Game Theory

Theory

Christian Rossi

Academic Year 2024-2025

Contents

CHAPTER 1

Introduction

1.1 Games

Games provide valuable models for simulating a variety of real-world situations.

Definition (*Game*)**.** A game is a structured process that includes the following components:

- A group of participants, referred to as players, with at least two members.
- An initial state or starting condition.
- A set of rules that define how players can act.
- A range of possible outcomes or end states.
- The preferences of each player concerning these potential outcomes.

1.2 Game Theory Assumptions

Game theory operates under the following key assumptions about the players involved:

1. *Self-interested.*
2. *Rational.*

1.2.1 Self-interest

Players are assumed to focus solely on their own preferences concerning the outcomes of the game. This is a mathematical assumption, not an ethical judgment. In fact, it is essential for defining what constitutes a rational choice within the framework of game theory.

1.2.2 Rationality

Definition (*Preference relation*)**.** Let X be a set. A preference relation on X is a binary relation $\preceq$ that satisfies the following properties for all $x, y, z \in X$:

- *Reflexive*: $x \preceq x$ (every element is at least as preferred as itself).
- *Complete*: $x \preceq y$ or $y \preceq x$ (any two elements can be compared).
- *Transitive*: if $x \preceq y$ and $y \preceq z$, then $x \preceq z$ (preferences are consistent across comparisons).

The transitive property ensures that preferences can be consistently ranked.

Definition (*Utility function*)**.** Given a preference relation $\preceq$ over a set X, a utility function representing $\preceq$ is a function $u : X \to \mathbb{R}$ such that:

$$u(x) \geq u(y) \Leftrightarrow x \preceq y$$

While a utility function may not always exist in specific cases, it does exist in general settings, particularly when X is finite. If a utility function does exist, there are infinitely many such functions, differing by any strictly increasing transformation of the original function.

Each player i is assigned a set X_i, representing all the choices available to them. Therefore, the set $X = xX_i$ over which the utility function u is defined represents the combined choices of all players.

Rationality assumptions The following assumptions define the rational behavior of players:

1. *Consistent preferences*: players can establish a preference relation over the game's outcomes, and this ordering is consistent.
2. *Utility representation*: players can define a utility function that represents their preference relations when needed.
3. *Consistent use of probability*: players apply the laws of probability consistently, including computing expected utilities and updating probabilities according to Bayes' rule.
4. *Understanding consequences*: players comprehend the outcomes of their actions, the impact on other players, and the resulting chain of consequences.
5. *Application of decision theory*: players use decision theory to maximize their utility. Given a set of alternatives X and a utility function u, each player seeks $\bar{x} \in X$ such that:

$$u(\bar{x}) \geq u(x) \qquad \forall x \in X$$

One significant consequence of these axioms is the principle of eliminating strictly dominated strategies: a player will not choose an action a if there exists another action b that yields a strictly better outcome, regardless of the actions of other players.

Example:
Consider the following games:

Gain	**Probability**
2500	33%
2400	66%
0	1%

Table 1.1: Game A

Gain	**Probability**
2500	0%
2400	100%
0	0%

Table 1.2: Game B

In a sample of 72 participants, 82% chose to play Game B, indicating a preference for certainty—characteristic of risk-averse individuals. According to expected utility theory, this decision is rational if:

$$u(2400) > \frac{33}{100}u(2500) + \frac{66}{100}u(2400)$$

This simplifies to:

$$\frac{34}{100}u(2400) > \frac{33}{100}u(2500)$$

Now consider the following alternatives:

Gain	**Probability**
2500	33%
0	67%

Table 1.3: Game C

Gain	**Probability**
2400	34%
0	66%

Table 1.4: Game D

In this new setup, 83% of participants preferred Game C, reflecting a preference for a larger gain even with a lower probability of success. Rationality in this scenario requires:

$$\frac{34}{100}u(2400) < \frac{33}{100}u(2500)$$

This contradicts the earlier experiment, where the opposite preference was observed. Such behavior violates the independence axiom in expected utility theory, which states that consistent preferences should hold under similar probabilistic transformations.

This contradiction is known as the Allais Paradox, demonstrating that individuals do not always act as fully rational decision-makers.

Example:
A group of players is asked to choose an integer between 1 and 100. The mean of all chosen numbers, M is then calculated. The objective of the game is to select the number closest to qM, where $0 < q < 1$.

A purely rational player would conclude that the optimal number to choose is 1, regardless of the value of q. However, this player is likely to lose.

For example, let $q = \frac{1}{2}$. Since $M \leq 100$, in the first step, it seems irrational to choose a number greater than $\frac{1}{2} \cdot 100$, as this is the initial target value based on the game's rules.

However, in the second step, assuming all players are rational and recognize that others are also rational, each player would realize that others will also choose numbers below 50. Therefore, the new logical step would be to pick a number less than $\left(\frac{1}{2}\right)^2 \cdot 100$.

This reasoning continues iteratively: at step n, it becomes irrational to choose a number greater than $\left(\frac{1}{2}\right)^n \cdot 100$. Ultimately, after enough steps, the only rational choice would appear to be selecting the smallest possible number (1).

Despite this reasoning, experiments show that the actual winning number is far higher than 1. In fact, the winning number tends to increase as the value of q increases, revealing that real-life behavior often deviates from purely rational game theory predictions.

1.3 Bimatrices

Conventionally, Player 1 selects a row, while Player 2 selects a column. This results in a pair of values that represent the utilities for Player 1 and Player 2, respectively. These options can be conveniently displayed in a bimatrix.

Example:
Consider the following bimatrix:

$$\begin{pmatrix} \begin{pmatrix} 8 & 8 \end{pmatrix} & \begin{pmatrix} 2 & 7 \end{pmatrix} \\ \begin{pmatrix} 7 & 2 \end{pmatrix} & \begin{pmatrix} 0 & 0 \end{pmatrix} \end{pmatrix}$$

In this example, Player 1's utilities are given by:

$$\begin{pmatrix} 8 & 2 \\ 7 & 0 \end{pmatrix}$$

Since the second row is strictly dominated by the first (i.e., Player 1's utility in the first row is higher for any choice by Player 2), Player 1 will rationally choose the first row. Similarly, Player 2 will select the first column, as it strictly dominates the second column.

While the principle of eliminating strictly dominated strategies may seem simplistic, it can lead to surprisingly powerful insights and outcomes.

Example:
Consider the following two games:

$$\begin{pmatrix} \begin{pmatrix} 10 & 10 \end{pmatrix} & \begin{pmatrix} 3 & 15 \end{pmatrix} \\ \begin{pmatrix} 15 & 3 \end{pmatrix} & \begin{pmatrix} 5 & 5 \end{pmatrix} \end{pmatrix}$$

$$\begin{pmatrix} \begin{pmatrix} 8 & 8 \end{pmatrix} & \begin{pmatrix} 2 & 7 \end{pmatrix} \\ \begin{pmatrix} 7 & 2 \end{pmatrix} & \begin{pmatrix} 0 & 0 \end{pmatrix} \end{pmatrix}$$

Note that in the first game, players have outcomes like $\begin{pmatrix} 10 & 10 \end{pmatrix}$ and $\begin{pmatrix} 15 & 3 \end{pmatrix}$, which individually seem to offer higher utilities than most outcomes in the second game. However, applying rational decision-making principles leads to a surprising result.

According to the principle of elimination of dominated strategies, players will end up choosing the outcome pair $\begin{pmatrix} 8 & 8 \end{pmatrix}$ in the second game because it dominates other available outcomes. This leads them to prefer the second game over the first game, despite the fact that the first game contains outcomes with higher individual utilities, like $\begin{pmatrix}10 & 10\end{pmatrix}$ and $\begin{pmatrix}15 & 3\end{pmatrix}$.

Now, consider the expanded form of the first game, which contains even more outcomes:

$$\begin{pmatrix} \begin{pmatrix}1 & 1\end{pmatrix} & \begin{pmatrix}11 & 0\end{pmatrix} & \begin{pmatrix}4 & 0\end{pmatrix} \\ \begin{pmatrix}0 & 11\end{pmatrix} & \begin{pmatrix}8 & 8\end{pmatrix} & \begin{pmatrix}2 & 7\end{pmatrix} \\ \begin{pmatrix}0 & 4\end{pmatrix} & \begin{pmatrix}7 & 2\end{pmatrix} & \begin{pmatrix}0 & 0\end{pmatrix} \end{pmatrix}$$

This expanded version of the first game includes all the outcomes from the second game, plus some additional options. However, rationality axioms suggest that in the first game, players should choose the outcome $\begin{pmatrix}10 & 10\end{pmatrix}$, which dominates the other possibilities.

Interestingly, in the second game, where fewer options are available, the players end up selecting $\begin{pmatrix}8 & 8\end{pmatrix}$. This leads to a paradoxical outcome: having fewer available actions can actually make players better off by simplifying the decision-making process and avoiding suboptimal choices.

Example:
Consider the rational outcomes of the following game.

$$\begin{pmatrix} \begin{pmatrix}0 & 0\end{pmatrix} & \begin{pmatrix}1 & 1\end{pmatrix} \\ \begin{pmatrix}1 & 1\end{pmatrix} & \begin{pmatrix}0 & 0\end{pmatrix} \end{pmatrix}$$

While we may not know the rational outcomes formally, it is clear that the preferred outcome for both players is $\begin{pmatrix}1 & 1\end{pmatrix}$. However, this leads to a coordination problem.

Both pairs of actions result in the same outcome $\begin{pmatrix}1 & 1\end{pmatrix}$, but there is no clear way for the players to distinguish between these two strategies. As a result, while the rational outcome is obvious, the players face difficulty coordinating on which specific actions to take to achieve it.

Example:
Consider a voting game with three players, each having the following preferences:

1. Player 1: $A \gneqq B \gneqq C$
2. Player 2: $B \gneqq C \gneqq A$
3. Player 3: $C \gneqq A \gneqq B$

Here, the notation $A \gneqq B$ indicates that Player 1 prefers $B \preceq A$, but not vice versa. The winner is determined by the alternative that receives the most votes. However, if there is a tie among three different votes, the alternative chosen by Player 1 will win.

Let's now analyse the rational outcome of the game through the elimination of dominated actions:

- Alternative A is a weakly dominant strategy for Player 1.
- Players 2 and 3 have their least preferred choice as a weakly dominated strategy.

To avoid their worst outcome, Player 2 retains options B and C (ordered in rows), while Player 3 keeps C and A (ordered in columns). Player 1 will consistently choose A. This simplifies the game to a 2×2 table with the following outcomes:

	C	**A**
B	A	A
C	C	A

Since $C \precneqq A$ for both Players 2 and 3, they will choose the outcome in the second row and first column, leading to the final result being C, which is the worst outcome for Player 1

CHAPTER 2

Extensive form games

2.1 Tree representation

2.1.1 Voting gmae

Three politicians are supposed to decide whether to raise their salaries or not. The vote is public and in sequence. They would prefer to receive a salary increase, yet they would also like to vote against it so as not to lose public support. Optimal result for each player: having salary increase while voting against it! Main features of the game: 1 The moves take place in sequence: the politicians vote one after the other 2 Every possible situation is known to the players: at any time they know the whole past history, as well as the possible developments 3 The final outcome is determined by the majority of votes This is an example of what is called a game with perfect information: each player has knowledge about all the events that have previously occurred. How can we represent such a game? And how can we solve it? We may use a tree in which Each player's vote is represented at a branch: YES on the left and NO on the right. Utilities are based on individual votes and possible final outcomes: 1= YES and no raise; 2= NO and no raise; 3= YES and raise; 4= N0 and raise The tree in this case will become:

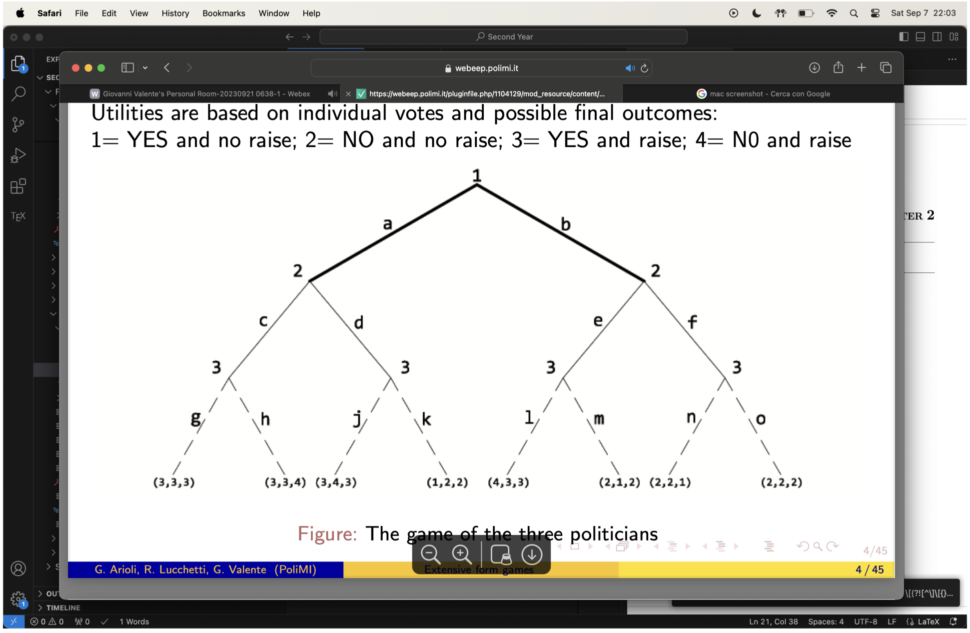


Figure 2.1: Voting game tree

2.1.2 A game with chance

Two players 1 and 2 must decide in sequence whether to play or not. If both of them decide to play, then a coin is tossed (random component R): the first player wins with heads, whereas the second one with tails.

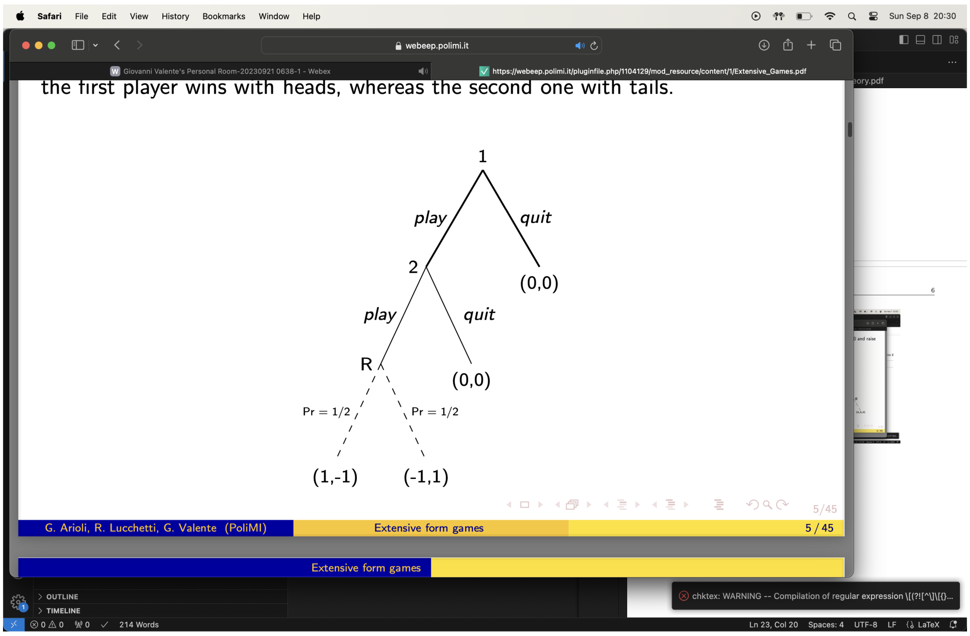


Figure 2.2: Chance game tree

2.1.3 Definitions

Definition (*Finite directed graph*)**.** A finite directed graph is a pair (V, E) where:

- V is a finite set, called the set of vertices.

- $E \subset V \times V$ is a set of ordered pairs of vertices called the set of the (directed) edges.

Definition (*Path*)**.** A path from a vertex v_1 to a vertex v_{k+1} is a finite sequence of vertices-edges $v_1, e_1, v_2, \ldots, v_k, e_k, v_{k+1}$ such that $e_i \neq e_j$ if $i \neq j$ and $e_j = (v_j, v_{j+1})$. k is called the length of the path.

Definition (*Oriented graph*)**.** An oriented graph is finite directed graph having no bidirected edges, that is for all j, k at most one between (v_j, v_k) and (v_k, v_j) may be arrows of the graph.

Definition (*Tree*)**.** A tree is a triple (V, E, x_0) where (V, E) is an oriented graph and $_x0$ is a vertex in V such that there is a unique path from x_0 to x, where x is any vertex in V.

Definition (*Child*)**.** A child of a vertex v is any vertex x such that $(v, x) \in E$. A vertex is called a leaf if it has no children. We say that the vertex x follows the vertex v if there is a path from v to x.

2.2 Extensive game

Definition (*Extensive form game with perfect information*)**.** An extensive form game with perfect information consists of:

1. A finite set $N = \{1, \ldots, n\}$ of players.
2. A game tree (V, E, x_0).
3. A partition of the vertices that are not leaves into sets $P1, P2, \ldots, Pn + 1$.
4. A probability distribution for each vertex in P_{n+1}, defined on the edges from the vertex to its children.

We have the following:

1. The set P_i , for $i \leq n$, is the set of the nodes v where Player i must choose a child of v, representing a possible move from him at v.
2. P_{n+1} is the set of the nodes where a chance move is present: that is $n + 1$ is the number of players plus the random component. P_{n+1} can be empty, meaning that the game does not admit any chance.
3. When P_{n+1} is empty, the n players have only preferences on the leaves: a utility function is not required.

2.2.1 Solving an extensive game

In order to find the optimal outcome, we employ the rationality axioms.

> **Example:**
> For the voting game described before:

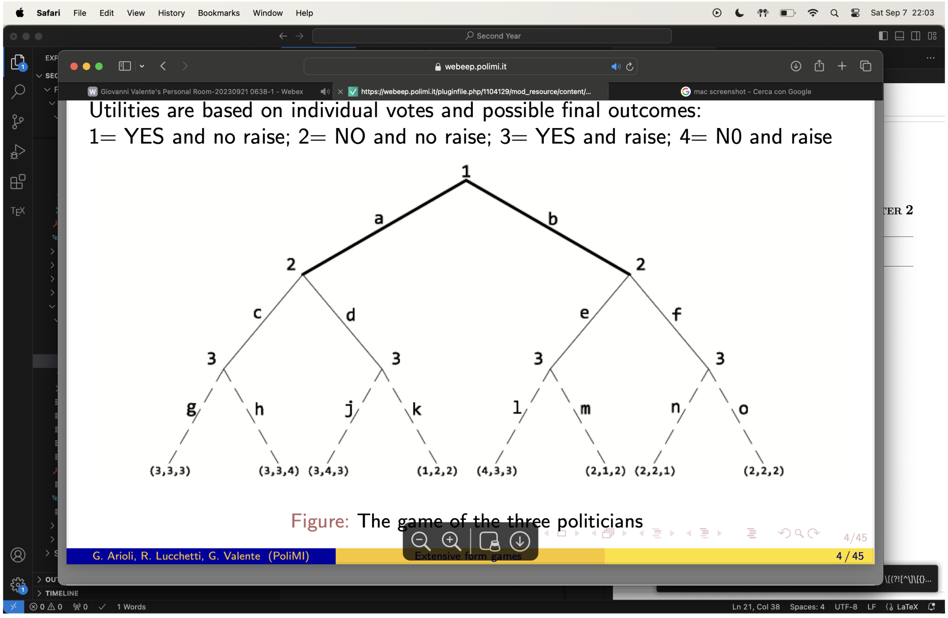


Figure 2.3: Voting game tree

We have the following:

- Player3: h = (3, 3, 4) over g = (3, 3, 3) ; j = (3, 4, 3) over k = (1, 2, 2); l = (4, 3, 3) over m= (2, 1, 2) ; o = (2, 2, 2) over n= (2, 2, 1)
- Player2: d =(3,4,3)overc =(3,3,4);e =(4,3,3)overf =(2,2,2)
- Player1: b = (4, 3, 3) over a = (3, 4, 3), thereby finding the optimal outcome

Definition (*Length*)**.** Define Length of the game as the length of the longest path in the game.

Decision theory, i.e. rationality assumption 5, enables us to solve games of length 1. Rationality assumption 4 allows us to solve a game of length $i+1$if the games of length at most i are solved. Thus, by repeated applications, we can solve games of any finite length. This method takes the name of backward induction: it is the process of reasoning backwards in time (that is from the leaves of the tree up to the root), so as to determine a sequence of actions leading one to the optimal outcome.

Theorem 2.2.1 (First rationality theorem)**.** *The rational outcomes of a finite, perfect information game are those given by the procedure of backward induction.*

The method of backwards induction can be applied since every vertex v of the game is the root of a new game, made by all followers of v in the initial game. Such a game is called a subgame of the original one.

Example:
For the second game:

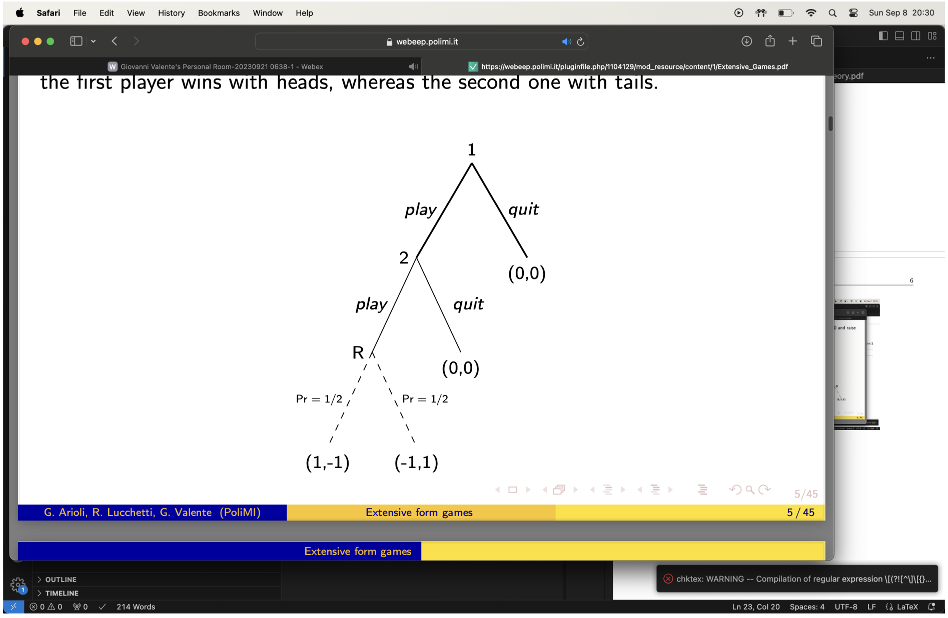


Figure 2.4: Chance game tree

The outcomes obtained by backward induction are: (4, 3) and (3, 4). In fact, Pl2 does not have any preference between (4, 3) and (0, 3)! Therefore, in general, uniqueness of solutions is not guaranteed.

2.3 The chess theorem

Theorem 2.3.1 (Von Neumann). *In the game of chess one and only one of the following alternatives holds:*

1. *The white has a way to win, no matter what the black does.*
2. *The black has a way to win, no matter what the white does.*
3. *The white has a way to force at least a draw, no matter what the black does, and the same holds for the black.*

Proof. Suppose the length of the game is $2K$ so each player has K choices to make. Call a_i the move of the White at her i-th stage and b_i the one of the Black. The first alternative in the chess theorem can be expressed as

$$\exists a_1 : \forall b_1 \exists a_2 : \forall b_2 \dots \exists a_K : \forall b_K \implies \text{white wins}$$

Now suppose this is not true. Then

$$\forall a_1 \exists b_1 : \forall a_2 : \exists b_2 : \dots \forall a_K : \exists b_K \implies \text{white does not win}$$

But this means exactly that Black has the possibility to get at least a draw. □

If White does not have a strategy to win no matter what Black does, then Black has the possibility to get at least the draw. Symmetrically, if Black does not have a strategy to win no matter what White does, then White has the possibility to get at least the draw Thus if the first and the second alternatives in the chess theorem are not true, necessarily the third one is true.

Von Neumann theorem extension

The von Neumann theorem applies to every finite game of perfect information where the possible result is either the victory of one player or a tie. Thus the following corollary holds:

Corollary 2.3.1.1. *Consider a finite perfect information game with two players, where the only possible outcomes are the victory of one or the other player. Then one and only one of the following alternative holds:*

1. *The first player can win, no matter what the second one does.*
2. *The second player can win, no matter what the first one does.*

The possible solutions of a game are the follwing:

- Very weak solution: the game has a rational outcome, but it is inaccessible, like in chess.
- Weak solution: the outcome of the game is known, but how to get to it is not (in general).
- Solution: it is possible to provide an algorithm to find a solution.

Example:
Consider the game of chomp. In this game we have a grid in which each player can remove a tile with all the others on the right and above it. In this game we have a solution if the grid is a square, and a weak solution if the grid is rectangular. In the latter case we may have the following situation:

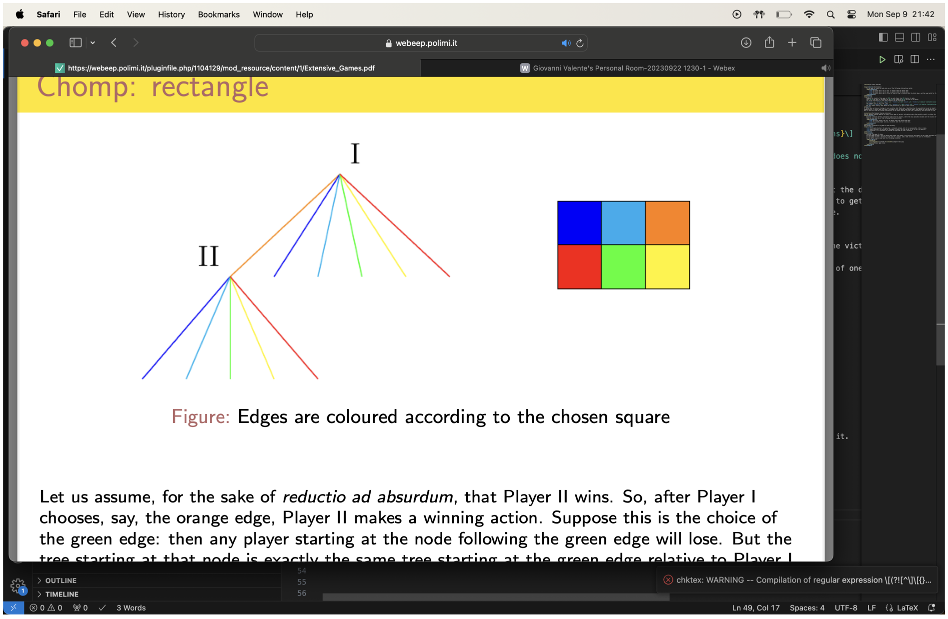


Figure 2.5: Rectangular chomp

Let us assume, for the sake of reductio ad absurdum, that Player II wins. So, after Player I chooses, say, the orange edge, Player II makes a winning action. Suppose this is the choice of the green edge: then any player starting at the node following the green edge will lose. But the tree starting at that node is exactly the same tree starting at the green edge relative to Player I. Thus Player I has a move available that guarantees victory, whereby Player II would have to lose. Since we derived a contradiction, the original assumption must be false: it is Player I that wins!

2.4 Impartial combinatorial games

Definition (*Impartial combinatorial game*)**.** An impartial combinatorial game is a game such that:

1. There are two players moving in alternate order.
2. There is a finite number of positions in the game.
3. The players follow the same rules.
4. The game ends when no further moves are possible.
5. The game does not involve chance.
6. In the classical version, the winner is the player leaving the other player with no available moves, in the mis‘ere version the opposite.

Example:
Examples of impartial combinatorial games are:

- k piles of cards. At her turn each player takes as many cards as she wants (at least one!) from one and only one pile.
- k piles of cards. At her turn each player takes as many cards as she wants (at least one) from no more than $j < k$ piles.
- k cards in a row. At her turn each player takes either j_1 or $\vdots$ or j_l cards.

In all these variants of the game, the player remaining without cards loses. In the first two cases the positions are $(n_1, \cdots, n_k)$ where n_i is a non negative integer for all i. In the last example positions can be seen as all non negative integers smaller or equal to k.

To solve this type of games we start by partitioning the set of all possible (finitely many) positions into two sets:

1. P-positions (i.e. previous player has a winning strategy): loosing.
2. N-positions (i.e. next player has a winning strategy): winning.

Note: it is the state of the game that matters, and not who is called to move. Rules for the partition (for the classical version):

- Terminal position $(0, 0, \cdots, 0)$ is a P-position (it is a losing position, as the player does not have any card left)
- From a P-position only N-positions are available
- From a N-position it is possible, yet not necessary, to go to a P-position.

Therefore, the player starting from a N-position wins.

2.4.1 Nim game

The Nim game is defined as $(n_1, \cdots, n_k)$ where n_i is a positive integer for all i. At her turn any player is supposed to take one (and only one) n_i and substitute it with $\hat{n}_i < n_i$. The winner is the player who arrives at the position $(0, \ldots, 0)$.

- Actions: taking away cards from one pile.
- Goal: to clear the whole table.

Theorem 2.4.1 (Bouton)**.** *In the Nim game the position $(n_1, n_2, \ldots, n_k)$ is a P-position if and only if:*

$$n_1 \oplus n_2 \oplus \cdots \oplus n_k = 0$$

Proof. Terminal position $(0, 0, \ldots, 0)$ is a P-position, with zero Nim-sum.

Positions with $n_1 \oplus n_2 \oplus \cdots \oplus n_N = 0$ go only to positions with non-zero Nim-sum. For, suppose that the next position $(\hat{n}_1, n_2, \ldots, n_N)$ is such that $\hat{n}_1 \oplus n_2 \oplus \cdots \oplus n_N = 0 = n_1 \oplus n_2 \oplus \cdots \oplus n_N$: then, by the cancelation law one has $\hat{n}_1 = n_1$, which is impossible since the game requires $\hat{n}_1 < n_1$.

Positions with $n_1 \oplus n_2 \oplus \cdots \oplus n_N \neq 0$ can go to positions with zero Nim-sum Let $z := n_1 \oplus n_2 \oplus \cdots \oplus n_N \neq 0$. Take a pile having 1 in the first column on the left of the expansion of z and put 0 there; then go right, leaving unchanged digits corresponding to 0 and changing them otherwise. Provably, the result is smaller than the original number. $\square$

Example:
From the following arrangement yielding non-zero Nim-sum: $4, 6, 5$. by taking a card out of the first row we go to the next one with zero Nim-sum $3, 6, 5$. So, there are three initial good moves, one for each row.

2.4.2 Conclusions

Games with perfect information can be solved by using backward induction. However backward induction is a concrete solution method only for very simple games, because of limited rationality. Depending on the game, we can reach different levels of solutions.

2.5 Strategies

In Backward induction a move must be specified at any node. Let P_i be the set of all the nodes where player i is called upon to make a move.

Definition (*Pure strategy*)**.** A pure strategy for player i is a function defined on the set P_i, associating to each node v in P_i a child x, or equivalently an edge (v, x).

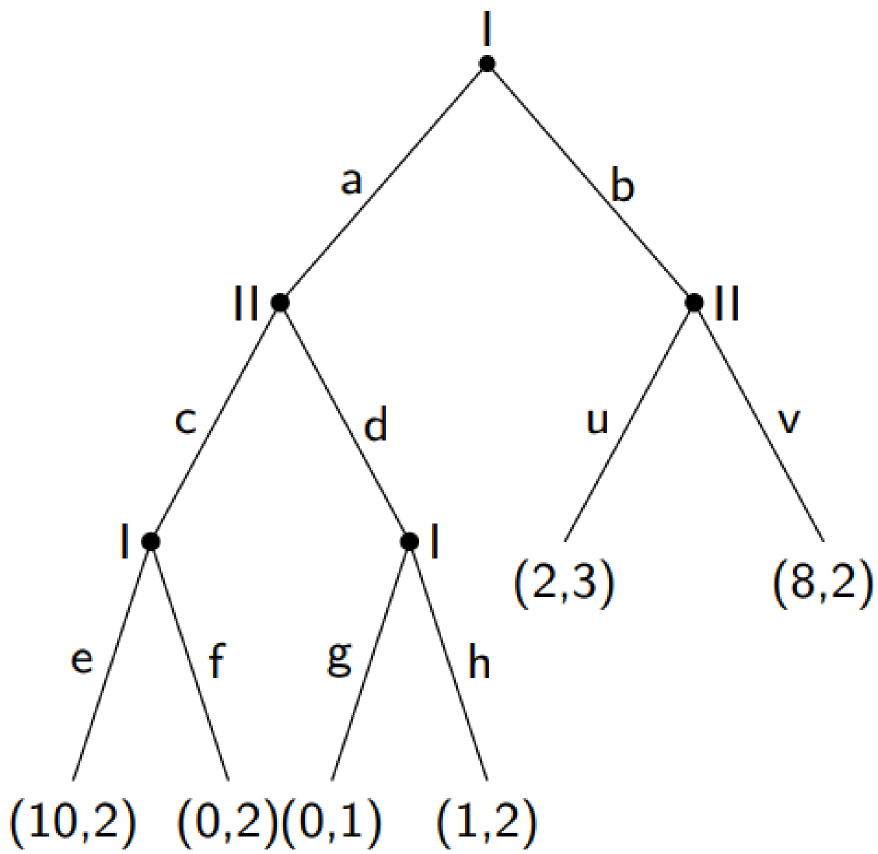
Definition (*Mixed strategy*)**.** A mixed strategy is a probability distribution on the set of the pure strategies.

When a player has n pure strategies, the set of her mixed strategies is:

$$\sum_n := \left\{ p = (p_1, \ldots, p_n) | p_i \geq 0 \text{ and } \sum p_i = 1 \right\}$$

$\sum_n$ is the fundamental simplex in n-dimensional space.

Example:
Consider the following tree:



The strategies in the tree are:

	cu	cv	du	dv
aeg	(10,2)	(10,2)	(0,1)	(0,1)
aeh	(10,2)	(10,2)	(1,2)	(1,2)
afg	(0,2)	(0,2)	(0,1)	(0,1)
afh	(0,2)	(0,2)	(1,2)	(1,2)
beg	(2,3)	(8,2)	(2,3)	(8,2)
beh	(2,3)	(8,2)	(2,3)	(8,2)
bfg	(2,3)	(8,2)	(2,3)	(8,2)
bfh	(2,3)	(8,2)	(2,3)	(8,2)

Remember that the first player is in the rows, while the second is in the columns. All combinations are listed even if equivalent (e.g. strategies b.. for Player II) The table has repeated pairs: different strategies can lead to the same outcomes

- Extensive form: the different moves of the players are presented in sequence
- Strategic form: all strategies of the players are presented at the same time

Theorem 2.5.1 (Von Neumann for strategies)**.** *In the chess game one of the following alternatives holds:*

1. *the white has a winning strategy.*
2. *the black has a winning strategy.*
3. *both players have a strategy leading them at least to a tie.*

We have the first outcome if there is a raw with all winning elements. We have the second outcome if there is a column with all winning elements. The third outcome has a mixed outcomes, comprising also ties (but not all three outcomes in the same row or column).

If $P_i = \{v_1, \ldots, v_k\}$ and v_j has n_j children, then the number of strategies of Player i is $n_1 \cdot n_2 \cdot \cdots \cdot n_k$. This shows that the number of strategies even in short games is usually very high.

Example:
If Tic-Tac-Toe is stopped after three moves, the first player has (without exploiting symmetries) $9 \cdot 7^{(8\times9)}$ strategies

2.6 Games with imperfect information

Sometimes players must make moves at the same time, and so they cannot have full knowledge of each other's moves. This can be still represented with a tree.

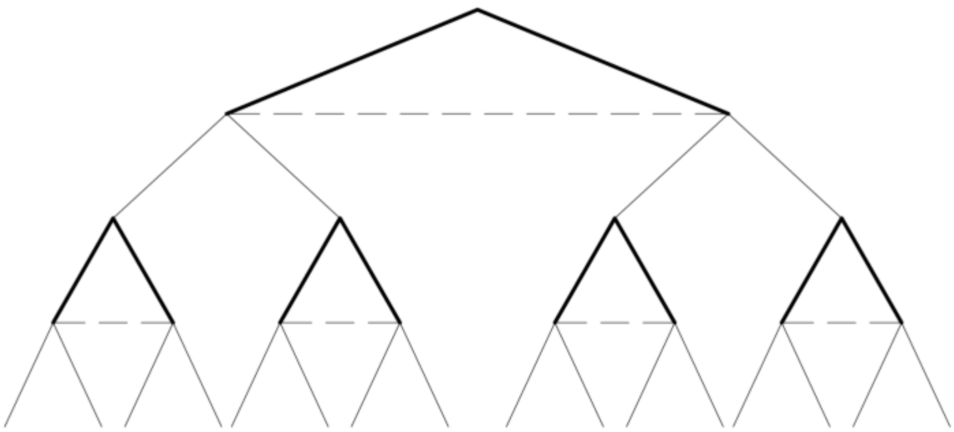


Figure 2.6: Tree with imperfect information

Dashed line: the player does not know exactly which vertex she finds herself in.

Definition (*Information set*)**.** An information set for a player i is a pair $(U_i, A(U_i))$ with the following properties:

1. $U_i \subset P_i$ is a nonempty set of vertices $v_1, \cdots, v_k$

2. each $v_j \in U_i$ has the same number of childre.
3. $A_i(U_i)$ is a partition of the children of $v_1 \cup \cdots \cup v_k$ with the property that each element of the partition contains exactly one child of each vertex v_j

Accordingly, player i knows to be in U_i, but not in which vertex she is. The partition yields the choice function, meaning that each set in $A_i(U_i)$ represents an available move for the player (graphically, it is the same choice, i.e. an edge, coming out of the different vertices).

Definition (*Extensive form game with imperfect information*)**.** An Extensive form Game with imperfect information is constituted by:

1. A finite set $N = \{1, \dots, n\}$ of players.
2. A game tree (V, E, x_0).
3. A partition made by sets $P_1, P_2, \dots, P_{n+1}$ of the vertices which are not leaves.
4. A partition $(U_i^j), j = 1, \dots, ki$ of the set P_i, for all i, with (U_i^j, A_i^j) information set for all players i for all vertices j (with the same number of children).
5. A probability distribution, for each vertex in P_{n+1}, defined on the edges going from the vertex to its children.
6. An n-dimensional vector attached to each leaf

Note that if the partition comprises just a single vertex, then a game with imperfect information becomes the same as a game with perfect information.

Definition (*Pure strategy*)**.** A pure strategy for player i in an imperfect information game is a function defined on the collection $\mathcal{U}$ of his information sets and assigning to each U_i in $\mathcal{U}$ an element of the partition $A(U_i)$. A mixed strategy is a probability distribution over the pure strategies.

A game of perfect information is a particular game of imperfect information where all information sets of all players are singletons.

CHAPTER 3

Zero sum games

3.1 Introduction

Definition (*Zero sum game*)**.** A two player zero sum game in strategic form is the triplet $(X, Y, f : X \times Y \to \mathbb{R})$. Here:

- X is the strategy space of Player 1.
- Y the strategy space of Player 2.
- $f(x, y)$ is what Player 1 gets from Player 2, when they play x, y respectively.

Given that f is the utility function of Player 1, by definition of zero sum games the utility function g of Player 2 must be:

$$g = -f$$

In the finite case $X = \{1, 2, \dots, n\}$, $Y = \{1, 2, \dots, m\}$ the game is described by a payoff matrix P, wherein Player 1 selects row i while Player 2 selects column j:

$$\begin{pmatrix} p_{11} & \cdots & p_{1m} \\ \cdots & p_{ij} & \cdots \\ p_{n1} & \cdots & p_{nm} \end{pmatrix}$$

Here, p_{ij} is the payment of Player 2 to Player 1 when they play i, j respectively. In order to choose the optimal strategy, each player can reason as follows:

- Player 1 can guarantee herself to get at least $v_1 = \max_i \min_j p_{ij}$.
- Player 2 can guarantee himself to pay at most $v_2 = \min_j max_i p_{ij}$.

v_1 and v_2 are said to be the conservative values of Player 1 and Player 2, respectively.

Example:
Consider the game:

$$\begin{pmatrix} 4 & 3 & 1 \\ 7 & 5 & 8 \\ 8 & 2 & 0 \end{pmatrix}$$

For the player 1 we pick the minimum for each row, that is : $\begin{pmatrix} 1 & 5 & 0 \end{pmatrix}$ and then we choose the maximum between them. Thus, the conservative value for the player 1 is $v_1 = 5$.

For the player 2 we pick the maximum for each column, that is : $\begin{pmatrix} 8 & 5 & 8 \end{pmatrix}$ and then we choose the mimimum between them. Thus, the conservative value for the player 2 is $v_2 = 5$.

Accordingly, the rational outcome is 5 and the rational behavior is $(\bar{i} = 2, \bar{j} = 2)$.

3.1.1 Rationality in zero sum games

Let us suppose the following:

- $v_1 = v_2 := v$,
- $\bar{i}$ the row such that $p_{\bar{i}\bar{j}} \geq v_1 = v$ for all j
- $\bar{j}$ the column such that $p_{\bar{i}\bar{j}} \leq v_2 = v$ for all i

Then $p_{\bar{i}\bar{j}} = v$ and $p_{\bar{i}\bar{j}} = v$ is the rational outcome of the game

$\bar{i}$ is an optimal strategy for Pl1, because she cannot get more than v_2, since v_2 is the conservative value of the second player $\bar{j}$ is an optimal strategy for Pl2, because he cannot pay less than v_1, since v_1 is the conservative value of the first player $\bar{i}$ maximizes the function $\alpha(i) = minjp_{ij}$; $\bar{j}$ minimizes the function $\beta(j) = \max_i p_{ij}$.

3.1.2 Extension to arbitrary games

Let the triplet $(X, Y, f : X \times Y \to \mathbb{R})$ represent a game between two players, wherein their respective strategy spaces X and Y may not be finite sets. For the sake of rational behaviour, the players can guarantee to themselves the following outcomes:

- Player 1: $v_1 = \sup_x \inf_y f(x, y)$
- Player 2: $v_2 = \inf_y \sup_x f(x, y)$

The outcomes v_1, v_2 are the conservative values of the players. If $v_1 = v_2$, we set $v = v_1 = v_2$ and we say that the game has value v.

Optimality Let X and Y be arbitrary sets. Suppose:

1. $v_1 = v_2 := v$
2. there exists strategy $\bar{x}$ such that $f(\bar{x}, y) \geq v$ for all $y \in Y$
3. there exists strategy $\bar{y}$ such that $f(x, \bar{y}) \leq v$ for all $x \in X$

(the last two conditions are needed if the sets are infinite and not compact). Then:

- v is the rational outcome of the game.
- $\bar{x}$ is an optimal strategy for Pl1.
- $\bar{y}$ is an optimal strategy for Pl2.

Observe $\bar{x}$ is optimal for Pl1 since it maximizes the function $\alpha(x) = \inf_y f(x, y)$ $\bar{y}$ is optimal for Pl2 since it minimizes the function $\beta(y) = \sup_x f(x, y)$ where $\alpha(x)$ is the value of the optimal choice of Pl2 if he knows that Pl1 plays x, and $\beta(y)$ is the value of the optimal choice of Pl1 if she knows that Pl2 plays y.

Proposition. Let X, Y be nonempty sets and let $f : X \times Y \to \mathbb{R}$ be an arbitrary real valued function. Then:

$$v_1 = \sup_x \inf_y f(x, y) \leq \inf_y \sup_x f(x, y) = v_2$$

Proof. By definition, for all x, y:

$$\inf_y f(x, y) \leq f(x, y) \leq \sup_x f(x, y)$$

Thus:

$$\alpha(x) = \inf_y f(x, y) \leq \sup_x f(x, y) = \beta(y)$$

Since for all $x \in X$ and $y \in Y$ it holds that

$$\alpha(x) \leq \beta(y)$$

it follows that

$$\sup_x \alpha(x) \leq \inf_y \beta(y)$$

$\square$

As a consequence, in every game $v_1 \leq v_2$.

Example:
Consider the rock, scissors, and paper game:

$$\begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$$

The conservative values are not the same: in fact, $v_1 = -1$ and $v_2 = 1$.

There is no winning strategy since each player always plays randomly. But if the game is repeated many times, the rational solution for both players is to play each option one-third of the times, so that in the long run their expected utility is zero. By extending the game with mixed strategies, both conservative values become 0.

Case $v_1 < v_2$ Consider the case of mixed strategies with a gmae with an $n \times m$ utility matrix P. In this case the strategy spaces are defined as:

$$\sum_k = \left\{ x = (x_1, \ldots, x_k) | x_i \geq 0 \text{ and } \sum_{i=1}^{k} x_i = 1 \right\}$$

with $k = n$ for Pl1 and $k = m$ for Pl2. The utility function is:

$$f(x, y) = \sum_{i=1,\ldots,n, j=i,\ldots,m} x_i y_j p_{ij} = (x, Py)$$

Here p_{ij} is an element of P corresponding to the utility of Pl1 when she plays row i and Pl2 plays column j (of course the utility of Pl2 is just the opposite). Thus, the mixed extension of the initial game is

$$\left(\sum_n, \sum_m, f(x, y) = (x, Py) \right)$$

CHAPTER 4

Mathematical concepts

4.1 Binary sum

Define an operation $\oplus$ on $\mathbb{N} = \{0, 1, 2, \dots, n, \dots\}$ in the following way: for $n_1, n_2 \in \mathbb{N}$:

1. Write n_1, $_n2$ in binary form, denoted by $[n_1]_2$, $[n_2]_2$.
2. Write the sum $[n_1]_2 \oplus [n_2]_2$ in binary form where $\oplus$ is the (usual) sum, but without carry.
3. What you get is the result in binary form.

> **Example:**
> The $\oplus$ operation applied to 1,2,4, and 1. In binary they are 001 010, 100, and 001. The sum is 110, thus 6.

4.2 Group

Definition (*Group*)**.** A nonempty set A with a binary operation $\cdot$ defined on it is called a group provided that:

1. for $a, b \in A$ the element $a \cdot b \in A$
2. $\cdot$ is associative: $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
3. there is a (unique) element e, called identity, such that $a \cdot e = e \cdot a = a$ for all $a \in A$
4. for every $a \in A$ there is $b \in A$ such that $a \cdot b = b \cdot a = e$: such an element isunique and called inverse of a

If $a \cdot b = b \cdot a$ for all $a, b \in A$ the group is called abelian.

> **Example:**
> Examples of abelian groups: The integers $\mathbb{Z}$, equipped with the usual sum The real numbers excluded 0, equipped with the usual product
>
> Examples of non-abelian groups: The $n \times n$ matrices with non-zero determinant, equipped with the usual product

Proposition. Let $(A, \cdot)$ be a group. Then the cancellation law holds:

$$a \cdot b = a \cdot c \implies b = c$$

Proof. By multiplying by a^{-1} both sides of the equation $a \cdot b = a \cdot c$, one obtains $a^{-1}a \cdot b = a^{-1}a \cdot c$. Then, insofar a^{-1} is the inverse of a, this expression reduces to $e \cdot b = e \cdot c$, which by the property of the identity e is exactly equal to $b = c$. □

Proposition. The set of the natural numbers with operation $\oplus$ is an abelian group.

Proof. The identity element is of course 0. The inverse of n is n itself. Associativity and commutativity of $\oplus$ are easy to show. □

Therefore, the cancellation law holds: $n_1 \oplus n_2 = n_1 \oplus n_3 \implies n_2 = n_3$.

4.3 Convexity

Definition (*Convex set*)**.** A set $C \subset \mathbb{R}^n$ is convex just in case for any $x, y \in C$, provided $\lambda \in [0, 1]$, one has:

$$\lambda x + (1 - \lambda)y \in C$$

The intersection of an arbitrary family of convex sets is convex. A closed convex set with nonempty interior coincides with the closure of its internal points.

Definition (*Convex combination*)**.** We shall call a convex combination of elements $x_1, \ldots, x_n$ any vector x of the form

$$x = \lambda_1 x_1 + \cdots + \lambda_n x_n$$

with $\lambda_1 \geq 0, \ldots, \lambda_n \geq 0$ and $\sum_{i=1}^{n} \lambda_i = 1$

Proposition. A set C is convex if and only if for every $\lambda_1 \geq 0, \ldots, \lambda_n \geq 0$ such that $\sum_{i=1}^{n} \lambda_i = 1$, for every $c_1, \ldots, c_n \in C$, for all n, then $\sum_{i=1}^{n} \lambda_i c_i \in C$.

If C is not convex, then there is a smallest convex set containing C: it is the intersection of all convex sets containing C.

Definition (*Convex hull*)**.** The convex hull of a set C, denoted by co C, is:

$$\text{co } C = \bigcap_{A \in \mathcal{C}} A$$

where $\mathcal{C} = \{A | C \subset A \text{ and } A \text{ is convex}\}$.

Proposition. Given a set C, then

$$\text{co } C = \left\{ \sum_{i=1}^{n} \lambda_i c_i | \lambda_i \geq 0, \sum_{i=1}^{n} \lambda_i = 1, c_i \in C \quad \forall i, n \in N \right\}$$

The convex hull of any set C is the set of all convex combinations of points in C. When C is a finite collection of points, its convex hull co C is called a polytope. E.g. if C contains three points, co C is the triangle (i.e. a polygone) with such points at its angles, which includes also all the points inside

Theorem 4.3.1. *Given a closed convex set C and a point x outside C, there is a unique element $p \in C$ such that, for all $c \in C$:*

$$\|p - x\| \leq \|c - x\|$$

The projection p is characterized by the following properties:

1. $p \in C$
2. $(x - p, c - p) \leq 0$ *for all* $c \in C$

That is, p is the closest point to x belonging to the set C, and by the last property it forms an obtuse angle between x and any $c \in C$.

Theorem 4.3.2. *Let C be a convex proper subset of the Euclidean space $\mathbb{R}^l$ and assume $\bar{x} \in$ cl C^c. Then there is an element $0 \neq x^* \in \mathbb{R}^l$ such that, $\forall c \in C$:*

$$(x^*, c) \geq (x^*, \bar{x})$$

This theorem means that a criterion to tell apart an external point from any internal point.

Proof. Suppose $\bar{x} \notin \text{cl } C$ and call p its projection on cl C. Based on the previous theorem, in follows that $(\bar{x} - p, c - p) \leq 0$ for all $c \in C$. By setting $x^* := p - \bar{x} \neq 0$, the inequality becomes $(-x^*, c - \bar{x} - x^*) = (-x^*, -x^*) + (-x^*, c - \bar{x}) \leq 0$, that is $(x^*, c - \bar{x}) \geq \|x^*\|^2$. Then, since $\|x^*\|^2 > 0$, by linearity one obtains

$$(x^*, c) \geq (x^*, \bar{x}) \quad \forall c \in C$$

As x^* appears in both sides, by renormalization we can choose $\|x^*\| = 1$. If $\bar{x} \in \text{cl } C \setminus C$, take a sequence $\{x_n\} \subset C^c$ such that $x_n \to \bar{x}$. From the first step of the proof, we can find some norm one x_n^* such that

$$(x_n^*, c) \geq (x_n^*, x_n) \quad \forall c \in C$$

So, given that for some sub-sequence one has $x_n^* \to x^*$, taking the limit of the above inequality yields

$$(x^*, c) \geq (x^*, \bar{x}) \quad \forall c \in C$$

□

Corollary 4.3.2.1. *Let C be a closed convex set in a Euclidean space, let x be on the boundary of C. Then there is a hyperplane containing x and leaving all of C in one of the halfspaces determined by the hyperplane.*

Such an hyperplane is said to be an hyperplane supporting C at x

Corollary 4.3.2.2. *Let C be a closed convex set in a Euclidean space. Then C is the intersection of all half-spaces containing it.*

Theorem 4.3.3. *Let A, C be closed convex subsets of $\mathbb{R}^l$ such that int$A$ is nonempty and int$A \cap C = \varnothing$. Then there are $0 \neq x^*$ and $b \in \mathbb{R}$ such that, $\forall a \in A$, $\forall c \in C$:*

$$(x^*, a) \geq b \geq (x^*, c)$$

This means a criterion to determine whether a point is in A or in C.

Proof. Since $\bar{x} = 0 \in (\text{int}A - C)^c$, from the previous separation theorem with $\bar{x} = 0$ there is $x^* \neq 0$ such that

$$(x^*, x) \geq 0 \quad \forall x \in \text{int}A - C$$

Thus, for $x = a - c$ by linearity we obtain

$$(x^*, a) \geq (x^*, c) \quad \forall a \in \text{int}A, \forall c \in C$$

By extension this implies

$$(x^*, a) \geq (x^*, c) \quad \forall a \in \text{cl int}A = A, \forall c \in C$$

□

$H = \{x : (x^*, x) = b\}$ is called the separating hyperplane: A and C are contained in the two different half-spaces generated by H.