(m+1)}$$

$$\frac{1}{H_m} = \frac{b(m+1)+m(a-b)}{ab(m+1)}$$

$$H_m - H_1 = \frac{ab(m+1)}{b+ma} - \frac{ab(m+1)}{(bm+a)}$$

$$H_m - H_1 = (ab(m+1))\left(\frac{bm+a-b-ma}{(ma+b)(bm+a)}\right)$$

$$H_m - H_1 = \frac{ab(m+1)((b-a)(m-1))}{(abm^2+(a^2+b^2)m+ab)}$$

$$H_m - H_1 = \frac{ab(b-a)(m^2-1)}{(m^2-1)} \longrightarrow \text{from Eq}^n \text{ (1)}$$

$$H_m - H_1 = ab(b-a)$$

6. If positive number a, b, c are in A.P. and a^2, b^2, c^2 are in H.P., then

(A) a = b = c (B) 2b = a + c (C) $b^2 = \sqrt{\frac{ac}{8}}$ (D) none of these

$$\Rightarrow \quad 2b = a + c$$

$$b^2 = \frac{2a^2c^2}{(a^2+c^2)}$$

$$\Rightarrow \quad \frac{(a+c)^2}{4} = \frac{2a^2c^2}{(a+c)^2 - 2ac}$$

$$= \quad (a+c)^2\left((a+c)^2 - 2ac\right) = 8a^2c^2$$

$$\Rightarrow \quad (a+c)^4 - 2ac(a+c)^2 + a^2c^2 = 9a^2c^2$$

$$\left((a+c)^2 - ac\right)^2 = (3ac)^2$$

$$(a+c)^2 - ac = 3ac$$

$$a^2 + c^2 + 2ac - 4ac = 0 \Rightarrow (a-c)^2 = 0$$

$$\boxed{a = c}$$

$$2b = a + c \Rightarrow 2b = 2c \Rightarrow b = c$$

$$a = b = c$$

7. If the pth term of an H.P. is qr and the qth term in rp, then the rth term of the H.P. is

(A) pqr (B) 1 (C) pq (D) pqr^2

Sol^n:

$$\frac{1}{T_p} = \frac{1}{qr} = a + (p-1)d$$

$$\frac{1}{T_q} = \frac{1}{rp} = a + (q-1)d$$

$$\frac{1}{qr} - \frac{1}{rp} = (p-q)d$$

$$= \frac{\cancel{(p-q)}}{pqr} = \cancel{(p-q)}d \Rightarrow \boxed{d = \frac{1}{pqr}}$$

$$\boxed{a = 1/pqr}$$

$$\frac{1}{T_r} = \frac{1}{pqr} + (r-1) \times \frac{1}{pqr} \Rightarrow \frac{1}{T_r} = \frac{1}{pq}$$

$$\boxed{T_r = pq}$$ Ans

8. If x, y, z are in A.P., a, b, c are in H.P. and ax, by, cz are in G.P., then $\frac{x}{z}+\frac{z}{x}$ is equal to

(A) $\frac{a}{c}-\frac{c}{a}$ (B) $\frac{a}{c}+\frac{c}{a}$ (C) $\frac{b}{a}+\frac{a}{b}$ (D) $\frac{b}{c}-\frac{c}{b}$

Sol^n:

$$b^2y^2 = (ac)(xz)$$

$$\Rightarrow \frac{\cancel{4}a^2c^2}{(a+c)^2} \times \frac{(x+z)^2}{\cancel{4}} = ac(xz)$$

$$\frac{ac}{(a+c)^2} \times (x+z)^2 = xz$$

$$\frac{(x+z)^2}{xz} = \frac{(a+c)^2}{ac}$$

$$\Rightarrow \frac{x^2+z^2+2xz}{xz} = \frac{a^2+c^2+2ac}{ac}$$

$$\Rightarrow \frac{x}{z}+\frac{z}{x}+\cancel{2} = \frac{a}{c}+\frac{c}{a}+\cancel{2}$$

9. If the first two terms of an H.P. are 2/5 and 12/13 respectively, then the largest term is

(A) 5th term (B) 6th term (C) 10th term (D) none of these.

Sol^n:

$$\frac{1}{T_2} = \frac{1}{T_1} + d$$

$$\frac{13}{12} = \frac{5}{2} + d \Rightarrow$$

$$\frac{13}{12} - \frac{5}{2} = d \Rightarrow d = -\frac{27}{12} = -\frac{9}{4}$$

$$\frac{1}{T_n} = \frac{1}{T_1} + (n-1)\left(-\frac{9}{4}\right)$$

$$\frac{1}{T_n} = T_n' = \frac{5}{2} - \frac{9(n-1)}{4} = 0$$

$$5 = \frac{9(n-1)}{2}$$

$$10 = 9n - 9$$

$$\boxed{n = 2}$$

2nd term will be largest.

10. If $H_1, H_2, \ldots H_n$ be n H.M.s between a and b, then $\dfrac{H_1+a}{H_1-a}+\dfrac{H_n+b}{H_n-b}$ is equal to

(A) n (B) 2n (C) 3n (D) 4n

Sol^n:

$a, H_1, H_2 --- H_n, b$ in H.P.

$\dfrac{1}{a}, \dfrac{1}{H_1}, \dfrac{1}{H_2} --- \dfrac{1}{H_n}, \dfrac{1}{b}$ in A.P.

$$\Rightarrow \frac{\frac{1}{a}+\frac{1}{H_1}}{\underbrace{\frac{1}{a}-\frac{1}{H_1}}_{-d}} + \frac{\frac{1}{b}+\frac{1}{H_n}}{\underbrace{\frac{1}{b}-\frac{1}{H_n}}_{d}}$$

$$= \frac{-\frac{1}{a}-\frac{1}{H_1}+\frac{1}{b}+\frac{1}{H_n}}{d}$$

$$= \frac{-\cancel{\frac{1}{a}}-\cancel{\frac{1}{a}}-d+\cancel{\frac{1}{a}}+(n+1)d+\cancel{\frac{1}{a}}+nd}{d}$$

$$\frac{2nd}{d} = \underline{\underline{2n}}.$$

11. If three numbers are in HP then the numbers obtained by subtracting half of the middle number from each of them are in

(A) AP (B) GP (C) HP (D) none of these

Solⁿ:

a, b, c in H.P.

$\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ in A.P.

$a-\frac{b}{2}, \frac{b}{2}, c-\frac{b}{2}$?

$a-\frac{ac}{a+c}, \frac{ac}{a+c}, c-\frac{ac}{a+c}$

$$\underset{T_1}{\frac{a^2}{a+c}}, \underset{T_2}{\frac{ac}{a+c}}, \underset{T_3}{\frac{c^2}{a+c}}$$

$(T_2)^2 = T_1 T_3$

So, T_1, T_2, T_3 in G.P.

12. If HM : GM = 4 : 5 for two positive numbers then the ratio of the numbers is

~~(A)~~ 4 : 1 (B) 3 : 2 (C) 3 : 4 (D) 2 :

Sol^n: Let numbers are a & b

$$\frac{\frac{2ab}{(a+b)}}{\sqrt{ab}} = \frac{4}{5}$$

$$\frac{\sqrt{ab}}{a+b} = \frac{2}{5} \Rightarrow 2(a+b) = 5\sqrt{ab}$$

$$\Rightarrow 4(a^2+b^2+2ab) = 25ab$$

$$4\frac{a}{b} + 4\frac{b}{a} - 17 = 0$$

$$a/b = t$$

$$4t + \frac{4}{t} - 17 = 0$$

$$\Rightarrow 4t^2 - 17t + 4 = 0$$

$$4t^2 - 16t - t + 4 = 0$$

$$(4t-1)(t-4) = 0$$

$$a/b = 1/4, 4$$

13. Insert two harmonic means between 5 and 11.

Solⁿ:

$$\frac{1}{5}, \frac{1}{H_1}, \frac{1}{H_2}, \frac{1}{11} \text{ in A.P.}$$

$$\frac{1}{11} = \frac{1}{5} + 3d$$

$$\frac{5-11}{55} = 3d \Rightarrow d = \frac{-2}{55}$$

$$\frac{1}{H_1} = \frac{1}{5} - \frac{2}{55} = \frac{9}{55} \qquad H_1 = \frac{55}{9}$$

$$\frac{1}{H_2} = \frac{1}{5} - \frac{4}{55} = \frac{7}{55} \qquad H_2 = \frac{55}{7}$$

14. If 12 and $9\frac{3}{5}$ are the geometric and harmonic means, respectively between two numbers, find them.

Solⁿ:

Let numbers are a & b

$$\sqrt{ab} = 12 \Rightarrow ab = 144$$

$$\frac{2ab}{a+b} = \frac{48}{5}$$

$$\Rightarrow \frac{2 \times 144}{a+b} = \frac{48}{5} \Rightarrow a+b = 30$$

Numbers are 24 & 6.

15. If between any two quantities there be inserted two arithmetic means A_1, A_2; two geometric means G_1, G_2; and two harmonic means H_1, H_2; show that $G_1G_2 : H_1H_2 = A_1 + A_2 : H_1 + H_2$.

Sol^n: To prove that

$$\frac{G_1G_2}{H_1H_2} = \frac{A_1 + A_2}{H_1 + H_2}$$

Numbers be a, b

$$G_1G_2 = ab \qquad A_1 + A_2 = a + b$$

$$\frac{1}{H_1} + \frac{1}{H_2} = \frac{1}{a} + \frac{1}{b}$$

$$\Rightarrow \frac{H_1 + H_2}{H_1H_2} = \frac{a+b}{ab}$$

$$\frac{G_1G_2}{A_1 + A_2} = \frac{H_1H_2}{H_1 + H_2}$$

$$= \frac{ab}{a+b} = \frac{ab}{a+b}$$

H.P.

16. (A) If A be the A.M. and H the H.M. between two numbers a and b, then $\frac{a-A}{a-H} \times \frac{b-A}{b-H} = \frac{A}{H}$.

Solⁿ:

$$\frac{a - \frac{(a+b)}{2}}{a - \frac{2ab}{(a+b)}} \times \frac{b - \left(\frac{a+b}{2}\right)}{b - \frac{2ab}{(a+b)}}$$

$$= \frac{\frac{2a-(a+b)}{2}}{\frac{a^2+ab-2ab}{(a+b)}} \times \frac{\frac{2b-(a+b)}{2}}{\frac{ab+b^2-2ab}{(a+b)}}$$

$$= \frac{\cancel{(a-b)} \times (a+b)}{2a\,\cancel{(a-b)}} \times \frac{\cancel{(b-a)}\,(a+b)}{2b \times \cancel{(b-a)}}$$

$$= \frac{(a+b)}{2 \times \frac{2ab}{(a+b)}} = \frac{A}{H}$$

(B) If 9 arithmetic and harmonic means be inserted between 2 and 3, prove that A + 6/H = 5 where A is any of the A.M.'s and H the corresponding H.M.

Sol^n:

$$A_n + \frac{6}{H_n} = 5 \qquad n = 1, 2 \ldots 9$$

$$2, A_1 \ldots A_n, 3$$

$$3 = 2 + (n+1)d \qquad d = \frac{1}{(n+1)}$$

$$\boxed{A_n = 2 + \frac{n}{(n+1)}}$$

$$\frac{1}{2}, \frac{1}{H_1} \ldots \frac{1}{H_n}, \frac{1}{3}$$

$$\frac{1}{3} = \frac{1}{2} + (n+1)d'$$

$$-\frac{1}{6} = d'(n+1) \Rightarrow d' = \frac{-1}{6(n+1)}$$

$$\frac{1}{H_n} = \frac{1}{2} - \frac{n}{6(n+1)} \Rightarrow \frac{1}{H_n} = \frac{2n+3}{6(n+1)}$$

$$A_n + \frac{6}{H_n} = 2 + \frac{n}{(n+1)} + \frac{2n+3}{(n+1)} = 2 + \frac{(3n+3)}{(n+1)} = 5.$$

17. Find $\sum_{i=1}^{100} \frac{1}{Hi}$. If $H_1, H_2 \ldots H_{100}$ are HMs between 1 and 1/100

Sol^n:

$$1, \frac{1}{H_1}, \frac{1}{H_2} --- \frac{1}{H_{100}}, 100 \rightarrow A.P.$$

$$1 + \frac{1}{H_1} + \frac{1}{H_2} + -- + \frac{1}{H_{100}} + 100 = \frac{102}{2}(1+100)$$

$$101 + \sum_{i=1}^{100} \frac{1}{H_i} = 51 \times 101$$

$$\sum_{i=1}^{100} \frac{1}{H_i} = 50 \times 101 = 5050$$