Ideals.

Thm (Hilbert). Let $I \subset k[x_1, \dots, x_s]$ be an ideal. Then there exists a finite generating set for I. That is, there exist $f_1, \dots, f_k$ such that

$$I = \langle f_1, \dots, f_k \rangle.$$

Analogy: Every subspace of k^s has a finite spanning set.

Caution 1: A minimal set of generators is not "polynomially linearly independent".

i.e. there may exist non trivial relations

$$\sum c_i f_i = 0 \qquad \text{where } c_i \in k[x_1, \dots, x_s]$$

Example: $I = \langle x^2, xy, y^2 \rangle$
Then
$$y \cdot (x^2) - x\,(xy) = 0$$

Such relations are called "syzygies".

Remark: For $s=1$, i.e. univariate polynomial rings, every ideal is generated by 1 element
$$I = \langle f \rangle.$$

Caution: Different (minimal) generating sets for the same I may have different cardinality.

Example: $\langle x, y \rangle = \langle x + xy,\ y + xy,\ x^2,\ y^2 \rangle$

Recall the Nullstellensatz: If $k = \mathbb{C}$, then we have a bijection

$$\left\{ \begin{array}{c} \text{Algebraic subsets} \\ \text{of } \mathbb{C}^s \end{array} \right\} \longleftrightarrow \begin{array}{c} \text{Radical ideals} \\ \text{of } \mathbb{C}[X_1, \dots, X_s]. \end{array}$$

In particular, consider the radical ideal $\langle 1 \rangle$. Then the corresponding algebraic subset is the empty set. The Nullstellensatz implies the converse!

<u>Thm</u> (conseq. of Nullstellensatz).
Consider a system of polynomial equations

$$f_1 = 0$$
$$\vdots$$
$$f_k = 0$$

Then either it has a solution (in $\mathbb{C}$) or there exist $c_1, \dots, c_k \in \mathbb{C}[X_1, \dots, X_s]$ such that

$$1 = c_1 f_1 + \dots + c_k f_k.$$

<u>Problem</u>: Devise an algorithm to either find a solution or express 1 as a linear combination.

<u>Example</u> 1) $I = \langle x^3 + x + 1, x^5 - 1 \rangle$

2) $I = \langle xy - 1, x^3 - y^3, x^2 + y^2 \rangle$

(compute Gröbner basis wrt lex order)

3) $I = \langle xy - 1, x^2 + y^2 \rangle$.

More on varieties & ideals

Variety	Ideal
$S_1 \cup S_2$	$I_1 \cap I_2$
$S_1 \cap S_2$	$I_1 + I_2$
$S_1 \subset S_2$	$I_1 \supset I_2$

Next time – Quotient rings &
0-dim systems of equations.