Model Identification And Data Analysis I
Exercises

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Abstract

The course delves into the fundamental concepts of stochastic processes, explores ARMA and ARMAX classes of parametric models for both time series and input-output systems, examines parameter identification techniques for ARMA and ARMAX models, analyzes various identification methods, and addresses model validation and pre-processing.

Contents

CHAPTER 1

Stochastic processes

1.1 Exercise 1

Consider an MA(2) process defined by the function:

$$y(t) = e(t) + \frac{1}{2}e(t-1) - e(t-2) \qquad e(t) \sim WN(0,1)$$

1. Find the transfer function for this system.
2. Calculate the expected value of the process $y(t)$.
3. Compute the covariance function of the process $y(t)$.

Solution

1. Utilizing the Z-transform, we express the MA(2) process as:

   $$y(t) = e(t) + \frac{1}{2}e(t)z^{-1} - e(t)z^{-2}$$

   Grouping the $e(t)$ factor, we obtain:

   $$y(t) = e(t)\left(1 + \frac{1}{2}z^{-1} - z^{-2}\right)$$

   This yields the polynomial:

   $$W(z) = 1 + \frac{1}{2}z^{-1} - z^{-2}$$

   In normal form, $W(z)$ becomes:

   $$W(z) = \frac{z^2 + \frac{1}{2}z - 1}{z^2}$$

2. The expected value is computed as follows:

$$\begin{aligned}
\mathbb{E}\left[y(t)\right] &= \mathbb{E}\left[e(t) + \frac{1}{2}e(t-1) - e(t-2)\right] \\
&= \mathbb{E}\left[e(t)\right] + \mathbb{E}\left[\frac{1}{2}e(t-1)\right] - \mathbb{E}\left[e(t-2)\right] \\
&= \underbrace{\mathbb{E}\left[e(t)\right]}_{0} + \frac{1}{2}\underbrace{\mathbb{E}\left[e(t-1)\right]}_{0} - \underbrace{\mathbb{E}\left[e(t-2)\right]}_{0} \\
&= 0
\end{aligned}$$

3. For the covariance:

$$\begin{aligned}
\gamma_y(0) &= \mathbb{E}\left[y(t)^2\right] \\
&= \mathbb{E}\left[\left(e(t) + \frac{1}{2}e(t-1) - e(t-2)\right)^2\right] \\
&= \mathbb{E}\left[e(t)^2 + \frac{1}{2}e(t-1)^2 + e(t-2)^2 + \text{cross products}\right] \\
&= \underbrace{\mathbb{E}\left[e(t)^2\right]}_{1} + \frac{1}{4}\underbrace{\mathbb{E}\left[e(t-1)^2\right]}_{1} + \underbrace{\mathbb{E}\left[e(t-2)^2\right]}_{1} + \underbrace{\mathbb{E}\left[\text{cross products}\right]}_{0} \\
&= 1 + \frac{1}{4} + 1 \\
&= \frac{9}{4}
\end{aligned}$$

The covariance at lag one is:

$$\gamma_y(1) = 0$$

We need to compute another time lag since we have two correlated time instants in the formula (square of the same time instant). The covariance of two is as follows:

$$\gamma_y(2) = -1$$

There is another correlation of the time instant $t-2$, but it is the only one, so for time instants after two, we have a null covariance. The final result is:

$$\begin{cases}
\gamma_y(0) = \frac{9}{4} \\
\gamma_y(1) = 0 \\
\gamma_y(2) = -1 \\
\gamma_y(\tau) = 0 \qquad \forall\,|\tau| \geq 3
\end{cases}$$

1.2 Exercise 2

Consider a process with the following covariance:

$$\gamma(0) = \frac{5}{2} \qquad \gamma(1) = 1 \qquad \gamma(\tau) = 0 \quad |\tau| > 1$$

1. Examine the process.
2. Determine the expression of the process.

Solution

- The process adheres to an MA(1) model.
- Utilizing the general system, we have:

$$y(t) = c_0 e(t) + c_1 e(t-1) \qquad e \sim WN(0, \lambda^2)$$

The coefficients can be found using the following system of equations:

$$\begin{cases} (c_0^2 + c_1^2)\,\lambda^2 = \frac{5}{2} \\ (c_0 c_1)\,\lambda^2 = 1 \end{cases}$$

To simplify, we set $c_0 = 1$ and solve the system:

$$\begin{cases} (1 + c_1^2)\,\lambda^2 = \frac{5}{2} \\ (1 c_1)\,\lambda^2 = 1 \end{cases}$$

Solving the system yields:

$$\begin{cases} c_{1,2} = 2, \frac{1}{2} \\ \lambda_{1,2} = \frac{1}{2}, 2 \end{cases}$$

1.3 Exercise 3

Consider an AR(2) process described by the following equation:

$$y(t) = \frac{1}{2} y(t-1) - \frac{1}{4} y(t-2) + e(t)$$

Here, $e(t) \sim WN(0, 1)$.

1. Determine the transfer function of the given system.
2. Calculate the expected value.
3. Compute the covariance.

Solution

1. Using the Z-transform, we get:

$$y(t) = \frac{1}{2} y(t) z^{-1} - \frac{1}{4} y(t) z^{-2} + e(t)$$

This yields:

$$y(t) = \frac{1}{1 - \frac{1}{2} z^{-1} + \frac{1}{4} z^{-2}} e(t)$$

2. The expected value is determined as follows:

$$\begin{aligned}\mathbb{E}\left[y(t)\right] &= \mathbb{E}\left[\frac{1}{2}y(t-1) - \frac{1}{4}y(t-2) + e(t)\right] \\ &= \frac{1}{2}\mathbb{E}\left[y(t-1)\right] + \frac{1}{4}\mathbb{E}\left[y(t-2)\right] - \underbrace{\mathbb{E}\left[e(t)\right]}_{0} \\ &= \frac{1}{2}\mathbb{E}\left[y(t-1)\right] + \frac{1}{4}\mathbb{E}\left[y(t-2)\right]\end{aligned}$$

Now, $y(t)$ is a stationary stochastic process because $e(t)$ is an SSP and $W(z)$ is asymptotically stable, we have $\mathbb{E}\left[y(t)\right] = m$ for all instants. Thus, rewriting the previous formula:

$$m = \frac{1}{2}m - \frac{1}{4}m \rightarrow m = 0$$

This value coincides with the expected value.

To confirm the hypothesis, we need to check if the input process is a stationary stochastic process (White Noise is a stationary stochastic process) and if the transfer function is stable:

$$W(x) = \frac{z^2}{z^2 - \frac{1}{2}z + \frac{1}{4}}$$

Stability requires that all the modules of the poles are inside the unit circle:

$$z^2 - \frac{1}{2}z + \frac{1}{4} = 0$$

The solutions to this equation are:

$$z_{1,2} = \frac{1}{4} \pm i\frac{\sqrt{3}}{4}$$

From which the modules are:

$$|z_{1,2}| = \frac{1}{2}$$

Thus, the system is stable, confirming the hypothesis.

3. The covariance at lag zero is calculated as follows:

$$\gamma_y(0) = \mathbb{E}\left[\left(\frac{1}{2}y(t-1) - \frac{1}{4}y(t-2) + e(t)\right)^2\right]$$

From this we have:

$$\begin{aligned}\gamma_y(0) = \frac{1}{4}\underbrace{\mathbb{E}\left[y(t-1)^2\right]}_{\gamma_y(0)} + \frac{1}{16}\underbrace{\mathbb{E}\left[y(t-2)^2\right]}_{\gamma_y(0)} + \underbrace{\mathbb{E}\left[e(t^2)\right]}_{1} + \frac{1}{4}\underbrace{\mathbb{E}\left[y(t-1)y(t-2)\right]}_{\gamma_y(1)} + \\ + \underbrace{\mathbb{E}\left[y(t-1)e(t)\right]}_{0} + \frac{1}{2}\underbrace{\mathbb{E}\left[y(t-2)e(t)\right]}_{0}\end{aligned}$$

The resulting equation is:

$$\frac{11}{16}\gamma_y(0) = \frac{1}{4}\gamma_y(1) + 1$$

To determine the covariance at lag one, we compute:

$$\begin{aligned}\gamma_y(1) &= \mathbb{E}\left[\left(\frac{1}{2}y(t-1) - \frac{1}{4}y(t-2) + e(t)\right)y(t-1)\right] \\ &= \frac{1}{2}\underbrace{\mathbb{E}\left[y(t-1)^2\right]}_{\gamma_y(0)} - \frac{1}{4}\underbrace{\mathbb{E}\left[y(t-2)y(t-1)\right]}_{\gamma_y(1)} + \underbrace{\mathbb{E}\left[e(t)y(t-1)\right]}_{0} \\ &= \frac{1}{2}\gamma_y(0) - \frac{1}{4}\gamma_y(1)\end{aligned}$$

This leads to the equation:

$$\gamma_y(1) = \frac{1}{2}\gamma_y(0) - \frac{1}{4}\gamma_y(1)$$

The resulting system of equations is:

$$\begin{cases}\frac{11}{16}\gamma_y(0) + \frac{1}{4}\gamma_y(1) = 1 \\ -\frac{1}{2}\gamma_y(0) + \frac{5}{4}\gamma_y(1) = 0\end{cases}$$

Solving this system yields:

$$\begin{cases}\gamma_y(0) = \frac{80}{63} \\ \gamma_y(1) = \frac{32}{63}\end{cases}$$

Now, we can compute the covariance at lag two:

$$\begin{aligned}\gamma_y(2) &= \mathbb{E}\left[\left(\frac{1}{2}y(t-1) - \frac{1}{4}y(t-2) + e(t)\right)y(t-2)\right] \\ &= \frac{1}{2}\underbrace{\mathbb{E}\left[y(t-1)y(t-2)\right]}_{\gamma_y(1)} - \frac{1}{4}\underbrace{\mathbb{E}\left[y(t-2)^2\right]}_{\gamma_y(0)} + \underbrace{\mathbb{E}\left[e(t)y(t-2)\right]}_{0} \\ &= \frac{1}{2}\gamma_y(1) - \frac{1}{4}\gamma_y(0) \\ &= -\frac{4}{63}\end{aligned}$$

The final result is:

$$\begin{cases}\gamma_y(0) = \frac{80}{63} \\ \gamma_y(1) = \frac{32}{63} \\ \gamma_y(\tau) = \frac{1}{2}\gamma_y(\tau-1) - \frac{1}{4}\gamma_y(\tau-2) \qquad \forall\,|\tau| \geq 2\end{cases}$$

1.4 Exercise 4

Consider the AR(1) process:

$$y(t) = \frac{1}{3}y(t-1) + e(t) + 2 \qquad e(t) \sim WN(1,1)$$

1. Determine the transfer function of the system and confirm its stationary stochastic nature.
2. Calculate the expected value.
3. Compute the covariance.

Solution

1. Applying the input delay operator yields:

$$y(t) = \frac{1}{3}z^{-1}y(t) + e(t) + 2$$

Rearranging terms, we get:

$$y(t) = \left[\frac{z}{z - \frac{1}{3}}\right](e(t) + 2)$$

As the input is a stationary stochastic process, the poles of the transfer function are:

$$z - \frac{1}{3} = 0 \to z = \frac{1}{3}$$

Since the pole is inside the unity circle, the process is stationary and stochastic.

2. The expected value is:

$$\mathbb{E}\left[y(t)\right] = \mathbb{E}\left[\frac{1}{3}y(t-1) + e(t) + 2\right] = \frac{1}{3}\mathbb{E}\left[y(t-1)\right] + 1 + 2$$

Given that we have a stationary stochastic process, the mean is constant:

$$m_y = \frac{1}{3}m_y + 3 \to m_y = \frac{9}{2}$$

3. We define the unbiased process:

$$\begin{cases} \tilde{y}(t) = y(t) - m_y \\ \tilde{e}(t) = e(t) - m_e \end{cases}$$

In our case, this yields:

$$\tilde{y}(t) + \frac{9}{2} = \frac{1}{3}\left(\tilde{y}(t-1) + \frac{9}{2}\right) + \tilde{e}(t) + 1 + 2 \to \tilde{y}(t) = \frac{1}{3}\tilde{y}(t-1) + \tilde{e}(t)$$

Finally, we compute the covariance function as:

$$\gamma_y(\tau) = \mathbb{E}\left[\tilde{y}(t)\tilde{y}(t-\tau)\right]$$

Beginning with the covariance at $\tau = 0$:

$$\gamma_{\tilde{y}}(0) = \mathbb{E}\left[\tilde{y}(t)^2\right] = \mathbb{E}\left[\left(\frac{1}{3}\tilde{y}(t-1) + \tilde{e}(t)\right)^2\right] = \frac{1}{9}\gamma_{\tilde{y}}(0) + 1 \to \gamma_{\tilde{y}}(0) = \frac{9}{8}$$

Next, we compute the covariance at $\tau = 1$:

$$\gamma_{\tilde{y}}(1) = \mathbb{E}\left[\tilde{y}(t)\tilde{y}(t-1)\right] = \mathbb{E}\left[\left(\frac{1}{3}\tilde{y}(t-1) + \tilde{e}(t)\right)\tilde{y}(t-1)\right] = \frac{1}{3}\gamma_{\tilde{y}}(0) \to \gamma_{\tilde{y}}(1) = \frac{3}{8}$$

For a generic τ:

$$\gamma_{\tilde{y}}(\tau) = \frac{1}{3}\gamma_{\tilde{y}}(\tau - 1) \qquad |\tau| \geq 1$$

1.5 Exercise 5

Consider the ARMA$(1,1)$ process described by the expression:

$$y(t) = \frac{1}{2}y(t-1) + e(t) - e(t-1) \qquad e(t) \sim WN(1,9)$$

1. Determine the transfer function and verify if it is a stationary stochastic process.
2. Calculate the expected value.
3. Compute the covariance function.

Solution

1. We express the formula in operatorial representation:

$$y(t) = \frac{1}{2}y(t)z^{-1} + e(t) - e(t)z^{-1} \rightarrow y(t) = \frac{z-1}{z-\frac{1}{2}}e(t)$$

   The system exhibits a zero at $z = 1$ and a pole in $z = \frac{1}{2}$, indicating asymptotic stability. As the input, White Noise, is a stationary stochastic process, $y(t)$ is also a stationary stochastic process.

2. The expected value is computed as:

$$\mathbb{E}\left[y(t)\right] = \mathbb{E}\left[\frac{1}{2}y(t-1) + e(t) - e(t-1)\right] = \frac{1}{2}\mathbb{E}\left[y(t-1)\right] + 1 - 1$$

   Since $y(t)$ is a stationary stochastic process, its mean is constant:

$$m_y = \frac{1}{2}m_y \rightarrow m_y = 0$$

   Alternatively, it can be computed using the theorem:

$$\mathbb{E}\left[y(t)\right] = W(1) \cdot \mathbb{E}\left[e(t)\right] = 0 \cdot 1 = 0$$

3. Define the unbiased process as:

$$\begin{cases} \tilde{y}(t) = y(t) - m_y \\ \tilde{e}(t) = e(t) - m_e \end{cases}$$

   In this case, we obtain:

$$\tilde{y}(t) + m_y = \frac{1}{2}\left(\tilde{y}(t-1) + m_y\right) + \tilde{e}(t) + m_e - \left(\tilde{e}(t-1) + m_e\right)$$

   Simplifying, we have:

$$\tilde{y}(t) = \frac{1}{2}\tilde{y}(t-1) + \tilde{e}(t) + 1 - \tilde{e}(t-1) - 1 \rightarrow \tilde{y}(t) = \frac{1}{2}\tilde{y}(t-1) + \tilde{e}(t) - \tilde{e}(t-1)$$

Starting with the covariance at $\tau = 0$:

$$\gamma_{\tilde{y}}(0) = \mathbb{E}\left[\tilde{y}(t)^2\right] = \mathbb{E}\left[\left(\frac{1}{2}\tilde{y}(t-1) + \tilde{e}(t) - \tilde{e}(t-1)\right)^2\right] = \frac{1}{4}\gamma_{\tilde{y}}(0) + 9 - 9 - 9 \rightarrow \gamma_{\tilde{y}}(0) = 12$$

Next, compute the covariance at $\tau = 1$:

$$\gamma_{\tilde{y}}(1) = \mathbb{E}\left[\tilde{y}(t)\tilde{y}(t-1)\right] = \mathbb{E}\left[\left(\frac{1}{2}\tilde{y}(t-1) + \tilde{e}(t) - \tilde{e}(t-1)\right) y(t-1)\right] \rightarrow \gamma_{\tilde{y}}(1) = -3$$

Then, compute the covariance at $\tau = 2$:

$$\gamma_{\tilde{y}}(2) = \mathbb{E}\left[\tilde{y}(t)\tilde{y}(t-2)\right] = \mathbb{E}\left[\left(\frac{1}{2}\tilde{y}(t-1) + \tilde{e}(t) - \tilde{e}(t-1)\right) y(t-2)\right] \rightarrow \gamma_{\tilde{y}}(1) = -\frac{3}{2}$$

For a generic τ:

$$\gamma_{\tilde{y}}(\tau) = \frac{1}{2}\gamma_{\tilde{y}}(\tau - 1) \qquad |\tau| \geq 2$$

1.6 Exercise 6

Consider the MA(2) process generated by the expression:

$$y(t) = e(t) + 0.5e(t-1) + 0.5e(t-2) \qquad e(t) \sim WN(2, 1)$$

1. Determine the transfer function and verify if it is a stationary stochastic process.
2. Calculate the expected value.
3. Compute the covariance function.

Solution

1. We express the formula in operatorial representation:

$$y(t) = e(t) + 0.5e(t)z^{-1} + 0.5e(t)z^{-2} \rightarrow y(t) = \frac{z^2 + 0.5z + 0.5}{z^2}e(t)$$

The system has two zeros at $z_{1,2} = -\frac{1}{4} \pm \frac{\sqrt{7}}{4}i$ and a pole at $z = 0$, indicating asymptotic stability. As the input, White Noise, is a stationary stochastic process, $y(t)$ is also a stationary stochastic process.

2. The expected value is computed as:

$$\mathbb{E}\left[y(t)\right] = \mathbb{E}\left[e(t) + 0.5e(t-1) + 0.5e(t-2)\right] = 2 + 1 + 1 = 4$$

Alternatively, it can be computed using the theorem:

$$\mathbb{E}\left[y(t)\right] = W(1) \cdot \mathbb{E}\left[e(t)\right] = 2 \cdot 2 = 4$$

3. Define the unbiased process as:

$$\begin{cases} \tilde{y}(t) = y(t) - m_y \\ \tilde{e}(t) = e(t) - m_e \end{cases}$$

In this case, we have:

$$\tilde{y}(t) + m_y = (\tilde{e}(t) + m_e) + 0.5\,(\tilde{e}(t-1) + m_e) + 0.5\,(\tilde{e}(t-2) + m_e)$$

Simplifying, we obtain:

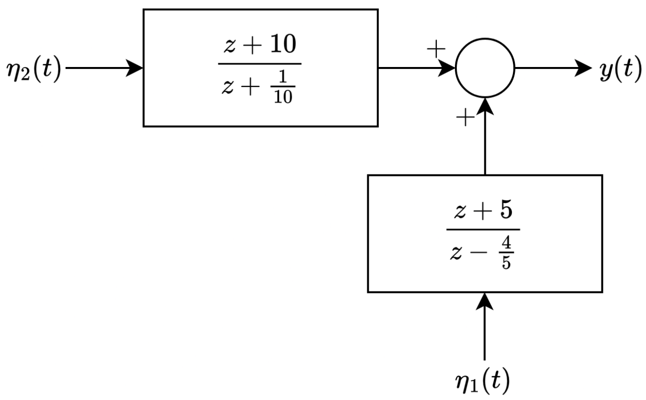
$$\tilde{y}(t) = \tilde{e}(t) + 0.5\tilde{e}(t-1) + 0.5\tilde{e}(t-2)$$

Since it is a Moving Average process, we can directly find the covariance as:

$$\begin{cases} (c_0^2 + c_1^2 + c_2^2)\,\lambda^2 & \tau = 0 \\ (c_0c_1 + c_1c_2)\,\lambda^2 & |\tau| = 1 \\ (c_0c_2)\,\lambda^2 & |\tau| = 2 \\ 0 & |\tau| \geq 3 \end{cases} \rightarrow \begin{cases} \frac{3}{2} & \tau = 0 \\ \frac{3}{4} & |\tau| = 1 \\ \frac{1}{2} & |\tau| = 2 \\ 0 & |\tau| \geq 3 \end{cases}$$

1.7 Exercise 7

Consider the stochastic process defined by the following diagram:



Here, $\eta_1 \sim WN(1,1)$ and $\eta_2 \sim WN(0,1)$ are uncorrelated.
Find the characteristic values of the given process $y(t)$.

Solution

Remember that for an ARMA(n_a, n_b) process:

- If $n_a > n_b$, the covariance becomes recursive for $\tau = n_a$.
- If $n_a \leq n_b$, the covariance becomes recursive for $\tau = n_b + 1$

The output process is composed of two uncorrelated processes because the White Noise sources are uncorrelated:

$$y(t) = y_1(t) + y_2(t)$$

Since both $y_1(t)$ and $y_2(t)$ are stationary, $y(t)$ is also stationary.

The mean is:

$$m_y = \mathbb{E}\left[y(t)\right] = \mathbb{E}\left[y_1(t) + y_2(t)\right] = W_1(1)\mathbb{E}\left[\eta_1(t)\right] + W_2(1)\mathbb{E}\left[\eta_2(t)\right] = \frac{15}{2}$$

The covariance can be computed as the sum of the covariances of $y_1(t)$ and $y_2(t)$ (since they are uncorrelated):

$$\gamma_y(\tau) = \gamma_{y_1}(\tau) + \gamma_{y_2}(\tau)$$

For the stochastic process $y_1(t)$ in the time domain:

$$y_1(t) = \frac{1}{5}y_1(t-1) + \eta_1(t) + 5\eta_1(t-1)$$

Define the unbiased process by:

$$\begin{cases} \tilde{y}_1(t) = y_1(t) - m_{y_1} \\ \tilde{\eta}_1(t) = \eta_1(t) - m_{\eta_1} \end{cases}$$

Then, the process becomes:

$$\tilde{y}_1(t) = \frac{1}{5}\tilde{y}_1(t-1) + \tilde{\eta}_1(t) + 5\tilde{\eta}_1(t-1)$$

The covariance at different time lags is:

$$\gamma_{y_1}(\tau) = \begin{cases} \frac{175}{6} & \tau = 0 \\ \frac{65}{6} & |\tau| = 1 \\ \frac{13}{6} & |\tau| = 2 \\ \frac{1}{5}\gamma_{y_1}(\tau - 1) & |\tau| \geq 3 \end{cases}$$

For the stochastic process $y_2(t)$:

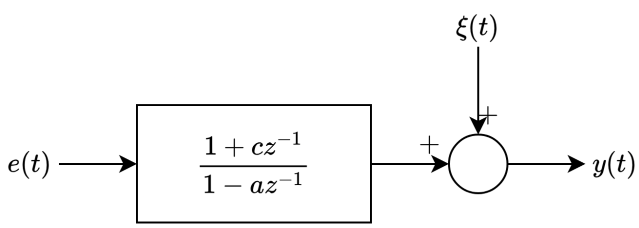
$$y_2(t) = -\frac{1}{10}y_2(t-1) + \eta_2(t) + 10\eta_2(t-1)$$

The covariance function is:

$$\gamma_{y_2}(\tau) = \begin{cases} 100 & \tau = 0 \\ 0 & |\tau| \geq 1 \end{cases}$$

1.8 Exercise 8

Consider the stochastic process defined by the following diagram:



Here, $e(t) \sim WN(1, 1)$ and $\xi(t) \sim WN(0, 1)$ are uncorrelated.

1. Determine when the process is stationary.
2. Given $\gamma_y(0) = 6$, $\gamma_y(1) = -2$, and $\gamma_y(\tau) = 0$ for $\tau \geq 2$, compute the values of a and c.

Solution

1. The process $y(t)$ is stationary when both $\xi(t)$ and $y_1(t)$ are stationary. Since $\xi(t)$ is a White Noise process, it is stationary by definition. The process $y_1(t)$ is stationary when $|a| < 1$.
2. Since $\gamma_y(\tau) = 0$ for $\tau \geq 2$, this implies that $y(t)$ is a Moving Average Process of order one. Hence, $a = 0$.

   The process in the time domain is:

$$y(t) = -ay(t-1) + e(t) + ce(t-1) + \xi(t)$$

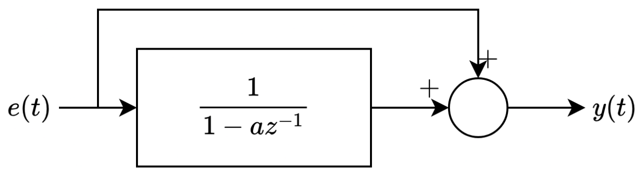
   We can compute the covariance at $\tau = 0$:

$$\gamma_y(0) = \mathbb{E}\left[y(t)^2\right] = 0$$

   From this, we obtain $c = \pm 2$.

1.9 Exercise 9

Consider the stochastic process defined by the following diagram:



Here, $e(t) \sim WN(0, \lambda^2)$, and $|a| < 1$.
Find the characteristic values of the given process $y(t)$.

Solution

To begin, let's compute the expected value of $y(t)$:

$$m_y = \mathbb{E}\left[y(t)\right] = \mathbb{E}\left[ay(t-1) + 2e(t)\right] = a\mathbb{E}\left[y(t-1)\right] \rightarrow m_y = 0$$

The covariance function at $\tau = 0$ is given by:

$$\gamma_y(0) = \mathbb{E}\left[y(t)^2\right] = \mathbb{E}\left[(y_1(t) + y_2(t))^2\right] = \frac{4-3a^2}{1-a^2}\lambda^2$$

The covariance function at $\tau = 1$ is given by:

$$\gamma_y(1) = \mathbb{E}\left[y(t)y(t-1)\right] = \frac{a\lambda^2(2-a^2)}{1-a^2}$$

Alternatively, noting that we have two processes in parallel with a transfer function equal to:

$$y(t) = \frac{1}{1 - az^{-1}} e(t) + e(t) = \frac{2 - az^{-1}}{1 - az^{-1}} e(t)$$

The canonical form becomes:

$$y(t) = \frac{1 - \frac{a}{2} z^{-1}}{1 - az^{-1}} e_1(t)$$

Here, $e_1(t) = 2e(t)$, implying that $e(t) \sim WN(0, 2^2\lambda^2)$. We can now find the time-domain representation, which is:

$$y(t) = ay(t-1) + \eta_1(t) - \frac{a}{2}\eta_1(t-1)$$

From this, we can compute the covariance in a more straightforward manner.

CHAPTER 2

Frequency analysis

2.1 Exercise 1

Consider the process described by the expression:

$$y(t) = \frac{1}{2}y(t-1) + e(t) - e(t-1) \qquad e(t) \sim WN(0,9)$$

Determine the spectral density function of the provided process.

Solution

For a stationary stochastic process, the following formula holds:

$$\Gamma_y(\omega) = \left|W(e^{j\omega})\right|^2 \Gamma_u(\omega) = \left|W(e^{j\omega})\right|^2 \lambda^2$$

We start by computing the transfer function:

$$y(t) = \frac{z-1}{z-\frac{1}{2}}$$

Since the pole is inside the unit circle and $e(t)$ is a stationary stochastic process (White Noise), $y(t)$ is also a stationary stochastic process. We can then use the fundamental theorem of spectral analysis:

$$\Gamma_y(\omega) = \left|\frac{e^{j\omega}-1}{e^{j\omega}-\frac{1}{2}}\right|^2 9$$

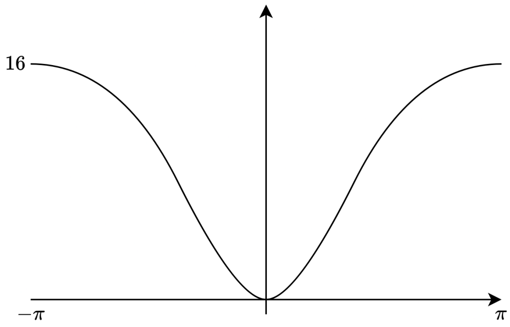
We compute the squares as follows:

- $\left|e^{j\omega}-1\right|^2 = \left(e^{j\omega}-1\right)\left(e^{-j\omega}-1\right) = 2(1-\cos\omega)$
- $\left|e^{j\omega}-\frac{1}{2}\right|^2 = \left(e^{j\omega}-\frac{1}{2}\right)\left(e^{-j\omega}-\frac{1}{2}\right) = \frac{5}{4} - \cos\omega$

Thus, the spectral density function is:

$$\Gamma_y(\omega) = \frac{1-\cos\omega}{\frac{5}{4}-\cos\omega}18$$

This allows us to generate the graph:



2.2 Exercise 2

Consider the process generated by the following expression:

$$y(t) = \left(1 - z^{-1} + z^{-2}\right)\left(1 + \frac{3}{2}z^{-1}\right)e(t) \qquad e(t) \sim N(0,1)$$

Find the spectral density function of the given process.

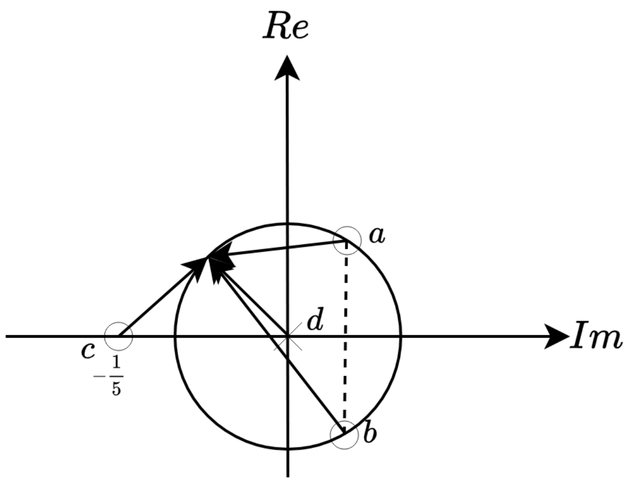
Solution

This can be rewritten as:

$$y(t) = \frac{\left(z^2 - z + 1\right)\left(z + \frac{3}{2}\right)}{z^2}e(t)$$

The poles are at $z = 0$, and the zeros are at $z_{1,2,3} = -\frac{3}{2}, \frac{1}{2} \pm j\frac{\sqrt{3}}{2}$

The simplest way to compute the spectral density function is by using the vectors that connect a generic point $e^{j\omega}$ to the poles (d) and the zeros (a, b, c):



In this case, the spectral density function is computed as:

$$\Gamma_y(\omega) = \frac{|a|^2 |b|^2 |c|^2}{|d|^2}\lambda^2$$

For e^{j0}:

- $|a|^2 = 1$
- $|b|^2 = 1$
- $|c|^2 = \frac{25}{4}$
- $|d|^2 = 1$

Thus, $\Gamma_y(0) = \frac{25}{4}$.

For $e^{j\frac{\pi}{2}}$:

- $|a|^2 = 2 - \sqrt{3}$
- $|b|^2 = 2 + \sqrt{3}$
- $|c|^2 = \frac{13}{4}$
- $|d|^2 = 1$

Therefore, $\Gamma_y\left(\frac{\pi}{2}\right) = \frac{13}{4}$.

For $e^{j\pi}$:

- $|a|^2 = 3$
- $|b|^2 = 3$
- $|c|^2 = \frac{1}{4}$
- $|d|^2 = 1$

Hence, $\Gamma_y(\pi) = \frac{9}{4}$.

Note that $\Gamma_y\left(\frac{\pi}{3}\right) = 0$.

2.3 Exercise 3

Consider the process described by the function:

$$y(t) = \frac{z^4}{\left(z - \frac{1}{2} - j\frac{1}{2}\right)\left(z - \frac{1}{2} + j\frac{1}{2}\right)\left(z + \frac{1}{2} - j\frac{1}{2}\right)\left(z + \frac{1}{2} + j\frac{1}{2}\right)} e(t)$$

Here, $e(t) \sim WN(0, 1)$. Find the spectral density function of the given process.

Solution

In this case, the spectral density function is computed as:

$$\Gamma_y(\omega) = \frac{1}{|a|^2 |b|^2 |c|^2 |d|^2} \lambda^2$$

Starting at e^{j0}, we have:

- $|a|^2 = \frac{1}{2}$

- $|b|^2 = \frac{5}{2}$
- $|c|^2 = \frac{5}{2}$
- $|d|^2 = \frac{1}{2}$

Thus, $\Gamma_y(0) = \dfrac{16}{25}$.

For $e^{j\frac{\pi}{2}}$ and $e^{j\pi}$, we have the same result.

Using the fundamental theorem of spectral analysis, we have:

$$\Gamma_y(\omega) = \left|W(e^{j\omega})\right|^2$$

This can be rewritten as:

$$y(t) = \frac{z^4}{z^4 + \frac{1}{4}} e(t)$$

Thus,

$$\Gamma_y(\omega) = \left|\frac{e^{j\omega 4}}{e^{j\omega 4} + \frac{1}{4}}\right|^2 \cdot 1 = \frac{16}{17 + 8\cos(4\omega)}$$

2.4 Exercise 4

Consider the following process:

$$y(t) = \frac{1}{2} y(t-1) + e(t) - e(t-1) \qquad e(t) \sim WN(0, 9)$$

We have that $\gamma_y(0) = 12$, $\gamma_y(\pm 1) = -3$, and $\gamma_y(\pm\tau) = \frac{1}{2}\gamma_y(\tau - 1)$ with $|\tau| \geq 2$. Find the spectral density function of the given process.

Solution

The spectrum is the sum of all covariances:

$$\begin{aligned}
\Gamma_y(\omega) &= \sum_{\tau=-\infty}^{+\infty} \gamma_y(\tau) e^{-j\omega\tau} \\
&= 12e^{-j\omega 0} - 3e^{-j\omega} - 3e^{j\omega} - \frac{3}{2}e^{-j\omega 2} - \frac{3}{2}e^{j\omega 2} + \dots \\
&= 12 - 6\left[\frac{1}{2}e^{j\omega} + \frac{1}{2}e^{-j\omega} + \frac{1}{4}e^{j\omega} + \frac{1}{4}e^{-j\omega} + \dots\right] \\
&= 12 - 6\left[-1 + 1 + \frac{1}{2}e^{-j\omega} + \frac{1}{4}e^{-j\omega} + \dots - 1 + 1 + \frac{1}{2}e^{j\omega} + \frac{1}{4}e^{j\omega}\right] \\
&= 24 - 6\left[\sum_{i=0}^{+\infty}\left(\frac{1}{2}e^{-j\omega}\right)^i + \sum_{i=0}^{+\infty}\left(\frac{1}{2}e^{j\omega}\right)^i\right] \\
&= 24 - 6\left[\frac{1}{1 - \frac{1}{2}e^{-j\omega}} + \frac{1}{1 - \frac{1}{2}e^{j\omega}}\right] \\
&= \frac{1 - \cos(\omega)}{\frac{5}{4} - \cos(\omega)} 18
\end{aligned}$$

CHAPTER 3

Prediction

3.1 Exercise 1

Consider the given process:

$$y(t) = \frac{1}{2}y(t-2) + \eta(t) + 4\eta(t-1) \qquad \eta(t) \sim WN(0,1)$$

1. Determine the transfer function.
2. Calculate $\hat{y}(t+1|t)$.
3. Validate the obtained predictor.
4. Find $\hat{y}(t+2|t)$.

Solution

1. The transfer function is given by:

$$y(t) = \left[\frac{z(z+4)}{z^2 - \frac{1}{2}}\right]\eta(t)$$

2. To find $\hat{y}(t+1|t)$, we need to follow these steps:

   - Check if the process is in canonical form:
     (a) Numerator and denominator are monic (coefficient of the highest power equal one): both are equal to one.
     (b) Numerator and denominator have the same degree: both are of second degree.
     (c) Numerator and denominator are co-prime: they have no common roots.
     (d) The singularities must be inside the unit circle: not satisfied. Therefore, redefine:

$$\left[\frac{z(z+4)}{z^2 - \frac{1}{2}}\right]\left[\frac{z+\frac{1}{4}}{z+4}\right] = \frac{z\left(z+\frac{1}{4}\right)}{z^2 - \frac{1}{2}}$$

     And we must redefine the White Noise as $e(t) \sim WN(0, 4^2 \cdot 1)$

- Compute the predictor via long division:

$$\hat{y}(t|t-1) = E(z)e(t) + \frac{F(z)}{A(z)}e(t-1) = e(t) + \frac{\frac{1}{4} - \frac{1}{2}z^{-1}}{1 - \frac{1}{2}z^{-2}}e(t-1)$$

  But we don't know the value of $e(t)$ because we have given only past samples until $t-1$. So the predictor is:

$$\hat{y}(t|t-1) = \frac{\frac{1}{4}z^{-1} - \frac{1}{2}z^{-2}}{1 - \frac{1}{2}z^{-2}} \frac{1 - \frac{1}{2}z^{-1}}{1 + \frac{1}{4}z^{-1}}y(t) = \frac{\frac{1}{4} - \frac{1}{2}z^{-1}}{1 + \frac{1}{4}z^{-1}}y(t-1)$$

  Equivalently:

$$\hat{y}(t+1|t) = \frac{\frac{1}{4} - \frac{1}{2}z^{-1}}{1 + \frac{1}{4}z^{-1}}y(t)$$

  In time domain, this becomes:

$$\hat{y}(t+1|t) = -\frac{1}{4}\hat{y}(t|t-1) + \frac{1}{4}y(t) + \frac{1}{2}y(t-1)$$

Note that predictors from noise can also be computed as:

$$\hat{y}(t|t-1) = \frac{C(z) - A(z)}{A(z)}e(t)$$

Predictors from data can be computed as:

$$\hat{y}(t|t-1) = \frac{C(z) - A(z)}{C(z)}y(t)$$

3. The prediction error is given by:

$$\varepsilon(t|t-1) = y(t) - \hat{y}(t|t-1) = E(z)e(t)$$

The variance is:

$$\mathrm{Var}\left[\varepsilon(t|t-1)\right] = \mathbb{E}\left[\varepsilon(t|t-1)^2\right] = \mathbb{E}\left[1 \cdot e(t)^2\right] = 16$$

Since the variance of the process is approximately 23, the predictor is optimal but not very good because the variance and covariance are similar.

4. The two-step ahead predictor can only be found via long division. After performing two steps in the division, we obtain:

$$\hat{y}(t|t-2) = \frac{F(z)}{A(z)}e(t) = \frac{\frac{1}{2}z^{-2} + \frac{1}{8}z^{-3}}{1 - \frac{1}{2}z^{-2}}e(t)$$

The predictor from data can be found knowing that:

$$e(t) = \frac{1 - \frac{1}{2}z^{-1}}{1 + \frac{1}{4}z^{-1}}y(t)$$

By substitution, we obtain:

$$\hat{y}(t|t-2) = \frac{\frac{1}{2} + \frac{1}{8}z^{-1}}{1 + \frac{1}{4}z^{-1}}y(t-2)$$

The prediction error is:

$$\varepsilon(t|t-2) = y(t) - \hat{y}(t|t-2) = E(z)e(t)$$

The variance is:

$$\mathrm{Var}\left[\varepsilon(t|t-2)\right] = \mathbb{E}\left[\varepsilon(t|t-2)^2\right] = \mathbb{E}\left[\left(e(t) + \frac{1}{4}e(t-1)\right)^2\right] = 17$$

Since the variance of the process is approximately 23, the predictor is optimal but not very good because the variance and covariance are similar.

3.2 Exercise 2

Consider the following process described by the expression:

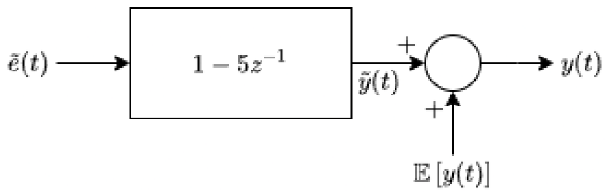
$$y(t) = e(t) + 5e(t-1) \qquad e(t) \sim WN(1,1)$$

The expected value of the process $y(t)$ is 6.

1. Determine the unbiased process.
2. Find the predictor $\hat{y}(t|t-1)$.

Solution

1. The given system can be represented as:



In the block diagram, we define:

$$\begin{cases} \tilde{y}(t) = y(t) - m_y \\ \tilde{e}(t) = e(t) - m_e \end{cases} \rightarrow \begin{cases} \tilde{y}(t) = y(t) - 6 \\ \tilde{e}(t) = e(t) - 1 \end{cases}$$

The process $y(t)$ is composed of:

$$y(t) = \tilde{y}(t) + 6 = \tilde{e}(t)\left(1 + 5z^{-1}\right) + 6 = (e(t) - 1)\left(1 + 5z^{-1}\right) + 6 = e(t) + 5e(t-1)$$

The unbiased process is:

$$\tilde{y}(t) = \tilde{e}(t) + 5\tilde{e}(t-1)$$

Since the unbiased process is not in canonical form, an all-pass filter must be used:

$$\tilde{y}(t) = \frac{1 + \frac{1}{5}z^{-1}}{1} \frac{1 + 5z^{-1}}{1 + \frac{1}{5}z^{-1}1}\eta(t)$$

Here, $\eta(t) \sim WN(0, 25)$.

In the time domain, this becomes:

$$\tilde{y}(t) = \eta(t) + \frac{1}{5}\eta(t-1)$$

2. The predictor from noise is:
$$\hat{\tilde{y}}(t|t-1) = \frac{1}{5}\eta(t-1)$$
The predictor from data is:
$$\hat{\tilde{y}}(t|t-1) = \frac{1}{5}z^{-1}\frac{1}{1+\frac{1}{5}z^{-1}}\tilde{y}(t) = -\frac{1}{5}\tilde{y}(t-1|t-2) + \frac{1}{5}\tilde{y}(t-1)$$
To find the predictor of the original process by substitution, as the prediction is linear, we have:
$$\begin{aligned}
\hat{\tilde{y}}(t+1|t) &= -\frac{1}{5}\tilde{y}(t|t-1) + \frac{1}{5}\tilde{y}(t) \rightarrow \\
\hat{y}(t+1|t) - 6 &= -\frac{1}{5}\left(y(t|t-1) - 6\right) + \frac{1}{5}\left(y(t) - 6\right) \rightarrow \\
\hat{y}(t+1|t) - 6 &= -\frac{1}{5}y(t|t-1) + \frac{6}{5} + \frac{1}{5}y(t) - \frac{6}{5} \rightarrow \\
\hat{y}(t+1|t) &= -\frac{1}{5}y(t|t-1) + \frac{1}{5}y(t) + 6
\end{aligned}$$

3.3 Exercise 3

Consider the given process:
$$\frac{1}{2}y(t) = -\frac{1}{3}y(t-1) - \frac{1}{18}y(t-2) + 3e(t-2) - 8e(t-3) - 3e(t-4)$$
Here, $e(t) \sim WN(0, 1)$. Let's compute the one-step ahead predictor.

Solution

The transfer function is:
$$y(t) = \frac{3z^{-2} - 8z^{-3} - 3z^{-4}}{\frac{1}{2} + \frac{1}{3}z^{-1} + \frac{1}{18}z^{-2}}e(t)$$
We need to rewrite this function in canonical form:
$$y(t) = \frac{z^2 - \frac{8}{3}z - 1}{z^2 + \frac{2}{3}z + \frac{1}{9}}\frac{3}{\frac{1}{2}}e(t)$$
To ensure the same degree, it becomes:
$$y(t) = \frac{z^2 - \frac{8}{3}z - 1}{z^2 + \frac{2}{3}z^1 + \frac{1}{9}}\frac{3}{\frac{1}{2}}z^{-2}e(t)$$
Now, define the new White Noise as:
$$\eta(t) = \frac{3}{\frac{1}{2}}z^{-2}e(t) \rightarrow \eta(t) \sim WN\left(0, 36\right)$$
Thus, we have:
$$y(t) = \frac{1 - \frac{8}{3}z^{-1} - z^{-2}}{1 + \frac{2}{3}z^{-1} + \frac{1}{9}z^{-2}}\eta(t) \qquad \eta(t) \sim WN\left(0, 36\right)$$

The poles are at $z_{1,2} = -\frac{1}{3}$, and the zeros are at $z_{1,2} = -\frac{1}{3}, 3$. We have a zero that is not inside the unit circle.

Next, factorize the numerator and denominator:

$$y(t) = \frac{\left(1+\frac{1}{3}z^{-1}\right)\left(1-3z^{-1}\right)}{\left(1+\frac{1}{3}z^{-1}\right)\left(1+\frac{1}{3}z^{-1}\right)}\eta(t) = \frac{1-3z^{-1}}{1+\frac{1}{3}z^{-1}}\eta(t)$$

Use an all-pass filter to remove the zero at three:

$$y(t) = \frac{1-3z^{-1}}{1+\frac{1}{3}z^{-1}}\frac{1-\frac{1}{3}z^{-1}}{1+\frac{1}{3}z^{-1}}\eta(t)$$

Redefined the White Noise as:

$$\xi(t) = \frac{1-3z^{-1}}{1+\frac{1}{3}z^{-1}}\eta(t) \rightarrow \xi(t) \sim WN(0, 324)$$

The canonical form is:

$$y(t) = \frac{1-\frac{1}{3}z^{-1}}{1+\frac{1}{3}z^{-1}}\xi(t)$$

Now, with the canonical representation, compute the one-step ahead predictor as:

$$\hat{y}(t|t-1) = \frac{C(z)-A(z)}{C(z)}y(t) = \frac{1+\frac{1}{3}z^{-1}-\left(1-\frac{1}{3}z^{-1}\right)}{1+\frac{1}{3}z^{-1}}y(t) = \frac{-\frac{2}{3}z^{-1}}{1-\frac{1}{3}z^{-1}}y(t)$$

3.4 Exercise 4

Consider the ARMAX process described by the expression:

$$y(t) = \frac{1}{3}y(t-1) + u(t-1) + 3e(t-1) + e(t-2) \qquad e(t) \sim WN(0,1)$$

Let's compute the one-step ahead predictor.

Solution

The ARMAX process can be rewritten as:

$$y(t) = \frac{C(z)}{A(z)}e(t) + \frac{B(z)}{A(z)}u(t-1) = \frac{3z^{-1}+z^{-2}}{1-\frac{1}{3}z^{-1}}e(t) + \frac{1}{1-\frac{1}{3}z^{-1}}u(t-1)$$

The transfer function we consider is the one multiplied by the noise $e(t)$:

$$W(z) = \frac{3z^{-1}+z^{-2}}{1-\frac{1}{3}z^{-1}}e(t) = \frac{3z+1}{z^2-\frac{1}{3}z}e(t)$$

By collecting $3z^{-1}$ at the numerator, we get:

$$W(z) = \frac{1+\frac{1}{3}z^{-1}}{1-\frac{1}{3}z^{-1}}\eta(t) \qquad \eta(t) \sim WN(0,9)$$

The canonical form of the full ARMAX is:

$$y(t) = \frac{1+\frac{1}{3}z^{-1}}{1-\frac{1}{3}z^{-1}}\eta(t) + \frac{1}{1-\frac{1}{3}z^{-1}}u(t-1)$$

The one-step ahead predictor for an ARMAX is:

$$\hat{y}(t|t-1) = \frac{F(z)}{C(z)}y(t) + \frac{B(z)E(z)}{C(z)}u(t-1) = \frac{\frac{2}{3}z^{-1}}{1+\frac{1}{3}z^{-1}}y(t) + \frac{1}{1+\frac{1}{3}z^{-1}}u(t-1)$$

3.5 Exercise 5

Consider the process:

$$y(t) = 3 + v(t)$$

Let's find the predictor $\hat{y}(t|t-k)$ for all k when:

1. $v(t) \sim WN(0,1)$
2. $v(t) = e(t) + \frac{1}{2}e(t-2) \quad e(t) \sim WN(0,1)$

Solution

1. In this case, the process becomes:

$$y(t) = 3 + v(t) \qquad v(t) \sim WN(0,1)$$

The only predictable part at any time different from zero is the constant, so:

$$\hat{y}(t|t-k) = 3$$

2. Here, the process becomes:

$$y(t) = 3 + e(t) + \frac{1}{2}e(t-2) \quad e(t) \sim WN(0,1)$$

The expected value of the process is three, so we consider the unbiased process:

$$\tilde{y}(t) = y(t) - 3$$

Thus,

$$\tilde{y}(t) = e(t) + \frac{1}{2}e(t-2) = \frac{1 + \frac{1}{2}z^{-2}}{1}e(t)$$

The process is in canonical form. With long division, we get $F_1(z) = \frac{1}{2}z^{-2}$, $F_2(z) = \frac{1}{2}z^{-2}$, and $F_{3\to\infty} = 0$.

For the one-step ahead predictor:

$$\hat{\tilde{y}}(t|t-1) = \frac{\frac{1}{2}z^{-2}}{1 + \frac{1}{2}z^{-2}}\tilde{y}(t) = -\frac{1}{2}\tilde{y}(t-2|t-3) + \frac{1}{2}\tilde{y}(t-2)$$

Thus,

$$\hat{y}(t|t-1) = -\frac{1}{2}y(t-2|t-3) + \frac{1}{2}y(t-2) + 3$$

For the two-step predictor:

$$\hat{y}(t|t-2) = -\frac{1}{2}y(t-2|t-4) + \frac{1}{2}y(t-2) + 3$$

For the general k:

$$\hat{y}(t|t-k) = 3$$

3.6 Exercise 6

Consider the process:

$$y(t) = \frac{1}{4}y(t-2) + \eta(t-2) + \frac{1}{3}\eta(t-3) \qquad \eta(t) \sim WN(0,1)$$

Let's compute the predictor $\hat{y}(t|t-2)$.

Solution

The transfer function of the expression is:

$$y(t) = \frac{z^{-1} + \frac{1}{3}z^{-3}}{1 - \frac{1}{4}z^{-2}}\eta(t) = \frac{z^3 + \frac{1}{3}z}{z^3 - \frac{1}{4}z}\eta(t)$$

In canonical form it becomes:

$$y(t) = \frac{1 + \frac{1}{3}z^{-2}}{1 - \frac{1}{4}z^{-2}}(t) \qquad e(t) \sim WN(0,1)$$

All the poles and zeros are inside the unit circle, so the transfer function is stable.

By performing the long division for two steps, we get $F_2(z) = \frac{7}{12}z^{-2}$ and $E(z) = 1$. The predictor is:

$$\hat{y}(t|t-2) = e(t) + \frac{\frac{7}{12}}{1 - \frac{1}{4}z^{-2}}e(t-2) = \frac{\frac{7}{12}z^{-2}}{1 - \frac{1}{4}z^{-2}}e(t)$$

In terms of data, it becomes:

$$\hat{y}(t|t-2) = \frac{\frac{7}{12}z^{-2}}{1 - 1\frac{1}{4}z^{-2}}\frac{1 - \frac{1}{4}z^{-2}}{1 + \frac{1}{3}z^{-2}}y(t) = \frac{\frac{7}{12}z^{-2}}{1 + \frac{1}{3}z^{-2}}y(t)$$

In the time domain:

$$\hat{y}(t|t-2) = \frac{7}{12}y(t-2) - \frac{1}{3}z^{-2}\hat{y}(t-2|t-4)$$

CHAPTER 4

Identification

4.1 Exercise 1

Consider the system:

$$\mathcal{S} : y(t) = e(t) + \frac{1}{2}e(t-1) \qquad e(t) \sim WN(0,1)$$

And the model:

$$\mathcal{M} : y(t) = ay(t-1) + \xi(t) \qquad \xi(t) \sim WN(0, \lambda^2)$$

Compute the value of a^* and λ^{*2}.

Solution

The steps are:

1. Compute the predictor:
$$\hat{y}(t|t-1) = ay(t-1)$$
2. Compute the prediction error (by substituting the real system to $y(t)$):
$$\varepsilon(t|t-1) = y(t) - ay(t-1) = \left(1 - az^{-1}\right) y(t) = \left(1 - az^{-1}\right) \left(1 + \frac{1}{2}z^{-1}\right) e(t)$$
That is:
$$\varepsilon(t|t-1) = e(t) + \left(\frac{1}{2} - a\right) e(t-1) - \frac{1}{2}ae(t-2)$$
3. Compute the variance of the prediction error:
$$\bar{J}(a^*) = \mathrm{Var}\left[\varepsilon\right] = \mathbb{E}\left[\varepsilon^2\right] = \frac{5}{4} + \frac{5}{4}a^2 - a$$
4. Derive with respect to the variable a^*:
$$\frac{d\bar{J}(a^*)}{da^*} = \frac{5}{2}a^* - 1$$
We want a minimum, so we set this derivative to zero:
$$\frac{5}{2}a^* - 1 \to a^* = \frac{2}{5}$$

5. The value of λ^{*2} can be computed by substituting the value of a^* into the variance function:
$$\lambda^{*2} = \frac{5}{4} + \frac{5}{4}\left(\frac{2}{5}\right)^2 - \frac{2}{5} = \frac{21}{20}$$
The prediction is good since it is similar to the variance of the White Noise.

The model is stable since the poles are inside the unit circle.

4.2 Exercise 2

Consider the system:

$$\mathcal{S} : y(t) = e(t) + \frac{1}{2}e(t-1) \qquad e(t) \sim WN(0,1)$$

And the model:
$$\mathcal{M} : y(t) = \eta(t) + b\eta(t-1) \qquad \eta(t) \sim WN(0,\lambda^2)$$

Find the value of b^* and λ^{*2}.

Solution

Since both the model and the system are of the same type (Moving Average of order one), we can conclude that:

- $b^* = \frac{1}{2}$.
- $\lambda^{*2} = 1$.

Thus, we obtain the same formulation for the system. The model is stable since the poles are inside the unit circle.

4.3 Exercise 3

Consider the system:

$$\mathcal{S} : y(t) = e(t) + \frac{1}{2}e(t-1) \qquad e(t) \sim WN(0,1)$$

And the model:

$$\mathcal{M} : y(t) = \frac{1}{1 + az^{-1} + bz^{-2}}\eta(t) \qquad \eta(t) \sim WN(0,\lambda^2)$$

Find the value of $\theta^* = \begin{bmatrix} a^* & b^* \end{bmatrix}$ and λ^{*2}.

Solution

The steps are:

1. Compute the predictor:
$$\hat{y}(t|t-1) = \frac{C(z) - A(z)}{C(z)} y(t)$$

2. Compute the prediction error (by substituting the real system to $y(t)$):
$$\varepsilon(t|t-1) = \frac{A(z)}{C(z)} y(t) = \frac{1 + az^{-1} + bz^{-2}}{1} \left(e(t) + \frac{1}{2} e(t-1) \right)$$

3. Compute the variance of the prediction error:
$$\bar{J}(\theta^*) = \mathrm{Var}\left[\varepsilon(t|t-1)\right] = \mathbb{E}\left[\varepsilon(t|t-1)^2\right] = \frac{5}{4}a^2 + \frac{5}{4}b^2 + a + ab + \frac{5}{4}$$

4. Derive with respect to the variable θ^*:
$$\begin{cases} \frac{\partial \theta^*}{\partial a^*} = \frac{5}{2}a^* + 1 + b^* \\ \frac{\partial \theta^*}{\partial b^*} = \frac{5}{2}b^* + a^* \end{cases}$$
We want a minimum, so we set those derivatives to zero:
$$\begin{cases} \frac{5}{2}a^* + 1 + b^* = 0 \\ \frac{5}{2}b^* + a^* = 0 \end{cases} \quad \rightarrow \begin{cases} a = -\frac{10}{21} \\ b = \frac{4}{21} \end{cases}$$

5. The value of λ^{*2} can be computed by substituting the value of θ^* into the variance function:
$$\lambda^{*2} = 1.011$$
The prediction is good since it is similar to the variance of the White Noise.

The model is stable since the poles are inside the unit circle.

4.4 Exercise 4

Consider the system:
$$\mathcal{S} : y(t) = 3e(t) + 9e(t-1) \qquad e(t) \sim WN(0, 1)$$
And the model:
$$\mathcal{M} : y(t) = \eta(t) + b\eta(t-1) \qquad \eta(t) \sim WN(0, \lambda^2)$$
Find the value of b^* and λ^{*2}.

Solution

The system is not written in canonical form, so rewrite it as:

$$\begin{aligned}
\mathcal{S} : y(t) &= 3\left(e(t) + 3e(t-1)\right) \\
&= 3\left(1 + 3z^{-1}\right) e(t) \\
&= 3\frac{\left(1 + 3z^{-1}\right) e(t)}{1 + \frac{1}{3}z^{-1}} \left(1 + \frac{1}{3}z^{-1}\right) e(t)
\end{aligned}$$

Now we obtain:

$$\xi(t) = \frac{1 + 3z^{-1}}{1 + \frac{1}{3}z^{-1}} e(t) \qquad \xi(t) \sim WN(0, 81)$$

And the system expression becomes:

$$\mathcal{S} : y(t) = \xi(t) + \frac{1}{3}\xi(t-1)$$

Now, with the same expression, we find:

- $b^* = \frac{1}{3}$.
- $\lambda^{*2} = 81$.

The model is stable since the poles are inside the unit circle.

4.5 Exercise 5

Consider the system:

$$\mathcal{S} : y(t) = e(t) + \frac{1}{3}e(t-1) \qquad e(t) \sim WN(0, 1)$$

And the model is:

$$\mathcal{M} : y(t) = -ay(t-1) + \eta(t) \qquad \eta(t) \sim WN(0, \lambda^2)$$

Find the value of a^* and λ^{*2}.

Solution

The steps are:

1. Compute the predictor:
$$\hat{y}(t|t-1) = -ay(t-1)$$

2. Compute the prediction error:
$$\varepsilon(t|t-1) = y(t) - \hat{y}(t|t-1) = (1 + az^{-1})y(t) = y(t) + ay(t-1)$$

3. Compute the variance of the prediction error:
$$\bar{J}(a^*) = \mathrm{Var}\left[\varepsilon(t)\right] = \mathbb{E}\left[\left(y(t) + ay(t-1)\right)^2\right] = \gamma_y(0) + a^2\gamma_y(0) + 2a\gamma_y(1)$$

4. Derive with respect to the variable a^*:

$$\frac{d\bar{J}(a^*)}{da^*} = 2a^*\gamma_y(0) + 2\gamma_y(1)$$

We want a minimum, so set this derivative to zero:

$$2a^*\gamma_y(0) + 2\gamma_y(1) = 0 \rightarrow a^* = -\frac{\gamma_y(1)}{\gamma_y(0)}$$

5. Find the value of the covariance from the system $\mathcal{S}$:

$$\gamma_y(0) = \mathbb{E}\left[\left(e(t) + \frac{1}{3}e(t-1)\right)^2\right] = \frac{10}{9}$$

$$\gamma_y(1) = \mathbb{E}\left[\left(e(t) + \frac{1}{3}e(t-1)\right)\left(e(t-1) + \frac{1}{3}e(t)\right)\right] = \frac{1}{3}$$

Thus,

$$a^* = -\frac{\gamma_y(1)}{\gamma_y(0)} = -\frac{3}{10}$$

6. The value of λ^{*2} can be computed by substituting the value of a^* into the variance function:

$$\lambda^{*2} = \gamma_y(0) + a^{*2}\gamma_y(0) + 2a^*\gamma_y(1) = \frac{10}{9} + \left(-\frac{3}{10}\right)^2\frac{10}{9} + 2\left(-\frac{3}{10}\right)\frac{1}{3} = \frac{91}{90}$$

This is similar to the variance of the White Noise, indicating good identification.

Since that absolute value of a is less than one, the system is in canonical form.

4.6 Exercise 6

Consider the system:

$$\mathcal{S}: y(t) = \frac{1}{3}y(t-1) + u(t-1) + \eta(t) + \frac{1}{2}\eta(t-1)$$

Here $\eta(t) \sim WN(0,1)$, $u(t) \sim WN(0,1)$ are two independent White Noises And the model:

$$\mathcal{M}: y(t) = -ay(t-1) + bu(t-1) + e(t) \qquad e(t) \sim WN(0,\lambda^2)$$

Find the value of θ^* and λ^{*2}.

Solution

The steps are:

1. Compute the predictor:

$$\hat{y}(t|t-1) = \frac{F(z)}{C(z)}y(t) + \frac{B(z)E(z)}{C(z)}u(t) = -\frac{a}{1}y(t-1) + \frac{bu(t-1)}{1}$$

2. Compute the prediction error:

$$\varepsilon(t|t-1) = y(t) - \hat{y}(t|t-1) = y(t) + ay(t-1) - bu(t-1)$$

3. Compute the variance of the prediction error:

$$\begin{aligned}\bar{J}(\theta^*) &= \mathrm{Var}\left[\varepsilon(t)\right] \\ &= \mathbb{E}\left[(y(t) + ay(t-1) - bu(t-1))^2\right] \\ &= \left(1 + a^{*2}\right)\gamma_y(0) + b^{*2}\gamma_u(0) + 2a^*\gamma_y(1) - 2b^*\mathbb{E}\left[y(t)u(t-1)\right]\end{aligned}$$

4. Derive with respect to the variables a^* and b^*:

$$\frac{\partial \bar{J}(\theta^*)}{\partial a^*} = 2a^*\gamma_y(0) + 2\gamma_y(1)$$

$$\frac{\partial \bar{J}(\theta^*)}{\partial b^*} = 2b^*\gamma_y(0) + 2\mathbb{E}\left[u(t-1)y(t)\right]$$

We want a minimum, so we impose those derivatives to be null:

$$2a^*\gamma_y(0) + 2\gamma_y(1) = 0 \to a^* = -\frac{\gamma_y(1)}{\gamma_y(0)}$$

$$2b^*\gamma_u(0) + 2\mathbb{E}\left[u(t-1)y(t)\right] = 0 \to b^* = \frac{\mathbb{E}\left[u(t-1)y(t)\right]}{\gamma_u(0)}$$

5. We may now find the value of the covariance from the system $\mathcal{S}$:

$$\gamma_y(0) = \frac{69}{32}$$

$$\gamma_y(1) = \frac{7}{32}$$

As a result:

$$a^* = -\frac{\gamma_y(1)}{\gamma_y(0)} = -\frac{7}{69}$$

$$b^* = \frac{\mathbb{E}\left[u(t-1)y(t)\right]}{\gamma_y(0)} = -\frac{\gamma_y(1)}{\gamma_y(0)} = 1$$

6. The value of λ^{*2} can be computed by substituting the value of a^* and b^* into the variance function:

$$\lambda^{*2} = \left(1 + a^{*2}\right)\gamma_y(0) + b^{*2}\gamma_u(0) + 2a^*\gamma_y(1) - 2b^*\mathbb{E}\left[y(t)u(t-1)\right] = 1.134$$

That is similar to the variance of the White Noise, so the identification is good.

4.7 Exercise 7

Consider the system:

$$\mathcal{S} : y(t) = -\frac{1}{2}y(t-1) + e(t) \qquad e(t) \sim WN(0,1)$$

And the model is:

$$\mathcal{M} : y(t) = -ay(t-2) + \eta(t) \qquad \eta(t) \sim WN(0,\lambda^2)$$

Find the value of a^* and λ^{*2}.

Solution

The steps are:

1. Compute the predictor:
$$\hat{y}(t|t-1) = \frac{C(z) - A(z)}{C(z)} y(t)$$

2. Compute the prediction error:
$$\varepsilon(t|t-1) = y(t) - \hat{y}(t|t-1) = y(t) - \frac{C(z) - A(z)}{C(z)} y(t) = y(t) + ay(t-2)$$

3. Compute the variance of the prediction error:
$$\bar{J}(a^*) = \text{Var}\left[\varepsilon(t)\right] = \gamma_y(0) + a^{*2}\gamma_y(0) + 2a^*\gamma_y(2)$$

4. Derive with respect to the variable a^*:
$$\frac{d\bar{J}(a^*)}{da^*} = 2a^*\gamma_y(0) + 2\gamma_y(2)$$
We want a minimum, so we impose those derivatives to be null:
$$2a^*\gamma_y(0) + 2\gamma_y(2) = 0 \rightarrow a^* = -\frac{\gamma_y(2)}{\gamma_y(0)}$$

5. We may now find the value of the covariance from the system $\mathcal{S}$:
$$\gamma_y(0) = \frac{4}{3}$$
$$\gamma_y(2) = \frac{1}{3}$$
As a result:
$$a^* = -\frac{\gamma_y(2)}{\gamma_y(0)} = -\frac{1}{4}$$
The system is stable since $|a^*| < 1$.

6. The value of λ^{*2} can be computed by substituting the value of a^* into the variance function:
$$\lambda^{*2} = \frac{5}{4}$$
That is similar to the variance of the White Noise, so the identification is good.

4.8 Exercise 8

Consider the system:
$$\mathcal{S} : y(t) = 3e(t) + e(t-2) \qquad e(t) \sim WN(0,1)$$
And the model:
$$\mathcal{M} : y(t) = a_1 y(t-1) + a_2 y(t-2) + \eta(t) \qquad \eta(t) \sim WN(0,\lambda^2)$$
Find the value of θ^* and λ^{*2}.

Solution

The steps are:

1. Compute the predictor:
$$\hat{y}(t|t-1) = \frac{C(z) - A(z)}{C(z)} y(t)$$

2. Compute the prediction error:
$$\varepsilon(t|t-1) = y(t) - \hat{y}(t|t-1) = y(t) - \frac{C(z) - A(z)}{C(z)} y(t) = y(t) - a_1 y(t-1) - a_2 y(t-2)$$

3. Compute the variance of the prediction error:
$$\begin{aligned}\bar{J}(\theta^*) &= \text{Var}\left[\varepsilon(t)\right] \\ &= \mathbb{E}\left[(y(t) - a_1^* y(t-1) - a_2^* y(t-2))^2\right] \\ &= \left(1 + a_1^{*2} + a_2^{*2}\right)\gamma_y(0) + 2a_1^*\left(a_2^* - 1\right)\gamma_y(1) - 2a_2^*\gamma_y(2)\end{aligned}$$

4. Derive with respect to the variables a_1^* and a_2^*:
$$\frac{\partial \bar{J}(\theta^*)}{\partial a_1^*} = 2a_1^*\gamma_y(0) + 2(a_2^* - 1)\gamma_y(1)$$
$$\frac{\partial \bar{J}(\theta^*)}{\partial a_2^*} = 2a_2^*\gamma_y(0) + 2a_1^*\gamma_y(1) - 2\gamma_y(2)$$
We want a minimum, so we impose those derivatives to be null:
$$\begin{cases} 2a_1^*\gamma_y(0) + 2(a_2^* - 1)\gamma_y(1) = 0 \\ 2a_2^*\gamma_y(0) + 2a_1^*\gamma_y(1) - 2\gamma_y(2) = 0 \end{cases}$$

5. We may now find the value of the covariance from the system $\mathcal{S}$:
$$\gamma_y(0) = 10$$
$$\gamma_y(1) = 0$$
$$\gamma_y(1) = 3$$
As a result:
$$a_1^* = 0$$
$$a_2^* = -\frac{3}{10}$$

6. The value of λ^{*2} can be computed by substituting the value of a_1^* and a_2^* into the variance function:
$$\lambda^{*2} = 9.1$$
That is similar to the variance of the White Noise (remember to consider the system in canonical form), so the identification is good.

4.9 Exercise 9

Consider a stationary process $y(t)$ of which we know:

$$y(1) = 1 \qquad y(2) = 0 \qquad y(3) = -1$$

And the model:

$$\mathcal{M} : y(t) = ay(t-1) + \xi(t) + a\xi(t-1) \qquad \xi(t) \sim WN(0, \lambda^2)$$

Let's compute the parameter a.

Solution

1. Check if the mean of the given samples is zero.
2. Compute the predictor of the model:
$$\hat{y}(t|t-1) = -a\hat{y}(t-1|t-2) + 2ay(t-1)$$
3. Compute the predictions on the given data applying the heuristic at time zero:

t	$y(t)$	$\hat{y}(t\|t-1)$
0	0	$\hat{y}(0\|-1) = 0$
1	1	$\hat{y}(1\|0) = 0$
2	0	$\hat{y}(2\|1) = 2a$
3	-1	$\hat{y}(3\|2) = -2a^2$

4. Compute the cost function:
$$\hat{J}_3 = \frac{1}{3}\sum_{i=1}^{3}\left(y(i) - \hat{y}(i|i-1)\right)^2 = \frac{1}{3}\left[(1-0)^2 + (1-2a)^2\left(-1+2a^2\right)^2\right]$$
Thus,
$$\hat{J}_3 = \frac{1}{3}\left(2 + 4a^2\right)$$
5. Find the derivative and equal to zero:
$$\frac{8}{3}\hat{a} = 0 \rightarrow \hat{a} = 0$$

4.10 Exercise 10

Consider an input defined as:

$$u(t) = 1$$

With the following output:

$$y(t) = \begin{cases} 1 & t \text{ is odd} \\ -1 & t \text{ is even} \end{cases}$$

We have the data from $t = 0$ to $t = 15$. And the model:

$$\mathcal{M} : y(t) = ay(t-1) + bu(t-1) + \xi(t)$$

Here $\xi(t) \sin WN(0, \lambda^2)$

Let's identify the parameter θ^*.

Solution

1. Compute the predictor:

$$\hat{y}(t|t-1) = ay(t-1) + bu(t-1)$$

2. The cost function is:

$$\hat{J}_a = \frac{1}{15}\sum_{i=1}^{15}(y(i) - \hat{y}(i|i-1))^2$$

3. Apply the Least Squares formula. We can rewrite the model as:

$$\mathcal{M} : y(t) = \theta^T\varphi(t) + \xi(t) \qquad \varphi(t) = \begin{bmatrix} y(t-1) \\ u(t-1) \end{bmatrix}$$

In this way, the predictor becomes:

$$\hat{y}(t|t-1) = \theta^T\varphi(t)$$

At this point, we have:

$$\hat{\theta}_{15} = \left[\sum_{i=1}^{15}\varphi(i)\varphi(i)^T\right]^{-1} + \left[\sum_{i=1}^{15}\varphi(i)y(i)\right]$$

4. Compute the formula:

$$\hat{\theta}_{15} = \left[\sum_{i=1}^{15}\begin{bmatrix} y(i-1) \\ u(i-1) \end{bmatrix}\begin{bmatrix} y(i-1) & u(i-1) \end{bmatrix}\right]^{-1} + \left[\sum_{i=1}^{15}\begin{bmatrix} y(i-1) \\ u(i-1) \end{bmatrix} y(i)\right]$$

We have:

$$\begin{aligned}
\hat{\theta}_{15} &= \begin{bmatrix} \sum_{i=1}^{15} y(i-1)^2 = 15 & \sum_{i=1}^{15} y(i-1)u(i-1) \\ \sum_{i=1}^{15} u(i-1)y(i-1) & \sum_{i=1}^{15} u(t-1)^2 \end{bmatrix}^{-1} \begin{bmatrix} \sum_{i=1}^{15} y(i-1)y(i) \\ \sum_{i=1}^{15} u(i-1)y(i) \end{bmatrix} \\
&= \begin{bmatrix} 15 & 1 \\ 1 & 15 \end{bmatrix}^{-1} \begin{bmatrix} -15 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}
\end{aligned}$$

4.11 Exercise 11

Consider a model:

$$\mathcal{M} : y(t) = \frac{1}{4}y(t-1) + \frac{1}{a}y(t-2) + e(t) \qquad e(t) \sim WN(0, \lambda^2)$$

We are given:

$$y(0) = 2 \qquad y(1) = 0 \qquad y(2) = -1$$

We also know that $y(t) = 0$ for all $t < 0$.

Let's find the parameter $\hat{a}$

Solution

1. Check if the mean of the given samples is zero. In this case, we don't need normalization since we have an infinite number of samples with a value of zero.

2. Compute the predictor of the model:

$$\hat{y}(t|t-1) = \frac{1}{4}y(t-1) + \frac{1}{a}y(t-2)$$

3. Compute the predictions on the given data applying the heuristic at time zero:

t	$y(t)$	$\hat{y}(t\|t-1)$
0	2	$\hat{y}(0\|-1) = 0$
1	0	$\hat{y}(1\|0) = \frac{1}{2}$
2	-1	$\hat{y}(2\|1) = \frac{2}{a}$

4. Compute the cost function:

$$\hat{J}_3 = \frac{1}{3}\sum_{i=1}^{3}(y(i) - \hat{y}(i|i-1))^2 = \frac{1}{3}\left[(2-0)^2 + \left(0 - \frac{1}{2}\right)^2\left(-1 + \frac{2}{a}\right)^2\right]$$

Thus,

$$\hat{J}_3 = \frac{7}{4} + \frac{4}{3a} + \frac{1}{a^2}$$

5. Find the derivative and equal to zero:

$$\frac{4}{3} - \frac{a^2 - 2a}{a^4} = 0 \to \hat{a} = -2$$

Roots and poles are inside the unit circle, so the system is stable.