# Quadratic Equation

## Lecture -1

# Quadratic Equations: →

## DEFINITION OF POLYNOMIAL :

An algebraic expression of the form $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots\ldots\ldots\ldots + a_{n-1}x + a_n$ is called polynomial of degree 'n' where $a_i \in R$, $i = 1, 2, \ldots, n$ & $a_0 \neq 0$

If n = 1 is taken then

$$f(x) = a_0x + a_1,\ a_0 \neq 0 \qquad \text{(Linear expression)}$$

**Note :** $f(x) = 0$ is a polynomial whose degree is not defined.

If n = 2 is taken then

$$f(x) = a_0x^2 + a_1x + a_2,\ a_0 \neq 0 \text{ (Quadratic expression)}$$

Hence, algebraic expression of degree two in one variable is called quadratic expression, which is generally represented as -

$$f(x) = ax^2 + bx + c;\ a, b, c \in R,\ a \neq 0$$

where, a = leading coefficient, b = middle term coefficient, c = absolute term

## QUADRATIC EQUATION :

Equation of the form $ax^2 + bx + c = 0$, $a, b, c \in R$ & $a \neq 0$ is known as quadratic equation.

## ROOTS OF QUADRATIC EQUATION :

Solving a quadratic equation would mean finding the value or values of x for which $ax^2 + bx + c$ vanishes and these values of x are also called the roots of the quadratic equation or zeros or solutions of the corresponding quadratic polynomial. Two methods of solving a quadratic equation are

(a) Graphical (not very useful) (b) Algebraic

$$f(x) = \underbrace{x^2 - 6x + 5}_{(x-5)(x-1)} = 0$$

$x = 5, 1$

$x = 5$ OR $1$

$\uparrow$ $x_1$ $\uparrow$ $x_2$

## Algebraic method :

$$ax^2 + bx + c = 0 \quad a \neq 0 \quad a, b, c \in R$$

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

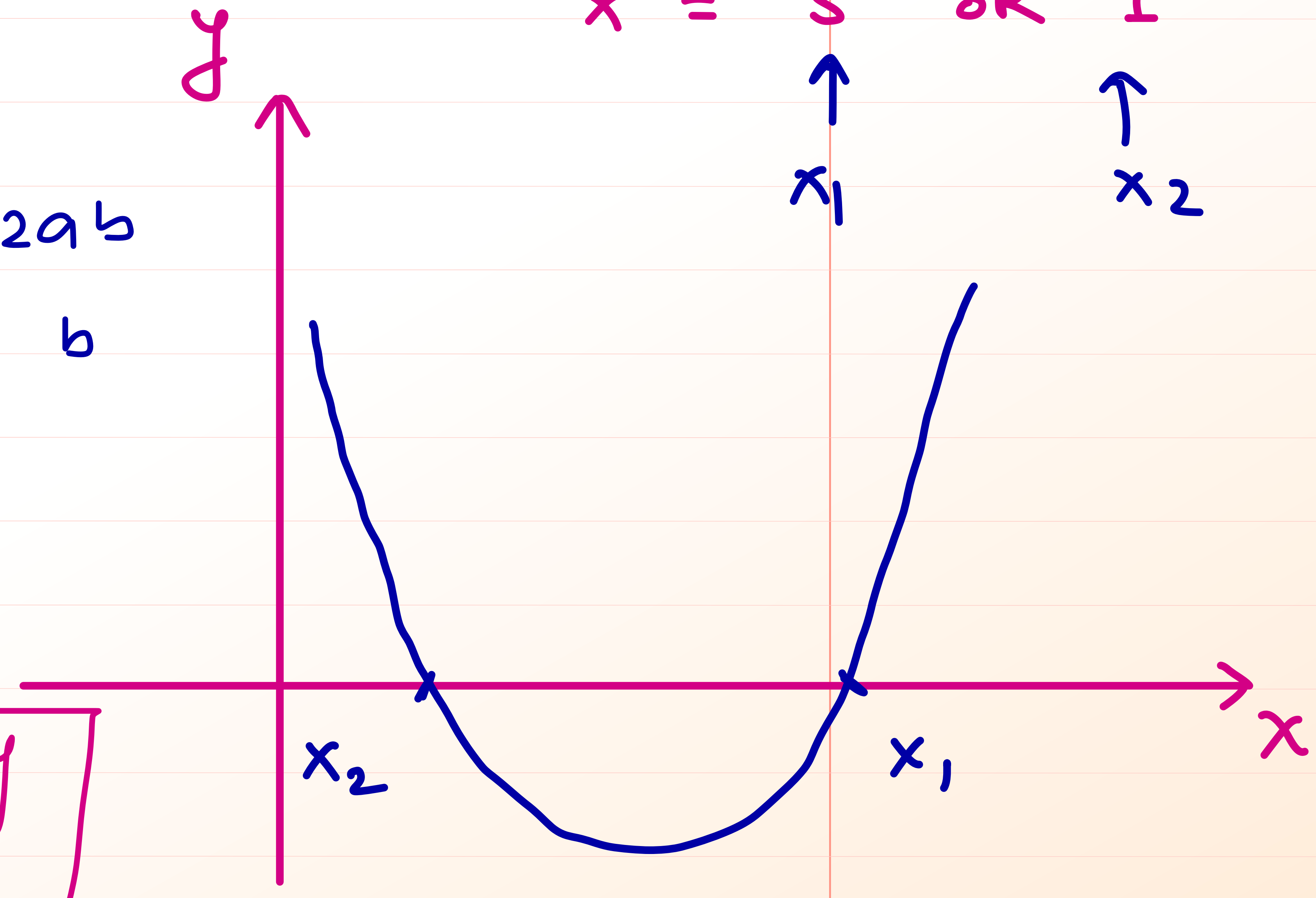
$$(a+b)^2 = a^2 + b^2 + 2ab$$

b

$a \rightarrow x$

$b \rightarrow \frac{b}{2a}$

$$x^2 + 2\cdot\left(\frac{b}{2a}\right)x + \boxed{\left(\frac{b}{2a}\right)^2} = -\frac{c}{a} + \boxed{\left(\frac{b}{2a}\right)^2}$$




$\boxed{\text{add square of half of coefficient of } x}$

$$x^2 + 2\cdot\frac{b}{2a}x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2$$

$$\left(x + \frac{b}{2a}\right)^2 = -\frac{c}{a} + \frac{b^2}{4a^2}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{-4ac + b^2}{4a^2}$$

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$\boxed{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

# RELATION BETWEEN ROOTS AND COEFFICIENT OF QUADRATIC EQUATION :

$$ax^2 + bx + c = 0 \quad a \neq 0 \quad a, b, c \in R$$

If $\alpha, \beta$ are the roots then $\underbrace{\alpha + \beta = -\frac{b}{a}}$ ; $\underbrace{\alpha\beta = \frac{c}{a}}$ ; $|\alpha - \beta| = \frac{\sqrt{D}}{|a|}$

Hence we can form the quadratic equation if the sum and product of its roots are known

$$ax^2 + bx + c = a(x-\alpha)(x-\beta)$$

roots $(\alpha, \beta)$

$\downarrow$

leading coefficient.

$$ax^2 + bx + c = a(x^2 \underbrace{- \alpha x - \beta x} + \alpha\beta)$$

$$ax^2 + bx + c = ax^2 + x \cdot a(-\alpha - \beta) + a\alpha\beta$$

$$b = a(-\alpha - \beta) \Rightarrow b = -a(\alpha + \beta) \Rightarrow \alpha + \beta = -\frac{b}{a}$$

$$c = a\alpha\beta \Rightarrow \alpha\beta = \frac{c}{a}$$

$x^2 - 6x + 5$

$(x-5)(x-1)$

$\boxed{5, 1} \rightarrow$ Roots

$(x-5) \quad (x-1)$

$$ax^2+bx+c=0$$

$$x^2+\frac{b}{a}x+\frac{c}{a}=0$$

$$x^2+\frac{b}{a}x+\frac{c}{a}=(x-\alpha)(x-\beta)$$

$$x^2+\frac{b}{a}x+\frac{c}{a}=x^2-\alpha x-\beta x+\alpha\beta$$

$$x^2+\frac{b}{a}x+\frac{c}{a}=\underbrace{x^2+x(-\alpha-\beta)+\alpha\beta}$$

$$-\alpha-\beta=\frac{b}{a}\Rightarrow \boxed{\alpha+\beta=-\frac{b}{a}}$$

$$\alpha\beta=\frac{c}{a}\Rightarrow \boxed{\alpha\beta=\frac{c}{a}}$$

quadratic Equation if sum of roots & product of roots are given

$$x^2-(\text{sum of roots})x+(\text{product of roots})=0$$

$$(\alpha-\beta)^2=\alpha^2+\beta^2-2\alpha\beta$$

$$=\alpha^2+\beta^2+2\alpha\beta-4\alpha\beta$$

$$(\alpha-\beta)^2=(\alpha+\beta)^2-4\alpha\beta$$

$$=\left(\frac{-b}{a}\right)^2-4\frac{c}{a}$$

$$=\frac{b^2}{a^2}-\frac{4c}{a}$$

$$(\alpha-\beta)^2=\frac{b^2-4ac}{a^2}$$

$$|(\alpha-\beta)|=\frac{\sqrt{b^2-4ac}}{|a|}$$

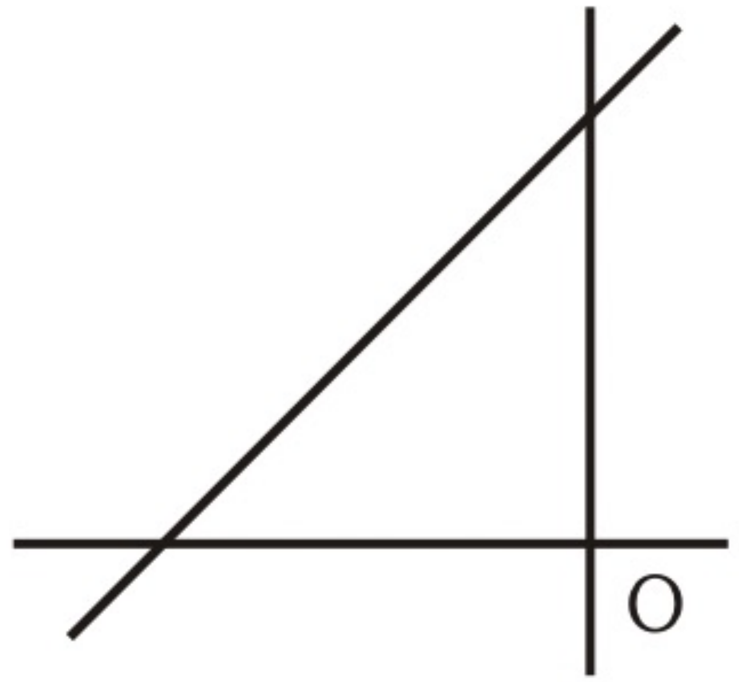
$$\boxed{|\alpha-\beta|=\frac{\sqrt{D}}{|a|}}$$

**Note :**

(a) If exactly one root of quadratic equation = 0, then P = 0 $\Rightarrow \frac{c}{a} = 0 \Rightarrow c = 0, b \neq 0$

(b) If both roots of the quadratic equation are zero then S = 0 and P = 0 $\Rightarrow$ b = c = 0.

(c) If one root is $\infty$ put $x = \frac{1}{y}$ in $ax^2 + bx + c = 0$, we get

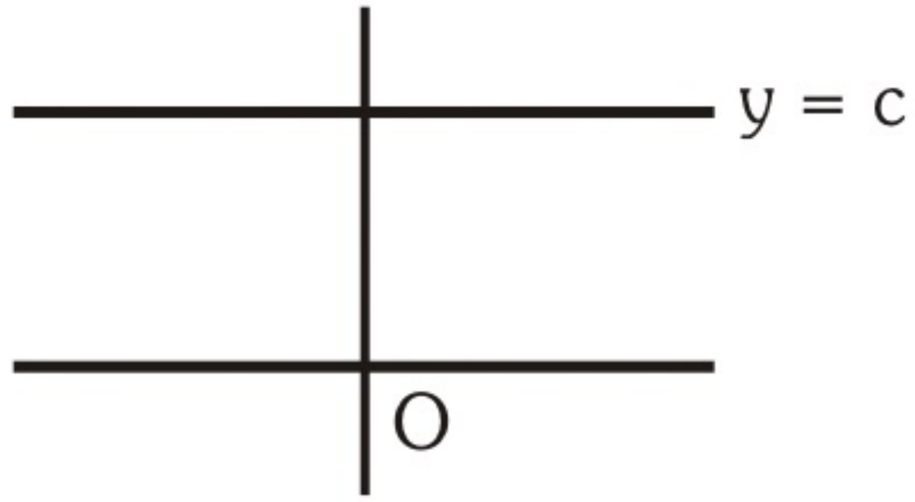



$cy^2 + by + a = 0$ must have one root zero $\Rightarrow$ P = 0 i.e. $\frac{a}{c} = 0$

Hence, a = 0 and $-\frac{b}{c} \neq 0 \Rightarrow b \neq 0$.

Original quadratic equation becomes $bx + c = 0$

(d) When both roots of the quadratic equation are infinity then.




The quadratic equation $cy^2 + by + a = 0$ must have both roots zero.

i.e. $= -\frac{b}{c} = 0$ and $\frac{a}{c} = 0 \Rightarrow b = 0$ ; $a = 0$ and $c \neq 0$.

(e) If roots of the quadratic equation are opposite in sign, then P < 0 $\Rightarrow$ a and c are of opposite in sign.

(f) If roots of the quadratic equation are numerically equal but opposite in sign, then b = 0.

$\alpha + \beta = 0 = -\frac{b}{a}$

(c) $ax^2+bx+c$

one root $\infty$ replace $\underline{x}$ by $\frac{1}{y}$

$x = \frac{1}{y}$

if $x \to \infty$

$\underline{\underline{y \to 0}}$

$$a\left(\frac{1}{y}\right)^2 + b\left(\frac{1}{y}\right) + c = 0$$

$$a + by + \underline{cy^2} = 0$$

$$cy^2 + by + a = 0$$

$y = 0$ $a = 0$

Product of roots $= 0 = \frac{a}{c}$

Sum of roots $= -\frac{b}{c} \neq 0 \Rightarrow b \neq 0$

(d) $ax^2 + bx + c = 0$

both root $\infty$

$x = \frac{1}{y}$

$a\left(\frac{1}{y}\right)^2 + b\left(\frac{1}{y}\right) + c = 0$

$cy^2 = 0$

$a + by + cy^2 = 0$

Both roots of quad $\underbrace{a + by} + cy^2 = 0$

$a = b = 0$

## NATURE OF ROOTS :

The quantity $D = b^2 - 4ac$ is called the discriminant of the quadratic equation and plays a very vital role in deciding the nature of roots of the equation without actually determining them. Now

If $D > 0$ then roots are real and distinct.

If $D = 0$ roots are coincident. (Roots are equal)

If $D \geq 0$ roots are real.

If $D < 0$ no real roots.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Discriminant $D = b^2 - 4ac$

**Note :**

(i) In case the coefficient of quadratic equation are rational then the roots are rational if $D > 0$ and is a perfect square.

(ii) Irrational roots of a quadratic equation with rational coefficients always occur in pair if one of the roots is $m + \sqrt{n}$ then other root will be $m - \sqrt{n}$.

(iii) Imaginary roots of a quadratic equation with real coefficients always occur in pair if one of the roots is $p + iq$ then other root will be $p - iq$.

(iv) If a, b, c are odd integers, then roots can not be rational.

irrational roots $\alpha + \sqrt{\beta}$ and $\alpha - \sqrt{\beta}$ $\Big\} \rightarrow$ if $a, b, c$ are rational.

imaginary roots $\alpha + i\beta$ and $\alpha - i\beta$

## EXAMPLES :

**E(1)** If $ax^2 + bx + c = 0 \begin{matrix} \nearrow \alpha \\ \searrow \beta \end{matrix}$, then find the value of $a\left(\frac{\alpha^2}{\beta}+\frac{\beta^2}{\alpha}\right)+b\left(\frac{\alpha}{\beta}+\frac{\beta}{\alpha}\right)$ [**Ans.** b]

**E(3)** If $\alpha$, $\beta$ are the roots of the quadratic equation $x^2 - 2x + 5 = 0$ then form a quadratic equation whose roots are $\alpha^3 + \alpha^2 - \alpha + 22$ and $\beta^3 + 4\beta^2 - 7\beta + 35$. [**Ans.** $x^2 - 12x + 35 = 0$]

**E(4)** If $x = 3 + \sqrt{5}$ find the value of $x^4 - 12x^3 + 44x^2 - 48x + 17$. [**Ans.** 1]

**E(5)** Prove that roots of $bx^2 + (b - c)x + b - c - a = 0$ are real, if those of $ax^2 + 2bx + b = 0$ are imaginary.

**E(1)** If $ax^2 + bx + c = 0 \begin{matrix} \alpha \\ \beta \end{matrix}$, then find the value of $a\left(\frac{\alpha^2}{\beta}+\frac{\beta^2}{\alpha}\right)+b\left(\frac{\alpha}{\beta}+\frac{\beta}{\alpha}\right)$

$$\alpha+\beta = \frac{-b}{a}$$

$$\alpha\beta = \frac{c}{a}$$

$$a\left(\frac{\alpha^3+\beta^3}{\alpha\beta}\right) + b\left(\frac{\alpha^2+\beta^2}{\alpha\beta}\right)$$

$$a\left(\frac{(\alpha+\beta)(\alpha^2+\beta^2-\alpha\beta)}{\alpha\beta}\right) + b\left(\frac{(\alpha+\beta)^2-2\alpha\beta}{\alpha\beta}\right)$$

$$a\left(\frac{(\alpha+\beta)((\alpha+\beta)^2-3\alpha\beta)}{\alpha\beta}\right) + b\left[\frac{(\alpha+\beta)^2-2\alpha\beta}{\alpha\beta}\right]$$

$$= a\left(\frac{\left(\frac{-b}{a}\right)\left(\frac{b^2}{a^2}-\frac{3c}{a}\right)}{\alpha\beta}\right) + b\,\frac{\left(\frac{b^2}{a^2}-2\frac{c}{a}\right)}{\alpha\beta}$$

$$=$$

**E(3)** If α, β are the roots of the quadratic equation $x^2 - 2x + 5 = 0$ then form a quadratic equation whose roots are $\alpha^3 + \alpha^2 - \alpha + 22$ and $\beta^3 + 4\beta^2 - 7\beta + 35$. [**Ans.** $x^2 - 12x + 35 = 0$]

$x^2 - 2x + 5 = 0 \rightarrow \alpha, \beta$

$\alpha^2 - 2\alpha + 5 = 0 \Rightarrow \alpha^2 = 2\alpha - 5$

$\beta^2 - 2\beta + 5 = 0 \Rightarrow \beta^2 = 2\beta - 5$

$A_1 = \alpha^3 + \alpha^2 - \alpha + 22$

$= \alpha(\alpha^2) + \alpha^2 - \alpha + 22$

$= \alpha(2\alpha - 5) + (2\alpha - 5) - \alpha + 22$

$= 2\alpha^2 - 5\alpha + 2\alpha - 5 - \alpha + 22$

$= 2\alpha^2 - 4\alpha + 17$

$= 2(\alpha^2 - 2\alpha) + 17$

$= 2(2\alpha - 5 - 2\alpha) + 17$

$\boxed{A_1 = 7}$

$A_2 = \beta^3 + 4\beta^2 - 7\beta + 35$

$= \beta(\beta^2) + 4\beta^2 - 7\beta + 35$

$= \beta(2\beta - 5) + 4(2\beta - 5) - 7\beta + 35$

$= 2\beta^2 - 5\beta + 8\beta - 20 - 7\beta + 35$

$= 2\beta^2 - 4\beta + 15$

$= 2(2\beta - 5) - 4\beta + 15 = 5$

$\boxed{A_2 = 5}$

$x^2 - (7+5)x + (35) = 0$

$\boxed{x^2 - 12x + 35 = 0}$

# Quadratic Equation

## Lecture -2

**If root of the equation $ax^2 + bx + c = 0$ are $\alpha$ & $\beta$ then finding equation whose roots are symmetric expressions of $\alpha$ and $\beta$.**

Sum of roots = SOR
Product of root = POR

If $f(\alpha, \beta) = f(\beta, \alpha)$ then $f(\alpha, \beta)$ denotes symmetric functions of roots.

**EXAMPLE :**

**E(1)** If roots of the equation $ax^2 + bx + c = 0$ are $\alpha$ & $\beta$ then find equation whose roots are :

(i) $1-2\alpha, 1-2\beta$  (ii) $\dfrac{\alpha+1}{\alpha-1}, \dfrac{\beta+1}{\beta-1}$

(i) $ax^2 + bx + c = 0$

MI

$\alpha+\beta = -\frac{b}{a}$

$\alpha\beta = \frac{c}{a}$

$SOR = 1-2\alpha + 1-2\beta = 2-2(\alpha+\beta)$
$= 2-2\left(\frac{-b}{a}\right)$
$= \frac{2a+2b}{a}$

$POR = (1-2\alpha)(1-2\beta)$
$= 1-2(\alpha+\beta)+4\alpha\beta$
$= 1-2\left(\frac{-b}{a}\right)+4\left(\frac{c}{a}\right)$
$=$

Eqn

$x^2 - (SOR)x + POR = 0$

$$\boxed{x^2 - \left(\frac{2a+2b}{a}\right)x + \left(\frac{a+2b+4c}{a}\right) = 0}$$

$ax^2 - 2x(a+b) + (a+2b+4c) = 0$

M II

$\underline{\alpha, \beta}$ $\qquad ax^2 + bx + c = 0$

roots $\alpha, \beta$

$\underline{\text{eqn}} \Rightarrow 1-2\alpha, \quad 1-2\beta$

$y = 1 - 2x \quad ; \quad x = \dfrac{1-y}{2}$

$$a\left(\frac{1-y}{2}\right)^2 + b\left(\frac{1-y}{2}\right) + c = 0$$

$$a\left(\frac{1+y^2-2y}{4}\right) + b\left(\frac{1-y}{2}\right) + c = 0$$

$$a + ay^2 - 2ay + 2b(1-y) + 4c = 0$$

$$ay^2 - 2ay - 2by + 2b + 4c + a = 0$$

$$\boxed{ay^2 - 2y(a+b) + a + 2b + 4c = 0}$$

(ii) $\dfrac{\alpha+1}{\alpha-1}, \quad \dfrac{\beta+1}{\beta-1}$

$$\frac{y}{1} = \frac{x+1}{x-1} \Rightarrow \boxed{x = \frac{y+1}{y-1}}$$

By componendo & dividendo

$$\frac{y+1}{y-1} = \frac{x+1+x-1}{(x+1)-(x-1)}$$

$$\frac{y+1}{y-1} = \frac{2x}{2} \Rightarrow x = \frac{y+1}{y-1}$$

$$ax^2 + bx + c = 0$$

$$a\left(\frac{y+1}{y-1}\right)^2 + b\left(\frac{y+1}{y-1}\right) + c = 0$$

$$a(y+1)^2 + b(y+1)(y-1) + c(y-1)^2 = 0$$

$$a(y^2+1+2y) + b(y^2-1) + c(y^2+1-2y) = 0$$

$$y^2(a+b+c) + y(2a-2c) + (a-b+c) = 0$$

## QUADRATIC EQUATION V/S IDENTITY :

A quadratic equation in x is satisfied by only two values of x but an identity in x is satisfied by all values of x.

A quadratic equation will become an identity if it has more than two roots & the required condition is $a = b = c = 0$

$ax^2 + bx + c = 0$ → $\alpha$, $\beta$, $\gamma$

$a\alpha^2 + b\alpha + c = 0$

$a\beta^2 + b\beta + c = 0$

$a\gamma^2 + b\gamma + c = 0$

subtract

$a(\alpha^2 - \beta^2) + b(\alpha - \beta) = 0$

$a(\alpha+\beta)(\alpha-\beta) + b(\alpha-\beta) = 0$

$(\alpha-\beta)(a(\alpha+\beta) + b) = 0$

$(\beta-\gamma)(a(\beta+\gamma) + b) = 0$

$a(\alpha+\beta) + b = 0$

$a(\beta+\gamma) + b = 0$

subtract

$a(\alpha - \gamma) = 0$

$\boxed{a = 0}$

but $a = 0$

$\boxed{b = 0}$

$a = 0,\ b = 0$

$\boxed{c = 0}$

## EXAMPLES :

**E(1)** For what values of p, the equation

$(p + 2)(p - 1)x^2 + (p - 1)(2p + 1)x + p^2 - 1 = 0$ has more than two roots. **[Ans. 1]**

**E(2)** If $\frac{(x-a)(x-b)}{(c-a)(c-b)} + \frac{(x-b)(x-c)}{(a-b)(a-c)} + \frac{(x-c)(x-a)}{(b-c)(b-a)} = 1$ then prove that it is an identity.

(1)
$a = 0 \Rightarrow (p+2)(p-1) = 0 \Rightarrow p = 1, -2$
$b = 0 \Rightarrow (p-1)(2p+1) = 0 \Rightarrow p, 1, -1/2$
$c = 0 \Rightarrow p^2 - 1 = 0 \Rightarrow p = 1, -1$
$\rightarrow \boxed{p = 1}$

(2)
at $x = a$ we get $1 = 1$
at $x = b$ we get $1 = 1$
at $x = c$ we get $1 = 1$

so $x = a, b, c$ satisfies given quadratic equation.

$x = a, b, c$ (more than two values)

so its an identity.

# CONDITION OF COMMON ROOTS :

Let $a_1x^2 + b_1x + c_1 = 0$ and $a_2x^2 + b_2x + c_2 = 0$ have a common root $\alpha$.

Hence $a_1\alpha^2 + b_1\alpha + c_1 = 0$

$a_2\alpha^2 + b_2\alpha + c_2 = 0$

$$\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}^2 = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$$

by cross multiplication $\dfrac{\alpha^2}{b_1c_2 - b_2c_1} = \dfrac{\alpha}{a_2c_1 - a_1c_2} = \dfrac{1}{a_1b_2 - a_2b_1}$

$$\therefore \quad \alpha = \frac{b_1c_2 - b_2c_1}{a_2c_1 - a_1c_2} = \frac{a_2c_1 - a_1c_2}{a_1b_2 - a_2b_1}$$

or $\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}^2 = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$

which is the required condition for at least one root in common

This is also the condition that the two quadratic expressions $a_1x^2 + b_1xy + c_1y^2$ and $a_2x^2 + b_2xy + c_2y^2$ may have a common factor.

**Note :** If both roots of the given equations are common then $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2}$

## Common roots

(i) One root common

(ii) Both roots common

(i) $a_1x^2 + b_1x + c_1 = 0$

$a_2x^2 + b_2x + c_2 = 0$ → common root $= \alpha$

$a_1\alpha^2 + b_1\alpha + c_1 = 0$

$a_2\alpha^2 + b_2\alpha + c_2 = 0$

$$\frac{\alpha^2}{\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}} = \frac{-\alpha}{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}} = \frac{1}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

$$\boxed{\frac{\alpha^2}{b_1c_2 - b_2c_1}} = \boxed{\frac{-\alpha}{a_1c_2 - a_2c_1} = \boxed{\frac{1}{a_1b_2 - a_2b_1}}}$$

$$\boxed{\alpha = \frac{a_2c_1 - a_1c_2}{a_1b_2 - a_2b_1}}$$

$$\frac{\alpha^2}{b_1c_2-b_2c_1} = \frac{1}{a_1b_2-a_2b_1}$$

$$\alpha^2 = \frac{b_1c_2-b_2c_1}{a_1b_2-a_2b_1}$$

$$\left(\frac{a_2c_1-a_1c_2}{a_1b_2-a_2b_1}\right)^2 = \left(\frac{b_1c_2-b_2c_1}{a_1b_2-a_2b_1}\right)$$

$$\frac{(a_2c_1-a_1c_2)^2}{(a_1b_2-a_2b_1)^{\cancel{2}}} = \frac{b_1c_2-b_2c_1}{\cancel{a_1b_2-a_2b_1}}$$

Condition when one root is common $\Rightarrow$ $\boxed{(a_2c_1-a_1c_2)^2 = (b_1c_2-b_2c_1)(a_1b_2-a_2b_1)}$

(ii) Both roots common ; –

$$a_1 x^2 + b_1 x + c_1 = 0$$
$$a_2 x^2 + b_2 x + c_2 = 0$$

Condition for both roots common $\Rightarrow$ $\boxed{\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2}}$

## Cross Multiplication Method :-

$$a_1x + b_1y + c_1 = 0$$
$$a_2x + b_2y + c_2 = 0$$

$$\frac{x}{\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}} = \frac{1}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

$$\frac{x}{b_1c_2 - b_2c_1} = \frac{-y}{a_1c_2 - c_1a_2} = \frac{1}{a_1b_2 - a_2b_1}$$

**EXAMPLES :**

**E(1)** Find the value of k for which the equations $3x^2 + 4kx + 2 = 0$ and $2x^2 + 3x - 2 = 0$ have a common root.

[**Ans.** $k = \frac{-11}{8}, \frac{7}{4}$]

**E(2)** If the quadratic equation $x^2 + bx + c = 0$ and $x^2 + cx + b = 0$ $(b \neq c)$ have a common root then prove that their uncommon roots are the roots of the equation $x^2 + x + bc = 0$.

**E(3)** If the equation $x^2 - 4x + 5 = 0$ and $x^2 + ax + b = 0$, $a, b \in R$ have a common root find a and b.

[**Ans.** a=–4, b=5]

**E(4)** If both the roots of the equation $x^2 + bx + b = 0$ are the roots of the equation $x^3 + 3x^2 + 3x + 2 = 0$, then find b. [**Ans.** 1]

1

$$(a_2c_1 - a_1c_2)^2 = (b_1c_2 - b_2c_1)(a_1b_2 - a_2b_1)$$

$a_1 = 3$
$b_1 = 4k$
$c_1 = 2$
$a_2 = 2$
$b_2 = 3$
$c_2 = -2$

$$[2(2) - 3(-2)]^2 = (4k(-2) - 3(2))(3(3) - 2(4k))$$

$$100 = (-8k-6)(9-8k)$$

$$64k^2 - 24k - 154 = 0$$

$$\boxed{32k^2 - 12k - 77}$$

**E(2)** If the quadratic equation $x^2 + bx + c = 0$ and $x^2 + cx + b = 0$ $(b \neq c)$ have a common root then prove that their uncommon roots are the roots of the equation $x^2 + x + bc = 0$.

$$\alpha^2 + b\alpha + c = 0$$
$$\alpha^2 + c\alpha + b = 0$$

Subtract

$$\alpha(b-c) + c - b = 0$$

common root $\rightarrow$ $\boxed{\alpha = 1}$

$$x^2 + bx + c = 0$$
$$\alpha + \beta = -b$$
$$1 + \beta = -b$$

uncom $\boxed{\beta = -b-1}$

$$\alpha\beta = c$$
$$\boxed{\beta = c}$$
$$b + 1 = -c$$

Eqn

$$x^2 + x + bc = 0$$

put $x = -b-1$

$$(-b-1)^2 - b - 1 + bc = 0$$
$$b^2 + 1 + 2b - b - 1 + bc = 0$$
$$b^2 + b + bc = 0$$
$$b(b+1) + bc = 0$$
$$b(-c) + bc = 0$$
$$0 = 0$$

Q 4

$$x^3 + 3x^2 + 3x + 2 = 0$$

$$\boxed{\alpha + \beta + \gamma = -3}$$

$$\alpha\beta + \underline{\beta\gamma + \gamma\alpha} = 3$$

$$6 + \gamma(-6) = 3$$

$$\gamma = \frac{3-6}{-6} = \boxed{\frac{6-3}{6}}$$

$$x^2 + 6x + 6 = 0$$

$$\alpha + \beta = -6$$

$$\underline{\alpha\beta} = 6$$

## RATIONAL ALGEBRAIC INEQUALITIES (Method of interval) :

**Type-1 :** Quadratic inequality involving non-repeated linear factors.

**E(1)** $2x^2 - 3x + 4 > 0$ **[Ans.** $x \in R$ ]

**E(2)** $(x^2 + x - 6)(x^2 - 2x - 8) \geq 0$ **[Ans.** $(-\infty, -3] \cup [-2, 2] \cup [4, \infty)$ ]

**D(3)** Solve $f'(x) \geq g'(x)$, where $f(x) = 5 - 3x + \dfrac{5}{2}x^2 - \dfrac{x^3}{3}$, $g(x) = 3x - 7$. **[Ans.** [2, 3]]

**Type-2 :** Quadratic inequality involving Repeated linear factors.

**E(1)** $(x + 1)(x - 3)(x - 2)^2 \geq 0$ **[Ans.** $(-\infty, -1] \cup [3, \infty) \cup \{2\}$]

**D(2)** $x(x + 6)(x + 2)^2(x - 3) > 0$ **[Ans.** $(-6, 0) \cup (3, \infty) - \{-2\}$]

**E(3)** $(x - 1)^2(x + 1)^3(x - 4) < 0$ **[Ans.** $(-1, 4) - \{1\}$]

**Type-3 :** Quadratic/algebraic inequality of the type of $\frac{f(x)}{g(x)}$. (Rational inequality)

**E(1)** $\frac{2x-3}{3x-7} > 0$ [**Ans.** $\left(-\infty, \frac{3}{2}\right) \cup \left(\frac{7}{3}, +\infty\right)$]

**E(2)** $\frac{x^2-5x+12}{x^2-4x+5} > 3$ [ **Ans.** $\left(\frac{1}{2}, 3\right)$ ]

**E(3)** $\frac{x^2-5x+6}{x^2+x+1} < 0$ [**Ans.** (2, 3)]

**E(4)** $\frac{(x-1)^2(x+1)^3}{x^4(x-2)} < 0$ [**Ans.** $(-1, 2) - \{0, 1\}$]

**D(5)** $\frac{x+1}{x-1} \geq \frac{x+5}{x+1}$ [**Ans.** $(-\infty, -1) \cup (1, 3]$]

**E(6)** $\frac{2(x-4)}{(x-1)(x-7)} \geq \frac{1}{x-2}$ [**Ans.** $(1, 2) \cup (7, +\infty)$]

**D(7)** $\frac{x^2+4x+4}{2x^2-x-1} > 0$ [**Ans.** $(-\infty, -2) \cup \left(-2, -\frac{1}{2}\right) \cup (1, \infty)$ ]

**D(8)** Number of positive integral solution of $\frac{x^3(2x-3)^2(x-4)^6}{(x-3)^3(3x-8)^4} \leq 0$ -

(A) only one (B) 2 •(C) 3 (D) 4

**Type-4 :** Double inequality and biquadratic inequality.

**E(1)** $1 < \frac{3x^2-7x+8}{x^2+1} \leq 2$ [**Ans.** [1, 6]]

**D(2)** $-1 \leq \frac{3x^2-7x+8}{x^2+1} \leq 1$ [**Ans.** $\phi$]

**E(3)** $(x^2 + 3x + 1)(x^2 + 3x - 3) \geq 5$ [**Ans.** $(-\infty, -4] \cup [-2, -1] \cup [1, \infty)$]

# Self Evaluation : →

## Part- 1 Inequalities involving mod

**E(1)** Find solution set of the inequality $|x+2| - |x-1| < x - \frac{3}{2}$.

Solve for x,

**E(2)** $\frac{|x-1|}{x+2} < 1 \; x \in R$ [**Ans.** $(-\infty, -2) \cup (-\frac{1}{2}, \infty)$]

**E(3)** $|x^3 - 1| \geq 1 - x$ [**Ans.** $(-\infty, -1] \cup [0, \infty)$]

**E(4)** $\frac{x^2 - 5x + 6}{|x| + 7} < 0$ [**Ans.** $(2, 3)$]

**E(5)** $\frac{x^2 + 6x - 7}{|x+4|} < 0$ [**Ans.** $(-7, -4) \cup (-4, 1)$]

**E(6)** $\frac{|x+2| - x}{x} < 2$ [**Ans.** $(-\infty, 0) \cup (1, \infty)$]

**E(7)** $\frac{1}{|x| - 3} < \frac{1}{2}$ [**Ans.** $(-\infty, -5) \cup (-3, 3) \cup (5, \infty)$]

## PART- 2 PROPERTIES OF MODULUS FUNCTION :

(a) $|x| \geq a \Rightarrow x \geq a$ or $x \leq -a$ where a is positive.

**(E) Example :** $\left|\dfrac{2x-1}{x-1}\right| > 2$ **[Ans.** $\left(\dfrac{3}{4},1\right) \cup (1, \infty)$]

(b) $|x| \leq a \Rightarrow x \in [-a, a]$. where a is positive

**(E) Example :** $\left|\dfrac{x^2-3x-1}{x^2+x+1}\right| < 3$ **[Ans.** $[(-\infty, -2) \cup (-1, +\infty)]$]

(c) $|x| > |y| \Rightarrow x^2 > y^2$

**(E) Example :** $|x^2 - 1| \leq |2x - 1|$ **[Ans.** $\left[-1-\sqrt{3},0\right] \cup \left[-1+\sqrt{3},2\right]$]

(d) $|x + y| = |x| + |y| \Rightarrow xy \geq 0$

**(E) Example :** $|x + 1| + |x - 1| = |2x|$ **[Ans.** $x \in (-\infty, -1] \cup [1, \infty)$]

(e) $|x - y| = |x| + |y| \Rightarrow xy \leq 0$

**(E) Example :** $|2x - 3| - |x^2 - 4x + 3| = |x^2 - 2x|$ **[Ans.** $x \in [0, 1] \cup [2, 3]$]

## PART-3 IRRATIONAL INEQUATIONS :

**Examples :**

**E(1)** $x - 3 < \sqrt{x^2+4x-5}$ **[Ans.** $(-\infty, -5] \cup [1, \infty)$]

**E(2)** $x + 1 > \sqrt{5-x^2}$ **[Ans.** $(1, \sqrt{5}]$]

## PART-4 LOGARITHMIC INEQUATIONS :

$$\log_a f(x) > \log_a g(x) = \begin{cases} f(x) > g(x) \ \& \ g(x) > 0, & \text{when } a > 1 \\ f(x) < g(x) \ \& \ f(x) > 0, & \text{when } 0 < a < 1 \end{cases}$$

**Examples :**

**E(1)** $\log_2 \log_4 \log_5 x > 0$ **[Ans.** $x > 625$]

**E(2)** $\log_3 |3 - 4x| > 2$ **[Ans.** $\left(-\infty, -\frac{3}{2}\right) \cup (3, \infty)$ ]

**E(3)** $\log_{0.5}(x^2 - 5x + 6) > -1$ **[Ans.** $(1, 2) \cup (3, 4)$]

**E(4)** $\log_{0.2}(x^2 - x - 2) > \log_{0.2}(-x^2 + 2x + 3)$ **[Ans.** $\left(2, \frac{5}{2}\right)$ ]

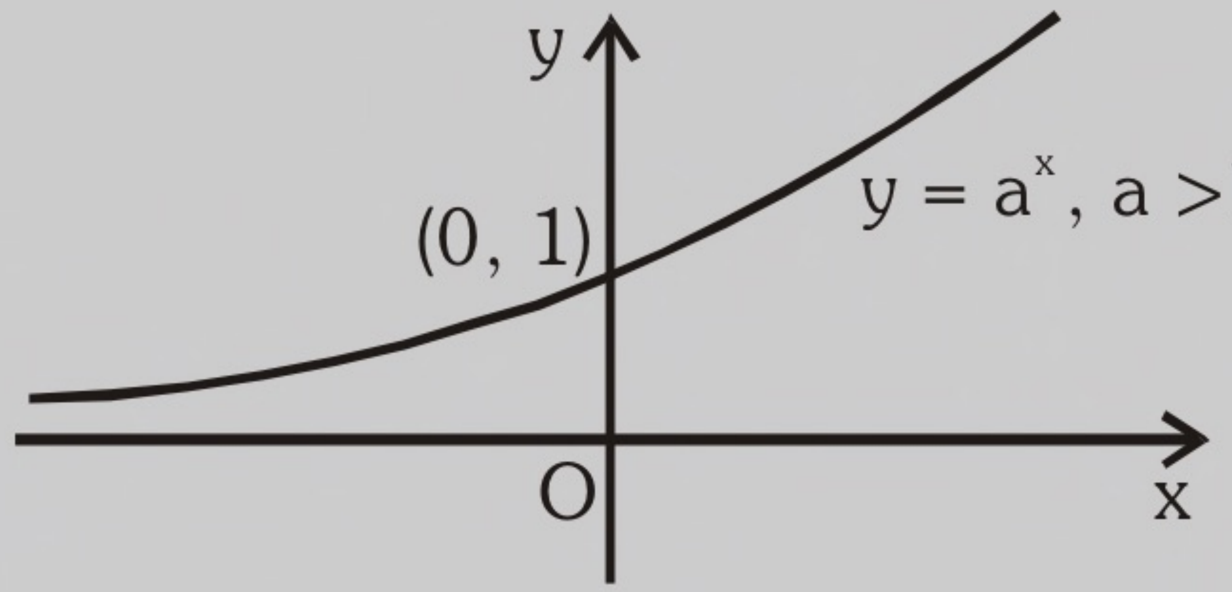
**E(5)** $\log_x\left(2x - \frac{3}{4}\right) > 2$ **[Ans.** $x \in \left(\frac{3}{8}, \frac{1}{2}\right) \cup \left(1, \frac{3}{2}\right)$ ]

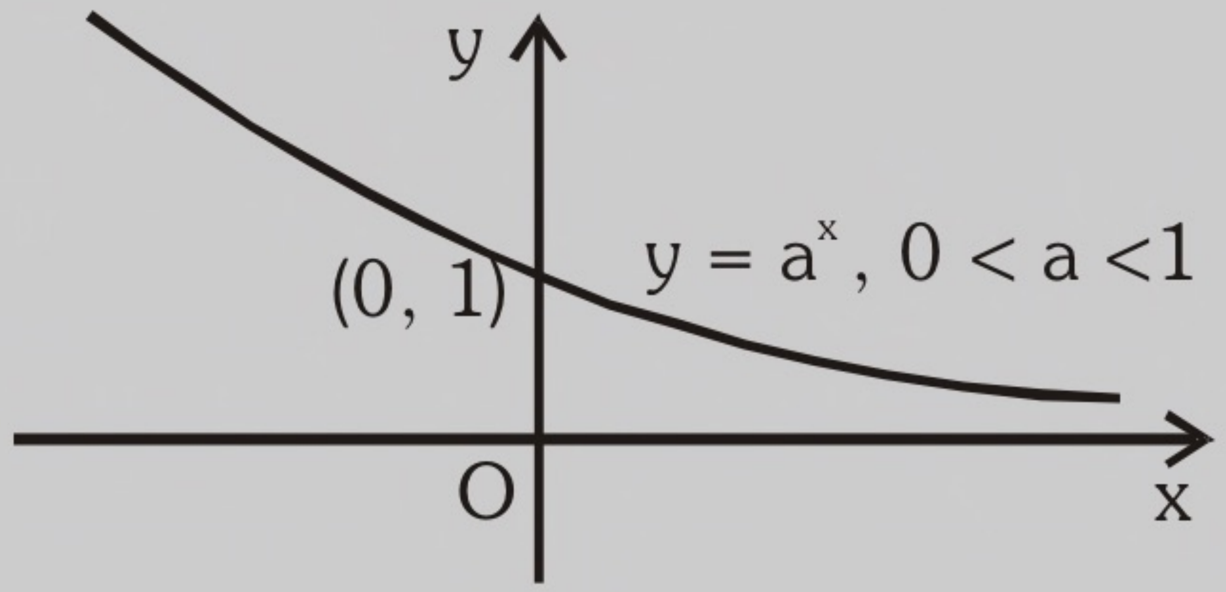
**E(6)** $\log_{\frac{x+6}{3}}\left(\log_2 \frac{x-1}{x+2}\right) > 0$ **[Ans.** $(-6, -5) \cup (-3, -2)$]

**E(7)** $\log_{0.5}\left(\log_6 \frac{x^2 + x}{x+4}\right) < 0$ **[Ans.** $(-4, -3) \cup (8, \infty)$]

**E(8)** $\log_2 x \le \frac{2}{\log_2 x - 1}$ **[Ans.** $\left(0, \frac{1}{2}\right] \cup (2, 4]$ ]

## PART-5 EXPONENTIAL EQUATIONS / INEQUATIONS :




If $a^{f(x)} > b \Rightarrow \begin{bmatrix} f(x) > \log_a b & \text{when} & a > 1 \\ f(x) < \log_a b & \text{when} & 0 < a < 1 \end{bmatrix}$

**Examples :**

**E(1)** $\left(\frac{2}{5}\right)^{\frac{6-5x}{2+5x}} < \frac{25}{4}$ [**Ans.** $(-\infty, -2) \cup \left(-\frac{2}{5}, \infty\right)$]

**E(2)** $\left(\frac{1}{3}\right)^{\frac{|x+2|}{2-|x|}} > 9$ [**Ans.** $(2, 6)$]]

## GRAPHS OF QUADRATIC EXPRESSIONS, $y = ax^2 + bx + c$.

$$y = ax^2 + bx + c$$

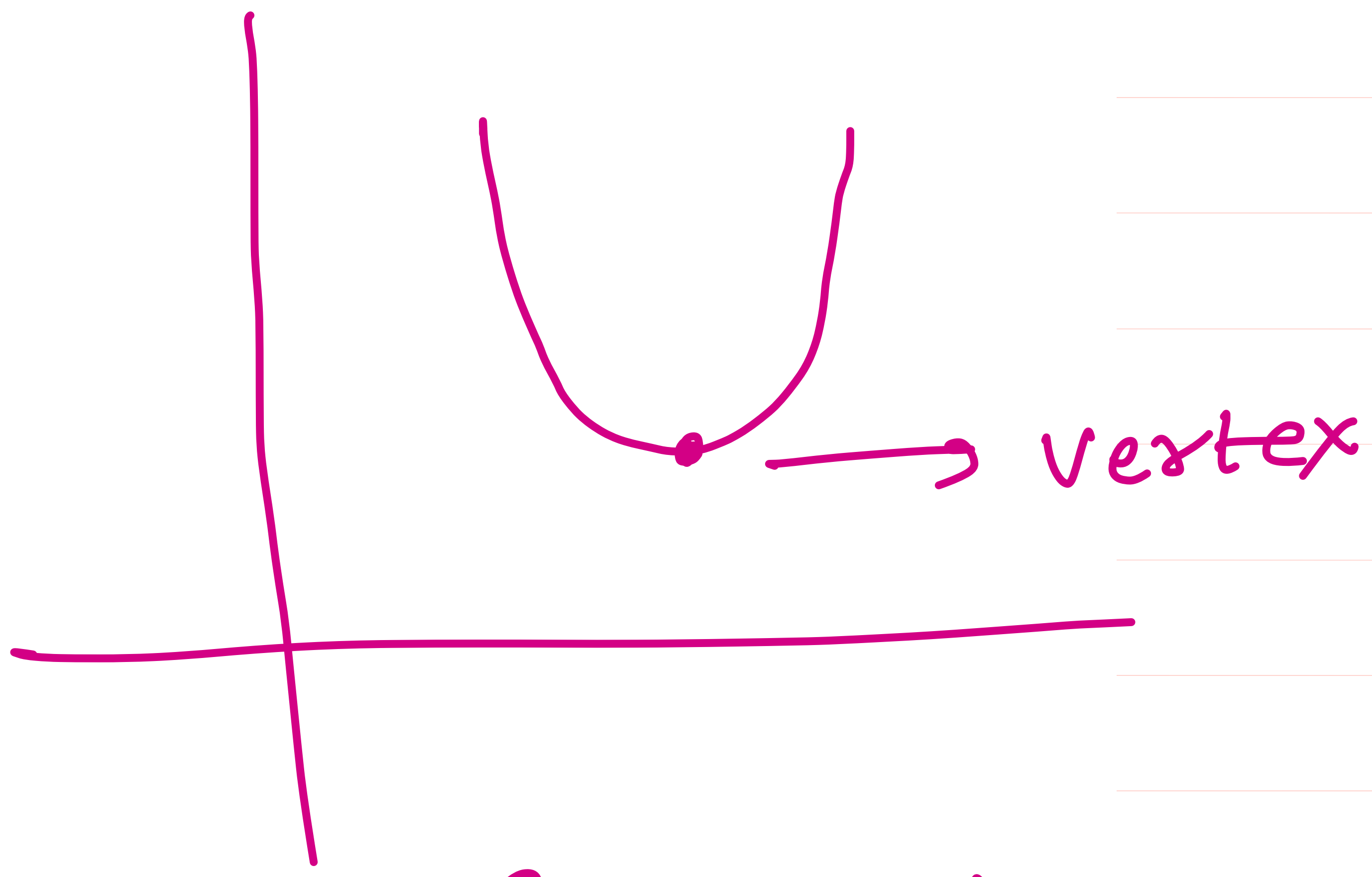
$$y = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right)$$

$$y = a\left[\left(x + \frac{b}{2a}\right)^2 + \frac{c}{a} - \frac{b^2}{4a^2}\right] = a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a}$$

$$\left(y - \left(\frac{-D}{4a}\right)\right) = a\left(x + \frac{b}{2a}\right)^2$$

$$X^2 = \frac{1}{a}Y.$$

$\therefore$ vertex of the parabola is $\left(\frac{-b}{2a}, \frac{-D}{4a}\right)$.




vertex $\left(\frac{-b}{2a}, \frac{-D}{4a}\right)$

Now for different values of a, b, c if graph of $y = ax^2 + bx + c$ is plotted then the following 6 different shapes are obtained. The graph is called a parabola.

If $a > 0$, then parabola open upwards.

If $a < 0$, then parabola open downwards.

$$y = ax^2 + bx + c$$

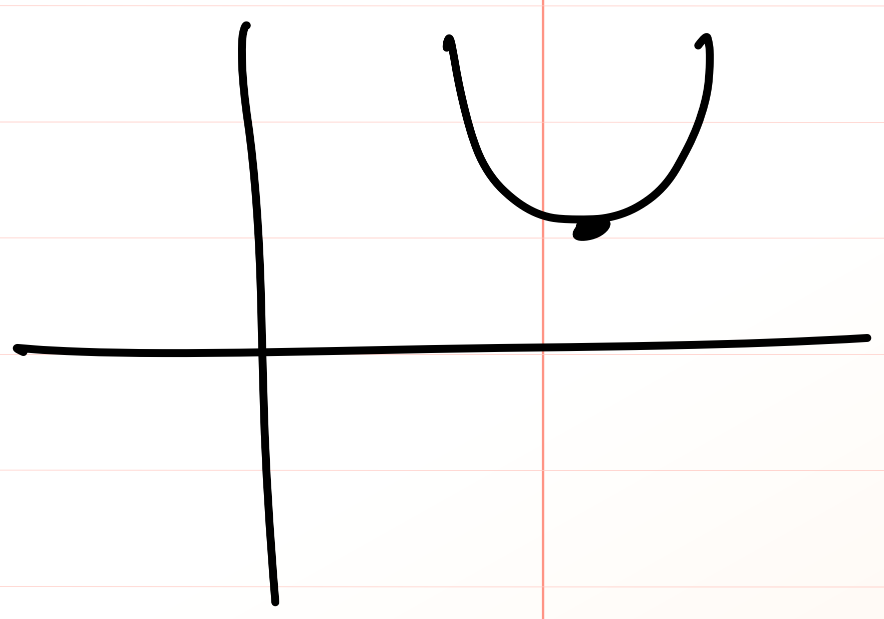
$$= a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right)$$

$$= \underline{a}\left(\underbrace{x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2} - \underline{\left(\frac{b}{2a}\right)^2}\right) + \frac{c}{a}\cdot(a)$$

$$= a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c$$

$$= a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a}$$

$$= \underbrace{a\left(x + \frac{b}{2a}\right)^2} - \underbrace{\frac{(b^2 - 4ac)}{4a}}$$



$$\text{Vertex} = \left(\frac{-b}{2a}, \frac{-D}{4a}\right)$$

# Quadratic Equation

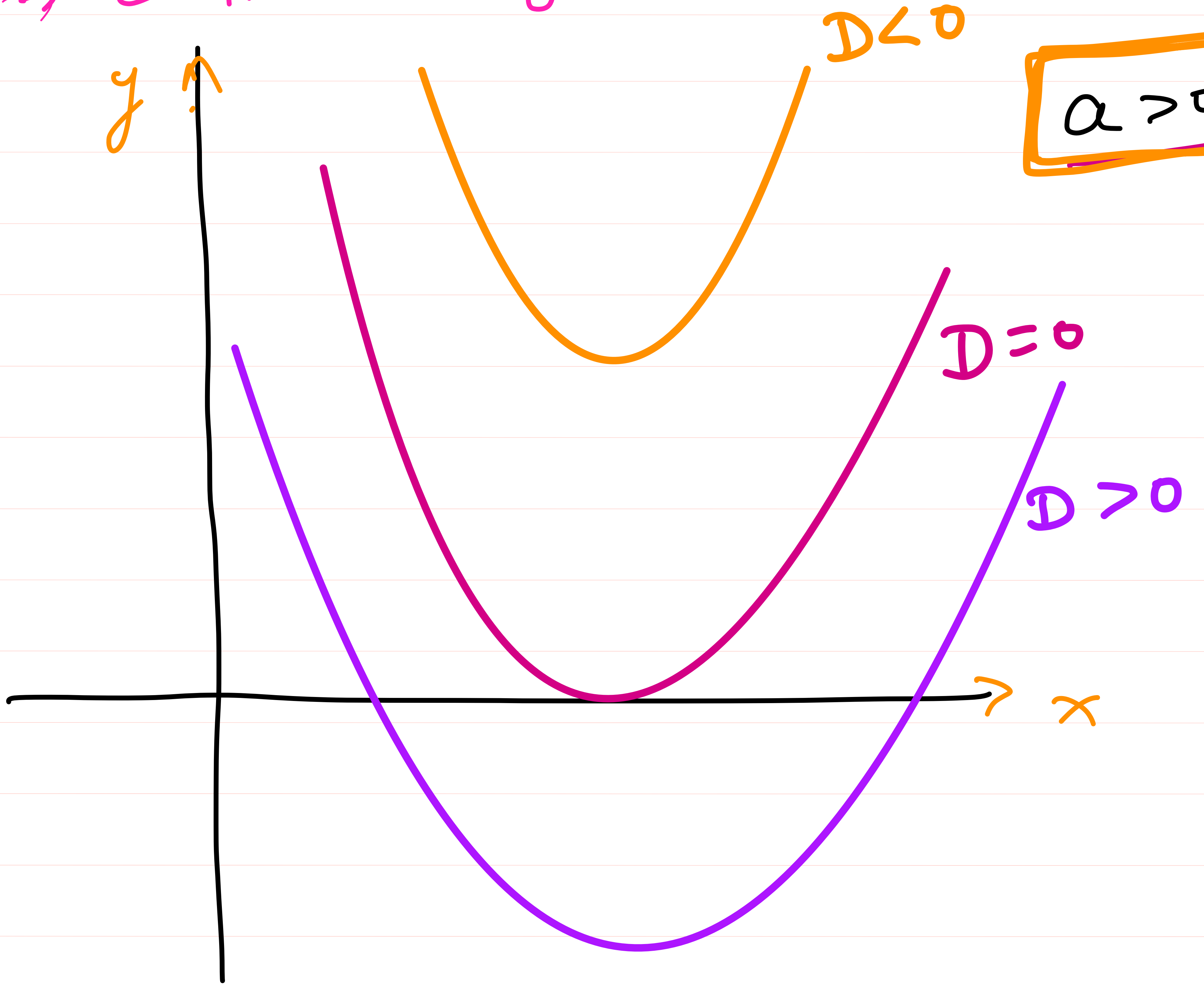
## Lecture -3

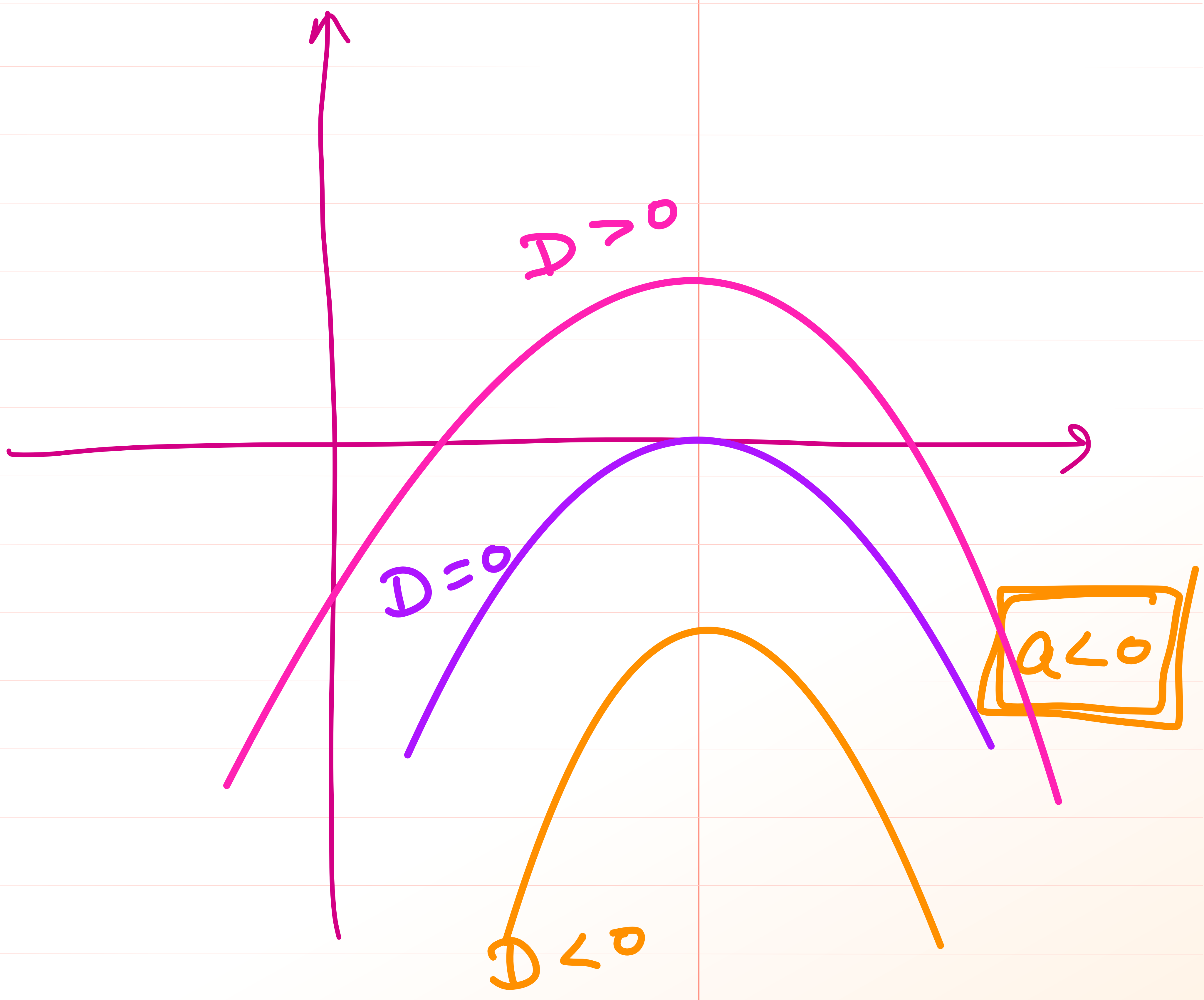
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\underline{\overset{y}{f(x)}} = ax^2 + bx + c$$

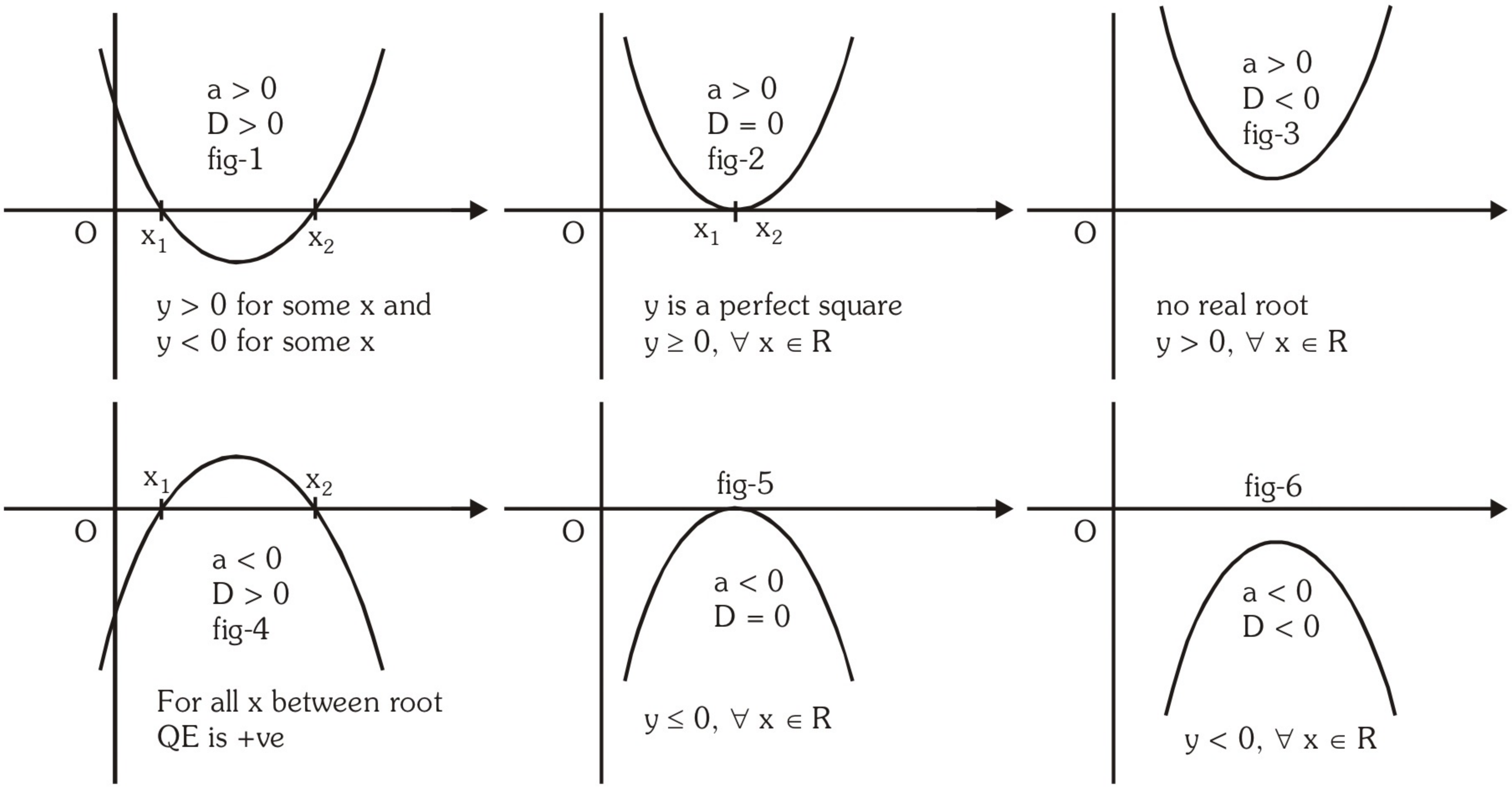
$a \rightarrow$ coefficient of $x^2$

$\boxed{a > 0}$




$$D = b^2 - 4ac$$


a > 0
D > 0
fig-1
O
x1
x2
y > 0 for some x and
y < 0 for some x
a > 0
D = 0
fig-2
O
x1
x2
y is a perfect square
y ≥ 0, ∀ x ∈ R
a > 0
D < 0
fig-3
O
no real root
y > 0, ∀ x ∈ R
x1
x2
O
a < 0
D > 0
fig-4
For all x between root
QE is +ve
fig-5
O
a < 0
D = 0
y ≤ 0, ∀ x ∈ R
fig-6
O
a < 0
D < 0
y < 0, ∀ x ∈ R

$\forall \rightarrow$ for all

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**Note :**

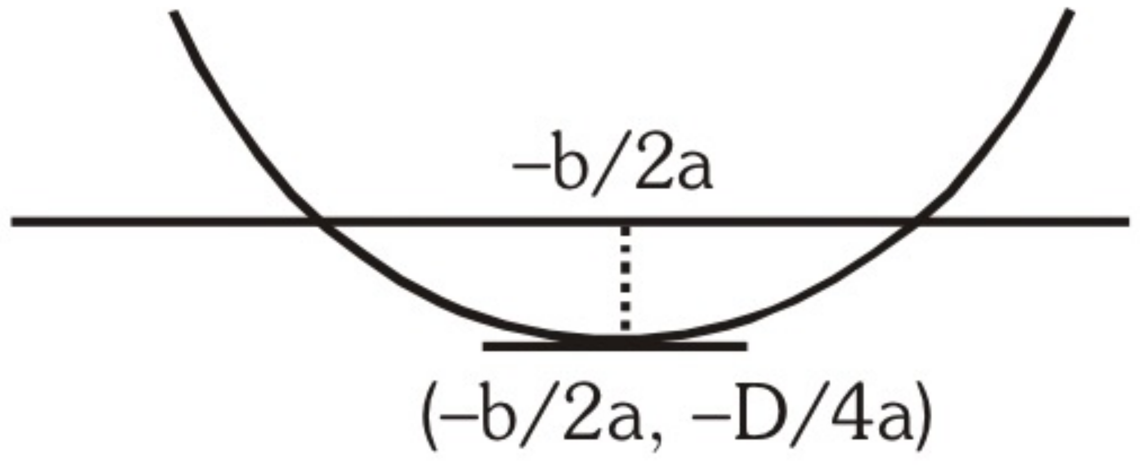
(a) Condition for which quadratic expression $y = ax^2 + bx + c$ is always positive is $a > 0$ and $D < 0$

(b) Condition for which quadratic expression $y = ax^2 + bx + c$ is always negative is $a < 0$ and $D < 0$

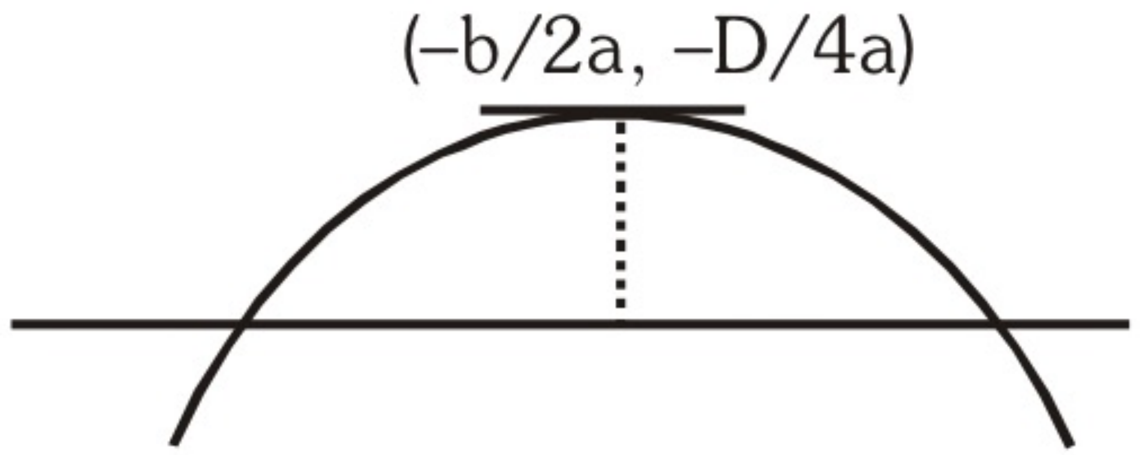
(c) $ax^2+bx+c \geq 0 \ \forall \ x \in R$, if $a>0$ & $D \leq 0$

(d) $ax^2+bx+c \leq 0 \ \forall \ x \in R$, if $a<0$ & $D \leq 0$

(e) If $a > 0$ then minimum value of y is $\frac{-D}{4a}$ at $x = \frac{-b}{2a}$




If $a < 0$ then maximum value of y is $\frac{-D}{4a}$ at $x = \frac{-b}{2a}$

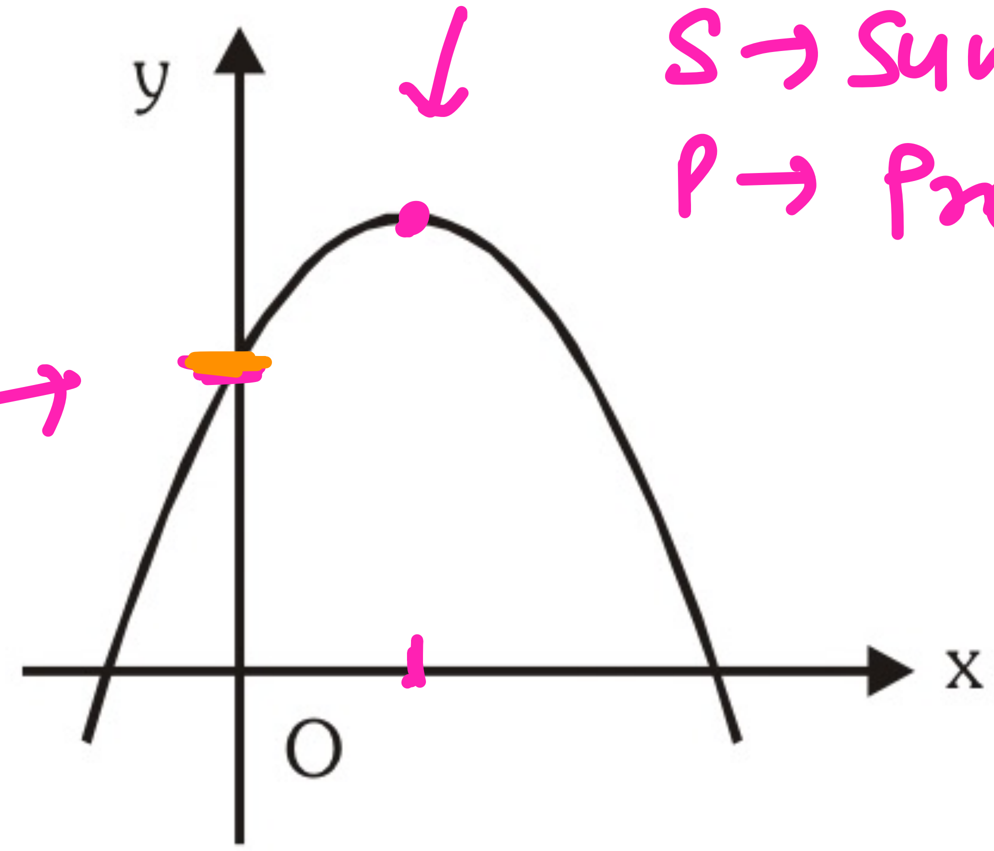



(f) The quadratic expression $ax^2+bx+c$ is a perfect square, then $a > 0$, $D = 0$

i.e. $x^2 - 2x + 1 = (x - 1)^2$ but $2x - x^2 - 1$ is not a perfect square.

$\forall \rightarrow$ for all

**E(1)** Graph of $y = ax^2 + bx + c$ is shown in the figure then




$c > 0$

S → Sum of roots
P → Product of roots

(i) $a < 0$ T  (ii) $D > 0$ T

(iii) $S > 0$ T  (iv) $P < 0$ T

(v) $-\dfrac{b}{a} > 0 \ (b > 0)$  (vi) $\dfrac{c}{a} < 0 \ (c > 0)$

(vii) b and c have the same sign and different than a.

**E(2)** Find the set of values of 'a' for which the quadratic equation.

(i) $(a + 4)x^2 - 2ax + 2a - 6 < 0 \ \ \forall\, x \in R$  [**Ans.** $(-\infty, -6)$]

or

(ii) $(a - 1)x^2 - (a + 1)x + (a + 1) > 0, \ \ \forall\, x \in R$  [**Ans.** $\left(\dfrac{5}{3}, \infty\right)$]

**E(3)** If $f(x) = ax^2 + bx + c > 0 \ \forall\, x \in R$ then prove that $g(x) = f(x) + f'(x) + f''(x) > 0 \ (\forall\, x \in R)$.

**E(4)** If $y = x^2 - 3x - 4$ then find the range of y when

(i) $x \in R$  (ii) $x \in [0, 3]$  (iii) $x \in [-2, 0]$

[**Ans.** (i) $[-25/4, \infty)$  (ii) $[-25/4, -4]$  (iii) $[-4, 6]$]

**E(2)** Find the set of values of 'a' for which the quadratic equation.

(i) $(a + 4)x^2 - 2ax + 2a - 6 < 0 \quad \forall x \in R$

or

(ii) $(a - 1)x^2 - (a + 1)x + (a + 1) > 0, \quad \forall x \in R$ (Try)

(i) $(a+4)x^2 - 2ax + 2a - 6 < 0$

$a + 4 < 0 \Rightarrow \boxed{a \in (-\infty, -4)}$

$D < 0$

$(-2a)^2 - 4(a+4)(2a-6) < 0$

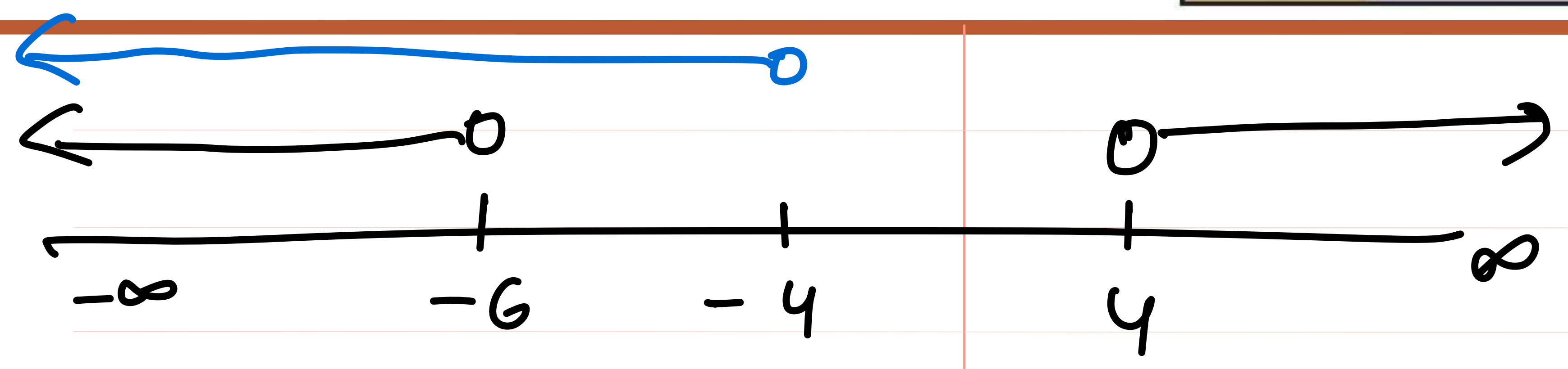
$4a^2 - 4(2a^2 + 2a - 24) < 0$

$4a^2 - 8a^2 - 8a + 96 < 0$

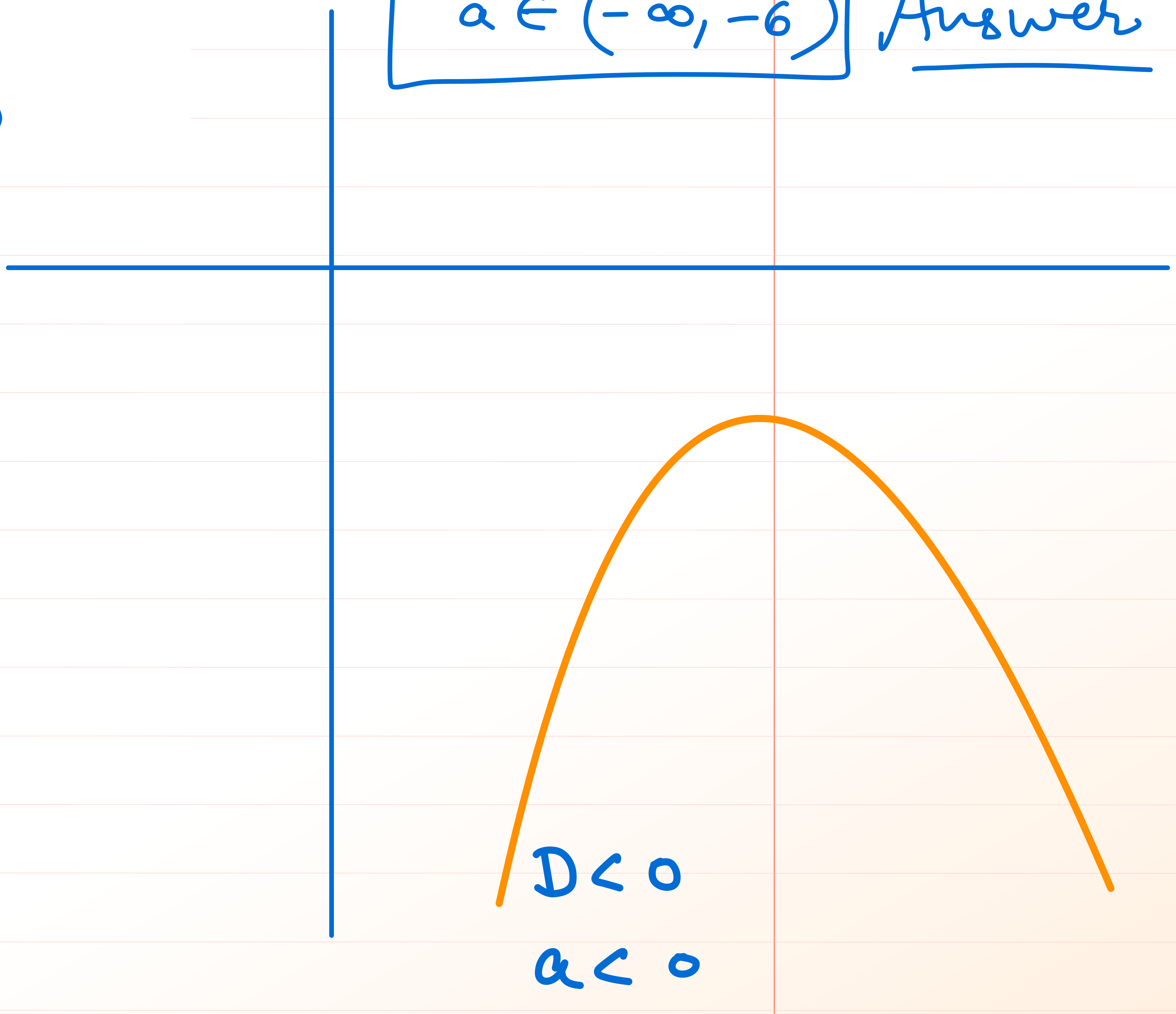
$-4a^2 - 8a + 96 < 0$

$a^2 + 2a - 24 > 0$

$(a+6)(a-4) > 0 \Rightarrow a \in (-\infty, -6) \cup (4, \infty)$




$\boxed{a \in (-\infty, -6)}$ Answer



**E(3)** If $f(x) = ax^2 + bx + c > 0\ \forall\ x \in R$ then prove that $g(x) = \underline{f(x) + f'(x) + f''(x) > 0}\ (\forall\ x \in R)$.

**E(4)** If $y = x^2 - 3x - 4$ then find the range of y when

(i) $x \in R$ (ii) $x \in [0, 3]$ (iii) $x \in [-2, 0]$

3

$f(x) = ax^2 + bx + c$

$f'(x) = 2ax + b$

$f''(x) = 2a$

$f(x) = ax^2 + bx + c > 0$

$D < 0,\ a > 0$

$\boxed{b^2 - 4ac < 0}$ ✓

$g(x) = ax^2 + bx + c + 2ax + b + 2a$

$a > 0$

$g(x) = ax^2 + x(2a+b) + (2a+b+c)$

$D = (2a+b)^2 - 4a(2a+b+c)$

$= 4a^2 + b^2 + \cancel{4ab} - 8a^2 - \cancel{4ab} - 4ac$

$= \underbrace{-4a^2}_{<0} + \underbrace{b^2 - 4ac}_{<0}$

$=$

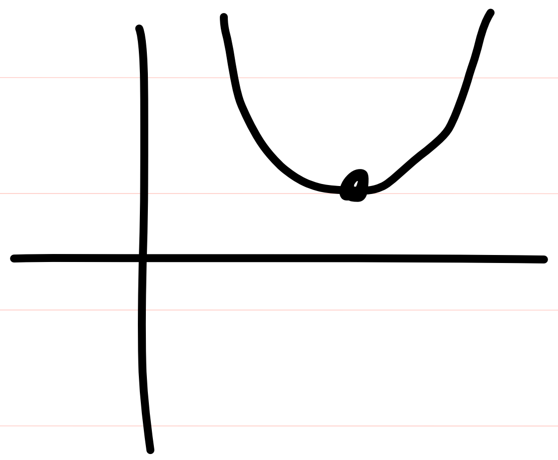
$\boxed{D < 0}$

$\therefore\ g(x) > 0$

(4) $y = 1 . x^2 - 3x - 4$

(i) $x \in R$

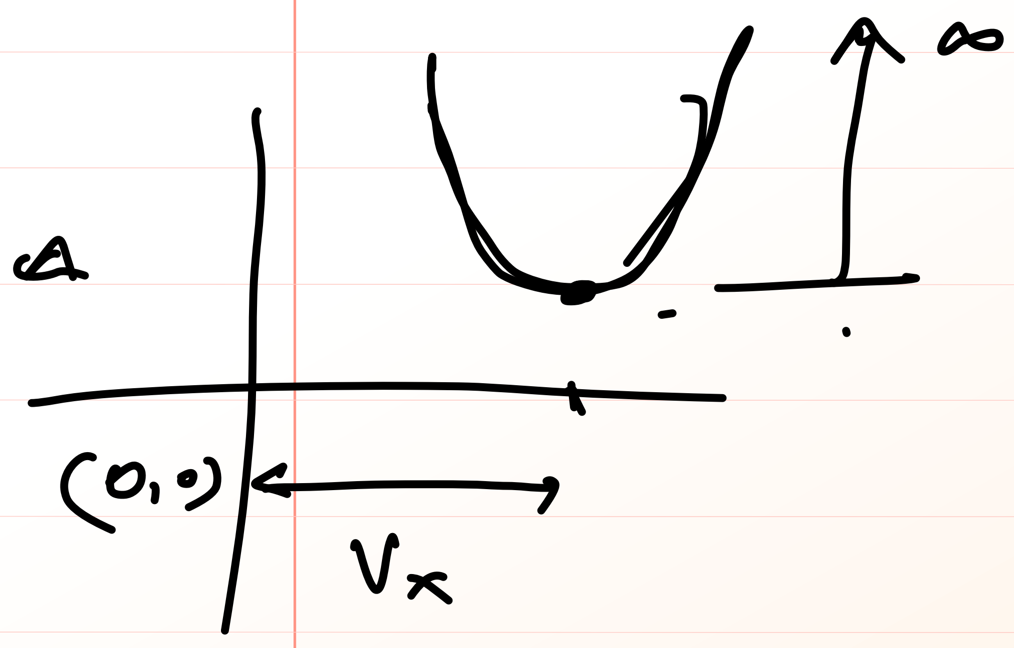
M-I



$V_x = -\frac{b}{2a} = -\frac{(-3)}{2(1)} = \frac{3}{2}$

$V_y = \frac{-D}{4a} = \frac{-\left((-3)^2 - 4(-4)\right)}{4(1)} = \frac{-25}{4}$

Range $\left[-\frac{25}{4}, \infty\right)$




M II

$y = x^2 - 3x - 4$

$x^2 - 3x - 4 - y = 0$

$D \geq 0$ $\qquad a \to 1 ; \quad b = -3 ; \quad c = -4 - y$

$(-3)^2 - 4(1)(-4-y) \geq 0$

$9 + 16 + 4y \geq 0 \Rightarrow y \geq -\frac{25}{4}$

$y \in \left[-\frac{25}{4}, \infty\right)$

M - III

$$y = x^2 - 3x - 4$$

$$y' = 2x - 3 = 0 \Rightarrow y'' = 2 \quad +ve \quad (\text{min})$$

$$x = 3/2$$

$$y = \left(\frac{3}{2}\right)^2 - 3\left(\frac{3}{2}\right) - 4 = -\frac{25}{4}$$

$$\boxed{y \in \left[-\frac{25}{4}, \infty\right)}$$

(ii) $x \in [0, 3]$ $y = x^2 - 3x - 4$

$y$ for $x \in R$ $y \in \left[-\frac{25}{4}, \infty\right)$

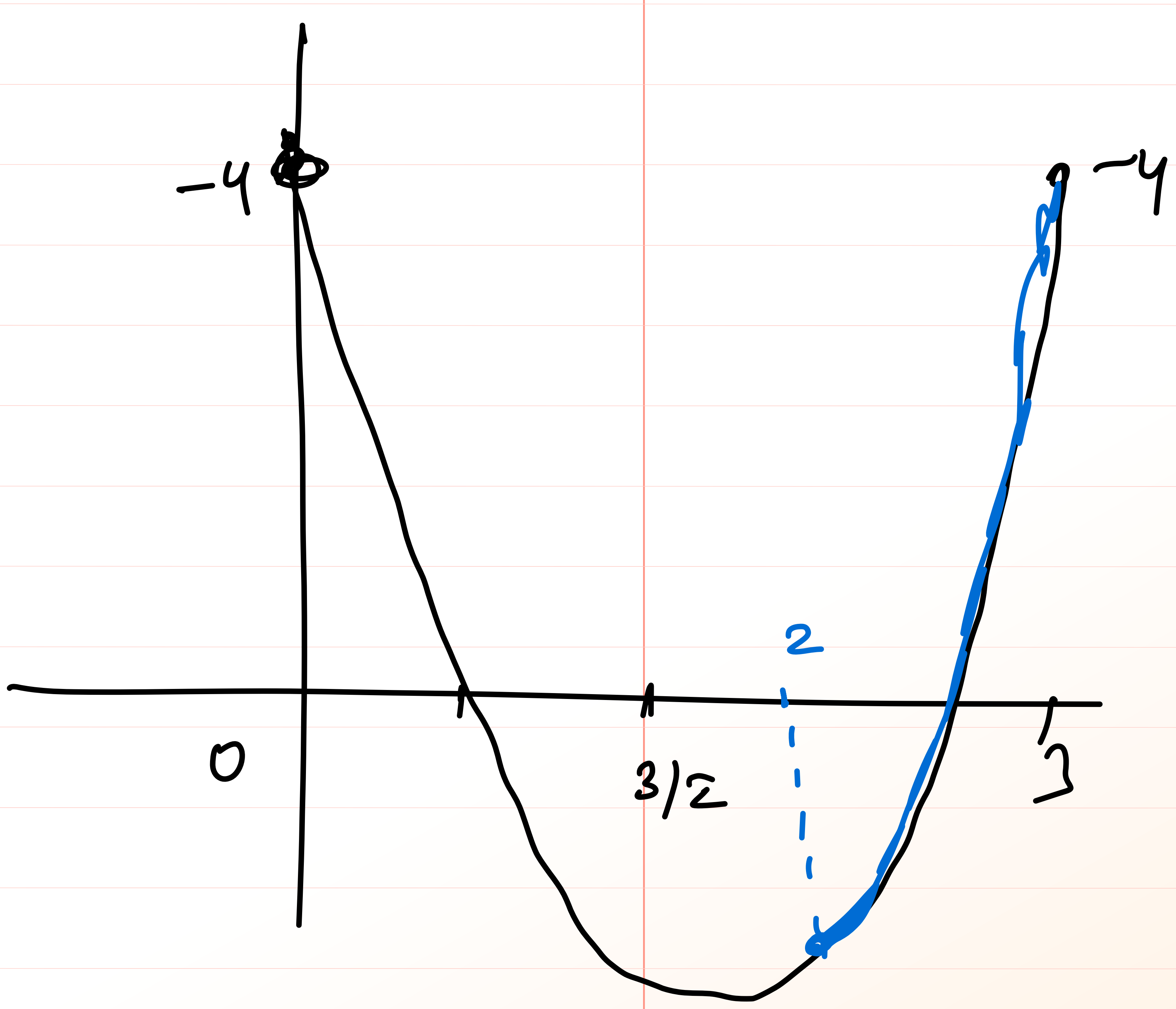
at $x = 0$ $y = -4$

at $x = 3$ $y = 3^2 - 3(3) - 4 = -4$

$y_{max} = -4$

$y_{min} = \left[-\frac{25}{4}, -4\right]$

$x = 3/2$



$y = x^2 - 3x - 4$

at $x = 0$ $\quad y = -4$

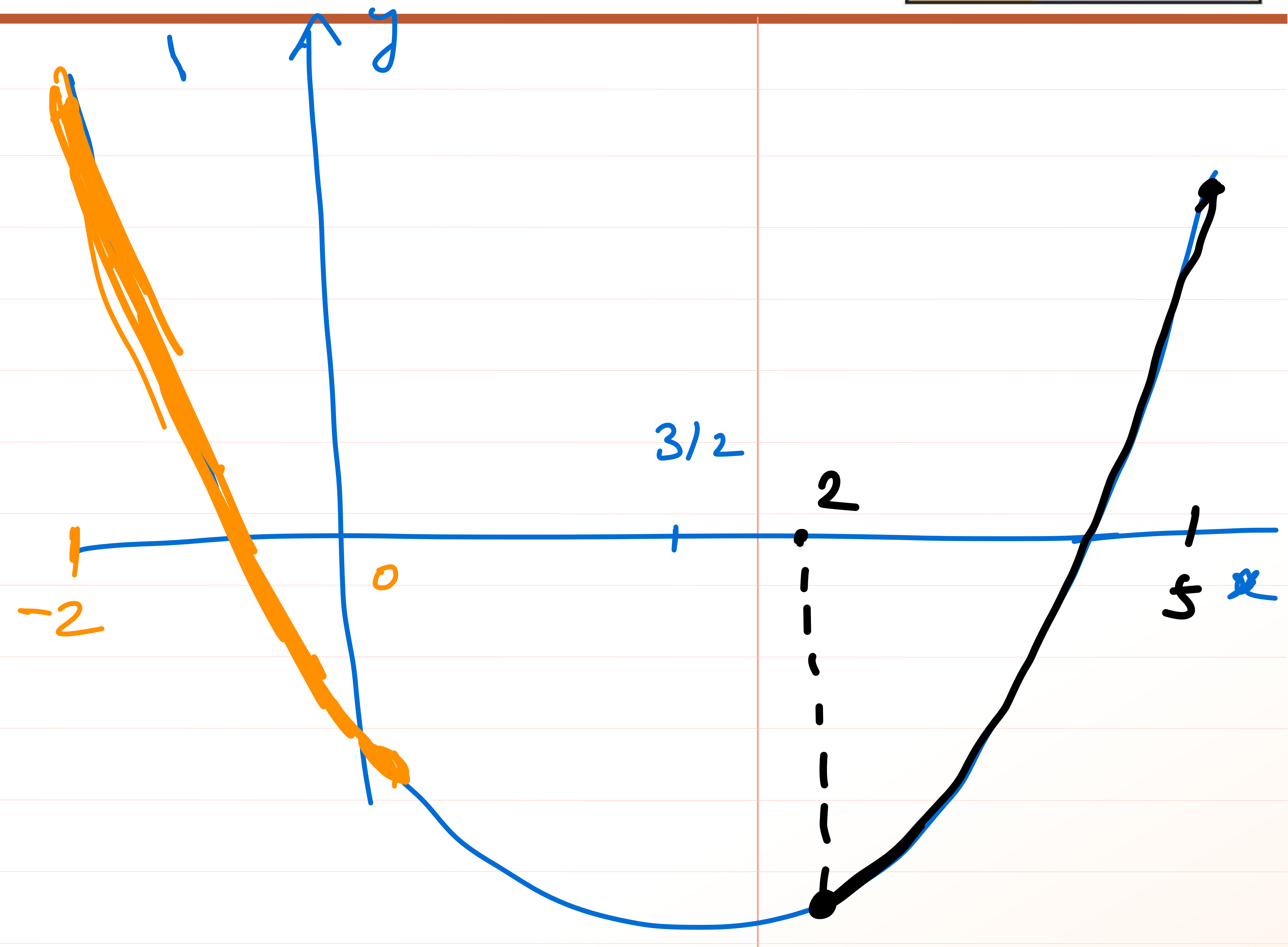
at $x = -2$ $\quad y = 4 + 6 - 4 = 6$

Range $[-4, 6]$

Q if $x \in [2, 5]$

at $x=2$ $\quad y = 4 - 6 - 4 = -6$

at $x=5$ $\quad y = 25 - 15 - 4 = 6$



# Quadratic Equation

## Lecture -4

## COMPUTING THE MAXIMUM OR MINIMUM VALUES OF RATIONAL FUNCTION :

## EXAMPLES :

**(1)** If x is real, then prove that $\frac{x^2 - x + 1}{x^2 + x + 1}$ lies from $\frac{1}{3}$ to 3.

**(2)** Prove that $y = \frac{(x+1)(x-2)}{x(x+3)}$ can have any value in $(-\infty, \infty)$ for $x \in R$.

**(3)** Find all possible value of 'a' for which the expression $\frac{ax^2 - 7x + 5}{5x^2 - 7x + a}$ may be capable of all values, x being any real quantity. [**Ans.** $a \in (-12, 2)$]

**(4)** If x is real, prove that the expression $y = \frac{x^2 + 2x - 11}{2(x-3)}$ can have all numerical values except which lie between 2 and 6.

**(5)** Find the maximum and minimum value of $y = \frac{x^2 + 14x + 9}{x^2 + 2x + 3}$ $\forall\, x \in R$. [**Ans.** 4,–5]

(1)

$$y = \frac{x^2 - x + 1}{x^2 + x + 1}$$

$$yx^2 + xy + y = x^2 - x + 1$$

$$yx^2 - x^2 + xy + x + y - 1 = 0$$

$$x^2(y-1) + x(y+1) + (y-1) = 0$$

$$D \geq 0$$

$$(y+1)^2 - 4(y-1)(y-1) \geq 0$$

$$(y^2 + 2y + 1) - 4(y^2 - 2y + 1) \geq 0$$

$$-3y^2 + 10y - 3 \geq 0$$

$$3y^2 - 10y + 3 \leq 0$$

$$3y^2 - 9y - y + 3 \leq 0 \Rightarrow (3y-1)(y-3) \leq 0$$

$$y \in \left[\frac{1}{3}, 3\right]$$

(2)

$$y = \frac{(x+1)(x-2)}{x(x+3)}$$

$$yx^2 + 3xy = x^2 - x - 2$$

$$x^2(y-1) + x(3y+1) + 2 = 0$$

$\underline{D \geq 0}$

$$(3y+1)^2 - 4(y-1)(2) \geq 0$$

$$9y^2 + 1 + 6y - 8y + 8 \geq 0$$

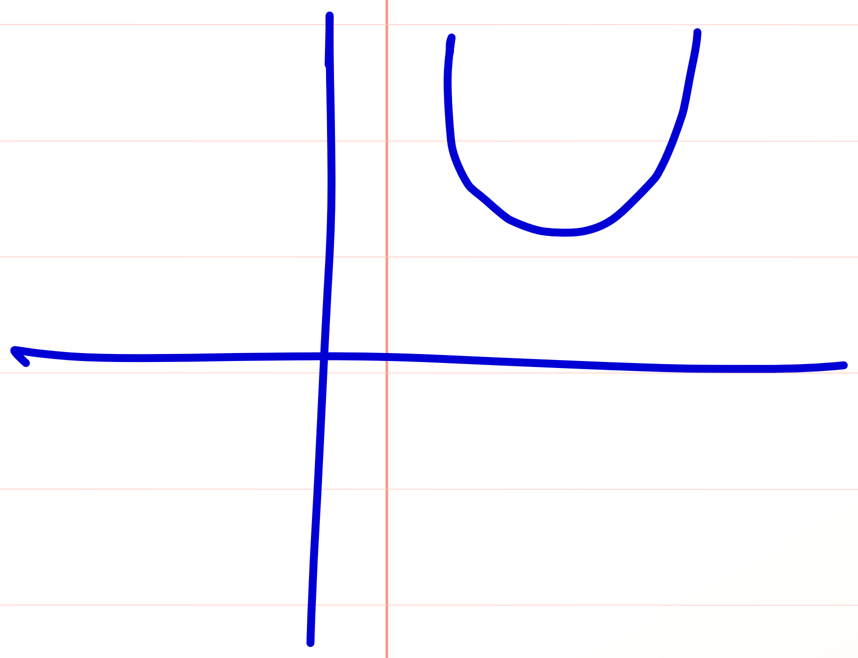
$$\underbrace{9y^2 - 2y + 9}\ \underline{\geq 0}$$

$$\boxed{y \in R}$$

$$\boxed{y \in (-\infty, \infty)}$$

$$9x^2 - 2x + 9 \geq 0$$

$x \in R$ ✓



Q 3

$$y = \frac{ax^2 - 7x + 5}{5x^2 - 7x + a}$$

$$5yx^2 - 7xy + ay = ax^2 - 7x + 5$$

$$x^2(5y - a) + x(-7y + 7) + (ay - 5) = 0$$

$D \geq 0$

$$(-7y + 7)^2 - 4(5y - a)(ay - 5) \geq 0$$

$$49y^2 - 98y + 49 - 4(5ay^2 - a^2y - 25y + 5a) \geq 0$$

$$y^2(49 - 20a) + y(-98 + 4a^2 + 100) + (49 - 20a) \geq 0$$

$D' \leq 0$

$$\boxed{49 - 20a > 0}$$

$$a < \frac{49}{20}$$

$$(4a^2+2)^2 - 4(49-20a)^2 \leq 0$$

$(a-4)$

$$(2a^2+1)^2 - (49-20a)^2 \leq 0$$

$$(2a^2+1+49-20a)(2a^2+1-49+20a) \leq 0$$

$$(2a^2-20a+50)(2a^2+20a-48) \leq 0$$

$$(a^2-10a+25)(a^2+10a-24) \leq 0$$

$$(a-5)^2 \quad (a+12)(a-2) \leq 0$$

$$\boxed{a \in [-12, 2] \cup \{5\}}$$

check for $-12, 2$ and $5$ whether it can be included.

**(4)** If x is real, prove that the expression $y = \dfrac{x^2+2x-11}{2(x-3)}$ can have all numerical values except which lie between 2 and 6.

$$2xy - 6y = x^2 + 2x - 11$$

$$x^2 + x(2-2y) + 6y - 11 = 0$$

$$(2-2y)^2 - 4(6y-11) \geq 0$$

$$(1-y)^2 - (6y-11) \geq 0$$

$$1 + y^2 - 2y - 6y + 11 \geq 0$$

$$y^2 - 8y + 12 \geq 0$$

$$(y-6)(y-2) \geq 0$$

$$y \in (-\infty, 2] \cup [6, \infty)$$

5. Find the maximum and minimum value of $y = \dfrac{x^2 + 14x + 9}{x^2 + 2x + 3}$ $\forall x \in R$.

$$x^2(y-1) + x(2y-14) + 3y-9 = 0$$

$$(2y-14)^2 - 4(y-1)(3y-9) \geq 0$$

$$(y-7)^2 - (y-1)(3y-9) \geq 0$$

$$y^2 - 14y + 49 - 3y^2 + 12y - 9 \geq 0$$

$$-2y^2 - 2y + 40 \geq 0$$

$$y^2 + y - 20 \leq 0$$

$$(y+5)(y-4) \leq 0$$

$$y \in [-5, 4]$$

Location of roots :→ $f(x) = ax^2 + bx + c$

Type 1 Both roots of QE are greater than a specific number (d).

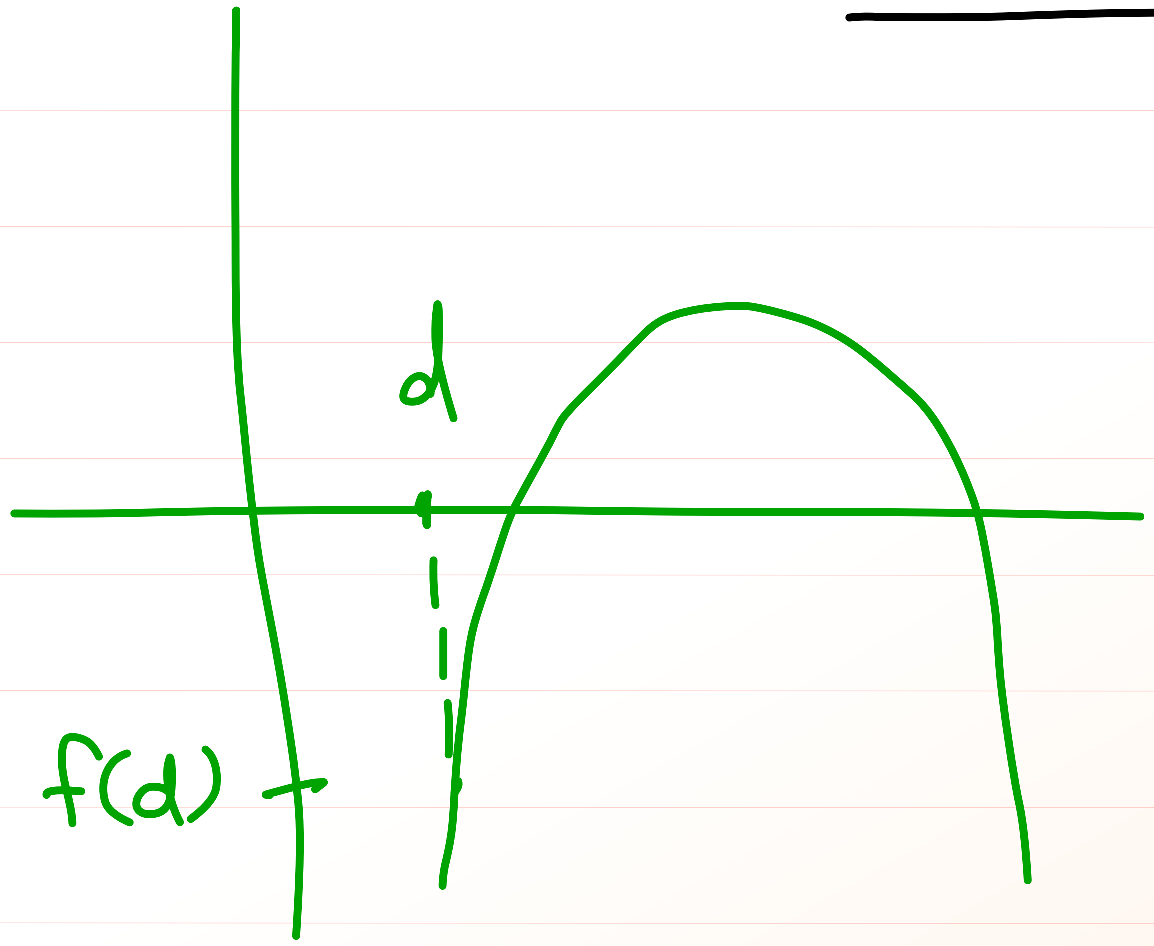
$\boxed{D \geq 0}$

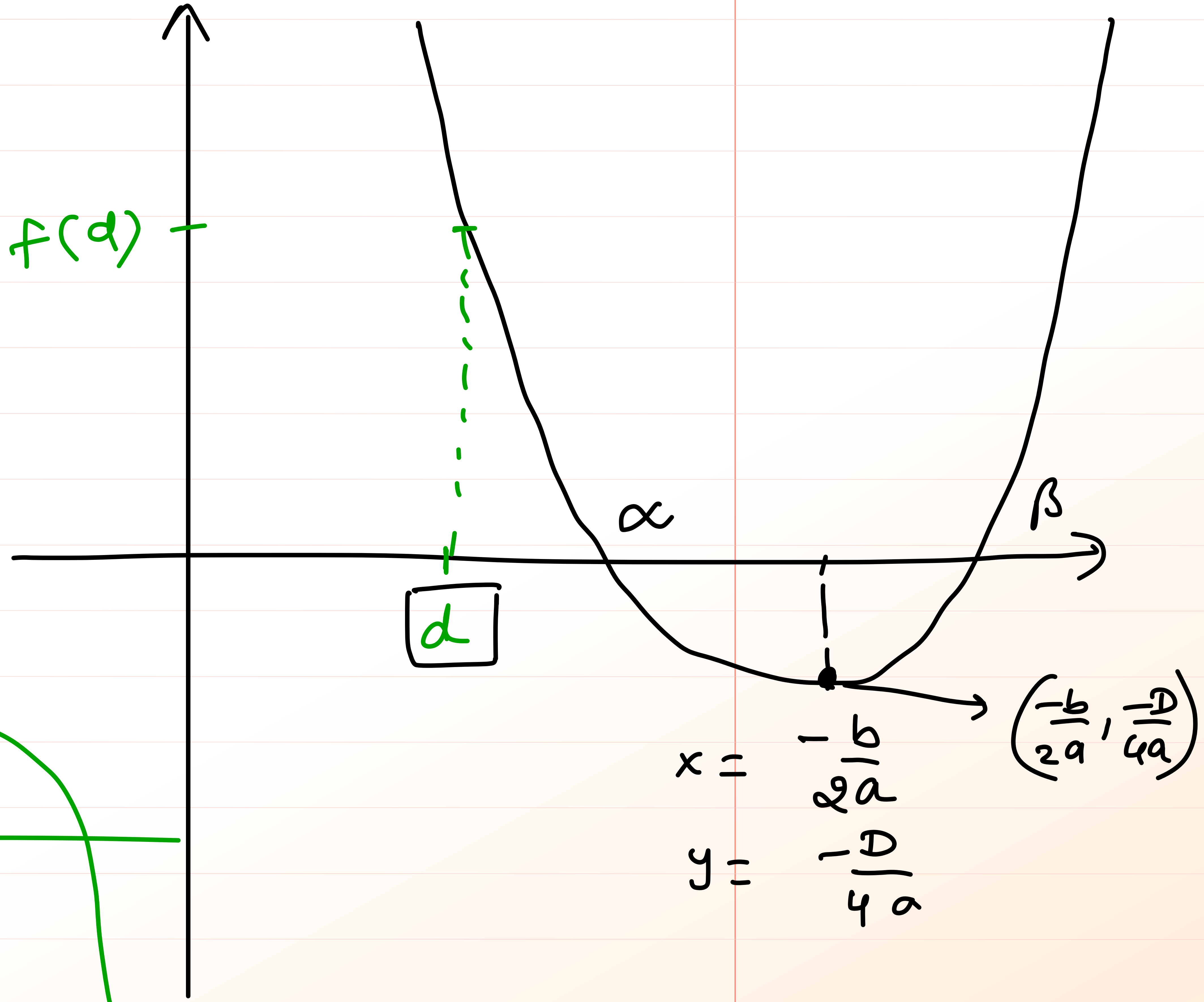
$a > 0 ;\ f(d) > 0$

$a < 0 ;\ f(d) < 0$

$\rightarrow a \cdot f(d) > 0$

$d < \frac{-b}{2a}$




$D \geq 0$

$a \cdot f(d) > 0$

$d < \frac{-b}{2a}$

# Quadratic Equation

## Lecture -5

## Location of roots : → $f(x) = ax^2+bx+c$

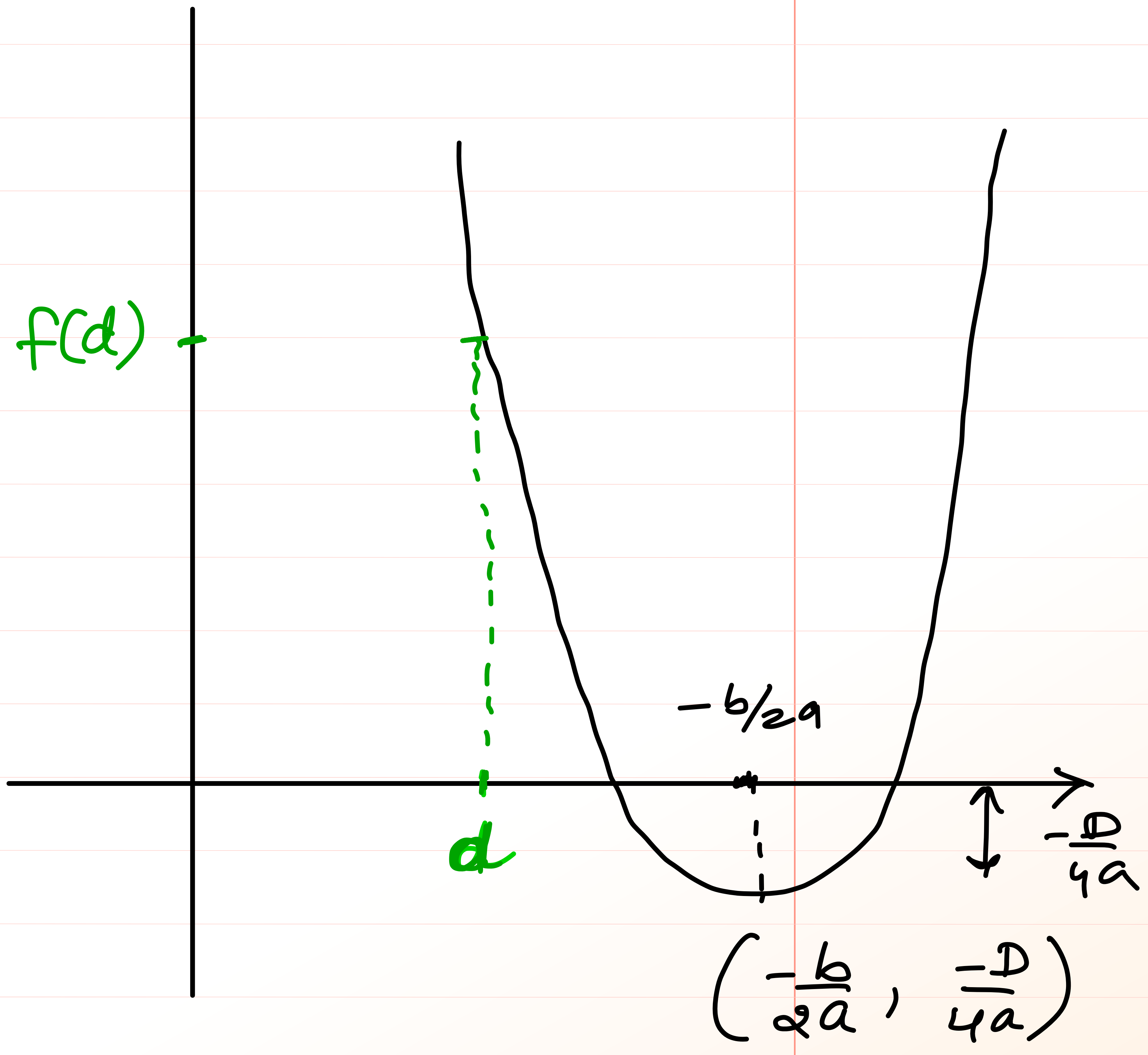
**Type (i)** Both roots of QE are greater than a specific number (d).

$$\begin{cases} \underline{a>0 \ ; \ f(d)>0} \longrightarrow a\cdot f(d)>0 \\ \underline{a<0 \ ; \ f(d)<0} \longrightarrow a\cdot f(d)>0 \end{cases}$$

$$\boxed{a\cdot f(d)>0}$$

$$\boxed{D \geq 0}$$

$$\boxed{-\frac{b}{2a} > d}$$

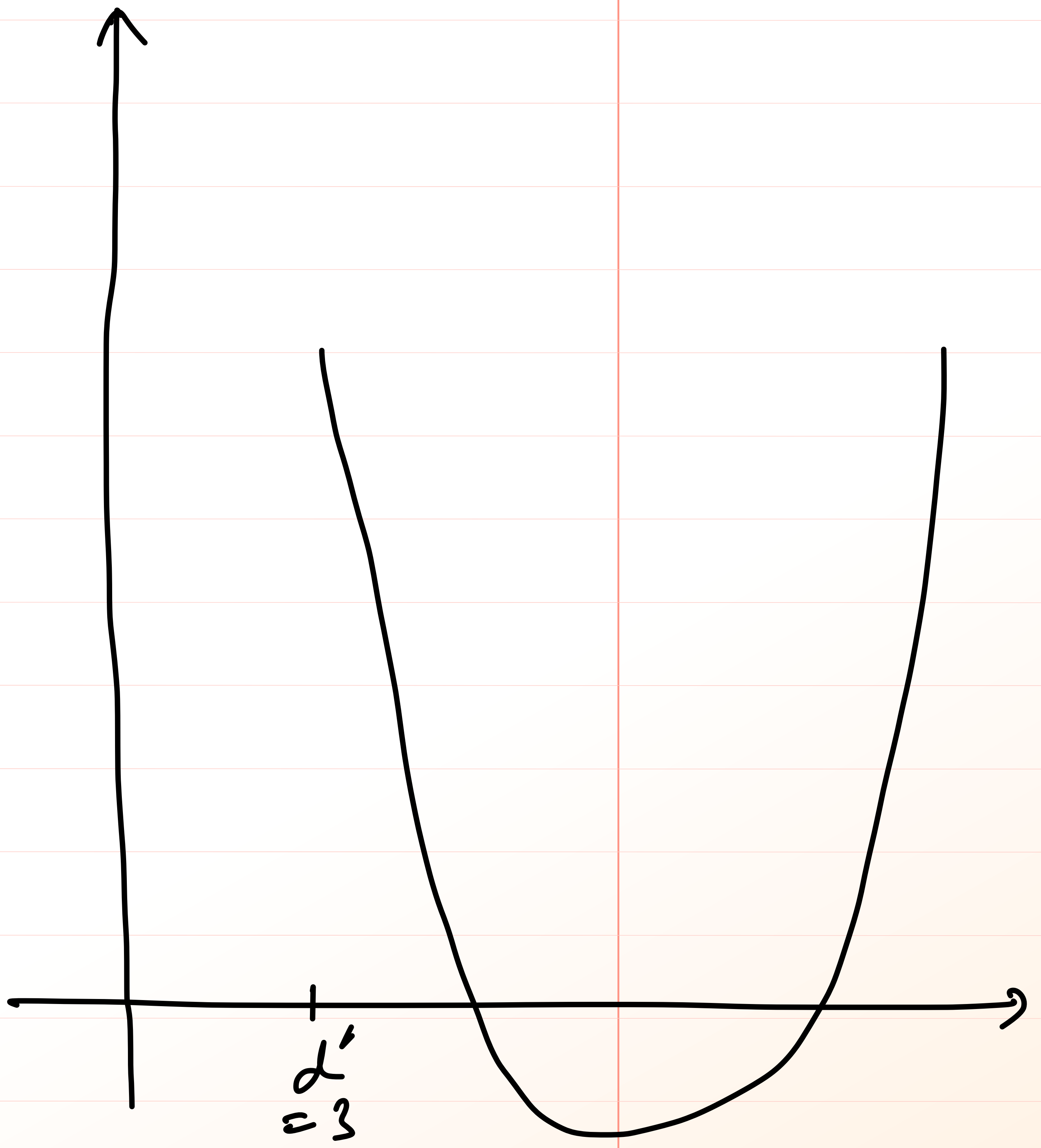


Q find all values of 'd' for which both root of the equation $1.x^2-6dx+(2-2d+9d^2)=0$ exceed the number 3.

$a\cdot f(d)>0 \Rightarrow$

$D \geq 0$

$\frac{-b}{2a} > d$




$f(3) = 3^2 - 6d(3) + (2-2d+9d^2)$

$= 9 - 18d + 2 - 2d + 9d^2$

$f(3) = 9d^2 - 20d + 11$

$a\cdot f(d)>0$

$1\cdot f(3)>0$

$1\cdot f(3) > 0$

$9d^2 - 20d + 11 > 0$

$\boxed{d \in (-\infty, 1) \cup \left(\frac{11}{9}, \infty\right)}$

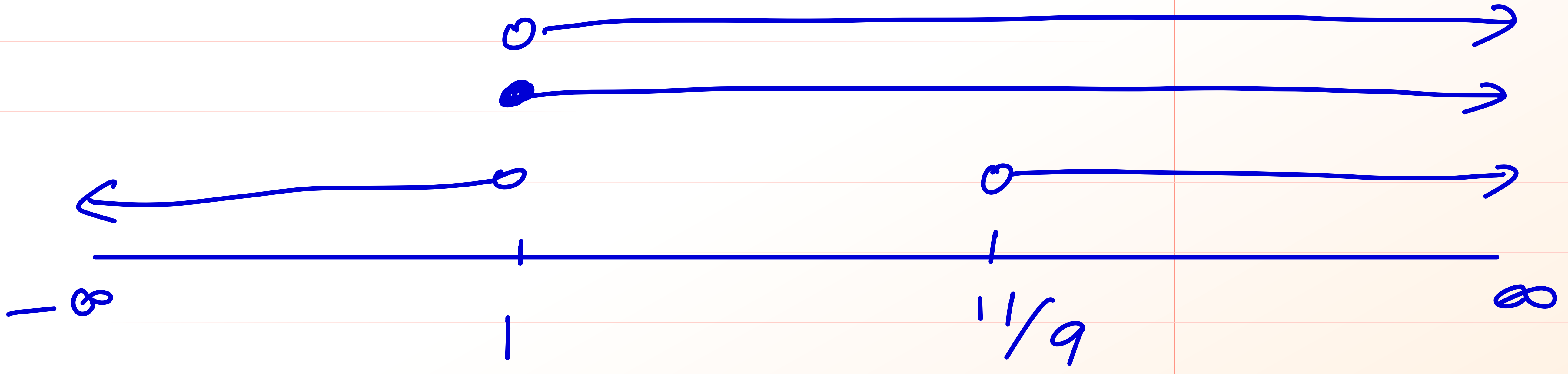
1. $x^2 - 6dx + (2 - 2d + 9d^2) = 0$

$$(36d^2) - 4(2 - 2d + 9d^2) \geq 0$$

$$-8 + 8d \geq 0$$

$$\boxed{d \geq 1}$$ ✓

$$\frac{-b}{2a} > 3 \Rightarrow -\frac{(-6d)}{2(1)} > 1 \Rightarrow \boxed{d > 1}$$ ✓



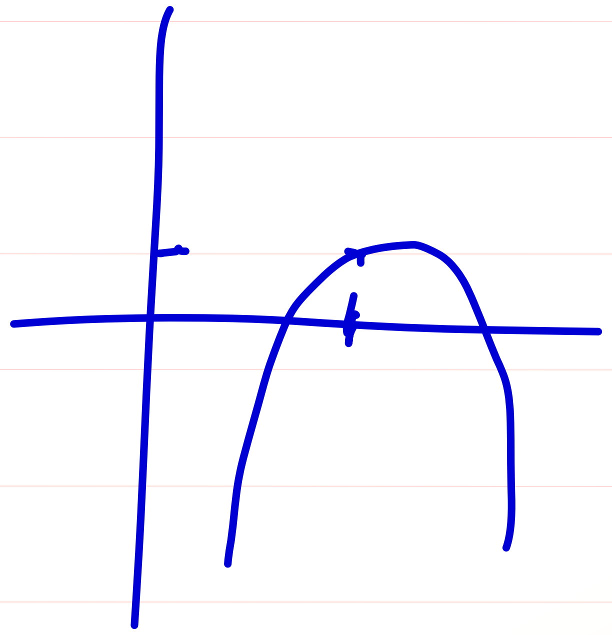

$$\boxed{d \in \left(\frac{11}{9}, \infty\right)}$$

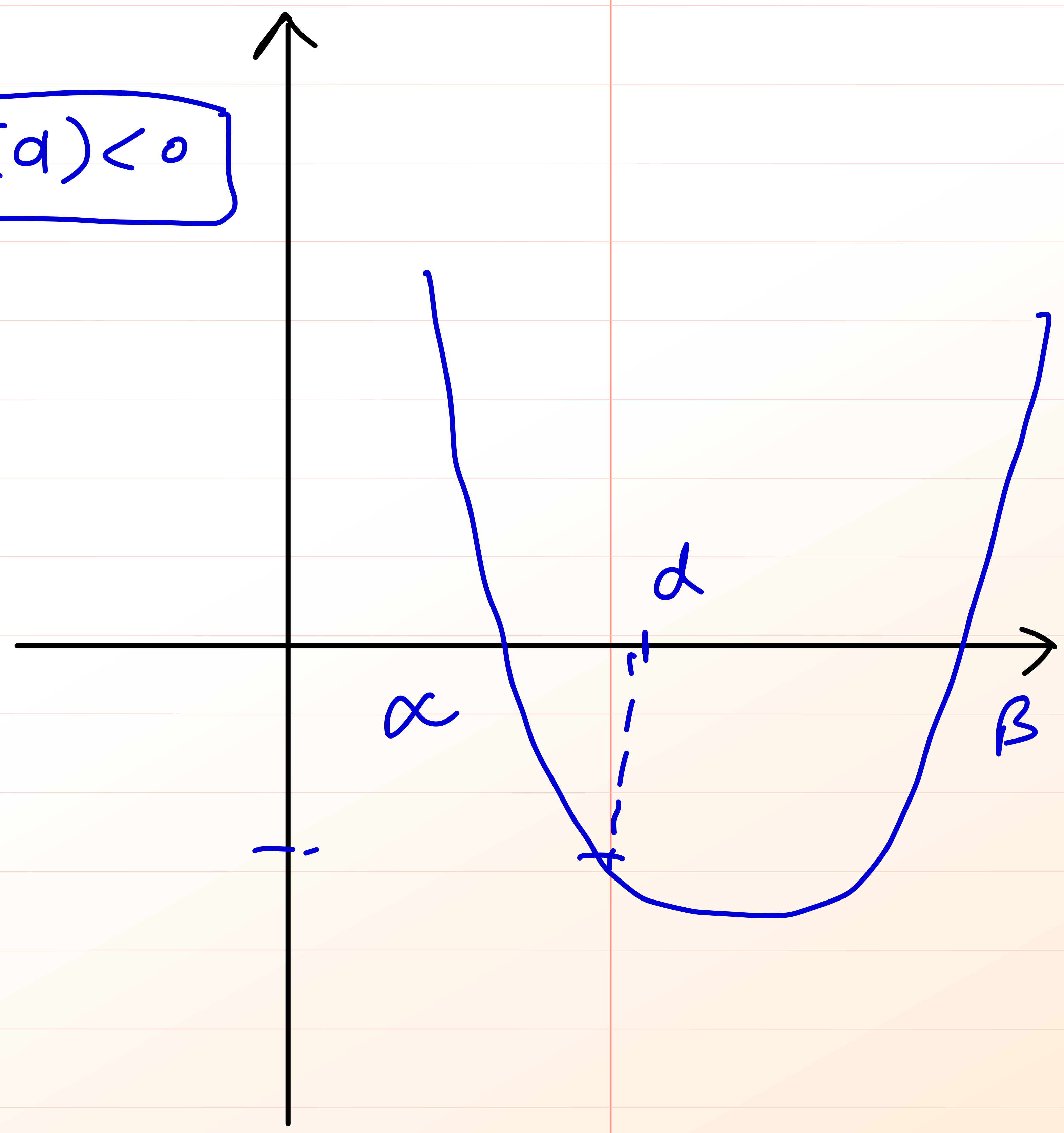
**Type-2** Both roots lie on either side of a fixed number ($d$). [Alternatively one root is greater than $d$ and other is less than $d$; or $d$ lies between the roots of given QE).

$a > 0$; $f(d) < 0 \longrightarrow a \cdot f(d) < 0$

$a < 0$; $f(d) > 0 \longrightarrow a \cdot f(d) < 0$

$\boxed{a \cdot f(a) < 0}$






**Note** Consideration of discriminant is not necessary.

Q find the value of a for which one root of the equation $x^2-(2a+3)x+a^2=0$ exceed 3 and other is smaller than 3.

$a \cdot f(d) < 0$

1. $f(3)<0$

$a^2-6a<0$

$\boxed{a \in (0,6)}$

Answer

$f(3)= 3^2-(2a+3)3+a^2$

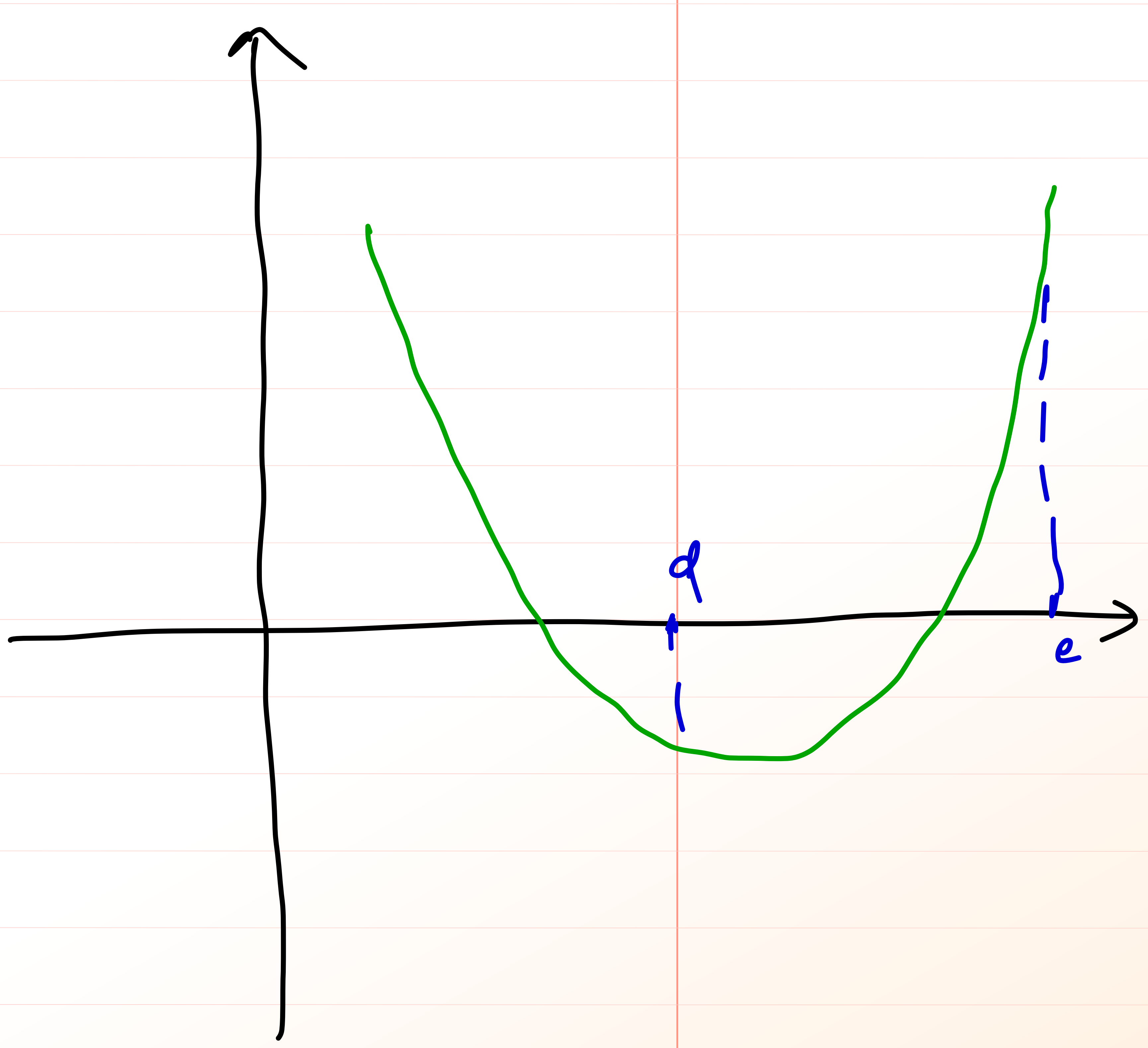
$= 9-6a-9+a^2$

$= a^2-6a$

## Type -3

Exactly one root lies in the interval $(d, e)$ (when $d<e$)

$f(d) < 0$ ; $f(e) > 0$ $\rightarrow$ $\boxed{f(d) \cdot f(e) < 0}$




**Note** In this case also check for end points. If interval is closed $[d, e]$ then $f(d) = 0$ or $f(e) = 0 \Rightarrow$ no other root should lie in $(d, e)$.

**Q** Find all possible values of 'K' for which exactly one root of QE $x^2 - (6K-1)x - 2K = 0$ lie in interval $(-1, 1)$.

$$f(-1) \cdot f(1) < 0$$

$$\left[(-1)^2 - (6K-1)(-1) - 2K\right]\left[1^2 - (6K-1) - 2K\right] < 0$$

$$(1 + 6K - 1 - 2K)(1 - 6K + 1 - 2K) \leq 0$$

$$(4K)(-8K + 2) < 0$$

$$(4K)(4K - 1) > 0$$

$$\boxed{K \in (-\infty, 0) \cup \left(\frac{1}{4}, \infty\right)}$$

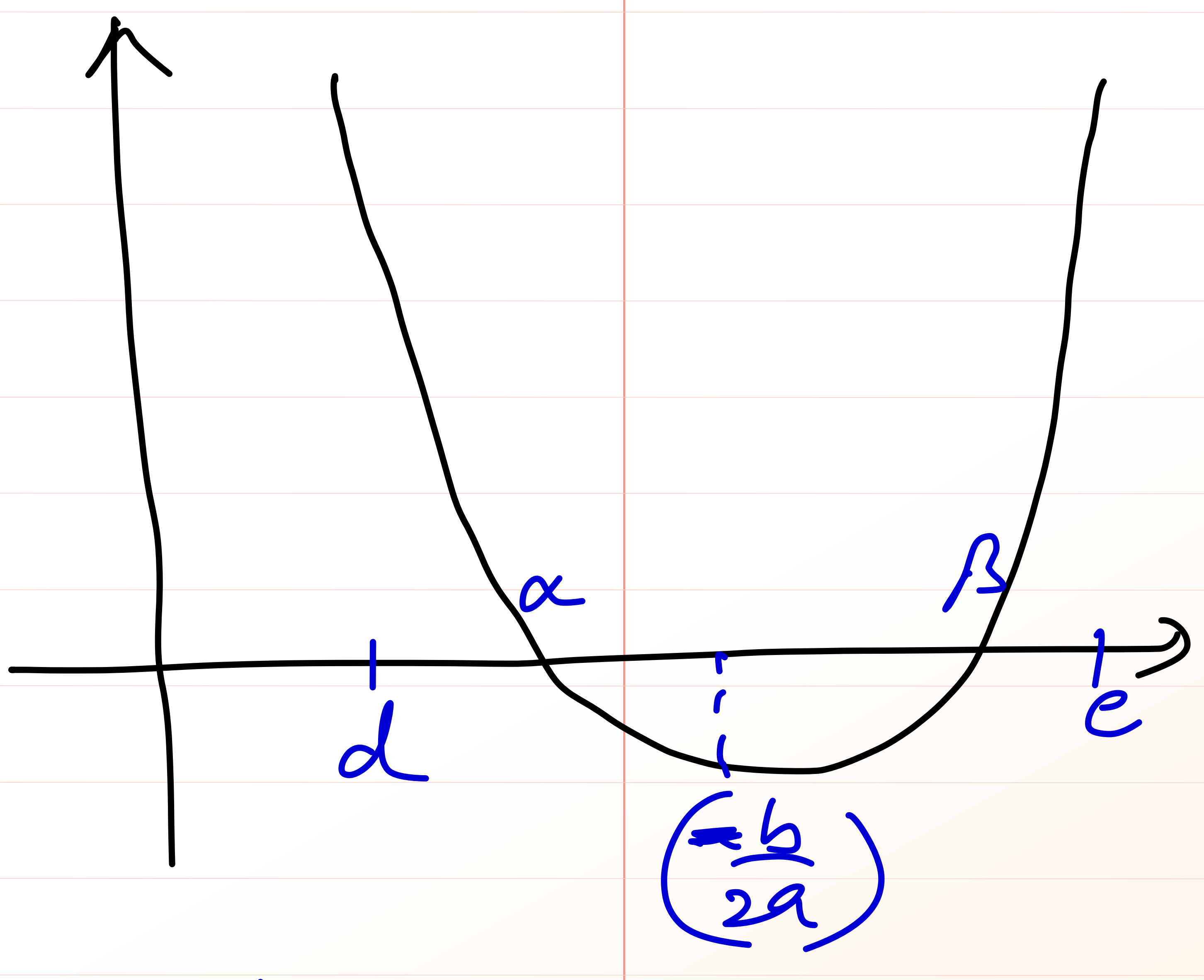
**Type-4** when both roots are confined between the number d and e ( d< e)

(i) $D \geq 0$

(ii) $a \cdot f(d) > 0$

(iii) $a \cdot f(e) > 0$

(iv) $d < \frac{-b}{2a} < e$



**Q** If $\alpha, \beta$ are the roots of the equation

$x^2 + 2(K-3)x + 9 = 0$ $(\alpha \neq \beta)$ If $(\alpha, \beta) \in (-6, 1)$

then find $K$.

(i) $D > 0$ $\quad 4(K-3)^2 - 4(9) > 0$

$(K-3)^2 - 9 > 0$

$(K-3+3)(K-3-3) > 0 \Rightarrow$ $K \in (-\infty, 0) \cup (6, \infty)$

(ii) $1 \cdot f(-6) > 0 \Rightarrow$ $36 + 2(-6)(K-3) + 9 > 0$

$-12K + 81 > 0$

$K < \frac{81}{12} \Rightarrow \boxed{K \in \left(-\infty, \frac{27}{4}\right)}$

(iii) $1 \cdot f(1) > 0 \Rightarrow$ $1 + 2(K-3)1 + 9 > 0$

$K - 3 > -5 \Rightarrow K > -2 \Rightarrow \boxed{K \in (-2, \infty)}$

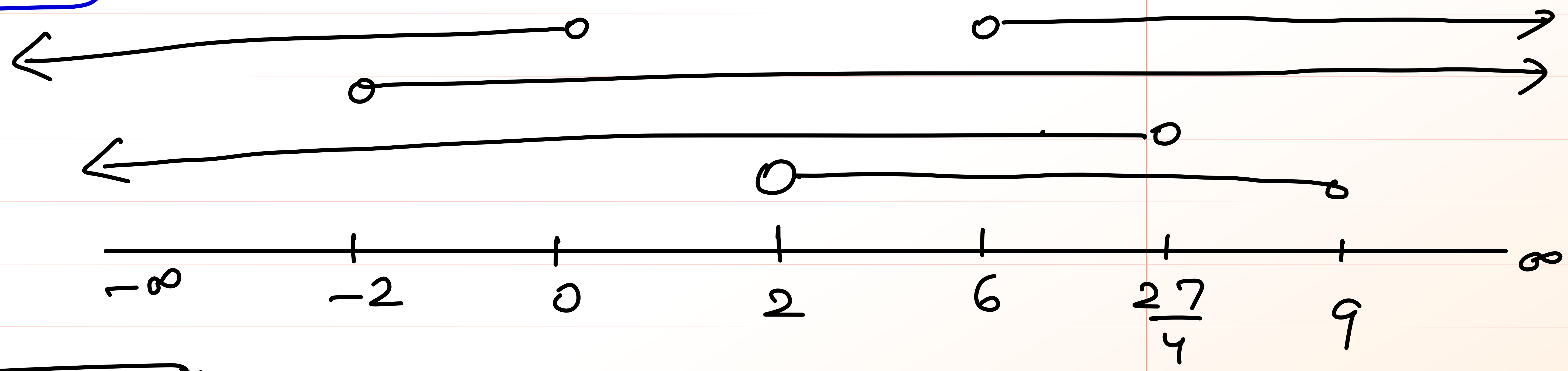
(iv) $-6 < \dfrac{-\cancel{2}(K-3)}{\cancel{2}(1)} < 1$

$$-6 < -K + 3 < 1$$
$$-3 \qquad -3 \qquad -3$$

$$-9 < -K < -2$$

$$\boxed{K \in (2, 9)}$$

Intersection



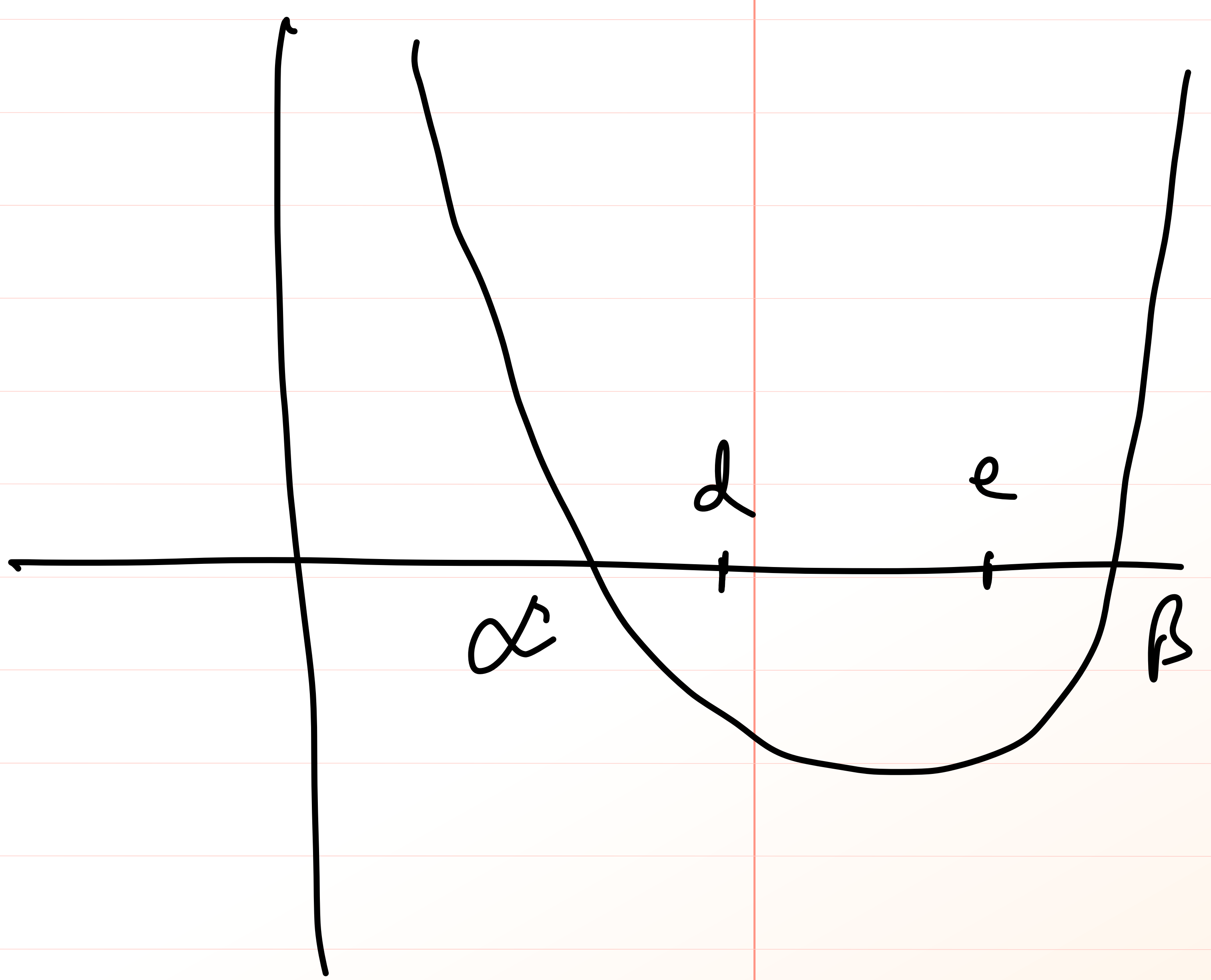

$$\boxed{K \in \left(6, \frac{27}{4}\right)}$$

Answer

Type 5 one root is greater than $e$ and the other root is less than $d$. $(d < e)$

(i) $a \cdot f(d) < 0$

(ii) $a \cdot f(e) < 0$



Q find all the values of k for which one root of QE $(k-5)x^2 - 2kx + k-4 = 0$ is smaller than 1 and other root exceed 2.

$a \cdot f(1) < 0 \Rightarrow (k-5)(-9) < 0 \Rightarrow (k-5) > 0$

$\boxed{k > 5}$ ✓

$f(1) = \cancel{k}-5-2\cancel{k}+\cancel{k}-4$

$= -9$

$a \cdot f(2) < 0$

$(k-5)(k-24) < 0$

$k \in (5, 24)$

$f(2) = 4(k-5) - 2k(2) + k-4$

$= 4k-20-4k+k-4$

$=$

# Quadratic Equation

## Lecture -6

## GENERAL AND MIXED PROBLEM :

For $y = f(x) = ax^2 + bx + c$

if $f(p) < 0$ and $f(q) > 0$

i.e. $f(p)\,f(q) < 0 \Rightarrow$ then the equation $ax^2 + bx + c = 0$ has one root lying between p and q.

1. If $a < b < c < d$, then show that the quadratic equation $(x - a)(x - c) + \lambda(x - b)(x - d) = 0$ has real roots for all real values of $\lambda$.
2. Prove that for any real value of a the inequality, $(a^2 + 3)x^2 + (a + 2)x - 5 < 0$ is true for at least one negative x.

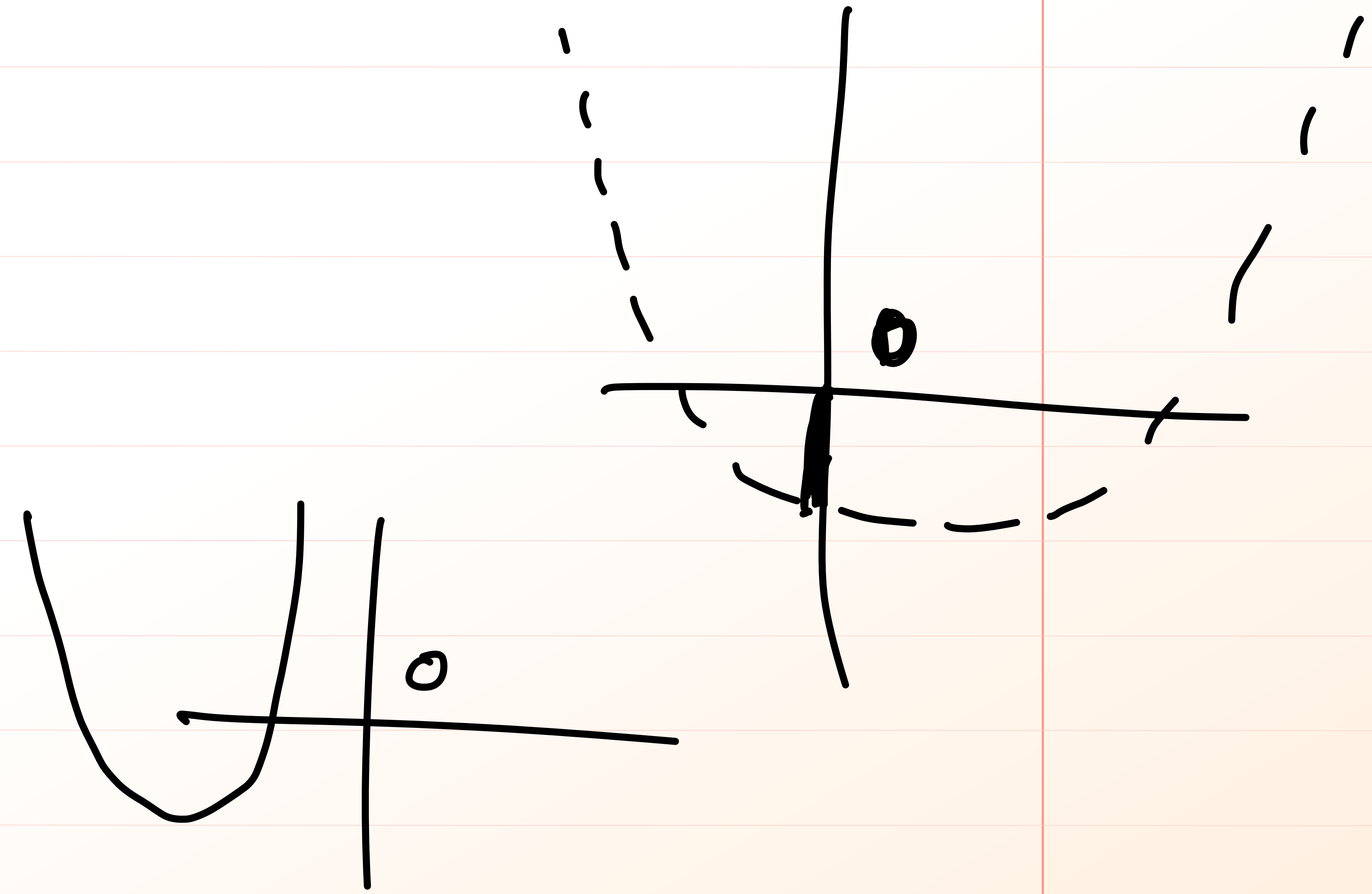
2

$a^2+3>0$ ✓

$f(x) = (a^2+3)x^2 + (a+2)x - 5$

$f(0) = -5$ ✓

$D > 0$



## FINDING THE CONDITION FOR WHICH A GENERAL TWO DEGREE EXPRESSION :

$$f(x, y) = ax^2 + 2hxy + by^2 + 2gx + 2fy + c$$ → general two degree exp.

**can be resolved as a product of two linear factors.**

Required condition is $\boxed{abc + 2fgh - af^2 - bg^2 - ch^2 = 0}$

$$f(x,y) = ax^2 + 2hxy + by^2 + 2gx + 2fy + c$$

$(a_1x + b_1y + c_1)(a_2x + b_2y + c_2)$

$(x + y - 2)(2x + 3y - 5)$

$$\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

$$= a(bc - f^2) - h(hc - gf) + g(hf - bg)$$

$$0 = abc + 2fgh - af^2 - bg^2 - ch^2$$

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Q. If $3x^2 + 2pxy + 2y^2 + 2ax - 4y + 1 = 0$ can be expressed as the product of two linear factors then prove that $p$ will be a root of

$t^2 + 4at + 2a^2 + 6 = 0$ ✓

$a = 3$

$b = 2$

$2h = 2p \Rightarrow h = p$

$2g = 2a \Rightarrow g = a$

$-4 = 2f \Rightarrow f = -2$

$c = 1$

$$abc + 2fgh - af^2 - bg^2 - ch^2 = 0$$

$$3(2)(1) + 2(-2)ap - 3(-2)^2 - 2a^2 - 1p^2 = 0$$

$$-6 - 4ap - 2a^2 - p^2 = 0$$

$$\boxed{p^2 + 4ap + 2a^2 + 6 = 0}$$

Q If the equation $x^2+16y^2-3x+2=0$ is satisfied by all real values of x and y then prove that

$1 \le x \le 2$ and $-\frac{1}{8} \le y \le \frac{1}{8}$

$x^2+16y^2-3x+2=0$

$x^2-3x+(16y^2+2)=0$

$a=1;\ b=-3;\ c=16y^2+2$

$D_1 \ge 0 \Rightarrow 9-4(16y^2+2) \ge 0 \Rightarrow 9-64y^2-8 \ge 0$

$64y^2-1 \le 0$

$\boxed{y \in \left[-\frac{1}{8}, \frac{1}{8}\right]}$

$a=16 \leftarrow 16y^2+\underbrace{x^2-3x+2}=0$

$b=0$

$c=x^2-3x+2$

$D_2 \ge 0 \Rightarrow 0^2-4(16)(x^2-3x+2) \ge 0$

$-64(x^2-3x+2) \ge 0$

$x^2-3x+2 \le 0$

$(x-1)(x-2) \le 0$

$\boxed{x \in [1,2]}$

(3) Prove that $2x^2+3xy+y^2+2y+3x+1$ can be factorized into two linear factors. Find the factors.

$a=2$; $b=1$; $2h=3 \Rightarrow h=\frac{3}{2}$; $2g=3 \Rightarrow g=\frac{3}{2}$; $2f=2 \Rightarrow f=1$; $c=1$

$abc+2fgh-af^2-bg^2-ch^2$

$= 2(1)(1)+2(1)\frac{3}{2}\left(\frac{3}{2}\right)-2(1)^2-1\left(\frac{3}{2}\right)^2-1\left(\frac{3}{2}\right)^2$

$= \cancel{2}+\cancel{\frac{9}{2}}-\cancel{2}-\cancel{\frac{9}{4}}-\cancel{\frac{9}{4}} = 0$

$2x^2+3xy+3x+y^2+2y+1=0$

$2x^2+x(3y+3)+(y^2+2y+1)=0$

HW

Solve after writing in terms of y

$$2x^2 + x(3y+3) + (y^2+2y+1) = 0$$

$$x = \frac{-(3y+3) \pm \sqrt{9(y+1)^2 - 4(2)(y+1)^2}}{2(2)}$$

$$= \frac{-3y-3 \pm (y+1)}{4}$$

$$x = \frac{-3y-3+y+1}{4}$$

$$x = \frac{-2y-2}{4}$$

$$x = \frac{-y-1}{2}$$

$$\boxed{2x = -y-1}$$

$$2x+y+1$$

$$x = \frac{-3y-3-y-1}{4}$$

$$x = \frac{-4y-4}{4}$$

$$\boxed{x = -y-1}$$

$$x+y+1=0$$

## 5. THEORY OF EQUATION :

The equation $a_0x^n + a_1x^{n-1} + a_2x^{n-2} + ..... + a_{n-1}x + a_n = 0$ when $a_0, a_1, a_2, ......, a_n$ are constants, but $a_0 \neq 0$, is a polynomial of degree n. It has n and only n roots. Let $\alpha_1, \alpha_2, \alpha_3, ......., \alpha_n$ be n roots then

$$\Sigma\alpha_1 = (-1)^1 \frac{a_1}{a_0}; \Sigma\alpha_1\alpha_2 = (-1)^2 \frac{a_2}{a_0}, \Sigma\alpha_1\alpha_2\alpha_3 = (-1)^3 \frac{a_3}{a_0}$$

In general $\Sigma\alpha_1\alpha_2\alpha_3.......\alpha_p = (-1)^n \frac{a_p}{a_0}$

**In particular**

If the roots of equation $ax^3 + bx^2 + cx + d = 0$ be $\alpha, \beta, \gamma$

then $\Sigma\alpha = \alpha + \beta + \gamma = -\frac{b}{a}; \Sigma\alpha\beta = \alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a}; \alpha\beta\gamma = -\frac{d}{a}$

If the roots of $ax^4 + bx^3 + cx^2 + dx + e = 0$ be $\alpha, \beta, \gamma, \delta$

then $\Sigma\alpha = \alpha + \beta + \gamma + \delta = -\frac{b}{a}; \Sigma\alpha\beta = (\alpha + \beta)(\gamma + \delta) + \alpha\beta + \gamma\delta = \frac{c}{a};$

$$\Sigma\alpha\beta\gamma = \alpha\beta(\gamma + \delta) + \gamma\delta(\alpha + \beta) = -\frac{d}{a}; \alpha\beta\gamma\delta = \frac{e}{a}$$

$ax^3 + bx^2 + cx + d = 0$

$\alpha + \beta + \gamma = -\frac{b}{a} = (-1)\frac{\text{coe of } x^2}{\text{coe of } x^3}$

$\Sigma\alpha\beta = \frac{c}{a} = (-1)^2\frac{\text{coe of } x}{\text{coe of } x^3}$

$\alpha\beta\gamma = -\frac{d}{a} = (-1)^3\frac{\text{constant term}}{\text{coe of } x^3}$

$$a_0x^n + a_1x^{n-1} + \underline{a_2x^{n-2}} + a_3x^{n-3} + ---- + a_{n-1}x + a_n = 0$$

$$\alpha_1, \alpha_2, \alpha_3, ---, \alpha_{n-1}, \alpha_n$$

$$\alpha_1 + \alpha_2 + \alpha_3 + ----- + \alpha_n = (-1)^1\frac{a_1}{a_0} \Rightarrow \boxed{\Sigma\alpha_1 = (-1)^1\frac{a_1}{a_0}}$$

$$\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_1\alpha_4 + ---- + \alpha_1\alpha_n$$
$$+ \alpha_2\alpha_3 + \alpha_2\alpha_4 + ----- + \alpha_2\alpha_n$$
$$+ ----$$

$$\Sigma\alpha_1\alpha_2 = (-1)^2\frac{\text{coe of } x^{n-2}}{\text{coe of } x^n} = (-1)^2\left(\frac{a_2}{a_0}\right)$$

$$\boxed{\Sigma\alpha_1\alpha_2 = (-1)^2\left(\frac{a_2}{a_0}\right)}$$

$$\boxed{\Sigma\alpha_1\alpha_2\alpha_3 = (-1)^3\left(\frac{a_3}{a_0}\right)}$$

$$\boxed{\Sigma\alpha_1\alpha_2\alpha_3\alpha_4 = (-1)^4\frac{a_4}{a_0}}$$

$$\boxed{\alpha_1\cdot\alpha_2\cdot ---- \alpha_n = \frac{a_n}{a_0}(-1)^n}$$

## Some Results on roots of a polynomial equation :

**(1)** **Factor theorem :** If $\alpha$ is a root of the equation $f(x) = 0$, then $f(x)$ is a exactly divisible by $(x - \alpha)$ and conversely, if $f(x)$ is exactly divisible by $(x - \alpha)$, then $\alpha$ is a root of the equation $f(x) = 0$ and the remainder obtained is $f(\alpha)$.

**(2)** Every equation of an odd degree has at least one real root, whose sign is opposite to that of its last term, provided that the coefficient of the first form is positive.

**(3)** Every equation of an even degree has at least two real roots, one positive and one negative, whose last term is negative, provided that the coefficient of the first term is positive.

**(4)** If an equation has no odd powers of x, then all roots of the equation are complex provided all the coefficients of the equation have positive sign.

**(5)** Let $f(x) = 0$ be a polynomial equation and $\lambda$, $\mu$ are two real numbers,

Then $f(x) = 0$ will have at least one real root or an odd number of roots between $\lambda$ and $\mu$ if $f(\lambda)$ and $f(\mu)$ are of opposite signs.

But if $f(\lambda)$ and $f(\mu)$ are of same signs, then either $f(x) = 0$ has no real roots or an even number of roots between $\lambda$ and $\mu$.

**Q** If $\alpha, \beta, \gamma$ are roots of $x^3 - px^2 + qx - r = 0$ then find $\alpha^2 + \beta^2 + \gamma^2$

$$(\alpha+\beta+\gamma)^2 = \alpha^2+\beta^2+\gamma^2 + 2(\alpha\beta+\beta\gamma+\gamma\alpha)$$

$$p^2 = \alpha^2+\beta^2+\gamma^2 + 2q$$

$$\alpha^2+\beta^2+\gamma^2 = p^2 - 2q$$

## Equation reducible to Quadratic eqn :-

(i) $(x+1)(x+2)(x+3)(x+4) = 120$

$$\frac{x^2+5x+4}{x}$$

$$x + 5 + \frac{4}{x}$$

$$(x^2+5x+4)(x^2+5x+6) = 120$$

Let $x^2+5x = t$

$$(t+4)(t+6) = 120$$

$$t^2 + 10t + 24 = 120$$

$$t^2 + 10t - 96 = 0$$

$$(t+16)(t-6) = 0$$

$t = -16$

$x^2+5x = -16$

$x^2+5x+16 = 0$

$D < 0$

$t = 6$

$x^2+5x = 6$

$x^2+5x-6 = 0$

$(x+6)(x-1) = 0$

$\boxed{x = -6, 1}$

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(2) $(x+1)(x+2)(x+3)(x+6) = 3x^2$

$$\frac{(x^2+7x+6)}{x}\,\frac{(x^2+5x+6)}{x} = \frac{3x^2}{x^2}$$

$$\left(x+\frac{6}{x}+7\right)\left(x+\frac{6}{x}+5\right) = 3$$

Let $x+\frac{6}{x} = t$

$$\frac{x^2}{x}+7x+\frac{6}{x}$$

$$\frac{x^2}{x}+5x+\frac{6}{x}$$

$-4 \pm \sqrt{10}$

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Q $(x-6)^4 + (x-8)^4 = 16$

$x-7 = t$

$(x-6) = t+1$

$x-8 = t-1$

$(t+1)^4 + (t-1)^4 = 16$

$t^4 + \cancel{4t^3} + 6t^2 + \cancel{4t} + 1$

$+ t^4 - \cancel{4t^3} + 6t^2 - \cancel{4t} + 1 = 16$

$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$

$2t^4 + 12t^2 + 2 = 16$

$$\boxed{t^4 + 6t^2 - 7 = 0}$$

$(t^2+7)(t^2-1) = 0$

$t^2 - 1 = 0$

$t = \pm 1$

8, 6

(iv) $\underline{x^8 - x^6} \underline{- x^2 + 1} = 0$

$x^6(x^2-1) - 1(x^2-1) = 0$

$(\underline{x^2-1})(\underline{x^6-1}) = 0$

(v) $16^{\sin^2 x} + 16^{\cos^2 x} = 10$ in $[0, 2\pi]$

$16^{\sin^2 x} + 16^{1-\sin^2 x} = 10$

$16^{\sin^2 x} + \dfrac{16}{16^{\sin^2 x}} = 10$

Let $16^{\sin^2 x} = t$

$t + \dfrac{16}{t} = 10 \Rightarrow t^2 - 10t + 16 = 0$

$(t-8)(t-2) = 0$

$t = 8 \quad t = 2$

$16^{\sin^2 x} = 8$

$2^{4\sin^2 x} = 2^3$

$4\sin^2 x = 3$

$\sin^2 x = \frac{3}{4} \Rightarrow \boxed{\sin x = \pm\frac{\sqrt{3}}{2}} = \frac{\pi}{3}, \frac{4\pi}{3}$

$2^{4\sin^2 x} = 2^1$

$4\sin^2 x = 1$

$\boxed{\sin x = \pm\frac{1}{2}} = \frac{\pi}{6}, \frac{\cancel{8}\pi}{6}$

$8 = 2^3$

$= \left[(16)^{1/4}\right]^3$

$= (16)^{3/4}$

## Miscelleneous Questions : —

1. $\alpha, \beta, \gamma, \delta$ are the roots of the equation $\tan\left(\frac{\pi}{4}+x\right) = 3\tan 3x$ no two of which have equal tangents, find the value of $\tan\alpha + \tan\beta + \tan\gamma + \tan\delta$. [**Ans.** 0]

2. Find the cubic each of whose roots is greater by unity than root of the equation $x^3 - 5x^2 + 6x - 3 = 0$.
[**Ans.** $y^3 - 8y^2 + 19y - 15 = 0$]

**3)** Show that in the equation $x^2 - 3xy + 2y^2 - 2x - 3y - 35 = 0$, for every real value of x there is a real value of y, and for every value of y there is a real value of x.

4. If $\frac{a^2(x-b)(x-c)}{(a-b)(a-c)} + \frac{b^2(x-c)(x-a)}{(b-c)(b-c)} + \frac{c^2(x-a)(x-b)}{(c-a)(c-b)} = x^2$ then prove that it is an identity.

5. If $\frac{(a+x)^2}{(a-b)(a-c)} + \frac{(b+x)^2}{(b-c)(b-a)} + \frac{(c+x)^2}{(c-a)(c-b)} = 1$ then prove that it is an identity.

6. If roots of the equation $2x^3+4x^2-5x+8=0$ are $\alpha, \beta, \gamma$, then find value of

(i) $(2-\alpha)(2-\beta)(2-\gamma)$ (ii) $(1-\alpha^2)(1-\beta^2)(1-\gamma^2)$ (Ans 15, $\frac{135}{4}$)

7. Let $P(x) = x^4+ax^3+bx^2+cx+d$ and $P(1)=1$; $P(2)=2$; $P(3)=3$; $P(4)=4$ then find $P(5)$

$\boxed{\text{Ans } 29}$