Trigonometric Ratios and Identities

Lecture - 9

Trigonometric Identities in a triangle (conditional Identities):-

If A, B, C are angles of a triangle $A+B+C = \pi$

(i) $\sin(\underbrace{A+B}) = \sin(\pi - C) = \sin C$

$$\sin(A+B) = \sin C$$

$A+B+C = \pi$

$2A+2B+2C = 2\pi$

$2A+2B = 2\pi - 2C$

(ii) $\cos(A+B) = \cos(\pi - C) = -\cos C$

$$\cos(A+B) = -\cos C$$

(iii) $\tan(A+B) = \tan(\pi - C) = -\tan C$

$$\tan(A+B) = -\tan C$$

(iv) $\sin(2A+2B) = \sin(2\pi - 2C) = -\sin 2C$

$$\sin(2A+2B) = -\sin 2C$$

(v) $\cos(2A+2B) = \cos(2\pi - 2C) = \cos 2C$

$$\cos(2A+2B) = \cos 2C$$

(vi) $\tan(2A+2B) = \tan(2\pi - 2C) = -\tan 2C$

$$\tan(2A+2B) = -\tan 2C$$

(vii) $\sin\left(\frac{A+B}{2}\right) = \sin\left(\frac{\pi - C}{2}\right) = \sin\left(\frac{\pi}{2} - \frac{C}{2}\right) = \cos\frac{C}{2}$

$$\sin\left(\frac{A+B}{2}\right) = \cos\frac{C}{2}$$

(viii) $\cos\left(\frac{A+B}{2}\right) = \cos\left(\frac{\pi - C}{2}\right) = \sin\frac{C}{2}$

$$\cos\frac{A+B}{2} = \sin\frac{C}{2}$$

(ix) $\tan\left(\frac{A+B}{2}\right) = \tan\left(\frac{\pi - C}{2}\right) = \cot\frac{C}{2}$

$$\tan\left(\frac{A+B}{2}\right) = \cot\frac{C}{2}$$

(x) $\underbrace{\sin 2A + \sin 2B} + \sin 2C = 4 \sin A \sin B \sin C$

$A + B + C = \pi$

$C = \pi - (A+B)$

$\underline{LHS} = 2 \sin\left(\frac{2A+2B}{2}\right) \cos\left(\frac{2A-2B}{2}\right) + \sin(2C)$

$= 2 \sin(A+B) \cos(A-B) + \sin 2C$

$= 2 \sin(\pi - C) \cos(A-B) + \sin 2C$

$= 2 \sin C \cos(A-B) + 2 \sin C \cos C$

$= 2 \sin C [\cos(A-B) + \cos C]$

$= 2 \sin C [\cos(A-B) + \cos(\pi - (A+B))]$

$= 2 \sin C [\underline{\cos(A-B)} - \cos(\underline{A+B})]$

$= 2 \sin C [2 \sin A \sin B] = 4 \sin A \sin B \sin C = RHS.$

Hence Proved

(xi) $\cos 2A + \cos 2B + \cos 2C = -1 - 4\cos A \cos B \cos C$ (HW)

$LHS = 2\cos(A+B)\cos(A-B) + \cos 2C$

$= 2\cos(\pi - C)\cos(A-B) + 2\cos^2 C - 1$

$= -2\cos C \cos(A-B) + 2\cos^2 C - 1$

$= 2\cos C\,[\cos C - \cos(A-B)] - 1$

$= 2\cos C\,[-\cos(A+B) - \cos(A-B)] - 1$

$= -2\cos C\,[2\cos A \cos B] - 1$

$= -4\cos A \cos B \cos C - 1$

$=$ RHS.

Hence Proved

$C = \pi - (A+B)$

$\cos C = \cos(\pi - (A+B))$

$= -\cos(A+B)$

(xii) $\sum \tan A = \prod \tan A$ $\qquad$ $A+B+C=\pi$

$\tan A + \tan B + \tan C = \tan A \tan B \tan C$

$A+B+C = \pi$

$A+B = \pi - C$

$\tan(A+B) = \tan(\pi - C)$

$\dfrac{\tan A + \tan B}{1-\tan A \tan B} = -\tan C$

$\tan A + \tan B = -\tan C + \tan A \tan B \tan C$

$\tan A + \tan B + \tan C = \tan A \tan B \tan C$

$\sum \tan A = \prod \tan A$

(xiii) $\sum \cot A \cot B = 1 \Rightarrow \cot A \cot B + \cot B \cot C + \cot C \cot A = 1$

(xiv) $\underbrace{\sum \tan\frac{A}{2}\tan\frac{B}{2}}$ $= 1$

(xv) $\sum \cot\left(\frac{A}{2}\right) = \prod \cot\left(\frac{A}{2}\right)$

$A+B+C = \pi$

$\frac{A+B+C}{2} = \frac{\pi}{2}$

$\frac{A+B}{2} = \left(\frac{\pi}{2} - \frac{C}{2}\right)$

$\tan\left(\frac{A+B}{2}\right) = \tan\left(\frac{\pi}{2} - \frac{C}{2}\right)$

$\frac{\tan\left(\frac{A}{2}\right) + \tan\frac{B}{2}}{1 - \tan\frac{A}{2}\tan\frac{B}{2}} = \frac{1}{\tan\frac{C}{2}}$

① $\sin(B+C-A) + \sin(C+A-B) + \sin(A+B-C) = 4\sin A \sin B \sin C$

where $A+B+C = \pi$

$B+C = \pi - A$

LHS $= \sin(\pi - A - A) + \sin(\pi - B - B) + \sin(\pi - C - C)$

$= \sin(\pi - 2A) + \sin(\pi - 2B) + \sin(\pi - 2C)$

$= \sin 2A + \sin 2B + \sin 2C$

$= 4 \sin A \sin B \sin C$

(2) $\sin 2A + \sin 2B - \sin 2C = 4\cos A \cos B \sin C \quad (A+B+C=\pi)$

LHS :

$$2\sin(A+B)\cos(A-B) - \sin 2C$$

$$= 2\sin(\pi - C)\cos(A-B) - \sin 2C$$

$$= 2\sin C\cos(A-B) - 2\sin C\cos C$$

$$= 2\sin C[\cos(A-B) - \cos C]$$

$$= 2\sin C[\cos(A-B) - \cos(\pi-(A+B))]$$

$$= 2\sin C[\cos(A-B) + \cos(A+B)]$$

$$= 2\sin C[2\cos A\cos B]$$

$$= 4\sin C\cos A\cos B$$

$$= \text{RHS.}$$

$A+B+C=\pi$

$C = \pi-(A+B)$

Hence Proved

(3) $\cos 2A + \cos 2B - \cos 2C = 1 - 4 \sin A \sin B \cos C \quad (A+B+C) = \pi$

④ If $A+B+C=\pi$, Prove that $\sin^2 A + \sin^2 B - \sin^2 C = 2\sin A \sin B \cos C$

$\underline{\text{LHS}} = \sin^2 A + \sin(B+C)\sin(B-C)$

$= \sin^2 A + \underline{\sin(\pi - A)}\sin(B-C)$

$= \sin A\left[\sin A + \sin(B-C)\right]$

$= \sin A\left[\sin(\pi-(B+C)) + \sin(B-C)\right]$

$= \sin A\left[\sin(B+C) + \sin(B-C)\right]$

$= \sin A \; 2\sin B\cos C$

$= 2\sin A \sin B \cos C$

$\sin^2 A - \sin^2 B$

$= \sin(A+B)\cdot\sin(A-B)$

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5 $\sin^2\left(\frac{A}{2}\right) + \sin^2\frac{B}{2} - \sin^2\frac{C}{2} = 1 - 2\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}$ $(A+B+C=\pi)$

6 If $A+B+C=2S$, prove that

$$\sin(S-A)\sin(S-B)+\sin S\left(\sin(S-C)\right)=\sin A\sin B.$$

(7) Solve $\dfrac{\sin 50^\circ + \sin 100^\circ + \sin 210^\circ}{\sin 25^\circ \sin 50^\circ \sin 105^\circ}$

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8 find $\dfrac{\sin 2A + \sin 2B + \sin 2C}{\sin A + \sin B + \sin C}$ $(A+B+C=\pi)$

Inequalities in a triangle

Q In a ΔABC, show that $\cot^2 A + \cot^2 B + \cot^2 C \geq 1$

AM $\geq$ GM

$$\frac{\cot^2 A + \cot^2 B}{2} \geq \sqrt{\cot^2 A \cot^2 B}$$

add

$$\frac{\cot^2 A + \cot^2 B}{2} \geq \cot A \cot B \checkmark$$

$$\frac{\cot^2 B + \cot^2 C}{2} \geq \cot B \cot C \checkmark$$

$$\frac{\cot^2 C + \cot^2 A}{2} \geq \cot C \cot A \checkmark$$

$$\cot^2 A + \cot^2 B + \cot^2 C \geq \cot A \cot B + \cot B \cot C + \cot C \cot A$$

$$\boxed{\cot^2 A + \cot^2 B + \cot^2 C \geq 1}$$

Hence Proved

② In ΔABC, P. T. $\cos A \cos B \cos C \leq \frac{1}{8}$

M-I LHS

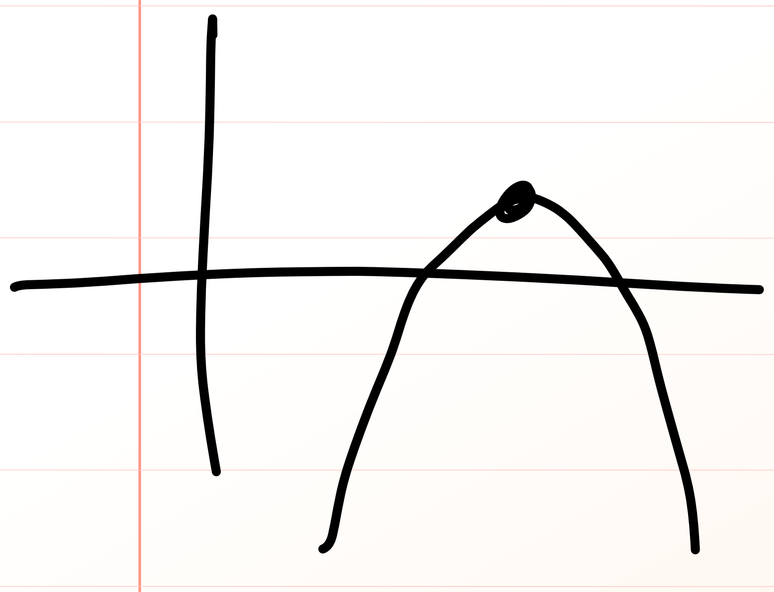
$$y = \cos A \cos B \cos C$$

$$= \frac{\cos A}{2}\left(2\cos B \cos C\right)$$

$$= \frac{\cos A}{2}\left[\cos(B+C) + \cos(B-C)\right]$$

$$= \frac{\cos A}{2}\left[\cos(\pi - A) + \cos(B-C)\right]$$

$$= \frac{\cos A}{2}\left[-\cos A + \cos(B-C)\right]$$



$$y = \frac{\cos A}{2}\left[\cos(B-C) - \cos A\right]$$

$$y \leq \frac{\cos A}{2}\left[1 - \cos A\right] \Rightarrow y \leq \underbrace{\left(\frac{\cos A}{2} - \frac{\cos^2 A}{2}\right)}$$

$$y \le \frac{\cos A}{2} - \frac{\cos^2 A}{2}$$

$$-\frac{t^2}{2} + \frac{t}{2}$$

vertex

$$V_x = \frac{-b}{2a} = \frac{-1/2}{2\left(-\frac{1}{2}\right)} = \frac{1}{2}$$

$$y \le \frac{1}{8}$$

$$V_y = -\frac{1}{8} + \frac{1}{4} = \frac{1}{8}$$

$$\cos A \cos B \cos C \le \frac{1}{8}$$

M-II

$$\underbrace{2\cos A \cos B}\ \frac{\cos C}{2} = y$$

$$\frac{1}{2}\left[\cos(A+B) + \cos(A-B)\right]\cos C = y$$

$$\left[\cos(\pi - C) + \cos(A-B)\right]\cos C = 2y$$

$$-\cos^2 C + \cos(A-B)\cos C = 2y$$

$$-\cos^2 C + \cos(A-B)\cos C = 2y$$

$$-\cos^2 C + \cos(A-B)\cos C - 2y = 0$$

$$\cos^2 C - \underline{\cos(A-B)}\cos C + 2y = 0$$

$$D \geq 0$$

$$[\cos(A-B)]^2 - 4(2y) \geq 0$$

$$\cos^2(A-B) \geq 8y$$

$$\frac{\cos^2(A-B)}{8} \geq y$$

$$y \leq \frac{\cos^2(A-B)}{8}$$

$$y \leq \frac{1}{8} \Rightarrow \boxed{\cos A \cos B \cos C \leq \frac{1}{8}}$$

Hence Proved