Informal Notes on Mathematics
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一些零碎的东西

1. 设二次函数$f(x)=ax^2+bx+c(a<b<c)$的图象过点(1,0)和B(m, $-a$).

(1)若函数的顶点坐标为$\left(-\frac{1}{4},\frac{25}{16}\right)$，求实数m的值；

(2)若函数图象的对称轴为$x=x_0$，求x_0的取值范围：

(3)若对$x\geq k$ (k是与a,b,c无关的常数)时，恒有$f(x)+a\leq 0$，试求实数的最小值.

解：设$f(x)+a$与x轴交于$(m,0)$

$\therefore am^2+bm-b=0$

$am^2=-b(m-1)$

$-\frac{b}{2a}=\frac{m^2}{2m-2}$

由(2)知, $-\frac{1}{2}<x_0\leq 0,$

即 $-\frac{1}{2}<\frac{m^2}{2m-2}\leq 0$

设$\frac{m^2}{2m-2}=x,\quad m^2=(2m-2)x$

$\therefore m^2-2mx+2x=0$

$m_1=\frac{2x-\sqrt{4x^2-8x}}{2}, m_2=\frac{2x+\sqrt{4x^2-8x}}{2}$

$\because m_2>m_1$

令 $m(x)=\frac{2x+\sqrt{4x^2-8x}}{2}$

$\therefore \frac{dm}{dx}=1-\frac{x-2}{\sqrt{4x^2-8x}}$

(i) x_0 为区间端点

$x_0=-\frac{1}{2}$ 或 $x_0=0$

(ii)x_0 处 $m'(x)$ 不存在，

即 $1-\frac{x-2}{\sqrt{4x^2-8x}}$ 无意义

$\because x_0\in(-\frac{1}{2},0]\therefore x_0=0$

(iii) x_0 处 $m'(x)=0$

即 $1-\frac{x-2}{\sqrt{4x^2-8x}}=0$

$x_0=2$(增根) 或 $x_0=-\frac{2}{3}$

$\because x_0\in(-\frac{1}{2},0]$ ∴舍

$\therefore x_0=0$ 或 $-\frac{1}{2}$ 时取得局部极值

$\because x_0=0, m=0;\ x_0=-\frac{1}{2}, m=\frac{\sqrt{5}-1}{2}$

$\therefore m<\frac{\sqrt{5}-1}{2}\ \therefore k=\frac{\sqrt{5}-1}{2}$

2.

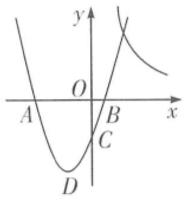
1. **万唯原创** 如图，在平面直角坐标系中，抛物线 $y=x^2-2ax+a^2-4$ 与 x 轴相交于 A,B 两点(点 A 在点 B 的左侧)，与 y 轴相交于点 C，点 D 是抛物线的顶点，双曲线 $y=\frac{k}{x}(k>0)$ 在第一象限内的图象记作 G.

(1)求线段 AB 的长；

(2)当抛物线的对称轴为 y 轴时，令抛物线 $y=x^2-2ax+a^2-4$ 与图象 G 的交点为 M，设点 M 的横坐标为 x_0，若 $3<x_0<4$，求 k 的取值范围；

(3)已知图象 G 经过点 $P(4,m-3)$ 和点 $Q(m,2)$，若存在抛物线 $y=x^2-2ax+a^2-4$ 与图象 G 的交点中至少有一个在 P,Q 两点同侧，直接写出 a 的取值范围.

作图区



第 1 题图

答题区

我们现在只关注 $f(x)=x^2-2ax+a^2-4$ 在对称轴右侧的部分与 $g(x)=\frac{12}{x}(x>0)$ 的交点，求两函数仅有一交点的情况。所以问题转化为解方程

roots	$\frac{1}{6}\left(i\left(\sqrt{3}+i\right)\sqrt[3]{-a^3+6\left(\sqrt{-3a^4-9a^3+24a^2+324a+681}+27\right)+36a}-\frac{i\left(\sqrt{3}-i\right)\left(a^2+12\right)}{\sqrt[3]{-a^3+6\left(\sqrt{-3a^4-9a^3+24a^2+324a+681}+27\right)+36a}}+4a\right)=\frac{1}{6}\left(-\left(1+i\sqrt{3}\right)\sqrt[3]{-a^3+6\left(\sqrt{-3a^4-9a^3+24a^2+324a+681}+27\right)+36a}+\frac{i\left(\sqrt{3}+i\right)\left(a^2+12\right)}{\sqrt[3]{-a^3+6\left(\sqrt{-3a^4-9a^3+24a^2+324a+681}+27\right)+36a}}+4a\right)$

解得

$$a=-\frac{3}{4}+\frac{1}{4\sqrt{\frac{3}{91-6752\sqrt[3]{\frac{2}{104663+4491\sqrt{1497}}}+2\times 2^{2/3}\sqrt[3]{104663+4491\sqrt{1497}}}}}+\frac{1}{2}\sqrt{\left(\frac{91}{6}+\frac{1688}{3}\sqrt[3]{\frac{2}{104663+4491\sqrt{1497}}}-\frac{1}{3}\sqrt[3]{\frac{1}{2}\left(104663+4491\sqrt{1497}\right)}+\frac{741}{2}\sqrt{\left(3\middle/\left(91-6752\sqrt[3]{\frac{2}{104663+4491\sqrt{1497}}}+2\times 2^{2/3}\sqrt[3]{104663+4491\sqrt{1497}}\right)\right)}\right)}$$

3. 已知$x^2+y^3=2$, 求$x+y$的极大值和极小值

解：根据拉格朗日乘数，我们显然有
$$\begin{cases}1+2\lambda x=0\\ 1+3\lambda y^2=0\\ x^2+y^3=2\end{cases}$$

解得分别在以下情况取得极大或极小值

$$x=\frac{1}{27}\left(2+\frac{1}{2}\sqrt{\left(1960+\frac{1}{3}\sqrt[3]{200\,195\,950\,464-1\,836\,660\,096\sqrt{217}}+324\sqrt[3]{2\left(109+\sqrt{217}\right)}\right)}-\frac{1}{2}\sqrt{\left(3920-\frac{1}{3}\sqrt[3]{200\,195\,950\,464-1\,836\,660\,096\sqrt{217}}-324\sqrt[3]{2\left(109+\sqrt{217}\right)}+23\,456\Big/\left(\sqrt{\left(1960+\frac{1}{3}\sqrt[3]{200\,195\,950\,464-1\,836\,660\,096\sqrt{217}}+324\sqrt[3]{2\left(109+\sqrt{217}\right)}\right)}\right)\right)}\right),$$

$$y=\frac{1}{18}\left(2+\frac{1}{2}\sqrt{\left(1960+\frac{1}{3}\sqrt[3]{200\,195\,950\,464-1\,836\,660\,096\sqrt{217}}+324\sqrt[3]{2\left(109+\sqrt{217}\right)}\right)}-\frac{1}{2}\sqrt{\left(3920-\frac{1}{3}\sqrt[3]{200\,195\,950\,464-1\,836\,660\,096\sqrt{217}}-324\sqrt[3]{2\left(109+\sqrt{217}\right)}+23\,456\Big/\left(\sqrt{\left(1960+\frac{1}{3}\sqrt[3]{200\,195\,950\,464-1\,836\,660\,096\sqrt{217}}+324\sqrt[3]{2\left(109+\sqrt{217}\right)}\right)}\right)\right)}\right)+\frac{1}{6561}$$

$$2\left(2+\frac{1}{2}\sqrt{\left(1960+\frac{1}{3}\sqrt[3]{200\,195\,950\,464-1\,836\,660\,096\sqrt{217}}+324\sqrt[3]{2\left(109+\sqrt{217}\right)}\right)}-\frac{1}{2}\sqrt{\left(3920-\frac{1}{3}\sqrt[3]{200\,195\,950\,464-1\,836\,660\,096\sqrt{217}}-324\sqrt[3]{2\left(109+\sqrt{217}\right)}+23\,456\Big/\left(\sqrt{\left(1960+\frac{1}{3}\sqrt[3]{200\,195\,950\,464-1\,836\,660\,096\sqrt{217}}+324\sqrt[3]{2\left(109+\sqrt{217}\right)}\right)}\right)\right)}\right)^2-\frac{1}{26\,244}$$

$$\left(2+\frac{1}{2}\sqrt{\left(1960+\frac{1}{3}\sqrt[3]{200\,195\,950\,464-1\,836\,660\,096\sqrt{217}}+324\sqrt[3]{2\left(109+\sqrt{217}\right)}\right)}-\frac{1}{2}\sqrt{\left(3920-\frac{1}{3}\sqrt[3]{200\,195\,950\,464-1\,836\,660\,096\sqrt{217}}-324\sqrt[3]{2\left(109+\sqrt{217}\right)}+23\,456\Big/\left(\sqrt{\left(1960+\frac{1}{3}\sqrt[3]{200\,195\,950\,464-1\,836\,660\,096\sqrt{217}}+324\sqrt[3]{2\left(109+\sqrt{217}\right)}\right)}\right)\right)}\right)^3,$$

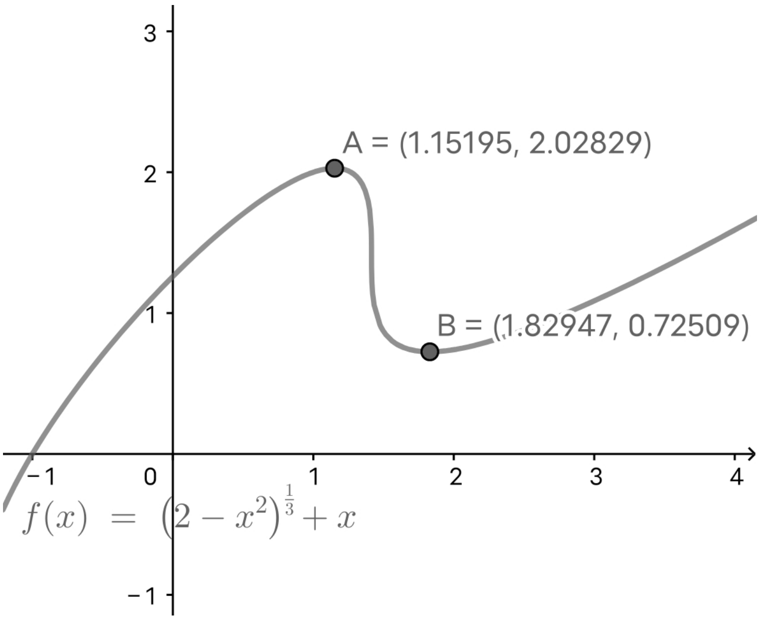
$$x=\frac{1}{27}\left(2+\frac{1}{2}\sqrt{\left(1960+\frac{1}{3}\sqrt[3]{200\,195\,950\,464-1\,836\,660\,096\sqrt{217}}+324\sqrt[3]{2\left(109+\sqrt{217}\right)}\right)}+\frac{1}{2}\sqrt{\left(3920-\frac{1}{3}\sqrt[3]{200\,195\,950\,464-1\,836\,660\,096\sqrt{217}}-324\sqrt[3]{2\left(109+\sqrt{217}\right)}+23\,456\Big/\left(\sqrt{\left(1960+\frac{1}{3}\sqrt[3]{200\,195\,950\,464-1\,836\,660\,096\sqrt{217}}+324\sqrt[3]{2\left(109+\sqrt{217}\right)}\right)}\right)\right)}\right),$$

$$y=\frac{1}{18}\left(2+\frac{1}{2}\sqrt{\left(1960+\frac{1}{3}\sqrt[3]{200\,195\,950\,464-1\,836\,660\,096\sqrt{217}}+324\sqrt[3]{2\left(109+\sqrt{217}\right)}\right)}+\frac{1}{2}\sqrt{\left(3920-\frac{1}{3}\sqrt[3]{200\,195\,950\,464-1\,836\,660\,096\sqrt{217}}-324\sqrt[3]{2\left(109+\sqrt{217}\right)}+23\,456\Big/\left(\sqrt{\left(1960+\frac{1}{3}\sqrt[3]{200\,195\,950\,464-1\,836\,660\,096\sqrt{217}}+324\sqrt[3]{2\left(109+\sqrt{217}\right)}\right)}\right)\right)}\right)+\frac{1}{6561}$$

$$2\left(2+\frac{1}{2}\sqrt{\left(1960+\frac{1}{3}\sqrt[3]{200\,195\,950\,464-1\,836\,660\,096\sqrt{217}}+324\sqrt[3]{2\left(109+\sqrt{217}\right)}\right)}+\frac{1}{2}\sqrt{\left(3920-\frac{1}{3}\sqrt[3]{200\,195\,950\,464-1\,836\,660\,096\sqrt{217}}-324\sqrt[3]{2\left(109+\sqrt{217}\right)}+23\,456\Big/\left(\sqrt{\left(1960+\frac{1}{3}\sqrt[3]{200\,195\,950\,464-1\,836\,660\,096\sqrt{217}}+324\sqrt[3]{2\left(109+\sqrt{217}\right)}\right)}\right)\right)}\right)^2-\frac{1}{26\,244}$$

$$\left(2+\frac{1}{2}\sqrt{\left(1960+\frac{1}{3}\sqrt[3]{200\,195\,950\,464-1\,836\,660\,096\sqrt{217}}+324\sqrt[3]{2\left(109+\sqrt{217}\right)}\right)}+\frac{1}{2}\sqrt{\left(3920-\frac{1}{3}\sqrt[3]{200\,195\,950\,464-1\,836\,660\,096\sqrt{217}}-324\sqrt[3]{2\left(109+\sqrt{217}\right)}+23\,456\Big/\left(\sqrt{\left(1960+\frac{1}{3}\sqrt[3]{200\,195\,950\,464-1\,836\,660\,096\sqrt{217}}+324\sqrt[3]{2\left(109+\sqrt{217}\right)}\right)}\right)\right)}\right)^3,$$

作图如下


A = (1.15195, 2.02829)
B = (1.82947, 0.72509)
$f(x) = (2 - x^2)^{\frac{1}{3}} + x$
3
2
1
0
−1
−1
1
2
3
4