GEOMETRIC PROGRESSION - I

1. If the p^{th}, q^{th}, r^{th} terms of a G.P. be a, b, c respectively, then prove that $a^{q-r}b^{r-p}c^{p-q}=1$.

2. The fifth term of a G.P. is 81, and the second term is 24; find the series.

3. Find the sum of the series : $3, -4, \frac{16}{3}, \ldots$ to 2n terms.

1. $T_p = [a_1 \cdot r_1^{p-1} = a]$ power $(q-r)$

$T_q = [a_1 \cdot r_1^{q-1} = b]$ power $(r-p)$

$T_r = [a_1 \cdot r_1^{r-1} = c]$ power $(p-q)$

$$a^{q-r} \cdot b^{r-p} \cdot c^{p-q} = a_1^{q-r} \cdot r_1^{(p-1)(q-r)} \; a_1^{r-p} \cdot r_1^{(q-1)(r-p)} \; a_1^{p-q} \cdot r_1^{(r-1)(p-q)}$$

$$= a_1^{q-r+r-p+p-q} \cdot r_1^{(p-1)(q-r)+(q-1)(r-p)+(r-1)(p-q)}$$

$$= a_1^{0} \cdot r_1^{0} = 1$$

$$\therefore \quad \boxed{a^{q-r} \cdot b^{r-p} \cdot c^{p-q} = 1}$$

2. $T_5 = 81 \Rightarrow a \cdot r^4 = 81$

$T_2 = 24 \Rightarrow ar = 24$ $\Big\} \rightarrow$ divide $\dfrac{ar^4}{ar} = \dfrac{\cancel{81}^{\,27}}{\cancel{24}_{\,8}}$

$$r^3 = \frac{27}{8} \Rightarrow \boxed{r = \frac{3}{2}}$$

$$a = 24\left(\frac{2}{3}\right) = \boxed{a = 16}$$

Series $16, 24, 36, 54, \ldots$

3. $a = 3$; $r = -\frac{4}{3}$

$$S_{2n} = \frac{a(1-r^{2n})}{1-r} = \frac{3\left(1-\left(-\frac{4}{3}\right)^{2n}\right)}{1-\left(-\frac{4}{3}\right)} = \frac{9}{7}\left(1-\left(\frac{-4}{3}\right)^{2n}\right)$$

$$= \frac{9}{7}\left(1-\left(\frac{4}{3}\right)^{2n}\right)$$

4. The sum of the first 6 terms of a G.P. is 9 times the sum of the first 3 terms; find the common ratio.

5. The sum of a G.P. whose common ratio is 3 is 728, and the last term is 486; find the first term.

6. In a G.P. the first term is 7, the last term 448, and the sum 889; find the common ratio.

(4) $S_6 = 9 S_3 \Rightarrow \quad \frac{a(r^6-1)}{r-1} = \frac{9a(r^3-1)}{(r-1)}$

$r^6 - 1 = 9r^3 - 9$

$r^6 - 9r^3 + 8 = 0$

$(r-2)(r-1)(r^2+r+1)(r^2+2r+4) = 0$

so $\boxed{r=2}$

(5) $r = 3 \; ; \quad S = 728 \; ; \quad T_n = 486$

$a \cdot r^{n-1} = 486$

$\boxed{a \cdot 3^{n-1} = 486}$

$S = \frac{a \cdot (r^n - 1)}{r-1}$

$728 = \frac{a \cdot (3^n - 1)}{3-1}$

$1456 = a(3^n - 1)$

$1456 = a \cdot (3 \cdot 3^{n-1}) - a$

$1456 = 3a \cdot 3^{n-1} - a$

$1456 = 3(486) - a$

$\boxed{a = 2}$

(6) $a = 7 \; ; \quad a \cdot r^{n-1} = 448 \; ;$

$\frac{a(r^n-1)}{r-1} = 889$

$a \cdot r \cdot r^{n-1} - a = 889(r-1)$

$448r - 7 = 889r - 889$

$882 = 441r$

$\boxed{r=2}$

7. The sum of infinite number of terms of a G.P. is 4 and the sum of their cubes is 192; find the series.

8. The sum of three numbers in G.P. is 38, and their product is 1728; find them.

(7) $\frac{a}{1-r} = 4$ and $\frac{a^3}{1-r^3} = 192$

$a = 4(1-r)$

$\frac{\cancel{4}^3(1-r)^3}{1-r^3} = \cancel{192}^{3}$

$1-r^3-3r(1-r) = 3-3r^3$

$2r^3-3r+3r^2-2=0$

$2r^3+3r^2-3r-2=0$

$(r-1)(r+2)(2r+1)=0$

$\therefore\ r = -2, -\frac{1}{2}, -1$

-2 & $-\frac{1}{2}$ accepted

-1 rejected

Series

at $r=-2$

$a = 4(1+2) = 12$

series

$12, -24, 48,$

$-96, +192, ----$

Series

at $r = -\frac{1}{2}$

$a = 4(1+\frac{1}{2}) = 6$

Series $6, -3, \frac{3}{2}, -\frac{3}{4}, ----$

(8) $\frac{a}{r} + a + ar = 38$

$\frac{a}{r}\cdot a\cdot ar = 1728$

$\boxed{a=12}$

$\frac{12}{r} + 12 + 12r = 38$

$12 + 12r + 12r^2 = 38r$

$12r^2 - 26r + 12 = 0$

$6r^2 - 13r + 6 = 0$

$6r^2 - 9r - 4r + 6 = 0 \Rightarrow r = \frac{2}{3}, \& \frac{3}{2}$

Number 8, 12, 18

9. The continued product of three nubmers in G.P. is 216, and the sum of the products of them in pairs is 156; find the numbers.

10. The sum of three numbers in G.P. is 70; if the two extremes be multipled each by 4, and the mean by 5, the products are in A.P., find the numbers.

(9) $\frac{a}{r} \cdot a \cdot ar = 216 \Rightarrow a = 6$

$\frac{a}{r} \cdot a + a \cdot ar + ar \cdot \frac{a}{r} = 156$

$\frac{36}{r} + 36r + 36 = 156$

$\frac{3}{r} + 3r + 3 = 13$

$3 + 3r^2 + 3r = 13r \Rightarrow 3r^2 - 10r + 3 = 0$

$3r^2 - 9r - r + 3 = 0$

$r = 3$ and $\frac{1}{3}$

Numbers $\frac{6}{3}, 6, 6(3)$

$2, 6, 18$

(10) $\frac{a}{r} + a + ar = 70$

$\frac{4a}{r}, 5a, 4ar$ are in AP

$\therefore 2(5a) = \frac{4a}{r} + 4ar$

$5 = \frac{2}{r} + 2r \Rightarrow 2r^2 - 5r + 2 = 0$

$\boxed{r = 2, \frac{1}{2}}$ $\boxed{a = 20}$

Numbers are $\boxed{10, 20, 40}$ or $\boxed{40, 20, 10}$

11. If the p^{th}, q^{th}, r^{th}, s^{th} terms of an A.P. are in G.P., show that $p - q$, $q - r$, $r - s$ are in G.P.

12. The sum of first three terms of a G.P. is to the sum of the first six terms as 125 : 152. Find the common ratio of the G.P.

(11)

$T_p = a + (p-1)d$

$T_q = a + (q-1)d$

$T_r = a + (r-1)d$

$T_s = a + (s-1)d$

$$\frac{T_q}{T_p} = \frac{T_r}{T_q}$$

$$\frac{a+(q-1)d}{a+(p-1)d} = \frac{a+(r-1)d}{a+(q-1)d}$$

subtrat 1 from both sides

$$\frac{d(q-p)}{a+(p-1)d} = \frac{d(r-q)}{a+(q-1)d}$$

$$\boxed{\frac{q-p}{r-q} = \frac{T_p}{T_q}}$$

$$\frac{T_r}{T_q} = \frac{T_s}{T_r}$$

$$\frac{a+(r-1)d}{a+(q-1)d} = \frac{a+(s-1)d}{a+(r-1)d}$$

subtract 1 from both sides

$$\frac{d(r-q)}{T_q} = \frac{d(s-r)}{T_r}$$

$$\frac{r-q}{s-r} = \frac{T_q}{T_r}$$

$\because$ T_p, T_q, T_r are in G.P.

$\therefore$ $(p-q), (q-r), (r-s)$ are in G.P.

(12)

$$\frac{S_3}{S_6} = \frac{125}{152} \Rightarrow \frac{a(r^3-1)}{(r-1)} \cdot \frac{(r-1)}{a(r^6-1)} = \frac{125}{152}$$

$$\frac{r^3-1}{(r^3+1)(r^3-1)} = \frac{125}{152}$$

$$r^3+1 = \frac{152}{125}$$

$$r^3 = \frac{27}{125} \Rightarrow \boxed{r = \frac{3}{5}}$$

13. Sum the series : (a) $\left(x+\frac{1}{x}\right)^2+\left(x^2+\frac{1}{x^2}\right)^2+\left(x^3+\frac{1}{x^3}\right)^2+\ldots\left(x^n+\frac{1}{x^n}\right)^2$

(b) $1+(1+x)+(1+x+x^2)+(1+x+x^2+x^3)+\ldots$ to n terms.

(a) $\left(x+\frac{1}{x}\right)^2+\left(x^2+\frac{1}{x^2}\right)^2+\left(x^3+\frac{1}{x^3}\right)^2+\text{----}+\left(x^n+\frac{1}{x^n}\right)^2$

$$= x^2 + x^4 + x^6 + x^8 + \text{----} + x^{2n}$$
$$+2+2+2+2+\text{----}+2$$
$$+\frac{1}{x^2}+\frac{1}{x^4}+\frac{1}{x^6}+\frac{1}{x^8}+\text{----}+\frac{1}{x^{2n}}$$

$$= \frac{x^2(x^{2n}-1)}{x^2-1} + 2n + \frac{\frac{1}{x^2}\left(\frac{1}{x^{2n}}-1\right)}{\left(\frac{1}{x^2}-1\right)}$$

$$= \frac{x^2(x^{2n}-1)}{x^2-1} + 2n + \frac{(1-x^{2n})}{x^{2n}(1-x^2)}$$

$$= \frac{(x^{2n}-1)}{(x^2-1)}\left[x^2+\frac{1}{x^{2n}}\right]+2n = \frac{x^{2n}-1}{x^2-1}\left(\frac{x^{2n+2}+1}{x^{2n}}\right)+2n$$

(b) $1+(1+x)+(1+x+x^2)+(1+x+x^2+x^3)+\text{----}$

multiply and divide by $(1-x)$

$$= \frac{1}{1-x}\left[(1-x)+(1-x^2)+(1-x^3)+\text{-----}\right]$$

$$= \frac{1}{1-x}\left[(1+1+1+\text{----})-(x+x^2+x^3+\text{----})\right]$$

$$= \frac{1}{1-x}\left[n-\frac{x(x^n-1)}{x-1}\right] = \frac{n(1-x)-x(x^n-1)}{(1-x)^2}$$

14. Find the sum of n terms of the following series

(a) $\cdot 7 + \cdot 77 + \cdot 777 + \ldots$

(b) $6 + 66 + 666 + \ldots$

(a)

$$S = 0\cdot7 + 0\cdot77 + 0\cdot777 + \text{----}$$

$$S = 7\left(0\cdot1 + 0\cdot11 + 0\cdot111 + \text{----}\right)$$

$$= \frac{7}{9}\left(0\cdot9 + 0\cdot99 + 0\cdot999 + \text{----}\right)$$

$$= \frac{7}{9}\left[(1-0\cdot1) + (1-0\cdot01) + (1-0\cdot001) + (1-0\cdot0001) + \text{------}\right]$$

$$= \frac{7}{9}\left[(1+1+1+\text{---}) - (0\cdot1 + 0\cdot01 + 0\cdot001 + \text{---})\right]$$

$$= \frac{7}{9}\left[n - \frac{(0\cdot1)(1-(0\cdot1)^n)}{(1-0\cdot1)}\right] = \frac{7n}{9} - \frac{7}{81}\left(1-\left(\frac{1}{10}\right)^n\right)$$

(b)

$$S = 6 + 66 + 666 + \text{------}$$

$$S = 6\left[1 + 11 + 111 + 1111 + \text{----}\right]$$

$$= \frac{6}{9}\left[9 + 99 + 999 + 9999 + \text{----}\right]$$

$$= \frac{2}{3}\left[(10-1) + (100-1) + (1000-1) + (10000-1) + \text{---}\right]$$

$$= \frac{2}{3}\left[(10 + 100 + 1000 + \text{----}) - (1+1+1+1+\text{----})\right]$$

$$= \frac{2}{3}\left[\frac{10(10^n-1)}{10-1} - n\right] = \frac{20}{27}(10^n-1) - \frac{2n}{3}$$

15. (a) Find the value of $\cdot 1\dot{2}\dot{3}$ regarding it as geometric series.

(b) Find the value of $\cdot 4\dot{2}\dot{3}$.

(a) $0.1\dot{2}\dot{3} = 0.1232323\ldots$

$= 0.1 + 0.23 + 0.0023 + 0.000023 + \ldots$

$= 0.1 + 23\times10^{-2} + 23\times10^{-4} + 23\times10^{-6} + \ldots$

$= 0.1 + 23\left[\frac{10^{-2}}{1-10^{-2}}\right] = 0.1 + \frac{23}{99}$

$= \frac{1}{10} + \frac{23}{99} = \frac{122}{990}$

(b) $0.4\dot{2}\dot{3} = 0.4232323\ldots$

$= 0.4 + 0.23 + 0.0023 + 0.000023 + \ldots$

$= 0.4 + 23\times10^{-2} + 23\times10^{-4} + 23\times10^{-6} + \ldots$

$= 0.4 + 23\left[\frac{10^{-2}}{1-10^{-2}}\right] = 0.4 + \frac{23}{99}$

$= \frac{4}{10} + \frac{23}{99} = \frac{419}{990}$

GEOMETRIC PROGRESSION - II

1. If p^{th}, q^{th} and r^{th} terms of an A.P. are in G.P., then the common ratio of G.P. is

(A) $\frac{q-r}{p-q}$ (B) $\frac{r-q}{p-q}$ (C) $\frac{q-r}{q-p}$ (D) $\frac{q-p}{q-r}$

① Method 1

$a+(p-1)d,\ a+(q-1)d,\ a+(r-1)d \longrightarrow$ G.P.

$[a+(q-1)d]^2 = (a+(p-1)d)(a+(r-1)d)$

$\cancel{a^2}+(q-1)^2d^2+2a(q-1)d = \cancel{a^2}+a(r-1)d + a(p-1)d + d^2(p-1)(r-1)$

$d^2[(q-1)^2-(p-1)(r-1)] + ad[2(q-1)-r+1-p+1]=0$

$d^2[q^2-2q+1-pr+r+p-1] + ad[2q-r-p]=0$

$d[q^2-2q-pr+r+p] + a[2q-r-p]=0$

$d = \left[\frac{(p+r-2q)a}{q^2-2q-pr+p+r}\right]$

$$\frac{d}{a} = \frac{(p+r-2q)}{q^2-2q-rp+p+r}$$

$$\frac{d}{a} = \frac{(p+r-2q)}{q^2-2q-rp+p+r}$$

Common ratio $= \dfrac{a+(q-1)d}{a+(p-1)d}$

$$= \frac{1+(q-1)\frac{d}{a}}{1+(p-1)\frac{d}{a}}$$ (divided by a in numerator & denominator)

$$= \frac{1+(q-1)\left(\frac{p+r-2q}{q^2-2q-pr+p+r}\right)}{1+(p-1)\left(\frac{p+r-2q}{q^2-2q-pr+p+r}\right)}$$

$$= \frac{(q^2-\cancel{2q}-pr+\cancel{p}+\cancel{r})+(pq+qr-2q^2-\cancel{p}-\cancel{r}+\cancel{2q})}{(q^2-\cancel{2q}-pr+\cancel{p}+\cancel{r})+(p^2+pr-2qp-\cancel{p}-\cancel{r}+\cancel{2q})}$$

$$= \frac{-q^2-pr+pq+qr}{q^2+p^2-2pq} = \frac{q(-q+p)-r(p-q)}{(p-q)^2}$$

$$= \frac{(q-r)(p-q)}{(p-q)^2} = \frac{q-r}{p-q}$$ Ans

Method 2

$$T_p = a_1 + (p-1)d = a$$

$$T_q = a_1 + (q-1)d = ar$$

$$T_r = a_1 + (r-1)d = ar^2$$

Subtract $\quad d(p-q) = a - ar$

$$\boxed{d = \frac{a-ar}{p-q}}$$

Subtract $\quad d(q-r) = ar - ar^2$

$$\boxed{d = \frac{(a-ar)r}{(q-r)}}$$

$$d \quad \frac{a-ar}{p-q} = \frac{(a-ar)r}{(q-r)}$$

$$\boxed{r = \frac{q-r}{p-q}}$$

2. If the roots of cubic equation $ax^3 + bx^2 + cx + d = 0$ are in G.P., then

(A) $c^3a = b^3d$ (B) $ca^3 = bd^3$ (C) $a^3b = c^3d$ (D) $ab^3 = cd^3$

3. If $\frac{p+q.5^x}{p-q.5^x} = \frac{q+r.5^x}{q-r.5^x} = \frac{r+s.5^x}{r-s.5^x}$ then p, q, r, s are in

A) A.P. B) G.P. C) H.P. D) none of these

(2) let roots are a, ar, ar^2

$a + ar + ar^2 = \frac{-b}{a} \Rightarrow a(1+r+r^2) = -\frac{b}{a}$

$a \cdot ar + ar \cdot ar^2 + a \cdot ar^2 = \frac{c}{a} \Rightarrow a^2r(1+r+r^2) = \frac{c}{a}$

$ar\left(\frac{-b}{a}\right) = \frac{c}{a}$

$ar = -\frac{c}{b}$

$a \cdot ar \cdot ar^2 = -\frac{d}{a}$

$a^3r^3 = -\frac{d}{a} \Rightarrow \left(-\frac{c}{b}\right)^3 = -\frac{d}{a}$

$\boxed{ac^3 = db^3}$

(3) $\frac{p+q.5^x}{p-q.5^x} = \frac{q+r.5^x}{q-r.5^x} = \frac{r+s.5^x}{r-s.5^x}$

apply componendo & dividendo

$\frac{2p}{2q\,5^x} = \frac{2q}{2r\,5^x} = \frac{2r}{2s\,5^x}$

$\frac{p}{q \cdot \cancel{5^x}} = \frac{q}{r \cdot \cancel{5^x}} = \frac{r}{s\,\cancel{5^x}} \Rightarrow \frac{p}{q} = \frac{q}{r} = \frac{r}{s}$

$\therefore$ p, q, r, s are in G.P.

4. If the sum of the series $\sum_{n=0}^{\infty} r^n$, $|r| < 1$, is S, then sum of the series $\sum_{n=0}^{\infty} r^{2n}$ is

(A) S^2 (B) $\frac{2S}{S^2-1}$ (C) $\frac{S^2}{2S+1}$ (D) $\frac{S^2}{2S-1}$

$$\sum_{n=0}^{\infty} r^n = r^0 + r^1 + r^2 + r^3 + \cdots$$

$$= 1 + r + r^2 + r^3 + \cdots$$

$$S = \frac{1}{1-r}$$

$$1 - r = \frac{1}{S} \Rightarrow r = 1 - \frac{1}{S}$$

$$\sum_{r=0}^{\infty} r^{2n} = r^0 + r^2 + r^4 + r^6 + \cdots$$

$$= \frac{1}{1-r^2} = \frac{1}{1-\left(1-\frac{1}{S}\right)^2}$$

$$= \frac{1}{1-\left(1+\frac{1}{S^2}-\frac{2}{S}\right)}$$

$$= \frac{S^2}{2S-1}$$

5. If S denotes the sum of infinity and S_n the sum of n terms of the series $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+$ ...such that $S - S_n < \frac{1}{1000}$, then the least value of n is

(A) 11 (B) 9 (C) 10 (D) 8

(5) Series $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+ - - - -$

$$S_\infty = \frac{1}{1-\frac{1}{2}} = 2$$

$$S_n = \frac{1\left(1-\left(\frac{1}{2}\right)^n\right)}{\left(1-\frac{1}{2}\right)} = 2\left(1-\left(\frac{1}{2}\right)^n\right)$$

$$S_\infty - S_n < \frac{1}{1000}$$

$$2 - 2\left(1-\frac{1}{2^n}\right) < \frac{1}{1000}$$

$$1 - 1 + \frac{1}{2^n} < \frac{1}{2000}$$

$$\frac{1}{2^n} < \frac{1}{2000}$$

$$2^n > 2000$$

$$n > 10....$$

least value of n $\boxed{n = 11}$

6. If a, b, c are in G.P. then the equations $ax^2 + 2bx + c = 0$ and $dx^2 + 2ex + f = 0$ have a common root if $\frac{d}{a}, \frac{e}{b}, \frac{f}{c}$ are in

(A) A.P. (B) G.P. (C) H.P (D) None of these

7. A certain number is inserted between the number 3 and the unknown number so that the three numbers form an A.P. If the middle term is diminished by 6 then the number are in G.P. The unknown number can be

(A) 3 (B) 15 (C) 18 (D) 27

(6) $ax^2 + 2bx + c = 0 \Rightarrow ax^2 + 2\sqrt{ac}\,x + c = 0$

$(\sqrt{a}x)^2 + 2\sqrt{a}\cdot\sqrt{c}\,x + (\sqrt{c})^2 = 0$

$(\sqrt{a}x + \sqrt{c})^2 = 0$

$\boxed{x = -\frac{\sqrt{c}}{\sqrt{a}}}$

$dx^2 + 2ex + f = 0$

$d\left(-\frac{\sqrt{c}}{\sqrt{a}}\right)^2 + 2e\left(-\frac{\sqrt{c}}{\sqrt{a}}\right) + f = 0$

$d\frac{c}{a} - 2e\sqrt{\frac{c}{a}} + f = 0$

$dc - 2e\sqrt{ac} + af = 0$

$dc + af = 2e\sqrt{ac}$

$dc + af = 2eb$

divide by b^2 to both sides

$\frac{dc}{b^2} + \frac{af}{b^2} = \frac{2e}{b}$

$\because b^2 = ac$

$\therefore \frac{dc}{ac} + \frac{af}{ac} = \frac{2e}{b}$

$\frac{d}{a} + \frac{f}{c} = \frac{2e}{b}$

$\therefore$ $\boxed{\frac{d}{a}, \frac{e}{b}, \frac{f}{c} \text{ are in AP}}$

(7) Let $a, b, 3$ are in AP. $\Rightarrow b = \frac{a+3}{2}$

$a, b-6, 3$ are in GP

$b - 6 = \sqrt{3a} \Rightarrow \frac{a+3}{2} - 6 = \sqrt{3a} \Rightarrow a - 9 = 2\sqrt{3a}$

$\Rightarrow a^2 + 81 - 18a - 12a = 0$

$a^2 - 30a + 81 = 0$

$(a-27)(a-3) = 0$

$a = 3, 27$

$\therefore a = 27;\ b = 9;$

8. Let the numbers a_1, a_2, a_3 ..... a_n constitute a geometric progression. If $S = a_1 + a_2 + ... + a_n$, $T = \frac{1}{a_1} + \frac{1}{a_2} + ...$ $+ \frac{1}{a_n}$ and $P = a_1 a_2 a_3 ..... a_n$ then P^2 is equal to

(A) $\left(\frac{S}{T}\right)^n$ ✓ (B) $\left(\frac{T}{S}\right)^n$ (C) $\left(\frac{2S}{T}\right)^n$ (D) $\left(\frac{2T}{S}\right)^n$

8) $S = a + ar + ar^2 + ar^3 + ----- + ar^{n-1}$

$S = \frac{a(1-r^n)}{1-r} \Rightarrow \frac{1-r^n}{1-r} = \frac{S}{a}$

$T = \frac{1}{a} + \frac{1}{ar} + \frac{1}{ar^2} + ---- + \frac{1}{ar^{n-1}}$

$T = \frac{\frac{1}{a}\left(1-\left(\frac{1}{r}\right)^n\right)}{\left(1-\frac{1}{r}\right)} = \frac{1}{a}\frac{(r^n-1)\,r}{r^n\,(r-1)}$

$T = \frac{1}{a \cdot r^{n-1}} \frac{(1-r^n)}{(1-r)} \Rightarrow T = \frac{1}{a \cdot r^{n-1}} \cdot \frac{S}{a}$

$\frac{T}{S} = \frac{1}{a^2 r^{n-1}} \Rightarrow \boxed{\frac{S}{T} = a^2 r^{n-1}}$

$P^2 = [a \cdot ar \cdot ar^2 --- ar^{n-1}]^2$

$= [a^n \cdot r^{1+2+3+----+(n-1)}]^2$

$= [a^n \cdot r^{\frac{n(n-1)}{2}}]^2 = [a \cdot r^{\frac{n-1}{2}}]^{2n}$

$= \left[\frac{S}{T}\right]^n$

9. Let α, β be the roots of $x^2 - x + p = 0$ and γ, δ be the roots of $x^2 - 4x + q = 0$. If α, β, γ, δ are in G.P., then the integral values of p and q respectively, are

(A) – 2, – 32 (B) – 2, 3 (C) – 6, 3 (D) – 6, – 32

9) Let $\alpha = a$, $\beta = ar$; $\gamma = ar^2$, $\delta = ar^3$

from $x^2 - x + p = 0$ $\quad a + ar = 1$

and from $x^2 - 4x + q = 0$ $\quad ar^2 + ar^3 = 4$

$r^2(a + ar) = 4$

$r^2(1) = 4$

$r = \pm 2$

$\therefore \quad a + a(2) = 1 \Rightarrow a = 1/3$

or $\quad a + a(-2) = 1 \Rightarrow a = -1$

$p = a \cdot ar = \frac{1}{3} \cdot \frac{1}{3}(2) = \frac{2}{9}$

or $\quad p = (-1)(-1)(-2) = -2$

$q = ar^2 \cdot ar^3 = (-1)^2(-2)^5 = -32$

10. a, b, c, d are in increasing G.P. If the AM between a and b is 6 and the AM between c and d is 54., then the AM of a and d is

(A) 15 (B) 48 (C) 44 (D) 42

11. Insert 3 geometric means between $\frac{9}{4}$ and $\frac{4}{9}$.

(10) $\frac{a+b}{2} = 6 \Rightarrow a+b = 12 \Rightarrow a + ar = 12$

$\frac{c+d}{2} = 54 \Rightarrow c+d = 108 \Rightarrow ar^2 + ar^3 = 108$

$\frac{a(1+r)}{ar^2(1+r)} = \frac{12}{108} \Rightarrow r^2 = 9 \Rightarrow r = \pm 3$

at $r = 3$, $a = \frac{12}{4}$

AM of a & d $= \frac{a+d}{2} = \frac{a+ar^3}{2}$

$= \frac{3 + 3(3^3)}{2} = 42$

(11) $\frac{9}{4}$ & $\frac{4}{9}$ $G_1, G_2\ G_3$

$\frac{9}{4} \cdot r^4 = \frac{4}{9} \Rightarrow r^4 = \frac{16}{81} \Rightarrow r = \pm \frac{2}{3}$

$G_1 = \frac{9}{4}\left(\frac{2}{3}\right) = \frac{3}{2}$

$G_2 = \frac{9}{4}\left(\frac{2}{3}\right)^2 = 1$

$G_3 = \frac{9}{4}\left(\frac{2}{3}\right)^3 = \frac{2}{3}$

$G_1 = \frac{9}{4}\left(-\frac{2}{3}\right) = -\frac{3}{2}$

$G_2 = \frac{9}{4}\left(-\frac{2}{3}\right)^2 = 1$

$G_3 = \frac{9}{4}\left(-\frac{2}{3}\right)^3 = \frac{2}{3}$

12. If the arithmetic mean between a and b is twice as great as the geometric mean, show that $a : b = 2+\sqrt{3} : 2-\sqrt{3}$.

13. If a, b, c, d be in G.P. Prove that

(a) $(a^2+ac+c^2)(b^2+bd+d^2) = (ab+bc+cd)^2$.

(b) $(a^2+b^2+c^2)(b^2+c^2+d^2) = (ab+bc+cd)^2$.

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$AM = 2GM$

$\frac{a+b}{2} = 2\sqrt{ab} \Rightarrow \frac{a+b}{\sqrt{ab}} = 4 \Rightarrow \frac{\sqrt{a}}{\sqrt{b}} + \frac{\sqrt{b}}{\sqrt{a}} = 4$

$\frac{a}{b} + \frac{b}{a} + 2 = 16$

Let $\frac{a}{b} = x$

$x + \frac{1}{x} - 14 = 0$

$x^2 - 14x + 1 = 0$

$x = \frac{14 \pm \sqrt{192}}{2}$

$x = 7 \pm 4\sqrt{3}$

$\frac{a}{b} = (7+4\sqrt{3})$

$= (2+\sqrt{3})^2$

$\boxed{\frac{a}{b} = \frac{(2+\sqrt{3})}{(2-\sqrt{3})}}$

$\frac{a}{b} = 7-4\sqrt{3}$

$\boxed{\frac{a}{b} = \frac{2-\sqrt{3}}{2+\sqrt{3}}}$

13 $a \rightarrow a$; $b = ar$; $c \rightarrow ar^2$, $d = ar^3$

(a) $(a^2+ac+c^2)(b^2+bd+d^2)$

$\boxed{= (a^2+b^2+c^2)(b^2+c^2+d^2)}$ $\because ac = b^2$ & $bd = c^2$

$= (a^2+a^2r^2+a^2r^4)(a^2r^2+a^2r^4+a^2r^6)$

$= a^2 \cdot a^2r^2(1+r^2+r^4)(1+r^2+r^4)$

$= [(a^2r)(1+r^2+r^4)]^2 = (a^2r + a^2r^3 + a^2r^5)^2$

$= (a \cdot ar + ar \cdot ar^2 + ar^2 \cdot ar^3)^2$

$= (ab+bc+cd)^2$

14. If a, b, c, d be in G.P. $(a \neq b \neq c \neq d)$. Prove that

(a) $(a-d)^2 = (b-c)^2 + (c-a)^2 + (d-b)^2$

(b) $a^2 - b^2, b^2 - c^2, c^2 - d^2$ are in G.P.

(a)

$a \rightarrow a$
$b \rightarrow ar$
$c \rightarrow ar^2$
$d \rightarrow ar^3$

$b^2 = ac$
$c^2 = bd$
$bc = ad$

$(b-c)^2 + (c-a)^2 + (d-b)^2$

$= b^2 + c^2 - 2bc + c^2 + a^2 - 2ac + d^2 + b^2 - 2bd$

$= \cancel{ac} + c^2 - 2ad + bd + a^2 - \cancel{2ac} + d^2 + \cancel{b^2} - 2c^2$

$= a^2 + \cancel{c^2} - 2ad + \cancel{bd} - \cancel{2c^2} + d^2 = (a-d)^2$

(b) $\because$ a, b, c, d are in GP

$a \rightarrow a, \quad b \rightarrow ar; \quad c \rightarrow ar^2, \quad d = ar^3$

series $\Rightarrow a^2 - b^2, \; b^2 - c^2, \; c^2 - d^2$

$\Rightarrow a^2 - a^2r^2, \; a^2r^2 - a^2r^4, \; a^2r^4 - a^2r^6$

$\Rightarrow a^2(1-r^2), \; a^2r^2(1-r^2), \; a^2r^4(1-r^2)$

divide by $1-r^2$

$\Rightarrow a^2, \; a^2r^2, \; a^2r^4$ It forms a GP.

Hence Proved

15. (a) If one geometric mean G and two arithmetic means p and q be inserted between any two given numbers, then show that $G^2 = (2p - q)(2q - p)$.

(b) If one arithmetic mean A and two geometric means p and q be inserted between any two given numbers, then show that $p^3 + q^3 = 2\ Apq$.

(a) Let a & b are the terms

$G = \sqrt{ab}$

$G^2 = ab$

a, p, q, b

$p = a + \frac{b-a}{3} = \frac{b}{3} + \frac{2a}{3}$

$q = a + 2\left(\frac{b-a}{3}\right) = \frac{2b}{3} + \frac{a}{3}$

RHS $= (2p - q)(2q - p)$

$= \left(2\left(\frac{b}{3} + \frac{2a}{3}\right) - \frac{2b}{3} - \frac{a}{3}\right)\left(2\left(\frac{2b}{3} + \frac{a}{3}\right) - \frac{b}{3} - \frac{2a}{3}\right)$

$= (a)(b) = G^2 =$ LHS.

Hence Proved

(b) $A = \frac{a+b}{2}$

a, p, q, b

$a \cdot r^3 = b \Rightarrow r^3 = \frac{b}{a}$

$r = \left(\frac{b}{a}\right)^{1/3}$

$p = a \cdot \left(\frac{b}{a}\right)^{1/3} = b^{1/3} \cdot a^{2/3}$

$q = a\left(\frac{b}{a}\right)^{2/3} = b^{2/3} \cdot a^{1/3}$

$p \cdot q = ab$

$p^3 + q^3$

$= ba^2 + ab^2$

$= ab(a+b)$

$= ab(2A)$

$= 2Aab$

$= 2Apq$

16. Find the $\prod_{i=1}^{3} Gi$ (Geometric means) inserted between 'a' and 'b' which satisfy the equation $(G_1+2)^4 + (G_2-4)^2 + |G_3 + 8| = 0$. Also find ab =

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$G_1 \cdot G_2 \cdot G_3 = (ab)^{3/2}$

$G_1 = -2$; $G_2 = 4$; $G_3 = -8$

$\therefore G_1 \cdot G_2 \cdot G_3 = (ab)^{3/2}$

$(-2)(4)(-8) = (ab)^{3/2}$

$\boxed{ab = 16}$