1 Empirical Polynomials

Before defining empirical polynomials, we must introduce what Stetter means by empirical data and vectors, as these lead us to the definition of empirical polynomials.

Definition 1.1. Empirical data $(\bar{\alpha}, \varepsilon)$ consists of specified value $\bar{\alpha} \in \mathbb{R}/\mathbb{C}$ and a tolerance $\varepsilon \in \mathbb{R}_+$, which indicate the range of potential values for the quantity as follows: ε provides the order of magnitude of the indetermination of $\bar{\alpha}$. Intuitively, this means that any $\tilde{\alpha}$ such that $|\tilde{\alpha} - \bar{\alpha}| \leq \varepsilon$ is a valid instance of $(\bar{\alpha}, \varepsilon)$.

Given an empirical data quantity, we have an associated family of neighbourhoods

$$N_\delta(\bar{\alpha}, \varepsilon) = \{\tilde{\alpha} \mid |\widetilde{alpha} - \bar{\alpha}| \leq \delta\varepsilon\},\ \delta \in \mathbb{R}_{0,+}$$

and if $\bar{\alpha}$ is in $\mathbb{R}$, we must specify if $\tilde{\alpha}$ is only in $\mathbb{R}$, or whether we allow $\tilde{\alpha} \in \mathbb{C}$.

To generalise this to vectors in $\mathbb{K}^m$, we first introduce a specific norm. Given $\varepsilon = (\varepsilon_1, \dots, \varepsilon_m)$, $\varepsilon_j > 0$ for all j, we define the following norm on $\mathbb{K}^m$

$$\|\tilde{\alpha} - \bar{\alpha}\|_\varepsilon^* = \left\|\left(\dots, \frac{|\tilde{\alpha}_j - \bar{\alpha}_j|}{\varepsilon_j}, \dots\right)\right\|^*$$

, with $\|\cdot\|^*$ the dual norm on $(\mathbb{K}^m)^*$.

From this we obtain empirical polynomials. An empircal polynomial $(\bar{p}, \varepsilon)$ consists of a polynomial $\bar{p} \in \mathcal{P}^s$, $s \geq 1$, and a tolerance vector $\varepsilon \in \mathbb{K}^M$, defined as follows. Let

$$\emptyset \neq \tilde{J} := \{j \in J \mid \bar{\alpha}_j \text{ is an empirical coefficient of } (\bar{p}, \varepsilon)\} \subset J$$

be the empirical support of $(\bar{p}, \varepsilon)$, with $|\tilde{J}| = M$. Then the components ε_j refer only subscripts in the empirical support $\tilde{J}$:

$$\varepsilon = (\varepsilon_j > 0 \mid j \in \tilde{J}) \in \mathbb{R}_+^m.$$

Remark 1.1. Given a $\delta = O(1)$, the $\tilde{p} \in N_\delta(\bar{p}, \varepsilon)$ are valid instances of the empirical polynomial.

Now, given an empirical polynomial $(\bar{p}, \varepsilon)$, we can ask what its set of zeroes is. Continuing with the notion of neighbourhoods of polynomials which are valid instances of a given empirical polynomial, we define the δ-pseudozero set of $(\bar{p}, \varepsilon)$ as

$$Z_\delta(\bar{p}, \varepsilon) := \{\zeta \in \mathbb{K} \mid |\bar{p}(\zeta)| \leq \left\|(\zeta)^\vee)\right\|_\varepsilon \delta\}$$

2 Divisors and Zeroes of Empirical Polynomials

Recall a polynomial s divides the polynomial p without remainder, $s|p$, iff all the zeroes of s are zeroes of p, iff $p \in \langle s \rangle$.

Remark 2.1. For univariate polynomials, we can assume that s is monic, and hence for $p = sq$ the leading coefficients of p and q are the equal, and $\frac{p}{lcmp}$ and 1 are the trivial divisors of p.

Definition 2.1. A monic polynomial $\tilde{s}$ is a pseudodivisor of $(\bar{p}, \varepsilon)$ if for $\delta = O(1)$, there is a polynomial $\tilde{p} \in N_\delta(\bar{p}, \varepsilon)$ such that

$$\tilde{p} = \bar{p} + \Delta p = q\tilde{s}$$

for Δp a small perturbation term.

As for non-empirical polynomials, we have a proposition relating zeroes of polynomials and their divisors.

Proposition 2.2. $\tilde{s}$ is a pseudozero of $(\bar{p}, \varepsilon)$ iff the zeroes of $\tilde{s}$ are simultaneous pseudozeroes of $(\bar{p}, \varepsilon)$.

Proof. If $\tilde{s}|(\bar{p}, \varepsilon)$, then all the zeros of $\tilde{s}$ are zeroes a $\tilde{p} \in N_\delta(\bar{p}, \varepsilon)$ with $\delta = O(1)$.

If the zeroes of $\tilde{s}$ are simultaneous pseudozeroes of $(\bar{p}, \varepsilon)$ then there is $\tilde{p} \in N_\delta(\bar{p}, \varepsilon)$ with $q\tilde{s} = \tilde{p} = \bar{p} + \Delta p$. □

Remark 2.2. This proposition allows us to switch between talking about pseudozeros and pseudodivisors at our will.

Remark 2.3. From a numerical perspective, we can form the backward error to verify that a monic polynomial $\tilde{s}$ of degree m is a pseudodivisor of a univariate empircal polynomial $(\bar{p}, \varepsilon)$.

3 Sylvester Matrices

Consider an exact factorisation of $p \in \mathcal{P}_n^1$

$$\sum_{j=0}^{n} \alpha_j x^j = p(x) = q(x)s(x) = \left(\sum_{i=0}^{n-m} \beta_i x^i\right)\left(\sum_{l=0}^{m} \gamma_l x^l\right)$$

where $s(x)$ need not be monic. The linearised effects of a perturbation term Δp can be found using $p + \Delta p = (q + \Delta q)(s + \Delta s)$ with $\Delta sq + \Delta qs = \Delta p(-\Delta q \Delta s)$.

We can write the linearisation using matrices as

$$\begin{bmatrix} (\Delta\gamma_l)_l & (\Delta\beta_j)_j \end{bmatrix} \begin{bmatrix} \beta_0 & beta_1 & \dots & \dots & \dots & \beta_n & 0 & 0 & 0 & 0 \\ 0 & beta_0 & beta_1 & \dots & \dots & \dots & \beta_n & 0 & 0 & 0 \\ & \dots & \dots & \dots & \dots & \dots & \dots & \dots & 0 & 0 \\ 0 & 0 & 0 & \beta_0 & \beta_1 & \dots & \dots & \beta_{n-1} & \beta_n & 0 \\ 0 & 0 & 0 & 0 & \beta_0 & \beta_1 & \dots & \dots & \beta_{n-1} & \beta_n \\ \gamma_0 & \gamma_1 & \dots & \dots & \dots & \gamma_{m-1} & \gamma_m & 0 & 0 & 0 \\ 0 & \gamma_0 & \gamma_1 & \dots & \dots & \dots & \gamma_{m-1} & \gamma_m & 0 & 0 \\ 0 & 0 & \dots & \dots & \dots & \dots & \dots & \dots & 0 & 0 \\ 0 & 0 & \gamma_0 & \gamma_1 & \dots & \dots & \dots & \gamma_{m-1} & \gamma_m & 0 \\ 0 & 0 & 0 & \gamma_0 & \gamma_1 & \dots & \dots & \dots & \gamma_{m-1} & \gamma_m \end{bmatrix} \begin{bmatrix} 1 \\ x \\ x^2 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ x^{n-1} \end{bmatrix} = \begin{bmatrix} (\Delta\alpha_i)_i \end{bmatrix} \begin{bmatrix} 1 \\ x \\ x^2 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ x^{n-1} \end{bmatrix}$$

or $(\Delta\gamma, \Delta\beta)S(q, s)\mathbf{x} = (\mathbf{\Delta}\alpha)\mathbf{x}$. We call the $n \times n$ matrix $S(q, s)$ the *Sylvestermatrix* of q and s.

3.1 Properties of the Sylvester Matrix

The determinant of the Sylvester matrix $S(q, s)$ is called the Resultant $R(q, s)$ of the polynomials q and s, and is a useful object for studying common divisors and zeroes of two polynomials:

Theorem 3.1. IF A is a unique factorisation domain, take $f, g \in A[x]$. Then f and g have a non-constant common divisor iff $R(f, g)$ vanishes.

Proof. The full details can be found in Section 4.2 (Divisibility properties of polynomials) in '*Plane Algebraic Cuvres*' by E. Brieskorn and H. Knörrer, translated by J. Stillwell.

The brief summary is that if f and g have a common divisor h, then $f = uh$ and $g = vh$, and so $vf - ug = 0$, for non-zero u, v. By looking at the system of linear equations the coefficients must satisfy, you obtain the matrix $S(f, g)$. □

Given $f(x) = \sum_{i=0}^{n} \alpha_i x^i$ and $g(x) = \sum_{j=0}^{m} \beta_j x^j$ with roots μ_i and ν_i respectively, we can write $R(f, g)$ as $\alpha_0^m \beta_0^n \Pi(\mu_i \nu_j)$. To see this, first consider the factorisations of f and g into linear components. As R vanishes when f and g have a common zero, there is some i and j s.t. $(\mu_i - \nu_j) | R(f, g)$. As the linear forms $(\mu_i - \nu_j)$ are relatively prime, the resultant is divisible by their product.

For the full proof see Section 35 in 'Algebra I' by B. L. van der Waerden.

Remark 3.1 (Fact). Given a polynomial f, let $f' = \frac{df}{dx}$, then $R(f, f') = 0$ iff f has a double root. This is the familiar object called the discriminant of f.

More generally, given two polynomials f and g, the rank of $S(f, g)$ is closely related to the relative positions of the zeroes of f and g. This is summarised by the following theorem:

Theorem 3.2.

- $S((f, g)$ is of full rank iff f and g have no common zeroes iff they are relatively prime
- $S(f, g)$ has rank deficiency $d \leq \min(\deg(f), \deg(g))$ iff f and g have exactly d common zeroes iff they have a common factor of degree d

Proof omitted in class due to time constraints.

We have another useful theorem (only part of it is stated):

Theorem 3.3. If f and g have a common divisor h of degree $d > 0$ then the last $\deg(f) + \deg(g) - d$ columns of $S(f, g)$ are linearly independent.

4 Multiples of Empirical Polynomials

Definition 4.1. A polynomial p is a pseudomultiple of $(\bar{s}, \varepsilon)$ if for $\delta = O(1)$ there are polynomials $\tilde{s} \in N_\delta(\bar{s}, \varepsilon)$ and $q \in \mathcal{P}$ such that $p = q\tilde{s}$.

Proposition 4.2. p is a pseudomultiple of $(\bar{s}, \varepsilon)$, $\bar{s} \in \mathcal{P}_m$ iff there is a set of m zeroes of p which are simultaneous pseudozeros of $\bar{s}$.