Chapter_4_2D_Steemson_triangle

May 27, 2019

1 Steemson Triangle Test

```
In [1]: import numpy as np
        from scipy.optimize import least_squares
        import matplotlib.pyplot as plt
        from PIL import Image
        from scipy import optimize

        #import extras for our examples
        import Steem_ex as se
        import Moment_Function as mf

        %matplotlib inline
```

Set-up Steemson triangle example by forming the IFS for a Steemson triangle with side lengths $1, 1.1, 0.85$. The actual functions used are in Python files, we simply import them here.

```
In [2]: #start Steemson attempt
        a   =   1.1
        b   =   0.85

        imgxy = 500

        #co ordinates of triangles third vertex
        Cy   =   (0.5*np.sqrt(-1 + 2*a**2 - a**4 + 2*b**2 + 2*a**2*b**2 - b**4))
        Cx   =   (0.5*(1 - a**2 + b**2))

        #angles for roatation
        cosA   =   (Cx/b)
        sinA   =   (Cy/b)
        cosB   =   ((1.0-Cx)/a)
        sinB   =   (Cy/a)

        #scaling factors
        alphaNFF   =   (b)/((a**2 + b**2))
        betaNFF    =   (a)/(a**2 + b**2)
        gammaNFF   =   -1.0*(1.0-a**2-b**2)/(a**2 + b**2)
```

```
alphaNNF   =  (b/(1.0 + b**2))
betaNNF    =  (1.0/(1.0 + b**2))
gammaNNF   =  (b**2/(1.0 + b**2))


alphaNFF   =  alphaNNF
betaNFF    =  betaNNF
gammaNFF   =  gammaNNF


NNF    =    [[cosA*alphaNNF,sinA*alphaNNF,sinA*alphaNNF,-cosA*alphaNNF,0.0,0.0],
             [betaNNF,0.0,0.0,betaNNF,(1.0-betaNNF),0],
             [gammaNNF,0.0,0.0,gammaNNF,(1.0-gammaNNF)*Cx,(1.0-gammaNNF)*Cy]]

#get the dimension
f_dim_steem  =   lambda D : np.abs(alphaNNF)**D +  np.abs(betaNNF)**D +  np.abs(gammaNl
D_Steem      =   optimize.root(f_dim_steem, 1.8,method='lm').x
```

Make a picture of our triangle to make sure we got it right

```
In [3]: IFS_STEEM = se.convertIFS(NNF)

        Steem_mom = se.mom_wrap_Steem(IFS_STEEM,D_Steem,n=21,it = 30)

        img, _ = se.chaos(IFS_STEEM,[1/3,1/3,1/3], it = 100000, imgxy=500)
        img
```

```
Probabilities:
[0.33232556 0.36557059 0.30210385]
```

Out[3]:
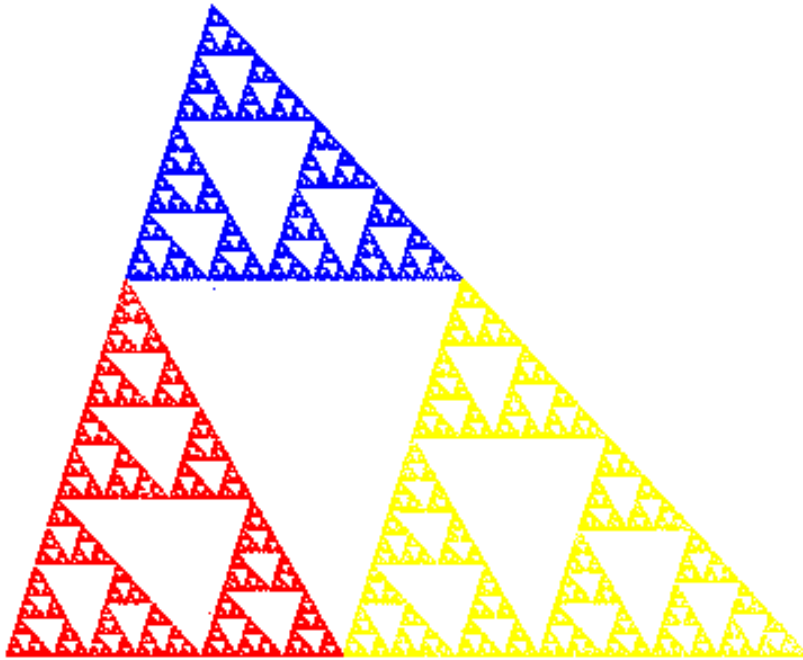
Make a function for our operator Φ, and wrap it so we can form pictures for our minimisation

```
In [4]: def Phi_wrap(x,D,it = 1,n=6, sim = False ):
            assert len(x)%6==0

            N = len(x)

            a_l = x[:N//6]

            b_l = x[1*(N//6):2*(N//6)]

            c_l = x[2*(N//6):3*(N//6)]

            d_l = x[3*(N//6):4*(N//6)]
```

```python
    e_l = x[4*(N//6):5*(N//6)]

    f_l = x[5*(N//6):]

    if sim:

        ad_l = (np.abs(a_l)+np.abs(d_l))/2

        a_l = np.sign(a_l)*ad_l

        d_l = np.sign(d_l)*ad_l

        bc_l = (np.abs(b_l)+np.abs(c_l))/2

        b_l = np.sign(b_l)*bc_l

        c_l = np.sign(c_l)*bc_l

    prob = np.power(np.sqrt(np.abs(a_l*d_l - b_l*c_l)),D-1)

    global p_l

    p_l =  prob/(np.sum(prob))


    global k

    if k % 20 == 0:
        print('Current maps:')
        img, _ = se.chaos(x,p_l,it = 20000)
        plt.imshow(img)
        plt.show()


    k = k + 1

    return mf.Phi(a_l,b_l,c_l,d_l,e_l,f_l,p_l,it = it,n=n)
```

Make a function to minimise and see that our IFS for the Steemson triangle makes the cost small.

Should the reader desire, the 'similitude' line can be changed to True. If this is done, at each step in the minimisation, the affine map is projected to its closest similitude function in the induced L^2 norm.

```python
In [24]: k = 0

         #can select to project to similitude transformation
```

```
        similitude = False

        def cost_Steem(x):

            return Phi_wrap(x,D_Steem,it = 1,n=len(Steem_mom),sim = similitude)@Steem_mom - St

        print(IFS_STEEM)

        cost_Steem(IFS_STEEM)
```

```
[ 0.14876633  0.58055152  0.41944848  0.47051039  0.          0.
  0.47051039  0.          0.          -0.14876633  0.58055152  0.41944848
  0.          0.41944848  0.14876633  0.          0.          0.47051039]
Current image:
```
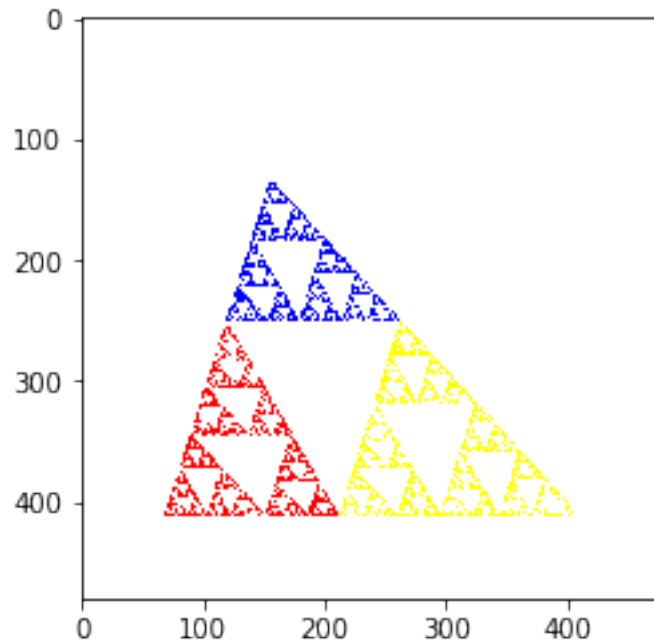


```
Out[24]: array([ 0.00000000e+00, -4.03820921e-10, -2.95844793e-10, -3.42571915e-10,
               -2.31741598e-10, -1.55687963e-10, -2.90381219e-10, -1.49346015e-10,
               -1.04951207e-10, -1.04374828e-10, -2.54266608e-10, -1.01293637e-10,
               -5.93118159e-11, -6.11033966e-11, -7.41286407e-11, -2.27984048e-10,
               -7.33204452e-11, -3.46715417e-11, -3.12596667e-11, -3.90244434e-11,
               -5.40108860e-11])
```

This starting point corresponds to a Sierpinski triangle with orientations with no flips. The Steemson triangle is also known as a FNN triangle, as it contains two maps which do not use conjugation and one that does.

```
In [25]: x0 = np.array([1/2,1/2,1/2,0,0,0,0,0,0,1/2,1/2,1/2,0,1/2,1/4,0,0,1/2])
```

We can now make a minimisation, whereby we also have plotted the attractor for the IFS parameters during the mininisation

```
In [26]: k=0
         t = time.time()
         sol_Steem = least_squares( cost_Steem, x0 , bounds=(-1,1) , verbose = 2)
         t =  time.time() - t
```
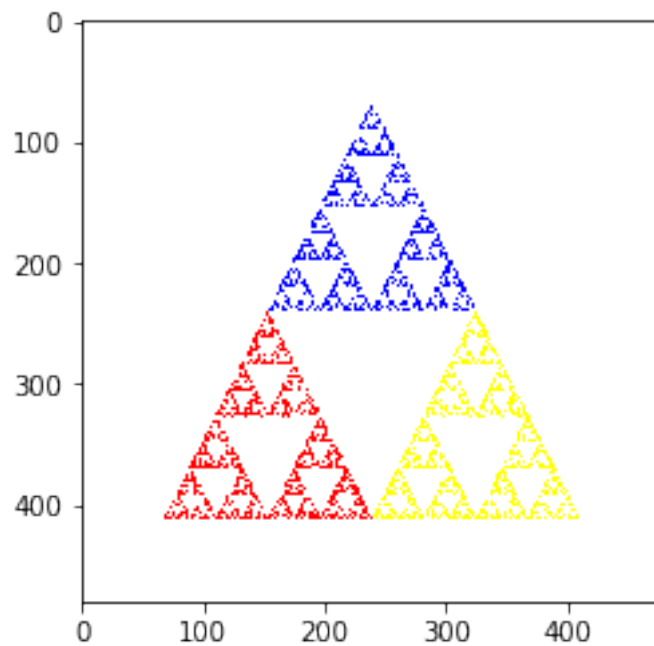
Current image:

Iteration	Total nfev	Cost	Cost reduction	Step norm	Optimality
0	1	6.9809e-03			1.23e-01

Current image:

```
     1              2          1.1720e-04      6.86e-03      1.30e-01      1.14e-02
Current image:
```
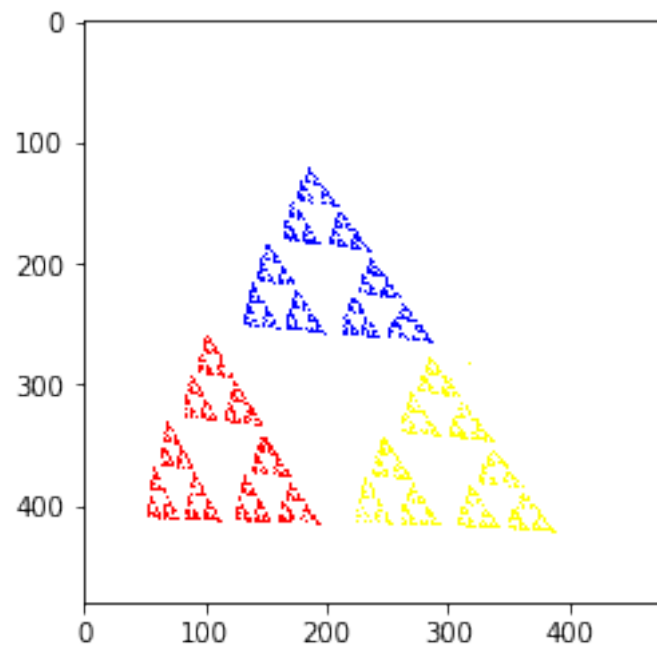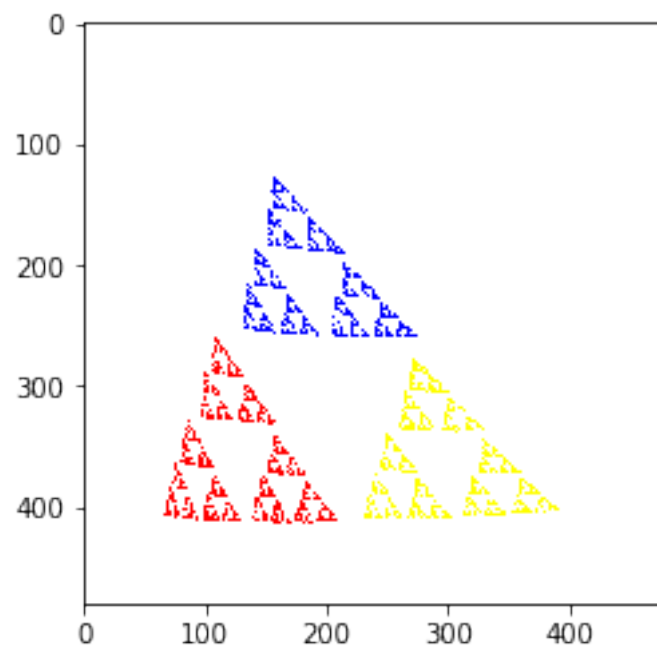
2	3	5.1008e-06	1.12e-04	6.71e-02	2.58e-03

Current image:
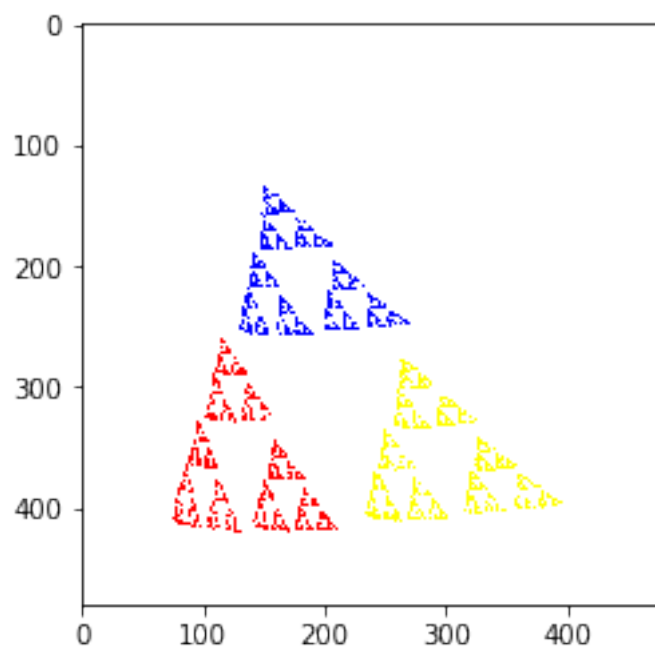
3	4	4.6443e-07	4.64e-06	6.48e-02	3.64e-04

Current image:
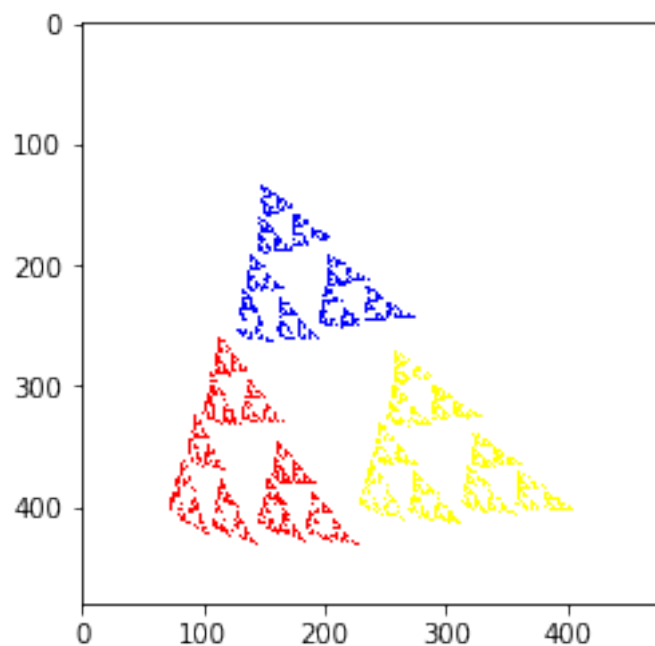
```
    4              5         1.4429e-07      3.20e-07      8.48e-02      4.07e-04
Current image:
```
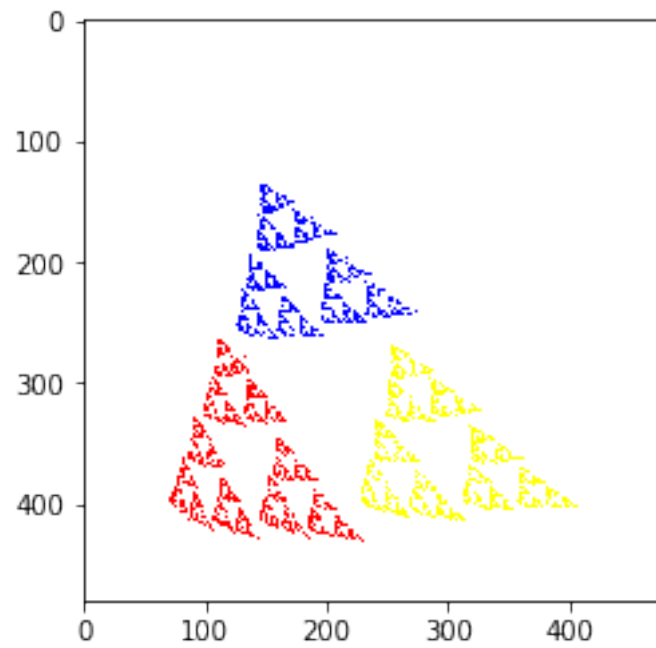


```
    5              6         3.7744e-08      1.07e-07      2.68e-02      4.67e-05
Current image:
```

| 6 | 7 | 2.0842e-08 | 1.69e-08 | 6.07e-02 | 1.34e-04 |
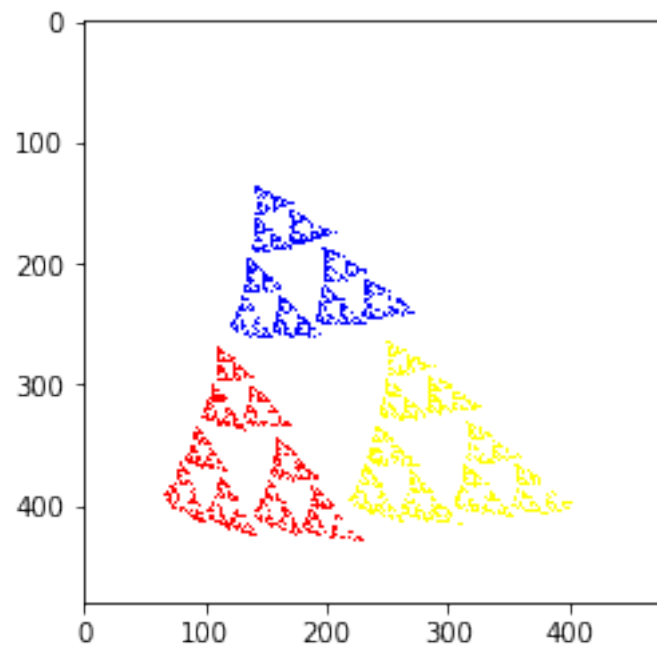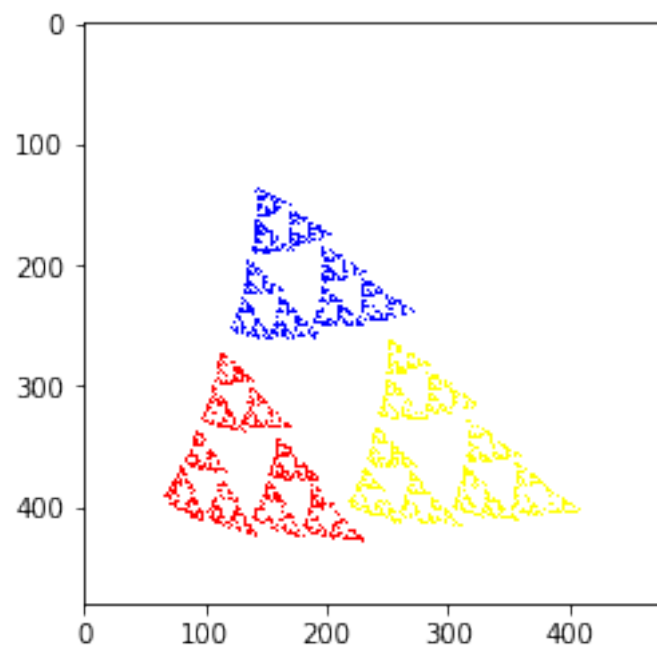
Current image:

| 7 | 8 | 5.3962e-09 | 1.54e-08 | 1.96e-02 | 1.60e-05 |

Current image:
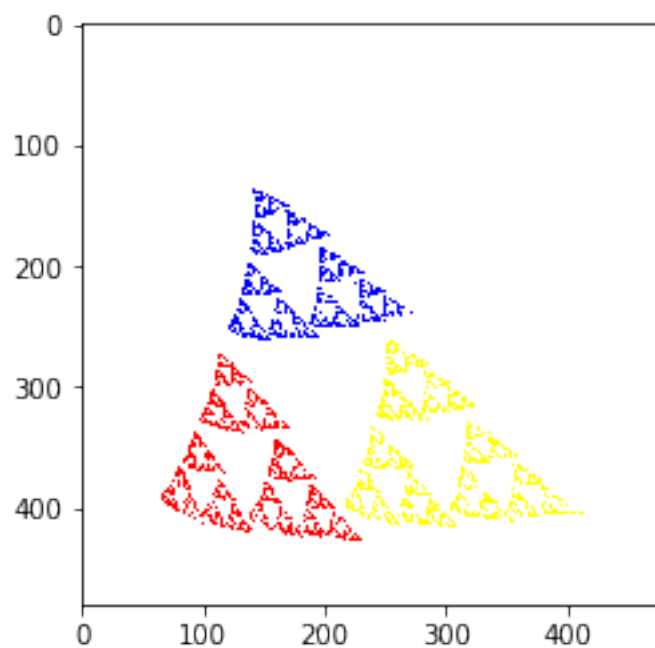
| 8 | 10 | 3.2061e-09 | 2.19e-09 | 1.54e-02 | 7.27e-06 |

Current image:
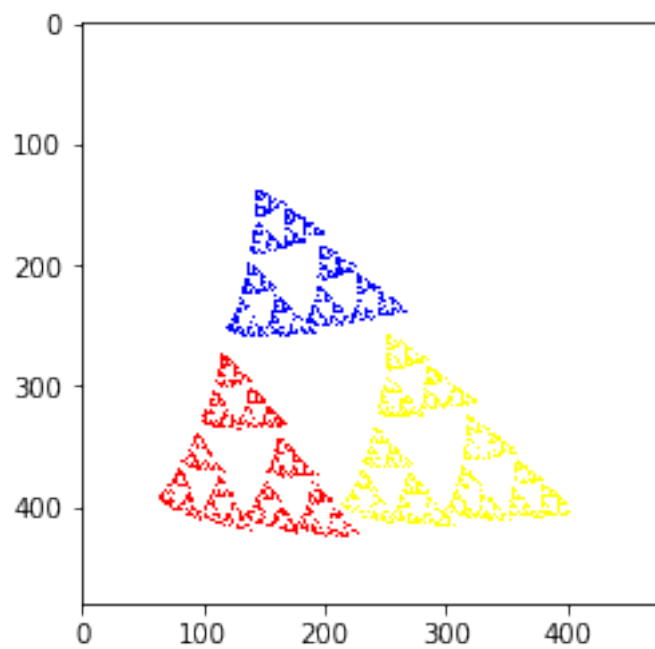
9   11   1.6310e-09   1.58e-09   2.90e-02   3.13e-05
Current image:
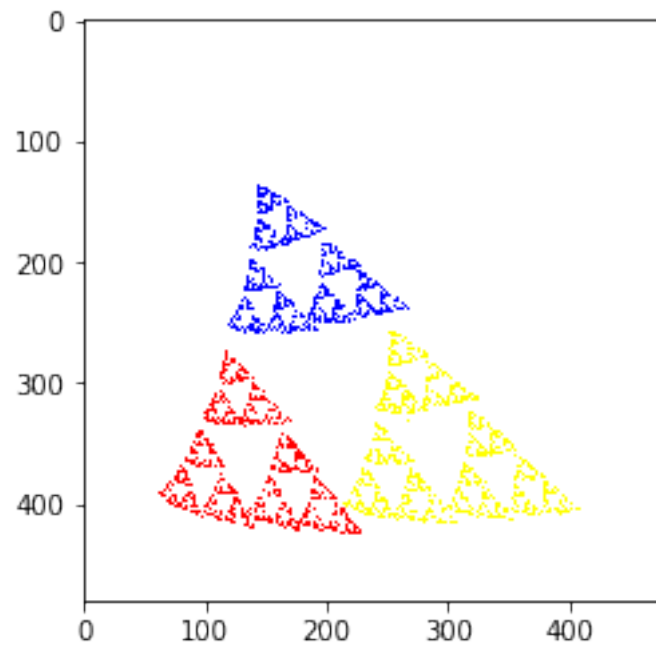
         10   12   7.2071e-10   9.10e-10   1.04e-02   8.76e-06
Current image:
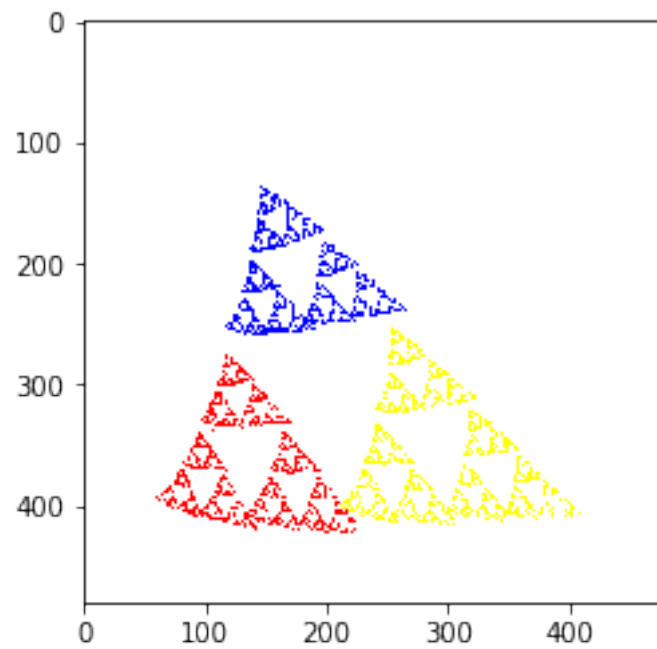
| 11 | 13 | 5.4347e-10 | 1.77e-10 | 1.73e-02 | 2.20e-05 |
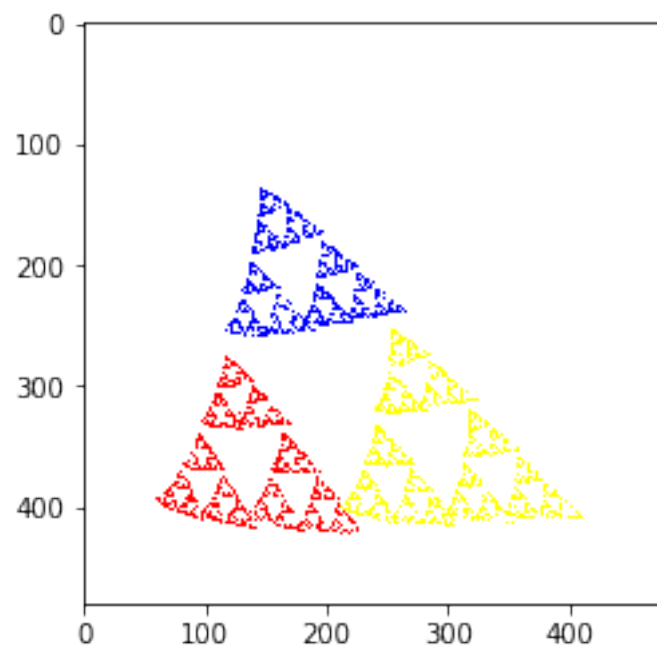
Current image:

| 12 | 14 | 3.2289e-10 | 2.21e-10 | 7.01e-03 | 2.61e-06 |

Current image:
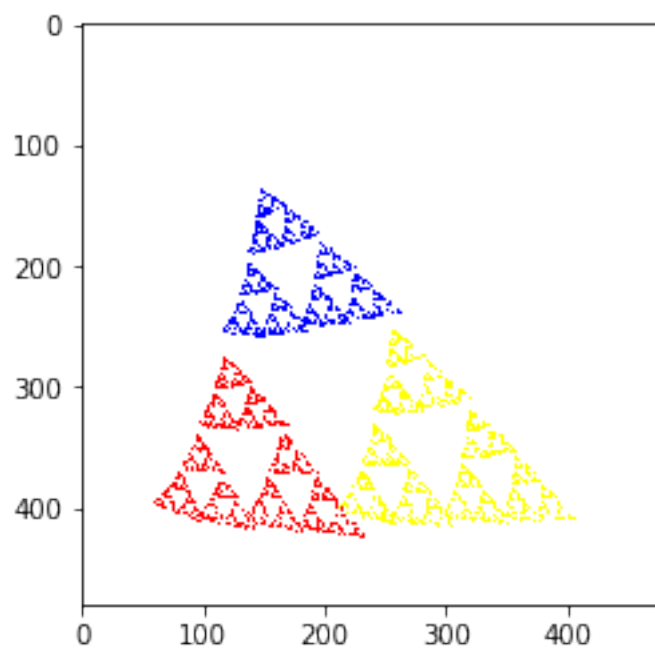
| 13 | 16 | 2.5892e-10 | 6.40e-11 | 7.25e-03 | 4.12e-06 |

Current image:
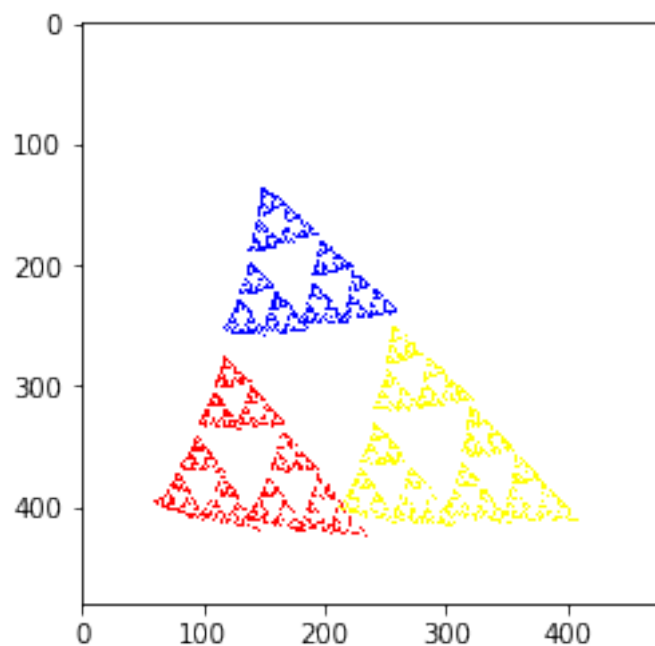
| 14 | 17 | 2.0629e-10 | 5.26e-11 | 1.44e-02 | 1.06e-05 |

Current image:
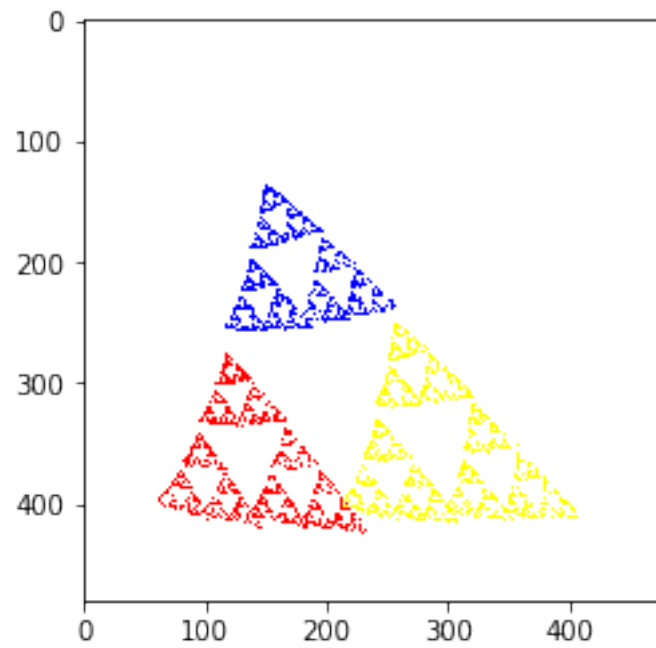
| 15 | 18 | 1.2893e-10 | 7.74e-11 | 5.31e-03 | 1.23e-06 |

Current image:
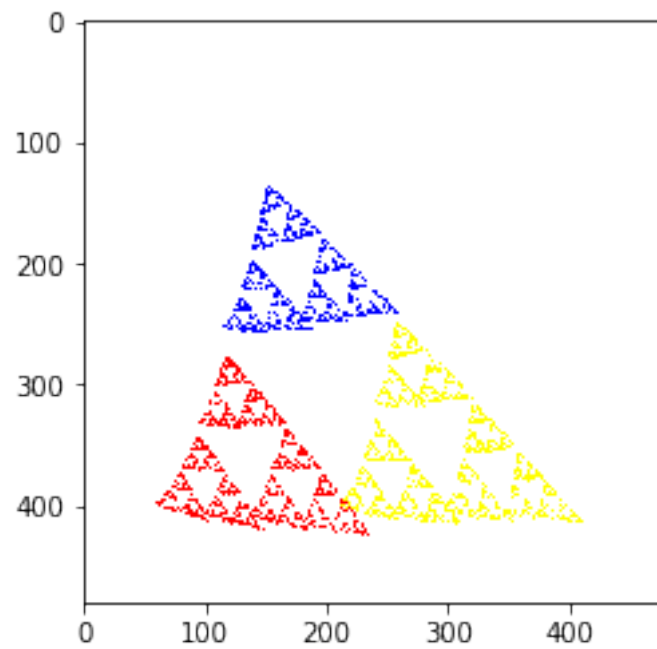
16   19   1.2227e-10   6.65e-12   1.44e-02   8.26e-06
Current image:
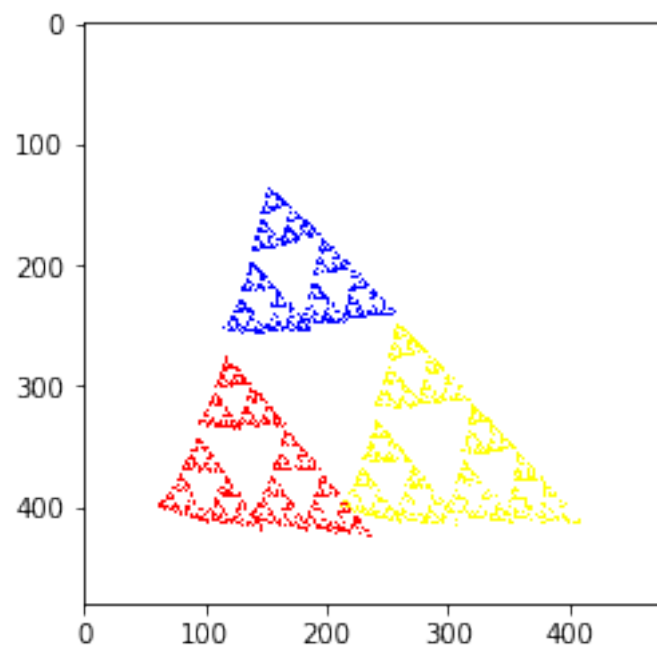
```
      17              20           6.5509e-11        5.68e-11        2.89e-03        2.74e-07
Current image:
```
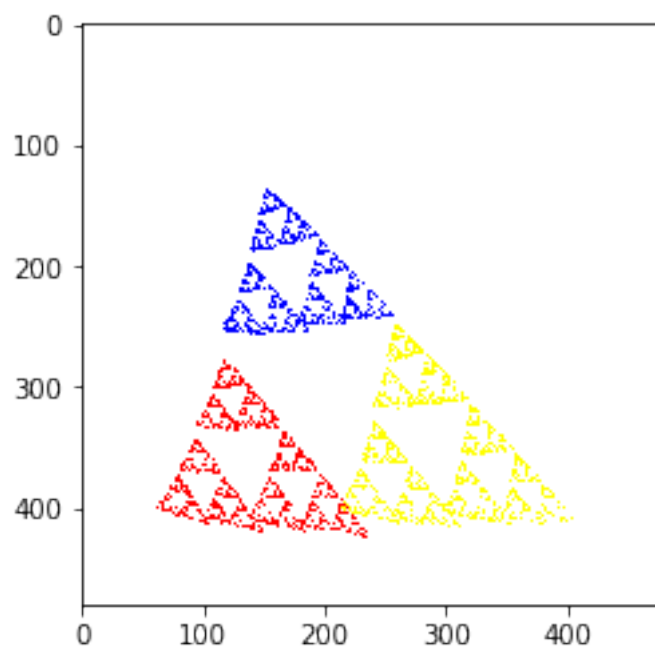


```
      18              21           5.7048e-11        8.46e-12        3.56e-03        8.61e-07
Current image:
```
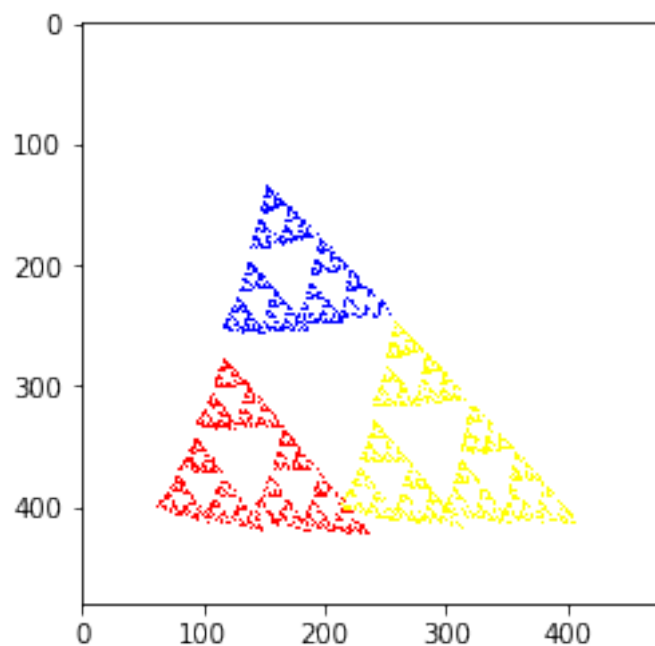
19	22	4.9511e-11	7.54e-12	7.05e-03	3.28e-06

Current image:
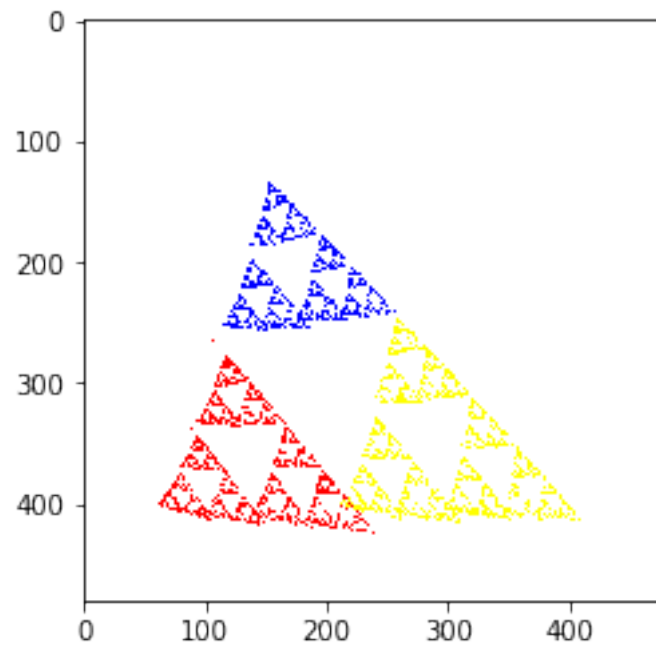
20	23	3.8847e-11	1.07e-11	6.16e-03	2.30e-06
21	24	3.3347e-11	5.50e-12	7.20e-03	2.81e-06

Current image:
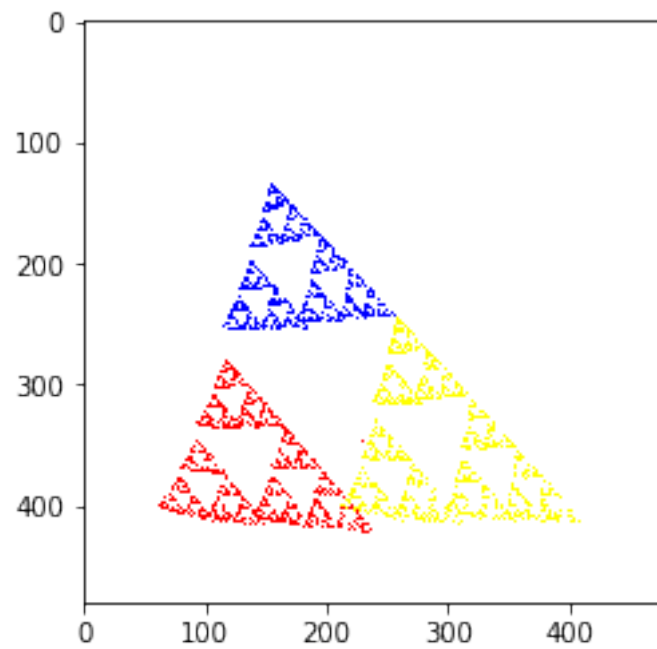
| 22 | 25 | 2.5960e-11 | 7.39e-12 | 4.98e-03 | 1.39e-06 |
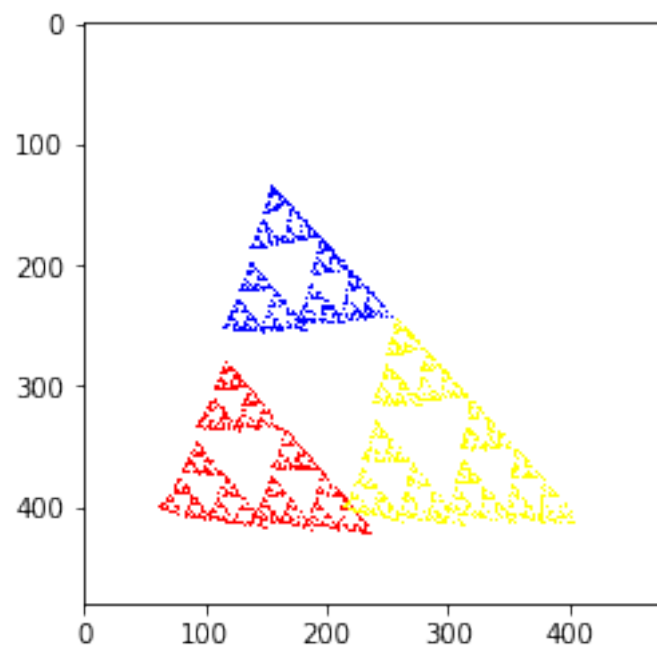
Current image:

23	26	2.4713e-11	1.25e-12	7.28e-03	3.00e-06
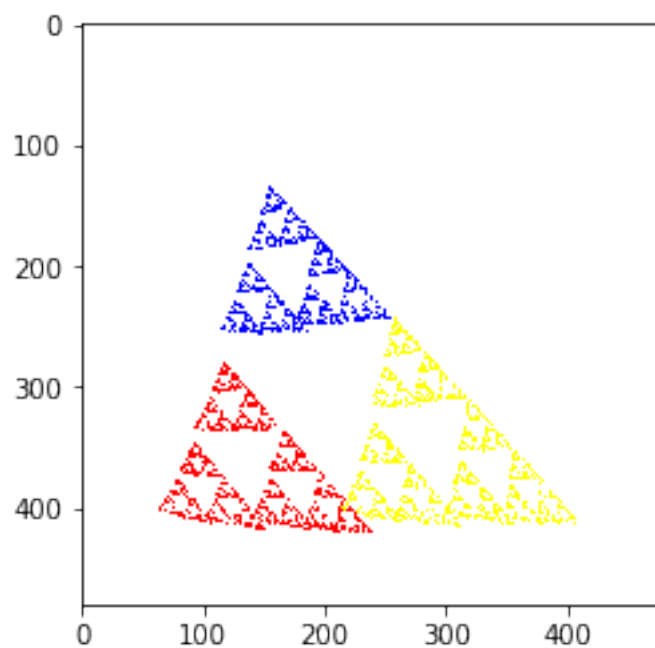
Current image:

24	27	1.8665e-11	6.05e-12	1.82e-03	2.01e-07
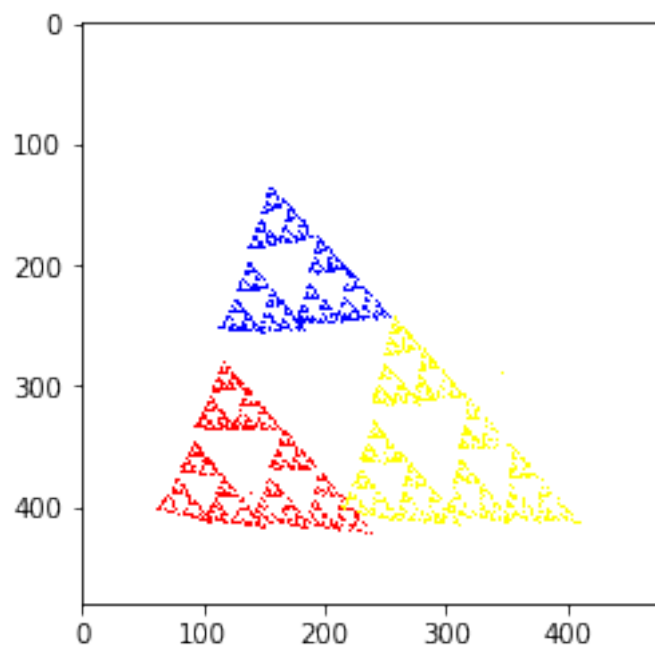
Current image:

```
        25              28          1.6652e-11      2.01e-12       3.56e-03       8.59e-07
Current image:
```
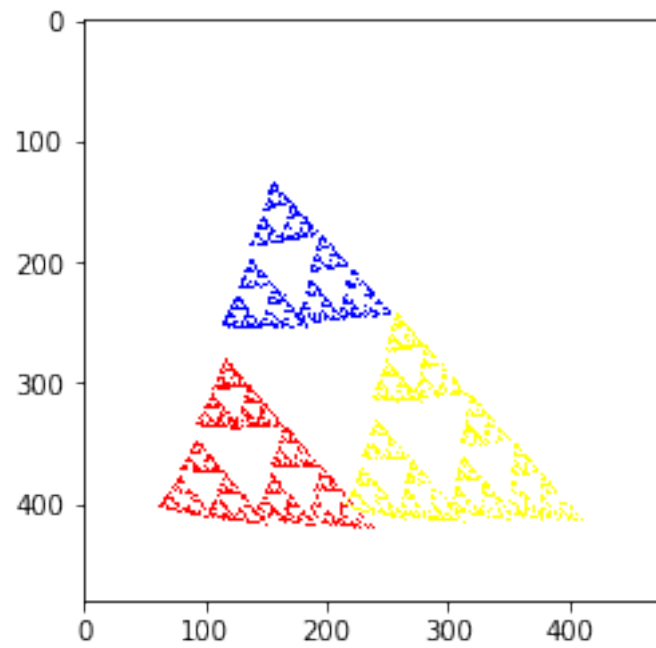


```
        26              29          1.6363e-11      2.89e-13       6.24e-03       2.49e-06
Current image:
```
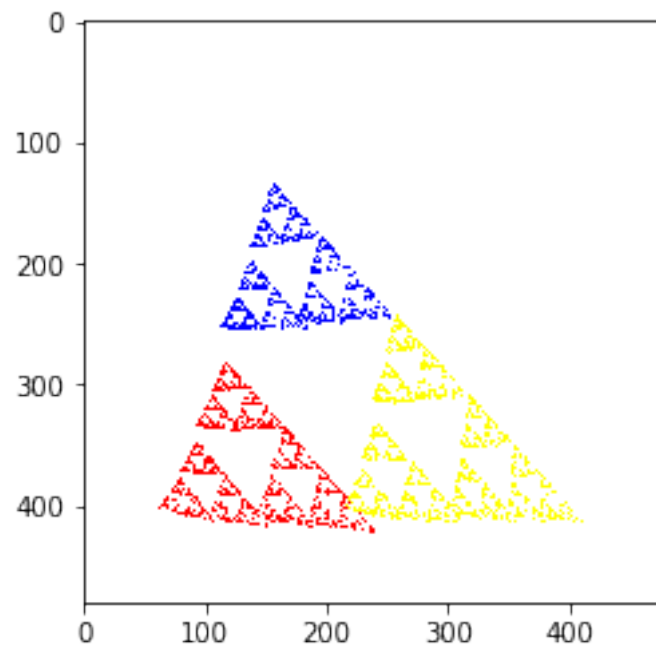
| 27 | 30 | 1.2435e-11 | 3.93e-12 | 1.57e-03 | 1.49e-07 |
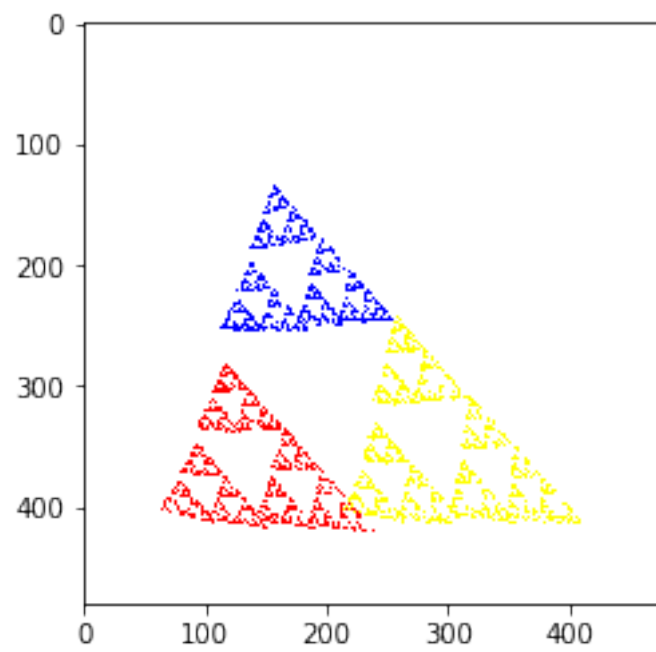
Current image:

| 28 | 31 | 1.1303e-11 | 1.13e-12 | 3.19e-03 | 6.96e-07 |

Current image:
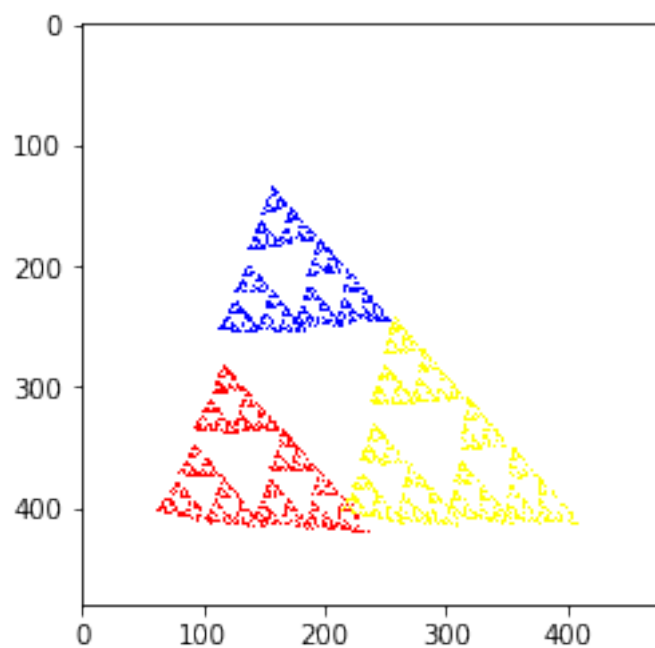
| 29 | 33 | 1.0516e-11 | 7.87e-13 | 1.46e-03 | 1.49e-07 |

Current image:
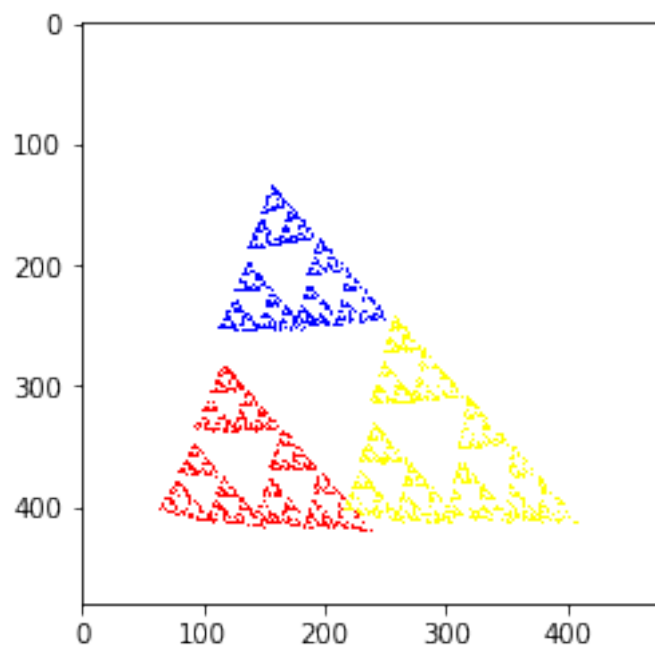
| 30 | 34 | 9.7373e-12 | 7.78e-13 | 2.92e-03 | 6.22e-07 |

Current image:
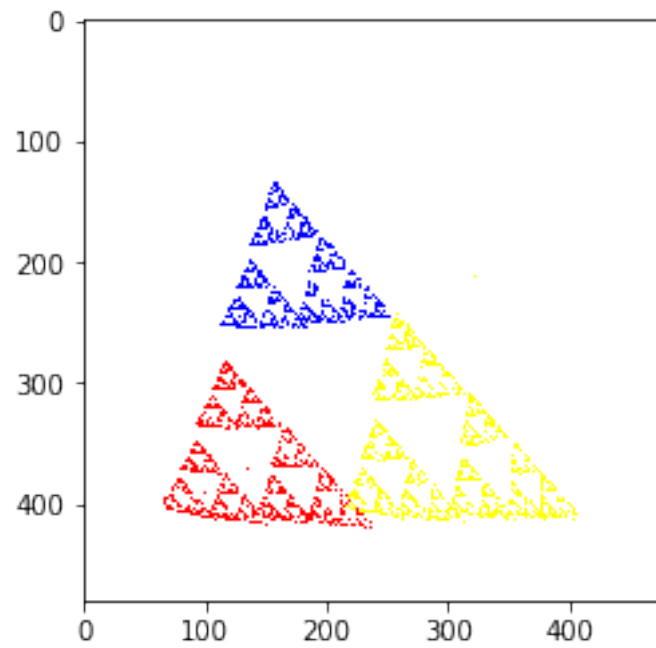
| 31 | 36 | 9.1955e-12 | 5.42e-13 | 1.22e-03 | 1.10e-07 |

Current image:
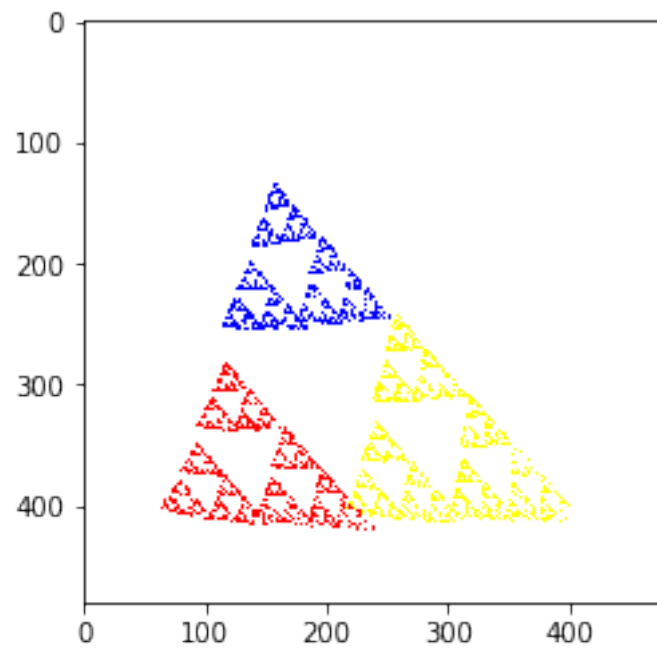
| 32 | 37 | 8.6893e-12 | 5.06e-13 | 2.45e-03 | 4.62e-07 |
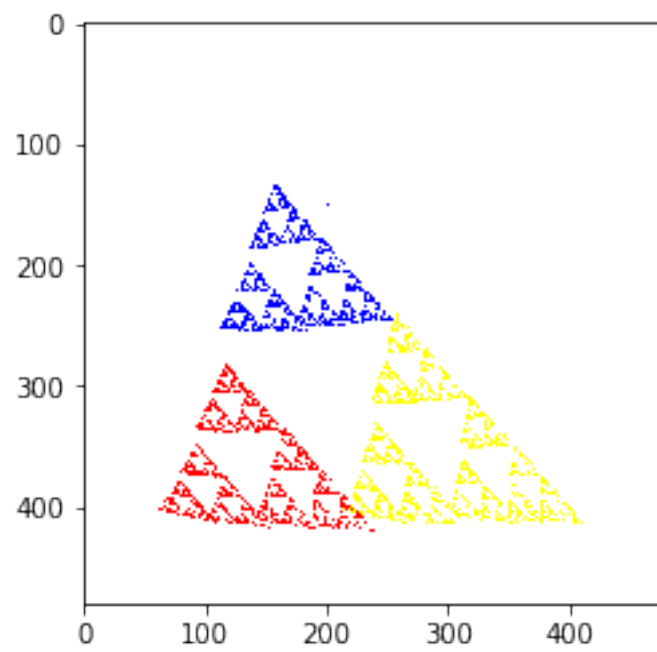
Current image:

| 33 | 39 | 8.3451e-12 | 3.44e-13 | 1.15e-03 | 9.98e-08 |

Current image:
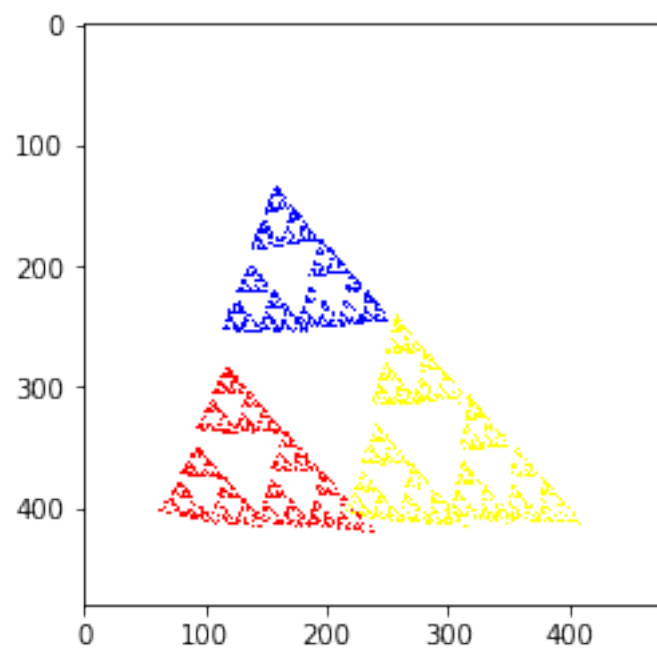
| 34 | 40 | 8.0252e-12 | 3.20e-13 | 2.29e-03 | 4.09e-07 |

Current image:
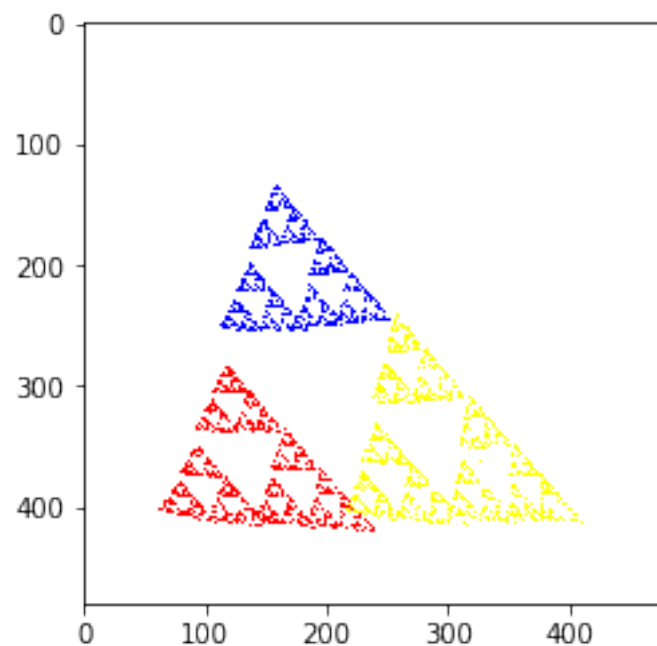
| 35 | 41 | 7.9312e-12 | 9.40e-14 | 3.65e-03 | 8.38e-07 |

Current image:
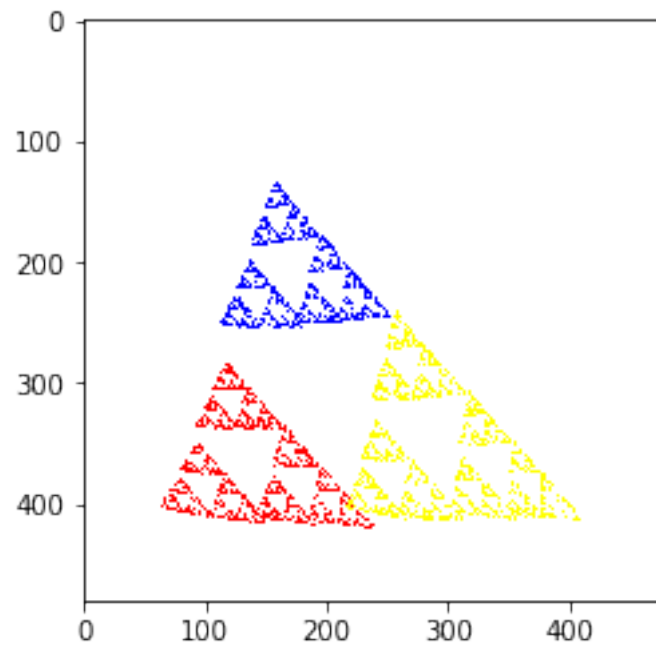
| 36 | 42 | 7.4824e-12 | 4.49e-13 | 9.45e-04 | 5.57e-08 |

Current image:
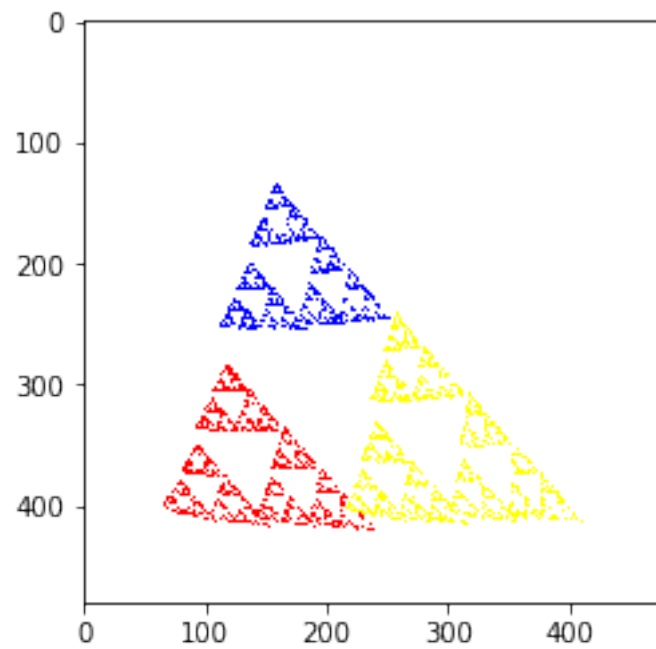
<pre>   37   43   7.3953e-12   8.72e-14   1.87e-03   2.39e-07</pre>
Current image:
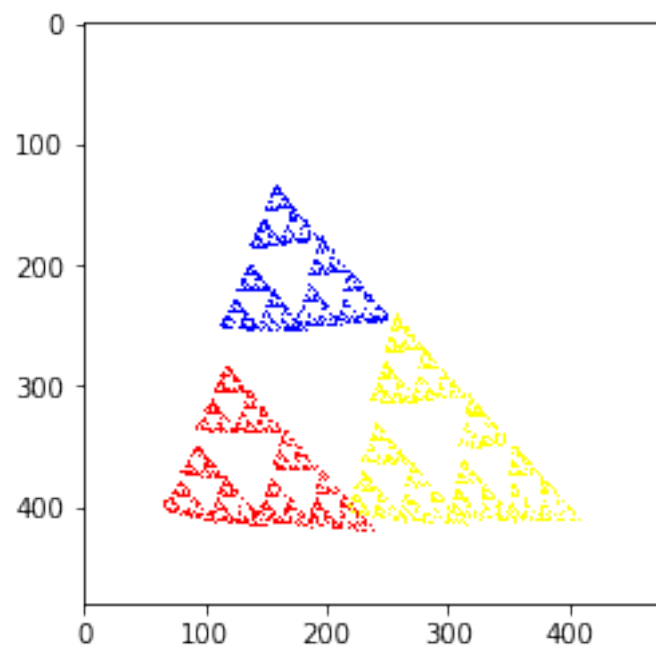
| 38 | 44 | 7.3224e-12 | 7.29e-14 | 1.83e-03 | 1.66e-07 |
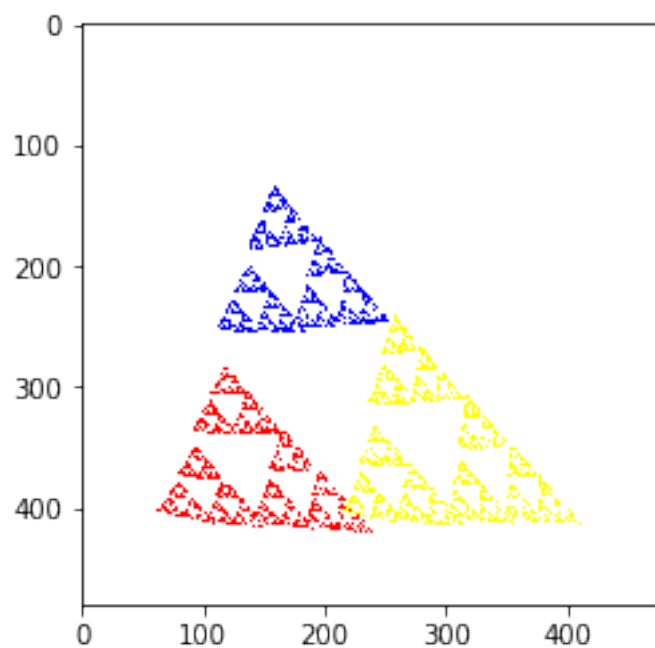
Current image:

| 39 | 45 | 7.2857e-12 | 3.66e-14 | 1.23e-03 | 5.20e-08 |
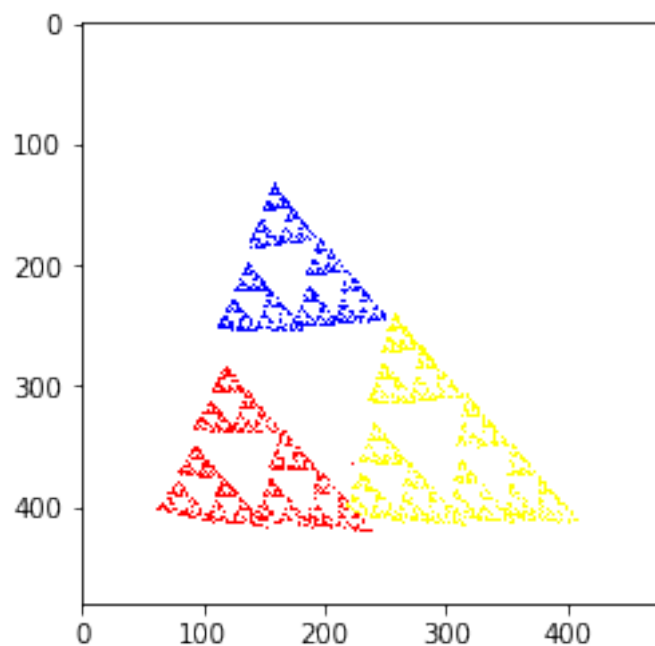
Current image:

| 40 | 46 | 7.2715e-12 | 1.42e-14 | 1.57e-03 | 9.31e-08 |

Current image:
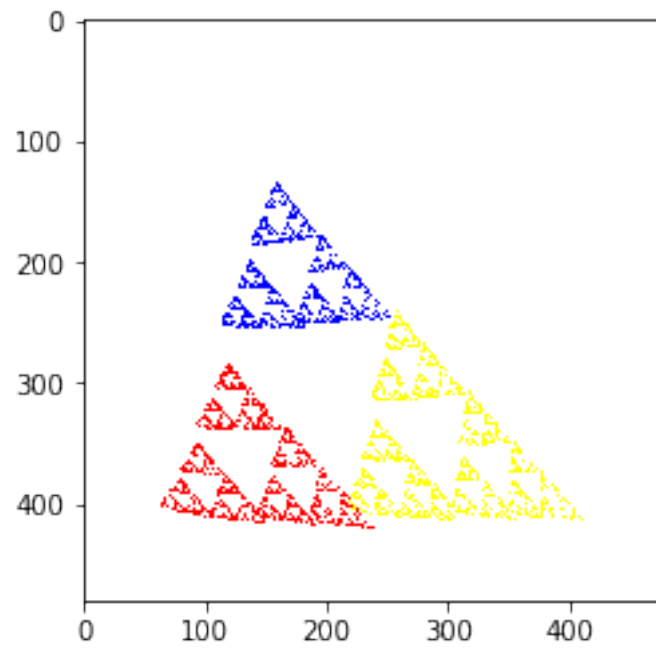
| 41 | 47 | 7.2621e-12 | 9.44e-15 | 6.28e-04 | 1.46e-08 |

Current image:
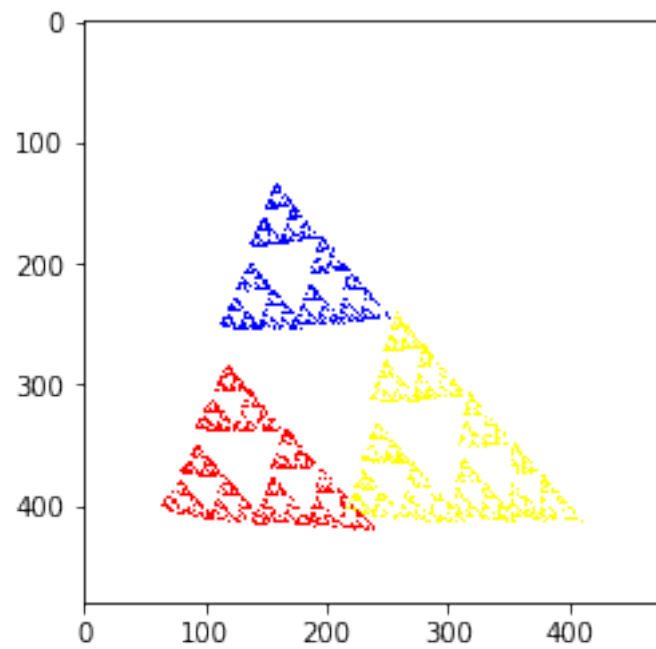
42   49   7.2583e-12   3.85e-15   5.26e-04   1.39e-08
Current image:
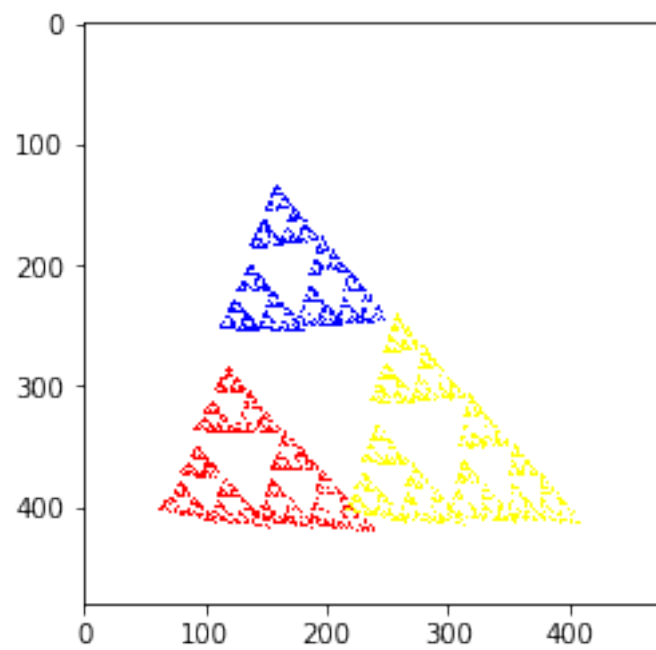
```
       43                50              7.2532e-12        5.01e-15         1.04e-03         5.59e-08
Current image:
```
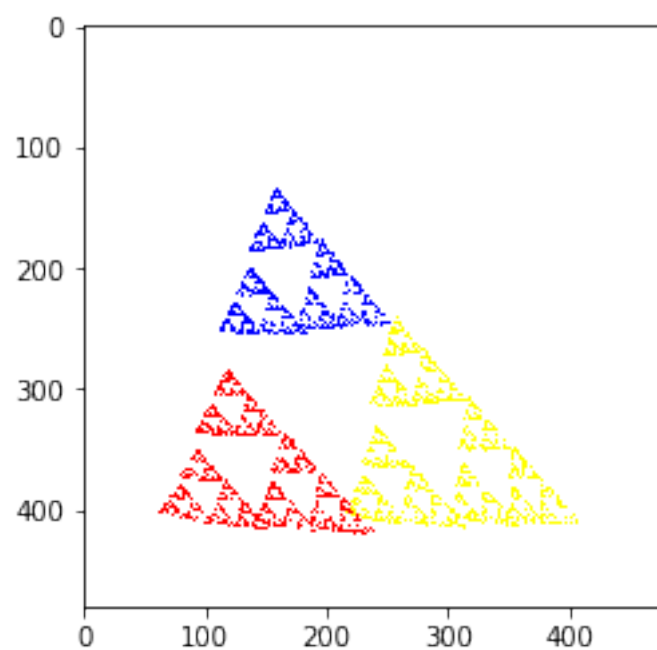


```
       44                51              7.2496e-12        3.62e-15         5.67e-04         1.50e-08
       45                52              7.2475e-12        2.15e-15         1.02e-03         5.43e-08
Current image:
```
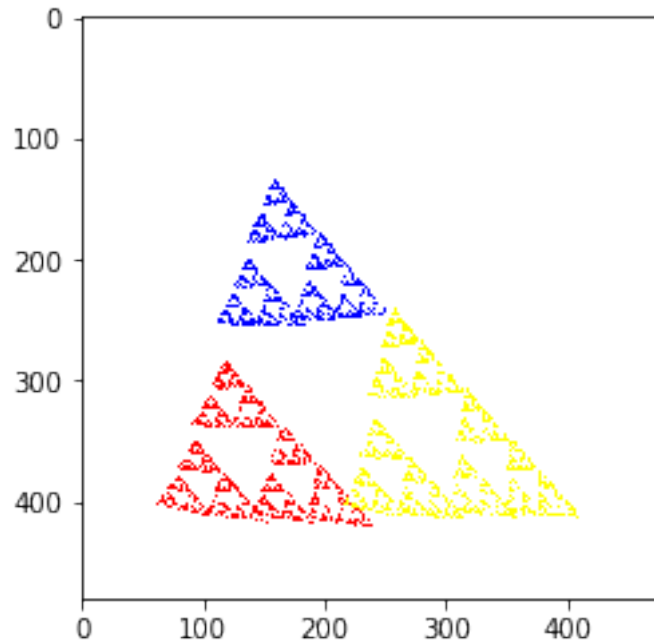
```
       46              53           7.2459e-12      1.58e-15       2.48e-04       2.68e-09
`gtol` termination condition is satisfied.
Function evaluations 53, initial cost 6.9809e-03, final cost 7.2459e-12, first-order optimality
```

As we can see, the algorithm did not find the global minimum, but the approximation for our local minima is interesting.

In fact this is the evidence for the Collage theorem for moments.

```
In [28]: img,_ = se.chaos(sol_Steem.x,p_l, it = 100000, imgxy=500)
         img
```

```
Out[28]:
```
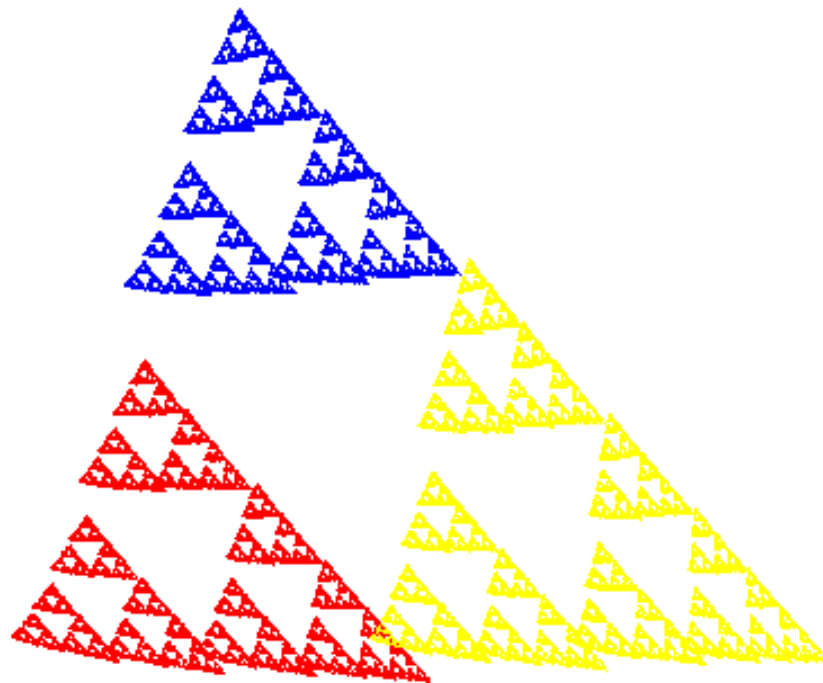
We now make a good starting point. This is still a Sierpinski triangle, but with the correct map oreientations to match a Steemson.

```
In [41]: #start Steemson Sierpinski attempt
         a  =  1
         b  =  1

         imgxy = 500

         #co ordinates of triangles third vertex
         Cy  =  (0.5*np.sqrt(-1 + 2*a**2 - a**4 + 2*b**2 + 2*a**2*b**2 - b**4))
         Cx  =  (0.5*(1 - a**2 + b**2))

         #angles for roatation
```

```python
        cosA   =   (Cx/b)
        sinA   =   (Cy/b)
        cosB   =   ((1.0-Cx)/a)
        sinB   =   (Cy/a)

        #scaling factors
        alphaNFF   =   (b)/((a**2 + b**2))
        betaNFF    =   (a)/(a**2 + b**2)
        gammaNFF   =   -1.0*(1.0-a**2-b**2)/(a**2 + b**2)

        alphaNNF   =   (b/(1.0 + b**2))
        betaNNF    =   (1.0/(1.0 + b**2))
        gammaNNF   =   (b**2/(1.0 + b**2))

        alphaNFF   =   alphaNNF
        betaNFF    =   betaNNF
        gammaNFF   =   gammaNNF

        NNF    =     [[cosA*alphaNNF,sinA*alphaNNF,sinA*alphaNNF,-cosA*alphaNNF,0.0,0.0],
                     [betaNNF,0.0,0.0,betaNNF,(1.0-betaNNF),0],
                     [gammaNNF,0.0,0.0,gammaNNF,(1.0-gammaNNF)*Cx,(1.0-gammaNNF)*Cy]]

        x0 = se.convertIFS(NNF)

In [42]: k=0
        t = time.time()
        sol_Steem = least_squares( cost_Steem, x0 , bounds=(-1,1) , verbose = 2)
        t =  time.time() - t
```
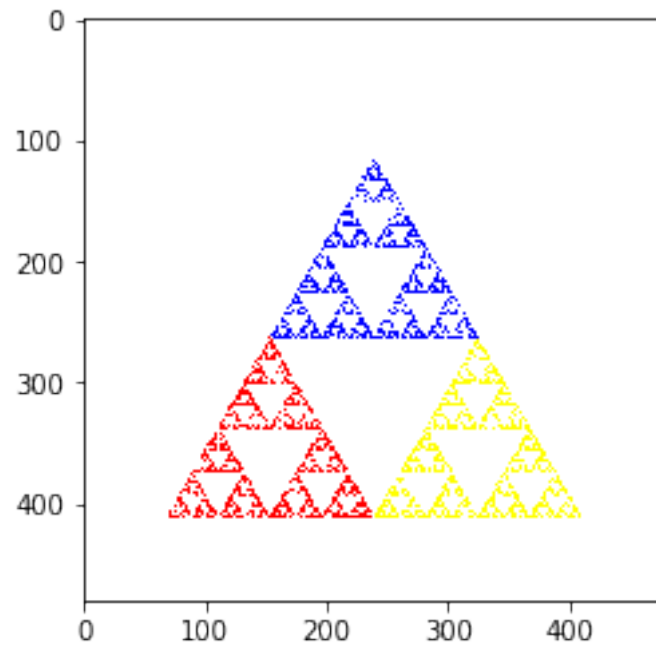
Current image:

Iteration	Total nfev	Cost	Cost reduction	Step norm	Optimality
0	1	4.9542e-03			1.27e-01

Current image:
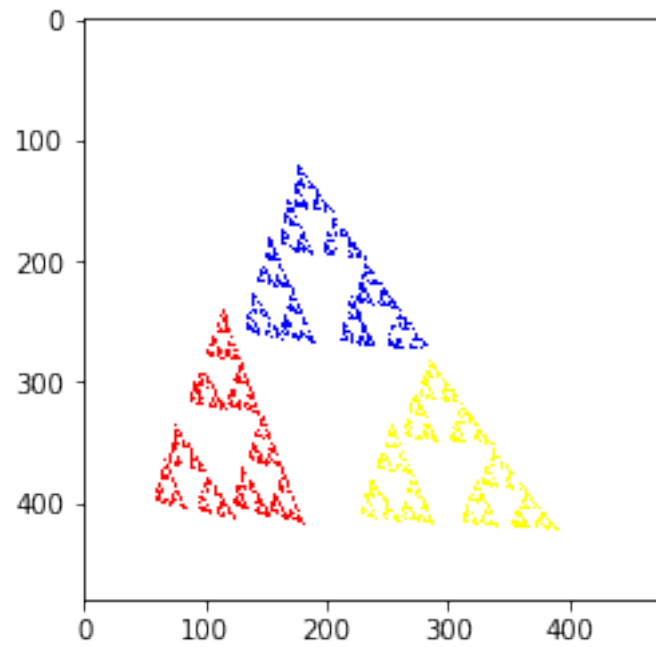
```
      1              2          6.2325e-05      4.89e-03      1.50e-01      1.27e-02
Current image:
```
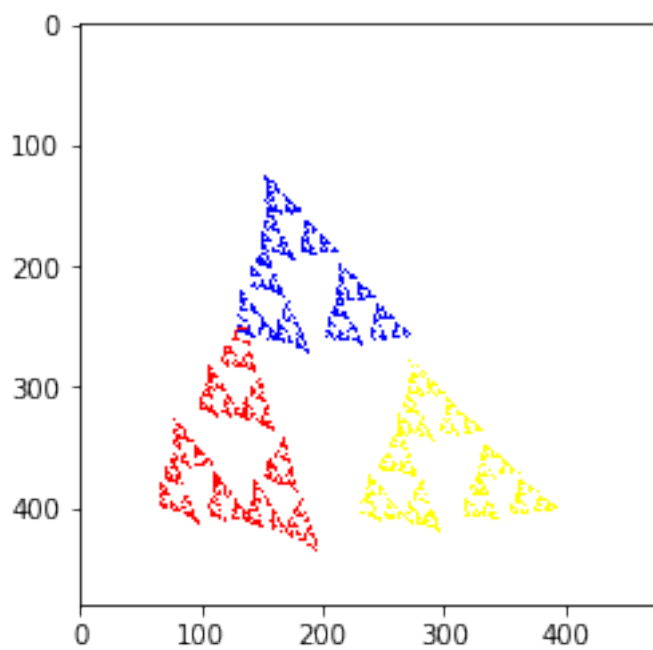


```
      2              3          4.1105e-06      5.82e-05      8.35e-02      2.96e-03
Current image:
```
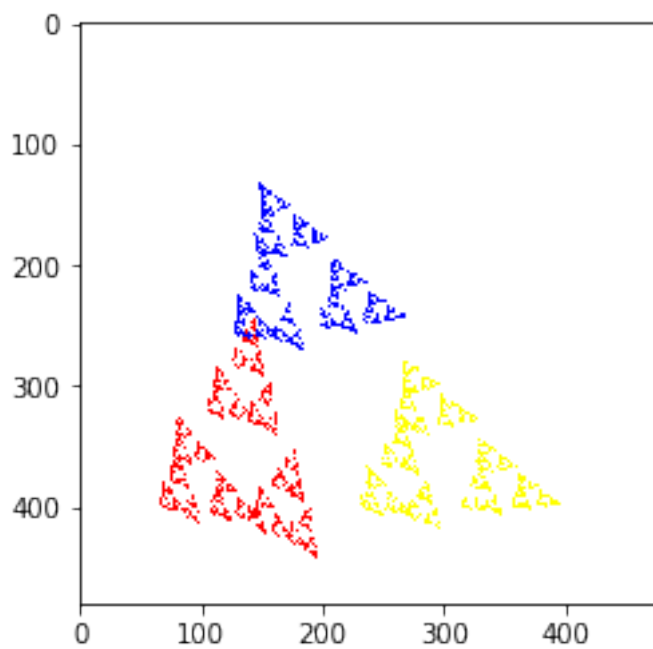
```
     3             4          3.1392e-07       3.80e-06        6.24e-02        3.07e-04
Current image:
```
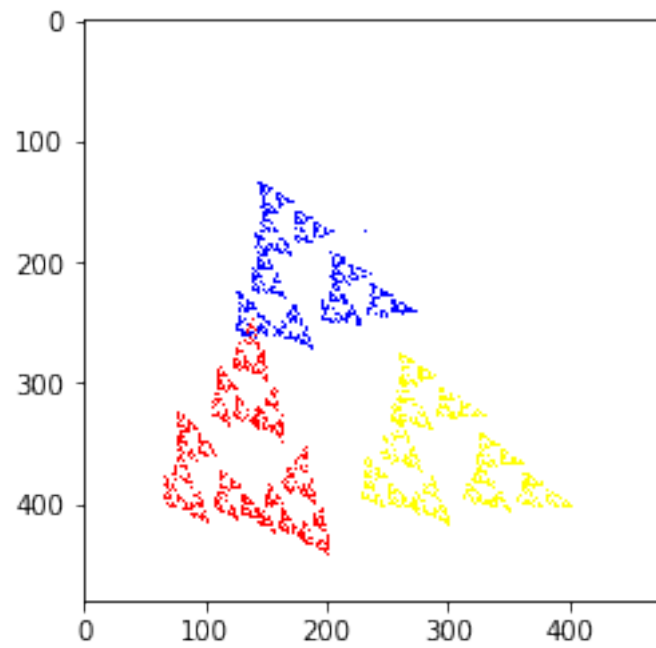


```
     4             5          7.9241e-08       2.35e-07        5.93e-02        1.18e-04
Current image:
```
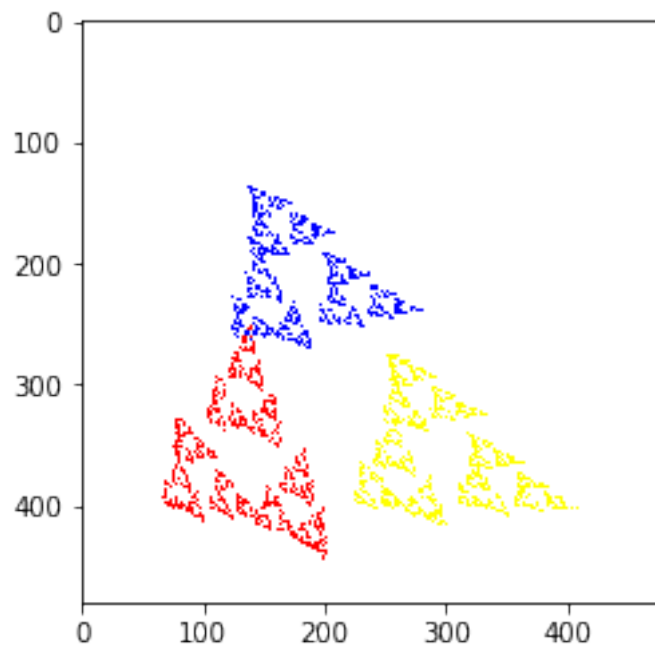
5   6   2.3343e-08   5.59e-08   4.60e-02   1.00e-04
Current image:

```
         6              7          1.0815e-08        1.25e-08        2.70e-02        3.65e-05
Current image:
```



```
         7              8          9.6149e-09        1.20e-09        5.51e-02        8.18e-05
Current image:
```
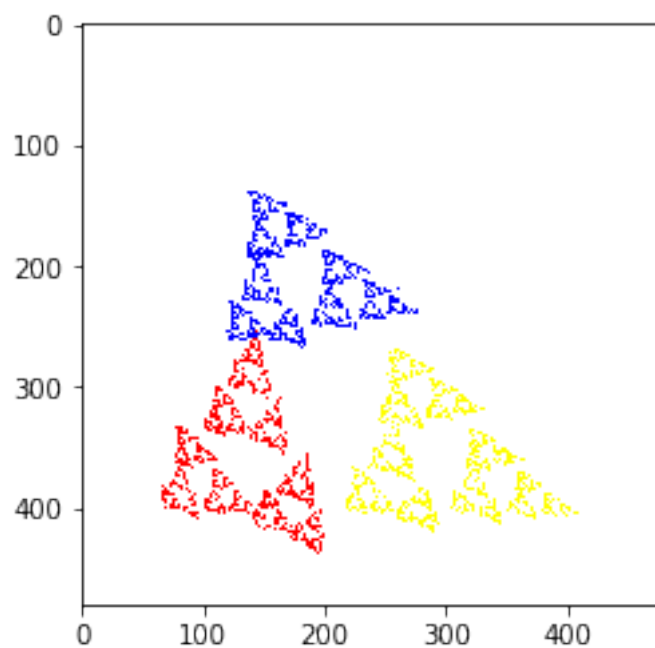
```
     8              9         2.8220e-09      6.79e-09       9.30e-03       5.41e-06
Current image:
```
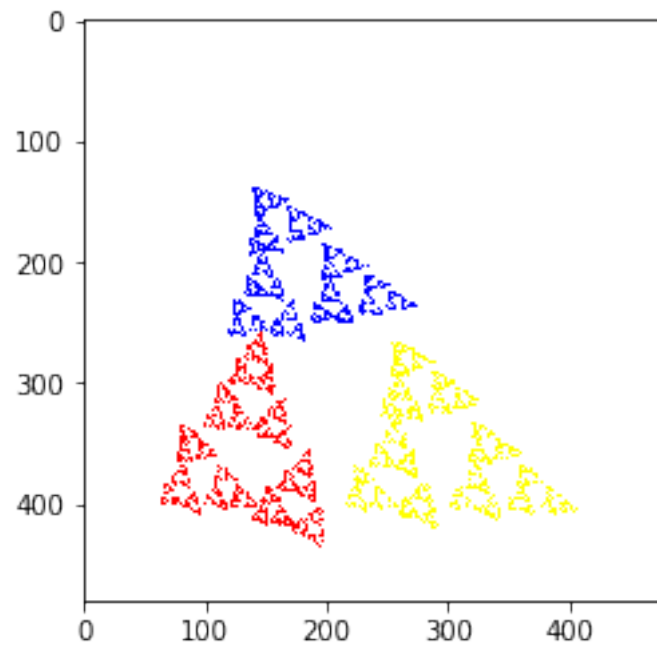


```
     9             10         1.8366e-09      9.85e-10       1.57e-02       5.86e-06
Current image:
```
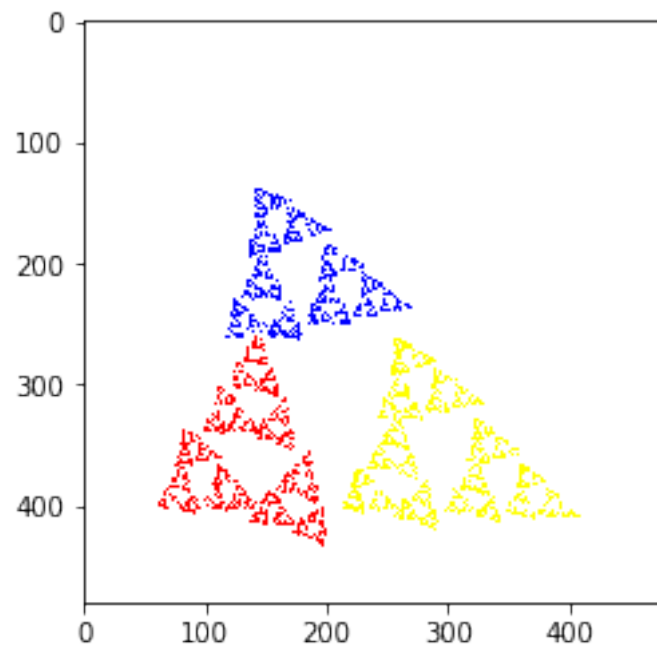
10   11   1.6099e-09   2.27e-10   2.86e-02   3.43e-05
Current image:
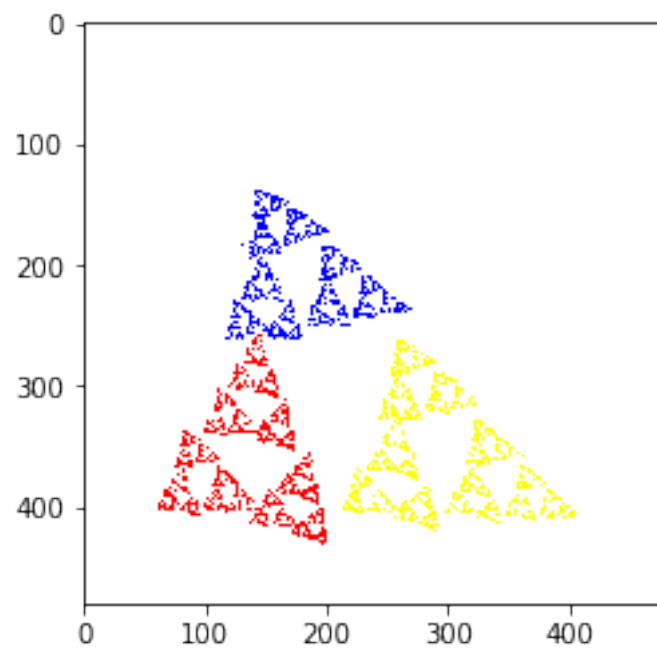
| 11 | 12 | 8.4952e-10 | 7.60e-10 | 4.66e-03 | 9.04e-07 |

Current image:
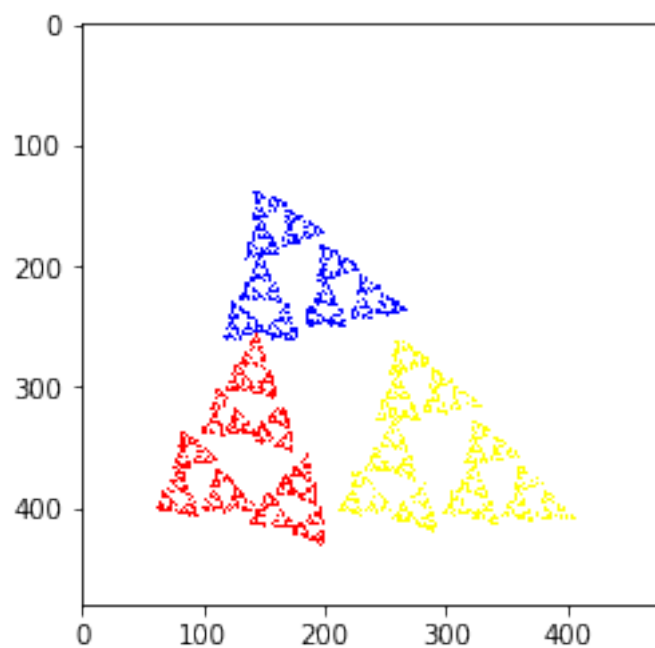
| 12 | 13 | 7.4631e-10 | 1.03e-10 | 7.43e-03 | 2.44e-06 |

Current image:
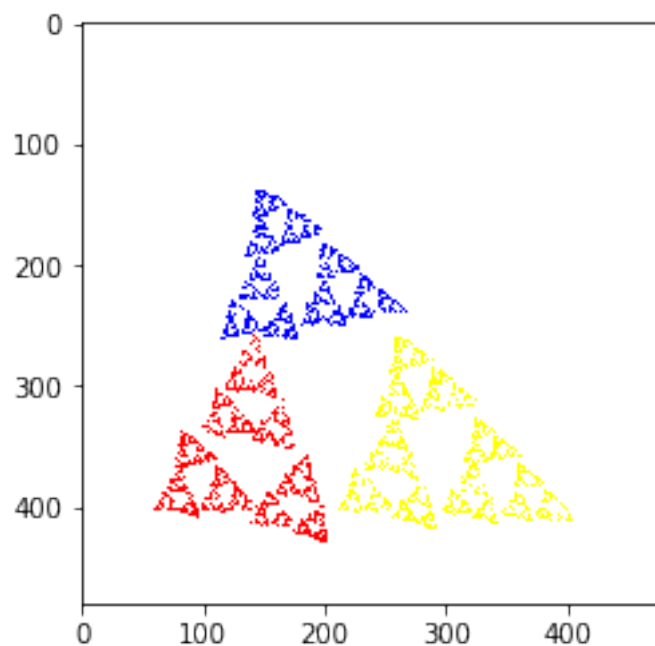
| 13 | 14 | 6.1154e-10 | 1.35e-10 | 1.38e-02 | 6.45e-06 |

Current image:
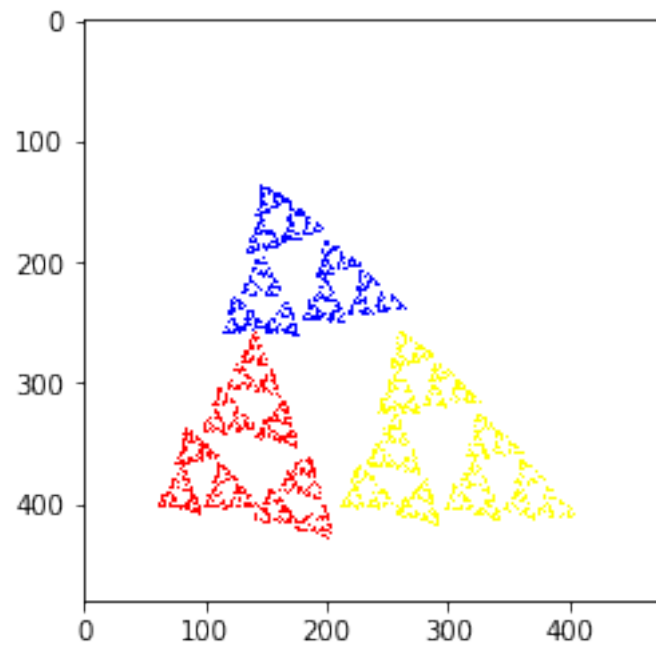
| 14 | 15 | 5.0578e-10 | 1.06e-10 | 1.41e-02 | 8.40e-06 |

Current image:
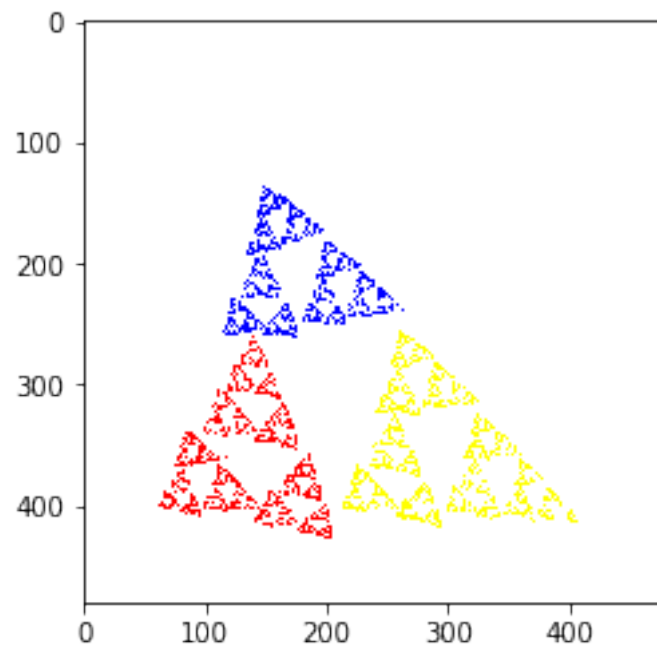
15   16   4.0441e-10   1.01e-10   9.72e-03   4.11e-06
Current image:
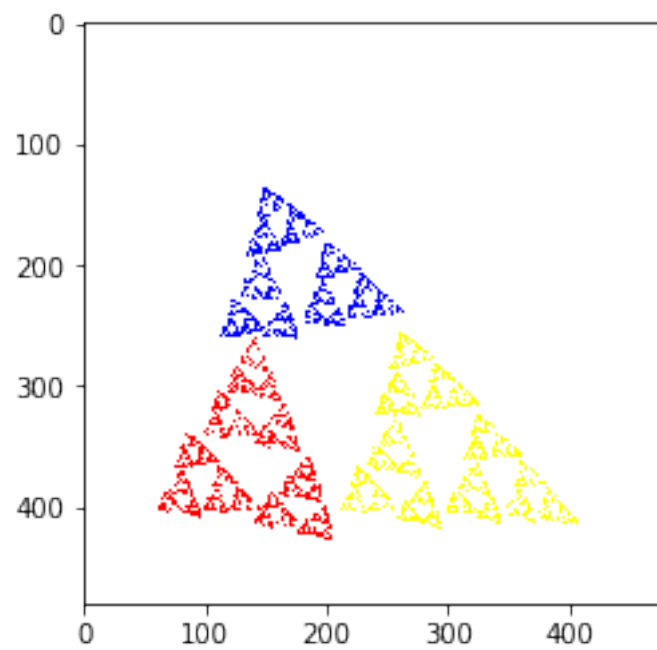
16	18	3.6510e-10	3.93e-11	4.47e-03	1.17e-06
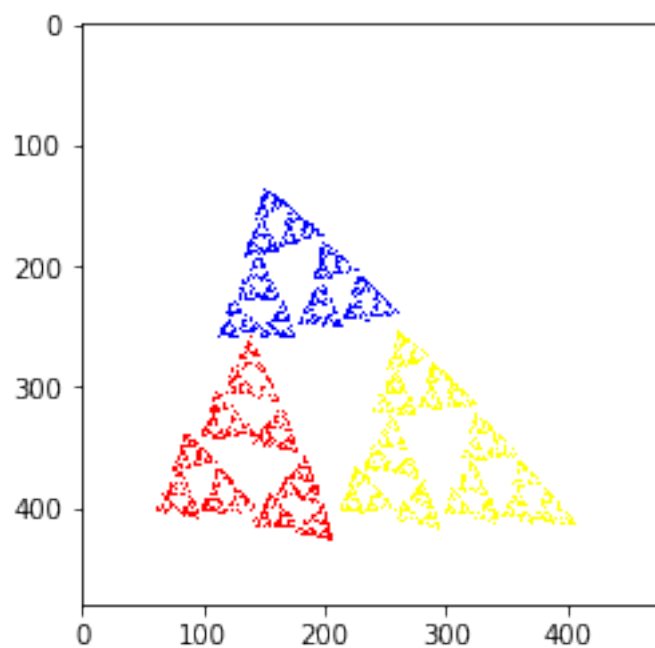
Current image:

17	19	3.2462e-10	4.05e-11	8.96e-03	4.81e-06
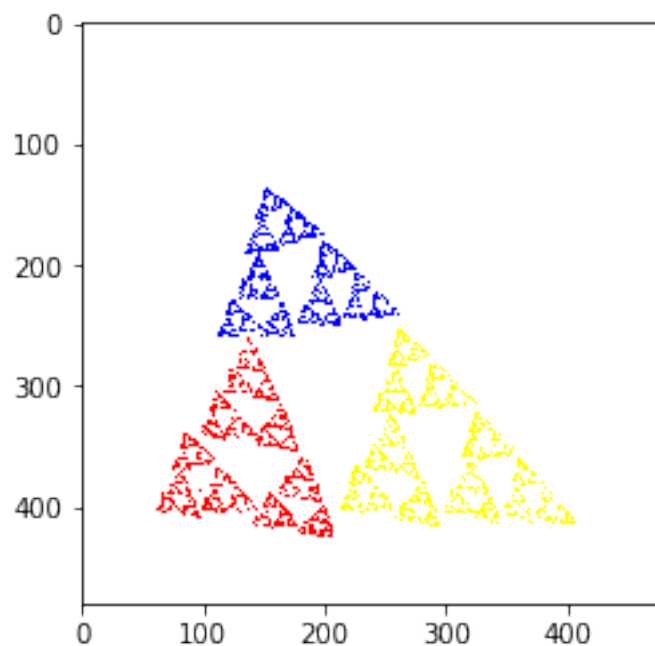
Current image:

| 18 | 20 | 2.7677e-10 | 4.78e-11 | 8.79e-03 | 3.67e-06 |

Current image:
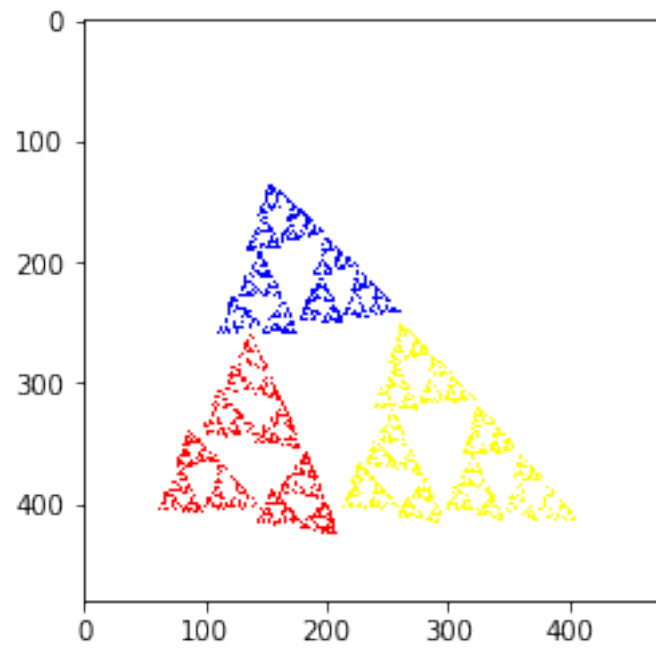
| 19 | 21 | 2.4984e-10 | 2.69e-11 | 1.32e-02 | 7.46e-06 |
| 20 | 22 | 2.0445e-10 | 4.54e-11 | 3.29e-03 | 3.91e-07 |

Current image:

21   23   1.8226e-10   2.22e-11   6.71e-03   1.75e-06
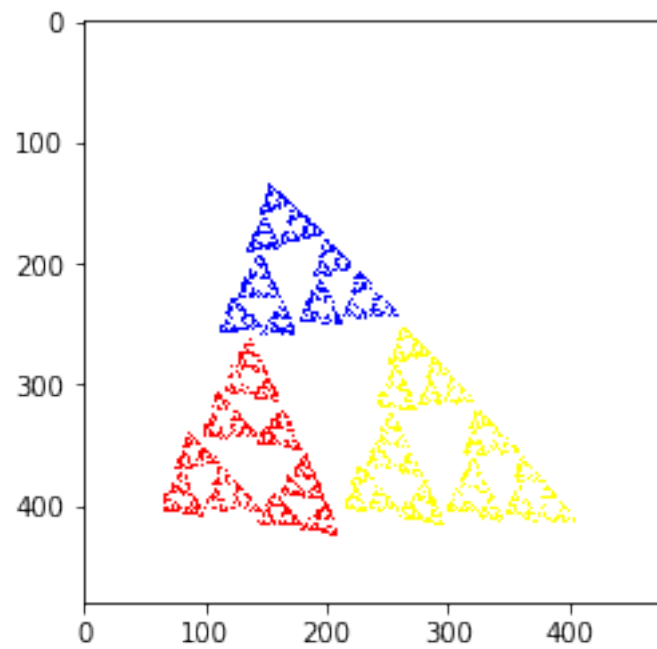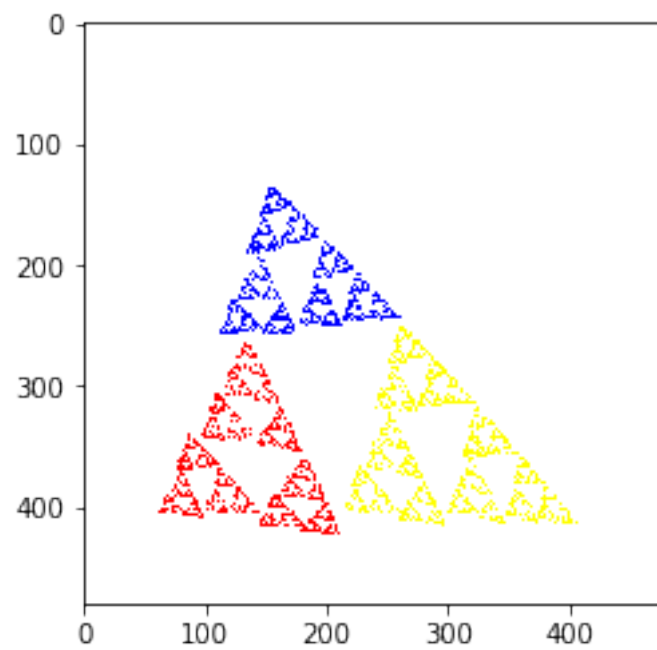
Current image:

22	24	1.5659e-10	2.57e-11	1.30e-02	2.47e-06

Current image:
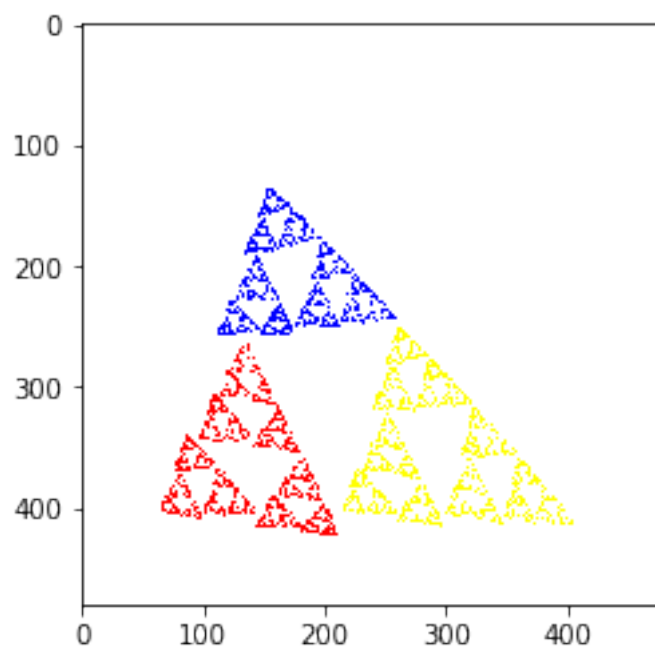
23	25	1.2518e-10	3.14e-11	5.44e-03	4.34e-07

Current image:
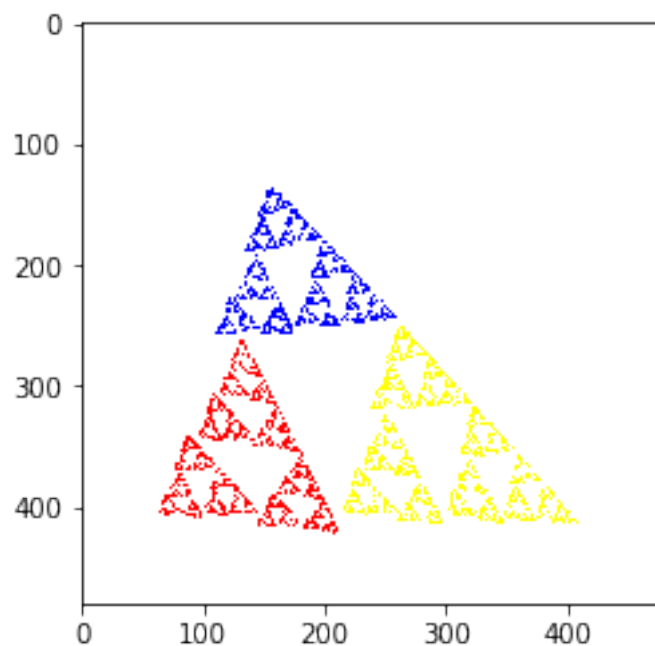
```
      24              26          1.1039e-10        1.48e-11         1.33e-02         2.75e-06
Current image:
```
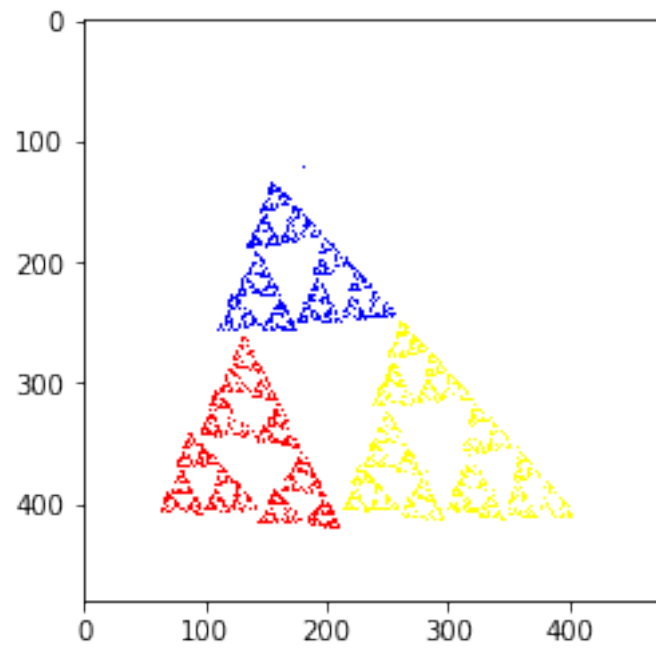


```
      25              27          8.3753e-11        2.66e-11         3.99e-03         2.54e-07
Current image:
```
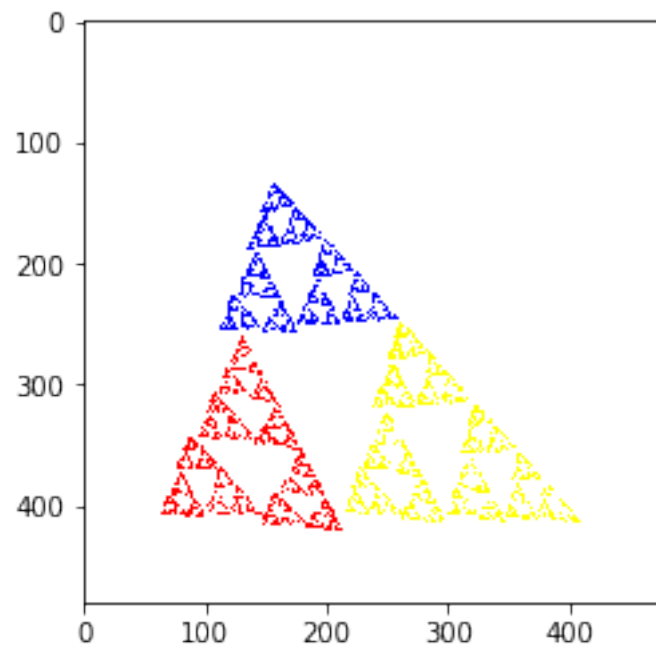
26   28   7.9454e-11   4.30e-12   1.33e-02   2.89e-06
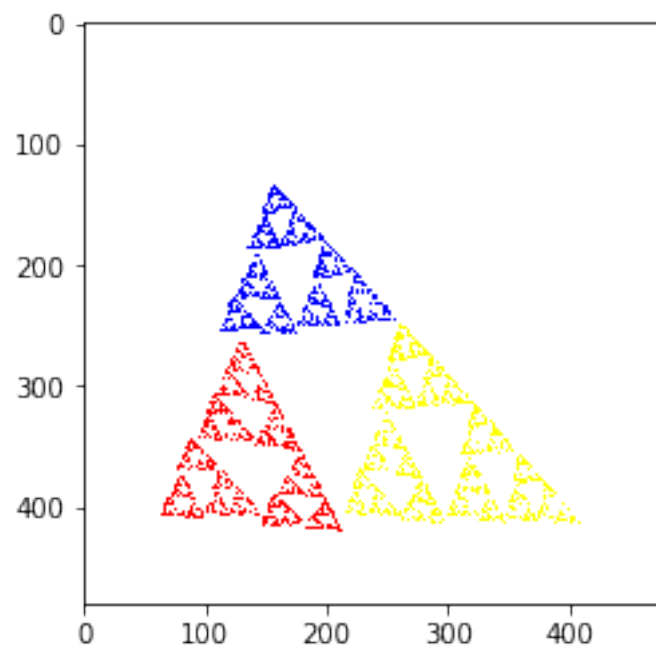
Current image:

| 27 | 29 | 5.4253e-11 | 2.52e-11 | 3.05e-03 | 1.59e-07 |
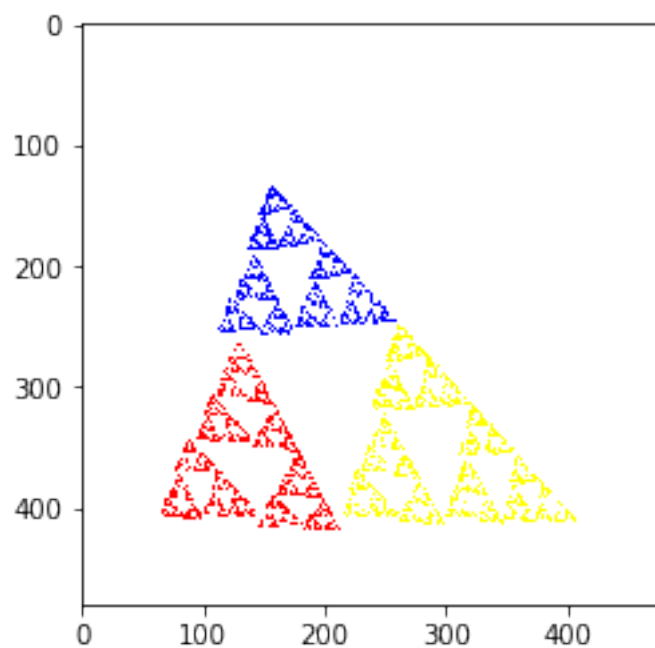
Current image:

| 28 | 30 | 4.9247e-11 | 5.01e-12 | 3.21e-03 | 1.51e-07 |
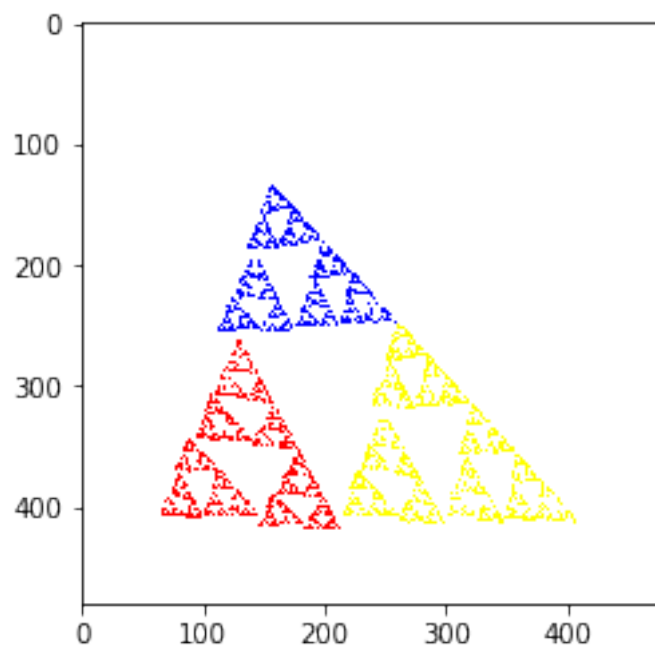
Current image:

```
      29              31           4.1542e-11      7.70e-12       6.48e-03       7.37e-07
Current image:
```
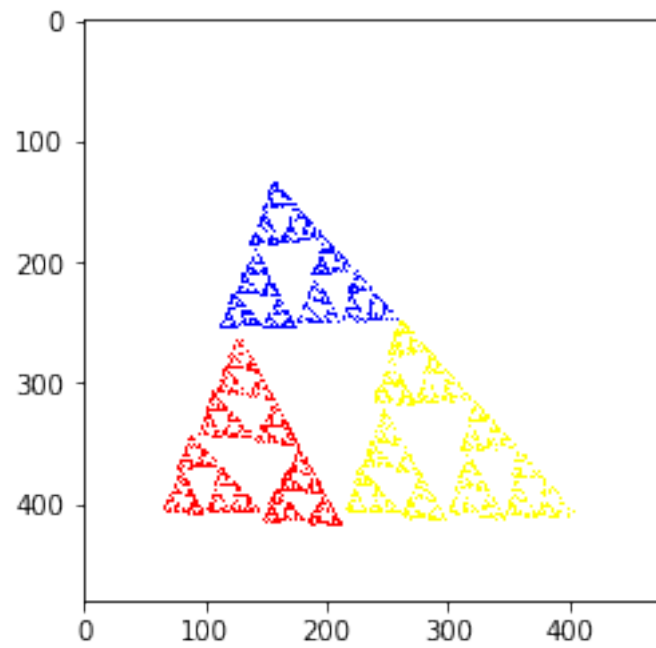


```
      30              32           3.3845e-11      7.70e-12       7.03e-03       8.80e-07
Current image:
```
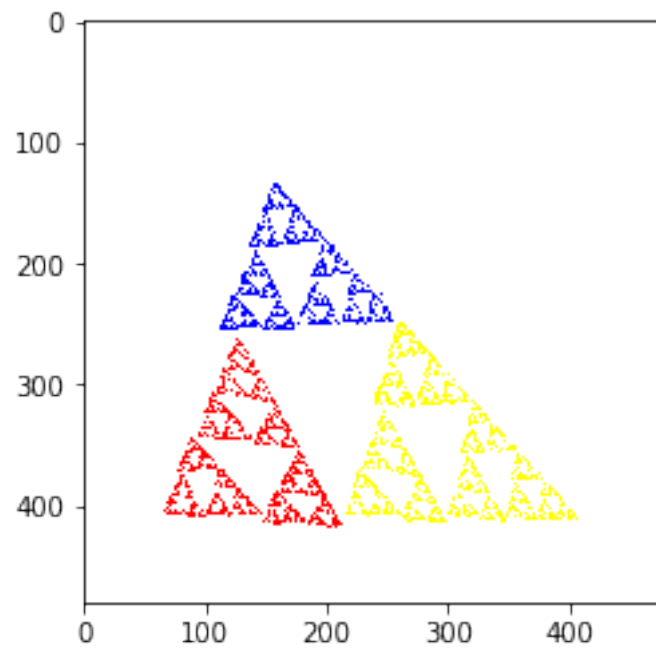
| | 31 | 33 | 2.7205e-11 | 6.64e-12 | 5.74e-03 | 5.76e-07 |
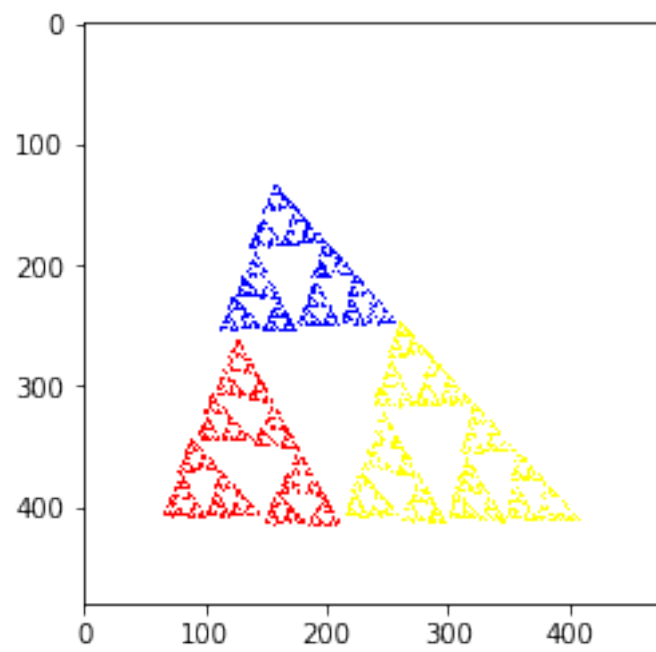
Current image:

32	34	2.2377e-11	4.83e-12	7.02e-03	8.80e-07

Current image:
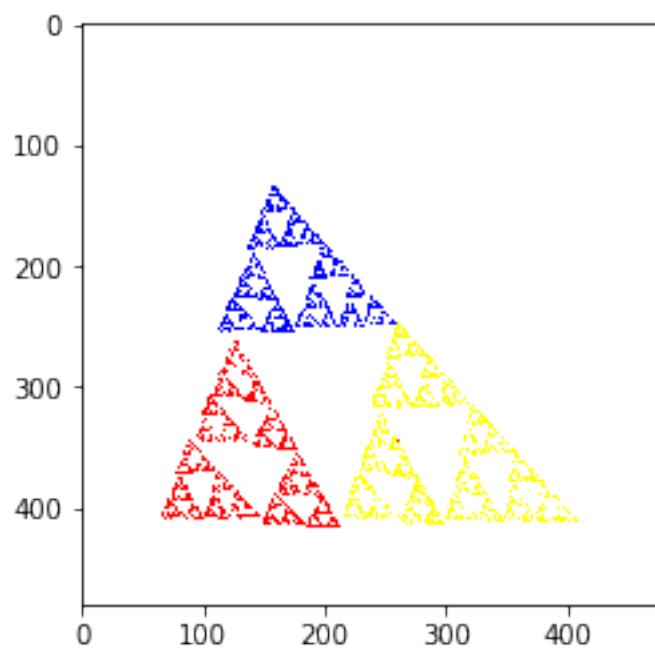
33	35	1.8193e-11	4.18e-12	3.45e-03	2.24e-07

Current image:
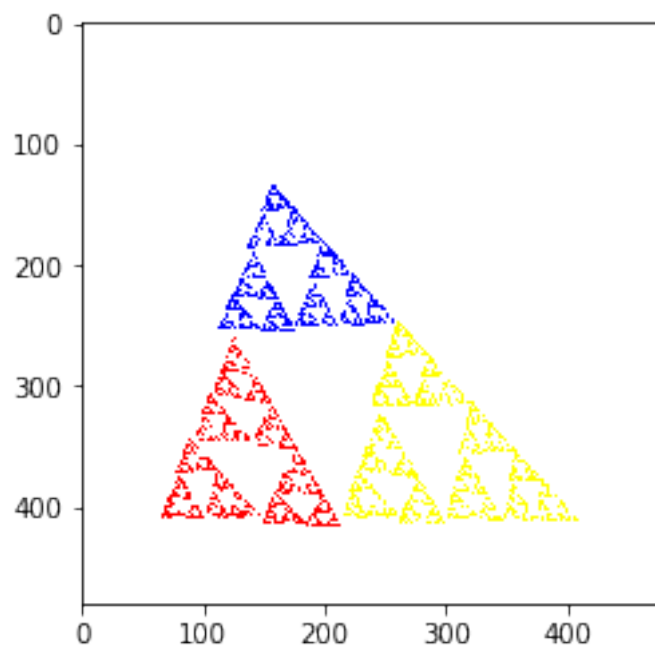
| 34 | 37 | 1.6611e-11 | 1.58e-12 | 2.08e-03 | 7.39e-08 |

Current image:
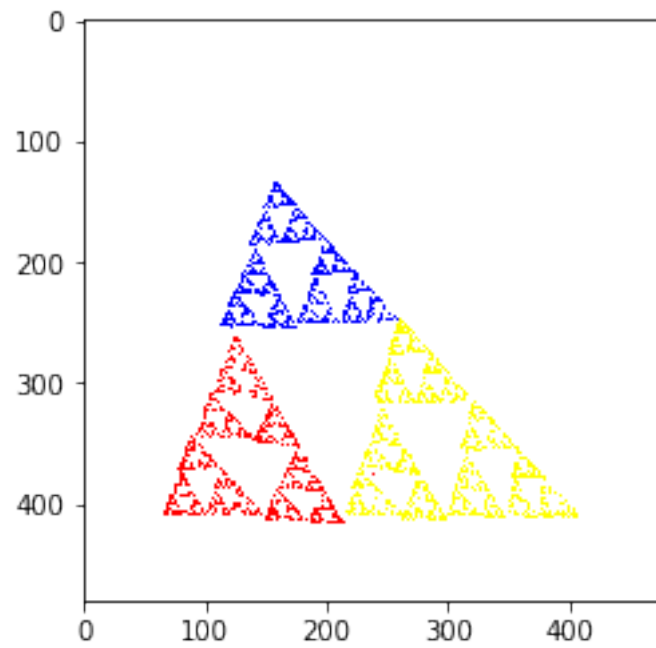
| 35 | 38 | 1.4332e-11 | 2.28e-12 | 4.13e-03 | 3.10e-07 |

Current image:
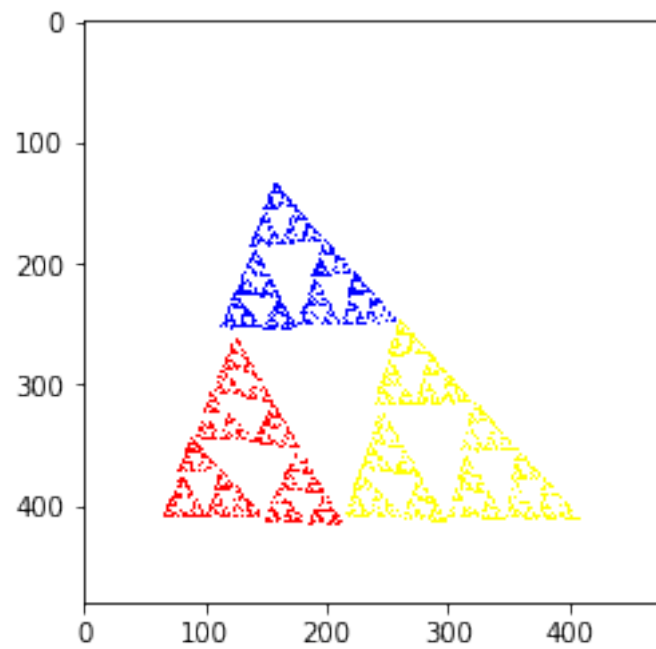
36   39   1.2074e-11   2.26e-12   4.83e-03   4.28e-07
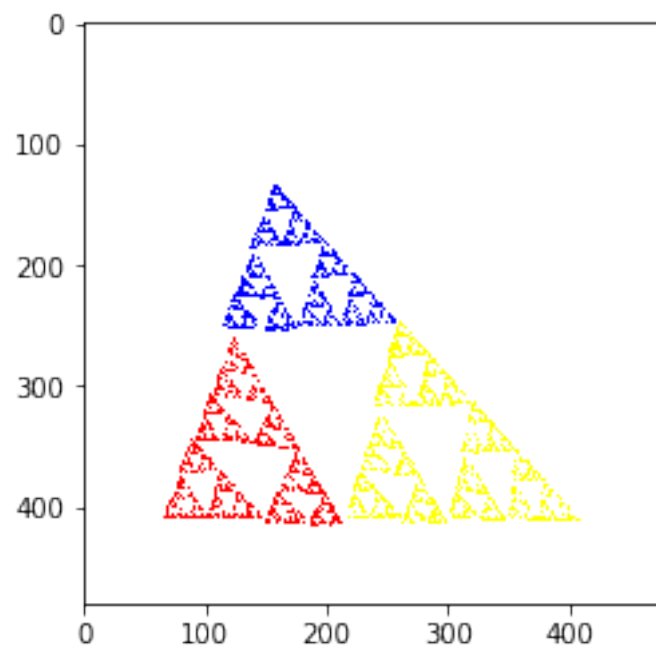
Current image:

| | 37 | 40 | 1.0001e-11 | 2.07e-12 | 3.02e-03 | 1.61e-07 |
| Current image: | | | | | | |

| | 38 | 41 | 9.1590e-12 | 8.42e-13 | 6.15e-03 | 6.43e-07 |
| Current image: | | | | | | |

| 39 | 42 | 6.8637e-12 | 2.30e-12 | 1.41e-03 | 3.33e-08 |
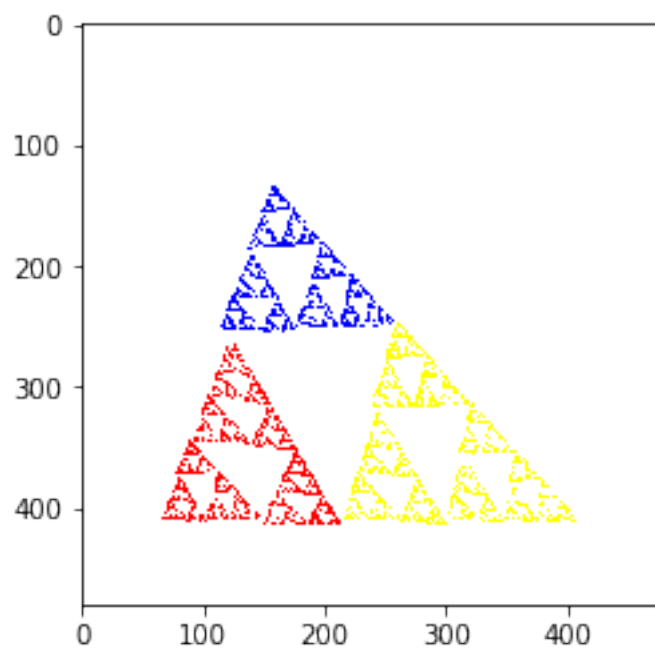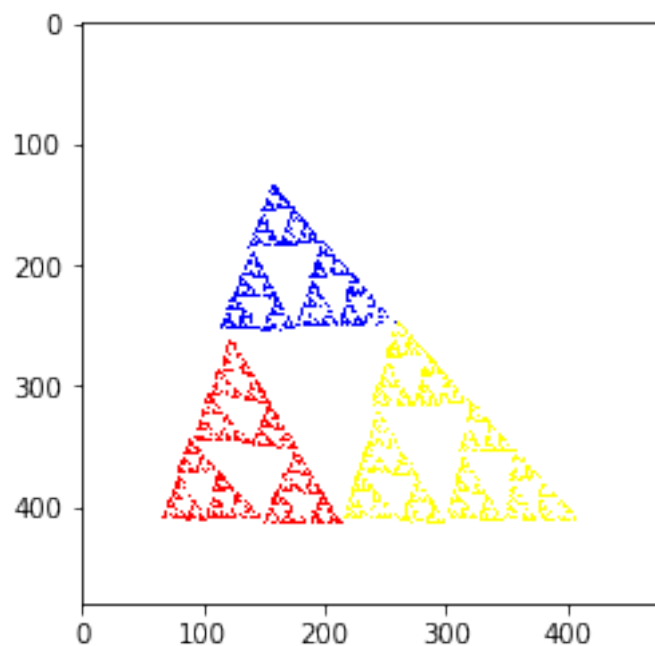
Current image:
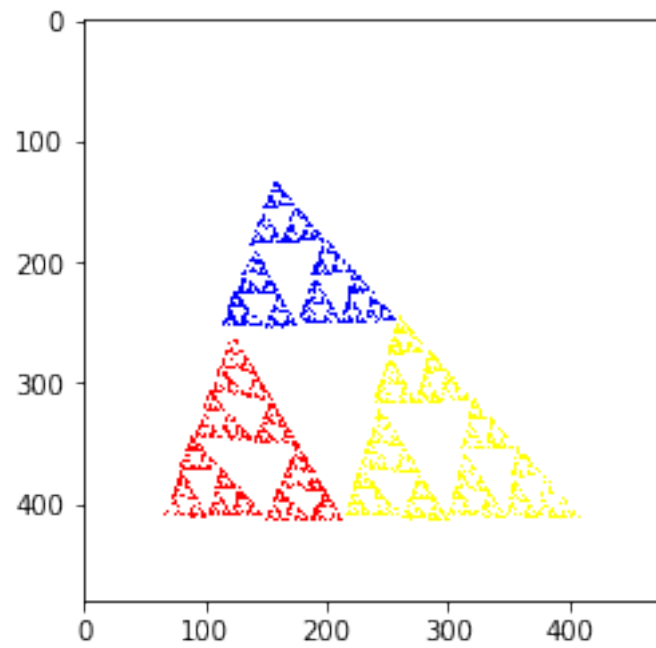
| 40 | 43 | 6.3401e-12 | 5.24e-13 | 1.53e-03 | 3.86e-08 |
| 41 | 44 | 5.5093e-12 | 8.31e-13 | 3.07e-03 | 1.54e-07 |

Current image:
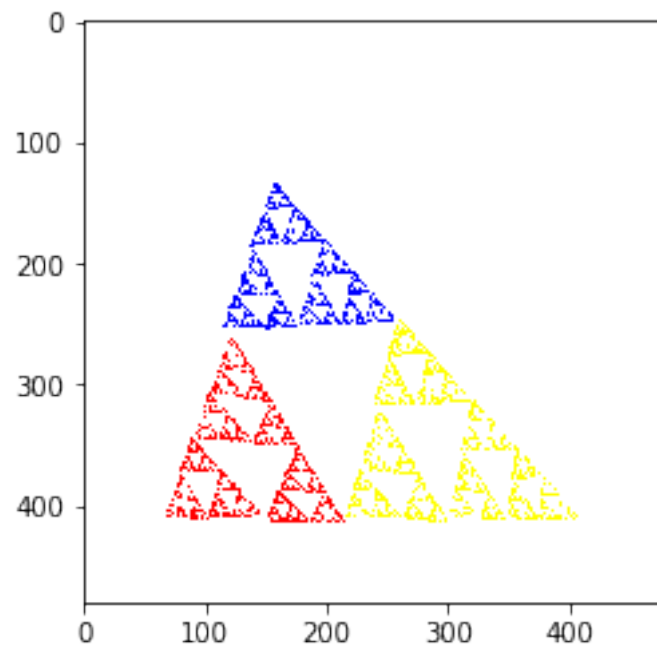
| 42 | 45 | 4.6705e-12 | 8.39e-13 | 3.86e-03 | 2.41e-07 |
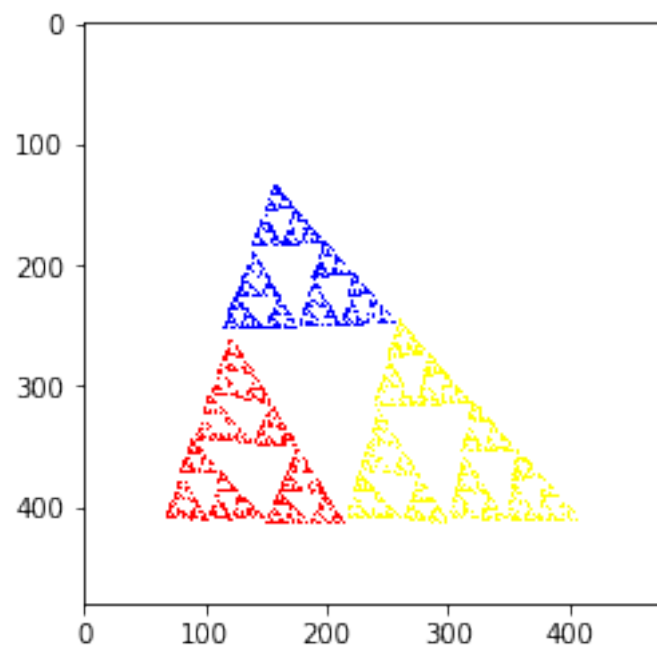
Current image:

| 43 | 46 | 3.9726e-12 | 6.98e-13 | 2.21e-03 | 7.52e-08 |

Current image:
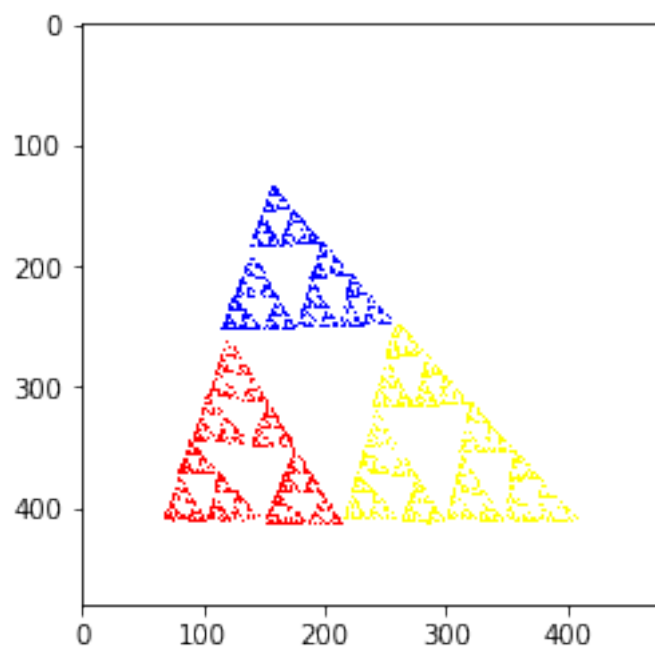
| 44 | 47 | 3.5451e-12 | 4.27e-13 | 4.88e-03 | 3.22e-07 |

Current image:
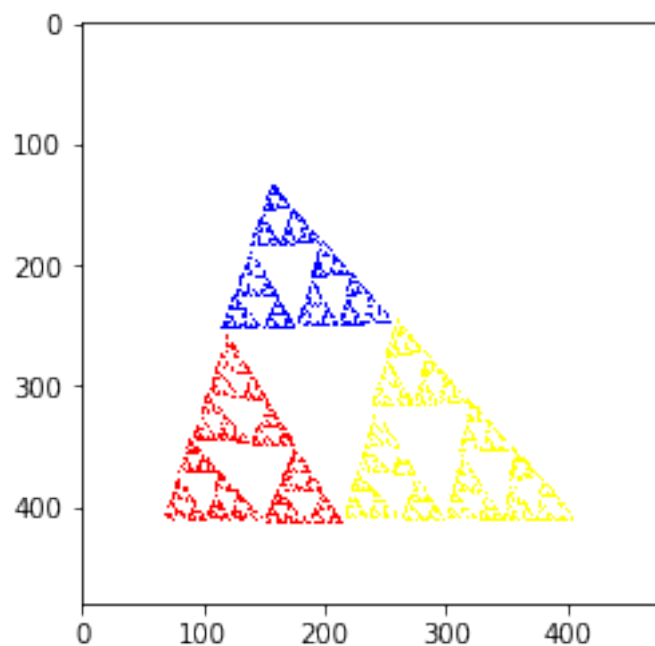
45	48	2.9224e-12	6.23e-13	1.04e-03	1.39e-08

Current image:
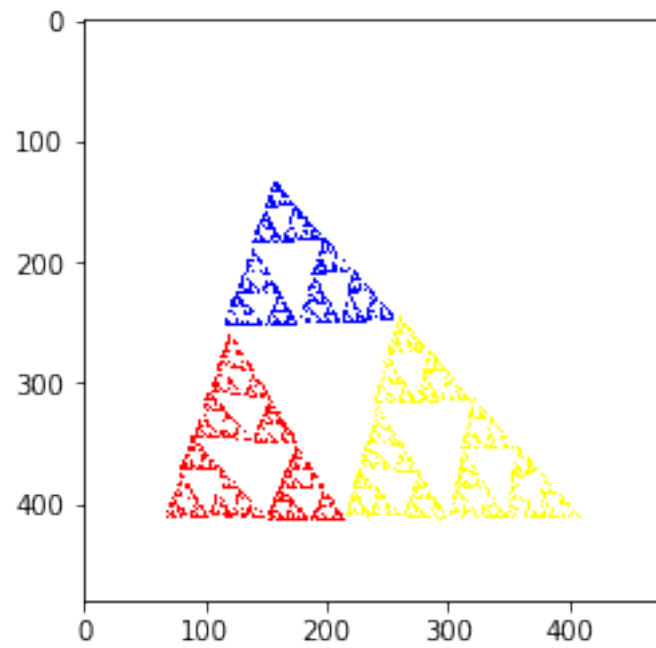
46	49	2.7513e-12	1.71e-13	1.20e-03	1.76e-08

Current image:
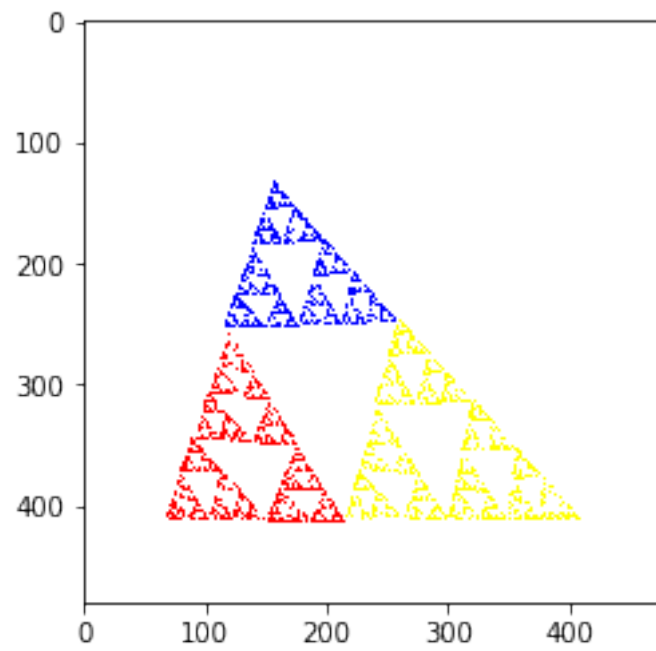
47   50   2.4615e-12   2.90e-13   2.39e-03   5.12e-08
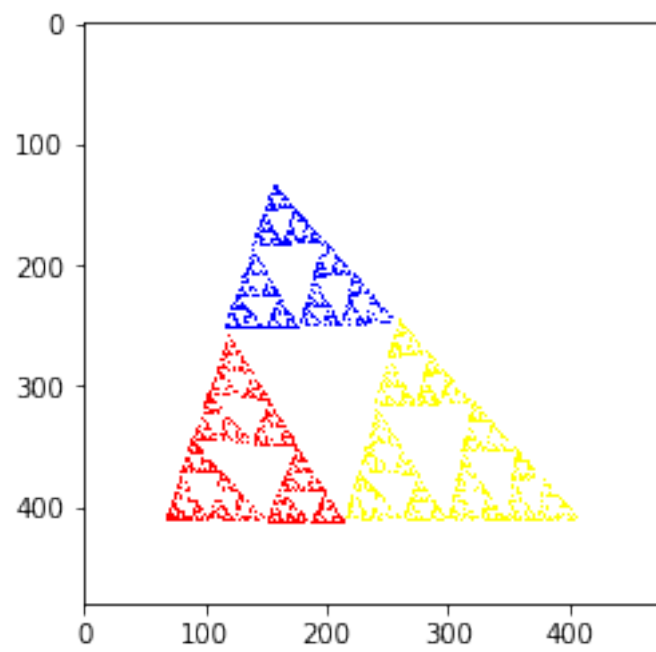
Current image:

48	51	2.0943e-12	3.67e-13	3.88e-03	8.11e-08

Current image:
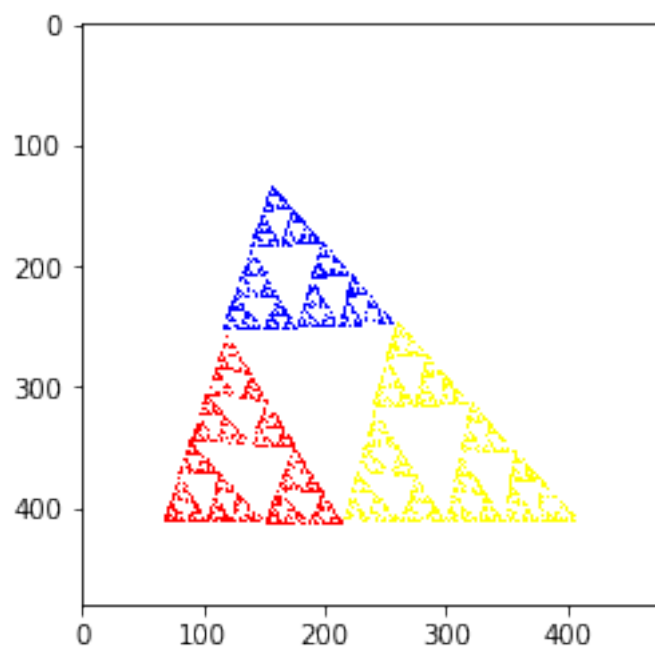
49	52	1.8256e-12	2.69e-13	2.74e-03	1.82e-07

Current image:
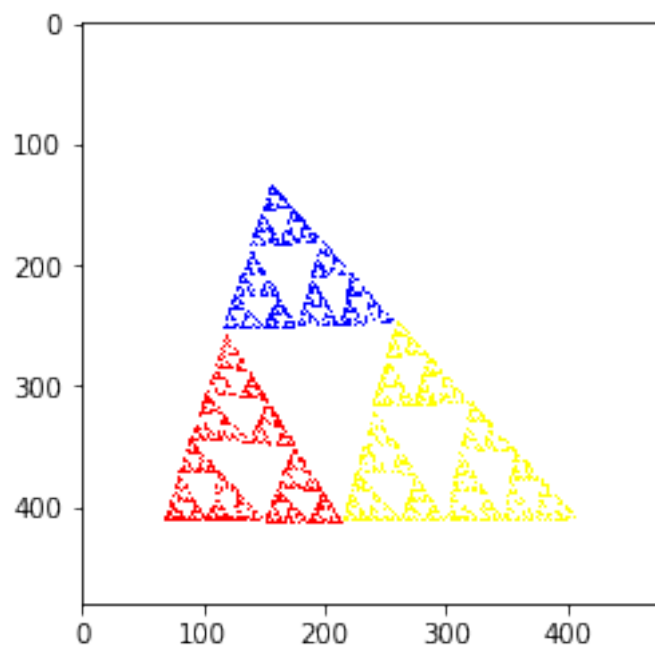
50	53	1.5661e-12	2.60e-13	4.18e-03	1.09e-07

Current image:
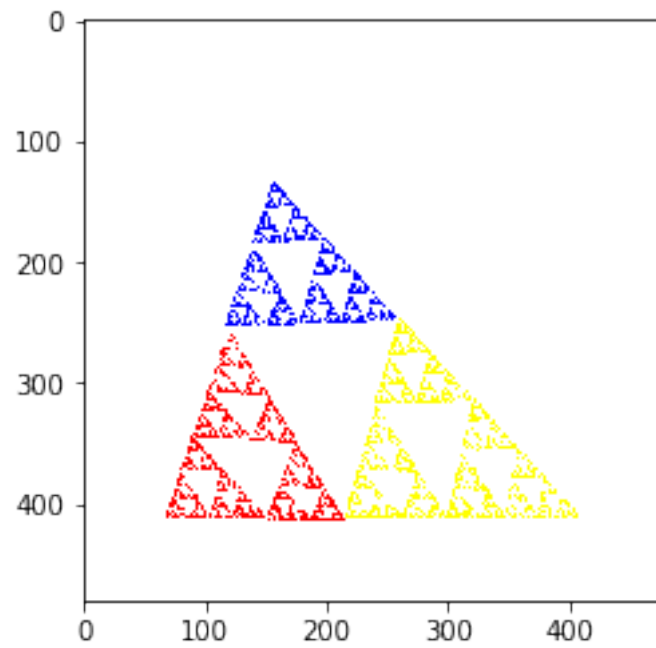
51	54	1.3731e-12	1.93e-13	3.02e-03	4.21e-07

Current image:
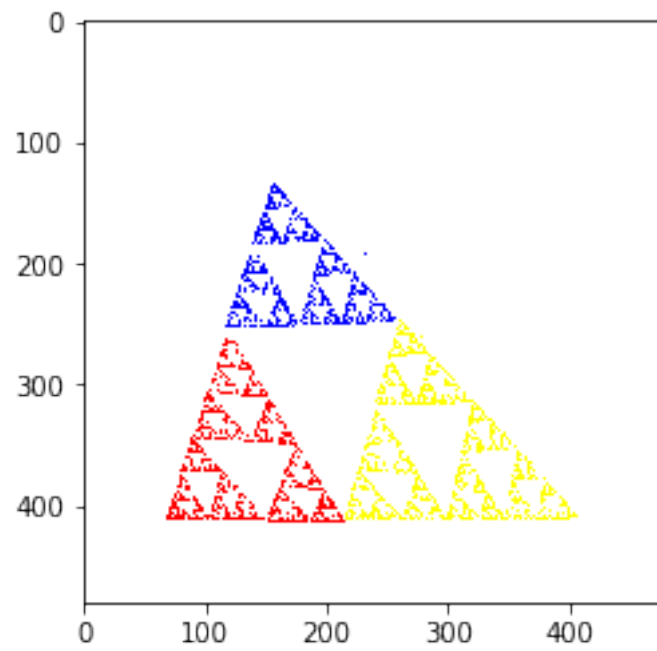
52   55   1.2258e-12   1.47e-13   1.34e-03   4.78e-08
Current image:
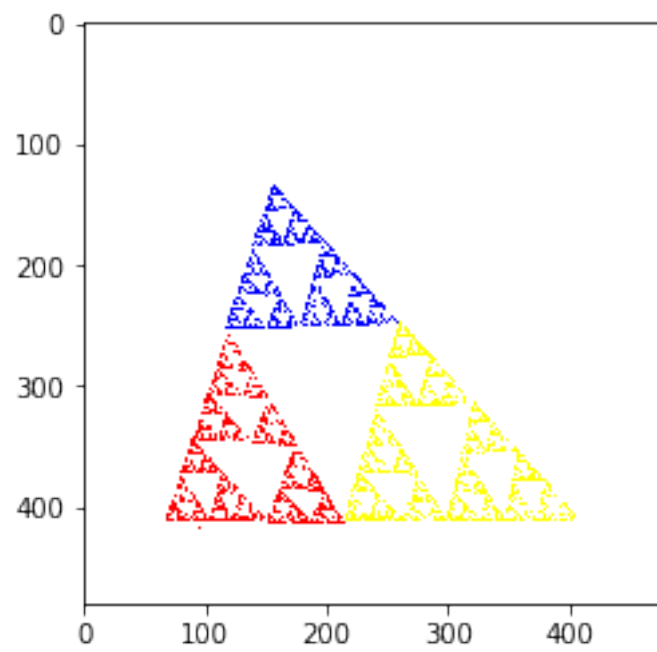
| 53 | 57 | 1.1478e-12 | 7.81e-14 | 1.31e-03 | 8.48e-08 |

Current image:
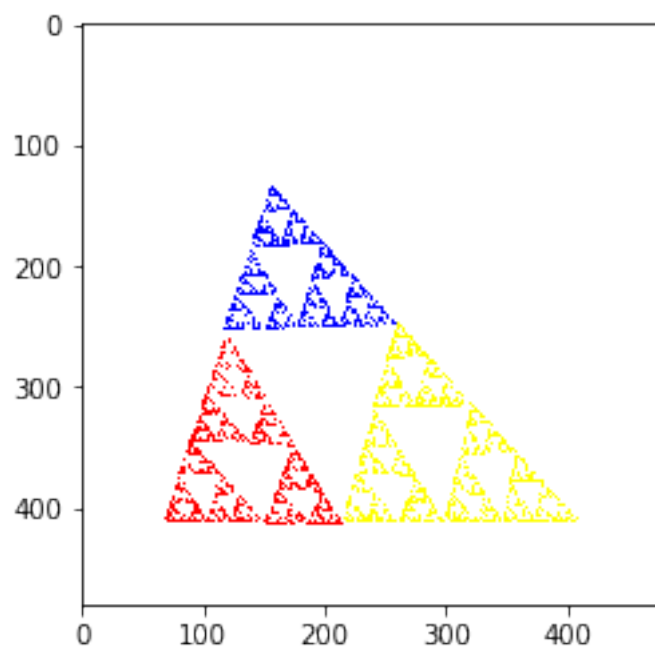
| 54 | 58 | 1.0450e-12 | 1.03e-13 | 2.70e-03 | 3.72e-07 |

Current image:
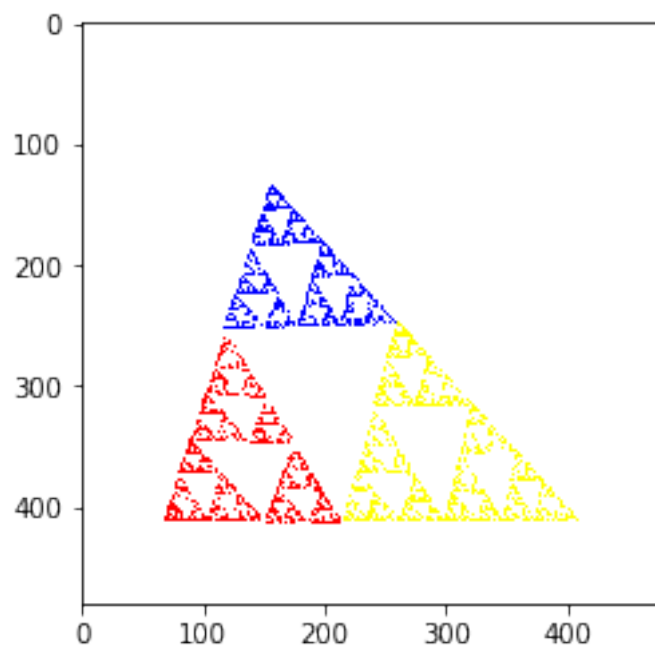
| 55 | 59 | 9.3334e-13 | 1.12e-13 | 1.17e-03 | 5.34e-08 |

Current image:
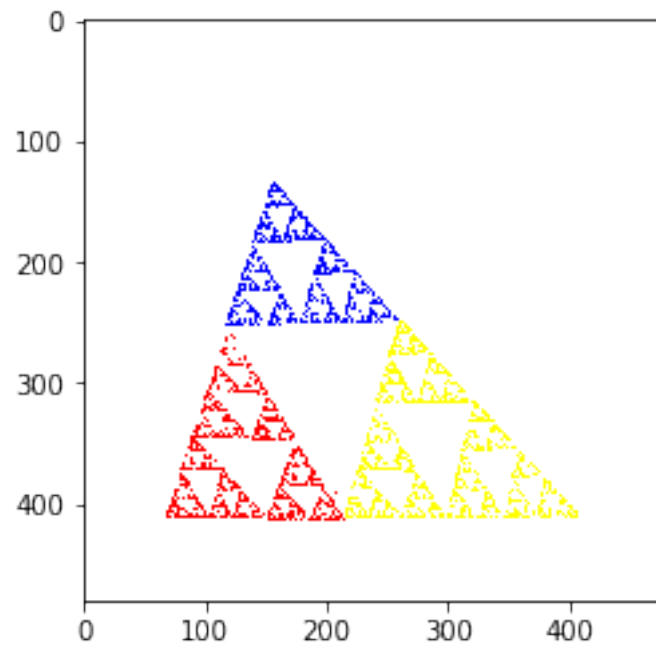
| 56 | 60 | 8.4563e-13 | 8.77e-14 | 2.60e-03 | 3.47e-07 |

Current image:
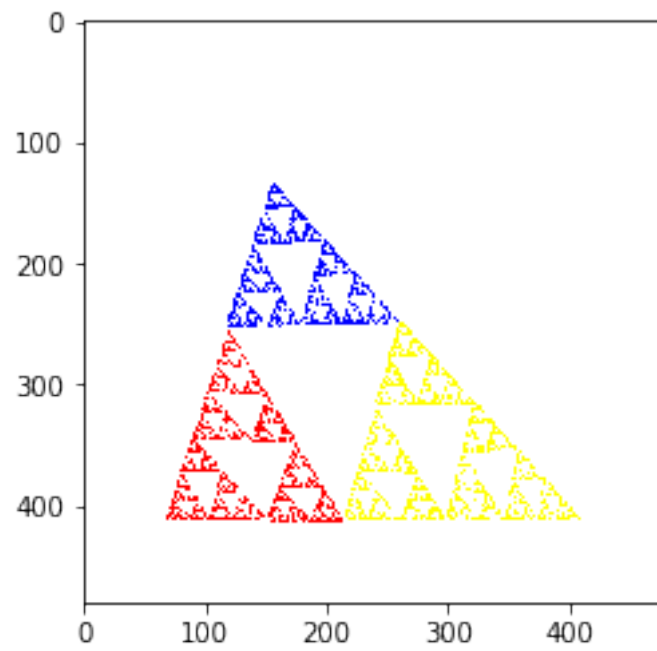
| 57 | 61 | 7.5064e-13 | 9.50e-14 | 1.10e-03 | 5.32e-08 |
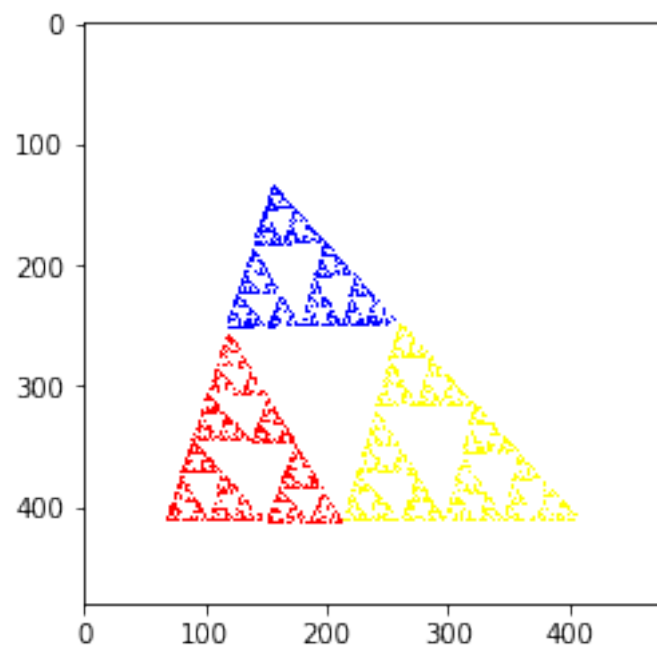
Current image:

| 58 | 62 | 6.7925e-13 | 7.14e-14 | 2.61e-03 | 3.50e-07 |

Current image:
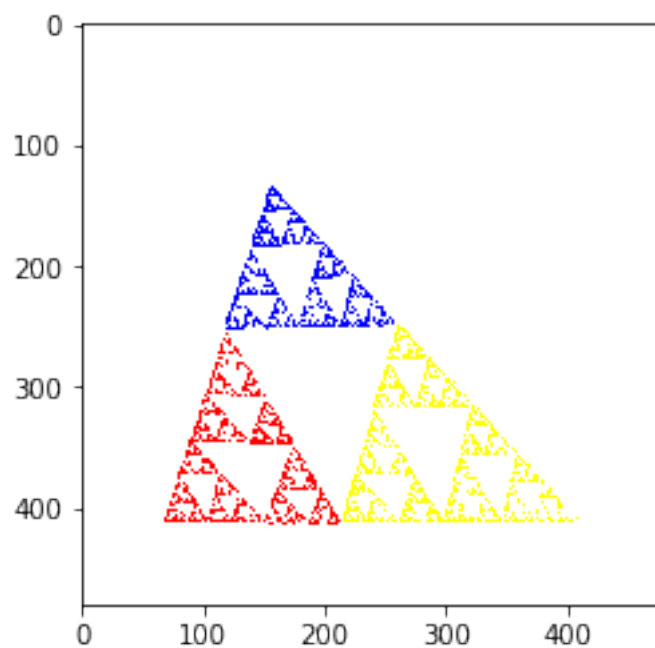
| 59 | 63 | 5.9513e-13 | 8.41e-14 | 9.47e-04 | 4.09e-08 |

Current image:
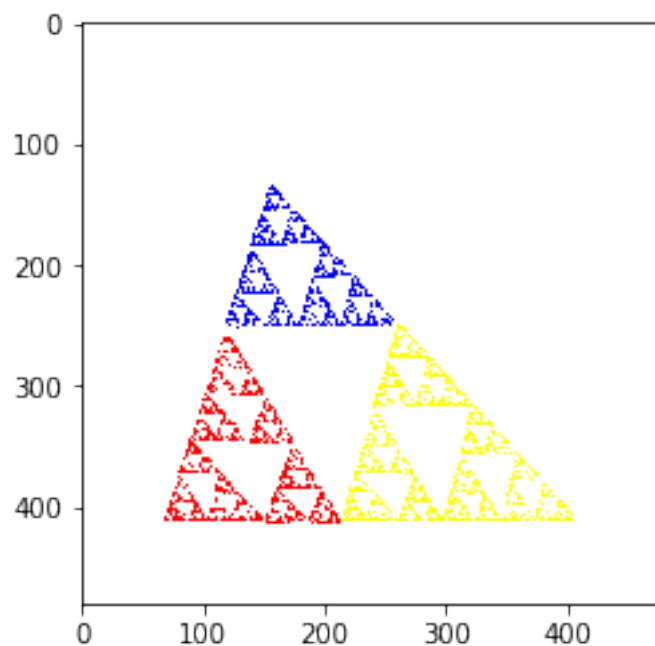
```
    60              64              5.3911e-13      5.60e-14        2.61e-03        3.52e-07
Current image:
```
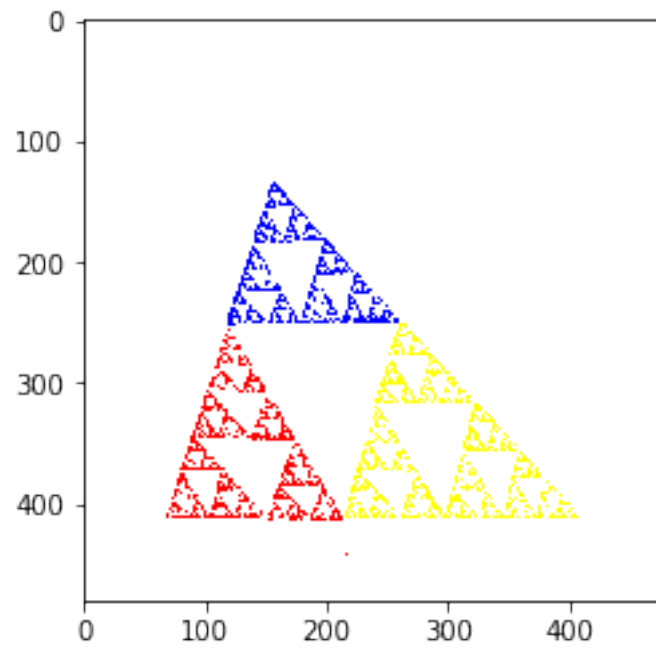


```
    61              65              4.6359e-13      7.55e-14        8.21e-04        3.09e-08
    62              66              4.2154e-13      4.20e-14        2.61e-03        3.53e-07
Current image:
```
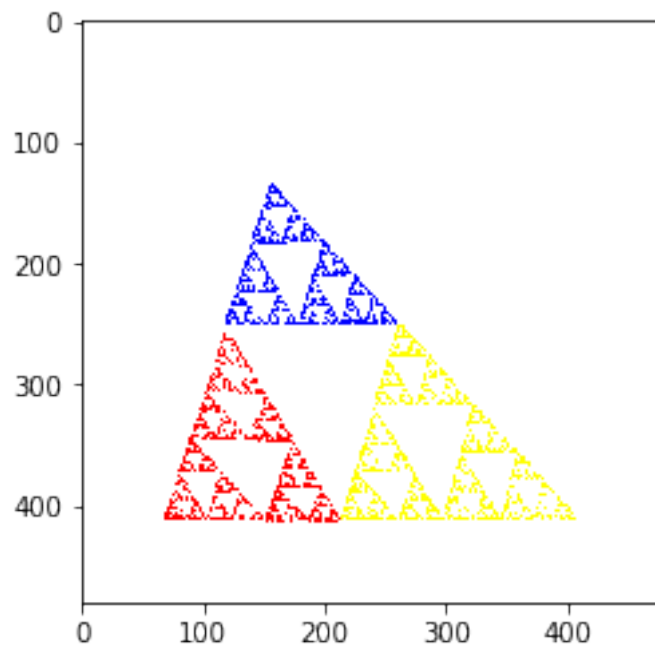
63	67	3.5315e-13	6.84e-14	7.09e-04	2.28e-08
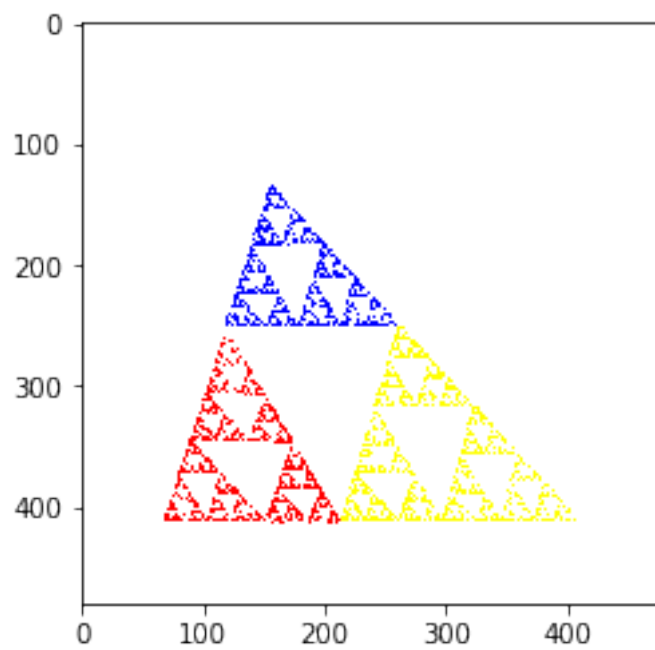
Current image:

64	68	3.2400e-13	2.92e-14	2.61e-03	3.53e-07
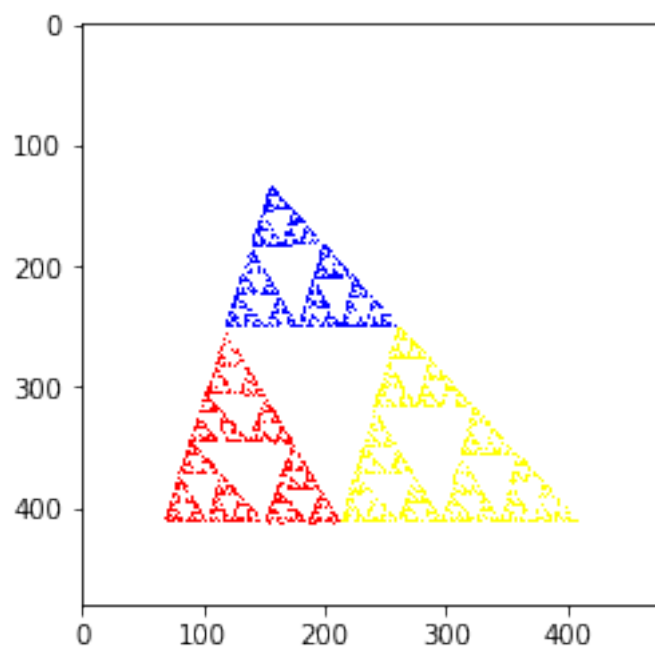
Current image:

65	69	2.6146e-13	6.25e-14	6.05e-04	1.64e-08

Current image:
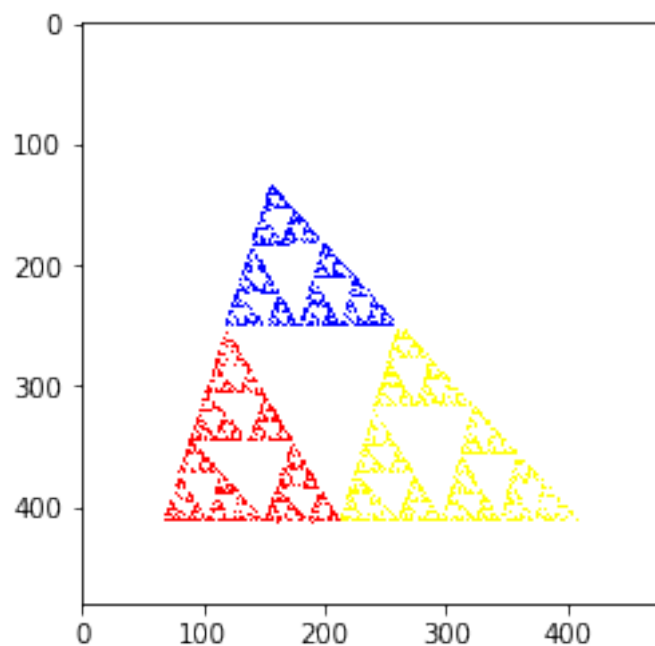
66   70   2.4428e-13   1.72e-14   2.61e-03   3.52e-07
Current image:
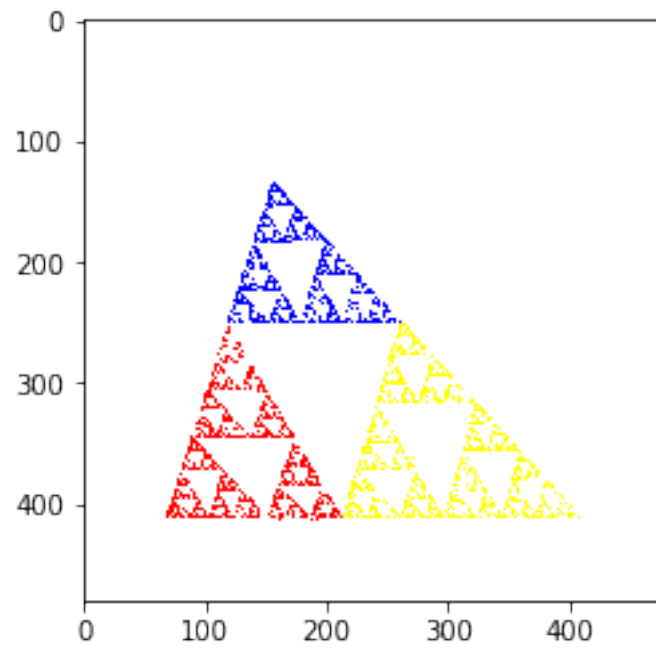
         67   71   1.8646e-13   5.78e-14   5.08e-04   1.14e-08
Current image:
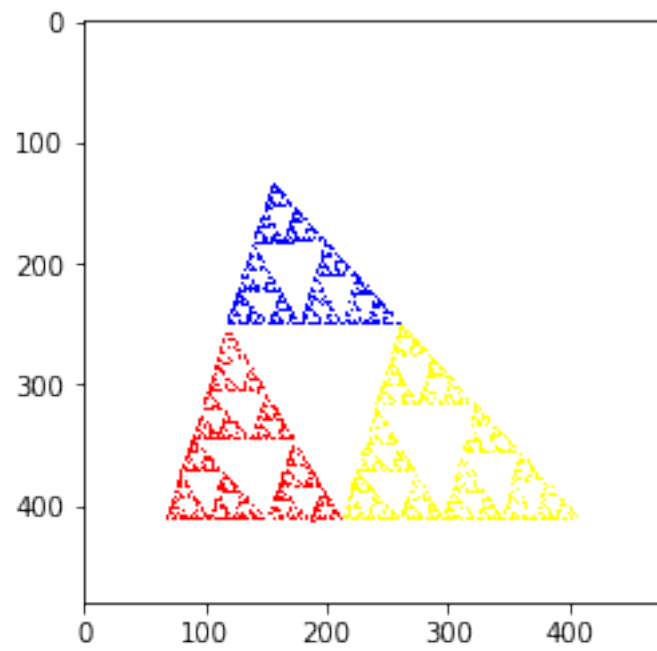
68   72   1.7235e-13   1.41e-14   6.57e-04   2.19e-08
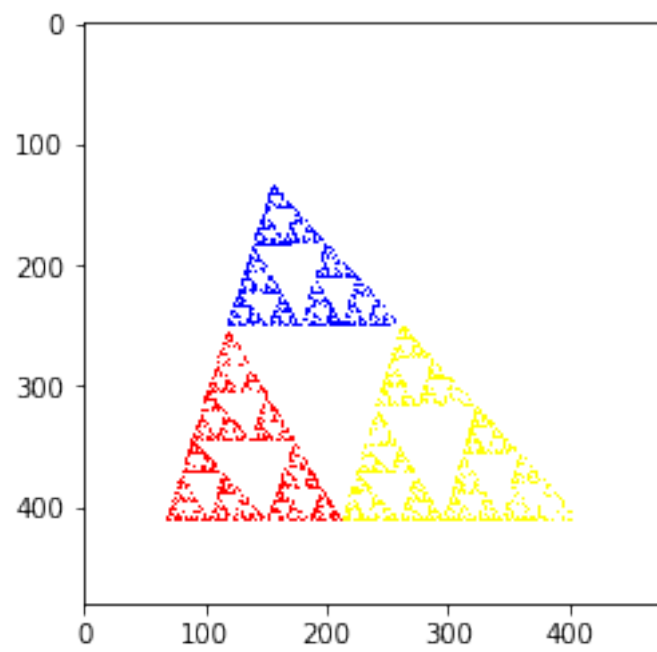
Current image:

69	73	1.4886e-13	2.35e-14	1.30e-03	8.83e-08

Current image:
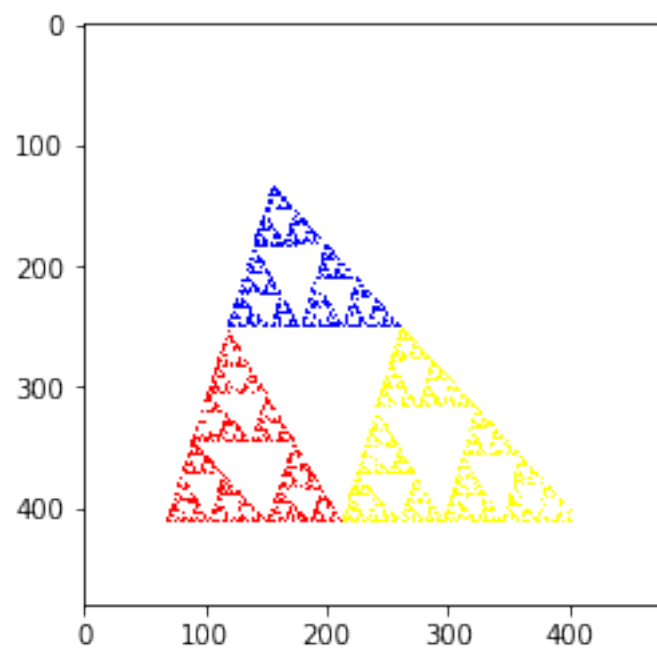
70	74	1.2325e-13	2.56e-14	1.52e-03	1.14e-07

Current image:
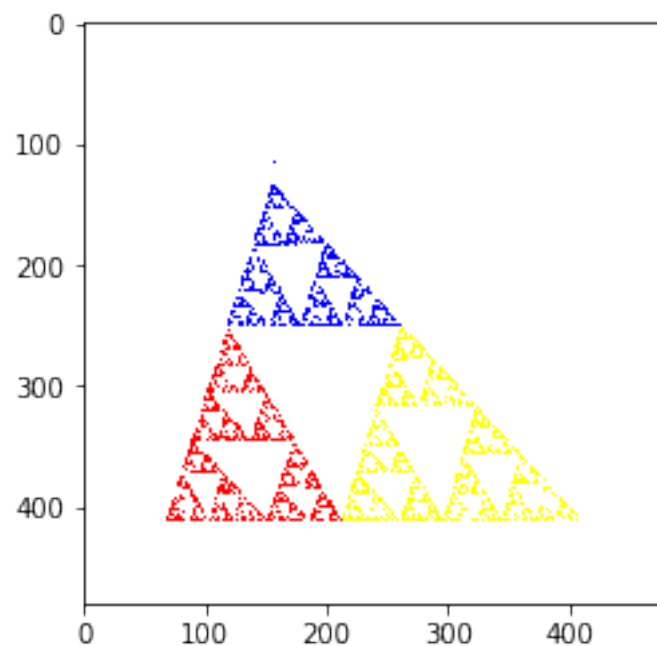
```
      71              75        1.0178e-13      2.15e-14      1.07e-03      5.75e-08
Current image:
```
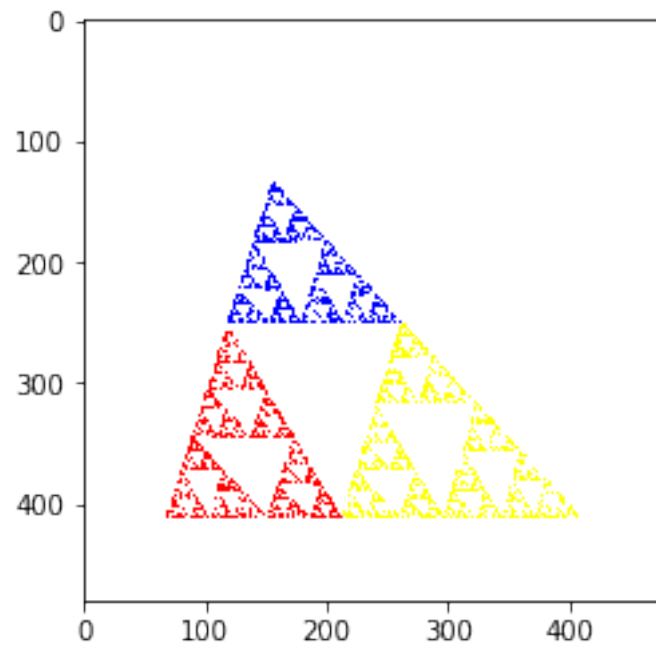


```
      72              76        8.4504e-14      1.73e-14      1.80e-03      1.66e-07
Current image:
```
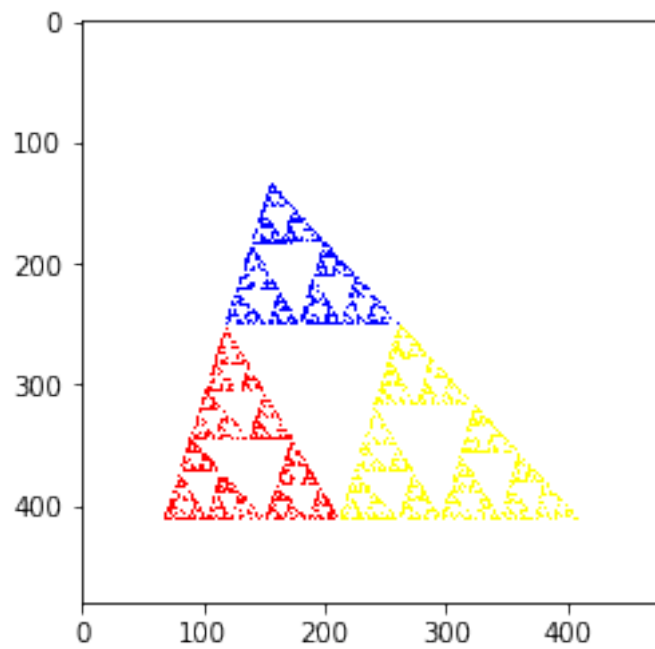
| 73 | 77 | 6.6398e-14 | 1.81e-14 | 5.85e-04 | 1.74e-08 |

Current image:
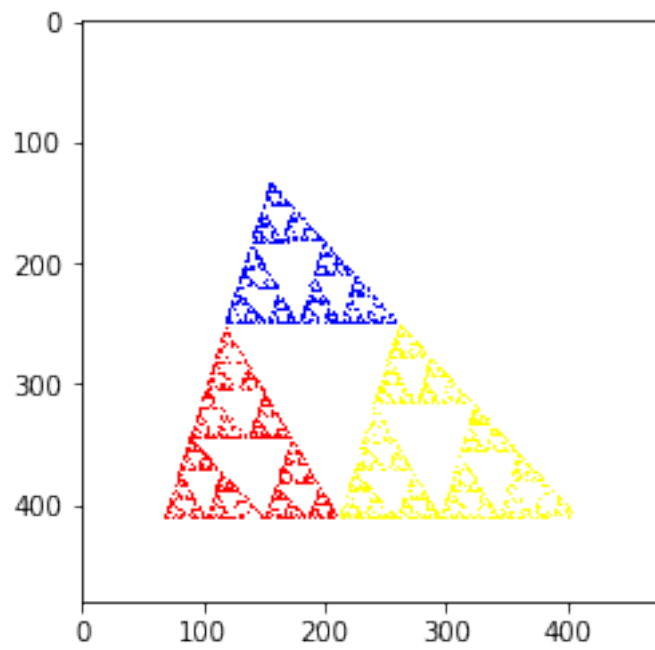
| 74 | 79 | 5.8288e-14 | 8.11e-15 | 6.57e-04 | 2.23e-08 |

Current image:
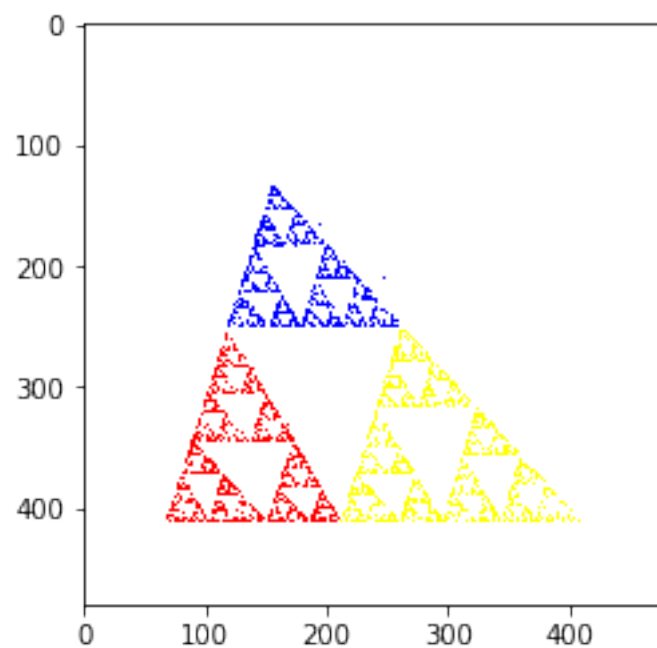
| 75 | 80 | 4.6556e-14 | 1.17e-14 | 1.30e-03 | 8.79e-08 |

Current image:
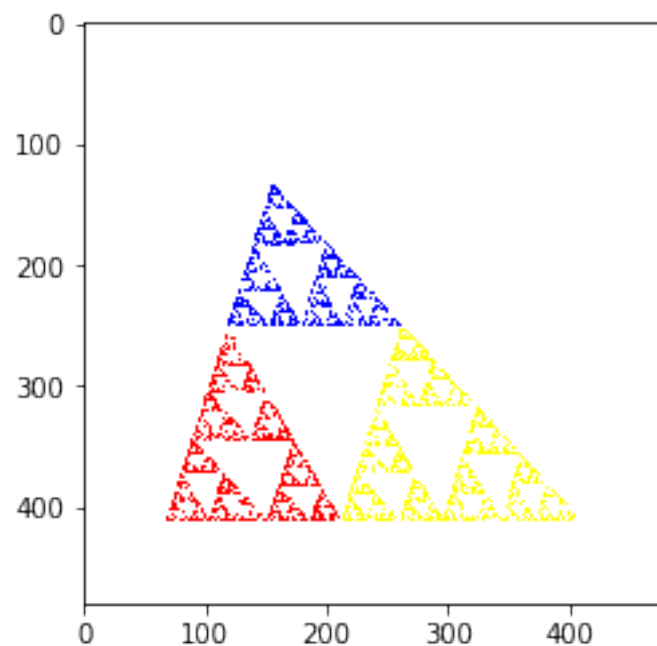
| 76 | 81 | 3.6116e-14 | 1.04e-14 | 8.13e-04 | 3.27e-08 |

Current image:
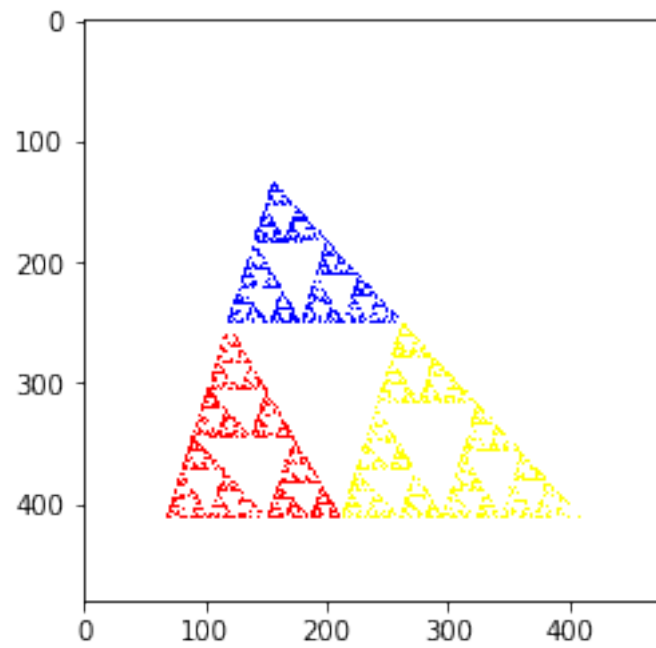
| 77 | 82 | 2.9801e-14 | 6.32e-15 | 1.68e-03 | 1.45e-07 |

Current image:
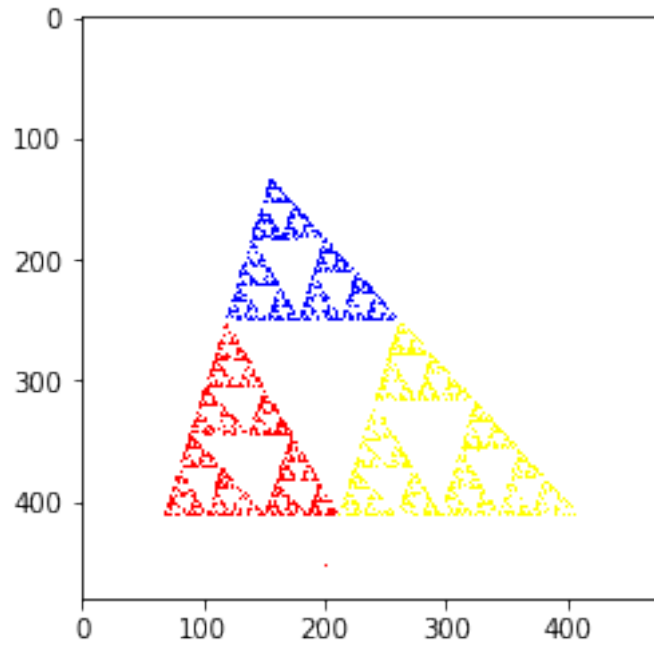
```
        78              83          1.9328e-14      1.05e-14       3.55e-04       6.40e-09
`gtol` termination condition is satisfied.
Function evaluations 83, initial cost 4.9542e-03, final cost 1.9328e-14, first-order optimality
```

As we can see, we found the global minimum of the function and recovered our Steemson triangle

```
In [43]: img,_  = se.chaos(sol_Steem.x,p_l, it = 100000, imgxy=500)
         img
```

Out[43]:
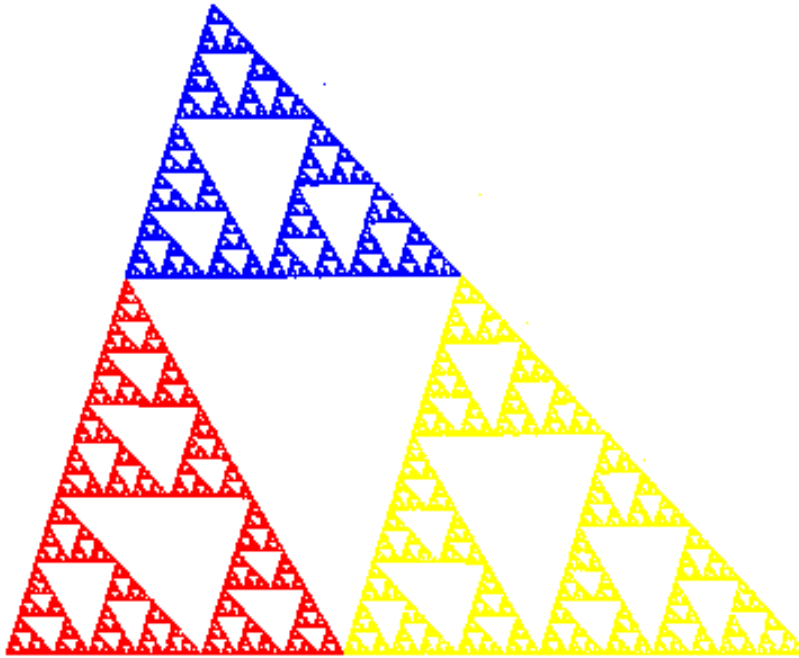
Now we may add some noise to our initial guess and find that the moments stil recover the Steemson triangle

```
In [46]: x0 = se.convertIFS(NNF) + 0.1*np.random.randn(len((se.convertIFS(NNF))))

In [47]: k=0
         t = time.time()
         sol_Steem = least_squares( cost_Steem, x0 , bounds=(-1,1) , verbose = 2)
         t =  time.time() - t

Current image:
```
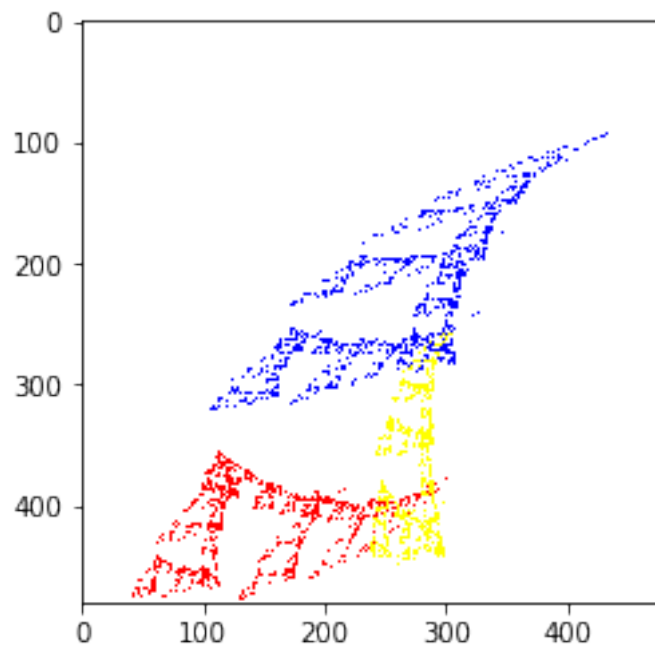
Iteration	Total nfev	Cost	Cost reduction	Step norm	Optimality
0	1	5.2846e-03			6.79e-02

Current image:
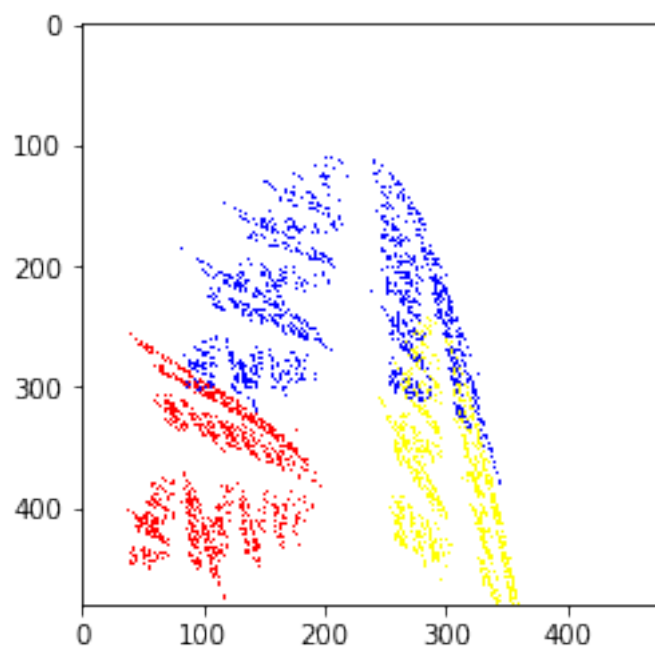
| 1 | 2 | 7.5633e-04 | 4.53e-03 | 3.47e-01 | 4.01e-02 |

Current image:
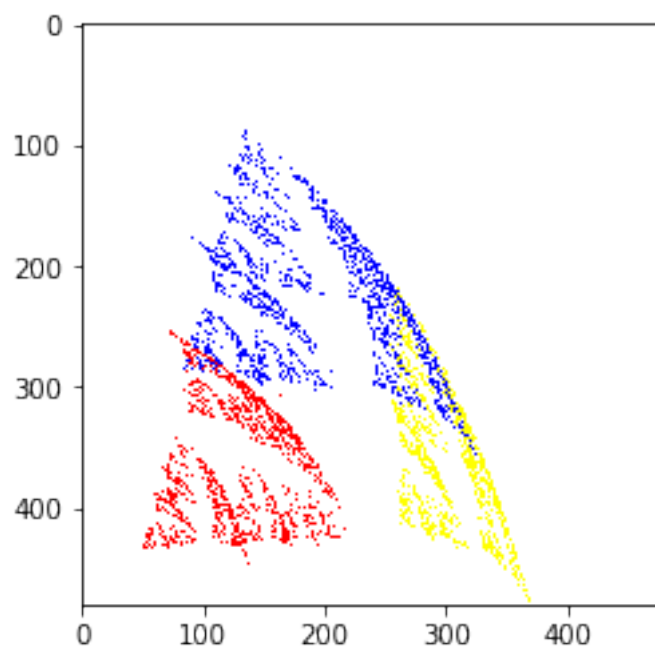
| 2 | 3 | 3.8558e-05 | 7.18e-04 | 1.48e-01 | 6.01e-03 |

Current image:
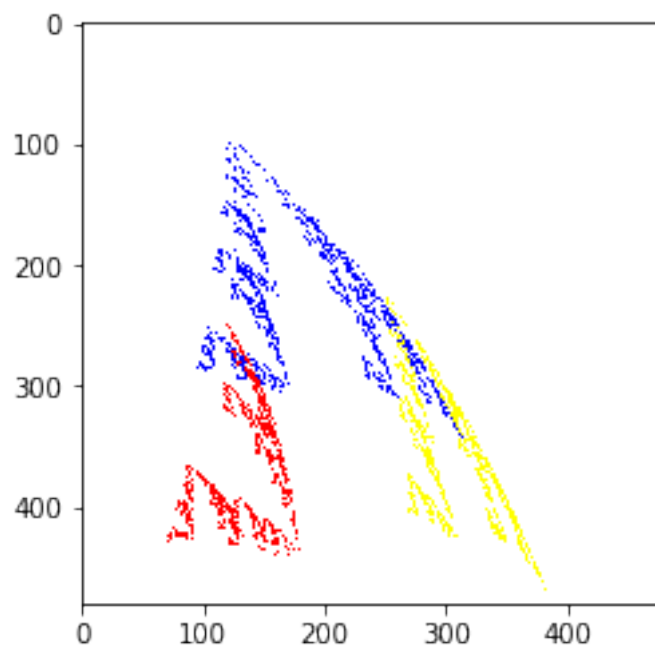
```
   3               4            1.4422e-05      2.41e-05       1.99e-01       3.97e-03
Current image:
```
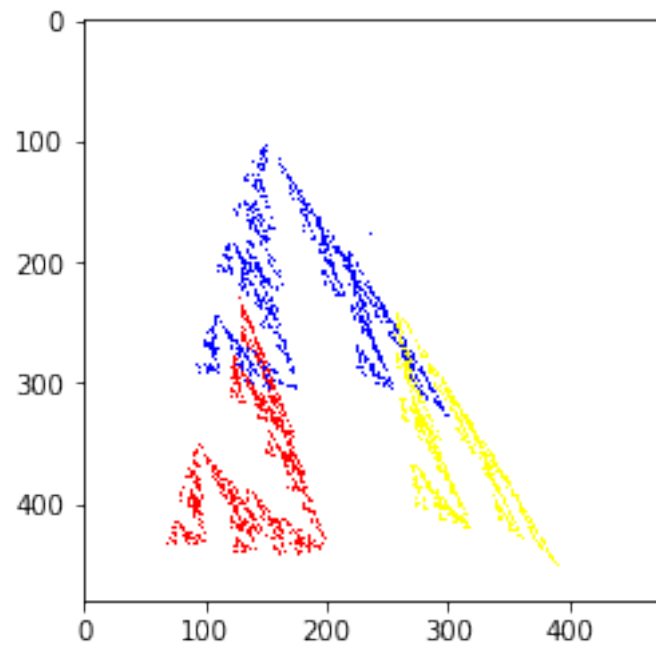


```
   4               5            4.1631e-06      1.03e-05       1.16e-01       1.47e-03
Current image:
```
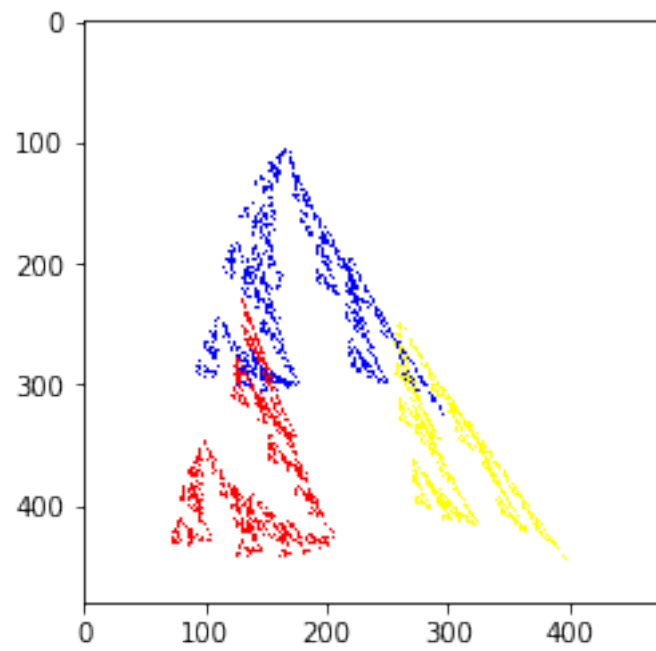
| 5 | 7 | 1.7136e-06 | 2.45e-06 | 6.28e-02 | 3.12e-04 |
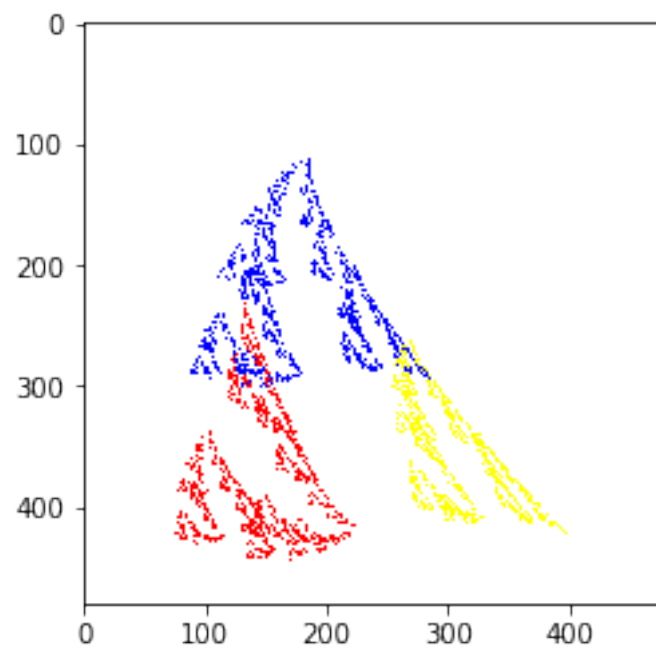
Current image:

| 6 | 8 | 8.7598e-07 | 8.38e-07 | 1.13e-01 | 8.43e-04 |

Current image:
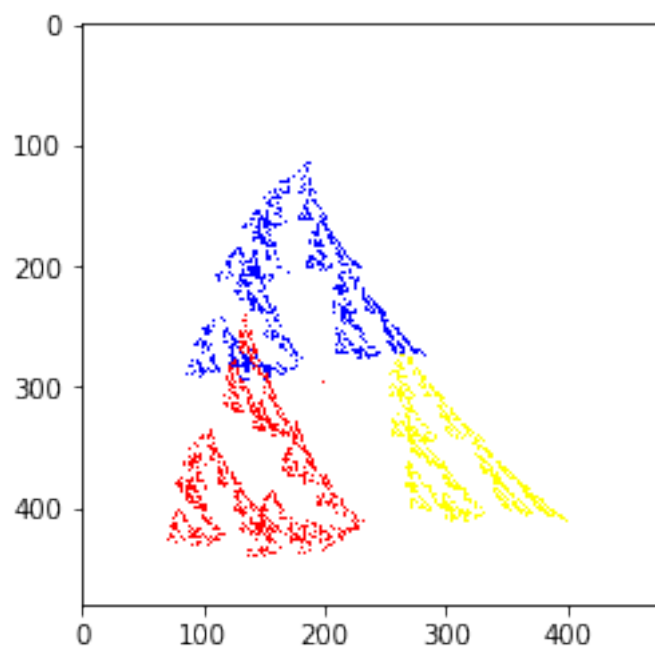
| 7 | 9 | 4.6111e-07 | 4.15e-07 | 8.05e-02 | 4.02e-04 |

Current image:
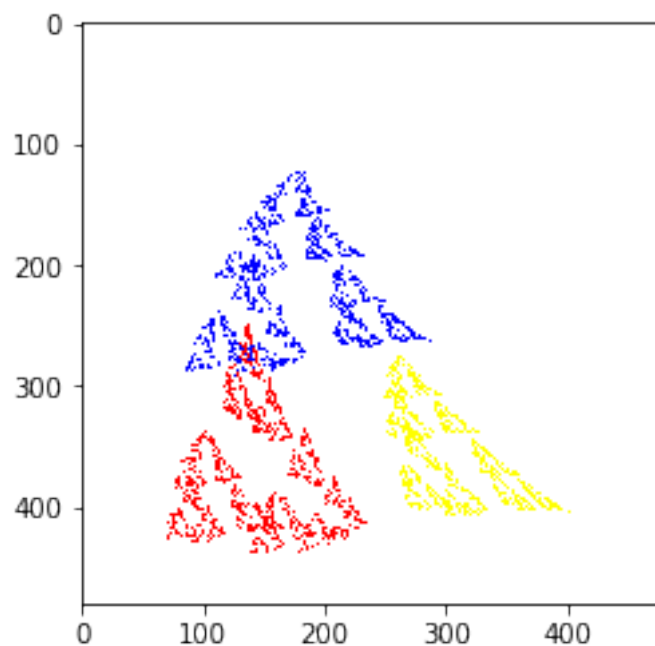
```
     8              10           2.8479e-07      1.76e-07       5.69e-02       1.44e-04
Current image:
```
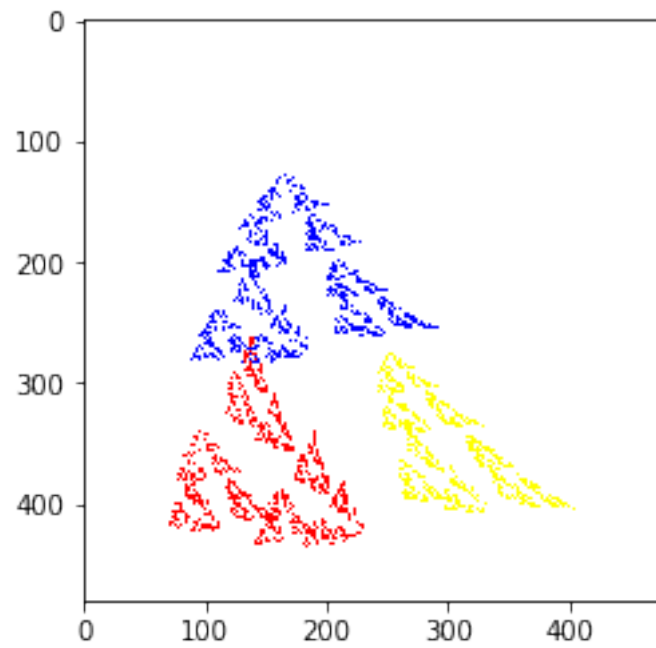


```
     9              11           2.3153e-07      5.33e-08       9.57e-02       4.13e-04
Current image:
```
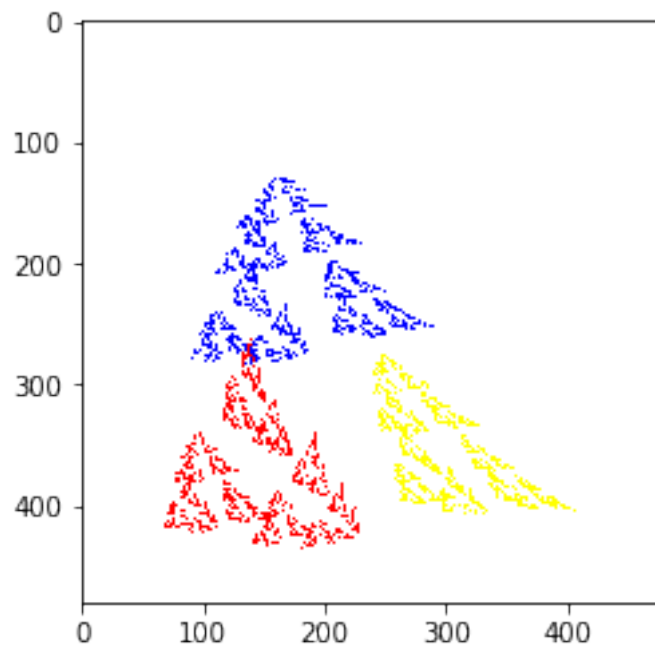
```
        10              12         1.2385e-07        1.08e-07        2.20e-02        1.50e-05
Current image:
```
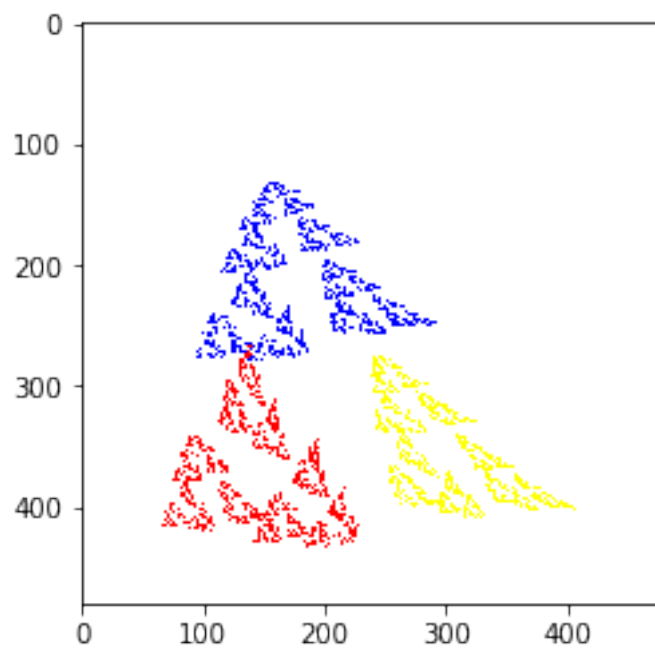
11	13	9.0211e-08	3.36e-08	4.53e-02	5.09e-05
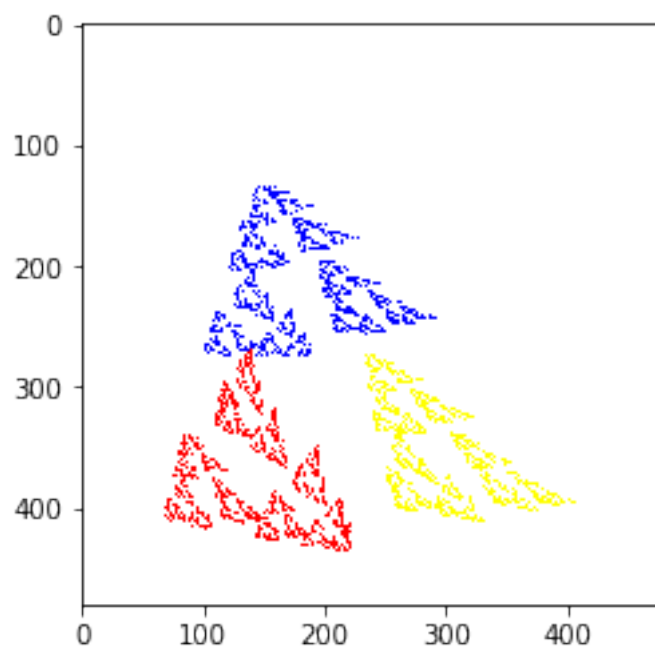
Current image:

12	14	5.5077e-08	3.51e-08	7.20e-02	6.18e-05
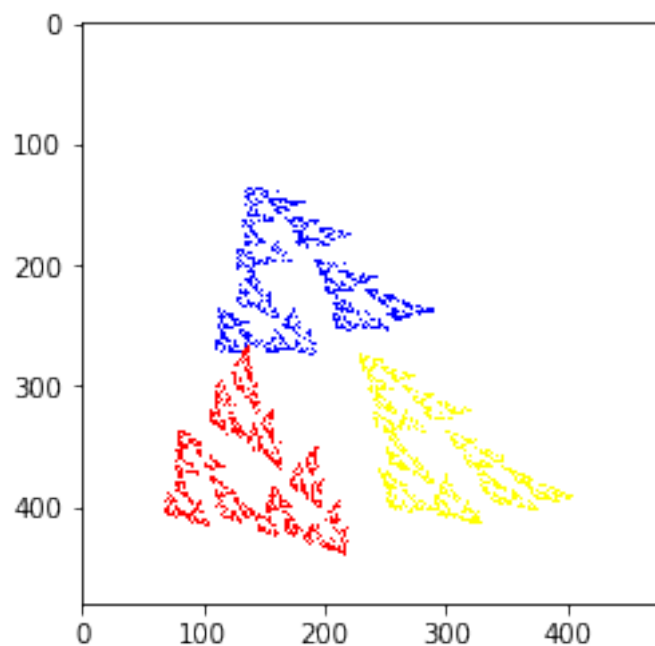
Current image:

13	15	3.5863e-08	1.92e-08	7.41e-02	6.14e-05

Current image:
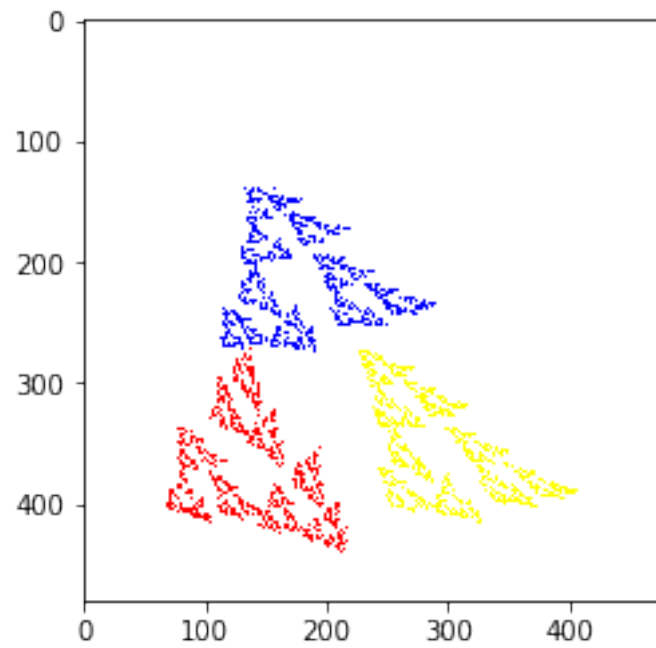
14	16	2.4355e-08	1.15e-08	2.99e-02	1.24e-05

Current image:
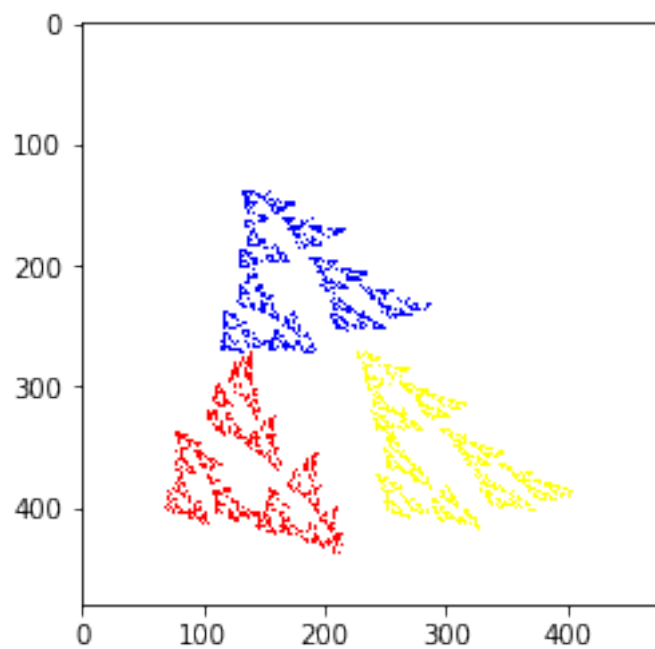
15	18	2.0021e-08	4.33e-09	2.23e-02	1.32e-05

Current image:

| 16 | 19 | 1.4426e-08 | 5.60e-09 | 3.58e-02 | 2.42e-05 |

Current image:

| 17 | 20 | 1.0317e-08 | 4.11e-09 | 4.31e-02 | 4.74e-05 |
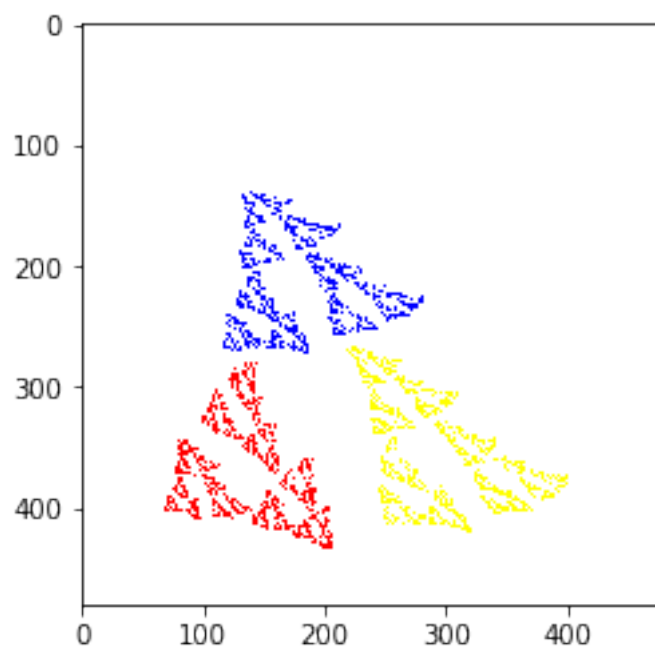
Current image:

| 18 | 21 | 7.1304e-09 | 3.19e-09 | 2.18e-02 | 2.01e-05 |

Current image:
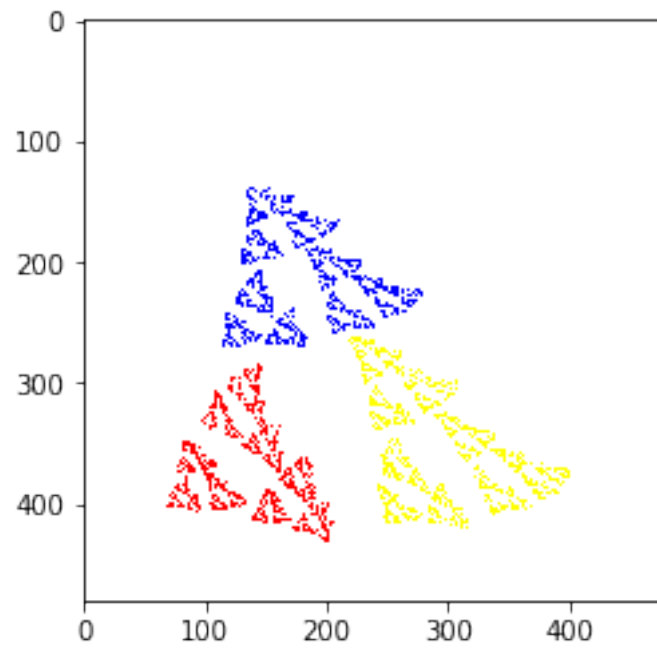
| 19 | 22 | 6.6211e-09 | 5.09e-10 | 2.96e-02 | 4.45e-05 |

Current image:
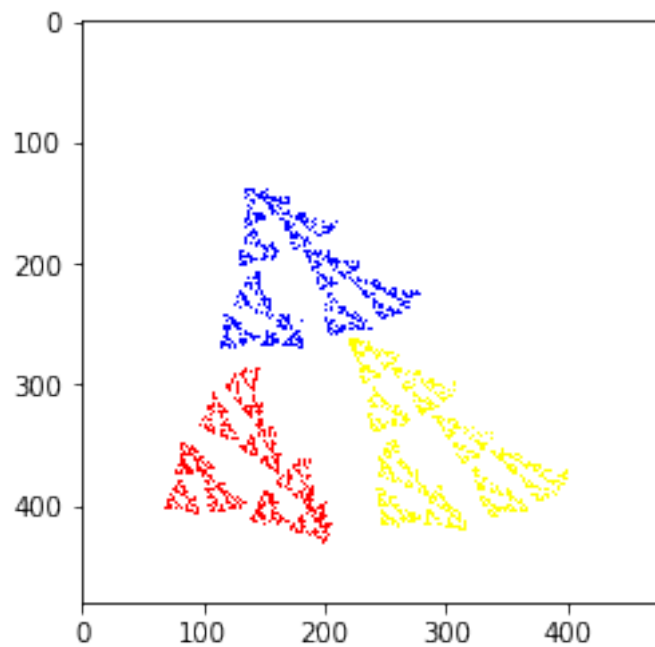
| 20 | 23 | 5.2756e-09 | 1.35e-09 | 7.50e-03 | 2.02e-06 |
| 21 | 24 | 4.8321e-09 | 4.43e-10 | 1.49e-02 | 8.30e-06 |

Current image:
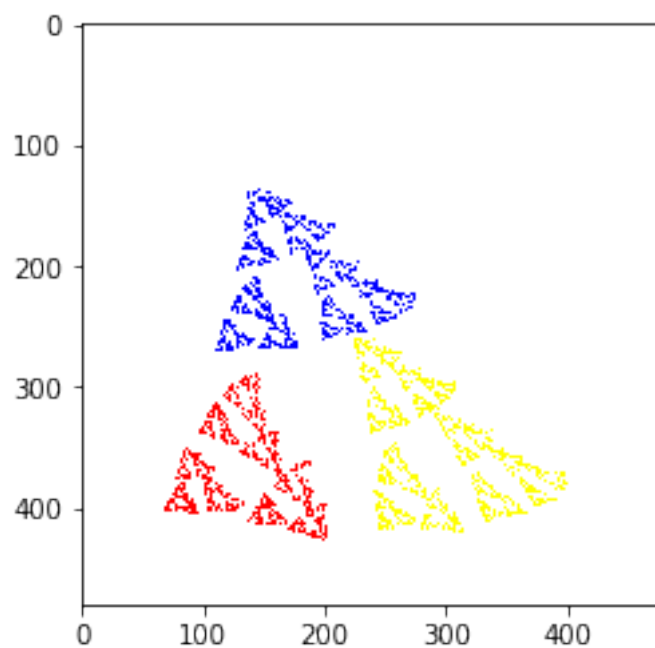
| 22 | 25 | 4.5261e-09 | 3.06e-10 | 2.13e-02 | 2.30e-05 |
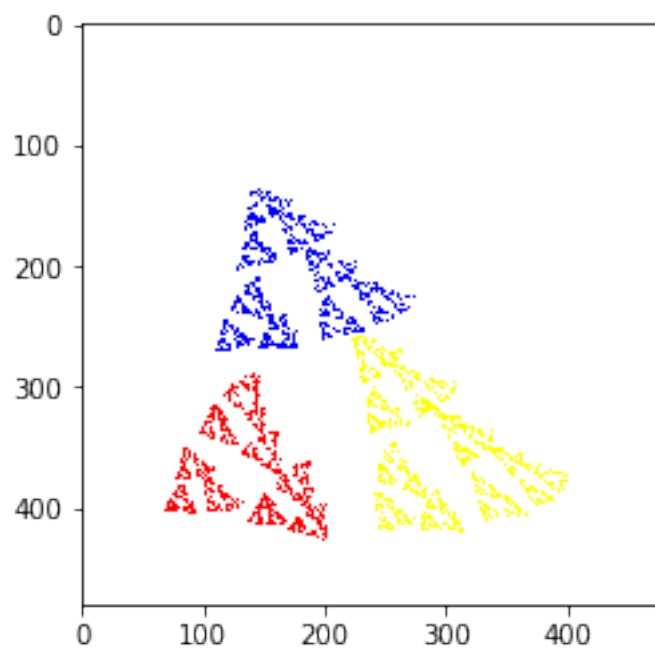
Current image:

| 23 | 26 | 4.1142e-09 | 4.12e-10 | 5.27e-03 | 2.02e-06 |
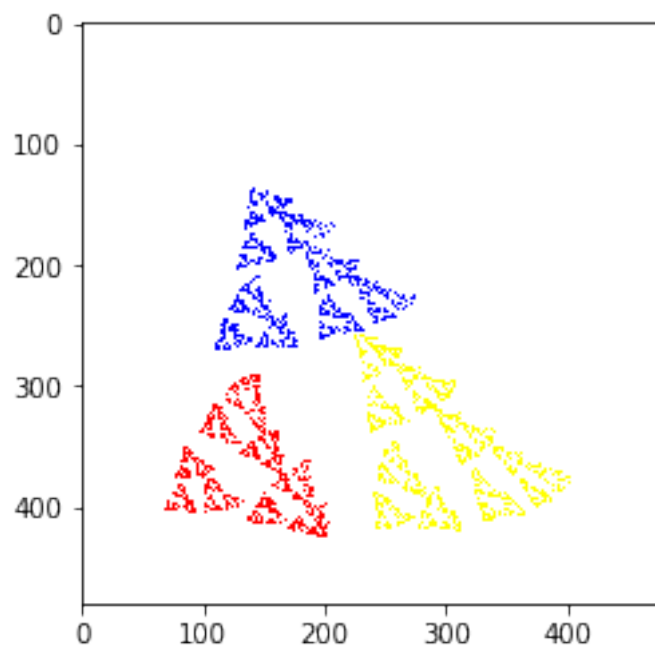
Current image:

```
      24              27          3.8710e-09      2.43e-10      1.03e-02      9.38e-06
Current image:
```
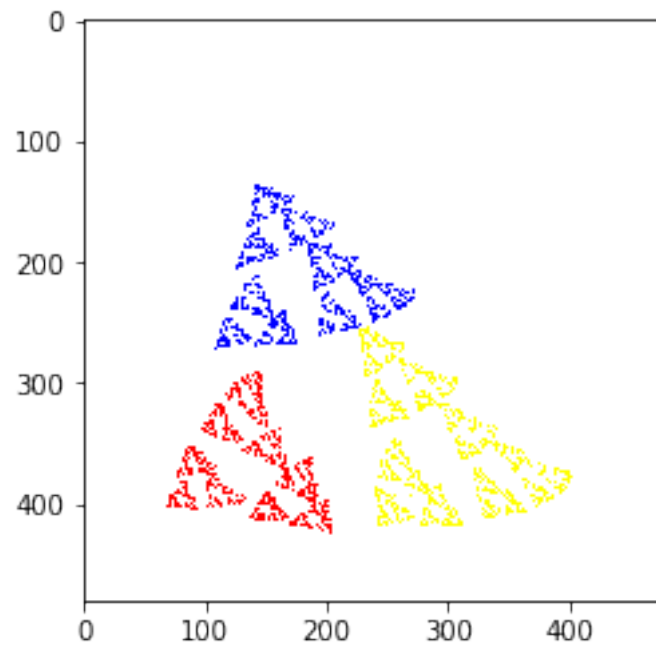


```
      25              28          3.7215e-09      1.50e-10      2.12e-02      3.44e-05
Current image:
```

26   29   3.2724e-09   4.49e-10   5.36e-03   2.81e-06
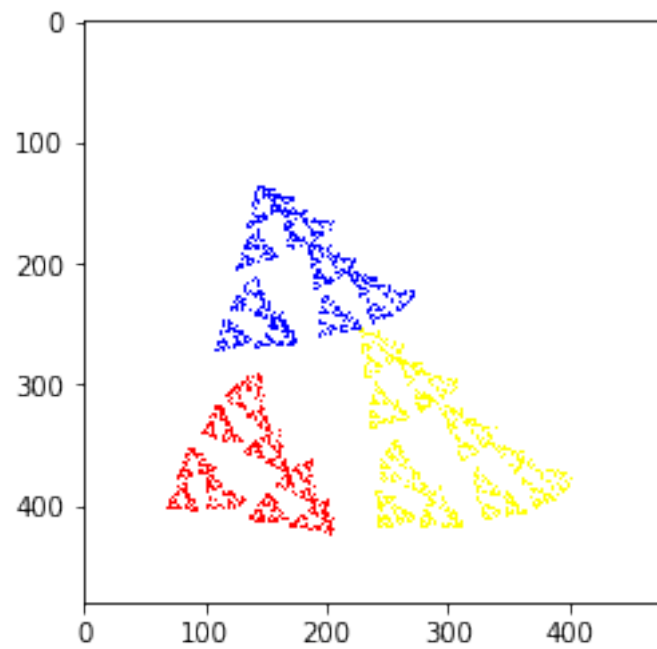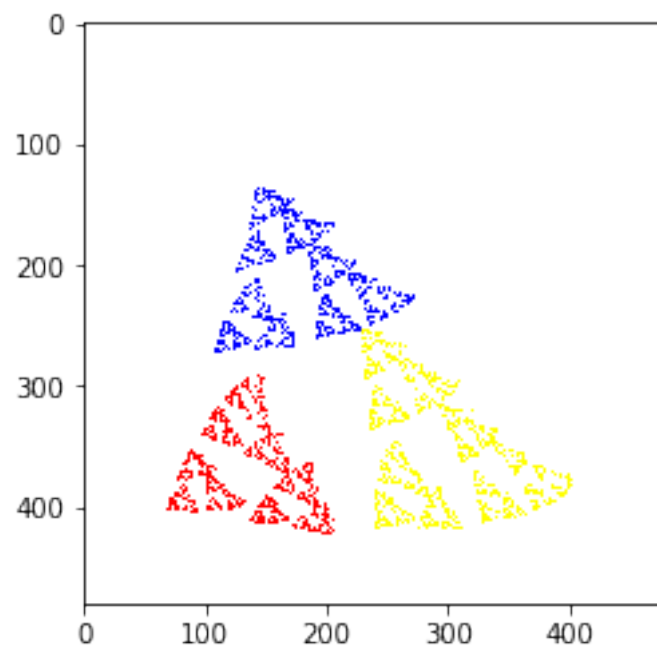
Current image:

| 27 | 30 | 3.0516e-09 | 2.21e-10 | 1.05e-02 | 1.27e-05 |
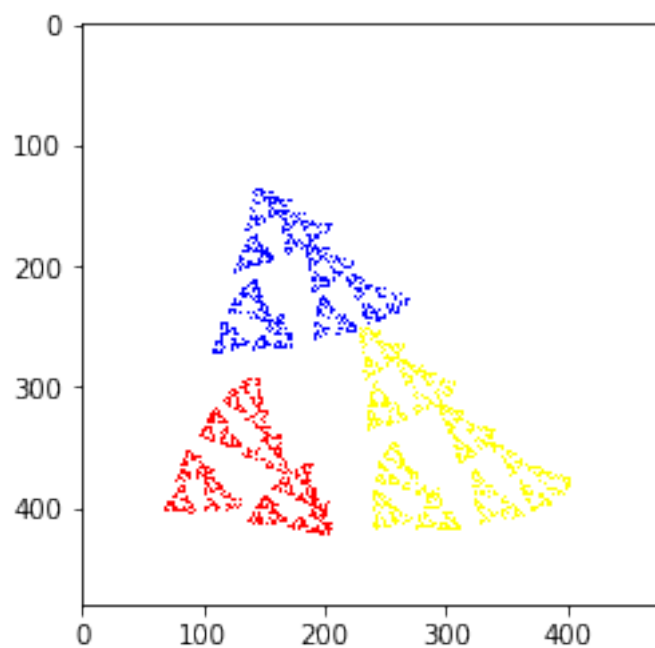
Current image:

| 28 | 32 | 2.8886e-09 | 1.63e-10 | 5.29e-03 | 3.47e-06 |
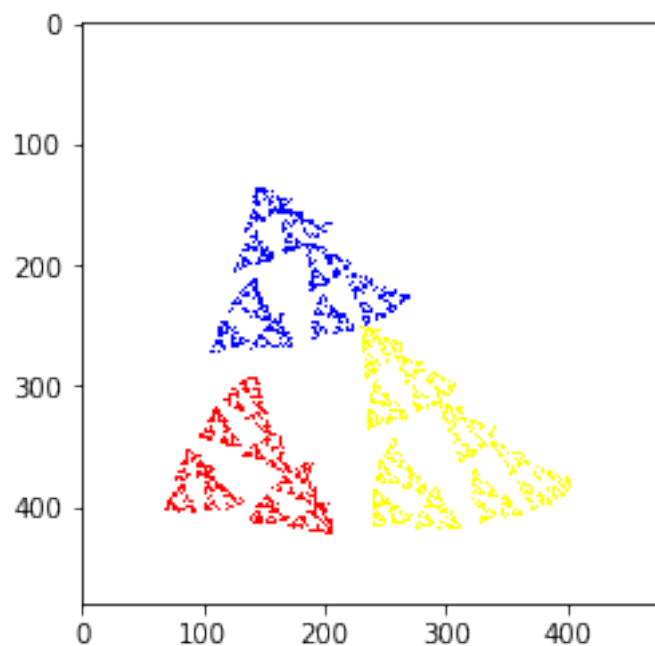
Current image:

| 29 | 33 | 2.6796e-09 | 2.09e-10 | 1.05e-02 | 1.39e-05 |

Current image:
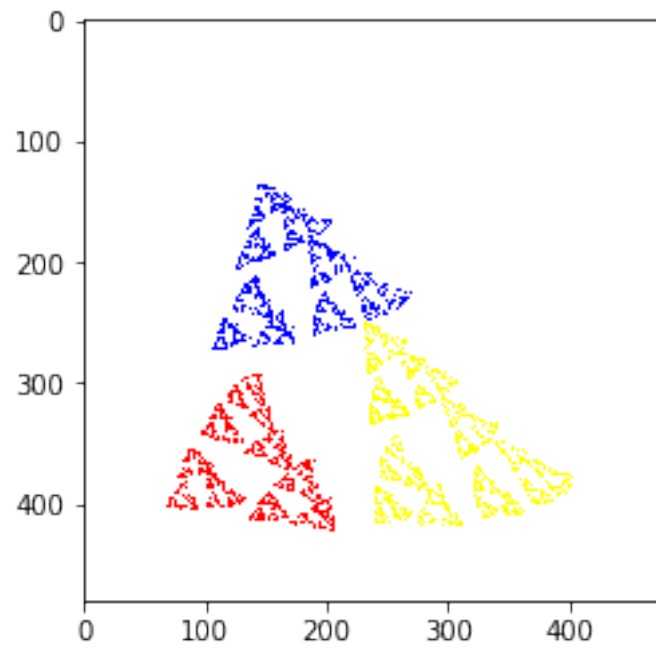
| 30 | 35 | 2.5123e-09 | 1.67e-10 | 5.35e-03 | 3.77e-06 |

Current image:
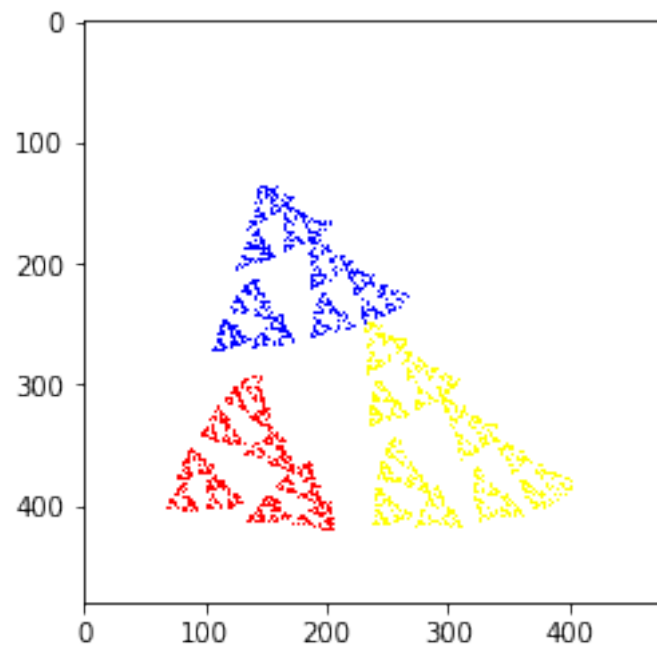
31 36 2.3168e-09 1.95e-10 1.07e-02 1.50e-05
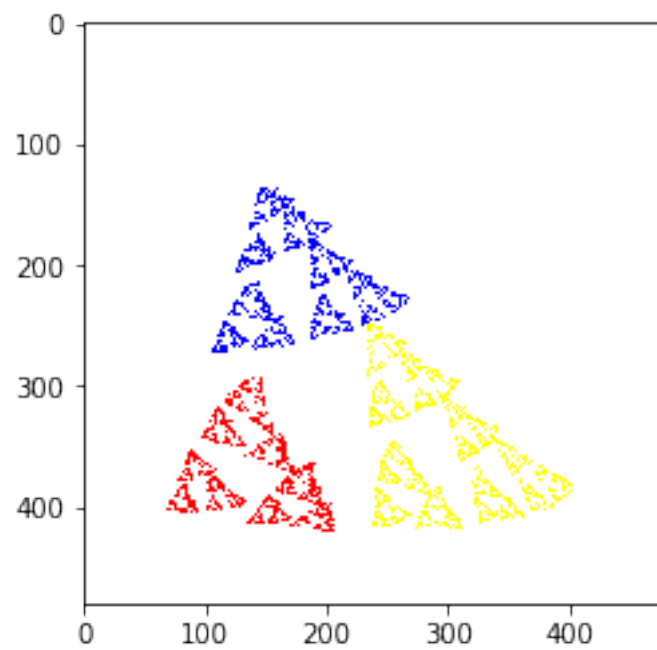
Current image:

```
    32              38           2.1522e-09       1.65e-10       5.08e-03       3.56e-06
Current image:
```
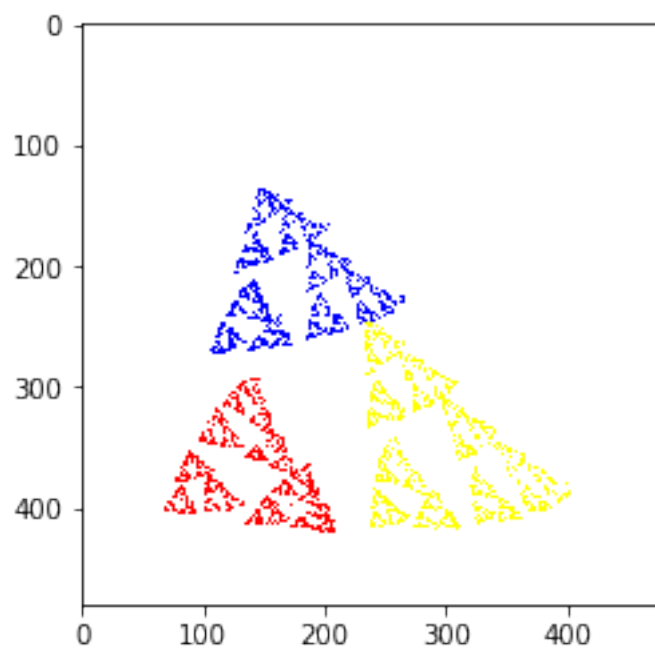


```
    33              39           1.9730e-09       1.79e-10       1.01e-02       1.40e-05
Current image:
```
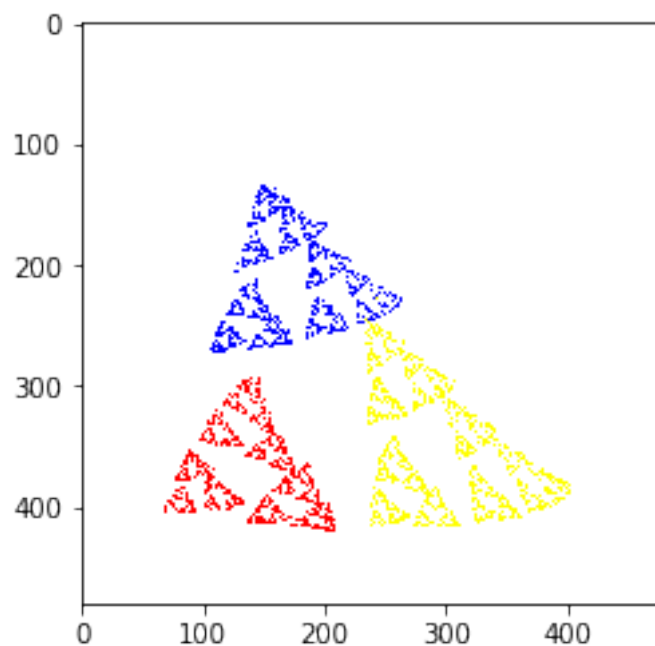
| 34 | 41 | 1.8207e-09 | 1.52e-10 | 5.12e-03 | 3.70e-06 |

Current image:
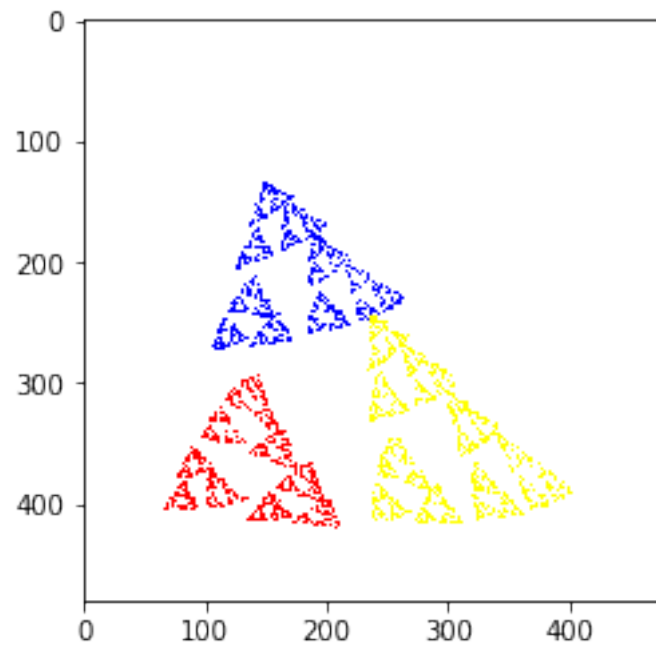
| 35 | 42 | 1.6575e-09 | 1.63e-10 | 1.02e-02 | 1.45e-05 |

Current image:
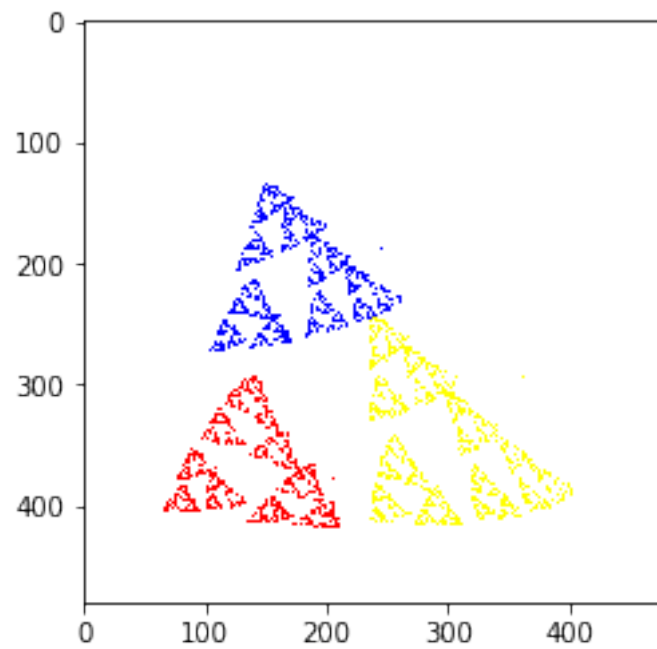
36   43   1.4761e-09   1.81e-10   1.05e-02   1.55e-05
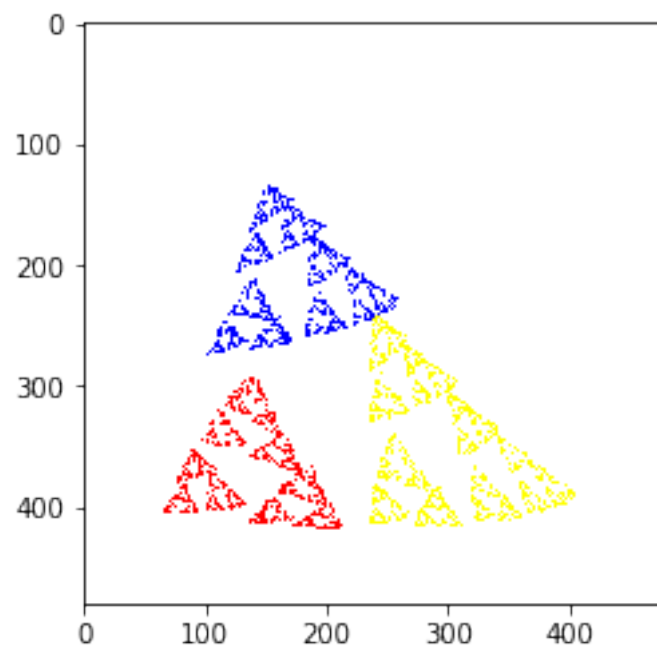
Current image:

| 37 | 44 | 1.3005e-09 | 1.76e-10 | 1.05e-02 | 1.58e-05 |
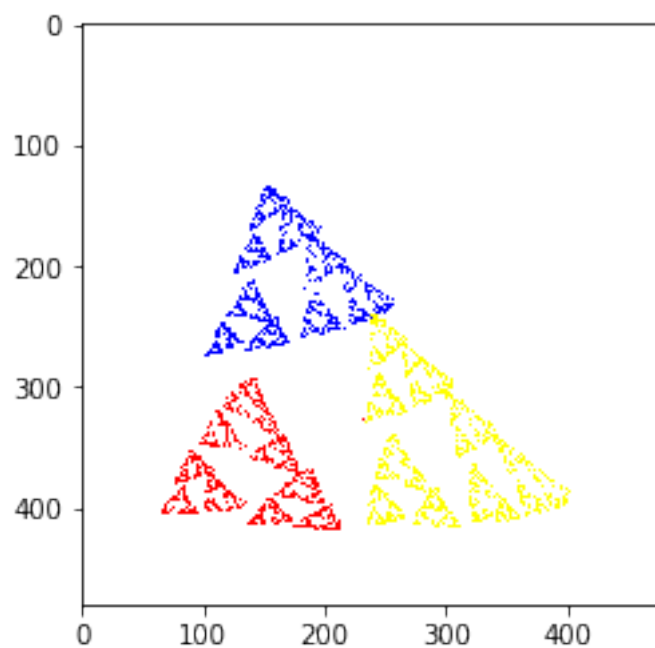
Current image:

| 38 | 45 | 1.1354e-09 | 1.65e-10 | 1.06e-02 | 1.60e-05 |
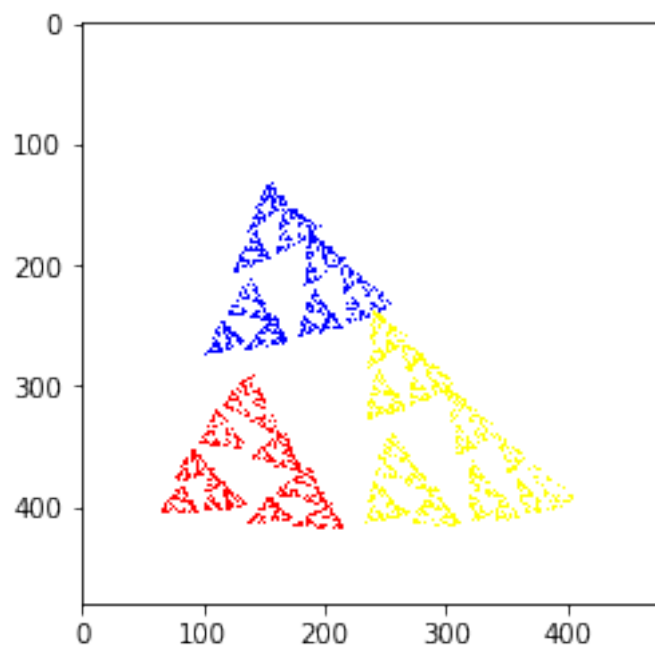
Current image:

```
     39              46          9.8172e-10      1.54e-10       1.06e-02       1.59e-05
Current image:
```
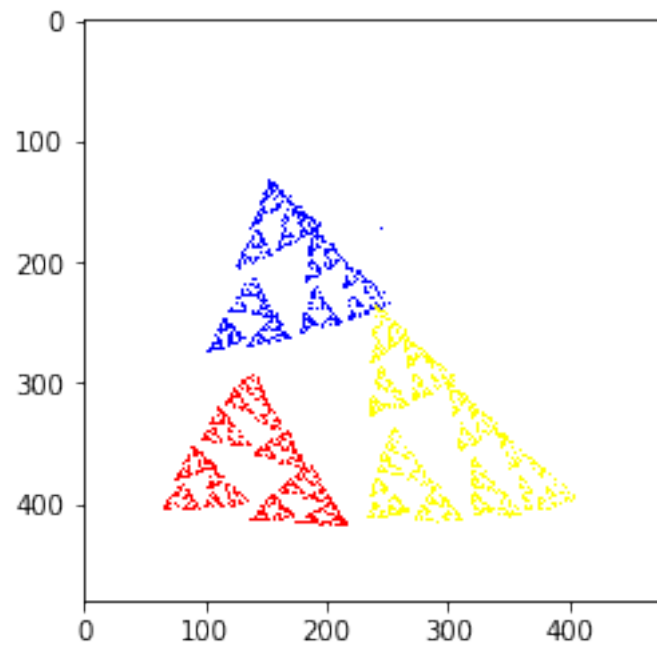


```
     40              47          8.4024e-10      1.41e-10       1.06e-02       1.57e-05
Current image:
```
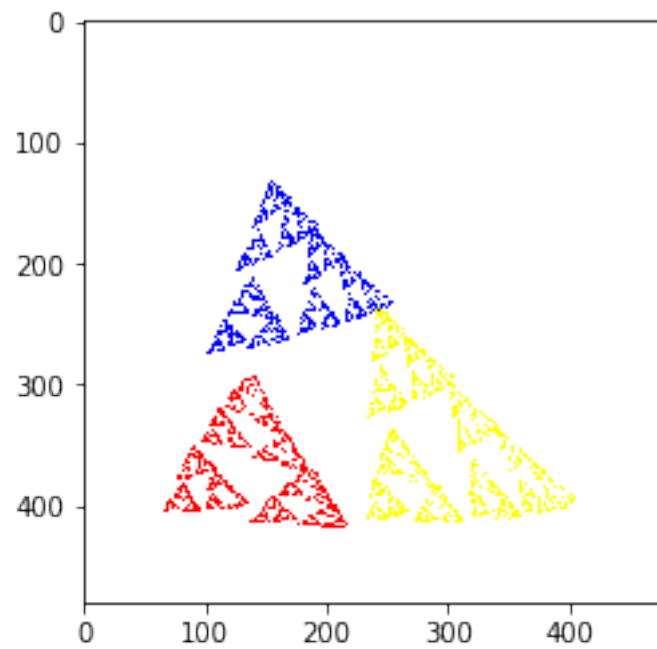
41	48	7.1191e-10	1.28e-10	1.06e-02	1.53e-05

Current image:

```
    42              49          5.9725e-10      1.15e-10        1.06e-02        1.48e-05
Current image:
```



```
    43              50          4.9681e-10      1.00e-10        1.05e-02        1.42e-05
Current image:
```

| 44 | 51 | 4.1112e-10 | 8.57e-11 | 1.05e-02 | 1.37e-05 |
| 45 | 52 | 3.3939e-10 | 7.17e-11 | 1.05e-02 | 1.32e-05 |

Current image:
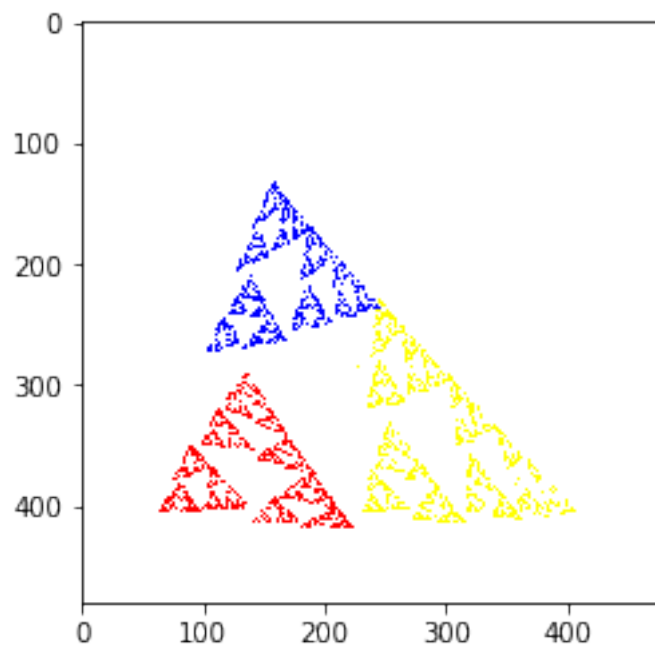
| 46 | 53 | 2.8123e-10 | 5.82e-11 | 1.05e-02 | 1.30e-05 |

Current image:

47   54   2.2255e-10   5.87e-11   9.34e-03   1.01e-05
Current image:
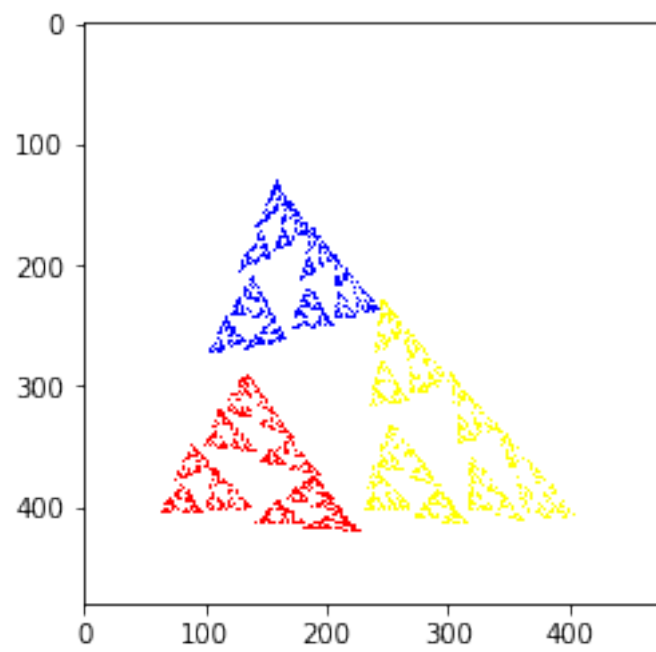
| 48 | 55 | 1.8418e-10 | 3.84e-11 | 8.72e-03 | 8.76e-06 |

Current image:
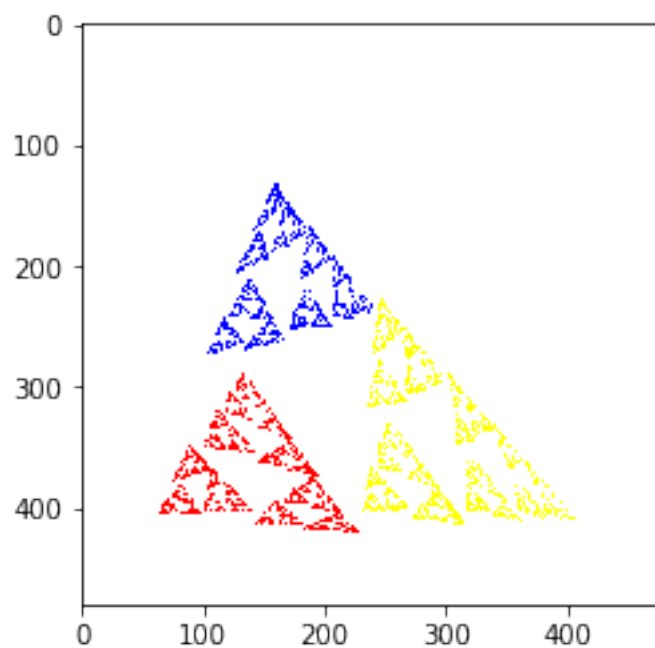
| 49 | 56 | 1.5254e-10 | 3.16e-11 | 7.49e-03 | 6.26e-06 |

Current image:
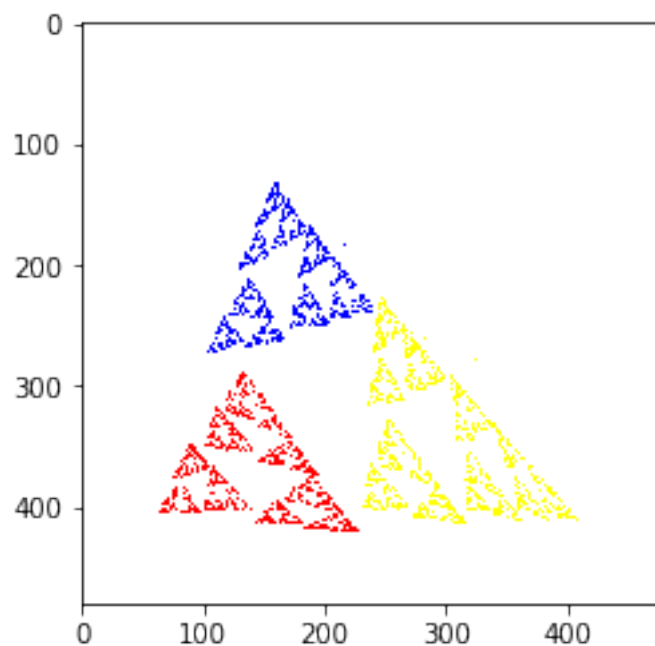
```
     50              57           1.3436e-10       1.82e-11        7.19e-03        5.60e-06
Current image:
```
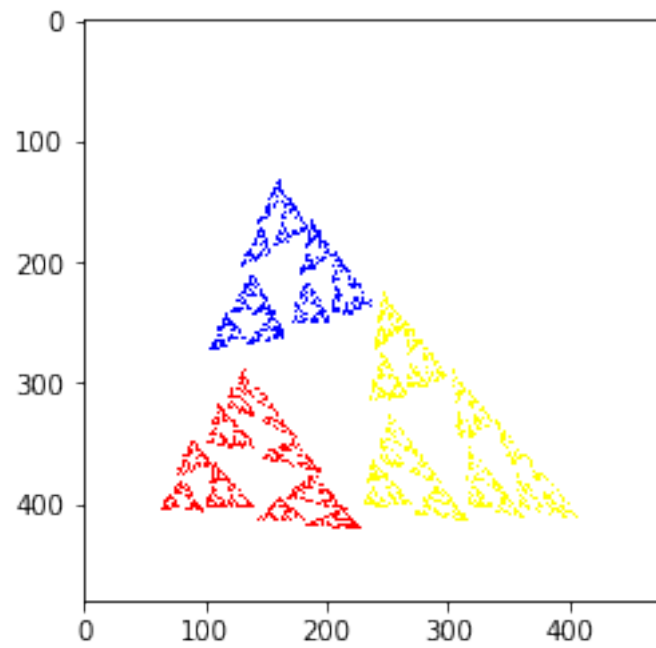


```
     51              58           1.1877e-10       1.56e-11        5.61e-03        3.23e-06
Current image:
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52   59   1.1052e-10   8.25e-12   5.99e-03   3.14e-06
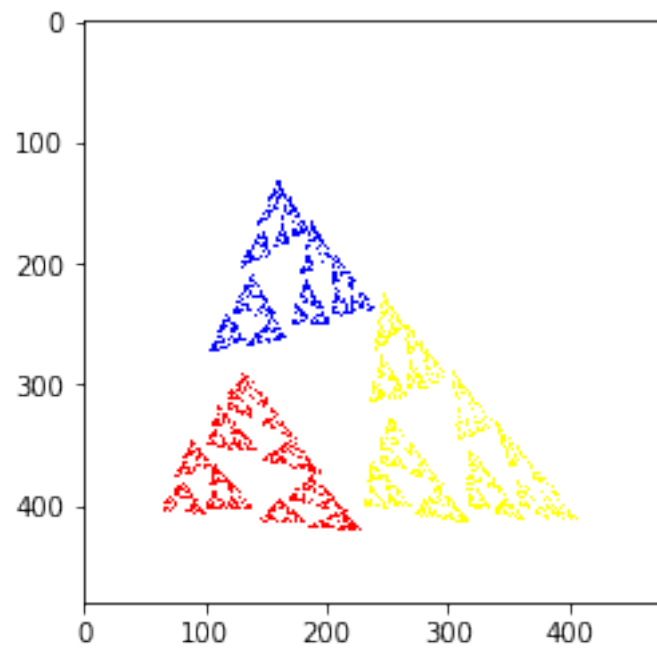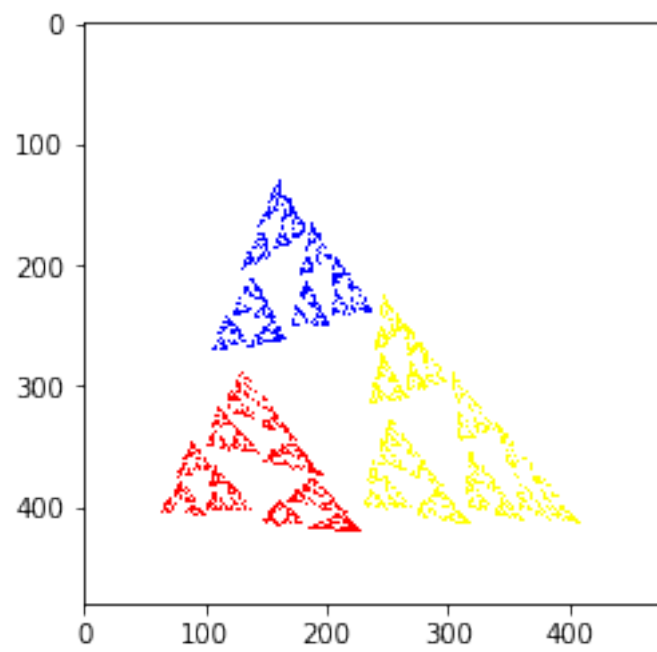
Current image:

53	60	1.0321e-10	7.31e-12	4.31e-03	1.13e-06

Current image:
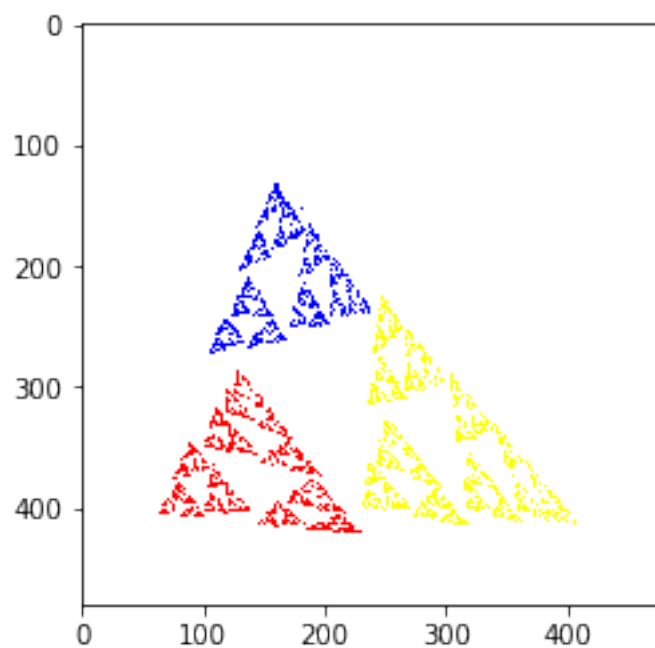
54	61	9.5990e-11	7.22e-12	7.85e-03	1.02e-06

Current image:
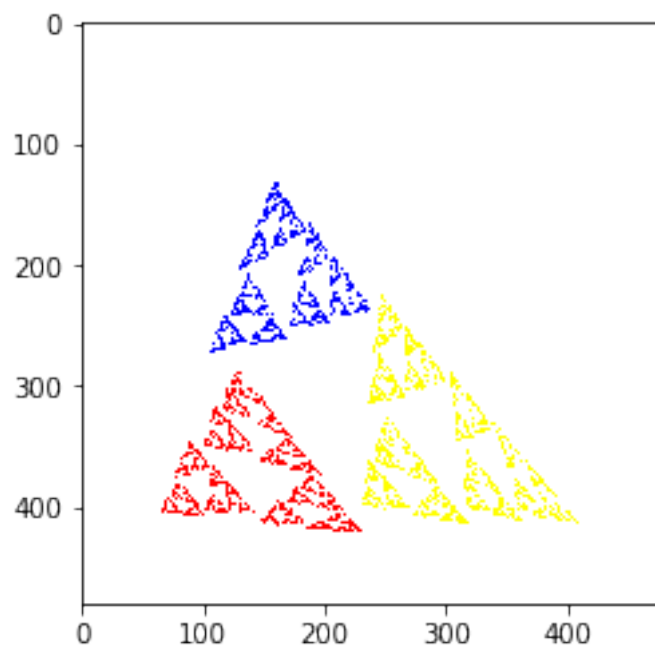
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      55              62           9.0129e-11       5.86e-12       8.65e-03       5.18e-07
Current image:
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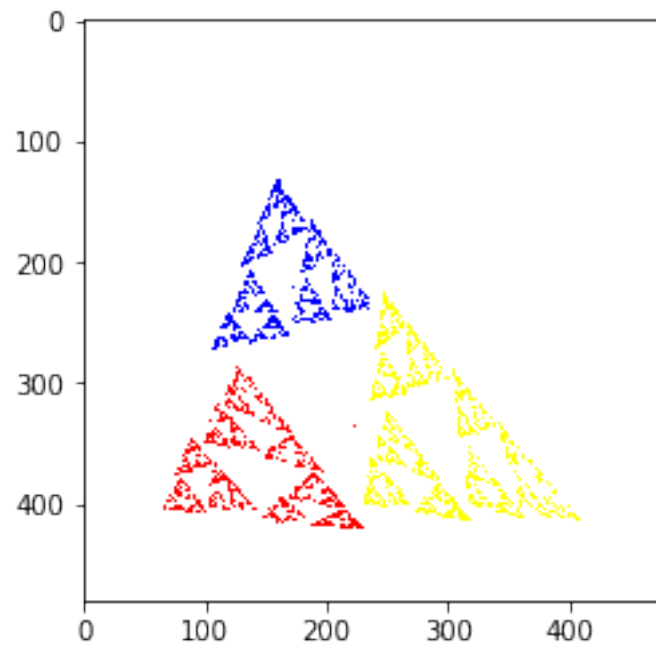


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      56              63           8.5920e-11       4.21e-12       3.62e-03       3.66e-07
Current image:
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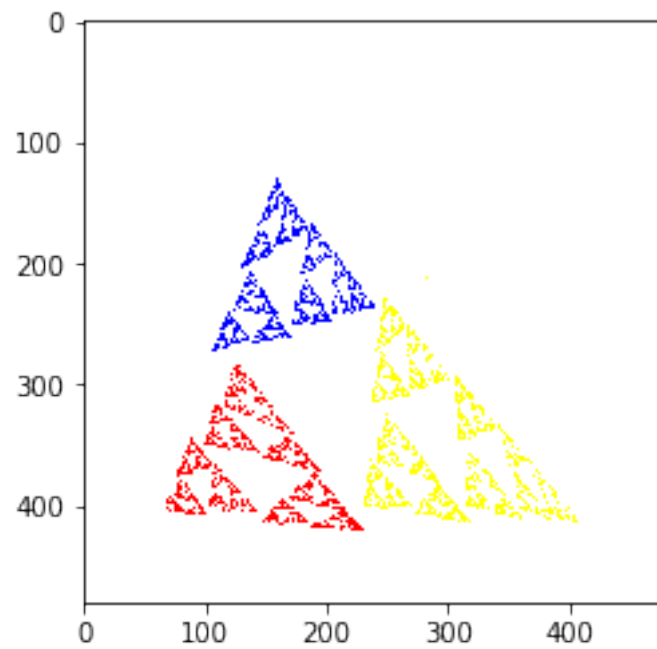
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          57              64          8.3489e-11      2.43e-12      1.06e-02      2.69e-06
Current image:
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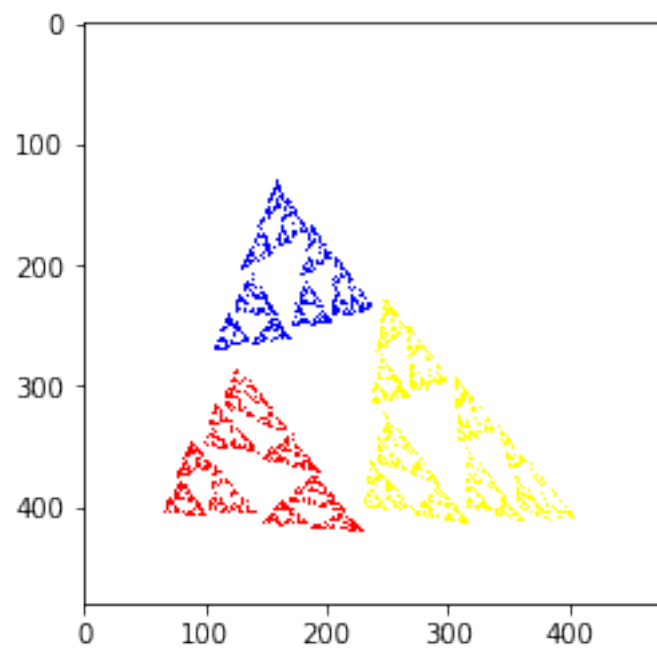
| 58 | 65 | 7.6903e-11 | 6.59e-12 | 2.25e-03 | 5.06e-08 |

Current image:
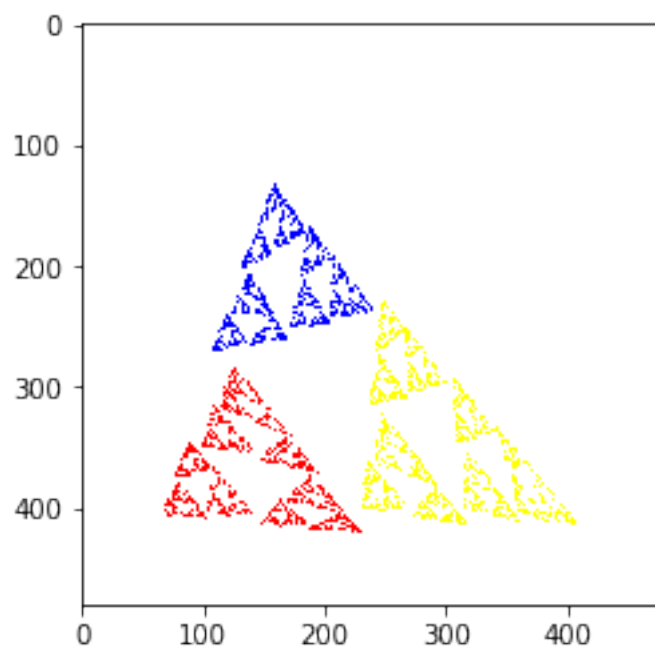
| 59 | 66 | 7.4002e-11 | 2.90e-12 | 4.75e-03 | 8.22e-07 |

Current image:
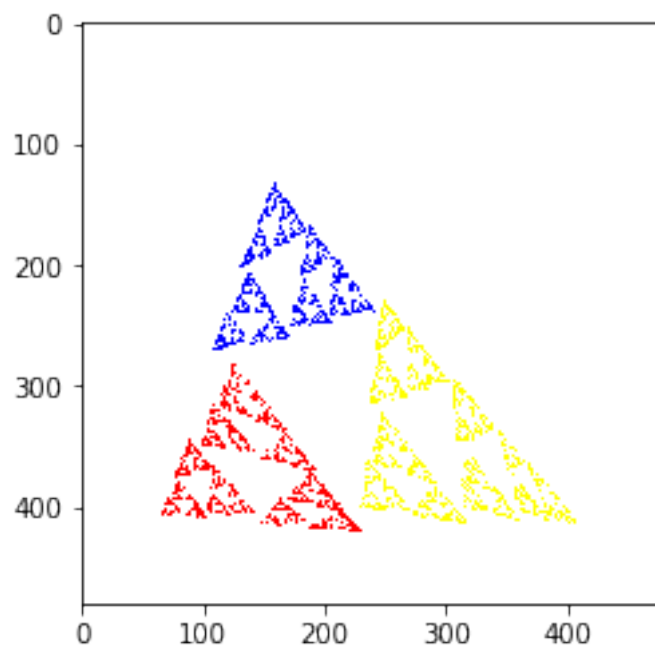
| 60 | 67 | 7.0189e-11 | 3.81e-12 | 8.11e-03 | 7.49e-07 |

Current image:
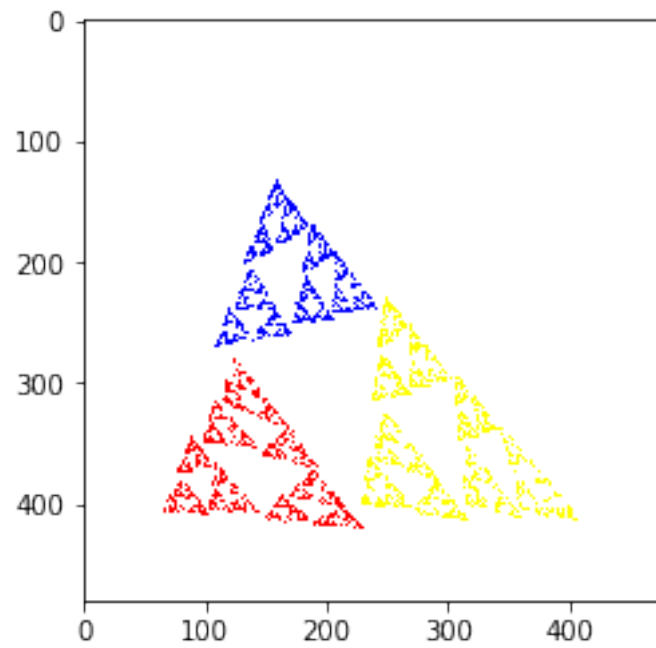
| 61 | 68 | 6.6503e-11 | 3.69e-12 | 5.87e-03 | 1.34e-06 |

Current image:
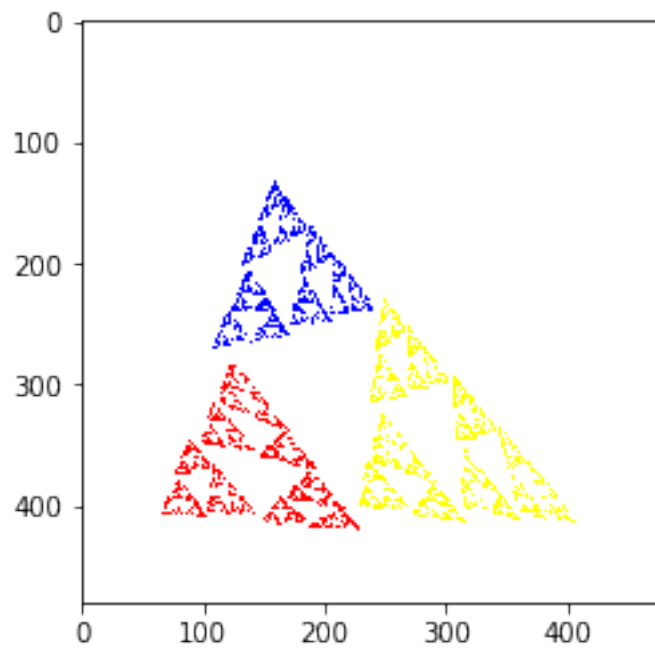
62	69	6.3675e-11	2.83e-12	4.12e-03	4.37e-07
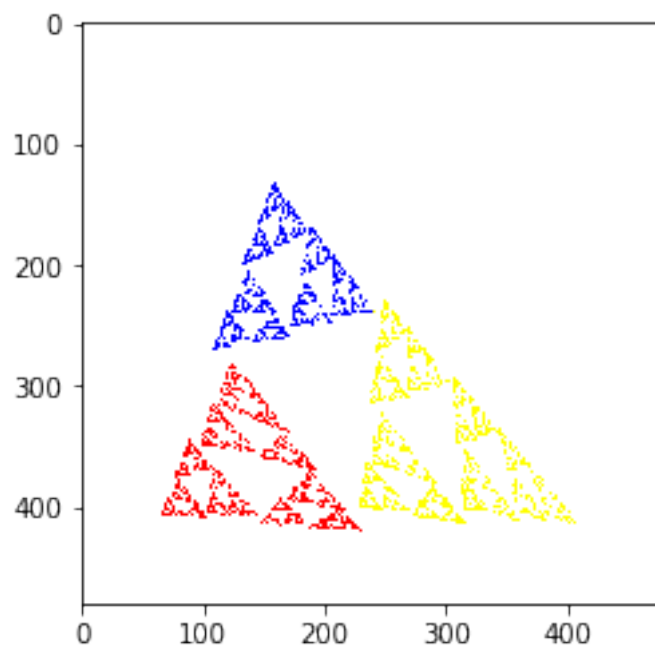
Current image:

| 63 | 71 | 6.2133e-11 | 1.54e-12 | 2.51e-03 | 2.36e-07 |

Current image:
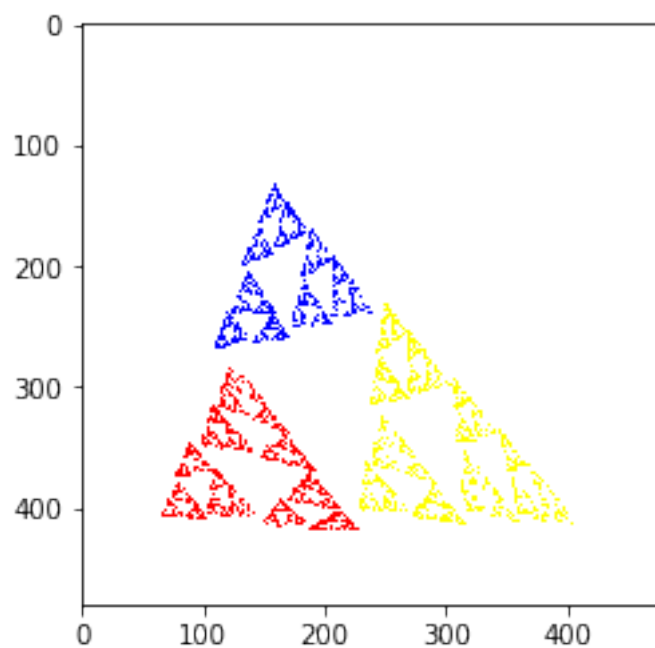
| 64 | 72 | 5.9835e-11 | 2.30e-12 | 4.96e-03 | 1.12e-06 |

Current image:
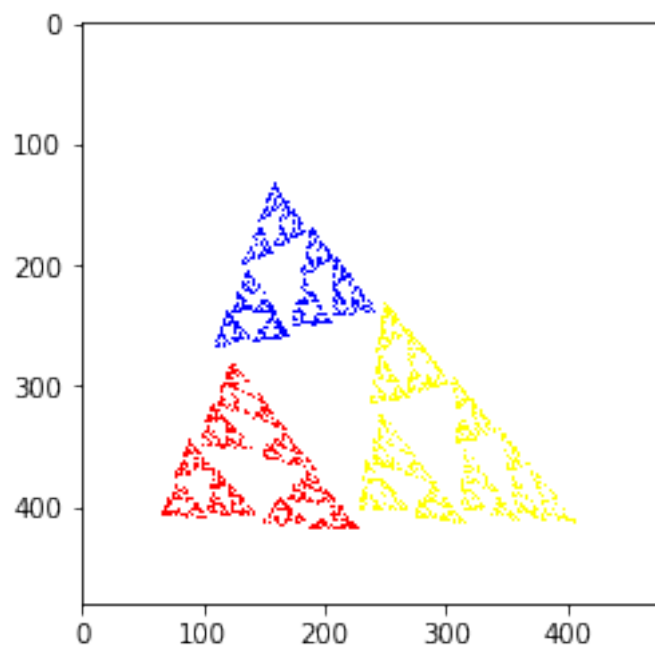
| 65 | 73 | 5.7306e-11 | 2.53e-12 | 4.44e-03 | 7.10e-07 |
| 66 | 74 | 5.4913e-11 | 2.39e-12 | 6.80e-03 | 1.74e-06 |

Current image:
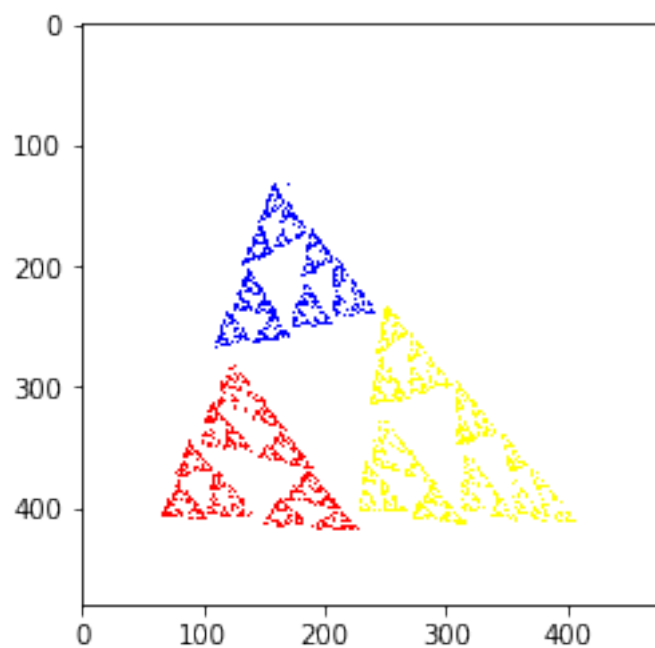
| 67 | 75 | 5.2317e-11 | 2.60e-12 | 2.56e-03 | 2.59e-07 |

Current image:
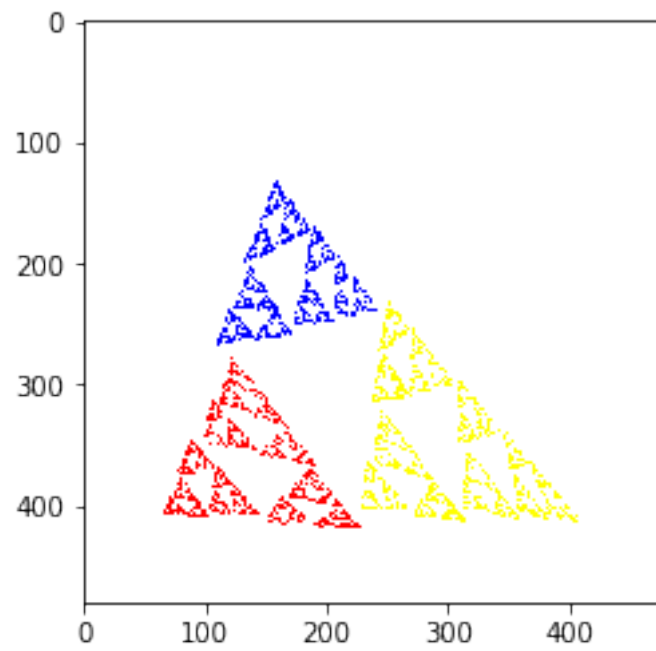
| 68 | 77 | 5.0977e-11 | 1.34e-12 | 2.52e-03 | 2.36e-07 |
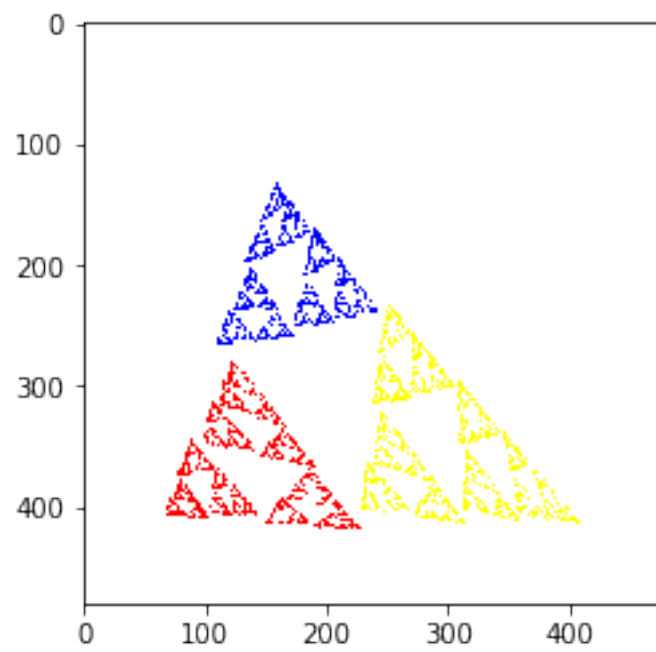
Current image:

| 69 | 78 | 4.8984e-11 | 1.99e-12 | 4.96e-03 | 1.17e-06 |

Current image:
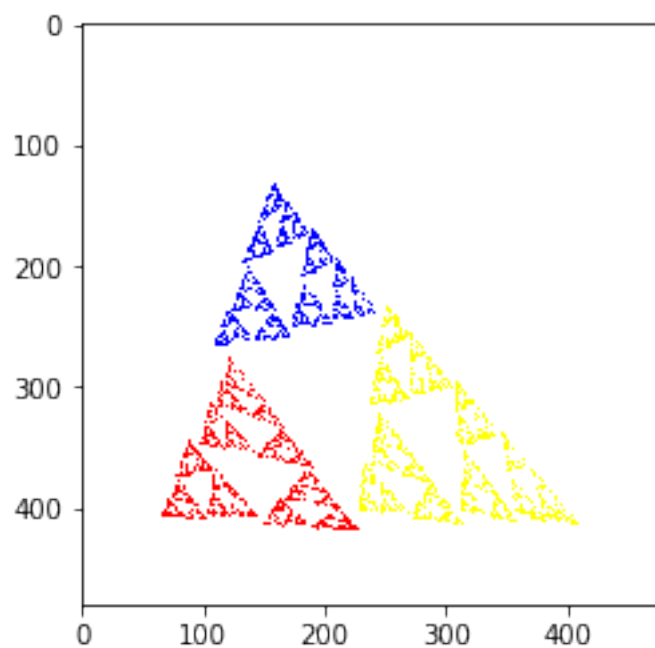
| 70 | 79 | 4.6860e-11 | 2.12e-12 | 3.73e-03 | 5.55e-07 |

Current image:
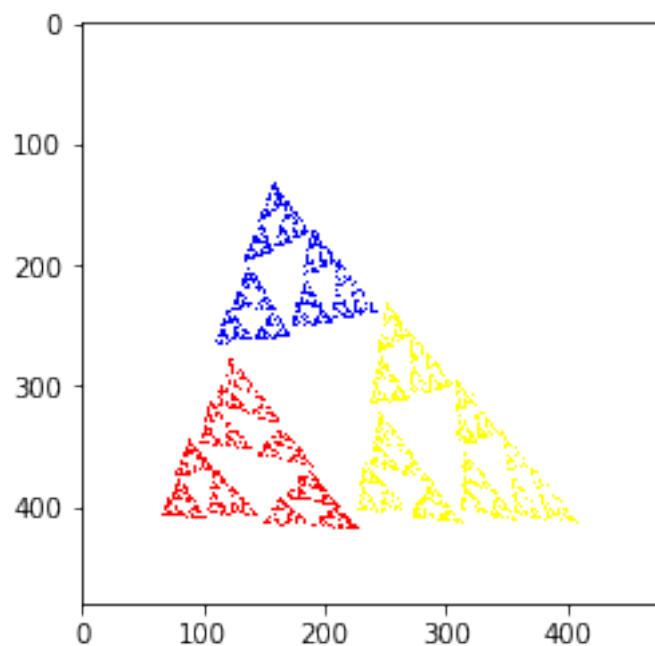
71	80	4.4785e-11	2.08e-12	5.01e-03	9.99e-07

Current image:
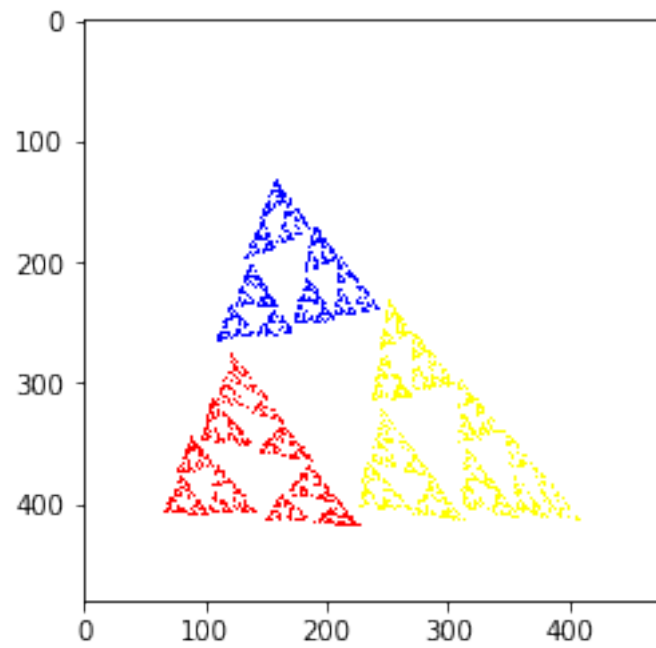
72	81	4.2758e-11	2.03e-12	3.90e-03	6.29e-07

Current image:

73   82   4.0763e-11   2.00e-12   4.97e-03   9.96e-07
Current image:
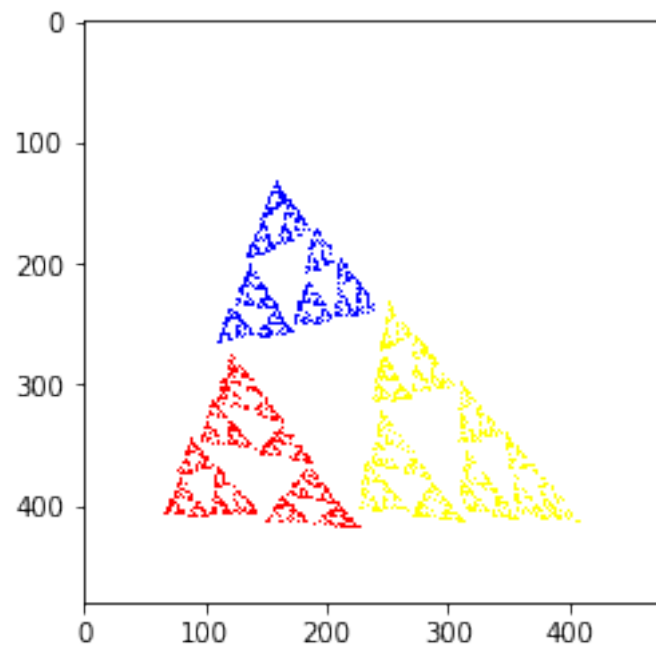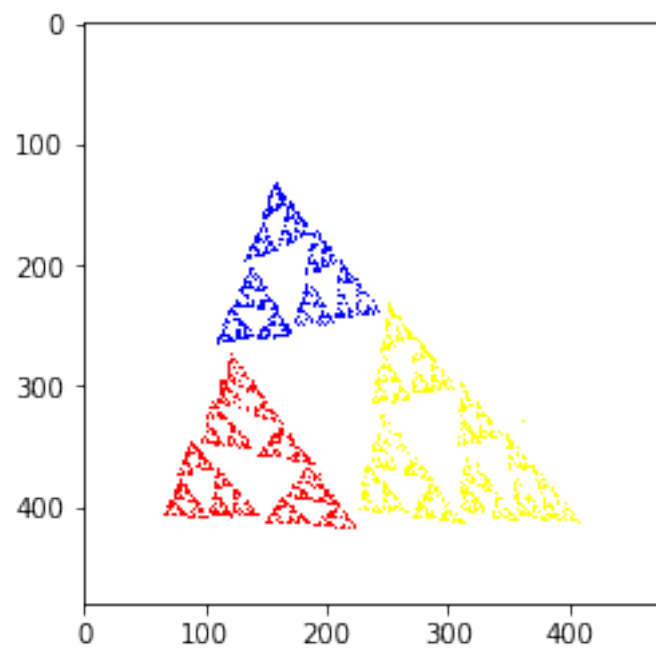
| 74 | 83 | 3.8869e-11 | 1.89e-12 | 3.67e-03 | 5.51e-07 |

Current image:
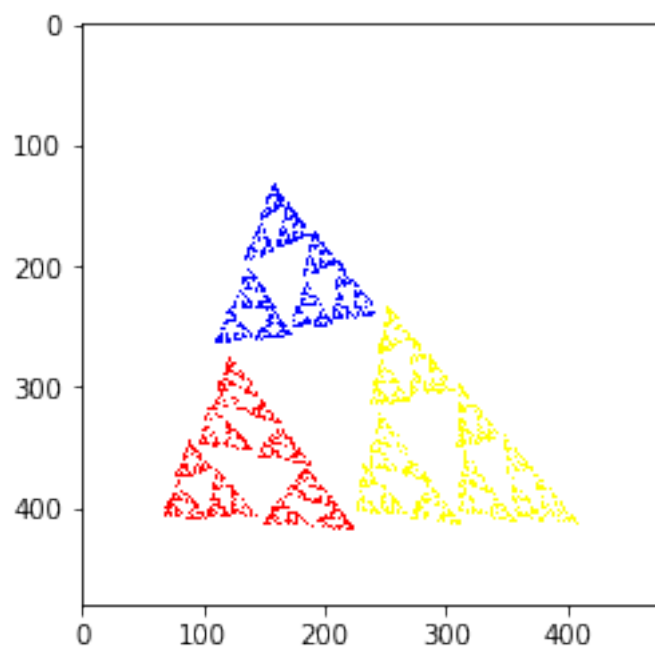
| 75 | 84 | 3.7030e-11 | 1.84e-12 | 4.96e-03 | 1.04e-06 |

Current image:
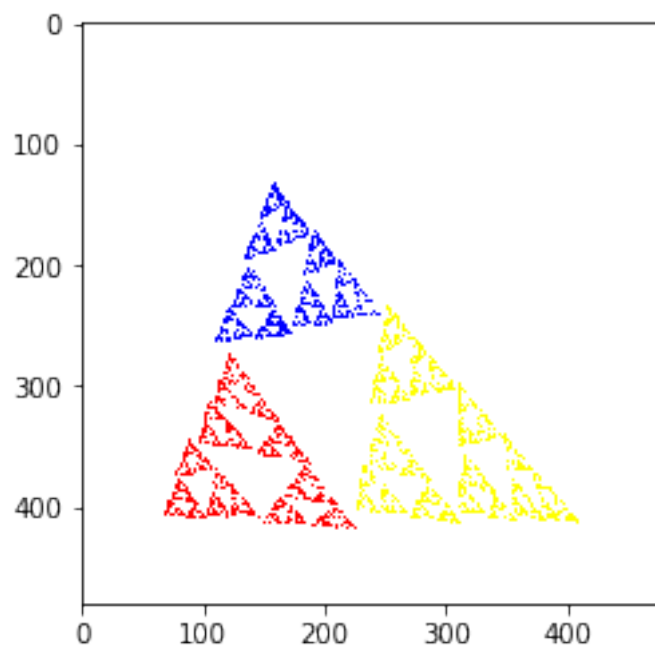
| 76 | 85 | 3.5280e-11 | 1.75e-12 | 3.36e-03 | 4.55e-07 |

Current image:
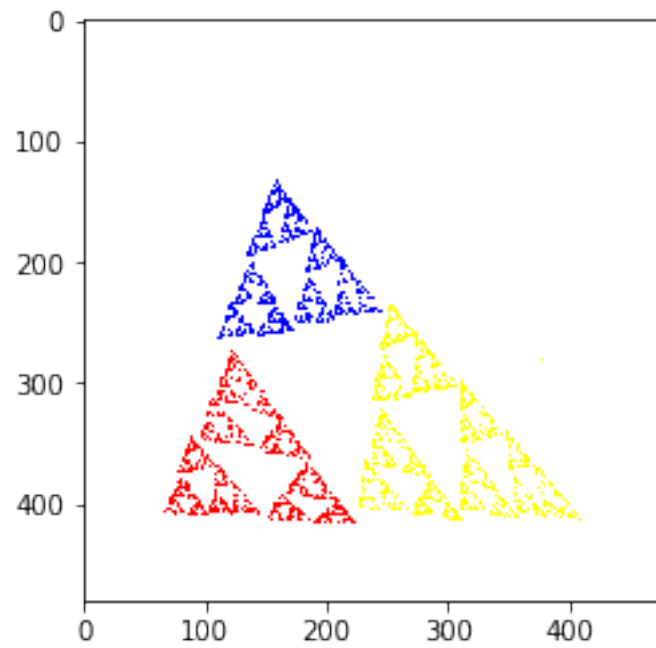
| 77 | 86 | 3.3599e-11 | 1.68e-12 | 4.97e-03 | 1.07e-06 |
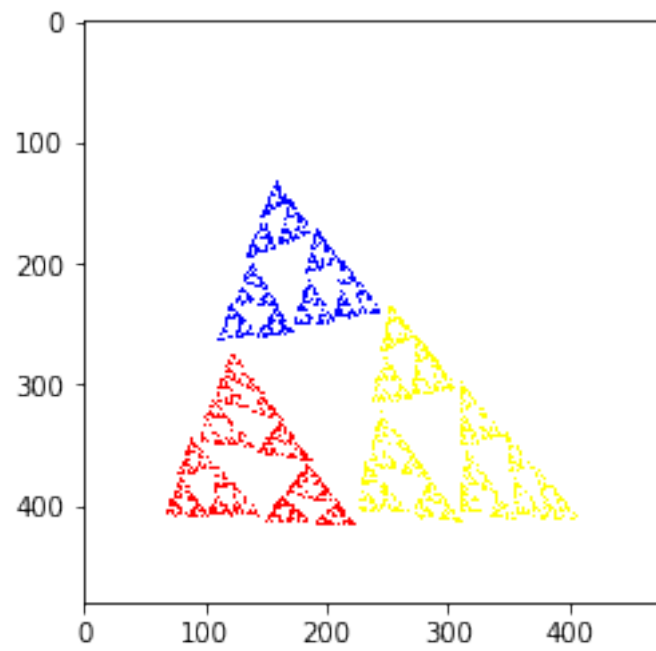
Current image:

78   87   3.1974e-11   1.62e-12   3.14e-03   3.96e-07
Current image:
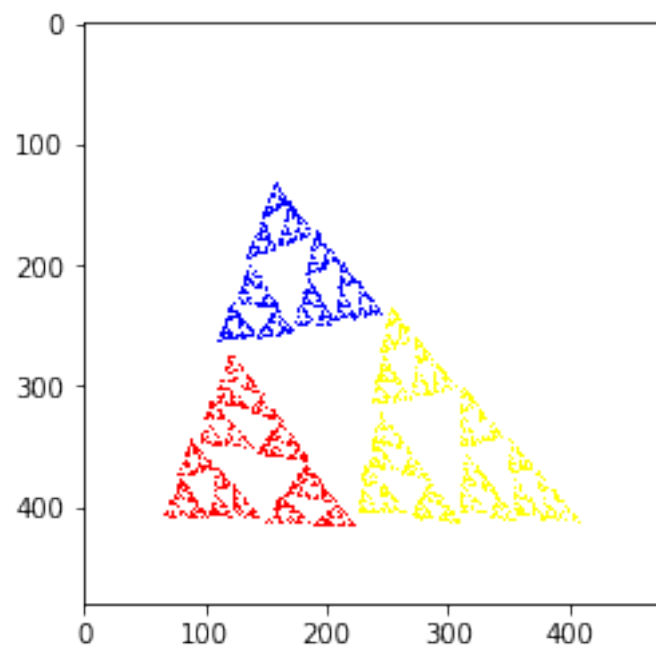
79	88	3.0428e-11	1.55e-12	4.97e-03	1.07e-06
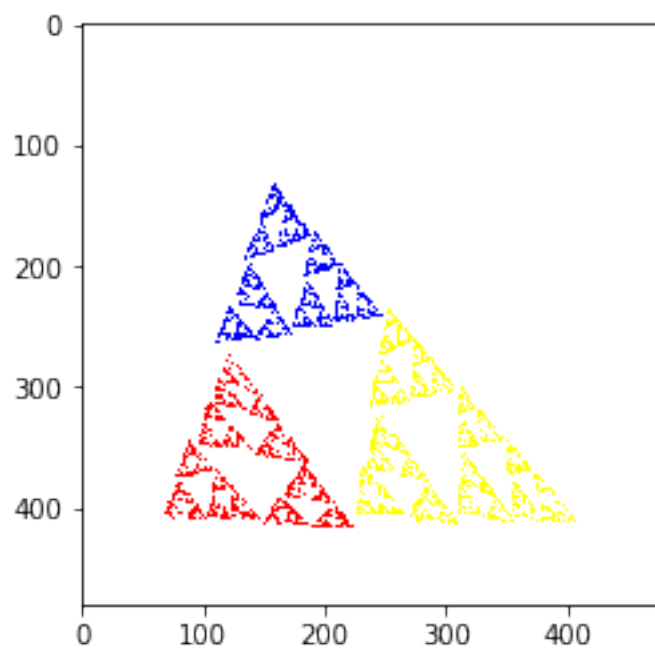
Current image:

80	89	2.8913e-11	1.51e-12	2.98e-03	3.52e-07

Current image:
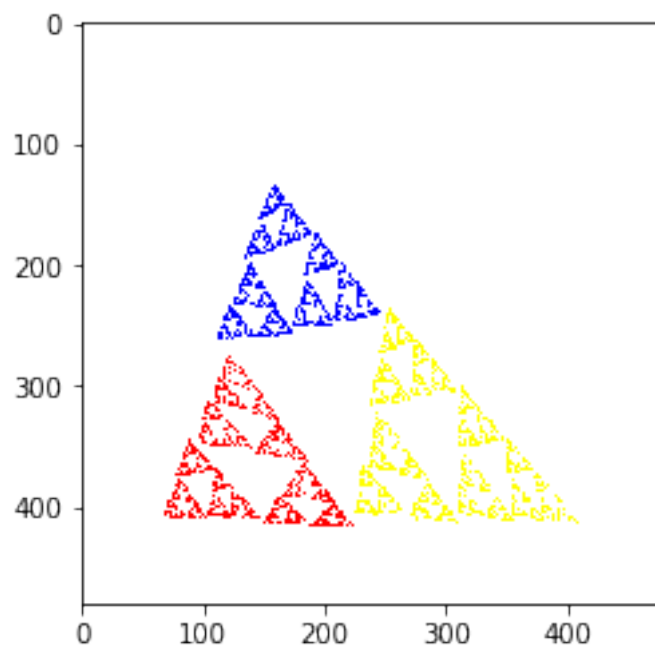
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    81              90          2.7488e-11      1.42e-12      4.97e-03      1.07e-06
Current image:
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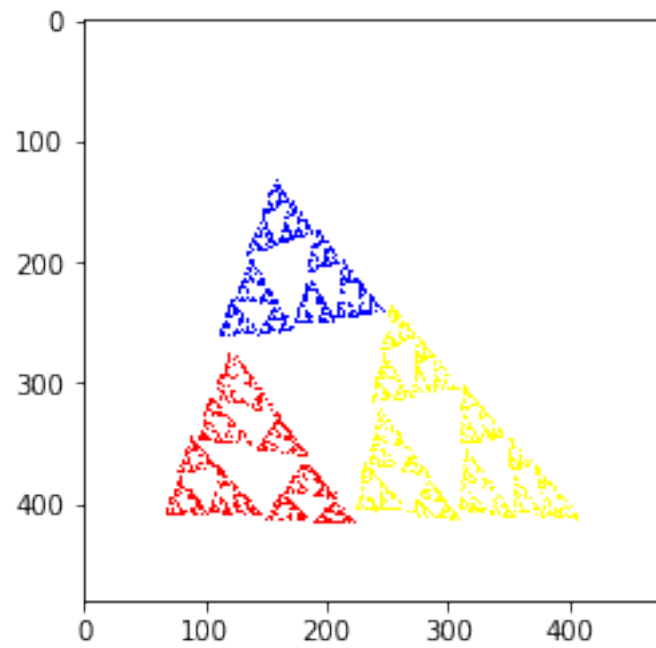


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    82              91          2.6076e-11      1.41e-12      2.84e-03      3.15e-07
Current image:
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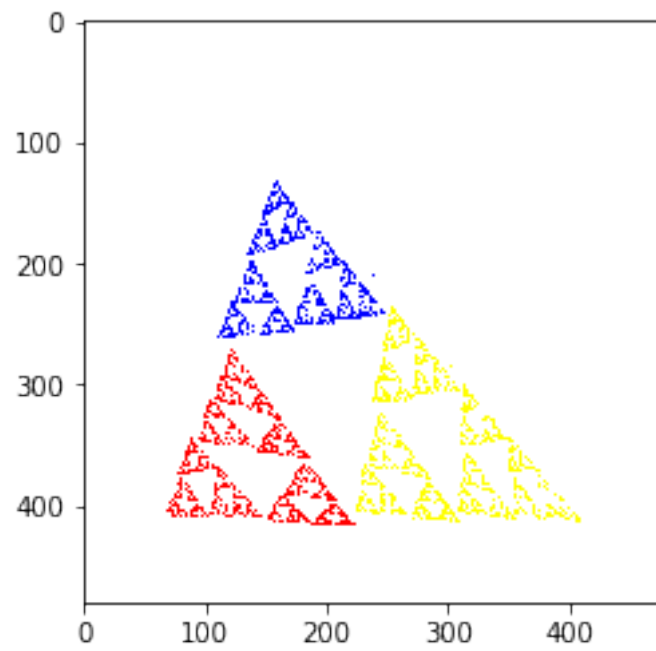
83	92	2.4774e-11	1.30e-12	4.96e-03	1.09e-06
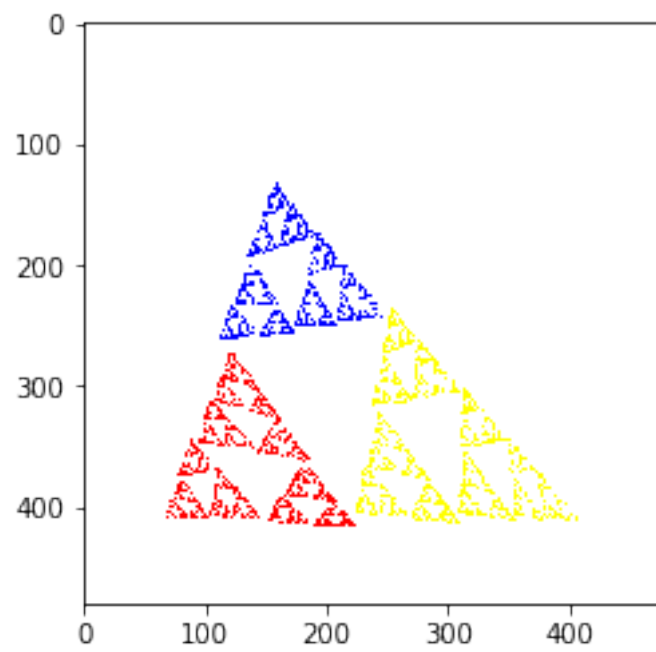
Current image:

| 84 | 93 | 2.3458e-11 | 1.32e-12 | 2.68e-03 | 2.74e-07 |

Current image:
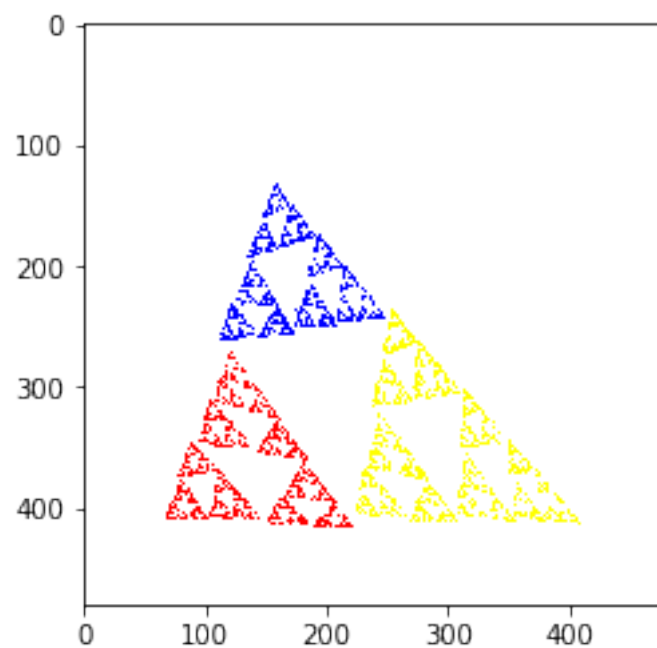
| 85 | 94 | 2.2276e-11 | 1.18e-12 | 4.95e-03 | 1.10e-06 |

Current image:
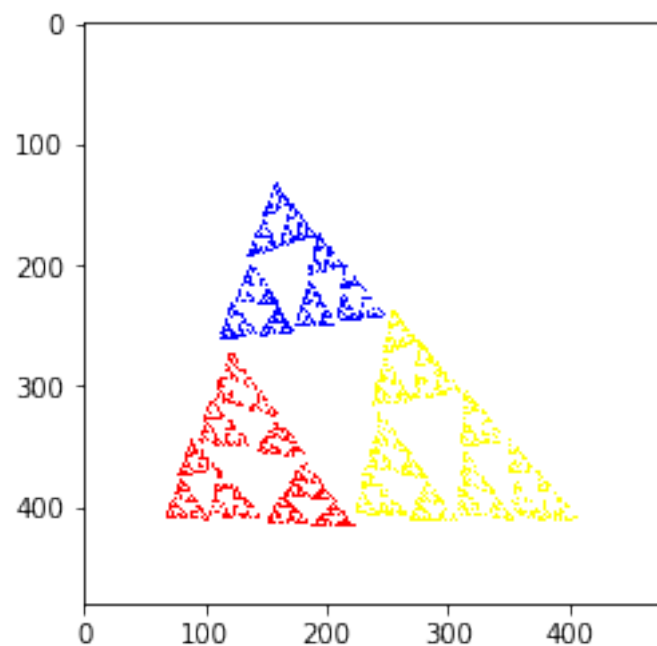
| 86 | 95 | 2.1048e-11 | 1.23e-12 | 2.52e-03 | 2.36e-07 |
| 87 | 96 | 1.9984e-11 | 1.06e-12 | 4.94e-03 | 1.12e-06 |

Current image:
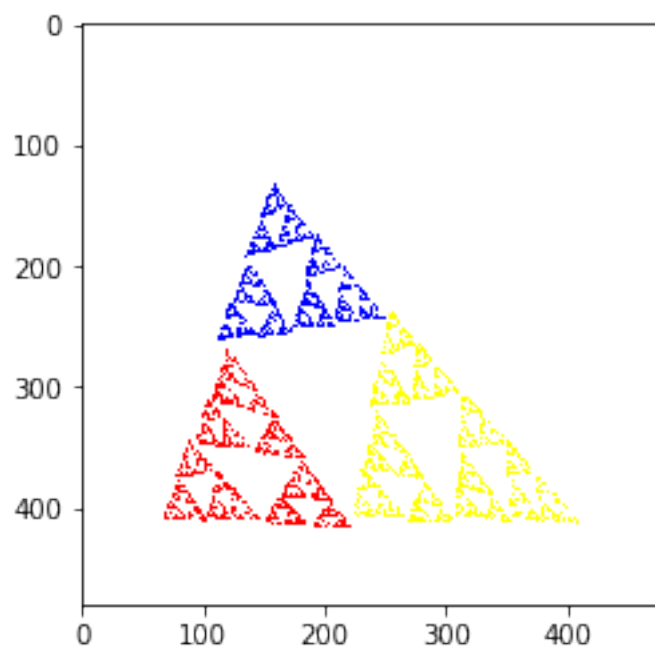
| 88 | 97 | 1.8836e-11 | 1.15e-12 | 2.37e-03 | 2.02e-07 |

Current image:
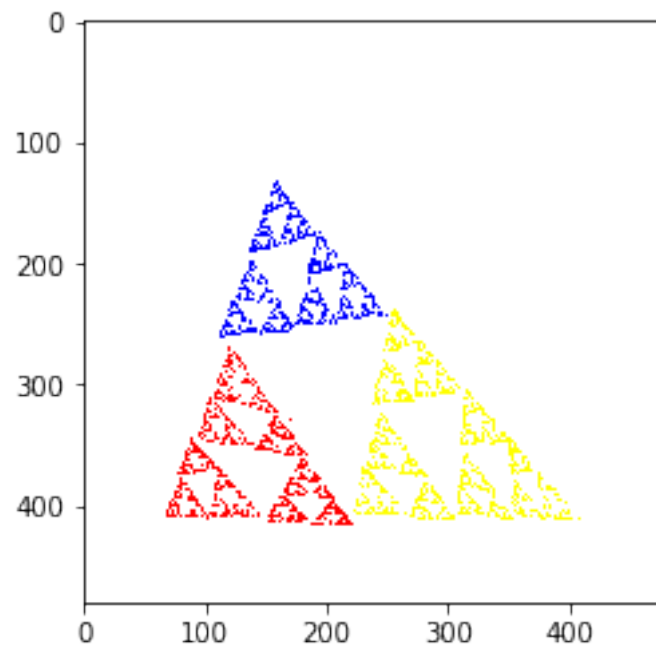
89   98   1.7883e-11   9.52e-13   4.93e-03   1.13e-06

Current image:

| 90 | 99 | 1.6808e-11 | 1.08e-12 | 2.22e-03 | 1.72e-07 |

Current image:

| 91 | 100 | 1.5962e-11 | 8.45e-13 | 4.91e-03 | 1.15e-06 |

Current image:

```
      92              101          1.4952e-11        1.01e-12        2.08e-03        1.45e-07
Current image:
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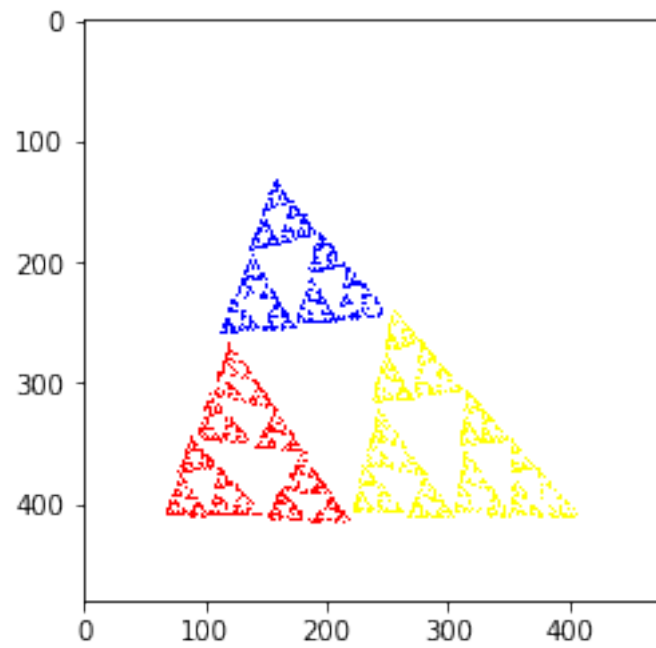


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      93              102          1.4208e-11        7.44e-13        4.90e-03        1.16e-06
Current image:
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94   103   1.3258e-11   9.50e-13   1.95e-03   1.22e-07
Current image:
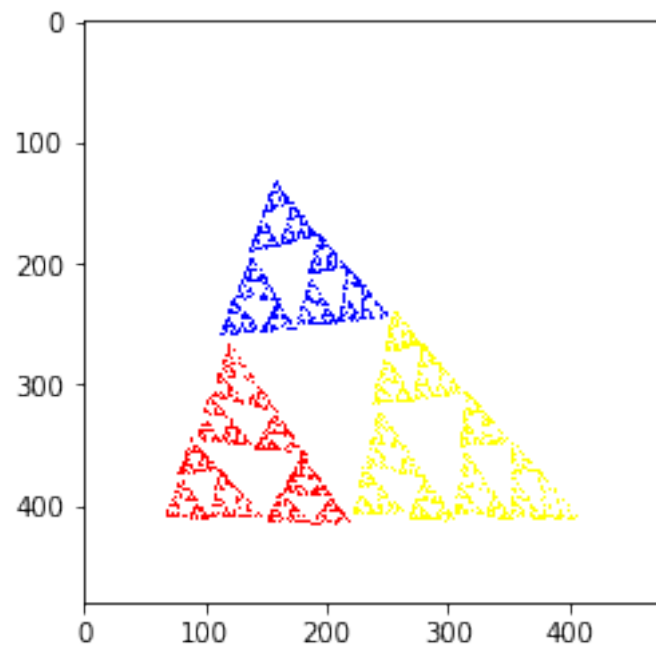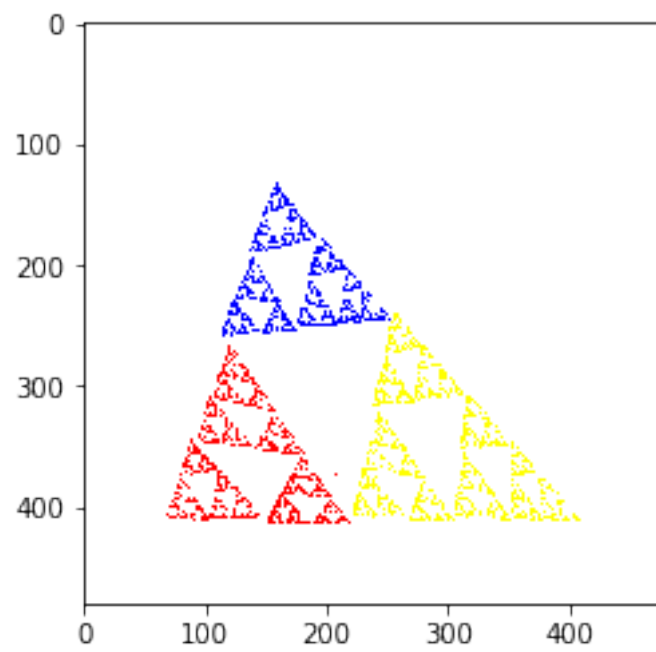
| 95 | 104 | 1.2610e-11 | 6.48e-13 | 4.89e-03 | 1.17e-06 |

Current image:
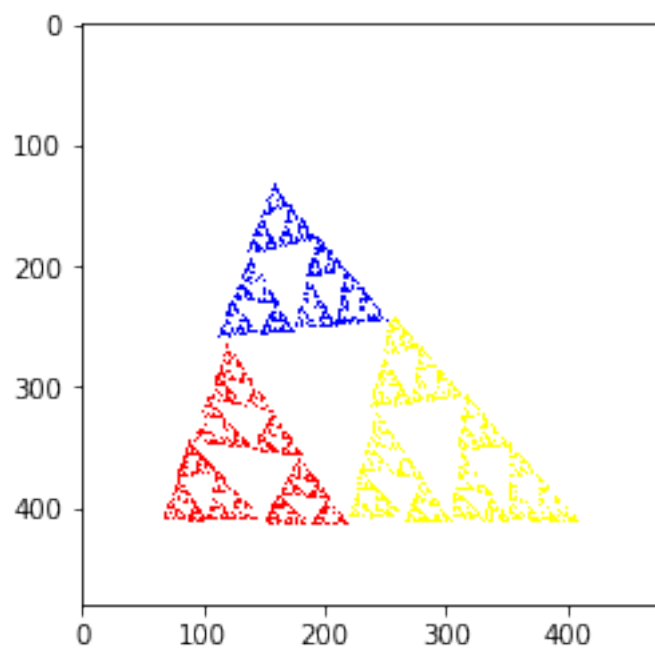
| 96 | 105 | 1.1714e-11 | 8.97e-13 | 1.82e-03 | 1.02e-07 |

Current image:
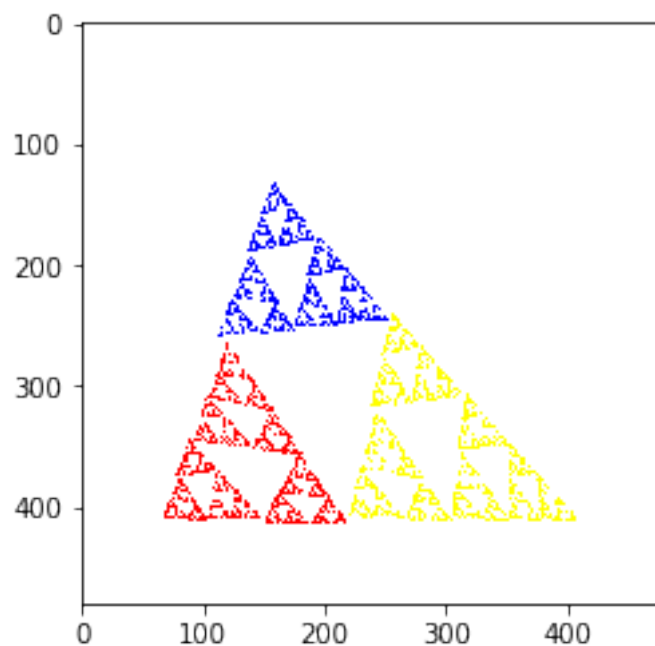
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    97              106         1.1157e-11      5.57e-13      4.88e-03      1.17e-06
Current image:
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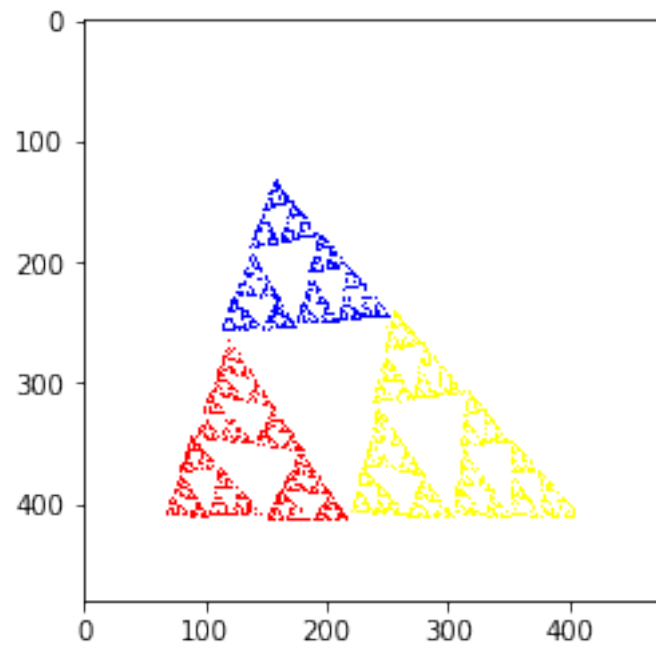


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    98              107         1.0308e-11      8.48e-13      1.69e-03      8.46e-08
Current image:
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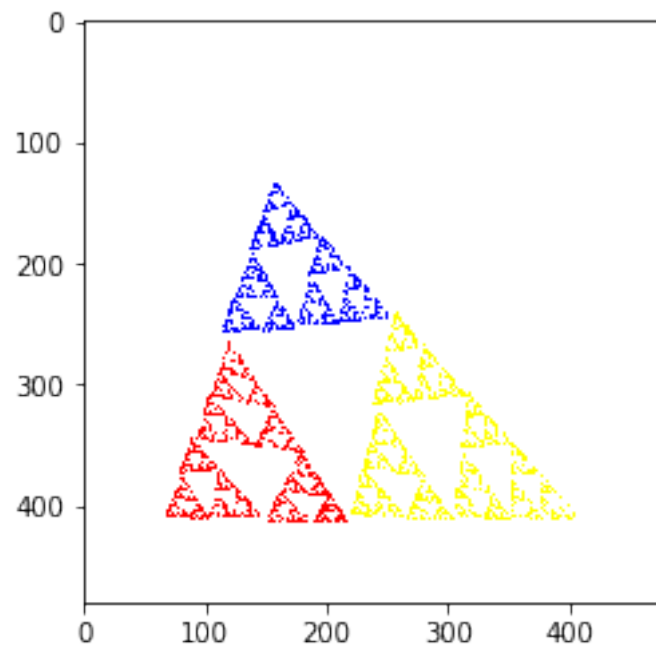
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        99              108         9.8372e-12        4.71e-13        4.87e-03        1.18e-06
Current image:
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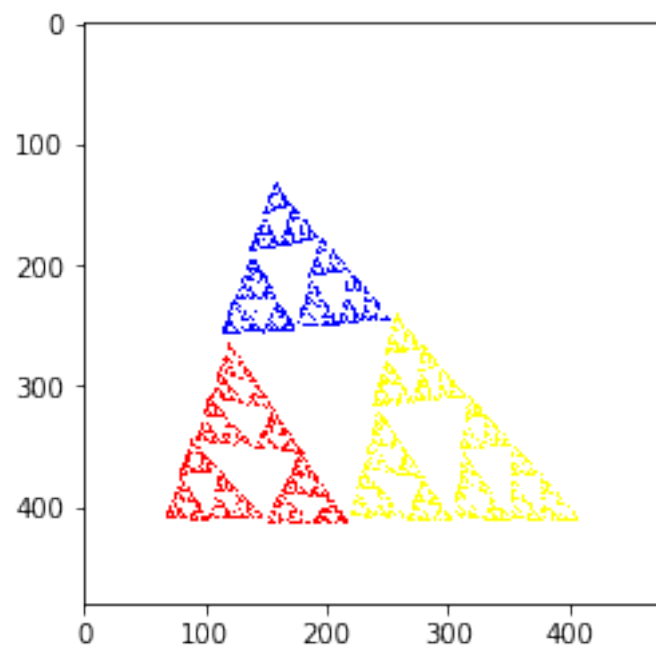
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    100            109        9.0321e-12      8.05e-13      1.57e-03      6.99e-08
Current image:
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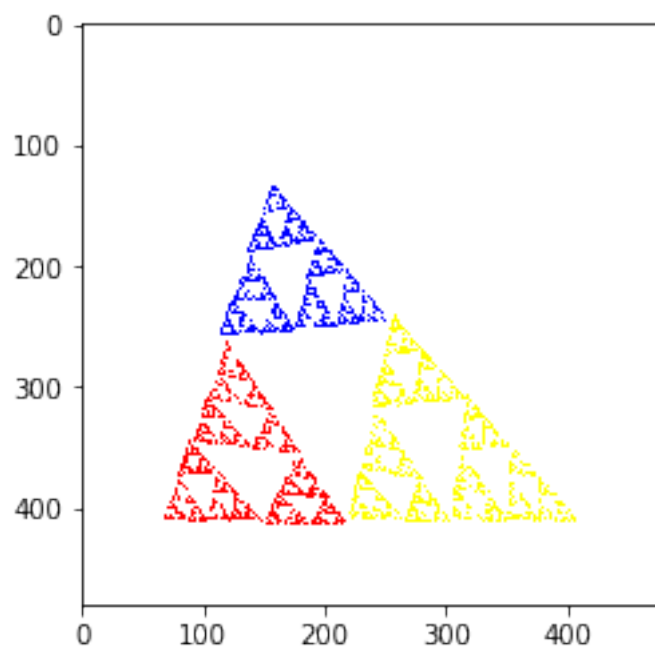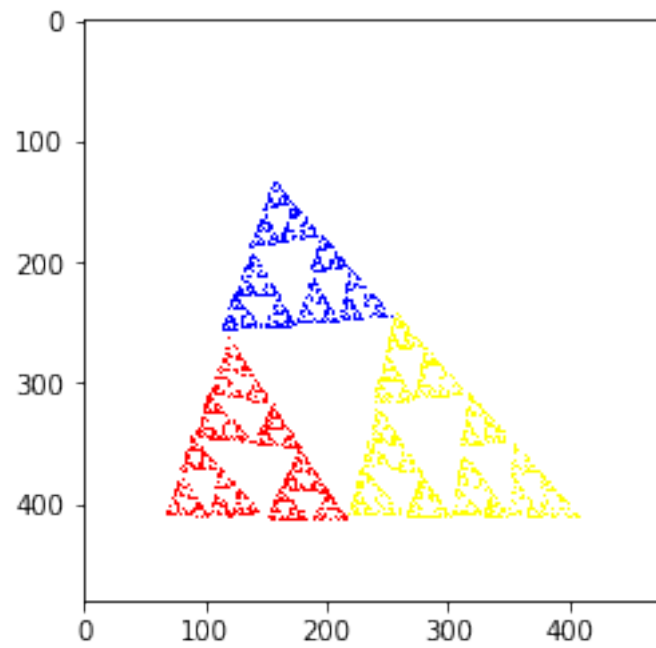


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    101            110        8.6420e-12      3.90e-13      4.86e-03      1.18e-06
Current image:
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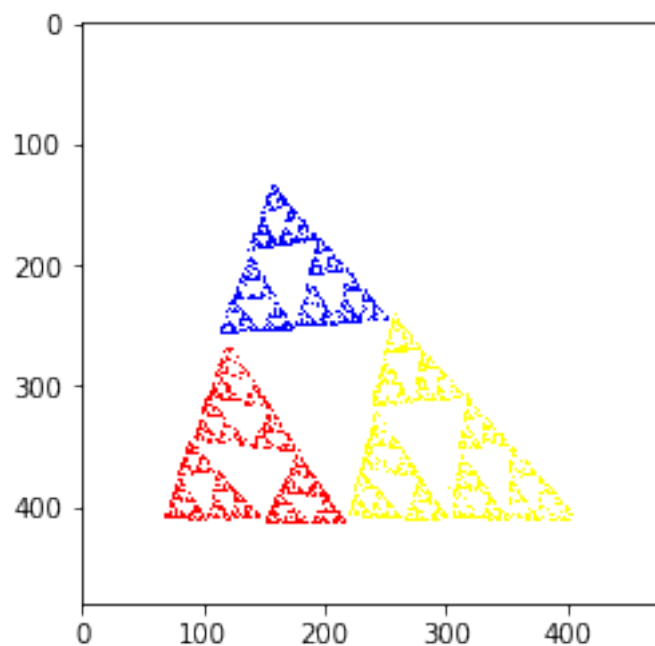
| 102 | 111 | 7.8759e-12 | 7.66e-13 | 1.46e-03 | 5.74e-08 |

Current image:

| 103 | 112 | 7.5621e-12 | 3.14e-13 | 4.85e-03 | 1.19e-06 |

Current image:
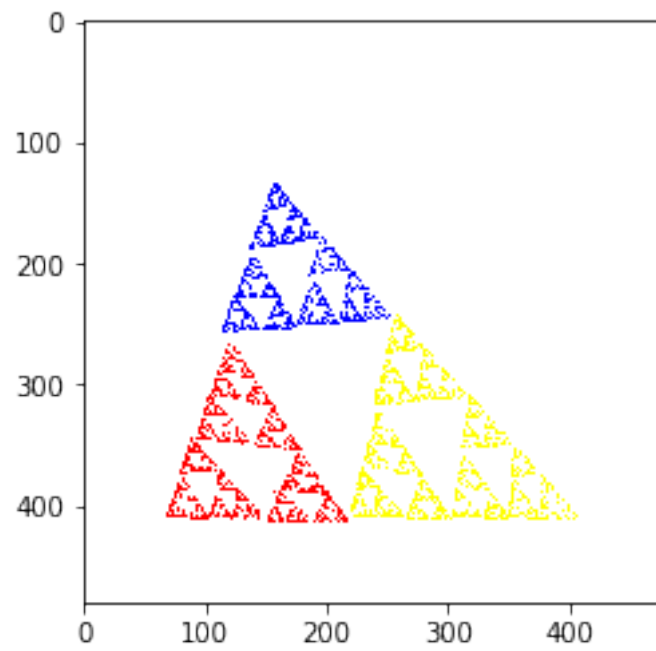
104   113   6.8308e-12   7.31e-13   1.35e-03   4.68e-08
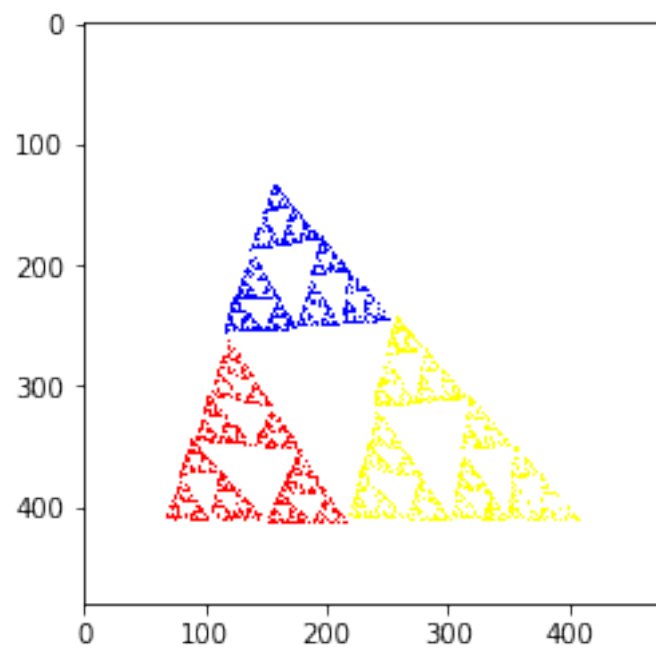
Current image:

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       105              114         6.4987e-12       3.32e-13       4.61e-03       1.04e-06
Current image:
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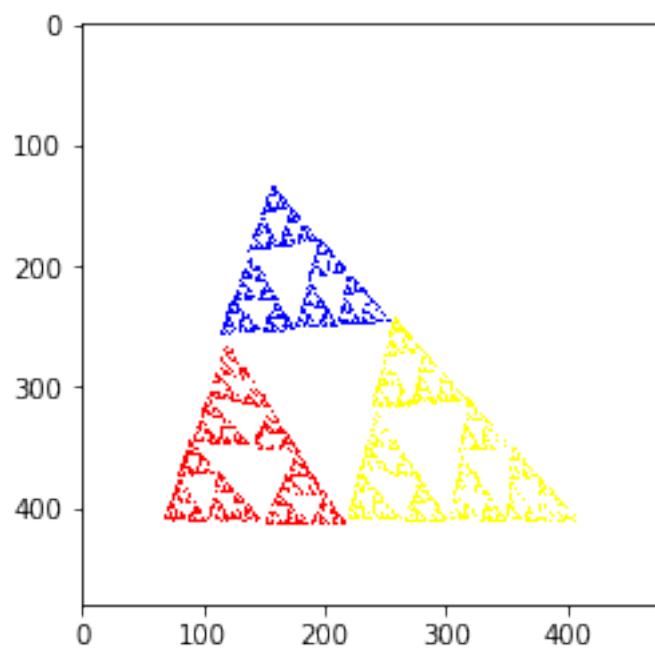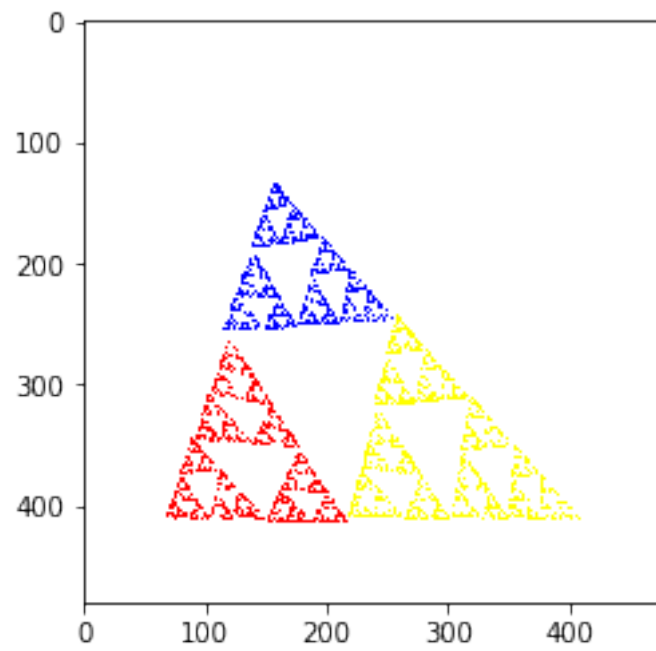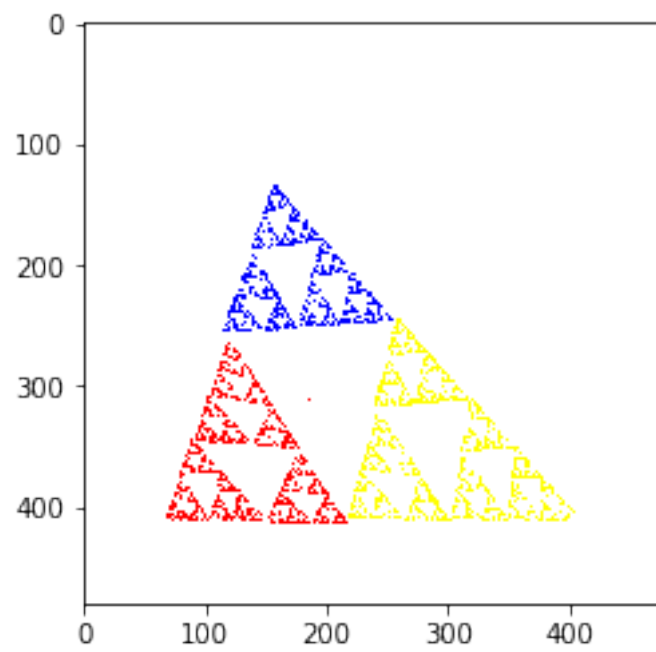


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       106              115         5.8943e-12       6.04e-13       1.43e-03       6.31e-08
       107              116         5.7223e-12       1.72e-13       4.84e-03       1.20e-06
Current image:
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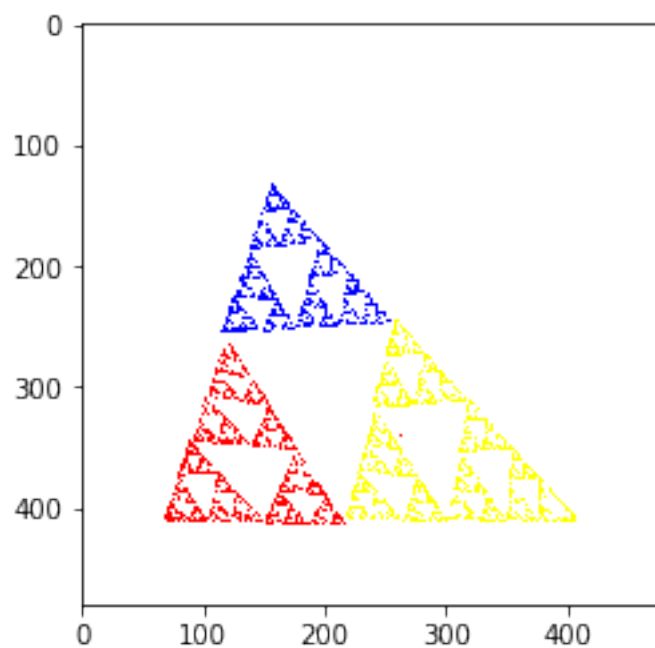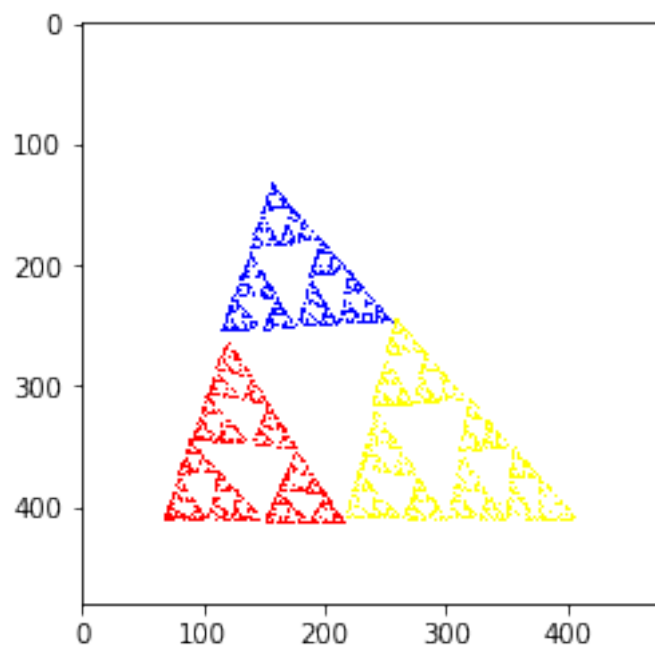
108	117	5.0486e-12	6.74e-13	1.10e-03	3.26e-08

Current image:

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     109            118         4.7559e-12      2.93e-13      2.32e-03      2.58e-07
Current image:
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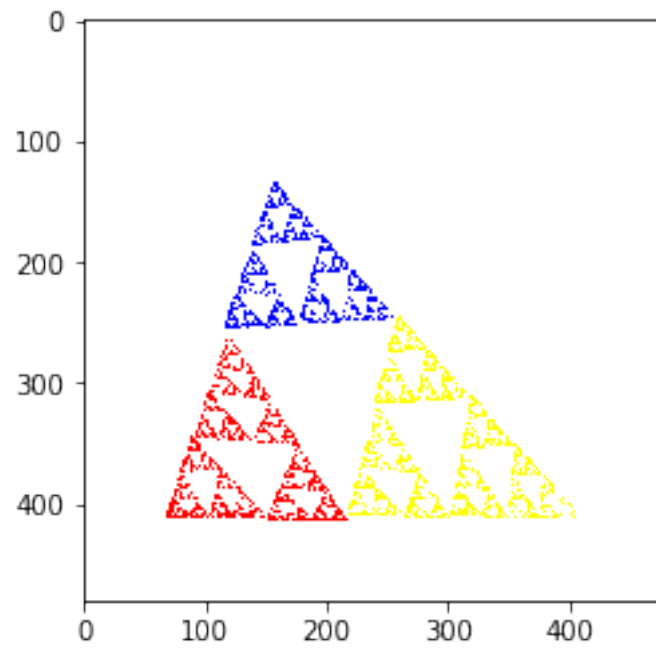


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     110            119         4.4309e-12      3.25e-13      4.15e-03      7.27e-07
Current image:
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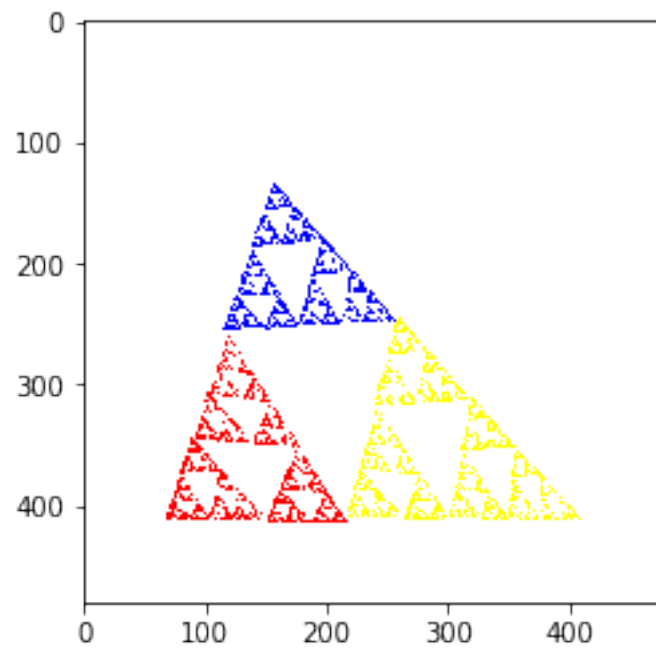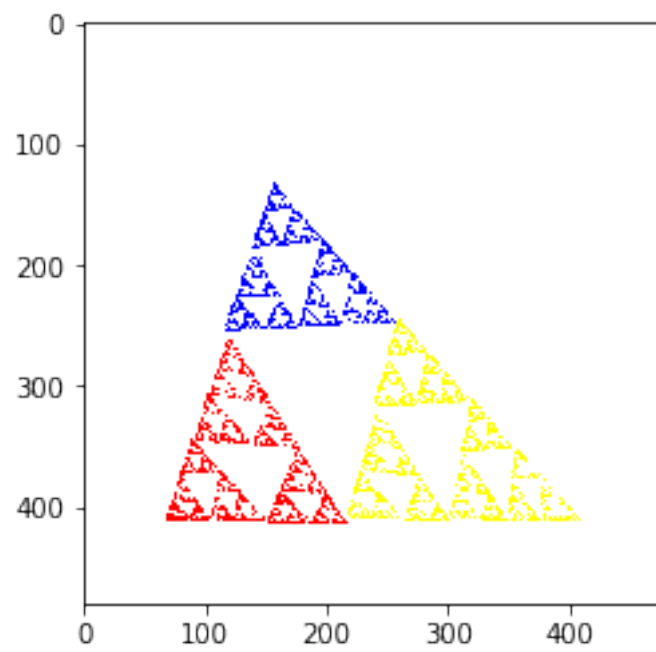
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        111              120           4.0498e-12      3.81e-13       1.40e-03       7.45e-08
Current image:
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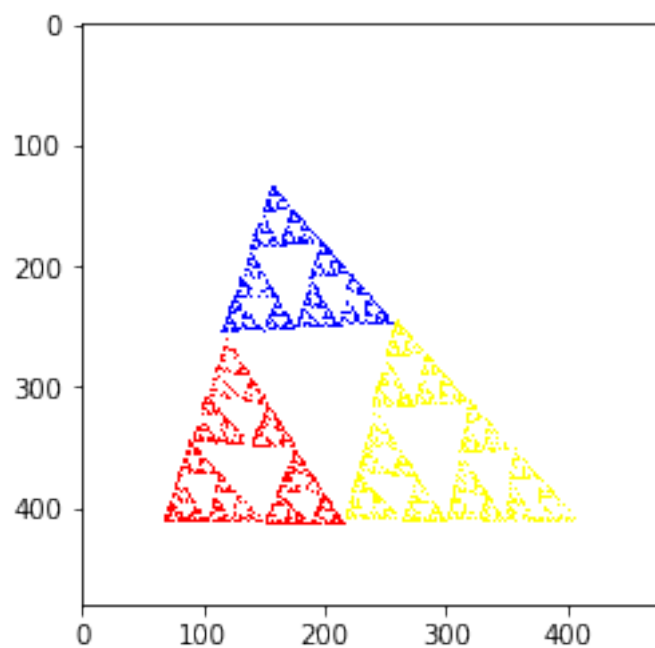


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        112              121           3.9488e-12      1.01e-13       4.61e-03       1.09e-06
Current image:
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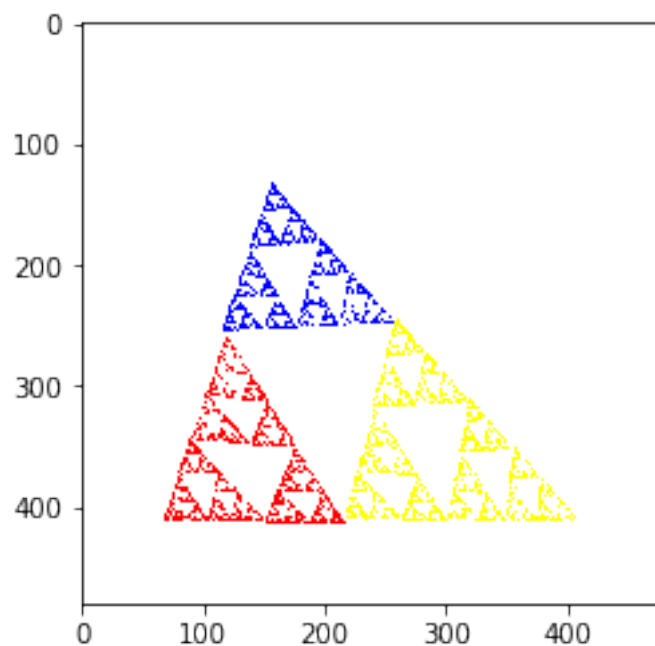
113   122   3.4128e-12   5.36e-13   9.04e-04   3.30e-08

Current image:

| 114 | 123 | 3.2803e-12 | 1.33e-13 | 1.22e-03 | 7.36e-08 |

Current image:

| 115 | 124 | 3.0609e-12 | 2.19e-13 | 2.42e-03 | 3.11e-07 |

Current image:

| 116 | 125 | 2.8062e-12 | 2.55e-13 | 2.62e-03 | 2.91e-07 |
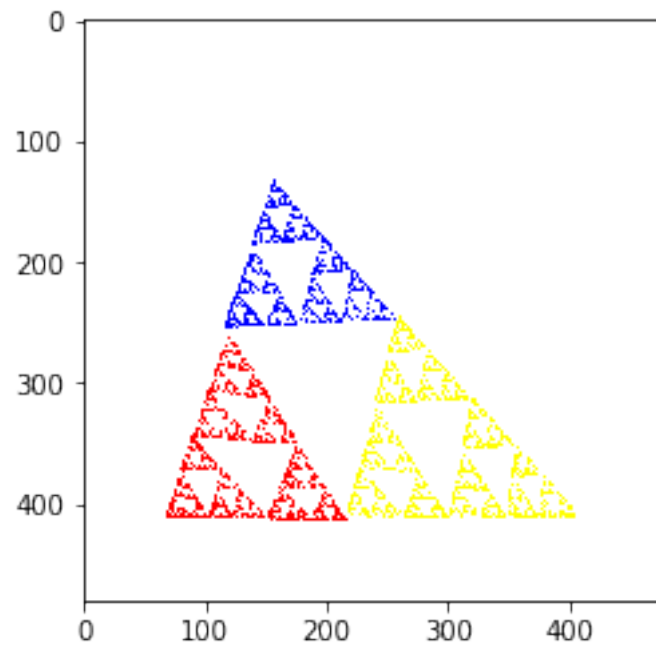
Current image:

| 117 | 126 | 2.5661e-12 | 2.40e-13 | 2.57e-03 | 3.01e-07 |
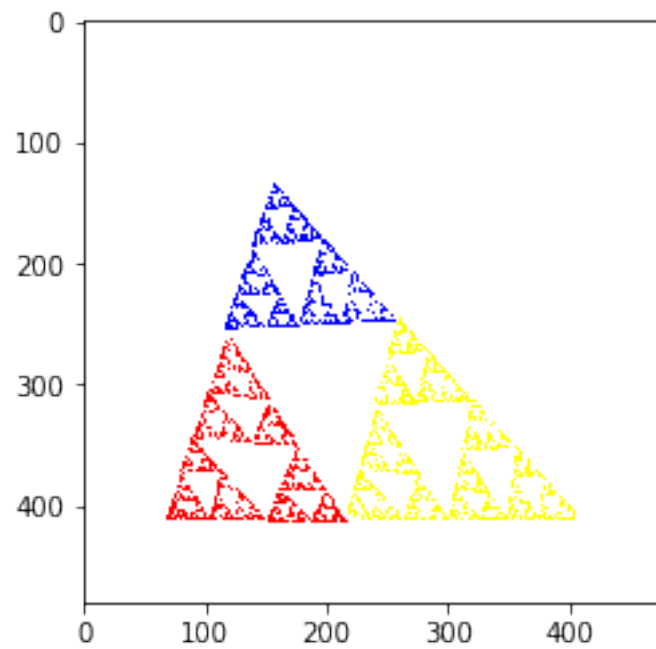
Current image:

| 118 | 127 | 2.3467e-12 | 2.19e-13 | 2.34e-03 | 2.62e-07 |
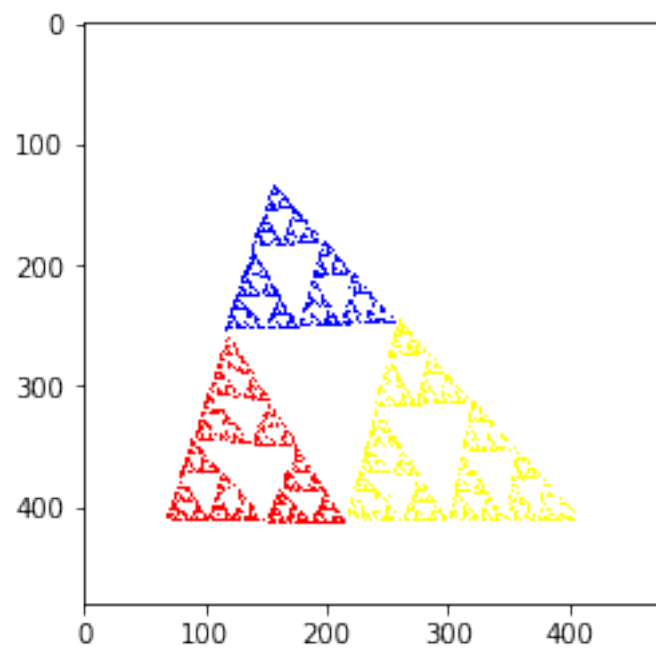
Current image:

| 119 | 128 | 2.1424e-12 | 2.04e-13 | 2.52e-03 | 3.15e-07 |

Current image:
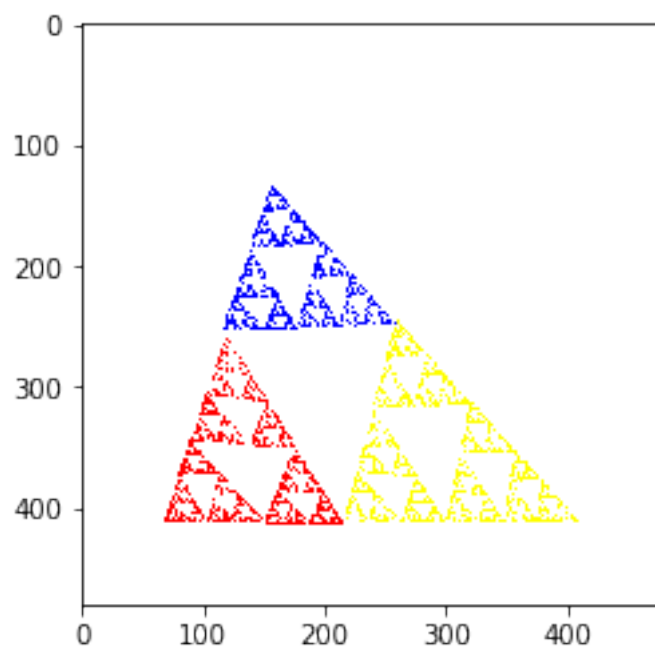
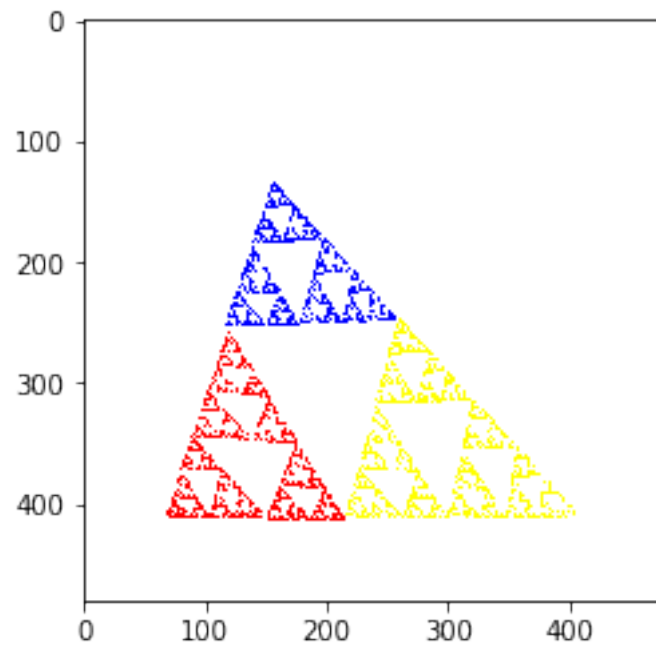| 120 | 129 | 1.9573e-12 | 1.85e-13 | 2.00e-03 | 1.98e-07 |

Current image:
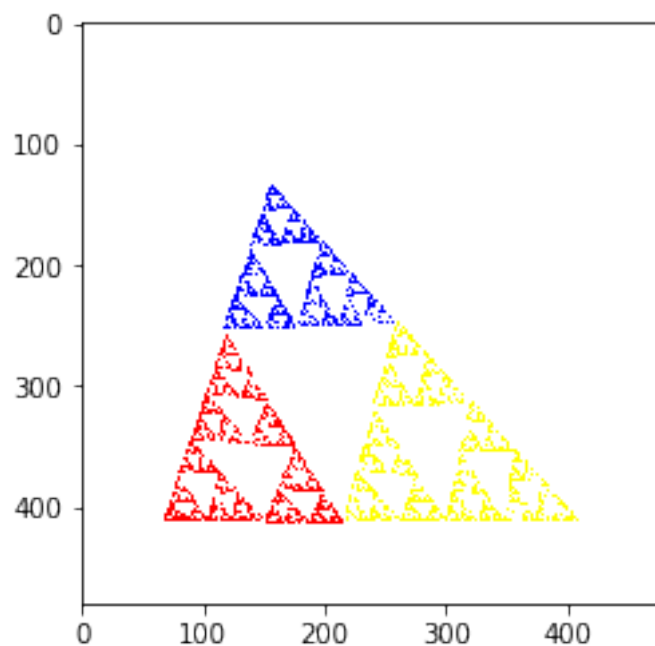
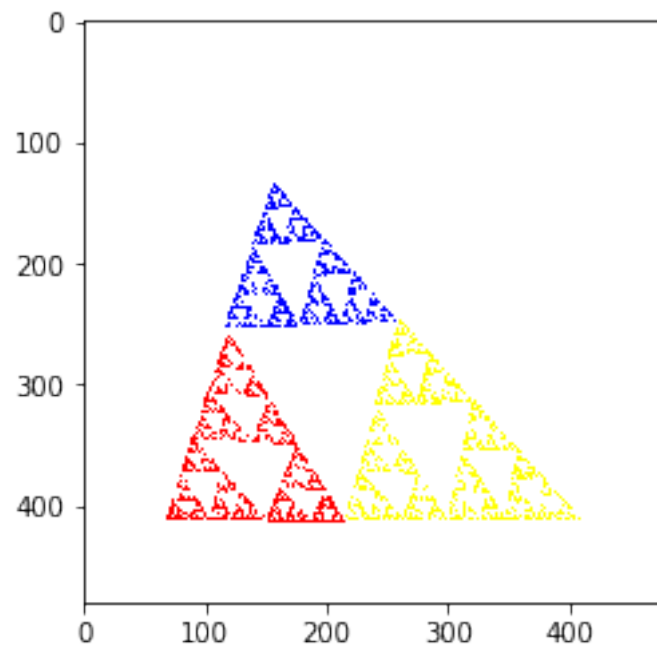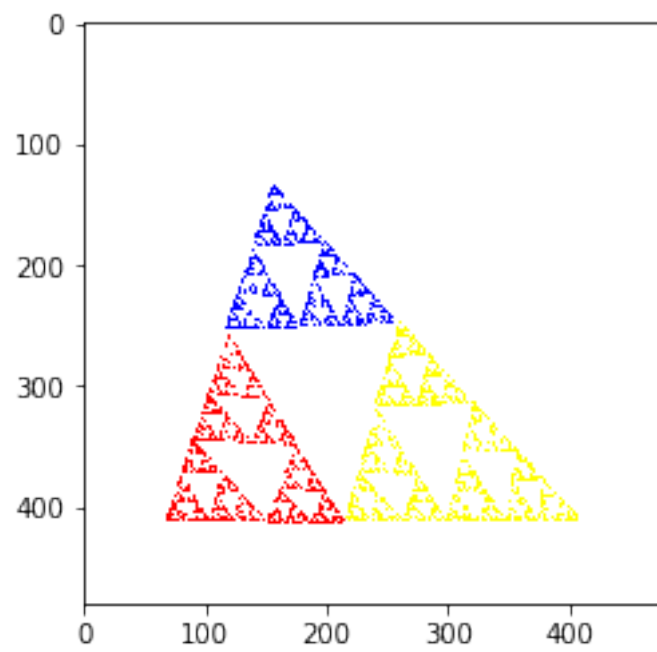121   130   1.7898e-12   1.68e-13   2.95e-03   4.43e-07
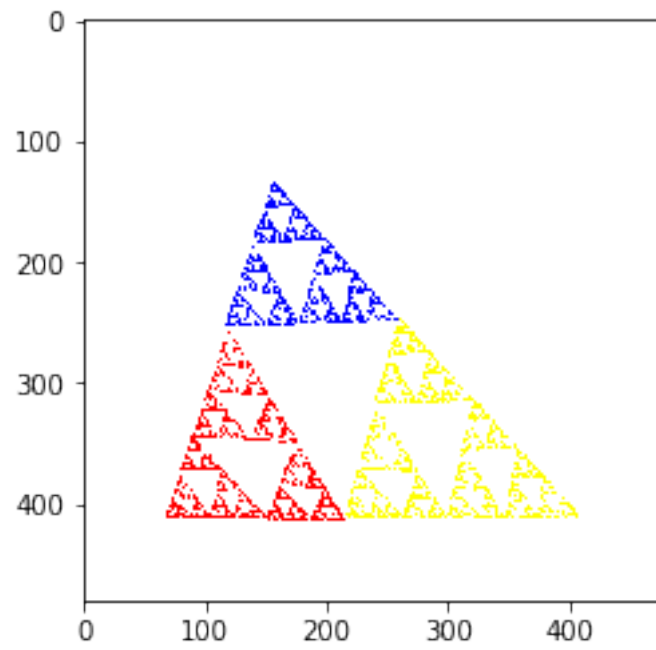Current image:

122   131   1.6253e-12   1.64e-13   1.27e-03   7.95e-08
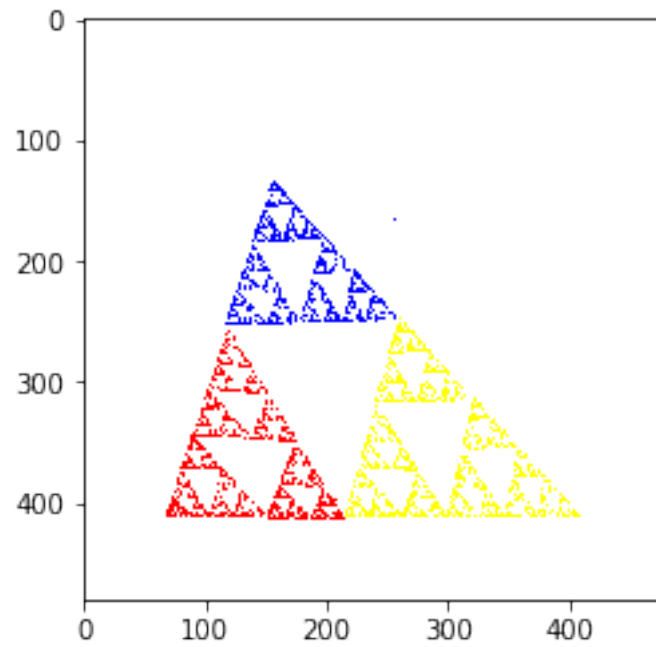Current image:

123   133   1.5374e-12   8.79e-14   1.22e-03   7.60e-08

Current image:

| 124 | 134 | 1.4088e-12 | 1.29e-13 | 2.42e-03 | 3.14e-07 |

Current image:

| 125 | 135 | 1.2775e-12 | 1.31e-13 | 1.61e-03 | 1.19e-07 |

Current image:

126	136	1.1596e-12	1.18e-13	2.40e-03	2.98e-07

Current image:
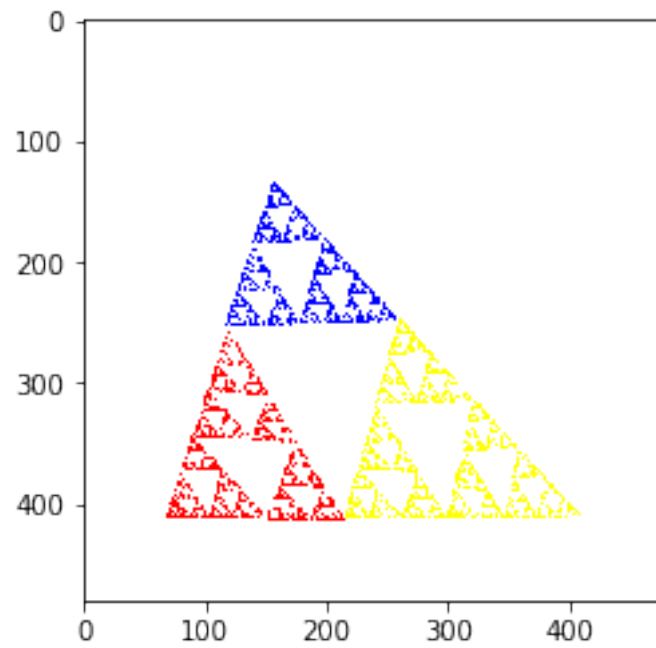
127	137	1.0463e-12	1.13e-13	1.50e-03	1.04e-07
128	138	9.4487e-13	1.01e-13	2.30e-03	2.71e-07

Current image:
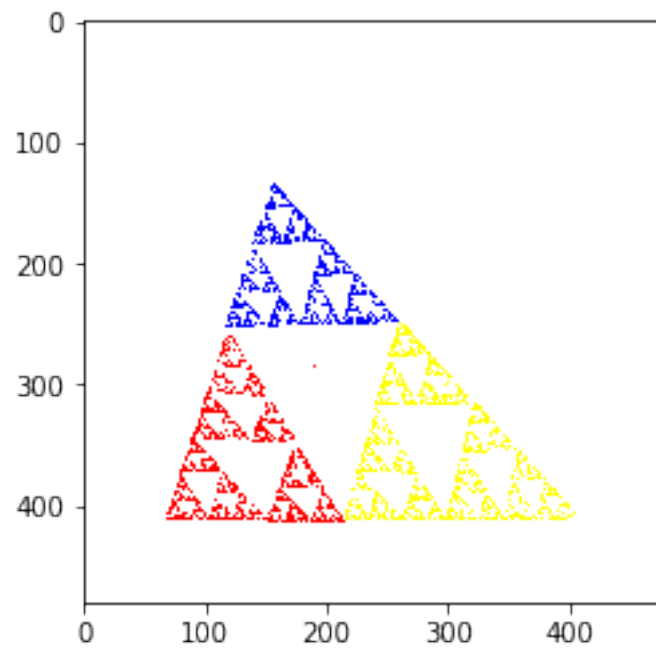
| 129 | 139 | 8.4808e-13 | 9.68e-14 | 1.46e-03 | 1.03e-07 |
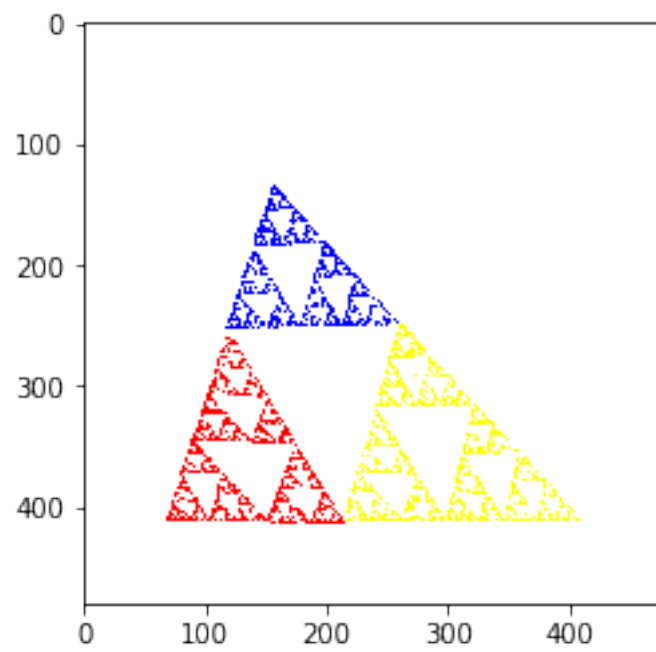
Current image:

| 130 | 140 | 7.6251e-13 | 8.56e-14 | 2.40e-03 | 2.99e-07 |
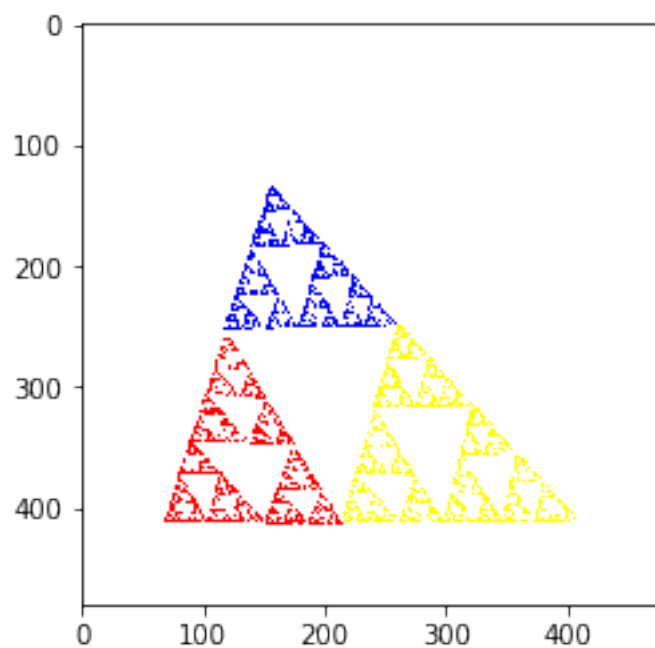
Current image:

| 131 | 141 | 6.7897e-13 | 8.35e-14 | 1.16e-03 | 6.42e-08 |
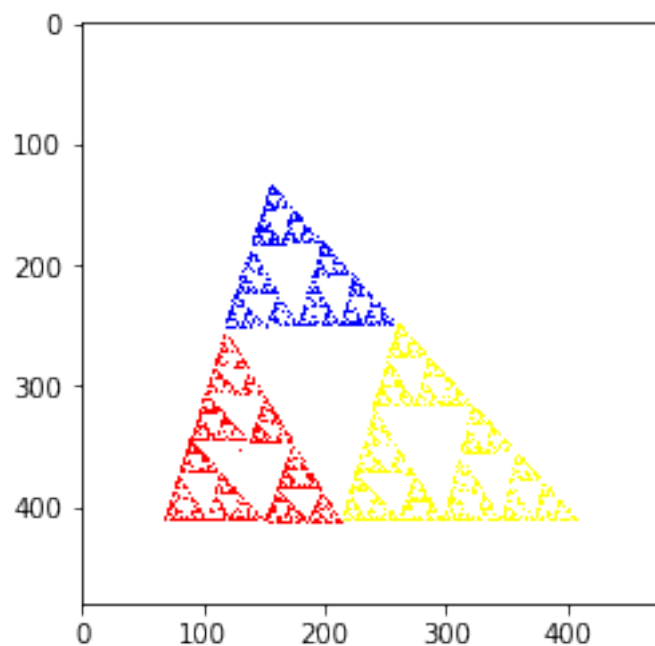
Current image:

```
      132            142         6.0792e-13      7.11e-14      2.30e-03      2.74e-07
Current image:
```
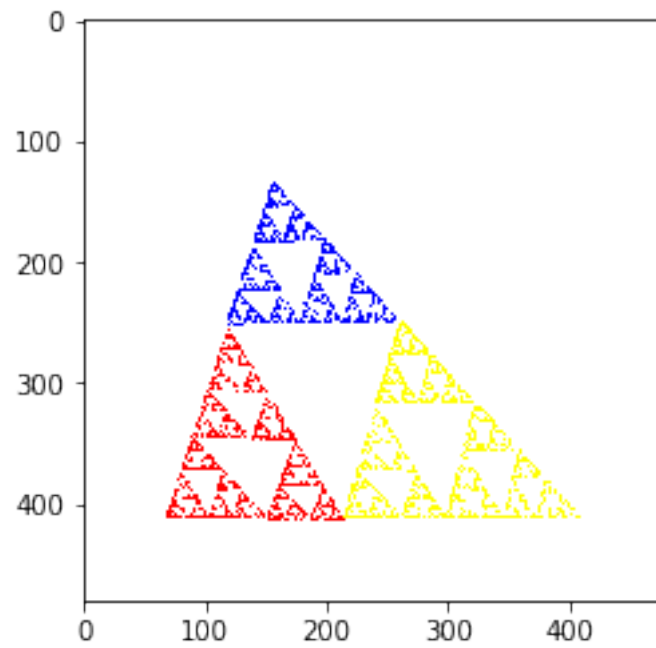


```
      133            143         5.3735e-13      7.06e-14      1.12e-03      6.06e-08
Current image:
```
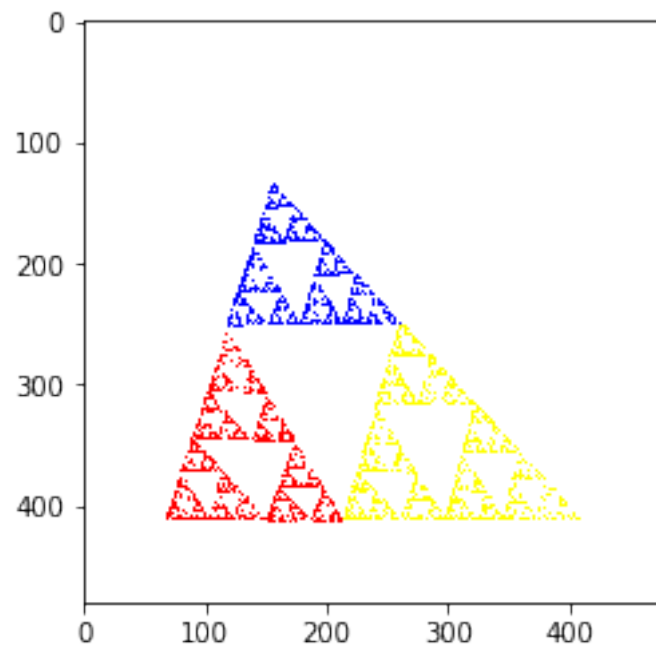
134   144   4.7868e-13   5.87e-14   2.30e-03   2.74e-07
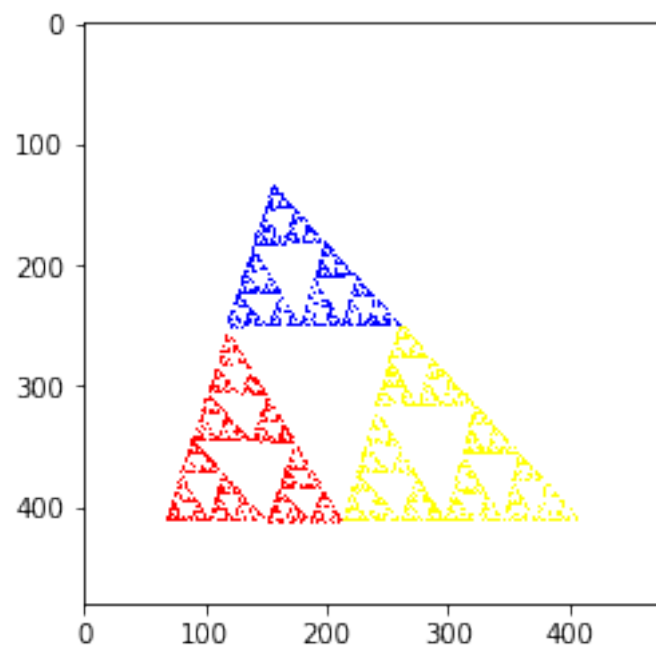
Current image:

```
   135              145           4.1805e-13        6.06e-14         9.79e-04         4.60e-08
Current image:
```
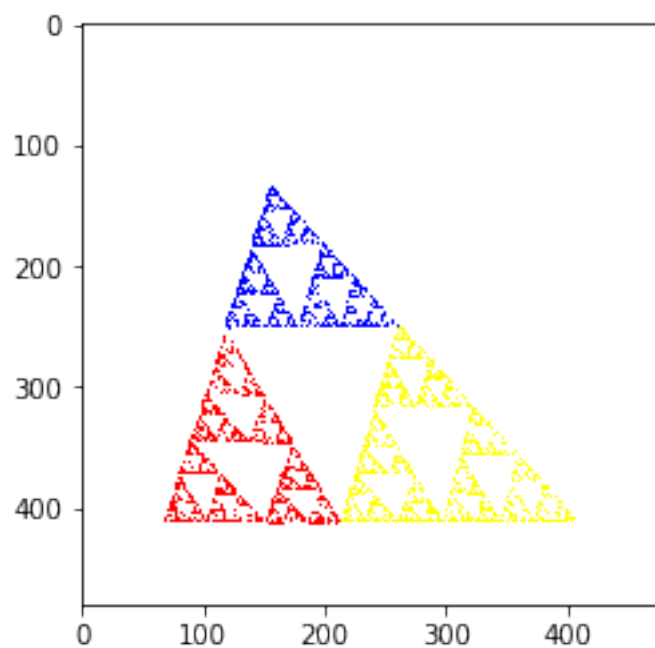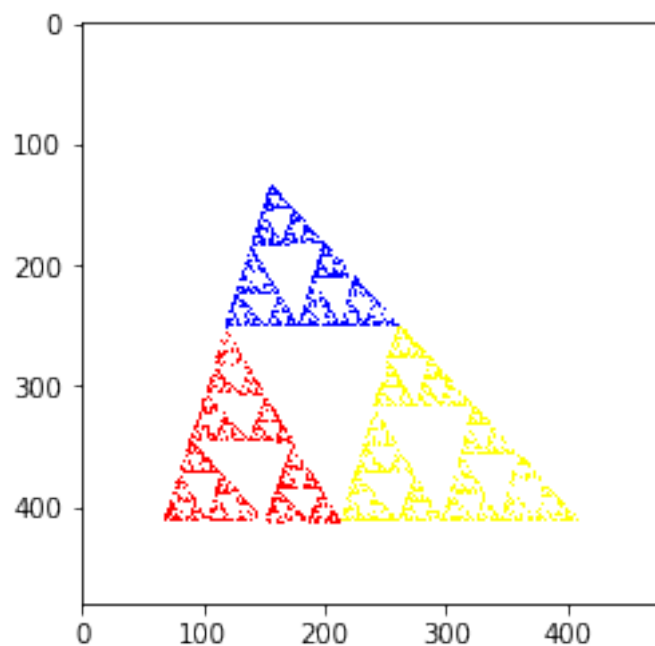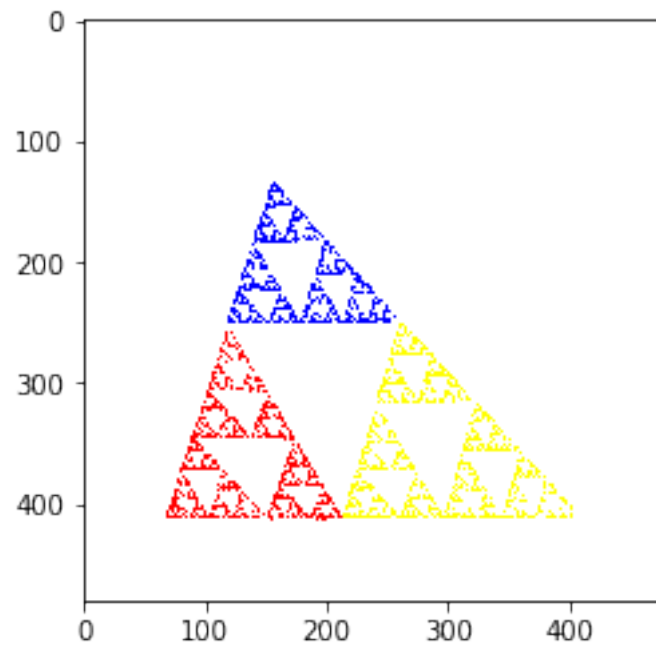


```
   136              146           3.7139e-13        4.67e-14         2.30e-03         2.74e-07
Current image:
```

```
   137              147          3.1875e-13        5.26e-14       8.45e-04       3.40e-08
Current image:
```



```
   138              148          2.8302e-13        3.57e-14       2.30e-03       2.74e-07
Current image:
```

139   149   2.3687e-13   4.62e-14   7.22e-04   2.46e-08

Current image:

140   150   2.1116e-13   2.57e-14   2.30e-03   2.74e-07
Current image:

141   151   1.7021e-13   4.09e-14   6.08e-04   1.72e-08
Current image:

```
       142              152         1.5376e-13      1.65e-14      2.30e-03      2.73e-07
Current image:
```
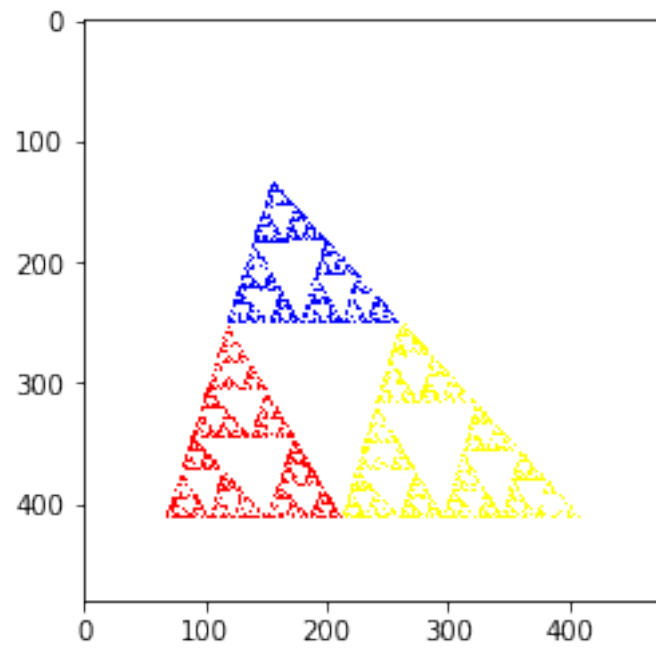


```
       143              153         1.1691e-13      3.68e-14      5.01e-04      1.15e-08
Current image:
```
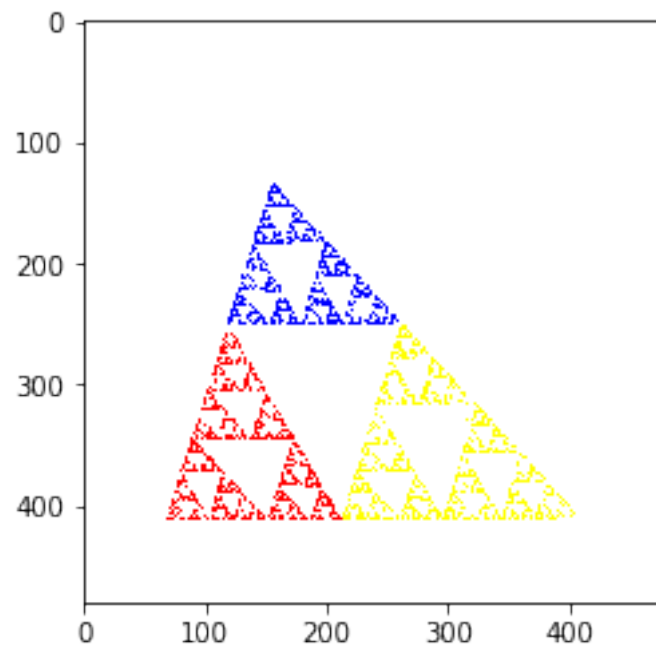
144   154   1.0902e-13   7.89e-15   2.30e-03   2.73e-07
Current image:
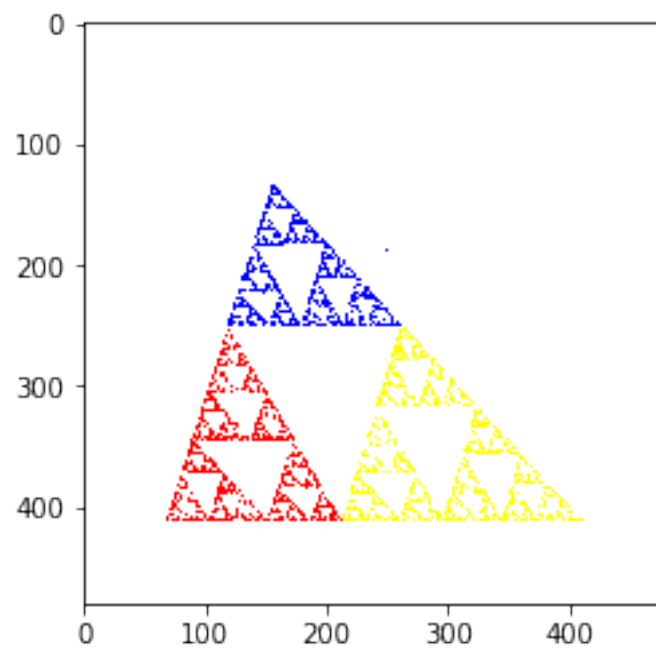
```
      145              155         7.5357e-14      3.37e-14      4.00e-04      7.22e-09
`gtol` termination condition is satisfied.
Function evaluations 155, initial cost 5.2846e-03, final cost 7.5357e-14, first-order optimalit
```

In [48]: img,_ = se.chaos(sol_Steem.x,p_l, it = 100000, imgxy=500)
         img

Out[48]: