ALLEN CAREER INSTITUTE
KOTA (RAJASTHAN)

CONTENTS

PERMUTATION COMBINATION

Recap of Early Classes

Suppose you have a suitcase with a number lock. The number lock has 4 wheels each labelled with 10 digits from 0 to 9. The lock can be opened if 4 specific digits are arranged in a particular sequence with no repetition. Some how, you have forgotten this specific sequence of digits. You remember only the first digit which is 7. In order to open the lock, how many sequences of 3-digits you may have to check with? To answer this question, you may, immediately, start listing all possible arrangements of 9 remaining digits taken 3 at a time. But, this method will be tedious, because the number of possible sequences may be large. Here, in this Chapter, we shall learn some basic counting techniques which will enable us to answer this question without actually listing 3-digit arrangements. In fact, these techniques will be useful in determining the number of different ways of arranging and selecting objects without actually listing them.

Index

PERMUTATION & COMBINATION

1.0 FUNDAMENTAL PRINCIPLE OF COUNTING

(counting without actual counting)

SL AL

If an event A can occur in 'm' different ways and another event B can occur in 'n' different ways, then the total number of different ways of-

(a) simultaneous occurrence of both events in a definite order is $m \times n$. This can be extended to any number of events (known as multiplication principle).

(b) happening exactly one of the events is $m + n$ (known as addition principle).

Example – There are 15 IITs in India and let each IIT has 10 branches, then the IITJEE topper can select the IIT and branch in $15 \times 10 = 150$ number of ways.

Example – There are 15 IITs & 20 NITs in India, then a student who cleared both IITJEE & AIEEE exams can select an institute in $(15 + 20) = 35$ number of ways.

Illustrations

Illustration 1. A college offers 6 courses in the morning and 4 in the evening. The possible number of choices with the student if he wants to study one course in the morning and one in the evening is-

(A) 24 (B) 2 (C) 12 (D) 10

Solution. The student has 6 choices from the morning courses out of which he can select one course in 6 ways.

For the evening course, he has 4 choices out of which he can select one in 4 ways.

Hence the total number of ways $6 \times 4 = 24$. ***Ans.(A)***

Illustration 2. A college offers 6 courses in the morning and 4 in the evening. The number of ways a student can select exactly one course, either in the morning or in the evening-

(A) 6 (B) 4 (C) 10 (D) 24

Solution. The student has 6 choices from the morning courses out of which he can select one course in 6 ways.

For the evening course, he has 4 choices out of which he can select one in 4 ways.

Hence the total number of ways $6 + 4 = 10$. ***Ans. (C)***

2.0 PERMUTATION & COMBINATION

SL AL

2.1 Factorial

A Useful Notation : $n! = n.(n-1).(n-2).............3.\ 2.\ 1$; $n! = n.\ (n-1)!$ where $n \in N$

2.2 Permutation

Each of the arrangements in a definite order which can be made by taking some or all of the things at a time is called a PERMUTATION. In permutation, order of appearance of things is taken into account; when the order is changed, a different permutation is obtained.

Generally, it involves the problems of arrangements (standing in a line, seated in a row), problems on digit, problems on letters from a word etc.

${}^nP_r$ denotes the number of permutations of n ***different*** things, taken r at a time ($n \in N$, $r \in W$, $r \leq n$)

$$ {}^nP_r = n\,(n-1)\,(n-2)\;............\;(n-r+1) = \frac{n!}{(n-r)!} $$

2.3 Combination

Each of the groups or selections which can be made by taking some or all of the things without considering the order of the things in each group is called a COMBINATION.

Generally, involves the problem of selections, choosing, distributed groups formation, committee formation, geometrical problems etc.

${}^nC_r$ denotes the number of combinations of n different things taken r at a time ($n \in N$, $r \in W$, $r \leq n$)

$$ {}^nC_r = \frac{n!}{r!(n-r)!} $$

Illustrations

Illustration 3. If a denotes the number of permutations of (x + 2) things taken all at a time, b the number of permutations of x things taken 11 at a time and c the number of permutations of (x – 11) things taken all at a time such that a = 182 bc, then the value of x is

(A) 15 (B) 12 (C) 10 (D) 18

Solution. $^{x+2}P_{x+2} = a \Rightarrow a = (x+2)!$

$^{x}P_{11} = b \Rightarrow b = \dfrac{x!}{(x-11)!}$

and $^{x-11}P_{x-11} = c \Rightarrow c = (x-11)!$

$\therefore$ $a = 182bc$

$(x+2)! = 182\dfrac{x!}{(x-11)!}(x-11)!$

$\Rightarrow$ $(x+2)(x+1) = 182 = 14 \times 13$

$\therefore$ $x+1 = 13 \quad \Rightarrow x = 12$ **Ans. (B)**

Illustration 4. A box contains 5 different red and 6 different white balls. In how many ways can 6 balls be drawn so that there are atleast two balls of each colour ?

Solution. The selections of 6 balls, consisting of atleast two balls of each colour from 5 red and 6 white balls, can be made in the following ways

Red balls (5)	White balls (6)	Number of ways
2	4	$^5C_2 \times {^6C_4} = 150$
3	3	$^5C_3 \times {^6C_3} = 200$
4	2	$^5C_4 \times {^6C_2} = 75$

Therefore total number of ways = 425 **Ans.**

Illustration 5. How many 4 letter words can be formed from the letters of the word 'ANSWER' ? How many of these words start with a vowel ?

Solution. Number of ways of arranging 4 different letters from 6 different letters are $^6C_4 4! = \dfrac{6!}{2!} = 360$.

There are two vowels (A & E) in the word 'ANSWER'.

Total number of 4 letter words starting with A : A _ _ _ = $^5C_3 3! = \dfrac{5!}{2!} = 60$

Total number of 4 letter words starting with E : E _ _ _ = $^5C_3 3! = \dfrac{5!}{2!} = 60$

$\therefore$ Total number of 4 letter words starting with a vowel = 60 + 60 = 120. **Ans.**

Illustration 6. If all the letters of the word 'RAPID' are arranged in all possible manner as they are in a dictionary, then find the rank of the word 'RAPID'.

Solution. First of all, arrange all letters of given word alphabetically : 'ADIPR'

Total number of words starting with A _ _ _ _ = 4! = 24
Total number of words starting with D _ _ _ _ = 4! = 24
Total number of words starting with I _ _ _ _ = 4! = 24
Total number of words starting with P _ _ _ _ = 4! = 24
Total number of words starting with RAD _ _ = 2! = 2
Total number of words starting with RAI _ _ = 2! = 2
Total number of words starting with RAPD _ = 1
Total number of words starting with RAPI _ = 1

$\therefore$ Rank of the word RAPID = 24 + 24 + 24 + 24 + 2 + 2 + 1 + 1 = 102 **Ans.**

3.0 PROPERTIES OF ^nP_r and ^nC_r

SL AL

(a) The number of permutation of n different objects taken r at a time, when p particular objects are always to be included is $r!.^{n-p}C_{r-p}$ $(p \le r \le n)$

(b) The number of permutations of n different objects taken r at a time, when repetition is allowed any number of times is n^r.

(c) Following properties of ^nC_r should be remembered :

(i) $^nC_r = {^nC_{n-r}}$; $^nC_0 = {^nC_n} = 1$

(ii) $^nC_x = {^nC_y} \Rightarrow x = y$ or $x + y = n$

(iii) $^nC_r + {^nC_{r-1}} = {^{n+1}C_r}$

(iv) $^nC_0 + {^nC_1} + {^nC_2} + \ldots\ldots\ldots\ldots + {^nC_n} = 2^n$

(v) $^nC_r = \dfrac{n}{r}\,{^{n-1}C_{r-1}}$

(vi) ^nC_r is maximum when $r = \dfrac{n}{2}$ if n is even & $r = \dfrac{n-1}{2}$ or $r = \dfrac{n+1}{2}$ if n is odd.

(d) The number of combinations of n different things taking r at a time,

(i) when p particular things are always to be included $= {^{n-p}C_{r-p}}$

(ii) when p particular things are always to be excluded $= {^{n-p}C_r}$

(iii) when p particular things are always to be included and q particular things are to be excluded $= {^{n-p-q}C_{r-p}}$

Illustrations

Illustration 7. There are 6 pockets in the coat of a person. In how many ways can he put 4 pens in these pockets?

(A) 360 (B) 1296 (C) 4096 (D) none of these

Solution. First pen can be put in 6 ways.

Similarly each of second, third and fourth pen can be put in 6 ways.

Hence total number of ways $= 6 \times 6 \times 6 \times 6 = 1296$ ***Ans.(B)***

Illustration 8. A delegation of four students is to be selected from a total of 12 students. In how many ways can the delegation be selected, if-

(a) all the students are equally willing ?

(b) two particular students have to be included in the delegation ?

(c) two particular students do not wish to be together in the delegation ?

(d) two particular students wish to be included together only ?

(e) two particular students refuse to be together and two other particular students wish to be together only in the delegation ?

Solution. ***(a)*** Formation of delegation means selection of 4 out of 12.

Hence the number of ways $= {^{12}C_4} = 495$.

(b) If two particular students are already selected. Here we need to select only 2 out of the remaining 10. Hence the number of ways $= {^{10}C_2} = 45$.

(c) The number of ways in which both are selected $= 45$. Hence the number of ways in which the two are not included together $= 495 - 45 = 450$

(d) There are two possible cases

(i) Either both are selected. In this case, the number of ways in which the selection can be made $= 45$.

(ii) Or both are not selected. In this case all the four students are selected from the remaining ten students. This can be done in $^{10}C_4 = 210$ ways.

Hence the total number of ways of selection $= 45 + 210 = 255$

(e) We assume that students A and B wish to be selected together and students C and D do not wish to be together. Now there are following 6 cases.

(i) (A, B, C) selected, (D) not selected

(ii) (A, B, D) selected, (C) not selected

(iii) (A, B) selected, (C, D) not selected

(iv) (C) selected, (A, B, D) not selected

(v) (D) selected, (A, B, C) not selected

(vi) A, B, C, D not selected

For (i) the number of ways of selection $= {}^{8}C_{1} = 8$

For (ii) the number of ways of selection $= {}^{8}C_{1} = 8$

For (iii) the number of ways of selection $= {}^{8}C_{2} = 28$

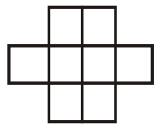
For (iv) the number of ways of selection $= {}^{8}C_{3} = 56$

For (v) the number of ways of selection $= {}^{8}C_{3} = 56$

For (vi) the number of ways of selection $= {}^{8}C_{4} = 70$

Hence total number of ways $= 8 + 8 + 28 + 56 + 56 + 70 = 226$. ***Ans.***

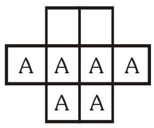
Illustration 9. In the given figure of squares, 6 A's should be written in such a manner that every row contains at least one 'A'. In how manynumber of ways is it possible ?



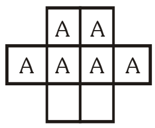
(A) 24 (B) 25 (C) 26 (D) 27

Solution. There are 8 squares and 6 'A' in given figure. First we can put 6 'A' in these 8 squares by ${}^{8}C_{6}$ number of ways.

(I)



(II)



According to question, atleast one 'A' should be included in each row. So after subtracting these two cases, number of ways are $= ({}^{8}C_{6} - 2) = 28 - 2 = 26$. ***Ans. (C)***

Illustration 10. There are three coplanar parallel lines. If any p points are taken on each of the lines, the maximum number of triangles with vertices at these points is :

(A) $3p^2 (p - 1) + 1$ (B) $3p^2 (p - 1)$ (C) $p^2 (4p - 3)$ (D) none of these

Solution. The number of triangles with vertices on different lines $= {}^{p}C_{1} \times {}^{p}C_{1} \times {}^{p}C_{1} = p^3$

The number of triangles with two vertices on one line and the third vertex on any one of the other two lines $= {}^{3}C_{1} \{{}^{p}C_{2} \times {}^{2p}C_{1}\} = 6p.\dfrac{p(p-1)}{2}$

So, the required number of triangles $= p^3 + 3p^2 (p - 1) = p^2 (4p - 3)$ ***Ans. (C)***

****Illustration 11.*** There are 10 points in a row. In how many ways can 4 points be selected such that no two of them are consecutive ?

Solution. Total number of remaining non-selected points = 6

. . . . . .

Total number of gaps made by these 6 points = 6 + 1 = 7

If we select 4 gaps out of these 7 gaps and put 4 points in selected gaps then the new points will represent 4 points such that no two of them are consecutive.

x . . ***x*** . ***x*** . . ***x*** .

Total number of ways of selecting 4 gaps out of 7 gaps $= {}^{7}C_{4}$ ***Ans.***

In general, total number of ways of selection of r points out of n points in a row such that no two of them are consecutive : ${}^{n-r+1}C_{r}$

4.0 FORMATION OF GROUPS

AL

(a) (i) The number of ways in which (m + n) different things can be divided into two groups such that one of them contains m things and other has n things, is $\dfrac{(m+n)!}{m!\ n!}$ $(m \neq n)$.

(ii) If m = n, it means the groups are equal & in this case the number of divisions is $\dfrac{(2n)!}{n!\ n!\ 2!}$. As in any one way it is possible to interchange the two groups without obtaining a new distribution.

(iii) If 2n things are to be divided equally between two persons then the number of ways : $\dfrac{(2n)!}{n!\ n!\ (2!)} \times 2!$.

(b) (i) Number of ways in which (m + n + p) different things can be divided into three groups containing m, n & p things respectively is : $\dfrac{(m+n+p)!}{m!\ n!\ p!}$, $m \neq n \neq p$.

(ii) If m = n = p then the number of groups = $\dfrac{(3n)!}{n!\ n!\ n!\ 3!}$.

(iii) If 3n things are to be divided equally among three people then the number of ways in which it can be done is $\dfrac{(3n)!}{(n!)^3}$.

(c) In general, the number of ways of dividing n distinct objects into ℓ groups containing p objects each and m groups containing q objects each is equal to $\dfrac{n!(\ell+m)!}{(p!)^{\ell}(q!)^{m}\ \ell!m!}$

Here $\ell p + mq = n$

Illustrations

Illustration 12. Find the number of ways of dividing 52 cards among 4 players equally such that each gets exactly one Ace.

Solution. Total number of ways of dividing 48 cards (Excluding 4Aces) in 4 groups $= \dfrac{48!}{(12!)^4 4!}$

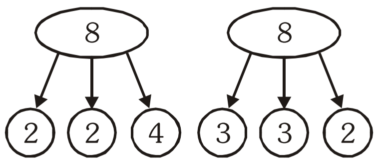
Now, distribute exactly one Ace to each group of 12 cards. Total number of ways $= \dfrac{48!}{(12!)^4 4!} \times 4!$

Now, distribute these groups of cards among four players

$$= \frac{48!}{(12!)^4 4!} \times 4!4! = \frac{48!}{(12!)^4} \times 4!$$ ***Ans.***

Illustration 13. In how many ways can 8 different books be distributed among 3 students if each receives at least 2 books ?

Solution. If each receives at least two books, then the division trees would be as shown below :



(i) (ii)

The number of ways of division for tree in figure (i) is $\left[\dfrac{8!}{(2!)^2 4!2!}\right]$.

The number of ways of division for tree in figure (ii) is $\left[\dfrac{8!}{(3!)^2 2!2!}\right]$.

The total number of ways of distribution of these groups among 3 students

is $\left[\dfrac{8!}{(2!)^2 4!2!} + \dfrac{8!}{(3!)^2 2!2!}\right] \times 3!$. ***Ans.***

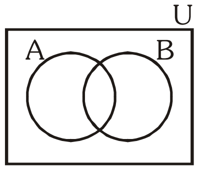
5.0 PRINCIPLE OF INCLUSION AND EXCLUSION

AL

In the Venn's diagram (i), we get

$n(A \cup B) = n(A) + n(B) - n(A \cap B)$

$n(A' \cap B') = n(U) - n(A \cup B)$



(i)

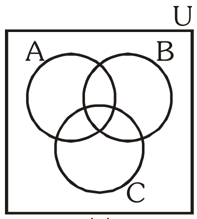
In the Venn's diagram (ii), we get

$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$

$n(A' \cap B' \cap C') = n(U) - n(A \cup B \cup C)$

In general, we have $n(A_1 \cup A_2 \cup \ldots\ldots\ldots \cup A_n)$

$$= \sum n(A_i) - \sum_{i \neq j} n(A_i \cap A_j) + \sum_{i \neq j \neq k} n(A_i \cap A_j \cap A_k) + \ldots\ldots + (-1)^n \sum n(A_1 \cap A_2 \cap \ldots \cap A_n)$$

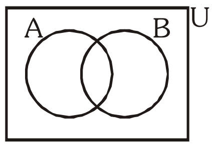


(ii)

Illustrations

Illustration 14. Find the number of permutations of letters a,b,c,d,e,f,g taken all at a time if neither 'beg' nor 'cad' pattern appear.

Solution. The total number of permutations without any restrictions; $n(U) = 7!$



beg a c d f

Let A be the set of all possible permutations in which 'beg' pattern always appears : $n(A) = 5!$

cad b e f g

Let B be the set of all possible permutations in which 'cad' pattern always appears : $n(B) = 5!$

cad beg f

$n(A \cap B)$: Number of all possible permutations when both 'beg' and 'cad' patterns appear.

$n(A \cap B) = 3!$.

Therefore, the total number of permutations in which 'beg' and 'cad' patterns do not appear

$n(A' \cap B') = n(U) - n(A \cap B) = n(U) - n(A) - n(B) + n(A \cap B)$

$= 7! - 5! - 5! + 3!$. ***Ans.***

BEGINNER'S BOX-1

TOPIC COVERED : FUNDAMENTAL PRINCIPLE OF COUNTING, FORMATION OF GROUPS, PRINCIPLE OF INCLUSION AND EXCLUSION

1. There are 3 ways to go from A to B, 2 ways to go from B to C and 1 way to go from A to C. In how many ways can a person travel from A to C ?

2. There are 2 red balls and 3 green balls. All balls are identical except colour. In how many ways can a person select two balls ?

3. Find the exponent of 10 in ${}^{75}C_{25}$.

4. If ${}^{10}P_r = 5040$, then find the value of r.

5. Find the number of ways of selecting 4 even numbers from the set of first 100 natural numbers.

6. If all letters of the word 'RANK' are arranged in all possible manner as they are in a dictionary, then find the rank of the word 'RANK'.

7. How many words can be formed using all letters of the word 'LEARN' ? In how many of these words vowels are together ?

8. Find the number of ways of selecting 5 members from a committee of 5 men & 2 women such that all women are always included.

9. Out of first 20 natural numbers, 3 numbers are selected such that there is exactly one even number. How many different selections can be made ?

10. How many four letter words can be made from the letters of the word 'PROBLEM'. How many of these start as well as end with a vowel ?

11. Find the number of ways in which 16 constables can be assigned to patrol 8 villages, 2 for each.

12. In how many ways can 6 different books be distributed among 3 students such that none gets equal number of books ?

13. n different toys are to be distributed among n children. Find the number of ways in which these toys can be distributed so that exactly one child gets no toy.

14. Find the number of n digit numbers formed using first 5 natural numbers, which contain the digits 2 & 4 essentially.

6.0 PERMUTATIONS OF ALIKE OBJECTS

SL AL

6.1 Taken all at a time

The number of permutations of n things taken all at a time : when p of them are similar of one type, q of them are similar of second type, r of them are similar of third type and the remaining n – (p + q+ r) are all different is : $\frac{n!}{p!\ q!\ r!}$.

Illustrations

Illustration 15. In how many ways the letters of the word "ARRANGE" can be arranged without altering the relative position of vowels & consonants.

Solution. The consonants in their positions can be arranged in $\frac{4!}{2!}$ = 12 ways.

The vowels in their positions can be arranged in $\frac{3!}{2!}$ = 3 ways

$\therefore$ Total number of arrangements = 12 × 3 = 36 ***Ans.***

Illustration 16. How many numbers can be formed with the digits 1, 2, 3, 4, 3, 2, 1 so that the odd digits always occupy the odd places?

(A) 17 (B) 18 (C) 19 (D) 20

Solution. There are 4 odd digits (1, 1, 3, 3) and 4 odd places (first, third, fifth and seventh). At these places the odd digits can be arranged in $\frac{4!}{2!2!} = 6$ ways

Then at the remaining 3 places, the remaining three digits (2, 2, 4) can be arranged in $\frac{3!}{2!} = 3$ ways

$\therefore$ The required number of numbers = 6 × 3 = 18. ***Ans. (B)***

Illustration 17. (a) How many permutations can be made by using all the letters of the word HINDUSTAN ?
(b) How many of these permutations begin and end with a vowel ?
(c) In how many of these permutations, all the vowels come together ?
(d) In how many of these permutations, none of the vowels come together ?
(e) In how many of these permutations, do the vowels and the consonants occupy the same relative positions as in HINDUSTAN ?

Solution. ***(a)*** The total number of permutations = Arrangements of nine letters taken all at a time $= \frac{9!}{2!} = 181440$.

(b) We have 3 vowels and 6 consonants, in which 2 consonants are alike. The first place can be filled in 3 ways and the last in 2 ways. The rest of the places can be filled in $\frac{7!}{2!}$ ways.

Hence the total number of permutations $= 3 \times 2 \times \frac{7!}{2!} = 15120$.

(c) Assume the vowels (I, U, A) as a single letter. The letters (IUA), H, D, S, T, N, N can be arranged in $\frac{7!}{2!}$ ways. Also IUA can be arranged among themselves in 3! = 6 ways.

Hence the total number of permutations $= \frac{7!}{2!} \times 6 = 15120$.

(d) Let us divide the task into two parts. In the first, we arrange the 6 consonants as shown below in $\frac{6!}{2!}$ ways.

$\times C \times C \times C \times C \times C \times C \times$ (Here C stands for a consonant and $\times$ stands for a gap between two consonants)

Now 3 vowels can be placed in 7 places (gaps between the consonants) in ${}^7C_3.3! = 210$ ways.

Hence the total number of permutations $= \frac{6!}{2!} \times 210 = 75600$.

(e) In this case, the vowels can be arranged among themselves in 3! = 6 ways.

Also, the consonants can be arranged among themselves in $\frac{6!}{2!}$ ways.

Hence the total number of permutations $= \frac{6!}{2!} \times 6 = 2160$. ***Ans.***

Illustration 18. If all the letters of the word 'PROPER' are arranged in all possible manner as they are in a dictionary, then find the rank of the word 'PROPER' .

Solution. First of all, arrange all letters of given word alphabetically : EOPPRR

Total number of words starting with-

E _ _ _ _ _ $= \frac{5!}{2!2!} = 30$ O _ _ _ _ _ $= \frac{5!}{2!2!} = 30$

PE _ _ _ _ $= \frac{4!}{2!} = 12$ PO _ _ _ _ $= \frac{4!}{2!} = 12$

PP _ _ _ _ $= \frac{4!}{2!} = 12$ PRE _ _ _ = 3! = 6

PROE _ _ = 2! = 2

PROPER = 1= 1

Rank of the word PROPER = 105 ***Ans.***

6.2 Taken some at a time

Illustrations

Illustration 19. Find the total number of 4 letter words formed using four letters from the word "PARALLELOPIPED'.

Solution. Given letters are PPP, LLL, AA, EE, R, O, I, D.

Cases	No.of ways of selection	No.of ways of arrangements	Total
All distinct	${}^{8}C_{4}$	${}^{8}C_{4} \times 4!$	1680
2 alike, 2 distinct	${}^{4}C_{1} \times {}^{7}C_{2}$	${}^{4}C_{1} \times {}^{7}C_{2} \times \frac{4!}{2!}$	1008
2 alike, 2 other alike	${}^{4}C_{2}$	${}^{4}C_{2} \times \frac{4!}{2!2!}$	36
3 alike, 1 distinct	${}^{2}C_{1} \times {}^{7}C_{1}$	${}^{2}C_{1} \times {}^{7}C_{1} \times \frac{4!}{3!}$	56
		Total	2780

Ans.

Illustration 20. Find the number of all 6 digit numbers such that all the digits of each number are selected from the set {1,2,3,4,5} and any digit that appears in the number appears at least twice.

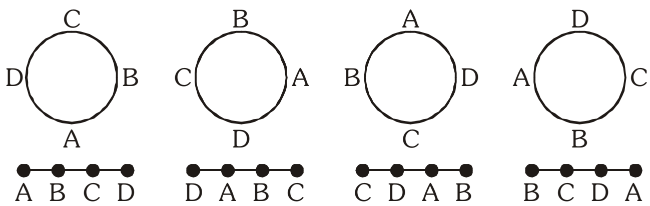
Solution.

Cases	No.of ways of selection	No.of ways of arrangements	Total
All alike	${}^{5}C_{1}$	${}^{5}C_{1} \times 1$	5
4 alike + 2 other alike	${}^{5}C_{2} \times 2!$	${}^{5}C_{2} \times 2 \times \frac{6!}{2!4!}$	300
3 alike + 3 other alike	${}^{5}C_{2}$	${}^{5}C_{2} \times \frac{6!}{3!3!}$	200
2 alike + 2 other alike +2 other alike	${}^{5}C_{3}$	${}^{5}C_{3} \times \frac{6!}{2!2!2!}$	900
		Total	1405

Ans.

7.0 CIRCULAR PERMUTATION

SL AL



(a) (b) (c) (d)

Let us consider that persons A,B,C,D are sitting around a round table. If all of them (A,B,C,D) are shifted by one place in anticlockwise order, then we will get Fig.(b) from Fig.(a). Now, if we shift A,B,C,D in anticlockwise order, we will get Fig.(c). Again, if we shift them, we will get Fig.(d) and in the next time, Fig.(a).

Thus, we see that if 4 persons are sitting at a round table, they can be shifted four times and the four different arrangements, thus obtained will be the same, because anticlockwise order of A,B,C,D does not change.

But if A,B,C,D are sitting in a row and they are shifted in such an order that the last occupies the place of first, then the four arrangements will be different.

Thus, if there are 4 things, then for each circular arrangement number of linear arrangements is 4.

Similarly, if n different things are arranged along a circle, for each circular arrangement number of linear arrangements is n.

Therefore, the number of linear arrangements of n different things is n $\times$ (number of circular arrangements of n different things). Hence, the number of circular arrangements of n different things is -

1/n × (number of linear arrangements of n different things) $= \frac{n!}{n} = (n-1)!$

Therefore note that –

(i) The number of circular permutations of n different things taken all at a time is : $(n-1)!$.

If clockwise & anti-clockwise circular permutations are considered to be same, then it is : $\frac{(n-1)!}{2}$.

(ii) The number of circular permutations of n different things taking r at a time distinguishing clockwise & anticlockwise arrangements is : $\frac{^nP_r}{r}$

Illustrations

Illustration 21. In how many ways can 5 boys and 5 girls be seated at a round table so that no two girls are together?

(A) $5! \times 5!$ (B) $5! \times 4!$ (C) $\frac{1}{2}(5!)^2$ (D) $\frac{1}{2}(5! \times 4!)$

Solution. Leaving one seat vacant between two boys, 5 boys may be seated in 4! ways. Then at remaining 5 seats, 5 girls sit in 5! ways. Hence the required number of ways $= 4! \times 5!$ ***Ans. (B)***

Illustration 22. The number of ways in which 7 girls can stand in a circle so that they do not have same neighbours in any two arrangements ?

(A) 720 (B) 380 (C) 360 (D) none of these

Solution. Seven girls can stand in a circle by $\frac{(7-1)!}{2!}$ number of ways, because there is no difference in anticlockwise and clockwise order of their standing in a circle.

$\therefore \frac{(7-1)!}{2!} = 360$ ***Ans. (C)***

***Illustration 23.** The number of ways in which 20 different pearls of two colours can be set alternately on a necklace, there being 10 pearls of each colour, is

(A) $9! \times 10!$ (B) $5(9!)^2$ (C) $(9!)^2$ (D) none of these

Solution. Ten pearls of one colour can be arranged in $\frac{1}{2}.(10-1)!$ ways. The number of arrangements of 10 pearls of the other colour in 10 places between the pearls of the first colour = 10!

$\therefore$ The required number of ways $= \frac{1}{2} \times 9! \times 10! = 5\,(9!)^2$ ***Ans. (B)***

Illustration 24. A person invites a group of 10 friends at dinner. They sit

(i) 5 on one round table and 5 on other round table,

(ii) 4 on one round table and 6 on other round table.

Find the number of ways in each case in which he can arrange the guests.

Solution. ***(i)*** The number of ways in which 10 persons can be divided into two groups of five person is

$$\frac{10!}{5! \times 5! \times 2!}$$

The total number of permutations of 5 guests at a round table is 4!. Hence, the total number of arrangements is $\frac{10!}{5! \times 5! \times 2!} \times 4! \times 4! = \frac{10!4!4!}{5!5!2!} = \frac{10!}{50}$

(ii) The number of ways of selection of 6 guests is $^{10}C_6$.

The number of ways of permutations of 6 guests on round table is 5!.

The number of permutations of 4 guests on round table is 3!

Therefore, total number of arrangements is : $^{10}C_6 5! \times 3! = \frac{(10)!}{6!4!}5!3! = \frac{(10)!}{24}$

Ans.

8.0 TOTAL NUMBER OF COMBINATIONS

SL AL

(a) Given n different objects , the number of ways of selecting atleast one of them is, ${}^{n}C_1 + {}^{n}C_2 + {}^{n}C_3 + \dots\dots + {}^{n}C_n = 2^n - 1$. This can also be stated as the total number of combinations of n distinct things.

(b) (i) Total number of ways in which it is possible to make a selection by taking some or all out of p + q + r +......things, where p are alike of one kind, q alike of a second kind, r alike of third kind & so on is given by : $(p + 1)(q + 1)(r + 1)\dots\dots - 1$.

(ii) The total number of ways of selecting one or more things from p identical things of one kind, q identical things of second kind, r identical things of third kind and n different things is given by : $(p + 1)(q + 1)(r + 1)\,2^n - 1$

Illustrations

****Illustration 25.*** A is a set containing n elements. A subset P of A is chosen. The set A is reconstructed by replacing the elements of P. A subset Q of A is again chosen. The number of ways of choosing P and Q so that $P \cap Q = \phi$ is :-

(A) $2^{2n} - {}^{2n}C_n$ (B) 2^n (C) $2^n - 1$ (D) 3^n

Solution. Let $A = \{a_1, a_2, a_3, \dots\dots a_n\}$. For $a_i \in A$, we have the following choices :

(i) $a_i \in P$ and $a_i \in Q$ ***(ii)*** $a_i \in P$ and $a_i \notin Q$

(iii) $a_i \notin P$ and $a_i \in Q$ ***(iv)*** $a_i \notin P$ and $a_i \notin Q$

Out of these only (ii), (iii) and (iv) imply $a_i \notin P \cap Q$. Therefore, the number of ways in which none of $a_1, a_2, \dots a_n$ belong to $P \cap Q$ is 3^n. ***Ans.(D)***

****Illustration 26.*** A student is allowed to select at most n books from a collection of (2n + 1) books. If the total number of ways in which he can select books is 63, find the value of n.

Solution. Given student selects at most n books from a collection of (2n + 1) books. It means that he selects one book or two books or three books or .... or n books. Hence, by the given condition-

$${}^{2n+1}C_1 + {}^{2n+1}C_2 + {}^{2n+1}C_3 + \dots\dots + {}^{2n+1}C_n = 63 \quad \dots(i)$$

But we know that

$${}^{2n+1}C_0 + {}^{2n+1}C_1 + {}^{2n+1}C_2 + {}^{2n+1}C_3 + \dots\dots + {}^{2n+1}C_{2n+1} = 2^{2n+1} \quad \dots(ii)$$

Since ${}^{2n+1}C_0 = {}^{2n+1}C_{2n+1} = 1$, equation (ii) can also be written as

$$2 + ({}^{2n+1}C_1 + {}^{2n+1}C_2 + {}^{2n+1}C_3 + \dots\dots + {}^{2n+1}C_n) + ({}^{2n+1}C_{n+1} + {}^{2n+1}C_{n+2} + {}^{2n+1}C_{n+3} + \dots\dots + {}^{2n+1}C_{2n-1} + {}^{2n+1}C_{2n}) = 2^{2n+1}$$

$$\Rightarrow 2 + ({}^{2n+1}C_1 + {}^{2n+1}C_2 + {}^{2n+1}C_3 + \dots\dots + {}^{2n+1}C_n) + ({}^{2n+1}C_n + {}^{2n+1}C_{n-1} + \dots\dots + {}^{2n+1}C_2 + {}^{2n+1}C_1) = 2^{2n+1}$$

$(\because {}^{2n+1}C_r = {}^{2n+1}C_{2n+1-r})$

$$\Rightarrow 2 + 2({}^{2n+1}C_1 + {}^{2n+1}C_2 + {}^{2n+1}C_3 + \dots\dots + {}^{2n+1}C_n) = 2^{2n+1} \quad \text{[from (i)]}$$

$$\Rightarrow 2 + 2.63 = 2^{2n+1} \qquad \Rightarrow 1 + 63 = 2^{2n}$$

$$\Rightarrow 64 = 2^{2n} \Rightarrow 2^6 = 2^{2n} \qquad \therefore 2n = 6$$

Hence, n = 3. ***Ans.***

Illustration 27. There are 3 different books of mathematics, 4 different books of science and 5 different books of english. How many different collections can be made such that each collection consists of-

(i) one book of each subject ?

(ii) at least one book of each subject ?

(iii) at least one book of english ?

Solution. ***(i)*** ${}^{3}C_1 \times {}^{4}C_1 \times {}^{5}C_1 = 60$

(ii) $(2^3 - 1)(2^4 - 1)(2^5 - 1) = 7 \times 15 \times 31 = 3255$

(iii) $(2^5 - 1)(2^3)(2^4) = 31 \times 128 = 3968$ ***Ans.***

Illustration 28. Find the number of groups that can be made from 5 red balls, 3 green balls and 4 black balls, if at least one ball of all colours is always to be included. Given that all balls are identical except colours.

Solution. After selecting one ball of each colour, we have to find total number of combinations that can be made from 4 red. 2 green and 3 black balls. These will be (4 + 1) (2 + 1) (3 + 1) = 60 **Ans.**

9.0 DIVISORS

AL

Let N = $p^a . q^b . r^c$ .... where p, q, r........ are distinct primes & a, b, c....... are natural numbers then :

(a) The total numbers of divisors of N including 1 & N is = (a + 1) (b + 1) (c + 1).......

(b) The sum of these divisors is

$= (p^0 + p^1 + p^2 + ... + p^a)\ (q^0 + q^1 + q^2 + ... + q^b)\ (r^0 + r^1 + r^2 + ... + r^c)...$

(c) Number of ways in which N can be resolved as a product of two factor is =

$\frac{1}{2}\ (a+1)\ (b+1)\ (c+1)......$ if N is not a perfect square

$\frac{1}{2}\ [(a+1)\ (b+1)\ (c+1)......+1]$ if N is a perfect square

(d) Number of ways in which a composite number N can be resolved into two factors which are relatively prime (or coprime) to each other is equal to 2^{n-1} where n is the number of different prime factors in N.

Illustrations

Illustration 29. Find the number of proper divisors of the number 38808. Also find the sum of these divisors.

Solution. ***(i)*** The number 38808 = $2^3 . 3^2 . 7^2 . 11$

Hence the total number of divisors (excluding 1 and itself i.e.38808)

= (3 + 1) (2 + 1) (2 + 1) (1 + 1) – 2 = 70

(ii) The sum of these divisors

$=(2^0 + 2^1 + 2^2 + 2^3)\ (3^0 + 3^1 + 3^2)\ (7^0 + 7^1 + 7^2)\ (11^0 + 11^1) - 1 - 38808$

= (15) (13) (57) (12) – 1 – 38808 = 133380 – 1 – 38808 = 94571.

Ans.

***Illustration 30.** In how many ways the number 18900 can be split in two factors which are relative prime (or coprime) ?

Solution. Here N = 18900 $= 2^2 . 3^3 . 5^2 . 7^1$

Number of different prime factors in 18900 = n = 4

Hence number of ways in which 18900 can be resolved into two factors which are relative prime (or coprime) = $2^{4-1} = 2^3 = 8$. **Ans.**

Illustration 31. Find the total number of proper factors of the number 35700. Also find

(i) sum of all these factors,

(ii) sum of the odd proper divisors,

(iii) the number of proper divisors divisible by 10 and the sum of these divisors.

Solution. $35700 = 5^2 \times 2^2 \times 3^1 \times 7^1 \times 17^1$

The total number of factors is equal to the total number of selections from (5,5), (2,2), (3), (7) and (17), which is given by 3 × 3 × 2 × 2 × 2 = 72.

These include 1 and 35700. Therefore, the number of proper divisors (excluding 1 and 35700) is 72 – 2 = 70

(i) Sum of all these factors (proper) is :

$(5^0 + 5^1 + 5^2)\ (2^0 + 2^1 + 2^2)\ (3^0 + 3^1)\ (7^0 + 7^1)\ (17^0 + 17^1) - 1 - 35700$

= 31 × 7 × 4 × 8 × 18 – 1 – 35700 = 89291

(ii) The sum of odd proper divisors is :

$(5^0 + 5^1 + 5^2)\ (3^0 + 3^1)\ (7^0 + 7^1)\ (17^0 + 17^1) - 1$

= 31 × 4 × 8 × 18 – 1 = 17856 – 1 = 17855

(iii) The number of proper divisors divisible by 10 is equal to number of selections from (5,5), (2,2), (3), (7), (17) consisting of at least one 5 and at least one 2 and 35700 is to be excluded and is given by $2 \times 2 \times 2 \times 2 \times 2 - 1 = 31$.

Sum of these divisors is :

$(5^1 + 5^2)(2^1 + 2^2)(3^0 + 3^1)(7^0 + 7^1)(17^0 + 17^1) - 35700$

$= 30 \times 6 \times 4 \times 8 \times 18 - 35700 = 67980$ ***Ans.***

BEGINNER'S BOX-2

TOPIC COVERED : PERMUTATIONS OF ALIKE OBJECTS, CIRCULAR PERMUTATION, TOTAL NUMBER OF COMBINATIONS, DIVISORS

1. In how many ways can the letters of the word 'ALLEN' be arranged ? Also find its rank if all these words are arranged as they are in dictionary.

2. How many numbers greater than 1000 can be formed from the digits 1, 1, 2, 2, 3 ?

3. In how many ways can 3 men and 3 women be seated around a round table such that all men are always together ?

4. Find the number of ways in which 10 different diamonds can be arranged to make a necklace.

5. Find the number of ways in which 6 persons out of 5 men & 5 women can be seated at a round table such that 2 men are never together.

6. In how many ways can 8 persons be seated on two round tables of capacity 5 & 3.

7. There are p copies each of n different books. Find the number of ways in which atleast one book can be selected ?

8. There are 10 questions in an examination. In how many ways can a candidate answer the questions, if he attempts atleast one question.

9. Find the number of ways in which the number 94864 can be resolved as a product of two factors.

10. Find the number of order pair of (x, y) of solution of xy = 1440.

10.0 TOTAL DISTRIBUTION

AL

(a) ***Distribution of distinct objects –*** Number of ways in which n distinct things can be distributed to p persons if there is no restriction to the number of things received by them is given by : p^n

(b) ***Distribution of alike objects –*** Number of ways to distribute n alike things among p persons so that each may get none, one or more thing(s) is given by ${}^{n+p-1}C_{p-1}$.

Illustrations

Illustration 32. In how many ways can 5 different mangoes, 4 different oranges & 3 different apples be distributed among 3 children such that each gets alteast one mango ?

Solution. 5 different mangoes can be distributed by following ways among 3 children such that each gets atleast 1 :

3 1 1

2 2 1

Total number of ways : $\left(\dfrac{5!}{3!1!1!2!} + \dfrac{5!}{2!2!2!}\right) \times 3!$

Now, the number of ways of distributing remaining fruits (i.e. 4 oranges + 3 apples) among 3 children = 3^7 (as each fruit has 3 options).

$\therefore$ Total number of ways $= \left(\dfrac{5!}{3!2!} + \dfrac{5!}{(2!)^3}\right) \times 3! \times 3^7$ ***Ans.***

Illustration 33. In how many ways can 12 identical apples be distributed among four children if each gets atleast 1 apple and not more than 4 apples.

Solution. Let x,y,z & w be the number of apples given to the children.

$\Rightarrow$ $x + y + z + w = 12$

Giving one-one apple to each

Now, $x + y + z + w = 8$ ... (i)

Here, $0 \le x \le 3,\ 0 \le y \le 3,\ 0 \le z \le 3,\ 0 \le w \le 3$

$x = 3 - t_1,\ y = 3 - t_2,\ z = 3 - t_3,\ w = 3 - t_4.$

Putting value of x, y, z, w in equation (i)

Put $12 - 8 = t_1 + t_2 + t_3 + t_4$

$\Rightarrow$ $t_1 + t_2 + t_3 + t_4 = 4$

(Here max. value that t_1, t_2, t_3 & t_4 can attain is 3, so we have to remove those cases when any of t_i getting value 4)

$= {}^7C_3 -$ (all cases when atleast one is 4)

$= {}^7C_3 - 4 = 35 - 4 = 31$ **Ans.**

Illustration 34. Find the number of non negative integral solutions of the inequation $x + y + z \le 20$.

Solution. Let w be any number $(0 \le w \le 20)$, then we can write the equation as :

$x + y + z + w = 20$ (here $x, y, z, w \ge 0$)

Total ways $= {}^{23}C_3$ **Ans.**

Illustration 35. Find the number of integral solutions of $x + y + z + w < 25$, where $x > -2$, $y > 1$, $z \ge 2$, $w \ge 0$.

Solution. Given $x + y + z + w < 25$

$x + y + z + w + v = 25$ ... (i)

Let $x = -1 + t_1,\ y = 2 + t_2,\ z = 2 + t_3,\ w = t_4,\ v = 1 + t_5$ where $(t_1, t_2, t_3, t_4 \ge 0)$

Putting value of x, y, z, w, v in equation (i)

$\Rightarrow$ $t_1 + t_2 + t_3 + t_4 + t_5 = 21.$

Number of solutions $= {}^{25}C_4$ **Ans.**

Illustration 36. Find the number of positive integral solutions of the inequation $x + y + z \ge 150$, where $0 < x \le 60,\ 0 < y \le 60,\ 0 < z \le 60$.

Solution. Let $x = 60 - t_1,\ y = 60 - t_2,\ z = 60 - t_3$ (where $0 \le t_1 \le 59,\ 0 \le t_2 \le 59,\ 0 \le t_3 \le 59$)

Given $x + y + z \ge 150$

or $x + y + z - w = 150$ (where $0 \le w \le 30$) ... (i)

Putting values of x, y, z in equation (i)

$60 - t_1 + 60 - t_2 + 60 - t_3 - w = 150$

$30 = t_1 + t_2 + t_3 + w$

Total solutions $= {}^{33}C_3$ **Ans.**

Illustration 37. Find the number of positive integral solutions of $xy = 12$

Solution. $xy = 12$

$xy = 2^2 \times 3^1$

(i) 3 has 2 ways either 3 can go to x or y

(ii) 2^2 can be distributed between x & y as distributing 2 identical things between 2 persons (where each person can get 0, 1 or 2 things). Let two person be ℓ_1 & ℓ_2

$\Rightarrow$ $\ell_1 + \ell_2 = 2$

$\Rightarrow$ ${}^{2+1}C_1 = {}^3C_1 = 3$

So total ways $= 2 \times 3 = 6.$

Alternatively –

$xy = 12 = 2^2 \times 3^1$

$x = 2^{a_1}3^{a_2}$ $\quad 0 \le a_1 \le 2$

$0 \le a_2 \le 1$

$y = 2^{b_1}3^{b_2}$ $\quad 0 \le b_1 \le 2$

$0 \le b_2 \le 1$

$2^{a_1+b_1}3^{a_2+b_2} = 2^2 3^1$

$\Rightarrow a_1 + b_1 = 2 \rightarrow$ $\quad {}^3C_1$ ways

$a_2 + b_2 = 1 \rightarrow$ $\quad {}^2C_1$ ways

Number of solutions $= {}^3C_1 \times {}^2C_1 = 3 \times 2 = 6$ **Ans.**

Illustration 38.** Find the number of solutions of the equation $xyz = 360$ when ***(i) $x,y,z \in N$ ***(ii)*** $x,y,z \in I$

Solution. ***(i)*** $xyz = 360 = 2^3 \times 3^2 \times 5$ $(x,y,z \in N)$

$x = 2^{a_1}3^{a_2}5^{a_3}$ (where $0 \le a_1 \le 3,\ 0 \le a_2 \le 2,\ 0 \le a_3 \le 1$)

$y = 2^{b_1}3^{b_2}5^{b_3}$ (where $0 \le b_1 \le 3,\ 0 \le b_2 \le 2,\ 0 \le b_3 \le 1$)

$z = 2^{c_1}3^{c_2}5^{c_3}$ (where $0 \le c_1 \le 3,\ 0 \le c_2 \le 2,\ 0 \le c_3 \le 1$)

$\Rightarrow 2^{a_1}3^{a_2}5^{a_3}.2^{b_1}3^{b_2}5^{b_3}.2^{c_1}3^{c_2}5^{c_3} = 2^3 \times 3^2 \times 5^1$

$\Rightarrow 2^{a_1+b_1+c_1}.3^{a_2+b_2+c_2}.5^{a_3+b_3+c_3} = 2^3 \times 3^3 \times 5^1$

$\Rightarrow a_1 + b_1 + c_1 = 3 \rightarrow {}^5C_2 = 10$

$a_2 + b_2 + c_2 = 2 \rightarrow {}^4C_2 = 6$

$a_3 + b_3 + c_3 = 1 \rightarrow {}^3C_2 = 3$

Total solutions $= 10 \times 6 \times 3 = 180$.

(ii) If $x,y,z \in I$ then, (a) all positive (b) 1 positive and 2 negative.

Total number of ways $= 180 + {}^3C_2 \times 180 = 720$ **Ans.**

Illustration 39. Find the exponent of 6 in 50!

Solution.

$$E_2(50!) = \left[\frac{50}{2}\right] + \left[\frac{50}{4}\right] + \left[\frac{50}{8}\right] + \left[\frac{50}{16}\right] + \left[\frac{50}{32}\right] + \left[\frac{50}{64}\right]$$ (where [ ] denotes integral part)

$E_2(50!) = 25 + 12 + 6 + 3 + 1 + 0 = 47$

$$E_3(50!) = \left[\frac{50}{3}\right] + \left[\frac{50}{9}\right] + \left[\frac{50}{27}\right] + \left[\frac{50}{81}\right]$$

$E_3(50!) = 16 + 5 + 1 + 0 = 22$

$\Rightarrow$ 50! can be written as $50! = 2^{47}.\,3^{22}$.........

Therefore exponent of 6 in 50! = 22 **Ans.**

11.0 DEARRANGEMENT

AL

There are n letters and n corresponding envelopes. The number of ways in which letters can be placed in the envelopes (one letter in each envelope) so that no letter is placed in correct envelope is

$$n!\left[1 - \frac{1}{1!} + \frac{1}{2!} + \ldots.. + \frac{(-1)^n}{n!}\right]$$

Proof – n letters are denoted by 1,2,3,........,n. Let A_i denote the set of distribution of letters in envelopes (one letter in each envelope) so that the i^{th} letter is placed in the corresponding envelope. Then,

$n(A_i) = 1 \times (n-1)!$ [since the remaining n–1 letters can be placed in n –1 envelops in (n–1)! ways]

Then, $n(A_i \cap A_j)$ represents the number of ways where letters i and j can be placed in their corresponding envelopes. Then,

$$n(A_i \cap A_j) = 1 \times 1 \times (n-2)!$$

Also $n(A_i \cap A_j \cap A_k) = 1 \times 1 \times 1 \times (n-3)!$

Hence, the required number is

$n(A_1' \cup A_2' \cup ..... \cup A_n') = n! - n(A_1 \cup A_2 \cup .... \cup A_n)$

$$= n! - \left[\sum n(A_i) - \sum n(A_i \cap A_j) + \sum n(A_i \cap A_j \cap A_k) + ... + (-1)^n \sum n(A_i \cap A_2 ..... \cap A_n)\right]$$

$$= n! - [{}^nC_1(n-1)! - {}^nC_2(n-2)! + {}^nC_3(n-3)! + ... + (-1)^{n-1} \times {}^nC_n 1]$$

$$= n! - \left[\frac{n!}{1!(n-1)!}(n-1)! - \frac{n!}{2!(n-2)!}(n-2)! + ... + (-1)^{n-1}\right]$$

$$= n!\left[1 - \frac{1}{1!} + \frac{1}{2!} + ... + \frac{(-1)^n}{n!}\right]$$

Illustrations

Illustration 40. A person writes letters to six friends and addresses the corresponding envelopes. In how many ways can the letters be placed in the envelopes so that–

(i) all the letters are in the wrong envelopes.

(ii) at least two of them are in the wrong envelopes.

Solution. ***(i)*** The number of ways is which all letters be placed in wrong envelopes

$$= 6!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!}\right) = 720\left(\frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120} + \frac{1}{720}\right)$$

$$= 360 - 120 + 30 - 6 + 1 = 265.$$

(ii) The number of ways in which at least two of them in the wrong envelopes

$$= {}^6C_4 . 2!\left(1 - \frac{1}{1!} + \frac{1}{2!}\right) + {}^6C_3 . 3!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!}\right) + {}^6C_2 . 4!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}\right)$$

$$+ {}^6C_1 . 5!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!}\right) + {}^6C_0\, 6!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!}\right)$$

$$= 15 + 40 + 135 + 264 + 265 = 719.$$ ***Ans.***

BEGINNER'S BOX-3

TOPIC COVERED : TOTAL DISTRIBUTION, DEARRANGEMENT

1. In how many ways can 12 identical apples be distributed among 4 boys. (a) If each boy receives any number of apples. (b) If each boy receives atleast 2 apples.

2. Find the number of non-negative integral solutions of the equation $x + y + z = 10$.

3. Find the number of integral solutions of $x + y + z = 20$, if $x \geq -4$, $y \geq 1$, $z \geq 2$

4. There are four balls of different colours and four boxes of colours same as those of the balls. Find the number of ways in which the balls, one in each box, could be placed in such a way that a ball does not go to box of its own colour.

5. Find the number of ways in which thirty five apples can be distributed among 3 boys so that each can have any number of apples.

6. Find the number of non negative integral solutions of $x_1 + x_2 + x_3 + ... + x_n \leq n$. (where n is positive integer).

GOLDEN KEY POINTS

- $0! = 1! = 1$
- Factorials of negative integers are not defined.
- $n!$ is also denoted by $\underline{|n}$
- $(2n)! = 2^n . n! \, [1. 3. 5. 7........(2n-1)]$
- Prime factorisation of $n!$: Let p be a prime number and n be a positive integer, then exponent of p in $n!$ is denoted by $E_p(n!)$ and is given by

  $$E_p(n!) = \left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \left[\frac{n}{p^3}\right] + ... + \left[\frac{n}{p^k}\right]$$

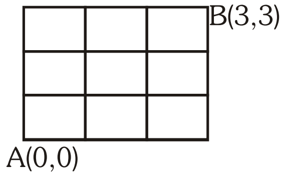
  where $p^k \le n < p^{k+1}$ and $[x]$ denotes the integral part of x.

  If we isolate the power of each prime contained in any number n, then n can be written as

  $n = 2^{\alpha_1} \cdot 3^{\alpha_2} \cdot 5^{\alpha_3} \cdot 7^{\alpha_4}$ .... where α_i are whole numbers.
- ${}^nP_n = n!,\quad {}^nP_0 = 1,\quad {}^nP_1 = n$
- Number of arrangements of n ***distinct*** things taken all at a time = n!
- ${}^nP_r$ is also denoted by A_r^n or $P(n,r)$.
- ${}^nC_r$ is also denoted by $\binom{n}{r}$ or $C(n, r)$.
- ${}^nP_r = {}^nC_r . r!$
- Every natural number except 1 has atleast 2 divisors. If it has exactly two divisors then it is called a prime. System of prime numbers begin with 2. All primes except 2 are odd.
- A number having more than 2 divisors is called composite. 2 is the only even number which is not composite.
- Two natural numbers are said to be relatively prime or coprime if their HCF is one. For two natural numbers to be relatively prime, it is not necessary that one or both should be prime. It is possible that they both are composite but still coprime, eg. 4 and 25.
- 1 is neither prime nor composite however it is co-prime with every other natural number.
- Two prime numbers are said to be twin prime numbers if their non-negative difference is 2 (e.g.5 & 7, 19 & 17 etc).
- All divisors except 1 and the number itself are called proper divisors.

SOME WORKED OUT ILLUSTRATIONS

Illustration 1. In how many ways can a person go from point A to point B if he can travel only to the right or upward along the lines (Grid Problem) ?



Solution. To reach the point B from point A, a person has to travel along 3 horizontal and 3 vertical strips. Therefore, we have to arrange 3H and 3V in a row. Total number of ways = $\frac{6!}{3!3!} = 20$ ways

Ans.

Illustration 2. Find sum of all numbers formed using the digits 2,4,6,8 taken all at a time and no digit being repeated.

Solution. All possible numbers = 4! = 24

If 2 occupies the unit's place then total numbers = 6

Hence, 2 comes at unit's place 6 times.

Sum of all the digits occuring at unit's place

$= 6 \times (2 + 4 + 6 + 8)$

Same summation will occur for ten's, hundred's & thousand's place. Hence required sum

$= 6 \times (2 + 4 + 6 + 8) \times (1 + 10 + 100 + 1000) = 133320$ ***Ans.***

Illustration 3. Find the sum of all the numbers greater than 1000 using the digits 0,1,2,2.

Solution.

(i) When 1 is at thousand's place, total numbers formed will be $= \frac{3!}{2!} = 3$

(ii) When 2 is at thousand's place, total numbers formed will be = 3! = 6

(iii) When 1 is at hundred's, ten's or unit's place then total numbers formed will be-

Thousand's place is fixed i.e. only the digit 2 will come here, remaining two places can be filled in 2! ways.

So total numbers = 2!

(iv) When 2 is at hundred's, ten's or unit's place then total numbers formed will be-

Thousand's place has 2 options and other two places can be filled in 2 ways.

So total numbers = $2 \times 2 = 4$

Sum $= 10^3 (1 \times 3 + 2 \times 6) + 10^2 (1 \times 2 + 2 \times 4) + 10^1(1 \times 2 + 2 \times 4)$

$+ (1 \times 2 + 2 \times 4) = 15 \times 10^3 + 10^3 + 10^2 + 10 = 16110$ ***Ans.***

Illustration 4. Find the number of positive integral solutions of $x + y + z = 20$, if $x \neq y \neq z$.

Solution.

$x \geq 1$

$y = x + t_1$ $\quad t_1 \geq 1$

$z = y + t_2$ $\quad t_2 \geq 1$

$x + x + t_1 + x + t_1 + t_2 = 20$

$3x + 2t_1 + t_2 = 20$

(i) $x = 1$ $\quad 2t_1 + t_2 = 17$

$t_1 = 1,2 \ldots\ldots\ldots 8 \Rightarrow 8$ ways

(ii) $x = 2$ $\quad 2t_1 + t_2 = 14$

$t_1 = 1,2 \ldots\ldots\ldots 6 \Rightarrow 6$ ways

(iii) $x = 3$ $\quad 2t_1 + t_2 = 11$

$t_1 = 1,2 \ldots\ldots\ldots 5 \Rightarrow 5$ ways

(vi) $x = 4$ $\quad 2t_1 + t_2 = 8$

$t_1 = 1,2,3 \Rightarrow 3$ ways

(v) $x = 5$ $\quad 2t_1 + t_2 = 5$

$t_1 = 1, 2 \Rightarrow 2$ ways

Total = 8 + 6 + 5 + 3 + 2 = 24

But each solution can be arranged by 3! ways.

So total solutions = $24 \times 3! = 144$. ***Ans.***

***Illustration 5.** A regular polygon of 15 sides is constructed. In how many ways can a triangle be formed using the vertices of the polygon such that no side of triangle is same as that of polygon ?

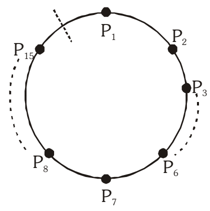
Solution. Select one point out of 15 point, therefore total number of ways = ${}^{15}C_1$

Suppose we select point P_1. Now we have to choose 2 more point which are not consecutive.

since we can not select P_2 & P_{15}.

Total points left are 12.

Now we have to select 2 points out of 12 points which are not consecutive



Total ways = ${}^{12-2+1}C_2 = {}^{11}C_2$

Every select triangle will be repeated 3 times.

So total number of ways $= \dfrac{{}^{15}C_1 \times {}^{11}C_2}{3} = 275$

Alternative –

First of all let us cut the polygon between points P_1 & P_{15}. Now there are 15 points on a straight line and we have to select 3 points out of these, such that the selected points are not consecutive.

x O y O z O w

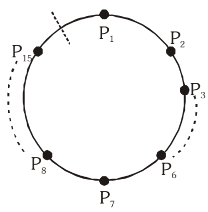
Here bubbles represents the selected points,

x represents the number of points before first selected point,

y represents the number of points between Ist & IInd selected point,

z represents the number of points between IInd & IIIrd selected point

and w represents the number of points after IIIrd selected point.



$x + y + z + w = 15 - 3 = 12$

here $x \geq 0,\ y \geq 1,\ z \geq 1,\ w \geq 0$

Put $y = 1 + y'$ & $z = 1 + z'$ $(y' \geq 0,\ z' \geq 0)$

$\Rightarrow$ $x + y' + z' + w = 10$

Total number of ways = ${}^{13}C_3$

These selections include the cases when both the points P_1 & P_{15} are selected. We have to remove those cases. Here a represents number of points between P_1 & 3^{rd} selected point & b represents number of points between 3^{rd} selected point and P_{15}

$\Rightarrow$ $a + b = 15 - 3 = 12 \quad (a \geq 1, b \geq 1)$

put $a = 1 + t_1$ & $b = 1 + t_2$

$t_1 + t_2 = 10$

Total number of ways = ${}^{11}C_1 = 11$

Therefore required number of ways = ${}^{13}C_3 - {}^{11}C_1 = 286 - 11 = 275$ ***Ans.***

***Illustration 6.** Find the number of ways in which three numbers can be selected from the set $\{5^1, 5^2, 5^3, ...5^{11}\}$ so that they form a G.P.

Solution. Any three selected numbers which are in G.P. have their powers in A.P.

Set of powers is = $\{1,2,.........6,7,.....11\}$

By selecting any two numbers from $\{1,3,5,7,9,11\}$, the middle number is automatically fixed.

Total number of ways = ${}^{6}C_2$

Now select any two numbers from $\{2,4,6,8,10\}$ and again middle number is automatically fixed.

Total number of ways = ${}^{5}C_2$

$\therefore$ Total number of ways are = ${}^{6}C_2 + {}^{5}C_2 = 15 + 10 = 25$ ***Ans.***

ANSWERS

BEGINNER'S BOX-1

1.	7	**2.**	3	**3.**	0	**4.**	r = 4	**5.**	${}^{50}C_4$
6.	20	**7.**	120, 48	**8.**	10	**9.**	450	**10.**	840, 40
11.	$\frac{16!}{(2!)^8 8!} \times 8!$	**12.**	360	**13.**	${}^{n}C_2.n!$	**14.**	$5^n - 4^n - 4^n + 3^n$		

BEGINNER'S BOX-2

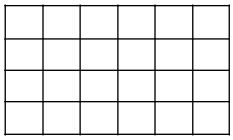
1.	60, 6^{th}	**2.**	60	**3.**	36	**4.**	$\frac{9!}{2} = 181440$
5.	5400	**6.**	2688	**7.**	$(p + 1)^n - 1$		
8.	$2^{10} - 1$	**9.**	23	**10.**	36		

BEGINNER'S BOX-3

1.	(a) ${}^{15}C_3$ (b) ${}^{7}C_3$	**2.**	${}^{12}C_2$	**3.**	${}^{23}C_2$	**4.**	9
5.	666	**6.**	${}^{2N}C_N$				

EXERCISE – 1 MCQ (SINGLE CHOICE CORRECT)

1. Number of rectangles in the grid shown which are not squares is



(A) 160 (B) 162 (C) 170 (D) 185

2. The number of different seven digit numbers that can be written using only three digits 1, 2 & 3 under the condition that the digit 2 occurs exactly twice in each number is -
(A) 672 (B) 640 (C) 512 (D) none of these

3. Out of seven consonants and four vowels, the number of words of six letters, formed by taking four consonants and two vowels is (Assume that each ordered group of letter is a word) -
(A) 210 (B) 462 (C) 151200 (D) 332640

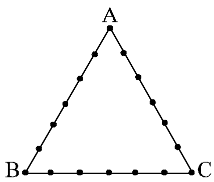
4. A 5 digit number divisible by 3 is to be formed using the numerals 0, 1, 2,3,4 & 5 without repetition. The total number of ways this can be done is -
(A) 3125 (B) 600 (C) 240 (D) 216

5. The number of ways in which 5 different books can be distributed among 10 people if each person can get at most one book is -
(A) 252 (B) 10^5 (C) 5^{10} (D) $^{10}C_5 \,.\, 5!$

6. Number of ways in which 9 different prizes can be given to 5 students, if one particular student receives 4 prizes and the rest of the students can get any numbers of prizes is -
(A) $^9C_4 \,.\, 2^{10}$ (B) $^9C_5 \,.\, 5^4$ (C) $4 \,.\, 4^5$ (D) none of these

7. Boxes numbered 1, 2, 3, 4 and 5 are kept in a row and they are necessarily to be filled with either a red or a blue ball such that no two adjacent boxes can be filled with blue balls. How many different arrangements are possible, given that the balls of a given colour are exactly identical in all respects ?
(A) 8 (B) 10 (C) 13 (D) 22

8. 18 points are indicated on the perimeter of a triangle ABC (see figure). How many triangles are there with vertices at these points?



(A) 331 (B) 408 (C) 710 (D) 711

9. If all the letters of the word "QUEUE" are arranged in all possible manner as they are in a dictionary, then the rank of the word QUEUE is -
(A) 15^{th} (B) 16^{th} (C) 17^{th} (D) 18^{th}

10. Number of ways in which 9 different toys can be distributed among 4 children belonging to different age groups in such a way that distribution among the 3 elder children is even and the youngest one is to receive one toy more is -

(A) $\dfrac{(5!)^2}{8}$ (B) $\dfrac{9!}{2}$ (C) $\dfrac{9!}{3!(2!)^3}$ (D) none of these

11. The number of ways in which 10 boys can take positions about a round table if two particular boys must not be seated side by side is :
(A) 10 (9) ! (B) 9 (8) ! (C) 7 (8) ! (D) none of these

12. Number of numbers greater than a million and divisible by 5 which can be formed by using only the digits 1, 2, 1, 2, 0, 5 & 2 is -
(A) 120 (B) 110 (C) 90 (D) none of these

13. The maximum number of different permutations of 4 letters of the word "EARTHQUAKE" is -
(A) 2910 (B) 2550 (C) 2190 (D) 2091

***14.** The number of ways in which we can arrange n ladies & n gentlemen at a round table so that 2 ladies or 2 gentlemen may not sit next to one another is -
(A) $(n-1)!\,(n-2)!$ (B) $(n)!\,(n-!)!$ (C) $(n+1)!\,(n)!$ (D) none of these

15. There are 10 red balls of different shades & 9 green balls of identical shades. Then the number of arranging them in a row so that no two green balls are together is
(A) $(10\,!)\,.\,{}^{11}P_9$ (B) $(10\,!)\,.\,{}^{11}C_9$ (C) $10\,!$ (D) $10\,!\;9\,!$

16. The sum of all numbers greater than 1000 formed by using the digits 1, 3, 5, 7 such that no digit is being repeated in any number is -
(A) 72215 (B) 83911 (C) 106656 (D) 114712

17. The number of way in which 10 identical apples can be distributed among 6 children so that each child receives atleast one apple is -
(A) 126 (B) 252 (C) 378 (D) none of these

***18.** Number of ways in which 25 identical pens can be distributed among Keshav, Madhav, Mukund and Radhika such that at least 1, 2, 3 and 4 pens are given to Keshav, Madhav, Mukund and Radhika respectively, is -
(A) ${}^{18}C_4$ (B) ${}^{28}C_3$ (C) ${}^{24}C_3$ (D) ${}^{18}C_3$

19. The number of ways in which a mixed double tennis game can be arranged from amongst 9 married couple if no husband & wife plays in the same game is -
(A) 756 (B) 3024 (C) 1512 (D) 6048

20. The number of words of four letters containing equal number of vowels and consonants, where repetition is allowed, is
(A) 105^2 (B) 210×243 (C) 105×243 (D) 150×21^2

21. In a group of 13 cricket players, four are bowlers. Find out in how many ways can they form a cricket team of 11 players in which at least 2 bowlers are included.
(A) 55 (B) 72 (C) 78 (D) None of these

EXERCISE – 2 MCQ (ONE OR MORE CHOICE CORRECT)

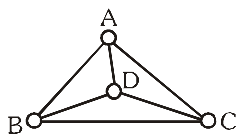
Select the correct alternatives (one or more than one correct answers)

1. 5 Indian & 5 American couples meet at a party & shake hands. If no wife shakes hands with her own husband & no Indian wife shakes hands with a male, then the number of hand shakes that takes place in the party is -
(A) 95 (B) 110 (C) 135 (D) 150

2. Number of different words that can be formed using all the letters of the word "DEEPMALA" if two vowels are together and the other two are also together but separated from the first two is -
(A) 960 (B) 1200 (C) 2160 (D) 1440

3. In a unique hockey series between India & Pakistan, they decide to play on till a team wins 5 matches. The number of ways in which the series can be won by India, if no match ends in a draw is -
(A) 126 (B) 252 (C) 225 (D) none of these

4. A road network as shown in the figure connect four cities. In how many ways can you start from any city (say A) and come back to it without travelling on the same road more than once ?



(A) 8 (B) 12 (C) 9 (D) 16

5. Six persons A, B, C, D, E and F are to be seated at a circular table. The number of ways this can be done if A must have either B or C on his right and B must have either C or D on his right is -
(A) 36 (B) 12 (C) 24 (D) 18

6. $N = 2^2 . 3^3 . 5^4 . 7$, then -
(A) Number of proper divisors of N(excluding 1 & N) is 118
(B) Number of proper divisors of N(excluding 1 & N) is 120
(C) Number of positive integral solutions of $xy = N$ is 60
(D) Number of positive integral solutions of $xy = N$ is 120

7. Sameer has to make a telephone call to his friend Harish, Unfortunately he does not remember the 7 digit phone number. But he remembers that the first three digits are 635 or 674, the number is odd and there is exactly one 9 in the number. The maximum number of trials that Sameer has to make to be successful is -
(A) 10,000 (B) 3402 (C) 3200 (D) 5000

***8.** The number of solutions of $x_1 + x_2 + x_3 = 51$ (x_1, x_2, x_3 being odd natural numbers) is : -
(A) 300 (B) 325 (C) 330 (D) 350

***9.** The number of positive integral solutions of the equation $x_1x_2x_3 = 60$ is : -
(A) 54 (B) 27 (C) 81 (D) none of these

10. Total number of even divisors of 2079000 which are divisible by 15 are -
(A) 54 (B) 128 (C) 108 (D) 72

11. The number of five digit numbers that can be formed using all the digits 0, 1, 3, 6, 8 which are -

(A) divisible by 4 is 30

(B) divisible by 4 is 60

(C) smaller than 60,000 when digit 8 always appears at ten's place is 6

(D) between 30,000 and 60,000 and divisible by 6 is 18.

12. There are (p + q) different books on different topics in Mathematics. ($p \neq q$)

If L = the number of ways in which these books are distributed between two students X and Y such that X get p books and Y gets q books.

M = The number of ways in which these books are distributed between two students X and Y such that one of them gets p books and another gets q books.

N = The number of ways in which these books are divided into two groups of p books and q books then -

(A) L = N (B) L = 2M = 2N (C) 2L = M (D) L = M

13. Number of dissimilar terms in the expansion of $(x_1 + x_2 + ... + x_n)^3$ is -

(A) $\frac{n^2(n+1)^2}{4}$ (B) $\frac{n(n+1)(n+2)}{6}$ (C) ${}^{n+1}C_2 + {}^{n+1}C_3$ (D) $\frac{n^3+3n^2}{4}$

14. A persons wants to invite one or more of his friend for a dinner party. In how many ways can he do so if he has eight friends : -

(A) 2^8 (B) $2^8 - 1$ (C) 8^2 (D) ${}^8C_1 + {}^8C_2 + ...+ {}^8C_8$

15. Which of the following statement(s) is/are true :-

(A) ${}^{100}C_{50}$ is not divisible by 10

(B) $n(n-1)(n-2)$ ....$(n-r+1)$ is always divisible by r! ($n \in N$ and $0 \leq r \leq n$)

(C) Morse telegraph has 5 arms and each arm moves on 6 different positions including the position of rest. Number of different signals that can be transmitted is $5^6 - 1$.

(D) There are 5 different books each having 5 copies. Number of different selections is $6^5 - 1$.

Match the Column

Following questions contains statements given in two columns, which have to be matched. The statements in ***Column-I*** are labelled as A, B, C and D while the statements in ***Column-II*** are labelled as p, q, r and s. Any given statement in ***Column-I*** can have correct matching with ***ONE*** statement in ***Column-II***.

16. 5 balls are to be placed in 3 boxes. Each box can hold all the 5 balls. Number of ways in which the balls can be placed so that no box remains empty, if :

	Column-I		***Column-II***
(A)	balls are identical but boxes are different	(p)	2
(B)	balls are different but boxes are identical	(q)	25
(C)	balls as well as boxes are identical	(r)	50
(D)	balls as well as boxes are identical but boxes are kept in a row	(s)	6

Comprehension Based Questions

Let p be a prime number and n be a positive integer, then exponent of p is n! is denoted by E_p (n!) and is given by

$$E_p(n!) = \left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \left[\frac{n}{p^3}\right] + \ldots.. + \left[\frac{n}{p^k}\right]$$

where $p^k < n < p^{k+1}$

and [x] denotes the integral part of x.

If we isolate the power of each prime contained in any number N, then N can be written as

$N = 2^{\alpha_1} \cdot 3^{\alpha_2} \cdot 5^{\alpha_3} \cdot 7^{\alpha_4}$ ....

where α_i are whole numbers.

On the basis of above information, answer the following questions

17. The number of zeros at the end of 108! is -
(A) 10 (B) 13 (C) 25 (D) 26

18. The exponent of 12 in 100! is -
(A) 32 (B) 48 (C) 97 (D) none of these

19. The exponent of 7 in $^{100}C_{50}$ is -
(A) 0 (B) 1 (C) 2 (D) 3

EXERCISE – 3 SUBJECTIVE

1. (a) Prove that : ${}^{n}P_r = {}^{n-1}P_r + r.\, {}^{n-1}P_{r-1}$

(b) If ${}^{20}C_{r+2} = {}^{20}C_{2r-3}$ find ${}^{12}C_r$

(c) Find r if ${}^{15}C_{3r} = {}^{15}C_{r+3}$

(d) Find the ratio ${}^{20}C_r$ to ${}^{25}C_r$ when each of them has the greatest value possible.

2. How many 4 digit numbers are there which contain not more than 2 different digits ?

3. An examination paper consists of 12 questions divided into parts A & B. Part-A contains 7 questions & Part - B contains 5 questions. A candidate is required to attempt 8 questions selecting atleast 3 from each part. In how many maximum ways can the candidate select the questions ?

4. The straight lines ℓ_1, ℓ_2, and ℓ_3 are parallel & lie in the same plane. A total of m points are taken on the line ℓ_1, n points on ℓ_2 and k points on ℓ_3. How many maximum number of triangles are there whose vertices are at these points ?

5. A man has 7 relatives, 4 of them are ladies & 3 gentlemen; his wife has also 7 relatives, 3 of them are ladies & 4 gentlemen. In how many ways can they invite a dinner party of 3 ladies & 3 gentlemen so that there are 3 of the man's relatives & 3 of the wife's relatives ?

6. 5 boys & 4 girls sit in a straight line. Find the number of ways in which they can be seated if 2 girls are together & the other 2 are also together but separated from the first 2.

7. In how many ways 8 persons can be seated on a round table

(a) If two of them (say A and B) must not sit in adjacent seats ?

(b) If 4 of the persons are men and 4 ladies and if no two men are to be in adjacent seats?

(c) If 8 persons constitute 4 married couples and if no husband and wife, as well as no two men are to be in adjacent seats ?

8. There are 2 women participating in a chess tournament. Every participant played 2 games with the other participants. The number of games that the men played between themselves exceeded by 66 as compared to the number of games that the men played with the women. Find the number of participants & the total number of games played in the tournament.

9. In how many ways can you divide a pack of 52 cards equally among 4 players. In how many ways the cards can be divided in 4 sets, 3 of them having 17 cards each & the 4^{th} with 1 card ?

10. (a) How many divisors are there of the number 21600 ? Also find the sum of these divisors.

(b) In how many ways the number 7056 can be resolved as a product of 2 factors.

(c) Find the number of ways in which the number 300300 can be split into 2 factors which are relatively prime.

11. How many different ways can 15 candy bars be distributed between Ram, Shyam, Ghanshyam and Balram, if Ram can not have more than 5 candy bars and Shyam must have at least two ? Assume all candy bars to be alike.

12. Find the sum of all numbers greater than 10000 formed by using the digits 0, 1, 2, 4, 5 & no digit being repeated in any number.

13. Find the number of ways in which the letters of the word 'MUNMUN' can be arranged so that no two alike letters are together.

14. Find the number of ways in which a selection of 100 balls, can be made out of 100 identical red balls, 100 identical blue balls & 100 identical white balls.

15. There are 5 balls of different colours & 5 boxes of colours same as those of the balls. The number of ways in which the balls, one in each box could be placed such that exactly one ball goes to the box of its own colour.

EXERCISE – 4 RECAP OF AIEEE/JEE (MAIN)

1. Numbers greater than 1000 but not greater than 4000 which can be formed with the digits 0, 1, 2, 3, 4 (repetition of digits is allowed), are **[AIEEE 2002]**

(A) 350 (B) 375 (C) 450 (D) 576

2. A five digit number divisible by 3 has to formed using the numerals 0, 1, 2, 3, 4 and 5 without repetition. The total number of ways in which this can be done is **[AIEEE 2002]**

(A) 216 (B) 240 (C) 600 (D) 3125

3. Total number of four digit odd numbers that can be formed using 0, 1, 2, 3, 5, 7 are **[AIEEE 2002]**

(A) 192 (B) 375 (C) 400 (D) 720

4. The number of ways in which 6 men and 5 women can dine at a round table if no two women are to sit together is given by **[AIEEE 2003]**

(A) $6! \times 5!$ (B) 30 (C) $5! \times 4!$ (D) $7! \times 5!$

5. A student is to answer 10 out of 13 questions in an examination such that he must choose at least 4 from the first five question. The number of choices available to him is **[AIEEE 2003]**

(A) 140 (B) 196 (C) 280 (D) 346

6. If ${}^nC_r$ denotes the number of combinations of n things taken r at a time, then the expression ${}^nC_{r+1} + {}^nC_{r-1} + 2 \times {}^nC_r$ equals **[AIEEE 2003]**

(A) ${}^{n+2}C_r$ (B) ${}^{n+2}C_{r+1}$ (C) ${}^{n+1}C_r$ (D) ${}^{n+1}C_{r+1}$

7. How many ways are there to arrange the letters in the word 'GARDEN' with the vowels in alphabetical order? **[AIEEE 2004]**

(A) 120 (B) 240 (C) 360 (D) 480

8. The number of ways of distributing 8 identical balls in 3 distinct boxes so that none of the boxes is empty is **[AIEEE 2004]**

(A) 5 (B) 21 (C) 3^8 (D) ${}^8C_3$

9. If the letters of the word 'SACHIN' are arranged in all possible ways and these words are written out as in dictionary, then the word 'SACHIN' appears at serial number **[AIEEE 2005]**

(A) 602 (B) 603 (C) 600 (D) 601

10. The value of ${}^{50}C_4 + \sum_{r=1}^{6} {}^{56-r}C_3$ is **[AIEEE 2005]**

(A) ${}^{56}C_4$ (B) ${}^{56}C_3$ (C) ${}^{55}C_3$ (D) ${}^{55}C_4$

11. At an election, a voter may vote for any number of candidates, not greater than the number to be elected. There are 10 candidated and 4 are to be elected. If a voter votes for at least one candidates, then the number of ways in which he can vote is **[AIEEE-2006]**

(A) 385 (B) 1110 (C) 5040 (D) 6210

***12.** The set S = {1, 2, 3,.....,12} is to partitioned into three sets A, B, C of equal size. Thus,$A \cup B \cup C = S$, $A \cap B = B \cap C = C \cap A = \phi$, then number of ways to partition S are- **[AIEEE-2007]**

(A) $\frac{12!}{3!(3!)^4}$ (B) $\frac{12!}{(4!)^3}$ (C) $\frac{12!}{(3!)^4}$ (D) $\frac{12!}{3!(4!)^3}$

13. In a shop there are five types of ice–creams available. A child buys six ice–creams. ***[AIEEE 2008]***

Statement –1 – The number of different ways the child can buy the six ice–creams is ${}^{10}C_5$.

Statement –2 – The number of different ways the child can buy the six ice–creams is equal to the number of different ways of arranging 6 A's and 4 B's in a row.

(A) Statement –1 is false, Statement –2 is true

(B) Statement–1 is true, Statement–2 is true; Statement–2 is a correct explanation for Statement–1

(C) Statement–1 is true, Statement–2 is true; Statement–2 is not a correct explanation for Statement–1

(D) Statement–1 is true, Statement–2 is false

14. From 6 different novels and 3 different dictionaries, 4 novels and 1 dictionary are to be selected and arranged in a row on a shelf so that the dictionary is always in the middle. Then the number of such arrangements is :- ***[AIEEE 2009]***

(A) At least 750 but less than 1000 (B) At least 1000

(C) Less than 500 (D) At least 500 but less than 750

****15.*** There are two urns. Urn A has 3 distinct red balls and urn B has 9 distinct blue balls. From each urn two balls are taken out at random and then transferred to the other. The number of ways in which this can be done is :- ***[AIEEE-2010]***

(A) 3 (B) 36 (C) 66 (D) 108

16. ***Statement - 1 –*** The number of ways of distributing 10 identical balls in 4 distinct boxes such that no box is empty is ${}^9C_3$. ***[AIEEE-2011]***

Statement - 2 – The number of ways of choosing any 3 places from 9 different places is ${}^9C_3$.

(A) Statement-1 is true, Statement-2 is false.

(B) Statement-1 is false, Statement-2 is true

(C) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1

(D) Statement-1 is true, Statement-2 is true; Statement-2 is ***not*** a correct explanation for Statement-1.

17. There are 10 points in a plane, out of these 6 are collinear. If N is the number of triangles formed by joining these points, then : ***[AIEEE-2011]***

(A) $N > 190$ (B) $N \leq 100$ (C) $100 < N \leq 140$ (D) $140 < N \leq 190$

18. Assuming the balls to be identical except for difference in colours, the number of ways in which one or more balls can be selected from 10 white, 9 green and 7 black balls is :- ***[AIEEE-2012]***

(A) 879 (B) 880 (C) 629 (D) 630

19. Let A and B be two sets containing 2 elements and 4 elements respectively. The number of subsets of $A \times B$ having 3 or more elements is ***[JEE (Main)-2013]***

(A) 256 (B) 220 (C) 219 (D) 211

20. Let T_n be the number of all possible triangles formed by joining vertices of an n-sided regular polygon. If $T_{n+1} - T_n = 10$, then the value of n is : ***[JEE (Main)-2013]***

(A) 7 (B) 5 (C) 10 (D) 8

21. The number of integers greater than 6,000 that can be formed, using the digits 3, 5, 6, 7 and 8, without repetition, is : ***[JEE (Main)-2015]***

(A) 216 (B) 192 (C) 120 (D) 72

22. If all the words (with or without meaning) having five letters, formed using the letters of the word SMALL and arranged as in a dictionary: then the position of the word SMALL is : ***[JEE (Main)-2016]***

(A) 46^{th} (B) 59^{th} (C) 52^{nd} (D) 58^{th}

23. A man X has 7 friends, 4 of them are ladies and 3 are men. His wife Y also has 7 friends, 3 of them are ladies and 4 are men. Assume X and Y have no common friends. Then the total number of ways in which X and Y together can throw a party inviting 3 ladies and 3 men, so that 3 friends of each of X and Y are in this party is

(A) 469 (B) 484 (C) 485 (D) 468 **[JEE (Main)-2017]**

24. From 6 different novels and 3 different dictionaries, 4 novels and 1 dictionary are to be selected and arranged in a row on a shelf so that the dictionary is always in the middle. The number of such arrangements is- **[JEE (Main)-2018]**

(A) less than 500 (B) at least 500 but less than 750
(C) at least 750 but less than 1000 (D) at least 1000

25. The number of natural numbers less than 7,000 which can be formed by using the digits 0, 1, 3, 7, 9 (repitition of digits allowed) is equal to: **[JEE (Main)-2019]**

(A) 250 (B) 374 (C) 372 (D) 375

26. Consider a class of 5 girls and 7 boys. The number of different teams consisting of 2 girls and 3 boys that can be formed from this class, if there are two specific boys A and B, who refuse to be the members of the same team, is: **[JEE (Main)-2019]**

(A) 200 (B) 300 (C) 500 (D) 350

27. The sum of all two digit positive numbers which when divided by 7 yield 2 or 5 as remainder is:

(A) 1365 (B) 1256 (C) 1465 (D) 1356 **[JEE (Main)-2019]**

28. The number of functions f from $\{1, 2, 3, 20\}$ onto $\{1,2,3, ..., 20\}$ such that f(k) is a multiple of 3, whenever k is a multiple of 4, is : **[JEE (Main)-2019]**

(A) $(15)! \times 6!$ (B) $5^6 \times 15$ (C) $5! \times 6!$ (D) $6^5 \times (15)!$

29. There are m men and two women participating in a chess tournament. Each participant plays two games with every other participant. If the number of games played by the men between themselves exceeds the number of games played between the men and the women by 84, then the value of m is : **[JEE (Main)-2019]**

(A) 9 (B) 11 (C) 12 (D) 7

30. If ${}^nC_4$, ${}^nC_5$ and ${}^nC_6$ are in A.P., then n can be: **[JEE (Main)-2019]**

(A) 14 (B) 11 (C) 9 (D) 12

31. Let S = $\{1, 2, 3, ..., 100\}$. The number of nonempty subsets A of S such that the product of elements in A is even is : **[JEE (Main)-2019]**

(A) $2^{50}(2^{50} - 1)$ (B) $2^{100} - 1$ (C) $2^{50} - 1$ (D) $2^{50} + 1$

32. Consider three boxes, each containing 10 balls labelled 1, 2, ..., 10. Suppose one ball is randomly drawn from each of the boxes. Denote by n_i, the label of the ball drawn from the i^{th} box, (i = 1, 2, 3). Then, the number of ways in which the balls can be chosen such that $n_1 < n_2 < n_3$ is : **[JEE (Main)-2019]**

(A) 82 (B) 240 (C) 164 (D) 120

33. All possible numbers are formed using the digits 1, 1, 2, 2, 2, 2, 3, 4, 4 taken all at a time. The number of such numbers in which the odd digits occupy even places is : **[JEE (Main)-2019]**

(A) 175 (B) 162 (C) 160 (D) 180

34. The number of four-digit numbers strictly greater than 4321 that can be formed using the digits 0,1,2,3,4,5 (repetition of digits is allowed) is : **[JEE (Main)-2019]**

(A) 288 (B) 306 (C) 360 (D) 31

35. A committee of 11 members is to be formed from 8 males and 5 females. If m is the number of ways the committee is formed with at least 6 males and n is the number of ways the committee is formed with at least 3 females, then : ***[JEE (Main)-2019]***

(A) $m = n = 78$ (B) $n = m - 8$ (C) $m + n = 68$ (D) $m = n = 68$

36. The number of 6 digit numbers that can be formed using the digits 0, 1, 2, 5, 7 and 9 which are divisible by 11 and no digit is repeated, is : ***[JEE (Main)-2019]***

(A) 36 (B) 60 (C) 48 (D) 72

37. Suppose that 20 pillars of the same height have been erected along the boundary of a circular stadium. If the top of each pillar has been connected by beams with the top of all its non-adjacent pillars, then the total number beams is : ***[JEE (Main)-2019]***

(A) 210 (B) 190 (C) 170 (D) 180

38. The number of ways of choosing 10 objects out of 31 objects of which 10 are identical and the remaining 21 are distinct, is : ***[JEE (Main)-2019]***

(A) 2^{20} (B) $2^{20} - 1$ (C) $2^{20} + 1$ (D) 2^{21}

39. A group of students comprises of 5 boys and n girls. If the number of ways, in which a team of 3 students can randomly be selected from this group such that there is at least one boy and at least one girl in each team, is 1750, then n is equal to : ***[JEE (Main)-2019]***

(A)25 (B) 28 (C) 27 (D) 24

EXERCISE – 5 RECAP OF IIT-JEE/JEE (ADVANCED)

1.** If r, s, t are the prime numbers and p, q are the positive integers such that the LCM of p & q is $r^2t^4s^2$, then the number of ordered pair (p, q) is : ***[JEE 2006]

(A) 252 (B) 254 (C) 225 (D) 224

2. The letters of the word COCHIN are permuted and all the permutations are arranged in an alphabetical order as in an English dictionary. The number of words that appear before the word COCHIN is - ***[JEE 2007]***

(A) 360 (B) 192 (C) 96 (D) 48

3. Consider all possible permutations of the letters of the word ENDEANOEL. ***[JEE 2008]***

Match the Statements / Expressions in Column I with the Statements / Expressions in Column II.

	Column I		***Column II***
(A)	The number of permutations containing the word ENDEA is	(p)	5!
(B)	The number of permutations in which the letter E occurs in the first and the last positions is	(q)	$2 \times 5!$
(C)	The number of permutations in which none of the letters D, L, N occurs in the last five positions is	(r)	$7 \times 5!$
(D)	The number of permutations in which the letters A, E, O occur only in odd positions is	(s)	$21 \times 5!$

4. The number of seven digit integers, with sum of the digits equal to 10 and formed by using the digits 1, 2 and 3 only is - ***[JEE 2009]***

(A) 55 (B) 66 (C) 77 (D) 88

5.** Let S = {1,2,3,4}. The total number of unordered pairs of disjoint subsets of S is equal to - ***[JEE 2010]

(A) 25 (B) 34 (C) 42 (D) 41

6. The total number of ways in which 5 balls of different colours can be distributed among 3 persons so that each person gets at least one ball is - ***[JEE 2012]***

(A) 75 (B) 150 (C) 210 (D) 243

Paragraph for Question 7 and 8

Let a_n denotes the number of all n-digit positive integers formed by the digits 0, 1 or both such that no consecutive digits in them are 0. Let b_n = the number of such n-digit integers ending with digit 1 and c_n = the number of such n-digit integers ending with digit 0.

7.** The value of b_6 is ***[JEE 2012]

(A) 7 (B) 8 (C) 9 (D) 11

8.** Which of the following is correct ? ***[JEE 2012]

(A) $a_{17} = a_{16} + a_{15}$ (B) $c_{17} \neq c_{16} + c_{15}$ (C) $b_{17} \neq b_{16} + c_{16}$ (D) $a_{17} = c_{17} + b_{16}$

9.** Let $n_1 < n_2 < n_3 < n_4 < n_5$ be positive integers such that $n_1 + n_2 + n_3 + n_4 + n_5 = 20$. Then the number of such distinct arrangements $(n_1, n_2, n_3, n_4, n_5)$ is ***[JEE 2014]

10. Let $n \geq 2$ be an integer. Take n distinct points on a circle and join each pair of points by a line segment. Colour the line segment joining every pair of adjacent points by blue and the rest by red. If the number of red and blue line segments are equal, then the value of n is : ***[JEE 2014]***

11. Six cards and six envelopes are numbered 1, 2,3,4,5,6 and cards are to be placed in envelopes so that each envelope contains exactly one card and no card is placed in the envelope bearing the same number and moreover the card numbered 1 is always placed in envelope numbered 2. Then the number of ways it can be done is ***[JEE 2014]***

(A) 264 (B) 265 (C) 53 (D) 67

12. Let n be the number of ways in which 5 boys and 5 girls can stand in a queue in such a way that all the girls stand consecutively in the queue. Let m be the number of ways in which 5 boys and 5 girls can stand in a queue in such a way that exactly four girls stand consecutively in the queue. Then the value of $\frac{m}{n}$ is ***[JEE 2015]***

13. A debate club consists of 6 girls and 4 boys. A team of 4 members is to be selected from this club including the selection of a captain (from among these 4 members) for the team. If the team has to include at most one boy, then the number of ways of selecting the team is ***[JEE 2016]***

(A) 380 (B) 320 (C) 260 (D) 95

14. Words of length 10 are formed using the letters A, B, C, D, E, F, G, H, I, J. Let x be the number of such words where no letter is repeated; and let y be the number of such words where exactly one letter is repeated twice and no other letter is repeated. Then, $\frac{y}{9x} =$ ***[JEE 2017]***

15. Let S = {1, 2, 3, ... 9}. For k = 1, 2, ..., 5, let N_k be the number of subsets of S, each containing five elements out of which exactly k are odd. Then $N_1 + N_2 + N_3 + N_4 + N_5 =$ ***[JEE 2017]***

(A) 125 (B) 252 (C) 210 (D) 126

16. The number of 5 digit numbers which are divisible by 4, with digits from the set {1, 2, 3, 4, 5} and the repetition of digits is allowed, is —— ***[JEE 2018]***

17. In a high school, a committee has to be formed from a group of 6 boys M_1, M_2, M_3, M_4, M_5, M_6 and 5 girls G_1, G_2, G_3, G_4, G_5.

(i) Let α_1 be the total number of ways in which the committee can be formed such that the committee has 5 members, having exactly 3 body and 2 girls.

(ii) Let α_2 be the total number of ways in which the committe can be formed such that the committee has at least 2 members, and having an equal number of boys and girls.

(iii) Let α_3 be the total number of ways in which the committe can be formed such that the committee has 5 members, at least 2 of them being girls.

(iv) Let α_4 be the total number of ways in which the committee can be formed such that the commitee has 4 members, having at least 2 girls and such that both M_1 and G_1 are ***NOT*** in the committee together.

LIST-I		**LIST-II**	
P.	The value of α_1 is	**1.**	136
Q.	The value of α_2 is	**2.**	189
R.	The value of α_3 is	**3.**	192
S.	The value of α_4 is	**4.**	200
		5.	381
		6.	461

The correct option is :-

(A) $\boldsymbol{P \to 4; Q \to 6, R \to 2; S \to 1}$

(B) $\boldsymbol{P \to 1; Q \to 4; R \to 2; S \to 3}$

(C) $\boldsymbol{P \to 4; Q \to 6, R \to 5; S \to 2}$

(D) $\boldsymbol{P \to 4; Q \to 2; R \to 3; S \to 1}$ ***[JEE 2018]***

18. Five person A,B,C,D and E are seated in a ciruclar arrangement. If each of them is given a hat of one of the three colours red, blue and green ,then the number of ways of distributing the hats such that the persons seated in adjacent seats get different coloured hats is ***[JEE 2019]***

ANSWER KEY

EXERCISE-1

Que.	1	2	3	4	5	6	7	8	9	10
Ans.	A	A	C	D	D	A	C	D	C	C
Que.	11	12	13	14	15	16	17	18	19	20
Ans.	C	B	C	B	B	C	A	D	C	D
Que.	21									
Ans.	C									

EXERCISE-2

Que.	1	2	3	4	5	6	7	8	9	10
Ans.	C	D	A	B	D	AD	B	B	A	C
Que.	11	12	13	14	15					
Ans.	BD	AC	BC	BD	ABD					

EXERCISE-3

- ***Match the Column*** **16.** (A) $\rightarrow$ (s), (B) $\rightarrow$ (q), (C) $\rightarrow$ (p), (D) $\rightarrow$ (s)
- ***Comprehension Based Questions*** **17.** C **18.** B **19.** A

EXERCISE-3

1. ***(b)*** 792; ***(c)*** $r = 3$; ***(d)*** $\frac{143}{4025}$ **2.** 576 **3.** 420

4. ${}^{m+n+k}C_3 - ({}^{m}C_3 + {}^{n}C_3 + {}^{k}C_3)$ **5.** 485 **6.** 43200 **7.** ***(a)*** 5.(6!); ***(b)*** 3!4!; ***(c)*** 12

8. 13, 156 **9.** $\frac{52!}{(13!)^4}; \frac{52!}{3!(17!)^3}$ **10.** ***(a)*** 72; 78120; ***(b)*** 23; ***(c)*** 32 **11.** 440

12. 3119976 **13.** 30 **14.** 5151 **15.** 45

EXERCISE-4

Que.	1	2	3	4	5	6	7	8	9	10
Ans.	B	A	D	A	B	B	C	B	D	A
Que.	11	12	13	14	15	16	17	18	19	20
Ans.	A	B	A	B	D	C	B	A	C	B
Que.	21	22	23	24	25	26	27	28	29	30
Ans.	B	D	C	D	B	B	D	A	C	A
Que.	31	32	33	34	35	36	37	38	39	
Ans.	A	D	D	D	A	B	C	A	A	

EXERCISE-5

1. (C) **2.** (C) **3.** (A)$\rightarrow$(p), (B)$\rightarrow$(s), (C)$\rightarrow$(q), (D)$\rightarrow$(q) **4.** (C) **5.** (D) **6.** (B)

7. (B) **8.** (A) **9.** (7) **10.** (5) **11.** (C) **12.** 5 **13.** (A) **14.** (5) **15.** (D) **16.** 625

17. (C) **18.** (30)

BINOMIAL THEOREM

Recap of Early Classes

In earlier classes, we have learnt how to find the squares and cubes of binomials like a + b and a – b. Using the m, we could evaluate the numerical values of numbers like $(98)^2 = (100-2)^2$, $(999)^3 = (1000-1)^3$, etc. However, for higher powers like $(98)^5$, $(101)^6$, etc., the calculations become difficult by using repeated multiplication. This difficulty was overcome by a theorem known as binomial theorem. It gives an easier way to expand $(a+b)^n$, where *n* is an integer or a rational number.

Index

BINOMIAL THEOREM

1.0 BINOMIAL EXPRESSION

SL AL

Any algebraic expression which contains two dissimilar terms is called binomial expression.

For example : $x - y$, $xy + \frac{1}{x}, \frac{1}{z} - 1, \frac{1}{(x-y)^{1/3}} + 3$ etc.

2.0 BINOMIAL THEOREM

SL AL

The formula by which any positive integral power of a binomial expression can be expanded in the form of a series is known as ***BINOMIAL THEOREM.***

If $x, y \in R$ and $n \in N$, then : $(x + y)^n$

$$= {}^nC_0x^n + {}^nC_1x^{n-1}y + {}^nC_2x^{n-2}y^2 + ... + {}^nC_rx^{n-r}y^r + ... + {}^nC_ny^n$$

$$= \sum_{r=0}^{n} {}^nC_rx^{n-r}y^r$$

This theorem can be proved by induction.

Properties

(a) The number of terms in the expansion is $(n+1)$ i.e. one more than the index.

(b) The sum of the indices of x & y in each term is n.

(c) The binomial coefficients of the terms $({}^nC_0, {}^nC_1.....)$ equidistant from the beginning and the end are equal. i.e. ${}^nC_r = {}^nC_{n-r}$

(d) Symbol ${}^nC_r$ can also be denoted by $\binom{n}{r}$, ***C(n, r)***

Some important expansions

(i) $(1 + x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + ... + {}^nC_nx^n$.

(ii) $(1 - x)^n = {}^nC_0 - {}^nC_1x + {}^nC_2x^2 + ... + (-1)^n . {}^nC_nx^n$.

Note – The coefficient of x^r in $(1+x)^n = {}^nC_r$ & that in $(1-x)^n = (-1)^r . {}^nC_r$

Illustrations

Illustration 1. Expand : $(y + 2)^6$.

Solution. ${}^6C_0y^6 + {}^6C_1y^5.2 + {}^6C_2y^4.2^2 + {}^6C_3y^3.2^3 + {}^6C_4y^2.2^4 + {}^6C_5y^1.2^5 + {}^6C_6.2^6$.

$= y^6 + 12y^5 + 60y^4 + 160y^3 + 240y^2 + 192y + 64$.

Illustration 2. Write first 4 terms of $\left(1 - \frac{2y^2}{5}\right)^7$

Solution. ${}^7C_0, {}^7C_1\left(-\frac{2y^2}{5}\right)$,

${}^7C_2\left(-\frac{2y^2}{5}\right)^2, {}^7C_3\left(-\frac{2y^2}{5}\right)^3$

Illustration 3. The value of $\dfrac{\left(18^3+7^3+3.18.7.25\right)}{3^6+6.243.2+15.81.4+20.27.8+15.9.16+6.3.32+64}$ is -

(A) 1 (B) 2 (C) 3 (D) 4

Solution. The numerator is of the form $a^3+b^3+3ab(a+b)=(a+b)^3$

Where, a = 18 and b = 7

$\therefore$ $N^r = (18+7)^3 = (25)^3$

Denominator can be written as

$$3^6+{}^6C_1.3^5.2^1+{}^6C_2.3^4.2^2+{}^6C_3.3^3.2^3+{}^6C_4\,3^2.2^4+{}^6C_5\,3.2^5+{}^6C_6 2^6=(3+2)^6=5^6=(25)^3$$

$\therefore$ $\dfrac{Nr}{Dr}=\dfrac{(25)^3}{(25)^3}=1$ **Ans.**

Illustration 4. If in the expansion of $(1+x)^m(1-x)^n$, the coefficients of x and x^2 are 3 and – 6 respectively then m is -

(A) 6 (B) 9 (C) 12 (D) 24

Solution. $(1+x)^m(1-x)^n=\left[1+mx+\dfrac{(m)(m-1).x^2}{2}+......\right]\left[1-nx+\dfrac{n(n-1)}{2}x^2+......\right]$

Coefficient of x = m – n = 3 ... (i)

Coefficient of $x^2 = -mn + \dfrac{n(n+1)}{2}+\dfrac{m(m-1)}{2}=-6$ ... (ii)

Solving (i) and (ii), we get

m = 12 and n = 9.

Pascal's triangle – A triangular arrangement of numbers as shown. The numbers give the coefficients for the expansion of $(x+y)^n$. The first row is for n = 0, the second for n = 1, etc. Each row has 1 as its first and last number. Other numbers are generated by adding the two numbers immediately to the left and right in the row above.

```
            1
          1   1
        1   2   1
      1   3   3   1
    1   4   6   4   1
  1   5  10  10   5   1
1   6  15  20  15   6   1
           etc.
```

3.0 IMPORTANT TERMS IN THE BINOMIAL EXPANSION

SL AL

3.1 General term

The general term or the $(r+1)^{th}$ term in the expansion of $(x+y)^n$ is given by $T_{r+1} = {}^nC_r\, x^{n-r}\, y^r$

3.2 Term independent of x

Term independent of x does not contain x ; Hence find the value of r for which the exponent of x is zero.

3.3 Middle term

The middle term(s) in the expansion of $(x+y)^n$ is (are) :

(i) If n is even, there is only one middle term which is given by $T_{(n+2)/2} = {}^nC_{n/2}.\, x^{n/2}.\, y^{n/2}$

(ii) If n is odd, there are two middle terms which are $T_{(n+1)/2}$ & $T_{[(n+1)/2]+1}$

Properties of Middle Term

(a) Middle term has greatest binomial coefficient and if there are 2 middle terms their coefficients will be equal.

(b) ${}^nC_r$ will be maximum
- When $r=\dfrac{n}{2}$ if n is even
- When $r=\dfrac{n-1}{2}$ or $\dfrac{n+1}{2}$ if n is odd

(c) The term containing greatest binomial coefficient will be middle term in the expansion of $(1+x)^n$

Illustrations

Illustration 5. Find :

(a) The coefficient of x^7 in the expansion of $\left(ax^2 + \frac{1}{bx}\right)^{11}$

(b) The coefficient of x^{-7} in the expansion of $\left(ax - \frac{1}{bx^2}\right)^{11}$

Also, find the relation between a and b, so that these coefficients are equal.

Solution. ***(a)*** In the expansion of $\left(ax^2 + \frac{1}{bx}\right)^{11}$, the general term is :

$$T_{r+1} = {}^{11}C_r (ax^2)^{11-r}\left(\frac{1}{bx}\right)^r = {}^{11}C_r . \frac{a^{11-r}}{b^r} . x^{22-3r}$$

putting $22 - 3r = 7$

$\therefore$ $3r = 15 \Rightarrow r = 5$

$\therefore$ $T_6 = {}^{11}C_5 \frac{a^6}{b^5} . x^7$

Hence the coefficient of x^7 in $\left(ax^2 + \frac{1}{bx}\right)^{11}$ is ${}^{11}C_5 a^6 b^{-5}$. ***Ans.***

Note that binomial coefficient of sixth term is ${}^{11}C_5$.

(b) In the expansion of $\left(ax - \frac{1}{bx^2}\right)^{11}$, general term is :

$$T_{r+1} = {}^{11}C_r (ax)^{11-r}\left(\frac{-1}{bx^2}\right)^r = (-1)^r \, {}^{11}C_r \frac{a^{11-r}}{b^r} . x^{11-3r}$$

putting $11 - 3r = -7$

$\therefore$ $3r = 18 \quad \Rightarrow \quad r = 6$

$\therefore$ $T_7 = (-1)^6 . \, {}^{11}C_6 \frac{a^5}{b^6} . x^{-7}$

Hence the coefficient of x^{-7} in $\left(ax - \frac{1}{bx^2}\right)^{11}$ is ${}^{11}C_6 a^5 b^{-6}$. ***Ans.***

Also given

Coefficient of x^7 in $\left(ax^2 + \frac{1}{bx}\right)^{11}$ = coefficient of x^{-7} in $\left(ax - \frac{1}{bx^2}\right)^{11}$

$\Rightarrow$ ${}^{11}C_5 a^6 b^{-5} = {}^{11}C_6 a^5 b^{-6}$

$\Rightarrow$ $ab = 1$ $\qquad (\because \; {}^{11}C_5 = {}^{11}C_6)$

which is the required relation between a and b. ***Ans.***

****Illustration 6.*** Find the number of rational terms in the expansion of $(9^{1/4} + 8^{1/6})^{1000}$.

Solution. The general term in the expansion of $(9^{1/4} + 8^{1/6})^{1000}$ is

$$T_{r+1} = {}^{1000}C_r \left(9^{\frac{1}{4}}\right)^{1000-r} \left(8^{\frac{1}{6}}\right)^r = {}^{1000}C_r \, 3^{\frac{1000-r}{2}} 2^{\frac{r}{2}}$$

The above term will be rational if exponents of 3 and 2 are integers

It means $\frac{1000 - r}{2}$ and $\frac{r}{2}$ must be integers

The possible set of values of r is {0, 2, 4, ..., 1000}

Hence, number of rational terms is 501 ***Ans.***

Illustration 7. Find the middle term in the expansion of $\left(3x-\frac{x^3}{6}\right)^9$

Solution. The number of terms in the expansion of $\left(3x-\frac{x^3}{6}\right)^9$ is 10 (even). So there are two middle terms.

i.e. $\left(\frac{9+1}{2}\right)^{th}$ and $\left(\frac{9+3}{2}\right)^{th}$ are two middle terms. They are given by T_5 and T_6

$$\therefore T_5 = T_{4+1} = {}^9C_4(3x)^5\left(-\frac{x^3}{6}\right)^4 = {}^9C_4 3^5 x^5 . \frac{x^{12}}{6^4}$$

$$= \frac{9.8.7.6}{1.2.3.4}.\frac{3^5}{2^4.3^4}x^{17} = \frac{189}{8}x^{17}$$

and $T_6 = T_{5+1} = {}^9C_5(3x)^4\left(-\frac{x^3}{6}\right)^5$

$$= -{}^9C_4 3^4.x^4.\frac{x^{15}}{6^5} = \frac{-9.8.7.6}{1.2.3.4}.\frac{3^4}{2^5.3^5}x^{19} = -\frac{21}{16}x^{19}$$ **Ans.**

Illustration 8. The term independent of x in $\left[\sqrt{\frac{x}{3}}+\sqrt{\left(\frac{3}{2x^2}\right)}\right]^{10}$ is -

(A) 1 (B) $\frac{5}{12}$ (C) ${}^{10}C_1$ (D) none of these

Solution. General term in the expansion is

$$ {}^{10}C_r\left(\frac{x}{3}\right)^{\frac{r}{2}}\left(\frac{3}{2x^2}\right)^{\frac{10-r}{2}} = {}^{10}C_r x^{\frac{3r}{2}-10}.\frac{3^{5-r}}{2^{\frac{10-r}{2}}}$$

For constant term, $\frac{3r}{2}=10 \Rightarrow r=\frac{20}{3}$

which is not an integer. Therefore, there will be no constant term. **Ans. (D)**

4.0 NUMERICALLY GREATEST TERM

AL

Let numerically greatest term in the expansion of $(a + b)^n$ be T_{r+1}.

$$\Rightarrow \begin{cases} |T_{r+1}| \ge |T_r| \\ |T_{r+1}| \ge |T_{r+2}| \end{cases} \text{ where } T_{r+1} = {}^nC_r a^{n-r} b^r$$

Solving above inequalities we get

$$\frac{n+1}{1+\left|\frac{a}{b}\right|}-1 \le r \le \frac{n+1}{1+\left|\frac{a}{b}\right|}$$

Case I – When $\frac{n+1}{1+\left|\frac{a}{b}\right|}$ is an integer equal to m, then T_m and T_{m+1} will be numerically greatest term.

Case II – When $\frac{n+1}{1+\left|\frac{a}{b}\right|}$ is not an integer and its integral part is m, then T_{m+1} will be the numerically greatest term.

Illustrations

Illustration 9. Find numerically greatest term in the expansion of $(3-5x)^{11}$ when $x = \frac{1}{5}$

Solution. Using $\frac{n+1}{1+\left|\frac{a}{b}\right|}-1 \le r \le \frac{n+1}{1+\left|\frac{a}{b}\right|}$

$$\frac{11+1}{1+\left|\frac{3}{-5x}\right|}-1 \le r \le \frac{11+1}{1+\left|\frac{3}{-5x}\right|}$$

solving we get $2 \le r \le 3$

$\therefore$ r = 2, 3

so, the greatest terms are T_{2+1} and T_{3+1}.

$\therefore$ Greatest term (when r = 2)

$T_3 = {}^{11}C_2.3^9\,(-5x)^2 = 55.3^9 = T_4$

From above we say that the value of both greatest terms are equal. ***Ans.***

Illustration 10. Given T_3 in the expansion of $(1-3x)^6$ has maximum numerical value. Find the range of 'x'.

Solution. Using $\frac{n+1}{1+\left|\frac{a}{b}\right|}-1 \le r \le \frac{n+1}{1+\left|\frac{a}{b}\right|}$

$$\frac{6+1}{1+\left|\frac{1}{-3x}\right|}-1 \le 2 \le \frac{7}{1+\left|\frac{1}{-3x}\right|}$$

Let $|x| = t$

$$\frac{21t}{3t+1}-1 \le 2 \le \frac{21t}{3t+1}$$

$$\left\{\begin{array}{l}\frac{21t}{3t+1} \le 3 \\ \frac{21t}{3t+1} \ge 2\end{array}\right. \Rightarrow \left\{\begin{array}{l}\frac{4t-1}{3t+1} \le 0 \Rightarrow t \in \left[-\frac{1}{3},\frac{1}{4}\right] \\ \frac{15t-2}{3t+1} \ge 0 \Rightarrow t \in \left(-\infty,-\frac{1}{3}\right] \cup \left[\frac{2}{15},\infty\right)\end{array}\right.$$

Common solution $t \in \left[\frac{2}{15},\frac{1}{4}\right]$

$$\Rightarrow x \in \left[-\frac{1}{4},-\frac{2}{15}\right] \cup \left[\frac{2}{15},\frac{1}{4}\right]$$

BEGINNER'S BOX-1

TOPIC COVERED : BINOMIAL EXPRESSION, BINOMIAL THEOREM, IMPORTANT TERMS IN THE BINOMIAL EXPANSION, NUMERICALLY GREATEST TERM

1. Expand $\left(3x^2-\frac{x}{2}\right)^5$

2. Expand $(y+x)^n$

3. Find the 7^{th} term of $\left(3x^2-\frac{1}{3}\right)^{10}$

4. Find the term independent of x in the expansion : $\left(2x^2-\frac{3}{x^3}\right)^{25}$

5. Find the middle term in the expansion of : (a) $\left(\frac{2x}{3}-\frac{3}{2x}\right)^6$ (b) $\left(2x^2-\frac{1}{x}\right)^7$

6. Find the numerically greatest term in the expansion of $(3-2x)^9$, when $x = 1$.

7. In the expansion of $\left(\frac{1}{2}+\frac{2x}{3}\right)^n$ when $x=-\frac{1}{2}$, it is known that 3^{rd} term is the greatest term. Find the possible integral values of n.

8. Find the 7th term in $\left(\frac{1}{y}+y^2\right)^{10}$.

5.0 PROPERTIES OF BINOMIAL COEFFICIENTS

SL AL

$(1+x)^n = C_0 + C_1x + C_2x^2 + C_3x^3 + \dots\dots\dots + C_nx^n = \sum_{r=0}^{n} {}^nC_r r^r \ ; n \in N$ ... (i)

where $C_0, C_1, C_2, \dots\dots\dots\dots C_n$ are called combinatorial (binomial) coefficients.

(a) **The sum of all the binomial coefficients is 2^n**

Put $x = 1$, in (i) we get

$C_0 + C_1 + C_2 + \dots\dots\dots\dots + C_n = 2^n \Rightarrow \sum_{r=0}^{n} {}^nC_r = 2^n$ ... (ii)

Put $x=-1$ in (i) we get

$C_0 - C_1 + C_2 - C_3 \dots\dots\dots\dots + (-1)^n C_n = 0 \Rightarrow \sum_{r=0}^{n} (-1)^r\, {}^nC_r = 0$ ... (iii)

(b) ***The sum of the binomial coefficients at odd position is equal to the sum of the binomial coefficients at even position and each is equal to 2^{n-1}.***

From (ii) & (iii), $C_0 + C_2 + C_4 + \dots\dots\dots\dots = C_1 + C_3 + C_5 + \dots\dots\dots = 2^{n-1}$

Illustrations

Illustration 11. Prove that : ${}^{25}C_{10} + {}^{24}C_{10} + \dots\dots\dots + {}^{10}C_{10} = {}^{26}C_{11}$

Solution. LHS $= {}^{10}C_{10} + {}^{11}C_{10} + {}^{12}C_{10} + \dots\dots\dots\dots\dots + {}^{25}C_{10}$

$\Rightarrow {}^{11}C_{11} + {}^{11}C_{10} + {}^{12}C_{10} + \dots\dots\dots + {}^{25}C_{10}$

$\Rightarrow {}^{12}C_{11} + {}^{12}C_{10} + \dots\dots\dots + {}^{25}C_{10}$

$\Rightarrow {}^{13}C_{11} + {}^{13}C_{10} + \dots\dots\dots {}^{25}C_{10}$

and so on. $\therefore$ LHS $= {}^{26}C_{11}$

Aliter

LHS = coefficient of x^{10} in $\{(1 + x)^{10} + (1 + x)^{11} + \ldots\ldots\ldots\ldots (1+ x)^{25}\}$

$\Rightarrow$ coefficient of x^{10} in $\left[(1+x)^{10}\dfrac{\{1+x\}^{16}-1}{1+x-1}\right]$

$\Rightarrow$ coefficient of x^{10} in $\dfrac{\left[(1+x)^{26}-(1+x)^{10}\right]}{x}$

$\Rightarrow$ coefficient of x^{11} in $\left[(1+x)^{26}-(1+x)^{10}\right] = {}^{26}C_{11} - 0 = {}^{26}C_{11}$

Illustration 12. Prove that –

(i) $C_1 + 2C_2 + 3C_3 + \ldots\ldots + nC_n = n \,.\, 2^{n-1}$

(ii) $C_0 + \dfrac{C_1}{2} + \dfrac{C_2}{3} + \ldots\ldots + \dfrac{C_n}{n+1} = \dfrac{2^{n+1}-1}{n+1}$

Solution. ***(i)*** L.H.S. $= \sum\limits_{r=1}^{n} r.\,{}^{n}C_r = \sum\limits_{r=1}^{n} r.\dfrac{n}{r}.\,{}^{n-1}C_{r-1}$

$= n\sum\limits_{r=1}^{n} {}^{n-1}C_{r-1} = n.\left[{}^{n-1}C_0 + {}^{n-1}C_1 + \ldots\ldots + {}^{n-1}C_{n-1}\right]$

$= n \,.\, 2^{n-1}$

Aliter – (Using method of differentiation)

$(1 + x)^n = {}^{n}C_0 + {}^{n}C_1x + {}^{n}C_2x^2 + \ldots\ldots + {}^{n}C_nx^n$ ... (A)

Differentiating (A), we get

$n(1 + x)^{n-1} = C_1 + 2C_2x + 3C_3x^2 + \ldots\ldots + n.C_nx^{n-1}$.

Put x = 1,

$C_1 + 2C_2 + 3C_3 + \ldots\ldots + n.C_n = n.2^{n-1}$

(ii) L.H.S. $= \sum\limits_{r=0}^{n}\dfrac{C_r}{r+1} = \dfrac{1}{n+1}\sum\limits_{r=0}^{n}\dfrac{n+1}{r+1}\,{}^{n}C_r$

$= \dfrac{1}{n+1}\sum\limits_{r=0}^{n}{}^{n+1}C_{r+1} = \dfrac{1}{n+1}\left[{}^{n+1}C_1 + {}^{n+1}C_2 + \ldots\ldots + {}^{n+1}C_{n+1}\right] = \dfrac{1}{n+1}\left[2^{n+1}-1\right]$

Aliter – (Using method of integration)

Integrating (A), we get

$\dfrac{(1+x)^{n+1}}{n+1} + C = C_0x + \dfrac{C_1x^2}{2} + \dfrac{C_2x^3}{3} + \ldots\ldots + \dfrac{C_nx^{n+1}}{n+1}$ (where C is a constant)

Put x = 0, we get, $C = -\dfrac{1}{n+1}$

$\therefore \dfrac{(1+x)^{n+1}-1}{n+1} = C_0x + \dfrac{C_1x^2}{2} + \dfrac{C_2x^3}{3} + \ldots\ldots + \dfrac{C_nx^{n+1}}{n+1}$

Put x = 1, we get $C_0 + \dfrac{C_1}{2} + \dfrac{C_2}{3} + \ldots\ldots \dfrac{C_n}{n+1} = \dfrac{2^{n+1}-1}{n+1}$

Put x = –1, we get $C_0 - \dfrac{C_1}{2} + \dfrac{C_2}{3} - \ldots\ldots = \dfrac{1}{n+1}$

Illustration 13. If $(1+x)^n = \sum_{r=0}^{n} {}^nC_r x^r$, then prove that $C_1^2 + 2.C_2^2 + 3.C_3^2 + \ldots\ldots\ldots + n.C_n^2 = \dfrac{(2n-1)!}{((n-1)!)^2}$

Solution. $(1+x)^n = C_0 + C_1x + C_2x^2 + C_2x^3 + \ldots\ldots\ldots + C_n x^n$ ... (i)

Differentiating both the sides, w.r.t. x, we get

$n(1+x)^{n-1} = C_1 + 2C_2x + 3C_2x^2 + \ldots\ldots\ldots + n.C_nx^{n-1}$ ... (ii)

also, we have

$(x+1)^n = C_0x^n + C_1x^{n-1} + C_2x^{n-2} + \ldots\ldots\ldots + C_n$ ... (iii)

Multiplying (ii) & (iii), we get

$(C_1 + 2C_2x + 3C_3x^2 + \ldots\ldots + C_nx^{n-1})(C_0x^n + C_1x^{n-1} + C_2x^{n-2} + \ldots\ldots + C_n) = n(1+x)^{2n-1}$

Equating the coefficients of x^{n-1}, we get

$$C_1^2 + 2C_2^2 + 3C_3^2 + \ldots\ldots\ldots + n.C_n^2 = n.\,{}^{2n-1}C_{n-1} = \frac{(2n-1)!}{((n-1)!)^2} \quad \textbf{Ans.}$$

Illustration 14. Prove that : $C_0 - 3C_1 + 5C_2 - \ldots\ldots\ldots(-1)^n(2n+1)C_n = 0$

Solution. $T_r = (-1)^r(2r+1)\,{}^nC_r = 2(-1)^r r\,.\,{}^nC_r + (-1)^r\,{}^nC_r$

$$\Sigma T_r = 2\sum_{r=1}^{n}(-1)^r.r.\frac{n}{r}.{}^{n-1}C_{r-1} + \sum_{r=0}^{n}(-1)^r\ {}^nC_r = 2\sum_{r=1}^{n}(-1)^r.{}^{n-1}C_{r-1} + \sum_{r=0}^{n}(-1)^r.{}^nC_r$$

$$= 2\left[{}^{n-1}C_0 - {}^{n-1}C_1 + \ldots\ldots\right] + \left[{}^nC_0 - {}^nC_1 + \ldots\ldots\ldots\right] = 0$$

Illustration 15. Prove that $({}^{2n}C_0)^2 - ({}^{2n}C_1)^2 + ({}^{2n}C_2)^2 - \ldots\ldots + (-1)^n({}^{2n}C_{2n})^2 = (-1)^n.\,{}^{2n}C_n$

Solution. $(1-x)^{2n} = {}^{2n}C_0 - {}^{2n}C_1x + {}^{2n}C_2x^2 - \ldots\ldots + (-1)^n\,{}^{2n}C_{2n}x^{2n}$ ... (i)

and $(x+1)^{2n} = {}^{2n}C_0x^{2n} + {}^{2n}C_1x^{2n-1} + {}^{2n}C_2x^{2n-2} + \ldots + {}^{2n}C_{2n}$ ... (ii)

Multiplying (i) and (ii), we get

$(x^2-1)^{2n} = ({}^{2n}C_0 - {}^{2n}C_1x + \ldots\ldots + (-1)^n\,{}^{2n}C_{2n}x^{2n}) \times ({}^{2n}C_0x^{2n} + {}^{2n}C_1x^{2n-1} + \ldots\ldots + {}^{2n}C_{2n})$ ... (iii)

Now, coefficient of x^{2n} in R.H.S.

$= ({}^{2n}C_0)^2 - ({}^{2n}C_1)^2 + ({}^{2n}C_2)^2 - \ldots\ldots + (-1)^n({}^{2n}C_{2n})^2$

$\because$ General term in L.H.S., $T_{r+1} = {}^{2n}C_r(x^2)^{2n-r}(-1)^r$

Putting $2(2n-r) = 2n$

$\therefore$ $r = n$

$\therefore$ $T_{n+1} = {}^{2n}C_nx^{2n}(-1)^n$

Hence coeffiecient of x^{2n} in L.H.S. $= (-1)^n.{}^{2n}C_n$

But (iii) is an identity, therefore coefficient of x^{2n} in R.H.S. = coefficient of x^{2n} in L.H.S.

$\Rightarrow ({}^{2n}C_0)^2 - ({}^{2n}C_1)^2 + ({}^{2n}C_2)^2 - \ldots\ldots + (-1)^n({}^{2n}C_{2n})^2 = (-1)^n.\,{}^{2n}C_n$

***Illustration 16.** Prove that : ${}^nC_0.{}^{2n}C_n - {}^nC_1.{}^{2n-2}C_n + {}^nC_2.{}^{2n-4}C_n + \ldots\ldots = 2^n$

Solution. L.H.S. = Coefficient of x^n in $[{}^nC_0(1+x)^{2n} - {}^nC_1(1+x)^{2n-2}\ \ldots\ldots]$

$=$ Coefficient of x^n in $[(1+x)^2 - 1]^n$

$=$ Coefficient of x^n in $x^n(x+2)^n = 2^n$

***Illustration 17.** If $(1+x)^n = C_0 + C_1x + C_2x^2 + \ldots\ldots + C_nx^n$ then show that the sum of the products of the C_i's taken two at a time represented by : $\sum\limits_{0\le i<j\le n}\sum C_iC_j$ is equal to $2^{2n-1} - \dfrac{2n!}{2.n!n!}$

Solution. Since $(C_0 + C_1 + C_2 + \ldots\ldots + C_{n-1} + C_n)^2$

$$= C_0^2 + C_1^2 + C_2^2 + \ldots\ldots + C_{n-1}^2 + C_n^2 + 2(C_0C_1 + C_0C_2 + C_0C_3 + \ldots + C_0C_n + C_1C_2 + C_1C_3 + \ldots + C_1C_n + C_2C_3 + C_2C_4 + \ldots + C_2C_n + \ldots\ldots + C_{n-1}C_n)$$

$$(2^n)^2 = {}^{2n}C_n + 2\sum_{0\le i<j\le n}\sum C_iC_j$$

Hence $\sum\limits_{0\le i<j\le n}\sum C_iC_j = 2^{2n-1} - \dfrac{2n!}{2.n!n!}$ ***Ans.***

Illustration 18. If $(1+x)^n = C_0 + C_1x + C_2x^2 + \ldots\ldots + C_nx^n$ then prove that

$$\sum_{0\le i<j\le n}\sum (C_i + C_j)^2 = (n-1)\,{}^{2n}C_n + 2^{2n}$$

Solution. L.H.S $\sum\limits_{0\le i<j\le n}\sum (C_i + C_j)^2$

$= (C_0 + C_1)^2 + (C_0 + C_2)^2 + \ldots + (C_0 + C_n)^2 + (C_1 + C_2)^2 + (C_1 + C_3)^2 + \ldots\ldots$
$\ldots + (C_1 + C_n)^2 + (C_2 + C_3)^2 + (C_2 + C_4)^2 + \ldots + (C_2 + C_n)^2 + \ldots + (C_{n-1} + C_n)^2$

$$= n(C_0^2 + C_1^2 + C_2^2 + \ldots + C_n^2) + 2\sum_{0\le i<j\le n}\sum C_iC_j$$

$$= n.{}^{2n}C_n + 2.\left\{2^{2n-1} - \frac{2n!}{2.n!n!}\right\} \quad \{\text{from Illustration 17}\}$$

$$= n.{}^{2n}C_n + 2^{2n} - {}^{2n}C_n = (n-1).{}^{2n}C_n + 2^{2n} = \text{R.H.S.}$$

BEGINNER'S BOX-2

TOPIC COVERED : PROPERTIES OF BINOMIAL COEFFICIENTS

1. $\dbinom{n}{0} + \dbinom{n}{1} + \dbinom{n}{2} + \ldots\ldots + \dbinom{n}{n} =$

(A) 2^{n-1} (B) ${}^{2n}C_n$ (C) 2^n (D) 2^{n+1}

If $(1+x)^n = C_0 + C_1x + C_2x^2 + \ldots\ldots + C_nx^n$, $n \in N$. Then prove the following :

2. $3C_0 - 8C_1 + 13C_2 - 18C_3 + \ldots\ldots$ upto $(n+1)$ terms $= 0$, if $n \ge 2$.

3. $2C_0 + 2^2\dfrac{C_1}{2} + 2^3\dfrac{C_2}{3} + 2^4\dfrac{C_3}{4} + \ldots\ldots + 2^{n+1}\dfrac{C_n}{n+1} = \dfrac{3^{n+1}-1}{n+1}$

4. $C_0^2 + \dfrac{C_1^2}{2} + \dfrac{C_2^2}{3} + \ldots\ldots + \dfrac{C_n^2}{n+1} = \dfrac{(2n+1)!}{((n+1)!)^2}$

5. $\sum\limits_{k=0}^{n}\dfrac{c_k}{k+1} = \dfrac{2^{n+1}-1}{n+1}$

6. $\sum\limits_{k=1}^{n}\left(\dfrac{c_{2k-1}}{2k}\right) = \dfrac{2^n-1}{n+1}$

6.0 MULTINOMIAL THEOREM

AL

Using binomial theorem, we have $(x + a)^n$

$$= \sum_{r=0}^{n} {}^nC_r x^{n-r} a^r,\ n \in N$$

$$= \sum_{r=0}^{n} \frac{n!}{(n-r)!r!} x^{n-r} a^r$$

$$= \sum_{r+s=n} \frac{n!}{r!s!} x^s a^r \text{ , where } s + r = n$$

This result can be generalized in the following form.

$$(x_1 + x_2 + \ldots\ldots + x_k)^n = \sum_{r_1+r_2+\ldots\ldots+r_k=n} \frac{n!}{r_1!r_2!\ldots\ldots r_k!} x_1^{r_1} x_2^{r_2} \ldots\ldots x_k^{r_k}$$

The general term in the above expansion is $\dfrac{n!}{r_1!r_2!r_3!\ldots\ldots r_k!} . x_1^{r_1} x_2^{r_2} x_3^{r_3} \ldots\ldots x_k^{r_k}$

The number of terms in the above expansion is equal to the number of non-negative integral solution of the equation $r_1 + r_2 + \ldots\ldots. + r_k = n$ because each solution of this equation gives a term in the above expansion. The number of such solutions is ${}^{n+k-1}C_{k-1}$

Particular cases

(i) $(x + y + z)^n = \sum_{r+s+t=n} \frac{n!}{r!s!t!} x^r y^s z^t$

The above expansion has ${}^{n+3-1}C_{3-1} = {}^{n+2}C_2$ terms

(ii) $(x + y + z + u)^n$

$$= \sum_{p+q+r+s=n} \frac{n!}{p!q!r!s!} x^p y^q z^r u^s$$

There are ${}^{n+4-1}C_{4-1} = {}^{n+3}C_3$ terms in the above expansion.

6.1 IMPORTANT CONCEPT USING PSEUDO FUNCTION

In order to find the integral and fractional part of irrational numbers of the form $\left(a + b\sqrt{c}\right)^n$.

Algorithm

Step 1 : Write the given expression equal to [N] + F, where [N] is its integral part and F is the fractional part.

Step 2 : Define f by Replacing + sign in the given expression by – . So that f always lies betwen 0 and 1.

Step 3 : Either add f to the expression in step 1 or subtract f from the expression in step 1 so that R.H.S is an integer.

Step 4 : If f is added to the expression in step1, then f + F will always come out to be equal to 1

i.e. f = 1 – F

If f is subtracted from the expression in step 1, then f will always come out to be equal to F.

Step 5 : Obtain the value of the desired expression after getting F in terms of f.

Illustrations

Illustration 19. (A) The greatest integer less than or equal to $\left(\sqrt{2}+1\right)^6$.

(B) Show that the integral part of $\left(5\sqrt{5}+11\right)^{2n+1}$ is even.

Solution. ***(A)*** $\left(\sqrt{2}+1\right)^6 = [N] + F$

$f = \left(\sqrt{2}-1\right)^6$

$[N] + F + f = \left(\sqrt{2}+1\right)^6 + \left(\sqrt{2}-1\right)^6 = 2\left\{{}^6C_0\left(\sqrt{2}\right)^6 + {}^6C_2\left(\sqrt{2}\right)^4 + ...\right\}$

[N] + F + f = an even integer

F + f = even – Integer = integer

Now $0 < F < 1$ and $0 < f < 1$

$\Rightarrow 0 < F + f < 2$

So f + g must be equal to 1, because this is the only integer $\in (0, 2)$.

$\therefore [N] + F + f = 2\left\{{}^6C_0\left(\sqrt{2}\right)^6 + {}^6C_2\left(\sqrt{2}\right)^4 + ...\right\}$

$[N] + 1 = 2\{8 + 15 \times 4 + 15 \times 2 + 1\}$

$[N] = 197$

(B) $[N] + F = \left(5\sqrt{5}+11\right)^{2n+1}$

$f = \left(5\sqrt{5}-11\right)^{2n+1}$

Now to make calculation free from $\sqrt{\ }$ sign, subtract

$[N] + F - f = \left(5\sqrt{5}+11\right)^{2n+1} - \left(5\sqrt{5}-11\right)^{2n+1}$

[N] + F – f = even

F – f = even – [N]

= Integer

Now

$0 < F < 1$ and also $0 < f < 1$

$\therefore -1 < F - f < 1$

So F – f = 0 is the only integer.

$\therefore F - f = 0 \Rightarrow F = f$

So [N] + F – f = even

[N] + 0 = even

[N] = even

Illustration 20. (A) Find the coefficient of $x^2 y^3 z^4 w$ in the expansion of $(x - y - z + w)^{10}$

(B) Find the total number of terms in the expansion of $(1 + x + y)^{10}$ and coefficient of x^2y^3.

Solution. ***(A)*** $(x - y - z + w)^{10} = \sum_{p+q+r+s=10} \frac{n!}{p!q!r!s!}(x)^p(-y)^q(-z)^r(w)^s$

We want to get $x^2y^3z^4w$ this implies that p = 2, q = 3, r = 4, s = 1

$\therefore$ Coefficient of $x^2y^3z^4w$ is $\frac{10!}{2!\,3!\,4!\,1!}(-1)^3(-1)^4 = -12600$ ***Ans.***

(B) Total number of terms $= {}^{10+3-1}C_{3-1} = {}^{12}C_2 = 66$

Coefficient of $x^2y^3 = \frac{10!}{2! \times 3! \times 5!} = 2520$ ***Ans.***

Illustration 21. Find the coefficient of x^5 in the expansion of $(2 - x + 3x^2)^6$.

Solution. The general term in the expansion of $(2 - x + 3x^2)^6 = \frac{6!}{r!s!t!}2^r(-x)^s(3x^2)^t$, where $r + s + t = 6$.

$= \frac{6!}{r!s!t!}2^r \times (-1)^s \times (3)^t \times x^{s+2t}$

For the coefficient of x^5, we must have $s + 2t = 5$.

But, $r + s + t = 6$,

$\therefore$ $s = 5 - 2t$ and $r = 1 + t$, where $0 \le r, s, t \le 6$.

Now $t = 0 \Rightarrow r = 1,\ s = 5$.

$t = 1 \Rightarrow r = 2,\ s = 3$.

$t = 2 \Rightarrow r = 3,\ s = 1$.

Thus, there are three terms containing x^5 and coefficient of x^5

$= \frac{6!}{1!\,5!\,0!} \times 2^1 \times (-1)^5 \times 3^0 + \frac{6!}{2!\,3!\,1!} \times 2^2 \times (-1)^3 \times 3^1 + \frac{6!}{3!\,1!\,2!} \times 2^3 \times (-1)^1 \times 3^2$

$= -12 - 720 - 4320 = -5052$. **Ans.**

Illustration 22. If $(6\sqrt{6}+14)^{2n+1} = [N] + F$ and $F = N - [N]$; where [.] denotes greatest integer function, then NF is equal to

(A) 20^{2n+1} (B) an even integer (C) odd integer (D) 40^{2n+1}

Solution. Since $(6\sqrt{6}+14)^{2n+1} = [N] + F$

Let us assume that $f = (6\sqrt{6}-14)^{2n+1}$; where $0 < f < 1$.

Now, $[N] + F - f = (6\sqrt{6}+14)^{2n+1} - (6\sqrt{6}-14)^{2n+1}$

$= 2\left[{}^{2n+1}C_1(6\sqrt{6})^{2n}(14) + {}^{2n+1}C_3(6\sqrt{6})^{2n-2}(14)^3 + \ldots\right]$

$\Rightarrow$ $[N] + F - f =$ even integer.

Now $0 < F < 1$ and $0 < f < 1$

so $-1 < F - f < 1$ and $F - f$ is an integer so it can only be zero

Thus $NF = (6\sqrt{6}+14)^{2n+1}(6\sqrt{6}-14)^{2n+1} = 20^{2n+1}$. **Ans. (A,B)**

Illustration 23. Find the last three digits in 11^{50}.

Solution. Expansion of $(10 + 1)^{50} = {}^{50}C_0 10^{50} + {}^{50}C_1 10^{49} + \ldots.. + {}^{50}C_{48}10^2 + {}^{50}C_{49}10 + {}^{50}C_{50}$

$= \underbrace{{}^{50}C_0 10^{50} + {}^{50}C_1 10^{49} + \ldots\ldots + {}^{50}C_{47}10^3}_{1000K} + 49 \times 25 \times 100 + 500 + 1$

$\Rightarrow$ $1000\,K + 123001$

$\Rightarrow$ Last 3 digits are 001.

***Illustration 24.** Prove that $2222^{5555} + 5555^{2222}$ is divisible by 7.

Solution. When 2222 is divided by 7 it leaves a remainder 3. So adding & subtracting 3^{5555}, we get :

$E = \underbrace{2222^{5555} - 3^{5555}}_{E_1} + \underbrace{3^{5555} + 5555^{2222}}_{E_2}$

For E_1 : Now since $2222-3 = 2219$ is divisible by 7, therefore E_1 is divisible by 7

($\because x^n - a^n$ is divisible by $x - a$)

For E_2 : 5555 when devided by 7 leaves remainder 4. So adding and subtracting 4^{2222}, we get :

$E_2 = 3^{5555} + 4^{2222} + 5555^{2222} - 4^{2222}$

$= (243)^{1111} + (16)^{1111} + (5555)^{2222} - 4^{2222}$

Again $(243)^{1111} + 16^{1111}$ and $(5555)^{2222} - 4^{2222}$ are divisible by 7

($\because x^n + a^n$ is divisible by $x + a$ when n is odd)

Hence $2222^{5555} + 5555^{2222}$ is divisible by 7.

BEGINNER'S BOX-3

TOPIC COVERED : MULTINOMIAL THEOREM, CONCEPT OF PSEUDO FUNCTION

1. Find the coefficient of x^2y^5 in the expansion of $(3 + 2x - y)^{10}$.

2. If $(1+x+x^2)^n = \sum_{r=0}^{2n} a_r x^r$, then prove that

(a) $a_r = a_{2n-r}$ (b) $\sum_{r=0}^{n-1} a_r = \frac{1}{2}(3^n - a_n)$

3. Prove that $5^{25} - 3^{25}$ is divisible by 2.

4. Find the remainder when the number 9^{100} is divided by 8.

5. Find last three digits in 19^{100}.

6. Let $R = (8+3\sqrt{7})^{20}$ and [.] denotes greatest integer function, then prove that :

(a) [R] is odd (b) $R - [R] = 1 - \frac{1}{(8+3\sqrt{7})^{20}}$

7. Find the digit at unit's place in the number $17^{1995} + 11^{1995} - 7^{1995}$

8. Find coefficient of x^4 in $(1 + x + x^3 + x^4)^{10}$.

7.0 BINOMIAL THEOREM FOR NEGATIVE OR FRACTIONAL INDICES

AL

If $n \in Q$, then $(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!} x^2 + \frac{n(n-1)(n-2)}{3!} x^3 + \ldots\ldots \infty$ provided $|x| < 1$.

Note –

(i) When the index n is a positive integer the number of terms in the expansion of $(1+x)^n$ is finite i.e. $(n+1)$ & the coefficient of successive terms are : ${}^nC_0, {}^nC_1, {}^nC_2, \ldots\ldots {}^nC_n$

(ii) When the index is other than a positive integer such as negative integer or fraction, the number of terms in the expansion of $(1+x)^n$ is infinite and the symbol ${}^nC_r$ cannot be used to denote the coefficient of the general term.

(iii) Following expansion should be remembered ($|x| < 1$).

(a) $(1 + x)^{-1} = 1 - x + x^2 - x^3 + x^4 - \ldots \infty$

(b) $(1 - x)^{-1} = 1 + x + x^2 + x^3 + x^4 + \ldots \infty$

(c) $(1 + x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \ldots \infty$

(d) $(1 - x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \ldots \infty$

(e) $(1 + x)^{-3} = 1 - 3x + 6x^2 - 10x^3 + \ldots + \frac{(-1)^r(r+1)(r+2)}{2!}x^r + \ldots\ldots$

(f) $(1 - x)^{-3} = 1 + 3x + 6x^2 + 10x^3 + \ldots + \frac{(r+1)(r+2)}{2!}x^r + \ldots\ldots$

(iv) The expansions in ascending powers of x are only valid if x is 'small'. If x is large i.e. $|x| > 1$ then we may find it convenient to expand in powers of $1/x$, which then will be small.

8.0 APPROXIMATIONS

AL

$(1 + x)^n = 1 + nx + \frac{n(n-1)}{1.2} x^2 + \frac{n(n-1)(n-2)}{1.2.3} x^3 \ldots\ldots$

If $x < 1$, the terms of the above expansion go on decreasing and if x be very small, a stage may be reached when we may neglect the terms containing higher powers of x in the expansion. Thus, if x be so small that its square and higher powers may be neglected then $(1 + x)^n = 1 + nx$, approximately.

This is an approximate value of $(1 + x)^n$

Illustrations

Illustration 25. If x is so small such that its square and higher powers may be neglected then find the approximate value of $\dfrac{(1-3x)^{1/2}+(1-x)^{5/3}}{(4+x)^{1/2}}$

Solution. $\dfrac{(1-3x)^{1/2}+(1-x)^{5/3}}{(4+x)^{1/2}}$

$$= \frac{1-\frac{3}{2}x+1-\frac{5x}{3}}{2\left(1+\frac{x}{4}\right)^{1/2}}$$

$$= \frac{1}{2}\left(2-\frac{19}{6}x\right)\left(1+\frac{x}{4}\right)^{-1/2}$$

$$= \frac{1}{2}\left(2-\frac{19}{6}x\right)\left(1-\frac{x}{8}\right)$$

$$= \frac{1}{2}\left(2-\frac{x}{4}-\frac{19}{6}x\right)$$

$$= 1-\frac{x}{8}-\frac{19}{12}x = 1-\frac{41}{24}x$$ ***Ans.***

****Illustration 26.*** The sum of $1+\frac{1}{4}+\frac{1.3}{4.8}+\frac{1.3.5}{4.8.12}+\ldots\infty$ is -

(A) $\sqrt{2}$ (B) $\frac{1}{\sqrt{2}}$ (C) $\sqrt{3}$ (D) $2^{3/2}$

Solution. Comparing with $1+nx+\frac{n(n-1)}{2!}x^2+\ldots$

$nx = 1/4$ ... (i)

and $\frac{n(n-1)x^2}{2!}=\frac{1.3}{4.8}$

or $\frac{nx(nx-x)}{2!}=\frac{3}{32}$

$\Rightarrow \frac{1}{4}\left(\frac{1}{4}-x\right)=\frac{3}{16}$ (by (i))

$\Rightarrow \left(\frac{1}{4}-x\right)=\frac{3}{4}\Rightarrow x=\frac{1}{4}-\frac{3}{4}=-\frac{1}{2}$ ... (ii)

putting the value of x in (i)

$n(-1/2) = 1/4 \Rightarrow n = -1/2$

$\therefore$ sum of series $= (1+x)^n = (1-1/2)^{-1/2} = (1/2)^{-1/2} = \sqrt{2}$ ***Ans. (A)***

9.0 EXPONENTIAL SERIES

AL

(a) e is an irrational number lying between 2.7 & 2.8. Its value correct upto 10 places of decimal is 2.7182818284.

(b) Logarithms to the base 'e' are known as the Napierian system, so named after Napier, their inventor. They are also called ***Natural Logarithm.***

(c) $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + ...\infty$; where x may be any real or complex number

(d) $a^x = 1 + \frac{x}{1!}\ell na + \frac{x^2}{2!}\ell n^2 a + \frac{x^3}{3!}\ell n^3 a + ...\infty$, where a > 0

(e) $e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + ...\infty$

10.0 LOGARITHMIC SERIES

AL

(a) $\ell n\,(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ...\infty$, where $-1 < x \le 1$

(b) $\ell n\,(1 - x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} + ...\infty$, where $-1 \le x < 1$

Remember

(i) $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + ...\infty = \ell n 2$

(ii) $e^{\ln x} = x$; for all x > 0

(iii) $\ell n2 = 0.693$

(iv) $\ell n10 = 2.303$

GOLDEN KEY POINTS

- ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$
- $\frac{{}^nC_r}{{}^nC_{r-1}} = \frac{n-r+1}{r}$
- ${}^nC_r = \frac{n}{r}\,{}^{n-1}C_{r-1} = \frac{n}{r}.\frac{n-1}{r-1}\,{}^{n-2}C_{r-2} = ... = \frac{n(n-1)(n-2).......(n-r+1)}{r(r-1)(r-2)..........1}$
- ${}^nC_r = \frac{r+1}{n+1}.\,{}^{n+1}C_{r+1}$

ANSWERS

BEGINNER'S BOX-1

1. $^5C_0\,(3x^2)^5 + {}^5C_1(3x^2)^4\left(-\frac{x}{2}\right) + {}^5C_2(3x^2)^3\left(-\frac{x}{2}\right)^2 + {}^5C_3(3x^2)^2\left(-\frac{x}{2}\right)^3 + {}^5C_4(3x^2)^1\left(-\frac{x}{2}\right)^4 + {}^5C_5\left(-\frac{x}{2}\right)^5$

2. $^nC_0y^n + {}^nC_1y^{n-1}.x + {}^nC_2.y^{n-2}.x^2 + \ldots\ldots\ldots + {}^nC_n.x^n$

3. $\frac{70}{3}x^8$;

4. $\frac{25!}{10!\ 15!}2^{15}3^{10}$;

5. (a) -20; (b) $-560x^5$, $280x^2$

6. 4^{th} & 5^{th} i.e. 489888

7. n = 4, 5, 6

8. $210\ y^2$

BEGINNER'S BOX-2

1. C

BEGINNER'S BOX-3

1. -272160 or $-{}^{10}C_5 \times {}^5C_2 \times 108$

4 1

5. 001

7. 1

8. 310

EXERCISE – 1 MCQ (SINGLE CHOICE CORRECT)

1. The two successive terms in the expansion of $(1+x)^{24}$ whose coefficients are in the ratio 4 :1 are
(A) 3^{rd} and 4^{th} (B) 4^{th} and 5^{th} (C) 5^{th} and 6^{th} (D) 6^{th} and 7^{th}

2. The sum of the binomial coefficients of $\left[2x+\frac{1}{x}\right]^n$ is equal to 256. The constant term in the expansion is -
(A) 1120 (B) 2110 (C) 1210 (D) none

3. The sum of the co–efficients in the expansion of $(1-2x+5x^2)^n$ is 'a' and the sum of the co–efficients in the expansion of $(1+x)^{2n}$ is b. Then -
(A) $a = b$ (B) $a = b^2$ (C) $a^2 = b$ (D) $ab = 1$

4. $\binom{35}{6}+\sum_{r=0}^{10}\binom{45-r}{5}=\binom{x}{y}$, then $x - y$ is equal to -
(A) 39 (B) 29 (C) 52 (D) 40

5. The expression $\frac{1}{\sqrt{4x+1}}\left[\left[\frac{1+\sqrt{4x+1}}{2}\right]^7-\left[\frac{1-\sqrt{4x+1}}{2}\right]^7\right]$ is a polynomial in x of degree -
(A) 7 (B) 5 (C) 4 (D) 3

6. In the binomial $(2^{1/3}+3^{-1/3})^n$, if the ratio of the seventh term from the beginning of the expansion to the seventh term from its end is 1/6, then n is equal to -
(A) 6 (B) 9 (C) 12 (D) 15

7. The term independent of x in the product $(4+x+7x^2)\left(x-\frac{3}{x}\right)^{11}$ is -
(A) $7.^{11}C_6$ (B) $3^6.\ ^{11}C_6$ (C) $3^5.\ ^{11}C_5$ (D) $-12.\ 2^{11}$

8. If ‘a’ be the sum of the odd terms & ‘b’ be the sum of the even terms in the expansion of $(1+x)^n$, then $(1-x^2)^n$ is equal to -
(A) a^2-b^2 (B) a^2+b^2 (C) b^2-a^2 (D) none

9. The sum of the co–efficients of all the even powers of x in the expansion of $(2x^2-3x+1)^{11}$ is -
(A) 2.6^{10} (B) 3.6^{10} (C) 6^{11} (D) none

10. The greatest terms of the expansion $(2x+5y)^{13}$ when $x = 10$, $y = 2$ is -
(A) $^{13}C_5 . 20^8 . 10^5$ (B) $^{13}C_6 . 20^7 . 10^4$ (C) $^{13}C_4 . 20^9 . 10^4$ (D) none of these

11. The number of distinct terms in the expansion of $(2x+3y-z+\omega-7\mu)^n$ is
(A) $n+1$ (B) $^{(n+4)}C_4$ (C) $^{(n+5)}C_5$ (D) $^nC_5$

12. If $\binom{p}{q}=0$ for $p < q$, where $p, q \in W$, then $\sum_{r=0}^{\infty}\binom{n}{2r}=$
(A) 2^n (B) 2^{n-1} (C) 2^{2n-1} (D) $^{2n}C_n$

13. $\binom{47}{4}+\sum_{j=1}^{5}\binom{52-j}{3}=\binom{x}{y}$, then $\frac{x}{y}=$

(A) 11 (B) 12 (C) 13 (D) 14

14. Let $R=(5\sqrt{5}+11)^{31}=I+f$, where I is an integer and f is the fractional part of R, then $R\cdot f$ is equal to -

(A) 2^{31} (B) 3^{31} (C) 2^{62} (D) 1

15. The value of $\sum_{r=0}^{10}\binom{10}{r}\binom{15}{14-r}$ is equal to -

(A) ${}^{25}C_{12}$ (B) ${}^{25}C_{15}$ (C) ${}^{25}C_{10}$ (D) ${}^{25}C_{11}$

16. Value of $\sum_{r=0}^{n} r.\left({}^{n}C_r\right)^2$ is equal to

(A) $n\,.\,{}^{2n}C_n$ (B) $\frac{n\,.\,{}^{2n}C_n}{2}$ (C) $n^2\,.\,{}^{2n}C_n$ (D) $\frac{n^2\,.\,{}^{2n}C_n}{2}$

17. If $a_n=\sum_{r=0}^{n}\frac{1}{{}^{n}C_r}$, then $\sum_{r=0}^{n}\frac{r}{{}^{n}C_r}$ equals -

(A) $(n-1)\,a_n$ (B) $n\,a_n$ (C) $n\,a_n/2$ (D) none of these

18. The last two digits of the number 3^{400} are -

(A) 81 (B) 43 (C) 29 (D) 01

19. Value of $\sum_{r=1}^{n+1}\left(\sum_{k=1}^{n}{}^{k}C_{r-1}\right)$ where $n, r, k \in N$ equals

(A) $2^{n+1}-1$ (B) $2^{n+1}-2$ (C) $2^{n+1}-3$ (D) None of these

20. If $(1+x+x^2)^{25}=a_0+a_1x+a_2x^2+.....+a_{50}\,.\,x^{50}$ then $a_0+a_2+a_4+.....+a_{50}$ is -

(A) even
(B) odd & of the form 3n
(C) odd & of the form $(3n-1)$
(D) odd & of the form $(3n+1)$

21. The co-efficient of x^4 in the expansion of $(1-x+2x^2)^{12}$ is -

(A) ${}^{12}C_3$ (B) ${}^{13}C_3$ (C) ${}^{14}C_4$ (D) ${}^{12}C_3+3\,{}^{13}C_3+{}^{14}C_4$

EXERCISE – 2 MCQ (ONE OR MORE CHOICE CORRECT)

Select the correct alternatives (one or more than one correct answers)

1. Let $(1 + x^2)^2 (1 + x)^n = A_0 + A_1x + A_2x^2 + \ldots\ldots$ If A_0, A_1, A_2 are in A.P. then the value of n is -
(A) 2 (B) 3 (C) 5 (D) 7

***2.** Co-efficient of α^t in the expansion of $(\alpha + p)^{m-1} + (\alpha + p)^{m-2} (\alpha + q) + (\alpha + p)^{m-3} (\alpha + q)^2 + \ldots .(\alpha+q)^{m-1}$ where $\alpha \neq -q$ and $p \neq q$ is -
(A) $\frac{{}^mC_t\left(p^t-q^t\right)}{p-q}$ (B) $\frac{{}^mC_t\left(p^{m-t}-q^{m-t}\right)}{p-q}$ (C) $\frac{{}^mC_t\left(p^t+q^t\right)}{p-q}$ (D) $\frac{{}^mC_t\left(p^{m-t}+q^{m-t}\right)}{p-q}$

3. The co-efficient of x^{401} in the expansion of $(1 + x + x^2 + \ldots\ldots + x^9)^{-1}$, $(|x| < 1)$ is -
(A) 1 (B) –1 (C) 2 (D) –2

4. The value r for which $\binom{30}{r}\binom{15}{0}+\binom{30}{r-1}\binom{15}{1}+\ldots\ldots+\binom{30}{0}\binom{15}{r}$ (where $\binom{p}{q}=0$ if $p < q$) is maximum is/are-
(A) 21 (B) 22 (C) 23 (D) 24

5. If the 6^{th} term in the expansion of $\left(\frac{3}{2}+\frac{x}{3}\right)^n$ when $x = 3$ is numerically greatest then the possible integral value(s) of n can be -
(A) 11 (B) 12 (C) 13 (D) 14

***6.** In the expansion of $\left(x^3+3.2^{-\log_{\sqrt{2}}\sqrt{x^3}}\right)^{11}$ -
(A) there appears a term with the power x^2
(B) there does not appear a term with the power x^2
(C) there appears a term with the power x^{-3}
(D) the ratio of the co-efficient of x^3 to that of x^{-3} is $\frac{1}{3}$

7. The binomial expansion of $\left(x^k+\frac{1}{x^{2k}}\right)^{3n}$, $n \in N$ contains a term independent of x -
(A) only if k is an integer
(B) only if k is a natural number
(C) only if k is rational
(D) for any real k

8. Let $n \in N$. If $(1 + x)^n = a_0 + a_1x + a_2x^2 + \ldots\ldots + a_nx^n$ and $a_{n-3}, a_{n-2}, a_{n-1}$ are in AP, then -
(A) a_1, a_2, a_3 are in AP (B) a_1, a_2, a_3 are in HP (C) $n = 7$ (D) $n = 14$

9. Set of values of r for which, ${}^{18}C_{r-2} + 2 . {}^{18}C_{r-1} + {}^{18}C_r \geq {}^{20}C_{13}$ contains -
(A) 4 elements (B) 5 elements (C) 7 elements (D) 10 elements

10. If the coefficients of three consecutive terms in the expansion of $(1 + x)^n$ are in the ratio of 1 : 7 : 42, then n is divisible by -
(A) 9 (B) 5 (C) 3 (D) 11

11. In the expansion of $\left(\sqrt[3]{4}+\frac{1}{\sqrt[4]{6}}\right)^{20}$ -

(A) the number of irrational terms = 19
(B) middle term is irrational
(C) the number of rational terms = 2
(D) 9^{th} term is rational

12. If $(1 + x + x^2 + x^3)^{100} = a_0 + a_1x + a_2x^2 + ... + a_{300}x^{300}$, then -
(A) $a_0 + a_1 + a_2 + a_3 + ... + a_{300}$ is divisible by 1024
(B) $a_0 + a_2 + a_4 + ... + a_{300} = a_1 + a_3 + ... + a_{299}$
(C) coefficients equidistant from beginning and end are equal
(D) $a_1 = 100$

13. The number $101^{100} - 1$ is divisible by -
(A) 100
(B) 1000
(C) 10000
(D) 100000

14. If $\left(9+\sqrt{80}\right)^n = I + f$ where I, n are integers and $0 < f < 1$, then -
(A) I is an odd integer
(B) I is an even integer
(C) $(I + f)(1 - f) = 1$
(D) $1 - f = \left(9-\sqrt{80}\right)^n$

15. In the expansion of $\left(x^{2/3}-\frac{1}{\sqrt{x}}\right)^{30}$, a term containing the power x^{13} -
(A) does not exist
(B) exists and the co-efficient is divisible by 29
(C) exists and the co-efficient is divisible by 63
(D) exists and the co-efficient is divisible by 65

16. The co-efficient of the middle term in the expansion of $(1 + x)^{2n}$ is -
(A) $\frac{1.3.5.7......(2n-1)}{n!}2^n$
(B) ${}^{2n}C_n$
(C) $\frac{(n+1)(n+2)(n+3)....(2n-1)(2n)}{1.2.3..........(n-1)n}$
(D) $\frac{2.6.10.14......(4n-6)(4n-2)}{1.2.3.4.....(n-1).n}$

17. Let $a_n = \left(1+\frac{1}{n}\right)^n$, then for each $n \in N$
(A) $a_n \geq 2$
(B) $a_n < 3$
(C) $a_n < 4$
(D) $a_n < 2$

Match the Column

Following question contains statements given in two columns, which have to be matched. The statements in ***Column-I*** are labelled as A, B, C and D while the statements in ***Column-II*** are labelled as p, q, r and s. Any given statement in ***Column-I*** can have correct matching with ***ONE*** statement in ***Column-II***.

18.

	Column-I		Column-II
(A)	$(2n + 1)(2n + 3)(2n + 5) .... (4n - 1)$ is equal to	(p)	$\frac{(n+1)^n}{n!}$
(B)	$\frac{C_1}{C_0}+\frac{2.C_2}{C_1}+\frac{3.C_3}{C_2}+......+\frac{n.C_n}{C_{n-1}}$ is equal to here C_r stand for ${}^nC_r$.	(q)	$n.2^n.(2^n-1)$
(C)	If $(C_0 + C_1)(C_1 + C_2)(C_2 + C_3) ...... (C_{n-1} + C_n)$	(r)	$\frac{(4n)!\ n!}{2^n.(2n)!(2n)!}$

$= m \,.\, C_1C_2C_3 \,....\, C_{n-1}$, then m is equal to

(D) If C_r are the binomial co–efficients in the expansion of $(1 + x)^n$, the value of $\sum_{i=1}^{n}\sum_{j=1}^{n}(i+j)C_iC_j$ is

(s) $\dfrac{n\,(n+1)}{2}$

Comprehension Based Questions

The 2^{nd}, 3^{rd} and 4^{th} terms in the expansion of $(x + a)^n$ are 240, 720 and 1080, respectively.

19. The value of $(x - a)^n$ can be
(A) 64 (B) –1 (C) –32 (D) None of these

20. The sum of odd-numbered terms is
(A) 1664 (B) 2376 (C) 1562 (D) 1486

EXERCISE - 3 SUBJECTIVE

1. If the coefficients of $(2r + 4)^{th}$, $(r - 2)^{th}$ terms in the expansion of $(1 + x)^{18}$ are equal, find r.

2. If the coefficients of the r^{th}, $(r+1)^{th}$ & $(r+2)^{th}$ terms in the expansion of $(1 + x)^{14}$ are in AP, find r.

3. Find the term independent of x in the expansion of :

(a) $\left[\sqrt{\frac{x}{3}}+\frac{\sqrt{3}}{2x^2}\right]^{10}$ (b) $\left[\frac{1}{2}x^{1/3}+x^{-1/5}\right]^{8}$

4. Prove that : ${}^{n-1}C_r + {}^{n-2}C_r + {}^{n-3}C_r + ... + {}^{r}C_r = {}^{n}C_{r+1}$.

5. If ${}^{40}C_1 . x(1-x)^{39} + 2 . {}^{40}C_2\, x^2 (1-x)^{38} + 3\, {}^{40}C_3\, x^3 (1-x)^{37} + ... + 40. {}^{40}C_{40}\, x^{40} = ax + b$, then find a & b.

6. If ${}^{n+1}C_2 + 2\,({}^2C_2 + {}^3C_2 + {}^4C_2 + ... + {}^nC_2) = 1^2 + 2^2 + 3^2 + ... + 100^2$, then find n.

***7.** Which is larger : $(99^{50} + 100^{50})$ or $(101)^{50}$.

8. Show that ${}^{2n-2}C_{n-2} + 2.\, {}^{2n-2}C_{n-1} + {}^{2n-2}C_n > \frac{4n}{n+1}$, $n \in N$, $n > 2$

9. Find the coefficient of x^4 in the expansion of :

(a) $(1 + x + x^2 + x^3)^{11}$ (b) $(2 - x + 3x^2)^6$

10. Find numerically the greatest term in the expansion of :

(a) $(2 + 3x)^9$ when $x = \frac{3}{2}$ (b) $(3 - 5x)^{15}$ when $x = \frac{1}{5}$

11. Prove that the ratio of the coefficient of x^{10} in $(1-x^2)^{10}$ & the term independent of x in $\left(x-\frac{2}{x}\right)^{10}$ is 1 : 32.

12. Find the term independent of x in the expansion of $(1 + x + 2x^3)\left(\frac{3x^2}{2}-\frac{1}{3x}\right)^9$.

13. Find the coefficient of :

(a) x^6 in the expansion of $(ax^2 + bx + c)^9$.

(b) $x^2 y^3 z^4$ in the expansion of $(ax - by + cz)^9$.

(c) $a^2 b^3 c^4 d$ in the expansion of $(a - b - c + d)^{10}$.

14. (a) $\sum_{r=0}^{20}\binom{20}{r}\binom{30}{25-r} = {}^xC_y$, then find x, y.

(b) Prove that : $\sum_{r=0}^{25}\binom{30}{r}\binom{70}{25-r} = {}^{100}C_{25}$

EXERCISE – 4 RECAP OF AIEEE/JEE (MAIN)

1. The sum of the coefficients in the expansion of $(x + y)^n$ is 4096. The greatest coefficient in the expansion is- ***[AIEEE 2002]***

(A) 1024 (B) 924 (C) 824 (D) 724

2. If for positive integers $r > 1$, $n > 2$ the coefficients of the $(3r)^{th}$ and $(r+2)^{th}$ powers of x in the expansion of $(1+x)^{2n}$ are equal, then- ***[AIEEE 2002]***

(A) $n = 2r$ (B) $n = 3r$ (C) $n = 2r + 1$ (D) $n = 2r - 1$

3. If $(1+x)^n = C_0 + C_1x + C_2x^2 + \ldots + \ldots C_nx^n$, then $\frac{C_1}{C_0} + \frac{2C_2}{C_1} + \frac{3C_3}{C_2} + \ldots + \frac{nC_n}{C_{n-1}} =$ ***[AIEEE-2002]***

(A) $\frac{n(n-1)}{2}$ (B) $\frac{n(n+2)}{2}$ (C) $\frac{n(n+1)}{2}$ (D) $\frac{(n-1)(n-2)}{2}$

4. The number of integral terms in the expansion of $\left(\sqrt{3}+\sqrt[8]{5}\right)^{256}$ is- ***[AIEEE 2003]***

(A) 32 (B) 33 (C) 34 (D) 35

5. The coefficient of the middle term in the binomial expansion in powers of x of $(1 + \alpha x)^4$ and of $(1 - \alpha x)^6$ is the same if α equals- ***[AIEEE 2004]***

(A) $-\frac{5}{3}$ (B) $\frac{10}{3}$ (C) $\frac{-3}{10}$ (D) $\frac{3}{5}$

6. The coefficient of x^n in expansion of $(1 + x)(1 - x)^n$ is- ***[AIEEE 2004]***

(A) $(n - 1)$ (B) $(-1)^n (1-n)$ (C) $(-1)^{n-1} (n-1)^2$ (D) $(-1)^{n-1}n$

7. If the coefficients of r^{th}, $(r + 1)^{th}$ and $(r + 2)^{th}$ terms in the binomial expansion $(1 + y)^m$ are in A.P., then m and r satisfy the equation- ***[AIEEE 2005]***

(A) $m^2 - m(4r - 1) + 4r^2 + 2 = 0$ (B) $m^2 - m(4r + 1) + 4r^2 - 2 = 0$
(C) $m^2 - m(4r + 1) + 4r^2 + 2 = 0$ (D) $m^2 - m(4r - 1) + 4r^2 - 2 = 0$

8. If the coefficient of x^7 in $\left[ax^2 + \left(\frac{1}{bx}\right)\right]^{11}$ equals the coefficient of x^{-7} in $\left[ax - \left(\frac{1}{bx^2}\right)\right]^{11}$, then a and b satisfy the relation- ***[AIEEE 2005]***

(A) $ab = 1$ (B) $\frac{a}{b} = 1$ (C) $a + b = 1$ (D) $a - b = 1$

9. For natural numbers m, n if $(1 - y)^m (1 + y)^n = 1 + a_1y + a_2y^2 + \ldots\ldots$, and $a_1 = a_2 = 10$, then (m, n) is- ***[AIEEE 2006]***

(A) (45, 35) (B) (35, 45) (C) (20, 45) (D) (35, 20)

10. The sum of the series ${}^{20}C_0 - {}^{20}C_1 + {}^{20}C_2 - {}^{20}C_3 + ...- ...... + {}^{20}C_{10}$ is - **[AIEEE 2007]**

(A) $\frac{1}{2}\,{}^{20}C_{10}$ (B) 0 (C) $-{}^{20}C_{10}$ (D) ${}^{20}C_{10}$

11. In the binomial expansion of $(a - b)^n$, $n \geq 5$, the sum of 5^{th} and 6^{th} terms is zero, then $\frac{a}{b}$ equals

(A) $\frac{6}{n-5}$ (B) $\frac{n-5}{6}$ (C) $\frac{n-4}{5}$ (D) $\frac{5}{n-4}$ **[AIEEE 2007]**

12. ***Statement -1*** – $\sum_{r=0}^{n}(r+1)\,{}^nC_r = (n+2)2^{n-1}$

Statement-2 – $\sum_{r=0}^{n}(r+1)\,{}^nC_r x^r = (1 + x)^n + nx\,(1+x)^{n-1}$ **[AIEEE 2008]**

(A) Statement –1 is false, Statement –2 is true

(B) Statement–1 is true, Statement–2 is true; Statement–2 is a correct explanation for Statement–1

(C) Statement–1 is true, Statement–2 is true; Statement–2 is not a correct explanation for Statement–1

(D) Statement–1 is true, Statement–2 is false

13. The remainder left out when $8^{2n} - (62)^{2n+1}$ is divided by 9 is :- **[AIEEE 2009]**

(A) 7 (B) 8 (C) 0 (D) 2

14. Let $S_1 = \sum_{j=1}^{10} j(j-1)\,{}^{10}C_j$, $S_2 = \sum_{j=1}^{10} j\,{}^{10}C_j$ and $S_3 = \sum_{j=1}^{10} j^2\,{}^{10}C_j$. **[AIEEE-2010]**

Statement-1 – $S_3 = 55 \times 2^9$.

Statement-2 – $S_1 = 90 \times 2^8$ and $S_2 = 10 \times 2^8$.

(A) Statement–1 is true, Statement–2 is true; Statement–2 is a correct explanation for Statement–1.

(B) Statement–1 is true, Statement–2 is true; Statement–2 is not a correct explanation for statement–1.

(C) Statement–1 is true, Statement–2 is false.

(D) Statement–1 is false, Statement–2 is true.

15. The coefficient of x^7 in the expansion of $(1 - x - x^2 + x^3)^6$ is :- **[AIEEE 2011]**

(A) –144 (B) 132 (C) 144 (D) – 132

16. If n is a positive integer, then $\left(\sqrt{3}+1\right)^{2n} - \left(\sqrt{3}-1\right)^{2n}$ is : **[AIEEE 2012]**

(A) a rational number other than positive integers (B) an irrational number

(C) an odd positive integer (D) an even positive integer

17. The term independent of x in expansion of $\left(\frac{x+1}{x^{2/3}-x^{1/3}+1} - \frac{x-1}{x-x^{1/2}}\right)^{10}$ is : **[JEE (Main)-2013]**

(A) 4 (B) 120 (C) 210 (D) 310

18. If the coefficients of x^3 and x^4 in the expansion of $(1 + ax + bx^2)(1 - 2x)^{18}$ in powers of x are both zero, then (a, b) is equal to **[JEE (Main)-2014]**

(A) $\left(14, \frac{272}{3}\right)$ (B) $\left(16, \frac{272}{3}\right)$ (C) $\left(16, \frac{251}{3}\right)$ (D) $\left(14, \frac{251}{3}\right)$

19. The sum of coefficients of integral power of x in the binomial expansion of $\left(1-2\sqrt{x}\right)^{50}$ is : ***[JEE (Main)-2015]***

(A) $\frac{1}{2}\left(3^{50}+1\right)$ (B) $\frac{1}{2}\left(3^{50}\right)$ (C) $\frac{1}{2}\left(3^{50}-1\right)$ (D) $\frac{1}{2}\left(2^{50}+1\right)$

20. The value of $\left({}^{21}C_1-{}^{10}C_1\right)+\left({}^{21}C_2-{}^{10}C_2\right)+\left({}^{21}C_3-{}^{10}C_3\right)+\left({}^{21}C_4-{}^{10}C_4\right)+....+\left({}^{21}C_{10}-{}^{10}C_{10}\right)$ is :

[JEE (Main)-2017]

(A) $2^{20}-2^{9}$ (B) $2^{20}-2^{10}$ (C) $2^{21}-2^{11}$ (D) $2^{21}-2^{10}$

21. The sum of the co-efficients of all odd degree terms in the expansion of $\left(x+\sqrt{x^3-1}\right)^5+\left(x-\sqrt{x^3-1}\right)^5$, (x > 1) is- ***[JEE (Main)-2018]***

(A) 0 (B) 1 (C) 2 (D) –1

22. The coefficient of t^4 in the expansion of $\left(\frac{1-t^6}{1-t}\right)^3$ is ***[JEE (Main)-2019]***

(A) 12 (B) 15 (C) 10 (D) 14

23. If the fractional part of the number $\frac{2^{403}}{15}$ is $\frac{k}{15}$, then k is equal to : ***[JEE (Main)-2019]***

(A) 14 (B) 6 (C) 4 (D) 8

24. The positive value of ***X*** for which the co–efficient of x^2 in the expression $x^2\left(\sqrt{x}+\frac{\lambda}{x^2}\right)^{10}$ is 720, is

(A) $\sqrt{5}$ (B) 4 (C) $2\sqrt{2}$ (D) 3 ***[JEE (Main)-2019]***

25. If $\sum_{r=0}^{25}\left\{{}^{50}C_r\cdot{}^{50-r}C_{25-r}\right\}=K\left({}^{50}C_{25}\right)$, then K is equal to : ***[JEE (Main)-2019]***

(A) $2^{25}-1$ (B) $(25)^2$ (C) 2^{25} (D) 2^{24}

26. $\sum_{i=1}^{20}\left(\frac{{}^{20}C_{i-1}}{{}^{20}C_i+{}^{20}C_{i-1}}\right)=\frac{k}{21}$, then k equals : ***[JEE (Main)-2019]***

(A) 200 (B) 50 (C) 100 (D) 400

27. If the third term in the binomial expansion of $\left(1+x^{\log_2 x}\right)^5$ equals 2560, then a possible value of x is :

(A) $2\sqrt{2}$ (B) $\frac{1}{8}$ (C) $4\sqrt{2}$ (D) $\frac{1}{4}$ ***[JEE (Main)-2019]***

28. Let $(x+10)^{50}+(x-10)^{50}=a_0+a_xx+a_2x^2+....+a_{50}x^{50}$, for all $x\in R$, then is equal to : ***[JEE (Main)-2019]***

(A) 12.50 (B) 12.00 (C) 12.75 (D) 12.25

29. Let $S_n = 1 + q + q^2 + ... + q^{11}$ and $T_n = 1 + \left(\frac{q+1}{2}\right) + \left(\frac{q+1}{2}\right)^2 + ... + \left(\frac{q+1}{2}\right)^n$ where q is a real number and $q \neq 1$.Tf ${}^{101}C_1 + {}^{101}C_1 \cdot S_1 + ... + {}^{101}C_1 \cdot S_{100} = \alpha T_{100}$, then α is equal to : **[JEE (Main)-2019]**

(A) 2^{100} (B) 200 (C) 2^{99} (D) 202

30. The sum of the real values of $\times$ for which the middle term in the binomial expansion of $\left(\frac{x^3}{3} + \frac{3}{x}\right)^8$ equals 5670 is : **[JEE (Main)-2019]**

(A) 6 (B) 8 (C) 0 (D) 4

31. The value of r for which ${}^{20}C_r\,{}^{20}C_0 + {}^{20}C_{r-1}\,{}^{20}C_1 + {}^{20}C_{r-2}\,{}^{20}C_2 + ... {}^{20}C_0$is maximum, is **[JEE (Main)-2019]**

(A) 20 (B) 15 (C) 11 (D) 10

32. The total number of irrational terms in the binomial expansion of $\left(7^{1/5} - 3^{1/10}\right)^{60}$ is : **[JEE (Main)-2019]**

(A) 55 (B) 49 (C) 48 (D) 54

33. A ratio of the 5^{th} term from the beginning to the 5^{th} term from the end in the binomial expansion of $\left(2^{\frac{1}{3}} + \frac{1}{2(3)^{\frac{1}{3}}}\right)^{10}$ is **[JEE (Main)-2019]**

(A) $1 : 4(16)^{\frac{1}{3}}$ (B) $1 : 2(6)^{\frac{1}{3}}$ (C) $2(36)^{\frac{1}{3}} : 1$ (D) $4(36)^{\frac{1}{3}} : 1$

34. The sum of the series $2.{}^{20}C_0 + 5.{}^{20}C_1 + 8.{}^{20}C_2 + 11.{}^{20}C_3 + ... + 62.{}^{20}C_{20}$ is equal to : **[JEE (Main)-2019]**

(A) 2^{24} (B) 2^{25} (C) 2^{26} (D) 2^{23}

35. The sum of the co-efficients of all even degree terms in x in the expansion of $\left(x + \sqrt{x^3 - 1}\right)^6 + \left(x - \sqrt{x^3 - 1}\right)^6, (x > 1)$ is equal to : **[JEE (Main)-2019]**

(A) 32 (B) 26 (C) 29 (D) 24

36. If the fourth term in the binomial expansion of $\left(\sqrt{\frac{1}{x^{1+\log_{10} x}}} + x^{\frac{1}{12}}\right)^6$ is equal to 200, and x > 1, then the value of x is : **[JEE (Main)-2019]**

(A) 10^3 (B) 100 (C) 10^4 (D) 10

37. If the fourth term in the binomial expansion of $\left(\frac{2}{x} + x^{\log_8 x}\right)^6$ $(x > 0)$ is 20×8^7, then a value of x is :

[JEE (Main)-2019]

(A) 8 (B) 8^2 (C) 8^{-2} (D) 8^3

38. If some three consecutive in the binomial expansion of $(x + 1)^n$ is powers of x are in the ratio 2 : 15 : 70, then the average of these three coefficient is :- ***[JEE (Main)-2019]***

(A) 964 (B) 625 (C) 227 (D) 232

39. If the coefficients of x^2 and x^3 are both zero, in the expansion of the expression $(1 + ax + bx^2)(1 - 3x)^{15}$ in powers of x, then the ordered pair (a, b) is equal to : ***[JEE (Main)-2019]***

(A) (28, 315) (B) (–54, 315) (C) (–21, 714) (D) (24, 861)

40. The smallest natural number n, such that the coefficient of x in the expansion of $\left(x^2 + \frac{1}{x^3}\right)^n$ is ${}^nC_{23}$, is :

[JEE (Main)-2019]

(A) 35 (B) 38 (C) 23 (D) 58

41. The coefficient of x^{18} in the product $(1 + x)(1 - x)^{10}(1 + x + x^2)^9$ is : ***[JEE (Main)-2019]***

(A) –84 (B) 84 (C) 126 (D) –126

42. If ${}^{20}C_1 + (2^2)\,{}^{20}C_2 + (3^2)\,{}^{20}C_3 + \ldots\ldots + (20^2)\,{}^{20}C_{20} = A(2^{\beta})$, then the ordered pair (A, β) is equal to :-

[JEE (Main)-2019]

(A) (420, 18) (B) (380, 19) (C) (380, 18) (D) (420, 19)

43. The term independent of x in the expansion of $\left(\frac{1}{60} - \frac{x^8}{81}\right)\cdot\left(2x^2 - \frac{3}{x^2}\right)^6$ is equal to : ***[JEE (Main)-2019]***

(A) 36 (B) –108 (C) –72 (D) –36

EXERCISE – 5 RECAP OF IIT-JEE/JEE (ADVANCED)

1. For r = 0, 1,....,10, let A_r, B_r and C_r denote, respectively, the coefficient of x^r in the expansions of $(1 + x)^{10}$, $(1 + x)^{20}$ and $(1 + x)^{30}$. Then $\sum_{r=1}^{10} A_r(B_{10}B_r - C_{10}A_r)$ is equal to - ***[JEE 10, 5M, –2M]***

(A) $B_{10} - C_{10}$ (B) $A_{10}\left(B_{10}^2 - C_{10}A_{10}\right)$ (C) 0 (D) $C_{10} - B_{10}$

2. The coefficients of three consecutive terms of $(1 + x)^{n+5}$ are in the ratio 5 : 10 : 14. Then n = ***[JEE-Advanced 2013, 4, (–1)]***

3. Coefficient of x^{11} in the expansion of $(1 + x^2)^4 (1 + x^3)^7 (1 + x^4)^{12}$ is ***[JEE-Advanced 2014]***

(A) 1051 (B) 1106 (C) 1113 (D) 1120

4. The coefficient of x^9 in the expansion of $(1+x)(1+x^2)(1+x^3) \ldots (1+x^{100})$ is ***[JEE-Advanced 2015]***

5. Let m be the smallest positive integer such that the coefficient of x^2 in the expansion of $(1 + x)^2 + (1 + x)^3 + \ldots + (1+x)^{49} + (1+mx)^{50}$ is $(3n + 1)\ {}^{51}C_3$ for some positive integer n. Then the value of n is ***[JEE-Advanced 2016]***

6. Let $X = \left({}^{10}C_1\right)^2 + 2\left({}^{10}C_2\right)^2 + 3\left({}^{10}C_3\right)^2 + \ldots + 10\left({}^{10}C_{10}\right)^2$, where ${}^{10}C_r$, $r \in \{1, 2, \ldots, 10\}$ denote binomial coefficients. Then, the value of $\frac{1}{1430}X$ is _____ . ***[JEE-Advanced 2018]***

ANSWER KEY

EXERCISE-1

Que.	1	2	3	4	5	6	7	8	9	10
Ans.	C	A	A	D	D	B	B	A	B	C
Que.	11	12	13	14	15	16	17	18	19	20
Ans.	B	B	C	C	D	B	C	D	B	A
Que.	21									
Ans.	D									

EXERCISE-2

Que.	1	2	3	4	5	6	7	8	9	10
Ans.	AB	B	B	BC	BCD	BCD	D	AC	C	BD
Que.	11	12	13	14	15	16	17			
Ans.	ABCD	ABCD	ABC	ACD	BCD	ABCD	ABC			

- ***Match the Column*** **18.** (A)→(r), (B)→(s), (C)→(p), (D)→(q)
- ***Comprehension Based Questions*** **19.** B **20.** C

EXERCISE-3

1. r = 6 **2.** r = 5 or 9 **3.** ***(a)*** $T_3 = \frac{5}{12}$; ***(b)*** $T_6 = 7$ **5.** a = 40, b = 0

6. 100 **7.** 101^{50} **9.** ***(a)*** 990 ***(b)*** 3660 **10.** ***(a)*** $T_7 = \frac{7.3^{13}}{2}$ ***(b)*** 455×3^{12} **12.** $\frac{17}{54}$

13. ***(a)*** $84\, b^6c^3 + 630\, ab^4c^4 + 756\, a^2b^2c^5 + 84\, a^3c^6$; ***(b)*** $-1260.a^2b^3c^4$; ***(c)*** -12600

14. ***(a)*** x = 50, y = 25

EXERCISE-4

Que.	1	2	3	4	5	6	7	8	9	10
Ans.	B	C	C	B	C	B	B	A	B	A
Que.	11	12	13	14	15	16	17	18	19	20
Ans.	C	B	D	C	A	B	C	B	A	B
Que.	21	22	23	24	25	26	27	28	29	30
Ans.	C	B	D	B	C	C	D	D	A	C
Que.	31	32	33	34	35	36	37	38	39	40
Ans.	A	D	D	B	D	BONUS	B	D	A	B
Que.	41	42	43							
Ans.	B	A	D							

EXERCISE-5

1. D **2.** 6 **3.** C **4.** 8 **5.** 5 **6.** 646

QUADRATIC EQUATION & EXPRESSION

Recap of Early Classes

You already have basic idea of solving quadratic equation in the previous class. This is the most important chapter of algebra and solve many purposes in other streams of mathematics. In this chapter, we will study systematic study of solving quadratic equation. Graphs of quadratic expression and its analysis plays important role while solving problems. Through the name of chapter is quadratic equation. We will also study higher degree polynomial equation under Theory of equations.

Index

CONTENTS

THEORY

1. Comprehensive theory covering all concepts & subtopics for *excellence* in both *school level* as well as *competitive exams*.

   Symbols used for categorization

   SL $\Rightarrow$ Topics required for *school level* preparations.

   AL $\Rightarrow$ Topics required for *Advance level* preparations useful for competitive exams.

2. *Golden Key Points :* Important points/formulaes or concepts summarized at the end to have a *quick revision* of the topic.

3. *Illustrations : Subtopic based solved questions* to get comfortable in problem solving. [Students should go through these after the topic is dealt]

4. *Solved examples :* A collection of *miscellaneous solved question* based on different concepts from the chapter at the end to be referred before exercise solving.

5. *Beginner Boxes :* Collection of *elementary sub–topic* based questions to be attempted on completion of each subtopic.

EXERCISE

6. *EXERCISE-1*
   EXERCISE-2
   EXERCISE-3
   EXERCISE-4
   EXERCISE-5

QUADRATIC EQUATION & EXPRESSION

1.0 INTRODUCTION

SL AL

The algebraic expression of the form $ax^2 + bx + c$, $a \neq 0$ is called a quadratic expression, because the highest order term in it is of second degree. Quadratic equation means, $ax^2 + bx + c = 0$. In general whenever one says zeroes of the expression $ax^2 + bx + c$, it implies roots of the equation $ax^2 + bx + c = 0$, unless specified otherwise.

A quadratic equation has exactly two roots which may be real (equal or unequal) or imaginary.

2.0 SOLUTION OF QUADRATIC EQUATION & RELATION BETWEEN ROOTS AND CO-EFFICIENTS

SL AL

(a) The general form of quadratic equation is $ax^2 + bx + c = 0$, $a \neq 0$.

The roots can be found in following manner :

$$a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) = 0 \quad \Rightarrow \quad \left(x + \frac{b}{2a}\right)^2 + \frac{c}{a} - \frac{b^2}{4a^2} = 0$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a} \quad \Rightarrow \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

This expression can be directly used to find the two roots of a quadratic equation.

(b) The expression $b^2 - 4ac \equiv D$ is called the discriminant of the quadratic equation.

(c) If α & β are the roots of the quadratic equation $ax^2 + bx + c = 0$, then :

(i) $\alpha + \beta = -b/a$ (ii) $\alpha\beta = c/a$ (iii) $|\alpha - \beta| = \sqrt{D}/|a|$

(d) A quadratic equation whose roots are α & β is $(x - \alpha)(x - \beta) = 0$ i.e.

$x^2 - (\alpha + \beta)x + \alpha\beta = 0$ i.e. $x^2 -$ (sum of roots) x + product of roots $= 0$.

Illustrations

Illustration 1. If α, β are the roots of a quadratic equation $x^2 - 3x + 5 = 0$, then the equation whose roots are $(\alpha^2 - 3\alpha + 7)$ and $(\beta^2 - 3\beta + 7)$ is -

(A) $x^2 + 4x + 1 = 0$ (B) $x^2 - 4x + 4 = 0$

(C) $x^2 - 4x - 1 = 0$ (D) $x^2 + 2x + 3 = 0$

Solution. Since α, β are the roots of equation $x^2 - 3x + 5 = 0$

So $\alpha^2 - 3\alpha + 5 = 0$

$\beta^2 - 3\beta + 5 = 0$

$\therefore$ $\alpha^2 - 3\alpha = -5$

$\beta^2 - 3\beta = -5$

Putting in $(\alpha^2 - 3\alpha + 7)$ & $(\beta^2 - 3\beta + 7)$ ...(i)

$-5 + 7, -5 + 7$

$\therefore$ 2 and 2 are the roots.

$\therefore$ The required equation is

$x^2 - 4x + 4 = 0$. ***Ans. (B)***

Illustration 2. If α and β are the roots of $ax^2 + bx + c = 0$, find the value of $(a\alpha + b)^{-2} + (a\beta + b)^{-2}$.

Solution. We know that $\alpha + \beta = -\frac{b}{a}$ & $\alpha\beta = \frac{c}{a}$

$$(a\alpha + b)^{-2} + (a\beta + b)^{-2} = \frac{1}{(a\alpha+b)^2} + \frac{1}{(a\beta+b)^2}$$

$$= \frac{a^2\beta^2 + b^2 + 2ab\beta + a^2\alpha^2 + b^2 + 2ab\alpha}{(a^2\alpha\beta + ba\beta + ba\alpha + b^2)^2} = \frac{a^2(\alpha^2+\beta^2) + 2ab(\alpha+\beta) + 2b^2}{(a^2\alpha\beta + ab(\alpha+\beta) + b^2)^2}$$

($\alpha^2 + \beta^2$ can always be written as $(\alpha + \beta)^2 - 2\alpha\beta$)

$$= \frac{a^2\left[(\alpha+\beta)^2 - 2\alpha\beta\right] + 2ab(\alpha+\beta) + 2b^2}{(a^2\alpha\beta + ab(\alpha+\beta) + b^2)^2}$$

$$= \frac{a^2\left[\frac{b^2-2ac}{a^2}\right] + 2ab\left(-\frac{b}{a}\right) + 2b^2}{\left(a^2\frac{c}{a} + ab\left(-\frac{b}{a}\right) + b^2\right)^2} = \frac{b^2-2ac}{a^2c^2}$$

Alternatively

Take $b = -(\alpha + \beta)a$

$$(a\alpha + b)^{-2} + (a\beta + b)^{-2} = \frac{1}{a^2}\left[\frac{1}{(\alpha-\alpha-\beta)^2} + \frac{1}{(\beta-\alpha-\beta)^2}\right]$$

$$= \frac{1}{a^2}\left[\frac{\alpha^2+\beta^2}{\alpha^2\beta^2}\right] = \frac{1}{a^2}\left[\frac{b^2-2ac}{a^2.\frac{c^2}{a^2}}\right] = \frac{b^2-2ac}{a^2c^2}$$

3.0 NATURE OF ROOTS

SL AL

3.1 Real and Imaginary Roots

Consider the quadratic equation $ax^2 + bx + c = 0$ where a, b , c $\in$ R & a $\neq$ 0 then $x = \frac{-b \pm \sqrt{D}}{2a}$

(i) $D > 0 \Leftrightarrow$ roots are real & distinct (unequal).

(ii) $D = 0 \Leftrightarrow$ roots are real & coincident (equal)

(iii) $D < 0 \Leftrightarrow$ roots are imaginary.

(iv) If $p + iq$ is one root of a quadratic equation, then the other root must be the conjugate $p - iq$ & vice versa. ($p, q \in R$ & $i = \sqrt{-1}$) .

3.2 Rational and Irrational Roots

Consider the quadratic equation $ax^2 + bx + c = 0$ where a, b, c $\in$ Q & a $\neq$ 0 then ;

(i) If D is a perfect square, then roots are rational.

(ii) If $\alpha = p + \sqrt{q}$ is one root in this case, (where p is rational & $\sqrt{q}$ is a surd) then other root will be $p - \sqrt{q}$.

3.3 Integral Roots

Consider the quadratic equation $ax^2 + bx + c = 0$ where a = 1, b, c $\in$ I & D = perfect square of integer then roots are integer.

Illustrations

Illustration 3. If the coefficient of the quadratic equation are rational & the coefficient of x^2 is 1, then find the equation one of whose roots is $\tan\frac{\pi}{8}$.

Solution. We know that $\tan\frac{\pi}{8} = \sqrt{2} - 1$

Irrational roots always occur in conjugational pairs.

Hence if one root is $(-1 + \sqrt{2})$, the other root will be $(-1 - \sqrt{2})$. Equation is

$(x - (-1 + \sqrt{2}))\,(x - (-1 - \sqrt{2})) = 0 \quad \Rightarrow \quad x^2 + 2x - 1 = 0$

Illustration 4. Find all the integral values of a for which the quadratic equation $(x - a)(x - 10) + 1 = 0$ has integral roots.

Solution. Here the equation is $x^2 - (a + 10)x + 10a + 1 = 0$. Since integral roots will always be rational it means D should be a perfect square.

From (i) $D = a^2 - 20a + 96$.

$\Rightarrow \quad D = (a - 10)^2 - 4 \quad \Rightarrow \quad 4 = (a - 10)^2 - D$

If D is a perfect square it means we want difference of two perfect square as 4 which is possible only when $(a - 10)^2 = 4$ and $D = 0$.

$\Rightarrow \quad (a - 10) = \pm 2 \quad \Rightarrow \quad a = 12, 8$ **Ans.**

***Illustration 5.** If equation $\dfrac{x^2 - bx}{ax - c} = \dfrac{k-1}{k+1}$ has roots equal in magnitude & opposite in sign, then the value of k is -

(A) $\dfrac{a+b}{a-b}$ (B) $\dfrac{a-b}{a+b}$ (C) $\dfrac{a}{b}+1$ (D) $\dfrac{a}{b}-1$

Solution. Let the roots are α & $-\alpha$.

given equation is

$(x^2 - bx)(k + 1) = (k - 1)(ax - c)$ {Considering, $x \neq c/a$ & $k \neq -1$}

$\Rightarrow \quad x^2(k + 1) - bx(k + 1) = ax(k - 1) - c(k - 1)$

$\Rightarrow \quad x^2(k + 1) - bx(k + 1) - ax(k - 1) + c(k - 1) = 0$

Now sum of roots $= 0$ $\quad (\because \ \alpha - \alpha = 0)$

$\therefore \quad b(k + 1) + a(k - 1) = 0 \quad \Rightarrow \quad k = \dfrac{a-b}{a+b}$ **Ans. (B)**

4.0 IDENTITY

AL

An equation which is true for every value of the variable within the domain is called an identity , for example : $5(a - 3) = 5a - 15$, $(a + b)^2 = a^2 + b^2 + 2ab$ for all $a, b \in R$.

Note – A quadratic equation cannot have three or more roots & if it has , it becomes an identity.

If $ax^2 + bx + c = 0$ is an identity $\Leftrightarrow a = b = c = 0$

Illustrations

Illustration 6. If the equation $(\lambda^2 - 5\lambda + 6)x^2 + (\lambda^2 - 3\lambda + 2)x + (\lambda^2 - 4) = 0$ has more than two roots, then find the value of λ ?

Solution. As the equation has more than two roots so it becomes an identity. Hence

$\lambda^2 - 5\lambda + 6 = 0 \quad \Rightarrow \quad \lambda = 2, 3$

and $\lambda^2 - 3\lambda + 2 = 0 \quad \Rightarrow \quad \lambda = 1, 2$

and $\lambda^2 - 4 = 0 \quad \Rightarrow \quad \lambda = 2, -2$

So $\lambda = 2$ **Ans. $\lambda = 2$**

BEGINNER'S BOX-1

TOPIC COVERED : BASICS OF QUADRATIC EQUATION

1. Find the roots of following equations :

(a) $x^2 + 3x + 2 = 0$ (b) $x^2 - 8x + 16 = 0$ (c) $x^2 - 2x - 1 = 0$

2. Find the roots of the equation $a(x^2 + 1) - (a^2 + 1)x = 0$, where $a \neq 0$.

****3.*** Solve : $\frac{6-x}{x^2-4} = 2 + \frac{x}{x+2}$

4. If the roots of $4x^2 + 5k = (5k + 1)x$ differ by unity, then find the values of k.

5. If $2+\sqrt{3}$ is a root of the equation $x^2 + bx + c = 0$, where $b, c \in Q$, find b, c.

6. For the following equations, find the nature of the roots (real & distinct, real & coincident or imaginary).

(a) $x^2 - 6x + 10 = 0$ (b) $x^2 - (7+\sqrt{3})x + 6(1+\sqrt{3}) = 0$ (c) $4x^2 + 28x + 49 = 0$

****7.*** If ℓ, m are real and $\ell \neq m$, then show that the roots of $(\ell - m)x^2 - 5(\ell + m)x - 2(\ell - m) = 0$ are real and unequal.

8. Consider $f(x) = x^2 + bx + c$.

(a) Find c if $x = 0$ is a root of $f(x) = 0$.

(b) Find c if $\alpha, \frac{1}{\alpha}$ are roots of $f(x) = 0$.

(c) Comment on sign of b & c, if $\alpha < 0 < \beta$ & $|\beta| > |\alpha|$, where α, β are roots of $f(x) = 0$.

****9.*** Let $\sin x$ and $\sin y$ be roots of the quadratic equation $a\sin^2\theta + b\sin\theta + c = 0$ ($a, b, c \in R$ and $a \neq 0$) such that $\sin x + 2\sin y = 1$, then the value of $(a^2 + 2b^2 + 3ab + ac)$ equals

(A) 0 (B) 1 (C) 2 (D) 4

****10.*** Let k be a real number such that $k \neq 0$. If α and β are non zero complex numbers satisfying $\alpha + \beta = -2k$ and $\alpha^2 + \beta^2 = 4k^2 - 2k$, then a quadratic equation having $\frac{\alpha+\beta}{\alpha}$ and $\frac{\alpha+\beta}{\beta}$ as its roots is equal to

(A) $4x^2 - 4kx + k = 0$ (B) $x^2 - 4kx + 4k = 0$ (C) $4kx^2 - 4x + k = 0$ (D) $4kx^2 - 4kx + 1 = 0$

5.0 COMMON ROOTS OF TWO QUADRATIC EQUATIONS

AL

5.1 Only one Common Root

Let α be the common root of $ax^2 + bx + c = 0$ & $a'x^2 + b'x + c' = 0$ then

$a\alpha^2 + b\alpha + c = 0$ & $a'\alpha^2 + b'\alpha + c' = 0$. By Cramer's Rule $\frac{\alpha^2}{bc'-b'c} = \frac{\alpha}{a'c-ac'} = \frac{1}{ab'-a'b}$

Therefore, $\alpha = \frac{ca'-c'a}{ab'-a'b} = \frac{bc'-b'c}{a'c-ac'}$

So the condition for a common root is $(ca' - c'a)^2 = (ab' - a'b)(bc' - b'c)$.

5.2 Both Root Common

If both roots are same then $\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}$.

Illustrations

Illustration 7. Find p and q such that $px^2 + 5x + 2 = 0$ and $3x^2 + 10x + q = 0$ have both roots in common.

Solution. $a_1 = p,\ b_1 = 5,\ c_1 = 2$

$a_2 = 3,\ b_2 = 10,\ c_2 = q$

We know that :

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

$$\Rightarrow \quad \frac{p}{3} = \frac{5}{10} = \frac{2}{q}$$

$$\Rightarrow \quad p = \frac{3}{2}\ ;\ q = 4$$

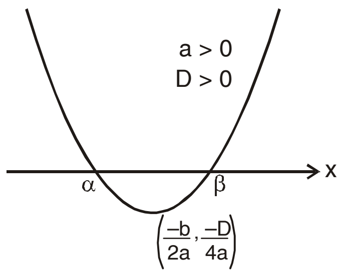
6.0 QUADRATIC EXPRESSION AND IT'S GRAPHS

Consider the quadratic expression, $y = ax^2 + bx + c$, $a \neq 0$ & $a, b, c \in R$ then ;

(a) The graph between x, y is always a parabola. If $a > 0$ then the shape of the parabola is concave upwards & if $a < 0$ then the shape of the parabola is concave downwards.

(b) The graph of $y = ax^2 + bx + c$ can be divided in 6 broad categories which are as follows : (Let the real roots of quadratic equation $ax^2 + bx + c = 0$ be α & β where $\alpha \leq \beta$).

Fig. 1

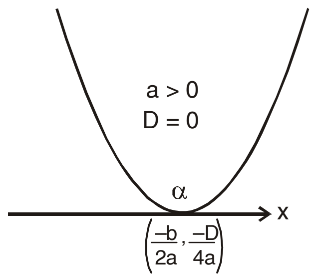


Roots are real & distinct

$ax^2 + bx + c > 0 \ \forall\ x \in (-\infty, \alpha) \cup (\beta, \infty)$

$ax^2 + bx + c < 0 \ \forall\ x \in (\alpha, \beta)$

Fig. 2

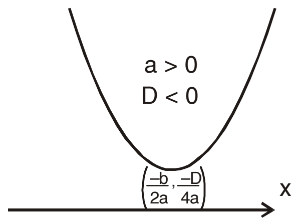


Roots are coincident

$ax^2 + bx + c > 0 \ \forall\ x \in R - \{\alpha\}$

$ax^2 + bx + c = 0$ for $x = \alpha = \beta$

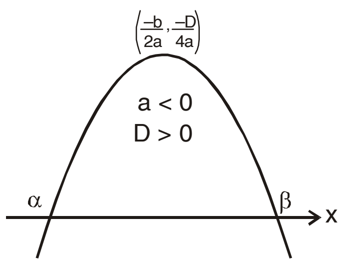
Fig.3



Roots are complex conjugate

$ax^2 + bx + c > 0 \ \forall\ x \in R$

Fig. 4

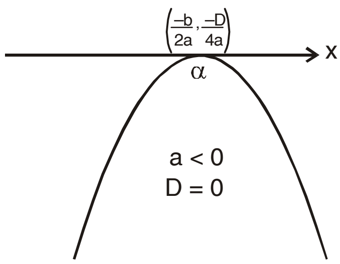


Roots are real & distinct

$ax^2 + bx + c > 0 \ \forall\ x \in (\alpha, \beta)$

$ax^2 + bx + c < 0 \ \forall\ x \in (-\infty, \alpha) \cup (\beta, \infty)$

Fig. 5

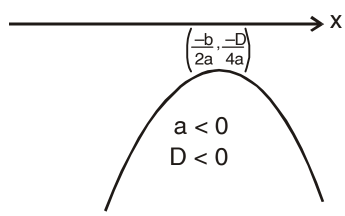


Roots are coincident

$ax^2 + bx + c < 0 \ \forall\ x \in R - \{\alpha\}$

$ax^2 + bx + c = 0$ for $x = \alpha = \beta$

Fig.6



Roots are complex conjugate

$ax^2 + bx + c < 0 \ \forall\ x \in R$

Important Note –

(i) The quadratic expression $ax^2 + bx + c > 0$ for each $x \in R \Rightarrow a > 0,\ D < 0$ & vice-versa (Fig. 3)

(ii) The quadratic expression $ax^2 + bx + c < 0$ for each $x \in R \Rightarrow a < 0,\ D < 0$ & vice-versa (Fig. 6)

7.0 MAXIMUM & MINIMUM VALUES OF QUADRATIC EXPRESSIONS $y = ax^2 + bx + c$

AL

We know that $y = ax^2 + bx + c$ takes following form : $y = a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{(b^2 - 4ac)}{4a^2}\right]$,

which is a parabola. $\therefore$ vertex $= \left(\frac{-b}{2a}, \frac{-D}{4a}\right)$

When $a > 0$, y will take a minimum value at vertex; $x = \frac{-b}{2a}$; $y_{min} = \frac{-D}{4a}$

When $a < 0$, y will take a maximum value at vertex; $x = \frac{-b}{2a}$; $y_{max} = \frac{-D}{4a}$.

If quadratic expression ax^2+bx+c is a perfect square, then $a > 0$ and $D = 0$

Illustrations

Illustration 8. If f(x) is a quadratic expression such that $f(x) > 0\ \forall\ x \in R$, and if $g(x) = f(x) + f'(x) + f''(x)$, then prove that $g(x) > 0\ \forall\ x \in R$.

Solution. Let $f(x) = ax^2 + bx + c$

Given that $f(x) > 0$ so $a > 0$, $b^2 - 4ac < 0$

Now $g(x) = ax^2 + bx + c + 2ax + b + 2a = ax^2 + (b + 2a)x + (b + c + 2a)$

For this quadratic expression $a > 0$ and discriminant

$D = (b + 2a)^2 - 4a(b + c + 2a) = b^2 + 4a^2 + 4ab - 4ab - 4ac - 8a^2 = b^2 - 4ac - 4a^2 < 0$

So $g(x) > 0\ \forall\ x \in R$.

Illustration 9. The value of the expression $x^2 + 2bx + c$ will be positive for all real x if -

(A) $b^2 - 4c > 0$ (B) $b^2 - 4c < 0$ (C) $c^2 < b$ (D) $b^2 < c$

Solution. As $a > 0$, so this expression will be positive if $D < 0$

so $4b^2 - 4c < 0$

$b^2 < c$ ***Ans. (D)***

Illustration 10. The minimum value of the expression $4x^2 + 2x + 1$ is -

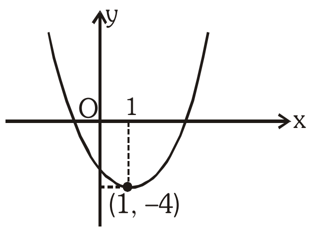
(A) 1/4 (B) 1/2 (C) 3/4 (D) 1

Solution. Since $a = 4 > 0$ therefore its minimum value is $= \frac{4(4)(1) - (2)^2}{4(4)} = \frac{16 - 4}{16} = \frac{12}{16} = \frac{3}{4}$ ***Ans. (C)***

****Illustration 11.*** If $y = x^2 - 2x - 3$, then find the range of y when :

(i) $x \in R$ (ii) $x \in [0,3]$ (iii) $x \in [-2,0]$

Solution. We know that minimum value of y will occur at



$x = -\frac{b}{2a} = -\frac{(-2)}{2 \times 1} = 1$

$y_{min} = -\frac{D}{4a} = \frac{-(4 + 3 \times 4)}{4} = -4$

(i) $x \in R$;

$y \in [-4, \infty)$ ***Ans.***

(ii) $x \in [0, 3]$

$f(0) = -3$, $f(1) = -4$, $f(3) = 0$

$\because$ $f(3) > f(0)$

$\therefore$ y will take all the values from minimum to f(3).

$y \in [-4, 0]$ ***Ans.***

(iii) $x \in [-2, 0]$

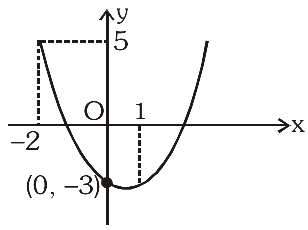
This interval does not contain the minimum value of y for $x \in R$.

y will take values from f(0) to f(–2)

$f(0) = -3$

$f(-2) = 5$

$y \in [-3, 5]$



Ans.

8.0 MAXIMUM & MINIMUM VALUES OF RATIONAL ALGEBRAIC EXPRESSIONS

AL

$$y = \frac{a_1x^2 + b_1x + c_1}{a_2x^2 + b_2x + c_2}, \frac{1}{ax^2 + bx + c}, \frac{a_1x + b_1}{a_2x^2 + b_2x + c_2}, \frac{a_1x^2 + b_1x + c_1}{a_2x + b_2}:$$

Sometime we have to find range of expression of form $\frac{a_1x^2 + b_1x + c_1}{a_2x^2 + b_2x + c_2}$. The following procedure is used :

Step 1 – Equate the given expression to y i.e. $y = \frac{a_1x^2 + b_1x + c_1}{a_2x^2 + b_2x + c_2}$

Step 2 – By cross multiplying and simplifying, obtain a quadratic equation in x.

$(a_1 - a_2y)x^2 + (b_1 - b_2y)x + (c_1 - c_2y) = 0$

Step 3 – Put Discriminant ≥ 0 and solve the inequality for possible set of values of y.

Illustrations

Illustration 12. For $x \in R$, find the set of values attainable by $\frac{x^2 - 3x + 4}{x^2 + 3x + 4}$.

Solution. Let $y = \frac{x^2 - 3x + 4}{x^2 + 3x + 4}$

$x^2(y - 1) + 3x(y + 1) + 4(y - 1) = 0$

Case- I : $y \neq 1$

For $y \neq 1$ above equation is a quadratic equation.

So for $x \in R$, $D \geq 0$

$\Rightarrow \quad 9(y + 1)^2 - 16(y - 1)^2 \geq 0$

$\Rightarrow \quad 7y^2 - 50y + 7 \leq 0$

$\Rightarrow \quad (7y - 1)(y - 7) \leq 0$

$\Rightarrow \quad y \in \left[\frac{1}{7}, 7\right] - \{1\}$

Case II : when $y = 1$

$\Rightarrow \quad 1 = \frac{x^2 - 3x + 4}{x^2 + 3x + 4}$

$\Rightarrow \quad x^2 + 3x + 4 = x^2 - 3x + 4$

$\Rightarrow \quad x = 0$

Hence $y = 1$ for real value of x.

so range of y is $\left[\frac{1}{7}, 7\right]$

BEGINNER'S BOX-2

TOPIC COVERED : COMMON ROOTS, QUADRATIC AND RATIONAL EXPRESSION

1. If $x^2 + bx + c = 0$ & $2x^2 + 9x + 10 = 0$ have both roots common then find b & c.

2. If $x^2 - 7x + 10 = 0$ & $x^2 - 5x + c = 0$ have a common root, find c.

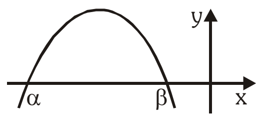
***3.** Show that $x^2 + (a^2 - 2)x - 2a^2 = 0$ and $x^2 - 3x + 2 = 0$ have exactly one common root for all $a \in R$.

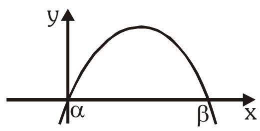
4. Find the minimum value of :

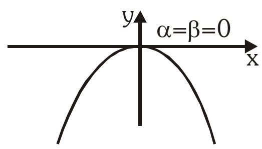
(a) $y = x^2 + 2x + 2$ (b) $y = 4x^2 - 16x + 15$

5. For following graphs of $y = ax^2 + bx + c$ with $a,b,c \in R$, comment on the sign of :

(i) a (ii) b (iii) c (iv) D (v) $\alpha + \beta$ (vi) $\alpha\beta$

(1) 

(2) 

(3) 

6. Given the roots of equation $ax^2 + bx + c = 0$ are real & distinct, where $a,b,c \in R^+$, then the vertex of the graph will lie in which quadrant.

***7.** Find the range of 'a' for which :

(a) $ax^2 + 3x + 4 > 0 \quad \forall\, x \in R$ (b) $ax^2 + 4x - 2 < 0 \quad \forall\, x \in R$

8. Prove that the expression $\dfrac{8x-4}{x^2+2x-1}$ cannot have values between 2 and 4, in its domain.

9. Find the range of $\dfrac{x^2+2x+1}{x^2+2x+7}$, where x is real

9.0 LOCATION OF ROOTS

AL

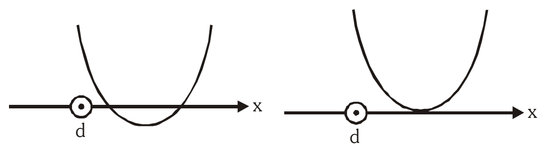
This article deals with an elegant approach of solving problems on quadratic equations when the roots are located / specified on the number line with variety of constraints :

Consider the quadratic equation $ax^2 + bx + c = 0$ with $a > 0$ and let $f(x) = ax^2 + bx + c$

Type-1 – Both roots of the quadratic equation are greater than a specific number (say d).

The necessary and sufficient condition for this are :

(i) $D \geq 0$; (ii) $f(d) > 0$; (iii) $-\dfrac{b}{2a} > d$



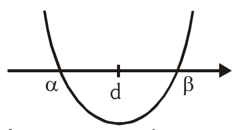
Note – When both roots of the quadratic equation are less than a specific number d than the necessary and sufficient condition will be :

(i) $D \geq 0$; (ii) $f(d) > 0$; (iii) $-\dfrac{b}{2a} < d$

Type-2 –

Both roots lie on either side of a fixed number say (d). Alternatively one root is greater than 'd' and other root less than 'd' or 'd' lies between the roots of the given equation.

The necessary and sufficient condition for this are : $f(d) < 0$



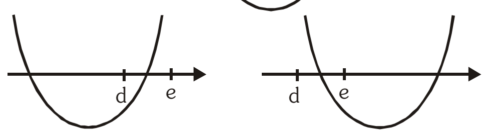
Note – Consideration of discriminant is not needed.

Type-3 –

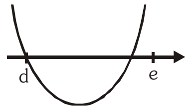
Exactly one root lies in the interval (d, e).

The necessary and sufficient condition for this are :

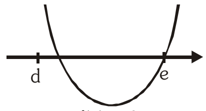
$f(d) \cdot f(e) < 0$



Note – The extremes of the intervals found by given conditions give 'd' or 'e' as the root of the equation. Hence in this case also check for end points.



f(d) = 0



f(e) = 0

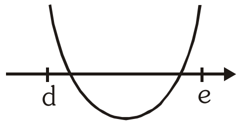
Type-4 –

When both roots are confined between the number d and e (d < e).

The necessary and sufficient condition for this are :

(i) $D \geq 0$; (ii) $f(d) > 0$; (iii) $f(e) > 0$

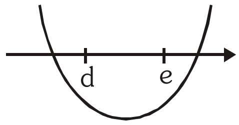
(iv) $d < -\frac{b}{2a} < e$



Type-5 –

One root is greater than e and the other roots is less than d (d < e).

The necessary and sufficient condition for this are : f(d) < 0 and f(e) < 0



Note – If a < 0 in the quadratic equation $ax^2 + bx + c = 0$ then we divide the whole equation by 'a'. Now assume $x^2 + \frac{b}{a}x + \frac{c}{a}$ as f(x). This makes the coefficient of x^2 positive and hence above cases are applicable.

Illustrations

***Illustration 13.** Find the values of the parameter 'a' for which the roots of the quadratic equation $x^2 + 2(a-1)x + a + 5 = 0$ are

(i) real and distinct (ii) equal

(iii) opposite in sign (iv) equal in magnitude but opposite in sign

(v) positive (vi) negative

(vii) greater than 3 (viii) smaller than 3

(ix) such that both the roots lie in the interval (1, 3)

Solution. Let $f(x) = x^2 + 2(a-1)x + a + 5 = Ax^2 + Bx + C$ (say)

$\Rightarrow$ $A = 1, B = 2(a-1), C = a + 5.$

Also $D = B^2 - 4AC = 4(a-1)^2 - 4(a+5) = 4(a+1)(a-4)$

(i) $D > 0$

$\Rightarrow$ $(a+1)(a-4) > 0 \Rightarrow \quad a \in (-\infty, -1)\cup(4, \infty).$

(ii) $D = 0$

$\Rightarrow$ $(a+1)(a-4) = 0 \Rightarrow \quad a = -1, 4.$

(iii) This means that 0 lies between the roots of the given equation.

$\Rightarrow$ $f(0) < 0$ and $D > 0$ i.e. $a \in (-\infty, -1) \cup (4, \infty)$

$\Rightarrow$ $a + 5 < 0 \Rightarrow a < -5 \Rightarrow a \in (-\infty, -5).$

(iv) This means that the sum of the roots is zero

$\Rightarrow$ $-2(a-1) = 0$ and $D > 0$ i.e. $a \in (-\infty, -1) \cup (4, \infty) \Rightarrow \quad a = 1$

which does not belong to $(-\infty, -1)\cup(4, \infty)$

$\Rightarrow$ $a \in \phi.$

(v) This implies that both the roots are greater than zero

$\Rightarrow$ $-\frac{B}{A} > 0, \frac{C}{A} > 0, D \geq 0. \Rightarrow \quad -(a-1) > 0, a + 5 > 0, a \in (-\infty, -1]\cup[4, \infty)$

$\Rightarrow$ $a < 1, -5 < a, a \in (-\infty, -1]\cup[4, \infty) \quad \Rightarrow \quad a \in (-5, -1].$

(vi) This implies that both the roots are less than zero

$\Rightarrow$ $-\frac{B}{A} < 0, \frac{C}{A} > 0, D \geq 0.$

$\Rightarrow$ $-(a-1) < 0, a + 5 > 0, a \in (-\infty, -1]\cup[4, \infty)$

$\Rightarrow$ $a > 1, a > -5, a \in (-\infty, -1]\cup[4, \infty)$

$\Rightarrow$ $a \in [4, \infty).$

(vii) In this case

$-\dfrac{B}{2A} > 3$, A.f(3) > 0 and D ≥ 0.

$\Rightarrow$ $-(a-1) > 3$, $7a + 8 > 0$ and $a \in (-\infty, -1] \cup [4, \infty)$

$\Rightarrow$ $a < -2$, $a > -8/7$ and $a \in (-\infty, -1] \cup [4, \infty)$

Since no value of 'a' can satisfy these conditions simultaneously, there can be no value of a for which both the roots will be greater than 3.

(viii) In this case

$-\dfrac{B}{2A} < 3$, A.f(3) > 0 and D ≥ 0.

$\Rightarrow$ $a > -2$, $a > -8/7$ and $a \in (-\infty, -1] \cup [4, \infty)$

$\Rightarrow$ $a \in (-8/7, -1] \cup [4, \infty)$

(ix) In this case

$1 < -\dfrac{B}{2A} < 3$, A.f(1) > 0, A.f(3) > 0, D ≥ 0.

$\Rightarrow$ $1 < -1(a-1) < 3$, $3a + 4 > 0$, $7a + 8 > 0$, $a \in (-\infty, -1] \cup [4, \infty)$

$\Rightarrow$ $-2 < a < 0$, $a > -4/3$, $a > -8/7$, $a \in (-\infty, -1] \cup [4, \infty)$

$\Rightarrow$ $a \in \left(-\dfrac{8}{7}, -1\right]$

Illustration 14. Find value of k for which one root of equation $x^2 - (k+1)x + k^2 + k - 8 = 0$ exceeds 2 & other is less than 2.

Solution. $4 - 2(k+1) + k^2 + k - 8 < 0$

$\Rightarrow$ $k^2 - k - 6 < 0$

$(k-3)(k+2) < 0$

$\Rightarrow$ $-2 < k < 3$

Taking intersection, $k \in (-2, 3)$.

****Illustration 15.*** Find all possible values of a for which exactly one root of $x^2 - (a+1)x + 2a = 0$ lies in interval (0,3).

Solution. $f(0) \cdot f(3) < 0$

$\Rightarrow$ $2a(9 - 3(a+1) + 2a) < 0$

$\Rightarrow$ $2a(-a + 6) < 0$

$\Rightarrow$ $a(a-6) > 0$

$\Rightarrow$ $a < 0$ or $a > 6$

Checking the extremes.

If $a = 0$, $x^2 - x = 0$

$x = 0, 1$

$1 \in (0, 3)$

If $a = 6$, $x^2 - 7x + 12 = 0$

$x = 3, 4$ But $4 \notin (0, 3)$

Hence solution set is

$a \in (-\infty, 0] \cup (6, \infty)$

10.0 GENERAL QUADRATIC EXPRESSION IN TWO VARIABLES

AL

$f(x, y) = ax^2 + 2hxy + by^2 + 2gx + 2fy + c$ may be resolved into two linear factors if ;

$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2 = 0$ OR $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$

Illustrations

Illustration 16. If $x^2 + 2xy + 2x + my - 3$ have two linear factor then m is equal to -

(A) 6, 2 (B) –6, 2 (C) 6, –2 (D) –6, –2

Solution. Here a =1, h =1, b = 0, g = 1, f = m/2, c = –3

So $\Delta = 0 \Rightarrow \begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & m/2 \\ 1 & m/2 & -3 \end{vmatrix} = 0$

$\Rightarrow \quad -\frac{m^2}{4} - (-3 - m/2) + m/2 = 0$

$\Rightarrow \quad -\frac{m^2}{4} + m + 3 = 0$

$\Rightarrow \quad m^2 - 4m - 12 = 0$

$\Rightarrow \quad m = -2, 6$ **Ans. (C)**

BEGINNER'S BOX-3

TOPIC COVERED : LOCATION OF ROOTS AND GENERAL EXPRESSION OF IInd DEGREE

1. If α, β are roots of $7x^2 + 9x - 2 = 0$, find their position with respect to following $(\alpha < \beta)$:

(a) –3 (b) 0 (c) 1

***2.** If $a > 1$, roots of the equation $(1 - a)x^2 + 3ax - 1 = 0$ are -

(A) one positive one negative (B) both negative

(C) both positive (D) both non-real

3. Find the set of all values of a for which the roots of the equation $x^2 - 2ax + a^2 + a - 3 = 0$ are less than 3.

4. If α, β are the roots of $x^2 - 3x + a = 0$, $a \in R$ and $\alpha < 1 < \beta$, then find the values of a.

***5.** If α, β are roots of $4x^2 - 16x + \lambda = 0$, $\lambda \in R$ such that $1 < \alpha < 2$ and $2 < \beta < 3$, then find the range of λ.

6. Find the value of k for which the expression $x^2 + 2xy + ky^2 + 2x + k = 0$ can be resolved into two linear factors.

7. Let a_1 and a_2 be two values of a for which the expression
$f(x, y) = 2x^2 + 3xy + y^2 + ay + 3x + 1$ can be factorised into two linear factors
then the product (a_1a_2) is equal to

(A) 1 (B) 3 (C) 5 (D) 7

***8.** If exactly one root of the quadratic equation $x^2 - \left(k + \frac{11}{3}\right) x - (k^2 + k + 1) = 0$ lies in (0, 3) then which one of the following relation is correct?

(A) $-8 < k < -4$ (B) $-3 < k < -1$ (C) $1 < k < 4$ (D) $6 < k < 10$

***9.** If roots of the quadratic equation $bx^2 - 2ax + a = 0$ are real and distinct, where $a, b \in R$ and $b \neq 0$, then

(A) atleast one root lies in the interval (0, 1).

(B) no root lies in the interval (0, 1).

(C) atleast one root lies in the interval (– 1, 0).

(D) none of the above.

11.0 THEORY OF EQUATIONS

AL

Let $\alpha_1, \alpha_2, \alpha_3, \ldots\ldots \alpha_n$ are roots of the equation, $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots.\ a_{n-1}x + a_n = 0$, where $a_0, a_1, \ldots\ldots\ldots a_n$ are constants and $a_0 \neq 0$.

$f(x) = a_0(x-\alpha_1)(x-\alpha_2)(x-\alpha_3) \ldots\ldots\ldots (x-\alpha_n)$

$\therefore \quad a_0x^n + a_1x^{n-1} + \ldots\ldots\ldots a_{n-1}x + a_n = a_0(x-\alpha_1)(x-\alpha_2) \ldots.. (x-\alpha_n)$

Comparing the coefficients of like powers of x, we get

$$\sum \alpha_i = -\frac{a_1}{a_0} = S_1 \quad \text{(say)}$$

or $\quad S_1 = -\dfrac{\text{coefficient of } x^{n-1}}{\text{coefficient of } x^n}$

$$S_2 = \sum_{i \neq j} \alpha_i \alpha_j = (-1)^2 \frac{a_2}{a_0}$$

$$S_3 = \sum_{i \neq j \neq k} \alpha_i \alpha_j \alpha_k = (-1)^3 \frac{a_3}{a_0}$$

$\vdots$

$$S_n = \alpha_1\alpha_2 \ldots\ldots \alpha_n = (-1)^n \frac{a_n}{a_0} = (-1)^n \frac{\text{constant term}}{\text{coefficient of } x^n}$$

where S_k denotes the sum of the product of root taken k at a time.

Quadratic equation – If α, β are the roots of the quadratic equation $ax^2 + bx + c = 0$, then

$$\alpha + \beta = -\frac{b}{a} \quad \text{and} \quad \alpha\beta = \frac{c}{a}$$

Cubic equation – If α, β, γ are roots of a cubic equation $ax^3 + bx^2 + cx + d = 0$, then

$$\alpha + \beta + \gamma = -\frac{b}{a}, \ \alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a} \ \text{ and } \ \alpha\beta\gamma = -\frac{d}{a}$$

NOTE

(i) If α is a root of the equation $f(x) = 0$, then the polynomial $f(x)$ is exactly divisible by $(x-\alpha)$ or $(x-\alpha)$ is a factor of $f(x)$ and conversely.

(ii) If the coefficients of the equation $f(x) = 0$ are all real and $\alpha + i\beta$ is its root, then $\alpha - i\beta$ is also a root. i.e. imaginary roots occur in conjugate pairs, where α, β are real numbers.

(iii) If the coefficients in the equation are all rational & $\alpha + \sqrt{\beta}$ is one of its roots, then $\alpha - \sqrt{\beta}$ is also a root where $\alpha, \beta \in Q$ & β is not a perfect square.

(iv) If there be any two real numbers 'a' & 'b' such that f(a) & f(b) are of opposite signs, then f(x)=0 must have atleast one real root between 'a' and 'b'.

(v) Every equation f(x) = 0 of degree odd has atleast one real root of a sign opposite to that of its last term.

Descartes rule of signs

The maximum number of positive real roots of polynomial equation $f(x) = 0$ is the number of changes of signs in $f(x)$.

Consider $x^3 + 6x^2 + 11x - 6 = 0$

The signs are : + + + –

As there is only one change of sign, the equation has atmost one positive real root.

The maximum number of negative real roots of a polynomial equation $f(x) = 0$ is the number of changes of signs in $f(-x)$

Consider $\quad f(x) = x^4 + x^3 + x^2 - x - 1 = 0$

$\qquad\qquad f(-x) = x^4 - x^3 + x^2 + x - 1 = 0$

3 sign changes, hence atmost 3 negative real roots.

Illustrations

***Illustration 17.** If two roots are equal, find the roots of $4x^3 + 20x^2 - 23x + 6 = 0$.

Solution. Let roots be α, α and β

$\therefore \quad \alpha + \alpha + \beta = -\frac{20}{4} \quad \Rightarrow \quad 2\alpha + \beta = -5 \quad$ .... (i)

$\therefore \quad \alpha . \alpha + \alpha\beta + \alpha\beta = -\frac{23}{4} \quad \Rightarrow \quad \alpha^2 + 2\alpha\beta = -\frac{23}{4}$ & $\alpha^2\beta = -\frac{6}{4}$

from equation (i)

$\alpha^2 + 2\alpha(-5 - 2\alpha) = -\frac{23}{4} \quad \Rightarrow \quad \alpha^2 - 10\alpha - 4\alpha^2 = -\frac{23}{4} \Rightarrow 12\alpha^2 + 40\alpha - 23 = 0$

$\therefore \quad \alpha = 1/2, -\frac{23}{6}$

when $\alpha = \frac{1}{2}$

$\alpha^2\beta = \frac{1}{4}(-5-1) = -\frac{3}{2}$

when $\alpha = -\frac{23}{6} \Rightarrow \alpha^2\beta = \frac{23 \times 23}{36}\left(-5 - 2x\left(-\frac{23}{6}\right)\right) \neq -\frac{3}{2}$

$\Rightarrow \quad \alpha = \frac{1}{2} \quad \beta = -6$

Hence roots of equation $= \frac{1}{2}, \frac{1}{2}, -6$ ***Ans.***

***Illustration 18.** If α, β, γ are the roots of $x^3 - px^2 + qx - r = 0$, find :

(i) $\sum \alpha^3$ (ii) $\alpha^2(\beta+\gamma) + \beta^2(\gamma+\alpha) + \gamma^2(\alpha+\beta)$

Solution. We know that $\alpha + \beta + \gamma = p$

$\alpha\beta + \beta\gamma + \gamma\alpha = q$

$\alpha\beta\gamma = r$

(i) $\alpha^3 + \beta^3 + \gamma^3 = 3\alpha\beta\gamma + (\alpha+\beta+\gamma)\{(\alpha+\beta+\gamma)^2 - 3(\alpha\beta+\beta\gamma+\gamma\alpha)\}$

$= 3r + p\{p^2 - 3q\} = 3r + p^3 - 3pq$

(ii) $\alpha^2(\beta+\gamma) + \beta^2(\alpha+\gamma) + \gamma^2(\alpha+\beta) = \alpha^2(p-\alpha) + \beta^2(p-\beta) + \gamma^2(p-\gamma)$

$= p(\alpha^2+\beta^2+\gamma^2) - 3r - p^3 + 3pq = p(p^2 - 2q) - 3r - p^3 + 3pq = pq - 3r$

***Illustration 19.** If q, r, s are positive, show that the equation $f(x) \equiv x^4 + qx^2 + rx - s = 0$ has one positive, one negative and two imaginary roots.

Solution. Product $= -s < 0$

let roots be $\alpha,\beta,\gamma,\delta$

$\Rightarrow \quad \alpha\beta\gamma\delta < 0$

this is possible when -

(i) one root is negative & three are positive

(ii) three roots are negative & one is positive

(iii) one root negative, one positive & two roots imaginary.

$f(x) \equiv x^4 + qx^2 + rx - s$

As there is only one change of sign, the equation has atmost one positive root.

$f(-x) \equiv x^4 + qx^2 - rx - s$

Again there is only one change of sign, the equation has atmost only one negative root.

so (i), (ii) can't be possible.

Hence there is only one negative root, one positive root & two imaginary roots.

12.0 TRANSFORMATION OF THE EQUATION

AL

Let $ax^2 + bx + c = 0$ be a quadratic equation with two roots α and β. If we have to find an equation whose roots are $f(\alpha)$ and $f(\beta)$, i.e. some expression in α & β, then this equation can be found by finding α in terms of y. Now as α satisfies given equation, put this α in terms of y directly in the equation.

$y = f(\alpha)$

By transformation, $\alpha = g(y)$

$a(g(y))^2 + b(g(y)) + c = 0$

This is the required equation in y.

Illustrations

***Illustration 20.** If the roots of $ax^2 + bx + c = 0$ are α and β, then find the equation whose roots are :

(a) $\frac{-2}{\alpha}, \frac{-2}{\beta}$ (b) $\frac{\alpha}{\alpha+1}, \frac{\beta}{\beta+1}$ (c) α^2, β^2

Solution. ***(a)*** $\frac{-2}{\alpha}, \frac{-2}{\beta}$

put, $y = \frac{-2}{\alpha} \Rightarrow \alpha = \frac{-2}{y}$

$a\left(-\frac{2}{y}\right)^2 + b\left(\frac{-2}{y}\right) + c = 0 \quad \Rightarrow \quad cy^2 - 2by + 4a = 0$

Required equation is $cx^2 - 2bx + 4a = 0$

(b) $\frac{\alpha}{\alpha+1}, \frac{\beta}{\beta+1}$

put, $y = \frac{\alpha}{\alpha+1} \Rightarrow \alpha = \frac{y}{1-y}$

$\Rightarrow \quad a\left(\frac{y}{1-y}\right)^2 + b\left(\frac{y}{1-y}\right) + c = 0 \quad \Rightarrow \quad (a + c - b)y^2 + (-2c + b)y + c = 0$

Required equation is $(a + c - b)x^2 + (b - 2c)x + c = 0$

(c) α^2, β^2

put $y = \alpha^2 \Rightarrow \alpha = \sqrt{y}$

$ay + b\sqrt{y} + c = 0$

$b^2y = a^2y^2 + c^2 + 2acy$

$\Rightarrow \quad a^2y^2 + (2ac - b^2)y + c^2 = 0$

Required equation is $a^2x^2 + (2ac - b^2)x + c^2 = 0$

***Illustration 21.** If the roots of $ax^3 + bx^2 + cx + d = 0$ are α, β, γ then find equation whose roots are $\frac{1}{\alpha\beta}, \frac{1}{\beta\gamma}, \frac{1}{\gamma\alpha}$.

Solution. Put $y = \frac{1}{\alpha\beta} = \frac{\gamma}{\alpha\beta\gamma} = -\frac{a\gamma}{d} \quad \left(\because \alpha\beta\gamma = -\frac{d}{a}\right)$

Put $x = -\frac{dy}{a}$

$\Rightarrow \quad a\left(-\frac{dy}{a}\right)^3 + b\left(-\frac{dy}{a}\right)^2 + c\left(-\frac{dy}{a}\right) + d = 0$

Required equation is $d^2x^3 - bdx^2 + acx - a^2 = 0$

BEGINNER'S BOX-4

TOPIC COVERED : THEORY OF EQUATION AND TRANSFORMATION OF ROOTS

1. Let α, β be two of the roots of the equation $x^3 - px^2 + qx - r = 0$. If $\alpha + \beta = 0$, then show that $pq = r$

2. If two roots of $x^3 + 3x^2 - 9x + c = 0$ are equal, then find the value of c.

***3.** If α, β, γ be the roots of $ax^3 + bx^2 + cx + d = 0$, then find the value of

(a) $\sum \alpha^2$ (b) $\sum \frac{1}{\alpha}$ (c) $\sum \alpha^2(\beta+\gamma)$

4. If α, β are the roots of $ax^2 + bx + c = 0$, then find the equation whose roots are

(a) $\frac{1}{\alpha^2}, \frac{1}{\beta^2}$ (b) $\frac{1}{a\alpha+b}, \frac{1}{a\beta+b}$ (c) $\alpha+\frac{1}{\beta}, \quad \beta+\frac{1}{\alpha}$

***5.** If α, β are roots of $x^2 - px + q = 0$, then find the quadratic equation whose root are $(\alpha^2 - \beta^2)(\alpha^3 - \beta^3)$ and $\alpha^2\beta^3 + \alpha^3\beta^2$.

***6.** If α, β, γ are the roots of the cubic $2010x^3 + 4x^2 + 1 = 0$, then the value of $(\alpha^{-2} + \beta^{-2} + \gamma^{-2})$ is equal to

(A) 8 (B) – 8 (C) 4 (D) – 4

7. Let a, b and c be three distinct real roots of the cubic $x^3 + 2x^2 - 4x - 4 = 0$. If the equation $x^3 + qx^2 + rx + s = 0$ has roots $\frac{1}{a}, \frac{1}{b}$ and $\frac{1}{c}$, then the value of $(q + r + s)$ is equal to

(A) $\frac{3}{4}$ (B) $\frac{1}{2}$ (C) $\frac{1}{4}$ (D) $\frac{1}{6}$

***8.** If α and β are the roots of equation $x^2 - a(x+1) - b = 0$ where $a, b \in R-\{0\}$ and $a + b \neq 0$ then the value of $\frac{1}{\alpha^2 - a\alpha} + \frac{1}{\beta^2 - a\beta} - \frac{2}{a+b}$ is equal to

(A) $\frac{4}{a+b}$ (B) $\frac{2}{a+b}$ (C) 0 (D) $\frac{1}{a+b}$

GOLDEN KEY POINTS

- If sum of coefficient of given polynomial equation is 0 then one root of the equation is unity.
- Every equation of nth degree ($n \geq 1$) has exactly n root & if the equation has more than n roots, it is an identity.

SOME WORKED OUT ILLUSTRATIONS

***Illustration 1.** A polynomial in x of degree greater than three, leaves remainders 2, 1 and –1 when divided, respectively, by (x – 1), (x + 2) and (x + 1). What will be the remainder when it is divided by (x – 1) (x + 2) (x + 1).

Solution. Let required polynomial be $f(x) = p(x)(x-1)(x+2)(x+1) + a_0x^2 + a_1x + a_2$

By remainder theorem, $f(1) = 2$, $f(-2) = 1$, $f(-1) = -1$.

$\Rightarrow \quad a_0 + a_1 + a_2 = 2$

$4a_0 - 2a_1 + a_2 = 1$

$a_0 - a_1 + a_2 = -1$

Solving we get, $a_0 = \frac{7}{6}$, $a_1 = \frac{3}{2}$, $a_2 = \frac{2}{3}$

Remainder when f(x) is divided by (x – 1) (x + 2) (x + 1)

will be $\frac{7}{6}x^2 + \frac{3}{2}x + \frac{2}{3}$.

***Illustration 2.** If α, β are the roots of $x^2 + px + q = 0$, and γ, δ are the roots of $x^2 + rx + s = 0$, evaluate $(\alpha - \gamma)(\alpha - \delta)(\beta - \gamma)(\beta - \delta)$ in terms of p, q, r and s. Deduce the condition that the equations have a common root.

Solution. α, β are the roots of $x^2 + px + q = 0$

$\therefore \quad \alpha + \beta = -p, \ \alpha\beta = q$ ...(1)

and γ, δ are the roots of $x^2 + rx + s = 0$

$\therefore \quad \gamma + \delta = -r, \ \gamma\delta = s$ ...(2)

Now, $(\alpha - \gamma)(\alpha - \delta)(\beta - \gamma)(\beta - \delta)$

$= [\alpha^2 - \alpha(\gamma + \delta) + \gamma\delta][\beta^2 - \beta(\gamma + \delta) + \gamma\delta]$

$= (\alpha^2 + r\alpha + s)(\beta^2 + r\beta + s)$

$= \alpha^2\beta^2 + r\alpha\beta(\alpha + \beta) + r^2\alpha\beta + s(\alpha^2 + \beta^2) + sr(\alpha + \beta) + s^2$

$= \alpha^2\beta^2 + r\alpha\beta(\alpha + \beta) + r^2\alpha\beta + s((\alpha + \beta)^2 - 2\alpha\beta) + sr(\alpha + \beta) + s^2$

$= q^2 - pqr + r^2q + s(p^2 - 2q) + sr(-p) + s^2$

$= (q - s)^2 - rpq + r^2q + sp^2 - prs$

$= (q - s)^2 - rq(p - r) + sp(p - r)$

$= (q - s)^2 + (p - r)(sp - rq)$

For a common root (Let α = γ or β = δ) ...(3)

then $(\alpha - \gamma)(\alpha - \delta)(\beta - \gamma)(\beta - \delta) = 0$ ...(4)

from (3) and (4), we get

$(q - s)^2 + (p - r)(sp - rq) = 0$

$\Rightarrow \quad (q - s)^2 = (p - r)(rq - sp)$, which is the required condition.

***Illustration 3.** If $(y^2 - 5y + 3)(x^2 + x + 1) < 2x$ for all $x \in R$, then find the interval in which y lies.

Solution. $(y^2 - 5y + 3)(x^2 + x + 1) < 2x, \ \forall \ x \in R$

$\Rightarrow \quad y^2 - 5y + 3 < \frac{2x}{x^2 + x + 1}$

Let $\frac{2x}{x^2 + x + 1} = P$

$\Rightarrow \quad px^2 + (p - 2)x + p = 0$

(1) Since x is real, $(p - 2)^2 - 4p^2 \geq 0$

$\Rightarrow \quad -2 \leq p \leq \frac{2}{3}$

(2) The minimum value of $2x/(x^2 + x + 1)$ is –2. So,

$y^2 - 5y + 3 < -2 \quad \Rightarrow \quad y^2 - 5y + 5 < 0$

$\Rightarrow \quad y \in \left(\frac{5 - \sqrt{5}}{2}, \frac{5 + \sqrt{5}}{2}\right)$

Illustration 4. If roots of the equation $(a-b)x^2 + (c-a)x + (b-c) = 0$ are equal, then a, b, c are in

(A) A.P. (B) H.P. (C) G.P. (D) none of these

Solution. $(a-b)x^2 + (c-a)x + (b-c) = 0$

As roots are equal so

$B^2 - 4AC = 0$

$\Rightarrow \quad (c-a)^2 - 4(a-b)(b-c) = 0$

$\Rightarrow \quad (c-a)^2 - 4ab + 4b^2 + 4ac - 4bc = 0$

$\Rightarrow \quad (c-a)^2 + 4ac - 4b(c+a) + 4b^2 = 0$

$\Rightarrow \quad (c+a)^2 - 2\,.\,(2b)(c+a) + (2b)^2 = 0$

$\Rightarrow \quad [c + a - 2b]^2 = 0$

$\Rightarrow \quad c + a - 2b = 0$

$\Rightarrow \quad c + a = 2b$

Hence a, b, c are in A. P.

Alternative method :

$\because$ Sum of the coefficients = 0

Hence one root is 1 and other root is $\dfrac{b-c}{a-b}$.

Given that both roots are equal, so

$$1 = \frac{b-c}{a-b}$$

$\Rightarrow \quad a - b = b - c$

$\Rightarrow \quad 2b = a + c$

Hence a, b, c are in A.P. ***Ans. (A)***

Illustration 5. The equations $5x^2 + 12x + 13 = 0$ and $ax^2 + bx + c = 0$ $(a,b,c \in R)$ have a common root, where a, b, c are the sides of the Δ ABC. Then find $\angle C$.

(A) 45° (B) 60° (C) 90° (D) 30°

Solution. As we can see discriminant of the equation $5x^2 + 12x + 13 = 0$ is negative so roots of the equation are imaginary. We know that imaginary roots always occurs in pair. So this equation can not have single common roots with any other equation having real coefficients. So both roots are common of the given equations.

Hence $\dfrac{a}{5} = \dfrac{b}{12} = \dfrac{c}{13} = \lambda(\text{let})$

then $a = 5\lambda,\ b = 12\lambda\ ,\ c = 13\lambda$

Now $\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$

$$= \frac{25\lambda^2 + 144\lambda^2 - 169\lambda^2}{2(5\lambda)(12\lambda)} = 0$$

$\therefore \quad \angle C = 90°$ ***Ans. (C)***

Illustration 6. If $ax^2 + bx + 10 = 0$ does not have real & distinct roots, find the minimum value of 5a–b.

Solution. Either $f(x) \geq 0\ \forall\ x \in R$

or $\quad f(x) \leq 0\ \ \forall\ x \in R$

$\because \quad f(0) = 10 > 0$

$\Rightarrow \quad f(x) \geq 0\ \forall\ x \in R$

$\Rightarrow \quad f(-5) = 25a - 5b + 10 \geq 0$

$\Rightarrow \quad 5a - b \geq -2$ ***Ans.***

***Illustration 7.** Find the values of a for which the expression $\frac{ax^2+3x-4}{3x-4x^2+a}$ assumes all real values for real values of x.

Solution. Let $y = \frac{ax^2+3x-4}{3x-4x^2+a}$

$x^2(a + 4y) + 3(1 - y)x - (4 + ay) = 0$

If $x \in R$, $D \geq 0$

$\Rightarrow$ $9(1 - y)^2 + 4(a + 4y)(4 + ay) \geq 0$

$\Rightarrow$ $(9 + 16a)y^2 + (4a^2 + 46)y + (9 + 16a) \geq 0$

for all $y \in R$, $(9 + 16a) > 0$ & $D \leq 0$

$\Rightarrow$ $(4a^2 + 46)^2 - 4(9 + 16a)(9 + 16a) \leq 0$

$\Rightarrow$ $4(a^2 - 8a + 7)(a^2 + 8a + 16) \leq 0$

$\Rightarrow$ $a^2 - 8a + 7 \leq 0$ $\Rightarrow$ $1 \leq a \leq 7$

$9 + 16a > 0$ & $1 \leq a \leq 7$

Taking intersection, $a \in [1, 7]$

Now, checking the boundary values of a

For $a = 1$

$$y = \frac{x^2+3x-4}{3x-4x^2+1} = -\frac{(x-1)(x+4)}{(x-1)(4x+1)}$$

$\because$ $x \neq 1 \Rightarrow y \neq -1$

$\Rightarrow$ $a = 1$ is not possible.

if $a = 7$

$$y = \frac{7x^2+3x-4}{3x-4x^2+7} = \frac{(7x-4)(x+1)}{(7-4x)(x+1)} \qquad \because x \neq -1 \Rightarrow y \neq -1$$

So y will assume all real values for some real values of x.

So $a \in (1,7)$

Illustration 8. If α is a root of the equation $ax^2 + bx + c = 0$ and β is a root of the equation $-ax^2+bx+c=0$, then prove that there will be a root of the equation $\frac{a}{2}x^2 + bx + c = 0$ lying between α and β.

Solution. Let $f(x) = \frac{a}{2}x^2 + bx + c$

$f(\alpha) = \frac{a}{2}\alpha^2 + b\alpha + c = a\alpha^2 + b\alpha + c - \frac{a}{2}\alpha^2$

$= -\frac{a}{2}\alpha^2$ (As α is a root of $ax^2 + bx + c = 0$)

And $f(\beta) = \frac{a}{2}\beta^2 + b\beta + c = -a\beta^2 + b\beta + c + \frac{3a}{2}\beta^2$

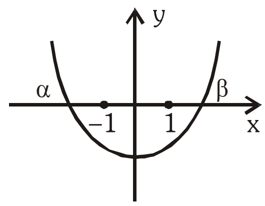
$= \frac{3a}{2}\beta^2$ (As β is a root of $-ax^2 + bx + c = 0$)

Now $f(\alpha) . f(\beta) = \frac{-3}{4}a^2\alpha^2\beta^2 < 0$

$\Rightarrow$ $f(x) = 0$ has one real root between α and β.

Illustration 9. Let a, b, c be real. If $ax^2 + bx + c = 0$ has two real roots α and β where $\alpha < -1$ and $\beta > 1$, then show that $1 + \frac{c}{a} + \left|\frac{b}{a}\right| < 0$.

Solution. Let $f(x) = x^2 + \frac{b}{a}x + \frac{c}{a}$



from graph $f(-1) < 0$ and $f(1) < 0$

$\Rightarrow \quad 1 + \frac{c}{a} - \frac{b}{a} < 0$ and $1 + \frac{c}{a} + \frac{b}{a} < 0$

Multiplying these two, we get $\left(1 + \frac{c}{a}\right)^2 - \frac{b^2}{a^2} > 0$

$\Rightarrow \quad \left|1 + \frac{c}{a}\right| > \left|\frac{b}{a}\right| \qquad \{\alpha\beta < -1 \Rightarrow \frac{c}{a} < -1\}$

$\Rightarrow \quad 1 + \frac{c}{a} + \left|\frac{b}{a}\right| < 0$

Illustration 10. If $b^2 < 2ac$ and a, b, c, d $\in$R, then prove that $ax^3 + bx^2 + cx + d = 0$ has exactly one real root.

Solution. Let α, β, γ be the roots of $ax^3 + bx^2 + cx + d = 0$

Then $\alpha + \beta + \gamma = -\frac{b}{a}$

$\alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a}$

$\alpha\beta\gamma = \frac{-d}{a}$

$\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha) = \frac{b^2}{a^2} - \frac{2c}{a} = \frac{b^2 - 2ac}{a^2}$

$\Rightarrow \quad \alpha^2 + \beta^2 + \gamma^2 < 0$, which is not possible if all α, β, γ are real. So atleast one root is non-real, but complex roots occurs in pair. Hence given cubic equation has two non-real and one real roots.

ANSWERS

BEGINNER'S BOX-1

1. (a) –1, –2; (b) 4; (c) $1 \pm \sqrt{2}$; **2.** a, $\frac{1}{a}$; **3.** $\frac{7}{3}$ **4.** 3, $-\frac{1}{5}$

5. b = –4, c = 1; **6.** (a) imaginary; (b) real & distinct ; ***(c)*** real & coincident

8. ***(a)*** c = 0; (b) c = 1; ***(c)*** b $\rightarrow$ negative, c $\rightarrow$ negative

9. (A) **10.** (B)

BEGINNER'S BOX-2

1. $b = \frac{9}{2}, c = 5$; **2.** c = 0, 6

4. ***(a)*** 1, x = –1; ***(b)*** –1, x = 2

5. (1) (i) a < 0 (ii) b < 0 (iii) c < 0 (iv) D > 0 (v) $\alpha+\beta<0$ (vi) $\alpha\beta > 0$

(2) (i) a < 0 (ii) b > 0 (iii) c = 0 (iv) D > 0 (v) $\alpha+\beta>0$ (vi) $\alpha\beta = 0$

(3) (i) a < 0 (ii) b = 0 (iii) c = 0 (iv) D = 0 (v) $\alpha+\beta=0$ (vi) $\alpha\beta = 0$

6. Third quadrant

7. ***(a)*** a > 9/16 ***(b)*** a < –2

9. least value = 0, greatest value = 1

BEGINNER'S BOX-3

1. $-3 < \alpha < 0 < \beta < 1$; **2.** C ; **3.** a < 2 ; **4.** a < 2 ; **5.** $12 < \lambda < 16$

6. 0, 2 **7.** (C) **8.** (B) **9.** (A)

BEGINNER'S BOX-4

2. –27, 5 ;

3. ***(a)*** $\frac{1}{a^2}(b^2 - 2ac)$,

(b) $-\frac{c}{d}$,

(c) $\frac{1}{a^2}(3ad - bc)$

4. ***(a)*** $c^2y^2 + y(2ac - b^2) + a^2 = 0$;

(b) $acx^2 - bx + 1 = 0$;

(c) $acx^2 + (a + c)bx + (a + c)^2 = 0$

5. $x^2 - p(p^4 - 5p^2q + 5q^2)x + p^2q^2(p^2 - 4q)(p^2 - q) = 0$

6. (B)

7. (C)

8. (C)

EXERCISE – 1 MCQ (SINGLE CHOICE CORRECT)

1. The roots of the quadratic equation $(a + b - 2c)x^2 - (2a - b - c)x + (a - 2b + c) = 0$ are -

(A) $a + b + c$ & $a - b + c$ (B) $1/2$ & $a - 2b + c$

(C) $a - 2b + c$ & $1/(a + b - 2c)$ (D) none of these

2. If the A.M. of the roots of a quadratic equation is $\frac{8}{5}$ and A.M. of their reciprocals is $\frac{8}{7}$, then the quadratic equation is -

(A) $5x^2 - 8x + 7 = 0$ (B) $5x^2 - 16x + 7 = 0$ (C) $7x^2 - 16x + 5 = 0$ (D) $7x^2 + 16x + 5 = 0$

3. If $\sin\alpha$ & $\cos\alpha$ are the roots of the equation $ax^2 + bx + c = 0$ then -

(A) $a^2 - b^2 + 2ac = 0$ (B) $a^2 + b^2 + 2ac = 0$

(C) $a^2 - b^2 - 2ac = 0$ (D) $a^2 + b^2 - 2ac = 0$

***4.** If one root of the quadratic equation $px^2 + qx + r = 0$ $(p \neq 0)$ is a surd $\frac{\sqrt{a}}{\sqrt{a}+\sqrt{a-b}}$, where p, q, r; a, b are all rationals then the other root is -

(A) $\frac{\sqrt{b}}{\sqrt{a}-\sqrt{a-b}}$ (B) $a + \frac{\sqrt{a(a-b)}}{b}$ (C) $\frac{a+\sqrt{a(a-b)}}{b}$ (D) $\frac{\sqrt{a}-\sqrt{a-b}}{\sqrt{b}}$

***5.** $ax^2 + bx + c = 0$ has real and distinct roots α and $\beta(\beta > \alpha)$. Further $a > 0$, $b < 0$ and $c < 0$, then-

(A) $0 < \beta < |\alpha|$ (B) $0 < |\alpha| < \beta$ (C) $\alpha + \beta < 0$ (D) $|\alpha|+|\beta|=\left|\frac{b}{a}\right|$

6. If the roots of $(a^2 + b^2)x^2 - 2b(a + c)x + (b^2 + c^2) = 0$ are equal then a, b, c are in

(A) A.P. (B) G.P. (C) H.P. (D) none of these

7. If $a(b - c)x^2 + b(c - a)x + c(a - b) = 0$ has equal root, then a, b, c are in

(A) A.P. (B) G.P. (C) H.P. (D) none of these

***8.** Let $p, q \in \{1, 2, 3, 4\}$. Then number of equation of the form $px^2 + qx + 1 = 0$, having real roots, is

(A) 15 (B) 9 (C) 7 (D) 8

9. If the roots of the quadratic equation $ax^2 + bx + c = 0$ are imaginary then for all values of a, b, c and $x \in R$, the expression $a^2x^2 + abx + ac$ is -

(A) positive (B) non-negative

(C) negative (D) may be positive, zero or negative

***10.** If x, y are rational number such that $x + y + (x - 2y)\sqrt{2} = 2x - y + (x - y - 1)\sqrt{6}$, then

(A) x and y connot be determined (B) $x = 2, y = 1$

(C) $x = 5, y = 1$ (D) none of these

11. The equation whose roots are the squares of the roots of the equation $ax^2 + bx + c = 0$ is -

(A) $a^2x^2 + b^2x + c^2 = 0$ (B) $a^2x^2 - (b^2 - 4ac)x + c^2 = 0$

(C) $a^2x^2 - (b^2 - 2ac)x + c^2 = 0$ (D) $a^2x^2 + (b^2 - ac)x + c^2 = 0$

***12.** If $\alpha \neq \beta$, $\alpha^2 = 5\alpha - 3$, $\beta^2 = 5\beta - 3$, then the equation whose roots are α/β & β/α, is

(A) $x^2 + 5x - 3 = 0$ (B) $3x^2 + 12x + 3 = 0$ (C) $3x^2 - 19x + 3 = 0$ (D) none of these

13. If α, β are the roots of the equation $x^2 - 3x + 1 = 0$, then the equation with roots $\frac{1}{\alpha - 2}, \frac{1}{\beta - 2}$ will be

(A) $x^2 - x - 1 = 0$ (B) $x^2 + x - 1 = 0$ (C) $x^2 + x + 2 = 0$ (D) none of these

14. If $x^2 - 11x + a$ and $x^2 - 14x + 2a$ have a common factor then 'a' is equal to

(A) 24 (B) 1 (C) 2 (D) 12

15. The value of 'a' for which the sum of the squares of the roots of $2x^2 - 2(a - 2)x - a - 1 = 0$ is least is -

(A) 1 (B) 3/2 (C) 2 (D) – 1

16. If the roots of the quadratic equation $x^2 + 6x + b = 0$ are real and distinct and they differ by atmost 4 then the least value of b is -

(A) 5 (B) 6 (C) 7 (D) 8

17. The expression $\frac{x^2 + 2x + 1}{x^2 + 2x + 7}$ lies in the interval ; $(x \in R)$ -

(A) $[0, -1]$ (B) $(-\infty, 0] \cup [1, \infty)$ (C) $[0, 1)$ (D) none of these

18. If the roots of the equation $x^2 - 2ax + a^2 + a - 3 = 0$ are real & less than 3 then -

(A) $a < 2$ (B) $2 \leq a \leq 3$ (C) $3 < a \leq 4$ (D) $a > 4$

19. The number of integral values of m, for which the roots of $x^2 - 2mx + m^2 - 1 = 0$ will lie between – 2 and 4 is -

(A) 2 (B) 0 (C) 3 (D) 1

***20.** If the roots of the equation, $x^3 + Px^2 + Qx - 19 = 0$ are each one more than the roots of the equation, $x^3 - Ax^2 + Bx - C = 0$, where A, B, C, P & Q are constants then the value of $A + B + C =$

(A) 18 (B) 19 (C) 20 (D) none

21. If $\alpha, \beta, \gamma, \delta$ are roots of $x^4 - 100x^3 + 2x^2 + 4x + 10 = 0$, then $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} + \frac{1}{\delta}$ is equal to -

(A) $\frac{2}{5}$ (B) $\frac{1}{10}$ (C) 4 (D) $-\frac{2}{5}$

***22.** Number of real solutions of the equation $x^4 + 8x^2 + 16 = 4x^2 - 12x + 9$ is equal to -

(A) 1 (B) 2 (C) 3 (D) 4

23. Let p(x) be the cubic polynomial $7x^3 - 4x^2 + K$. Suppose the three zeroes of p(x) form an arithmetic progression. Then the value of K, is -

(A) $\frac{4}{21}$ (B) $\frac{16}{147}$ (C) $\frac{16}{441}$ (D) $\frac{128}{1323}$

24. If α, β are the roots of the quadratic equation $(p^2 + p + 1)x^2 + (p - 1)x + p^2 = 0$ such that unity lies between the roots then the set of values of p is -

(A) ϕ (B) $p \in (-\infty, -1) \cup (0, \infty)$ (C) $p \in (-1, 0)$ (D) $(-1, 1)$

EXERCISE – 2 MCQ (ONE OR MORE CHOICE CORRECT)

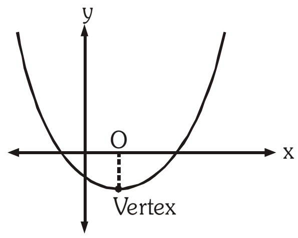
Select the correct alternatives (one or more than one correct answers)

***1.** The equation whose roots are $\sec^2\alpha$ & $\text{cosec}^2\alpha$ can be -
(A) $2x^2 - x - 1 = 0$ (B) $x^2 - 3x + 3 = 0$ (C) $x^2 - 9x + 9 = 0$ (D) $x^2 + 3x + 3 = 0$

2. If $\cos\alpha$ is a root of the equation $25x^2 + 5x - 12 = 0$, $-1 < x < 0$, then the value of $\sin 2\alpha$ is -
(A) 12/25 (B) – 12 / 25 (C) – 24 / 25 (D) 24 / 25

***3.** If the roots of the equation $\frac{1}{x+p} + \frac{1}{x+q} = \frac{1}{r}$ are equal in magnitude and opposite in sign, then -
(A) $p + q = r$ (B) $p + q = 2r$
(C) product of roots $= -\frac{1}{2}(p^2 + q^2)$ (D) sum of roots = 1

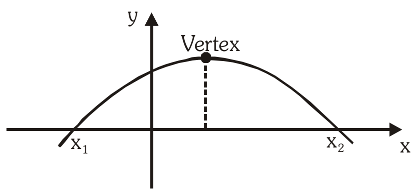
4. Graph of $y = ax^2 + bx + c = 0$ is given adjacently. What conclusions can be drawn



from this graph -
(A) $a > 0$ (B) $b < 0$ (C) $c < 0$ (D) $b^2 - 4ac > 0$

5. If a, b, c are real distinct numbers satisfying the condition a + b + c = 0 then the roots of the quadratic equation $3ax^2 + 5bx + 7c = 0$ are -
(A) positive (B) negative (C) real and distinct (D) imaginary

6. The adjoining figure shows the graph of $y = ax^2 + bx + c$. Then -



(A) $a > 0$ (B) $b > 0$ (C) $c > 0$ (D) $b^2 < 4ac$

***7.** If $x^2 + Px + 1$ is a factor of the expression $ax^3 + bx + c$ then -
(A) $a^2 + c^2 = -ab$ (B) $a^2 - c^2 = -ab$ (C) $a^2 - c^2 = ab$ (D) none of these

***8.** If p & q are distinct reals, then $2\{(x-p)(x-q) + (p-x)(p-q) + (q-x)(q-p)\} = (p-q)^2 + (x-p)^2 + (x-q)^2$ is satisfied by -
(A) no value of x (B) exactly one value of x (C) exactly two values of x (D) infinite values of x

9. Set of values of 'K' for which roots of the quadratic $x^2 - (2K-1)x + K(K-1) = 0$ are -
(A) both less than 2 is $K \in (2, \infty)$ (B) of opposite sign is $K \in (-\infty, 0) \cup (1, \infty)$
(C) of same sign is $K \in (-\infty, 0) \cup (1, \infty)$ (D) both greater than 2 is $K \in (2, \infty)$

10. Equation $2x^2 - 2(2a + 1)x + a(a + 1) = 0$ has one root less than 'a' and other root greater than 'a', if

(A) $0 < a < 1$ (B) $-1 < a < 0$ (C) $a > 0$ (D) $a < -1$

11. The value(s) of 'b' for which the equation, $2\log_{1/25}(bx + 28) = -\log_5(12 - 4x - x^2)$ has coincident roots, is/are -

(A) $b = -12$ (B) $b = 4$ (C) $b = 4$ or $b = -12$ (D) $b = -4$ or $b = 12$

***12.** If α is a root of the equation $2x(2x + 1) = 1$, then the other root is -

(A) $3\alpha^3 - 4\alpha$ (B) $-2\alpha(\alpha + 1)$ (C) $4\alpha^3 - 3\alpha$ (D) none of these

***13.** If $b^2 \geq 4ac$ for the equation $ax^4 + bx^2 + c = 0$, then all roots of the equation will be real if -

(A) $b > 0, a < 0, c > 0$ (B) $b < 0, a > 0, c > 0$

(C) $b > 0, a > 0, c > 0$ (D) $b > 0, a < 0, c < 0$

14. Let α, β be the roots of $x^2 - ax + b = 0$, where a & $b \in R$. If $\alpha + 3\beta = 0$, then -

(A) $3a^2 + 4b = 0$ (B) $3b^2 + 4a = 0$ (C) $b < 0$ (D) $a < 0$

15. For $x \in [1, 5]$, $y = x^2 - 5x + 3$ has -

(A) least value $= -1.5$ (B) greatest value $= 3$

(C) least value $= -3.25$ (D) greatest value $= \dfrac{5+\sqrt{13}}{2}$

Match the Column

Following question contains statements given in two columns, which have to be matched. The statements in ***Column-I*** are labelled as A, B, C and D while the statements in ***Column-II*** are labelled as p, q, r and s. Any given statement in ***Column-I*** can have correct matching with ***ONE OR MORE*** statement(s) in ***Column-II***.

16. Consider the equation $x^2 + 2(a - 1)x + a + 5 = 0$, where 'a' is a parameter. Match of the real values of 'a' so that the given equation has

	Column-I		***Column-II***
(A)	imaginary roots	(p)	$\left(-\infty, -\frac{8}{7}\right)$
(B)	one root smaller than 3 and other root greater than 3	(q)	$(-1, 4)$
(C)	exactly one root in the interval (1, 3) & 1 and 3 are not the root of the equation	(r)	$\left(-\frac{4}{3}, -\frac{8}{7}\right)$
(D)	one root smaller than 1 and other root greater than 3	(s)	$\left(-\infty, -\frac{4}{3}\right)$

Comprehension Based Questions

If α, β, γ be the roots of the equation $ax^3 + bx^2 + cx + d = 0$. To obtain the equation whose roots are $f(\alpha), f(\beta), f(\gamma)$, where f is a function, we put $y = f(\alpha)$ and simplify it to obtain $\alpha = g(y)$ (some function of y). Now, α is a root of the equation $ax^3 + bx^2 + cx + d = 0$, then we obtain the desired equation which is $a\{g(y)\}^3 + b\{g(y)\}^2 + c\{g(y)\} + d = 0$

For example, if α, β, γ are the roots of $ax^3 + bx^2 + cx + d = 0$. To find equation whose roots are

$\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$ we put $y = \frac{1}{\alpha} \Rightarrow \alpha = \frac{1}{y}$

As α is a root of $ax^3 + bx^2 + cx + d = 0$

we get $\frac{a}{y^3} + \frac{b}{y^2} + \frac{c}{y} + d = 0 \Rightarrow dy^3 + cy^2 + by + a = 0$

This is desired equation.

On the basis of above information, answer the following questions

17. If α, β are the roots of the equation $ax^2 + bx + c = 0$, then the roots of the equation $a(2x + 1)^2 + b(2x + 1)(x - 1) + c(x - 1)^2 = 0$ are-

(A) $\frac{2\alpha+1}{\alpha-1}, \frac{2\beta+1}{\beta-1}$ (B) $\frac{2\alpha-1}{\alpha+1}, \frac{2\beta-1}{\beta+1}$ (C) $\frac{\alpha+1}{\alpha-2}, \frac{\beta+1}{\beta-2}$ (D) $\frac{2\alpha+3}{\alpha-1}, \frac{2\beta+3}{\beta-1}$

18. If α, β are the roots of the equation $2x^2 + 4x - 5 = 0$, the equation whose roots are the reciprocals of $2\alpha - 3$ and $2\beta - 3$ is -

(A) $x^2 + 10x - 11 = 0$ (B) $11x^2 + 10x + 1 = 0$

(C) $x^2 + 10x + 11 = 0$ (D) $11x^2 - 10x + 1 = 0$

19. If α, β are the roots of the equation $px^2 - qx + r = 0$, then the equation whose roots are $\alpha^2 + \frac{r}{p}$ and $\beta^2 + \frac{r}{p}$ is-

(A) $p^3x^2 + pq^2x + r = 0$ (B) $px^2 - qx + r = 0$

(C) $p^3x^2 - pq^2x + q^2r = 0$ (D) $px^2 + qx - r = 0$

20. If α, β, γ are the roots of the equation $x^3 - x - 1 = 0$, then the value of $\Pi\left(\frac{1+\alpha}{1-\alpha}\right)$ is equal to -

(A) –7 (B) –5 (C) –3 (D) –1

EXERCISE – 3 SUBJECTIVE

***1.** If α, β are the roots of the equation $x^2 - 2x + 3 = 0$ obtain the equation whose roots are $\alpha^3 - 3\alpha^2 + 5\alpha - 2$, $\beta^3 - \beta^2 + \beta + 5$.

2. If one root of the equation $ax^2 + bx + c = 0$ be the square of the other, prove that $b^3 + a^2c + ac^2 = 3abc$.

3. Show that if p, q , r & s are real numbers & $pr = 2(q + s)$, then at least one of the equations $x^2 + px + q = 0$, $x^2 + rx + s = 0$ has real roots.

***4.** Find the product of the real roots of the equation, $x^2 + 18x + 30 = 2\sqrt{x^2 + 18x + 45}$

5. Find the values of 'a' for which $-3 < \dfrac{x^2 + ax - 2}{x^2 + x + 1} < 2$ is valid for all real x .

***6.** The equation $x^2 - ax + b = 0$ & $x^3 - px^2 + qx = 0$, where $b \neq 0, q \neq 0$, have one common root & the second equation has two equal roots. Prove that $2(q + b) = ap$.

7. Find all values of a for which the inequality $(a + 4)x^2 - 2ax + 2a - 6 < 0$ is satisfied for all $x \in R$.

8. Find all values of a for which both roots of the equation $x^2 - 6ax + 2 - 2a + 9a^2 = 0$ are greater than 3.

9. Find all the values of the parameter 'a' for which both roots of the quadratic equation $x^2 - ax + 2 = 0$ belong to the interval (0, 3).

10. Find the values of K so that the quadratic equation $x^2 + 2(K - 1)x + K + 5 = 0$ has atleast one positive root.

EXERCISE - 4 RECAP OF AIEEE/JEE (MAIN)

1. If the roots of the equation $x^2 - 5x + 16 = 0$ are α, β and the roots of the equation $x^2 + px + q = 0$ are $(\alpha^2 + \beta^2)$ and $\frac{\alpha\beta}{2}$, then- ***[AIEEE-2002]***

(A) $p = 1$ and $q = 56$ (B) $p = 1$ and $q = -56$
(C) $p = -1$ and $q = 56$ (D) $p = -1$ and $q = -56$

2. If α and β be the roots of the equation $(x - a)(x - b) = c$ and $c \neq 0$, then roots of the equation $(x - \alpha)(x - \beta) + c = 0$ are - ***[AIEEE-2002]***

(A) a and c (B) b and c (C) a and b (D) a + b and b + c

3. If $\alpha^2 = 5\alpha - 3$, $\beta^2 = 5\beta - 3$ then the value of $\frac{\alpha}{\beta} + \frac{\beta}{\alpha}$ (where $\alpha \neq \beta$) is- ***[AIEEE-2002]***

(A) 19/3 (B) 25/3 (C) –19/3 (D) none of these

4.** The value of a for which one roots of the quadratic equation $(a^2 - 5a + 3)x^2 + (3a - 1)x + 2 = 0$ is twice as large as the other is ***[AIEEE-2003]

(A) – 2/3 (B) 1/3 (C) – 1/3 (D) 2/3

5, If the sum of the roots of the quadratic equation $ax^2 + bx + c = 0$ is equal to the sum of the square of their reciprocals, then $\frac{a}{c}, \frac{b}{a}$ and $\frac{c}{b}$ are in ***[AIEEE-2003]***

(A) geometric progression (B) harmonic progression
(C) arithmetic-geometric progression (D) arithmetic progression

6. The number of real solutions of the equation $x^2 - 3|x| + 2 = 0$, is- ***[AIEEE-2003]***

(A) 4 (B) 1 (C) 3 (D) 2

7. The real number x when added to its inverse gives the minimum value of the sum at x equal to-

(A) 1 (B) –1 (C) –2 (D) 2 ***[AIEEE-2003]***

8. Let two numbers have arithmetic mean 9 and geometric mean 4. Then these numbers are the roots of the quadratic equation- ***[AIEEE-2004]***

(A) $x^2 + 18x - 16 = 0$ (B) $x^2 - 18x + 16 = 0$
(C) $x^2 + 18x + 16 = 0$ (D) $x^2 - 18x - 16 = 0$

9. If $(1 - p)$ is a root of quadratic equation $x^2 + px + (1 - p) = 0$ then its roots are- ***[AIEEE-2004]***

(A) 0, – 1 (B) – 1, 1 (C) 0, 1 (D) – 1, 2

10. If one root of the equation $x^2 + px + 12 = 0$ is 4, while the equation $x^2 + px + q = 0$ has equal roots, then the value of 'q' is- ***[AIEEE-2004]***

(A) 3 (B) 12 (C) 49/4 (D) 4

11. If value of a for which the sum of the squares of the roots of the equation $x^2 - (a - 2)x - a - 1 = 0$ assume the least value is- ***[AIEEE-2005]***

(A) 2 (B) 3 (C) 0 (D) 1

12. If the roots of the equation $x^2 - bx + c = 0$ be two consecutive integers, then $b^2 - 4c$ equals- ***[AIEEE-2005]***

(A) 1 (B) 2 (C) 3 (D) –2

13. If both the roots of the quadratic equation $x^2 - 2kx + k^2 + k - 5 = 0$ are less than 5, then k lies in the interval-

(A) [4, 5] (B) $(-\infty, 4)$ (C) $(6, \infty)$ (D) (5, 6) ***[AIEEE-2005]***

14.** All the values of m for which both roots of the equation $x^2 - 2mx + m^2 - 1 = 0$ are greater than –2 but less than 4, lie in the interval- ***[AIEEE-2006]

(A) $-1 < m < 3$ (B) $1 < m < 4$ (C) $-2 < m < 0$ (D) $m > 3$

15. If the roots of the quadratic equation $x^2 + px + q = 0$ are $\tan 30°$ and $\tan 15°$, respectively then the value of $2 + q - p$ is- ***[AIEEE-2006]***

(A) 0 (B) 1 (C) 2 (D) 3

16. If x is real, then maximum value of $\frac{3x^2+9x+17}{3x^2+9x+7}$ is- ***[AIEEE-2006]***

(A) 1 (B) $\frac{17}{7}$ (C) $\frac{1}{4}$ (D) 41

17.** If the difference between the roots of the equation $x^2 + ax + 1 = 0$ is less than $\sqrt{5}$, then the set of possible values of a is ***[AIEEE-2007]

(A) $(-3, \infty)$ (B) $(3, \infty)$ (C) $(-\infty, -3)$ (D) $(-3, -2) \cup (2, 3)$

18. The quadratic equations $x^2 - 6x + a = 0$ and $x^2 - cx + 6 = 0$ have one root in common. The other roots of the first and second equations are integers in the ratio 4:3. Then the common root is ***[AIEEE-2008]***

(A) 1 (B) 4 (C) 3 (D) 2

19.** If the roots of the equation $bx^2 + cx + a = 0$ be imaginary, then for all real values of x, the expression $3b^2x^2 + 6bcx + 2c^2$ is :- ***[AIEEE-2009]

(A) Greater than –4ab (B) Less than –4ab (C) Greater than 4ab (D) Less than 4ab

20.** Let for $a \neq a_1 \neq 0$, $f(x) = ax^2 + bx + c$, $g(x) = a_1x^2 + b_1x + c_1$ and $p(x) = f(x) - g(x)$. If $p(x) = 0$ only for $x = -1$ and $p(-2) = 2$, then the value of p(B) is: ***[AIEEE-2011]

(A) 18 (B) 3 (C) 9 (D) 6

21. Sachin and Rahul attempted to solve a quadratic equation. Sachin made a mistake in writing down the constant term and ended up in roots (4, 3). Rahul made a mistake in writing down coefficient of x to get roots (3, 2). The correct roots of equation are: ***[AIEEE-2011]***

(A) –4, –3 (B) 6, 1 (C) 4, 3 (D) –6, –1

22.** The equation $e^{\sin x} - e^{-\sin x} - 4 = 0$ has : ***[AIEEE-2012]

(A) exactly four real roots. (B) infinite number of real roots.

(C) no real roots. (D) exactly one real root.

23.** Let α and β be the roots of equation $px^2 + qx + r = 0$, $p \neq 0$. If p, q, r are in A.P. and $\frac{1}{\alpha} + \frac{1}{\beta} = 4$, then the value of $|\alpha - \beta|$ is ***[JEE 2014 Mains]

(A) $\frac{\sqrt{34}}{9}$ (B) $\frac{2\sqrt{13}}{9}$ (C) $\frac{\sqrt{61}}{9}$ (D) $\frac{2\sqrt{17}}{9}$

24. The sum of all real values of x satisfying the equation $(x^2-5x+5)^{x^2+4x-60} = 1$ is : ***[JEE 2016 Mains]***

(A) 3 (B) –4 (C) 6 (D) 5

25. If both the roots of the quadratic equation $x^2 - mx + 4 = 0$ are real and distinct and they lie in the interval [1,5], then m lies in the interval: ***[JEE 2019 Mains]***

(A) (4, 5) (B) (3, 4) (C) (5, 6) (D) (–5, –4)

26. The number of all possible positive integral values of a for which the roots of the quadratic equation, $6x^2 - 11x + \alpha = 0$ are rational numbers is : ***[JEE 2019 Mains]***

(A) 2 (B) 5 (C) 3 (D) 4

27. The values of λ such that sum of the squares of the roots of the quadratic equation, $x^2 + (3 - \lambda)x + 2 = \lambda$ has the least value is : ***[JEE 2019 Mains]***

(A) 2 (B) $\frac{4}{9}$ (C) $\frac{15}{8}$ (D) 1

28. Consider the quadratic equation $(c - 5)x^2 - 2cx + (c - 4) = 0$, $c \neq 5$. Let S be the set of all integral values of c for which one root of the equation lies in the interval (0,2) and its other root lies in the interval (2,3). Then the number of elements in S is : ***[JEE 2019 Mains]***

(A) 11 (B) 18 (C) 10 (D) 12

29. Let α and β be the roots of the quadratic equation $x^2 \sin\theta - x(\sin\theta\cos\theta + 1) + \cos\theta = 0$, $(0 < \theta < 45°)$, and $\alpha < \beta$. Then $\sum_{n=0}^{\infty}\left(\alpha^n + \frac{(-1)^n}{\beta^n}\right)$ is equal to : ***[JEE 2019 Mains]***

(A) $\frac{1}{1-\cos\theta} + \frac{1}{1+\sin\theta}$ (B) $\frac{1}{1+\cos\theta} + \frac{1}{1-\sin\theta}$ (C) $\frac{1}{1-\cos\theta} - \frac{1}{1+\sin\theta}$ (D) $\frac{1}{1+\cos\theta} - \frac{1}{1-\sin\theta}$

30. If one real root of the quadratic equation $81x^2 + kx + 256 = 0$ is cube of the other root, then a value of k is ***[JEE 2019 Mains]***

(A) –81 (B) 100 (C) –300 (D) 144

31. The number of integral values of m for which the quadratic expression $(1 + 2m)x^2 - 2(1 + 3m)x + 4(1 + m)$, xeR, is always positive, is : ***[JEE 2019 Mains]***

(A) 8 (B) 7 (C) 6 (D) 3

32. If λ be the ratio of the roots of the quadratic equation in x, $3m^2x^2 + m(m - 4)x + 2 = 0$, then the least value of m for which $\lambda + \frac{1}{\lambda} = 1$, is : ***[JEE 2019 Mains]***

(A) $2-\sqrt{3}$ (B) $4-3\sqrt{2}$ (C) $-2+\sqrt{2}$ (D) $4-2\sqrt{3}$

33. If α and β be the roots of the equation $x^2 - 2x + 2 = 0$, then the least value of n for which $\left(\frac{\alpha}{\beta}\right)^n = 1$ is : ***[JEE 2019 Mains]***

(A) 2 (B) 3 (C) 4 (D) 5

34. The number of integral values of m for which the equation $(1 + m^2)x^2 - 2(1 + 3m)x + (1 + 8m) = 0$ has no real root is : ***[JEE 2019 Mains]***

(A) infinitely many (B) 2 (C) 3 (D) 1

35. Let p, q $\in$ R. If $2-\sqrt{3}$ is a root of the quadratic equation, $x^2 + px + q = 0$, then : ***[JEE 2019 Mains]***

(A) $q^2 + 4p + 14 = 0$ (B) $p^2 - 4q - 12 = 0$ (C) $q^2 - 4p - 16 = 0$ (D) $p^2 - 4q + 12 = 0$

36. If m is chosen in the quadratic equation $(m^2 + 1) x^2 - 3x + (m^2 + 1)^2 = 0$ such that the sum of its roots is greatest, then the absolute difference of the cubes of its roots is :- ***[JEE 2019 Mains]***

(A) $8\sqrt{3}$ (B) $4\sqrt{3}$ (C) $10\sqrt{5}$ (D) $8\sqrt{5}$

37. If α and β are the roots of the quadratic equation, $x^2 + x\sin\theta - 2\sin\theta = 0$, $\theta \in \left(0,\frac{\pi}{2}\right)$, then

$\dfrac{\alpha^{12}+\beta^{12}}{\left(\alpha^{-12}+\beta^{-12}\right)(\alpha-\beta)^{24}}$ is equal to : ***[JEE 2019 Mains]***

(A) $\dfrac{2^6}{(\sin\theta+8)^{12}}$ (B) $\dfrac{2^{12}}{(\sin\theta-8)^6}$ (C) $\dfrac{2^{12}}{(\sin\theta-4)^{12}}$ (D) $\dfrac{2^{12}}{(\sin\theta+8)^{12}}$

EXERCISE – 5 RECAP OF IIT-JEE/JEE (ADVANCED)

1.** (a) Let a, b, c be the sides of a triangle. No two of them are equal and $\lambda \in R$. If the roots of the equation $x^2 + 2(a + b + c)x + 3\lambda(ab + bc + ca) = 0$ are real, then ***[JEE 2006]

(A) $\lambda < \frac{4}{3}$ (B) $\lambda > \frac{5}{3}$ (C) $\lambda \in \left(\frac{1}{3}, \frac{5}{3}\right)$ (D) $\lambda \in \left(\frac{4}{3}, \frac{5}{3}\right)$

(b) If roots of the equation $x^2 - 10cx - 11d = 0$ are a, b and those of $x^2 - 10ax - 11b = 0$ are c, d, then find the value of $a + b + c + d$. (a, b, c and d are distinct numbers) ***[JEE 2006]***

***2.** (a) Let α, β be the roots of the equation $x^2 - px + r = 0$ and $\alpha/2, 2\beta$ be the roots of the equation $x^2 - qx + r = 0$. Then the value of 'r' is

(A) $\frac{2}{9}(p-q)(2q-p)$ (B) $\frac{2}{9}(q-p)(2p-q)$ (C) $\frac{2}{9}(q-2p)(2q-p)$ (D) $\frac{2}{9}(2p-q)(2q-p)$

(b) Let $f(x) = \frac{x^2 - 6x + 5}{x^2 - 5x + 6}$

Match the expressions / statements in ***Column I*** with expressions / statements in ***Column II***.

Column I	***Column II***
(A) If $-1 < x < 1$, then $f(x)$ satisfies	(P) $0 < f(x) < 1$
(B) If $1 < x < 2$, the $f(x)$ satisfies	(Q) $f(x) < 0$
(C) If $3 < x < 5$, then $f(x)$ satisfies	(R) $f(x) > 0$
(D) If $x > 5$, then $f(x)$ satisfies	(S) $f(x) < 1$

[JEE 2007]

***3.** Let a, b, c, p, q be real numbers. Suppose α, β are the roots of the equation $x^2 + 2px + q = 0$ and $\alpha, 1/\beta$ are the roots of the equation $ax^2 + 2bx + c = 0$, where $\beta^2 \notin \{-1, 0, 1\}$

STATEMENT-1– $(p^2 - q)(b^2 - ac) \geq 0$

and

STATEMENT-2– $b \neq pa$ or $c \neq qa$

(A) Statement-1 is True, Statement-2 is True; Statement-2 is a correct explanation for Statement-1

(B) Statement-1 is True, Statement-2 is True; Statement-2 is ***NOT*** a correct explanation for Statement-1

(C) Statement-1 is True, Statement-2 is False

(D) Statement-1 is False, Statement-2 is True ***[JEE 2008]***

4. The smallest value of k, for which both the roots of the equation, $x^2 - 8kx + 16(k^2 - k + 1) = 0$ are real, distinct and have values at least 4, is ***[JEE 2009]***

5.** Let p and q be real numbers such that $p \neq 0$, $p^3 \neq q$ and $p^3 \neq -q$. If α and β are nonzero complex numbers satisfying $\alpha + \beta = -p$ and $\alpha^3 + \beta^3 = q$, then a quadratic equation having $\frac{\alpha}{\beta}$ and $\frac{\beta}{\alpha}$ as its roots is ***[JEE 2010]

(A) $(p^3 + q)x^2 - (p^3 + 2q)x + (p^3 + q) = 0$ (B) $(p^3 + q)x^2 - (p^3 - 2q)x + (p^3 + q) = 0$

(C) $(p^3 - q)x^2 - (5p^3 - 2q)x + (p^3 - q) = 0$ (D) $(p^3 - q)x^2 - (5p^3 + 2q)x + (p^3 - q) = 0$

***6.** Let α and β be the roots of $x^2 - 6x - 2 = 0$, with $\alpha > \beta$. If $a_n = \alpha^n - \beta^n$ for $n \geq 1$, then the value of $\frac{a_{10} - 2a_8}{2a_9}$ is **[JEE 2011]**

(A) 1 (B) 2 (C) 3 (D) 4

7. A value of b for which the equations : $x^2 + bx - 1 = 0$, $x^2 + x + b = 0$, have one root in common is - **[JEE 2011]**

(A) $-\sqrt{2}$ (B) $-i\sqrt{3}$ (C) $i\sqrt{5}$ (D) $\sqrt{2}$

8. If the equations $x^2 + 2x + 3 = 0$ and $ax^2 + bx + c = 0$, $a, b, c \in R$, have a common root, then a : b : c is: **[JEE 2013]**

(A) 3 : 1 : 2 (B) 1 : 2 : 3 (C) 3 : 2 : 1 (D) 1 : 3 : 2

***9.** A pack contains n cards numbered from 1 to n. Two consecutive numbered cards are removed from the pack and the sum of the numbers on the remaining cards is 1224. If the smaller of the numbers on the removed cards is k, then k –20 = **[JEE 2013]**

***10.** The quadratic equation $p(x) = 0$ with real coefficients has purely imaginary roots. Then the equation $p(p(x)) = 0$ has **[JEE 2014]**

(A) only purely imaginary roots (B) all real roots
(C) two real and two purely imaginary roots (D) neither real nor purely imaginary roots

***11.** Let S be the set of all non-zero real numbers α such that the quadratic equation $\alpha x^2 - x + \alpha = 0$ has two distinct real roots x_1 and x_2 satisfying the inequality $|x_1 - x_2| < 1$. Which of the followign intervals is(are) a subset(s) of S? **[JEE 2015]**

(A) $\left(-\frac{1}{2}, -\frac{1}{\sqrt{5}}\right)$ (B) $\left(-\frac{1}{\sqrt{5}}, 0\right)$ (C) $\left(0, \frac{1}{\sqrt{5}}\right)$ (D) $\left(\frac{1}{\sqrt{5}}, \frac{1}{2}\right)$

12. Let $-\frac{\pi}{6} < \theta < -\frac{\pi}{12}$. Suppose α_1 and β_1 are the roots of the equation $x^2 - 2x \sec\theta + 1 = 0$ and α_2 and β_2 are the roots of the equation $x^2 + 2x \tan\theta - 1 = 0$. If $\alpha_1 > \beta_1$ and $\beta_2 > \beta_2$, then $\alpha_1 + \beta_2$ equals **[JEE 2016]**

(A) $2\tan\theta$ (B) $2\sec\theta$ (C) $-2\tan\theta$ (D) $-2\sec\theta$

PARAGRAPH

Let p,q be integers and let α,β be the roots of the equation, $x^2 - x - 1 = 0$, where $\alpha \neq \beta$. For $n = 0,1,2,....$, let $a_n = p\alpha^n + q\beta^n$.

FACT : If a and b are rational numbers and $a + b\sqrt{5} = 0$, then $a = 0 = b$. **[JEE 2017]**

13. If $a_4 = 28$, then $p + 2q =$

(A) 14 (B) 7 (C) 12 (D) 21

14. $a_{12} =$

(A) $2a_{11} + a_{10}$ (B) $a_{11} - a_{10}$ (C) $a_{11} + a_{10}$ (D) $a_{11} + 2a_{10}$

ANSWERS

EXERCISE-1

Que.	1	2	3	4	5	6	7	8	9	10
Ans.	D	B	A	C	B	B	C	C	A	B
Que.	11	12	13	14	15	16	17	18	19	20
Ans.	C	C	A	A	B	A	C	A	C	A
Que.	21	22	23	24						
Ans.	D	A	D	C						

EXERCISE-2

Que.	1	2	3	4	5	6	7	8	9	10
Ans.	C	CD	BC	ABCD	C	BC	C	D	C	ACD
Que.	11	12	13	14	15					
Ans.	B	BC	BD	AC	BC					

- ***Match the Column*** 16. (A) $\rightarrow$ (q), (B) $\rightarrow$ (p, r, s), (C) $\rightarrow$ (r), (D) $\rightarrow$ (s)
- ***Comprehension Based Questions*** ***17.*** (C) ***18.*** (B) ***19.*** (C) ***20.*** (D)

EXERCISE-3

1. $x^2 - 3x + 2 = 0$ ***4.*** 20

5. $-2 < a < 1$ ***7.*** For all $a \in (-\infty, -6)$ ***8.*** For all $a \in (11/9, +\infty)$

9. $2\sqrt{2} \le a < \dfrac{11}{3}$ ***10.*** $K \le -1$

EXERCISE-4

Que.	1	2	3	4	5	6	7	8	9	10
Ans.	D	C	A	D	B	A	A	B	A	C
Que.	11	12	13	14	15	16	17	18	19	20
Ans.	D	A	B	A	D	D	D	D	A	A
Que.	21	22	23	24	25	26	27	28	29	30
Ans.	B	C	B	A	A	C	A	A	A	C
Que.	31	32	33	34	35	36	37			
Ans.	B	B	C	A	A,B,C	D	D			

EXERCISE-5

1. ***(a)*** A; ***(b)*** 1210

2. ***(a)*** D; ***(b)*** (A) P, R, S; (B) Q, S; (C) Q, S; (D) P, R, S

3. (B) ***4.*** 2 ***5.*** (B) ***6.*** (C) ***7.*** (B) ***8.*** (B)

9. 5 ***10.*** (D) ***11.*** (AD) ***12.*** (C) ***13.*** (C) ***14.*** (C)

IMPORTANT NOTES

SEQUENCE AND SERIES

Recap of Early Classes

In mathematics, the word, "sequence" is used in much the same way as it is in ordinary English. When we say that a collection of objects is listed in a sequence, we usually mean that the collection is ordered in such a way that it has an identified first member, second member, third member and so on. For example, population of human beings or bacteria at different times form a sequence. Sequences, following specific patterns are called progressions. In previous class, we have studied about arithmetic progression (A.P).

Index

CONTENTS

THEORY

1. Comprehensive theory covering all concepts & subtopics for *excellence* in both *school level* as well as *competitive exams*.

Symbols used for categorization

SL ⇒ Topics required for *school level* preparations.

AL ⇒ Topics required for *Advance level* preparations useful for competitive exams.

2. *Golden Key Points :* Important points/formulaes or concepts summarized at the end to have a *quick revision* of the topic.

3. *Illustrations : Subtopic based solved questions* to get comfortable in problem solving.
[Students should go through these after the topic is dealt]

4. *Solved examples :* A collection of *miscellaneous solved question* based on different concepts from the chapter at the end to be referred before exercise solving.

5. *Beginner Boxes :* Collection of *elementary sub–topic* based questions to be attempted on completion of each subtopic.

EXERCISE

6. *EXERCISE-1*
EXERCISE-2
EXERCISE-3
EXERCISE-4
EXERCISE-5

SEQUENCE & SERIES

1.0 DEFINITION

SL AL

- ***Sequence***

A succession of terms a_1, a_2, a_3, a_4........ formed according to some rule or law.

Examples are : $1, 4, 9, 16, 25$

$-1, 1, -1, 1,........$

$\frac{x}{1!}, \frac{x^2}{2!}, \frac{x^3}{3!}, \frac{x^4}{4!}, ...$

A finite sequence has a finite (i.e. limited) number of terms, as in the first example above. An infinite sequence has an unlimited number of terms, i.e. there is no last term, as in the second and third examples.

- ***Series***

The indicated sum of the terms of a sequence. In the case of a finite sequence $a_1, a_2, a_3,................,a_n$ the corresponding series is $a_1 + a_2 + a_3 + ... + a_n = \sum_{k=1}^{n} a_k$. This series has a finite or limited number of terms and is called a finite series.

2.0 ARITHMETIC PROGRESSION (A.P.)

SL AL

A.P. is a sequence whose terms differ by a fixed number. This fixed number is called the common difference. If a is the first term & d the common difference, then A.P. can be written as

$a, a + d, a + 2d, ..., a + (n - 1) d , ...$

(a) n^{th} term of AP $T_n = a + (n-1)d$, where $d = t_n - t_{n-1}$

(b) The sum of the first n terms : $S_n = \frac{n}{2}[a + \ell] = \frac{n}{2}[2a + (n-1)d]$

where ℓ is the last term.

NOTE

(i) n^{th} term of an A.P. is of the form An + B i.e. a linear expression in 'n', in such a case the coefficient of n is the common difference of the A.P. i.e. A.

(ii) Sum of first 'n' terms of an A.P. is of the form $An^2 + Bn$ i.e. a quadratic expression in 'n', in such case the common difference is twice the coefficient of n^2. i.e. 2A

(iii) Also n^{th} term $T_n = S_n - S_{n-1}$

Illustrations

Illustration 1. If $(x + 1)$, $3x$ and $(4x + 2)$ are first three terms of an A.P. then its 5^{th} term is -

(A) 14 (B) 19 (C) 24 (D) 28

Solution. $(x + 1), 3x, (4x + 2)$ are in AP

$\Rightarrow$ $3x - (x + 1) = (4x + 2) - 3x$

$\Rightarrow$ $x = 3$

$\therefore$ $a = 4, d = 9 - 4 = 5$

$\Rightarrow$ $T_5 = 4 + 4(5) = 24$ ***Ans. (C)***

Illustration 2. The sum of first four terms of an A.P. is 56 and the sum of it's last four terms is 112. If its first term is 11 then find the number of terms in the A.P.

Solution.

$a + a + d + a + 2d + a + 3d = 56$

$4a + 6d = 56$

$44 + 6d = 56$ (as $a = 11$)

$6d = 12$ hence $d = 2$

Now sum of last four terms.

$a + (n-1)d + a + (n-2)d + a + (n-3)d + a + (n-4)d = 112$

$\Rightarrow$ $4a + (4n-10)d = 112$

$\Rightarrow$ $44 + (4n-10)2 = 112$

$\Rightarrow$ $4n - 10 = 34$

$\Rightarrow$ $n = 11$ ***Ans.***

****Illustration 3.*** The sum of first n terms of two A.Ps. are in ratio $\frac{7n+1}{4n+27}$. Find the ratio of their 11^{th} terms.

Solution. Let a_1 and a_2 be the first terms and d_1 and d_2 be the common differences of two A.Ps respectively then

$$\frac{\frac{n}{2}[2a_1+(n-1)d_1]}{\frac{n}{2}[2a_2+(n-1)d_2]} = \frac{7n+1}{4n+27}$$

$$\Rightarrow \quad \frac{a_1+\left(\frac{n-1}{2}\right)d_1}{a_2+\left(\frac{n-1}{2}\right)d_2} = \frac{7n+1}{4n+27}$$

For ratio of 11^{th} terms

$$\frac{n-1}{2} = 10 \quad \Rightarrow n = 21$$

so ratio of 11^{th} terms is $\frac{7(21)+1}{4(21)+27} = \frac{148}{111} = \frac{4}{3}$ ***Ans.***

3.0 PROPERTIES OF A.P.

SL AL

(a) If each term of an A.P. is increased, decreased, multiplied or divided by the same nonzero number, then the resulting sequence is also an A.P.

(b) Three numbers in A.P. : $a - d, a, a + d$

Four numbers in A.P. : $a - 3d, a - d, a + d, a + 3d$

Five numbers in A.P. : $a - 2d, a - d, a, a + d, a + 2d$

Six numbers in A.P. : $a - 5d, a - 3d, a - d, a + d, a + 3d, a + 5d$ etc.

(c) The common difference can be zero, positive or negative.

(d) k^{th} term from the last = $(n - k + 1)^{th}$ term from the beginning.

(e) The sum of the two terms of an AP equidistant from the beginning & end is constant and equal to the sum of first & last terms. $\Rightarrow T_k + T_{n-k+1}$ = constant = $a + \ell$.

(f) Any term of an AP (except the first) is equal to half the sum of terms which are equidistant from it.

$a_n = (1/2)(a_{n-k} + a_{n+k}), k < n$

For $k = 1$, $a_n = (1/2)(a_{n-1} + a_{n+1})$; For $k = 2$, $a_n = (1/2)(a_{n-2} + a_{n+2})$ and so on.

(g) If a, b, c are in AP, then $2b = a + c$.

Illustrations

Illustration 4. Four numbers are in A.P. If their sum is 20 and the sum of their squares is 120, then the middle terms are -

(A) 2, 4 (B) 4, 6 (C) 6, 8 (D) 8, 10

Solution. Let the numbers are $a-3d,\ a-d,\ a+d,\ a+3d$

given, $a-3d+a-d+a+d+a+3d = 20$

$\Rightarrow \quad 4a = 20 \Rightarrow a = 5$

and $\quad (a-3d)^2 + (a-d)^2 + (a+d)^2 + (a+3d)^2 = 120$

$\Rightarrow \quad 4a^2 + 20d^2 = 120$

$\Rightarrow \quad 4 \times 5^2 + 20d^2 = 120$

$\Rightarrow \quad d^2 = 1 \Rightarrow d = \pm 1$

Hence numbers are 2, 4, 6, 8 ***Ans. (B)***

Illustration 5. Four different integers form an increasing A.P. One of these numbers is equal to the sum of the squares of the other three numbers. Find the numbers.

Solution. Let the numbers be $a-d,\ a,\ a+d,\ a+2d$

where $\quad a, d \in I,\ d > 0$

according to the question

$$(a-d)^2 + a^2 + (a+d)^2 = a + 2d$$

i.e., $\quad 2d^2 - 2d + 3a^2 - a = 0$

$$\therefore \quad d = \frac{1}{2}[1 \pm \sqrt{(1+2a-6a^2)}\,]$$

Since, d is positive integer,

$$\Rightarrow \quad 1 + 2a - 6a^2 > 0 \Rightarrow \quad a^2 - \frac{a}{3} - \frac{1}{6} < 0$$

$$\Rightarrow \quad \left(a - \frac{1-\sqrt{7}}{6}\right)\left(a - \frac{1+\sqrt{7}}{6}\right) < 0$$

$$\therefore \quad \left(\frac{1-\sqrt{7}}{6}\right) < a < \left(\frac{1+\sqrt{7}}{6}\right)$$

$\because \quad a \in I$

$\therefore \quad a = 0$

then $\quad d = \frac{1}{2}[1 \pm 1] = 1$ or 0. Since, $d > 0$ $\quad \therefore \ d = 1$

Hence, the numbers are –1, 0, 1, 2

****Illustration 6.*** If $a_1, a_2, a_3, \ldots\ldots\ldots\ldots, a_n$ are in A.P. where $a_i > 0$ for all i, show that :

$$\frac{1}{\sqrt{a_1}+\sqrt{a_2}} + \frac{1}{\sqrt{a_2}+\sqrt{a_3}} + \ldots\ldots + \frac{1}{\sqrt{a_{n-1}}+\sqrt{a_n}} = \frac{(n-1)}{\sqrt{a_1}+\sqrt{a_n}}$$

Solution.

$$\text{L.H.S.} = \frac{1}{\sqrt{a_1}+\sqrt{a_2}} + \frac{1}{\sqrt{a_2}+\sqrt{a_3}} + \ldots\ldots + \frac{1}{\sqrt{a_{n-1}}+\sqrt{a_n}}$$

$$= \frac{1}{\sqrt{a_2}+\sqrt{a_1}} + \frac{1}{\sqrt{a_3}+\sqrt{a_2}} + \ldots\ldots + \frac{1}{\sqrt{a_n}+\sqrt{a_{n-1}}}$$

$$= \frac{\sqrt{a_2}-\sqrt{a_1}}{(a_2-a_1)} + \frac{\sqrt{a_3}-\sqrt{a_2}}{(a_3-a_2)} + \ldots\ldots + \frac{\sqrt{a_n}-\sqrt{a_{n-1}}}{a_n - a_{n-1}}$$

Let 'd' is the common difference of this A.P.

then $a_2 - a_1 = a_3 - a_2 = ... = a_n - a_{n-1} = d$

Now L.H.S.

$$= \frac{1}{d}\left\{\sqrt{a_2} - \sqrt{a_1} + \sqrt{a_3} - \sqrt{a_2} + ... + \sqrt{a_{n-1}} - \sqrt{a_{n-2}} + \sqrt{a_n} - \sqrt{a_{n-1}}\right\}$$

$$= \frac{1}{d}\left\{\sqrt{a_n} - \sqrt{a_1}\right\} = \frac{a_n - a_1}{d\left(\sqrt{a_n} + \sqrt{a_1}\right)} = \frac{a_1 + (n-1)d - a_1}{d\left(\sqrt{a_n} + \sqrt{a_1}\right)}$$

$$= \frac{1}{d}\frac{(n-1)d}{(\sqrt{a_n} + \sqrt{a_1})} = \frac{n-1}{\sqrt{a_n} + \sqrt{a_1}} = \text{R.H.S.}$$

4.0 GEOMETRIC PROGRESSION (G.P.)

SL AL

G.P. is a sequence of non zero numbers each of the succeeding term is equal to the preceeding term multiplied by a constant. Thus in a GP the ratio of successive terms is constant. This constant factor is called the COMMON RATIO of the sequence & is obtained by dividing any term by the immediately previous term. Therefore a, ar, ar^2, ar^3, ar^4, ... is a GP with 'a' as the first term & 'r' as common ratio.

(a) n^{th} term ; $T_n = a\,r^{n-1}$

(b) Sum of the first n terms

$$S_n = \frac{a(r^n - 1)}{r - 1}, \text{ if } r \neq 1$$

(c) Sum of infinite G.P. , $S_\infty = \frac{a}{1-r};\ 0 < |r| < 1$

5.0 PROPERTIES OF (G.P.)

SL AL

(a) If each term of a G.P. be multiplied or divided by the some non-zero quantity, then the resulting sequence is also a G.P.

(b) Three consecutive terms of a GP : a/r, a, ar ;
Four consecutive terms of a GP : a/r^3, a/r, ar, ar^3 & so on.

(c) If a, b, c are in G.P. then $b^2 = ac$.

(d) If in a G.P, the product of two terms which are equidistant from the first and the last term, is constant and is equal to the product of first and last term. $\Rightarrow T_k.\ T_{n-k+1} = \text{constant} = a.\ell$

(e) If each term of a G.P. be raised to the same power, then resulting sequence is also a G.P.

(f) In a G.P., $T_r^2 = T_{r-k}.\ T_{r+k},\ k < r,\ r \neq 1$

(g) If the terms of a given G.P. are chosen at regular intervals, then the new sequence is also a G.P.

(h) If $a_1, a_2, a_3.....a_n$ is a G.P. of positive terms, then $\log a_1, \log a_2, ...\log a_n$ is an A.P. and vice-versa.

(i) If $a_1, a_2, a_3...$ and $b_1, b_2, b_3...$ are two G.P.'s then $a_1b_1, a_2b_2, a_3b_3...$ & $\frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3}$.......... is also in G.P.

Illustrations

Illustration 7. If a, b, c, d and p are distinct real numbers such that

$$\left(a^2 + b^2 + c^2\right)p^2 - 2p(ab + bc + cd) + \left(b^2 + c^2 + d^2\right) \leq 0 \text{ then a, b, c, d are in}$$

(A) A.P. (B) G.P. (C) H.P. (D) none of these

Solution. Here, the given condition

$$\left(a^2+b^2+c^2\right)p^2-2p\left(ab+bc+ca\right)+b^2+c^2+d^2\le 0$$

$$\Rightarrow \left(ap-b\right)^2+\left(bp-c\right)^2+\left(cp-d\right)^2\le 0$$

$\because$ a square can not be negative

$\therefore$ $ap-b=0, bp-c=0, cp-d=0$

$\Rightarrow p=\frac{b}{a}=\frac{c}{b}=\frac{d}{c}\Rightarrow$ a, b, c, d are in G.P. **Ans. (B)**

Illustration 8. If a, b, c are in G.P., then the equations $ax^2+2bx+c=0$ and $dx^2+2ex+f=0$ have a common root if $\frac{d}{a},\frac{e}{b},\frac{f}{c}$ are in -

(A) A.P. (B) G.P. (C) H.P. (D) none of these

Solution. a, b, c are in G.P $\Rightarrow b^2=ac$

Now the equation $ax^2+2bx+c=0$ can be rewritten as $ax^2+2\sqrt{ac}x+c=0$

$$\Rightarrow \left(\sqrt{a}x+\sqrt{c}\right)^2=0\Rightarrow x=-\sqrt{\frac{c}{a}},-\sqrt{\frac{c}{a}}$$

If the two given equations have a common root, then this root must be $-\sqrt{\frac{c}{a}}$.

Thus $d\frac{c}{a}-2e\sqrt{\frac{c}{a}}+f=0$

$$\Rightarrow \frac{d}{a}+\frac{f}{c}=\frac{2e}{c}\sqrt{\frac{c}{a}}=\frac{2e}{\sqrt{ac}}=\frac{2e}{b}$$

$\Rightarrow \frac{d}{a},\frac{e}{b},\frac{f}{c}$ are in A.P. **Ans. (A)**

Illustration 9. A number consists of three digits which are in G.P. the sum of the right hand and left hand digits exceeds twice the middle digit by 1 and the sum of the left hand and middle digits is two third of the sum of the middle and right hand digits. Find the numbers.

Solution. Let the three digits be a, ar and ar^2 then number is

$100a + 10ar + ar^2$ ...(i)

Given, $a + ar^2 = 2ar + 1$

or $a(r^2 - 2r + 1) = 1$

or $a(r - 1)^2 = 1$ ...(ii)

Also given $a + ar = \frac{2}{3}(ar + ar^2)$

$\Rightarrow$ $3 + 3r = 2r + 2r^2$

$\Rightarrow$ $2r^2 - r - 3 = 0$

$\Rightarrow$ $(r + 1)(2r - 3) = 0$

$\therefore$ $r = -1, 3/2$

for $r = -1,\ a = \dfrac{1}{(r-1)^2} = \dfrac{1}{4} \notin I$ $\qquad \therefore r \neq -1$

for $r = 3/2, a = \dfrac{1}{\left(\dfrac{3}{2}-1\right)^2} = 4$ $\qquad$ {from (ii)}

From (i), number is $400 + 10.4.\dfrac{3}{2} + 4.\dfrac{9}{4} = 469$ **Ans.**

Illustration 10. Find the value of $0.32\overline{58}$

Solution. Let $R = 0.32\overline{58}$

$\Rightarrow \quad R = 0.32585858....$ ... (i)

Here number of figures which are not recurring is 2 and number of figures which are recurring is also 2.

then $\quad 100\,R = 32.585858......$ ... (ii)

and $\quad 10000\,R = 3258.5858.....$ ... (iii)

Subtracting (ii) from (iii) , we get

$9900\,R = 3226 \Rightarrow R = \dfrac{1613}{4950}$

Aliter Method : $R = .32 + .0058 + .0058 + .000058 + ...$

$$= .32 + \frac{58}{10^4}\left(1 + \frac{1}{10^2} + \frac{1}{10^4} + ...\infty\right)$$

$$= .32 + \frac{58}{10^4}\left(\frac{1}{1-\dfrac{1}{100}}\right)$$

$$= \frac{32}{100} + \frac{58}{9900} = \frac{3168+58}{9900} = \frac{3226}{9900} = \frac{1613}{4950}$$

BEGINNER'S BOX-1

TOPIC COVERED : A.P., G.P. AND THEIR PROPERTIES

1. Write down the sequence whose n^{th} terms is : (a) $\dfrac{2^n}{n}$ (b) $\dfrac{3+(-1)^n}{3^n}$

2. For an A.P, show that $t_m + t_{2n+m} = 2t_{m+n}$

3. If the sum of p terms of an A.P. is q and the sum of its q terms is p, then find the sum of its (p + q) term.

4. Find the sum of first 24 terms of the A.P. $a_1, a_2, a_3......$, if it is know that

$a_1 + a_5 + a_{10} + a_{15} + a_{20} + a_{24} = 225$.

5. Find the number of terms common to the two A.P.'s 3, 7, 11, ... 407 and 2, 9, 16, ..., 709

6. Find a three digit number whose consecutive digits form a G.P. If we subtract 792 from this number, we get a number consisting of the same digits written in the reverse order. Now, if we increase the second digit of the required number by 2, then the resulting digits will form an A.P.

7. If the third term of G.P. is 4, then find the product of first five terms.

***8.** If a, b, c are respectively the p^{th}, q^{th} and r^{th} terms of the given G.P., then show that $(q-r)\log a + (r-p)\log b + (p-q)\log c = 0$, where a, b, c > 0.

9. Find three numbers in G.P., whose sum is 52 and the sum of whose products in pairs is 624.

10. The rational number which equals the number $2.\overline{357}$ with recurring decimal is -

(A) $\frac{2357}{999}$ (B) $\frac{2379}{997}$ (C) $\frac{785}{333}$ (D) $\frac{2355}{1001}$

6.0 HARMONIC PROGRESSION (H.P.)

SL AL

A sequence is said to be in H.P. if the reciprocal of its terms are in AP.

If the sequence $a_1, a_2, a_3, ..., a_n$ is an HP then $1/a_1, 1/a_2, ..., 1/a_n$ is an AP . Here we do not have the formula for the sum of the n terms of an HP. The general form of a harmonic progression is

$$\frac{1}{a}, \frac{1}{a+d}, \frac{1}{a+2d}, ...\frac{1}{a+(n-1)d}$$

Note – No term of any H.P. can be zero.

(i) If a, b, c are in HP, then $b = \frac{2ac}{a+c}$ or $\frac{a}{c} = \frac{a-b}{b-c}$

Illustrations

Illustration 11. If $\frac{1}{a}+\frac{1}{c}+\frac{1}{a-b}+\frac{1}{c-b} = 0$, prove that a, b, c are in H.P. or b = a + c

Solution. We have $\frac{1}{a}+\frac{1}{c}+\frac{1}{a-b}+\frac{1}{c-b} = 0$,

$$\Rightarrow \frac{a+c}{ac} + \frac{c-b+a-b}{(a-b)(c-b)}$$

$$\Rightarrow \frac{a+c}{ac} + \frac{(a+c)-2b}{ac-b(a+c)+b^2} = 0$$

Let a + c = λ

$$\therefore \frac{\lambda}{ac} + \frac{\lambda-2b}{ac-b\lambda+b^2} = 0$$

$$\Rightarrow \frac{ac\lambda - b\lambda^2 + b^2\lambda + ac\lambda - 2abc}{ac(ac-b\lambda+b^2)} = 0$$

$$\Rightarrow 2ac\lambda - b\lambda^2 + b^2\lambda - 2abc = 0$$

$$\Rightarrow 2ac(\lambda-b) - b\lambda(\lambda-b) = 0$$

$$\Rightarrow (2ac-b\lambda)(\lambda-b) = 0$$

$$\Rightarrow \lambda = b \text{ or } \lambda = \frac{2ac}{b}$$

$$\Rightarrow a+c = b \text{ or } a+c = \frac{2ac}{b} \qquad (\because a+c=\lambda)$$

$$\Rightarrow a+c = b \text{ or } b = \frac{2ac}{a+c}$$

$\therefore$ a, b, c are in H.P. or a + c = b.

Illustration 12. The sum of three numbers are in H.P. is 37 and the sum of their reciprocals is $\frac{1}{4}$. Find the numbers.

Solution. Three numbers are in H.P. can be taken as

$$\frac{1}{a-d}, \frac{1}{a}, \frac{1}{a+d}$$

then $\quad \frac{1}{a-d} + \frac{1}{a} + \frac{1}{a+d} = 37 \qquad \text{...(i)}$

and $\quad a - d + a + a + d = \frac{1}{4} \quad \Rightarrow \quad a = \frac{1}{12}$

from (i), $\quad \frac{12}{1-12d} + 12 + \frac{12}{1+12d} = 37 \quad \Rightarrow \quad \frac{12}{1-12d} + \frac{12}{1+12d} = 25$

$\Rightarrow \quad \frac{24}{1-144d^2} = 25 \quad \Rightarrow \quad 1 - 144d^2 = \frac{24}{25}$

$\Rightarrow \quad d^2 = \frac{1}{25 \times 144} \quad \therefore \quad d = \pm\frac{1}{60}$

$\therefore \quad a - d, a, a + d$ are $\frac{1}{15}, \frac{1}{12}, \frac{1}{10}$ or $\frac{1}{10}, \frac{1}{12}, \frac{1}{15}$

Hence, three numbers in H.P. are 15, 12, 10 or 10, 12, 15 **Ans.**

***Illustration 13.** Suppose a is a fixed real number such that $\frac{a-x}{px} = \frac{a-y}{qy} = \frac{a-z}{rz}$

If p, q, r are in A.P., then prove that x, y, z are in H.P.

Solution. $\because \quad$ p, q, r are in A.P.

$\therefore \quad q - p = r - q \qquad \text{.... (i)}$

$\Rightarrow \quad p - q = q - r = k$ (let)

given $\quad \frac{a-x}{px} = \frac{a-y}{qy} = \frac{a-z}{rz} \Rightarrow \frac{\frac{a}{x}-1}{p} = \frac{\frac{a}{y}-1}{q} = \frac{\frac{a}{z}-1}{r}$

$\Rightarrow \quad \frac{\left(\frac{a}{x}-1\right)-\left(\frac{a}{y}-1\right)}{p-q} = \frac{\left(\frac{a}{y}-1\right)-\left(\frac{a}{z}-1\right)}{q-r}$ (by law of proportion)

$\Rightarrow \quad \frac{\frac{a}{x}-\frac{a}{y}}{k} = \frac{\frac{a}{y}-\frac{a}{z}}{k} \quad \Rightarrow a\left(\frac{1}{x}-\frac{1}{y}\right) = a\left(\frac{1}{y}-\frac{1}{z}\right) \quad \Rightarrow \frac{1}{x}-\frac{1}{y} = \frac{1}{y}-\frac{1}{z}$

$\therefore \quad \frac{2}{y} = \frac{1}{x} + \frac{1}{z}$

$\therefore \quad \frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ are in A.P.

Hence x, y, z are in H.P.

7.0 MEANS

SL AL

7.1 Arithmetic Mean

If three terms are in A.P. then the middle term is called the A.M. between the other two, so if a, b, c are in A.P., b is A.M. of a & c. So A.M. of a and c = $\frac{a+c}{2}=b$.

- ***n- Arithmetic Means Between Two Numbers***

If a,b be any two given numbers & a, A_1, A_2, ..., A_n, b are in AP, then A_1, A_2,........A_n are the 'n' A.M's between a & b then. $A_1 = a + d$, $A_2 = a + 2d$,......., $A_n = a + nd$ or $b - d$, where $d=\frac{b-a}{n+1}$

$\Rightarrow A_1=a+\frac{b-a}{n+1}$, $A_2=a+\frac{2(b-a)}{n+1}$,.......

Note : Sum of n A.M's inserted between a & b is equal to n times the single A.M. between a & b i.e. $\sum_{r=1}^{n}A_r=nA$ where A is the single A.M. between a & b.

7.2 Geometric Mean

If a, b, c are in G.P., then b is the G.M. between a & c, $b^2 = ac$. So G.M. of a and c = $\sqrt{ac}=b$

- ***n- Geometric Means Between Two Numbers***

If a, b are two given positive numbers & a, G_1, G_2, ..., G_n, b are in G.P. Then G_1, G_2, G_3 ,.......G_n are 'n' G.Ms between a & b.

$G_1 = a(b/a)^{1/n+1} = ar$, $G_2 = a(b/a)^{2/n+1} = ar^2$, ..., $G_n = a(b/a)^{n/n+1} = ar^n = b/r$,

where $r = (b/a)^{1/n+1}$

Note – The product of n G.Ms between a & b is equal to n^{th} power of the single G.M. between a & b i.e.

$\prod_{r=1}^{n} G_r = (G)^n$ where G is the single G.M. between a & b

7.3 Harmonic Mean

If a, b, c are in H.P., then b is H.M. between a & c. So H.M. of a and c = $\frac{2ac}{a+c}$ = b.

- ***Insertion of 'n' HM's between a and b***

a, H_1, H_2, H_3 ,........., H_n, $b \rightarrow$ H.P

$\frac{1}{a},\frac{1}{H_1},\frac{1}{H_2},\frac{1}{H_3},..........\frac{1}{H_n},\frac{1}{b}\rightarrow$ A.P.

$\frac{1}{b}=\frac{1}{a}+(n+1)D \Rightarrow \quad D=\frac{\frac{1}{b}-\frac{1}{a}}{n+1}$

$$\frac{1}{H_n}=\frac{1}{a}+n\left(\frac{\frac{1}{b}-\frac{1}{a}}{n+1}\right)$$

BEGINNER'S BOX-2

TOPIC COVERED : H.P. AND MEANS

1. If the 7^{th} term of a H.P. is 8 and the 8^{th} term is 7. Then find the 28^{th} term.

2. In a H.P., if 5^{th} term is 6 and 3^{rd} term is 10. Find the 2^{nd} term.

3. If the p^{th}, q^{th} and r^{th} terms of a H.P. are a,b,c respectively, then prove that $\frac{q-r}{a}+\frac{r-p}{b}+\frac{p-q}{c}=0$.

***4.** If $a+b+c=6$ and $a, b, c \in R^+$ prove that $\frac{1}{a}+\frac{4}{b}+\frac{9}{c} \geq 6$

***5.** If $a+b+c=6$ and $a, b, c \in R^+$ prove that $\frac{1}{6-a}+\frac{1}{6-b}+\frac{1}{6-c} \geq \frac{3}{4}$

6. If x, y, z are in H.P. then the value of expression $\log(x+z)+\log(x-2y+z)$ will be :

(A) $\log|x-z|$ (B) $2\log|x-z|$ (C) $3\log|x-z|$ (D) $4\log|x-z|$

7. If H is the harmonic mean between p and q, then the value of $\frac{H}{p}+\frac{H}{q}$ is :

(A) 2 (B) $\frac{pq}{p+q}$ (C) $\frac{p+q}{pq}$ (D) None of these

8. Harmonic mean between the roots of the equation $x^2-10x+11$ is :

(A) $\frac{1}{5}$ (B) $\frac{5}{21}$ (C) $\frac{21}{20}$ (D) $\frac{11}{5}$

9. If $\frac{a^n+b^n}{a^{n-1}+b^{n-1}}$ is the G.M. between a & b then find the value of 'n'.

10. If b is the harmonic mean between a and c, then prove that $\frac{1}{b-a}+\frac{1}{b-c}=\frac{1}{a}+\frac{1}{c}$.

Illustrations

***Illustration 14.** If $2x^3+ax^2+bx+4=0$ (a and b are positive real numbers) has 3 real roots, then prove that $a+b \geq 6(2^{1/3}+4^{1/3})$.

Solution Let α, β, γ be the roots of $2x^3+ax^2+bx+4=0$. Given that all the coefficients are positive, so all the roots will be negative.

Let $\alpha_1=-\alpha,\ \alpha_2=-\beta,\ \alpha_3=-\gamma$

$$\Rightarrow \quad \alpha_1+\alpha_2+\alpha_3=\frac{a}{2}$$

$$\alpha_1\alpha_2+\alpha_2\alpha_3+\alpha_3\alpha_1=\frac{b}{2}$$

$$\alpha_1\alpha_2\alpha_3=2$$

Applying AM $\geq$ GM, we have

$$\frac{\alpha_1+\alpha_2+\alpha_3}{3} \geq (\alpha_1\alpha_2\alpha_3)^{1/3} \quad \Rightarrow \quad a \geq 6\times 2^{1/3}$$

Also $\frac{\alpha_1\alpha_2+\alpha_2\alpha_3+\alpha_1\alpha_3}{3} > (\alpha_1\alpha_2\alpha_3)^{2/3} \Rightarrow b \geq 6\times 4^{1/3}$

Therefore $a+b \geq 6(2^{1/3}+4^{1/3})$.

Illustration 15. If $a_i > 0\ \forall\ i \in N$ such that $\prod_{i=1}^{n} a_i = 1$, then prove that $(1 + a_1)(1 + a_2)(1 + a_3).....(1 + a_n) \geq 2^n$

Solution Using A.M. $\geq$ G.M.

$1 + a_1 \geq 2\sqrt{a_1}$

$1 + a_2 \geq 2\sqrt{a_2}$

$\vdots$

$1 + a_n \geq 2\sqrt{a_n}$

$\Rightarrow \quad (1 + a_1)(1 + a_2)........(1 + a_n) \geq 2^n(a_1a_2a_3.....a_n)^{1/2}$

As $a_1a_2a_3.....a_n = 1$

Hence $(1 + a_1)(1 + a_2)..........(1 + a_n) \geq 2^n$.

****Illustration 16.*** If a, b, x, y are positive natural numbers such that $\frac{1}{x} + \frac{1}{y} = 1$ then prove that $\frac{a^x}{x} + \frac{b^y}{y} \geq ab$.

Solution Consider the positive numbers $a^x, a^x,.......$y times and $b^y, b^y,......$x times

For all these numbers,

$$AM = \frac{\{a^x + a^x + ...y \text{ time}\} + \{b^y + b^y + ...x \text{ times}\}}{x + y} = \frac{ya^x + xa^y}{(x + y)}$$

$$GM = \left\{\left(a^x.a^x......y \text{ times}\right)\left(b^y.b^y......x \text{ times}\right)\right\}^{\frac{1}{(x+y)}} = \left[\left(a^{xy}\right).\left(b^{xy}\right)\right]^{\frac{1}{(x+y)}} = (ab)^{\frac{xy}{(x+y)}}$$

As $\frac{1}{x} + \frac{1}{y} = 1$, $\frac{x+y}{xy} = 1$, i.e, $x + y = xy$

So using AM $\geq$ GM $\frac{ya^x + xa^y}{x + y} \geq (ab)^{\frac{xy}{x+y}}$

$$\therefore \quad \frac{ya^x + xa^y}{xy} \geq ab \quad \text{or} \quad \frac{a^x}{x} + \frac{a^y}{y} \geq ab.$$

8.0 ARITHMETICO - GEOMETRIC SERIES

SL AL

A series, each term of which is formed by multiplying the corresponding term of an A.P. & G.P. is called the Arithmetico-Geometric Series , e.g. $1 + 3x + 5x^2 + 7x^3 + ...$

Here 1, 3, 5, ... are in A.P. & 1, x, x^2, x^3 .... are in G.P.

(a) Sum of n terms of an Arithmetico-Geometric Series

Let $S_n = a + (a + d)r + (a + 2d)r^2 + ... + [a + (n - 1)d]r^{n-1}$

then $S_n = \frac{a}{1-r} + \frac{dr(1 - r^{n-1})}{(1-r)^2} - \frac{[a + (n-1)d]\ \ r^n}{1-r}, \quad r \neq 1$

(b) Sum to Infinity

If $0 < |r| < 1$ & $n \to \infty$, then $\lim_{n\to\infty} r^n = 0$, $S_\infty = \frac{a}{1-r} + \frac{dr}{(1-r)^2}$

Illustrations

***Illustration 17.** Find the sum of series $4 - 9x + 16x^2 - 25x^3 + 36x^4 - 49x^5 + \ldots\ldots\ldots \infty$.

Solution Let $S = 4 - 9x + 16x^2 - 25x^3 + 36x^4 - 49x^5 + \ldots\ldots\ldots \infty$

$-Sx = -4x + 9x^2 - 16x^3 + 25x^4 - 36x^5 + \ldots\ldots\ldots \infty$

On subtraction, we get

$S(1 + x) = 4 - 5x + 7x^2 - 9x^3 + 11x^4 - 13x^5 + \ldots\ldots \infty$

$-S(1 + x)x = -4x + 5x^2 - 7x^3 + 9x^4 - 11x^5 + \ldots\ldots \infty$

On subtraction, we get

$S(1 + x)^2 = 4 - x + 2x^2 - 2x^3 + 2x^4 - 2x^5 + \ldots\ldots \infty$

$$= 4 - x + 2x^2 (1 - x + x^2 - \ldots\ldots\ldots \infty) = 4 - x + \frac{2x^2}{1+x} = \frac{4+3x+x^2}{1+x}$$

$$S = \frac{4+3x+x^2}{(1+x)^3}$$ ***Ans.***

Illustration 18. Find the sum of series upto n terms $\left(\frac{2n+1}{2n-1}\right) + 3\left(\frac{2n+1}{2n-1}\right)^2 + 5\left(\frac{2n+1}{2n-1}\right)^3 + \ldots\ldots\ldots$.

Solution For $x \neq 1$, let

$S = x + 3x^2 + 5x^3 + \ldots\ldots + (2n-3)x^{n-1} + (2n-1)x^n$ ... (i)

$\Rightarrow \quad xS = x^2 + 3x^3 + \ldots\ldots + (2n-5)x^{n-1} + (2n-3)x^n + (2n-1)x^{n+1}$ ... (ii)

Subtracting (ii) from (i), we get

$(1 - x)S = x + 2x^2 + 2x^3 + \ldots\ldots\ldots + 2x^{n-1} + 2x^n - (2n-1)x^{n+1}$

$$= x + \frac{2x^2(1-x^{n-1})}{1-x} - (2n-1)x^{n+1}$$

$$= \frac{x}{1-x}[1 - x + 2x - 2x^n - (2n-1)x^n + (2n-1)x^{n+1}]$$

$$\Rightarrow \quad S = \frac{x}{(1-x)^2}[(2n-1)x^{n+1} - (2n+1)x^n + 1 + x]$$

Thus $\left(\frac{2n+1}{2n-1}\right) + 3\left(\frac{2n+1}{2n-1}\right)^2 + \ldots\ldots + (2n-1)\left(\frac{2n+1}{2n-1}\right)^n$

$$= \left(\frac{2n+1}{2n-1}\right)\left(\frac{2n-1}{2}\right)^2\left[(2n-1)\left(\frac{2n+1}{2n-1}\right)^{n+1} - (2n+1)\left(\frac{2n+1}{2n-1}\right)^n + 1 + \frac{2n+1}{2n-1}\right]$$

$$= \frac{4n^2-1}{4}\cdot\frac{4n}{2n-1} = n(2n+1)$$ ***Ans.***

9.0 SIGMA NOTATIONS (Σ)

SL AL

- ***Theorems***

(a) $\sum_{r=1}^{n}(a_r \pm b_r) = \sum_{r=1}^{n} a_r \pm \sum_{r=1}^{n} b_r$ ***(b)*** $\sum_{r=1}^{n} ka_r = k\sum_{r=1}^{n} a_r$ ***(c)*** $\sum_{r=1}^{n} k = nk$; where k is a constant.

10.0 RESULTS

SL AL

(a) $\sum_{r=1}^{n} r = \frac{n(n+1)}{2}$ (sum of the first n natural numbers)

(b) $\sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$ (sum of the squares of the first n natural numbers)

(c) $\sum_{r=1}^{n} r^3 = \frac{n^2(n+1)^2}{4} = \left[\sum_{r=1}^{n} r\right]^2$ (sum of the cubes of the first n natural numbers)

(d) $\sum_{r=1}^{n} r^4 = \frac{n}{30}(n+1)(2n+1)(3n^2+3n-1)$

(e) $\sum_{r=1}^{n} (2r-1) = n^2$ (sum of first n odd natural numbers)

(f) $\sum_{r=1}^{n} 2r = n(n+1)$ (sum of first n even natural numbers)

Illustrations

Illustration 19. Sum up to 16 terms of the series $\frac{1^3}{1} + \frac{1^3+2^3}{1+3} + \frac{1^3+2^3+3^3}{1+3+5} + \ldots\ldots$ is

(A) 450 (B) 456 (C) 446 (D) none of these

Solution $t_n = \frac{1^3+2^3+3^3+\ldots.+n^3}{1+3+5+\ldots.(2n-1)}$

$$= \frac{\left\{\frac{n(n+1)}{2}\right\}^2}{\frac{n}{2}\{2+2(n-1)\}} = \frac{\frac{n^2(n+1)^2}{4}}{n^2} = \frac{(n+1)^2}{4}$$

$$= \frac{n^2}{4} + \frac{n}{2} + \frac{1}{4}$$

$$\therefore \quad S_n = \Sigma t_n = \frac{1}{4}\Sigma n^2 + \frac{1}{2}\Sigma n + \frac{1}{4}\Sigma 1$$

$$= \frac{1}{4}.\frac{n(n+1)(2n+1)}{6} + \frac{1}{2}.\frac{n(n+1)}{2} + \frac{1}{4}.n$$

$$\therefore \quad S_{16} = \frac{16.17.33}{24} + \frac{16.17}{4} + \frac{16}{4} = 446$$

Ans. (C)

11.0 METHOD OF DIFFERENCE

AL

Some times the n^{th} term of a sequence or a series can not be determined by the method, we have discussed earlier. So we compute the difference between the successive terms of given sequence for obtained the n^{th} terms.

If $T_1, T_2, T_3, \ldots\ldots, T_n$ are the terms of a sequence then some times the terms $T_2 - T_1, T_3 - T_2, \ldots\ldots$ constitute an AP/GP. n^{th} term of the series is determined & the sum to n terms of the sequence can easily be obtained.

- ***Case–1***
  - (a) If difference series are in A.P., then
    Let $T_n = an^2 + bn + c$, where a, b, c are constant
  - (b) If difference of difference series are in A.P.
    Let $T_n = an^3 + bn^2 + cn + d$, where a, b, c, d are constant
- ***Case–2***
  - (a) If difference are in G.P., then
    Let $T_n = ar^n + b$, where r is common ratio & a, b are constant
  - (b) If difference of difference are in G.P., then
    Let $T_n = ar^n + bn + c$, where r is common ratio & a, b, c are constant

Determine constant by putting n = 1, 2, 3 .... n and putting the value of T_1, T_2, T_3 ......

and sum of series $(S_n) = \sum T_n$

BEGINNER'S BOX-3

TOPIC COVERED : A.G.P., SIGMA NOTATION, METHOD OF DIFFERENCE

1. Find sum to n terms of the series $3 + 5 \times \frac{1}{4} + 7 \times \frac{1}{4^2} + \ldots\ldots$

2. If the sum to the infinity of the series $3 + 5r + 7r^2 + \ldots\ldots$ is $\frac{44}{9}$, then find the value of r.

3. If the sum to infinity of the series $3 + (3+d).\frac{1}{4} + (3+2d).\frac{1}{4^2} + \ldots\ldots$ is $\frac{44}{9}$ then find d.

4. Find the sum of the series upto n terms $1 + \frac{1+2}{2} + \frac{1+2+3}{3} + \frac{1+2+3+4}{4} + \ldots\ldots$

5. Find the sum of 'n' terms of the series whose n^{th} term is $t_n = 3n^2 + 2n$.

***6.** If $|x| < 1$, then the sum of the series $1 + 2x + 3x^2 + 4x^3 + \ldots\ldots \infty$, will be

(A) $\frac{1}{1-x}$ (B) $\frac{1}{1+x}$ (C) $\frac{1}{(1+x)^2}$ (D) $\frac{1}{(1-x)^2}$

7. sum of the squares of "n" natural numbers exceed their sum by 330, then n =

(A) 8 (B) 10 (C) 15 (D) 20

8. The sum of series 1 + (1+2) + (1+2+3) + ...up to n terms, will be :

(A) $n^2 - 2n + 6$ (B) $\frac{n(n+1)(2n-1)}{6}$ (C) $n^2 - 2n + 6$ (D) $\frac{n(n+1)(n+2)}{6}$

GOLDEN KEY POINTS

- If A, G, H, are respectively A.M., G.M., H.M. between two positive number a & b then
  (a) $G^2 = AH$ (A, G, H constitute a GP) (b) $A \geq G \geq H$ (c) $A = G = H \Leftrightarrow a = b$
- Let $a_1, a_2, \ldots\ldots, a_n$ be n positive real numbers, then we define their arithmetic mean (A), geometric mean (G) and harmonic mean (H) as $A = \dfrac{a_1 + a_2 + \ldots.. + a_n}{n}$

  $G = (a_1 a_2 \ldots\ldots\ldots a_n)^{1/n}$ and $H = \dfrac{n}{\left(\dfrac{1}{a_1} + \dfrac{1}{a_2} + \dfrac{1}{a_3} + \ldots. \dfrac{1}{a_n}\right)}$

  It can be shown that $A \geq G \geq H$. Moreover equality holds at either place if and only if $a_1 = a_2 = \ldots\ldots = a_n$
- If n^{th} term of a sequence is given by $T_n = an^3 + bn^2 + cn + d$ where a, b, c, d are constants,
  then sum of n terms $S_n = \Sigma T_n = a\Sigma n^3 + b\Sigma n^2 + c\Sigma n + \Sigma d$
  This can be evaluated using the above results.

SOME WORKED OUT ILLUSTRATIONS

Illustration 1. If $\sum_{r=1}^{n} T_r = \frac{n}{8}(n+1)(n+2)(n+3)$, then find $\sum_{r=1}^{n} \frac{1}{T_r}$.

Solution $\because \quad T_n = S_n - S_{n-1}$

$$= \sum_{r=1}^{n} T_r - \sum_{r=1}^{n-1} T_r$$

$$= \frac{n(n+1)(n+2)(n+3)}{8} - \frac{(n-1)n(n+1)(n+2)}{8}$$

$$= \frac{n(n+1)(n+2)}{8}[(n+3)-(n-1)]$$

$$T_n = \frac{n(n+1)(n+2)}{8}(4) = \frac{n(n+1)(n+2)}{2}$$

$$\Rightarrow \quad \frac{1}{T_n} = \frac{2}{n(n+1)(n+2)} = \frac{(n+2)-n}{n(n+1)(n+2)} = \frac{1}{n(n+1)} - \frac{1}{(n+1)(n+2)} \quad \text{.... (i)}$$

Let $V_n = \frac{1}{n(n+1)}$

$$\therefore \quad \frac{1}{T_n} = V_n - V_{n+1}$$

Putting $n = 1, 2, 3, ... n$

$$\Rightarrow \quad \frac{1}{T_1} + \frac{1}{T_2} + \frac{1}{T_3} + ... + \frac{1}{T_n} = (V_1 - V_{n+1})$$

$$\Rightarrow \quad \sum_{r=1}^{n} \frac{1}{T_r} = \frac{n^2+3n}{2(n+1)(n+2)}$$

Illustration 2. Find the sum of n terms of the series $1 . 3 . 5 + 3 . 5 . 7 + 5 . 7 . 9 + ...$

Solution The n^{th} term is $(2n - 1)(2n + 1)(2n + 3)$

$T_n = (2n - 1)\ (2n + 1)\ (2n + 3)$

$$T_n = \frac{1}{8}(2n-1)\ (2n+1)\ (2n+3)\ \{(2n+5) - (2n-3)\}$$

$$= \frac{1}{8}(V_n - V_{n-1}) \quad [\text{Let } V_n = (2n-1)\ (2n+1)\ (2n+3)\ (2n+5)]$$

$$S_n = \sum T_n = \frac{1}{8}[V_n - V_0]$$

$$\therefore \quad S_n = \frac{(2n-1)(2n+1)(2n+3)(2n+5)}{8} + \frac{15}{8}$$

$$= n\ (2n^3 + 8n^2 + 7n - 2)$$ ***Ans.***

Illustration 3. Find the sum of n terms of the series 3 + 7 + 14 + 24 + 37 + ...

Solution Clearly here the differences between the successive terms are

7 – 3, 14 – 7, 24 – 14, ... i.e. 4, 7, 10, 13,........., which are in A.P.

Let $S = 3 + 7 + 14 + 24 + ... + T_n$

$S = 3 + 7 + 14 + ... + T_{n-1} + T_n$

Subtracting, we get

$0 = 3 + [4 + 7 + 10 + 13 + ... (n-1) \text{ terms}] - T_n$

$\therefore \quad T_n = 3 + S_{n-1}$ of an A.P. whose a = 4 and d = 3.

$$\therefore \quad T_n = 3 + \left(\frac{n-1}{2}\right)(2.4 + (n-2)3) = \frac{6 + (n-1)(3n+2)}{4}$$

or $\quad T_n = \frac{1}{2}(3n^2 - n + 4)$

Now putting n = 1, 2, 3,........, n and adding

$$\therefore \quad S_n = \frac{1}{2}[3\Sigma n^2 - \Sigma n + 4n] = \frac{1}{2}\left[3\frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2} + 4n\right] = \frac{n}{2}(n^2 + n + 4)$$ ***Ans.***

Aliter Method

Let $T_n = an^2 + bn + c$

Now, $T_1 = 3 = a + b + c$ ...(i)

$T_2 = 7 = 4a + 2b + c$ ...(ii)

$T_3 = 14 = 8a + 3b + c$ ...(iii)

Solving (i), (ii) & (iii) we get

$a = \frac{3}{2}, b = -\frac{1}{2}$ & c = 2

$\therefore \quad T_n = \frac{1}{2}(3n^2 - n + 4)$

$$\Rightarrow \quad s_n = \Sigma T_n = \frac{1}{2}[3\Sigma n^2 - \Sigma n + 4n] = \frac{1}{2}\left[3\frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2} + 4n\right] = \frac{n}{2}(n^2 + n + 4)$$

Ans.

Illustration 4. Find the sum of n-terms of the series 1 + 4 + 10 + 22 + ...

Solution Let $S = 1 + 4 + 10 + 22 + ... + T_n$ ... (i)

$S = 1 + 4 + 10 + ... + T_{n-1} + T_n$ ... (ii)

(i) – (ii) $\Rightarrow T_n = 1 + (3 + 6 + 12 + ... + T_n - T_{n-1})$

$T_n = 1 + 3\left(\frac{2^{n-1} - 1}{2 - 1}\right)$

$T_n = 3 . 2^{n-1} - 2$

So $\quad S_n = \Sigma T_n = 3\Sigma 2^{n-1} - \Sigma 2 = 3\left(\frac{2^n - 1}{2 - 1}\right) - 2n = 3.2^n - 2n - 3$ ***Ans.***

Aliter Method

Let $T_n = ar^n + b$, where r = 2

Now $T_1 = 1 = ar + b$ ...(i)

$T_2 = 4 = ar^2 + b$ ...(ii)

Solving (i) & (ii), we get

$a = \frac{3}{2}, b = -2 \qquad \therefore T_n = 3.2^{n-1} - 2$

$\Rightarrow \quad S_n = \Sigma T_n = 3\Sigma 2^{n-1} - \Sigma 2 = 3\left(\frac{2^n - 1}{2 - 1}\right) - 2n = 3.2^n - 2n - 3$ ***Ans.***

Illustration 5. The series of natural numbers is divided into groups (1), (2, 3, 4), (5, 6, 7, 8, 9) .... and so on. Show that the sum of the numbers in n^{th} group is $n^3 + (n-1)^3$

Solution The groups are (1), (2, 3, 4), (5, 6, 7, 8, 9) ....

The number of terms in the groups are 1, 3, 5......

$\therefore$ The number of terms in the n^{th} group $= (2n-1)$

the last term of the n^{th} group is n^2

If we count from last term common difference should be -1

So the sum of numbers in the n^{th} group $= \left(\dfrac{2n-1}{2}\right)\left\{2n^2 + (2n-2)(-1)\right\}$

$= (2n-1)(n^2 - n + 1) = 2n^3 - 3n^2 + 3n - 1 = n^3 + (n-1)^3$

****Illustration 6.*** Find the natural number 'a' for which $\sum\limits_{k=1}^{n} f(a+k) = 16(2^n - 1)$, where the function f satisfied $f(x+y) = f(x).\, f(y)$ for all natural number x,y and further $f(1) = 2$.

Solution It is given that

$f(x+y) = f(x)\, f(y)$ and $f(1) = 2$

$f(1+1) = f(1)\, f(1) \Rightarrow f(2) = 2^2,\ f(1+2)$

$= f(1)\, f(2) \Rightarrow f(3) = 2^3,\quad f(2+2) = f(2)\, f(2) \Rightarrow f(4) = 2^4$

Similarly $f(k) = 2^k$ and $f(a) = 2^a$

Hence, $\sum\limits_{k=1}^{n} f(a+k) = \sum\limits_{k=1}^{n} f(a)f(k) = f(a)\sum\limits_{k=1}^{n} f(k)$

$= 2^a \sum\limits_{k=1}^{n} 2^k = 2^a\{2^1 + 2^2 + \ldots\ldots\ldots + 2^n\} = 2^a\left\{\dfrac{2(2^n-1)}{2-1}\right\} = 2^{a+1}(2^n - 1)$

But $\sum\limits_{k=1}^{n} f(a+k) = 16(2^n - 1)$

$2^{a+1}(2^n - 1) = 16(2^n - 1)$

$\therefore\quad 2^{a+1} = 2^4$

$\therefore\quad a+1 = 4 \quad\Rightarrow\quad a = 3$ ***Ans.***

ANSWER KEY

BEGINNER'S BOX-1

1. ***(a)*** $\dfrac{2}{1}, \dfrac{4}{2}, \dfrac{8}{3}, \dfrac{16}{4}, \ldots\ldots,$ ***(b)*** $\dfrac{2}{3}, \dfrac{4}{9}, \dfrac{2}{27}, \dfrac{4}{81}, \ldots\ldots;$ **3.** $-(p+q)$ **4.** 900

5. 14 **6.** 931 **7.** 4^5 **9.** 4, 12, 36 **10.** C

BEGINNER'S BOX-2

1. 2 **2.** 15 **6.** (B) **7.** (A) **8.** (D) **9.** $\dfrac{1}{2}$

BEGINNER'S BOX-3

1. $4 + \dfrac{8}{9}\left(1 - \dfrac{1}{4^{n-1}}\right) - \left(\dfrac{2n+1}{3 \times 4^{n-1}}\right)$ **2.** $\dfrac{1}{4}$ **3.** 2 **4.** $\dfrac{n(n+3)}{4}$

5. $\dfrac{n(n+1)(2n+3)}{2}$ **6.** (D) **7.** (B) **8.** (D)

EXERCISE – 1 MCQ (SINGLE CHOICE CORRECT)

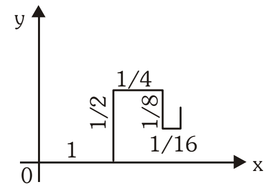
1. The maximum value of the sum of the A.P. 50, 48, 46, 44, ... is -
(A) 325 (B) 648 (C) 650 (D) 652

2. Let T_r be the r^{th} term of an A.P. for r = 1, 2, 3, ... If for some positive integers m, n we have $T_m = \frac{1}{n}$ & $T_n = \frac{1}{m}$, then T_{mn} equals -
(A) $\frac{1}{mn}$ (B) $\frac{1}{m}+\frac{1}{n}$ (C) 1 (D) 0

***3.** The interior angles of a convex polygon are in AP. The smallest angle is 120° & the common difference is 5°. Find the number of sides of the polygon -
(A) 9 (B) 16 (C) 12 (D) None of these

4. The first term of an infinitely decreasing G.P. is unity and its sum is S. The sum of the squares of the terms of the progression is -
(A) $\frac{S}{2S-1}$ (B) $\frac{S^2}{2S-1}$ (C) $\frac{S}{2-S}$ (D) S^2

5. A particle begins at the origin and moves successively in the following manner as shown, 1 unit to the right, 1/2 unit up, 1/4 unit to the right, 1/8 unit down, 1/16 unit to the right etc. The length of each move is half the length of the previous move and movement continues in the 'zigzag' manner indefinitely. The co-ordinates of the point to which the 'zigzag' converges is -



(A) (4/3, 2/3) (B) (4/3, 2/5) (C) (3/2, 2/3) (D) (2, 2/5)

6. The sum to n terms of the series, $\frac{1}{2}+\frac{3}{4}+\frac{7}{8}+\frac{15}{16}+$...... is equal to :
(A) $2^n - n - 1$ (B) $1 - 2^{-n}$ (C) $2^{-n} + n - 1$ (D) $2^n - 1$

7. If p, q, r in harmonic progression and p & r be different having same sign then the roots of the equation $px^2 + qx + r = 0$ are -
(A) real and equal (B) real and distinct (C) irrational (D) imaginary

8. Consider the ten numbers
$ar, ar^2, ar^3, ... ar^{10}$.
If their sum is 18 and the sum of their reciprocals is 6 then the product of these ten numbers, is
(A) 324 (B) 343 (C) 243 (D) 729

***9.** If the $(m + 1)^{th}$, $(n + 1)^{th}$ & $(r + 1)^{th}$ terms of an AP are in GP & m, n, r are in HP, then the ratio of the common difference to the first term of the AP is
(A) $\frac{1}{n}$ (B) $\frac{2}{n}$ (C) $-\frac{2}{n}$ (D) none of these

10. The sum of roots of the equation $ax^2 + bx + c = 0$ is equal to the sum of squares of their reciprocals. Then bc^2, ca^2 and ab^2 are in -
(A) AP (B) GP (C) HP (D) none of these

11. The quadratic equation whose roots are the A.M. and H.M. between the roots of the equation, $2x^2 - 3x + 5 = 0$ is -

(A) $4x^2 - 25x + 10 = 0$ (B) $12x^2 - 49x + 30 = 0$

(C) $14x^2 - 12x + 35 = 0$ (D) $2x^2 + 3x + 5 = 0$

12. If the sum of the first n natural numbers is 1/5 times the sum of the their squares, then the value of n is -

(A) 5 (B) 6 (C) 7 (D) 8

13. If $S_n = \frac{1}{1^3} + \frac{1+2}{1^3+2^3} + \ldots\ldots + \frac{1+2+3+\ldots\ldots+n}{1^3+2^3+3^3+\ldots\ldots+n^3}$, $n = 1, 2, 3, \ldots\ldots$ Then S_n is not greater than

(A) 1/2 (B) 1 (C) 2 (D) 4

14. If $3 + \frac{1}{4}(3 + d) + \frac{1}{4^2}(3 + 2d) + \ldots\ldots +$ upto $\infty = 8$, then the value of d is :

(A) 9 (B) 5 (C) 1 (D) none of these

15. The sum $\sum_{k=1}^{\infty} \frac{2^{k+2}}{3^k}$ equal to

(A) 12 (B) 8 (C) 6 (D) 4

***16.** If $\sum_{s=1}^{n}\left\{\sum_{r=1}^{s} r\right\} = an^3 + bn^2 + cn$, then find the value of a + b + c.

(A) 1 (B) 0 (C) 2 (D) 3

17. If the sum of the first 11 terms of an arithmetical progression equals that of the first 19 terms, then the sum of its first 30 terms, is

(A) equal to 0 (B) equal to – 1 (C) equal to 1 (D) non unique

18. Consider a decreasing G.P. : $g_1, g_2, g_3, \ldots\ldots g_n \ldots\ldots.$ such that $g_1 + g_2 + g_3 = 13$ and $g_1^2 + g_2^2 + g_3^2 = 91$ then which of the following does not hold?

(A) The greatest term of the G.P. is 9. (B) $3g_4 = g_3$

(C) $g_1 = 1$ (D) $g_2 = 3$

***19.** $\frac{1+3+5+\ldots\text{upto } n \text{ terms}}{4+7+10+\ldots\text{upto } n \text{ terms}} = \frac{20}{7\log_{10} x}$ and $n = \log_{10} x + \log_{10} x^{\frac{1}{2}} + \log_{10} x^{\frac{1}{4}} + \log_{10} x^{\frac{1}{8}} + \ldots\ldots + \infty$, then x is equal to

(A) 10^3 (B) 10^5 (C) 10^6 (D) 10^7

20. The sum to n terms of the series $\frac{3}{1^2} + \frac{5}{1^2+2^2} + \frac{7}{1^2+2^2+3^2} + \ldots\ldots$ is -

(A) $\frac{3n}{n+1}$ (B) $\frac{6n}{n+1}$ (C) $\frac{9n}{n+1}$ (D) $\frac{12n}{n+1}$

EXERCISE – 2 MCQ (ONE OR MORE CHOICE CORRECT)

1. Consider an A.P. with first term 'a' and the common difference d. Let S_k denote the sum of the first K terms. Let $\frac{S_{kx}}{S_x}$ is independent of x, then -

(A) a = d/2 (B) a = d (C) a = 2d (D) none of these

2. $\sum_{r=1}^{\infty}(2r-1)\left(\frac{9}{11}\right)^r$ is equal to -

(A) 45 (B) 55

(C) sum of first nine natural numbers (D) sum of first ten natural numbers

3. For the A.P. given by $a_1, a_2, \ldots\ldots, a_n, \ldots\ldots$, with non-zero common difference, the equations satisfied are-

(A) $a_1 + 2a_2 + a_3 = 0$ (B) $a_1 - 2a_2 + a_3 = 0$

(C) $a_1 + 3a_2 - 3a_3 - a_4 = 0$ (D) $a_1 - 4a_2 + 6a_3 - 4a_4 + a_5 = 0$

***4.** If a, $a_1, a_2, \ldots, a_{10}$, b are in A.P. and a, $g_1, g_2, \ldots g_{10}$, b are in G.P. and h is the H.M. between a and b, then $\frac{a_1+a_2+\ldots+a_{10}}{g_1g_{10}}+\frac{a_2+a_3+\ldots+a_9}{g_2g_9}+\ldots\ldots+\frac{a_5+a_6}{g_5g_6}$ is -

(A) $\frac{10}{h}$ (B) $\frac{15}{h}$ (C) $\frac{30}{h}$ (D) $\frac{5}{h}$

***5.** The sum of the first n terms of the series $1^2 + 2.2^2 + 3^2 + 2.4^2 + 5^2 + 2.6^2 + \ldots\ldots$ is $\frac{n(n+1)^2}{2}$, when n is even. When n is odd, the sum is -

(A) $\frac{n^2(n+1)}{2}$ (B) $\frac{n(n+1)(2n+1)}{6}$ (C) $\frac{n(n+1)^2}{2}$ (D) $\frac{n^2(n+1)^2}{2}$

6. The sum of the first 100 terms common to the series 17, 21, 25, ... and 16, 21, 26, ...is -

(A) 101100 (B) 111000 (C) 110010 (D) 100101

***7.** If a, b, c are positive such that $ab^2c^3 = 64$ then least value of $\left(\frac{1}{a}+\frac{2}{b}+\frac{3}{c}\right)$ is -

(A) 6 (B) 2 (C) 3 (D) 32

8. If the AM of two positive numbers be three times their geometric mean then the ratio of the numbers is -

(A) $3\pm2\sqrt{2}$ (B) $\sqrt{2}\pm1$ (C) $17+12\sqrt{2}$ (D) $\left(3-2\sqrt{2}\right)^{-2}$

9. The p^{th} term T_p of H.P. is q(q + p) and q^{th} term T_q is p(p + q) when p > 1, q > 1, then -

(A) $T_{p+q} = pq$ (B) $T_{pq} = p + q$ (C) $T_{p+q} > T_{pq}$ (D) $T_{pq} > T_{p+q}$

10. We inscribe a square in a circle of unit radius and shade the region between them. Then we inscribe another circle in the square and another square in the new circle and shade the region between the new circle and the square. If the process is repeated infinitely many times, the area of the shaded region.

(A) 2π (B) $3(\pi-2)$ (C) $2(\pi-2)$ (D) None of these

Match the column

Following questions contains statements given in two columns, which have to be matched. The statements in ***Column-I*** are labelled as A, B, C and D while the statements in ***Column-II*** are labelled as p, q, r and s. Any given statement in ***Column-I*** can have correct matching with ***ONE*** statement in ***Column-II***.

****11.***

	Column-I		Column-II
(A)	n^{th} term of the series 4, 11, 22, 37, 56, 79,.......	(p)	$2n^2 + n$
(B)	$\lvert 1^2 - 2^2 + 3^2 - 4^2 .... 2n \text{ terms}\rvert$ is equal to	(q)	$2n^2 + n + 1$
(C)	sum to n terms of the series 3, 7, 11, 15,....... is	(r)	$-(n^2 + n)$
(D)	coefficient of x^n in $2x(x-1)(x-2)$ .... $(x-n)$ is	(s)	$\frac{1}{2}(n^2+n)$

Comprehension Based Questions

Consider two different infinite geometric progressions with their sums S_1 and S_2 as

$S_1 = a + ar + ar^2 + ar^3 + ... \infty$; $S_2 = b + bR + bR^2 + bR^3 + ... \infty$

If $S_1 = S_2 = 1$, $ar = bR$ and $ar^2 = \frac{1}{8}$ then answer the following:

12. The sum of their common ratios is

(A) $\frac{1}{2}$ (B) $\frac{3}{4}$ (C) 1 (D) $\frac{3}{2}$

13. The sum of their first terms is

(A) 1 (B) 2 (C) 3 (D) none of these

14. Common ratio of the first G.P. is

(A) $\frac{1}{2}$ (B) $\frac{1-\sqrt{5}}{4}$ (C) $\frac{\sqrt{5}-1}{4}$ (D) $\frac{\sqrt{5}+1}{4}$

15. Common ratio of the second G.P. is

(A) $\frac{3+\sqrt{5}}{4}$ (B) $\frac{3-\sqrt{5}}{4}$ (C) $\frac{1}{2}$ (D) none of these

EXERCISE - 3 SUBJECTIVE

1. Given that $a^x = b^y = c^z = d^u$ & a, b, c, d are in GP, show that x, y, z, u are in HP.

2. There are n AM's between 1 & 31 such that 7th mean : (n – 1)th mean = 5 : 9, then find the value of n.

3. Find the sum of the series, 7 + 77 + 777 + ... to n terms.

***4.** If the p^{th}, q^{th} & r^{th} terms of an AP are in GP. Show that the common ratio of the GP is $\frac{q-r}{p-q}$.

5. Express the recurring decimal $0.1\overline{576}$ as a rational number using concept of infinite geometric series.

6. If one AM 'a' & two GM's p & q be inserted between any two given numbers then show that $p^3 + q^3 = 2apq$.

***7.** Find three numbers a, b, c between 2 & 18 which satisfy following conditions :

(i) their sum is 25

(ii) the numbers 2, a, b are consecutive terms of an AP &

(iii) the numbers b, c, 18 are consecutive terms of a GP.

8. Find the sum of the first n terms of the series : $1 + 2\left(1+\frac{1}{n}\right) + 3\left(1+\frac{1}{n}\right)^2 + 4\left(1+\frac{1}{n}\right)^3 + \ldots\ldots\ldots$

***9.** Let $a_1, a_2, a_3 \ldots\ldots a_n$ be an AP. Prove that :

$$\frac{1}{a_1 a_n} + \frac{1}{a_2 a_{n-1}} + \frac{1}{a_3 a_{n-2}} + \ldots\ldots\ldots + \frac{1}{a_n a_1} = \frac{2}{a_1 + a_n}\left[\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \ldots\ldots\ldots + \frac{1}{a_n}\right]$$

10. Find all x such that $\sum_{k=1}^{\infty} k \cdot x^k = 20$.

11. x_1, x_2 are the roots of the equation $x^2 - 3x + A = 0$; x_3, x_4 are roots of the equation $x^2 - 12x + B = 0$, such that x_1, x_2, x_3, x_4 form an increasing G.P., then find the value of A+B

12. Let S be the set of integers which are divisible by 5, and let T be the set of integer which are divisible by 7. Find the number of positive integers less than 1000 and not in $(S \cup T)$.

EXERCISE – 4 RECAP OF AIEEE/JEE (MAIN)

1. The sum of first 20 terms of the sequence 0.7, 0.77, 0.777, ..., is : **[JEE-MAIN 2013]**

(A) $\frac{7}{81}(179-10^{-20})$ (B) $\frac{7}{9}(99-10^{-20})$ (C) $\frac{7}{81}(179+10^{-20})$ (D) $\frac{7}{9}(99-10^{-20})$

***2.** If m is the A.M. of two distinct real numbers l and n (l, n> 1) and G_1, G_2 and G_3 are three geometric means between l and n, then $G_1^4+2G_2^4+G_3^4$ equals : **[JEE-MAIN 2015]**

(A) $4l^2mn$ (B) $4lm^2n$ (C) $4lmn^2$ (D) $4l^2m^2n^2$

3. The sum of first 9 terms of the series $\frac{1^3}{1}+\frac{1^3+2^3}{1+3}+\frac{1^3+2^3+3^3}{1+3+5}+....$ is : **[JEE-MAIN 2015]**

(A) 71 (B) 96 (C) 142 (D) 192

4. If the 2nd, 5th and 9th terms of a non-constant A.P. are in G.P., then the common ratio of this G.P. is : **[JEE-MAIN 2016]**

(A) $\frac{8}{5}$ (B) $\frac{4}{3}$ (C) 1 (D) $\frac{7}{4}$

5. If the sum of the first ten terms of the series $\left(1\frac{3}{5}\right)^2+\left(2\frac{2}{5}\right)^2+\left(3\frac{1}{5}\right)^2+4^2+\left(4\frac{4}{5}\right)^2+.........$ is $\frac{16}{5}$m, then m is equal to : **[JEE-MAIN 2016]**

(A) 102 (B) 101 (C) 100 (D) 99

6. For any three positive real numbers a, b and c $9(25a^2+b^2)+25(c^2–3ac) = 15b(3a+c)$. Then : **[JEE-MAIN 2017]**

(A) a, b and c are in A.P (B) a, b and c are in G.P
(C) b, c and a are in G.P (D) b, c and a are in A.P

7. Let $a_1, a_2, a_3, ... , a_{49}$ be in A.P. such that $\sum_{k=0}^{12} a_{4k+1}=416$ and $a_9 + a_{43} = 66$. If $a_1^2+a_2^2+......+a_{17}^2$ = 140m, then m is equal to- **[JEE-MAIN 2018]**

(A) 68 (B) 34 (C) 33 (D) 66

8. If 19th term of a non–zero A.P. is zero, then its (49th term) : (29th term) is : **[JEE-MAIN 2019]**

(A) 3 : 1 (B) 4 : 1 (C) 2 : 1 (D) 1 : 3

9. The sum of an infinite geometric series with positive terms is 3 and the sum of the cubes of its terms is $\frac{237}{19}$. Then the common ratio of this series is : **[JEE-MAIN 2019]**

(A) $\frac{4}{9}$ (B) $\frac{2}{9}$ (C) $\frac{2}{3}$ (D) $\frac{1}{3}$

10. Let $a_1, a_2, ... , a_{10}$ be a G.P. If $\frac{a_3}{a_1}=25$, then $\frac{a_9}{a_5}$ equals : **[JEE-MAIN 2019]**

(A) $2(5^2)$ (B) $4(5^2)$ (C) 5^4 (D) 5^3

11. The sum of the follwing series $1+6+\frac{9(1^2+2^2+3^2)}{7}+\frac{12(1^2+2^2+3^2+4^2)}{9}+\frac{15(1^2+2^2+...+5^2)}{11}+...$ up to 15 terms, is : ***[JEE-MAIN 2019]***

(A) 7820 (B) 7830 (C) 7520 (D) 7510

12. Let a, b and c be the 7th, 11th and 13th terms respectively of a non–constant A.P. If these are also the three consecutive terms of a G.P., then $\frac{a}{c}$ is equal to : ***[JEE-MAIN 2019]***

(A) $\frac{1}{2}$ (B) 4 (C) 2 (D) $\frac{7}{13}$

13. If a, b and c be three distinct real numbers in G. P. and a + b + c = xb, then x cannot be ***[JEE-MAIN 2019]***

(A) 4 (B) –3 (C) –2 (D) 2

14. Let $a_1, a_2, \ldots, a_{30}$ be an A. P., $S=\sum_{i=1}^{30} a_i$ and $T=\sum_{i=1}^{15} a_{(2i-1)}$. If $a_5 = 27$ and $S - 2T = 75$, then a_{10} is equal to ***[JEE-MAIN 2019]***

(A) 57 (B) 47 (C) 42 (D) 52

15. If the sum of the first 15 terns of the series $\left(\frac{3}{4}\right)^3+\left(1\frac{1}{2}\right)^3+\left(2\frac{1}{4}\right)+3^3+\left(3\frac{3}{4}\right)^3+...$ is equal to 225 k, then k is equal to : ***[JEE-MAIN 2019]***

(A) 9 (B) 27 (C) 108 (D) 54

16. The product of three consecutive terms of a G.P. is 512. If 4 is added to each of the first and the second of these terms, the three terms now from an A.P. Then the sum of the original three terms of the given G.P. is

(A) 36 (B) 24 (C) 32 (D) 28

17. Let $S_k=\frac{1+2+3+...+k}{k}$. If $S_1^2+S_2^2+S_3^2+...+S_{10}^2=\frac{5}{12}A$, then A is equal to: ***[JEE-MAIN 2019]***

(A) 303 (B) 283 (C) 156 (D) 301

18. Let x, y be positive real numbers and m, n positive integers. The maximum value of the expression $\frac{x^m y^n}{(1+x^{2m})(1+y^{2n})}$ is : ***[JEE-MAIN 2019]***

(A) $\frac{1}{2}$ (B) $\frac{1}{4}$ (C) $\frac{m+n}{6mn}$ (D) 1

19. The sum of all natural numbers 'n' such that 100 < n < 200 and H.C.F. (91, n) > 1 is : ***[JEE-MAIN 2019]***

(A) 3221 (B) 3121 (C) 3203 (D) 3303

20. The sum $\sum_{k=1}^{20} k\frac{1}{2^k}$ is equal to- ***[JEE-MAIN 2019]***

(A) $2-\frac{3}{2^{17}}$ (B) $2-\frac{11}{2^{19}}$ (C) $1-\frac{11}{2^{20}}$ (D) $2-\frac{21}{2^{20}}$

21. If three distinct numbers a,b,c are in G.P. and the equations $ax^2 + 2bx + c = 0$ and $dx^2 + 2ex + f = 0$ have a common root, then which one of the following statements is correct? ***[JEE-MAIN 2019]***

(A) d,e,f are in A.P. (B) $\frac{d}{a},\frac{e}{b},\frac{f}{c}$ are in G.P. (C) $\frac{d}{a},\frac{e}{b},\frac{f}{c}$ are in A.P. (D) d, e, f are in G.P.

22. Let the sum of the first n terms of a non-constant A.P., $a_1, a_2, a_3, \ldots$ be $50n+\frac{n(n-7)}{2}A$, wherre A is a constant. If d is the common difference of this A.P., then the ordered pair (d, a_{50}) is equal to ***[JEE-MAIN 2019]***

(A) (A, 50+46A) (B) (A, 50+45A) (C) (50, 50+46A) (D) (50, 50+45A)

23. Some identical balls are arranged in rows to form an equilateral triangle. The first row consists of one ball, the second row consists of two balls and so on. If 99 more identical balls are addded to the total number of balls used in forming the equilaterial triangle, then all these balls can be arranged in a square whose each side contains exactly 2 balls less than the number of balls each side of the triangle contains. Then the number of balls used to form the equilateral triangle is :- ***[JEE-MAIN 2019]***

(A) 190 (B) 262 (C) 225 (D) 157

24. If the sum and product of the first three term in an A.P. are 33 and 1155, respectively, then a value of its 11^{th} term is :- ***[JEE-MAIN 2019]***

(A) –25 (B) 25 (C) –36 (D) –35

25. The sum of the series $1 + 2 \times 3 + 3 \times 5 + 4 \times 7 + \ldots$ upto 11^{th} term is :- ***[JEE-MAIN 2019]***

(A) 915 (B) 946 (C) 945 (D) 916

26. The sum $\frac{3\times 1^3}{1^2}+\frac{5\times\left(1^3+2^3\right)}{1^2+2^2}+\frac{7\times\left(1^3+2^3+3^3\right)}{1^2+2^2+3^2}+\ldots\ldots$ upto 10 terms,is ***[JEE-MAIN 2019]***

(A) 660 (B) 620 (C) 680 (D) 600

27. If $a_1, a_2, a_3, \ldots\ldots\ldots, a_n$ are in A.P. and $a_1 + a_4 + a_7 + \ldots\ldots\ldots + a_{16} = 114$, then $a_1 + a_6 + a_{11} + a_{16}$ is equal to : ***[JEE-MAIN 2019]***

(A) 38 (B) 98 (C) 76 (D) 64

28. The sum $1 + \frac{1^3+2^3}{1+2}+\frac{1^3+2^3+3^3}{1+2+3}+\ldots+\frac{1^3+2^3+3^3+\ldots+15^3}{1+2+3+\ldots+15}-\frac{1}{2}(1+2+3+\ldots+15)$] is ***[JEE-MAIN 2019]***

(A) 1240 (B) 1860 (C) 660 (D) 620

29. Let a, b and c be in G. P. with common ratio r, where $a \neq 0$ and $0 < r \leq \frac{1}{2}$. If 3a, 7b and 15c are the first three terms of an A. P., then the 4^{th} term of this A. P. is : ***[JEE-MAIN 2019]***

(A) $\frac{7}{3}a$ (B) a (C) $\frac{2}{3}a$ (D) 5a

30. Let $a_1, a_2, a_3, \ldots\ldots$ be an A. P. with $a_6 = 2$. Then the common difference of this A. P., which maximises the produce $a_1a_4a_5$, is : ***[JEE-MAIN 2019]***

(A) $\frac{6}{5}$ (B) $\frac{8}{5}$ (C) $\frac{2}{3}$ (D) $\frac{3}{2}$

31. If α and β are the roots of the equation $375x^2 - 25x - 2 = 0$, then $\lim\limits_{n\to\infty}\sum\limits_{r=1}^{n}\alpha^r + \lim\limits_{n\to\infty}\sum\limits_{r=1}^{n}\beta^r$ is equal to :

[JEE-MAIN 2019]

(A) $\frac{21}{346}$ (B) $\frac{29}{358}$ (C) $\frac{1}{12}$ (D) $\frac{7}{116}$

32. Let S_n denote the sum of the first n terms of an A.P. If $S_4 = 16$ and $S_6 = -48$, then S_{10} is equal to :

[JEE-MAIN 2019]

(A) –320 (B) –260 (C) –380 (D) –410

33. If $a_1, a_2, a_3, \ldots\ldots$ are in A.P. such that $a_1 + a_7 + a_{16} = 40$, then the sum of the first 15 terms of this A.P. is :

[JEE-MAIN 2019]

(A) 200 (B) 280 (C) 120 (D) 150

34. If α, β and γ are three consecutive terms of a non-constant G.P. such that the equations $\alpha x^2 + 2\beta x + \gamma = 0$ and $x^2 + x - 1 = 0$ have a common root, then $\alpha(\beta + \gamma)$ is equal to : ***[JEE-MAIN 2019]***

(A) $\beta\gamma$ (B) 0 (C) $\alpha\gamma$ (D) $\alpha\beta$

EXERCISE – 5 RECAP OF IIT-JEE/JEE (ADVANCED)

1. In quadratic equation $ax^2 + bx + c = 0$, if α, β are roots of equation, $\Delta = b^2 - 4ac$ and $\alpha + \beta, \alpha^2 + \beta^2, \alpha^3 + \beta^3$ are in G.P. then ***[JEE 2005 (screening)]***

(A) $\Delta \neq 0$ (B) $b\Delta = 0$ (C) $c\Delta = 0$ (D) $\Delta = 0$

2.** If $a_n = \frac{3}{4} - \left(\frac{3}{4}\right)^2 + \left(\frac{3}{4}\right)^3 + \ldots (-1)^{n-1}\left(\frac{3}{4}\right)^n$ and $b_n = 1 - a_n$ then find the minimum natural number n_0 such that $b_n > a_n \ \forall \ n \geq n_0$ ***[JEE 2006]

Comprehension Based Question

****Comprehension–1***

Let V_r denote the sum of first r terms of an arithmetic progression (A.P.) whose first term is r and the common difference is $(2r - 1)$.

Let $T_r = V_{r+1} - V_r - 2$ and $Q_r = T_{r+1} - T_r$ for r = 1,2,........

3. The sum $V_1 + V_2 + \ldots + V_n$ is : ***[JEE 2007]***

(A) $\frac{1}{12}n(n + 1)(3n^2 - n + 1)$ (B) $\frac{1}{12}n(n + 1)(3n^2 + n + 2)$

(C) $\frac{1}{2}n(2n^2 - n + 1)$ (D) $\frac{1}{3}(2n^3 - 2n + 3)$

4. T_r is always : ***[JEE 2007]***

(A) an odd number (B) an even number (C) a prime number (D) a composite number

5. Which one of the following is a correct statement ? ***[JEE 2007]***

(A) $Q_1, Q_2, Q_3, \ldots$ are in A.P. with common difference 5 (B) $Q_1, Q_2, Q_3, \ldots$ are in A.P. with common difference 6

(C) $Q_1, Q_2, Q_3, \ldots$ are in A.P. with common difference 11 (D) $Q_1 = Q_2 = Q_3 = \ldots$

****Comprehension–2***

Let A_1, G_1, H_1 denote the arithmetic, geometric and harmonic means, respectively, of two distinct positive numbers. For $n \geq 2$, let A_{n-1} and H_{n-1} has arithmetic, geometric and harmonic means as A_n, G_n, H_n respectively.

6. Which one of the following statements is correct ? ***[JEE 2007]***

(A) $G_1 > G_2 > G_3 > \ldots$ (B) $G_1 < G_2 < G_3 < \ldots$

(C) $G_1 = G_2 = G_3 = \ldots$ (D) $G_1 < G_2 < G_3 < \ldots$ and $G_4 > G_5 > G_6 > \ldots$

7. Which one of the following statements is correct ? ***[JEE 2007]***

(A) $A_1 > A_2 > A_3 > \ldots$ (B) $A_1 < A_2 < A_3 < \ldots$

(C) $A_1 > A_3 > A_5 > \ldots$ and $A_2 < A_4 < A_6 < \ldots$ (D) $A_1 < A_3 < A_5 < \ldots$ and $A_2 > A_4 > A_6 > \ldots$

8. Which one of the following statements is correct ? ***[JEE 2007]***

(A) $H_1 > H_2 > H_3 > \ldots$ (B) $H_1 < H_2 < H_3 < \ldots$

(C) $H_1 > H_3 > H_5 > \ldots$ and $H_2 < H_4 < H_6 > \ldots$ (D) $H_1 < H_3 < H_5 < \ldots$ and $H_2 > H_4 > H_6 > \ldots$

9. Suppose four distinct positive numbers a_1, a_2, a_3, a_4 are in G.P. Let $b_1 = a_1$, $b_2 = b_1 + a_2$, $b_3 = b_2 + a_3$ and $b_4 = b_3 + a_4$.

Statement–I– The numbers b_1, b_2, b_3, b_4 are neither in A.P. nor in G.P.

and

Statement–II– The numbers b_1, b_2, b_3, b_4 are in H.P. ***[JEE 2008]***

(A) Statement-I is true, Statement-II is true ; Statement-II is correct explanation for Statement-I.

(B) Statement-I is true, Statement-II is true ; Statement-II is NOT a correct explanation for statement-I.

(C) Statement-I is true, Statement-II is false.

(D) Statement-I is false, Statement-II is true.

10. If the sum of first n terms of an A.P. is cn^2, then the sum of squares of these n terms is ***[JEE 2009]***

(A) $\frac{n(4n^2-1)c^2}{6}$ (B) $\frac{n(4n^2+1)c^2}{3}$ (C) $\frac{n(4n^2-1)c^2}{3}$ (D) $\frac{n(4n^2+1)c^2}{6}$

11.** Let S_k, k = 1,2,.......,100, denote the sum of the infinite geometric series whose first term is $\frac{k-1}{k!}$ and the common ratio is $\frac{1}{k}$. Then the value of $\frac{100^2}{100!}+\sum_{k=1}^{100}\left|(k^2-3k+1)S_k\right|$ is ***[JEE 2010]

***12.** Let $a_1,a_2,a_3,.........,a_{11}$ be real numbers satisfying

$a_1 = 15$, $27 - 2a_2 > 0$ and $a_k = 2a_{k-1} - a_{k-2}$ for k = 3,4,........,11.

If $\frac{a_1^2+a_2^2+....+a_{11}^2}{11}=90$, then the value of $\frac{a_1+a_2+...+a_{11}}{11}$ is equal to ***[JEE 2010]***

13. The minimum value of the sum of real numbers a^{-5}, a^{-4}, $3a^{-3}$, 1, a^8 and a^{10} with $a > 0$ is ***[JEE 2011]***

14. Let $a_1,a_2,a_3,.........,a_{100}$ be an arithmetic progression with $a_1 = 3$ and $S_p=\sum_{i=1}^{p}a_i, 1\le p\le 100$. For any integer n with $1 \le n \le 20$, let m = 5n. If $\frac{S_m}{S_n}$ does not depend on n, then a_2 is ***[JEE 2011]***

15. Let a_1, a_2, a_3, ... be in harmonic progression with $a_1 = 5$ and $a_{20} = 25$. The least positive integer n for which $a_n < 0$ is ***[JEE 2012]***

(A) 22 (B) 23 (C) 24 (D) 25

16.** Let $S_n=\sum_{k=1}^{4n}(-1)^{\frac{k(k+1)}{2}}k^2$. Then S_n can take value(s) ***[JEE-Advanced 2013]

(A) 1056 (B) 1088 (C) 1120 (D) 1332

17.** A pack contains n cards numbered from 1 to n. Two consecutive numbered cards are removed from the pack and the sum of the numbers on the remaining cards is 1224. If the smaller to the numbers on the removed cards is k, then k – 20 = ***[JEE-Advanced 2013]

18. Let the harmonic mean of two positive real number a and b be 4. If q is a positive real number such that a, 5, q, b is an arithmetic progression, then the value(s) of $|q-a|$ is (are) ***[JEE-Advanced 2015]***

19. Suppose that all the terms of an arithmetic progression (A. P.) are natural numbers. If the ratio of the sum of the first seven terms to the sum of the first eleven terms is 6 : 11 and the seventh term lies in between 130 and 140, then the common difference of this A.P. is **[JEE-Advanced 2015]**

20. Let $b_i > 1$ for $i = 1, 2, ..., 101$. Suppose $\log_e b_1, \log_e b_2, ..., \log_e b_{101}$ are in Arithmetic Progression (A.P.) with the common difference $\log_e 2$. Suppose $a_1, a_2, ..., a_{101}$ are in A.P. such that $a_1 = b_1$ and $a_{51} = b_{51}$. If $t = b_1 + b_2 + ... + b_{51}$ and $s = a_1 + a_2 + ... + a_{51}$, then **[JEE-Advanced 2016]**

(A) $s > t$ and $a_{101} > b_{101}$ (B) $s > t$ and $a_{101} < b_{101}$

(C) $s < t$ and $a_{101} > b_{101}$ (D) $s < t$ and $a_{101} < b_{101}$

21. The sides of a right angled triangle are in arithmetic progression. If the triangle has area 24, then what is the length of its smallest side? **[JEE-Advanced 2017]**

22. Let X be the set consisting of the first 2018 terms of the arithmetic progression 1, 6, 11, ... , and Y be the set consisting of the first 2018 terms of the arithmetic progression 9, 16, 23, ... Then, the number of elements in the set $X \cup Y$ is —— **[JEE 2018]**

23. Let α and β be the roots of $x^2 - x - 1 = 0$, with $\alpha > \beta$. For all positive integers n, define $a_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$, $n \geq 1$,

$b_1 = 1$ and $b_n = a_{n-1} + a_{n+1}$, $n \geq 2$. Then which of the following options is/are correct ? **[JEE 2019]**

(1) $a_1 + a_2 + a_3 + ... + a_n = a_{n+2} - 1$ for all $n \geq 1$

(2) $\sum_{n=1}^{\infty} \frac{a_n}{10^n} = \frac{10}{89}$

(3) $\sum_{n=1}^{\infty} \frac{b_n}{10^n} = \frac{8}{89}$

(4) $b_n = \alpha^n + \beta^n$ for all $n \geq 1$

24. Let AP (a ; d) denote the set of all the terms of an infinite arithmetic progression with first term a and common difference $d > 0$. If $AP(1 ; 3) \cap AP(2 ; 5) \cap AP(3 ; 7) = AP(a; d)$ then $a + d$ equals __ **[JEE 2019]**

ANSWER KEY

EXERCISE-1

Que.	1	2	3	4	5	6	7	8	9	10
Ans.	C	C	A	B	B	C	D	C	C	A
Que.	11	12	13	14	15	16	17	18	19	20
Ans.	B	C	C	A	B	A	A	C	B	D

EXERCISE-2

Que.	1	2	3	4	5	6	7	8	9	10
Ans.	A	AC	BD	C	A	A	C	CD	ABC	C

- ***Match the Column*** **11.** (A) $\rightarrow$ (q), (B) $\rightarrow$ (p), (C) $\rightarrow$ (p), (D) $\rightarrow$ (r)
- ***Comprehension Based Questions***

12. (C) **13.** (A) **14.** (D) **15.** (B)

EXERCISE-3

2. 14 **3.** $S = (7/81)(10^{n+1} - 9n - 10)$ **5.** 35/222 **7.** $a = 5, b = 8, c = 12$ **8.** n^2

10. $\frac{4}{5}$ **11.** 34 **12.** 686

EXERCISE-4

Que.	1	2	3	4	5	6	7	8	9	10
Ans.	C	B	B	B	B	D	B	A	C	C
Que.	11	12	13	14	15	16	17	18	19	20
Ans.	A	B	D	D	B	D	A	B	B	B
Que.	21	22	23	24	25	26	27	28	29	30
Ans.	C	A	A	A	B	A	C	D	B	B
Que.	31	32	33	34						
Ans.	C	A	A	A						

EXERCISE-5

1. (C) **2.** 6 **3.** (B) **4.** (D) **5.** (B) **6.** (C)
7. (A) **8.** (B) **9.** (C) **10.** (C) **11.** 3 **12.** 0
13. 8 **14.** 9 or 3 **15.** (D) **16.** (A,D) **17.** 5 **18.** (2,5)
19. 9 **20.** (B) **21.** 6 **22.** 3748 **23.** 1,2,4 **24.** 157

Important Notes

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