Model Identification And Data Analysis I
Exercises

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Abstract

The course topics are:

- Basic concepts of stochastic processes.
- ARMA and ARMAX classes of parametric models for time series and for Input/Output systems.
- Parameter identification of ARMA and ARMAX models.
- Analysis of identification methods.
- Model validation and pre-processing.

Contents

CHAPTER 1

Exercise session I

1.1 Exercise one

We are examining an MA(2) process defined by the function:

$$y(t) = e(t) + \frac{1}{2}e(t-1) - e(t-2)$$

Here, $e(t)$ follows a white noise distribution with mean 0 and variance 1.

1. Determine the transfer function for this system.
2. Calculate the expected value of the process $y(t)$.
3. Compute the covariance of the process $y(t)$ at different time lags.

1.1.1 Solution

1. Using the Z-transform, we express the MA(2) process as:

   $$y(t) = e(t) + \frac{1}{2}e(t)z^{-1} - e(t)z^{-2}$$

   Grouping the $e(t)$ factor, we have:

   $$y(t) = e(t)\left(1 + \frac{1}{2}z^{-1} - z^{-2}\right)$$

   This yields the polynomial:

   $$P(z) = 1 + \frac{1}{2}z^{-1} - z^{-2}$$

   In normal form, $P(z)$ becomes:

   $$P(z) = \frac{z^2 + \frac{1}{2}z - 1}{z^2}$$

2. The expected value is computed as follows:

$$\begin{aligned}\mathbb{E}\left[y(t)\right] &= \mathbb{E}\left[e(t) + \frac{1}{2}e(t-1) - e(t-2)\right] \\ &= \mathbb{E}\left[e(t)\right] + \mathbb{E}\left[\frac{1}{2}e(t-1)\right] - \mathbb{E}\left[e(t-2)\right] \\ &= \underbrace{\mathbb{E}\left[e(t)\right]}_{0} + \frac{1}{2}\underbrace{\mathbb{E}\left[e(t-1)\right]}_{0} - \underbrace{\mathbb{E}\left[e(t-2)\right]}_{0} \\ &= 0\end{aligned}$$

3. For the covariance:

$$\begin{aligned}\gamma_y(0) &= \mathbb{E}\left[y(t)^2\right] \\ &= \mathbb{E}\left[\left(e(t) + \frac{1}{2}e(t-1) - e(t-2)\right)^2\right] \\ &= \mathbb{E}\left[e(t)^2 + \frac{1}{2}e(t-1)^2 + e(t-2)^2 + \text{cross products}\right] \\ &= \underbrace{\mathbb{E}\left[e(t)^2\right]}_{1} + \frac{1}{4}\underbrace{\mathbb{E}\left[e(t-1)^2\right]}_{1} + \underbrace{\mathbb{E}\left[e(t-2)^2\right]}_{1} + \underbrace{\mathbb{E}\left[\text{cross products}\right]}_{0} \\ &= 1 + \frac{1}{4} + 1 \\ &= \frac{9}{4}\end{aligned}$$

The covariance at lag 1 is:

$$\gamma_y(1) = 0$$

We need to compute another time lag since we have two correlated time instants in the formula (square of the same time instant). The covariance of two is as follows:

$$\gamma_y(2) = -1$$

There is another correlation of the time instant $t-2$, but it is the only one, so for time instants after two, we have a null covariance. The final result is:

$$\begin{cases} \gamma_y(0) = \frac{9}{4} \\ \gamma_y(1) = 0 \\ \gamma_y(2) = -1 \\ \gamma_y(\tau) = 0 \qquad \forall\, |\tau| \geq 3 \end{cases}$$

1.2 Exercise two

Consider a process with the following covariance:

$$\gamma(0) = \frac{5}{2} \qquad \gamma(1) = 1 \qquad \gamma(\tau) = 0 \quad |\tau| > 1$$

1. Analyze the process.
2. Find the expression of the process.

1.2.1 Solution

- The process follows an AR(1) model.
- Utilizing the general system, we have:

$$y(t) = c_0 e(t) + c_1 e(t-1) \qquad e \sim WN(0, \lambda^2)$$

The coefficients can be determined using the following system of equations:

$$\begin{cases} (c_0^2 + c_1^2)\,\lambda^2 = \frac{5}{2} \\ (c_0 c_1)\,\lambda^2 = 1 \end{cases}$$

To simplify, we set $c_0 = 1$ and solve the system:

$$\begin{cases} (1 + c_1^2)\,\lambda^2 = \frac{5}{2} \\ (1 c_1)\,\lambda^2 = 1 \end{cases}$$

Solving the system yields:

$$\begin{cases} c_{1,2} = 2, \frac{1}{2} \\ \lambda_{1,2} = \frac{1}{2}, 2 \end{cases}$$

1.3 Exercise three

Consider an AR(2) process described by the following equation:

$$y(t) = \frac{1}{2} y(t-1) - \frac{1}{4} y(t-2) + e(t)$$

Here, $e(t) \sim WN(0, 1)$.

1. Determine the transfer function of the given system.
2. Calculate the expected value.
3. Compute the covariance.

1.3.1 Solution

1. Using the Z-transform, we have:

$$y(t) = \frac{1}{2} y(t) z^{-1} - \frac{1}{4} y(t) z^{-2} + e(t)$$

This yields:

$$y(t) = \frac{1}{1 - \frac{1}{2} z^{-1} + \frac{1}{4} z^{-2}} e(t)$$

2. The expected value is determined as follows:

$$\begin{aligned}\mathbb{E}[y(t)] &= \mathbb{E}\left[\frac{1}{2}y(t-1) - \frac{1}{4}y(t-2) + e(t)\right] \\ &= \frac{1}{2}\mathbb{E}[y(t-1)] + \frac{1}{4}\mathbb{E}[y(t-2)] - \underbrace{\mathbb{E}[e(t)]}_{0} \\ &= \frac{1}{2}\mathbb{E}[y(t-1)] + \frac{1}{4}\mathbb{E}[y(t-2)]\end{aligned}$$

Now, assuming that $y(t)$ is a stationary stochastic process, we have $\mathbb{E}[y(t)] = m$ for all instants. Thus, rewriting the previous formula:

$$m = \frac{1}{2}m + \frac{1}{4}m \to m = 0$$

This value coincides with the expected value.

To confirm the hypothesis, we need to check if the input process is a stationary stochastic process (white noise is a stationary stochastic process) and if the transfer function is stable:

$$W(x) = \frac{z^2}{z^2 - \frac{1}{2}z + \frac{1}{4}}$$

Stability requires that all the modules of the poles are inside the unit circle:

$$z^2 - \frac{1}{2}z + \frac{1}{4} = 0$$

The solutions to this equation are:

$$z_{1,2} = \frac{1}{4} \pm i\frac{\sqrt{3}}{4}$$

From which the modules are:

$$|z_{1,2}| = \frac{1}{2}$$

Thus, the system is stable, confirming the hypothesis.

3. The covariance at lag zero is calculated as follows:

$$\gamma_y(0) = \mathbb{E}\left[\frac{1}{2}y(t-1) - \frac{1}{4}y(t-2) + e(t)\right]$$

From this we have:

$$\begin{aligned}\gamma_y(0) = \frac{1}{4}\underbrace{\mathbb{E}\left[y(t-1)^2\right]}_{\gamma_y(0)} + \frac{1}{16}\underbrace{\mathbb{E}\left[y(t-2)^2\right]}_{\gamma_y(0)} + \underbrace{\mathbb{E}\left[e(t^2)\right]}_{1} + \frac{1}{4}\underbrace{\mathbb{E}[y(t-1)y(t-2)]}_{\gamma_y(1)} + \\ + \underbrace{\mathbb{E}[y(t-1)e(t)]}_{0} + \frac{1}{2}\underbrace{\mathbb{E}[y(t-2)e(t)]}_{0}\end{aligned}$$

The resulting equation is:

$$\frac{11}{16}\gamma_y(0) + \frac{1}{4}\gamma_y(1) = 1$$

To determine the covariance at lag one, we compute:

$$\begin{aligned}
\gamma_y(1) &= \mathbb{E}\left[\left(\frac{1}{2}y(t-1) - \frac{1}{4}y(t-2) + e(t)\right) y(t-1)\right] \\
&= \frac{1}{2}\underbrace{\mathbb{E}\left[y(t-1)^2\right]}_{\gamma_y(0)} - \frac{1}{4}\underbrace{\mathbb{E}\left[y(t-2)y(t-1)\right]}_{\gamma_y(1)} + \underbrace{\mathbb{E}\left[e(t)y(t-1)\right]}_{0} \\
&= \frac{1}{2}\gamma_y(0) - \frac{1}{4}\gamma_y(1)
\end{aligned}$$

This leads to the equation:

$$\gamma_y(1) = \frac{1}{2}\gamma_y(0) - \frac{1}{4}\gamma_y(1)$$

The resulting system of equations is:

$$\begin{cases} \dfrac{11}{16}\gamma_y(0) + \frac{1}{4}\gamma_y(1) = 1 \\ -\dfrac{1}{2}\gamma_y(0) + \frac{5}{4}\gamma_y(1) = 0 \end{cases}$$

Solving this system yields:

$$\begin{cases} \gamma_y(0) = \frac{80}{63} \\ \gamma_y(1) = \frac{32}{63} \end{cases}$$

Now, we can compute the covariance at lag two:

$$\begin{aligned}
\gamma_y(2) &= \mathbb{E}\left[\left(\frac{1}{2}y(t-1) - \frac{1}{4}y(t-2) + e(t)\right) y(t-2)\right] \\
&= \frac{1}{2}\underbrace{\mathbb{E}\left[y(t-1)y(t-2)\right]}_{\gamma_y(1)} - \frac{1}{4}\underbrace{\mathbb{E}\left[y(t-2)^2\right]}_{\gamma_y(0)} + \underbrace{\mathbb{E}\left[e(t)y(t-2)\right]}_{0} \\
&= \frac{1}{2}\gamma_y(1) - \frac{1}{4}\gamma_y(0) \\
&= -\frac{4}{63}
\end{aligned}$$

The final result is:

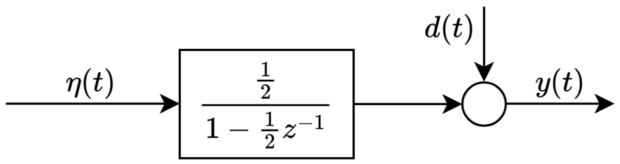
$$\begin{cases} \gamma_y(0) = \frac{80}{63} \\ \gamma_y(1) = \frac{32}{63} \\ \gamma_y(\tau) = \frac{1}{2}\gamma_y(\tau-1) - \frac{1}{4}\gamma_y(\tau-2) \qquad \forall\, |\tau| \geq 2 \end{cases}$$

CHAPTER 2

Exercise session II

2.1 Exercise one

Consider the stochastic process defined by the following diagram:



Where:

- $v(t) = \dfrac{1}{2}v(t-1) + \dfrac{1}{2}\eta(t) \quad \eta(\cdot) \sim WN(0,1)$.
- $d(\cdot) \sim WN(0,1)$.
- $\eta(\cdot)$ and $d(\cdot)$ are two independent white processes (uncorrelated).

Calculate the spectrum of process $y(\cdot)$.

Now consider that $d(t) = \eta(t)$ and compute the spectrum.

2.1.1 Solution

Let's compute the expected value of the process:

$$\mathbb{E}\left[y(t)\right] = 0$$

Because it is the sum of two processes with expected values equal to zero.

Now we compute the covariance at a generic time instant τ:

$$\begin{aligned}
\gamma_y(\tau) &= \mathbb{E}\left[(v(t)+d(t))\left(v(t+\tau)+d(t+\tau)\right)\right] \\
&= \mathbb{E}\left[v(t)v(t+\tau)+v(t)d(t+\tau)+d(t)v(t+\tau)+d(t)d(t+\tau)\right] \\
&= \underbrace{\mathbb{E}\left[v(t)v(t+\tau)\right]}_{\gamma_v(\tau)} + \underbrace{\mathbb{E}\left[v(t)d(t+\tau)\right]}_{\eta\perp d} + \underbrace{\mathbb{E}\left[d(t)v(t+\tau)\right]}_{\eta\perp d} + \underbrace{\mathbb{E}\left[d(t)d(t+\tau)\right]}_{\gamma_d(\tau)} \\
&= \gamma_v(\tau) + \gamma_d(\tau)
\end{aligned}$$

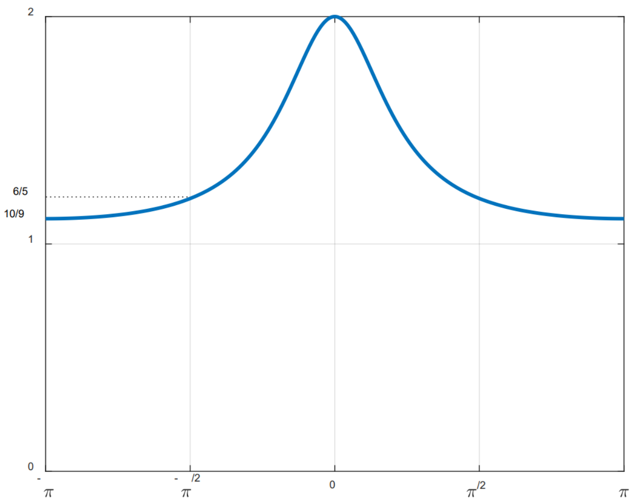
At this point, we have:

$$\Gamma_y(\omega) = \sum_{-\infty}^{\infty} \gamma_y(\tau)e^{-j\omega\tau} = \sum_{-\infty}^{\infty} \gamma_v(\tau)e^{-j\omega\tau} + \sum_{-\infty}^{\infty} \gamma_d(\tau)e^{-j\omega\tau} = \Gamma_v(\omega) + \Gamma_d(\omega)$$

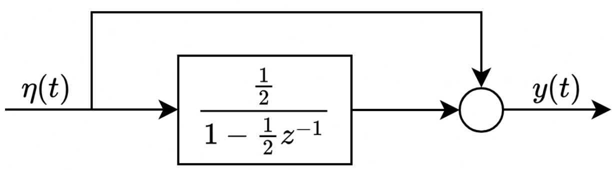
So, we can obtain the spectrum as the sum of the spectra of the two processes:

$$\begin{aligned}
\Gamma_y(\omega) &= \left|W(e^{j\omega})\right|^2 \cdot \underbrace{\lambda_\eta^2}_{1} + \underbrace{\lambda_d^2}_{1} \\
&= \left|W(e^{j\omega})\right|^2 + 1 \\
&= \frac{\frac{1}{4}}{\left|1 - \frac{1}{2}e^{-j\omega}\right|^2} + 1 \\
&= \frac{\frac{1}{4}}{\left|1 - \frac{1}{2}\cos(\omega) + j\frac{1}{2}\sin(\omega)\right|^2} + 1 \\
&= \frac{\frac{1}{4}}{\left(1 - \frac{1}{2}\cos(\omega)\right)^2 + \left(\frac{1}{2}\sin(\omega)\right)^2} + 1 \\
&= \frac{\frac{1}{4}}{\frac{5}{4} - \cos(\omega)} + 1 \\
&= \frac{6 - 4\cos(\omega)}{5 - 4\cos(\omega)}
\end{aligned}$$

This result in the following graph:



In the second case the white noises are correlated, so we have the following block diagram:



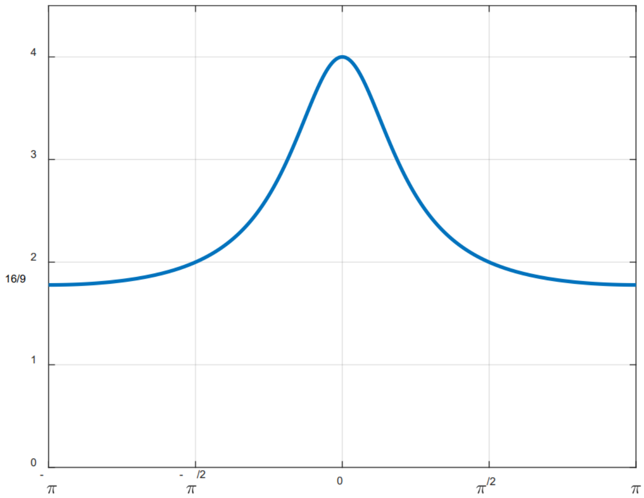
And the transfer function from $\eta(\cdot)$ to $y(\cdot)$ equals:

$$W(z) = \frac{\frac{1}{2}}{1 - \frac{1}{2}z^{-1}} + 1 = \frac{\frac{3}{2} - \frac{1}{2}z^{-1}}{1 - \frac{1}{2}z^{-1}}$$

The spectrum would be:

$$\begin{aligned}
\Gamma_{yy}(\omega) &= \left|W(e^{j\omega})\right|^2 \lambda_n^2 \\
&= \left|W(e^{j\omega})\right|^2 \\
&= \frac{10 - 6\cos(\omega)}{5 - 4\cos(\omega)}
\end{aligned}$$

This result in the following graph:



2.2 Exercise two

Consider the ARMA(1, 1) process:

$$v(t) = \frac{1}{2}v(t-1) + \eta(t) - \eta(t-1) \qquad \eta(\cdot) \sim WN(1, 9)$$

Compute the expected value and the variance.

2.2.1 Solution

The expected value is zero. The variance is:

$$\begin{aligned}
\gamma(0) = \text{Var}\left[v(t)\right] &= \mathbb{E}\left[(v(t) - 0)^2\right] \\
&= \mathbb{E}\left[(\frac{1}{2}v(t-1) + \eta(t) - \eta(t-1))^2\right]
\end{aligned}$$

Expanding, we get:

$$\begin{aligned}
\text{Var}\left[v(t)\right] = \frac{1}{4}\underbrace{\mathbb{E}\left[v(t-1)^2\right]}_{\text{Var}[v(t)]} + \underbrace{\mathbb{E}\left[\eta(t)^2\right]}_{10} - \underbrace{\mathbb{E}\left[\eta(t-1)^2\right]}_{10} + \underbrace{\mathbb{E}\left[v(t-1)\eta(t)\right]}_{0} - \\
- \underbrace{\mathbb{E}\left[v(t-1)\eta(t-1)\right]}_{9} - 2\underbrace{\mathbb{E}\left[\eta(t)\eta(t-1)\right]}_{1}
\end{aligned}$$

Given that $\text{Var}[v(t)] = \text{Var}[v(t-1)] = \gamma(0)$ and $\text{Var}[\eta(t)] = \text{Var}[\eta(t-1)] = 9$, we have:

$$\text{Var}\left[v(t)\right] = \text{Var}\left[v(t)\right] + 10 - 10 - 9 - 2 \rightarrow \text{Var}\left[v(t)\right] = 12$$

Additionally, we have:

- $\gamma(1) = -3$.
- $\gamma(2) = -\frac{3}{2}$.

For $|\tau| \geq 2$, we have:

$$\gamma(\tau) = \left(\frac{1}{2}\right)^{|\tau|-1} \gamma(1)$$

The correlation function and the spectrum remain unchanged when we depolarize the process. Defining:

$$\tilde{v}(t) = v(t) - \mathbb{E}\left[v(t)\right] = v(t)$$

$$\tilde{\eta}(t) = \eta(t) - \mathbb{E}\left[\eta(t)\right] = \eta(t) - 1$$

We obtain:

$$\tilde{v}(t) = \frac{1}{2}\tilde{v}(t-1) + \tilde{\eta}(t) - \tilde{\eta}(t-1) \qquad \tilde{\eta}(\cdot) \sim WN(0,9)$$

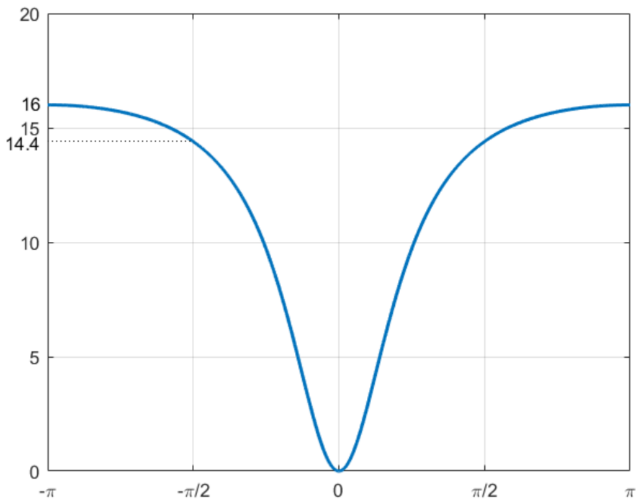
Using the transfer function:

$$W(z) = \frac{1 - z^{-1}}{1 - \frac{1}{2}z^{-1}}$$

We can calculate the spectrum using the formula:

$$\Gamma_{\tilde{v}\tilde{v}} = \left|W(e^{j\omega})\right|^2 \Gamma_{\eta}(\omega) = \frac{1 - 1\cos(\omega)}{1.25 - \cos(\omega)} 18$$

This result in the following graph:



The system blocks the component at zero frequency. Indeed, although the input has a non-zero mean, the output has null expected value.