FUNCTION

Recap of Early Classes

In earlier classes, we have studied about sets, a collection of elements. We also had a brief idea about functions and relations. In this chapter, we will systematically discuss "Functions". Their properties and application will lead us into further understanding differential and integral calculus and their wider practical applications.

Index

FUNCTION

1.0 CARTESIAN PRODUCT OF TWO SETS

SL AL

The cartesian product of two non-empty sets A & B is the set of all possible ordered pair of the form (a, b) where the first element comes from set A & second comes from set B.

$A \times B = \{(a, b) \mid a \in A, b \in B\}$

e.g. $A = \{1, 2, 3\}$, $B = \{p, q\}$

$A \times B = \{(1, p)(1, q), (2, p), (2, q), (3, p), (3, q)\}$

NOTE

(i) If either A or B is the null set, then $A \times B$ will also be empty set, i.e. $A \times B = \phi$

(ii) If $n(A) = p$ & $n(B) = q$, then $n(A \times B) = p \times q$, where $n(X)$ (cardinal number) denotes the number of elements in set X.

2.0 RELATION

SL AL

A relation R from a non-empty set A to non-empty set B is subset of cartesian product $A \times B$. The subset is derived by describing a relationship between the first element and the second element of the ordered pairs in $A \times B$. The first element is called pre-image of second element. The second element is called the image of the first element.

NOTE

(i) The set of all first elements of the ordered pairs in a relation R from a set A to a set B is called the domain of the relation R.

(ii) The set of all second elements in a relation R from a set A to a set B is called the range of the relation R. The whole set B is called the co-domain of the relation R.

(iii) Range is a subset of co-domain.

3.0 FUNCTION

SL AL

A relation R from a set A to a set B is called a function if each element of A has unique image in B.

It is denoted by the symbol. $f : A \to B$ or $A \xrightarrow{f} B$ which reads 'f' is a function from A to B 'or' f maps A to B,

If an element $a \in A$ is associated with an element $b \in B$, then b is called 'the f image of a' or 'image of a under f 'or' the value of the function f at a'. Also a is called the pre-image of b or argument of b under the function f. We write it as $b = f(a)$ or $f : a \to b$ or $f : (a, b)$

Thus a function 'f' from a set A to a set B is a subset of $A \times B$ in which each 'a' belonging to A appears in one and only one ordered pair belonging to f.

Representation of Function

(a) ***Ordered pair –*** Every function from $A \to B$ satisfies the following conditions :

(i) $f \subseteq A \times B$ (ii) $\forall\ a \in A$ there exist $b \in B$ and (iii) $(a, b) \in f$ & $(a, c) \in f \Rightarrow b = c$

(b) ***Formula based (uniformly/nonuniformly)*** e.g.

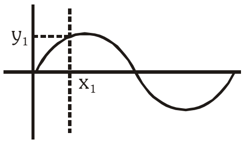
(i) $f : R \to R,\ y = f(x) = 4x$ (uniformly defined)

(ii) $f(x) = x^2$ (uniformly defined)

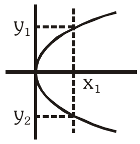
(iii) $f(x) = \begin{cases} x+1 & -1 \le x < 4 \\ -x & 4 \le x < 7 \end{cases}$ (non-uniformly defined)

(iv) $f(x) = \begin{cases} x^2 & x \ge 0 \\ -x-1 & x < 0 \end{cases}$ (non-uniformly defined)

(c) Graphical representation



Graph (1)



Graph (2)

Graph(1) represent a function but graph(2) does not represent a function.

NOTE

(i) If a vertical line cuts a given graph at more than one point then it can not be the graph of a function.

(ii) Every function is a relation but every relation is not necessarily a function.

4.0 DOMAIN, CO-DOMAIN & RANGE OF A FUNCTION

SL AL

Let $f: A \to B$, then the set A is known as the domain of f & the set B is known as co-domain of f. The set of f images of all the elements of A is known as the range of f.

Thus Domain of $f = \{a \mid a \in A, (a, f(a)) \in f\}$

Range of $f = \{f(a) \mid a \in A, f(a) \in B\}$

NOTE :

(i) It should be noted that range is a subset of co-domain.

(ii) If only the rule of function is given then the domain of the function is the set of those real numbers, where function is defined. For a continuous function, the interval from minimum to maximum value of a function gives the range

5.0 ALGEBRAIC OPERATIONS ON FUNCTIONS

SL AL

If f & g are real valued functions of x with domain set A, B respectively, f + g, f – g, (f. g) & (f/g) as follows.

(a) $(f \pm g)(x) = f(x) \pm g(x)$ domain in each case is $A \cap B$

(b) $(f.g)(x) = f(x).g(x)$ domain is $A \cap B$

(c) $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ domain $A \cap B - \{x \mid g(x) = 0\}$

Illustrations

Illustration 1. Let $A = \{2, 3, 5, 8\}$

$B = \{4, 6, 7, 15, 20\}$

There is a relation R : x is a divisor of y defined from set A to set B where $x \in A$ and $y \in B$. Find relation R in the form of ordered pair

Solution. $R = \{(x, y), x \in A \text{ and } y \in R, x \text{ is a divisor of } y\}$

$= \{(2, 4), (2, 6), (2, 20), (3, 6), (3, 15), (5, 15), (5, 20)\}$

Illustration 2. If set A has 4 element, set B has 3 elements then how many relation $R : A \to B$ are possible?

Solution. Number of subsets of $A \times B = 2^{4\times 3}$

$\therefore$ Number of relations $= 2^{12}$

Illustration 3. Which of the following relations are function from

$A = \{1, 3, 5, 7, 9\}$ to $B = \{1, 2, 3, 4\}$

(i) $f = \{(3, 1), (5, 1), (7, 1), (9, 1)\}$

(ii) $f = \{(1, 2), (3, 3), (5, 3), (1, 4)\}$

(iii) $f = \{(1, 3), (3, 1), (5, 3), (7, 1), (9, 2)\}$

Solution. (i) Not a function since element 1 has no image
(ii) Not a function
(iii) function

Illustration 4. Find domain of following function

(a) $f(x) = 5\sin\left(\sqrt{\frac{\pi^2}{16} - x^2}\right)$ (b) $f(x) = \sqrt{\log_{10}\left(\frac{3-x}{x}\right)}$

(c) $f(x) = \frac{1}{\sqrt{|\sin x| + \sin x}}$ *(d) $f(x) = \sqrt{\sin x} + \sqrt{16 - x^2}$

(e) $y = \log_{(x-4)}(x^2 - 11x + 24)$ *(f) $f(x) = \log_2\left(-\log_{1/2}\left(1 + \frac{1}{\sqrt[4]{x}}\right) - 1\right)$

Solution. ***(a)*** $\frac{\pi^2}{16} - x^2 \ge 0$

$\therefore$ $x \in \left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$

(b) $\sqrt{\log_{10}\left(\frac{3-x}{x}\right)}$ is defined for $\log_{10}\left(\frac{3-x}{x}\right) \ge 0$

$\frac{3-x}{x} \ge 1 \Rightarrow x \in \left(0, \frac{3}{2}\right]$ ... (1)

Also, $\log_{10}\left(\frac{3-x}{x}\right)$ is defined for $\frac{3-x}{x} > 0$

$\Rightarrow$ $x \in (0, 3)$ ... (2)

$(1) \cap (2) \Rightarrow x \in \left(0, \frac{3}{2}\right]$

(c) f(x) is defined if $|\sin x| + \sin x > 0$
$\sin x > 0 \Rightarrow x \in (2n\pi, 2n\pi + \pi)$

(d) $\sin x \ge 0$ and $16 - x^2 \ge 0$ $\Rightarrow$ $2n\pi \le x \le (2n + 1)\pi$ and $-4 \le x \le 4$

$\therefore$ Domain is $[-4, -\pi] \cup [0, \pi]$

(e) $y = \log_{(x-4)}(x^2 - 11x + 24)$
Here 'y' would assume real value if,
$x - 4 > 0$ and $\ne 1$, $x^2 - 11x + 24 > 0$ $\Rightarrow$ $x > 4$ and $\ne 5$, $(x - 3)(x - 8) > 0$

$\Rightarrow$ $x > 4$ and $\ne 5$, $x < 3$ or $x > 8 \Rightarrow x > 8$ $\Rightarrow$ Domain $= (8, \infty)$

(f) We have $f(x) = \log_2\left(-\log_{1/2}\left(1 + \frac{1}{\sqrt[4]{x}}\right) - 1\right)$

f(x) is defined if $-\log_{1/2}\left(1 + \frac{1}{\sqrt[4]{x}}\right) - 1 > 0$

or if $\log_{1/2}\left(1 + \frac{1}{\sqrt[4]{x}}\right) < -1$ or if $\left(1 + \frac{1}{\sqrt[4]{x}}\right) > (1/2)^{-1}$

or if $1 + \frac{1}{\sqrt[4]{x}} > 2$ or if $\frac{1}{\sqrt[4]{x}} > 1$ or if $x^{1/4} < 1$ or if $0 < x < 1$

$\therefore$ $D(f) = (0, 1)$

Illustration 5. Find the range of following functions :

(i) $f(x) = \log_2\left(\frac{\sin x - \cos x + 3\sqrt{2}}{\sqrt{2}}\right)$ *(ii) $f(x) = \log_{\sqrt{2}}\left(2 - \log_2(16\sin^2 x + 1)\right)$

Solution. (i) Let $y = \log_2\left(\frac{\sin x - \cos x + 3\sqrt{2}}{\sqrt{2}}\right)$

$\Rightarrow$ $2^y = \sin\left(x - \frac{\pi}{4}\right) + 3 \Rightarrow -1 \le 2^y - 3 \le 1$

$\Rightarrow$ $2 \le 2^y \le 4 \Rightarrow y \Rightarrow [1, 2]$

(ii) $f(x) = \log_{\sqrt{2}}\left(2 - \log_2(16\sin^2 x + 1)\right)$

$1 \le 16\sin^2 x + 1 \le 17$

$\therefore$ $0 \le \log_2(16\sin^2 x + 1) \le \log_2 17$

$\therefore$ $2 - \log_2 17 \le 2 - \log_2(16\sin^2 x + 1) \le 2$

Now consider $0 < 2 - \log_2(16\sin^2 x + 1) \le 2$

$\therefore$ $-\infty < \log_{\sqrt{2}}[2 - \log_2(16\sin^2 x + 1)] \le \log_{\sqrt{2}} 2 = 2$

$\therefore$ the range is $(-\infty, 2]$

BEGINNER'S BOX-1

TOPIC COVERED : CARTESIAN PRODUCT OF TWO SETS, RELATION, FUNCTION, DOMAIN, CO-DOMAIN & RANGE OF A FUNCTION, ALGEBRAIC OPERATIONS ON FUNCTIONS

1. Check the following whether a function from R to R or NOT

(a) $f(x) = \sqrt{x}$ (b) $f(x) = \sqrt[3]{x}$ (c) $f(x) = \sqrt{x-2} + \frac{1}{\sqrt{2-x}}$ (d) $f^2(x) = x$

2. What is vertical line test ?

Find the domain of following functions :

3. (a) $y = 1 - \log_{10} x$ (b) $y = \sqrt{5 - 2x}$ (c) $y = \frac{1}{\sqrt{x^2 - 4x}}$ (d) $y = \frac{1}{\sqrt{|x| - x}}$

4. (a) $y = \log_2 \log_3 \log_4 x$ (b) $y = \sqrt{3-x} + \cos^{-1}\left(\frac{3-2x}{5}\right)$ (c) $\frac{\sqrt{9-x^2}}{\sin^{-1}(3-x)}$

5. ***Find the range of the following function :***

(a) $f(x) = \sin 3x$ (b) $f(x) = \frac{1}{3 - \cos x}$ (c) $f(x) = e^{-3x}$ (d) $f(x) = \cos\left(2x + \frac{\pi}{4}\right)$

6. (a) $f(x) = 2\sin\left(2x + \frac{\pi}{4}\right)$ (b) $f(x) = \frac{x^2}{x^4 + 1}$ (c) $y = \frac{x}{1 + x^2}$ (d) $y = \ln(3x^2 - 4x + 5)$

7. Given 'n' real numbers $a_1, a_2 .....a_n$. Determine the value of x at which the function $f(x) = (x - a_1)^2 + (x - a_2)^2 + ...+ (x - a_n)^2$ takes the minimum value.

8. Find the range $f(x) = \frac{1}{8 - 3\sin x}$

6.0 IMPORTANT TYPES OF FUNCTION

AL

(a) ***Polynomial function***

If a function 'f' is called by $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + ...+ a_{n-1}x + a_n$ where n is a non negative integer and $a_0, a_1, a_2, ...a_n$ are real numbers and $a_0 \neq 0$, then f is called a polynomial function of degree n.

NOTE

(i) A polynomial of degree one with no constant term is called an odd linear function. i.e. $f(x) = ax, a \neq 0$

(ii) There are two polynomial functions, satisfying the relation; $f(x). f(1/x) = f(x) + f(1/x)$. They are (1) $f(x) = x^n + 1$ & (2) $f(x) = 1 - x^n$, where n is a positive integer.

(iii) Domain of a polynomial function is R

(iv) Range of odd degree polynomial is R whereas range of an even degree polynomial is never R.

(b) Algebraic function

A function 'f' is called an algebraic function if it can be constructed using algebraic operations (such as addition, subtraction, multiplication, division, and taking radicals) starting with polynomials.

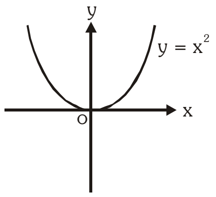
Examples – $f(x) = \sqrt{x^2+1}$; $g(x) = \dfrac{x^4-16x^2}{x+\sqrt{x}} + (x-2)\sqrt[3]{x+1}$

If y is an algebraic function of x, then it satisfies a polynomial equation of the form $P_0(x)y^n + P_1(x)y^{n-1} + ... + P_{n-1}(x)y + P_n(x) = 0$, where 'n' is a positive integer and $P_0(x), P_1(x),.......$ are polynomial in x.

Note that all polynomial functions are Algebraic but the converse in not true. A functions that is not algebraic is called ***TRANSCEDENTAL*** function.

Basic algebraic function

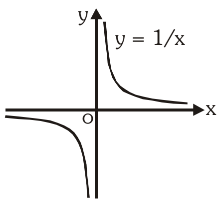
(i) $y = x^2$



Domain : R

Range : $R^+ \cup \{0\}$ or $[0,\infty)$

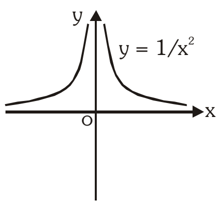
(ii) $y = \dfrac{1}{x}$



Domain : $R - \{0\}$ or R_0

Range : $R - \{0\}$

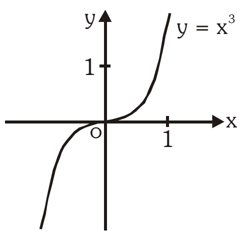
(iii) $y = \dfrac{1}{x^2}$



Domain : R_0

Range : R^+ or $(0, \infty)$

(iv) $y = x^3$



Domain : R

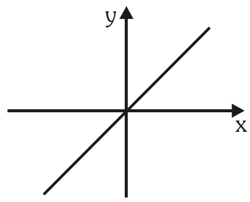
Range : R

(c) Rational function

A rational function is a function of the form $y = f(x) = \dfrac{g(x)}{h(x)}$, where g(x) & h(x) are polynomials & $h(x) \neq 0$, ***Domain –*** $R-\{x \mid h(x)=0\}$

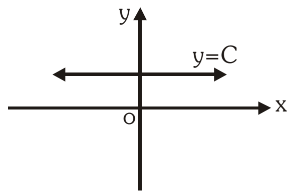
Any rational function is automatically an algebraic function.

(d) *Identity function*



The function f– A → A defined by f(x) = x ∀ x ∈ A is called the identity function on A and is denoted by I_A. It is easy to observe that identity function is a bijection.

(e) *Constant function*

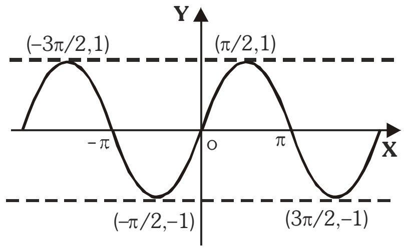


f : A → B is said to be constant function if every element of A has the same f image in B. Thus f : A → B ; f(x) = c, ∀ x ∈ A, c ∈ B is constant function. Note that the range of a constant function is a singleton set.

Domain – R ***Range*** – {C}

(f) *Trigonometric functions*

(i) *Sine function*

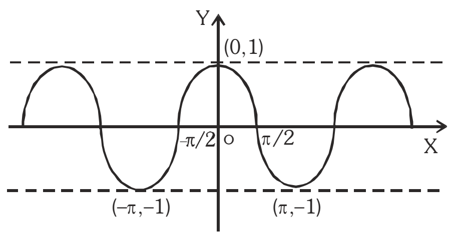


f(x) = sin x

Domain – R

Range – [–1, 1], period 2π

(ii) *Cosine function*

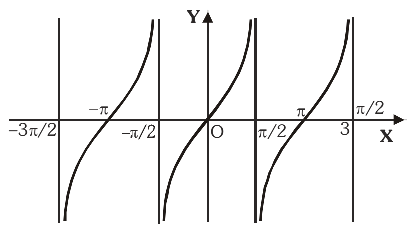


f(x) = cos x

Domain : R

Range – [–1, 1], period 2π

(iii) *Tangent function*

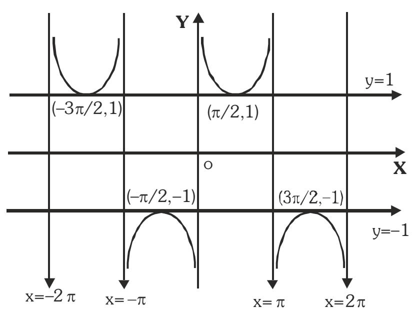


f(x) = tan x

Domain – $R-\left\{x \mid x=\frac{(2n+1)\pi}{2}, n \in I\right\}$

Range – R , period π

(iv) *Cosecant function*

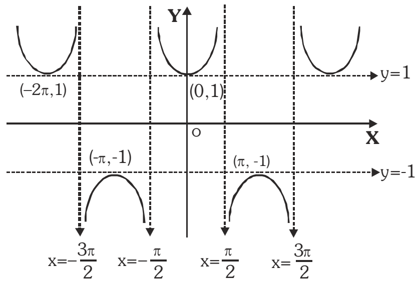


f(x) = cosec x

Domain – R – {x|x = nπ, n ∈I}

Range – R – (–1, 1), period 2π

(v) Secant function

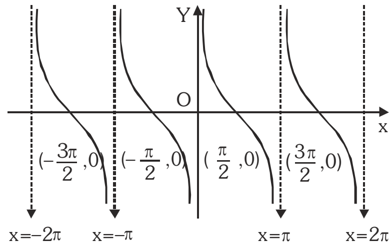


f(x) = sec x

Domain – R – {x|x = (2n + 1) π/2 : n ∈ I}

Range – R – (–1, 1), period 2π

(vi) Cotangent function



f(x) = cot x

Domain – R – {x|x = nπ, n ∈I}

Range – R, period π

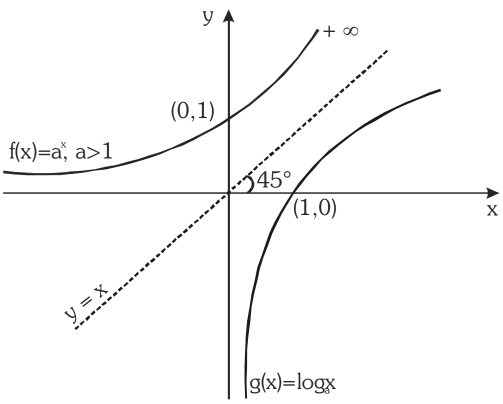
(g) Exponential and Logarithmic Function

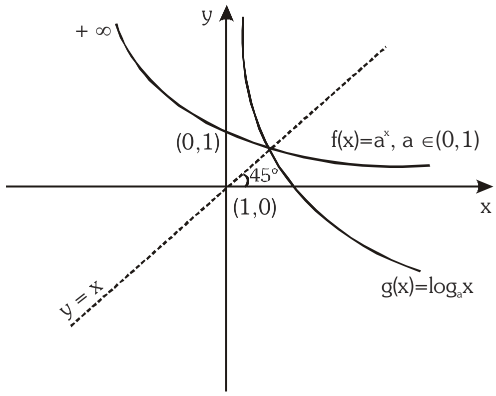
A function $f(x) = a^x (a > 0)$, $a \neq 1$, $x \in R$ is called an exponential function. The inverse of the exponential function is called the logarithmic function, i.e. $g(x) = \log_a x$.

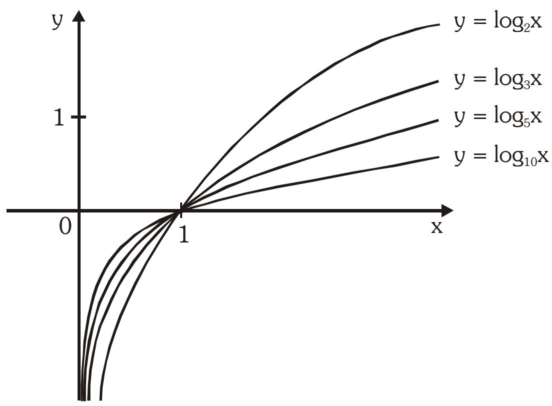
Note that f(x) & g(x) are inverse of each other & their graphs are as shown. (If functions are mirror image of each other about the line y = x)

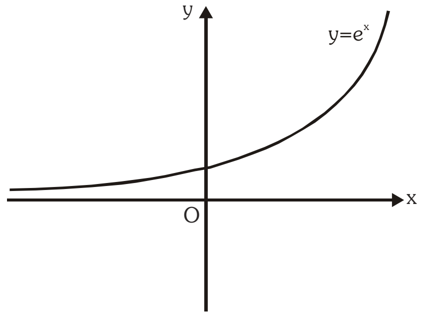
Domain of a^x is R   ***Range*** R^+

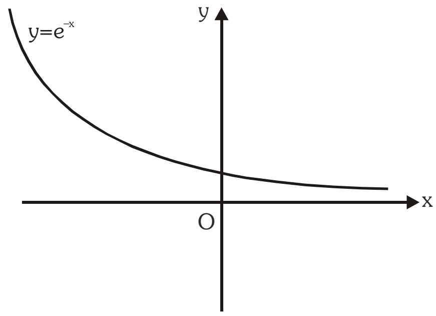
Domain of $\log_a x$ is R^+ ***Range*** R











Note-1 – $f(x) = a^{1/x}$, $a > 0$ ***Domain*** – $R - \{0\}$ ***Range*** – $R^+ - \{1\}$

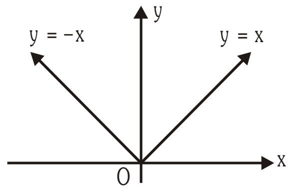
Note-2 – $f(x) = \log_x a = \dfrac{1}{\log_a x}$ ***Domain*** – $R^+ - \{1\}$ ***Range*** – $R - \{0\}$

$(a > 0)$ $(a \neq 1)$

(h) Absolute value function

The absolute value (or modulus) of a real number x (written $|x|$) is a non negative real number that satisfies the conditions.

$$|x| = \begin{cases} x & \text{if} \quad x \geq 0 \\ -x & \text{if} \quad x < 0 \end{cases}$$



Domain : R
Range : $[0, \infty)$

The properties of absolute value function are

(i) The inequality $|x| \leq \alpha$ means that $-\alpha \leq x \leq \alpha$; if $\alpha > 0$

(ii) The inequality $|x| \geq \alpha$ means that $x \geq \alpha$ or $x \leq -\alpha$ if $\alpha > 0$

(iii) $|x \pm y| \leq |x| + |y|$ ***(Triangle Inequality)*** Equality holds when $x.y \geq 0$

(iv) $|x \pm y| \geq ||x| - |y||$ ***(Triangle Inequality)*** Equality holds when $x.y \geq 0$

(v) $|xy| = |x|.|y|$

(vi) $\left|\dfrac{x}{y}\right| = \dfrac{|x|}{|y|}$, $(y \neq 0)$

Note – $f(x) = \dfrac{1}{|x|}$, ***Domain*** : $R - \{0\}$, ***Range*** : R^+

Illustrations

Illustration 6. Determine the values of x satisfying the equality.

$|(x^2 + 4x + 9) + (2x - 3)| = |x^2 + 4x + 9| + |2x - 3|$.

Solution. The equality $|a + b| = |a| + |b|$ is valid if and only if both summands have the same sign,

$\because$ $x^2 + 4x + 9 = (x + 2)^2 + 5 > 0$ at any values of x, the equality is satisfied at those values of x at which $2x - 3 \geq 0$, i.e. at $x \geq \dfrac{3}{2}$.

Illustration 7. Determine the values of x satisfying the equality $|x^4 - x^2 - 6| = |x^4 - 4| - |x^2 + 2|$.

Solution. The equality $|a - b| = |a| - |b|$ holds true if and only if a and b have the same sign and $|a| \geq |b|$.

In our case the equality will hold true for the value of x at which $x^4 - 4 \geq x^2 + 2$.

Hence $x^2 - 2 \geq 1$; $|x| \geq \sqrt{3}$.

BEGINNER'S BOX-2

TOPIC COVERED : TYPE OF FUNCTION, DOMAIN, GRAPH OF FUNCTIONS

1. Solve for x : $2|x|^2 - 5|x| + 2 = 0$

Find set of real values of x satisfying

2. $||x-1| - 1| \le 1$

3. $x^2 - |x| - 2 \ge 0$

4. Number of solution of $x^2 = 2^x$

5. Least positive solution of tan x = x lies in which quadrant.

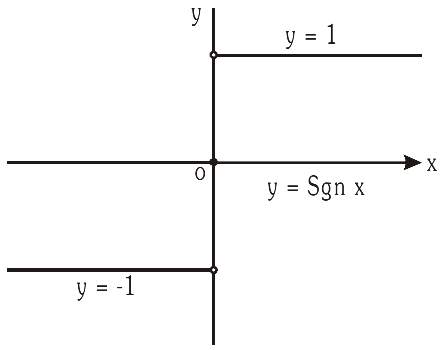
6. Solve for x : $|x^2 - 1| + (x-1)^2 + \sqrt{x^2 - 3x + 2} = 0$

7. If $f(x)f\left(\frac{1}{x}\right) = f(x) + f\left(\frac{1}{x}\right)$ $\forall$ x $\in$ R – {0}, where f(x) be a polynomial function and f(5) = 126, find f(3)=

(i) Signum function

Signum function y = sgn (x) is defined as follows

$$y = \begin{cases} \frac{|x|}{x}, x \ne 0 \\ 0, x = 0 \end{cases} = \begin{cases} 1 & \text{for } x > 0 \\ 0 & \text{for } x = 0 \\ -1 & \text{for } x < 0 \end{cases}$$



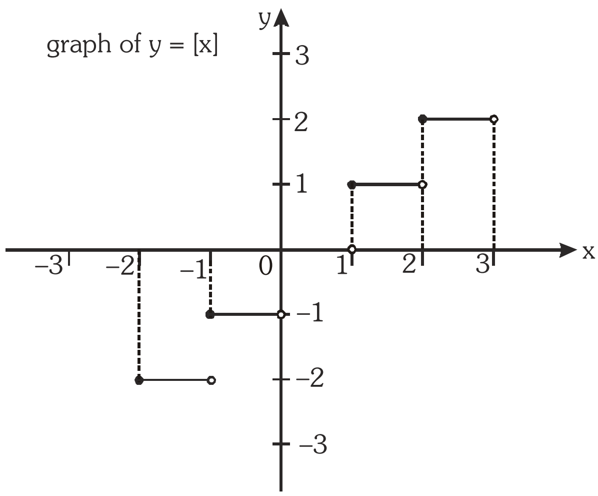
Domain – R

Range – {–1, 0, 1}

(j) Greatest integer or step up function

The function y = f(x) = [x] is called the greatest integer function where [x] denotes the greatest integer less than or equal to x. Note that for :

x	[x]
[–2,–1)	–2
[–1,0)	–1
[0,1)	0
[1,2)	1



Domain – R

Range – I

Properties of greatest integer function

(i) $[x] \le x < [x] + 1$ and $x - 1 < [x] \le x$, $0 \le x - [x] < 1$

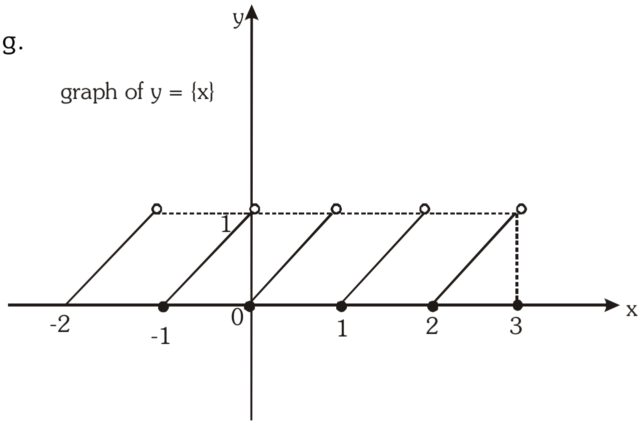
(ii) $[x + m] = [x] + m$ if m is an integer.

(iii) $[x] + [-x] = \begin{cases} 0, & x \in I \\ -1, & x \notin I \end{cases}$

Note – $f(x) = \dfrac{1}{[x]}$ ***Domain*** – R – [0, 1) ***Range*** – $\{x \mid x = \frac{1}{n}, n \in I_0\}$

(k) Fractional part function

It is defined as – g(x) = {x} = x – [x] e.g. the fractional part of the number 2.1 is 2.1–2 = 0.1 and the fractional part of –3.7 is 0.3 The period of this function is 1 and graph of this function is as shown.



x	{x}
[–2,–1)	x+2
[–1,0)	x+1
[0,1)	x
[1,2)	x–1

Domain – R **Range** – [0,1)

Note – $f(x)=\dfrac{1}{\{x\}}$ **Domain** – R – I **Range** – $(1, \infty)$

- ***Properties of Fractional part function***

(i) $0 \le \{x\} < 1$ ***(ii)*** $\{[x]\} = [\{x\}] = 0$

(iii) $\{\{x\}\} = \{x\}$ ***(iv)*** $\{x+m\} = \{x\},\ m \in I$

(v) $\{x\} + \{-x\} = \begin{cases} 1, & x \notin I \\ 0, & x \in I \end{cases}$ ***(vi)*** $[x+y] = \begin{cases} [x]+[y], & \text{if } \{x\}+\{y\} < 1 \\ [x]+[y]+1, & \text{if } \{x\}+\{y\} \ge 1 \end{cases}$

Illustrations

****Illustration 8.*** If y = 2[x] + 3 & y = 3[x – 2] + 5 then find [x + y] where [.] denotes greatest integer function.

Solution. y = 3[x – 2] + 5 = 3[x] – 1

so 3 [x] – 1 = 2 [x] + 3

$[x] = 4 \Rightarrow 4 \le x < 5$

then y = 11

so x + y will lie in the interval [15, 16)

so [x + y] = 15

Illustration 9. Find the value of $\left[\frac{1}{2}\right]+\left[\frac{1}{2}+\frac{1}{1000}\right]+\ldots\ldots\left[\frac{1}{2}+\frac{2946}{1000}\right]$ where [.] denotes greatest integer function ?

Solution.

$$\left[\frac{1}{2}\right]+\left[\frac{1}{2}+\frac{1}{1000}\right]+\ldots\ldots\left[\frac{1}{2}+\frac{499}{1000}\right]+\left[\frac{1}{2}+\frac{500}{1000}\right]+\ldots\ldots\left[\frac{1}{2}+\frac{1499}{1000}\right]+\left[\frac{1}{2}+\frac{1500}{1000}\right]+\ldots\ldots$$

$$+\left[\frac{1}{2}+\frac{2499}{1000}\right]+\left[\frac{1}{2}+\frac{2500}{1000}\right]+\ldots\ldots\left[\frac{1}{2}+\frac{2946}{1000}\right]$$

$$= 0 + 1 \times 1000 + 2 \times 1000 + 3 \times 447 = 3000 + 1341 = 4341$$ ***Ans.***

Illustration 10. Find the domain $f(x) = \dfrac{1}{\sqrt{|[|x|-5]|-11}}$ where [.] denotes greatest integer function.

Solution. $|[|x|-5]| > 11$

so $[|x|-5] > 11$ or $[|x|-5] < -11$

$[|x|] > 16$ $\quad$ $[|x|] < -6$

$|x| \geq 17$ or $|x| < -6$ (Not Possible)

$\Rightarrow$ $x \leq -17$ or $x \geq 17$

so $x \in (-\infty, -17] \cup [17, \infty)$

Illustration 11. Find the range of $f(x) = \dfrac{x-[x]}{1+x-[x]}$, where [.] denotes greatest integer function.

Solution. $y = \dfrac{x-[x]}{1+x-[x]} = \dfrac{\{x\}}{1+\{x\}}$

$\therefore \quad \dfrac{1}{y} = \dfrac{1}{\{x\}} + 1 \Rightarrow \dfrac{1}{\{x\}} = \dfrac{1-y}{y} \quad \Rightarrow \quad \{x\} = \dfrac{y}{1-y}$

$0 \leq \{x\} < 1 \Rightarrow 0 \leq \dfrac{y}{1-y} < 1$

Range = [0, 1/2)

***Illustration 12.** Solve the equation $|2x-1| = 3[x] + 2\{x\}$ where [.] denotes greatest integer and {.} denotes fractional part function.

Solution. We are given that, $|2x-1| = 3[x] + 2\{x\}$

Let, $2x - 1 \leq 0$ i.e. $x \leq \dfrac{1}{2}$. The given equation yields.

$1 - 2x = 3[x] + 2\{x\}$

$\Rightarrow \quad 1 - 2[x] - 2\{x\} = 3[x] + 2\{x\} \quad \Rightarrow \quad 1 - 5[x] = 4\{x\} \Rightarrow \{x\} = \dfrac{1-5[x]}{4}$

$\Rightarrow \quad 0 \leq \dfrac{1-5[x]}{4} < 1 \Rightarrow 0 \leq 1 - 5[x] < 4 \Rightarrow -\dfrac{3}{5} < [x] \leq \dfrac{1}{5}$

Now, $[x] = 0$ as zero is the only integer lying between $-\dfrac{3}{5}$ and $\dfrac{1}{5}$

$\Rightarrow \quad \{x\} = \dfrac{1}{4} \Rightarrow x = \dfrac{1}{4}$ which is less than $\dfrac{1}{2}$,

Hence $\dfrac{1}{4}$ is one solution.

Now, let $2x - 1 > 0$ i.e $x > \dfrac{1}{2}$

$\Rightarrow \quad 2x - 1 = 3[x] + 2\{x\}$

$\Rightarrow \quad 2[x] + 2\{x\} - 1 = 3[x] + 2\{x\}$

$\Rightarrow \quad [x] = -1 \Rightarrow -1 \leq x < 0$ which is not a solution as $x > \dfrac{1}{2}$

$\Rightarrow \quad x = \dfrac{1}{4}$ is the only solution.

7.0 EQUAL OR IDENTICAL FUNCTION

AL

Two function f & g are said to be equal if.

(a) The domain of f = the domain of g

(b) The range of f = range of g and

(c) f(x) = g (x), for every x belonging to their common domain (i.e. should have the same graph)

e.g. $f(x) = \frac{1}{x}$ & $g(x) = \frac{x}{x^2}$ are identical functions.

Illustrations

Illustration 13. The functions $f(x) = \log(x-1) - \log(x-2)$ and $g(x) = \log\left(\frac{x-1}{x-2}\right)$ are identical when x lies in the interval

(A) $[1, 2]$ (B) $[2, \infty)$ (C) $(2, \infty)$ (D) $(-\infty, \infty)$

Solution. Since $f(x) = \log(x-1) - \log(x-2)$.

Domain of f(x) is $x > 2$ or $x \in (2, \infty)$ ...(i)

$g(x) = \log\left(\frac{x-1}{x-2}\right)$ is defined if $\frac{x-1}{x-2} > 0$

$\Rightarrow x \in (-\infty, 1) \cup (2, \infty)$ ...(ii)

From (i) and (ii), $x \in (2, \infty)$. ***Ans. (C)***

BEGINNER'S BOX-3

TOPIC COVERED : IMPORTANT TYPES OF FUNCTION, EQUAL OR IDENTICAL FUNCTION

****1.*** Determine the values of x satisfying the equality $|x^2-8| = |3x^2-5| - |2x^2+3|$.

2. Let {x} & [x] denotes the fraction and integral part of a real number x respectively, then match the column.

	Column-I		**Column-II**
(A)	$[x^2] > 3$	(p)	$x \in [2, 4)$
(B)	$[x]^2 - 5[x] + 6 = 0$	(q)	$x \in (-\infty, -2] \cup [2, \infty)$
(C)	$x = \{x\}$	(r)	$x \in \{0\}$
*(D)	$\{x\} = [x]$	(s)	$x \in (-\infty, -5)$
(E)	$[x] < -5.2$	(t)	$x \in \{-2\}$
(F)	$1 + x = \text{sgn}(x)$	(u)	$x \in [0, 1)$

3. Are the following functions identical ?

(a) $f(x) = \frac{x}{x^2}$ & $\phi(x) = \frac{x^2}{x}$ (b) $f(x) = x$ & $\phi(x) = \sqrt{x^2}$ (c) $f(x) = \log_{10}x^2$ & $\phi(x) = 2\log_{10}|x|$

4. Draw the graph of the function $f(x) = \left|x^2 - 4|x| + 3\right|$ and also find the set of values of 'a' for which the equation f(x) = a has exactly four distinct real roots.

5. Draw the graph of following functions where [.] denotes greatest integer function and { .} denotes fractional part function.

(i) $y = \{\sin x\}$ (ii) $y = [x] + \sqrt{\{x\}}$

6. Solve the following equation for x (where [x] & {x} denotes integral and fractional part of x)

$2x + 3[x] - 4\{-x\} = 4$

7. Area of region enclosed by solution set of $[x] \cdot [y] = 2$ is

8. Write the decreasing order of sec1, sec2, sec3, sec4, sec5, sec6.

8.0 ODD & EVEN FUNCTIONS

AL

If a function is such that whenever 'x' is in it's domain '–x' is also in it's domain & it satisfies

f(–x) = f(x) it is an even function

f(–x) = –f(x) it is an odd function

Note :

(i) A function may neither be odd nor even.

(ii) Inverse of an even function is not defined, as it is many – one function.

(iii) Every even function is symmetric about the *y*-axis & every odd function is symmetric about the origin.

(iv) Every function which has '–x' in it's domain whenever 'x' is in it's domain, can be expressed as the sum of an even & an odd function .

$$\text{e.g. } f(x) = \underbrace{\frac{f(x)+f(-x)}{2}}_{\text{EVEN}} + \underbrace{\frac{f(x)-f(-x)}{2}}_{\text{ODD}}$$

(v) The only function which is defined on the entire number line & even and odd at the same time is f(x)=0

f(x)	g(x)	*f* (x) + g(x)	*f*(x) – g(x)	*f* (x) . g(x)	*f* (x)/g(x)	(go*f*) (x)	(*f*og)(x)
odd	odd	odd	odd	even	even	odd	odd
even	even	even	even	even	even	even	even
odd	even	neither odd nor even	neither odd nor even	odd	odd	even	even
even	odd	neither odd nor even	neither odd nor even	odd	odd	even	even

Illustrations

Illustration 14. Which of the following functions is (are) even, odd or neither :

(i) $f(x) = x^2\sin x$ (ii) $f(x) = \sqrt{1+x+x^2} - \sqrt{1-x+x^2}$

Solution. (i) $f(-x) = (-x)^2 \sin(-x) = -x^2 \sin x = -f(x)$. Hence f(x) is odd.

(ii) $f(-x) = \sqrt{1+(-x)+(-x)^2} - \sqrt{1-(-x)+(-x)^2}$

$= \sqrt{1-x+x^2} - \sqrt{1+x+x^2} = -f(x)$. Hence f(x) is odd.

Illustration 15. Which of the following functions is (are) even, odd or neither :

(i) $f(x) = \log\left(\frac{1-x}{1+x}\right)$ (ii) $f(x) = \sin x - \cos x$ (iii) $f(x) = \frac{e^x + e^{-x}}{2}$

Solution. (i) $f(-x) = \log\left(\frac{1-(-x)}{1+(-x)}\right) = \log\left(\frac{1+x}{1-x}\right) = -f(x)$. Hence f(x) is odd

(ii) $f(-x) = \sin(-x) - \cos(-x) = -\sin x - \cos x$.

Hence f(x) is neither even nor odd.

(iii) $f(-x) = \frac{e^{-x} + e^{-(-x)}}{2} = \frac{e^{-x}+e^x}{2} = f(x)$. Hence f(x) is even

Illustration 16. Identify the given functions as odd, even or neither :

(i) $f(x) = \frac{x}{e^x - 1} + \frac{x}{2} + 1$

(ii) $f(x + y) = f(x) + f(y)$ for all $x, y \in R$

Solution. (i) $f(x) = \frac{x}{e^x - 1} + \frac{x}{2} + 1$

Clearly domain of f(x) is $R \sim \{0\}$. We have,

$$f(-x) = \frac{-x}{e^{-x}-1} - \frac{x}{2} + 1 = \frac{-e^x.x}{1-e^x} - \frac{x}{2} + 1 = \frac{(e^x-1+1)x}{(e^x-1)} - \frac{x}{2} + 1$$

$$= x + \frac{x}{e^x-1} - \frac{x}{2} + 1 = \frac{x}{e^x-1} + \frac{x}{2} + 1 = f(x)$$

Hence f(x) is an even function.

(ii) $f(x + y) = f(x) + f(y)$ for all $x, y \in R$

Replacing x, y by zero, we get $f(0) = 2f(0)$

$\Rightarrow \quad f(0) = 0$

Replacing y by –x, we get $f(x) + f(-x) = f(0) = 0$

$\Rightarrow \quad f(x) = -f(-x)$

Hence f(x) is an odd function.

9.0 CLASSIFICATION OF FUNCTIONS

SL AL

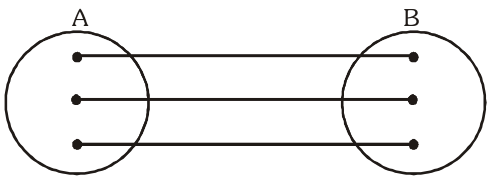
(a) *One-One function (Injective mapping)*

A function $f : A \to B$ is said to be a one-one function or injective mapping if different elements of A have different f images in B. Thus for $x_1, x_2 \in A$ & $f(x_1), f(x_2) \in B$, $f(x_1) = f(x_2) \Leftrightarrow x_1 = x_2$ or $x_1 \neq x_2 \Leftrightarrow f(x_1) \neq f(x_2)$.

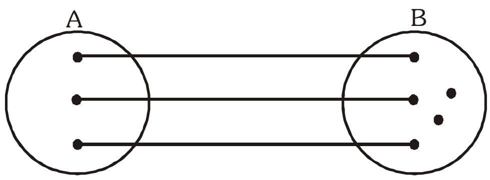
Note– (i) Any continuous function which is entirely increasing or decreasing in whole domain is one-one.

(ii) If a function is one-one, any line parallel to x-axis cuts the graph of the function at atmost one point

Diagramatically an injective mapping can be shown



or

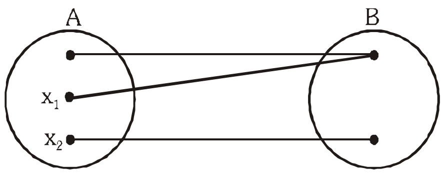


(b) *Many-one function*

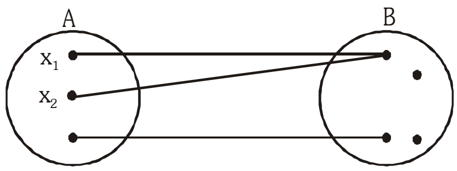
A function $f - A \to B$ is said to be a many one function if two or more elements of A have the same f image in B.

Thus $f - A \to B$ is many one if $\exists\ x_1, x_2 \in A$, $f(x_1) = f(x_2)$ but $x_1 \neq x_2$

Diagramatically a many one mapping can be shown



or

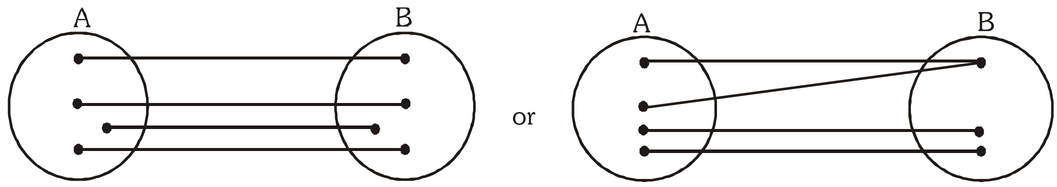


Note – If a continuous function has local maximum or local minimum, then f(x) is many-one because atleast one line parallel to x-axis will intersect the graph of function atleast twice.

(c) *Onto function (Surjective mapping)*

If the function $f : A \to B$ is such that each element in B (co-domain) is the 'f' image of atleast one element in A, then we say that f is a function of A 'onto' B. Thus $f : A \to B$ is surjective if $\forall\ b \in B$, $\exists$ some $a \in A$ such that $f(a) = b$

Diagramatically surjective mapping can be shown

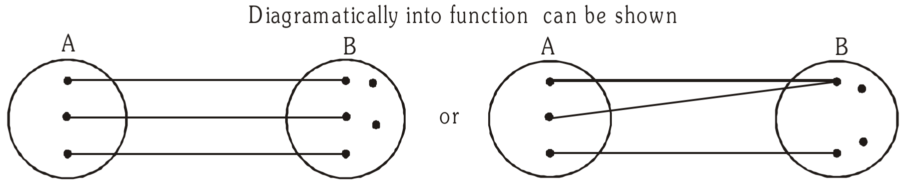


Note that – If range = co-domain, then f(x) is onto.

(d) Into function

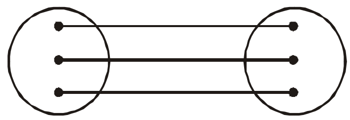
If f : A → B is such that there exists atleast one element in co-domain which is not the image of any element in domain, then f(x) is into.

Diagramatically into function can be shown

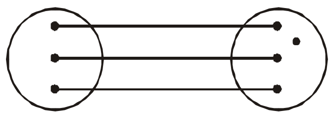


- ***Thus a function can be one of these four types***

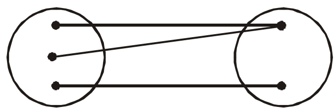
(i) one-one onto (injective & surjective)



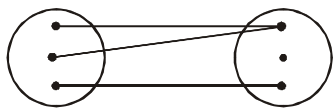
(ii) one-one into (injective but not surjective)



(iii) many-one onto (surjective but not injective)



(iv) many-one into (neither surjective nor injective)



Note – (i) If 'f' is both injective & surjective, then it is called a ***Bijective*** mapping. The bijective functions are also named as invertible, non singular or biuniform functions.

(ii) If a set A contains n distinct elements then the number of different functions defined from $A \to A$ is n^n & out of it n! are one one and rest are many one.

(iii) $f\text{–}R \to R$ is a polynomial

(a) Of even degree, then it will neither be injective nor surjective.

(b) Of odd degree, then it will always be surjective, no general comment can be given on its injectivity.

Illustrations

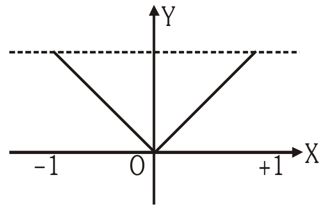
Illustration 17. Let $A = \{x : -1 \le x \le 1\} = B$ be a mapping $f : A \to B$. For each of the following functions from A to B, find whether it is surjective or bijective.

(a) $f(x) = |x|$ (b) $f(x) = x|x|$ (c) $f(x) = x^3$

(d) $f(x) = [x]$ (e) $f(x) = \sin\dfrac{\pi x}{2}$

Solution. ***(a)*** $f(x) = |x|$

Graphically ;



Which shows many one, as the straight line is parallel to x-axis and cuts at two points.

Here range for $f(x) \in [0, 1]$

Which is clearly subset of co-domain i.e., $[0, 1] \subseteq [-1,1]$

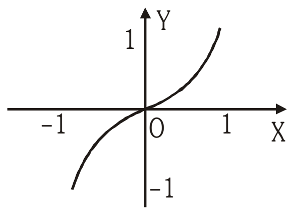
Thus, into.

Hence, function is many-one-into

$\therefore$ Neither injective nor surjective

(b) $f(x) = x|x| = \begin{Bmatrix} -x^2, & -1 < x < 0 \\ x^2, & 0 \le x < 1 \end{Bmatrix}$,

Graphically,



The graph shows f(x) is one-one, as the straight line parallel to x-axis cuts only at one point.

Here, range

$f(x) \in [-1, 1]$

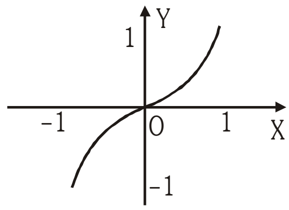
Thus, range = co-domain

Hence, onto.

Therefore, f(x) is one-one onto or (Bijective).

(c) $f(x) = x^3$,

Graphically;



Graph shows f(x) is one-one onto

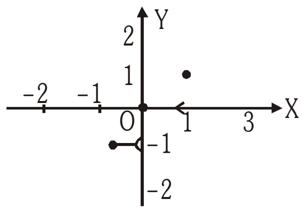
(i.e. Bijective)

[as explained in above example]

(d) $f(x) = [x]$,

Graphically;

Which shows f(x) is many-one, as the straight line



parallel to x-axis meets at more than one point.

Here, range

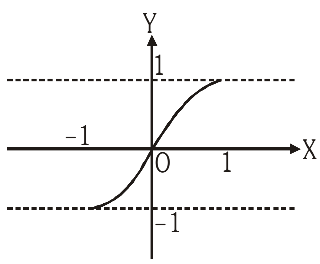
$f(x) \in \{-1, 0, 1\}$

which shows into as range $\subseteq$ co-domain

Hence, many-one-into

(e) $f(x) = \sin\dfrac{\pi x}{2}$

Graphically;



Which shows f(x) is one-one and onto as range = co-domain.

Therefore, f(x) is bijective.

****Illustration 18.*** Let $f : R \to R$ be a function defined by $f(x) = x + \sqrt{x^2}$, then f is

(A) injective (B) surjective (C) bijective (D) None of these

Solution. We have, $f(x) = x + \sqrt{x^2} = x + |x|$

Clearly, f is not one-one as $f(-1) = f(-2) = 0$ and $-1 \neq -2$

Also, f is not onto as $f(x) \ge 0\ \forall\ x \in R$

$\therefore$ range of $f = (0, \infty) \subset R$ ***Ans.(D)***

***Illustration 19.** Let $f(x) = \dfrac{x^2+3x+a}{x^2+x+1}$, where $f : R \to R$. Find the value of parameter 'a' so that the given function is one-one.

Solution. $f(x) = \dfrac{x^2+3x+a}{x^2+x+1}$

$$f'(x) = \frac{(x^2+x+1)(2x+3)-(x^2+3x+a)(2x+1)}{(x^2+x+1)^2} = \frac{-2x^2+2x(1-a)+(3-a)}{(x^2+x+1)^2}$$

Let, $g(x) = -2x^2 + 2x(1-a) + (3-a)$

$g(x)$ will be negative if $4(1-a)^2 + 8(3-a) < 0$

$\Rightarrow$ $1 + a^2 - 2a + 6 - 2a < 0 \Rightarrow (a-2)^2 + 3 < 0$

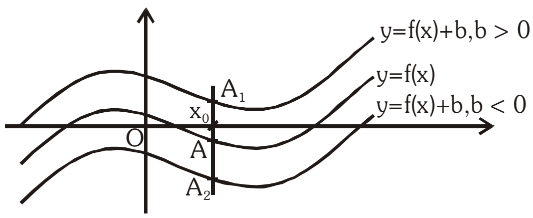
which is not possible. Therefore function is not monotonic.

Hence, no value of a is possible.

10.0 BASIC TRANSFORMATIONS ON GRAPHS

AL

(i) Drawing the graph of $y = f(x) + b$, $b \in R$, from the known graph of $y = f(x)$



It is obvious that domain of f(x) and f(x) + b are the same. Let us take any point x_0 in the domain of f(x). $y|_{x=x_0} = f(x_0)$.

The corresponding point on f(x) + b would be $f(x_0) + b$.

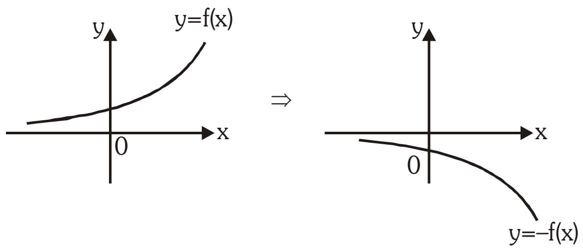
For $b > 0 \Rightarrow f(x_0) + b > f(x_0)$ it means that the corresponding point on f(x) + b would be lying at a distance 'b' units above the point on f(x).

For $b < 0 \Rightarrow f(x_0) + b < f(x_0)$ it means that the corresponding point on f(x) + b would be lying at a distance 'b' units below the point on f(x).

Accordingly the graph of f(x) + b can be obtained by translating the graph of f(x) either in the positive y-axis direction (if b > 0) or in the negative y-axis direction (if b < 0), through a distance |b| units.

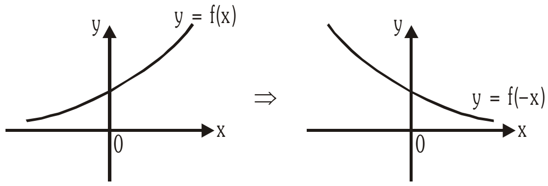
(ii) Drawing the graph of $y = -f(x)$ from the known graph of $y = f(x)$

To draw $y = -f(x)$, take the image of the curve $y = f(x)$ in the x-axis as plane mirror.



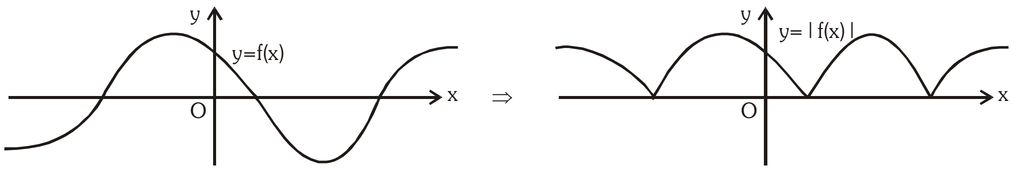
(iii) Drawing the graph of $y = f(-x)$ from the known graph of $y = f(x)$

To draw $y = f(-x)$, take the image of the curve $y = f(x)$ in the y-axis as plane mirror.



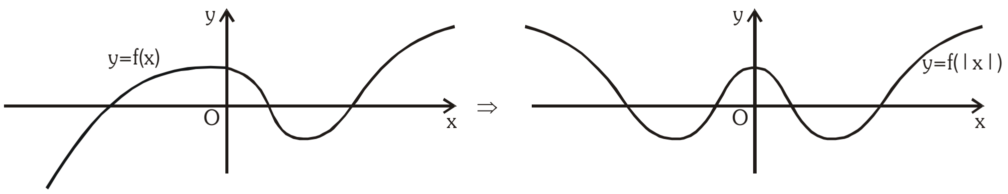
(iv) Drawing the graph of $y = |f(x)|$ from the known graph of $y = f(x)$

$|f(x)| = f(x)$ if $f(x) \geq 0$ and $|f(x)| = -f(x)$ if $f(x) < 0$. It means that the graph of f(x) and $|f(x)|$ would coincide if $f(x) \geq 0$ and for the portions where $f(x) < 0$ graph of $|f(x)|$ would be image of $y = f(x)$ in x-axis.



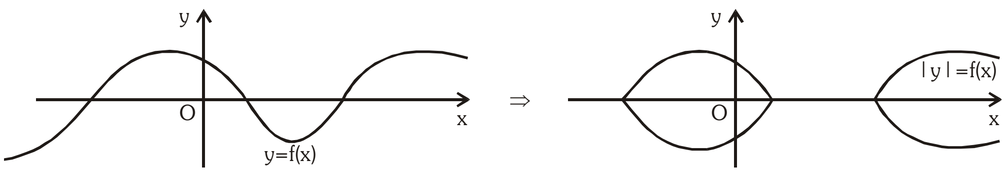
(v) Drawing the graph of $y = f(|x|)$ from the known graph of $y = f(x)$

It is clear that, $f(|x|) = \begin{cases} f(x), & x \geq 0 \\ f(-x), & x < 0 \end{cases}$. Thus $f(|x|)$ would be a even function, graph of $f(|x|)$ and f(x) would be identical in the first and the fourth quadrants (ax $x \geq 0$) and as such the graph of $f(|x|)$ would be symmetric about the y-axis (as $(|x|)$ is even).

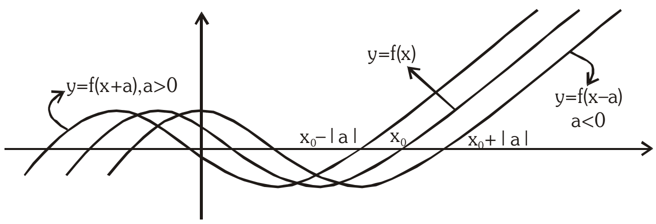


(vi) Drawing the graph of $|y| = f(x)$ from the known graph of $y = f(x)$

Clearly $|y| \geq 0$. If $f(x) < 0$, graph of $|y| = f(x)$ would not exist. And if $f(x) \geq 0$, $|y| = f(x)$ would give $y = \pm f(x)$. Hence graph of $|y| = f(x)$ would exist only in the regions where f(x) is non-negative and will be reflected about the x-axis only in those regions.



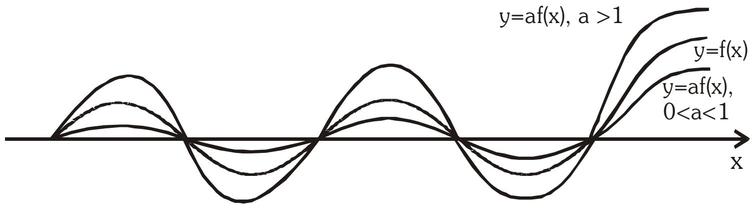
(vii) Drawing the graph of $y = f(x + a)$, $a \in R$ from the known graph of $y = f(x)$



(i) If $a > 0$, shift the graph of f(x) through 'a' units towards left of f(x).

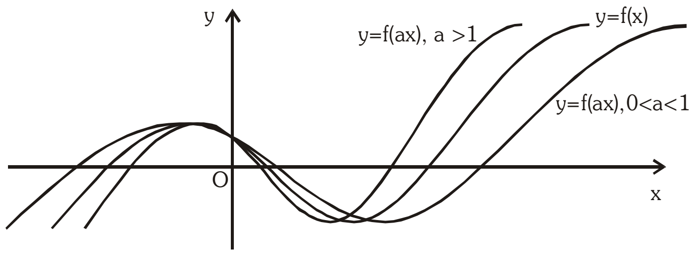
(ii) If $a < 0$, shift the graph of f(x) through 'a' units towards right of f(x).

(viii) Drawing the graph of y = af(x) from the known graph of y = f(x)



It is clear that the corresponding points (points with same x co-ordinates) would have their ordinates in the ratio of 1 : a.

(ix) Drawing the graph of y = f(ax) from the known graph of y = f(x).



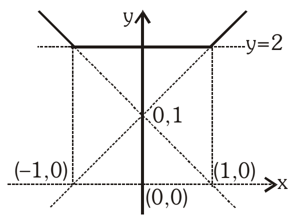
Let us take any point $x_0 \in$ domain of f(x). Let $ax = x_0$ or $x = \frac{x_0}{a}$.

Clearly if $0 < a < 1$, then $x > x_0$ and $f(x)$ will stretch by $\frac{1}{a}$ units along the y-axis and if $a > 1$, $x < x_0$, then f(x) will compress by 'a' units along the y-axis.

Illustrations

Illustration 20. Find f(x) = max {1 + x, 1 – x, 2}.

Solution. From the graph it is clear that



$$f(x) = \begin{cases} 1-x\ ; & x<-1 \\ 2\ ; & -1 \le x \le 1 \\ 1+x\ ; & x>1 \end{cases}$$

Illustration 21. Draw the graph of y = |2 –|x –1||.

Solution.

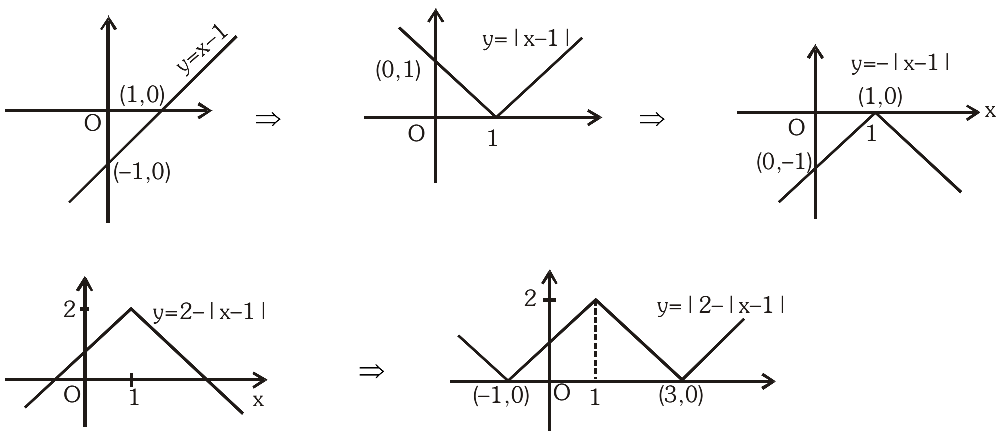


Illustration 22. Draw the graph of $y = 2 - \frac{4}{|x-1|}$

Solution.

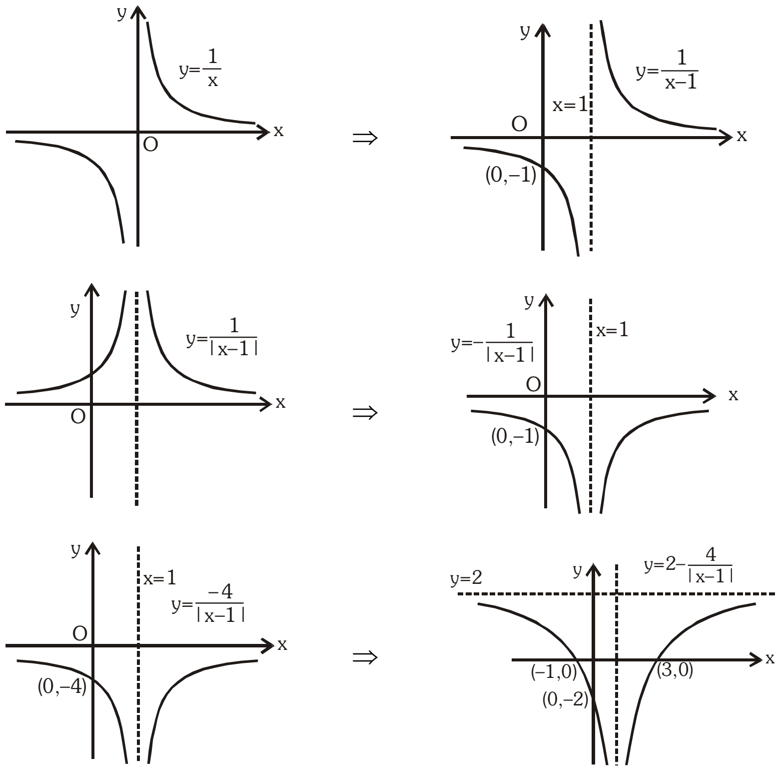


Illustration 23. Draw the graph of $y = |e^{|x|} - 2|$

Solution.

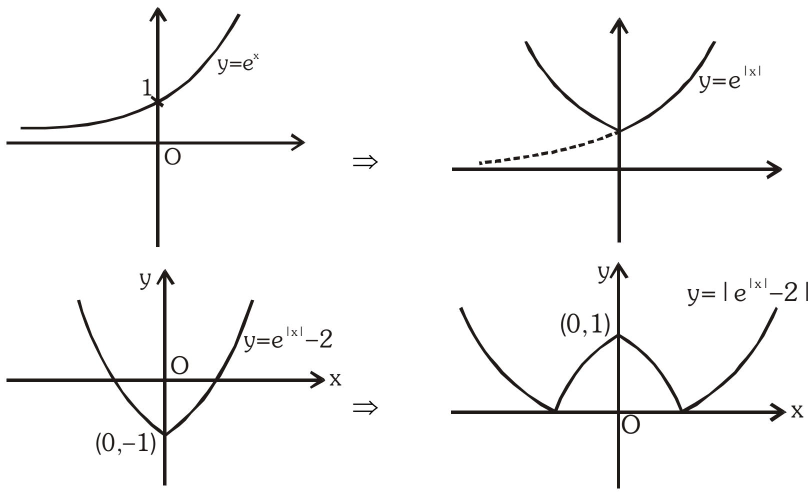
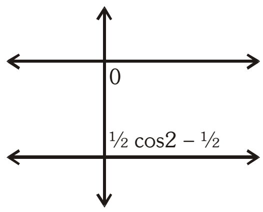


Illustration 24. Draw the graph of $f(x) = \cos x \cos(x + 2) - \cos^2(x + 1)$.

Solution. $f(x) = \cos x \cos(x + 2) - \cos^2(x + 1)$

$$= \frac{1}{2}\left[\cos(2x+2)+\cos 2\right] - \frac{1}{2}\left[\cos(2x+2)+1\right]$$

$$= \frac{1}{2}\cos 2 - \frac{1}{2} < 0.$$



BEGINNER'S BOX-4

TOPIC COVERED : ODD & EVEN FUNCTION, BASIC TRANSFORMATIONS ON GRAPHS

1. Determine the nature (odd/even) of function

(a) $f(x)=\dfrac{\sin^4 x+\cos^4 x}{x+x^2\tan x}$ (b) $f(x)=\log\left(\dfrac{1-x}{1+x}\right)$ (c) $f(x)=x\left(\dfrac{a^x-1}{a^x+1}\right)$

2. Draw the graph of following function

(a) $y = |x-1| + 2$ (b) $y=\sin\left(x+\dfrac{\pi}{4}\right)$ (c) $y = \sin 2x$

3. Draw the graph of following functions

(a) $y = \log_e(-x)$ (b) $y = |\ln |x||$ (c) $y = |x^2 - 3x + 2|$ (d) $y = |x|^2 - 2|x| + 3$

4. Find number of solution of $|\ln|x|| = k$, $k \in R$, for following values of k

(a) $k > 0$ (b) $k = 0$ (c) $k < 0$

5. Plot $|x| + |y| = 1$

***6.** Plot $y=\left|e^{-|x|}-\dfrac{1}{2}\right|$

7. Find number of values of x satisfying $e^x \ln x = 1$

8. The number of roots of equation $x+2\tan x=\dfrac{\pi}{2}$ in interval $[0, 2\pi]$ is

***9.** The total number of solution of the equation $|\ln |x|| = |\sin x|$ in $[-\pi,\pi]$ is

11.0 COMPOSITE OF UNIFORMLY & NON-UNIFORMLY DEFINED FUNCTION

SL AL

Let $f - A \to B$ & $g : B \to C$ be two functions. Then the function gof : $A \to C$ defined by (gof) (x) $= g(f(x))\ \forall\, x \in A$ is called the composite of the two functions f & g.

Diagramatically $\xrightarrow{x} \boxed{f} \xrightarrow{f(x)} \boxed{g} \longrightarrow g\,(f(x))$

Thus the image of every $x \in A$ under the function gof is the g-image of f-image of x.

Note that gof is defined only if $\forall\, x \in A$, f(x) is an element of the domain of 'g' so that we can take its g-image. Hence in gof(x) the range of 'f' must be a subset of the domain of 'g'.

• ***Properties of composite functions***

(a) In general composite of functions is not commutative i.e. gof $\neq$ fog.

(b) The composite of functions is associative i.e. if f, g, h are three functions such that fo(goh) & (fog)oh are defined, then fo(goh) = (fog)oh.

(c) The composite of two bijections is a bijection i.e. if f & g are two bijections such that gof is defined, then gof is also a bijection.

Illustrations

Illustration 25. If f be the greatest integer function and g be the modulus function, then $(gof)\left(-\dfrac{5}{3}\right)-(fog)\left(-\dfrac{5}{3}\right) =$

(A) 1 (B) –1 (C) 2 (D) 4

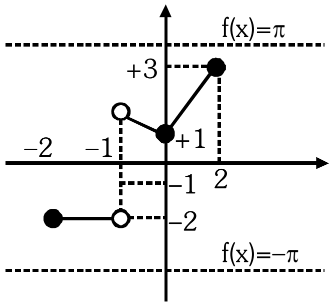
Solution. Given $(gof)\left(\dfrac{-5}{3}\right)-(fog)\left(\dfrac{-5}{3}\right)=g\left\{f\left(\dfrac{-5}{3}\right)\right\}-f\left\{g\left(\dfrac{-5}{3}\right)\right\}=g(-2)-f\left(\dfrac{5}{3}\right)=2-1=1$

Ans.(A)

***Illustration 26.** Find the domain and range of h(x) = g(f(x)), where

$$f(x)=\begin{cases}[x], & -2\le x\le -1\\ |x|+1, & -1<x\le 2\end{cases} \text{ and } g(x)=\begin{cases}[x], & -\pi\le x\le 0\\ \sin x, & 0\le x\le \pi\end{cases}$$, [.] denotes the greatest integer function.

Solution. $h(x)=g(f(x))=\begin{cases}[f(x)], & -\pi\le f(x)<0\\ \sin(f(x)), & 0\le f(x)\le\pi\end{cases}$



From graph of f(x), we get

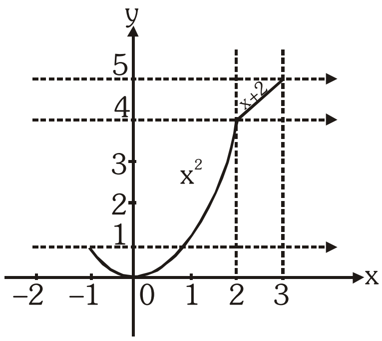
$$h(x)=\begin{cases}[[x]], & -2\le x\le -1\\ \sin(|x|+1), & -1<x\le 2\end{cases}$$

$\Rightarrow$ Domain of h(x) is [–2, 2]

and Range of h(x) is $\{-2, 1\}\cup[\sin 3, 1]$

Illustration 27. Let $f(x)=\begin{cases}x+1, & x\le 1\\ 2x+1, & 1<x\le 2\end{cases}$ and $g(x)=\begin{cases}x^2, & -1\le x<2\\ x+2, & 2\le x\le 3\end{cases}$, find (fog)

Solution. $f(g(x))=\begin{cases}g(x)+1, & g(x)\le 1\\ 2g(x)+1, & 1<g(x)\le 2\end{cases}$



Here, g(x) becomes the variable that means we should draw the graph.

It is clear that $g(x)\le 1$; $\forall\ x\in[-1, 1]$

and $1<g(x)\le 2$; $\forall\ x\in(1, \sqrt{2}\,]$

$$\Rightarrow \quad f(g(x))=\begin{cases}x^2+1, & -1\le x\le 1\\ 2x^2+1, & 1<x\le\sqrt{2}\end{cases}$$

12.0 HOMOGENEOUS FUNCTIONS

AL

A function is said to be homogeneous with respect to any set of variables when each of its terms is of the same degree with respect to those variables.

For examples $5x^2+3y^2-xy$ is homogenous in x & y. Symbolically if, $f(tx, ty)=t^n f(x, y)$ then f(x, y) is homogeneous function of degree n.

Illustrations

Illustration 28. Which of the following function is not homogeneous ?

(A) $x^3+8x^2y+7y^3$ (B) y^2+x^2+5xy

(C) $\dfrac{xy}{x^2+y^2}$ (D) $\dfrac{2x-y+1}{2y-x+1}$

Solution. It is clear that (D) does not have the same degree in each term. ***Ans. (D)***

13.0 BOUNDED FUNCTION

AL

A function is said to be bounded if $|f(x)|\le M$, where M is a finite quantity.

Illustrations

Illustration 29. If x, y ∈ [0, 10], then the number of solution (x, y) of the inequality $3^{\sec^2 x-1}\sqrt{9y^2-6y+2} \le 1$ are

Solution. If x, y ∈ [0, 10], then the.......

We have, $3^{\sec^2 x-1}\sqrt{9y^2-6y+2} \le 1$

$$\Rightarrow \quad 3^{\sec^2 x}\sqrt{\left(y-\frac{1}{3}\right)^2+\frac{1}{9}} \le 1$$

but $\quad 3^{\sec^2 x} \ge 3$ and $\sqrt{\left(y-\frac{1}{3}\right)^2+\frac{1}{9}} \ge \frac{1}{3}$

So, using boundness we must have $\sec^2 x = 1$ and $y-\frac{1}{3}=0$

$\Rightarrow \quad x = 0, \pi\ 2\pi, 3\pi$ and $y=\frac{1}{3}$

$\Rightarrow$ There are 4 solution.

BEGINNER'S BOX-5

TOPIC COVERED : ODD & EVEN FUNCTIONS, CLASSIFICATION OF FUNCTIONS, BASIC TRANSFORMATIONS ON GRAPHS, COMPOSITE OF UNIFORMLY & NON-UNIFORMLY DEFINED FUNCTION, HOMOGENEOUS FUNCTIONS, BOUNDED FUNCTION

1. Which of the following functions is (are) even, odd or neither.

(a) $f(x) = x^3 \sin 3x$ (b) $f(x) = \frac{e^{x^2}+e^{-x^2}}{2x}$ (c) $f(x) = \frac{e^x-e^{-x}}{e^x+e^{-x}}$ (d) $f(x) = x^2 + 2^x$

2. Is the function $f - N \to N$ (the set of natural numbers) defined by $f(x) = 2x + 3$ surjective ?

3. Let $A = R - \{3\}$, $B = R - \{1\}$ and let $f : A \to B$ defined by $f(x)=\frac{x-2}{x-3}$. Check whether the function f(x) is bijective or not.

4. The function $f - R \to R$, where R is the set of all real numbers defined by $f(x) = 2x + 3$ is -

(A) f is both one-one and onto (B) f is one-one but not onto

(C) f is onto but not one-one (D) f is neither one-one nor onto

5. $f(x) = x^3 - x$ & $g(x) = \sin 2x$, find

(a) $f(f(1))$ (b) $f(f(-1))$ (c) $f\left(g\left(\frac{\pi}{2}\right)\right)$

(d) $f\left(g\left(\frac{\pi}{4}\right)\right)$ (e) $g(f(1))$ (f) $g\left(g\left(\frac{\pi}{2}\right)\right)$

6. If $f(x) = \begin{cases} x+1; & 0 \le x < 2 \\ |x|; & 2 \le x < 3 \end{cases}$, then find fof(x).

7. Find the boundness of the function $f(x) = \frac{x^2}{x^4+1}$

***8.** If X = {1, 2, 3, 4, 5} and Y = {a, b, c, d, e, f} and $f : X \to Y$, then find total number of

(a) Functions (b) One to one function (c) Many one function (d) Constant function

(e) Onto function (f) Into function

14.0 IMPLICIT & EXPLICIT FUNCTION

AL

A function defined by an equation not solved for the dependent variable is called an ***implicit function***. e.g. the equations $x^3 + y^3 = 1$ & $x^y = y^x$, defines y as an implicit function. If y has been expressed in terms of x alone then it is called an ***Explicit function.***

Illustrations

Illustration 30. Which of the following function is implicit function ?

(A) $y = \dfrac{x^2 + e^x + 5}{\sqrt{1 - \cos^{-1} x}}$ (B) $y = x^2$

(C) $xy - \sin(x + y) = 0$ (D) $y = \dfrac{x^2 \log x}{\sin x}$

Solution. It is clear that in (C) y is not clearly expressed in x. ***Ans. (C)***

15.0 INVERSE OF A FUNCTION

AL

Let $f - A \to B$ be a one-one & onto function, then their exists a unique function $g : B \to A$ such that $f(x) = y \Leftrightarrow g(y) = x$, $\forall x \in A$ & $y \in B$. Then g is said to be inverse of f.

Thus $g = f^{-1} : B \to A = \{(f(x), x)) | (x, f(x)) \in f\}$.

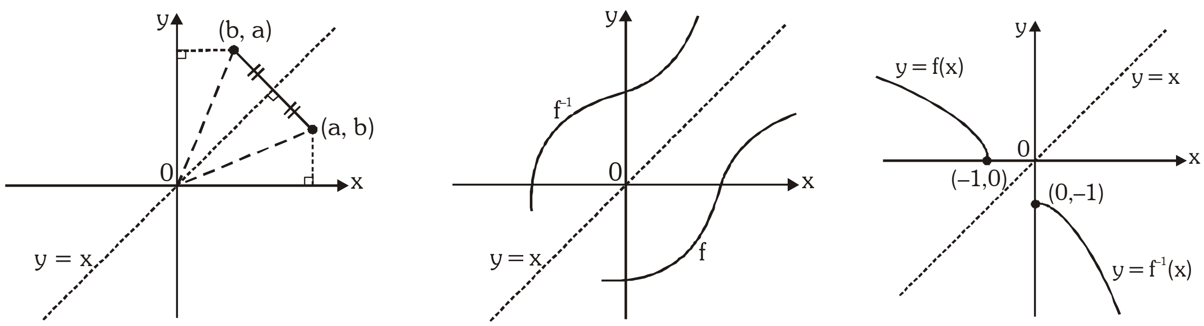
- ***Properties of inverse function***

(a) The inverse of a bijection is unique.

(b) If $f - A \to B$ is a bijection & $g : B \to A$ is the inverse of f, then $fog = I_B$ and $gof = I_A$, where I_A & I_B are identity functions on the sets A & B respectively. If $fof = I$, then f is inverse of itself.
(c) The inverse of a bijection is also a bijection.

(d) If f & g are two bijections $f : A \to B$, $g : B \to C$ then the inverse of gof exists and $(gof)^{-1} = f^{-1} \text{ o } g^{-1}$.

(e) Since $f(a) = b$ if and only if $f^{-1}(b) = a$, the point (a, b) is on the graph of 'f' if and only if the point (b, a) is on the graph of f^{-1}. But we get the point (b, a) from (a, b) by reflecting about the line $y = x$.



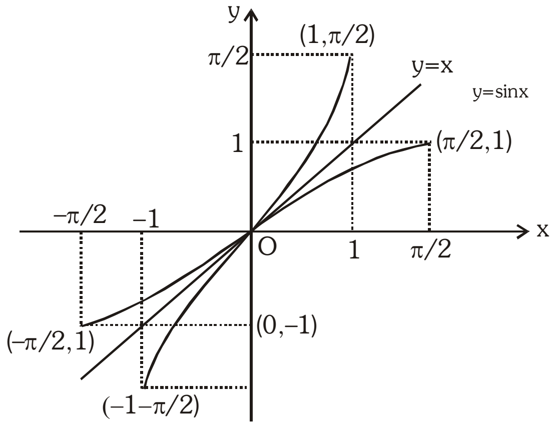
The graph of f^{-1} is obtained by reflecting the graph of f about the line $y = x$.

Drawing the graph of $y = f^{-1}(x)$ from the known graph of $y = f(x)$

For drawing the graph of $y = f^{-1}(x)$ we have to first of all find the interval in which the function is bijective (invertible). Then take the reflection of $y = f(x)$ (within the invertible region) about the line $y = x$. The reflected part would give us the graph of $y = f^{-1}(x)$.
e.g. let us draw the graph of $y = \sin^{-1}x$. We know that $y = f(x)$

$= \sin x$ is invertible if $f - \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to [-1, 1]$

$\Rightarrow$ the inverse mapping would be $f^{-1} - [-1, 1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.



Illustrations

Illustration 31. Let $f - R \to R$ be defined by $f(x) = (e^x - e^{-x})/2$. Is $f(x)$ invertible ? If so, find its inverse.

Solution. Let us check for invertibility of $f(x)$:

(a) One-One

Let $x_1, x_2 \in R$ and $x_1 < x_2$

$\Rightarrow$ $e^{x_1} < e^{x_2}$ (Because base $e > 1$) ... (i)

Also $x_1 < x_2 \Rightarrow -x_2 < -x_1$

$\Rightarrow$ $e^{-x_2} < e^{-x_1}$ (Because base $e > 1$) ... (ii)

(i) + (ii) $\Rightarrow e^{x_1} + e^{-x_2} < e^{x_2} + e^{-x_1}$

$\Rightarrow \frac{1}{2}(e^{x_1} - e^{-x_1}) < \frac{1}{2}(e^{x_2} - e^{-x_2}) \Rightarrow f(x_1) < f(x_2)$ i.e. f is one-one.

(b) Onto

As x tends to larger and larger values so does f(x) and

when $x \to \infty$, $f(x) \to \infty$.

Similarly as $x \to -\infty$, $f(x) \to -\infty$ i.e. $-\infty < f(x) < \infty$ so long as $x \in (-\infty, \infty)$

Hence the range of f is same as the set R. Therefore f(x) is onto.

Since f(x) is both one-one and onto, f(x) is invertible.

(c) To find f^{-1}

Let f^{-1} be the inverse function of f, then by rule of identity $fof^{-1}(x) = x$

$$\frac{e^{f^{-1}(x)} - e^{-f^{-1}(x)}}{2} = x \Rightarrow e^{2f^{-1}(x)} - 2xe^{f^{-1}(x)} - 1 = 0$$

$$\Rightarrow \quad e^{f^{-1}(x)} = \frac{2x \pm \sqrt{4x^2 + 4}}{2} \Rightarrow e^{f^{-1}(x)} = x \pm \sqrt{1 + x^2}$$

Since $e^{f^{-1}(x)} > 0$, hence negative sign is ruled out and

Hence $e^{f^{-1}(x)} = x + \sqrt{1 + x^2}$

Taking logarithm, we have $f^{-1}(x) = \ell n(x + \sqrt{1 + x^2})$.

Illustration 32. Find the inverse of the function $f(x) = \begin{cases} x; < 1 \\ x^2; 1 \le x \le 4 \\ 8\sqrt{x};\ \ x > 4 \end{cases}$

Solution. Given $f(x) = \begin{cases} x; < 1 \\ x^2; 1 \le x \le 4 \\ 8\sqrt{x};\ \ x > 4 \end{cases}$

Let $f(x) = y \Rightarrow x = f^{-1}(y)$

$$\therefore \quad x = \begin{cases} y, & y < 1 \\ \sqrt{y}, & 1 \le \sqrt{y} \le 4 \\ \dfrac{y^2}{64}, & \dfrac{y^2}{64} > 4 \end{cases}$$

$$\Rightarrow \quad f^{-1}(y) = \begin{cases} y, & y < 1 \\ \sqrt{y}, & 1 \le y \le 16 \\ \dfrac{y^2}{64}, & y > 16 \end{cases}$$

Hence $f^{-1}(x) = \begin{cases} x; < 1 \\ \sqrt{x}; 1 \le x \le 16 \\ \dfrac{x^2}{64}; x > 16 \end{cases}$ ***Ans.***

16.0 PERIODIC FUNCTION

AL

A function f(x) is called periodic if there exists a least positive number T(T >0) called the period of the function such that f(x + T) = f(x), for all values of x within the domain of f(x).

e.g. The function sinx & cosx both are periodic over 2π & tan x is periodic over π.

Note – For periodic function

(i) f(T) = f(0) = f(–T), where 'T' is the period.

(ii) Inverse of a periodic function does not exist.

(iii) Every constant function is periodic, but its period is not defined.

(iv) If f(x) has a period T & g(x) also has a period T then it does not mean that f(x) + g(x) must have a period T. e.g. f(x) = |sin x| + |cos x|.

(v) If f(x) has period p, then $\dfrac{1}{f(x)}$ and $\sqrt{f(x)}$ also has a period p.

(vi) If f(x) has period T then f(ax + b) has a period T/|a| (a ≠ 0).

17.0 GENERAL

AL

If x, y are independent variables, then

(a) $f(xy) = f(x) + f(y) \Rightarrow f(x) = k\ell n\ x$

(b) $f(xy) = f(x)\ .\ f(y) \Rightarrow f(x) = x^n,\ n \in R$ or $f(x) = 0$

(c) $f(x + y) = f(x)\ .\ f(y) \Rightarrow f(x) = a^{kx}$ or $f(x) = 0$

(d) $f(x + y) = f(x) + f(y) \Rightarrow f(x) = kx$, where k is a constant.

Illustrations

Illustration 33. Find the periods (if periodic) of the following functions, where [.] denotes the greatest integer function

(i) $f(x) = e^{\ell n(\sin x)} + \tan^3 x - \operatorname{cosec}(3x - 5)$

(ii) $f(x) = x - [x - b],\ b \in R$

(iii) $f(x) = \dfrac{|\sin x + \cos x|}{|\sin x| + |\cos x|}$

(iv) $f(x) = \tan \dfrac{\pi}{2}[x]$

(v) $f(x) = \cos(\sin x) + \cos(\cos x)$

(vi) $f(x) = \dfrac{(1+\sin x)(1+\sec x)}{(1+\cos x)(1+\operatorname{cosec} x)}$

(vii) $f(x) = e^{x-[x]+|\cos \pi x|+|\cos 2\pi x|+........+|\cos n\pi|}$

Solution. (i) $f(x) = e^{\ell n(\sin x)} + \tan^3 x - \operatorname{cosec}(3x - 5)$

Period of $e^{\ell n \sin x} = 2\pi,\ \tan^3 x = \pi$

$\operatorname{cosec}\ (3x - 5) = \dfrac{2\pi}{3}$

$\therefore$ Period $= 2\pi$

(ii) $f(x) = x - [x - b] = b + \{x - b\}$

$\therefore$ Period $= 1$

(iii) $f(x) = \dfrac{|\sin x + \cos x|}{|\sin x| + |\cos x|}$

Since period of $|\sin x + \cos x| = \pi$ and period of $|\sin x| + |\cos x|$ is $\dfrac{\pi}{2}$. Hence f(x) is periodic with π as its period

(iv) $f(x) = \tan \dfrac{\pi}{2}[x]$

$\tan \dfrac{\pi}{2}[x + T] = \tan \dfrac{\pi}{2}[x] \Rightarrow \dfrac{\pi}{2}[x + T] = n\pi + \dfrac{\pi}{2}[x]$

$\therefore$ $T = 2$

$\therefore$ Period $= 2$

(v) Let f(x) is periodic then $f(x + T) = f(x)$

$\Rightarrow$ $\cos(\sin(x + T)) + \cos(\cos(x + T)) = \cos(\sin x) + \cos(\cos x)$

If $x = 0$ then $\cos(\sin T) + \cos(\cos T)$

$=$ $\cos(0) + \cos(1) = \cos\left(\cos\dfrac{\pi}{2}\right) + \cos\left(\sin\dfrac{\pi}{2}\right)$

On comparing $T = \dfrac{\pi}{2}$

(vi) $f(x) = \dfrac{(1+\sin x)(1+\sec x)}{(1+\cos x)(1+\operatorname{cosec} x)}$

$= \dfrac{(1+\sin x)(1+\cos x)\sin x}{\cos x(1+\sin x)(1+\cos x)}$

$\Rightarrow$ $f(x) = \tan x$

Hence f(x) has period π.

(vii) $f(x) = e^{x-[x]+|\cos \pi x|+|\cos 2\pi x|+........+|\cos n\pi|}$

Period of $x - [x] = 1$

Period of $|\cos \pi x| = 1$

Period of $|\cos 2\pi x| = \dfrac{1}{2}$

....

Period of $|\cos n\pi x| = \dfrac{1}{n}$

So period of f(x) will be L.C.M. of all period $= 1$

Illustration 34. Find the periods (if periodic) of the following functions, where [.] denotes the greatest integer function

(i) $f(x) = e^{x-[x]} + \sin x$

(ii) $f(x) = \sin\dfrac{\pi x}{\sqrt{2}} + \cos\dfrac{\pi x}{\sqrt{3}}$

(iii) $f(x) = \sin\dfrac{\pi x}{\sqrt{3}} + \cos\dfrac{\pi x}{2\sqrt{3}}$

Solution. (i) Period of $e^{x-[x]} = 1$

period of $\sin x = 2\pi$

$\because$ L.C.M. of rational and an irrational number does not exist.

$\therefore$ not periodic.

(ii) Period of $\sin\dfrac{\pi x}{\sqrt{2}} = \dfrac{2\pi}{\pi/\sqrt{2}} = 2\sqrt{2}$

Period of $\cos\dfrac{\pi x}{\sqrt{3}} = \dfrac{2\pi}{\pi/\sqrt{3}} = 2\sqrt{3}$

$\because$ L.C.M. of two different kinds of irrational number does not exist.

$\therefore$ not periodic.

(iii) Period of $\sin\dfrac{\pi x}{\sqrt{3}} = \dfrac{2\pi}{\pi/\sqrt{3}} = 2\sqrt{3}$

Period of $\cos\dfrac{\pi x}{2\sqrt{3}} = \dfrac{2\pi}{\pi/2\sqrt{3}} = 4\sqrt{3}$

$\because$ L.C.M. of two similar irrational number exist.

$\therefore$ Periodic with period $= 4\sqrt{3}$ ***Ans.***

BEGINNER'S BOX-6

TOPIC COVERED : IMPLICIT & EXPLICIT FUNCTION, INVERSE OF A FUNCTION, PERIODIC FUNCTION, GENERAL

1. Which of the following function is implicit function ?

(A) $xy - \cos(x + y) = 0$ (B) $y = x^3$

(C) $y = \log(x^2 + x + 1)$ (D) $y = |x|$

2. Convert the implicit form into the explicit function.

(a) $xy = 1$ (b) $x^2y = 1$.

3. Let $f\ [-1, 1] \to [-1, 1]$ defined by $f(x) = x|x|$, find $f^{-1}(x)$.

4. Find the periods (if periodic) of the following functions.

(a) $f(x) = 2^{\ell n(\cos x)} + \tan^3 x$. (b) $f(x) = \sin(\sin x) + \sin(\cos x)$

(c) $f(x) = e^{x-[x]}$, [.] denotes greatest integer function (d) $f(x) = \left|\sin\frac{x}{2}\right| + \left|\cos\frac{x}{2}\right|$

(e) $f(x) = \log(\sin^{-1}(x - [x]))$ (f) $f(x) = \sin^4 x + \cos^4 x$

5. If $f : R \to R$ is a function satisfying the property $f(2x + 3) + f(2x + 7) = 2\ \forall\ x \in R$, then find the period of f(x)

Find inverse of the function

6. $f(x) = \dfrac{a^x - a^{-x}}{a^x + a^{-x}}$

7. $y = [1 - (x - 3)^4]^{1/7}$

8. Let $f : A \to B$ be a function defined by $f(x) = \sqrt{3}\sin x + \cos x + 4$. If f is invertible, then

(A) $A = \left[-\frac{2\pi}{3}, \frac{\pi}{3}\right]$, B = [2, 6] (B) $A = \left[\frac{\pi}{6}, \frac{5\pi}{6}\right]$, B = [–2, 2]

(C) $A = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, B = [2, 6] (D) $A = \left[-\frac{\pi}{3}, \frac{\pi}{3}\right]$, B = [2, 6]

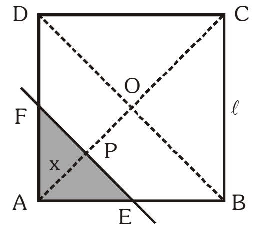
GOLDEN KEY POINTS

- To check whether relation is a function, vertical line test can be applied on its graph.
- Domain of f_1, f_2 be D_1, D_2 then domain of $f_1 + f_2$, $f_1 - f_2$, $f_1 \times f_2$ will be $D_1 \cap D_2$.
- f and g one are two functions defined for same domain, range of f is R and g is a bounded in domain. Then range of f(x) + g(x) is R.
- If domain consist of discrete number of elements range can be found by direct substituting the values of x.
- f & g are two one-one function defined for all real values then both fog & gof are one-one.
- $f : R \to R$, $G : R \to R$ and range of f is A then range of fog is A.

SOME WORKED OUT ILLUSTRATIONS

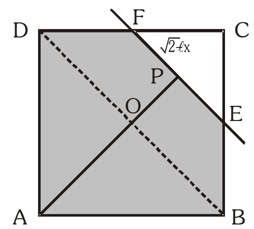
Illustration 1. ABCD is a square of side ℓ. A line parallel to the diagonal BD at a distance 'x' from the vertex A cuts two adjacent sides. Express the area of the segment of the square with A at a vertex, as a function of x. Find this area at $x = 1/\sqrt{2}$ and at x = 2, when $\ell = 2$.

Solution. There are two different situations



Case-I when x = AP ≤ OA, i.e., $x \le \dfrac{\ell}{\sqrt{2}}$

$$ar(\Delta AEF) = \frac{1}{2}x.2x = x^2 \qquad (\because PE = PF = AP = x)$$



Case-II when x = AP > OA, i.e., $x > \dfrac{\ell}{\sqrt{2}}$ but $x \le \sqrt{2}\ell$

ar(ABEFDA) = ar(ABCD) – ar(ΔCFE)

$$= \ell^2 - \frac{1}{2}\left(\sqrt{2}\ell - x\right).2\left(\sqrt{2}\ell - x\right) \quad [\because \; CP = \sqrt{2}\ell - x\,]$$

$$= \ell^2 - \left(2\ell^2 + x^2 - 2\sqrt{2}\ell x\right) = 2\sqrt{2}\ell x - x^2 - \ell^2$$

∴ the required function s(x) is as follows :

$$s(x) = \begin{cases} x^2, & 0 \le x \le \dfrac{\ell}{\sqrt{2}} \\ 2\sqrt{2}\,\ell x - x^2 - \ell^2, & \dfrac{\ell}{\sqrt{2}} < x \le \sqrt{2}\ell \end{cases} ;$$

$$\text{area of } s(x) = \begin{cases} \dfrac{1}{2} & \text{at } x = \dfrac{1}{\sqrt{2}} \\ 8(\sqrt{2}-1) & \text{at } x = 2 \end{cases} \qquad \textbf{\textit{Ans.}}$$

Illustration 2. If the function f(x) satisfies the functional rule, f(x + y) = f(x) + f(y) ∀ x,y ∈ R & f(1) = 5, then find $\sum_{n=1}^{m} f(n)$ and also prove that f(x) is odd function.

Solution. Here, f(x + y) = f(x) + f(y); put x = t – 1, y = 1

f(t) = f(t – 1) + f(1) ....(1)

∴ f(t) = f(t – 1) + 5

⇒ f(t) = {f(t – 2) + 5} + 5

⇒ f(t) = f(t – 2) + 2(5)

⇒ f(t) = f(t – 3) + 3(5)

....

....

⇒ f(t) = f{t – (t –1)} + (t – 1)5

⇒ f(t) = f(1)+ (t – 1)5

⇒ f(t) = 5 + (t – 1)5

⇒ f(t) = 5t

$$\therefore \quad \sum_{n=1}^{m} f(n) = \sum_{n=1}^{m}(5n) = 5[1+2+3+.....+m] = \frac{5m(m+1)}{2}$$

$$\text{Hence, } \sum_{n=1}^{m} f(n) = \frac{5m(m+1)}{2} \qquad .....(i)$$

Now putting x=0, y=0 in the given function, we have

$f(0 + 0) = f(0) + f(0)$

$\therefore \quad f(0) = 0$

Also putting (–x) for (y) in the given function.

$f(x - x) = f(x) + f(- x)$

$\Rightarrow \quad f(0) = f(x) + f(-x)$

$\Rightarrow \quad 0 = f(x) + f(-x)$

$\Rightarrow \quad f(-x) = -f(x)$ ....(ii)

Thus, $\sum_{n=1}^{m} f(n) = \frac{5m(m+1)}{2}$ and f(x) is odd function.

Illustration 3. Range of $f(x) = 4^x + 2^x + 1$ is

Solution. $f(x) = 4^x + 2^x + 1$

Let $2^x = t > 0, \forall x \in R$

$\therefore \quad f(x) = g(t) = t^2 + t + 1 \quad t > 0$

$$g(t) = \left(t+\frac{1}{2}\right)^2 + \frac{3}{4}$$

$$\left(t+\frac{1}{2}\right) > \frac{1}{2} \Rightarrow \left(t+\frac{1}{2}\right)^2 > \frac{1}{4} \Rightarrow \left(t+\frac{1}{2}\right)^2 + \frac{3}{4} > 1$$

Range is $(1, \infty)$

Illustration 4. Let $f : N \to N$, where $f(x) = x + (-1)^{x-1}$, then the inverse of f is

Solution. $f(x) = x + (-1)^{x-1}$

$$y = f(x) = \begin{cases} x+1 \ , & x \text{ is an odd natural number} \\ x-1 \ , & x \text{ is an even natural number} \end{cases}$$

$$x = \begin{cases} y-1 \ , & y \text{ is an even natural number} \\ y+1 \ , & y \text{ is an odd natural number} \end{cases}$$

$$f^{-1}(x) = \begin{cases} x-1 \ , & x \text{ is an even natural number} \\ x+1 \ , & x \text{ is an odd natural number} \end{cases}$$

$\therefore f^{-1}(x) = x + (-1)^{x-1}, x \in N$

Illustration 5. The domain of $f(x) = \sqrt{x^4 - x^3 + 1}$ is

Solution. $f(x) = \sqrt{x^4 - x^3 + 1}$

For f(x) to be defined.

$x^4 - x^3 + 1 \geq 0$ i.e. $(1 + x^4) - x^3 \geq 0$, which is true $\forall x \in R$

$\therefore$ Domain is $(-\infty, \infty)$

Illustration 6. The value of $\left[\frac{3}{4}+\frac{1}{100}\right]+\left[\frac{3}{4}+\frac{2}{100}\right]+\left[\frac{3}{4}+\frac{3}{100}\right]+.......\left[\frac{3}{4}+\frac{99}{100}\right]$, where [.] represents greatest integer function:

Solution.

$$\left[\frac{3}{4}+\frac{1}{100}\right]+\left[\frac{3}{4}+\frac{2}{100}\right]+\left[\frac{3}{4}+\frac{3}{100}\right]+.......\left[\frac{3}{4}+\frac{24}{100}\right]$$

$$+\left[\frac{3}{4}+\frac{25}{100}\right]+\left[\frac{3}{4}+\frac{26}{100}\right]+\left[\frac{3}{4}+\frac{27}{100}\right]+.......\left[\frac{3}{4}+\frac{99}{100}\right]$$

$= 0 + 75 = 75$

***Illustration 7.** Fundamental period of the function $f(x) = \dfrac{1}{\left||\sin 4x| - |\cos 4x|\right|} + \cos(\cos 6x)$ is

Solution. As $|\sin 4x| - |\cos 4x|$ has period $\dfrac{\pi}{4}$

But on taking $\left||\sin 4x| - |\cos 4x|\right|$ as g(x)

we get $g\left(x+\dfrac{\pi}{8}\right) = \left|\left|\sin\left(\dfrac{\pi}{2}+4x\right)\right| - \left|\cos\left(\dfrac{\pi}{2}+4x\right)\right|\right|$

$= \left||\cos 4x| - |\sin 4x|\right| = g(x)$

$\therefore$ Fundamental period of g(x) is $\dfrac{\pi}{8}$.

Now h(x) = cos (cos 6x)

then $h\left(x+\dfrac{\pi}{6}\right) = \cos(\cos(\pi + 6x))$

$= \cos(-\cos 6x) = \cos(\cos 6x)$ $\quad\therefore$ Period is $\dfrac{\pi}{6}$

Taking L.C.M. of $\dfrac{\pi}{6}, \dfrac{\pi}{8}$ we get $\dfrac{\pi}{2}$

Illustration 8. If f(x) = px + sinx is bijective function then complete set of values of p is

Solution. f(x) is one-one $\Rightarrow$ f(x) is monotonic function $\Rightarrow$ $p \in (-\infty, -1] \cup [1, \infty]$

f(x) is onto $\Rightarrow p \in R - \{0\}$

$\therefore$ f(x) is bijective if $p \in (-\infty, -1] \cup [1, \infty]$

Illustration 9. Let f (x) and g (x) be bijective functions where f: $\{7, 8, 9, 10\} \to \{1, 2, 3, 4\}$ and g:$\{3, 4, 5, 6\} \to \{13, 14, 15, 16\}$ respectively then find the number of elements in domain and range of gof(x) ?

Solution. domain of gof is $\{x \in \{7, 8, 9, 10\} : f(x) \in \{3,4\}\}$

$\therefore$ there are 2 elements in the domain of gof

Since gof is one-one, therefore, there are 2 elements in the range of gof.

ANSWERS

BEGINNER'S BOX-1

1. ***(a)*** Not a function ***(b)*** function ***(c)*** not a function ***(d)*** not a function

2. on sweeping a vertical line parallely if it cuts the graphs of relation on atmost one point, relation is a function.

3. ***(a)*** $x \in (0, \infty)$ ***(b)*** $x \in (-\infty, 5/2]$ ***(c)*** $x \in (-\infty, 0) \cup (4, \infty)$ ***(d)*** $x \in (-\infty, 0)$

4. ***(a)*** $(4, \infty)$ ***(b)*** $[-1, 3]$ ***(c)*** $[2, 3)$

5. ***(a)*** $[-1, 1]$ ***(b)*** $\left[\frac{1}{4}, \frac{1}{2}\right]$ ***(c)*** $(0, \infty)$ ***(d)*** $[-1, 1]$

6. ***(a)*** $[-2, 2]$ ***(b)*** $\left[0, \frac{1}{2}\right]$ ***(c)*** $\left[-\frac{1}{2}, \frac{1}{2}\right]$ ***(d)*** $\left[\ln\frac{11}{3}, \infty\right)$

7. $x = \dfrac{a_1 + a_2 + \ldots + a_n}{n}$ **8.** Range of $f = \left[\frac{1}{11}, \frac{1}{5}\right]$

BEGINNER'S BOX-2

1. $\pm 1/2, \pm 2$ **2.** $x \in [-1, 3]$ **3.** $x \in (-\infty, -2] \cup [2, \infty)$ **4.** 3

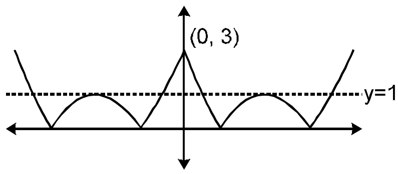
5. 3 **6.** $x = 1$ **7.** 28

BEGINNER'S BOX-3

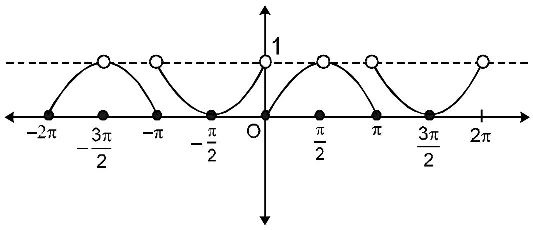
1. $x \in (-\infty, -\sqrt{8}) \cup (\sqrt{8}, \infty)$

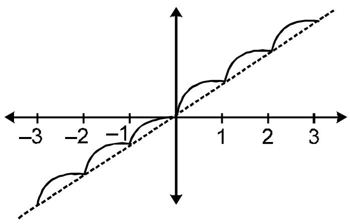
2. (A) $\to$ (q), (B) $\to$ (p), (C) $\to$ (u), (D) $\to$ (r), (E) $\to$ (s), (F) $\to$ (t)

3. (a) No (b) No (c) Yes

4. 

$a \in (1, 3) \cup \{0\}$

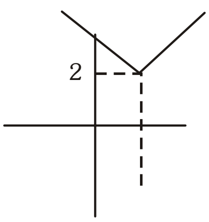
5. (i) 

(ii) 

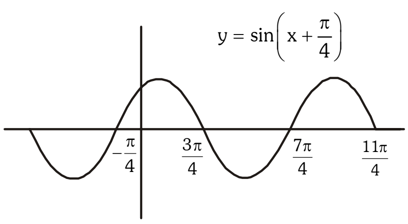
6. $\left\{\frac{3}{2}\right\}$ **7.** 4 **8.** sec 5 > sec1 > sec6 > sec3 > seec 4 > sec 2

BEGINNER'S BOX-4

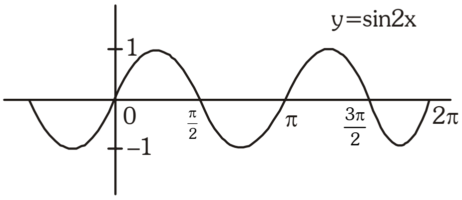
1. (a) odd function (b) odd function (c) even function

2. (a) 

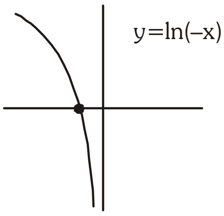
$y = |x - 1| + 2$

(b) 

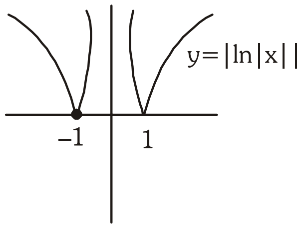
(c)



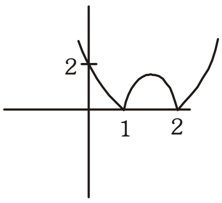
3. (a)



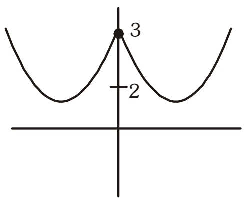
(b)



(c) $y = |x^2 - 3x + 2|$

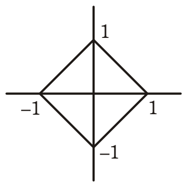


(d) $y = |x|^2 - 2|x| + 3$

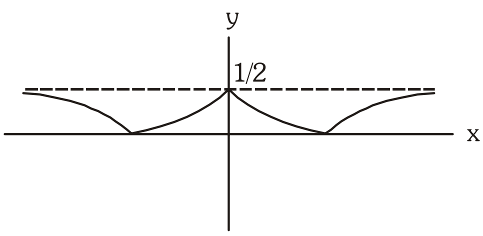


4. (a) 4 (b) 2 (c) 0

5. $|x| + |y| = 1$



6.



7. 1 **8.** 3 **9.** 4

BEGINNER'S BOX-5

1. ***(a)*** even ***(b)*** odd ***(c)*** odd ***(d)*** neither even nor odd

2. not onto **3.** yes **4.** A **5.** ***(a)*** 0 ***(b)*** 0 ***(c)*** 0 ***(d)*** 0 ***(e)*** 0 ***(f)*** 0

6. $\begin{cases} x+2, & 0 \le x < 1 \\ x+1, & 1 \le x < 2 \\ x, & 2 \le x < 3 \end{cases}$ **7.** $\left[0, \frac{1}{2}\right]$ **8.** (a) 6^5; (b) ${}^6C_5 \times 5!$; (c) $6^5 - 6!$; (d) 6; (e) 0; (f) 6^5

BEGINNER'S BOX-6

1. A **2.** ***(a)*** $y = \frac{1}{x}$ ***(b)*** $y = \frac{1}{x^2}$ **3.** $f^{-1}(x) = \begin{cases} -\sqrt{-x}, & -1 \le x \le 0 \\ \sqrt{x}, & 0 \le x \le 1 \end{cases}$

4. ***(a)*** 2π ***(b)*** 2π ***(c)*** 1 ***(d)*** π ***(e)*** 1 ***(f)*** $\frac{\pi}{2}$

5. 8 **6.** $\frac{1}{2}\log_a\left(\frac{1+x}{1-x}\right)$ **7.** $3 + (1 - x^7)^{1/4}$ **8.** (A)

EXERCISE – 1 MCQ (SINGLE CHOICE CORRECT)

1. If f : R→ R, which of the following rule is ***NOT*** a real function-

(A) $y = 4 - x^2$ (B) $y = 3x^2$ (C) $y = \sqrt{x} - |x|$ (D) $y = 3x^2 + 5$

2. The domain of $f(x) = \log_e |\log_e x|$ is-

(A) $(0, \infty)$ (B) $(1, \infty)$ (C) $(0, 1) \cup (1, \infty)$ (D) $(-\infty, 1)$

3. The domain of definition of the function : $f(x) = \ln(\sqrt{x^2 - 5x - 24} - x - 2)$ is

(A) $(-\infty, -3]$ (B) $(-\infty, -3] \cup [8, \infty)$ (C) $\left(-\infty, -\frac{28}{9}\right)$ (D) None of these

4. The domain of the function $f(x) = \frac{1}{\log_{10}(1-x)} + \sqrt{x+2}$, is -

(A) $[-2, 0) \cup (0, 1)$ (B) $(-2, 0) \cup (0, 1]$ (C) $(-2, 0) \cup (0, 1]$ (D) $(-2, 0) \cup [0, 1]$

5. The domain of the function $f(x) = \frac{1}{\sqrt{[x]^2 - [x] - 6}}$ is-

(A) $(-\infty, -2) \cup [4, \infty)$ (B) $(-\infty, -2] \cup [4, \infty)$ (C) $(-\infty, -2) \cup (4, \infty)$ (D) none of these

6. The range of the function $y = \frac{8}{9 - x^2}$ is

(A) $(-\infty, \infty) - \{\pm 3\}$ (B) $\left[\frac{8}{9}, \infty\right)$ (C) $\left(0, \frac{8}{9}\right)$ (D) $(-\infty, 0) \cup \left[\frac{8}{9}, \infty\right)$

7. Let [x] denote the greatest integer in x . Then in the interval [0, 3] the number of solutions of the equation, $x^2 - 3x + [x] = 0$ is :

(A) 6 (B) 4 (C) 2 (D) 0

8. If $f(x) = \frac{4^x}{4^x + 2}$, then $f(x) + f(1 - x)$ is equal to-

(A) 0 (B) –1 (C) 1 (D) 4

***9.** If $f(x) = \{x\} + \{x + 1\} + \{x + 2\}........\{x + 99\}$, then the value of $[f(\sqrt{2})]$ is, where {.} denotes fractional part function & [.] denotes the greatest integer function

(A) 5050 (B) 4950 (C) 41 (D) 14

10. Which of the following function(s) is identical to $g(x) = |x - 2|$

(A) $f(x) = \sqrt{x^2 - 4x + 4}$ (B) $f(x) = |x| - |2|$ (C) $f(x) = \frac{|x-2|^2}{|x-2|}$ (D) $f(x) = \left|\frac{x^2 - x + 2}{x - 1}\right|$

11. Function $f(x) = \log_e(x^3 + \sqrt{1 + x^6})$ is-

(A) even (B) odd (C) neither even nor odd (D) none of these

12. If $f : R \to R$, $f(x) = x^3 + x$, then f is-

(A) one-one onto (B) one-one into (C) many-one onto (D) many-one into

13. If S be the set of all triangles and $f : S \to R^+$, $f(\Delta)$ = Area of Δ, then f is-

(A) one-one onto (B) one-one into (C) many-one onto (D) many-one into

14. The function $f : R \to R$, defined as $f(x) = \dfrac{x^2 - 6x + 10}{3x - 3 - x^2}$ is :

(A) injective but not surjective (B) surjective but not injective

(C) injective as well as surjective (D) neither injective nor surjective

***15.** Let $f: R \to R$ be a function defined by $f(x) = \dfrac{x^2 - 3x + 4}{x^2 + 3x + 4}$ then f is -

(A) one – one but not onto (B) onto but not one – one

(C) onto as well as one – one (D) neither onto nor one – one

16. The values of the parameter α, for which the function $f(x) = 1 + \alpha x$, $\alpha \neq 0$ is the inverse of itself, is-

(A) 1 (B) 2 (C) –1 (D) 0

17. If $f : R \to R$, $f(x) = (x + 1)^2$ and $g : R \to R$, $g(x) = x^2 + 1$ then $(fog)(-3)$ is equal to-

(A) 121 (B) 144 (C) 112 (D) 11

18. If $f(x) = [x]$ and $g(x) = \cos(\pi x)$, then the range of gof is-

(A) $\{0\}$ (B) $\{-1, 1\}$ (C) $\{-1, 0, 1\}$ (D) $[-1, 1]$

***19.** If $f_0(x) = x/(x + 1)$ and $f_{n+1} = f_0 \text{ o } f_n$ for n = 0, 1, 2,......, then $f_n(x)$ is -

(A) $\dfrac{x}{(n+1)x+1}$ (B) $f_0(x)$ (C) $\dfrac{nx}{nx+1}$ (D) $\dfrac{x}{nx+1}$

20. The fundamental period of $\sin^4x + \cos^4x$ is-

(A) $\pi/2$ (B) π (C) 2π (D) none of these

21. The fundamental period of the function $f(x) = \sin\left(\dfrac{x}{3}\right) + \cos\left(\dfrac{x}{2}\right)$ is-

(A) 2π (B) 3π (C) 6π (D) 12π

22. Let f (x) be a function whose domain is [–5, 7]. Let $g(x) = |\,2x + 5\,|$, then the domain of fog(x) is :

(A) [–5, 1] (B) [–4, 0] (C) [–6, 1] (D) none of these

23. Let $f(x) = \sin\sqrt{[a]}\ x$ (where [ ] denotes the greatest integer function). If f is periodic with fundamental period π, then a belongs to -

(A) [2, 3) (B) {4, 5} (C) [4, 5] (D) [4, 5)

24. If $f : R \to R$ is a function satisfying the property $f(x+1) + f(x+3) = K$ (constant) $\forall\ x \in R$ then the period of $f(x)$ is -

(A) 4 (B) K (C) 1 (D) π

25. The inverse of the function $y = [1 - (x-3)^5]^{1/7}$ is- ***[JEE ADV]***

(A) $3 + (1 - x^7)^{1/5}$ (B) $3 - (1 - x^7)^{1/5}$ (C) $3 - (1 + x^7)^{1/5}$ (D) None of these

***26.** If the function $f : [1, \infty) \to [1, \infty)$ is defined by $f(x) = 2^{x(x-1)}$, then $f^{-1}(x)$ is-

(A) $\left(\frac{1}{2}\right)^{x(x-1)}$ (B) $\frac{1}{2}\left(1+\sqrt{1+4\log_2 x}\right)$ (C) $\frac{1}{2}\left(1-\sqrt{1+4\log_2 x}\right)$ (D) Not defined

27. Let $f(x) = \frac{4^x}{4^x + 2}$. Then value of $f\left(\frac{1}{1997}\right) + f\left(\frac{2}{1997}\right) + \cdots + \left(\frac{1996}{1997}\right)$ is :

(A) 898 (B) 998 (C) 991 (D) None of these

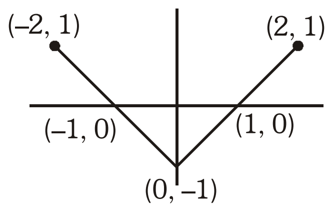
28. Let n(A) = 4 and n(B) = 6 then the number of one-one functions from A to B is :

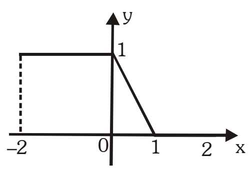
(A) 120 (B) 360 (C) 24 (D) None of these

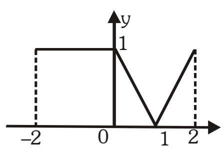
29. Set A has 3 elements and set B has 4 elements. The number of injective mappings that can be defined from A to B is-

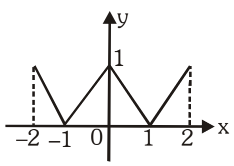
(A) 144 (B) 12 (C) 24 (D) 64

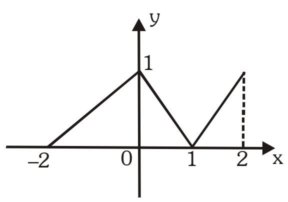
30. The graph of the function $y = f(x)$ is as shown in the figure. Then which of the following could represent the graph of the function $y = |f(x)|$?



(A)


(B)


(C)


(D)


EXERCISE – 2 MCQ (ONE OR MORE CHOICE CORRECT)

1. The function cot(sinx) -

(A) is not defined for $x=(4n+1)\frac{\pi}{2}$
(B) is not defined for $x = n\pi$
(C) lies between –cot 1 and cot 1
(D) can't lie between –cot 1 and cot 1

***2.** Range of the function f(x) = cos (K sinx) is [–1, 1], then the positive integral value of K can be -

(A) 1 (B) 2 (C) 5 (D) 4

3. The graph of the function $\cos x \cos(x + 2) - \cos^2(x + 1)$ can not be

(A) a straight line passing through $(0, -\sin^2 1)$ with slope 2
(B) a straight line passing through (0,0)
(C) a parabola with vertex $(1, -\sin^2 1)$
(D) a straight line passing through the point $\left(\frac{\pi}{2},-\sin^2 1\right)$ and parallel to the x-axis

4. If $f(x + ay,\ x - ay) = axy$ then f (x, y) is equal to -

(A) $\frac{x^2-y^2}{4}$ (B) $\frac{x^2+y^2}{4}$ (C) $\frac{x^4-y^4}{4(x^2+y^2)}$ (D) None of these

5. A function whose graph is ***NOT*** symmetrical about the origin is given by -

(A) $f(x) = e^x + e^{-x}$
(B) f(x) = sin(sin(cos(sinx)))
(C) f(x + y) = f(x) + f(y)
(D) sinx + sin|x|

6. Which of the functions are even -

(A) $\log\left(\frac{1+x^2}{1-x^2}\right)$ (B) $\sin^2 x + \cos^2 x$ (C) $\log\left(\frac{1+x^3}{1-x^3}\right)$ (D) $\frac{(1+2^x)^2}{2^x}$

7. If $f : (e, \infty) \to R$ & f(x) = log [log(log x)], then f is -

(A) one-one (B) many one (C) onto (D) into

8. Let $f(x) = \begin{cases} 4, & x < -1 \\ -4x, & -1 \le x \le 0 \end{cases}$

If f(x) is an even function in R then the definition of f(x) in $(0, \infty)$ is -

(A) $f(x) = \begin{cases} 4x, & 0 < x \le 1 \\ 4, & x > 1 \end{cases}$
(B) $f(x) = \begin{cases} 4x, & 0 < x \le 1 \\ -4, & x > 1 \end{cases}$
(C) $f(x) = \begin{cases} 4, & 0 < x \le 1 \\ 4x, & x > 1 \end{cases}$
(D) none of these

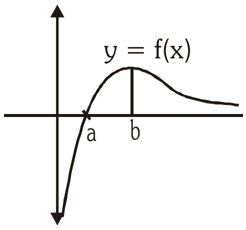
9. Which of the functions defined below are NOT one-one function(s)?

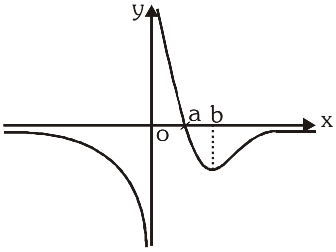
(A) $f(x) = 5(x^2 + 4),\ (x \in R)$
(B) $g(x) = 2x+(1/x)$
(C) $h(x) = \ell n(x^2+x+1),\ (x \in R)$
(D) $f(x) = e^{-x}$

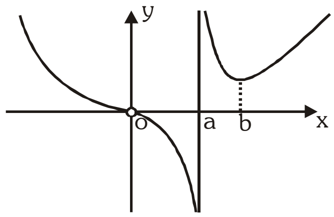
10. Let $f : R \rightarrow R$ be a function defined by $f(x) = \dfrac{e^{|x|} - e^{-x}}{e^{x} + e^{-x}}$ then -

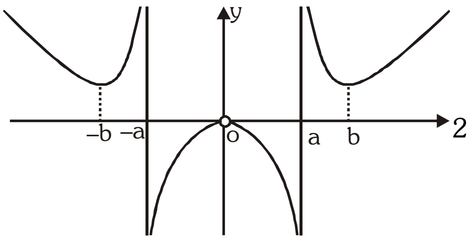
(A) injective (B) not injective (C) surjective (D) not surjective

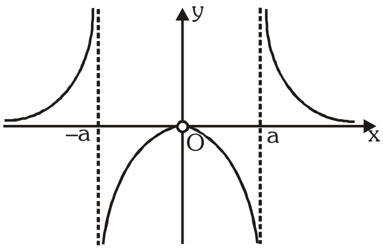
***11.** The graph of function f(x) is as shown, adjacently. Then the graph of $\dfrac{1}{f(|x|)}$ is -



(A)


(B)


(C)


(D)


12. $f(x) = \begin{cases} 2x+3 & x \le 1 \\ a^2x+1 & x > 1 \end{cases}$ values of 'a' for which f(x) is injective is -

(A) –3 (B) 3 (C) 0 (D) 1

13. Which of the following functions are periodic -

(A) sinx + cosx (B) $\cos x + \left\{\dfrac{x}{\pi}\right\}$ (C) $\cos\pi x + \{2x\}$ (D) $ln\{x\} + \sin 2x$

({x} denotes the fractional part of x)

14. f(x) and g(x) are two functions defined for all real values of x. f(x) is an even function and g(x) is periodic function, then -

(A) f [g(x)] is a periodic function
(B) g [f(x)] is a periodic function
(C) f [g(x)] is an even function
(D) g [f(x)] is an even function

15. Which of the following function(s) is/are periodic ?

(A) f(x) = 3x – [3x]
(B) g(x) = sin(1/x), x ≠ 0 & g(0) = 0
(C) h(x) = x cosx
(D) w(x) = sin(sin(sinx))

16. The period of the function $f(x) = \sin\left(\cos\dfrac{x}{2}\right) + \cos(\sin x)$ equal -

(A) 8π (B) 4π (C) π (D) $\dfrac{\pi}{2}$

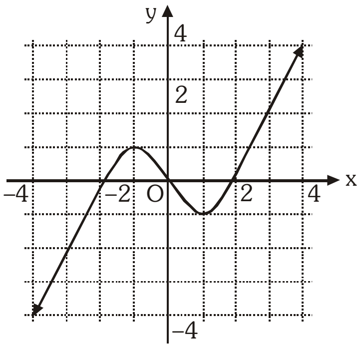
***17.** If g(x) is a polynomial satisfying $g(x)\,g(y) = g(x) + g(y) + g(xy) - 2$ for all real x and y and $g(2) = 5$ then g(3) is equal to -

(A) 10 (B) 24 (C) 21 (D) none of these

***18.** If f(x) is defined on (0, 1) then the domain of definition of $f(e^x) + f(\ln|x|)$ is subset of -

(A) $(-e, -1)$ (B) $(-e, -1) \cup (1, e)$ (C) $(-\infty, -1) \cup (1, \infty)$ (D) $(-e, e)$

19. The graph of $\phi(x)$ is given then the number of positive solution of $\left||\phi(x)|-1\right| = 1$ is less than-



(A) 5 (B) 2 (C) 3 (D) 1

Match the column

Following question contains statements given in two columns, which have to be matched. The statements in ***Column-I*** are labelled as A, B, C and D while the statements in ***Column-II*** are labelled as p, q, r and s. Any given statement in ***Column-I*** can have correct matching with ***ONE OR MORE*** statement(s) in ***Column-II***.

20.

Column - I	***Column - II***
(A) $f : R \to R$, $f(x) = (x-1)(x-2)........(x-11)$	(p) one one
(B) $f : R - \{-4/3\} \to R$, $f(x) = \dfrac{2x+1}{3x+4}$	(q) onto
(C) $f : R \to R$, $f(x) = e^{\sin x} + e^{-\sin x}$	(r) many one
(D) $f : R \to R$, $f(x) = \log(x^2 + 2x + 3)$	(s) into

Comprehension Based Questions

Comprehension – 1

If $f(x) = \begin{cases} x+1, & \text{if } x \le 1 \\ 5-x^2, & \text{if } x > 1 \end{cases}$ & $g(x) = \begin{cases} x, & \text{if } x \le 1 \\ 2-x, & \text{if } x > 1 \end{cases}$

On the basis of above information, answer the following questions

21. The range of $f(x)$ is -

(A) $(-\infty, 4)$ (B) $(-\infty, 5)$ (C) R (D) $(-\infty, 4]$

22. If $x \in (1, 2)$, then $g(f(x))$ is equal to -

(A) $x^2 + 3$ (B) $x^2 - 3$ (C) $5 - x^2$ (D) $1 - x$

23. Number of negative integral solutions of $g(f(x)) + 2 = 0$ are -

(A) 0 (B) 3 (C) 1 (D) 2

EXERCISE – 3 SUBJECTIVE LEVEL-I

1. Find a function that represents the amount of air required to inflate the spherical balloon from a radius of r inches to a radius of r + 1 inches.

2. Solve the following problems from (a) to (d) on functional equation :

(a) The function f(x) defined on the real numbers has the property that $f(f(x)).(1+f(x))=-f(x)$ for all x in the domain of f. If the number 3 is in the domain and range of f, compute the value of f(3).

(b) Suppose f is a real function satisfying $f(x + f(x)) = 4f(x)$ and $f(1) = 4$. Find the value of f(21).

*(c) Let f be function defined from $R^+ \to R^+$. If $[f(xy)]^2 = x(f(y))^2$ for all positive numbers x and y and $f(2) = 6$, find the value of f(50).

(d) Let f(x) be a function with two properties

(i) for any two real number x and y, $f(x + y) = x + f(y)$ and (ii) $f(0) = 2$.Find the value of f(100).

***3.** Let f be a function such that $f(3) = 1$ and $f(3x) = x + f(3x - 3)$ for all x. Then find the value of f(300).

4. Let $f(x) = \dfrac{9^x}{9^x + 3}$ then find the value of the sum $f\left(\dfrac{1}{2008}\right) + f\left(\dfrac{2}{2008}\right) + f\left(\dfrac{3}{2008}\right) + ...f\left(\dfrac{2007}{2008}\right)$

5. Find the domains of definitions of the following functions :

(Read the symbols [*] and {*} as greatest integers and fractional part functions respectively)

(a) $f(x) = \sqrt{\cos 2x} + \sqrt{16 - x^2}$

(b) $f(x) = \log_7 \log_5 \log_3 \log_2 (2x^3 + 5x^2 - 14x)$

(c) $f(x) = \dfrac{1}{\sqrt{4x^2 - 1}} + \ell n(x^2 - 1)$

*(d) $f(x) = \sqrt{\log_{\frac{1}{2}} \dfrac{x}{x^2 - 1}}$

(e) $f(x) = \log_x \sin x$

*(f) $f(x) = \sqrt{(5x - 6 - x^2)[\{\ell n\{x\}\}]} + \sqrt{(7x - 5 - 2x^2)} + \left(\ell n\left(\dfrac{7}{2} - x\right)\right)^{-1}$

(g) If $f(x) = \sqrt{x^2 - 5x + 4}$ & $g(x) = x + 3$, then find the domain of $\dfrac{f}{g}(x)$

*(h) $f(x) = \dfrac{1}{[x]} + \log_{1-\{x\}}(x^2 - 3x + 10) + \dfrac{1}{\sqrt{2 - |x|}} + \dfrac{1}{\sqrt{\sec(\sin x)}}$

6. The function f(x) is defined on the interval [0, 1]. Find the domain of definition of the functions.

(a) $f(\sin x)$

(b) $f(2x + 3)$

7. Find the domain & range of the following functions.

(a) $y = \log_{\sqrt{5}}\left(\sqrt{2}(\sin x - \cos x) + 3\right)$ (b) $y = \frac{2x}{1+x^2}$

(c) $y = \sqrt{2-x} + \sqrt{1+x}$ (d) $y = \sin(x^2 - 3x + 2)$

(e) $y = \tan(\sin x + \cos x)$ (f) $y = \cos(\log x)$

(g) $y = \sin^2 x + 4\sin x + 5$ *(h) $y = \sin(4\cos x)$

*(i) $y = \sin\sqrt{\frac{\pi^2}{9} - x^2}$

8. Classify the following function f(x) defined in $R \to R$ as injective, surjective, both or none

(a) $f(x) = \frac{x^2+x+1}{x^2-x+1}$ (b) $f(x) = x^3 + 6x^2 + 11x + 6$

*(c) $f(x) = (x^2 + x + 5)(x^2 + x - 3)$

9. Let $f(x) = \begin{cases} 1+x, & 0 \le x \le 2 \\ 3-x, & 2 < x \le 3 \end{cases}$, then find (fof)(x).

10. If $g(x) = 2x + 1$ and $h(x) = 4x^2 + 4x + 7$, find a function f such that $f \circ g = h$.

11. Find two distinct linear functions which map the interval [–1, 1] onto [0, 2].

12. Find whether the following functions are even or odd or none :

(a) $f(x) = \log\left(x + \sqrt{1+x^2}\right)$ (b) $f(x) = \frac{x(a^x+1)}{a^x-1}$

(c) $f(x) = \sin x + \cos x$ (d) f(x) = K, where K is constant

(e) $f(x) = \frac{(1+2^x)^2}{2^x}$ (f) $f(x) = [(x+1)^2]^{1/3} + [(x-1)^2]^{1/3}$

13. Find the period for each of the following functions :

(a) $f(x) = \sin^4 x + \cos^4 x$ (b) $f(x) = |\cos x|$ (c) $f(x) = |\sin x| + |\cos x|$ (d) $f(x) = \cos\frac{3}{5}x - \sin\frac{2}{7}x$

14. Prove that the functions;

(a) $f(x) = \cos\sqrt{x}$ (b) $f(x) = \sin\sqrt{x}$ (c) $f(x) = x + \sin x$ (d) $f(x) = \cos x^2$

are not periodic.

15. Write explicitly, functions of y defined by the following equations and also find the domains of definition of the given implicit functions :

(a) $10^x + 10^y = 10$ *(b) $x + |y| = 2y$

16. Compute the inverse of the functions :

(a) $f(x) = \ell n\left(x + \sqrt{x^2+1}\right)$ (b) $f(x) = 2^{\frac{x}{x-1}}$ (c) $y = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

***17.** Show if $f(x) = \sqrt[n]{a - x^n}$, $x > 0$ $n \ge 2$, $n \in N$, then (fof) (x) = x. Find also the inverse of f(x).

EXERCISE - 4 RECAP OF AIEEE/JEE (MAIN)

1. Which of the following is not a periodic function- **[AIEEE 2002]**

(A) $\sin 2x + \cos x$ (B) $\cos\sqrt{x}$ (C) $\tan 4x$ (D) $\log\cos 2x$

2. The period of $\sin^2 x$ is- **[AIEEE 2002]**

(A) $\pi/2$ (B) π (C) $3\pi/2$ (D) 2π

3. The function $f : R \to R$ defined by $f(x) = \sin x$ is- **[AIEEE 2002]**

(A) into (B) onto (C) one-one (D) many-one

4. The range of the function $f(x) = \dfrac{2+x}{2-x}$, $x \neq 2$ is- **[AIEEE 2002]**

(A) R (B) $R - \{-1\}$ (C) $R - \{1\}$ (D) $R - \{2\}$

5. The domain of $\sin^{-1}\left[\log_3\left(\dfrac{x}{3}\right)\right]$ **[AIEEE 2002]**

(A) [1, 9] (B) [-1, 9] (C) [-9, 1] (D) [-9, -1]

6. The function $f(x) = \log(x + \sqrt{x^2+1})$, is- **[AIEEE 2003]**

(A) neither an even nor an odd function (B) an even function
(C) an odd function (D) a periodic function

7. Domain of definition of the function $f(x) = \dfrac{3}{4-x^2} + \log_{10}(x^3 - x)$, is- **[AIEEE 2003]**

(A) $(-1, 0) \cup (1, 2) \cup (2, \infty)$ (B) $(1, 2)$ (C) $(-1, 0) \cup (1, 2)$ (D) $(1, 2) \cup (2, \infty)$

8. If $f : R \to R$ satisfies $f(x + y) = f(x) + f(y)$, for all $x, y \in R$ and $f(1) = 7$, then $\sum_{r=1}^{n} f(r)$ is- **[AIEEE 2003]**

(A) $\dfrac{7n(n+1)}{2}$ (B) $\dfrac{7n}{2}$ (C) $\dfrac{7(n+1)}{2}$ (D) $7n(n + 1)$

***9.** A function f from the set of natural numbers to integers defined by $f(n) = \begin{cases} \dfrac{n-1}{2}, & \text{when n is odd} \\ -\dfrac{n}{2}, & \text{when n is even} \end{cases}$ is- **[AIEEE 2003]**

(A) neither one-one nor onto (B) one-one but not onto
(C) onto but not one-one (D) one-one and onto both

10. The domain of the function $f(x) = \dfrac{\sin^{-1}(x-3)}{\sqrt{9-x^2}}$ is **[AIEEE - 2004]**

(A) [1, 2) (B) [2, 3) (C) [1, 2] (D) [2, 3]

***11.** The range of the function $f(x) = {}^{7-x}P_{x-3}$ is- **[AIEEE 2004]**

(A) $\{1, 2, 3, 4, 5\}$ (B) $\{1, 2, 3, 4, 5, 6\}$ (C) $\{1, 2, 3\}$ (D) $\{1, 2, 3, 4\}$

12. If $f : R \to S$ defined by $f(x) = \sin x - \sqrt{3}\cos x + 1$ is onto, then the interval of S is- **[AIEEE 2004]**

(A) $[-1, 3]$ (B) $[-1, 1]$ (C) $[0, 1]$ (D) $[0, -1]$

13. Let $f : (-1, 1) \to B$, be a function defined by $f(x) = \tan^{-1}\frac{2x}{1-x^2}$, then f is both one-one and onto when B is the interval- **[AIEEE 2005]**

(A) $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ (B) $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ (C) $\left(0, \frac{\pi}{2}\right)$ (D) $\left[0, \frac{\pi}{2}\right)$

***14.** A real valued function $f(x)$ satisfies the function equation $f(x-y) = f(x)f(y) - f(a-x)f(a+y)$ where a is a given constant and $f(0) = 1$, $f(2a-x)$ is equal to **[AIEEE 2005]**

(A) $f(A) + f(a-x)$ (B) $f(-x)$ (C) $-f(x)$ (D) $f(x)$

15. If x is real, the maximum value of $\frac{3x^2+9x+17}{3x^2+9x+7}$ is- **[AIEEE 2006]**

(A) 41 (B) 1 (C) $\frac{17}{7}$ (D) $\frac{1}{4}$

16. The largest internal lying in $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$ for which the function $\left[f(x) = 4^{-x^2} + \cos^{-1}\left(\frac{x}{2}-1\right) + \log(\cos x)\right]$ is defined, is **[AIEEE 2007]**

(A) $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ (B) $\left[-\frac{\pi}{4}, \frac{\pi}{2}\right)$ (C) $\left[0, \frac{\pi}{2}\right)$ (D) $[0, \pi]$

17. Let $f : R \to R$ be a function defined by $f(x) = \text{Min}\{x+1, |x|+1\}$. Then which of the following is true ?

(A) f (x) is not differentiable at x = 1 (B) f (x) is differentiable everywhere

(C) f (x) is not differentiable at x = 0 (D) $f(x) \geq 1$ for all $x \in R$ **[AIEEE 2007]**

18. Let $f : N \to Y$ be a function defined as $f(x) = 4x + 3$ where **[AIEEE 2008]**

$Y = \{y \in N : y = 4x + 3 \text{ for some } x \in N\}$. So that f is invertible and its inverse is

(A) $g(y) = \frac{3y+4}{3}$ (B) $g(y) = 4 + \frac{y+3}{4}$ (C) $g(y) = \frac{y+3}{4}$ (D) $g(y) = \frac{y-3}{4}$

***19.** For real x, let $f(x) = x^3 + 5x + 1$, then :- **[AIEEE 2009]**

(A) f is one-one and onto R (B) f is neither one-one nor onto R

(C) f is one-one but not onto R (D) f is onto R but not one-one

20. Let $f(x) = (x+1)^2 - 1$, $f : [-1, \infty) \to [-1, \infty)$. **[AIEEE 2009]**

Statement–1 – The set $\{x : f(x) = f^{-1}(x)\} = \{0, -1\}$.

Statement–2 – f is a bijection.

(A) Statement–1 is true, Statement–2 is false.

(B) Statement–1 is false, Statement–2 is true.

(C) Statement–1 is true, Statement–2 is true ; Statement–2 is a correct explanation for Statement–1.

(D) Statement–1 is true, Statement–2 is true ; Statement–2 is not a correct explanation for statement–1.

21. The domain of the function $f(x) = \dfrac{1}{\sqrt{|x|-x}}$ is :- **[AIEEE 2011]**

(A) $(-\infty, 0)$ (B) $(-\infty, \infty) - \{0\}$ (C) $(-\infty, \infty)$ (D) $(0, \infty)$

22. Let f be a function defined by $f(x) = (x-1)^2 + 1, f : [1, \infty) \to [1, \infty)$. **[AIEEE 2011]**

Statement - 1– The set $\{x : f(x) = f^{-1}(x)\} = \{1,2\}$

Statement - 2– f is bijection and

$f^{-1}(x) = 1 + \sqrt{x-1}, x \geq 1$.

(A) Statement–1 is true, Statement–2 is false.
(B) Statement–1 is false, Statement–2 is true.
(C) Statement–1 is true, Statement–2 is true ; Statement–2 is a correct explanation for Statement–1.
(D) Statement–1 is true, Statement–2 is true ; Statement–2 is not a correct explanation for statement–1.

23. If $f : R \to R$ is a function defined by $f(x) = [x] \cos\left(\dfrac{2x-1}{2}\right)\pi$, where [x] denotes the greatest integer function, then f is : **[AIEEE 2012]**

(A) continuous only at x = 0.
(B) continuous for every real x.
(C) discontinuous only at x = 0
(D) discontinuous only at non-zero integral values of x.

***24.** If g is the inverse of a function f and $f'(x) = \dfrac{1}{1+x^5}$, then $g'(x)$ is equal to **[JEE MAINS 2014]**

(A) $\dfrac{1}{1+\{g(x)\}^5}$ (B) $1+\{g(x)\}^5$ (C) $1 + x^5$ (D) $5x^4$

25. Let $f_k(x) = \dfrac{1}{k}(\sin^k x + \cos^k x)$ where $x \in \boldsymbol{R}$ and $K \geq 1$. Then $f_4(x) - f_6(x)$ equals: **[JEE MAINS 2014]**

(A) $\dfrac{1}{4}$ (B) $\dfrac{1}{12}$ (C) $\dfrac{1}{6}$ (D) $\dfrac{1}{3}$

26. If $f(x) + 2f\left(\dfrac{1}{x}\right) = 3x, x \neq 0$, and $S = \{x \in R : f(x) = f(-x)\}$; then S : **[JEE MAINS 2016]**

(A) is an empty set (B) contains exactly one element
(C) contains exactly two elemwnts (D) contains more than two elements

27. Let $a, b, c \in R$. If $f(x) = ax^2 + bx + c$ is such that $a + b + c = 3$ and $f(x+y) = f(x) + f(y) + xy, \forall\ x, y \in R$, then $\sum_{n=1}^{10} f(n)$ is euqal to : **[JEE MAINS 2017]**

(A) 190 (B) 255 (C) 330 (D) 165

28. The function $f : R \to \left[-\dfrac{1}{2}, \dfrac{1}{2}\right]$ defined as $f(x) = \dfrac{x}{1+x^2}$ is : **[JEE MAINS 2017]**

(A) surjective but not injective (B) neither injective nor surjective
(C) ivertible (D) injective but not surjective

29. Let A = {xεR : x is not a positive integer} Define a function f :A → R as $f(x)=\frac{2x}{x-1}$ then f is

(A) injective but not surjective
(B) not injective **[JEE MAINS 2019]**
(C) surjective but not injective
(D) neither injective nor surjective

30. For $x \in R - \{0, 2\}$ let $f_1(x)=\frac{1}{x}$, $f_2(x) = 1 - x$ and $f_3(x)=\frac{1}{1-x}$ given functions. If a function, J(x) satisfies $(f_2 \circ J \circ f_1)(x) = f_3(x)$ then J(x) is equal to : **[JEE MAINS 2019]**

(A) $f_3(x)$
(B) $f_1(x)$
(C) $f_2(x)$
(D) $\frac{1}{x}f_3(x)$

31. Let N be the set of natural numbers and two functions f and g be defined as f,g : N → N such that :

$$f(n)=\begin{cases}\frac{n+1}{2} & \text{if n is odd}\\ \frac{n}{2} & \text{if n is even}\end{cases}$$ and $g(n) = n-(-1)^n$. The fog is : **[JEE MAINS 2019]**

(A) Both one–one and onto
(B) One–one but not onto
(C) Neither one–one nor onto
(D) onto but not one–one

32. Let a function f : (0, ∞) → (0, ∞) be defined by $f(x)=\left|1-\frac{1}{x}\right|$. Then f is : **[JEE MAINS 2019]**

(A) Injective only
(B) Not injective but it is surjective
(C) Both injective as well as surjective
(D) Neither injective nor surjective

33. All x satisfying the inequality $(\cot^{-1} x)^2 - 7(\cot^{-1} x) + 10 > 0$, lie in the interval : **[JEE MAINS 2019]**

(A) $(-\infty, \cot 5) \cup (\cot 4, \cot 2)$
(B) $(\cot 5, \cot 4)$
(C) $(\cot 2, \infty)$
(D) $(-\infty, \cot 5) \cup (\cot 2, \infty)$

34. Let f : R → R be defined by $f(x)=\frac{x}{1+x^2}$, $x \in R$. Then the range of f is : **[JEE MAINS 2019]**

(A) $(-1, 1) - \{0\}$
(B) $\left[-\frac{1}{2}, \frac{1}{2}\right]$
(C) $R-\left[-\frac{1}{2}, \frac{1}{2}\right]$
(D) $R - [-1, 1]$

35. If $f(x)=\log_e\left(\frac{1-x}{1+x}\right), |x|<1$, then $f\left(\frac{2x}{1+x^2}\right)$ is equal to : **[JEE MAINS 2019]**

(A) $2f(x)$
(B) $2f(x^2)$
(C) $(f(x))^2$
(D) $-2f(x)$

36. Let $f(x) = a^x$ $(a > 0)$ be written as $f(x) = f_1(x) + f_2(x)$, where $f_1(x)$ is an even function of $f_2(x)$ is an odd function. Then $f_1(x + y) + f_1(x - y)$ equals **[JEE MAINS 2019]**

(A) $2f_1(x)f_1(y)$
(B) $2f_1(x)f_2(y)$
(C) $2f_1(x + y)f_2(x - y)$
(D) $2f_1(x + y)f_1(x - y)$

37. Let $\sum_{k=1}^{10} f(a+k)=16\left(2^{10}-1\right)$, where the function f satisfies $f(x + y) = f(x)f(y)$ for all natural numbers x, y and $f(A) = 2$. then the natural number 'a' is **[JEE MAINS 2019]**

(A) 4
(B) 3
(C) 16
(D) 2

38. If the function $f : R - \{1, -1\} \rightarrow A$ defined by $f(x) = \frac{x^2}{1-x^2}$, is surjective, then A is equal to

[JEE MAINS 2019]

(A) $R - [-1, 0)$ (B) $R - (-1, 0)$ (C) $R - \{-1\}$ (D) $[0, \infty)$

39. The domain of the definition of the function $f(x) = \frac{1}{4-x^2} + \log_{10}(x^3 - x)$ is :- **[JEE MAINS 2019]**

(A) $(1, 2) \cup (2, \infty)$ (B) $(-1, 0) \cup (1, 2) \cup (3, \infty)$

(C) $(-1, 0) \cup (1, 2) \cup (2, \infty)$ (D) $(-2, -1) \cup (-1, 0) \cup (2, \infty)$

40. Le $f(x) = x^2$, $x \in R$. For any $A \subseteq R$, define $g(A) = \{x \in R, f(x) \in A\}$. If $S = [0, 4]$, then which one of the following statements is not true ? **[JEE MAINS 2019]**

(A) $f(g(S)) \neq f(S)$ (B) $f(g(S)) = S$ (C) $g(f(S)) = g(S)$ (D) $g(f(S)) \neq S$

41. Let $f(x) = \log_e(\sin x)$, $(0 < x < \pi)$ and $g(x) = \sin^{-1}(e^{-x})$, $(x \geq 0)$. If α is a positive real number such that $a = (fog)'(\alpha)$ and $b = (fog)(\alpha)$, then : **[JEE MAINS 2019]**

(A) $a\alpha^2 - b\alpha - a = 0$ (B) $a\alpha^2 + b\alpha - a = -2\alpha^2$ (C) $a\alpha^2 + b\alpha + a = 0$ (D) $a\alpha^2 - b\alpha - a = 1$

42. For $x \in \left(0, \frac{3}{2}\right)$, let $f(x) = \sqrt{x}$, $g(x) = \tan x$ and $h(x) = \frac{1-x^2}{1+x^2}$. If $\phi(x) = ((hof)og)(x)$, then $\phi = \left(\frac{\pi}{3}\right)$ is equal to : **[JEE MAINS 2019]**

(A) $\tan\frac{\pi}{12}$ (B) $\tan\frac{7\pi}{12}$ (C) $\tan\frac{11\pi}{12}$ (D) $\tan\frac{5\pi}{12}$

43. For $x \in R$, let $[x]$ denote the greatest integer $\leq x$, then the sum of the series

$\left[-\frac{1}{3}\right] + \left[-\frac{1}{3} - \frac{1}{100}\right] + \left[-\frac{1}{3} - \frac{2}{100}\right] + \ldots\ldots + \left[-\frac{1}{3} - \frac{99}{100}\right]$ is **[JEE MAINS 2019]**

(A) –153 (B) –133 (C) –131 (D) –135

44. If $[x]$ denotes the greatest integer $\leq x$, then the system of linear equations $[\sin\theta]\,x + [-\cos\theta]y = 0$
$[\cot\theta]x + y = 0$ **[JEE MAINS 2019]**

(A) have infinitely many solutions if $\theta \in \left(\frac{\pi}{2}, \frac{2\pi}{3}\right) \cup \left(\pi, \frac{7\pi}{6}\right)$

(B) have infinitely many solutions if $\theta \in \left(\frac{\pi}{2}, \frac{2\pi}{3}\right)$ and has a unique solution if $\theta \in \left(\pi, \frac{7\pi}{6}\right)$

(C) has a unique solution if $\theta \in \left(\frac{\pi}{2}, \frac{2\pi}{3}\right)$ and have infinitely many solutions if $\theta \in \left(\pi, \frac{7\pi}{6}\right)$

(D) has a unique solution if $\theta \in \left(\frac{\pi}{2}, \frac{2\pi}{3}\right) \cup \left(\pi, \frac{7\pi}{6}\right)$

EXERCISE – 5 RECAP OF IIT-JEE/JEE (ADVANCED)

***1.** Let $f(x) = x^2$ and $g(x) = \sin x$ for all $x \in R$. Then the set of all x satisfying $(f \circ g \circ g \circ f)(x) = (g \circ g \circ f)(x)$, where $(f \circ g)(x) = f(g(x))$, is- **[JEE 2011, 3, (–1)]**

(A) $\pm\sqrt{n\pi}, n \in \{0,1,2,....\}$

(B) $\pm\sqrt{n\pi}, n \in \{1,2,...\}$

(C) $\frac{\pi}{2}+2n\pi, n \in \{....,-2,-1,0,1,2,.....\}$

(D) $2n\pi, n \in \{....,-2,-1,0,1,2,.....\}$

2. The function $f : [0,3] \to [1,29]$, defined by $f(x) = 2x^3 - 15x^2 + 36x + 1$, is : **[JEE 2012, 3, –1]**

(A) one-one and onto

(B) onto but not one-one

(C) one-one but not onto

(D) neither one-one nor onto

***3.** Let $f : (-1,1) \to IR$ be such that $f(\cos 4\theta) = \frac{2}{2-\sec^2\theta}$ for $\theta \in \left(0,\frac{\pi}{4}\right)\cup\left(\frac{\pi}{4},\frac{\pi}{2}\right)$. Then the value(s) of $f\left(\frac{1}{3}\right)$ is (are) - **[JEE 2012, 4]**

(A) $1-\sqrt{\frac{3}{2}}$

(B) $1+\sqrt{\frac{3}{2}}$

(C) $1-\sqrt{\frac{2}{3}}$

(D) $1+\sqrt{\frac{2}{3}}$

4. Let $f:\left(-\frac{\pi}{2},\frac{\pi}{2}\right)\to\mathbb{R}$ be given by $f(x) = (\log(\sec x + \tan x))^3$. Then **[JEE 2014, 4]**

(A) f(x) is an odd function

(B) f(x) is a one-one function

(C) f(x) is an onto function

(d) f(x) is an even function

5. Let X be a set with exactly 5 elements and Y be a set with exactly 7 elements. If α is the number of one-one functions from X to Y and β is the number of onto functions from Y to X, then the value of $\frac{1}{5!}(\beta-\alpha)$ is _____

[JEE 2018]

ANSWER KEY

EXERCISE-1

Que.	1	2	3	4	5	6	7	8	9	10
Ans.	C	C	A	A	A	D	C	C	C	A
Que.	11	12	13	14	15	16	17	18	19	20
Ans.	B	A	C	D	D	C	A	B	A	A
Que.	21	22	23	24	25	26	27	28	29	30
Ans.	D	C	D	A	A	B	B	B	C	C

EXERCISE-2

Que.	1	2	3	4	5	6	7	8	9	10
Ans.	BD	CD	ABC	AC	ABD	ABD	AC	A	ABC	BD
Que.	11	12	13	14	15	16	17	18	19	
Ans.	C	AB	ABC	AD	AD	AB	A	ABCD	AC	

- ***Match the Column*** **20.** (A) $\rightarrow$ (r, q); (B) $\rightarrow$ (p, s); (C) $\rightarrow$ (r, s); (D) $\rightarrow$ (r, s)

- ***Comprehension Based Questions***
  Comprehension – 1 **21.** (A) **22.** (B) **23.** (C)

EXERCISE-3

1. $f(r) = \frac{4}{3}\pi(3r^2 + 3r + 1)$ **2.** ***(a)*** –3/4, ***(b)*** 64, ***(c)*** 30, ***(d)*** 102,

3. 5050 **4.** 1003.5

5. ***(a)*** $\left[-\frac{5\pi}{4}, \frac{-3\pi}{4}\right] \cup \left[-\frac{\pi}{4}, \frac{\pi}{4}\right] \cup \left[\frac{3\pi}{4}, \frac{5\pi}{4}\right]$ ***(b)*** $\left(-4, -\frac{1}{2}\right) \cup (2, \infty)$

(c) $(-1 < x < -1/2)$ U $(x > 1)$ ***(d)*** $\left[\frac{1-\sqrt{5}}{2}, 0\right) \cup \left[\frac{1+\sqrt{5}}{2}, \infty\right)$

(e) $2K\pi < x < (2K + 1)\pi$ but $x \neq 1$ where K is non-negative integer

(f) $(1, 2) \cup \left(2, \frac{5}{2}\right)$

(g) $(-\infty, -3) \cup (-3, 1] \cup [4, \infty)$
(h) $(-2, -1) \cup (-1, 0) \cup (1, 2)$

6. ***(a)*** $2K\pi \le x \le 2K\pi + \pi$ where $K \in I$ ***(b)*** $[-3/2, -1]$

7. ***(a)*** D : $x \in R$ R : [0, 2] ***(b)*** D = R ; R : [–1, 1] ***(c)*** D : $-1 \le x \le 2$ R : $\left[\sqrt{3}, \sqrt{6}\right]$

(d) D = R ;R : [–1, 1] ***(e)*** D = R ; R : $\left[-\tan\sqrt{2}, \tan\sqrt{2}\right]$

(f) D : $(0, \infty)$; R : [–1, 1] ***(g)*** D = R ; R : [2, 10] ***(h)*** D = R ; R : [–1, 1]

(i) D : $\left[-\frac{\pi}{3}, \frac{\pi}{3}\right]$; R : $\left[0, \frac{\sqrt{3}}{2}\right]$

8. ***(a)*** neither surjective nor injective ***(b)*** surjective but not injective ***(c)*** neither injective nor surjective

9. $(fof)(x) = \begin{cases} 2+x, & 0 \le x \le 1 \\ 2-x, & 1 < x \le 2 \\ 4-x, & 2 < x \le 3 \end{cases}$

10. $f(x) = x^2 + 6$

11. $f(x) = 1 + x$ or $1 - x$

12. ***(a)*** odd ***(b)*** even ***(c)*** neither odd nor even ***(d)*** even ***(e)*** even ***(f)*** even

13. ***(a)*** $\pi/2$, ***(b)*** π , ***(c)*** $\pi/2$, ***(d)*** 70π

15. ***(a)*** $y = + \log_{10}(10 - 10^x), -\infty < x < 1$
(b) $y = x/3$ when $-\infty < x < 0$ & $y = x$ when $0 \le x < +\infty$

16. ***(a)*** $\frac{e^x - e^{-x}}{2}$ ***(b)*** $\frac{\log_2 x}{\log_2 x - 1}$ ***(c)*** $\frac{1}{2} \ell n \frac{1+x}{1-x}$

17. $f^{-1}(x) = (a - x^n)^{1/n}$

EXERCISE-4

Que.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Ans	B	B	A,D	B	A	C	A	A	D	B	C	A	B	C	A
Que.	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
Ans	C	B	D	A	D	A	B	B	B	B	C	C	A	A	A
Que.	31	32	33	34	35	36	37	38	39	40	41	42	43	44	
Ans	D	B	D	B	D	A	B	A	C	C	D	C	B	B	

EXERCISE-5

1. (A) ***2.*** (B) ***3.*** (zero marks to all) ***4.*** (A,B,C) ***5.*** (119)