Model Identification And Data Analysis I
Exercises

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Abstract

The course topics are:

- Basic concepts of stochastic processes.
- ARMA and ARMAX classes of parametric models for time series and for Input/Output systems.
- Parameter identification of ARMA and ARMAX models.
- Analysis of identification methods.
- Model validation and pre-processing.

Contents

CHAPTER 1

Exercise session I

1.1 Exercise one

We are examining an MA(2) process defined by the function:

$$y(t) = e(t) + \frac{1}{2}e(t-1) - e(t-2) \qquad e(t) \sim WN(0,1)$$

1. Determine the transfer function for this system.
2. Calculate the expected value of the process $y(t)$.
3. Compute the covariance of the process $y(t)$ at different time lags.

1.1.1 Solution

1. Using the Z-transform, we express the MA(2) process as:

$$y(t) = e(t) + \frac{1}{2}e(t)z^{-1} - e(t)z^{-2}$$

Grouping the $e(t)$ factor, we have:

$$y(t) = e(t)\left(1 + \frac{1}{2}z^{-1} - z^{-2}\right)$$

This yields the polynomial:

$$P(z) = 1 + \frac{1}{2}z^{-1} - z^{-2}$$

In normal form, $P(z)$ becomes:

$$P(z) = \frac{z^2 + \frac{1}{2}z - 1}{z^2}$$

2. The expected value is computed as follows:

$$\begin{aligned}
\mathbb{E}\left[y(t)\right] &= \mathbb{E}\left[e(t) + \frac{1}{2}e(t-1) - e(t-2)\right] \\
&= \mathbb{E}\left[e(t)\right] + \mathbb{E}\left[\frac{1}{2}e(t-1)\right] - \mathbb{E}\left[e(t-2)\right] \\
&= \underbrace{\mathbb{E}\left[e(t)\right]}_{0} + \frac{1}{2}\underbrace{\mathbb{E}\left[e(t-1)\right]}_{0} - \underbrace{\mathbb{E}\left[e(t-2)\right]}_{0} \\
&= 0
\end{aligned}$$

3. For the covariance:

$$\begin{aligned}
\gamma_y(0) &= \mathbb{E}\left[y(t)^2\right] \\
&= \mathbb{E}\left[\left(e(t) + \frac{1}{2}e(t-1) - e(t-2)\right)^2\right] \\
&= \mathbb{E}\left[e(t)^2 + \frac{1}{2}e(t-1)^2 + e(t-2)^2 + \text{cross products}\right] \\
&= \underbrace{\mathbb{E}\left[e(t)^2\right]}_{1} + \frac{1}{4}\underbrace{\mathbb{E}\left[e(t-1)^2\right]}_{1} + \underbrace{\mathbb{E}\left[e(t-2)^2\right]}_{1} + \underbrace{\mathbb{E}\left[\text{cross products}\right]}_{0} \\
&= 1 + \frac{1}{4} + 1 \\
&= \frac{9}{4}
\end{aligned}$$

The covariance at lag 1 is:

$$\gamma_y(1) = 0$$

We need to compute another time lag since we have two correlated time instants in the formula (square of the same time instant). The covariance of two is as follows:

$$\gamma_y(2) = -1$$

There is another correlation of the time instant $t-2$, but it is the only one, so for time instants after two, we have a null covariance. The final result is:

$$\begin{cases}
\gamma_y(0) = \frac{9}{4} \\
\gamma_y(1) = 0 \\
\gamma_y(2) = -1 \\
\gamma_y(\tau) = 0 \qquad \forall \left|\tau\right| \geq 3
\end{cases}$$

1.2 Exercise two

Consider a process with the following covariance:

$$\gamma(0) = \frac{5}{2} \qquad \gamma(1) = 1 \qquad \gamma(\tau) = 0 \quad |\tau| > 1$$

1. Analyze the process.
2. Find the expression of the process.

1.2.1 Solution

- The process follows an AR(1) model.
- Utilizing the general system, we have:

$$y(t) = c_0 e(t) + c_1 e(t-1) \qquad e \sim WN(0, \lambda^2)$$

The coefficients can be determined using the following system of equations:

$$\begin{cases} (c_0^2 + c_1^2)\,\lambda^2 = \frac{5}{2} \\ (c_0 c_1)\,\lambda^2 = 1 \end{cases}$$

To simplify, we set $c_0 = 1$ and solve the system:

$$\begin{cases} (1 + c_1^2)\,\lambda^2 = \frac{5}{2} \\ (1 c_1)\,\lambda^2 = 1 \end{cases}$$

Solving the system yields:

$$\begin{cases} c_{1,2} = 2, \frac{1}{2} \\ \lambda_{1,2} = \frac{1}{2}, 2 \end{cases}$$

1.3 Exercise three

Consider an AR(2) process described by the following equation:

$$y(t) = \frac{1}{2} y(t-1) - \frac{1}{4} y(t-2) + e(t)$$

Here, $e(t) \sim WN(0, 1)$.

1. Determine the transfer function of the given system.
2. Calculate the expected value.
3. Compute the covariance.

1.3.1 Solution

1. Using the Z-transform, we have:

$$y(t) = \frac{1}{2} y(t) z^{-1} - \frac{1}{4} y(t) z^{-2} + e(t)$$

This yields:

$$y(t) = \frac{1}{1 - \frac{1}{2} z^{-1} + \frac{1}{4} z^{-2}} e(t)$$

2. The expected value is determined as follows:

$$\begin{aligned}\mathbb{E}\left[y(t)\right] &= \mathbb{E}\left[\frac{1}{2}y(t-1) - \frac{1}{4}y(t-2) + e(t)\right] \\ &= \frac{1}{2}\mathbb{E}\left[y(t-1)\right] + \frac{1}{4}\mathbb{E}\left[y(t-2)\right] - \underbrace{\mathbb{E}\left[e(t)\right]}_{0} \\ &= \frac{1}{2}\mathbb{E}\left[y(t-1)\right] + \frac{1}{4}\mathbb{E}\left[y(t-2)\right]\end{aligned}$$

Now, $y(t)$ is a stationary stochastic process because $e(t)$ is an SSP and $W(z)$ is asymptotically stable, we have $\mathbb{E}\left[y(t)\right] = m$ for all instants. Thus, rewriting the previous formula:

$$m = \frac{1}{2}m + \frac{1}{4}m \to m = 0$$

This value coincides with the expected value.

To confirm the hypothesis, we need to check if the input process is a stationary stochastic process (white noise is a stationary stochastic process) and if the transfer function is stable:

$$W(x) = \frac{z^2}{z^2 - \frac{1}{2}z + \frac{1}{4}}$$

Stability requires that all the modules of the poles are inside the unit circle:

$$z^2 - \frac{1}{2}z + \frac{1}{4} = 0$$

The solutions to this equation are:

$$z_{1,2} = \frac{1}{4} \pm i\frac{\sqrt{3}}{4}$$

From which the modules are:

$$|z_{1,2}| = \frac{1}{2}$$

Thus, the system is stable, confirming the hypothesis.

3. The covariance at lag zero is calculated as follows:

$$\gamma_y(0) = \mathbb{E}\left[\frac{1}{2}y(t-1) - \frac{1}{4}y(t-2) + e(t)\right]$$

From this we have:

$$\begin{aligned}\gamma_y(0) = \frac{1}{4}\underbrace{\mathbb{E}\left[y(t-1)^2\right]}_{\gamma_y(0)} + \frac{1}{16}\underbrace{\mathbb{E}\left[y(t-2)^2\right]}_{\gamma_y(0)} + \underbrace{\mathbb{E}\left[e(t^2)\right]}_{1} + \frac{1}{4}\underbrace{\mathbb{E}\left[y(t-1)y(t-2)\right]}_{\gamma_y(1)} + \\ + \underbrace{\mathbb{E}\left[y(t-1)e(t)\right]}_{0} + \frac{1}{2}\underbrace{\mathbb{E}\left[y(t-2)e(t)\right]}_{0}\end{aligned}$$

The resulting equation is:

$$\frac{11}{16}\gamma_y(0) + \frac{1}{4}\gamma_y(1) = 1$$

To determine the covariance at lag one, we compute:

$$
\begin{aligned}
\gamma_y(1) &= \mathbb{E}\left[\left(\frac{1}{2}y(t-1) - \frac{1}{4}y(t-2) + e(t)\right) y(t-1)\right] \\
&= \frac{1}{2}\underbrace{\mathbb{E}\left[y(t-1)^2\right]}_{\gamma_y(0)} - \frac{1}{4}\underbrace{\mathbb{E}\left[y(t-2)y(t-1)\right]}_{\gamma_y(1)} + \underbrace{\mathbb{E}\left[e(t)y(t-1)\right]}_{0} \\
&= \frac{1}{2}\gamma_y(0) - \frac{1}{4}\gamma_y(1)
\end{aligned}
$$

This leads to the equation:

$$\gamma_y(1) = \frac{1}{2}\gamma_y(0) - \frac{1}{4}\gamma_y(1)$$

The resulting system of equations is:

$$
\begin{cases}
\dfrac{11}{16}\gamma_y(0) + \frac{1}{4}\gamma_y(1) = 1 \\
-\dfrac{1}{2}\gamma_y(0) + \frac{5}{4}\gamma_y(1) = 0
\end{cases}
$$

Solving this system yields:

$$
\begin{cases}
\gamma_y(0) = \frac{80}{63} \\
\gamma_y(1) = \frac{32}{63}
\end{cases}
$$

Now, we can compute the covariance at lag two:

$$
\begin{aligned}
\gamma_y(2) &= \mathbb{E}\left[\left(\frac{1}{2}y(t-1) - \frac{1}{4}y(t-2) + e(t)\right) y(t-2)\right] \\
&= \frac{1}{2}\underbrace{\mathbb{E}\left[y(t-1)y(t-2)\right]}_{\gamma_y(1)} - \frac{1}{4}\underbrace{\mathbb{E}\left[y(t-2)^2\right]}_{\gamma_y(0)} + \underbrace{\mathbb{E}\left[e(t)y(t-2)\right]}_{0} \\
&= \frac{1}{2}\gamma_y(1) - \frac{1}{4}\gamma_y(0) \\
&= -\frac{4}{63}
\end{aligned}
$$

The final result is:

$$
\begin{cases}
\gamma_y(0) = \frac{80}{63} \\
\gamma_y(1) = \frac{32}{63} \\
\gamma_y(\tau) = \frac{1}{2}\gamma_y(\tau-1) - \frac{1}{4}\gamma_y(\tau-2) \qquad \forall\, |\tau| \geq 2
\end{cases}
$$

CHAPTER 2

Exercise session II

2.1 Exercise one

Consider the AR(1) process:

$$y(t) = \frac{1}{3}y(t-1) + e(t) + 2 \qquad e(t) \sim WN(1,1)$$

1. Find the transfer function of the system, and verify that it is a stationary stochastic process.
2. Compute the expected value.
3. Compute the covariance.

2.1.1 Solution

1. We apply the input delay operator, obtaining:

   $$y(t) = \frac{1}{3}z^{-1}y(t) + e(t) + 2$$

   By rearranging the terms we have:

   $$y(t) = \left[\frac{z}{z-\frac{1}{3}}\right](e(t) - 2)$$

   The input is a stationary stochastic process, the poles of the transfer function are:

   $$z - \frac{1}{3} = 0 \rightarrow z = \frac{1}{3}$$

   The pole is inside the unity circle, and so this is a stationary stochastic process.

2. The expected value is:

   $$\mathbb{E}\left[y(t)\right] = \mathbb{E}\left[\frac{1}{3}y(t-1) + e(t) + 2\right] = \frac{1}{3}\mathbb{E}\left[y(t-1)\right] + 1 + 2$$

   But we have a stationary stochastic process, so the mean is constant:

   $$m_y = \frac{1}{3}m_y + 3 \rightarrow m_y = \frac{9}{2}$$

3. We define the unbiased process:

$$\begin{cases} \tilde{y}(t) = y(t) - m_y \\ \tilde{e}(t) = e(t) - m_e \end{cases}$$

In our case we have:

$$\tilde{y}(t) + \frac{9}{2} = \frac{1}{3}\left(\tilde{y}(t-1) + \frac{9}{2}\right) + \tilde{e}(t) + 1 + 2 \rightarrow \tilde{y}(t) = \frac{1}{3}\tilde{y}(t-1) + \tilde{e}(t)$$

We can finally compute the covariance function as:

$$\gamma_y(\tau) = \mathbb{E}\left[\tilde{y}(t)\tilde{y}(t-\tau)\right]$$

We start by the covariance in $\tau = 0$:

$$\gamma_{\tilde{y}}(0) = \mathbb{E}\left[\tilde{y}(t)^2\right] = \mathbb{E}\left[\left(\frac{1}{3}\tilde{y}(t-1) + \tilde{e}(t)\right)^2\right] = \frac{1}{9}\gamma_{\tilde{y}}(0) + 1 \rightarrow \gamma_{\tilde{y}}(0) = \frac{9}{8}$$

We now compute the covariance in $\tau = 1$:

$$\gamma_{\tilde{y}}(1) = \mathbb{E}\left[\tilde{y}(t)\tilde{y}(t-1)\right] = \mathbb{E}\left[\left(\frac{1}{3}\tilde{y}(t-1) + \tilde{e}(t)\right)\tilde{y}(t-1)\right] = \frac{1}{3}\gamma_{\tilde{y}}(0) \rightarrow \gamma_{\tilde{y}}(1) = \frac{3}{8}$$

For a generic τ we have:

$$\gamma_{\tilde{y}}(\tau) = \frac{1}{3}\gamma_{\tilde{y}}(\tau - 1) \qquad |\tau| \geq 1$$

2.2 Exercise two

Consider the ARMA$(1,1)$ process described by the expression:

$$y(t) = \frac{1}{2}y(t-1) + e(t) - e(t-1) \qquad e(t) \sim WN(1,9)$$

1. Find the transfer function and verify if it is a stationary stochastic process.
2. Compute the expected value.
3. Compute the covariance function.

2.2.1 Solution

1. We can rewrite the formula in operatorial representation:

$$y(t) = \frac{1}{2}y(t)z^{-1} + e(t) - e(t)z^{-1} \rightarrow y(t) = \frac{z-1}{z-\frac{1}{2}}e(t)$$

We have a zero in $z = 1$, and a pole in $z = \frac{1}{2}$, so the transfer function is asymptotically stable. Since the White Noise is the input, and it is a stationary stochastic process, also $y(t)$ is a stationary stochastic process.

2. We can compute the expected value as:
$$\mathbb{E}\left[y(t)\right] = \mathbb{E}\left[\frac{1}{2}y(t-1) + e(t) - e(t-1)\right] = \frac{1}{2}\mathbb{E}\left[y(t-1)\right] + 1 - 1$$
We have that $y(t)$ is a stationary stochastic process, and so the mean is constant:
$$m_y = \frac{1}{2}m_y \rightarrow m_y = 0$$
It can also be computed by the theorem:
$$\mathbb{E}\left[y(t)\right] = W(1) \cdot \mathbb{E}\left[e(t)\right] = 0 \cdot 1 = 0$$
3. We define the unbiased process as:
$$\begin{cases} \tilde{y}(t) = y(t) - m_y \\ \tilde{e}(t) = e(t) - m_e \end{cases}$$
In our case we have:
$$\tilde{y}(t) + m_y = \frac{1}{2}\left(\tilde{y}(t-1) + m_y\right) + \tilde{e}(t) + m_e - \left(\tilde{e}(t-1) + m_e\right)$$
That becomes:
$$\tilde{y}(t) = \frac{1}{2}\tilde{y}(t-1) + \tilde{e}(t) + 1 - \tilde{e}(t-1) - 1 \rightarrow \tilde{y}(t) = \frac{1}{2}\tilde{y}(t-1) + \tilde{e}(t) - \tilde{e}(t-1)$$
We start by the covariance in $\tau = 0$:
$$\gamma_{\tilde{y}}(0) = \mathbb{E}\left[\tilde{y}(t)^2\right] = \mathbb{E}\left[\left(\frac{1}{2}\tilde{y}(t-1) + \tilde{e}(t) - \tilde{e}(t-1)\right)^2\right] = \frac{1}{4}\gamma_{\tilde{y}}(0) + 9 - 9 - 9 \rightarrow \gamma_{\tilde{y}}(0) = 12$$
We now compute the covariance in $\tau = 1$:
$$\gamma_{\tilde{y}}(1) = \mathbb{E}\left[\tilde{y}(t)\tilde{y}(t-1)\right] = \mathbb{E}\left[\left(\frac{1}{2}\tilde{y}(t-1) + \tilde{e}(t) - \tilde{e}(t-1)\right)y(t-1)\right] = \frac{1}{2}\gamma_{\tilde{y}}(0) - 9 \rightarrow \gamma_{\tilde{y}}(1) = -3$$
We now compute the covariance in $\tau = 2$:
$$\gamma_{\tilde{y}}(1) = \mathbb{E}\left[\tilde{y}(t)\tilde{y}(t-1)\right] = \mathbb{E}\left[\left(\frac{1}{2}\tilde{y}(t-1) + \tilde{e}(t) - \tilde{e}(t-1)\right)y(t-2)\right] \rightarrow \gamma_{\tilde{y}}(1) = -\frac{3}{2}$$
For a generic τ we have:
$$\gamma_{\tilde{y}}(\tau) = \frac{1}{2}\gamma_{\tilde{y}}(\tau-1) \qquad |\tau| \geq 2$$

2.3 Exercise three

Consider the MA(2) process generated by the expression:
$$y(t) = e(t) + 0.5e(t-1) + 0.5e(t-2) \qquad e(t) \sim WN(2,1)$$
1. Find the transfer function and verify if it is a stationary stochastic process.
2. Compute the expected value.
3. Compute the covariance function.

2.3.1 Solution

1. We can rewrite the formula in operatorial representation:

$$y(t) = e(t) + 0.5e(t)z^{-1} + 0.5e(t)z^{-2} \rightarrow y(t) = \frac{z^2 + 0.5z + 0.5}{z^2} e(t)$$

   We have two zeros in $z_{1,2} = -frac14 \pm \frac{\sqrt{7}}{4} i$, and a pole in $z = 0$, so the transfer function is asymptotically stable. Since the White Noise is the input, and it is a stationary stochastic process, also $y(t)$ is a stationary stochastic process.

2. We can compute the expected value as:

$$\mathbb{E}\left[y(t)\right] = \mathbb{E}\left[e(t) + 0.5e(t-1) + 0.5e(t-2)\right] = 2 + 1 + 1 = 4$$

   It can also be computed by the theorem:

$$\mathbb{E}\left[y(t)\right] = W(1) \cdot \mathbb{E}\left[e(t)\right] = 2 \cdot 2 = 4$$

3. We define the unbiased process as:

$$\begin{cases} \tilde{y}(t) = y(t) - m_y \\ \tilde{e}(t) = e(t) - m_e \end{cases}$$

   In our case we have:

$$\tilde{y}(t) + m_y = \left(\tilde{e}(t) + m_e\right) + 0.5\left(\tilde{e}(t-1) + m_e\right) + 0.5\left(\tilde{e}(t-2) + m_e\right)$$

   That becomes:

$$\tilde{y}(t) = \tilde{e}(t) + 0.5\tilde{e}(t-1) + 0.5\tilde{e}(t-2)$$

   Since it is a Moving Average process we can directly find the covariance as:

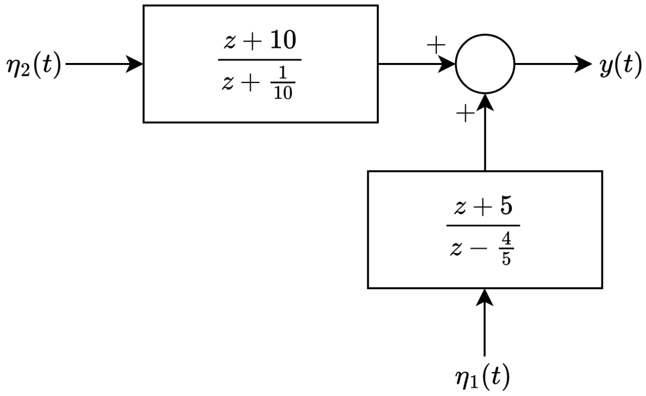
$$\begin{cases} \left(c_0^2 + c_1^2 + c_2^2\right)\lambda^2 & \tau = 0 \\ \left(c_0 c_1 + c_1 c_2\right)\lambda^2 & |\tau| = 1 \\ \left(c_0 c_2\right)\lambda^2 & |\tau| = 2 \\ 0 & |\tau| \geq 3 \end{cases} \rightarrow \begin{cases} \frac{3}{2} & \tau = 0 \\ \frac{3}{4} & |\tau| = 1 \\ \frac{1}{2} & |\tau| = 2 \\ 0 & |\tau| \geq 3 \end{cases}$$

CHAPTER 3

Exercise session III

3.1 Exercise one

Consider the stochastic process defined by the following diagram:



Here, $\eta_1 \sim WN(1,1)$ and $\eta_2 \sim WN(0,1)$ are uncorrelated.
Find the characteristic values of the given process $y(t)$.

3.1.1 Solution

Remember that if we have an ARMA(n_a, n_b) we have that:

- If $n_a > n_b$, the covariance becomes recursive for $\tau = n_a$.
- If $n_a \leq n_b$, the covariance becomes recursive for $\tau = n_b + 1$

The output process is composed by two process that are uncorrelated because the White Noise is uncorrelated:

$$y(t) = y_1(t) + y_2(t)$$

The processes $y_1(t)$ and $y_2(t)$ are stationary, and so also $y(t)$ is stationary.
The mean is:

$$m_y = \mathbb{E}\left[y(t)\right] = \mathbb{E}\left[y_1(t) + y_2(t)\right] = W_1(1)\lambda_1^2 + W_2(1)\lambda_2^2 = \frac{15}{2}$$

The covariance can be computed as the sum of the two covariances (uncorrelated):

$$\gamma_y(\tau) = \gamma_{y_1}(\tau) + \gamma_{y_2}(\tau)$$

The stochastic process $y_1(t)$ in the time domain is:

$$y_1(t) = \frac{1}{5}y_1(t-1) + \eta_1(t) + 5\eta_1(t-1)$$

We can define the unbiased process by defining:

$$\begin{cases} \tilde{y}_1(t) = y_1(t) - m_{y_1} \\ \tilde{\eta}_1(t) = \eta_1(t) - m_{\eta_1} \end{cases}$$

The process becomes:

$$\tilde{y}_1(t) = \frac{1}{5}\tilde{y}_1(t-1) + \tilde{\eta}_1(t) + 5\tilde{\eta}_1(t-1)$$

From this we can compute the covariance ad different time lags as:

$$\gamma_{y_1}(\tau) = \begin{cases} \frac{175}{6} & \tau = 0 \\ \frac{65}{6} & |\tau| = 1 \\ \frac{13}{6} & |\tau| = 2 \\ \frac{1}{5}\gamma_{y_1}(\tau-1) & |\tau| \geq 3 \end{cases}$$

The stochastic process $y_2(t)$ in the time domain is:

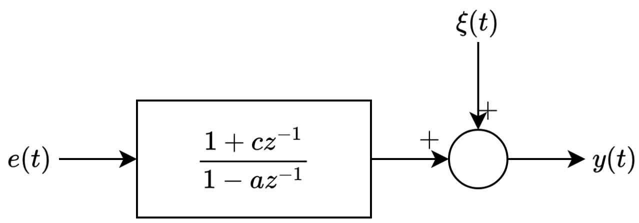
$$y_2(t) = -\frac{1}{10}y_2(t-1) + \eta_2(t) + 10\eta_2(t-1)$$

The process is already unbiased, and we can finally compute the covariance function:

$$\gamma_{y_2}(\tau) = \begin{cases} 100 & \tau = 0 \\ 0 & |\tau| \geq 1 \end{cases}$$

3.2 Exercise two

Consider the stochastic process defined by the following diagram:



Here, $e(t) \sim WN(1,1)$ and $\xi(t) \sim WN(0,1)$ are uncorrelated.

1. Find when the process is stationary.
2. Given $\gamma_y(0) = 6$, $\gamma_y(1) = -2$, and $\gamma_y(\tau) = 0$ for $\tau \geq 2$, we want to compute the values of a and c.

3.2.1 Solution

1. The process $y(t)$ is stationary when $\xi(t)$ and $y_1(t)$ are both stationary. The process $\xi(t)$ is a White Noise, so it is stationary by definition. The process $y_1(t)$ is stationary when $|a| < 1$.

2. Since we have that $\gamma_y(\tau) = 0$ for $\tau \geq 2$, this is a Moving Average Process of order 1. Thus, $a = 0$.

   The process in the time domain is:

$$y(t) = - + e(t) + ce(t-1) + \xi(t)$$

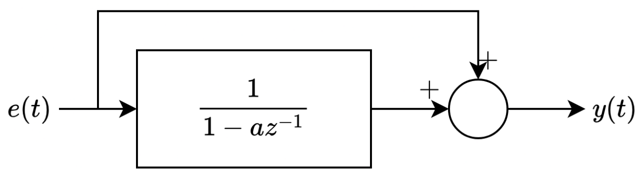
   We can compute the covariance in $\tau = 0$

$$\gamma_y(0) = \mathbb{E}\left[y(t)^2\right] = 0$$

   From which we obtain $c = -2$.

3.3 Exercise three

Consider the stochastic process defined by the following diagram:



Here, $e(t) \sim WN(0, \lambda^2)$, and $|a| < 1$.
Find the characteristic values of the given process $y(t)$.

3.3.1 Solution

We start by computing the expected value of $y(t)$:

$$m_y = \mathbb{E}\left[y(t)\right] = \mathbb{E}\left[ay(t-1) + 2e(t)\right] = a\mathbb{E}\left[y(t-1)\right] \to m_y = 0$$

The covariance function at $\tau = 0$ is equal to:

$$\gamma_y(0) = \mathbb{E}\left[y(t)^2\right] = \mathbb{E}\left[(y_1(t) + y_2(t))^2\right] = \frac{4 - 3a^2}{1 - a^2}\lambda^2$$

The covariance function at $\tau = 1$ is equal to:

$$\gamma_y(1) = \mathbb{E}\left[y(t)y(t-1)\right] = \frac{a\lambda^2(2 - a^2)}{1 - a^2}$$

Alternatively, we can note that we have two processes in parallel with a transfer function equal to:

$$y(t) = \frac{1}{1 - az^{-1}}e(t) + e(t) = \frac{2 - az^{-1}}{1 - az^{-1}}e(t)$$

The canonical form will become:

$$y(t) = \frac{1 - \frac{a}{2}z^{-1}}{1 - az^{-1}} e_1(t)$$

Here, $e_1(t) = 2e(t)$, that is $e(t) \sim WN(0, 2^2\lambda^2)$. We can now find the time domain representation, that is:

$$y(t) = ay(t-1) + \eta_1(t) - \frac{a}{2}\eta_1(t-1)$$

From this we can compute the covariance in a more simple way.

CHAPTER 4

Exercise session IV

4.1 Exercise one

Consider the process described by the expression:

$$y(t) = \frac{1}{2}y(t-1) + e(t) - e(t-1) \qquad e(t) \sim WN(0,9)$$

4.1.1 Solution

In case of a stationary stochastic process, the following formula holds:

$$\Gamma_y(\omega) = \left|W(e^{j\omega})\right|^2 \Gamma_u(\omega) = \left|W(e^{j\omega})\right|^2 \lambda^2$$

So, we start by computing the transfer function:

$$y(t) = \frac{z-1}{z-\frac{1}{2}}$$

The pole is inside the unit circle, $e(t)$ is a stationary stochastic process since it is a White Noise. Thus, $y(t)$ is a stationary stochastic process, and we can use the fundamental theorem of the spectral analysis:

$$\Gamma_y(\omega) = \left|\frac{e^{j\omega}-1}{e^{j\omega}-\frac{1}{2}}\right|^2 9$$

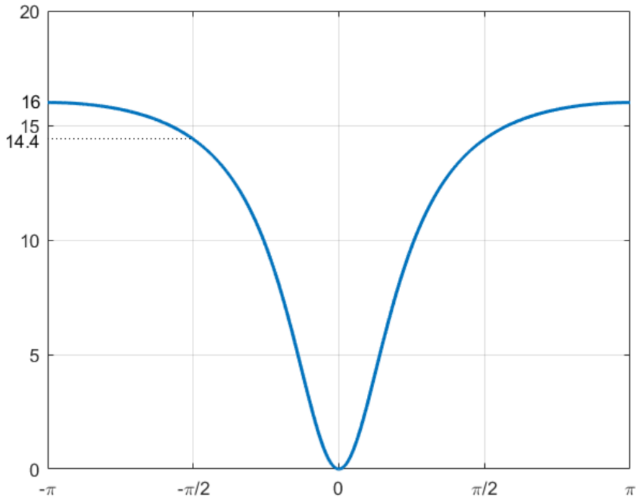
We compute the squares as:

- $\left|e^{j\omega}-1\right|^2 = \left(e^{j\omega}-1\right)\left(e^{-j\omega}-1\right) = 2(1-\cos\omega)$
- $\left|e^{j\omega}-\frac{1}{2}\right|^2 = \left(e^{j\omega}-\frac{1}{2}\right)\left(e^{-j\omega}-\frac{1}{2}\right) = \frac{5}{4}-\cos\omega$

And so the spectral density function is:

$$\Gamma_y(\omega) = \frac{1-\cos\omega}{\frac{5}{4}-\cos\omega}18$$

From which we can find the graph:



4.2 Exercise two

Consider the process generated by the following expression:

$$y(t) = \left(1 - z^{-1} + z{-}2\right)\left(1 + \frac{3}{2}z^{-1}\right)e(t) \qquad WN(0,1)$$

Find the spectral density function.

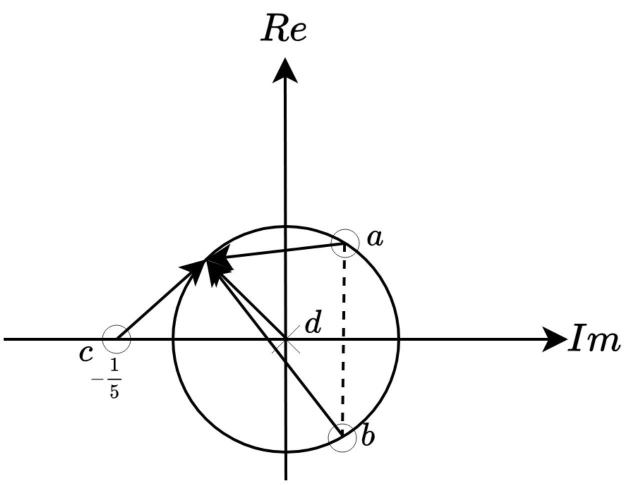
4.2.1 Solution

This can be rewritten as:

$$y(t) = \frac{\left(z^2 - z + 1\right)\left(z - \frac{3}{2}\right)}{z^2}e(t)$$

The poles are in $z = 0$, and the zeros are: $-\frac{3}{2}, \frac{1}{2} \pm j\frac{\sqrt{3}}{2}$

The simplest way to compute the spectral density function is by using the vectors that connect a generic point $e^{j\omega}$ to the poles (d) and the zeros (a, b, c):



In this case we have that the spectral density function is computed as:

$$\Gamma_y(\omega) = \frac{|a|^2\,|b|^2\,|c|^2}{|d|^2}\lambda^2$$

We start in e^{j0}, here we have:

- $|a|^2 = 1$
- $|b|^2 = 1$
- $|c|^2 = \frac{25}{4}$
- $|d|^2 = 1$

As a result we have that:

$$\Gamma_y(0) = \frac{25}{4}$$

We start in $e^{j\frac{\pi}{2}}$, here we have:

- $|a|^2 = 2 - \sqrt{3}$
- $|b|^2 = 2 + \sqrt{3}$
- $|c|^2 = \frac{13}{4}$
- $|d|^2 = 1$

As a result we have that:

$$\Gamma_y\left(\frac{\pi}{2}\right) = \frac{13}{4}$$

We start in $e^{j\pi}$, here we have:

- $|a|^2 = 3$
- $|b|^2 = 3$
- $|c|^2 = \frac{1}{4}$
- $|d|^2 = 1$

As a result we have that:

$$\Gamma_y(\pi) = \frac{9}{4}$$

Note that

$$\Gamma_y\left(\frac{\pi}{3}\right) = 0$$

Now it is possible to plot the graph.

4.3 Exercise one

Consider the process described by the expression:

$$y(t) = \frac{1}{2}y(t-1) + e(t) - e(t-1) \qquad e(t) \sim WN(0,9)$$

4.3.1 Solution

In case of a stationary stochastic process, the following formula holds:

$$\Gamma_y(\omega) = \left|W(e^{j\omega})\right|^2 \Gamma_u(\omega) = \left|W(e^{j\omega})\right|^2 \lambda^2$$

So, we start by computing the transfer function:

$$y(t) = \frac{z-1}{z-\frac{1}{2}}$$

The pole is inside the unit circle, $e(t)$ is a stationary stochastic process since it is a White Noise. Thus, $y(t)$ is a stationary stochastic process, and we can use the fundamental theorem of the spectral analysis:

$$\Gamma_y(\omega) = \left|\frac{e^{j\omega}-1}{e^{j\omega}-\frac{1}{2}}\right|^2 9$$

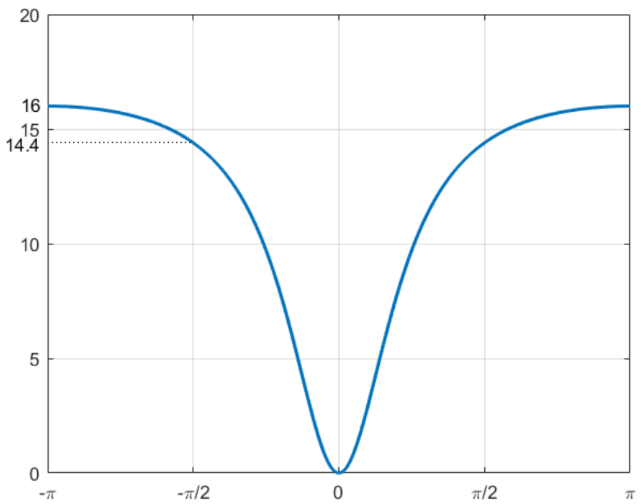
We compute the squares as:

- $\left|e^{j\omega}-1\right|^2 = \left(e^{j\omega}-1\right)\left(e^{-j\omega}-1\right) = 2(1-\cos\omega)$
- $\left|e^{j\omega}-\frac{1}{2}\right|^2 = \left(e^{j\omega}-\frac{1}{2}\right)\left(e^{-j\omega}-\frac{1}{2}\right) = \frac{5}{4}-\cos\omega$

And so the spectral density function is:

$$\Gamma_y(\omega) = \frac{1-\cos\omega}{\frac{5}{4}-\cos\omega}18$$

From which we can find the graph:



4.4 Exercise two

Consider the process generated by the following expression:

$$y(t) = \left(1 - z^{-1} + z{-}2\right)\left(1 + \frac{3}{2}z^{-1}\right)e(t) \qquad WN(0,1)$$

Find the spectral density function.

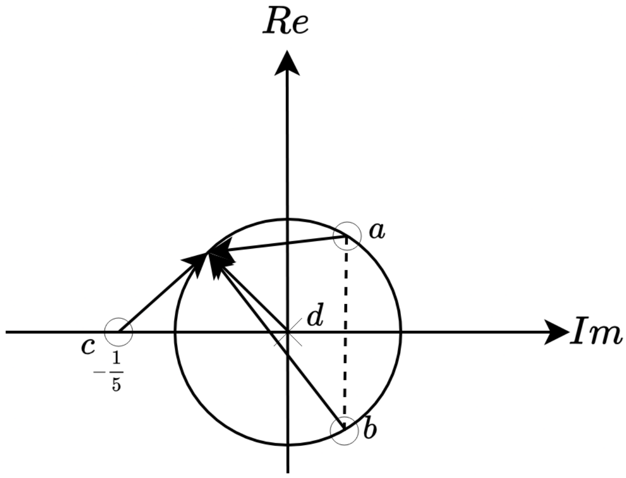
4.4.1 Solution

This can be rewritten as:

$$y(t) = \frac{(z^2 - z + 1)\left(z - \frac{3}{2}\right)}{z^2} e(t)$$

The poles are in $z = 0$, and the zeros are: $-\frac{3}{2}, \frac{1}{2} \pm j\frac{\sqrt{3}}{2}$

The simplest way to compute the spectral density function is by using the vectors that connect a generic point $e^{j\omega}$ to the poles (d) and the zeros (a, b, c):



In this case we have that the spectral density function is computed as:

$$\Gamma_y(\omega) = \frac{|a|^2 |b|^2 |c|^2}{|d|^2} \lambda^2$$

We start in e^{j0}, here we have:

- $|a|^2 = 1$
- $|b|^2 = 1$
- $|c|^2 = \frac{25}{4}$
- $|d|^2 = 1$

As a result we have that:

$$\Gamma_y(0) = \frac{25}{4}$$

We start in $e^{j\frac{\pi}{2}}$, here we have:

- $|a|^2 = 2 - \sqrt{3}$
- $|b|^2 = 2 + \sqrt{3}$
- $|c|^2 = \frac{13}{4}$
- $|d|^2 = 1$

As a result we have that:

$$\Gamma_y\left(\frac{\pi}{2}\right) = \frac{13}{4}$$

We start in $e^{j\pi}$, here we have:

- $|a|^2 = 3$
- $|b|^2 = 3$
- $|c|^2 = \frac{1}{4}$
- $|d|^2 = 1$

As a result we have that:

$$\Gamma_y\left(\pi\right) = \frac{9}{4}$$

Note that

$$\Gamma_y\left(\frac{\pi}{3}\right) = 0$$

Now it is possible to plot the graph.

CHAPTER 5

Exercise session V