The Axiom of Choice, Cardinal Numbers, etc.

This chapter includes Section 0.7 to 0.8 of the original book.

Axiom 3.0.1 (Axiom of Choice)**.** The product of a family of nonempty sets indexed by a nonempty set is nonempty.

Note 5 — Here's the alternate version: Let S be a set. A *choice function* for S is a function f from the set of all nonempty subsets of S to S such that $f(A) \in A$ for all $A \neq \emptyset$, $A \subset S$. The Axiom of Choice is equivalent to the claim: Every set S has a choice function.

Definition 3.0.2. A *partially ordered* set is a nonempty set A together with a relation R on $A \times A$ (called a *partial ordering*) of A which is reflexive, transitive and **anitsymmetric**. Antisymmetry means that if $(a, b), (b, a) \in R$ then $a = b$. For partial ordering R, when $(a, b) \in R$ we denote this as $a \leq b$. Elements $a, b \in A$ are *comparable* if either $a \leq b$ or $b \leq a$. A partial ordering of a set A such that any two elements are comparable is called a *total ordering*(or *linear* or *simple ordering*).

Example 3.0.3

Let A be the power set of $\{1, 2, 3, 4, 5\}$ and define $C \leq D$ if $C \subset D$.This is a partial ordering, but not a total ordering. For example, $\{1, 2\}$ and $\{2, 3\}$ are not comparable.

Let $(A, \leq)$ be a partially ordered set. An element $a \in A$ is *maximal* in A if for every $c \in A$ which is comparable to a, we have $c \leq a$. An *upper bound* of a nonempty subset B of A is an element $d \in A$ such that $b \leq d$ for every $b \in B$. A nonempty subset B of A that is totally ordered by $\leq$ is a *chain* in A.

Theorem 3.0.4 (Zorn's lemma)

Let A be a nonempty partially ordered set. If every chain has an upper bound, then A has a local maximum.

Definition 3.0.5. Let B be a nonempty subset of a partially ordered set A (under $\leq$). If every nonempty subset of A has a least element[1], then A is *well ordered.*

Proposition 3.0.6 (The Well Ordering Principle) If A is a nonempty set, then there exists a total ordering $\leq$ of A such that A is well ordered under $\leq$.

This allows us to extend the Principle of Mathematical Induction (2.0.10) to any well ordered set.

[1] Similar to the definition of maximum.

Definition 3.0.7.
- Two sets A and B are *equipollent* if there exists a bijective map from A to B, in which case we denote this as $A \sim B$.
- If set A is equipollent to a set $I_n = \{1, 2, ..., n\}$ for some$n \in \mathbb{N}$ or to the set $I_0 = \emptyset$ then set A is *finite*. Otherwise set A is *infinite*.

Definition 3.0.8. The *cardinal number* of a set A, denoted $|A|$, isthe equivalence class of A under the equivalence relation of equipollence. $|A|$ is an infinite or finite cardinal according as to whether A is an infinite or finite set.

Let A and B be disjoint sets such that $|A| = \aleph$ and $|B| = \beta$. The *sum* $\alpha + \beta$ is the cardinal number $|A \cup B|$. The *product* $\alpha\beta$ is the cardinal number $|A \times B|$

Theorem 3.0.9

If A is a set and $P(A)$ is its power set, then $|A| < |P(A)|$.

Proof. The assignment $a \mapsto \{a\}$ defines an injective map $A \to P(A)$ so that $|A| \leq |P(A)|$. If there were a bijective map $f : A \to P(A)$, then for some $a_0 \in A$, $f(a_0) = B$, where $B = \{a \in A \mid a \notin f(a)\} \subset A$. But this yields a contradiction: $a_0 \in B$ and $a_0 \notin B$. Therefore $|A| \neq |P(A)|$ and hence $|A| < |P(A)|$. ☐

*Because the mouse is broken , next we'll only list the theorems.

Theorem 3.0.10

If A and B are sets such that $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$.

Theorem 3.0.11

The class of all cardinal numbers is linearly ordered by $\leq$. If α and β are cardinal numbers, then exactly on of the following is true:

$$\alpha < \beta \;;\; \alpha = \beta \;;\; \beta < \alpha$$

Note 6 — This is called the *Law of Trichotomy.* A family of functions partially ordered as in the proof of 3.0.11 is said to *be ordered by extension.*

Theorem 3.0.12

Every infinite set has a denumerable subset. In particular, $\aleph_0 \leq \alpha$ for every infinite cardinal number α.

Lemma 3.0.13

If A is an infinite set and F is a finite set then $|A \cup F| = |A|$. In particular, $\alpha + n = \alpha$ for every infinite cardinal number α and every finite cardinal number n.

Theorem 3.0.14

If α and β are cardinal numbers such that $\beta \leq \alpha$ and α is infinite, then $\alpha + \beta = \alpha$.

Theorem 3.0.15

If α and β are cardinal numbers such that $0 \neq \beta \leq \alpha$ and α is infinite, then $\alpha\beta = \alpha$; in particular, $\alpha\aleph_0 = \alpha$ and if B is finite then $\aleph_0\beta = \aleph_0$.

Theorem 3.0.16

Let A be a set and for each integer $n \geq 1$ let $A^n = A \times A \times \ \times A$ (n factors).

(i) If A is finite, then $|A^n| = |A|^n$, and if A is infinite then $|A^n| = |A|$.

(ii) $|\bigcup_{n\in\mathbb{N}} A^n| = \aleph_0|A|$

Corollary 3.0.17 If A is an infinite set and $F(A)$ is the set of all finite subsets of A, then $|F(A)| = |A|$.