Chapter_5

May 27, 2019

```
In [1]: from PIL import Image
        import matplotlib.pyplot as plt
        import random
        import numpy as np
        from scipy.optimize import newton
        #this file contains the moment functions made in Chapter 3
        import Moment_Function as mf
        %matplotlib inline
```

1 Generation of a Graph Directed IFS

This is a function to generate the IFS for a Generalised Sierpinski Triangle

```
In [2]: def GSP(a = 1.1, b = 0.85, par = 'FFF', imgxy = 480, it = 50000, IFS = False):

            #co ordinates of triangles third vertex
            Cy=(float)(0.5*np.sqrt(-1 + 2*a**2 - a**4 + 2*b**2 + 2*a**2*b**2 - b**4))
            Cx=(float)(0.5*(1 - a**2 + b**2))

            #angles for roatation
            cosA=(float)(Cx/b)
            sinA=(float)(Cy/b)
            cosB=(float)((1.0-Cx)/a)
            sinB=(float)(Cy/a)

            #view box
            xa = min(-0.2,Cx-0.2)
            xb = max(1.2,Cx+0.2)
            ya =-0.2
            yb = max(1.2,Cy+0.2)

            #starting values
            x=0.0
            y=0.0

            #image
            image = Image.new("RGB", (imgxy, imgxy),"white")
```

```python
if par == 'FFF':

    alphaFFF=(float)(-(-1 + a**2 - b**2))/(float)(2*b)
    betaFFF=(float)(-(-1 - a**2 + b**2))/(float)(2*a)
    gammaFFF=(float)(-(1 - a**2 - b**2))/(float)(2*a*b)

    detaFFF=alphaFFF*alphaFFF
    detbFFF=betaFFF*betaFFF
    detgFFF=gammaFFF*gammaFFF
    pnormFFF=(detaFFF+detbFFF+detgFFF)
    paFFF=(float)(detaFFF)/(float)(pnormFFF)
    pbFFF=(float)(detbFFF)/(float)(pnormFFF)
    pgFFF=(float)(detgFFF)/(float)(pnormFFF)

    FFF=[[cosA*alphaFFF,sinA*alphaFFF,sinA*alphaFFF,-cosA*alphaFFF,0.0,0.0,paFFF],

    mat=FFF

    def f(d) :
        return [alphaFFF**d[0]+betaFFF**d[0]+gammaFFF**d[0]-1]

    alpha = alphaFFF

if par == 'FFN':

    alphaNFF=(float)(b)/float((a**2 + b**2))
    betaNFF=(float)(a)/(float)(a**2 + b**2)
    gammaNFF=-1.0*(float)(1.0-a**2-b**2)/(float)(a**2 + b**2)

    detaNFF=alphaNFF*alphaNFF
    detbNFF=betaNFF*betaNFF
    detgNFF=gammaNFF*gammaNFF
    pnormNFF=(detaNFF+detbNFF+detgNFF)
    paNFF=(float)(detaNFF)/(float)(pnormNFF)
    pbNFF=(float)(detbNFF)/(float)(pnormNFF)
    pgNFF=(float)(detgNFF)/(float)(pnormNFF)

    NFF=[[cosA*alphaNFF,sinA*alphaNFF,sinA*alphaNFF,-cosA*alphaNFF,0.0,0.0,paNFF],

    mat=NFF

    def f(d) :
        return [alphaNFF**d[0]+betaNFF**d[0]+gammaNFF**d[0]-1]

    alpha = alphaFFN

if par == 'FNN':
```

```
        alphaNNF=(float)(b/(1.0 + b**2))
        betaNNF=(float)(1.0/(1.0 + b**2))
        gammaNNF=(float)(b**2/(1.0 + b**2))


        detaNNF=alphaNNF*alphaNNF
        detbNNF=betaNNF*betaNNF
        detgNNF=gammaNNF*gammaNNF
        pnormNNF=(detaNNF+detbNNF+detgNNF)
        paNNF=(float)(detaNNF)/(float)(pnormNNF)
        pbNNF=(float)(detbNNF)/(float)(pnormNNF)
        pgNNF=(float)(detgNNF)/(float)(pnormNNF)

        NNF=[[cosA*alphaNNF,sinA*alphaNNF,sinA*alphaNNF,-cosA*alphaNNF,0.0,0.0,paNNF],

        mat=NNF

        def f(d) :
            return [alphaNNF**d[0]+betaNNF**d[0]+gammaNNF**d[0]-1]

        alpha = alphaNNF

    if par == 'NNN':
        NNN=[[0.5,0.0,0.0,0.5,0.0,0.0,1/3],[0.5,0.0,0.0,0.5,0.5,0,1/3],[0.5,0.0,0.0,0.5

        mat=NNN

        def f(d) :
            return [(0.5)**d[0]+(0.5)**d[0]+(0.5)**d[0]-1]

        alpha = 1/2


    for k in range(it):

        p=random.random()
        if p <= mat[0][6]:
            i=0
        elif p <= mat[0][6] + mat[1][6]:
            i=1
        else:
            i=2

        x0 = x * mat[i][0] + y * mat[i][1] + mat[i][4]
        y  = x * mat[i][2] + y * mat[i][3] + mat[i][5]
        x = x0
        jx = int((x - xa) / (xb - xa) * (imgxy - 1))
```

```
            jy = (imgxy - 1) - int((y - ya) / (yb - ya) * (imgxy - 1))

            if i==2:
                image.putpixel((jx, jy), (255,0,0,255))
            elif i==1:
                image.putpixel((jx, jy), (255,255,0,255))
            elif i==0:
                image.putpixel((jx, jy), (0,0,255,255))

        if IFS:

            return image, mat ,alpha

        return image
```

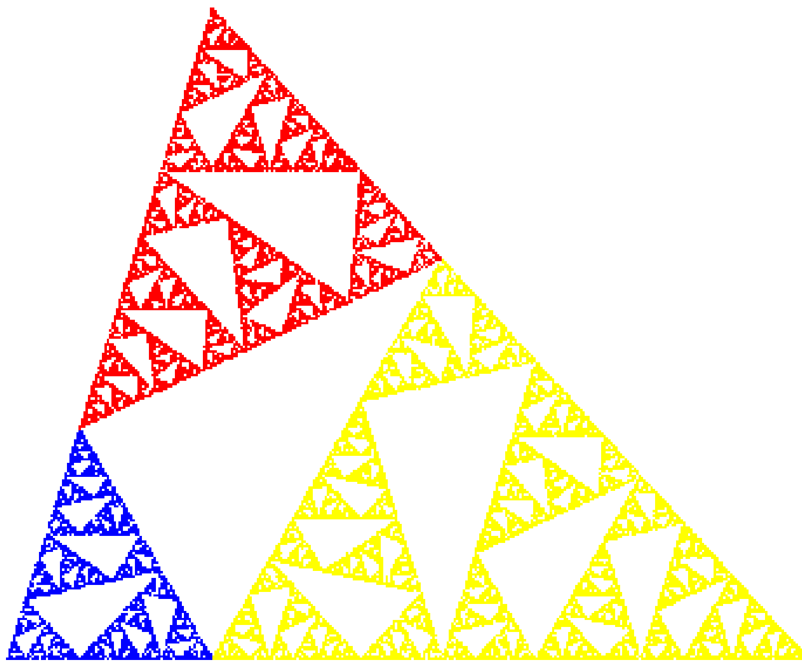
Make a Pedal triangle (FFF)

```
In [3]: img_p, IFS_p, alpha_p = GSP(par = 'FFF', IFS = True)
        print(IFS_p)
        print(alpha_p)
```

```
[[0.09088451557093417, 0.28744481279085693, 0.28744481279085693, -0.09088451557093417, 0.0, 0.0
0.301470588235294
```

```
In [4]: img_p
```

```
Out[4]:
```



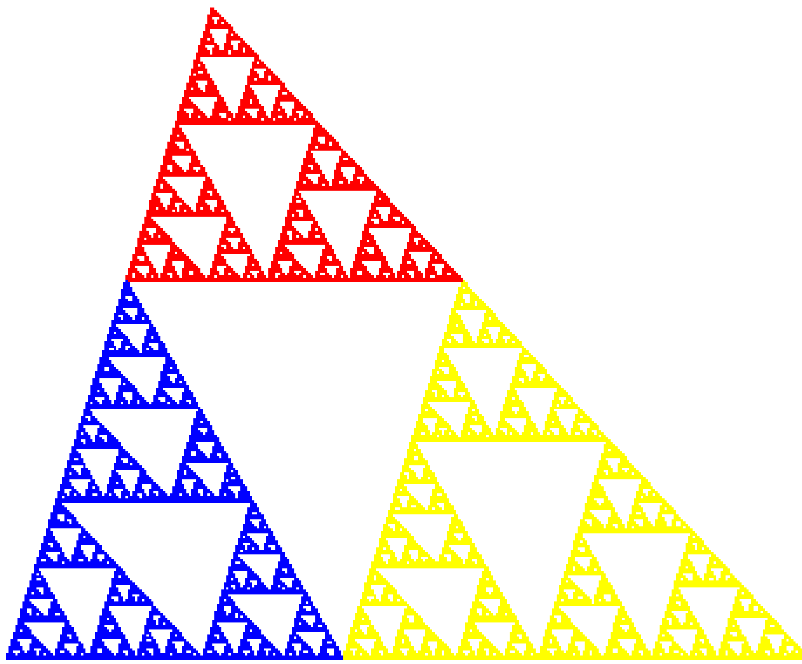
Make a Steemson triangle (FNN) with the same side lengths as above

In [5]:
```
img_s, IFS_s,alpha_s = GSP(par = 'FNN', IFS = True,it = 1000000)
print(IFS_s)
print(alpha_s)
```

```
[[0.14876632801161097, 0.4705103948263383, 0.4705103948263383, -0.14876632801161097, 0.0, 0.0,
0.4934687953555878
```

In [6]:
```
img_s
```

Out[6]:



Create a graph directed system by combining these two

In [7]:
```
f1 = np.hstack(((alpha_s/alpha_p)*np.array([IFS_p[0][:2] , IFS_p[0][2:4]]), np.array([
f2 = np.array([IFS_s[1][:2] + [IFS_s[1][4]], IFS_s[1][2:4] + [IFS_s[1][5]] ])
f3 = np.array([IFS_s[2][:2] + [IFS_s[2][4]], IFS_s[2][2:4] + [IFS_s[2][5]] ])

g1 = np.hstack(((alpha_p/alpha_s)*np.array([IFS_s[0][:2] , IFS_s[0][2:4]]), np.array([
g2 = np.array([IFS_p[1][:2] + [IFS_p[1][4]], IFS_p[1][2:4] + [IFS_p[1][5]] ])
g3 = np.array([IFS_p[2][:2] + [IFS_p[2][4]], IFS_p[2][2:4] + [IFS_p[2][5]] ])

lam_f1 = np.sqrt(abs(np.linalg.det(f1[:,:2])))
lam_f2 = np.sqrt(abs(np.linalg.det(f2[:,:2])))
lam_f3 = np.sqrt(abs(np.linalg.det(f3[:,:2])))
```

```
lam_g1 = np.sqrt(abs(np.linalg.det(g1[:,:2])))
lam_g2 = np.sqrt(abs(np.linalg.det(g2[:,:2])))
lam_g3 = np.sqrt(abs(np.linalg.det(g3[:,:2])))

f_dim = lambda D : lam_f1**D + lam_f2**D + lam_f3**D - 1

f_D = newton(f_dim, 1.5)

g_dim = lambda D : lam_g1**D + lam_g2**D + lam_g3**D - 1

g_D = newton(g_dim, 1.5)

pf1 = lam_f1**f_D
pf2 = lam_f2**f_D
pf3 = lam_f3**f_D

pg1 = lam_g1**g_D
pg2 = lam_g2**g_D
pg3 = lam_g3**g_D
```

Check that the dimension equation is correct.

```
In [8]: print(pf1+pf2+pf3)
        print(pg1+pg2+pg3)
```

```
1.0
1.0
```

```
In [9]: F = [f1,f2,f3]
        G = [g1,g2,g3]
```

The graph for this system can be coded as a row stochastic matrix

```
In [10]: P = np.array([[0,pf2,pf3,pf1,0,0],\
                       [0,pf2,pf3,pf1,0,0],\
                       [0,pf2,pf3,pf1,0,0],\
                       [pg1,0,0,0,pg2,pg3],\
                       [pg1,0,0,0,pg2,pg3],\
                       [pg1,0,0,0,pg2,pg3],\
                      ])
```

Make a version of the chaos game with place dependent probabilities

```
In [11]: def GraphIFS(F ,G ,P ,x0 = np.array([0,0]), it = 100000, imgxy=500, xa = -0.05, xb = 

              F_img = Image.new("RGB", (imgxy, imgxy),"white")
              G_img = Image.new("RGB", (imgxy, imgxy),"white")
              nf = len(F)
```

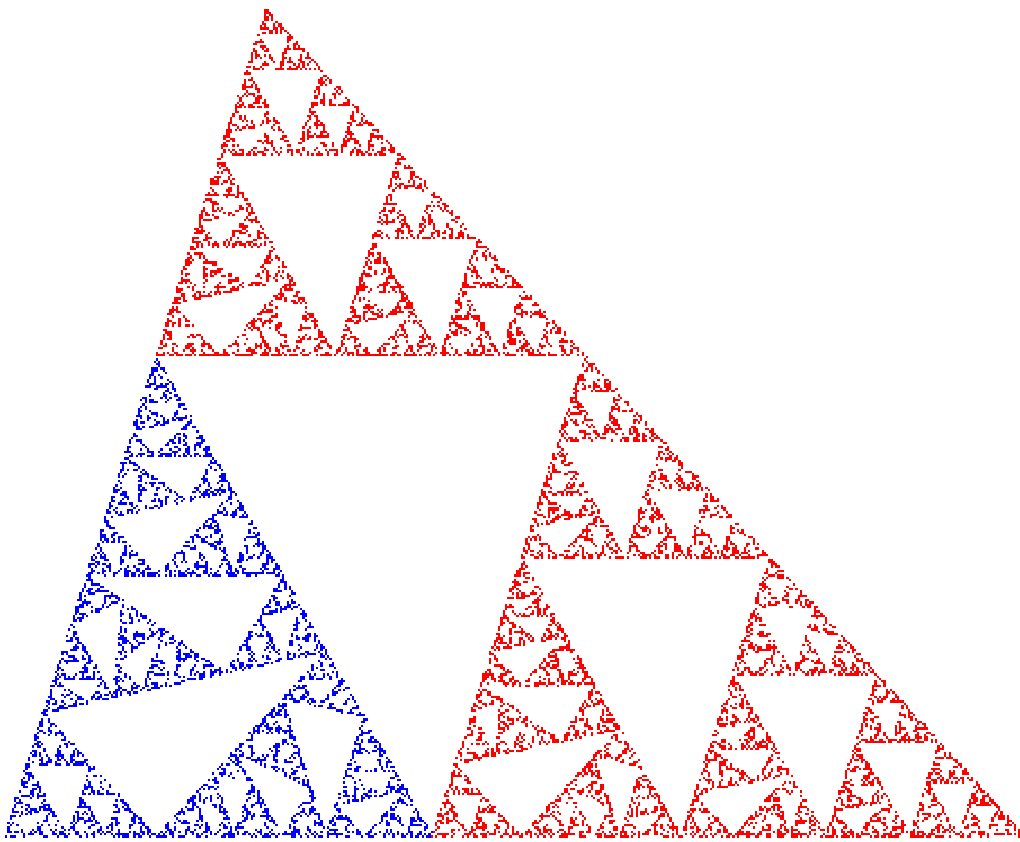
```
    ng = len(G)

    A = F[0][0:2,0:2]
    b = F[0][:,2]

    x = A @ x0 + b

    X1 = []
    X2 = []

    j = 0

    for i in range(it):

        p = P[j,:]
        j_old = j
        j = np.random.choice(nf+ng, p=p)

        if (j < nf):
            A = F[j][0:2,0:2]
            b = F[j][:,2]
            x = A @ x + b
            jx = int((x[0] - xa) / (xb - xa) * (imgxy - 1))
            jy = (imgxy - 1) - int((x[1] - ya) / (yb - ya) * (imgxy - 1))
            if (j_old < nf):
                F_img.putpixel((jx, jy), (255,0,0,255))
            else:
                F_img.putpixel((jx, jy), (0,0,255,255))

            X1.append(x)

        else:
            k = j - nf
            A = G[k][0:2,0:2]
            b = G[k][:,2]
            x = A @ x + b
            jx = int((x[0] - xa) / (xb - xa) * (imgxy - 1))
            jy = (imgxy - 1) - int((x[1] - ya) / (yb - ya) * (imgxy - 1))
            if (j_old < nf):
                G_img.putpixel((jx, jy), (255,0,0,255))
            else:
                G_img.putpixel((jx, jy), (0,0,255,255))

            X2.append(x)

    return F_img, G_img, np.array(X1), np.array(X2)
```

```
In [12]: F_img, G_img, A1, A2 = GraphIFS(F ,G ,P,it = 50000 ,imgxy = 500)
```

Make an image of the two metric spaces and their respective attractors

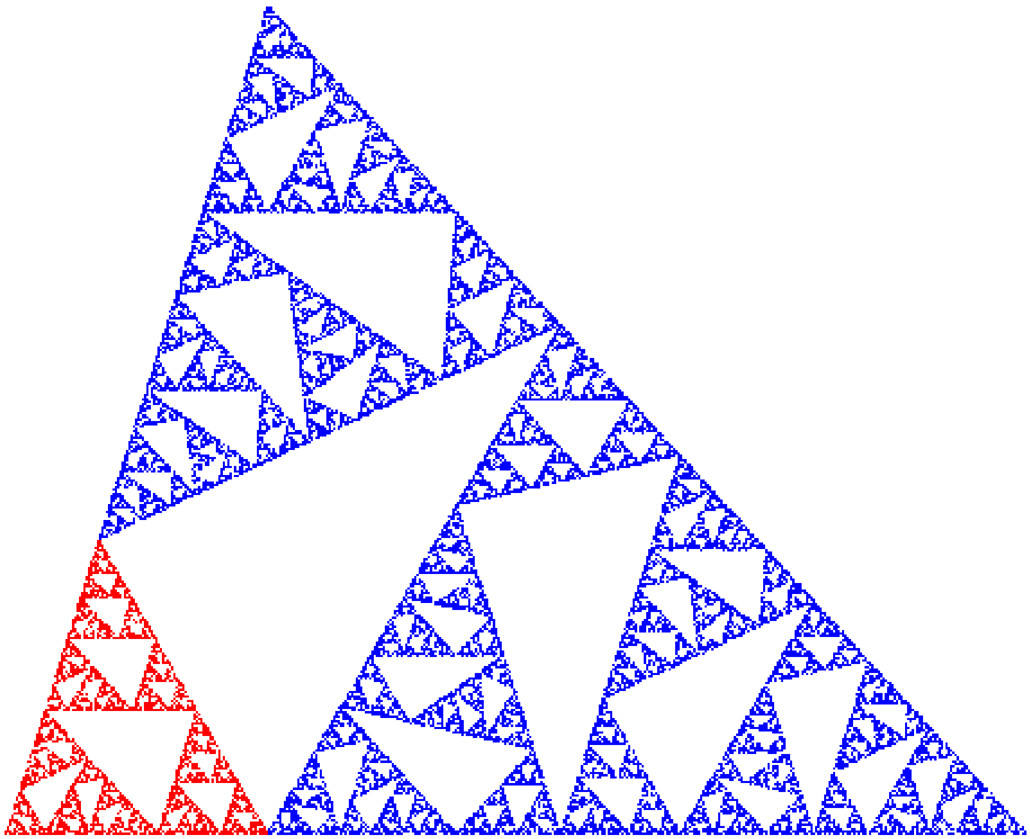
In [13]: F_img

Out[13]:



In [14]: G_img

Out[14]:



2 Moment Calculations

We aim to simply calculate the moments of the above vectorised system. There is a clear way to impliment this generally, we only present the outline to how this is done.

In [15]:
```
#number of moments
n = 6

#create stacked moment vector
M1 = np.zeros((n,1))
M1[0] = 1
M2 = np.zeros((n,1))
```

```
        M2[0] = 1
        M = np.vstack((M1,M2))
```

```
In [16]: M.T
```

```
Out[16]: array([[1., 0., 0., 0., 0., 0., 1., 0., 0., 0., 0., 0.]])
```

```
In [17]: def A(f,n=n):

             a_l,b_l,c_l,d_l,e_l,f_l = np.hstack((f[:2,:2].flatten(),f[:,2].flatten()))

             return mf.A_r(a_l,b_l,c_l,d_l,e_l,f_l,N = n)
```

Get probabilites

```
In [18]: p_1_1 = P[0,:3]
         p_1_2 = P[0,3:]
         p_2_1 = P[3,:3]
         p_2_2 = P[3,3:]
```

Define the individual moment matrices

```
In [19]: Phi_1_1 = p_1_1[0]*A(g1) + p_1_1[1]*A(f2) + p_1_1[2]*A(f3)
         Phi_1_2 = p_1_2[0]*A(f1) + p_1_2[1]*A(g2) + p_1_2[2]*A(g3)
         Phi_2_1 = p_2_1[0]*A(g1) + p_2_1[1]*A(f2) + p_2_1[2]*A(f3)
         Phi_2_2 = p_2_2[0]*A(f1) + p_2_2[1]*A(g2) + p_2_2[2]*A(g3)
```

```
In [20]: Phi_1 = np.hstack((Phi_1_1,Phi_1_2 ))
         Phi_2 = np.hstack((Phi_2_1,Phi_2_2 ))
```

Define the overall moment operator

```
In [21]: Phi = np.vstack((Phi_1,Phi_2))
```

Check iterative formula

```
In [22]: Mom = np.linalg.matrix_power(Phi,20)@M
```

Check with Eltons Theorem to see that our iterative formula works.

```
In [23]: print(mf.elt(A1,size = n))
         print(Mom[:n].flatten())

[1.         0.42700472 0.26243938 0.24228712 0.09667855 0.11652503]
[1.         0.42654595 0.26458401 0.24100212 0.09792307 0.11796502]
```

```
In [24]: print(mf.elt(A2,size = n))
         print(Mom[n:].flatten())

[1.         0.43306497 0.27073038 0.24496596 0.10231689 0.12116946]
[1.         0.43267849 0.27051691 0.24464886 0.10215002 0.12083241]
```