Informal Notes on Mathematics

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EXERCISES 1.A

1 Suppose a and b are real numbers, not both 0. Find real numbers c and d such that

$$1/(a+bi) = c + di.$$

Considering $(a+bi)(a-bi) = a^2+b^2$, one has $\frac{a-bi}{a^2+b^2} = c+di$. Thus, $\boxed{c = \frac{a}{a^2+b^2}}$, $\boxed{d = -\frac{b}{a^2+b^2}}$. □

2 Show that

$$\frac{-1+\sqrt{3}i}{2}$$

is a cube root of 1 (meaning that its cube equals 1).

Note that $(a+bi)+(a-bi) = 2a$, and $(a+bi)(a-bi) = a^2+b^2$, so $\frac{-1+\sqrt{3}i}{2} + \frac{-1-\sqrt{3}i}{2} = -1$ and $\frac{-1+\sqrt{3}i}{2} \cdot \frac{-1-\sqrt{3}i}{2} = 1$, so $\frac{-1+\sqrt{3}i}{2}$ is a root of equation $x^2+x+1=0$. Because $x^3-1 = (x-1)(x^2+x+1) = 0$, so $\left(\frac{-1+\sqrt{3}i}{2}\right)^3 = 1$.

$\left(\frac{-1+\sqrt{3}i}{2}\right)^3 = \left(\frac{-1+\sqrt{3}i}{2}\right)^2 \cdot \frac{-1+\sqrt{3}i}{2} = \frac{-1-\sqrt{3}i}{2} \cdot \frac{-1+\sqrt{3}i}{2} = \frac{(-1)^2+(\sqrt{3})^2}{4} = \boxed{1}$ □

3 Find two distinct square roots of i.

Let $(a+bi)^2 = i$ ∵ $a^2 - b^2 + 2abi = i$ ∴ $\begin{cases} a^2-b^2=0 \\ 2abi = i \end{cases}$ ∴ ① $a=b$, $a=\frac{1}{\sqrt{2}}$, $b=\frac{1}{\sqrt{2}}$, $a+bi = \boxed{\frac{\sqrt{2}(1+i)}{2}}$; ② $a=-b$, $a=\frac{1}{\sqrt{2}}i$, $b=\sqrt{2}i$, $a+bi = \frac{\sqrt{2}(i-1)}{2} = \boxed{\frac{\sqrt{2}(\ldots)}{2}}$

$i = e^{\pi i/2}$, so two roots are $e^{\pi i/4}$, $e^{5\pi i/4}$. Use $e^{xi} = \cos x + i\sin x$ $(x \in \mathbf{R})$ to solve.

4 Show that $\alpha + \beta = \beta + \alpha$ for all $\alpha, \beta \in \mathbf{C}$.

Let α be $(a+bi)$, β be $(c+di)$, $\alpha+\beta = (a+bi)+(c+di) = (a+c)+(b+d)i = (c+a)+(d+b)i = (c+di)+(a+bi) = \beta+\alpha$ □

5 Show that $(\alpha + \beta) + \lambda = \alpha + (\beta + \lambda)$ for all $\alpha, \beta, \lambda \in \mathbf{C}$.

Let α be (x_1+y_1i), β be (x_2+y_2i), λ be (x_3+y_3i), use associativity of addition in $\mathbf{R}$ to proof. Process is similar to Exercise 4. □

6 Show that $(\alpha\beta)\lambda = \alpha(\beta\lambda)$ for all $\alpha, \beta, \lambda \in \mathbf{C}$.

Let $\alpha = x_1+y_1i$, $\beta = x_2+y_2i$, $\lambda = x_3+y_3i$, use distributive property and others in $\mathbf{R}$ to proof □

7 Show that for every $\alpha \in \mathbf{C}$, there exists a unique $\beta \in \mathbf{C}$ such that $\alpha + \beta = 0$.

Let $\alpha = a+bi$ and $\beta = c+di$, so $\alpha+\beta = (a+bi)+(c+di) = (a+c)+(b+d)i$. For each number $x \in \mathbf{R}$, we can find a number written $-x$ to make $x+(-x)=0$. So $a+c$ and $b+d$ can be 0. So do $\alpha+\beta$.

8 Show that for every $\alpha \in \mathbf{C}$ with $\alpha \neq 0$, there exists a unique $\beta \in \mathbf{C}$ such that $\alpha\beta = 1$.

Let $\alpha = a+bi$, $\beta = c+di$. We had the result of Exercise 1, so we now should show its uniqueness. If $\alpha\beta = 1$, then $\beta = 1 \cdot \beta = (\frac{1}{\alpha}\cdot\alpha)\cdot\beta = \frac{1}{\alpha}\cdot(\alpha\cdot\beta) = \frac{1}{\alpha}$ (use the result of Exercise 6). □

9 Show that $\lambda(\alpha + \beta) = \lambda\alpha + \lambda\beta$ for all $\lambda, \alpha, \beta \in \mathbf{C}$.

Let $\lambda = a+bi$, $\alpha = x_1+y_1i$, $\beta = x_2+y_2i$, then, $\lambda(\alpha+\beta) = (a(x_1+x_2) - b(y_1+y_2)) + (a(y_1+y_2)+b(x_1+x_2))i = [(ax_1-by_1)+(ay_1+bx_1)i] + [(ax_2-by_2)+(ay_2+bx_2)i] = \lambda\alpha + \lambda\beta$ □

10 Find $x \in \mathbf{R}^4$ such that

$$(4, -3, 1, 7) + 2x = (5, 9, -6, 8).$$

Because $(4,-3,1,7)+2x = (5,9,-6,8)$, so $2x = (5,9,-6,8)-(4,-3,1,7) = (1,12,-7,1)$, so $x = \frac{1}{2}(1,12,-7,1) = \boxed{(\frac{1}{2}, 6, -\frac{7}{2}, \frac{1}{2})}$

11 Explain why there does not exist $\lambda \in \mathbf{C}$ such that

$$\lambda(2 - 3i, 5 + 4i, -6 + 7i) = (12 - 5i, 7 + 22i, -32 - 9i).$$

If the number λ is exist, then $\lambda(2-3i) = (12-5i)$, $\lambda(5+4i) = 7+22i$. So $(2-3i)(7+22i) = (12-5i)(5+4i)$. It means $80-17i = 80+23i$. Thus such $\lambda \in \mathbf{C}$ does not exist. □

12 Show that $(x + y) + z = x + (y + z)$ for all $x, y, z \in \mathbf{F}^n$.

Suppose $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$, $z = (z_1, \ldots, z_n)$. Process is similar to the proof of the conclusion in $\mathbf{C}$. □

13 Show that $(ab)x = a(bx)$ for all $x \in \mathbf{F}^n$ and all $a, b \in \mathbf{F}$.

Suppose $x = (x_1, \ldots, x_n)$. So $(ab)x = (abx_1, abx_2, \ldots, abx_n) = (a(bx_1), \ldots, a(bx_n)) = a(bx_1, \ldots, bx_n) = a(bx)$ □

14 Show that $1x = x$ for all $x \in \mathbf{F}^n$.

Suppose $x = (x_1, \ldots, x_n)$, so $1x = (1\cdot x_1, \ldots, 1\cdot x_n) = (x_1, \ldots, x_n) = x$ □

15 Show that $\lambda(x + y) = \lambda x + \lambda y$ for all $\lambda \in \mathbf{F}$ and all $x, y \in \mathbf{F}^n$.

Suppose $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$, $\lambda(x+y) = (\lambda x_1 + \lambda y_1, \ldots, \lambda x_n + \lambda y_n) = (\lambda x_1, \ldots, \lambda x_n) + (\lambda y_1, \ldots, \lambda y_n) = \lambda x + \lambda y$ □

16 Show that $(a + b)x = ax + bx$ for all $a, b \in \mathbf{F}$ and all $x \in \mathbf{F}^n$.

Suppose $x = (x_1, \ldots, x_n)$, so $(a+b)x = ((a+b)x_1, \ldots, (a+b)x_n) = (ax_1+bx_1, \ldots, ax_n+bx_n) = a(x_1, \ldots, x_n) + b(x_1, \ldots, x_n) = ax + bx$ □

1.31 The number -1 times a vector

$(-1)v = -v$ for every $v \in V$.

Proof For $v \in V$, we have

$$v + (-1)v = 1v + (-1)v = \big(1 + (-1)\big)v = 0v = 0.$$

This equation says that $(-1)v$, when added to v, gives 0. Thus $(-1)v$ is the additive inverse of v, as desired. ■

EXERCISES 1.B

By definition we know $(-v) + (-(-v)) = 0$, $(-v) + v = 0$, so $(-(-v)) = v$ □

1 Prove that $-(-v) = v$ for every $v \in V$.

① If $a = 0$, we have proofed. ② If $a \neq 0$, Suppose $v \neq 0$, then $a^{-1} \cdot av = a^{-1} \cdot 0 = 0$, but $a^{-1} \cdot av$ also $= (a^{-1} \cdot a) \cdot v = v$.

2 Suppose $a \in \mathbf{F}$, $v \in V$, and $av = 0$. Prove that $a = 0$ or $v = 0$.

Can be as an independent proof contradiction. So $v = 0$. □

3 Suppose $v, w \in V$. Explain why there exists a unique $x \in V$ such that $v + 3x = w$. let $x = \frac{w-v}{3}$, so $v + 3x = v + (w - v) = w$. This shows the existence.

Suppose we have another vector x' such that $v + 3x' = w$, so $3(x - x') = (w-v) - (w-v) = 0$. This shows the uniqueness. □

4 The empty set is not a vector space. The empty set fails to satisfy only one of the requirements listed in 1.19. Which one? In fact, all the requirements are not satisfy.

Additive identity. 'There exists an element $0 \in V$ such that $v + 0 = v$ for all $v \in V$' that shows

5 Show that in the definition of a vector space (1.19), the additive inverse condition can be replaced with the condition that V can't be empty □

We assume $0v = 0$ for all $v \in V$, then: $v + ((-1)v) = (1v) + (-1)v) = 0v = 0$.

$$0v = 0 \text{ for all } v \in V.$$

This shows the existence of additive inverse, i.e. the additive inverse condition. □

Here the 0 on the left side is the number 0, and the 0 on the right side is the additive identity of V. (The phrase "a condition can be replaced" in a definition means that the collection of objects satisfying the definition is unchanged if the original condition is replaced with the new condition.)

6 Let ∞ and $-\infty$ denote two distinct objects, neither of which is in $\mathbf{R}$. Define an addition and scalar multiplication on $\mathbf{R} \cup \{\infty\} \cup \{-\infty\}$ as you could guess from the notation. Specifically, the sum and product of two real numbers is as usual, and for $t \in \mathbf{R}$ define

If it is a vector over $\mathbb{R}$, we will have $\infty = (2 + (-1))\infty = 2\infty + ((-1)\infty) = \infty + (-\infty) = 0$

Thus, for any $t \in \mathbb{R}$, one has $t = 0 + t = \infty + t = \infty$, so the 0 is not unique, contradiction □

$$t\infty = \begin{cases} -\infty & \text{if } t < 0, \\ 0 & \text{if } t = 0, \\ \infty & \text{if } t > 0, \end{cases} \qquad t(-\infty) = \begin{cases} \infty & \text{if } t < 0, \\ 0 & \text{if } t = 0, \\ -\infty & \text{if } t > 0, \end{cases}$$

$$t + \infty = \infty + t = \infty, \qquad t + (-\infty) = (-\infty) + t = -\infty,$$

$$\infty + \infty = \infty, \qquad (-\infty) + (-\infty) = -\infty, \qquad \infty + (-\infty) = 0.$$

Is $\mathbf{R} \cup \{\infty\} \cup \{-\infty\}$ a vector space over $\mathbf{R}$? Explain. Isn't.

Associativity: Note that $\infty + \infty + (-\infty) = (\infty + \infty) + (-\infty) = \infty + (-\infty) = 0$

But also $\infty + \infty + (-\infty) = \infty + (\infty + (-\infty)) = \infty + 0 = \infty$

so $0 = \infty$, contradiction. □

If $u \in U$, then $-u$ [which equals $(-1)u$ by 1.31] is also in U by the third condition above. Hence every element of U has an additive inverse in U.

The other parts of the definition of a vector space, such as associativity and commutativity, are automatically satisfied for U because they hold on the larger space V. Thus U is a vector space and hence is a subspace of V. ■

The three conditions in the result above usually enable us to determine quickly whether a given subset of V is a subspace of V. You should verify all the assertions in the next example.

1.35 Example *subspaces*

$(0,0,0,0)$ is in it if and only if ~~it~~ $b=0$.

(a) If $b \in \mathbf{F}$, then Obviously, it satisfies closed under addition and scalar multiplication. □

$$\{(x_1, x_2, x_3, x_4) \in \mathbf{F}^4 : x_3 = 5x_4 + b\}$$

is a subspace of $\mathbf{F}^4$ if and only if $b = 0$.

(b) The set of continuous real-valued functions on the interval $[0, 1]$ is a subspace of $\mathbf{R}^{[0,1]}$. Similar to (a)

(c) The set of differentiable real-valued functions on $\mathbf{R}$ is a subspace of $\mathbf{R}^{\mathbf{R}}$.

Similar to (a)

(d) The set of differentiable real-valued functions f on the interval $(0, 3)$ such that $f'(2) = b$ is a subspace of $\mathbf{R}^{(0,3)}$ if and only if $b = 0$.

Similar to (a)

(e) The set of all sequences of complex numbers with limit 0 is a subspace of $\mathbf{C}^{\infty}$. Obviously $(0, 0, \ldots)$ is an element of it.

Closed under addition: Consider $(a_1, a_2, \ldots)$ and $(b_1, b_2, \ldots)$

The limit of them are both 0, Thus, the limit of $(a_1+b_1, a_2+b_2, \ldots) = \lim_{n\to\infty}(a_n+b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n = 0$

Closed under scalar multiplication: Consider $(a_1, a_2, \ldots)$ such that $\lim_{n\to\infty} a_n = 0$

Thus, $\lim_{n\to\infty} \lambda a_n \; (\lambda \in \mathbf{R}) = \lambda \lim_{n\to\infty} a_n = \lambda \cdot 0 = 0$ □

Verifying some of the items above shows the linear structure underlying parts of calculus. For example, the second item above requires the result that the sum of two continuous functions is continuous. As another example, the fourth item above requires the result that for a constant c, the derivative of cf equals c times the derivative of f.

> *Clearly $\{0\}$ is the smallest subspace of V and V itself is the largest subspace of V. The empty set is not a subspace of V because a subspace must be a vector space and hence must contain at least one element, namely, an additive identity.*

The subspaces of $\mathbf{R}^2$ are precisely $\{0\}$, $\mathbf{R}^2$, and all lines in $\mathbf{R}^2$ through the origin. The subspaces of $\mathbf{R}^3$ are precisely $\{0\}$, $\mathbf{R}^3$, all lines in $\mathbf{R}^3$ through the origin, and all planes in $\mathbf{R}^3$ through the origin. To prove that all these objects are indeed subspaces is easy—the hard part is to show that they are the only subspaces of $\mathbf{R}^2$ and $\mathbf{R}^3$. That task will be easier after we introduce some additional tools in the next chapter.