Polynomials and ideals

Notation -

$P^s_{\mathbb{R}}$ = Set of polynomials with coefficients in $\mathbb{R}$ in s variables.
$\quad\|$
$\mathbb{R}[x_1, \dots, x_s]$

Similarly, $P^s_{\mathbb{C}}$ or $\mathbb{C}[x_1, \dots, x_s]$ or more generally $k[x_1, \dots, x_s]$ where k is any field.

$k[x_1, \dots, x_d]$ is a ring (has $+$ and $\times$).

Ideals

Analogy with systems of linear equations.
Suppose we are solving -

$$(*) \quad \begin{array}{l} L_1(x_1, \dots, x_s) = 0 \\ \quad \vdots \\ L_r(x_1, \dots, x_s) = 0 \end{array} \qquad L_i \in P^s_1 \text{ (linear poly).}$$

Suppose $x = (x_1, \dots, x_s)$ is a solution.
Then $L(x) = 0$ for all linear combinations

$$L = a_1 L_1 + \dots + a_r L_r.$$

Consider $\Lambda = \{ a_1 L_1 + \dots + a_r L_r \mid a_i \in k \}$
$\Lambda \subset P^s_1$ a subspace.

Then zeros of $(*)$ = Zeros of Λ
$= \{ x \mid L(x) = 0 \ \forall L \in \Lambda \}$.

Polynomial equations

$$(\circledast)\begin{cases} P_1(x_1, \dots, x_s) = 0 \\ \vdots \\ P_r(x_1, \dots, x_s) = 0 \end{cases} \qquad \text{e.g.} \quad \begin{aligned} x^2 + x + y^3 &= 0 \\ 3x - 2y &= 0 \end{aligned}$$

Suppose x is a solution.
Then $P(x) = 0$ for all polynomial linear combinations

$$P = a_1 P_1 + \dots + a_r P_r$$

where $a_1, \dots, a_r \in k[x_1, \dots, x_s]$.

So, from $\circledast$, we get

$$I = \{ a_1 P_1 + \dots + a_r P_r \mid a_i \in k[x_1, \dots, x_s] \}$$

$I \subset k[x_1, \dots, x_s]$.

- a subspace (closed under $+$ & scalar mult)
- more (closed under $+$ & polynomial mult).

Def: An ideal of $k[x_1, \dots, x_s]$ is a subset I such that

$$f, g \in I \Rightarrow f + g \in I.$$
$$f \in I \Rightarrow af \in I \quad \forall\, a \in k[x_1, \dots, x_s]$$

Remark: If $P_1, \dots, P_s$ are polynomials then

$$I = \{ \sum a_i P_i \mid a_i \in k[x_1, \dots, x_s] \}$$

is an ideal, called ideal generated by $P_1, \dots, P_s$, denoted by $\langle P_1, \dots, P_s \rangle$

(Think :- Subspace spanned by a set of vectors)

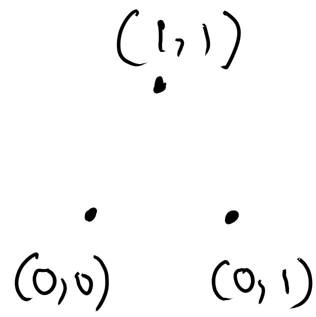
Ideals from geometry :

Let $Z \subset K^s$ be a subset.

Consider
$$I(Z) = \{ p \in k[X_1, \dots, X_s] \mid p(z) = 0 \text{ for all } z \in Z \}.$$

Then $I(Z)$ is an ideal.

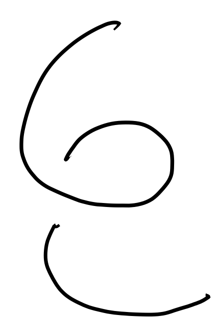
Example:

① $Z =$ 

$\subset \mathbb{R}^2.$

$$I(Z) = \langle y^2 - y,\ xy - x,\ x^2 - x \rangle$$

② $Z = \{ (1, t, t^2) \subset \mathbb{R}^3 \mid t \in \mathbb{R} \}.$

$I(Z) = \langle z - y^2,\ x - 1 \rangle$  $\subset \mathbb{R}^3.$

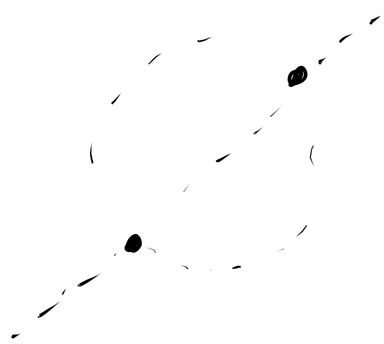
$$\text{Subsets of } k^s \longrightarrow \text{Ideals of } k[X_1, \dots, X_s]$$
$$Z \longmapsto I(Z)$$

s

$$\text{Ideals of } k[X_1, \dots, X_s] \longrightarrow \text{Subsets of } k^s$$
$$I \longmapsto V(I) = \{ x \in k^s \mid f(x) = 0 \ \forall f \in I \}$$

Example.

$$I = \{x^2+y^2-1,\ x-y\}$$

$$V(I) =$$  $$\subset \mathbb{R}^2.$$

$$\begin{array}{ccc} \text{Algebraic Subsets of } K^s & \longrightarrow & \text{Radical Ideals of } k[x_1, \dots, x_s] \\ & \longleftarrow & \end{array}$$

$k = \mathbb{C}$

Fundamental thm (Nullstellensatz) - the above is a one-one correspondance if we are careful.

① On the right, only consider radical ideals.

I is radical if $f^n \in I \Rightarrow f \in I$.

② On the left, only consider algebraic subsets. (subsets cut out by polynomial equations)

③ Take $k = \mathbb{C}$ (or any algebraically closed field).

Familiar from linear algebra -

$$\begin{array}{ccc} \text{Subset of } K^s & \longrightarrow & \text{Subspaces of } P_1^s \\ \text{(Subspace)} & \longleftarrow & \end{array}$$

Caution: Not true with $k = \mathbb{R}$.