Model Identification And Data Analysis II
Exercises

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Abstract

The course encompasses a diverse array of topics, including non-parametric system identification utilizing the subspace-based state-space approach, Kalman Filter applications for prediction, virtual-sensing, and gray-box system identification. It also covers the analysis and design of closed-loop systems using the minimum-variance approach, as well as non-linear system identification involving parametric nonlinear fitting, NARMAX models, and the optimal design of basis functions using principal component analysis. Furthermore, the course includes frequency-domain parametric estimation of models from data and extends into recursive system identification, which broadens the scope to encompass time-varying systems.

Contents

CHAPTER 1

Non-parametric system modeling

1.1 Exercise one

Consider a second-order system in state-space representation:

$$F = \begin{bmatrix} 0 & 2 \\ \frac{1}{2} & 3 \end{bmatrix} \quad G = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \quad H = \begin{bmatrix} 1 & 0 \end{bmatrix} \quad D = \begin{bmatrix} 0 \end{bmatrix}$$

1. Write the system of difference equations.
2. Compute the system transfer function.
3. Compute the poles.
4. Compute five samples of the impulse response.
5. Check the observability and reachability of the system.
6. Compute the Hankel matrix.

Solution

1. The system of difference equations for the given matrices are:

$$\begin{cases} x_1(t+1) = x_2(t) + \frac{1}{2}u(t) \\ x_2(t+1) = \frac{1}{2}x_1 + 3x_2(t) + \frac{1}{2}u(t) \\ y(t) = x_1(t) \end{cases}$$

2. The transfer function can be derived using the formula:

$$W(z) = H\left(zI - F\right)^{-1} G + D$$

In this scenario, it simplifies to:

$$W(z) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} z & -2 \\ -\frac{1}{2} & z-3 \end{bmatrix}^{-1} \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = \frac{1}{2} \frac{z-1}{z^2 - 3z - 1}$$

Alternatively, we can compute the transfer function using the shift operators as follows:

$$\begin{cases} zx_1(t) = x_2(t) + \frac{1}{2}u(t) \\ zx_2(t) = \frac{1}{2}x_1 + 3x_2(t) + \frac{1}{2}u(t) \end{cases}$$

Reformulating the system yields:

$$\begin{cases} x_1(t) = \frac{x_2(t)}{x} + \frac{1}{2z}u(t) \\ x_2(t) = \frac{\frac{1}{2}x_1 + \frac{1}{2}u(t)}{z-3} \end{cases}$$

Upon substitution, we obtain:

$$\begin{aligned} x_2(t) &= \frac{\frac{1}{2}\left(\frac{x_2(t)}{x} + \frac{1}{2z}u(t)\right) + \frac{1}{2}u(t)}{z-3} \\ &= \frac{\frac{1}{4} + \frac{1}{2}z}{z^2 - 3z - 1}u(t) \end{aligned}$$

Now, the expression for x_1 can be derived as:

$$x_1(t) = \frac{\frac{1}{4} + \frac{1}{2}zu(t)}{z^2 - 3z - 1} + \frac{1}{2z}u(t) = \frac{1}{2}\frac{z-1}{z^2 - 3z - 1}u(t)$$

Thus, the transfer function is given by:

$$y(t) = \frac{1}{2}\frac{z-1}{z^2 - 3z - 1}u(t)$$

3. To find the poles of the function, we equate the denominator to zero:

$$z^2 - 3z - 1 = 0 \to \lambda_{1,2} = \frac{3 \pm \sqrt{13}}{2}$$

As a result, one pole lies outside the unit circle, indicating that the system is unstable.

4. We can approach solving this problem through two methods:

   - Direct calculation of Impulse Response coefficients: compute the impulse response coefficients iteratively.

$$\begin{cases} \omega(0) = 0 \\ \omega(1) = HG = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = \frac{1}{2} \\ \omega(2) = HFG = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ \frac{1}{2} & 3 \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = 1 \\ \omega(3) = HF^2G = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ \frac{1}{2} & 3 \end{bmatrix}^2 \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = \frac{7}{2} \\ \omega(4) = HF^3G = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ \frac{1}{2} & 3 \end{bmatrix}^3 \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = \frac{23}{2} \end{cases}$$

   Note the increasing trend of the coefficients, indicating instability in the system.

- The alternative approach involves employing long division. Beginning with the transfer function expressed in negative power notation as $\frac{\frac{1}{2}z^{-1}-\frac{1}{2}z^{-2}}{1-3z^{-1}-z^{-2}}$, we divide the numerator by the denominator through five steps to yield the quotient:

$$E(z) = \frac{1}{2}z^{-1} + z^{-2} + \frac{7}{2}z^{-3} + \frac{23}{2}z^{-4}$$

In this representation, the coefficients correspond to the impulse response of the system:

$$\begin{cases} \omega(0) = 0 \\ \omega(1) = \frac{1}{2} \\ \omega(2) = 1 \\ \omega(3) = \frac{7}{2} \\ \omega(4) = \frac{23}{2} \end{cases}$$

5. The observability matrix, denoted as O, is structured as:

$$O = \begin{bmatrix} H \\ HF \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

This matrix possesses full rank, implying that the system is entirely observable.

On the other hand, the reachability matrix, denoted as R, takes the form:

$$R = \begin{bmatrix} G \\ FG \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 1 \\ \frac{1}{2} & \frac{7}{4} \end{bmatrix}$$

Similar to the observability matrix, this matrix also exhibits full rank, indicating that the system is fully reachable.

6. The Hankel matrix, H_n, can be computed in two ways:

- Utilizing the previously computed impulse response coefficients:

$$H_n = \begin{bmatrix} \frac{1}{2} & 1 \\ 1 & \frac{7}{2} \end{bmatrix}$$

- Deriving it from the observability and reachability matrices:

$$H_2 = O_2R_2 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 1 \\ \frac{1}{2} & \frac{7}{4} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 1 \\ 1 & \frac{7}{2} \end{bmatrix}$$

1.2 Exercise two

Given the impulse response:

$$\omega(t) = \begin{cases} 0 & \text{if } t \leq 1 \\ (-2)^{2-t} & \text{if } t > 1 \end{cases}$$

1. Compute the transfer function.

2. Write a state space representation of the system.

3. Apply the change of coordinates $\tilde{x} = Tx$, where $T = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$.
4. Compute the transfer function in the new coordinates.
5. Compute the Hankel matrix H_3 and verify that $H_3 = O_3 R_3$.
6. Verify that $F = O_3(1:2,:)$ and $F = R_3(2:3,:)$.

Solution

1. In general, the output of a system can be represented in the Z-domain as:

$$y(t) = \left[\sum_{k=0}^{+\infty} \omega(k) z^{-k}\right] u(t)$$

In our specific case, we start with the impulse response $\omega(t)$ and compute the transfer function $W(z)$ as follows:

$$\begin{aligned} W(z) &= \sum_{k=0}^{+\infty} \omega(k) z^{-k} \\ &= \sum_{k=0}^{1} \underbrace{\omega(k) z^{-k}}_{0} + \sum_{k=2}^{1} \underbrace{\omega(k) z^{-k}}_{(-2)^{2-k}} \\ &= \sum_{k=2}^{1} (-2)^{2-k} z^{-k} \\ &= z^{-2} \sum_{k=2}^{1} (-2)^{-(k-2)} z^{-(k-2)} \end{aligned}$$

We then redefine the index $t = k - 2$, leading to:

$$W(z) = z^{-2} \sum_{t=0}^{1} (-2)^{-t} z^{-t} = z^{-2} \sum_{t=0}^{1} \left(-\frac{1}{2} z^{-1}\right)^{t}$$

By recalling the geometric series formula (applicable for $|a| < 1$):

$$\sum_{k=0}^{+\infty} s^k = \frac{1}{1-a}$$

We rewrite our expression as:

$$\begin{aligned} W(z) &= z^{-2} \sum_{t=0}^{1} \left(-\frac{1}{2} z^{-1}\right)^{t} \\ &= z^{-2} \frac{1}{1 + \frac{1}{2} z^{-1}} \end{aligned}$$

2. To obtain the transfer function $W(z)$ in canonical form, we rewrite it as:

$$W(z) = \frac{0z + 1}{z^2 + \frac{1}{2}z + 0}$$

The matrix F, which represents the coefficients of the denominator polynomial, is extracted from the lowest terms coefficient of the numerator and denominator:

$$F = \begin{bmatrix} 0 & 1 \\ 0 & -\frac{1}{2} \end{bmatrix}$$

The matrix G, related to the input polynomial, depends on the highest order monomials of the numerator and denominator:

$$G = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The H contains the numerator coefficients read backwards:

$$H = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

Finally, since there's no direct feedthrough term, D is equal to 0.

3. The original system undergoes a change of coordinates according to the following equations:

$$\begin{cases} \tilde{x}(t+1) = TFT^{-1}\tilde{x}(t) + TG\tilde{u}(t) \\ y(t) = HT^{-1}\tilde{x}(t) + Du(t) \end{cases}$$

Substituting the matrices into the above equations, we obtain the transformed system as follows:

- $\tilde{F} = TFT^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1} = \frac{1}{4} \begin{bmatrix} 1 & -1 \\ 3 & -3 \end{bmatrix}$
- $\tilde{G} = TG = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$
- $\tilde{H} = HT^{-1} = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1} = \frac{1}{2} \begin{bmatrix} 1 & 1 \end{bmatrix}$
- $\tilde{D} = 0$

4. The transfer function now takes the form:

$$\tilde{W} = \tilde{H}(zI - \tilde{F})^{-1}\tilde{G} + \tilde{D} = W(z)$$

This remains the same because the input-output relation remains unchanged despite the change in coordinates.

5. The third-order Hankel matrix, H_3, is given by:

$$H_3 = \begin{bmatrix} \omega(1) & \omega(2) & \omega(3) \\ \omega(2) & \omega(3) & \omega(4) \\ \omega(3) & \omega(4) & \omega(5) \end{bmatrix}$$

Given the values of $\omega(t)$ from the provided formula:

$$\begin{cases} \omega(0) = 0 \\ \omega(1) = 0 \\ \omega(2) = 1 \\ \omega(3) = -\frac{1}{2} \\ \omega(4) = \frac{1}{4} \\ \omega(5) = -\frac{1}{8} \end{cases}$$

The Hankel matrix becomes:

$$H_3 = \begin{bmatrix} 0 & 1 & -\frac{1}{2} \\ 1 & -\frac{1}{2} & \frac{1}{4} \\ -\frac{1}{2} & \frac{1}{4} & -\frac{1}{8} \end{bmatrix}$$

The extended reachability matrix, R_3, and the extended observability matrix, O_3, are computed as follows:

$$R_3 = \begin{bmatrix} G & FG & F^2G \end{bmatrix} = \begin{bmatrix} 0 & 1 & -\frac{1}{2} \\ 1 & -\frac{1}{2} & \frac{1}{4} \end{bmatrix}$$

$$O_3 = \begin{bmatrix} H \\ HF \\ HF^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & -\frac{1}{2} \end{bmatrix}$$

To demonstrate their equality, we can multiply these matrices.

6. Upon visual inspection, we can confirm that the equality holds true between the extended reachability matrix R_3 and the extended observability matrix O_3, as they both match the form of the Hankel matrix H_3.

1.3 Exercise three

Given the transfer function:

$$W(z) = \frac{z+2}{(z+2)\left(z+\frac{1}{2}\right)}$$

1. Derive the state space representation in controllable canonical form.
2. Verify observability and reachability.
3. Compute the first five samples of both $x(t)$ and $y(t)$ for the impulse response.
4. Determine an analytical expression for $y(t)$ when the input is an impulse.
5. Identify matrices F, G, and H using the noise-free 4SID method.
6. Calculate the transfer function from $\hat{F}$, $\hat{G}$, $\hat{H}$, and $\hat{D}$.

Solution

1. The canonical form of the transfer function is:

$$W(z) = \frac{z+2}{z^2 + \frac{5}{3}z + 1}$$

The corresponding state space representation is:

$$F = \begin{bmatrix} 0 & 1 \\ -1 & -\frac{5}{2} \end{bmatrix} \quad G = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad H = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad D = \begin{bmatrix} 0 \end{bmatrix}$$

2. The observability matrix is given by:

$$O_2 = \begin{bmatrix} H \\ HF \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -1 & -\frac{1}{2} \end{bmatrix}$$

Since the rank is one, the system is not fully observable.

The reachability matrix is:

$$R_2 = \begin{bmatrix} F \\ FG \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -\frac{5}{2} \end{bmatrix}$$

With a rank of two, the system is fully reachable.

3. The state space representation can also be expressed as follows:

$$\begin{cases} x_1(t+1) = x_2(t) \\ x_2(t+2) = -x_1(t) - \frac{5}{2}x_2(t) + u(t) \\ y(t) = 2x_1(t) + x_2(t) \end{cases}$$

We can obtain the impulse response using the difference equation. Note that $u(0) = 1$ and is null otherwise since it represents an impulse.

t	$x_1(t)$	$x_2(t)$	$y(t)$
0	0	0	0
1	0	1	1
2	1	$-\frac{5}{2}$	$-\frac{1}{2}$
3	$-\frac{5}{2}$	$\frac{21}{4}$	$\frac{1}{4}$
4	$\frac{21}{4}$	$-\frac{85}{8}$	$-\frac{1}{8}$

4. The simplified transfer function is:

$$W(z) = \frac{z^{-1}}{1 + \frac{1}{2}z^{-1}}$$

Using the geometric series, we have:

$$\begin{aligned} W(z) &= z^{-1} \sum_{k=0}^{+\infty} \left(-\frac{1}{2}z^{-1}\right)^k \\ &= z^{-1} \sum_{k=0}^{+\infty} \left(-\frac{1}{2}\right)^k z^{-k} \\ &= \sum_{k=0}^{+\infty} \left(-\frac{1}{2}\right)^k z^{-(k+1)} \end{aligned}$$

By defining $t = k + 1$, we obtain:

$$
\begin{aligned}
W(z) &= \sum_{k=0}^{+\infty} \left(-\frac{1}{2}\right)^{k} z^{-(k+1)} \\
&= \sum_{t=1}^{+\infty} \left(-\frac{1}{2}\right)^{t-1} z^{-t} \\
&= 0z^{-1} \sum_{t=1}^{+\infty} \left(-\frac{1}{2}\right)^{t-1} z^{-t}
\end{aligned}
$$

The added term is to compensate for the sum starting from one instead of zero. The final result is:

$$
\omega(t) = \begin{cases} 0 & \text{if } t = 0 \\ \left(-\frac{1}{2}\right)^{t-1} & \text{if } t > 0 \end{cases}
$$

5. Here are the steps to follow:

   (a) Compute the Hankel matrix of increasing order until it loses full rank:

   $$H_1 = \begin{bmatrix} \omega(1) \end{bmatrix} = \begin{bmatrix} 1 \end{bmatrix}$$

   Since this matrix is full rank, we proceed:

   $$H_2 = \begin{bmatrix} \omega(1) & \omega(2) \\ \omega(2) & \omega(3) \end{bmatrix} = \begin{bmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{4} \end{bmatrix}$$

   This matrix is not full rank, indicating that the system order is $n = 1$.

   (b) Factorize H_2. Using the first row of H_2 as R_2:

   $$R_2 = \begin{bmatrix} 1 & -\frac{1}{2} \end{bmatrix}$$

   From the relation $H_2 = O_2 R_2$, we find:

   $$O_2 = \begin{bmatrix} 1 \\ -\frac{1}{2} \end{bmatrix}$$

   (c) Compute the system matrices.

   $$
   \begin{aligned}
   \hat{H} &= O_2(1,:) = \begin{bmatrix} 1 \end{bmatrix} \\
   \hat{G} &= R_2(:,1) = \begin{bmatrix} 1 \end{bmatrix} \\
   \hat{F} &= O_2(1,:)^{-1} O_2(2,:) = \begin{bmatrix} -\frac{1}{2} \end{bmatrix} \\
   \hat{D} &= \begin{bmatrix} 0 \end{bmatrix}
   \end{aligned}
   $$

6. The transfer function is:

$$W(z) = \hat{H}\left(zI - \hat{F}\right)^{-1} \hat{G} + \hat{D} = \frac{1}{z + \frac{1}{2}}$$

1.4 Exercise four

We are provided with the impulse response:

$$\omega(0) = 0 \quad \omega(1) = 0 \quad \omega(2) = 2 \quad \omega(3) = 0 \quad \omega(4) = 1 \quad \omega(5) = 0$$

We're tasked with:

1. Identifying the system order.
2. Identifying the system matrices using the 4SID method.
3. Computing the transfer function.
4. Finding an analytical expression for $\omega(t)$.

Solution

1. We compute the Hankel matrix in increasing order until we obtain a matrix with no full rank:
$$H_1 = \begin{bmatrix} \omega(1) \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$$
Since this matrix is not full rank, but the rank cannot be $n = -1$, we increase the order:
$$H_2 = \begin{bmatrix} \omega(1) & \omega(2) \\ \omega(2) & \omega(3) \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$$
The rank is two, so we continue:
$$H_3 = \begin{bmatrix} \omega(1) & \omega(2) & \omega(3) \\ \omega(2) & \omega(3) & \omega(4) \\ \omega(3) & \omega(4) & \omega(5) \end{bmatrix} = \begin{bmatrix} 0 & 2 & 0 \\ 2 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$
The rank is two again, so we stop. The order of the system is determined to be $n = 2$.

2. Let's proceed with the second and third steps:
   - Factorize H_3. The reachability matrix is:
   $$R_3 = H_3(1:2,:) = \begin{bmatrix} 0 & 2 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$
   From the relation $O_3 R_3 = H_3$, we obtain:
   $$0_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \frac{1}{2} & 0 \end{bmatrix}$$
   - Now, we can determine the matrices:
   $$\hat{G} = R_3(:,1) = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$
   $$\hat{H} = O_3(1,:) = \begin{bmatrix} 1 & 0 \end{bmatrix}$$
   $$\hat{F} = O_3(1:2,:)^{-1} O_3(2:3,:) = \begin{bmatrix} 1 & 0 \\ \frac{1}{2} & 0 \end{bmatrix}$$
   $$\hat{D} = \begin{bmatrix} 0 \end{bmatrix}$$

3. The transfer function is computed as:

$$W(z) = \hat{H}\left(zI - \hat{F}\right)^{-1}\hat{G} + \hat{D} = \frac{2}{z^2 + \frac{1}{2}}$$

4. We have the transfer function:

$$W(z) = \frac{2z^{-2}}{1 - \frac{1}{2}z^{-2}}$$

Using the geometric series, we can express it as:

$$\begin{aligned} W(z) &= 2z^{-2}\sum_{t=0}^{+\infty}\left(\frac{1}{2}z^{-2}\right)^k \\ &= 2z^{-2}\sum_{t=0}^{+\infty}\left(\frac{1}{2}\right)^k z^{-2k} \end{aligned}$$

From this, we can derive the analytical expression for $\omega(t)$:

$$\begin{cases} 0 & \text{if } t = 0 \\ 0 & \text{if } t = 2k-1 \\ \left(\frac{1}{2}\right)^{\frac{t}{2}-2} & \text{if } t = 2k \end{cases}$$

1.5 Exercise five

Given the system:

$$\begin{cases} x(t+1) = \frac{1}{2}x(t) + 2u(t) \\ y(t) = 3x(t) \end{cases}$$

We are tasked with:

1. Computing the first five samples of the impulse response.
2. Checking the rank of H_2 and justifying it.
3. Identifying the system matrices using the 4SID method.

Solution

1. Here are the first five samples of the impulse response:

$$\begin{cases} \omega(0) = 0 \\ \omega(1) = HG = 6 \\ \omega(2) = HFG = 3 \\ \omega(3) = HF^2G = \frac{3}{2} \\ \omega(4) = HF^3G = \frac{3}{4} \end{cases}$$

2. The matrix H_2 is given by:

$$H_2 = \begin{bmatrix} \omega(1) & \omega(2) \\ \omega(2) & \omega(3) \end{bmatrix} = \begin{bmatrix} 6 & 3 \\ 3 & \frac{3}{2} \end{bmatrix}$$

   Upon evaluating its rank, we find that it is one. This is correct, as expected for a first-order system.

3. Let's proceed with the second and third steps:

   - Factorize H_2. The reachability matrix is:

   $$R_2 = H_2(1,:) = \begin{bmatrix} 6 & 3 \end{bmatrix}$$

   From the relation $O_2 R_2 = H_2$, we obtain:

   $$0_2 = \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix}$$

   - Now, we can determine the matrices:

   $$\hat{G} = R_2(:,1) = \begin{bmatrix} 1 \end{bmatrix}$$

   $$\hat{H} = O_2(1,:) = \begin{bmatrix} 6 \end{bmatrix}$$

   $$\hat{F} = O_2(1,:)^{-1} O_2(2,:) = \begin{bmatrix} \frac{1}{2} \end{bmatrix}$$

   $$\hat{D} = \begin{bmatrix} 0 \end{bmatrix}$$

1.6 Exercise six

Given the impulse response:

$$\omega(t) = \frac{4}{3}\left(-\frac{1}{3}\right)^t \qquad t \geq 0$$

We are tasked with:

1. Determining if the data-generating system is stable and strictly proper.
2. Finding the transfer function.
3. Identifying the system matrices using the 4SID algorithm.
4. Identifying the system matrices using the controllable canonical form.

Solution

1. The system is asymptotically stable because as time tends to infinity, the impulse response tends to zero. Since $\omega(0) \neq 0$, the system is just proper and not strictly proper.

2. We can rewrite the transfer function as:

$$W(z) = \sum_{t=0}^{+\infty} \omega(t) z^{-t} = \sum_{t=0}^{+\infty} \frac{4}{3}\left(-\frac{1}{3}\right)^{t} z^{-t}$$

Using the geometric series formula, we get:

$$W(z) = \frac{4}{3}\left(-\frac{1}{3}z^{-1}\right)^{t} = \frac{4}{3}\frac{1}{1+\frac{1}{3}z^{-1}}$$

3. The 4SID method applies to proper systems for matrices $\hat{F}$, $\hat{G}$, and $\hat{H}$. The only difference is that $\hat{D} = \omega(0)$. The method involves the following steps:

   (a) Identify the order using the Hankel matrix:

   $$H_1 = \begin{bmatrix}\omega(1)\end{bmatrix} = \begin{bmatrix}-\frac{1}{9}\end{bmatrix}$$

   Since the rank is one, proceed to H_2:

   $$H_2 = \begin{bmatrix}\omega(1) & \omega(2) \\ \omega(2) & \omega(3)\end{bmatrix} = \begin{bmatrix}-\frac{4}{9} & \frac{4}{27} \\ \frac{4}{27} & -\frac{4}{81}\end{bmatrix}$$

   The rank is still one, indicating the system order is $n = 1$.

   (b) Factorize $H_2 = O_2 R_2$.

   $$R_2 = H_2(1,:) = \begin{bmatrix}-\frac{4}{9} & \frac{4}{27}\end{bmatrix}$$

   From the relation, also obtain:

   $$O_2 = \begin{bmatrix}1 \\ -\frac{1}{3}\end{bmatrix}$$

   (c) Determine the matrices:

   $$\hat{G} = R_2(:,1) = \begin{bmatrix}-\frac{4}{9}\end{bmatrix}$$

   $$\hat{H} = O_2(1,:) = \begin{bmatrix}1\end{bmatrix}$$

   $$\hat{F} = O_2(1,:)^{-1} O_2(2,:) = \begin{bmatrix}-\frac{1}{3}\end{bmatrix}$$

   $$\hat{D} = \begin{bmatrix}\frac{4}{3}\end{bmatrix}$$

4. Using the positive power transfer function, we have:

$$W(z) = \frac{4}{3}\frac{z}{z+\frac{1}{3}} = \frac{\frac{4}{3}\left(z+\frac{1}{3}\right) - \frac{4}{3}\frac{1}{3}}{z+\frac{1}{3}} = \frac{4}{3} + \frac{-\frac{4}{9}}{z+\frac{1}{3}}$$

From this expression, we identify the system matrices as:

$$F = -\frac{1}{3} \qquad G = 1 \qquad H = -\frac{4}{9} \qquad D = \frac{4}{3}$$

CHAPTER 2

Software sensing with Kalman Filter

2.1 Exercise one

Given the system:

$$\begin{cases} x(t+1) = 3x(t) + 2v(t) \\ y(t) = 2x(t) - v(t) \end{cases} \qquad v(t) \sim WN(0,1)$$

Assuming $v(t)$ is the initial position $x(1)$, we are asked to:

1. Compute the time variant one-step ahead Kalman predictor.
2. Compute the asymptotic Kalman gain of the one-step ahead predictor using the graphical method.
3. Check the existence of the Kalman predictor using asymptotic Kalman filter theorems.
4. Compute the transfer function from $y(t)$ to $\hat{y}(t+1|t)$.
5. Compute the variance of $e_y(t) = y(t) - \hat{y}(t|t-1)$.
6. Find the state equation of the Kalman filter $\hat{x}(t|t)$.

Solution

1. We define two White Noises as $v_1(t) = 2v(t)$, and $v_2(t) = -v(t)$:

$$V_1 = \mathrm{Var}\left[v_1(t)\right] = \mathbb{E}\left[v_1(t)v_1(t)^T\right] = 4\mathbb{E}\left[v(t)v(t)^T\right] = 4$$

$$V_2 = \mathrm{Var}\left[v_2(t)\right] = \mathbb{E}\left[v_2(t)v_2(t)^T\right] = (-1)^2\mathbb{E}\left[v(t)v(t)^T\right] = 1$$

$$V_{12} = \mathbb{E}\left[v_1(t)v_2(t)^T\right] = -2\mathbb{E}\left[v(t)v(t)^T\right] = -2$$

We can now write the Differential Riccati Equation for scalar problems:

$$\begin{aligned}
P(t+1) &= F^2P(t) + V_1 - (FHP(t) + V_{12})(H^2P(t) + V_2)^{-1}(FHP(t) + V_{12}) \\
&= F^2P(t) + V_1 - \frac{(FHP(t) + V_{12})^2}{(H^2P(t) + V_2)^{-1}} \\
&= \frac{F^2H^2P^2(t) + (H^2V_1 + F^2V_2)P(t) + V_1V_2 - F^2H^2P^2(t) - V_{12}^2 - 2FHV_{12}P(t)}{(H^2P(t) + V_2)^{-1}} \\
&= \frac{(H^2V_1 + F^2V_2)P(t) + V_1V_2 - V_{12}^2 - 2FHV_{12}P(t)}{(H^2P(t) + V_2)^{-1}} \\
&= \frac{49P(t)}{4P(t) + 1}
\end{aligned}$$

The Kalman gain is computed as:

$$k(t) = (FHP(t) + V_{12})\left(H^2P(t) + V_2\right)^{-1} = \frac{6P(t) - 2}{4P(t) + 1}$$

So the predictor becomes:

$$\begin{cases}
\hat{x}(t+1|t) = 3\hat{x}(t|t-1) + k(t)\left(y(t) - \hat{y}(t|t-1)\right) \\
\hat{y}(t|t-1) = 2\hat{x}(t|t-1) \\
k(t) = \dfrac{6P(t) - 2}{4P(t) + 1} \\
P(t+1) = \dfrac{49P(t)}{4P(t) + 1}
\end{cases}$$

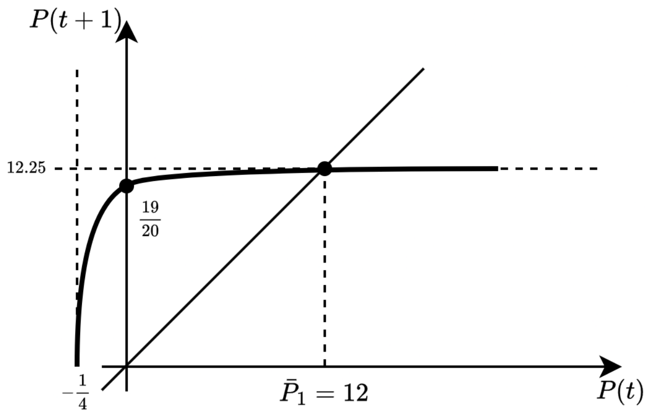
The state equation can be rewritten as:

$$\hat{x}(t+1|t) = (3 - 2k(t))\hat{x}(t|t-1) + k(t)y(t)$$

2. We compute the solutions of the Algebraic Riccati Equation

$$\bar{P} = \frac{49\bar{P}}{4\bar{P} + 1} \rightarrow \bar{P}_{1,2} = 0, 12$$

Both solutions are greater than or equal to zero, making them acceptable. The graph of $P(t+1)$ is shown below:



Next, we check the convergence in three cases:

- $P_0 = 0$: converges to $P = 0$. In this case, the Kalman static gain is $\bar{K} = -2$. With this gain, $F - \bar{K}H = 7 > 1$, indicating that the system is unstable.
- $P_0 > 0 \land P_0 \leq \bar{P}$: converges to $\bar{P} = 12$. The Kalman static gain in this case is $\bar{K} = \frac{10}{7}$. With this gain, $F - \bar{K}H = \frac{1}{7} < 1$, showing that the system is asymptotically stable.
- $P_0 > \bar{P}$: converges to $\bar{P} = 12$. The stability is the same as in the previous case.

In conclusion, for the system to be asymptotically stable, P_0 must be greater than zero.

3. One requirement for both theorems is that $V_{12} = 0$. In our case, $V_{12} \neq 0$, so we cannot apply the theorems.

4. The predictor is (with the gain found before):

$$\begin{cases} \hat{x}(t+1|t) = 3\hat{x}(t|t-1) + \dfrac{10}{7}y(t) - \dfrac{10}{7}\hat{y}(t|t-1) \\ \hat{y}(t|t-1) = 2\hat{x}(t|t-1) \end{cases}$$

By substituting the second equation into the first one, we obtain:

$$\begin{cases} \hat{x}(t+1|t) = \dfrac{1}{7}\hat{x}(t|t-1) + \dfrac{10}{7}y(t) \\ \hat{y}(t|t-1) = 2\hat{x}(t|t-1) \end{cases}$$

The transfer function from $y(t)$ to $\hat{y}(t|t-1)$ is computed as:

$$W(z) = \tilde{H}(zI - \tilde{F})^{-1}\tilde{G} = \frac{\frac{20}{7}}{z - \frac{1}{7}}$$

But since we need the transfer function from $y(t)$ to $\hat{y}(t+1|t)$ we multiply by z, yielding:

$$W(z) = \frac{\frac{20}{7}z}{z - \frac{1}{7}}$$

5. We have that:

$$\begin{aligned} \mathrm{Var}[e_y(t)] &= \mathrm{Var}\left[y(t) - \hat{t}(t|t-1)\right] \\ &= \mathrm{Var}\left[H\left((x|t) - \hat{x}(t|t-1)\right) + v_2(t)\right] \end{aligned}$$

Since the White Noise $v_2(t)$ only affects $y(t)$, it remains independent from the state $x(t)$. At time t, $\hat{x}(t|t-1)$ is influenced by all White Noise $v_2(t-\tau)$ with $\tau \geq 1$, thereby being independent from $v_2(t)$. Considering these factors, we derive:

$$\begin{aligned} \mathrm{Var}[e_y(t)] &= \mathrm{Var}\left[H\left((x|t) - \hat{x}(t|t-1)\right) + v_2(t)\right] \\ &= \mathrm{Var}\left[H\left((x|t) - \hat{x}(t|t-1)\right)\right] + \mathrm{Var}\left[v_2(t)\right] \\ &= \mathrm{Var}\left[He_x(t)\right] + V_2 \\ HP(t)H^T + V_2 &= 4P(t) + 1 \end{aligned}$$

In the asymptotic scenario where $P(t) \to 12$, we obtain:

$$\mathrm{Var}[e_y(t)] = 48 + 1$$

6. Based on the provided information:

$$F = 3 \quad G = 0 \quad H = 2 \quad V_1 = 4 \quad V_2 = 1 \quad V_{12} = -2$$

and also:

$$\bar{P} = 12 \quad \bar{K} = \frac{10}{7} \quad F - \bar{K}H = \frac{1}{7}$$

Since the matrix F is invertible, we can deduce:

$$\hat{x}(t|t) = \frac{1}{3}\hat{x}(t+1|t) = \frac{1}{21}\hat{x}(t|t-1) + \frac{10}{21}y(t)$$

Recalling that $3\hat{x}(t-1|t-1) = \hat{x}(t|t-1)$, we derive:

$$\hat{x}(t|t) = \frac{1}{7}\hat{x}(t-1|t-1) + \frac{10}{21}y(t)$$

2.2 Exercise two

Consider the system:

$$\begin{cases} x_1(t+1) = \frac{1}{2}x_1(t) + v_{11}(t) \\ x_2(t+1) = 2x_2(t) + v_{12}(t) \\ y(t) = x_2(t) + v_2(t) \end{cases}$$

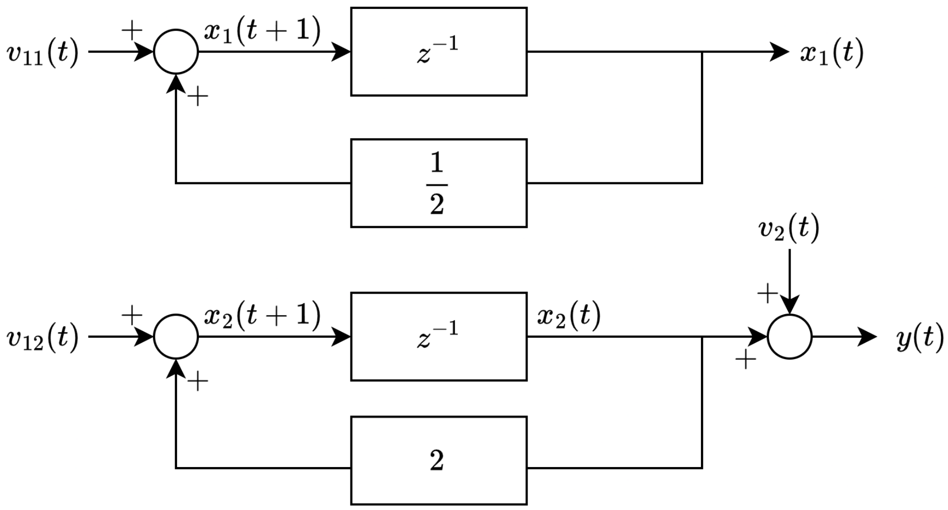
Where:

- $v_1(t) = \begin{bmatrix} v_{11}(t) \\ v_{12}(t) \end{bmatrix} \sim WN\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)$.
- $v_2(t) \sim WN(0, 1)$.
- $v_1(t)$ is independent from $v_2(t)$.

1. Draw the block scheme.
2. Check if the asymptotic state prediction error is bounded.
3. Compute the steady state predictor using the two subsystems.
4. Find the transfer function from $y(t-1)$ to $\hat{x}_2(t|t)$.

Solution

1. The block scheme is depicted below:



2. A system is considered bounded if it is either asymptotically stable or simply stable. Let's analyze the system using the two theorems for Kalman filters. For the first theorem we have:

   - $V_{12} = 0$
   - The system is asymptotically stable if its poles are inside the unit circle. In this case the poles are $z_{1,2} = \frac{1}{2}, 2$, indicating that the system is not asymptotically stable.

   Due to one unmet hypothesis, the first theorem cannot be applied.

   For the second theorem we have:

   - $V_{12} = 0$
   - (F, H) is observable. The observability matrix is:

   $$O = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}$$

   So the system is not fully observable.

   Since one hypothesis is not satisfied, the second theorem cannot be applied.

   Given that the system's states are uncoupled, we can decompose the system into two subsystems:

   $$\begin{cases} x_1(t+1) = \frac{1}{2}x_1(t) + v_{1A}(t) \\ y_A(t) = 0x_1(t) + \underbrace{v_{2A}(t)}_{\text{fictitious}} \end{cases} \qquad \begin{cases} x_2(t+1) = 2x_2(t) + v_{1B}(t) \\ y_B(t) = x_2(t) + v_{2B}(t) \end{cases}$$

   For Subsystem A:

   $$F_A = \frac{1}{2} \quad V_{1A} = 1 \quad H_A = 0 \quad V_{2A} = \varepsilon > 0 \quad V_{12A} = 0$$

   For Subsystem B:

   $$F_B = 2 \quad V_{1B} = 1 \quad H_B = 1 \quad V_{2B} = 1 \quad V_{12B} = 0$$

   We can now reapply the theorems to the two subsystems. Beginning with theorem one for subsystem A:

- $V_{12A} = 0$.
- The system is asymptotically stable if the poles are inside the unit circle. In this case, the pole is $z = \frac{1}{2}$, indicating that the system is asymptotically stable.

Since all the hypotheses are satisfied, we can apply the first theorem, which asserts the existence of an asymptotic predictor.

For subsystem B, we directly apply the second theorem since the first theorem cannot be applied due to the pole at $z = 2$. Here's the analysis:

- $V_{12B} = 0$
- (F, H) is observable. The observability matrix is:

$$O = \begin{bmatrix} 1 \end{bmatrix}$$

  This indicates that the system is fully observable.
- To satisfy the condition $\Gamma\Gamma^T = V_{1B} = 1$, we need $\Gamma = 1$. Therefore, the reachability matrix from noise is:

$$R = \begin{bmatrix} 1 \end{bmatrix}$$

  Since it is full rank, it is reachable.

Since all the hypotheses are satisfied, we can apply the second theorem, which confirms the existence of an asymptotic predictor for subsystem B.

In conclusion, the prediction error is bounded because each subsystem is asymptotically stable and has an asymptotic predictor.

3. For the subsystem A:

$$\begin{cases} x_1(t+1) = \frac{1}{2}x_1(t) + v_{1A}(t) \\ y_A(t) = 0x_1(t) + \underbrace{v_{2A}(t)}_{\text{fictitious}} \end{cases}$$

We find the Kalman gain:

$$\hat{K}_A = (F_A\bar{P}_AH_A + V_{12A})(H_A\bar{P}_AH_A^T + V_{2A}) = \frac{0}{\varepsilon}$$

Since the gain is null, no correction is applied to the prediction.

For subsystem B:

$$\begin{cases} x_2(t+1) = 2x_2(t) + v_{1B}(t) \\ y_B(t) = x_2(t) + v_{2B}(t) \end{cases}$$

We solve the Algebraic Riccati Equation:

$$\bar{P}_B = F_B\bar{P}_BF_B^T + V_{1B} - \frac{\left(F_B\bar{P}_BH_B^T + V_{12B}\right)^2}{H_B\bar{P}_BH_B^T + V_{2B}} = \frac{5\bar{P}_B + 1}{\bar{P}_B + 1}$$

The solutions are:

$$\bar{P}_B = 2 \pm \sqrt{5}$$

The second solution is invalid because it's smaller than zero.

We then compute the Kalman gain:

$$\bar{K}_B = \left(F_B \bar{P}_B H_B^T + V_{12B}\right)\left(H_B \bar{P}_B H_B^T + V_{2B}\right) = \frac{4+2\sqrt{5}}{3+\sqrt{5}}$$

Since $F_B - \bar{K}_B H_B = \dfrac{2}{3+\sqrt{5}} \approx 0.4$, the system is asymptotically stable.

The predictor is:

$$\hat{x}_2(t+1|t) = \frac{2}{3+\sqrt{5}}\hat{x}_2(t|t-1) + \frac{4+2\sqrt{5}}{3+\sqrt{5}}y(t)$$

4. To find the filter from the predictor, we start with:

$$\hat{x}(t+1|t) = (F-KH)\hat{x}(t|t-1) + Ky(t)$$

Expanding this, we get:

$$F\hat{x}(t|t) = (F-KH)F\hat{x}(t-1|t-1) + Ky(t)$$

If F is invertible, we can express $\hat{x}(t|t)$ as:

$$\hat{x}(t|t) = (I - F^{-1}KH)F\hat{x}(t-1|t-1) + F^{-1}Ky(t)$$

Substituting the given values, we have:

$$\begin{cases} \hat{x}(t|t) = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{2}{3+\sqrt{5}} \end{bmatrix} \hat{x}(t-1|t-1) + \begin{bmatrix} 0 \\ \frac{4+2\sqrt{5}}{3+\sqrt{5}} \end{bmatrix} y(t) \\ \hat{x}_2(t-1|t-1) = \begin{bmatrix} 0 & 1 \end{bmatrix} \hat{x}(t-1|t-1) \end{cases}$$

The transfer function from $y(t)$ to $\hat{x}_2(t-1|t-1)$ is:

$$W(z) = \tilde{H}(zI - \tilde{F})^{-1}\tilde{G} = \frac{4+2\sqrt{5}}{3+\sqrt{5}} \frac{1}{z - \frac{2}{3+\sqrt{5}}}$$

So, we have:

$$\hat{x}_2(t-1|t-1) = \frac{4+2\sqrt{5}}{3+\sqrt{5}} \frac{1}{z - \frac{2}{3+\sqrt{5}}} y(t)$$

However, we need $y(t-1)$, so:

$$W(z) = \frac{4+2\sqrt{5}}{3+\sqrt{5}} \frac{z}{z - \frac{2}{3+\sqrt{5}}}$$

CHAPTER 3

Minimum Variance Control

3.1 Exercise one

Consider the following system:

$$\begin{cases} x(t+1) = \frac{1}{2}x(t) + \frac{1}{4}u(t) - \frac{3}{2}v_1(t) \\ y(t) = x(t) + v_2(t) \end{cases} \qquad v_1(t) \sim WN(0,1) \quad v_2(t) \sim WN(0,1)$$

1. Assuming $v_1(t)$ and $v_2(t)$ are uncorrelated, find an equivalent model in the form:

$$y(t) = \frac{B(z)}{A(z)}u(t-k) + \frac{C(z)}{A(z)}e(t) \qquad e(t) \sim WN(\mu, \lambda^2)$$

2. Assuming $v_1(t) = v_2(t) = v(t)$, where $v(t) \sim WN(0,1)$, find an equivalent model in the form:

$$y(t) = \frac{B(z)}{A(z)}u(t-k) + \frac{C(z)}{A(z)}e(t) \qquad e(t) \sim WN(\mu, \lambda^2)$$

3. Given the reference signal $y^0(t)$, find a controller using the principles of Minimum Variance Control for both models found in the previous points.

4. Consider $y^0(t) = \frac{1}{2}$ for all t.

Solution

1. We rearrange the state equation as follows:

$$zx(t) = \frac{1}{2}x(t) + \frac{1}{4}u(t) - \frac{3}{2}v_1(t)$$

Isolating the term $x(t)$, we find:

$$x(t) = \frac{\frac{1}{4}}{z - \frac{1}{2}}u(t) - \frac{\frac{3}{2}}{z - \frac{1}{2}}v_1(t)$$

Upon substitution, we derive:

$$y(t) = \frac{\frac{1}{4}}{1-\frac{1}{2}z^{-1}}u(t-1) - \frac{\frac{3}{2}z^{-1}}{1-\frac{1}{2}z^{-1}}v_1(t) + v_2(t)$$

To achieve the requested representation, we aim for a single noise. For equivalence, matching spectra is necessary:

$$-\frac{\frac{3}{2}z^{-1}}{1-\frac{1}{2}z^{-1}}v_1(t) + v_2(t)$$

We define:

$$\tilde{v}_1(t) = -\frac{3}{2}z^{-1}v_1(t) \rightarrow \tilde{v}_1(t) \sim WN\left(0, \frac{9}{4}\right)$$

This new White Noise remains uncorrelated with $v_2(t)$. At this point:

$$\frac{1}{1-\frac{1}{2}z^{-1}}\tilde{v}_1(t) + v_2(t)$$

The spectrum of the two White Noises is the sum of their spectra:

$$\Gamma_d(\omega) = \left|\frac{1}{1-\frac{1}{2}e^{-j\omega}}\right|^2 \cdot \frac{9}{4} + 1 = \left(\frac{\frac{7}{2} - \frac{1}{2}(z^{-1}+z)}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{2}z\right)}\right)_{z=e^{j\omega}}$$

Given $A(z) = 1 - \frac{1}{2}z^{-1}$ from the system, and considering the numerator's degree cannot exceed the denominator's, $C(z) = 1 + cz^{-1}$ with $e(t) \sim WN(0, \lambda^2)$. The spectrum of $\frac{C(z)}{A(z)}$ is:

$$\Gamma(\omega) = \left|\frac{1+cz^{-1}}{1-\frac{1}{2}e^{-j\omega}}\right|^2 \cdot \lambda^2 = \left(\frac{(1+c^2)\lambda^2 + c\lambda^2(z^{-1}-z)}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{2}z\right)}\right)_{z=e^{j\omega}}$$

We then impose:

$$\begin{cases} \frac{7}{2} - \frac{1}{2}(z^{-1}+z) = (1+c^2)\lambda^2 + c\lambda^2(z^{-1}-z) \\ c\lambda^2 = -\frac{1}{2} \end{cases} \rightarrow \begin{cases} c_{1,2} = -0.146, -6.854 \\ c\lambda^2 = -\frac{1}{2} \end{cases}$$

As it's a zero, we require the modulus to be less than one, thus selecting $c = -0.146$, with $\lambda^2 = 3.427$.

In conclusion:

$$y(t) = \frac{\frac{1}{4}}{1-\frac{1}{2}z^{-1}}u(t-k) + \frac{1-\frac{73}{500}z^{-1}}{1-\frac{1}{2}z^{-1}}e(t) \qquad e(t) \sim WN\left(0, \frac{3427}{1000}\right)$$

2. The given system is:

$$\begin{cases} x(t+1) = \frac{1}{2}x(t) + \frac{1}{4}u(t) - \frac{3}{2}v(t) \\ y(t) = x(t) + v(t) \end{cases}$$

Shifting the state equation, we have:

$$zx(t) = \frac{1}{2}x(t) + \frac{1}{4}u(t) - \frac{3}{2}v(t)$$

Isolating the $x(t)$ term, we find:

$$x(t) = \frac{\frac{1}{4}}{z - \frac{1}{2}} u(t) - \frac{\frac{3}{2}}{z - \frac{1}{2}} v(t)$$

Substituting, we obtain the desired form:

$$y(t) = \frac{\frac{1}{4}}{1 - \frac{1}{2}z^{-1}} u(t-1) - \frac{1 - 2z^{-1}}{1 - \frac{1}{2}z^{-1}} v(t)$$

This representation fulfills the request.

We can transform it into canonical form while retaining the denominator:

$$y(t) = \frac{\frac{1}{4}}{1 - \frac{1}{2}z^{-1}} u(t-1) - \frac{1 - \frac{1}{2}z^{-1}}{1 - \frac{1}{2}z^{-1}} \eta(t) \qquad \eta(t) \sim WN(0, 4)$$

3. The initial model equations are:

$$y_1(t) = \frac{\frac{1}{4}}{1 - \frac{1}{2}z^{-1}} u(t-k) + \frac{1 - 0.146z^{-1}}{1 - \frac{1}{2}z^{-1}} e(t) \qquad e(t) \sim WN(0, 3.427)$$

$$y_2(t) = \frac{\frac{1}{4}}{1 - \frac{1}{2}z^{-1}} u(t-1) - \frac{1 - \frac{1}{2}z^{-1}}{1 - \frac{1}{2}z^{-1}} \eta(t) \qquad \eta(t) \sim WN(0, 4)$$

The following steps are required:

(a) Verification of assumptions:

- $b_0 \neq 0$: in both cases $b_0 = \frac{1}{4}$, satisfying the condition.
- $B(z)$ is minimum phase: Both models have no roots, hence minimum phase.
- $\frac{C(z)}{A(z)}$ is in canonical form: both models are constructed in canonical form.
- Assumption that $y^0(t)$ is independent of $e(t)$, and $y^0(t)$ is unpredictable.

(b) Determination of the one-step ahead predictors:

$$\hat{y}(t|t-1) = \frac{B(z)}{C(z)} u(t-k) + \frac{C(z) - A(z)}{C(z)} y(t)$$

In our case, we have:

$$\hat{y}_1(t|t-1) = \frac{\frac{1}{4}}{1 - 0.146z^{-1}} u(t-1) + \frac{0.354}{1 - 0.146z^{-1}} y(t-1)$$

$$\hat{y}_2(t|t-1) = \frac{\frac{1}{4}}{1 - \frac{1}{2}z^{-1}} u(t-1)$$

(c) Calculation of the Minimum Variance Control for both systems:

$$u(t) = \frac{1}{B(z)E(z)} \left(C(z)y^0(t) - \tilde{R}(z)y(t) \right)$$

In our case, we have:

$$u_1(t) = 4 \left(\left(1 - 0.146z^{-1}\right) y^0(t) - 0.354y(t) \right)$$

$$u_2(t) = 4 \left(\left(1 - \frac{1}{2}z^{-1}\right) y^0(t) \right)$$

4. Substituting the values into the found controllers, we get:

$$u_1(t) = 4\left(\left(1 - 0.146z^{-1}\right)\frac{1}{2} - 0.354y(t)\right) = 1.708 - 1.416y(t)$$

$$u_2(t) = 4\left(\left(1 - \frac{1}{2}z^{-1}\right)\frac{1}{2} - 0.354y(t)\right) = 1$$

3.2 Exercise two

Consider the ARMAX system:

$$y(t) = \frac{1}{2}y(t-1) + u(t-1) + e(t) + \frac{1}{3}e(t-1) \qquad e(t) \sim WN(0,1)$$

1. Check the assumptions for designing a Minimum Variance Controller.
2. Compute the one-step ahead predictor.
3. Compute the Minimum Variance Controller.
4. Draw the closed-loop schema.
5. Find the transfer function from $y^0(t)$ to $y(t)$.
6. Find the transfer function from $e(t)$ to $y(t)$.
7. Check the closed-loop stability.

Solution

1. The system in canonical form is:

$$y(t) = \frac{1}{1 - \frac{1}{2}z^{-1}}u(t-1) + \frac{1 + \frac{1}{3}z^{-1}}{1 - \frac{1}{2}z^{-1}}e(t-1)$$

   The assumptions are:

   - $b_0 \neq 0$: $b_0 = 1$, satisfying the condition.
   - $B(z)$ is minimum phase: both models have no roots, thus minimum phase.
   - $\frac{C(z)}{A(z)}$ is in canonical form: both models are constructed in canonical form.
   - We assume that $y^0(t)$ is independent of $e(t)$, and $y^0(t)$ is unpredictable.

2. The one-step ahead predictor is given by:

$$\hat{y}(t|t-1) = \frac{B(z)}{C(z)}u(t-k) + \frac{C(z) - A(z)}{C(z)}y(t)$$

   In our case, we have:

$$\hat{y}(t|t-1) = \frac{1}{1 + \frac{1}{3}z^{-1}}u(t-1) + \frac{\frac{5}{6}}{1 + \frac{1}{3}z^{-1}}y(t-1)$$

3. To find the Minimum Variance Control for both systems, we start with the formula:

$$u(t) = \frac{1}{B(z)E(z)}\Big(C(z)y^0(t) - \tilde{R}(z)y(t)\Big)$$

In our case, we have:

$$u_(t) = \left(\left(1 + \frac{1}{3}z^{-1}\right)y^0(t) - \frac{5}{6}y(t)\right)$$

We can also derive the control formula from the predictor. Starting with:

$$\hat{y}(t|t-1) = \frac{1}{1+\frac{1}{3}z^{-1}}u(t-1) + \frac{\frac{5}{6}}{1+\frac{1}{3}z^{-1}}y(t-1)$$

After shifting one sample, we get:

$$\hat{y}(t+1|t) = \frac{1}{1+\frac{1}{3}z^{-1}}u(t) + \frac{\frac{5}{6}}{1+\frac{1}{3}z^{-1}}y(t)$$

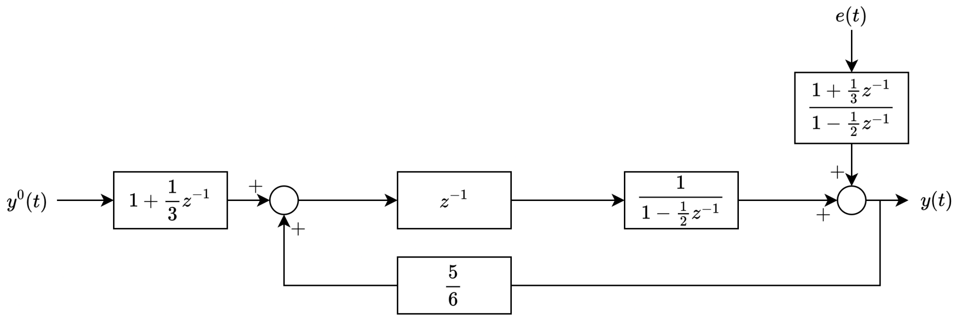
We impose that $\hat{y}(t+k|t) = y^0(t)$:

$$y^0(t) = \frac{1}{1+\frac{1}{3}z^{-1}}u(t) + \frac{\frac{5}{6}}{1+\frac{1}{3}z^{-1}}y(t)$$

By isolating the the term $y^0(t)$, we obtain:

$$u_(t) = \left(\left(1 + \frac{1}{3}z^{-1}\right)y^0(t) - \frac{5}{6}y(t)\right)$$

4. The closed-loop schema is represented by the following diagram:



5. The system equations are given by:

$$\begin{cases} y(t) = \dfrac{1}{1-\frac{1}{2}z^{-1}}u(t-1) + \dfrac{1+\frac{1}{3}z^{-1}}{1-\frac{1}{2}z^{-1}}e(t-1) \\ u(t) = \left(\left(1 + \dfrac{1}{3}z^{-1}\right)y^0(t) - \dfrac{5}{6}y(t)\right) \end{cases}$$

By substituting $u(t)$ into the equation for $y(t)$, we get:

$$y(t) = \frac{1}{1-\frac{1}{2}z^{-1}}z^{-1}\left(\left(1 + \frac{1}{3}z^{-1}\right)y^0(t) - \frac{5}{6}y(t)\right) + \frac{1+\frac{1}{3}z^{-1}}{1-\frac{1}{2}z^{-1}}e(t-1)$$

Isolating the $y(t)$ term, we simplify to:

$$y(t) = z^{-1}y^0(t) + e(t)$$

6. From the block scheme, we find that:

$$W_{ey} = \frac{\frac{C(z)}{A(z)}}{1 + \frac{1}{B(z)E(z)} z^{-1} \frac{B(z)}{A(z)} \tilde{R}(z)} = E(z)$$

7. Given that:

$$\chi(z) = B(z)C(z) = 1\left(1 + \frac{1}{3}z^{-1}\right) = 1 + \frac{1}{3}z^{-1}$$

   we conclude that the system exhibits asymptotic stability because the root lies inside the unit circle.

3.3 Exercise three

Consider the system:

$$y(t) = \frac{1}{2}y(t-1) + u(t-2) + e(t-1) + 2e(t-2) \qquad e(t) \sim WN(0,1)$$

1. Check the assumptions for designing a Minimum Variance Controller.
2. Compute the two-step ahead predictor.
3. Compute the Minimum Variance Controller.
4. Draw the closed-loop scheme.
5. Check the closed-loop stability.
6. Find the transfer function from $y^0(t)$ to $y(t)$.
7. Find the transfer function from $\eta(t)$ to $y(t)$.

Solution

1. For the given system with canonical representation:

$$y(t) = \frac{1}{1 - \frac{1}{2}z^{-1}} u(t-2) + \frac{1 - \frac{1}{2}z^{-1}}{1 - \frac{1}{2}z^{-1}} \eta(t) \qquad \eta(t) \sim WN(0,4)$$

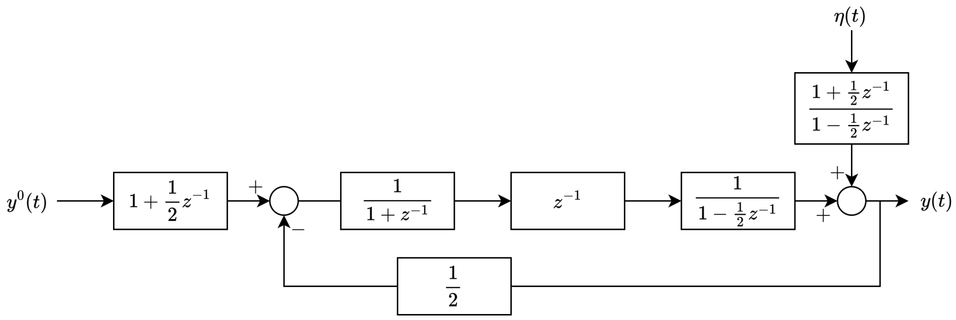
   The assumptions are as follows:

   - $b_0 \neq 0$: since $b_0 = 1$. this condition is satisfied.
   - $B(z)$ is minimum phase: As there are no roots, the system is minimum phase.
   - $\frac{C(z)}{A(z)}$ is in canonical form: both models are naturally in canonical form.
   - We assume $y^0(t)$ is independent of $\eta(t)$, and $y^0(t)$ is unpredictable.

2. Let's consider the transfer function from $\eta(t)$ to $y(t)$. After two steps of long division, we obtain $R(z) = \frac{1}{2}z^{-2}$ and $E(z) = 1 + z^{-1}$. The general formula for the k step ahead predictor is:
$$\hat{y}(t|t-k) = \frac{B(z)E(z)}{C(z)}u(t-k) + \frac{R(z)}{C(z)}y(t)$$
In our case, this becomes:
$$\hat{y}(t|t-k) = \frac{1+z^{-1}}{1-\dfrac{1}{2}z^{-1}}u(t-2) + \frac{\frac{1}{2}}{1-\dfrac{1}{2}z^{-1}}y(t-2)$$

3. To find the Minimum Variance Control controller, we set $y^0(t) = \hat{y}(t+2|t)$:
$$y^0(t) = \frac{1+z^{-1}}{1-\dfrac{1}{2}z^{-1}}u(t-k) + \frac{\frac{1}{2}}{1-\dfrac{1}{2}z^{-1}}y(t-2)$$
Next, we isolate $u(t)$:
$$u(t) = \frac{1}{1+z^{-1}}\left(\left(1-\frac{1}{2}z^{-1}\right)y^0(t) - \frac{1}{2}y(t)\right)$$
This simplifies to:
$$u(t) = -u(t-1) + y^0(t) + \frac{1}{2}y^0(t-1) - \frac{1}{2}y(t)$$

4. The closed-loop diagram is depicted below:



5. Given the loop transfer function:
$$L(z) = \frac{\frac{1}{2}z^{-2}}{(1+z^{-2})\left(1-\frac{1}{2}z^{-1}\right)}$$
From this, we obtain:
$$\chi(z) = L_D(z) + L_N(z) = 1 + \frac{1}{2}z^{-1}$$
Since the root is inside the unit circle, the system is stable.

6. We calculate the transfer function $W_{y^0 u}$ as:

$$W_{y^0 u} = \frac{\text{direct path from } y^0(t) \text{ to } u(t)}{1 + \text{loop function}}$$

From the block diagram, we find:

$$W_{y^0 u} = \frac{\frac{C(z)}{B(z)E(z)}}{1 + \frac{1}{B(z)E(z)} z^{-2} \frac{B(z)}{A(z)} \tilde{R}(z)} = \frac{A(z)}{B(z)}$$

7. From the block diagram, we determine:

$$W_{\eta u} = \frac{-\frac{C(z)\tilde{R}(z)}{B(z)E(z)A(z)}}{\frac{C(z)}{E(z)A(z)}} = -\frac{\tilde{R}(z)}{B(z)}$$

In general, we express the control input $u(t)$ as:

$$u(t) = -\frac{\tilde{R}(z)}{B(z)}\eta(t) + \frac{A(z)}{B(z)}y^0(t)$$

3.4 Exercise four

Consider the ARMAX system:

$$y(t) = \frac{4}{1 + \frac{1}{3}} u(t-1) + \frac{1 + 2z^{-1}}{\left(1 + \frac{1}{2}z^{-1}\right)\left(1 + \frac{1}{3}z^{-1}\right)} e(t) \qquad e(t) \sim WN(\mu, 1)$$

1. Check the assumptions for designing a Minimum Variance Controller.
2. Compute the one-step ahead predictor.
3. Compute the Minimum Variance Controller.

Solution

1. The system in canonical form is:

$$y(t) = \frac{4}{1 + \frac{1}{3}} u(t-1) + \frac{1}{1 + \frac{1}{3}z^{-1}} \eta(t) \qquad \eta(t) \sim WN(2\mu, 4)$$

The assumptions are:

- $b_0 \neq 0$: $b_0 = 4$, satisfying the condition.
- $B(z)$ is minimum phase: both models have no roots, thus minimum phase.
- $\frac{C(z)}{A(z)}$ is in canonical form: both models are constructed in canonical form.
- We assume that $y^0(t)$ is independent of $e(t)$, and $y^0(t)$ is unpredictable.

2. The one-step ahead predictor is given by:

$$\hat{y}(t|t-1) = 4u(t-1) - \frac{1}{3}y(t-1) + 2\mu$$

To find this predictor we have removed the mean from the process and the White Noise, and then added it again in the end.

3. To find the Minimum Variance Control for both systems, we do not use the formula since we have the bias term.

Starting with:

$$\hat{y}(t|t-1) = 4u(t-1) - \frac{1}{3}y(t-1) + 2\mu$$

We impose that $\hat{y}(t+k|t) = y^0(t)$:

$$y^0(t) = 4u(t-1) - \frac{1}{3}y(t-1) + 2\mu$$

By isolating the the term $y^0(t)$, we obtain:

$$u_(t) = \frac{1}{12}y(t) + \frac{1}{4}y^0(t) - \frac{1}{2}\mu$$

3.5 Exercise five

Consider the ARMAX system:

$$y(t) = \frac{1}{1-\frac{1}{2}}u(t-1) + \frac{1+\frac{1}{3}z^{-1}}{1-\frac{1}{2}}e(t) \qquad e(t) \sim WN(0,1)$$

1. Find the one-step ahead and two-step ahead predictors.
2. Find the Minimum Variance Control both using the one-step ahead and two-step ahead predictors.
3. Study the stability.

Solution

1. The one-step ahead predictor is given by:

$$\hat{y}(t|t-1) = \frac{1}{1+\frac{1}{3}z^{-1}}u(t-1) + \frac{\frac{5}{6}}{1+\frac{1}{3}z^{-1}}y(t-1)$$

The two-step ahead predictor is given by:

$$\hat{y}(t|t-2) = \frac{1+\frac{5}{6}z^{-1}}{1+\frac{1}{3}z^{-1}}u(t-2) + \frac{\frac{5}{12}}{1+\frac{1}{3}z^{-1}}y(t-2)$$

2. We set $\hat{y}(t+k|t) = y^0(t)$ in both predictors, yielding:

$$u(t) = \left(1+\frac{1}{3}z^{-1}\right)y^0(t) - \frac{5}{6}y(t)$$

$$u(t) = \left(\frac{1+\frac{1}{3}z^{-1}}{1+\frac{5}{6}z^{-1}}\right)y^0(t) - \frac{\frac{5}{12}}{1+\frac{5}{6}z^{-1}}y(t)$$

3. For the one-step ahead predictor, we have:

$$\chi(z) = B(z)C(z) = 1 + \frac{1}{3}z^{-1}$$

The root $z = \frac{1}{3}$ is inside the unit circle, indicating stability for the closed-loop system.

For the two-step ahead predictor:

$$\chi(z) = B(z)\left(E(z)A(z) + z^{-1}\tilde{R}(z)\right) = 1 + \frac{3}{4}z^{-1} - \frac{5}{12}z^{-2}$$

The roots are $z_{1,2} = 0.37, -1.12$. Only the first root is inside the unit circle, indicating instability for the closed-loop system.