The Trace Finite Element Method for PDEs on Surfaces

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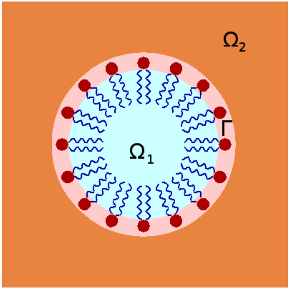
General setting



- two fluids Ω_1, Ω_2 contained in an open domain, seperated by Γ

Problem

For given initial data c_0, model the evolution of c.

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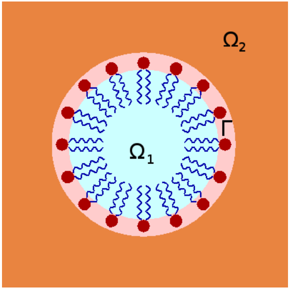

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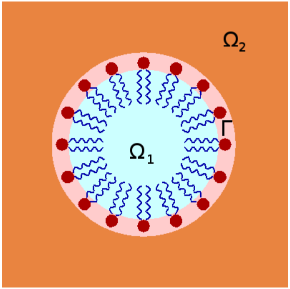

- two fluids Ω_1, Ω_2 contained in an open domain, seperated by Γ
- given velocity field $\vec{v}$
- concentration c of surfactant agent on Γ

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Surface gradient and divergence

- C^2-hypersurface $\Gamma \subseteq \mathbb{R}^d$, outward normal $\vec{n}_\Gamma$

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Surface gradient and divergence

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Definition

$$\nabla_\Gamma f := P\nabla f := \left(I - \vec{n}_\Gamma \vec{n}_\Gamma^T\right)\nabla f$$

$$\mathrm{div}_\Gamma \vec{g} := \nabla_\Gamma \cdot \vec{g} := \sum_{i=1}^{d} \vec{e}_i \sum_{j=1}^{d} p_{ij} \frac{\partial g_i}{\partial x_j}$$

Reynolds transport theorem on an interface

Theorem (Reynold's transport theorem on an interface)

The rate of change for a smooth function $f(x,t)$ on $W(t) \subseteq \Gamma$ with a given velocity field $\vec{v}$ can be described by

$$\frac{d}{dt}\int_{W(t)} f(x,t)\,ds = \int_{W(t)} \dot{f}(x,t) + f(x,t)\,\mathrm{div}_\Gamma(\vec{v})\,ds \tag{1}$$

with the material derivative $\dot{f}$ defined as

$$\dot{f} := \frac{\partial f}{\partial t} + \vec{v}\cdot\nabla f \tag{2}$$

Conservation of mass

- source function f

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$$\frac{d}{dt}\int_{W(t)} c\, ds = -\int_{\partial W(t)} \vec{q}\cdot n_W\, d\tilde{s} + \int_{W(t)} f\, ds$$

Conservation of mass

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$$= \int_{W(t)} \operatorname{div}_\Gamma (\alpha \nabla_\Gamma c) \, ds + \int_{W(t)} f \, ds$$

Conservation of mass

- source function f
- velocity $\vec{q} := -\alpha \nabla_\Gamma c$

$$
\begin{aligned}
\frac{d}{dt} \int_{W(t)} c \, ds &= -\int_{\partial W(t)} \vec{q} \cdot n_W \, d\tilde{s} + \int_{W(t)} f \, ds \\
&= \int_{W(t)} \mathrm{div}_\Gamma(\alpha \nabla_\Gamma c) \, ds + \int_{W(t)} f \, ds \\
&= \int_{W(t)} \dot{c} + c\,\mathrm{div}_\Gamma(\vec{v}) \, ds \\
&= \int_{W(t)} \frac{\partial c}{\partial t} + \vec{v} \cdot \nabla c + c\,\mathrm{div}_\Gamma(\vec{v}) \, ds
\end{aligned}
$$

Model equation in strong form

since $W(t)$ was arbitrary and we assume Γ to be stationary:

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$$\frac{\partial c}{\partial t} + \vec{v} \cdot \nabla c + c \operatorname{div}_\Gamma(\vec{v}) - \operatorname{div}_\Gamma(\alpha \nabla_\Gamma c) = f. \tag{3}$$

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$\vec{v}$ is tangential to Γ $\quad \Rightarrow \vec{v} \cdot \nabla c = P\vec{v} \cdot \nabla c = \vec{v} \cdot P\nabla c = \vec{v} \cdot \nabla_\Gamma c$

Model equation in strong form

since $W(t)$ was arbitrary and we assume Γ to be stationary:

$$\frac{\partial c}{\partial t} + \mathrm{div}_\Gamma(c\vec{v}) - \mathrm{div}_\Gamma(\alpha \nabla_\Gamma c) = f. \qquad (3)$$

Model equation in strong form

For a given (tangential) velocity field $\vec{v}$, a source function f and initial data c_0 find c such that

$$\frac{\partial c}{\partial t} + \mathrm{div}_\Gamma(c\vec{v}) - \mathrm{div}_\Gamma(\alpha \nabla_\Gamma c) = f. \tag{3}$$