Image Analysis And Computer Vision

Theory

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Abstract

The course begins with an introduction to camera sensors, including their transduction, optics, geometry, and distortion characteristics. It then covers the basics of projective geometry, focusing on modeling fundamental primitives such as points, lines, planes, conic sections, and quadric surfaces, as well as understanding projective spatial transformations and projections.

The course continues with an exploration of camera geometry and single-view analysis, addressing topics like calibration, image rectification, and the localization of 3D models. This is followed by a study of multi-view analysis techniques, which includes 3D shape reconstruction, self-calibration, and 3D scene understanding.

Students will also learn about linear filters and convolutions, including space-invariant filters, the Fourier Transform, and issues related to sampling and aliasing. Nonlinear filters are discussed as well, with a focus on image morphology and operations such as dilation, erosion, opening, and closing, as well as median filters.

The course further explores edge detection and feature detection techniques, along with feature matching and tracking in image sequences. It addresses methods for inferring parametric models from noisy data and outliers, including contour segmentation, clustering, the Hough Transform, and RANSAC (random sample consensus).

Finally, the course applies these concepts to practical problems such as object tracking, recognition, and classification.

Contents

CHAPTER 1

Optical sensors

1.1 Camera

Definition (*Camera*)**.** A camera is an optical sensor that generates data using electric transducers. It features an optical system designed to direct incoming light to its millions of photosensitive elements. Modern cameras are typically capable of recording 30 to 60 frames per second.

For simplicity, we will consider the optical system of a camera as a single lens with the following characteristics:

- *Spherical*: the lens is formed by the intersection of two spherical surfaces.
- *Thin*: the distance between the centers of the two spheres is nearly equal to the sum of their radii.
- *Small angles*: the light rays make only slight angles with respect to the optical axis.

These assumptions simplify the calculations involved in determining the path of a light ray as it passes through the lens. Specifically, the refraction of light at the boundary between two media is described by Snell's law:

$$\frac{\sin \theta_2}{\sin \theta_1} = \frac{n_1}{n_2}$$

Here:

- θ_1 and θ_2 are the angles between the normal at the surface and the direction of the light ray before and after crossing the boundary, respectively.
- n_1 and n_2 are the refractive indices of the two materials.

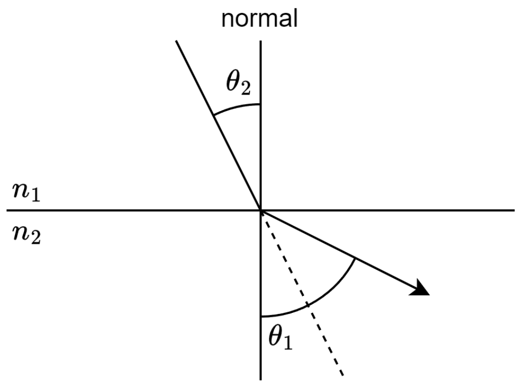
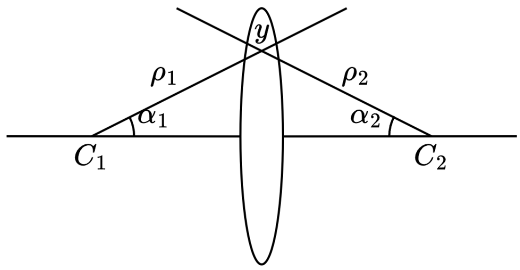


Figure 1.1: Snell's law

Definition (*Optical axis*)**.** The optical axis is the straight line that connects the centers of the two spheres that form the lens.

The angles of a ray passing through the centers of the spheres can be expressed as follows:

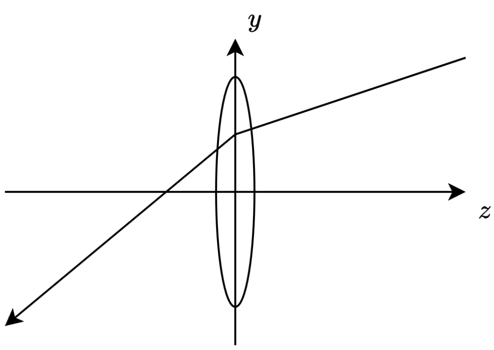
$$\alpha_1 = \frac{y_1}{\rho_1} \qquad \alpha_2 = -\frac{y_2}{\rho_2}$$



In this context, with the simplified lens, it is reasonable to assume:

$$y_1 = y_2 = y$$

1.2 Light rays deviation



For a lens with a refractive index n, the following equations apply:

$$\frac{\theta-\alpha_1}{\theta'-\alpha_1} \Rightarrow \frac{\sin(\theta-\alpha_1)}{\sin(\theta'-\alpha_1)} = n$$

$$\frac{\theta''-\alpha_2}{\theta'-\alpha_2} \Rightarrow \frac{\sin(\theta''-\alpha_2)}{\sin(\theta'-\alpha_2)} = n$$

Here:

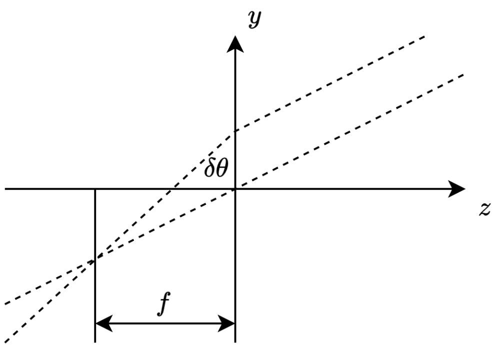
- θ is the angle of the incoming ray before entering the lens.
- θ' is the angle of the ray within the lens (not visible in the image).
- θ'' is the angle of the ray after exiting the lens.

By comparing these two equations, we can express the difference between the input angle θ and the output angle θ'' as:

$$\delta\theta = y(n-1)\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right)$$

Here, the term $n-1$ reflects the influence of the lens material, while the term $\frac{1}{\rho_1}+\frac{1}{\rho_2}$ is determined by the curvature of the lens surfaces.

1.3 Focalization of parallel light rays



In the image, we observe two rays: one passing through the center of the lens and another that remains parallel to the first ray but passes through a different point. From this, we can make the following observations:

- When $Y = 0$, the ray experiences no deviation and continues straight without deflection.
- Using the relationship $Y = f \cdot \delta\theta$, we can express the focal length of the lens as follows:

$$f = \frac{1}{(n-1)\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right)}$$

This indicates that all parallel rays converge at a common point known as the focal point, denoted as Z. The distance from the focal point to the y axis is given by:

$$Z = -f$$

1.4 Path of a light ray

To determine the trajectory of a light ray as it passes through a lens at any given position, you can follow these steps:

1. Draw a line parallel to the selected ray, passing through the center of the lens.
2. Identify the intersection point of this line with the focal plane.
3. The ray will travel from the point where it crosses the lens to the point on the focal plane.

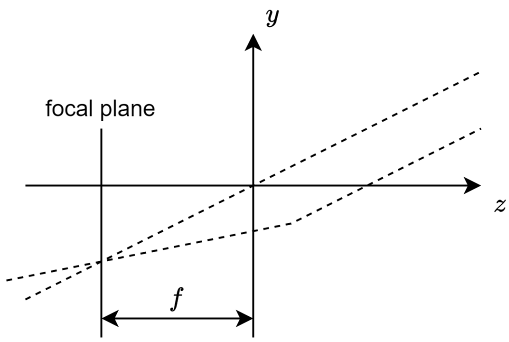


Figure 1.2: Path of a light ray through a lens

1.5 Pin-hole camera

To achieve a sharply focused image, it's crucial that each light ray converges precisely onto a single pixel on the camera's focal plane. To ensure this, the following conditions must be met:

- The distance between the lens and the light source, denoted as $Z(P)$, should be significantly greater than the lens aperture a ideally at least 1000 times larger.
- By positioning the screen at a distance Z from the lens, all rays can maintain parallel trajectories as they pass through the lens, resulting in a well-focused image.

The camera described is commonly known as a pin-hole camera, and it requires the following characteristics:

1. A thin spherical lens.
2. Utilization of small angles.
3. Ensuring that $Z(P) \gg a$.
4. Maintaining $Z = f$, where f represents the focal length.

1.6 From real world to two-dimensional images

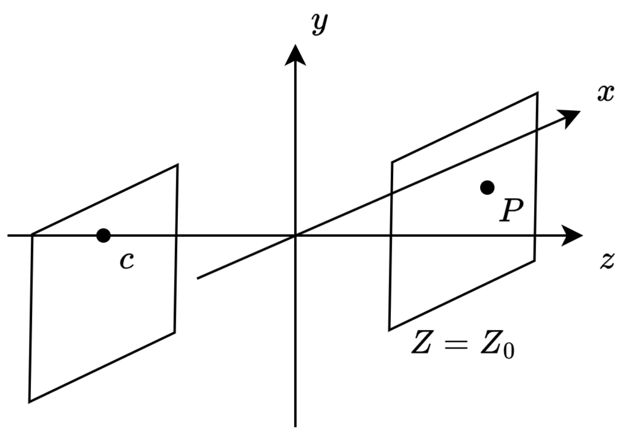
Images exist on a 2D plane, while the real world is three-dimensional, leading to a reduction of information compared to the original subject. This reduction is a result of perspective projection, which exhibits the following characteristics: nonlinearity, lack of shape preservation, and failure to maintain length ratios.

Using the triangle equality, we can express this perspective projection as:

$$x = f\frac{X}{Z} \qquad y = f\frac{Y}{Z}$$

To minimize information loss, one effective approach is to capture an image of a planar scene on a plane that is parallel to the image plane. This requires that:

$$Z = Z_0 = \text{constant}$$



In this scenario, the only difference between reality and the projection is a uniform downscaling, while other dimensions are preserved, yielding:

$$x = f\frac{X}{Z_0} = kX \qquad y = f\frac{Y}{Z_0} = kY$$

1.7 Perspective and vanishing point

When considering all lines parallel to the direction parameters $\begin{bmatrix}\alpha & \beta & 1\end{bmatrix}$, we can establish the following system of equations:

$$\begin{cases} X = X_0 + \alpha Z \\ Y = Y_0 + \beta Z \\ Z = 1 \cdot Z \end{cases}$$

To project these lines onto the 2D image using the triangle equality, we derive the following expressions:

$$x = f\frac{X}{Z} = f\frac{X_0 + \alpha Z}{Z} = f\alpha + \frac{X_0}{Z} \qquad y = f\frac{Y}{Z} = f\frac{Y_0 + \beta Z}{Z} = f\beta + \frac{Y_0}{Z}$$

Next, we find the image of the point at infinity along these lines, which results in the point:

$$\begin{bmatrix} f\alpha & f\beta \end{bmatrix}$$

Remarkably, this image point is independent of the values of X_0 and Y_0. Therefore, all parallel lines share the same image of their points at infinity.

Definition (*Vanishing point*)**.** The image of the point at infinity of the lines is known as the vanishing point.

Consequently, we observe that all parallel lines in the real world project onto converging lines in the image.

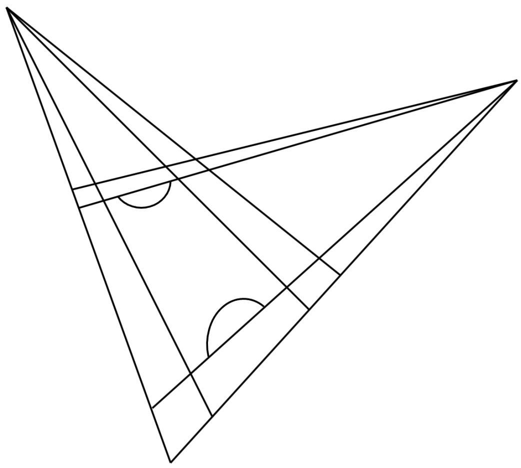


Figure 1.3: Vanishing point

CHAPTER 2

Two-dimensional planar projective geometry

2.1 Introduction

In the realm of planar geometry, the foundational elements consist of points, lines, conics, and dual conics. The transformations allowed within this geometry include projectivities, affinities, similarities, and isometries.

2.2 Points

To define points in Cartesian coordinates, we establish a Euclidean plane with a designated origin. Each point is uniquely represented by a pair of Cartesian coordinates, $\begin{bmatrix} x & y \end{bmatrix}$.

For image analysis, it is advantageous to use homogeneous coordinates. This involves constructing a 3D space with axes labeled $\begin{bmatrix} x & y & w \end{bmatrix}$. To represent a point, we assign three values, which allows for an infinite number of representations by varying the value of w. The relationship between Cartesian and homogeneous coordinates can be expressed as follows:

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ w \end{bmatrix} = w \begin{bmatrix} X \\ Y \\ 1 \end{bmatrix}$$

Consequently, a vector $\mathbf{x} = \begin{bmatrix} x & y & w \end{bmatrix}^T$ and all its nonzero multiples, including $\begin{bmatrix} \frac{x}{w} & \frac{y}{w} & 1 \end{bmatrix}^T$, represent the same point in Cartesian coordinates $\begin{bmatrix} X & Y \end{bmatrix}^T = \begin{bmatrix} \frac{x}{w} & \frac{y}{w} \end{bmatrix}^T$ on the Euclidean plane.

Property 2.2.1 (Homogeneity)**.** Any vector $\mathbf{x}$ is equivalent to all its nonzero multiples $\lambda\mathbf{x}$, where $\lambda \neq 0$, as they denote the same point.

The null vector does not represent any point.

Definition (*Projective plane*)**.** We define the projective plane as:

$$\mathbb{P}^2 = \left\{ \begin{bmatrix} x & y & w \end{bmatrix}^T \in \mathbb{R}^3 \right\} \setminus \left\{ \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T \right\}$$

Example:
The origin of the plane is defined as:

$$\begin{bmatrix}0 & 0 & 1\end{bmatrix}^T$$

A generic point in homogeneous coordinates can easily be transformed into Cartesian coordinates by simple division. For instance, the point:

$$\begin{bmatrix}0 & 8 & 4\end{bmatrix}^T$$

in Cartesian coordinates is:

$$\begin{bmatrix}\frac{x}{w} & \frac{y}{w}\end{bmatrix}^T = \begin{bmatrix}\frac{0}{4} & \frac{8}{4}\end{bmatrix} = \begin{bmatrix}0 & 4\end{bmatrix}$$

Consider a point $\mathbf{x} = \begin{bmatrix}x & y & w\end{bmatrix}^T$, and let w slowly decrease from $w = 1$. As w decreases, the point moves in a constant direction $\begin{bmatrix}x & y\end{bmatrix}$, distancing itself from the origin. As w approaches 0, the point tends toward infinity along the direction $\begin{bmatrix}x & y\end{bmatrix}$.

Definition (*Point at the infinity along the direction*)**.** We define the point at the infinity along the direction $\begin{bmatrix}x & y\end{bmatrix}$ as:

$$\mathbf{x} = \begin{bmatrix}x \\ y \\ w\end{bmatrix}$$

Points at infinity, representing directions, exist outside the Euclidean plane and are well-defined within the projective plane. Thus, the projective plane encompasses not only the Euclidean plane but also these points at infinity.

2.3 Lines

In the Euclidean plane, a line is typically defined by the equation:

$$aX + bY + c = 0$$

In the homogeneous plane, lines are represented as:

$$a\frac{x}{w} + b\frac{y}{w} + c = 0 \Longrightarrow ax + by + cw = 0$$

This equation can also be expressed using two vectors, denoted as $\mathbf{l}^T$ and $\mathbf{x}$, as follows:

$$\begin{bmatrix}a & b & c\end{bmatrix}\begin{bmatrix}x \\ y \\ w\end{bmatrix} = 0$$

Here, the vector $\mathbf{l} = \begin{bmatrix}a & b & c\end{bmatrix}^T$ epresents a line, with all its nonzero multiples also representing the same line.

Property 2.3.1 (Homogeneity)**.** Any vector $\mathbf{l}$ is equivalent to all its nonzero multiples, denoted as $\lambda\mathbf{l}$ (where $\lambda \neq 0$), since they denote the same line.

The coefficients a, b, and c are known as the homogeneous parameters of the line.

Similar to numbers, there are multiple equivalent representations for a single line, specifically all nonzero multiples of the unit normal vector. However, the null vector does not represent any lines.

Definition (*Projective dual plane*)**.** The projective dual plane is defined as:

$$\mathbb{P}^2 = \left\{ \begin{bmatrix} a & b & c \end{bmatrix}^T \in \mathbb{R}^3 \right\} \setminus \left\{ \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T \right\}$$

Property 2.3.2. If the third parameter is zero, denoted as $\mathbf{l} = \begin{bmatrix} a & b & 0 \end{bmatrix}^T$, then the line passes through the point $\begin{bmatrix} 0 & 0 \end{bmatrix}$.

Property 2.3.3. In the Euclidean plane, the direction $\begin{bmatrix} a & b \end{bmatrix}$ is perpendicular to the line represented by $\mathbf{l} = \begin{bmatrix} a & b & c \end{bmatrix}^T$.

Property 2.3.4. Two lines, $\mathbf{l} = \begin{bmatrix} a & b & c \end{bmatrix}^T$ and $\mathbf{l} = \begin{bmatrix} a & b & c' \end{bmatrix}^T$, are considered parallel if they share the same direction, represented by $\begin{bmatrix} b & -a \end{bmatrix}$.

> **Example:**
> The Cartesian axes are defined as:
>
> $$\mathbf{l}_x = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T$$
>
> $$\mathbf{l}_y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$$

In this context, the incidence relation of a line $\mathbf{l}^T \mathbf{x} = 0$ is defined when the point $\mathbf{x}$ lies on the line $\mathbf{l}$ or when the line $\mathbf{l}$ goes through the point $\mathbf{x}$.

Definition (*Line at the infinity*)**.** The line

$$\begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix} = w = 0$$

is called the line at the infinity, denoted as $\mathbf{l}_\infty = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$.

The principle of duality between points and lines states that the incidence relation is commutative, as the dot product is commutative.

To find the intersection of two lines, $\mathbf{l}_1$ and $\mathbf{l}_2$, the following condition is imposed:

$$\begin{bmatrix} \mathbf{l}_1^T \\ \mathbf{l}_2^T \end{bmatrix} \mathbf{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This equation leads to finding the right null space of the matrix formed by stacking the line vectors:

$$\mathbf{x} = \text{RNS}\left(\begin{bmatrix} \mathbf{l}_1^T \\ \mathbf{l}_2^T \end{bmatrix} \right)$$

The system is under-determined, meaning there is only one intersection point between the two lines, which can be represented in multiple ways using homogeneous coordinates. In 2D projective geometry, the vector $\mathbf{x}$, orthogonal to both lines, can be found using the cross product:

$$\mathbf{x} = \mathbf{l}_1 \times \mathbf{l}_2$$

Example:

Suppose we have two parallel lines $\mathbf{l}_1 = \begin{bmatrix} a & b & c_1 \end{bmatrix}^T$ and $\mathbf{l}_2 = \begin{bmatrix} a & b & c_2 \end{bmatrix}^T$. The point common to both lines can be found using the system:

$$\begin{cases} ax + by + c_1 w = 0 \\ ax + by + c_2 w = 0 \end{cases}$$

The solution is $\mathbf{x} = \begin{bmatrix} b & -a & 0 \end{bmatrix}^T$, which represents the point at infinity in the direction of both lines.

The line passing through two points can be found using the dual of the previous problem, expressed as:

$$\begin{bmatrix} \mathbf{x}_1^T \\ \mathbf{x}_2^T \end{bmatrix} \mathbf{l} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

In 2D, this simplifies to:

$$\mathbf{l} = \mathbf{x}_1 \times \mathbf{x}_2$$

Property 2.3.5. A point $\mathbf{x}$, which is a linear combination $\mathbf{x} = \alpha\mathbf{x}_1 + \beta\mathbf{x}_2$ of two points $\mathbf{x}_1$ and $\mathbf{x}_2$, lies on the line $\mathbf{l}$ through $\mathbf{x}_1$ and $\mathbf{x}_2$.

Proof. The line $\mathbf{l}$ passing through both points satisfies $\mathbf{l}^T\mathbf{x}_1 = 0$ and $\mathbf{l}^T\mathbf{x}_2 = 0$. By adding α times the first equation to β times the second one, we obtain:

$$0 = \mathbf{l}^T (\alpha\mathbf{x}_1 + \beta\mathbf{x}_2) = \mathbf{l}^T\mathbf{x} = 0$$

□

This establishes the duality between collinearity and concurrence.

Theorem 2.3.1. *For any true sentence containing the terms: point, line, is on, goes through, co-linear, and concurrent, there exists a dual statement (also true) obtained by making the following replacements:*

- *Point* ⇔ *line.*
- *Is on* ⇔ *goes through.*
- *Co-linear* ⇔ *concurrent.*

In the Euclidean plane, the direction normal to the line $\mathbf{l} = \begin{bmatrix} a & b & c \end{bmatrix}^T$ is represented by $\begin{bmatrix} a & b \end{bmatrix}$. The relationship between lines can be understood by recognizing that the angle between two lines is equal to the angle between their normal vectors. The formula for the angle between two vectors is:

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|}$$

This applies to the angle between two lines $\mathbf{l}_1 = \begin{bmatrix} a_1 & b_1 & c_1 \end{bmatrix}^T$ and $\mathbf{l}_2 = \begin{bmatrix} a_2 & b_2 & c_2 \end{bmatrix}^T$. In this context, it is the angle between their respective normal directions $\begin{bmatrix} a_1 & b_1 \end{bmatrix}$ and $\begin{bmatrix} a_2 & b_2 \end{bmatrix}$, calculated as:

$$\cos\theta = \frac{a_1 a_2 + b_1 b_2}{\sqrt{(a_1^2 + b_1^2)(a_2^2 + b_2^2)}}$$

2.3.1 Cross ratio

Now, consider a line with four points related as follows:

$$\mathbf{x}_1 = \alpha_1 \mathbf{y} + \beta_1 z \qquad \mathbf{x}_2 = \alpha_2 \mathbf{y} + \beta_2 \mathbf{z}$$

The cross ratio is given by:

$$CR_{\mathbf{x}_1,\mathbf{x}_2,\mathbf{y},\mathbf{z}} = \frac{\left(\dfrac{c-a}{c-b}\right)}{\left(\dfrac{d-a}{d-b}\right)} = \frac{\left(\dfrac{\beta_1}{\alpha_1}\right)}{\left(\dfrac{\beta_2}{\alpha_2}\right)} = \frac{\beta_1\alpha_2}{\beta_2\alpha_1}$$

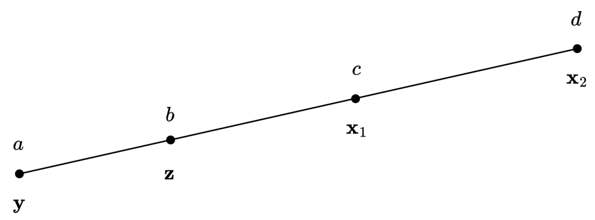
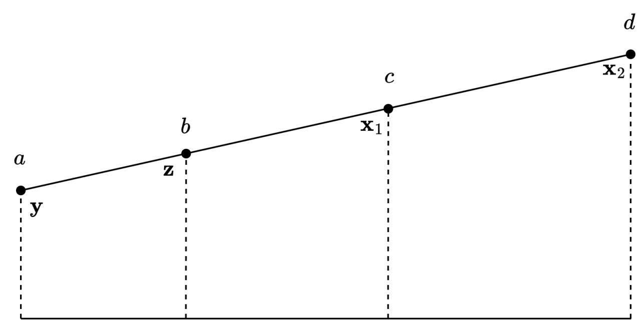


Figure 2.1: Line with the point of previous problem

Proof. Since the abscissae are proportional, the abscissae can be replaced by the x-coordinate, as shown in the figure.



The relation:

$$CR_{\mathbf{x}_1,\mathbf{x}_2,\mathbf{y},\mathbf{z}} = \frac{\left(\dfrac{c-a}{c-b}\right)}{\left(\dfrac{d-a}{d-b}\right)}$$

still holds. If we consider $\mathbf{y} = \begin{bmatrix} y & * & v \end{bmatrix}^T$ and $\mathbf{z} = \begin{bmatrix} z & * & w \end{bmatrix}^T$, we can determine that:

$$\mathbf{x}_1 = \begin{bmatrix} \alpha_1 y + \beta_1 z \\ * \\ \alpha_1 v + \beta_1 w \end{bmatrix} \qquad \mathbf{x}_2 = \begin{bmatrix} \alpha_2 y + \beta_2 z \\ * \\ \alpha_2 v + \beta_2 w \end{bmatrix}$$

The difference between the x coordinates of $\mathbf{x}_1$ and $\mathbf{y}$ is:

$$c - a = \frac{\beta_1(zv - yw)}{(\alpha_1 y + \beta_1 z)v}$$

Similarly, the difference between the x coordinates of $\mathbf{x}_1$ and $\mathbf{z}$ is:

$$c - b = \frac{-\alpha_1(zv - yw)}{(\alpha_1 y + \beta_1 z)w}$$

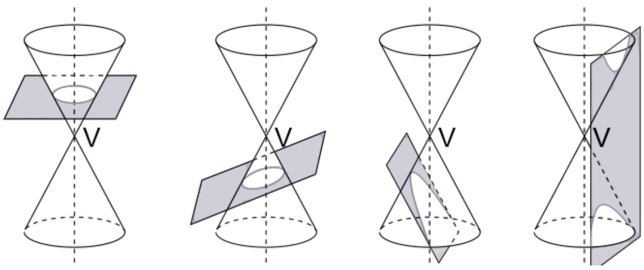
Substituting these expressions yields:

$$\frac{c-a}{c-b} = -\frac{\beta_1 w}{\alpha_1 v} \qquad \frac{d-a}{d-b} = -\frac{\beta_2 w}{\alpha_2 v}$$

$\square$

2.4 Conics

Conics are the geometric shapes formed by the intersection of a cone and a plane. These shapes include circles, ellipses, parabolas, and hyperbolas, as illustrated in the figure below:



Definition (*Conic*)**.** A point $\mathbf{x}$ lies on a conic $\mathbf{C}$ if it satisfies the homogeneous quadratic equation $\mathbf{x}^T C\mathbf{x} = 0$, where $\mathbf{C}$ is a symmetric matrix, a standard convention in defining conics.

Conics are curves that can be described by second-degree equations in the plane. In Euclidean coordinates, a conic is expressed as:

$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

In homogeneous coordinates, this equation becomes:

$$ax^2 + bxy + cy^2 + dxw + eyw + fw^2 = 0$$

Alternatively, conics can be represented in matrix form as:

$$\mathbf{x}^T \begin{bmatrix} a & \frac{b}{2} & \frac{d}{2} \\ \frac{b}{2} & c & \frac{e}{2} \\ \frac{d}{2} & \frac{e}{2} & f \end{bmatrix} \mathbf{x} = 0$$

Conics have five degrees of freedom, meaning that five points are sufficient to uniquely determine a conic.

Example:
A circle in Cartesian coordinates is represented as:

$$(x - x_0)^2 + (y - y_0)^2 - r^2 = 0$$

In homogeneous coordinates, it is given by:

$$\begin{bmatrix} x & y & w \end{bmatrix} \begin{bmatrix} 1 & 0 & -x_0 \\ 0 & 1 & -y_0 \\ -x_0 & -y_0 & x_0^2 + y_0^2 - r^2 \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix} = 0$$

When a conic is represented by a quadratic equation and a line by a linear equation, their intersection results in a second-degree equation in the point $\mathbf{x}$. Therefore, a line and a conic always intersect at two points, which can be categorized as follows:

- *Real and distinct*: two separate real points.
- *Real and coincident*: one real point, a repeated or double root.
- *Complex and distinct*: two distinct complex conjugate points.
- *Complex and coincident*: one complex point, a repeated or double root.

This is due to the fundamental theorem of algebra, which guarantees two solutions to a second-degree equation when considering complex numbers.

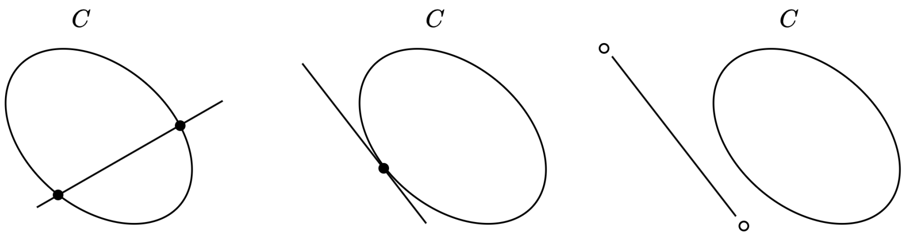


Figure 2.2: Intersection with two real roots, two coincident roots and two imaginary roots

The intersection of the line at infinity with a conic yields the following results:

- *Parabola*: two coincident solutions, representing the point at infinity along the axis.
- *Ellipse*: two complex-conjugate solutions, indicating no real solutions.
- *Hyperbola*: two real and distinct solutions, corresponding to the asymptotes.

2.4.1 Circular points

Example:
When a circle is intersected with the line at infinity, the system becomes:

$$\begin{cases} x^2 - 2x_0 w + x_0^2 w^2 + y^2 - 2y_0 w + y_0^2 w^2 - r^2 w^2 = 0 \\ w = 0 \end{cases}$$

This simplifies to:

$$x^2 + y^2 = 0$$

Here, the parameters of the circle (center and radius) disappear, indicating that the two intersection points are the same for all circles.

Definition (*Circular points*). The two intersection points obtained by intersecting any circle with the line at infinity are called the circular points. These points are:

$$\mathbf{I} = \begin{bmatrix} 1 \\ i \\ 0 \end{bmatrix} \qquad \mathbf{J} = \begin{bmatrix} 1 \\ -i \\ 0 \end{bmatrix}$$

2.4.2 Polar line

Definition (*Polar line*). Given a point $\mathbf{y}$ and a conic $\mathbf{C}$, the line $\mathbf{l} = \mathbf{C}\mathbf{y}$ is called the polar line of the point $\mathbf{y}$ with respect to the conic $\mathbf{C}$.

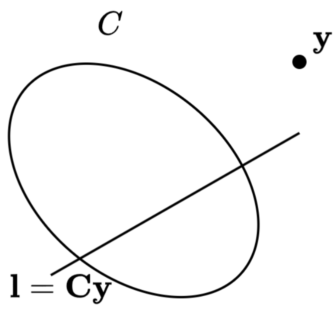


Figure 2.3: Polar line

2.4.3 Harmonic tuples

Definition (*Harmonic tuple*). A 4-tuple of collinear points $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ is called a harmonic tuple if their cross ratio is equal to -1.

A harmonic cross ratio value is shared by other collinear 4-tuples. A common example is:

$$(\mathbf{y}, \mathbf{z}, \text{midPoint}(\mathbf{y}, \mathbf{z}), P(\infty))$$

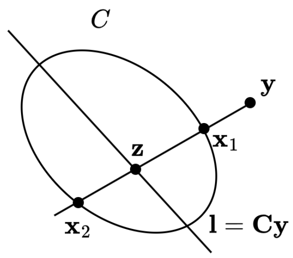
Moreover, if $(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d})$ is harmonic, then $(\mathbf{c}, \mathbf{d}, \mathbf{a}, \mathbf{b})$ is also harmonic.

Definition (*Conjugate points*). In a harmonic tuple $(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d})$, the points $\mathbf{a}$ and $\mathbf{b}$ are said to be conjugate with respect to $\mathbf{c}$ and $\mathbf{d}$.

Since the cross ratio of a harmonic tuple is negative, conjugate points $\mathbf{a}$ and $\mathbf{b}$ lie on opposite sides of the segment formed by $\mathbf{c}$ and $\mathbf{d}$, with one point inside and the other outside the segment.

2.4.4 Polar line and harmonic tuples

Given any point $\mathbf{z}$ on the polar line $\mathbf{l} = \mathbf{C}\mathbf{y}$, consider the line passing through the points $\mathbf{y}$ and $\mathbf{z}$. Let $\mathbf{x}_1$ and $\mathbf{x}_2$ represent the points at which this line intersects the conic $\mathbf{C}$.



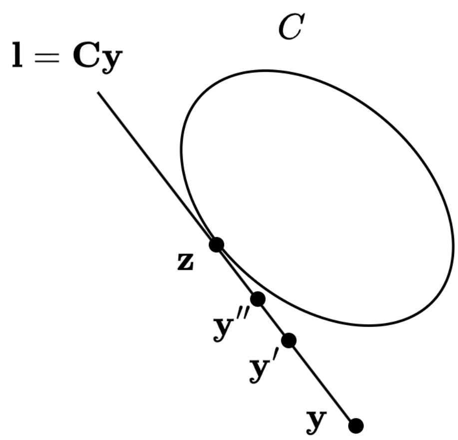
Theorem 2.4.1. *If* $\mathbf{x}_1$ *and* $\mathbf{x}_2$ *are the points where the line passing through* $\mathbf{y}$ *and* $\mathbf{z}$ *intersects the conic* $\mathbf{C}$*, then* $\mathbf{y}$ *and* $\mathbf{z}$ *are conjugate with respect to* $\mathbf{x}_1$ *and* $\mathbf{x}_2$*.*

The polar line $\mathbf{l} = \mathbf{C}\mathbf{y}$ contains all points that are conjugate to $\mathbf{y}$ with respect to the conic $\mathbf{C}$. Specifically, it includes points that are conjugate with respect to the intersection points of $\mathbf{C}$ with any line passing through $\mathbf{y}$.

2.4.5 Polar line and tangency points

As the line through $\mathbf{y}$ approaches tangency with the conic $\mathbf{C}$, the points $\mathbf{x}_1$ and $\mathbf{x}_2$ merge into the point of tangency. Consequently, the conjugate point $\mathbf{z}$ also coincides with the tangency point, applying to any line tangent to $\mathbf{C}$ from point $\mathbf{y}$.

Therefore, the polar line $\mathbf{l} = \mathbf{C}\mathbf{y}$ passes through the tangency points where lines from $\mathbf{y}$ meet the conic $\mathbf{C}$. If a point $\mathbf{z}$ lies on the conic, $\mathbf{y}$ is one of its conjugates with respect to the same conic. The tangent line $\mathbf{lz}$ to the conic at point $\mathbf{z}$ contains points conjugate to $\mathbf{z}$, making $\mathbf{lz}$ the polar line of $\mathbf{z}$ with respect to the conic.



In the illustration, the polar line $\mathbf{lz} = \mathbf{Cz}$ for point $\mathbf{z}$ on the conic $\mathbf{C}$ corresponds to the tangent line to the conic at point $\mathbf{z}$.

Example:

Consider a circle with radius r centered at the origin and the point $\mathbf{y} = \begin{bmatrix} x & 0 & 1 \end{bmatrix}^T$. The equation for the polar line is:

$$\mathbf{l} = \mathbf{C}\mathbf{y} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -r^2 \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{x} \\ 0 \\ -r^2 \end{bmatrix}$$

The Cartesian equation of the polar line becomes:

$$\mathbf{x}x - r^2 = 0 \to x = \frac{r^2}{\mathbf{x}}$$

This represents a vertical line.

From this, we deduce that the polar of a point $\mathbf{p}$ with respect to a circle is a line perpendicular to the line segment connecting the center of the circle to $\mathbf{p}$.

Example:

For a circle with radius r centered at the origin and the point $\mathbf{y} = \begin{bmatrix} x & 0 & 0 \end{bmatrix}^T$, the polar line equation is:

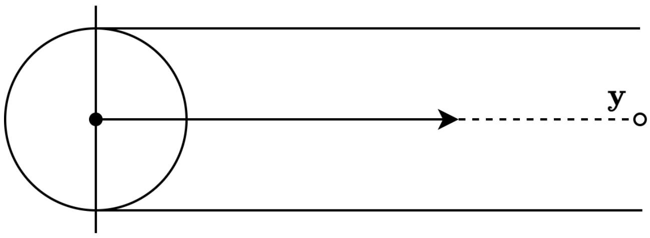
$$\mathbf{l} = \mathbf{C}\mathbf{y} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -r^2 \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \mathbf{x} \\ 0 \\ 0 \end{bmatrix}$$

The Cartesian equation becomes:

$$\mathbf{x}x = 0 \to \mathbf{x} = 0$$

This equation describes the diameter of the circle perpendicular to the direction of $\mathbf{y}$.

Parallel tangent lines from a point at infinity will have tangency points lying along a diameter that is perpendicular to the direction of the tangents.



Example:

For a circle with radius r centered at the origin and the point $\mathbf{y} = \begin{bmatrix} x & 0 & 0 \end{bmatrix}^T$, the polar line equation is:

$$\mathbf{l} = \mathbf{C}\mathbf{y} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -r^2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -r^2 \end{bmatrix}$$

The Cartesian equation becomes:

$$-r^2 w = 0 \to \mathbf{x} = 0$$

This equation describes the line at infinity.

Property 2.4.1. The polar line of any point at infinity is a diameter.

Property 2.4.2. Any diameter passes through the center of the circle.

Property 2.4.3. The center is conjugate to every point at infinity.

Property 2.4.4. All points at infinity are conjugate to the center.

Property 2.4.5. The polar of the center is the line that includes all points at infinity.

Property 2.4.6. The polar line of the center is the line at infinity.

2.4.6 Degenerate conics

Definition (*Non-degerate conic*)**.** A non-degenerate conic has a non-singular matrix $\mathbf{C}$, meaning:

$$\text{rank}(\mathbf{C}) = 3$$

Definition (*Degerate conic*)**.** A degenerate conic has a singular matrix $\mathbf{C}$, meaning:

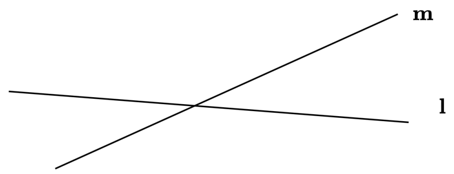
$$\text{rank}(\mathbf{C}) < 3$$

There are two types of degenerate conic:

- *Rank 2*: when $\text{rank}(\mathbf{C}) = 2$, $\mathbf{C}$ can be written as:

  $$\mathbf{C} = \mathbf{l}\mathbf{m}^T + \mathbf{m}\mathbf{l}^T$$

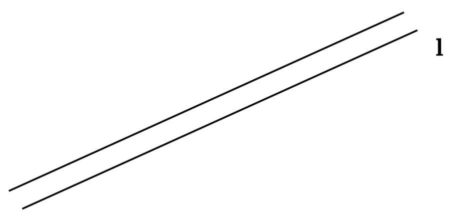
  Here, $\mathbf{x}$ satisfies $\mathbf{x}^T\mathbf{C}\mathbf{x} = 0$ when either $\mathbf{x}^T\mathbf{l} = 0$ or $\mathbf{m}^T\mathbf{x} = 0$, meaning $\mathbf{x}$ lies on the union of lines $\mathbf{l}$ and $\mathbf{m}$.



- *Rank 1*: when $\text{rank}(\mathbf{C}) = 1$, $\mathbf{C}$ can be written as:

  $$\mathbf{C} = \mathbf{l}\mathbf{l}^T$$

  The conic consists of points $\mathbf{x}$ that satisfy $\mathbf{x}^T\mathbf{C}\mathbf{x} = 0$, meaning $\mathbf{x}$ lies on the repeated line $\mathbf{l}$.



2.5 Dual conics

Definition (*Dual conic*)**.** A dual conic is a set of lines $\mathbf{l}$ that satisfy equation:

$$\mathbf{l}^T\mathbf{C}^*\mathbf{l} = 0$$

where $\mathbf{C}^*$ is a 3×3 symmetric matrix.

Definition (*Non degenrate dual conic*)**.** A non degenerate dual conic is a dual conic whose matrix $\mathbf{C}^*$ is non-singular:

$$\text{rank}(\mathbf{C}^*) = 3$$

Consider a non-degenerate conic, denoted as $\mathbf{C}$, and the collection of all lines $\mathbf{l}$ that are tangents to it. For each point $\mathbf{c}$ on the conic $\mathbf{C}$, there exists a line $\mathbf{l}$ that is tangent to $\mathbf{C}$. Since $\mathbf{l}$ is the polar line of $\mathbf{x}$ with respect to $\mathbf{C}$, we can express it as $\mathbf{l} = \mathbf{C}\mathbf{c}$. Consequently, we can represent $\mathbf{x}$ as:

$$\mathbf{x} = \mathbf{C}^{-1}\mathbf{l}$$

Moreover, given that $\mathbf{C}$ is a symmetric matrix, we have:

$$\mathbf{x}^T = \mathbf{l}^T\mathbf{l}^{-T} = \mathbf{l}^T\mathbf{C}^{-1}$$

Now, considering that the point $\mathbf{x}$ lies on the conic $\mathbf{C}$, we have:

$$\mathbf{x}^T\mathbf{C}\mathbf{x} = 0$$

By substituting the previously derived expressions, we arrive at:

$$\mathbf{l}^T\mathbf{C}^{-1}\mathbf{l} = 0$$

This equation represents a quadratic homogeneous equation on $\mathbf{l}$. Therefore, we can conclude that for the dual conic holds $\mathbf{C}^* = \mathbf{C}^{-1}$. We can also note that a non-degenerate dual conic $\mathbf{C}^*$ is the collection of lines that are tangent to a non-degenerate conic $\mathbf{C}$.

2.5.1 Degenerate dual conics

Definition (*Degerate dual conic*)**.** A degenerate dual conic is a conic where the matrix $\mathbf{C}^*$ is singular:

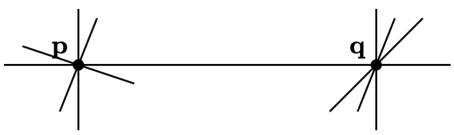
$$\text{rank}(\mathbf{C}^*) < 3$$

There are two possible scenarios to consider:

- When $\text{rank}(\mathbf{C}^*) = 2$, any symmetric 3×3 matrix $\mathbf{C}^*$ can be expressed as:

  $$\mathbf{C}^* = \mathbf{p}\mathbf{q}^T + \mathbf{q}\mathbf{p}^T$$

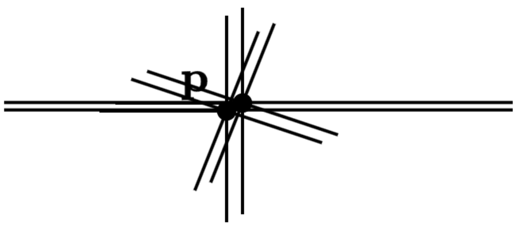
  In this case, the conic represents the line $\mathbf{l}$ passing through point $\mathbf{p}$ or the line $\mathbf{l}$ passing through point $\mathbf{q}$.



- When $\text{rank}(\mathbf{C}^*) = 1$, any symmetric 3×3 matrix $\mathbf{C}^*$ can be expressed as:

  $$\mathbf{C}^* = \mathbf{p}\mathbf{p}^T$$

  In this situation, the conic corresponds to the line $\mathbf{l}$ going through point $\mathbf{p}$ repeated twice.



Definition (*Conic dual to the circular points*). The degenerate dual conic $\mathbf{C}^* = \mathbf{p}\mathbf{q}^T + \mathbf{q}\mathbf{p}^T$ going through two circular point $\mathbf{p}$ and $\mathbf{q}$ is known as the conic dual to the circular points, and it can be expressed as:

$$\mathbf{C}^*_\infty = \mathbf{I}\mathbf{J}^T + \mathbf{J}\mathbf{I}^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

2.6 Transformations

Definition (*Projective mapping*). A projective mapping between a projective plane $\mathbb{P}^2$ and another projective plane $\mathbb{P}'^2$ is an invertible mapping which preserves co-linearity:

$$h : \mathbb{P}^2 \to \mathbb{P}'^2, x' = h(\mathbf{x}), \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \text{ are colinear}$$

$$\Leftrightarrow$$

$$\mathbf{x}'_1 = h(\mathbf{x}_1), \mathbf{x}'_2 = h(\mathbf{x}_2), \mathbf{x}'_3 = h(\mathbf{x}_3) \text{ are colinear}$$

Projective mapping is also called projectivity or homography.

Theorem 2.6.1. *A mapping $h : \mathbb{P}^2 \to \mathbb{P}'^2$ is projective if and only if there exists an invertible 3×3 matrix $\mathbf{H}$ such that for any point in $\mathbb{P}^2$ represented by the vector $\mathbf{x}$, is $h(\mathbf{x}) = \mathbf{H}\mathbf{x}$, where:*

$$\mathbf{H} = \begin{bmatrix} h_{11} & h_{12} & h13 \\ h_{21} & h_{22} & h23 \\ h_{31} & h_{32} & h33 \end{bmatrix}$$

Projective mappings are linear when expressed in homogeneous coordinates, but they do not exhibit linearity when represented in Cartesian coordinates.

According to the theorem, if we have $h(\mathbf{x}) = \mathbf{x}' = \mathbf{H}\mathbf{x}$, then multiplying the matrix $\mathbf{H}$ by any nonzero scalar λ still satisfies the relation for the same points, giving us $\mathbf{x}' = \lambda\mathbf{H}\mathbf{x}$. Therefore, any nonzero scalar multiple of the matrix $\mathbf{H}$ represents the same projective mapping as $\mathbf{H}$. As a result, we can conclude that $\mathbf{H}$ is a homogeneous matrix. Despite having nine entries, it possesses only eight degrees of freedom, specifically the ratios between its elements. Consequently, we can estimate $\mathbf{H}$ using just four point correspondences. Each point correspondence, expressed as $\mathbf{x}' = \mathbf{H}\mathbf{x}$, provides two independent equations in this estimation process.

Definition (*Homography*). A homography transforms various geometric entities as follows:

1. It maps a point $\mathbf{x}$ to a point $\mathbf{x}'$, where the transformation is expressed as:

$$\mathbf{x} \to \mathbf{H}\mathbf{x} = \mathbf{x}'$$

2. It maps a line $\mathbf{l}$ to a line $\mathbf{l}'$, and this transformation is represented as:

$$\mathbf{l} \to \mathbf{H}^{-T}\mathbf{l} = \mathbf{l}'$$

3. It maps a conic $\mathbf{C}$ to a conic $\mathbf{C}'$, and the transformation is given by:

$$\mathbf{C} \to \mathbf{H}^{-T}\mathbf{C}\mathbf{H}^{-1} = \mathbf{C}'$$

4. It maps a dual conic $\mathbf{C}^*$ to a dual conic $C^{*\prime}$, with the transformation being:

$$\mathbf{C}^* \rightarrow \mathbf{H}\mathbf{C}^*\mathbf{H}^T = \mathbf{C}^{*\prime}$$

Proof of mapping two. To transform the equation of the line in terms of $\mathbf{x}$, given by $\mathbf{l}^T\mathbf{x} = 0$, into a constraint on $\mathbf{x}' = \mathbf{H}\mathbf{x}$, we combine the two equations, resulting in a linear equation on $\mathbf{x}'$:

$$\mathbf{l}'^T\mathbf{x}' = 0$$

Here, $\mathbf{l}'^T = \mathbf{l}^T\mathbf{H}^{-1}$. Thus, we have:

$$\mathbf{l}' = \mathbf{H}^{-T}\mathbf{l}$$

□

Proof of mapping three. To transform the equation of the conic in terms of $\mathbf{x}$, given by $\mathbf{x}^T\mathbf{C}\mathbf{x} = 0$, into a constraint on $\mathbf{x}' = \mathbf{H}\mathbf{x}$, we have $\mathbf{x} = \mathbf{H}^{-1}\mathbf{x}'$ and $\mathbf{x}^T = \mathbf{x}'^T\mathbf{H}^{-T}$. Combining these three equations, we obtain a linear equation on $\mathbf{x}'$:

$$\mathbf{x}'^T\mathbf{C}'\mathbf{x}' = 0$$

Hence, we have:

$$\mathbf{C}' = \mathbf{H}^{-T}\mathbf{C}\mathbf{H}^{-1}$$

□

Proof of mapping four. For the transformation of a dual conic, we apply the same idea, yielding:

$$\mathbf{C}^{*\prime} = \mathbf{H}\mathbf{C}^*\mathbf{H}^T$$

□

The point-line incidence is preserved.

Proof. Let $\mathbf{x}$ be a point on the line $\mathbf{l}$. This is expressed as $\mathbf{l}^T\mathbf{x} = 0$. When we apply the projective transformation $\mathbf{H}$ to both $\mathbf{x}$ and $\mathbf{l}$, resulting in $\mathbf{H}\mathbf{x} = \mathbf{x}'$ and $\mathbf{H}^{-1}\mathbf{l} = \mathbf{l}'$, they remain incident if $\mathbf{l}'^T\mathbf{x}' = 0$:

$$\mathbf{l}'^T\mathbf{x}' = \mathbf{l}^T\mathbf{H}^{-1}\mathbf{x}' = \mathbf{l}^T\mathbf{H}^{-1}\mathbf{H}\mathbf{x} = \mathbf{l}^T x = 0$$

□

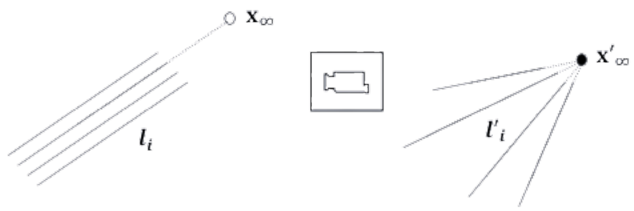
2.6.1 Vanishing points and vanishing line

The point that is common to both parallel lines $\mathbf{l}_1 = \begin{bmatrix} a & b & c_1 \end{bmatrix}^T$ and $\mathbf{l}_2 = \begin{bmatrix} a & b & c_2 \end{bmatrix}^T$ is the point $\mathbf{x} = \begin{bmatrix} b & -a & 0 \end{bmatrix}^T$. This point is situated at infinity along the direction of both lines. When seeking the common point of the infinite lines $\mathbf{l}_i$, we find that they all share the same point:

$$\mathbf{x}_\infty = \begin{bmatrix} b & -a & 0 \end{bmatrix}^T$$

Hence, it becomes apparent that all these lines converge at $\begin{bmatrix} b & -a & 0 \end{bmatrix}^T$.

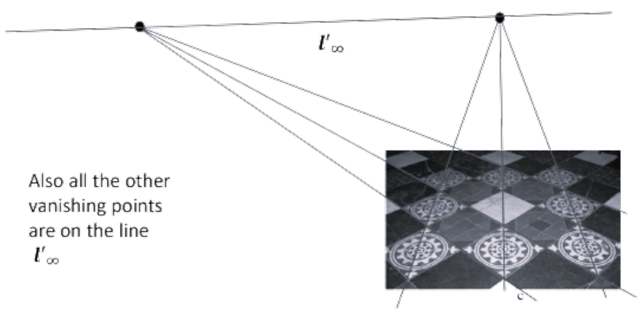
If we apply a projective transformation to all the aforementioned parallel lines $\mathbf{l}_i$, we obtain the transformed lines $\mathbf{l}'_i$. The common point $\mathbf{x}_\infty$, shared by all lines $\mathbf{l}_i$, is mapped to a point $\mathbf{x}'_\infty$ which belongs to each of the lines $\mathbf{l}'_i$.



Therefore, we can assert that all lines $\mathbf{l}'_i$ intersect at the point $\mathbf{x}'_\infty = \mathbf{H}\mathbf{x}_\infty$, referred to as the vanishing point associated with the direction $(b, -a)$ of the parallel lines.

Theorem 2.6.2. *The image of a set of parallel lines $\mathbf{l}_i$ is a set of lines $\mathbf{l}'_i$ concurrent at a common point $\mathbf{x}'$ known as the vanishing point of the direction of lines $\mathbf{l}_i$.*

By applying a projective transformation to the line at infinity $\mathbf{l}_\infty$, we obtain a line $\mathbf{l}'_\infty$. This line intersects the image all the points at the infinity $\mathbf{x}_\infty$ from the original plane. Consequently, the vanishing line $\mathbf{l}'_\infty$ can be determined from two vanishing points.



2.6.2 Polarity

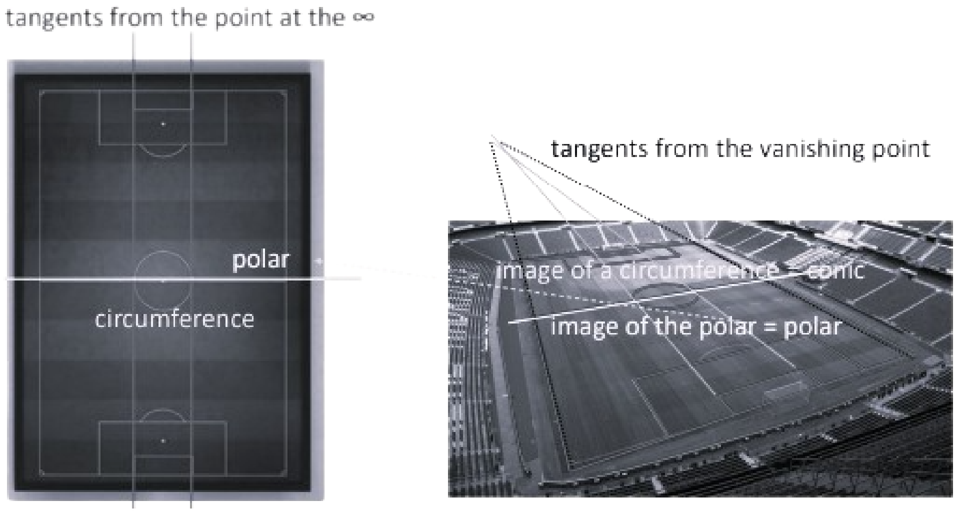
Polarity remains unaltered in the presence of projective mappings. The polar line $\mathbf{l} = \mathbf{C}\mathbf{x}$ corresponding to a point $\mathbf{x}$ with respect to a conic $\mathbf{C}$ gets mapped to the polar line of the transformed point $\mathbf{x}' = \mathbf{H}\mathbf{x}$ with respect to the transformed conic:

$$\mathbf{C}' = \mathbf{H}^{-T}\mathbf{C}\mathbf{H}^{-1}$$

Proof. This property holds because:

$$\mathbf{C}'\mathbf{x}' = \mathbf{H}^{-T}\mathbf{C}\mathbf{H}^{-1}\mathbf{H}\mathbf{x} = \mathbf{H}^{-T}\mathbf{C}\mathbf{x} = \mathbf{H}^{-T}\mathbf{l} = \mathbf{l}'$$

Therefore, the polar line of the transformed point aligns with the polar line of the original point. □



In conclusion, as polarity remains intact under projective mappings, conjugacy is similarly preserved, and the relationship CR $= -1$ is also upheld.

2.6.3 Cross ratio

Given a line defined by four points with the following relationships:

$$\mathbf{x}_1 = \propto_1 \mathbf{y} + \beta_1 \mathbf{z}$$

$$\mathbf{x}_2 = \propto_2 \mathbf{y} + \beta_2 \mathbf{z}$$

The cross ratio is expressed as:

$$\mathrm{CR}_{\mathbf{x}_1,\mathbf{x}_2,\mathbf{y},\mathbf{z}} = \frac{\frac{\beta_1}{\alpha_1}}{\frac{\beta_2}{\alpha_2}}$$

Upon applying a projective transformation $\mathbf{H}$ to these four points:

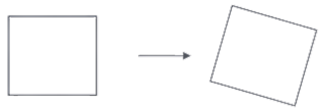
$$\begin{cases} \mathbf{y}' = \mathbf{H}\mathbf{y} \\ \mathbf{z}' = \mathbf{H}\mathbf{z} \\ \mathbf{x}_1' = \mathbf{H}\mathbf{x}_1 \propto_1 \mathbf{y}' + \beta_1 \mathbf{z}' \\ \mathbf{x}_2' = \mathbf{H}\mathbf{x}_2 \propto_2 \mathbf{y}' + \beta_2 \mathbf{z}' \end{cases}$$

The coefficients of the linear combination remain the same. Hence, the cross ratio is conserved, maintaining its original value:

$$\mathrm{CR}_{\mathbf{x}_1',\mathbf{x}_2',\mathbf{y}',\mathbf{z}'} = \frac{\frac{\beta_1}{\alpha_1}}{\frac{\beta_2}{\alpha_2}} = \mathrm{CR}_{\mathbf{x}_1,\mathbf{x}_2,\mathbf{y},\mathbf{z}}$$

2.6.4 Isometries

Isometries possess three degrees of freedom, which include translation denoted as t and the rotation angle represented by ϑ. Consequently, the invariants of this transformation encompass lengths, distances, and areas.



Definition. The *orthogonal matrix* $\mathbf{R}_\perp$ is defined as follows:

$$\mathbf{R}_\perp^{-1} = \mathbf{R}_\perp^T$$

Hence, the matrix $\mathbf{H}_I$ for isometries takes the following form:

$$\mathbf{H}_I = \begin{bmatrix} \cos\vartheta & -\sin\vartheta & t_x \\ \sin\vartheta & \cos\vartheta & t_y \\ 0 & 0 & 1 \end{bmatrix}$$

Here, $\begin{bmatrix} \cos\vartheta & -\sin\vartheta \\ \sin\vartheta & \cos\vartheta \end{bmatrix} = \mathbf{R}_\perp$

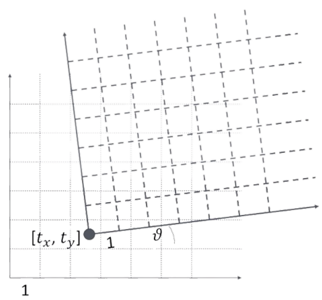
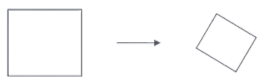


Figure 2.4: Isometry

2.6.5 Similarities

Similarities are characterized by four degrees of freedom, encompassing the translation, denoted as t; the scale, represented by s; and the rotation angle, expressed as ϑ. Consequently, the invariants of this transformation encompass the ratio of lengths and angles. Furthermore, the circular points **I** and **J** remain invariant throughout this transformation.



Hence, the matrix $\mathbf{H}_S$ for similarities is as follows:

$$\mathbf{H}_I = \begin{bmatrix} s\cos\vartheta & -s\sin\vartheta & t_x \\ s\sin\vartheta & s\cos\vartheta & t_y \\ 0 & 0 & 1 \end{bmatrix}$$

Here, $\begin{bmatrix} s\cos\vartheta & -s\sin\vartheta \\ s\sin\vartheta & s\cos\vartheta \end{bmatrix} = s\mathbf{R}_\perp$

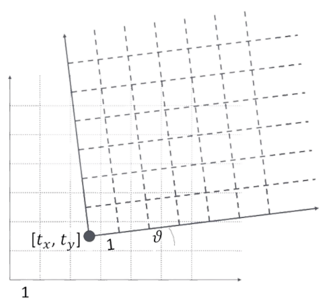
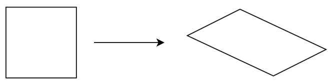


Figure 2.5: Similarity

2.6.6 Affinities

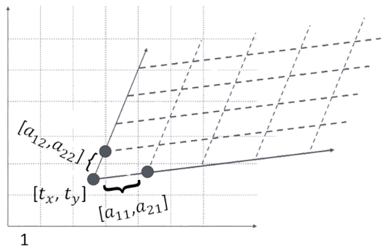
Affinities exhibit six degrees of freedom, consisting of the sub-matrix $\mathbf{A}$ and the translation component. As a result, the invariants of this transformation encompass parallelism, the ratio of parallel lengths, and the ratio of areas. The matrix $\mathbf{A}$ is defined as a 2×2 matrix with a rank of two. Additionally, the line at infinity, denoted as $\mathbf{l}_\infty$, remains invariant throughout the transformation.



Hence, the matrix $\mathbf{H}_A$ for affinities takes the following form:

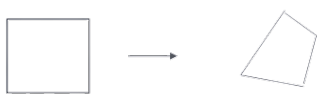
$$\mathbf{H}_I = \begin{bmatrix} a_{11} & a_{21} & t_x \\ a_{12} & a_{21} & t_y \\ 0 & 0 & 1 \end{bmatrix}$$

Here, $\begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{21} \end{bmatrix} = \mathbf{A}$



2.6.7 Projectivities

Projectivities possess eight degrees of freedom, encompassing the sub-matrix $\mathbf{A}$, the vector $\mathbf{v}$, and the translation component. Therefore, the invariants of this transformation include colinearity, incidence, and the order of contact. The matrix $\mathbf{A}$ is defined as a 2×2 matrix with a rank of two. Furthermore, the cross ratio remains invariant throughout this transformation.



Hence, the matrix $\mathbf{H}_P$ for projectivities takes the following form:

$$\mathbf{H}_I = \begin{bmatrix} a_{11} & a_{21} & t_x \\ a_{12} & a_{21} & t_y \\ v_1 & v_2 & 1 \end{bmatrix}$$

Here, $\begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{21} \end{bmatrix} = \mathbf{A}$

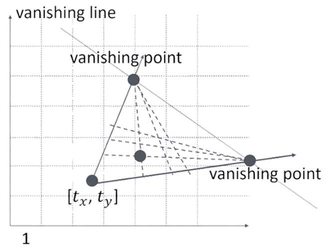


Figure 2.6: Affinity

CHAPTER 3

Two-dimensional reconstruction

3.1 Introduction

Recovering a model of an unknown planar scene from a single image, where the image is a projective transformation of the scene, presents a challenging problem. This transformation is represented by the equation $\mathbf{x}'_i = \mathbf{H}\mathbf{x}_i$, where $\mathbf{x}_i$ denotes the scene points and $\mathbf{H}$ is the transformation matrix. The difficulty arises from the fact that while the scene points $\mathbf{x}_i$ are known, the transformation matrix $\mathbf{H}$ is unknown, making a direct inversion of the mapping impossible.

The complexity of this task stems from the large number of unknown variables in the transformation matrix $\mathbf{H}$. In its general form, this problem is underdetermined, meaning that without additional constraints or simplifications, it cannot be uniquely solved. To address this challenge, two primary strategies are typically employed:

1. *Reducing the number of unknowns*: in many practical cases, the goal is not to recover the exact original configuration of the scene but rather to retrieve its overall geometric structure, a process known as shape reconstruction. By focusing on the shape rather than the full projective transformation, the number of unknowns can be reduced from eight to four. In this case, the transformation matrix $\mathbf{H}$ simplifies to:

$$\mathbf{H} = \begin{bmatrix} s\cos\vartheta & -s\sin\vartheta & t_x \\ s\sin\vartheta & s\cos\vartheta & t_y \\ 0 & 0 & 1 \end{bmatrix}$$

2. *Adding constraints*: another approach is to incorporate additional information to constrain the reconstruction. This extra information often involves parameters that remain invariant under the specific type of transformation being sought, but vary under more general classes of transformations. By leveraging these invariants, we can narrow down the possible solutions and recover a more accurate model of the scene.

Reconstruction methods can be broadly classified into two categories:

- *Affine reconstruction*: in this approach, the reconstructed scene is related to the original through an affine transformation, preserving parallelism but not necessarily angles or distances.

- *Shape reconstruction*: this method seeks to recover the overall shape of the scene using a similarity transformation, which preserves both angles and relative distances, providing a more faithful representation of the scene's geometry while simplifying the problem.

3.2 Affine reconstruction

Theorem 3.2.1. *A projective transformation* $\mathbf{H}$ *that maps the line at infinity* $\mathbf{l}_\infty$ *onto itself implies that* $\mathbf{H}$ *is affine.*

Proof. A point at infinity, represented as $\mathbf{x}_\infty = \begin{bmatrix} x & y & 0 \end{bmatrix}^T$, is mapped to another point $\mathbf{x}' = \mathbf{H}\mathbf{x}_\infty$ to remain a point at infinity, its third coordinate must be zero. This condition can be expressed as:

$$\begin{bmatrix} v_1 & v_2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} = 0$$

Which simplifies to:

$$\begin{bmatrix} v_1 & v_2 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$$

Thus, the matrix $\mathbf{H}$ has the structure of an affine transformation, confirming that $\mathbf{H}$ is affine. □

In an image produced by a general projective transformation of a scene, the line at infinity $\mathbf{l}'_\infty$ in the image will not coincide with the original line at infinity $\mathbf{l}_\infty$. However, this difference can be exploited by using $\mathbf{l}'_\infty$ as additional information. By applying a corrective projective transformation $\mathbf{H}_{\mathrm{AR}}$ that maps $\mathbf{l}'_\infty$ back to $\mathbf{l}_\infty$, a modified image is obtained. In this new image, the line at infinity $\mathbf{l}_\infty$ is preserved.

According to the theorem, this resulting transformation produces a model that is an affine mapping of the original scene. Therefore, the modified image is an affine reconstruction of the scene.

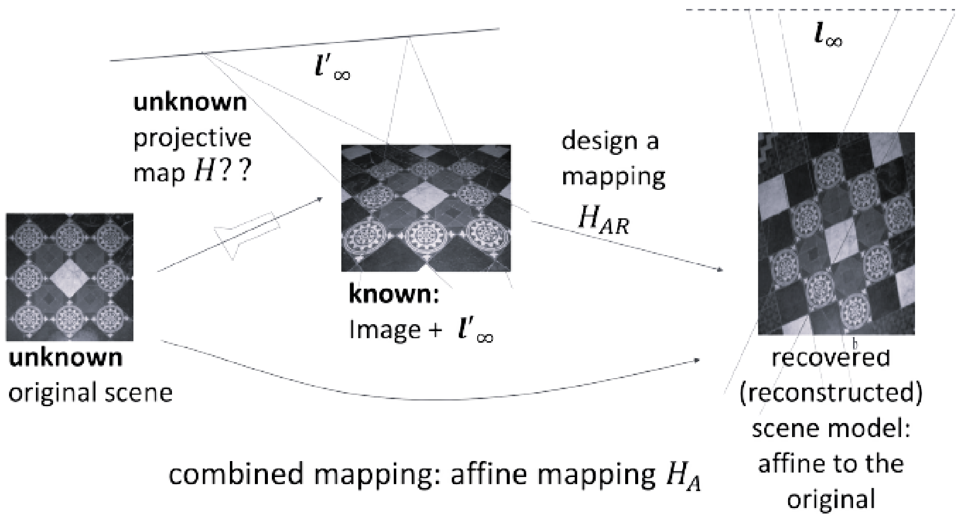


Figure 3.1: Affine transformation

The key challenges in this approach are:

- Determining the projective transformation $\mathbf{H}_{\mathrm{AR}}$ that maps $\mathbf{l}'_\infty$ back to $\mathbf{l}_\infty$.
- Identifying the vanishing line in the image, which corresponds to $\mathbf{l}'_\infty$.

3.2.1 Projective transformation determination

o find the corrective projective transformation $\mathbf{H}_{\mathrm{AR}}$ that restores $\mathbf{l}'_\infty$ to $\mathbf{l}_\infty$, the mapping must satisfy the condition that any point $\mathbf{x}' \in \mathbf{l}'_\infty$ is mapped to a point at infinity. Mathematically, this can be written as:

$$\mathbf{H}_{\mathrm{AR}}\mathbf{x}' = \begin{bmatrix} * \\ * \\ 0 \end{bmatrix}$$

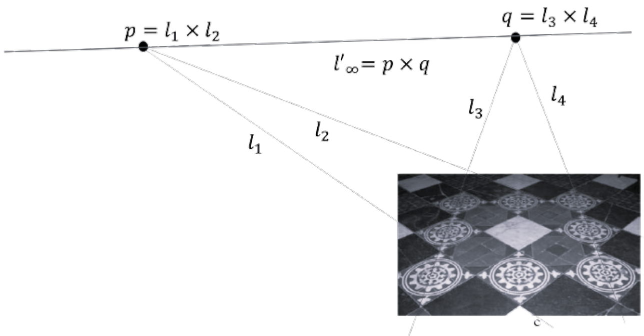
The transformation $\mathbf{H}_{\mathrm{AR}}$ can be represented in matrix form as:

$$\mathbf{H}_{\mathrm{AR}} = \begin{bmatrix} * & * & * \\ * & * & * \\ & \mathbf{l}'^{T}_\infty & \end{bmatrix}$$

which ensures that any point $\mathbf{x}' \in \mathbf{l}'_\infty$ is correctly mapped to the line at infinity.

3.2.2 Vanishing line identification

To identify the vanishing line $\mathbf{l}'_\infty$, additional geometric information can be used, such as the images of parallel lines in the scene. These parallel lines intersect at points on the vanishing line in the projective image.



By leveraging such information, the vanishing line can be accurately determined, enabling the projective transformation $\mathbf{H}_{\mathrm{AR}}$ to be applied for an affine reconstruction of the scene.

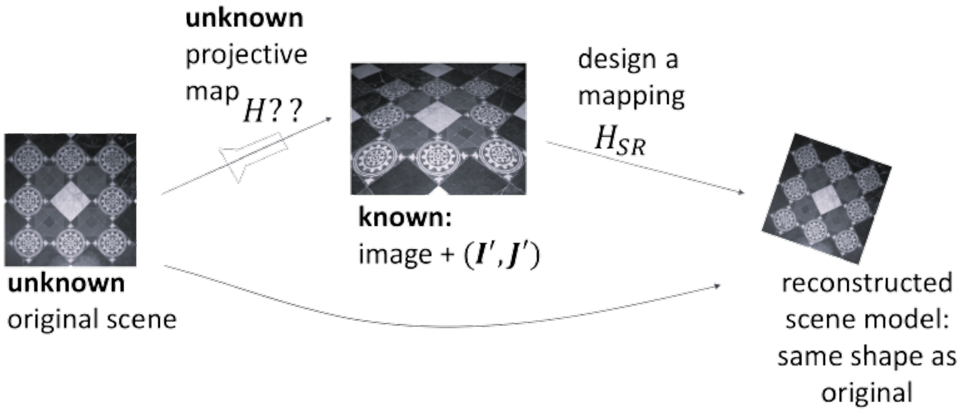
3.3 Shape reconstruction

Theorem 3.3.1. *A projective transformation* $\mathbf{H}$ *that maps the circular points* $\mathbf{I}$ *and* $\mathbf{J}$ *onto themselves implies that* $\mathbf{H}$ *is a similarity transformation.*

Proof. When a similarity transformation matrix $\mathbf{H}_{\mathrm{S}}$ is applied to the circular point $\mathbf{I}$, it produces a scalar multiple of $\mathbf{I}$. The same holds true for the other circular point, $\mathbf{J}$. Since both $\mathbf{I}$ and $\mathbf{J}$ remain unchanged under this transformation, $\mathbf{H}_{\mathrm{S}}$ is indeed a similarity transformation. □

In a general projective mapping of the original scene, the images of the circular points, denoted as $(\mathbf{I}', \mathbf{J}')$, do not coincide with the original circular points $\mathbf{I}$ and $\mathbf{J}$. To perform shape reconstruction, we apply a corrective projective transformation $\mathbf{H}_{\mathrm{SR}}$ that maps $\mathbf{I}'$ and $\mathbf{J}'$ back to $\mathbf{I}$ and $\mathbf{J}$, respectively. This results in a modified image where the circular points are restored to their original positions.

According to the theorem, this new transformation results in a similarity transformation of the original scene. Hence, the reconstructed model is a shape reconstruction, maintaining the overall proportions and geometry of the original scene.



The main challenges in this approach are:

- Finding the projective transformation $\mathbf{H}_{\mathrm{SR}}$ that restores the points $\mathbf{I}'$ and $\mathbf{J}'$ to $\mathbf{I}$ and $\mathbf{J}$.
- Determining the vanishing line in the image to aid in finding the circular points.

3.3.1 Projective transformation determination

Finding the projective transformation $\mathbf{H}_{\mathrm{SR}}$ that maps $\mathbf{I}'$ and $\mathbf{J}'$ back to $\mathbf{I}$ and $\mathbf{J}$ is equivalent to solving for one of the infinitely many matrices that satisfy:

$$\begin{cases} \mathbf{H}_{\mathrm{SR}}\mathbf{I}' = \mathbf{I} \\ \mathbf{H}_{\mathrm{SR}}\mathbf{J}' = \mathbf{J} \end{cases}$$

This task is non-trivial, but it can be simplified by leveraging additional geometric information, such as the degenerate conic dual to the circular points.

The degenerate conic dual to $\mathbf{I}', \mathbf{J}'$ is given by:

$$\mathbf{C}'^*_\infty = \mathbf{I}'\mathbf{J}'^T + \mathbf{J}'\mathbf{I}'^T$$

This conic is the image of the original conic dual to the circular points $\mathbf{I}$ and $\mathbf{IJ}$, denoted by:

$$\mathbf{C}^*_\infty = \mathbf{I}\mathbf{J}^T + \mathbf{J}\mathbf{I}^T$$

Since $\mathbf{C}'^*_\infty$ is the projective image of $\mathbf{C}^*_\infty$, any projective transformation $\mathbf{H}_{\mathrm{SR}}$ that restores $\mathbf{I}'$ and $\mathbf{J}'$ to $\mathbf{I}$ and $\mathbf{J}$ will also restore $\mathbf{C}'^*_\infty$ to $\mathbf{C}^*_\infty$. Using the transformation rule for dual conics, we have:

$$\mathbf{C}^*_\infty = \mathbf{H}_{\mathrm{SR}}\mathbf{C}'^*_\infty\mathbf{H}_{\mathrm{SR}}^T$$

Reversing this relationship gives:

$$\mathbf{C}'^*_\infty = \mathbf{H}_{\mathrm{SR}}^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mathbf{H}_{\mathrm{SR}}^{-T}$$

By applying singular value decomposition (SVD) to the equation above, we find that $\mathbf{H}_{\mathrm{SR}}^{-1}$ and $\mathbf{H}_{\mathrm{SR}}^{-T}$ are orthogonal matrices. This leads to:

$$\mathrm{SVD}\left(\mathbf{C}_{\infty}^{\prime *}\right) = \mathbf{U}_{\perp} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mathbf{U}_{\perp}^{T}$$

Thus, the solution for $\mathbf{H}_{\mathrm{SR}}$ is:

$$\mathbf{H}_{\mathrm{SR}} = \mathbf{U}_{\perp}^{-1} = \mathbf{U}_{\perp}^{T}$$

To ensure proper image rectification and scaling, the matrix $\mathbf{H}_{\mathrm{SR}}$ can be adjusted to:

$$\mathbf{H}_{\mathrm{SR}} = \begin{bmatrix} \frac{1}{\sqrt{a}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{b}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{U}^{T}$$

This transformation not only maps the circular points back to their original positions but also ensures that the final image is a faithful similarity reconstruction of the scene.

3.3.2 Vanishing line identification

To determine the vanishing line, one can leverage additional information from the observed scene. This information can be used to establish the following constraints:

1. Known angles between lines: when the angles between lines in the scene are known, these angles can be used to constrain the vanishing line. The angle between two lines is related to the angle between their normal directions and is independent of parameters c_1 and c_2. Mathematically, this relationship is expressed as:

$$\cos\vartheta = \frac{a_1 a_2 + b_1 b_2}{\sqrt{(a_1^2 + b_1^2)(a_2^2 + b_2^2)}}$$

   Here, a_1, b_1, a_2, and b_2 are coefficients of the normal vectors of the lines. By rewriting the terms, this equation can be expressed as:

$$\cos\vartheta = \frac{l^T C_{\infty}^{*} m}{\sqrt{(l^T C_{\infty}^{*} l)(m^T C_{\infty}^{*} m)}}$$

   This equation can be further simplified by using the rules obtaining $C_{\infty}^{*} = H^{-1} C_{\infty}^{*\prime} H^{-T}$. Now, we can rewrite $l^T C_{\infty}^{*} m$ as $l'^T C_{\infty}^{*\prime} m'$. With these transformations, the equation becomes:

$$\cos\vartheta = \frac{l'^T C_{\infty}^{*\prime} m'}{(l'^T C_{\infty}^{*\prime} l')(m'^T C_{\infty}^{*\prime} m')}$$

   In this case, m' and l' are obtained from the image. Since the angle is known, this equation provides a linear constraint on $C_{\infty}^{*\prime}$, that is linear when the lines are perpendicular ($\cos\vartheta = 0$). The unknown matrix C'_{∞} is symmetric, homogeneous, and singular, providing four independent constraints.

2. Known shape of objects: if the shape of objects in the scene is known, the reconstruction matrix $\mathbf{H}_{\mathrm{SR}}$ can be determined. The transformation matrix is defined as:

$$\mathbf{H}_{\mathrm{SR}} = \begin{bmatrix} \frac{1}{\sqrt{a}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{b}} & 0 \\ 0 & 0 & 1 \end{bmatrix} U^{T}$$

   The Euclidean reconstructed image is calculated as $M_S = \mathbf{H}_{\mathrm{SR}} \cdot \mathrm{image}$

3. Combinations of constraints: it is also possible to use a combination of known angles between lines and the shape of objects for additional constraints.

4. Observation of rigid planar motion: when observing rigid planar motion, which is a similarity transformation, the circular points remain invariant. The object has three degrees of freedom, and the center of rotation and the rotation angle can be determined. Given a matrix H, the eigenvectors of H correspond to fixed points, and the eigenvectors of H^{-T} correspond to fixed lines of the transformation. The eigenvectors can be used to extract important information:

   - Eigenvectors $\mathbf{I}', \mathbf{J}'$ correspond to complex eigenvalues.
   - The phase of these eigenvectors is the rotation angle.
   - Eigenvector O' correspond to real eigenvalues.

   The three eigenvectors of H are proportional to three distinct values: 1, $e^{i\theta}$, and $-e^{i\theta}$. The eigenvector corresponding to the eigenvalue 1 represents the image of the center of rotation, denoted as O, and the angle θ corresponds to the rotation angle. The eigenvectors associated with the complex eigenvalues represent the images of the circular points $\mathbf{I}', \mathbf{J}'$. Therefore, using the relationship $C_\infty'^* = \mathbf{I}'J'^T + \mathbf{J}'I'^T$, the singular value decomposition can be applied to obtain $\mathrm{SVD}(C_\infty'^*) = UC_\infty^* U^T$, where U^T is the rectification matrix. Two methods can be used to address this:

   - Direct method:
     (a) Find $C_\infty^{*'}$.
     (b) Compute H_{rect} for rectification.
   - Stratified method:
     (a) Perform affine reconstruction from projective to affine.
     (b) Perform shape reconstruction from affine to metric.

   In some cases, the stratified method reduces numerical errors, providing a more accurate result.

3.4 Accuracy issues

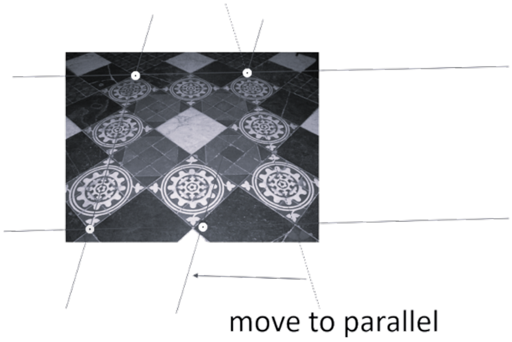
There are various accuracy issues when doing image rectification:

1. Noise and numerical errors: noise and numerical errors in the input data can affect the accuracy of the rectification process. It's essential to preprocess and filter the data to minimize these issues.

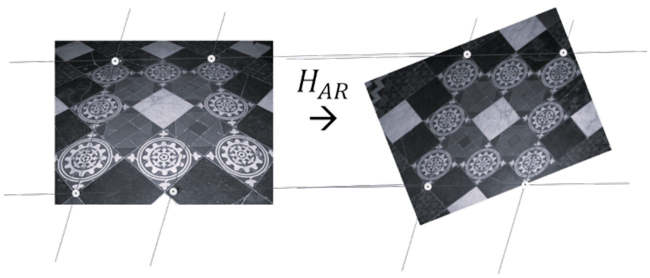
2. Little information: when choosing lines to identify vanishing points, it's crucial to select lines that are sufficiently far apart. Choosing lines that are too close to each other can lead to inaccuracies in vanishing point estimation and rectification.

3. Vanishing point near infinity: in cases where the vanishing point is nearly at infinity, it can be challenging to perform accurate affine rectification. To address this issue:

   - Draw two lines in the scene that are perpendicular to the given lines in the image. If the two new lines are not parallel, adjust one of the intersection points to make them parallel.



   Finally, apply affine reconstruction to obtain accurate results.



   - When dealing with sets of parallel lines, you can choose one line from each set and randomly select another pair of lines, making sure they are perpendicular. With these four lines, you can compute the matrix product of K and its transpose, denoted as KK^T, and then derive K through Cholesky factorization. Afterward, you can apply the rectifying transformation using the matrix:

$$H_{rect} = \begin{bmatrix} K & t \\ 0 & 1 \end{bmatrix}^{-1}$$

The accuracy of image rectification is crucial for various computer vision and image processing applications, and addressing these issues is essential for obtaining reliable results.

CHAPTER 4

Three-dimensional space projective geometry

4.1 Introduction

In space geometry, the fundamental elements required for defining the geometry include points, planes, quadrics, and dual quadrics. The allowable transformations within this type of geometry encompass projectivities, affinities, similarities, and isometries.

4.2 Points

Points in space geometry can be represented in Cartesian coordinates by defining a Euclidean space with its origin. This approach allows for the unambiguous definition of every point using three Cartesian coordinates (x, y, z).

However, when analyzing images, it is more convenient to employ homogeneous coordinates. The relationship between Cartesian and homogeneous coordinates is as follows:

$$X = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = w \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

This representation exhibits homogeneity, meaning any vector x is equivalent to all its non-zero multiples λx, where $\lambda \neq 0$, since they all represent the same point. The null vector, however, does not represent any point.

Definition. The *projective space* is defined as:

$$\mathbb{P}^3 = \{\begin{bmatrix} x & y & z & w \end{bmatrix}^T \in \mathbb{R}^4\} - \{\begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}^T\}$$

4.3 Planes

In the homogeneous coordinates, planes are defined by using the matrix:

$$\pi = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

Here, the direction normal to the plane is given by (a, b, c), and the distance from the origin to the plane is calculated as:

$$\text{distance} = -\frac{d}{\sqrt{a^2 + b^2 + c^2}}$$

Similar to homogeneous point coordinates, this representation of planes also exhibits the homogeneity property. Any vector π is equivalent to all its non-zero multiples $\lambda\pi$, where $\lambda \neq 0$, as they all represent the same plane. The parameters a, b, c, d are referred to as the homogeneous parameters of the plane. As with points, there are an infinite number of equivalent representations for a single plane, which includes all non-zero multiples of the unit normal vector. The null vector does not represent any plane. If $d = 0$, it signifies that the plane π passes through the origin of space.

To determine whether a point lies on a plane or if a plane passes through a point, you can solve the following system of equations:

$$\begin{cases} ax + by + cz + dw = 0 \\ \pi^T X = X^T \pi = 0 \end{cases}$$

Definition. The plane

$$\begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = w = 0$$

is called the *plane at the infinity* $\pi_\infty = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}^T$.

It's important to note that this plane has an undefined normal direction.