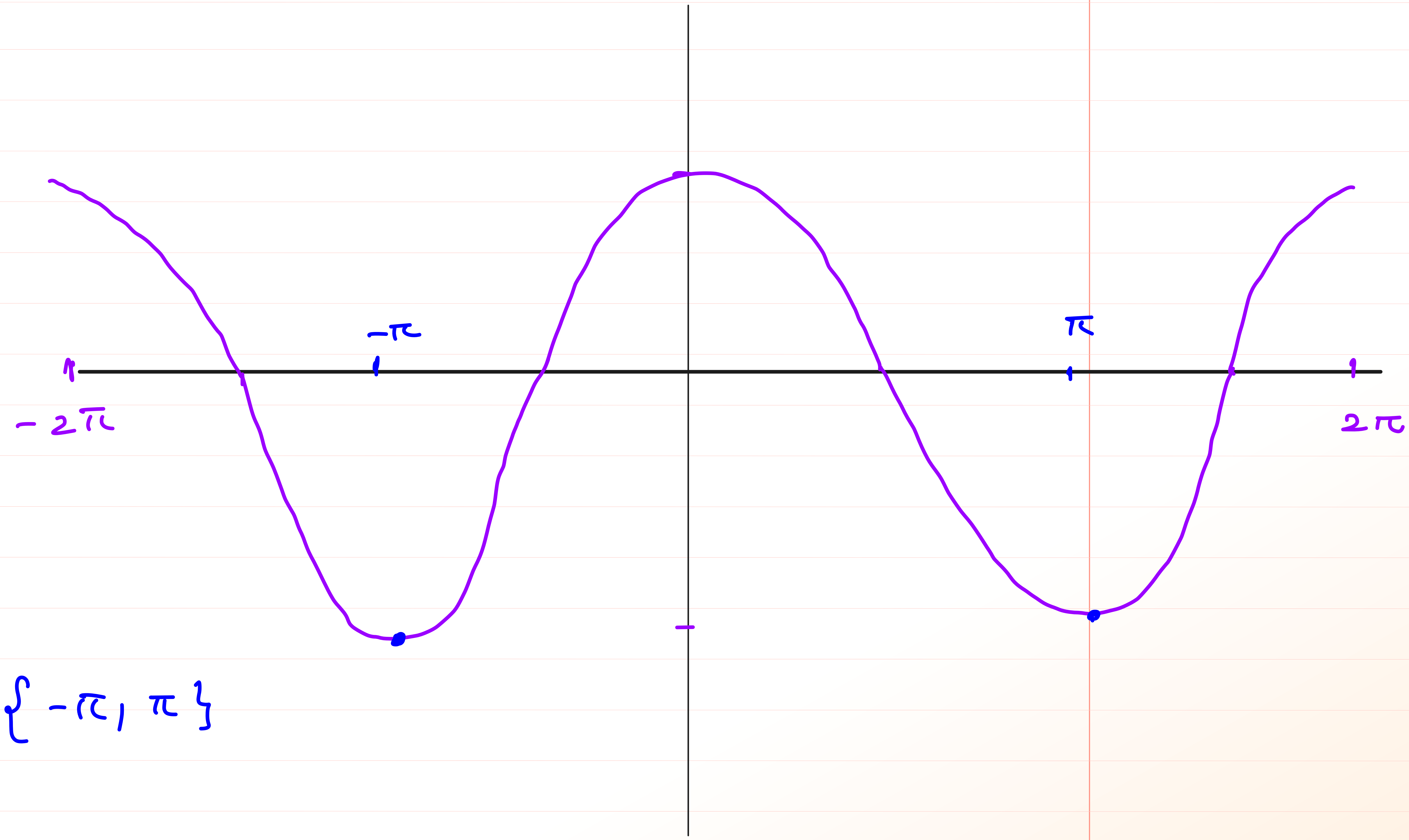
Trigonometric Ratios and Identities

Lecture - 5



$\theta \in \{-\pi, \pi\}$

TRIGONOMETRIC RATIO OF COMPOUND ANGLES :

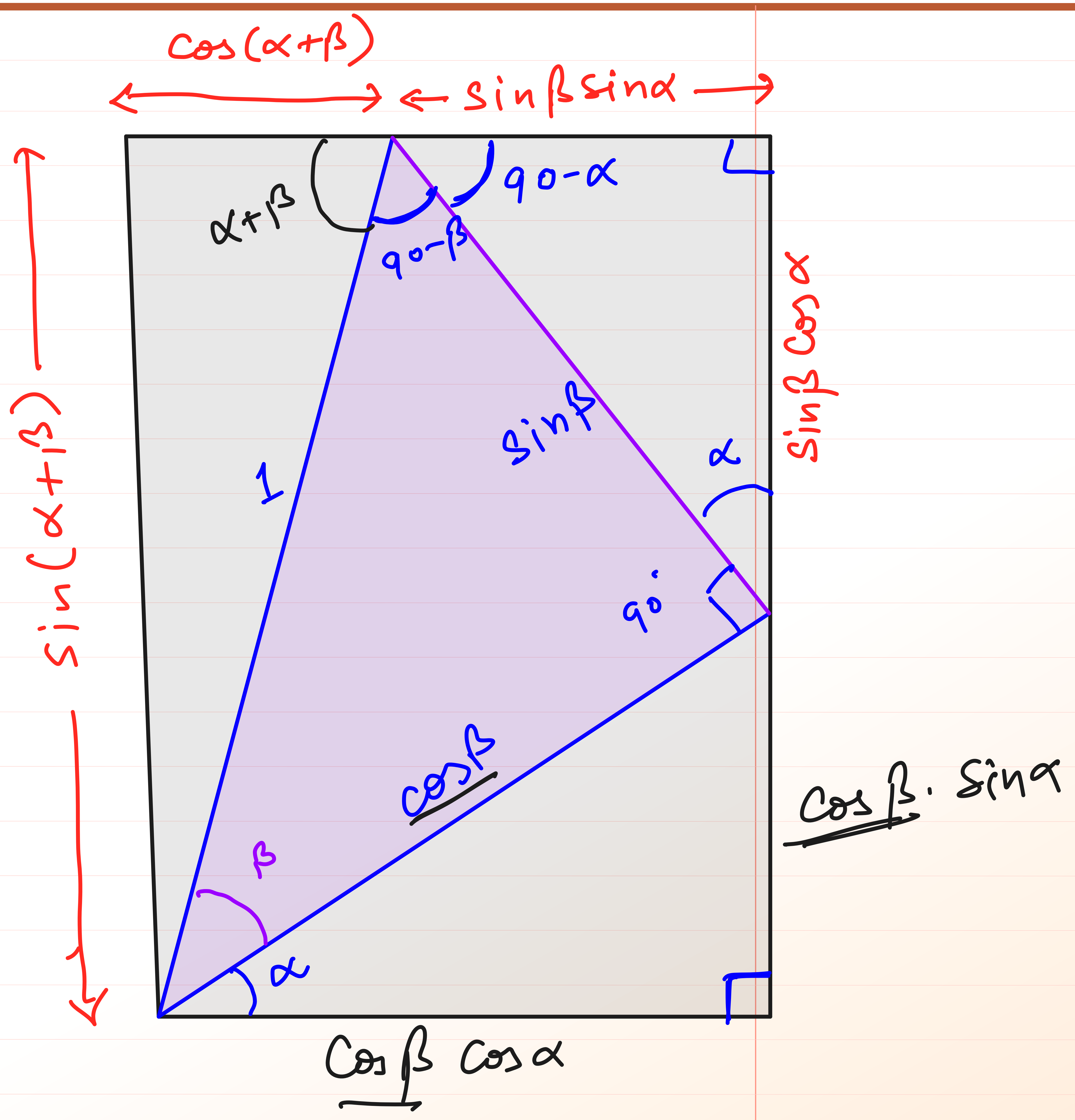
$\cos(\alpha+\beta) + \sin\beta\sin\alpha = \cos\beta\cos\alpha$

✓ $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

✓ $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$(\sin(\alpha+\beta))^2 = \sin^2(\alpha+\beta)$

$\sin(\alpha+\beta)\cdot\sin(\alpha+\beta)$



(i) $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

(ii) $\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

(iii) $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

(iv) $\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

① $\sin 99^\circ \cos 21^\circ + \cos 99 \sin 21^\circ = ?$

$= \sin(99+21) = \sin 120 = \sin(90+30) = \frac{\sqrt{3}}{2}$

② $\sin 15^\circ$

$= \sin(45-30)$

$= \sin 45 \cos 30 - \cos 45 \sin 30$

$= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \cdot \frac{1}{2}$

$= \frac{\sqrt{3}-1}{2\sqrt{2}}$

③ $\cos 15^\circ$

$= \cos(45-30)$

$= \cos 45 \cos 30 + \sin 45 \sin 30$

$= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \cdot \frac{1}{2}$

$= \frac{\sqrt{3}+1}{2\sqrt{2}}$

④ $\sin 75 = \sin(90-15)$

$= \cos 15 = \frac{\sqrt{3}+1}{2\sqrt{2}}$

⑤ $\cos 75 = \cos(90-15)$

$= \sin 15 = \frac{\sqrt{3}-1}{2\sqrt{2}}$

Formulae & Identities :

(a) $\sin(A+B) \cdot \sin(A-B) = \sin^2 A - \sin^2 B = \cos^2 B - \cos^2 A$

(b) $\cos^2 A - \sin^2 B = \cos(A+B)\cos(A-B) = \cos^2 B - \sin^2 A$

(c) **Formula to transform the product into sum or difference :**

(1) $2\sin A\cos B = \sin(A+B) + \sin(A-B)$

(2) $2\cos A\sin B = \sin(A+B) - \sin(A-B)$

(3) $2\cos A\cos B = \cos(A+B) + \cos(A-B)$

(4) $2\sin A\sin B = \cos(A-B) - \cos(A+B)$

(d) (1) $\sin\theta\sin(60^\circ+\theta)\sin(60^\circ-\theta) = \frac{1}{4}\sin 3\theta$

(2) $\cos\theta\cos(60^\circ+\theta)\cos(60^\circ-\theta) = \frac{1}{4}\cos 3\theta$

(3) $\tan\theta\tan(60^\circ+\theta)\tan(60^\circ-\theta) = \tan 3\theta$

(a) $\sin(A+B) \cdot \sin(A-B)$

$= (\sin A\cos B + \cos A\sin B)(\sin A\cos B - \cos A\sin B)$

$= \sin^2 A\cos^2 B - \cos^2 A\sin^2 B$

$= \sin^2 A(1-\sin^2 B) - (1-\sin^2 A)\sin^2 B$

$= \sin^2 A - \sin^2 A\sin^2 B - \sin^2 B + \sin^2 A\sin^2 B$

$= \sin^2 A - \sin^2 B$

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(c) **Formula to transform the product into sum or difference**

(1) $2\sin A\cos B = \sin(A+B) + \sin(A-B)$

(2) $2\cos A\sin B = \sin(A+B) - \sin(A-B)$

(3) $2\cos A\cos B = \cos(A+B) + \cos(A-B)$

(4) $2\sin A\sin B = \cos(A-B) - \cos(A+B)$

(A) $\sin(A+B) = \sin A\cos B + \cos A\sin B$

(B) $\sin(A-B) = \sin A\cos B - \cos A\sin B$

(C) $\cos(A+B) = \cos A\cos B - \sin A\sin B$

(D) $\cos(A-B) = \cos A\cos B + \sin A\sin B$

add (A) & (B) to get formula 1

Subtract (B) from (A) to get 2

add (C) & (D) to get (3)

Subtract (C) from (D) to get (4)

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(d)

(1) $\sin\theta\sin(60^\circ+\theta)\sin(60^\circ-\theta)=\frac{1}{4}\sin 3\theta$

(2) $\cos\theta\cos(60^\circ+\theta)\cos(60^\circ-\theta)=\frac{1}{4}\cos 3\theta$

(3) $\tan\theta\tan(60^\circ+\theta)\tan(60^\circ-\theta)=\tan 3\theta$

1 $\sin\theta\,(\sin(60+\theta))\sin(60-\theta)$

$= \sin\theta\cdot(\sin^2 60-\sin^2\theta)$

$= \sin\theta\left(\frac{3}{4}-\sin^2\theta\right)$

$= \frac{1}{4}\sin\theta\,(3-4\sin^2\theta)$

$= \frac{1}{4}\cdot(3\sin\theta-4\sin^3\theta)$

$= \frac{1}{4}\cdot\sin 3\theta.$

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(A) $\sin(A+B) = \sin A \cos B + \cos A \sin B$

(B) $\sin(A-B) = \sin A \cos B - \cos A \sin B$

(C) $\cos(A+B) = \cos A \cos B - \sin A \sin B$

(D) $\cos(A-B) = \cos A \cos B + \sin A \sin B$

divide (A) by (C)

$$\frac{\sin(A+B)}{\cos(A+B)} = \frac{(\sin A \cos B + \cos A \sin B)/\cos A \cos B}{(\cos A \cos B - \sin A \sin B)/\cos A \cos B}$$

$$\boxed{\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}}$$

$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

(d) Identities for converting sum to product :

Let $A + B = C$ & $A - B = D$

$\therefore \quad A = \dfrac{C+D}{2}, \quad B = \dfrac{C-D}{2}$

1. $\sin C + \sin D = 2\sin\left(\dfrac{C+D}{2}\right)\cos\left(\dfrac{C-D}{2}\right)$

2. $\sin C - \sin D = 2\cos\left(\dfrac{C+D}{2}\right)\sin\left(\dfrac{C-D}{2}\right)$

3. $\cos C + \cos D = 2\cos\left(\dfrac{C+D}{2}\right)\cos\left(\dfrac{C-D}{2}\right)$

4. $\cos C - \cos D = -2\sin\left(\dfrac{C+D}{2}\right)\sin\left(\dfrac{C-D}{2}\right)$

$= 2\sin\left(\dfrac{C+D}{2}\right)\sin\left(\dfrac{D-C}{2}\right)$

(1) $2\sin A\cos B = \sin(A+B) + \sin(A-B)$

(2) $2\cos A\sin B = \sin(A+B) - \sin(A-B)$

(3) $2\cos A\cos B = \cos(A+B) + \cos(A-B)$

(4) $2\sin A\sin B = \cos(A-B) - \cos(A+B)$

take $A + B = C$

$A - B = D$

$A = \dfrac{C+D}{2}$; $B = \dfrac{C-D}{2}$

$\sin C + \sin D = 2\sin\dfrac{C+D}{2}\cos\dfrac{C-D}{2}$

Trigonometric Ratios and Identities

Lecture - 6

E(1) $\dfrac{\sin 7\theta - \sin 5\theta}{\cos 7\theta + \cos 5\theta} = \tan\theta$

LHS $= \dfrac{2\cos 6\theta \sin\theta}{2\cos 6\theta \cos\theta} = \tan\theta$

E(2) $\dfrac{(\cos\theta - \cos 3\theta)(\sin 8\theta + \sin 2\theta)}{(\sin 5\theta - \sin\theta)(\cos 4\theta - \cos 6\theta)} = 1$

E(3) Prove that $\dfrac{\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10}{\cos 5\theta + 5\cos 3\theta + 10\cos\theta} = 2\cos\theta$

E(4) $\dfrac{\sin A + 2\sin 3A + \sin 5A}{\sin 3A + 2\sin 5A + \sin 7A} = \dfrac{\sin 3A}{\sin 5A}$

LHS $= \dfrac{2\sin 3A\cos 2A + 2\sin 3A}{2\sin 5A\cos 2A + 2\sin 5A} = \dfrac{\sin 3A}{\sin 5A}$

E(5) If $\alpha = \dfrac{\pi}{19}$ Find the value of $\dfrac{\sin 23\alpha - \sin 3\alpha}{\sin 16\alpha + \sin 4\alpha}$

$= \dfrac{2\cos 13\alpha \sin 10\alpha}{2\sin 10\alpha \cos 6\alpha} = \dfrac{\cos 13\alpha}{\cos 6\alpha} = \dfrac{-\cos\frac{6\pi}{19}}{\cos\frac{6\pi}{19}}$

$= -1$

$\cos 13\alpha = \cos\dfrac{13\pi}{19}$

$= \cos\left(\dfrac{19\pi - 6\pi}{19}\right)$

$= \cos\left(\pi - \dfrac{6\pi}{19}\right) = -\cos\dfrac{6\pi}{19}$

E(7) Find the value $\dfrac{\sin 8\theta\cos\theta - \sin 6\theta\cos 3\theta}{\cos 2\theta\cos\theta - \sin 3\theta\sin 4\theta}$ given $\theta = 7.5^\circ$

Ans $(2-\sqrt{3})$

E(8) If $\sin\alpha = \dfrac{15}{17}$, $\cos\beta = -\dfrac{5}{13}$, then find $\sin(\alpha - \beta)$.

E(3) Prove that $\dfrac{\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10}{\cos 5\theta + 5\cos 3\theta + 10\cos\theta} = 2\cos\theta$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta = 2\cos^2\theta - 1$$

$$\cos 2\theta + 1 = 2\cos^2\theta$$

$$\text{LHS} = \frac{\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10}{\cos 5\theta + 5\cos 3\theta + 10\cos\theta}$$

$$= \frac{\cos 6\theta + \cos 4\theta + 5\cos 4\theta + 5\cos 2\theta + 10\cos 2\theta + 10}{\cos 5\theta + 5\cos 3\theta + 10\cos\theta}$$

$$= \frac{\left(2\cos\frac{6\theta+4\theta}{2}\cos\frac{6\theta-4\theta}{2}\right) + 5\left(2\cos\frac{4\theta+2\theta}{2}\cos\frac{4\theta-2\theta}{2}\right) + 10(\cos 2\theta + 1)}{\cos 5\theta + 5\cos 3\theta + 10\cos\theta}$$

$$= \frac{2\cos 5\theta\cos\theta + 5(2\cos 3\theta\cos\theta) + 10(2\cos^2\theta)}{\cos 5\theta + 5\cos 3\theta + 10\cos\theta}$$

$$= \frac{2\cos\theta(\cos 5\theta + 5\cos 3\theta + 10\cos\theta)}{\cos 5\theta + 5\cos 3\theta + 10\cos\theta} = 2\cos\theta = \text{RHS.}$$

Hence Proved

E(8) If $\sin\alpha = \dfrac{15}{17}$, $\cos\beta = -\dfrac{5}{13}$, then find $\sin(\alpha - \beta)$.

$\alpha \rightarrow$ Ist quad or 2nd quad.

$\beta \rightarrow$ 2nd quad or 3rd quad.

C–I $\quad \alpha \rightarrow \mathrm{I}, \quad \beta \rightarrow$ 2nd

$\sin\alpha = \dfrac{15}{17}$; $\quad \sin\beta = \dfrac{12}{13}$

$\cos\alpha = \dfrac{8}{17}$; $\quad \cos\beta = -\dfrac{5}{13}$

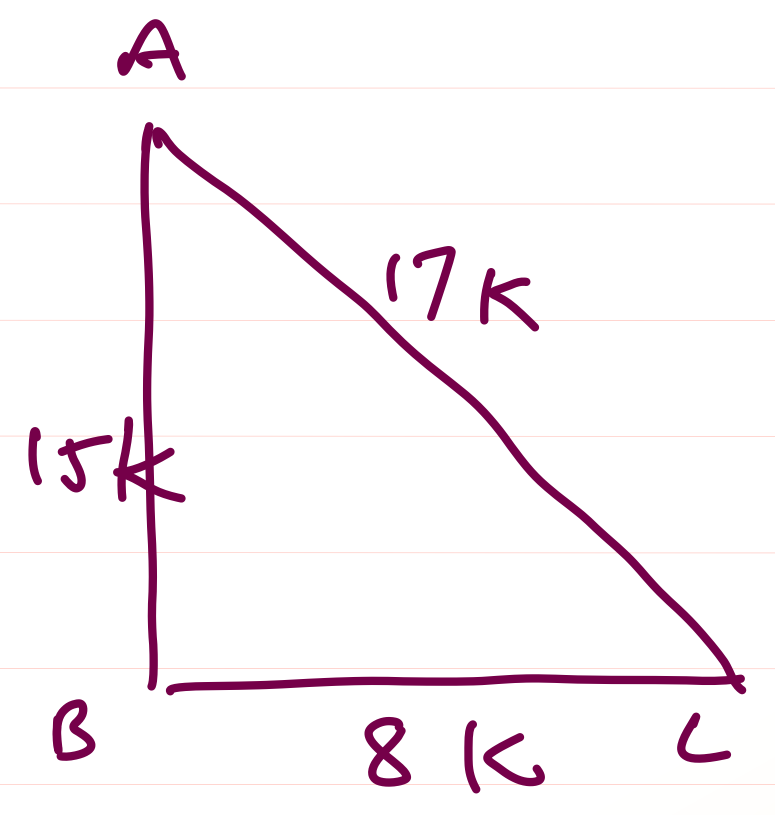
$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

$= \dfrac{15}{17}\cdot\left(\dfrac{-5}{13}\right) - \dfrac{8}{17}\cdot\dfrac{12}{13}$

$\sin(\alpha-\beta) = \dfrac{-171}{221}$ ✓

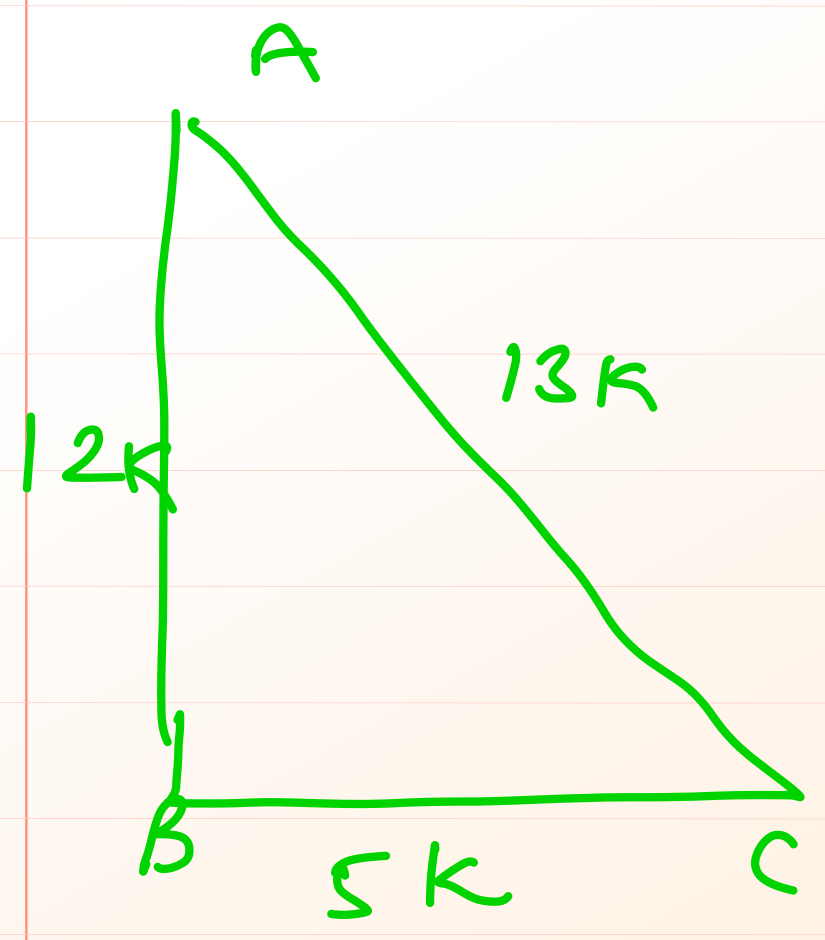
$\sin\alpha = \dfrac{15}{17}$

$\cos\alpha = \dfrac{8}{17}$



$\sin\beta = \dfrac{12}{13}$

$\cos\beta = \dfrac{-5}{13}$



C II $\alpha \to$ I quad $\beta \to$ 3rd quad

$\sin\alpha = \frac{15}{17}$; $\sin\beta = -\frac{12}{13}$

$\cos\alpha = \frac{8}{17}$; $\cos\beta = -\frac{5}{13}$

$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

$= \frac{15}{17}\cdot\left(-\frac{5}{13}\right) - \frac{8}{17}\cdot\left(\frac{-12}{13}\right) = \frac{21}{221}$

C III $\alpha \to$ 2nd, $\beta \to$ 2nd qu

$\sin\alpha = +\frac{15}{17}$; $\sin\beta = +\frac{12}{13}$

$\cos\alpha = -\frac{8}{17}$; $\cos\beta = -\frac{5}{13}$

$\sin(\alpha-\beta) =$

C IV $\alpha \to$ 2nd ; $\beta \to$ 3rd qua

$\sin(\alpha-\beta) =$

$\sin\alpha = +\frac{15}{17}$; $\sin\beta = -\frac{12}{13}$

$\cos\alpha = -\frac{8}{17}$; $\cos\beta = -\frac{5}{13}$

(e) $$\tan(A+B)=\frac{\tan A+\tan B}{1-\tan A\tan B}$$

(f) Trigonometrical ratios of the sum of more than 2 angles :

$$\sin(A+B+C)=\sin(A+B)\cos C+\cos(A+B)\sin C$$

$$=[\sin A\cos B+\cos A\sin B]\cos C+[\cos A\cos B-\sin A\sin B]\sin C$$

$$=\sin A\cos B\cos C+\cos A\sin B\cos C+\cos A\cos B\sin C-\sin A\sin B\sin C$$

HW

$$\cos(A+B+C)=\cos(A+B)\cos C-\sin(A+B)\sin C$$

$$=(\cos A\cos B-\sin A\sin B)\cos C-(\sin A\cos B+\cos A\sin B)\sin C$$

$$=\cos A\cos B\cos C-\cos A\sin B\sin C-\sin A\cos B\sin C-\sin A\sin B\cos C.$$

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

$$\sin(\underline{A}+\underline{B+C}) = \sin A\cos(B+C) + \cos A\sin(B+C)$$

$$= \sin A\cdot(\cos B\cos C - \sin B\sin C) + \cos A(\sin B\cos C + \cos B\sin C)$$

$$= \sin A\cos B\cos C - \sin A\sin B\sin C + \cos A\sin B\cos C + \cos A\cos B\sin C$$

$$\tan(A+B+C) = \frac{\tan A + \tan(B+C)}{1-\tan A\ \tan(B+C)}$$

$$= \frac{\tan A + \dfrac{\tan B+\tan C}{1-\tan B\tan C}}{1-\tan A\cdot\left(\dfrac{\tan B+\tan C}{1-\tan B\tan C}\right)}$$

$$= \frac{\tan A\,(1-\tan B\tan C)+\tan B+\tan C}{1\,(1-\tan B\tan C)-\tan A\,(\tan B+\tan C)}$$

$$\tan(A+B+C) = \frac{\tan A+\tan B+\tan C-\tan A\tan B\tan C}{1-(\tan A\tan B+\tan B\tan C+\tan C\tan A)}$$

$$\boxed{\tan(A+B+C)=\frac{S_1-S_3}{1-S_2}}$$

$S_1 = \tan A + \tan B + \tan C$

$S_2 = \tan A\tan B + \tan B\tan C + \tan C\tan A$

$S_3 = \tan A\tan B\tan C$

$\tan(A+B+C+D)$

Also, $\tan(A+B+C) = \dfrac{\tan(A+B)+\tan C}{1-\tan(A+B)\tan C} = \dfrac{\dfrac{\tan A+\tan B}{1-\tan A\tan B}+\tan C}{1-\dfrac{\tan A+\tan B}{1-\tan A\tan B}\tan C}$

$$= \frac{\tan A+\tan B+\tan C-\tan A\tan B\tan C}{1-\tan B\tan C-\tan C\tan A-\tan A\tan B.} = \frac{s_1-s_3}{1-s_2}$$

$\tan A_1 + \tan A_2 + \dots + \tan A_n$

$\tan A_1 \tan A_2 \tan A_3 + \tan A_1 \tan A_2 \tan A_4 + \tan A_1 \tan A_2 \tan A_5 + \dots$

$$\tan(A_1+A_2+\ldots+A_n) = \frac{s_1-s_3+s_5-s_7+\ldots.}{1-s_2+s_4-s_6+\ldots.}$$

$= \sum_{i=1<j<k}^{n} \tan A_i \tan A_j \tan A_k$

where $s_1 = \tan A_1 + \tan A_2 + \ldots.. + \tan A_n$ = the sum of the tangent of the separate angles,

$s_2 = \tan A_1 \tan A_2 + \tan A_1 \tan A_3 + \ldots..$ = the sum of the tangents taken two at a time,

$s_3 = \tan A_1 \tan A_2 \tan A_3 + \tan A_2 \tan A_3 \tan A_4 + \ldots$

= the sum of the tangents taken three at a time, and so on.

(A) $\sin(A+B) = \sin A \cos B + \cos A \sin B$

(B) $\sin(A-B) = \sin A \cos B - \cos A \sin B$

(C) $\cos(A+B) = \cos A \cos B - \sin A \sin B$

(D) $\cos(A-B) = \cos A \cos B + \sin A \sin B$

(1) $\sin 2A = 2 \sin A \cos A$

2 $\cos 2A = \cos^2 A - \sin^2 A$

$= 1 - 2\sin^2 A$

$= 2\cos^2 A - 1$

3 $\tan 2A = \dfrac{2\tan A}{1-\tan^2 A}$

$\sin(A+A) = \sin A \cos A + \cos A \sin A$

$\sin 2A = 2 \sin A \cos A$

$\cos 2A = \cos^2 A - \sin^2 A$

$= \cos^2 A - (1-\cos^2 A)$

$= \cos^2 A - 1 + \cos^2 A$

$\cos 2A = 2\cos^2 A - 1$

$= 1 - 2\sin^2 A$

$\tan(A+A) = \dfrac{\tan A + \tan A}{1 - \tan A \tan A} = \dfrac{2\tan A}{1-\tan^2 A}$

$$\sin 2A = 2\sin A\cos A = \frac{2\sin A\cos A}{1} = \frac{2\sin A\cos A}{\sin^2 A + \cos^2 A}$$

divide by $\cos^2 A$

$$\boxed{\sin 2A = \frac{2\tan A}{1+\tan^2 A}}$$

$$\cos 2A = \frac{\cos^2 A - \sin^2 A}{1} = \frac{\cos^2 A - \sin^2 A}{\cos^2 A + \sin^2 A} = \frac{1-\tan^2 A}{1+\tan^2 A}$$

$$\cos 2A = \frac{1-\tan^2 A}{1+\tan^2 A}$$

(A) $\sin(A+B) = \sin A\cos B + \cos A\sin B$

(B) $\sin(A-B) = \sin A\cos B - \cos A\sin B$

(C) $\cos(A+B) = \cos A\cos B - \sin A\sin B$

(D) $\cos(A-B) = \cos A\cos B + \sin A\sin B$

$B \to 2A \;;\; A \to A$

$$\sin(A+2A) = \sin A\cos 2A + \cos A\sin 2A$$

$$= \sin A(1-2\sin^2 A) + \cos A(2\sin A\cos A)$$

$$= \sin A - 2\sin^3 A + 2\sin A(1-\sin^2 A)$$

$$= \sin A - 2\sin^3 A + 2\sin A - 2\sin^3 A$$

$$\sin 3A = 3\sin A - 4\sin^3 A$$

$$\cos 3A = 4\cos^3 A - 3\cos A$$

T-RATIOS OF MULTIPLE & SUBMULTIPLE ANGLES :

Multiple angles are 2A, 3A, 4A etc & submultiple angles are $\frac{A}{2}, \frac{A}{4}, \frac{A}{8}$ etc.

$$\sin 2A = \sin(A + A) = 2\sin A \cos A = \frac{2\sin A\cos A}{\sin^2 A + \cos^2 A}$$

$$\Rightarrow \quad \sin 2A = \frac{2\tan A}{1 + \tan^2 A}$$

$$\cos(A + A) = \cos A \cos A - \sin A \,.\, \sin A$$

$$= \cos^2 A - \sin^2 A = \frac{\cos^2 A - \sin^2 A}{\cos^2 A + \sin^2 A}$$

$$\Rightarrow \quad \cos 2A = \frac{1 - \tan^2 A}{1 + \tan^2 A}$$

Also $\cos 2A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$

which gives $\sin A = \pm\sqrt{\frac{1 - \cos 2A}{2}}$ & $\cos A = \pm\sqrt{\frac{1 + \cos 2A}{2}}$

$$\tan 2A = \tan(A + A) = \frac{\tan A + \tan A}{1 - \tan^2 A} = \frac{2\tan A}{1 - \tan^2 A}$$

$$\cos 2A = 2\cos^2 A - 1$$

$$2A = y \Rightarrow A = \frac{y}{2}$$

$$\cos(y) = 2\cos^2\left(\frac{y}{2}\right) - 1$$

$$\cos y + 1 = 2\cos^2\left(\frac{y}{2}\right)$$

$$\pm\sqrt{\frac{\cos y + 1}{2}} = \cos\left(\frac{y}{2}\right)$$

$$\boxed{\cos\left(\frac{y}{2}\right) = \pm\sqrt{\frac{\cos y + 1}{2}}}$$

$$\cos 2A = 1 - 2\sin^2 A$$

$$\sin\left(\frac{y}{2}\right) = \pm\sqrt{\frac{1 - \cos y}{2}}$$

Trigonometric Ratios and Identities

Lecture - 7

E(1) Find the value of $-6\sin 40° + 8\sin^3 40°$ $[-\sqrt{3}]$

E(2) Find the value of $8\sin^3 10° - 6\sin 10°$ (-1)

E(3) Let $f(\theta) = 16\cos^3 2\theta - 32\sin^3\theta - 12\cos 2\theta + 24\sin\theta$ and if $f\left(\frac{\pi}{30}\right) = 3\sqrt{a} - b,\ (a, b \in N)$, then value of $(a + b)$ is **[Ans. 6]**

E(4) P. T. $(4\cos^2 9° - 3)(4\cos^2 27° - 3) = \tan 9°$ $\longrightarrow$ LHS $= \frac{\cos 9(4\cos^2 9 - 3)}{(\cos 9)} (4\cos^2 27 - 3)\frac{\cos 27}{\cos 27}$

$= \frac{(4\cos^3 9 - 3\cos 9)}{\cos 9} \cdot \frac{(4\cos^3 27 - 3\cos 27)}{(\cos 27)}$

$= \frac{\cancel{\cos 27}}{\cos 9} \cdot \frac{\cos 81}{\cancel{\cos 27}}$

$= \frac{\cos(90 - 9)}{\cos 9}$

$= \frac{\sin 9}{\cos 9} = \tan 9$

E(5) Express cos 5A in terms of cos A.

E(6) P.T. tan 3A . tan2A . tan A = tan 3A – tan 2A – tanA

E(7) Find the exact value of $\sqrt{3}$ cosec 20° – sec 20°.

E(8) Find the value of cosec 10° – $\sqrt{3}$ sec 10°

E(9) Given $\sin\theta + \sin\phi = a$, $\cos\theta + \cos\phi = b$. Find the value of $\tan\left(\frac{\theta - \phi}{2}\right)$, $\sin(\theta + \phi)$, $\cos(\theta + \phi)$.

Hint (4) $\cos 5A = \cos(2A + 3A)$

E(6) P.T. tan 3A . tan2A . tan A = tan 3A – tan 2A – tanA

E(7) Find the exact value of $\sqrt{3}$ cosec 20° – sec 20°.

E(8) Find the value of cosec 10° – $\sqrt{3}$ sec 10°

6

$$\tan(A+2A) = \frac{\tan A + \tan 2A}{1 - \tan A \tan 2A}$$

$$(\tan 3A)(1 - \tan A \tan 2A) = \tan A + \tan 2A$$

$$\tan 3A - \tan 3A \tan A \tan 2A = \underline{\tan A + \tan 2A}$$

$$\boxed{\tan 3A - \tan A - \tan 2A = \tan A \tan 2A \tan 3A}$$

Homework

Race 10 & 11 complete

Exercise 1 first 10 ques

B.B 1

E(3) Let $f(\theta) = 16\cos^3 2\theta - 32\sin^3\theta - 12\cos 2\theta + 24\sin\theta$ and if $f\left(\frac{\pi}{30}\right) = 3\sqrt{a} - b, (a, b \in N)$, then value of (a + b) is **[Ans. 6]**

$$f(\theta) = \underbrace{16\cos^3 2\theta}\ - \underbrace{32\sin^3\theta}\ - \underbrace{12\cos 2\theta}\ + \underbrace{24\sin\theta}$$

$$= 4[4\cos^3 2\theta - 3\cos 2\theta] + 8[-4\sin^3\theta + 3\sin\theta]$$

$$f(\theta) = 4[\cos 3(2\theta)] + 8[\sin 3\theta]$$

$$f(\theta) = 4[\cos 6\theta] + 8(\sin 3\theta)$$

$$f\left(\frac{\pi}{30}\right) = 4\cos\left(6\frac{\pi}{30}\right) + 8\sin\left(\frac{3\pi}{30}\right)$$

$$= 4\cos(36^\circ) + 8\sin 18^\circ$$

$$= 4\left(\frac{\sqrt{5}+1}{4}\right) + 8\left(\frac{\sqrt{5}-1}{4}\right) = \sqrt{5}+1+2\sqrt{5}-2$$

$$= 3\sqrt{5}-1$$

$$\cos(5A) = \cos(3A+2A)$$

$$= \cos 3A \cos 2A - \sin 3A \sin 2A$$

$$= (4\cos^3 A - 3\cos A)(2\cos^2 A - 1) - (3\sin A - 4\sin^3 A)$$

$$(2\sin A \cos A)$$

$$= (8\cos^5 A - 6\cos^3 A - 4\cos^3 A + 3\cos A) - (3 - 4\sin^2 A)$$

$$(2\sin^2 A \cos A)$$

$$= (8\cos^5 A - 10\cos^3 A + 3\cos A) - (3 - 4(1-\cos^2 A))$$

$$(2\cos A(1-\cos^2 A))$$

$$= (8\cos^5 A - 10\cos^3 A + 3\cos A) - (3 - 4 + 4\cos^2 A)$$

$$(2\cos A - 2\cos^3 A)$$

$$= (8\cos^5 A - 10\cos^3 A + 3\cos A) - (-1 + 4\cos^2 A)(2\cos A - 2\cos^3 A)$$

$$= (8\cos^5 A - 10\cos^3 A + 3\cos A) - (-2\cos A + 2\cos^3 A + 8\cos^3 A - 8\cos^5 A)$$

$$= \left(8\cos^5 A - 10\cos^3 A + 3\cos A\right) - \left(-2\cos A + 2\cos^3 A + 8\cos^3 A - 8\cos^5 A\right)$$

$$= 8\cos^5 A - 10\cos^3 A + 3\cos A + 2\cos A - 10\cos^3 A + 8\cos^5 A$$

$$\cos 5A = 16\cos^5 A - 20\cos^3 A + 5\cos A$$

$$\sin 5A = 16\sin^5 A - 20\sin^3 A + 5\sin A$$

(7)

$$\sqrt{3}\operatorname{cosec}20^\circ - \sec 20^\circ = \frac{\sqrt{3}\,(2)}{(2)\sin 20} - \frac{1}{\cos 20}$$

$$= \frac{(\sqrt{3}/2)}{\left(\frac{1}{2}\right)\sin 20^\circ} - \frac{1}{\cos 20^\circ}$$

$$= \frac{\sin 60^\circ}{\cos 60 \sin 20} - \frac{1}{\cos 20}$$

$$= \frac{\sin 60 \cos 20 - \cos 60 \sin 20}{\cos 60 \sin 20 \cos 20}$$

$$= \frac{(2)\sin(60-20)}{(2)\cos 60 \underbrace{\sin 20 \cos 20}} = \frac{2\sin 40}{\cos 60 \sin(2(20))}$$

$$= \frac{2\sin 40}{\cos 60 \sin 40} = \frac{2}{\cos 60} = 4$$ Answer

(9)

$$\boxed{\begin{aligned} \sin\theta + \sin\phi &= a \\ \cos\theta + \cos\phi &= b \end{aligned}}$$

$\xrightarrow{\text{square}}$ $\sin^2\theta + \sin^2\phi + 2\sin\theta\sin\phi = a^2$

$\xrightarrow{\text{square}}$ $\cos^2\theta + \cos^2\phi + 2\cos\theta\cos\phi = b^2$

add $\quad 1 + 1 + 2\sin\theta\sin\phi + 2\cos\theta\cos\phi = a^2+b^2$

$$2 + 2(\sin\theta\sin\phi + \cos\theta\cos\phi) = a^2+b^2$$

$$2(\cos(\theta-\phi)) = a^2+b^2-2$$

$$\cos(\theta-\phi) = \frac{a^2+b^2-2}{2}$$

$$\tan\left(\frac{\theta-\phi}{2}\right) = \frac{\sin\left(\frac{\theta-\phi}{2}\right)}{\cos\left(\frac{\theta-\phi}{2}\right)}$$

$$= \sqrt{\frac{1-\cos(\theta-\phi)}{1+\cos(\theta-\phi)}}$$

$$\tan\left(\frac{\theta-\phi}{2}\right) = \sqrt{\frac{1-\cos(\theta-\phi)}{1+\cos(\theta-\phi)}}$$

$$= \sqrt{\frac{1-\frac{a^2+b^2-2}{2}}{1+\frac{a^2+b^2-2}{2}}}$$

$$= \sqrt{\frac{(2-a^2-b^2+2)/\cancel{2}}{(2+a^2+b^2-2)/\cancel{2}}}$$

$$\tan\left(\frac{\theta-\phi}{2}\right) = \sqrt{\frac{4-a^2-b^2}{a^2+b^2}}$$

$\sin\theta + \sin\phi = a \Rightarrow 2\sin\left(\frac{\theta+\phi}{2}\right)\underline{\cos\left(\frac{\theta-\phi}{2}\right)} = a$

$\cos\theta + \cos\phi = b \Rightarrow 2\cos\left(\frac{\theta+\phi}{2}\right)\underline{\cos\left(\frac{\theta-\phi}{2}\right)} = b$

↓ divide

$$\boxed{\tan\left(\frac{\theta+\phi}{2}\right) = \frac{a}{b}}$$

$$\sin(\theta+\phi) = \frac{2\tan\left(\frac{\theta+\phi}{2}\right)}{1+\tan^2\left(\frac{\theta+\phi}{2}\right)}$$

$$\cos(\theta+\phi) = \frac{1-\tan^2\left(\frac{\theta+\phi}{2}\right)}{1+\tan^2\left(\frac{\theta+\phi}{2}\right)}$$

$\sin(\theta+\phi) = \frac{2ab}{a^2+b^2}$

$\cos(\theta+\phi) = \frac{b^2-a^2}{b^2+a^2}$

T- ratios of some standard angles ! → (18°, 36°, 72°, 54°)

$$\sin 18^\circ = \sin\left(\frac{\pi}{10}\right) = \frac{\sqrt{5}-1}{4} = \cos 72^\circ = \cos\left(\frac{2\pi}{5}\right)$$

Let $\theta = 18^\circ$

$5\theta = 90^\circ$

$2\theta + 3\theta = 90^\circ$

$2\theta = 90 - 3\theta$

take sine to both sides

$\sin 2\theta = \sin(90 - 3\theta)$

$2\sin\theta\cos\theta = \cos 3\theta$

$2\sin\theta\cos\theta = 4\cos^3\theta - 3\cos\theta$

$2\sin\theta\cancel{\cos\theta} = \cancel{\cos\theta}(4\cos^2\theta - 3)$

$2\sin\theta = 4\cos^2\theta - 3$

$2\sin\theta = 4(1-\sin^2\theta) - 3$

$2\sin\theta = 4 - 4\sin^2\theta - 3$

$2\sin\theta = 1 - 4\sin^2\theta$

$4\sin^2\theta + 2\sin\theta - 1 = 0$

$$\sin\theta = \frac{-2 \pm \sqrt{4+16}}{2(4)}$$

$$\sin\theta = \frac{-2 \pm 2\sqrt{5}}{8}$$

$\boxed{\sin\theta = \frac{-1+\sqrt{5}}{4}}$ OR $\boxed{\frac{-1-\sqrt{5}}{4}}$

Answer　　　Rejected

$$\boxed{\sin(18^\circ) = \frac{\sqrt{5}-1}{4}}$$

$$\sin(90-72) = \frac{\sqrt{5}-1}{4}$$

$$\boxed{\cos 72 = \frac{\sqrt{5}-1}{4}}$$

$$\boxed{\cos 36 = \frac{\sqrt{5}+1}{4}}$$

$$\cos 36 = \cos(2(18))$$

$$= 1 - 2\sin^2 18$$

$$= 1 - 2\left(\frac{\sqrt{5}-1}{4}\right)^2$$

$$= 1 - \frac{\cancel{2}(5+1-2\sqrt{5})}{\cancel{16}\,8}$$

$$\cos 36 = \frac{8-6+2\sqrt{5}}{8} = \frac{\sqrt{5}+1}{4}$$

$$\cos 18 = \sqrt{1-\sin^2 18}$$

$$= \sqrt{1 - \left(\frac{\sqrt{5}-1}{4}\right)^2}$$

$$= \sqrt{1 - \frac{5+1-2\sqrt{5}}{16}}$$

$$= \sqrt{\frac{16-6+2\sqrt{5}}{16}}$$

$$\boxed{\cos 18 = \frac{\sqrt{10+2\sqrt{5}}}{4}}$$

$$\cos(90-72) = \frac{\sqrt{10+2\sqrt{5}}}{4}$$

$$\boxed{\sin 72 = \frac{\sqrt{10+2\sqrt{5}}}{4}}$$

$$\boxed{\cos 36 = \frac{\sqrt{5}+1}{4}}$$

$$\cos(90-54) = \frac{\sqrt{5}+1}{4}$$

$$\boxed{\sin 54 = \frac{\sqrt{5}+1}{4}}$$

$$\sin 36 = \sqrt{1-\cos^2 36}$$

$$= \sqrt{1-\left(\frac{\sqrt{5}+1}{4}\right)^2}$$

$$= \sqrt{1-\frac{5+1+2\sqrt{5}}{16}}$$

$$= \sqrt{\frac{16-6-2\sqrt{5}}{16}}$$

$$\boxed{\sin 36 = \frac{\sqrt{10-2\sqrt{5}}}{4}}$$

$\sin(90-54)$

$$\boxed{\cos 54 = \frac{\sqrt{10-2\sqrt{5}}}{4}}$$

① find $\cos^2 48 - \sin^2 12$

$\cos^2 A - \sin^2 B$

$= \cos(A+B)\cdot\cos(A-B)$

$= \cos(48+12)\cdot\cos(48-12)$

$= \cos 60 \cdot \cos 36$

$= \frac{\sqrt{5}+1}{8}$

② find $\sin(132^\circ)\ \sin(12^\circ)$

$= \frac{1}{2}\left[-\cos(144) + \underbrace{\cos(120)}\right]$

$= \frac{1}{2}\left[-\cos(180-36) + \cos 120\right]$

$\boxed{\frac{\sqrt{5}-1}{8}}$

Ans

(3) Prove that $\tan 6^\circ \tan 42^\circ \tan 66^\circ \tan 78^\circ = 1$

$$\text{LHS} = \tan 6^\circ \tan(60+6) \cdot \frac{\tan(60-6)}{\tan(60-6)} \cdot \frac{\boxed{\tan(60-18)\tan(60+18)\tan 18}}{\tan 18}$$

$$= \frac{\tan 18}{\tan 54} \cdot \frac{\tan 54}{\tan 18} = 1$$

formula

$$\tan\theta \tan(60-\theta)\tan(60+\theta) = \tan 3\theta$$

Note :→ If $A+B=45$

$$A = 45 - B$$

$$\tan A = \tan(45 - B)$$

$$\tan A = \frac{\tan 45 - \tan B}{1+\tan 45 \tan B}$$

$$\tan A = \frac{1-\tan B}{1+\tan B}$$

$$\tan A + \tan A \tan B = 1 - \tan B$$

$$\tan A(1+\tan B) + \tan B = 1$$

$$+1 \qquad +1$$

$$\tan A(\tan B + 1) + 1(\tan B + 1) = 2$$

$$\boxed{(1+\tan B)(1+\tan A) = 2}$$

$$\left(1+\frac{1}{\cot B}\right)\left(1+\frac{1}{\cot A}\right) = 2$$

$$\frac{(\cot B + 1)(1+\cot A)}{\cot A \cot B} = 2$$

$$1 + \cot B + \cot A + \cot A \cot B = 2\cot A \cot B$$

$$\underbrace{1+\cot B}_{-2} + \cot A - \cot A \cot B = \underset{-2}{0}$$

$$\cot A(1-\cot B)$$

$$-1 + \cot B + \cot A(1-\cot B) = -2$$

$$-1(1-\cot B) + \cot A(1-\cot B) = -2$$

$$(1-\cot B)(-1+\cot A) = -2$$

$$(1-\cot B)(1-\cot A) = 2$$

Q $\sin(22.5^\circ) = ?$

$\sin\left(\frac{45}{2}\right)$

$$\boxed{\cos(22.5^\circ) = \frac{\sqrt{2+\sqrt{2}}}{2} = \sin(67.5^\circ)}$$

$\cos 2\theta = 2\cos^2\theta - 1$

$\cos 2\theta = 1 - 2\sin^2\theta$

$\cos 45 = 1 - 2\sin^2\left(\frac{45}{2}\right)$ | $2\theta = 45$, $\theta = \frac{45}{2}$

$\frac{1}{\sqrt{2}} = 1 - 2\sin^2\left(\frac{45}{2}\right)$

$2\sin^2\left(\frac{45}{2}\right) = 1 - \frac{1}{\sqrt{2}}$

$2\sin^2\left(\frac{45}{2}\right) = \frac{\sqrt{2}-1}{\sqrt{2}}$

$\sin^2\left(\frac{45}{2}\right) = \frac{\sqrt{2}-1}{2\sqrt{2}}$

$\sin\left(\frac{45}{2}\right) = \sqrt{\frac{\sqrt{2}-1}{2\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}}}$

$$\cos(67.5^\circ) = \boxed{\sin(22.5^\circ) = \frac{\sqrt{2-\sqrt{2}}}{2}}$$

Trigonometric Ratios and Identities

Lecture - 9

Trigonometric Identities in a triangle (conditional Identities):-

If A, B, C are angles of a triangle $A+B+C = \pi$

(i) $\sin(\underbrace{A+B}) = \sin(\pi - C) = \sin C$

$\sin(A+B) = \sin C$

$A+B+C = \pi$

$2A+2B+2C = 2\pi$

$2A+2B = 2\pi - 2C$

(ii) $\cos(A+B) = \cos(\pi - C) = -\cos C$

$\cos(A+B) = -\cos C$

(iii) $\tan(A+B) = \tan(\pi - C) = -\tan C$

$\tan(A+B) = -\tan C$

(iv) $\sin(2A+2B) = \sin(2\pi - 2C) = -\sin 2C$

$\sin(2A+2B) = -\sin 2C$

(v) $\cos(2A+2B) = \cos(2\pi-2C) = \cos 2C$

$$\boxed{\cos(2A+2B) = \cos 2C}$$

(vi) $\tan(2A+2B) = \tan(2\pi-2C) = -\tan 2C$

$$\boxed{\tan(2A+2B) = -\tan 2C}$$

(vii) $\sin\left(\frac{A+B}{2}\right) = \sin\left(\frac{\pi-C}{2}\right) = \sin\left(\frac{\pi}{2}-\frac{C}{2}\right) = \cos\frac{C}{2}$

$$\boxed{\sin\left(\frac{A+B}{2}\right) = \cos\frac{C}{2}}$$

(viii) $\cos\left(\frac{A+B}{2}\right) = \cos\left(\frac{\pi-C}{2}\right) = \sin\frac{C}{2}$

$$\boxed{\cos\frac{A+B}{2} = \sin\frac{C}{2}}$$

(ix) $\tan\left(\frac{A+B}{2}\right) = \tan\left(\frac{\pi-C}{2}\right) = \cot\frac{C}{2}$

$$\boxed{\tan\left(\frac{A+B}{2}\right) = \cot\frac{C}{2}}$$

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(x) $\underbrace{\sin 2A + \sin 2B} + \sin 2C = 4 \sin A \sin B \sin C$

$$A + B + C = \pi$$
$$C = \pi - (A+B)$$

LHS $= 2 \sin\left(\dfrac{2A+2B}{2}\right) \cos\left(\dfrac{2A-2B}{2}\right) + \sin(2C)$

$= 2 \sin(A+B) \cos(A-B) + \sin 2C$

$= 2 \sin(\pi - C) \cos(A-B) + \sin 2C$

$= 2 \sin C \cos(A-B) + 2 \sin C \cos C$

$= 2 \sin C [\cos(A-B) + \cos C]$

$= 2 \sin C [\cos(A-B) + \cos(\pi - (A+B))]$

$= 2 \sin C [\underline{\cos(A-B)} - \cos(\underline{A+B})]$

$= 2 \sin C [2 \sin A \sin B] = 4 \sin A \sin B \sin C$ = RHS.

Hence Proved

(xi) $\cos 2A + \cos 2B + \cos 2C = -1 - 4\cos A \cos B \cos C$ (HW)

$LHS = 2\cos(A+B)\cos(A-B) + \cos 2C$

$= 2\cos(\pi - C)\cos(A-B) + 2\cos^2 C - 1$

$= -2\cos C\cos(A-B) + 2\cos^2 C - 1$

$= 2\cos C\left[\cos C - \cos(A-B)\right] - 1$

$= 2\cos C\left[-\cos(A+B) - \cos(A-B)\right] - 1$

$= -2\cos C\left[2\cos A\cos B\right] - 1$

$= -4\cos A\cos B\cos C - 1$

$= RHS.$

Hence Proved

$C = \pi - (A+B)$

$\cos C = \cos(\pi - (A+B))$

$= -\cos(A+B)$

(xii) $\Sigma \tan A = \prod \tan A$ $\qquad A+B+C=\pi$

$$\tan A + \tan B + \tan C = \tan A \tan B \tan C$$

$$A+B+C = \pi$$

$$A+B = \pi - C$$

$$\tan(A+B) = \tan(\pi - C)$$

$$\frac{\tan A + \tan B}{1 - \tan A \tan B} = -\tan C$$

$$\tan A + \tan B = -\tan C + \tan A \tan B \tan C$$

$$\tan A + \tan B + \tan C = \tan A \tan B \tan C$$

$$\Sigma \tan A = \prod \tan A$$

(xiii) $\sum \cot A \cot B = 1 \Rightarrow \cot A \cot B + \cot B \cot C + \cot C \cot A = 1$

(xiv) $\underbrace{\sum \tan \frac{A}{2} \tan \frac{B}{2}} = 1$

(xv) $\sum \cot\left(\frac{A}{2}\right) = \prod \cot\left(\frac{A}{2}\right)$

$A+B+C = \pi$

$\frac{A+B+C}{2} = \frac{\pi}{2}$

$\frac{A+B}{2} = \left(\frac{\pi}{2} - \frac{C}{2}\right)$

$\tan\left(\frac{A+B}{2}\right) = \tan\left(\frac{\pi}{2} - \frac{C}{2}\right)$

$\frac{\tan\left(\frac{A}{2}\right) + \tan\frac{B}{2}}{1 - \tan\frac{A}{2}\tan\frac{B}{2}} = \frac{1}{\tan\frac{C}{2}}$

① $\sin(B+C-A) + \sin(C+A-B) + \sin(A+B-C) = 4\sin A \sin B \sin C$

where $A+B+C = \pi$

$B+C = \pi - A$

$LHS = \sin(\pi - A - A) + \sin(\pi - B - B) + \sin(\pi - C - C)$

$= \sin(\pi - 2A) + \sin(\pi - 2B) + \sin(\pi - 2C)$

$= \sin 2A + \sin 2B + \sin 2C$

$= 4 \sin A \sin B \sin C$

② $\sin 2A + \sin 2B - \sin 2C = 4\cos A \cos B \sin C$ $(A+B+C=\pi)$

<u>LHS</u> : $2\sin(A+B)\cos(A-B) - \sin 2C$

$= 2\sin(\pi - C)\cos(A-B) - \sin 2C$

$= 2\sin C\cos(A-B) - 2\sin C\cos C$

$= 2\sin C\,[\cos(A-B) - \cos C]$

$= 2\sin C\,[\cos(A-B) - \cos(\pi - (A+B))]$

$= 2\sin C\,[\cos(A-B) + \cos(A+B)]$

$= 2\sin C\,[2\cos A\cos B]$

$= 4\sin C\cos A\cos B$

$=$ <u>RHS.</u>

<u>Hence</u> <u>Proved</u>

$A+B+C=\pi$

$C = \pi - (A+B)$

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(3) $\cos 2A + \cos 2B - \cos 2C = 1 - 4\sin A \sin B \cos C$ $\quad (A+B+C) = \pi$

$= 2\cos(A+B)\cos(A-B) - 2\cos^2 C + 1$

$= -2\cos C \cos(A-B) - 2\cos^2 C + 1$

$= -2\cos C[\cos(A-B) + \cos C] + 1$

$= -2\cos C[\cos(A-B) + \cos(\pi-(A+B))] + 1$

$= -2\cos C[\cos(A-B) - \cos(A+B)] + 1$

$= -2\cos C[2\sin A \sin B] + 1$

$= -4\sin A \sin B \cos C + 1$

ALLEN CAREER INSTITUTE KOTA (RAJASTHAN) path to success

(4) If $A+B+C=\pi$, Prove that $\sin^2 A + \sin^2 B - \sin^2 C = 2\sin A \sin B \cos C$

$\underline{LHS} = \sin^2 A + \sin(B+C)\sin(B-C)$

$= \sin^2 A + \underline{\sin(\pi-A)}\ \sin(B-C)$

$= \sin A[\sin A + \sin(B-C)]$

$= \sin A[\sin(\pi-(B+C)) + \sin(B-C)]$

$= \sin A[\sin(B+C) + \sin(B-C)]$

$= \sin A\ 2\sin B\cos C$

$= 2\sin A\sin B\cos C$

$\sin^2 A - \sin^2 B$

$= \sin(A+B)\cdot\sin(A-B)$

(5) $\sin^2\left(\frac{A}{2}\right) + \sin^2\frac{B}{2} - \sin^2\frac{C}{2} = 1 - 2\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}$ $(A+B+C=\pi)$

(6) If $A+B+C=2S$, prove that

$$\sin(S-A)\sin(S-B)+\sin S\left(\sin(S-C)\right)=\sin A\sin B.$$

$$LHS=\frac{1}{2}\left[\underbrace{2\sin(S-A)\sin(S-B)}+\underbrace{2\sin(S)\sin(S-C)}\right]$$

$$=\frac{1}{2}\Big[$$

(7) Solve

$$\frac{\sin 50^\circ + \sin 100^\circ + \sin 210^\circ}{\sin 25^\circ \sin 50^\circ \sin 105^\circ}$$

8 find $\dfrac{\sin 2A + \sin 2B + \sin 2C}{\sin A + \sin B + \sin C}$ $(A+B+C=\pi)$

Inequalities in a triangle

Q In a ΔABC, show that $\cot^2 A + \cot^2 B + \cot^2 C \geq 1$

AM $\geq$ GM

$$\frac{\cot^2 A + \cot^2 B}{2} \geq \sqrt{\cot^2 A \cot^2 B}$$

add

$$\frac{\cot^2 A + \cot^2 B}{2} \geq \cot A \cot B$$

$$\frac{\cot^2 B + \cot^2 C}{2} \geq \cot B \cot C$$

$$\frac{\cot^2 C + \cot^2 A}{2} \geq \cot C \cot A$$

$$\cot^2 A + \cot^2 B + \cot^2 C \geq \cot A \cot B + \cot B \cot C + \cot C \cot A$$

$$\boxed{\cot^2 A + \cot^2 B + \cot^2 C \geq 1}$$

Hence Proved

(2) In $\triangle ABC$, P. T. $\cos A \cos B \cos C \le \frac{1}{8}$

M-I

LHS

$$y = \cos A \cos B \cos C$$

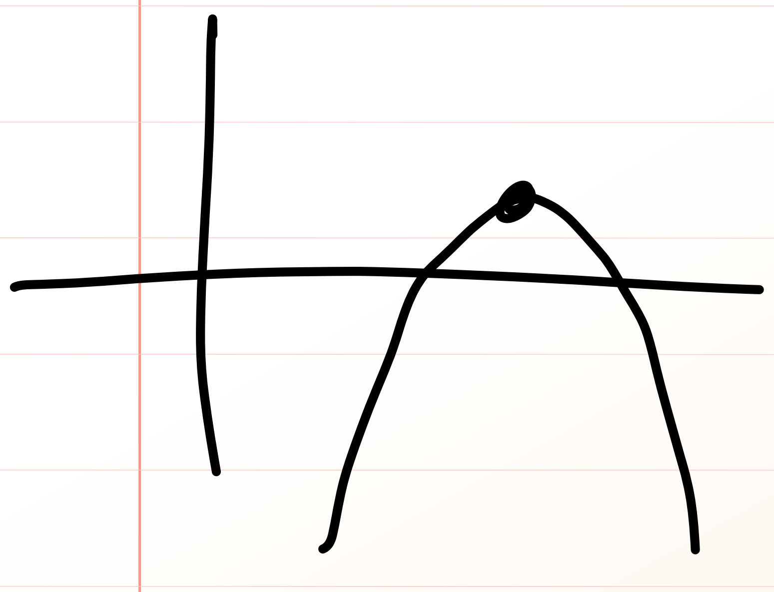
$$= \frac{\cos A}{2}(2\cos B \cos C)$$

$$= \frac{\cos A}{2}\left[\cos(B+C) + \cos(B-C)\right]$$

$$= \frac{\cos A}{2}\left[\cos(\pi - A) + \cos(B-C)\right]$$

$$= \frac{\cos A}{2}\left[-\cos A + \cos(B-C)\right]$$

$$y = \frac{\cos A}{2}\left[\cos(B-C) - \cos A\right]$$



$$y \le \frac{\cos A}{2}\left[1 - \cos A\right] \Rightarrow y \le \underbrace{\left(\frac{\cos A}{2} - \frac{\cos^2 A}{2}\right)}$$

$$y \le \frac{\cos A}{2} - \frac{\cos^2 A}{2}$$

$$-\frac{t^2}{2} + \frac{t}{2}$$

vertex

$$V_x = \frac{-b}{2a} = \frac{-1/2}{2\left(-\frac{1}{2}\right)} = \frac{1}{2}$$

$$y \le \frac{1}{8}$$

$$V_y = -\frac{1}{8} + \frac{1}{4} = \frac{1}{8}$$

$$\cos A \cos B \cos C \le \frac{1}{8}$$

M-II

$$\underbrace{2\cos A \cos B}\ \frac{\cos C}{2} = y$$

$$\frac{1}{2}\left[\cos(A+B) + \cos(A-B)\right] \cos C = y$$

$$\left[\cos(\pi - C) + \cos(A-B)\right] \cos C = 2y$$

$$-\cos^2 C + \cos(A-B)\cos C = 2y$$

$$-\cos^2 C + \cos(A-B)\cos C = 2y$$

$$-\cos^2 C + \cos(A-B)\cos C - 2y = 0$$

$$\cos^2 C - \underbrace{\cos(A-B)}\cos C + 2y = 0$$

$$D \geq 0$$

$$\left[\cos(A-B)\right]^2 - 4(2y) \geq 0$$

$$\cos^2(A-B) \geq 8y$$

$$\frac{\cos^2(A-B)}{8} \geq y$$

$$y \leq \frac{\cos^2(A-B)}{8}$$

$$y \leq \frac{1}{8} \Rightarrow \boxed{\cos A \cos B \cos C \leq \frac{1}{8}}$$

<u>Hence</u> <u>Proved</u>

Trigonometric Ratios and Identities

Lecture - 10 & 11

Application of Trigonometry in maximising & minimizing : →

Type I : →

$$-1 \le \sin x \le 1$$

$$-1 \le \cos x \le 1$$

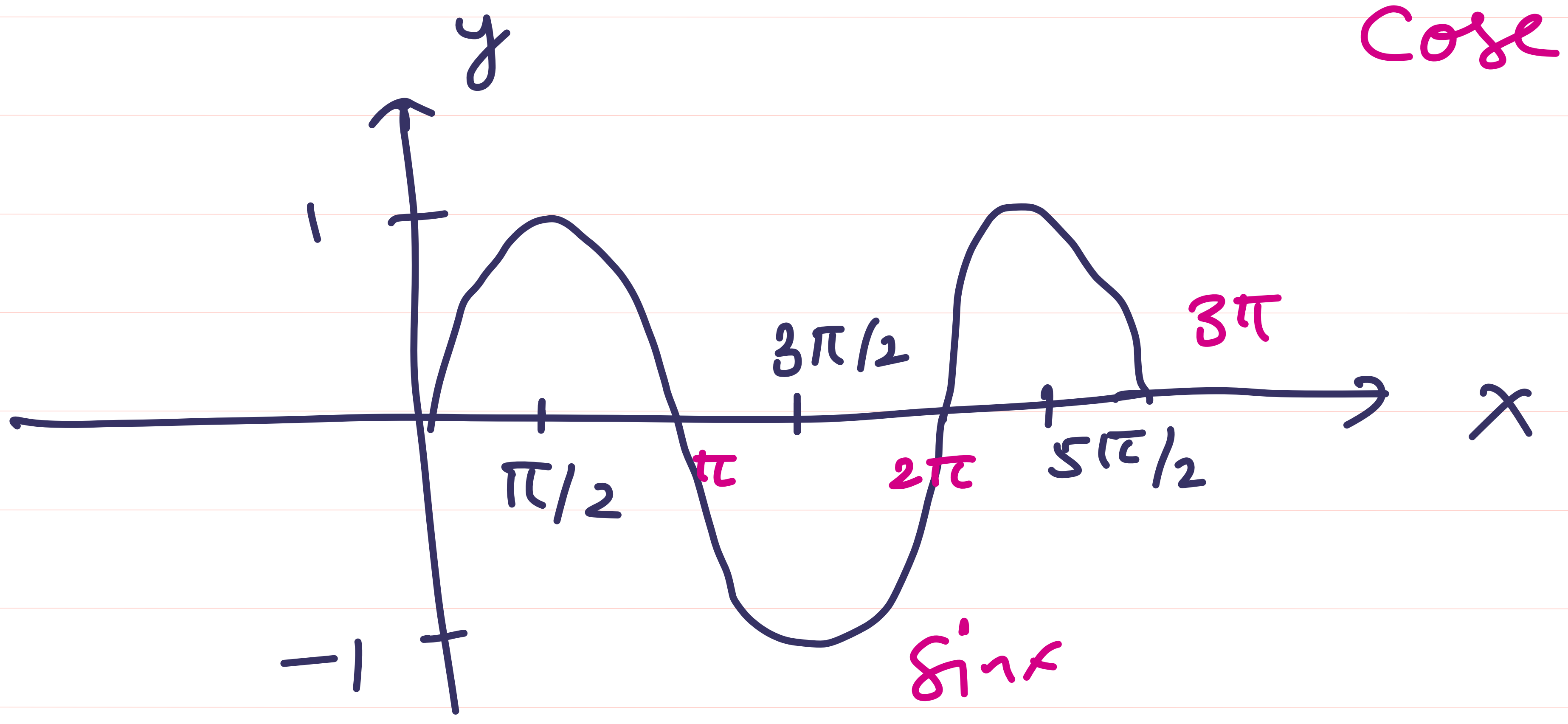
$$\tan x \in (-\infty, \infty)$$

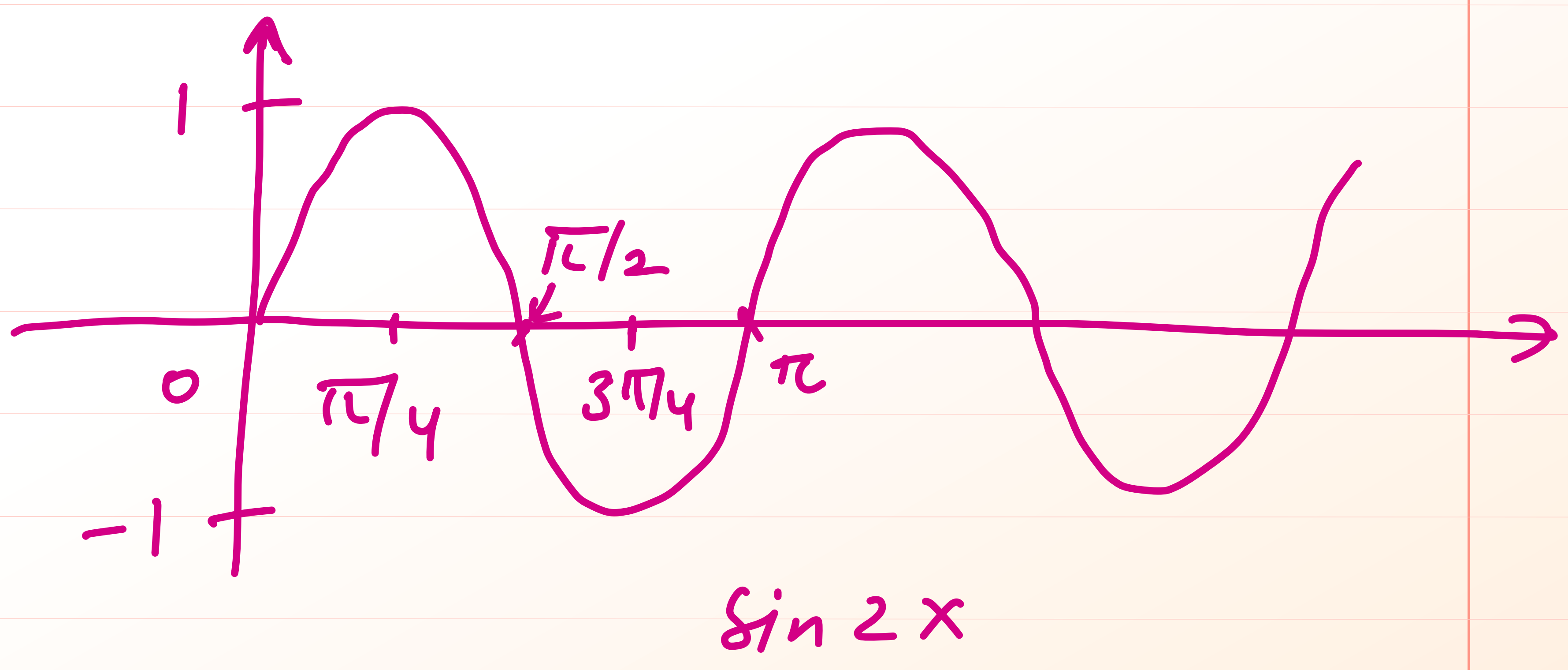
$$\cot x \in (-\infty, \infty)$$

$$\sec x \in (-\infty, -1] \cup [1, \infty)$$

$$\operatorname{cosec} x \in (-\infty, -1] \cup [1, \infty)$$

Range of $\sin x$ is $[-1, 1]$





① find range of $y = \cos^4 x - \sin^4 x$

$$= (\cos^2 x + \sin^2 x)(\cos^2 x - \sin^2 x)$$

$$= 1\,(\underbrace{\cos^2 x - \sin^2 x})$$

$$= \cos 2x$$

$$y \in [-1, 1]$$

(2) (a)

$$y = 4\tan x \cos x$$

$$y = 4\frac{\sin x}{\cos x}(\cos x)$$

$$y = 4\sin x$$

$$\cancel{y \in [-4, 4]}$$

$$y \in (-4, 4)$$

$$y = 4\sin x$$

$$y \in [-4, 4]$$

③ ⓐ $y = \sin(\sqrt{x})$

$\boxed{y \in [-1, 1]}$

ⓑ $y = \sin^2 x$

$-1 \le \sin x \le 1$

$0 \le \sin^2 x \le 1$

$\boxed{y \in [0, 1]}$

ⓒ $y = \sin 3x$

$\boxed{y \in [-1, 1]}$

Q $y = \sin^2\left(\frac{15\pi}{8} - 4x\right) - \sin^2\left(\frac{17\pi}{8} - 4x\right)$

$\sin^2(A) - \sin^2 B$

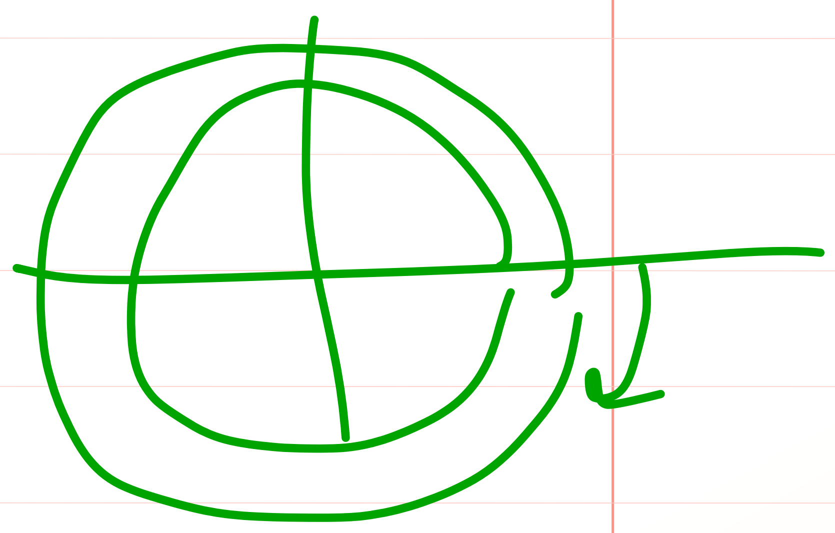
$= \sin(A+B)\sin(A-B)$

$y = \underbrace{\sin(4\pi - 8x)} \cdot \underbrace{\sin\left(-\frac{\pi}{4}\right)}$

$= +\underbrace{\sin(8x)}\ \frac{1}{\sqrt{2}}$

$=$

$$\boxed{y \in \left[-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]}$$



Type II when argument of sine & cosine are same :→

$a \sin \underline{x} + b \cos \underline{x}$

$$y = \left(\frac{a \sin x}{\sqrt{a^2+b^2}} + \frac{b \cos x}{\sqrt{a^2+b^2}}\right) \sqrt{a^2+b^2}$$

$$y = \sqrt{a^2+b^2}\left[\frac{a}{\sqrt{a^2+b^2}} \sin x + \frac{b}{\sqrt{a^2+b^2}} \cos x\right]$$

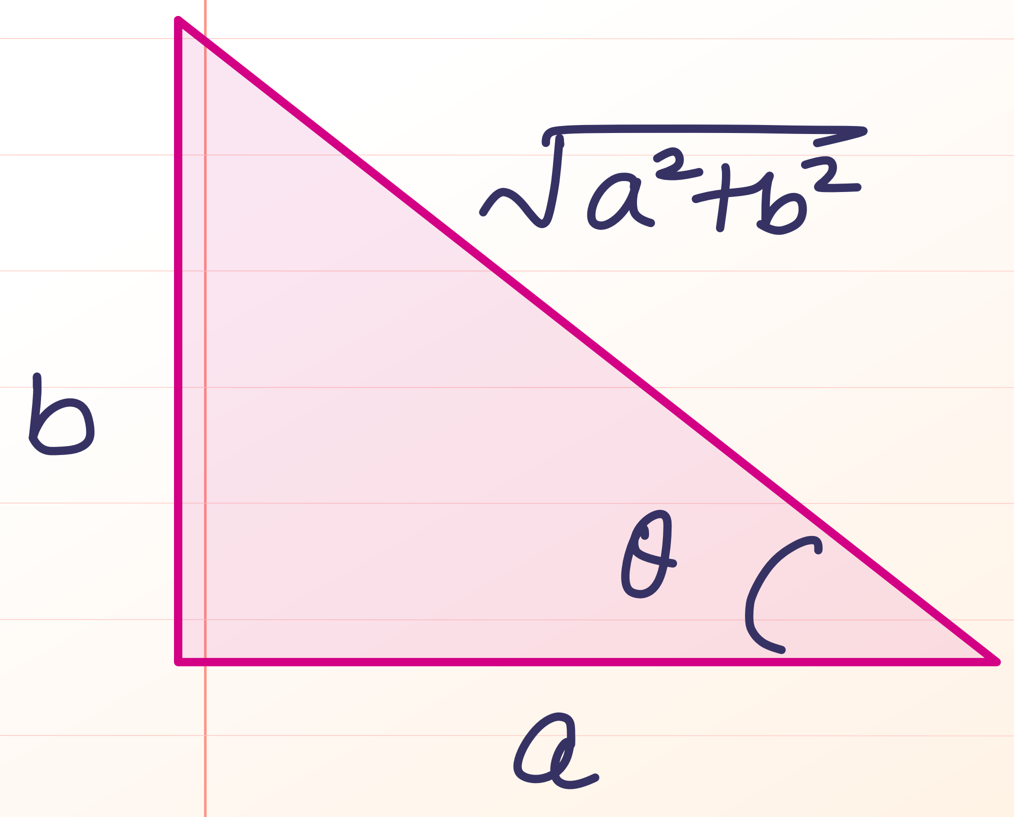
$$= \sqrt{a^2+b^2}\left[\cos\theta \cdot \sin x + \sin\theta \cos x\right]$$

$$= \sqrt{a^2+b^2}\left[\sin(x+\theta)\right]$$

$$-1 \leq \sin(x+\theta) \leq 1$$

$$-\sqrt{a^2+b^2} \leq \underbrace{\sqrt{a^2+b^2}\,(\sin(x+\theta))} \leq \sqrt{a^2+b^2}$$

$$\boxed{y \in \left[-\sqrt{a^2+b^2},\ +\sqrt{a^2+b^2}\right]}$$



$$\cos\theta = \frac{a}{\sqrt{a^2+b^2}}$$

$$\sin\theta = \frac{b}{\sqrt{a^2+b^2}}$$

find range

Q ① $y = \sin x + \cos x$

$-\sqrt{1^2+1^2} \le \underbrace{\sin x + \cos x} \le \sqrt{1^2+1^2}$

$-\sqrt{2} \le y \le \sqrt{2}$

$\boxed{y \in [-\sqrt{2}, \sqrt{2}]}$

Q2 $y = 3\sin x + 4\cos x + 5$

$-\sqrt{3^2+4^2} \le 3\sin x + 4\cos x \le +\sqrt{3^2+4^2}$

$-5 \le \underbrace{3\sin x + 4\cos x + 5} \le +5$

$+5 \qquad\qquad +5$

$0 \le y \le 10$

$y \in [0, 10]$

Q-3

$$y = \log_2\left(\frac{3\sin x - 4\cos x + 15}{10}\right)$$

$$-\sqrt{3^2+4^2} \le 3\sin x - 4\cos x \le \sqrt{3^2+4^2}$$

$$15 - 5 \le 3\sin x - 4\cos x + 15 \le 5 + 15$$

$$\frac{10}{10} \le \frac{3\sin x - 4\cos x + 15}{10} \le \frac{20}{10}$$

$$\underbrace{\log_2(1)} \le \log_2\left(\frac{3\sin x - 4\cos x + 15}{10}\right) \le \log_2 2$$

$$0 \le y \le 1$$

$$\boxed{y \in [0,1]}$$

Q 4 If $b \le 3\sin^2 x + 6\cos^2 x \underbrace{-4\sin x\cos x} + 5 \le a$ find a & b.

$$y = 3\sin^2 x + 6\cos^2 x \underbrace{-4\sin x\cos x} + 5$$

$$= \underbrace{3\sin^2 x + 3\cos^2 x} + 3\cos^2 x - 2\sin 2x$$

$$= 3(\sin^2 x + \cos^2 x) + 3\cos^2 x - 2\sin 2x + 5$$

$$= 8 + 3\cos^2 x - 2\sin 2x$$

$$= 8 + 3\left(\frac{\cos 2x + 1}{2}\right) - 2\sin 2x$$

$$y = 8 + \frac{3}{2}\cos 2x + \frac{3}{2} - 2\sin 2x$$

$$\boxed{y = \frac{19}{2} + \frac{3}{2}\cos 2x - 2\sin 2x}$$

$\cos 2x$
$= 2\cos^2 x - 1$
$\cos^2 x = \frac{\cos 2x + 1}{2}$

$$\boxed{y = \frac{19}{2} + \frac{3}{2}\cos 2x - 2\sin 2x}$$

$$-\sqrt{\left(\frac{3}{2}\right)^2 + (2)^2} \le \frac{3}{2}\cos 2x - 2\sin 2x \le \sqrt{\left(\frac{3}{2}\right)^2 + (2)^2}$$

$$-\frac{5}{2} \le \frac{3}{2}\cos 2x - 2\sin 2x \le \frac{5}{2}$$

$$+\frac{19}{2} \qquad +\frac{19}{2} \qquad +\frac{19}{2}$$

$$7 \le y \le 12$$

$$y \in [7, 12]$$

Q 6 find range of $y = 5\sin\left(x + \frac{\pi}{6}\right) + 3\cos x$ Ans $[-7, 7]$

Q-7 find range of $y = \sin\left(x + \frac{\pi}{6}\right) + 3\cos\left(x - \frac{\pi}{3}\right)$ Ans $[-4, 4]$

Q-8 find maximum & minimum value of

$$y = \frac{17 + \boxed{(5\sin x + 12\cos x)}}{17 - (5\sin x + 12\cos x)} \rightarrow [-13, 13]$$

$$y_{max} = \frac{17 + 13}{17 - 13} = \frac{15}{2}$$

$$y_{min} = \frac{17 - 13}{17 - (-13)} = \frac{2}{15}$$

Type-3 Argument of sine & cosine are different.

(1) $y = \underbrace{\cos 2x} + 3\underbrace{\sin x}$

$= \underbrace{1 - 2\sin^2 x} + 3\sin x$

$= \underbrace{-2\sin^2 x + 3\sin x} + 1$

$= -2\left[\sin^2 x - \frac{3}{2}\sin x\right] + 1$

$= -2\left[\sin^2 x - 2\cdot\sin x\cdot\left(\frac{3}{4}\right) + \left(\frac{3}{4}\right)^2\right] - \left(\frac{3}{4}\right)^2 \underline{(-2)} + 1$

$= -2\left[\sin x - \frac{3}{4}\right]^2 + \frac{9}{8} + 1$

$y = -2\left[\sin x - \frac{3}{4}\right]^2 + \frac{17}{8}$

$$y = -2\left[\sin x - \frac{3}{4}\right]^2 + \frac{17}{8}$$

$$y = \frac{17}{8} - 2\left[\sin x - \frac{3}{4}\right]^2$$

when $\sin x = \frac{3}{4}$

$$y_{max} = \frac{17}{8} - 0 = \frac{17}{8}$$

when $\sin x = -1$

$$y_{min} = \frac{17}{8} - 2\left[-1 - \frac{3}{4}\right]^2$$

$$= \frac{17}{8} - 2\left[-\frac{7}{4}\right]^2$$

$$= \frac{17}{8} - \frac{49}{8} = -4$$

$$\boxed{y \in \left[-4, \frac{17}{8}\right]}$$

Q

$$y = \sin^2 x - 20\cos x + 1$$

$$y = 1 - \cos^2 x - 20\cos x + 1$$

$$= -\cos^2 x - 20\cos x + 2$$

$$= -1\left[\cos^2 x + 20\cos x\right] + 2$$

$$= -1\left[\cos^2 x + 20\cos x + (10)^2\right] - (10)^2(-1) + 2$$

$$= -1\left[\cos x + 10\right]^2 + 102$$

$$y = 102 - (10 + \cos x)^2$$

$\cos x = -1 \Rightarrow$ $$y_{max} = 102 - (10-1)^2 = 21$$

$\cos = 1 \Rightarrow$ $$y_{min} = 102 - (11)^2 = -19$$

Q

$$y = \cos^2 x - 4\cos x + 13$$

$$= \underbrace{\cos^2 x - 4\cos x + 4} + \underbrace{13 - 4}$$

$$y = (\cos x - 2)^2 + 9$$

$$y_{min} = (1-2)^2 + 9 = 10$$

$$y_{max} = (-1-2)^2 + 9 = 18$$

Type - 4

Q $y = a^2\tan^2\theta + b^2\cot^2\theta$; $(a, b \geq 0)$

$a^2+b^2 = (a+b)^2 - 2ab$

$a^2+b^2 = (a-b)^2 + 2ab$

$= (a\tan\theta - b\cot\theta)^2 + 2ab\cot\theta\tan\theta$

$= (a\tan\theta - b\cot\theta)^2 + 2ab$

$y_{min} = 2ab$ when $a\tan\theta - b\cot\theta = 0$

$a\tan\theta = b\cot\theta$

$a\tan\theta = \frac{b}{\tan\theta}$

$\tan^2\theta = \frac{b}{a}$

$\boxed{\tan\theta = \sqrt{\frac{b}{a}}}$

② $y = a^2\sec^2\theta + b^2\operatorname{cosec}^2\theta$

$= a^2(1+\tan^2\theta) + b^2(1+\cot^2\theta)$

$y = a^2+b^2 + \underbrace{a^2\tan^2\theta + b^2\cot^2\theta}$

$y = a^2+b^2 + (a\tan\theta - b\cot\theta)^2 + 2ab$

$y_{min} = a^2+b^2+2ab$ when $a\tan\theta - b\cot\theta = 0$

$\Rightarrow \tan\theta = \sqrt{\dfrac{b}{a}}$

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(3) $y = 8\sec^2\theta + 18\cos^2\theta$

$$y = (2\sqrt{2}\sec\theta)^2 + (3\sqrt{2}\cos\theta)^2$$

$$= (2\sqrt{2}\sec\theta - 3\sqrt{2}\cos\theta)^2 + 2\cdot 2\sqrt{2}\cdot\sec\theta\cdot 3\sqrt{2}\cos\theta$$

$$= (2\sqrt{2}\sec\theta - 3\sqrt{2}\cos\theta)^2 + 24$$

$$\boxed{y_{min} = 24}$$

when $2\sqrt{2}\sec\theta - 3\sqrt{2}\cos\theta = 0$

$$\frac{2\sqrt{2}}{\cos\theta} = 3\sqrt{2}\cos\theta$$

$$\cos^2\theta = \frac{2\sqrt{2}}{3\sqrt{2}}$$

$$\boxed{\cos\theta = \sqrt{\frac{2}{3}}}$$

find range

(4) $y = 4\sin^2\theta + \operatorname{cosec}^2\theta$ $[y_{min} = 4]$

(5) $y = \underline{\sin^2\theta} + \underline{4\operatorname{cosec}^2\theta}$ $[y_{min} = 5]$

$y = (\sin\theta - 2\operatorname{cosec}\theta)^2 + 4$

$\sin\theta - 2\operatorname{cosec}\theta = 0$

$\sin\theta = 2\operatorname{cosec}\theta$

$\boxed{\sin^2\theta = 2}$ ✗

(6) $y = 4\sin^2 x + \underline{27}\operatorname{cosec}^2 x$ $[y_{min} = 31]$

sol (7) $y = \underline{18\sec^2 x} + \underline{8\cos^2 x}$ $[y_{min} = 26]$

(5) $y = (\sin^2\theta + \operatorname{cosec}^2\theta) + 3\operatorname{cosec}^2\theta$

$y = (\underbrace{\sin\theta - \operatorname{cosec}\theta}))^2 + 2 + 3\operatorname{cosec}^2\theta$

$y_{min} = 2 + 3(1) = 5$

$\sin\theta - \operatorname{cosec}\theta = 0$

$\sin\theta = \operatorname{cosec}\theta$

$\sin^2\theta = 1$

$\sin\theta = \pm 1$

$\underbrace{4\sin^2 x + 4\operatorname{cosec}^2 x} + \underbrace{23\operatorname{cosec}^2 x}$

$\underbrace{4(\underline{\sin x - \operatorname{cosec} x})^2} + \underline{2(4)} + \underline{23\operatorname{cosec}^2 x}$

$\alpha+\beta = 60^\circ$

Note : $\rightarrow$

(i) If $\alpha, \beta \in \left(0, \frac{\pi}{2}\right)$ and $\alpha+\beta = \sigma$ (constant) then

ⓐ max. value of the expression $\cos\alpha \cos\beta$, $\cos\alpha+\cos\beta$, $\sin\alpha \cdot \sin\beta$ or $\sin\alpha+\sin\beta$ occurs when $\alpha = \beta = \sigma/2$

ⓑ Minimum value of $\sec\alpha+\sec\beta$, $\tan\alpha+\tan\beta$, $\operatorname{cosec}\alpha+\operatorname{cosec}\beta$ occurs when $\alpha=\beta=\sigma/2$

(ii) If A, B, C are angles of a triangle then maximum value of $\sin A+\sin B+\sin C$ and $\sin A \sin B \sin C$ occurs when $A = B = C = 60^\circ$.

Q If $\underline{x^2+y^2=4}$; $\underline{a^2+b^2=8}$; find minimum & maximum value of $ax+by$.

$$x^2+y^2=(2)^2$$
$$x = 2\cos\theta$$
$$y = 2\sin\theta$$

$$a^2+b^2=(2\sqrt{2})^2$$
$$a = 2\sqrt{2}\cos\phi$$
$$b = 2\sqrt{2}\sin\phi$$

If $x^2+y^2=r^2$
then
$$x = r\cos\theta$$
$$y = r\sin\theta$$

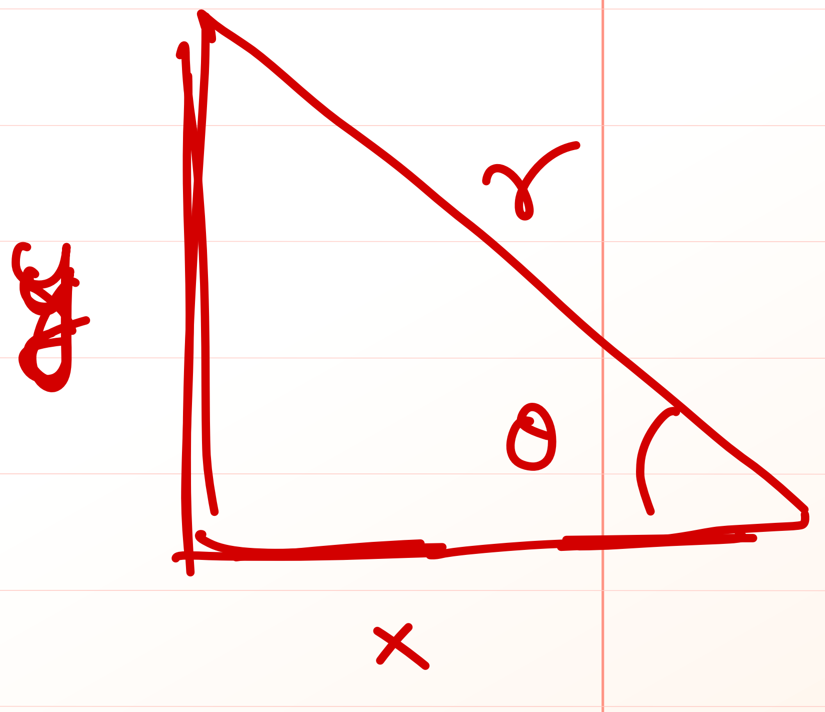
$$ax+by = (2\sqrt{2}\cos\phi)(2\cos\theta) + (2\sqrt{2}\sin\phi)(2\sin\theta)$$

$$= 4\sqrt{2}\left[\cos\theta\cos\phi + \sin\theta\sin\phi\right]$$

$$ax+by = 4\sqrt{2}\left[\cos(\theta-\phi)\right]$$

$$ax+by\Big|_{min} = -4\sqrt{2}$$

$$ax+by\Big|_{max} = +4\sqrt{2}$$



$$\cos\theta = \frac{x}{r}$$
$$x = r\cos\theta$$
$$\sin\theta = \frac{y}{r} \Rightarrow y = r\sin\theta$$

Q Prove that $\frac{\tan 3x}{\tan x}$ can not lies from $\frac{1}{3}$ to 3.

$$y = \frac{\tan 3x}{\tan x}$$

$$y = \frac{3\tan x - \tan^3 x}{(1 - 3\tan^2 x)(\tan x)}$$

$$y = \frac{3 - \tan^2 x}{1 - 3\tan^2 x}$$

$$y - 3y\tan^2 x = 3 - \tan^2 x$$

$$y - 3 = \tan^2 x\,(3y - 1)$$

$$\tan^2 x = \frac{y-3}{3y-1} \Rightarrow$$

$$\frac{y-3}{3y-1} = \tan^2 x$$

$$\frac{y-3}{3y-1} \geq 0$$

$$\boxed{y \in \left(-\infty, \frac{1}{3}\right) \cup [3, \infty)}$$

Trigonometric Ratios and Identities

Lecture - 12

Continued product of sine & cosine Series :—

$$\prod_{r=1}^{n} \sin(r\theta) = \sin\theta \cdot \sin 2\theta \cdot \sin 3\theta \cdot \sin 4\theta \text{ - - - - } \sin n\theta$$

$$\sum_{r=1}^{n} \sin(r\theta) = \sin\theta + \sin 2\theta + \sin 3\theta + \text{ - - - - } + \sin n\theta$$

Q

$$P = \frac{2\sin\theta\cos\theta}{2\sin\theta} \cdot \cos 2\theta \cdot \cos 2^2\theta \cdot \cos 2^3\theta \cdots\cdots \cos 2^{n-1}\theta$$

$$= \frac{2\sin 2\theta}{2(2)\sin\theta} \cos 2\theta \cdot \cos 2^2\theta \cdot \cos 2^3\theta \cdots\cdots \cos 2^{n-1}\theta$$

$$= \frac{2\sin 4\theta}{2(2^2)\sin\theta} \cos 2^2\theta \cdot \cos 2^3\theta \cdots\cdots \cos 2^{n-1}\theta$$

$$= \frac{\sin(2^3\theta)}{2^3\sin\theta} \cdot \cos 2^3\theta \cdot \cdots\cdots \cos 2^{n-1}\theta.$$

$$= \frac{\sin(2^n\theta)}{2^n\sin\theta}$$

$$\boxed{\begin{aligned} &2^{n-1}\cdot 2 \\ &= 2^{n-1}\cdot 2^1 \\ &= 2^n \end{aligned}}$$

Q $P = \sin\left(\frac{\pi}{14}\right)\sin\left(\frac{3\pi}{14}\right)\sin\left(\frac{5\pi}{14}\right)\underline{\sin\left(\frac{7\pi}{14}\right)}\sin\left(\frac{9\pi}{14}\right)\sin\left(\frac{11\pi}{14}\right)\sin\left(\frac{13\pi}{14}\right)$

$$= \sin\left(\frac{\pi}{14}\right)\sin\left(\frac{3\pi}{14}\right)\sin\left(\frac{5\pi}{14}\right)\sin\left(\frac{\pi}{2}\right)\sin\left(\pi-\frac{5\pi}{14}\right)\sin\left(\pi-\frac{3\pi}{14}\right)\sin\left(\pi-\frac{\pi}{14}\right)$$

$$= \sin^2\frac{\pi}{14}\cdot\sin^2\left(\frac{3\pi}{14}\right)\sin^2\left(\frac{5\pi}{14}\right)$$

$$= \left(\sin\frac{\pi}{14}\cdot\sin\frac{3\pi}{14}\cdot\sin\frac{5\pi}{14}\right)^2$$

$$= \left[\sin\left(\frac{7\pi-6\pi}{14}\right)\sin\left(\frac{7\pi-4\pi}{14}\right)\cdot\sin\left(\frac{7\pi-2\pi}{14}\right)\right]^2$$

$$= \left[\sin\left(\frac{\pi}{2}-\frac{6\pi}{14}\right)\cdot\sin\left(\frac{\pi}{2}-\frac{4\pi}{14}\right)\cdot\sin\left(\frac{\pi}{2}-\frac{2\pi}{14}\right)\right]^2$$

$$= \left[\left(\cos\frac{3\pi}{7}\cdot\cos\frac{2\pi}{7}\cdot\cos\frac{\pi}{7}\cdot\frac{2\sin\left(\frac{\pi}{7}\right)}{2\sin(\pi/7)}\right]^2 =$$

$$= \left[\left(\cos\frac{3\pi}{7}\cdot\cos\frac{2\pi}{7}\cdot\cos\frac{\pi}{7}\cdot\frac{2\sin\left(\frac{\pi}{7}\right)}{2\sin(\pi/7)}\right)\right]^2$$

$$= \left[\cos\frac{3\pi}{7}\cdot\frac{\boxed{\cos\frac{2\pi}{7}\cdot\left(2\sin 2\pi/7\right)}}{2(2)\sin\pi/7}\right]^2$$

$$= \left[\underbrace{\cos\left(\frac{3\pi}{7}\right)}\cdot\frac{\sin\left(\frac{4\pi}{7}\right)}{4\sin(\pi/7)}\right]^2$$

$$\cos\frac{3\pi}{7} = \cos\left(\pi-\frac{4\pi}{7}\right) = -\cos\frac{4\pi}{7}$$

$$= \left[\frac{-\cos(4\pi/7)\cdot(2)\sin\frac{4\pi}{7}}{(2)\,4\sin(\pi/7)}\right]^2 = \left(\frac{-\sin\frac{8\pi}{7}}{8\sin\pi/7}\right)^2$$

$$= \left[\frac{-\sin\left(\pi+\frac{\pi}{7}\right)}{8\sin(\pi/7)}\right]^2 = \left[+\frac{\sin\pi/7}{8\sin(\pi/7)}\right]^2 = \boxed{\frac{1}{64}}$$

Ans

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Q $\quad P = \sin\left(\frac{\pi}{16}\right) \cdot \sin\left(\frac{3\pi}{16}\right) \cdot \sin\left(\frac{5\pi}{16}\right) \cdot \sin\left(\frac{7\pi}{16}\right) \qquad \frac{8\pi}{16}$

$= \sin\left(\frac{\pi}{16}\right) \sin\left(\frac{3\pi}{16}\right) \sin\left(\frac{8\pi-3\pi}{16}\right) \sin\left(\frac{8\pi-\pi}{16}\right)$

$= \sin\left(\frac{\pi}{16}\right) \sin\frac{3\pi}{16} \sin\left(\frac{\pi}{2}-\frac{3\pi}{16}\right) \sin\left(\frac{\pi}{2}-\frac{\pi}{16}\right)$

$= \frac{2\sin\frac{\pi}{16}}{2} \cdot \frac{1}{2}\left(2\sin\frac{3\pi}{16} \cdot \cos\frac{3\pi}{16}\right) \cos\frac{\pi}{16}$

$= \frac{1}{4}\left[\sin\left(\frac{\pi}{8}\right) \cdot \sin\left(\frac{3\pi}{8}\right)\right]$

$= \frac{1}{4} \cdot \left[\sin\left(\frac{\pi}{8}\right) \sin\left(\frac{4\pi-\pi}{8}\right)\right] = \frac{1}{4}\left[\frac{2\sin\frac{\pi}{8} \cdot \cos\frac{\pi}{8}}{2}\right]$

$= \frac{1}{8}\left[\sin\frac{\pi}{4}\right] = \frac{1}{8\sqrt{2}}$ Ans

Summation of Trigonometric Series :→

Type I Sum of sine / cosine of n angles which are in AP. (i.e. successive argument of sine or cosine have the same difference.).

Q

$$2\sin\left(\frac{\beta}{2}\right) S = 2\sin\frac{\beta}{2}\left[\sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \sin(\alpha+3\beta) + \text{-----} + \sin(\alpha+(n-1)\beta)\right]$$

$$= \underbrace{2\sin\frac{\beta}{2}\sin\alpha} + \underbrace{2\sin\frac{\beta}{2}\sin(\alpha+\beta)} + \underbrace{2\sin\frac{\beta}{2}\cdot\sin(\alpha+2\beta)} + \text{----} + \underbrace{2\sin\frac{\beta}{2}\sin(\alpha+(n-1)\beta)}$$

$$= \left[\cos\left(\alpha-\frac{\beta}{2}\right) - \cos\left(\alpha+\frac{\beta}{2}\right)\right] + \left[\cos\left(\alpha+\beta-\frac{\beta}{2}\right) - \cos\left(\alpha+\beta+\frac{\beta}{2}\right)\right]$$

$$+ \left[\cos\left(\alpha+2\beta-\frac{\beta}{2}\right) - \cos\left(\alpha+2\beta+\frac{\beta}{2}\right)\right] + \text{----}$$

$$+ \left[\cos\left(\alpha+\underline{(n-1)\beta}-\frac{\beta}{2}\right) - \cos\left(\alpha+(n-1)\beta+\frac{\beta}{2}\right)\right]$$

$$\left(2\sin\frac{\beta}{2}\right)S = \cos\left(\alpha-\frac{\beta}{2}\right) - \cancel{\cos\left(\alpha+\frac{\beta}{2}\right)} + \cancel{\cos\left(\alpha+\frac{\beta}{2}\right)} - \cancel{\cos\left(\alpha+\frac{3\beta}{2}\right)}$$

$$+ \cancel{\cos\left(\alpha+\frac{3\beta}{2}\right)} - \cancel{\cos\left(\alpha+\frac{5\beta}{2}\right)} + - - - -$$

$$+ \cancel{\cos\left[\alpha+\left(n-\frac{3}{2}\right)\beta\right]} - \cos\left(\alpha+\left(n-\frac{1}{2}\right)\beta\right)$$

$$\left(2\sin\frac{\beta}{2}\right)\cdot S = \cos\left(\alpha-\frac{\beta}{2}\right) - \cos\left(\alpha+\left(n-\frac{1}{2}\right)\beta\right)$$

$$\left(\cancel{2}\sin\frac{\beta}{2}\right)S = \cancel{2}\sin\left(\alpha+\frac{\beta(n-1)}{2}\right)\sin\left(\frac{n\beta}{2}\right)$$

$$\boxed{S = \frac{\sin\left(\frac{n\beta}{2}\right)\cdot\sin\left(\alpha+\frac{(n-1)\beta}{2}\right)}{\sin\frac{\beta}{2}}}$$

$$\alpha-\frac{\beta}{2}+\alpha+n\beta-\frac{\beta}{2}$$

$$\frac{2\alpha+n\beta-\beta}{2}$$

$$\cancel{\alpha}+n\beta-\cancel{\frac{\beta}{2}}-\cancel{\alpha}+\cancel{\frac{\beta}{2}}$$

$$S = \cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + ---- + \cos(\alpha+(n-1)\beta)$$

$$S = \frac{\sin\left(\frac{n\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)} \cdot \cos\left(\alpha + (n-1)\frac{\beta}{2}\right)$$

Q $S = \sin\theta + \sin 2\theta + \sin 3\theta + \text{-----} + \sin(n-1)\theta + \sin n\theta$

$\alpha = \theta$

$\alpha + \beta = 2\theta \Rightarrow \theta + \beta = 2\theta \Rightarrow \beta = \theta$

$$S = \frac{\sin\left(\frac{n\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)} \cdot \sin\left(\alpha + (n-1)\frac{\beta}{2}\right)$$

$$= \frac{\sin\left(\frac{n\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)} \sin\left(\theta + (n-1)\frac{\theta}{2}\right)$$

$$S = \frac{\sin\left(\frac{n\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)} \cdot \sin\left((n+1)\frac{\theta}{2}\right) \checkmark$$

Given $\theta = 2\pi/n$

$$S = \frac{\boxed{\sin\left(\frac{n}{2}\cdot\frac{2\pi}{n}\right)} \cdot \sin\left((n+1)\frac{2\pi}{2n}\right)}{\sin\left(\frac{2\pi}{2n}\right)}$$

$S = 0$

Trigonometric Ratios and Identities

Lecture - 13

Q

$$S = \cos\theta + \cos 2\theta + \cos 3\theta + \cos 4\theta + \text{------} + \cos n\theta$$

$\alpha \to \theta$

$\beta = \theta$

$$S = \frac{\sin\left(\frac{n\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)} \cdot \cos\left(\alpha + (n-1)\frac{\beta}{2}\right)$$

$$S = \frac{\sin\left(\frac{n\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)} \cdot \cos\left((n+1)\frac{\theta}{2}\right)$$

Q $S = \cos\frac{\pi}{19} + \cos\frac{3\pi}{19} + \cos\frac{5\pi}{19} + ---- + \cos\left(\frac{17\pi}{19}\right)$

$n = 9$; $\alpha = \frac{\pi}{19}$; $\beta = \frac{3\pi}{19} - \frac{\pi}{19} = \frac{2\pi}{19}$

1, 3, 5, ---, 17

$17 = 1 + (n-1)2$

$n = 9$

$$S = \frac{\sin\left(\frac{9}{2}\cdot\frac{2\pi}{19}\right)}{\sin\left(\frac{2\pi}{2(19)}\right)} \cdot \cos\left(\frac{\pi}{19} + \frac{(8)}{2}\frac{2\pi}{(19)}\right)$$

$$= \frac{2\sin\left(\frac{9\pi}{19}\right)\cdot\cos\left(\frac{9\pi}{19}\right)}{2\sin(\pi/19)} = \frac{\sin\left(\frac{18\pi}{19}\right)}{2\sin(\pi/19)} = \frac{\sin\left(\pi - \frac{\pi}{19}\right)}{2\sin(\pi/19)}$$

$$= \frac{\sin(\pi/19)}{2\sin(\pi/19)} = \frac{1}{2}$$

Answer

Q

$$S = \frac{1}{2}\left[2\sin^2\theta + 2\sin^2 2\theta + 2\sin^2 3\theta + \text{------} + 2\sin^2 n\theta\right]$$

$\cos 2\theta = 1 - 2\sin^2\theta$

$2\sin^2\theta = \underline{1-\cos 2\theta}$

$$= \frac{1}{2}\left[(1-\cos 2\theta) + (1-\cos 4\theta) + (1-\cos 6\theta) + (1-\cos 8\theta) + \text{-------} + (1-\cos 2n\theta)\right]$$

$$= \frac{1}{2}\left[\underbrace{(1+1+1+\text{---}+1)}_{n \text{ times}} - (\cos 2\theta + \cos 4\theta + \cos 6\theta + \text{----} + \cos 2n\theta)\right]$$

$\alpha \to 2\theta$; $\beta = 2\theta$

$$= \frac{1}{2}\left[n - \frac{\sin\left(\frac{n\cdot 2\theta}{2}\right)\cdot \cos\left(2\theta + (n-1)\frac{\cancel{2}\theta}{\cancel{2}}\right)}{\sin\left(\frac{2\theta}{2}\right)}\right]$$

$$= \frac{1}{2}\left[n - \frac{\sin(n\theta)\cdot \cos(n\theta + \theta)}{\sin\theta}\right]$$

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T-2 Splitting the sum of series as difference of 2 terms :→

① $S = \operatorname{cosec} x + \operatorname{cosec} 2x + \operatorname{cosec} 4x + \operatorname{cosec} 8x + \text{------} + \operatorname{cosec}(2^n x)$.

$$T_1 = \operatorname{cosec} x = \frac{1}{\sin x}\cdot\frac{\sin\left(\frac{x}{2}\right)}{\sin\left(\frac{x}{2}\right)} = \frac{\sin\left(\frac{2x-x}{2}\right)}{\sin x\cdot\sin\left(\frac{x}{2}\right)} = \frac{\sin\left(x-\frac{x}{2}\right)}{\sin x\cdot\sin\left(\frac{x}{2}\right)}$$

$$= \frac{\sin x\cos\left(\frac{x}{2}\right) - \cos x\sin\left(\frac{x}{2}\right)}{\sin x\cdot\sin\left(\frac{x}{2}\right)} = \cot\left(\frac{x}{2}\right) - \cot x$$

$$\boxed{T_1 = \cot\left(\frac{x}{2}\right) - \cot x}$$

$$T_2 = \operatorname{cosec}(2x) = \frac{1}{\sin(2x)}\cdot\frac{\sin x}{\sin x} = \frac{\sin(2x-x)}{\sin x\cdot\sin 2x} = \frac{\sin 2x\cos x - \cos 2x\sin x}{\sin x\sin 2x}$$

$$\boxed{T_2 = \cot x - \cot 2x}$$

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$$T_3 = \operatorname{cosec} 4x = \frac{1}{\sin 4x} \cdot \frac{\sin 2x}{\sin 2x} = \frac{\sin(4x-2x)}{\sin 4x \cdot \sin 2x}$$

$$T_3 = \frac{\sin 4x \cos 2x - \cos 4x \sin 2x}{\sin 4x \cdot \sin 2x}$$

$$\boxed{T_3 = \cot 2x - \cot 4x}$$

Last term $= \cot 2^{n-1}x - \cot 2^n x$

$$\boxed{S = \cot\left(\frac{x}{2}\right) - \cot 2^n x} = \underline{RHS.}$$

Hence Proved

(2) $S = \tan\left(\frac{x}{2}\right)\sec x + \tan\left(\frac{x}{2^2}\right)\sec\left(\frac{x}{2}\right) + \tan\left(\frac{x}{2^3}\right)\sec\left(\frac{x}{2^2}\right) + \text{----}$

upto n terms.

$T_1 = \tan\frac{x}{2}\cdot\sec x$

$= \frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}\cdot\frac{1}{\cos x} = \frac{\sin\left(x-\frac{x}{2}\right)}{\cos x\cdot\cos\frac{x}{2}} = \frac{\sin x\cos\frac{x}{2} - \cos x\sin\frac{x}{2}}{\cos x\cos\frac{x}{2}}$

$T_1 = \tan x - \tan\left(\frac{x}{2}\right)$

$T_2 = \tan\frac{x}{2^2}\cdot\sec\frac{x}{2} = \frac{\sin\left(\frac{x}{2^2}\right)}{\cos\left(\frac{x}{2^2}\right)\cos\frac{x}{2}} = \frac{\sin\left(\frac{x}{2}-\frac{x}{2^2}\right)}{\cos\left(\frac{x}{2^2}\right)\cdot\cos\left(\frac{x}{2}\right)}$

$= \frac{\sin\frac{x}{2}\cos\left(\frac{x}{2^2}\right) - \cos\frac{x}{2}\sin\left(\frac{x}{2^2}\right)}{\cos\left(\frac{x}{2^2}\right)\cdot\cos\left(\frac{x}{2}\right)} = \left(\tan\frac{x}{2} - \tan\frac{x}{2^2}\right)$

$T_2 = \tan\left(\frac{x}{2}\right) - \tan\left(\frac{x}{2^2}\right)$

$\boxed{S = \tan(x) - \tan\left(\frac{x}{2^n}\right)}$

(3) If $f_n(\theta) = \sum_{r=1}^{n} \dfrac{\sin(4r-1)\theta}{\cos((4r+1)\theta).\cos((4r-3)\theta)}$ then find $\left[f_{11}\left(\frac{\pi}{4}\right)\right]^2$.

$$T_r = \frac{\sin(4r-1)\theta \cdot 2\sin 2\theta}{(2\sin 2\theta)\cos(4r+1)\theta \cdot \cos(4r-3)\theta}$$

$$= \frac{1}{2\sin 2\theta}\left[\frac{\cos(4r-3)\theta - \cos(4r+1)\theta}{\cos(4r+1)\theta \cdot \cos(4r-3)\theta}\right]$$

$$T_r = \frac{1}{2\sin 2\theta}\left[\frac{1}{\cos(4r+1)\theta} - \frac{1}{\cos(4r-3)\theta}\right]$$

$$T_1 = \frac{1}{2\sin 2\theta}\left[\cancel{\frac{1}{\cos 5\theta}} - \frac{1}{\cos\theta}\right]$$

$$T_2 = \frac{1}{2\sin 2\theta}\left[\cancel{\frac{1}{\cos 9\theta}} - \cancel{\frac{1}{\cos 5\theta}}\right]$$

$$T_3 = \frac{1}{2\sin 2\theta}\left[\cancel{\frac{1}{\cos 13\theta}} - \cancel{\frac{1}{\cos 9\theta}}\right] \quad - -$$

$$T_n = \frac{1}{2\sin 2\theta}\left[\frac{1}{\cos(4n+1)\theta} - \cancel{\frac{1}{\cos(4n-3)\theta}}\right]$$

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$$f_n(\theta) = S = \frac{1}{2\sin 2\theta}\left[\frac{1}{\cos(4r+1)\theta} - \frac{1}{\cos\theta}\right]$$

$$f_{11}\left(\frac{\pi}{4}\right) = \frac{1}{2\sin\left(2\left(\frac{\pi}{4}\right)\right)}\left[\frac{1}{\cos\left(4(11)+1\right)\frac{\pi}{4}} - \frac{1}{\cos\left(\frac{\pi}{4}\right)}\right]$$

$$= \frac{1}{2\sin\left(\frac{\pi}{2}\right)}\left[\frac{1}{\cos\left(\frac{45\pi}{4}\right)} - \frac{1}{\cos(\pi/4)}\right]$$

$$= \frac{1}{2(1)}\left[-\sqrt{2} - \sqrt{2}\right]$$

$$f_{11}\left(\frac{\pi}{4}\right) = \frac{1}{2}\left(-2\sqrt{2}\right) = -\sqrt{2}$$

$$\boxed{\left[f_{11}\left(\frac{\pi}{4}\right)\right]^2 = 2}$$

Answer

$$\cos\frac{45\pi}{4}$$
$$= \cos\left(10\pi + \frac{5\pi}{4}\right)$$
$$= \cos\left(\frac{5\pi}{4}\right)$$
$$= \cos\left(\pi + \frac{\pi}{4}\right)$$
$$= -\cos\left(\frac{\pi}{4}\right)$$
$$= -\frac{1}{\sqrt{2}}$$