Homework 2 Robotics and Control 1

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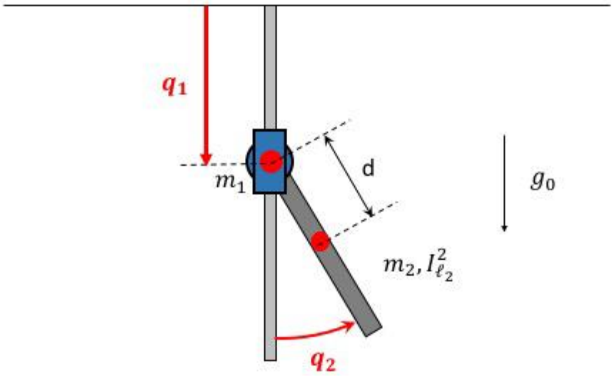
Exercise 1

Derivation of the dynamic model of the robot in the form:

$$\boldsymbol{B}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{\tau}$$

Two-link planar arm with a prismatic joint and a revolute joint moving in a vertical plane:



Dynamics parameters:

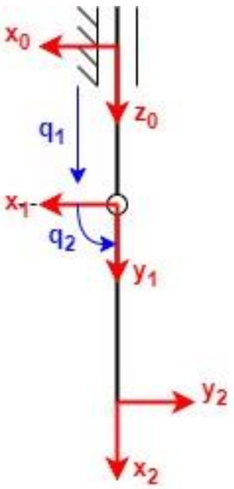
- m_1, m_2 masses of the two links
- a_1, a_2 lengths of the two links [added by me]
- $I_{l_2}^2$ inertia tensor of Link 2:

$$I_{l_2}^2 = \begin{bmatrix} I_{l_2xx}^2 & -I_{l_2xy}^2 & -I_{l_2xz}^2 \\ * & I_{l_2yy}^2 & -I_{l_2yz}^2 \\ * & * & I_{l_2zz}^2 \end{bmatrix}$$

- d distance of the center of mass of Link 2 w.r.t. the joint axis around which Link 2 rotates

Solution:

Reference frames according to the Denavit-Hartenberg convention:



where y_0 exiting the plane, z_1 entering the plane, z_2 entering the plane.

Note that the origins of the frames (w.r.t. Frame 0) are:

$$p_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad p_1 = \begin{bmatrix} 0 \\ 0 \\ q_1 \end{bmatrix} \qquad p_2 = \begin{bmatrix} a_2 c_2 \\ 0 \\ q_1 + a_2 s_2 \end{bmatrix}$$

and the centers of mass of the two links (w.r.t. Frame 0) are:

$$p_{l_1} = \begin{bmatrix} 0 \\ 0 \\ q_1 - \frac{1}{2} a_1 \end{bmatrix} \qquad p_{l_2} = \begin{bmatrix} \frac{1}{2} a_2 c_2 \\ 0 \\ q_1 + \frac{1}{2} a_2 s_2 \end{bmatrix} = \begin{bmatrix} d c_2 \\ 0 \\ q_1 + d s_2 \end{bmatrix}$$

where:

$$\sqrt{\left(\frac{1}{2} a_2 c_2\right)^2 + \left(\frac{1}{2} a_2 s_2\right)^2} = d \quad \rightarrow \quad \sqrt{\frac{1}{4} a_2^2 (c_2^2 + s_2^2)} = d \quad \rightarrow \quad d = \frac{1}{2} a_2 \quad \rightarrow \quad a_2 = 2d$$

The rotation matrices of the two links (w.r.t. Frame 0) are:

$$R_1 = R_1^0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \qquad R_2 = R_2^0 = R_1^0 R_2^1 = \begin{bmatrix} c_2 & -s_2 & 0 \\ 0 & 0 & -1 \\ s_2 & c_2 & 0 \end{bmatrix}$$

because $x_1 = x_0, y_1 = z_0, z_1 = -y_0$, and R_2^1 is given by the rotation of Frame 2 w.r.t Frame 1 around the z axis:

$$R_2^1 = \begin{bmatrix} c_2 & -s_2 & 0 \\ s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Now let's compute the partial Jacobians relative to the links, considering that Joint 1 is prismatic and Joint 2 is revolute:

$$J_P^{(l_1)} = \begin{bmatrix} J_{P_1}^{(l_1)} & 0 \end{bmatrix} = [z_0 \quad 0] = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$J_O^{(l_1)} = \begin{bmatrix} J_{O_1}^{(l_1)} & 0 \end{bmatrix} = [0 \quad 0] = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

and

$$J_P^{(l_2)} = \begin{bmatrix} J_{P_1}^{(l_2)} & J_{P_2}^{(l_2)} \end{bmatrix} = \begin{bmatrix} z_0 & z_1 \times (p_{l_2} - p_1) \end{bmatrix} = \begin{bmatrix} 0 & -d s_2 \\ 0 & 0 \\ 1 & d c_2 \end{bmatrix}$$

$$J_O^{(l_2)} = \begin{bmatrix} J_{O_1}^{(l_2)} & J_{O_2}^{(l_2)} \end{bmatrix} = [0 \quad z_1] = \begin{bmatrix} 0 & 0 \\ 0 & -1 \\ 0 & 0 \end{bmatrix}$$

where $z_1 = [0 \quad -1 \quad 0]^T$.

The total kinetic energy of the system is given by:

$$\mathcal{T} = \frac{1}{2} \dot{q}^T B(q) \dot{q}$$

where:

$$B(q) = \sum_{i=1}^{2} \left(m_{l_i} J_P^{(l_i)\,T} J_P^{(l_i)} + J_O^{(l_i)\,T} R_i I_{l_i}^{l_i} R_i^T J_O^{(l_i)} \right)$$

with:

$$J_P^{(l_1)\,T} J_P^{(l_1)} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$J_P^{(l_2)\,T} J_P^{(l_2)} = \begin{bmatrix} 1 & d c_2 \\ d c_2 & d^2 \end{bmatrix}$$

$$J_O^{(l_1)\,T} R_1 I_{l_1}^{l_1} R_1^T J_O^{(l_1)} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$
$$J_O^{(l_2)\,T} R_2 I_{l_2}^{l_2} R_2^T J_O^{(l_2)} = \begin{bmatrix} 0 & 0 \\ 0 & \bar{I} \end{bmatrix}$$
where $\bar{I} = I_{l_2 zz}^2$ is the only component of the inertia tensor $I_{l_2}^2$ playing a role in the dynamical model.

Summing up all the contributions, the inertial matrix is:

$$\begin{aligned} B(q) &= \begin{bmatrix} b_{11} & b_{12}(\theta_2) \\ b_{21}(\theta_2) & b_{22} \end{bmatrix} = \\ &= \left(m_{l_1} J_P^{(l_1)\,T} J_P^{(l_1)} + J_O^{(l_1)\,T} R_1 I_{l_1}^{l_1} R_1^T J_O^{(l_1)} \right) + \left(m_{l_2} J_P^{(l_2)\,T} J_P^{(l_2)} + J_O^{(l_2)\,T} R_2 I_{l_2}^{l_2} R_2^T J_O^{(l_2)} \right) = \\ &= \left(m_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right) + \left(m_2 \begin{bmatrix} 1 & dc_2 \\ dc_2 & d^2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & I_{l_2 zz}^2 \end{bmatrix} \right) = \\ &= \begin{bmatrix} m_1 + m_2 & m_2 dc_2 \\ m_2 dc_2 & m_2 d^2 + \bar{I} \end{bmatrix} \end{aligned}$$

Next, let's compute the Christoffel symbols. Recalling the definition:

$$c_{ijk} = \frac{1}{2}\left(\frac{\partial b_{ij}}{\partial q_k} + \frac{\partial b_{ik}}{\partial q_j} - \frac{\partial b_{jk}}{\partial q_i} \right)$$

and, noticing that, in view of symmetry of B, it is:

$$c_{ijk} = c_{ikk}$$

we get:

$$\begin{aligned} c_{111} &= \frac{1}{2}\frac{\partial b_{11}}{\partial q_1} = 0 \\ c_{112} &= c_{121} = \frac{1}{2}\frac{\partial b_{11}}{\partial q_2} = 0 \\ c_{122} &= \frac{1}{2}\left(\frac{\partial b_{12}}{\partial q_2} + \frac{\partial b_{12}}{\partial q_2} - \frac{\partial b_{22}}{\partial q_1} \right) = \frac{1}{2}(-m_2 ds_2 + (-m_2 ds_2) - 0) = -m_2 ds_2 \\ c_{211} &= \frac{1}{2}\left(\frac{\partial b_{21}}{\partial q_1} + \frac{\partial b_{21}}{\partial q_1} - \frac{\partial b_{11}}{\partial q_2} \right) = \frac{1}{2}(0 + 0 - 0) = 0 \\ c_{212} &= c_{221} = \frac{1}{2}\frac{\partial b_{22}}{\partial q_1} = 0 \\ c_{222} &= \frac{1}{2}\frac{\partial b_{22}}{\partial q_2} = 0 \end{aligned}$$

From the Christoffel symbols, the elements of the matrix C can be computed as:

$$c_{ij} = \sum_{k=1}^{n} c_{ijk} \dot{q}_k$$

obtaining:

$$\begin{aligned} c_{11} &= c_{111}\dot{q}_1 + c_{112}\dot{q}_2 = 0 + 0 = 0 \\ c_{12} &= c_{121}\dot{q}_1 + c_{122}\dot{q}_2 = 0 + (-m_2 ds_2)\dot{q}_2 = -m_2 ds_2 \dot{q}_2 \\ c_{21} &= c_{211}\dot{q}_1 + c_{212}\dot{q}_2 = 0 + 0 = 0 \\ c_{22} &= c_{221}\dot{q}_1 + c_{222}\dot{q}_2 = 0 + 0 = 0 \end{aligned}$$

Therefore, we can write the matrix C as follows:

$$C(q, \dot{q}) = \begin{bmatrix} 0 & -m_2 ds_2 \dot{q}_2 \\ 0 & 0 \end{bmatrix}$$

As, for the gravitational terms, recall that:

$$g_i(q) = \frac{\partial \mathcal{U}}{\partial q_i} = -\sum_{j=1}^{n} m_{l_j} g_0^T J_{P_i}^{(l_j)}$$

Since in our case $g_0 = [0 \quad 0 \quad g]^T$ (because g_0 has the same direction and orientation as z_0), where g is the gravity acceleration and from the computation of the above Jacobians, it follows:

$$g_1 = -\left(m_{l_1} g_0^T J_{P_1}^{(l_1)} + m_{l_2} g_0^T J_{P_1}^{(l_2)}\right) = -\left(m_1 [0 \quad 0 \quad g] \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + m_2 [0 \quad 0 \quad g] \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) =$$
$$= -(m_1 + m_2) g$$
$$g_2 = -\left(m_{l_1} g_0^T J_{P_2}^{(l_1)} + m_{l_2} g_0^T J_{P_2}^{(l_2)}\right) = -\left(m_1 [0 \quad 0 \quad g] \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + m_2 [0 \quad 0 \quad g] \begin{bmatrix} -ds_2 \\ 0 \\ dc_2 \end{bmatrix}\right) =$$
$$= -m_2 dc_2 g$$

In absence of friction and tip contact forces, the equations of the motion with the Lagrangian formulation can be written in compact matrix form as:

$$B(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = \tau$$

Therefore, from the previous computation of B, C and g. The joint model of the planar arm is:

$$\begin{bmatrix} m_1 + m_2 & m_2 dc_2 \\ m_2 dc_2 & m_2 d^2 + \bar{I} \end{bmatrix} \ddot{q} + \begin{bmatrix} 0 & -m_2 ds_2 \dot{q}_2 \\ 0 & 0 \end{bmatrix} \dot{q} + \begin{bmatrix} -(m_1 + m_2) g \\ -m_2 dc_2 g \end{bmatrix} = \tau$$

Now, we prove that the dynamical model satisfies the property:

$$\dot{q}^T N(q,\dot{q}) \dot{q} = 0, \quad \forall \dot{q}$$

Observe that:

$$\dot{B} = \begin{bmatrix} 0 & -m_2 ds_2 \dot{q}_2 \\ -m_2 ds_2 \dot{q}_2 & 0 \end{bmatrix}$$

Hence:

$$N(q,\dot{q}) = \dot{B}(q) - 2C(q,\dot{q}) = \begin{bmatrix} 0 & -m_2 ds_2 \dot{q}_2 \\ -m_2 ds_2 \dot{q}_2 & 0 \end{bmatrix} - 2 \begin{bmatrix} 0 & -m_2 ds_2 \dot{q}_2 \\ 0 & 0 \end{bmatrix} =$$
$$= \begin{bmatrix} 0 & m_2 ds_2 \dot{q}_2 \\ -m_2 ds_2 \dot{q}_2 & 0 \end{bmatrix}$$

From which we deduce that N is skew symmetric.

Exercise 2

Design of a PD + constant gravity compensation law that asymptotically stabilizes the robot to the desired configuration $\boldsymbol{q_d} = [\boldsymbol{0}, \boldsymbol{\pi}]$.

Solution:

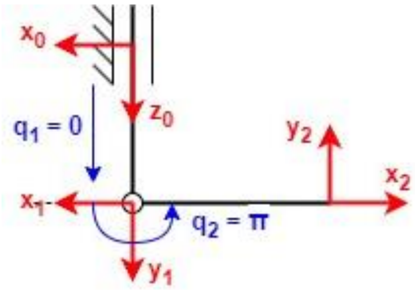
As seen in the previous exercise, in absence of friction and tip contact forces, the equations of the motion with the Lagrangian formulation is:

$$B(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = \tau$$

In our analysis we consider τ as input u:

$$B(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = u$$

so, we want to design u in order to make the robot reach the desired configuration $q_d = [0, \pi]$:



As required by the problem, we design u as a PD controller + constant gravity compensation:

$$u = K_p(q_d - q) - K_D\dot{q} + g(q_d)$$

where we assume to have perfect knowledge of the gravitational term $g(q_d)$ only in the desired configuration q_d (a.k.a. constant equilibrium posture).

Note: as we saw in the theory part, this result was obtained by the application of the direct Lyapunov method, with candidate Lyapunov function:

$$V(\dot{\boldsymbol{q}}, \tilde{\boldsymbol{q}}) = \frac{1}{2}\dot{\boldsymbol{q}}^T \boldsymbol{B}(\boldsymbol{q})\dot{\boldsymbol{q}} + \frac{1}{2}\tilde{\boldsymbol{q}}^T \boldsymbol{K}_P \tilde{\boldsymbol{q}} > 0, \quad \forall \dot{\boldsymbol{q}}, \tilde{\boldsymbol{q}} \neq \boldsymbol{0}$$

with the only differences that we add an additional differential term (in addition to the proportional one), and that we use $g(q_d)$ in place of $g(q)$.

Assuming K_P and K_D diagonal definite positive matrices, to ensure global asymptotic stabilization in the desired configuration q_d we exploit the result of the following theorem:

> **Theorem:** if we pick $K_{P,m} > \alpha$, then the state $(\boldsymbol{q}_d, \boldsymbol{0})$ of the robot under joint-space PD control + constant gravity compensation at $\boldsymbol{q}_d$ is **globally asymptotically stable**.

Hence, we have to choose K_P in such a way that its smallest element $K_{P,m}$ on the diagonal is greater than α in order to guarantee global asymptotic stability.

Notice, however, that the theorem does not say anything about K_D. As a matter of fact, K_D is not necessary for the convergence of the error $\tilde{q} = q_d - q$, instead, it is useful in order to speed up it. In particular, the larger the elements on the diagonal of K_D, the faster will be the convergence.

To derive the structural property α, we study the system at the equilibrium in the desired configuration q_d.
We suppose that at the equilibrium $g(q_d) = 0$, so the law that we have designed becomes:

$$u = K_p \begin{bmatrix} -q_1 \\ \pi - q_2 \end{bmatrix} - K_D \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}$$

Now, exploiting the property:

$$\exists \alpha > 0: \left\| \frac{\partial^2 U}{\partial q^2} \right\| = \left\| \frac{\partial g}{\partial q} \right\| \leq \alpha, \quad \forall q$$

we get:

$$\max_q \left\| \frac{\partial g}{\partial q} \right\| = \max_q \left\| \frac{\partial}{\partial q} \begin{bmatrix} -(m_1 + m_2)g \\ -m_2 d c_2 g \end{bmatrix} \right\| = \max_q \left\| \begin{bmatrix} 0 & 0 \\ 0 & m_2 d g s_2 \end{bmatrix} \right\| =$$
$$= \max_q \{0, m_2 d g s_2\} \leq m_2 d g = \alpha$$

Therefore, by the previous theorem, in order to guarantee global asymptotic stabilization, it must be $K_{P,m} > \alpha = m_2 dg$, and then $K_P > \alpha \mathbb{I}_{2\times 2}$. No additional constraints on $K_D > 0$.

Exercise 3

Linear parametrization of the dynamic model of the type $\boldsymbol{Y}(\boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}})\boldsymbol{\pi} = \boldsymbol{\tau}$, where:

$$\boldsymbol{\pi} = \begin{bmatrix} m_1 + m_2 \\ m_2 d \\ \bar{I} + m_2 d^2 \end{bmatrix}$$

with $\bar{I}$ the only component of the inertia tensor $\boldsymbol{I}_{l_2}^{2}$ playing a role in the dynamic model.

Solution:

In order to find the linear parametrization of the dynamic model found before:

$$\begin{bmatrix} m_1 + m_2 & m_2 d c_2 \\ m_2 d c_2 & m_2 d^2 + \bar{I} \end{bmatrix} \ddot{q} + \begin{bmatrix} 0 & -m_2 d s_2 \dot{q}_2 \\ 0 & 0 \end{bmatrix} \dot{q} + \begin{bmatrix} -(m_1 + m_2) g \\ -m_2 d c_2 g \end{bmatrix} = \tau$$

we just need to perform some calculations in order to rewrite it in the form $Y(q, \dot{q}, \ddot{q})\pi = \tau$:

$$\begin{bmatrix} m_1 + m_2 & m_2 d c_2 \\ m_2 d c_2 & m_2 d^2 + \bar{I} \end{bmatrix} \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} + \begin{bmatrix} 0 & -m_2 d s_2 \dot{q}_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} + \begin{bmatrix} -(m_1 + m_2) g \\ -m_2 d c_2 g \end{bmatrix} = \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix}$$

$$\begin{bmatrix} (m_1 + m_2)\ddot{q}_1 + (m_2 d c_2)\ddot{q}_2 \\ (m_2 d c_2)\ddot{q}_1 + (m_2 d^2 + \bar{I})\ddot{q}_2 \end{bmatrix} + \begin{bmatrix} -m_2 d s_2 \dot{q}_2^2 \\ 0 \end{bmatrix} + \begin{bmatrix} -(m_1 + m_2) g \\ -m_2 d c_2 g \end{bmatrix} = \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix}$$

$$\begin{bmatrix} (m_1 + m_2)\ddot{q}_1 + (m_2 d) c_2 \ddot{q}_2 - (m_2 d) s_2 \dot{q}_2^2 - (m_1 + m_2) g \\ (m_2 d) c_2 \ddot{q}_1 + (\bar{I} + m_2 d^2)\ddot{q}_2 - (m_2 d) c_2 g \end{bmatrix} = \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix}$$

$$\begin{bmatrix} \ddot{q}_1 - g & c_2 \ddot{q}_2 - s_2 \dot{q}_2^2 & 0 \\ 0 & c_2 \ddot{q}_1 - c_2 g & \ddot{q}_2 \end{bmatrix} \begin{bmatrix} m_1 + m_2 \\ m_2 d \\ \bar{I} + m_2 d^2 \end{bmatrix} = \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix}$$

Hence, we have shown that it is possible to provide the required linear parametrization of the dynamic model, and it is the following:

$$Y(q, \dot{q}, \ddot{q}) = \begin{bmatrix} \ddot{q}_1 - g & c_2 \ddot{q}_2 - s_2 \dot{q}_2^2 & 0 \\ 0 & c_2 \ddot{q}_1 - c_2 g & \ddot{q}_2 \end{bmatrix}$$

Exercise 4

Desing of an adaptive controller to track a desired trajectory $(\boldsymbol{q}_d, \dot{\boldsymbol{q}}_d, \ddot{\boldsymbol{q}}_d)$.

Solution:

To design an adaptive controller for tracking a desired trajectory $(q_d, \dot{q}_d, \ddot{q}_d)$, while dealing with the uncertainty, given the motion equation:

$$B(q)\ddot{q} + C(q, \dot{q})\dot{q} + g(q) = u$$

first of all, we need to define a new reference velocity:

$$\dot{q}_r = \dot{q}_d + \Lambda \tilde{q}$$

where $\Lambda = K_D^{-1} K_p$, $\tilde{q} = q_d - q$.

Then, we need to define the reference velocity error as:

$$\sigma = \dot{q}_r - \dot{q} = \dot{\tilde{q}} + \Lambda \tilde{q}$$

In general, we can design our adaptive control law as:

$$u = Y(q, \dot{q}, \dot{q}_r, \ddot{q}_r)\hat{\pi} + K_D \sigma$$

where $\hat{\pi}$ is the current parameter estimation, which is updated/adapted (in order to drive the pose error to 0) over the iterations of the adaptive control law algorithm according to the following update rule:

$$\dot{\hat{\pi}} = K_\pi^{-1} Y^T(q, \dot{q}, \dot{q}_r, \ddot{q}_r)\sigma$$

where K_π is a definite positive (in general diagonal) matrix whose parameters define the "adaptation strength" of the update law.

At this point, we need to compute the regressor $Y(q, \dot{q}, \dot{q}_r, \ddot{q}_r)$, starting from the linear regressor $Y(q, \dot{q}, \ddot{q})$ found in the previous exercise, and remembering that:

- $\ddot{q}$ always becomes $\ddot{q}_r$
- $\dot{q}$ for linear parameters becomes $\dot{q}_r$
- $\dot{q}$ for nonlinear parameters remains $\dot{q}$
- q remains q

So, by applying these rules, we get the following regressor:

$$Y(q, \dot{q}, \dot{q}_r, \ddot{q}_r) = \begin{bmatrix} \ddot{q}_{1r} - g & \ddot{q}_{2r}\cos q_2 - \dot{q}_2^2 \sin q_2 & 0 \\ 0 & \ddot{q}_{1r}\cos q_2 - g\cos q_2 & \ddot{q}_{2r} \end{bmatrix}$$

for the following estimated initial parameter vector:

$$\hat{\pi} = \begin{bmatrix} \widehat{m_1 + m_2} \\ \widehat{m_2 d} \\ \widehat{\bar{I} + m_2 d^2} \end{bmatrix}$$

Finally, we can rewrite the adaptive control law for our adaptive controller as:

$$u = \begin{bmatrix} \ddot{q}_{1r} - g & \ddot{q}_{2r}\cos q_2 - \dot{q}_2^2 \sin q_2 & 0 \\ 0 & \ddot{q}_{1r}\cos q_2 - g\cos q_2 & \ddot{q}_{2r} \end{bmatrix} \begin{bmatrix} \widehat{m_1 + m_2} \\ \widehat{m_2 d} \\ \widehat{\bar{I} + m_2 d^2} \end{bmatrix} + K_D \sigma$$

with update rule:

$$\dot{\hat{\pi}} = \begin{bmatrix} \dot{\widehat{m_1 + m_2}} \\ \dot{\widehat{m_2 d}} \\ \dot{\widehat{\bar{I} + m_2 d^2}} \end{bmatrix} = K_\pi^{-1} Y^T(q, \dot{q}, \dot{q}_r, \ddot{q}_r)\sigma$$

where, in order to guarantee global asymptotic stability, K_π, K_P, K_D (and then also Λ) are defined as definite positive matrices (generally diagonal), and they must be tuned in order to optimize the performance of the algorithm.