Game Theory

Christian Rossi

Academic Year 2024-2025

Abstract

The theory begins by examining the main assumptions that distinguish decision theory from interactive decision theory. While decision theory focuses on individual decision-making in isolation, interactive decision theory explores how multiple decision-makers interact, considering each other's potential actions.

In the context of non-cooperative games, the discussion extends to games represented in extensive form, where players make decisions at various points, and games with perfect information, where all players are fully informed of prior moves. The technique of backward induction is key in solving such games. Additionally, combinatorial games are explored, emphasizing their strategic complexity.

Zero-sum games are analyzed in terms of conservative values, where each player seeks to minimize potential losses. The concept of equilibrium in pure strategies is introduced, and this is extended to mixed strategies in finite games, invoking von Neumann's theorem. Finding optimal strategies and determining the value of finite games is achieved through the use of linear programming techniques.

The Nash non-cooperative model plays a central role in understanding strategic interactions. Nash equilibrium is discussed, focusing on the existence of equilibria in both pure and mixed strategies within finite games. Examples of potential games are provided, along with methods for identifying potential functions. Notable examples include congestion games, routing games, network games, and location games. Concepts such as the price of stability, price of anarchy, and correlated equilibria are explored to analyze the efficiency and stability of these systems.

Finally, the discussion shifts to cooperative games, defining key concepts such as the core, nucleolus, Shapley value, and power indices. Examples of cooperative scenarios illustrate how these concepts help to determine fair outcomes and power distribution among players.

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CHAPTER 1

Introduction

1.1 Games

Game theory is a decision theory with many decision maker, so it is more complex wrt the case in which we have to make the best choice with only one decision maker. Games are efficient models for an enormous amount of everyday life situations.

Definition (*Game*)**.** A game is a process consisting in:

- *Players* (at least two).
- *Initial situation*.
- *Rules* (for each player).
- *Outcomes* (all possible final situations).
- *Preferences* (different for each player wrt to the possible outcomes).

1.2 Players

Players are supposed to be selfish and rational.

1.2.1 Selfish player

The player only car about their own preferences wrt to the outcome of the game. This is a mathematical assumption to define the meaning of rational choice.

1.2.2 Rational player

Definition (*Preference relation*)**.** A preference relation on a set X is a binary relation $\succeq$ that satisfies the following properties for all $x, y, z \in X$:

- *Reflexive*: $x \succeq x$.
- *Complete*: $x \succeq y$ or $y \succeq x$.

- *Transitive*: if $x \succeq y$ and $y \succeq z$, then $x \succeq z$.

Definition (*Utility function*)**.** Given a preference relation $\succeq$ over a set X, a utility function representing $\succeq$ is a function $u : X \to \mathbb{R}$ such that:

$$u(x) \geq u(y) \Leftrightarrow x \succeq y$$

While a utility function may not always exist in specific cases, it does exist in general settings, particularly when X is finite. If a utility function does exist, there are infinitely many such functions, differing by any strictly increasing transformation of the original function. Each player i is assigned a set X_i, representing all the choices available to them. Therefore, the set $X = xX_i$ over which the utility function u is defined represents the combined choices of all players.

1.2.2.1 Rationality assumptions

The following assumptions define the rational behavior of players:

1. The players are able to provide a preference relation over the outcomes of the game, and the order must be consistent.
2. The players are able to provide a utility function representing their preferences relations, whenever it is necessary.
3. The players use consistently the laws of probability.
4. The players are able to understand the consequences of all their actions.
5. The players are able to use decision theory, whenever it is possible

Therefore, given a set of alternatives X and a utility function u, each player seeks $\bar{x} \in X$ such that:

$$u(\bar{x}) \geq u(x) \qquad \forall x \in X$$

From the axioms we can derive the principle of elimination of strictly dominated strategies: a player does not take an action if he has another action providing him a strictly better result, no matter what the other players do.

1.2.3 Actions

The set of actions can be represented as a set of pair of values that represent the utilities for Player 1 and Player 2, respectively. These options can be conveniently displayed in a bi-matrix. Conventionally, Player 1 selects a row, while Player 2 selects a column.

CHAPTER 2

Extensive games

2.1 Introduction

An extensive game can be represented with a tree, where each node represents the possible choice for each player. The nodes can also represent random outcomes of an action.

2.2 Extensive games

An extensive game with perfect information consists of:

1. A finite set $N = \{1, \dots, n\}$ of players.
2. A game tree (V, E, x_0).
3. A partition of the non-leaf vertices into sets $P_1, P_2, \dots, P_{n+1}$.
4. A probability distribution for each vertex in P_{n+1}, defined on the edges from that vertex to its children.
5. A n-dimensional vector attached to each leaf (list of possible outcomes).

The set P_i, for $i \leq n$, consists of nodes where Player i must choose a child of v, representing a possible move by that player. P_{n+1} is the set of nodes where chance plays a role. When P_{n+1} is empty, the game does not admit any random event.

2.2.1 Solution

To determine the optimal outcome, we apply the axioms of rationality to the game tree, finding the solution to the extensive game.

Definition (*Length*)**.** The length of a game is defined as the length of the longest path in the game tree.

The fifth rationality assumption allows us to solve games of length 1, while the fourth rationality assumption allows us to solve games of length $i+1$ if all games of length at most i have already been solved. The iterative process where we work backwards from the leaves of the tree to the root to determine the optimal sequence of actions is called backward induction.

Theorem 2.2.1 (First rationality theorem). *The rational outcomes of a finite game with perfect information are those determined by the backward induction procedure.*

Backward induction can be applied because every vertex v in the game is the root of a sub-game consisting of all the vertices that follow v.

2.2.2 Possible outcomes

In extensive games there may be more than one possible outcome.

Theorem 2.2.2 (Von Neumann). *In the game of chess, one and only one of the following alternatives holds:*

1. *The white has a way to win, no matter what the black does.*
2. *The black has a way to win, no matter what the white does.*
3. *Both white and black can force at least a draw, regardless of the opponent's actions.*

Proof. Assume the game has a finite length of $2k$ moves (each player makes k moves). Let a_i represent white's moves, and b_i represent black's moves. The first possibility in the theorem can be formulated as follows:

$$\exists a_1 \mid \forall b_1 \exists a_2 \mid \forall b_2 \ldots \exists a_k \mid \forall b_k \implies \text{white wins}$$

Now, suppose this is false, resulting in its negation:

$$\forall a_1 \exists b_1 \mid \forall a_2 \mid \exists b_2 \mid \ldots \forall a_k \mid \exists b_k \implies \text{white does not win}$$

This means black has the possibility to prevent White from winning, ensuring at least a draw. This applies also to the second possibility in the same way. □

Corollary 2.2.2.1. *Consider a finite, perfect information game with two players, where the only possible outcomes are a win for one of the players or a tie. Then, exactly one of the following holds:*

1. *Player 1 has a winning strategy, no matter what Player 2 does.*
2. *Player 2 has a winning strategy, no matter what Player 1 does.*

Therefore, the possible solutions for a game are classified as follows:

- *Very weak solution*: the game has a rational outcome, but it is inaccessible.
- *Weak solution*: the outcome of the game is known, but the method to achieve it is not.
- *Solution*: there exists an algorithm that can determine the outcome.

2.3 Combinatorial games

Definition (*Impartial combinatorial game*)**.** An impartial combinatorial game is defined by the following characteristics:

1. There are two players in alternate order.
2. There is a finite number of positions in the game.
3. The players follow the same rules.
4. The game concludes when no further moves are possible.
5. The game does not involve chance.
6. In the classical version, the winner is the player leaving the other player with no available moves.

To solve impartial combinatorial games, we begin by partitioning the set of all possible positions (which are finite in number) into two distinct categories:

1. *P-positions*: loosing.
2. *N-positions*: winning.

It is important to note that the current state of the game is what matters, rather than which player is designated to move.

We have the following rules:

- The terminal position $(0, 0, \cdots, 0)$ is classified as a P-position. This is a losing position because the player has no cards left to play.
- From any P-position, only N-positions can be reached.
- From any N-position it is possible (but not obligatory) to move to a P-position.

Therefore, the player who starts from an N-position is assured of a victory, given that they play optimally.

2.3.1 Nim game

The Nim game is characterized by a tuple $(n_1, \cdots, n_k)$, where each n_i is a positive integer. During their turn, each player must choose one pile n_i and replace it with $\hat{n}_i$, ensuring that $\hat{n}_i < n_i$. The player who reduces the position to $(0, \ldots, 0)$ wins. Therefore, each player's action involves removing cards from a single pile with the objective of clearing the entire table.

Theorem 2.3.1 (Bouton)**.** *A position $(n_1, n_2, \ldots, n_k)$ in the Nim game is a P-position if and only if:*

$$n_1 \oplus n_2 \oplus \cdots \oplus n_k = 0$$

Proof. The terminal position $(0, 0, \ldots, 0)$ is a P-position corresponding to a Nim-sum of zero.

If the Nim-sum $n_1 \oplus n_2 \oplus \cdots \oplus n_k = 0$, any subsequent position will have a non-zero Nim-sum. Assume the next position is $(\hat{n}_1, n_2, \ldots, n_k)$ such that $\hat{n}_1 \oplus n_2 \oplus \cdots \oplus n_k = 0$. Then we would have:

$$n_1 \oplus n_2 \oplus \cdots \oplus n_k = 0$$

Which, by the cancellation law, implies $\hat{n}_1 = n_1$. This is a contradiction, as the game rules stipulate that $\hat{n}_1 < n_1$.

Conversely, if $n_1 \oplus n_2 \oplus \cdots \oplus n_k \neq 0$, it is possible to move to a position with a zero Nim-sum. Let $z = n_1 \oplus n_2 \oplus \cdots \oplus n_k \neq 0$. Identify a pile where the binary representation of z has a 1 in its leftmost column. Change that digit to 0 and adjust the digits to the right, leaving unchanged the digits that correspond to 0. This operation produces a new number that is smaller than the original. □

Games with perfect information can typically be resolved through backward induction. However, this method is primarily effective for relatively simple games due to the constraints of limited rationality. Depending on the specifics of the game, we may arrive at varying degrees of solutions.

2.4 Strategy

In backward induction, a specific move must be identified at every node. Let P_i denote the set of all nodes at which player i is required to make a decision.

Definition (*Pure strategy*)**.** A pure strategy for player i is defined as a function on the set P_i, which associates each node v in P_i with a child node x, or equivalently, an edge (v, x).

Definition (*Mixed strategy*)**.** A mixed strategy refers to a probability distribution over the set of pure strategies.

When a player possesses n pure strategies, the collection of their mixed strategies is represented as:

$$\sum_n = \left\{ p = (p_1, \ldots, p_n) | p_i \geq 0 \text{ and } \sum p_i = 1 \right\}$$

Here, $\sum_n$ forms the fundamental simplex in n-dimensional space.

Theorem 2.4.1 (Von Neumann (strategies))**.** *In the game of chess, one of the following scenarios must hold:*

1. *White has a winning strategy.*
2. *Black has a winning strategy.*
3. *Both players possess a strategy that guarantees at least a tie.*

The first outcome occurs when there exists a row containing all winning elements. The second outcome arises when there is a column consisting of all winning elements. The third outcome features mixed results, including ties, but does not encompass all three outcomes in a single row or column.

If $P_i = \{v_1, \ldots, v_k\}$ and v_j has n_j children, then the total number of strategies available to Player i is $n_1 \cdot n_2 \cdot \cdots \cdot n_k$.

2.5 Imperfect information games

In certain scenarios, players must make their moves simultaneously, which prevents them from having complete knowledge of each other's actions. This situation can still be represented using a game tree.

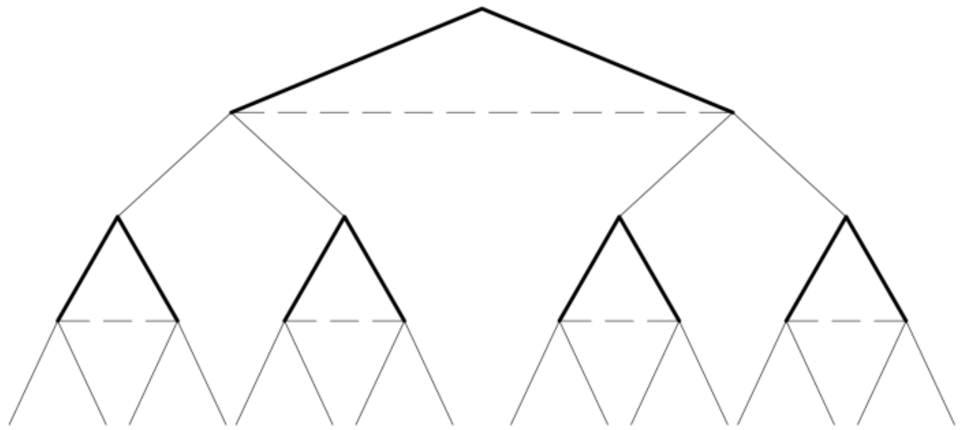

Figure 2.1: Tree with imperfect information

The dashed lines in the figure indicate that a player does not know exactly which vertex they occupy.

Definition (*Information set*)**.** An information set for Player i is a pair $(U_i, A(U_i))$ satisfying the following conditions:

1. $U_i \subset P_i$ is a non-empty set of vertices $v_1, \cdots, v_k$.
2. Each vertex $v_j \in U_i$ has the same number of children.
3. $A_i(U_i)$ is a partition of the children of $v_1 \cup \cdots \cup v_k$ such that each element of the partition contains exactly one child from each vertex v_j.

Thus, Player i knows they are in U_i but cannot determine the exact vertex. The partition defines the choice function, indicating that each set in $A_i(U_i)$ corresponds to an available move for the player (graphically, this represents the same choice, or edge, emanating from different vertices).

Definition (*Extensive game with imperfect information*)**.** An extensive form game with imperfect information is characterized by the following components:

1. A finite set $N = \{1, \ldots, n\}$ of players.
2. A game tree (V, E, x_0).
3. A partition comprising sets $P_1, P_2, \ldots, P_{n+1}$ of the non-leaf vertices.
4. A partition $(U_i^j), j = 1, \ldots, ki$ of the set P_i, for all i, with (U_i^j, A_i^j) being the information set for all players i at all vertices j (having the same number of children).
5. A probability distribution defined for each vertex in P_{n+1} on the edges leading to its children.

6. An n-dimensional vector assigned to each leaf.

It is important to note that if the partition consists of only a single vertex, then a game with imperfect information effectively becomes a game with perfect information.

Definition (*Pure strategy*). A pure strategy for player i in an imperfect information game is a function defined over the collection $\mathcal{U}$ of their information sets, assigning to each $U_i \in \mathcal{U}$ an element from the partition $A(U_i)$.

Definition (*Mixed strategy*). A mixed strategy is defined as a probability distribution over the pure strategies.

A game of perfect information is a specific type of imperfect information game where all information sets for all players are singletons.

CHAPTER 3

Zero sum games

3.1 Introduction

Definition (*Zero sum game*). A two-player zero-sum game in strategic form can be described as a triplet $(X, Y, f : X \times Y \to \mathbb{R})$, where:

- X is the strategy space of Player 1.
- Y is the strategy space of Player 2.
- $f(x, y)$ represents the payoff Player 1 receives from Player 2 when they play strategies x and y, respectively.

Since this is a zero-sum game, Player 2's utility function g is defined as the negative of Player 1's utility function:

$$g(x, y) = -f(x, y)$$

In the case where the strategy spaces are finite the game can be represented by a payoff matrix P. In this matrix, Player 1 chooses a row i, and Player 2 chooses a column j:

$$\begin{pmatrix} p_{11} & \cdots & p_{1m} \\ \cdots & p_{ij} & \cdots \\ p_{n1} & \cdots & p_{nm} \end{pmatrix}$$

Here, p_{ij} denotes the payment Player 2 makes to Player 1 when they select strategies i and j, respectively.

To determine the optimal strategy, both players can employ conservative reasoning:

- Player 1 can ensure a minimum payoff of $v_1 = \max_i \min_j p_{ij}$.
- Player 2 can limit their losses to at most $v_2 = \min_j \max_i p_{ij}$.

These values, v_1 and v_2, are known as the conservative values for Player 1 and Player 2, respectively.

In more general cases where the strategy spaces X and Y are not finite, a similar reasoning applies. Let $(X, Y, f : X \times Y \to \mathbb{R})$ describe the game, where X and Y are arbitrary strategy sets. The conservative values can be defined as follows:

- Player 1: $v_1 = \sup_x \inf_y f(x,y)$.
- Player 2: $v_2 = \inf_y \sup_x f(x,y)$.

These values, v_1 and v_2, are known as the conservative values for Player 1 and Player 2, respectively.

3.2 Rationality

Now, suppose the following holds:

- $v_1 = v_2 = v$.
- There exists a row $\bar{i}$ such that $p_{\bar{i}\bar{j}} \geq v_1 = v$ for all j.
- There exists a column $\bar{j}$ such that $p_{\bar{i}\bar{j}} \leq v_2 = v$ for all i.

In this case, $p_{\bar{i}\bar{j}} = v$, and this value represents the rational outcome of the game. Thus, $\bar{i}$ maximizes the function $\alpha(i) = \min_j p_{ij}$, and $\bar{j}$ minimizes the function $\beta(j) = \max_i p_{ij}$.

3.2.1 Existence of a rational outcome

To demonstrate the existence of a rational outcome in a zero-sum game, we need to establish the following:

1. *Equality of conservative values*: the conservative values of both players agree, i.e., $v_1 = v_2$.
2. *Existence of an optimal strategy for Player 1*: there exists a strategy $\bar{x}$ such that:
$$v_1 = \inf_y f(\bar{x}, y)$$
This ensures that $\bar{x}$ is an optimal strategy for Player 1.
3. *Existence of an optimal strategy for Player 2*: there exists a strategy $\bar{y}$ such that:
$$v_2 = \sup_x f(x, \bar{y})$$
This ensures that $\bar{y}$ is an optimal strategy for Player 2.

In the case where the strategy spaces are finite, such optimal strategies $\bar{x}$ and $\bar{y}$ always exist. Therefore, proving the existence of a rational outcome is equivalent to demonstrating the equality of the conservative values, i.e., $v_1 = v_2$.

Theorem 3.2.1 (Von Neumann)**.** *There always exists a rational outcome for a finite two-player zero-sum game, as described by its payoff matrix P.*

This fundamental result, known as the Minimax theorem, guarantees that in every finite zero-sum game, the conservative values for both players coincide, and optimal strategies exist for both players, leading to a rational outcome.

Proof. Suppose, without loss of generality, that all p_{ij} in the matrix P are positive. Consider the column vectors $p_1, \dots, p_m \in \mathbb{R}^n$, and let C denote their convex hull. Define the set

$$Q_t = \{x \in \mathbb{R}^n : x_i \le t\}$$

and

$$v = \sup\{t \ge 0 : Q_t \cap C = \emptyset\}$$

Since $\operatorname{int} Q_v \cap C = \emptyset$, the sets Q_v and C can be separated by a hyperplane. Hence, there exist coefficients $\bar{x}_1, \dots, \bar{x}_n$, with some $\bar{x}_i \neq 0$, and $b \in \mathbb{R}$ such that:

$$(\bar{x}, u) = \sum_{i=1}^{n} \bar{x}_i u_i \le b \le \sum_{i=1}^{n} \bar{x}_i w_i = (\bar{x}, w)$$

for all $u = (u_1, \dots, u_n) \in Q_v$ and $w = (w_1, \dots, w_n) \in C$.

Since all $\bar{x}_i$'s must be non-negative, we can assume $\sum \bar{x}_i = 1$. Additionally, $b = v$, since $\bar{v} := (v, \dots, v) \in Q_v$, and

$$(\bar{x}, \bar{v}) = \sum_i \bar{x}_i v = v \sum_i \bar{x}_i = v$$

Therefore, $b \ge v$. If $b > v$, by choosing a small $a > 0$ such that $b \ge v + a$, we would have:

$$\sup\left\{\sum_{i=1}^{n} \bar{x}_i u_i : u \in Q_{v+a}\right\} < b$$

which would imply $Q_{v+a} \cap C = \emptyset$, contradicting the definition of v.

Next, since $Q_v \cap C \neq \emptyset$, let $\bar{w} = \sum_{j=1}^{m} \bar{y}_j p_j$ (as C is convex) for some $\bar{y} = (\bar{y}_1, \dots, \bar{y}_m) \in \Sigma_m$. Since $\bar{w} \in Q_v$, we have $\bar{w}_i \le v$ for all i.

We now show that $\bar{x}$ is optimal for Player 1, $\bar{y}$ is optimal for Player 2, and v is the value of the game.

For Player 1, since $(\bar{x}, w) \ge v$ for every $w \in C$ by the separation result, and since each column $p_{\cdot j} \in C$, we have:

$$(\bar{x}, p_{\cdot j}) \ge v, \quad \text{for all } j$$

For Player 2, consider $w = \sum_{j=1}^{m} \bar{y}_j p_j \in Q_v \cap C$ as before. Then, $w_i = \bar{y} p_{i\cdot}$, and since $w \in Q_v$, it follows that $w_i \le v$ for every i. Hence, we have:

$$v \ge w_i = \bar{y} p_{i\cdot}$$

□

Von Neumann's theorem guarantees that even when a finite zero-sum game has no solutions in pure strategies, the following holds:

- For Player 1, there exists a mixed strategy, represented as a probability distribution $\mathbf{x} = \begin{pmatrix} x_1 & \dots & x_n \end{pmatrix}$, over her pure strategies. For every column j:

$$(x, p_{\cdot j}) = \sum_{i=1}^{n} x_i p_{ij} = x_1 p_{1j} + x_2 p_{2j} + \cdots + x_n p_{nj} \ge v$$

- For Player 2, there exists a mixed strategy, represented as a probability distribution $\mathbf{y} = \begin{pmatrix} y_1 & \dots & y_n \end{pmatrix}$, over her pure strategies. For every row i:

$$(y, p_{i\cdot}) = \sum_{j=1}^{n} y_j p_{ij} = y_j p_{i1} + y_2 p_{i2} + \cdots + y_m p_{im} \le v$$

The constant v is the value of the game under mixed strategies. Player 1 aims to maximize v, while Player 2 seeks to minimize it.

3.3 Optimality

Let X and Y be arbitrary sets. Suppose:

1. $v_1 = v_2 := v$.
2. There exists a strategy $\bar{x}$ such that $f(\bar{x}, y) \geq v$ for all $y \in Y$.
3. There exists a strategy $\bar{y}$ such that $f(x, \bar{y}) \leq v$ for all $x \in X$.

Then:

- v is the rational outcome of the game.
- $\bar{x}$ is an optimal strategy for Player 1.
- $\bar{y}$ is an optimal strategy for Player 2.

It follows that $\bar{x}$ is optimal for Player 1 since it maximizes $\alpha(x) = \inf_y f(x, y)$, while $\bar{y}$ is optimal for Player 2 since it minimizes $\beta(y) = \sup_x f(x, y)$. The values $\alpha(x)$ and $\beta(y)$ represent the best responses for the players if they knew the opponent's strategy.

3.3.1 Conservative values different or equal

Proposition. Let X and Y be nonempty sets, and let $f : X \times Y \to \mathbb{R}$ be an arbitrary real-valued function. Then:

$$v_1 = \sup_x \inf_y f(x, y) \leq \inf_y \sup_x f(x, y) = v_2$$

Proof. By definition, for all $x \in X$ and $y \in Y$:

$$\inf_y f(x, y) \leq f(x, y) \leq \sup_x f(x, y)$$

Thus, for all x and y, it holds that:

$$\alpha(x) = \inf_y f(x, y) \leq \sup_x f(x, y) = \beta(y)$$

Taking the supremum over x and the infimum over y, we conclude:

$$\sup_x \alpha(x) \leq \inf_y \beta(y)$$

□

As a result, it follows that for any game, $v_1 \leq v_2$.

3.3.2 Conservative values not equal

When the conservative values differ, mixed strategies must be considered. In this case, the strategy spaces for both players are probability distributions:

$$\sum_k = \left\{ x = (x_1, \ldots, x_k) | x_i \geq 0 \text{ and } \sum_{i=1}^{k} x_i = 1 \right\}$$

Here, $k = n$ for Player 1 and $k = m$ for Player 2. The utility function is extended to:

$$f(x, y) = \sum_{i=1,\ldots,n, j=i,\ldots,m} x_i y_j p_{ij} = (x, Py)$$

Thus, the mixed extension of the original game is given by:

$$\left(\sum_n \sum_m f(x, y) = (x, Py) \right)$$

3.3.3 Pure strategies optimality

Theorem 3.3.1. *If a player knows the strategy being used by the opposing player, they can always adopt a pure strategy to achieve the best possible outcome.*

This means that once one player's choice is fixed, the optimization problem reduces to a linear problem over a simplex, given that the utility function in such a game is bilinear.

Proof. Consider Player 2, who knows that Player 1 is using a mixed strategy $\bar{x}$. Player 2's task is then to minimize the function:

$$f(\bar{x}, y) = (\bar{x}, Py)$$

over the simplex $\sum_m$ (the set of mixed strategies for Player 2). The optimal value will be attained at one of the vertices e_j of the simplex, which corresponds to a pure strategy. Thus, Player 2 can use a pure strategy to achieve the optimal outcome. $\square$

Given a payoff matrix P, let the column vector corresponding to the j-th pure strategy be denoted as $p_{\cdot j}$, and the row vector corresponding to the i-th pure strategy as $p_{i\cdot}$, respectively. The payoff of the first player in the mixed extension of the game is given by:

$$f(x, y) = (x, Py)$$

The previous theorem implies that, to verify the existence of a rational outcome for the game, we need to show the existence of mixed strategies $\bar{x}$ and $\bar{y}$, as well as a value v, such that:

- $(\bar{x}, P_{e_j}) = (\bar{x}, p_{\cdot j})$ for every column j.
- $(e_i, p_{i\cdot}\bar{y}) \leq v$ for every row i.

Here, e_j is the j-th strategy of Player 2, and e_i is the i-th strategy of Player 1.

3.3.4 General case optimality

Von Neumann proof can be efficiently used to find rational outcome of payoff matrices that can be reduced to matrices where ine player has only two strategies. However, in higher dimensions this procedure becomes more complicated, since it is not clear when and where the set Q_t meets C. Therefore, we need to use Linear Programming.

Player one Player 1 must choose a probability distribution $\mathbf{x} = \begin{pmatrix} x_1 & \cdots & x_n \end{pmatrix} \in \sum_n$ in order to maximize v with the following constraints:

$$\begin{cases} (x, p_{\cdot,1}) = x_1 p_{11} + \cdots + x_n p_{n1} \geq v \\ \cdots \\ (x, p_{\cdot,j}) = x_1 p_{1j} + \cdots + x_n p_{nj} \geq v \\ \cdots \\ (x, p_{\cdot,m}) = x_1 p_{1m} + \cdots + x_n p_{nm} \geq v \end{cases}$$

It is a linear maximization problem where we need to find the value v and we do not know the vector $\mathbf{x}$. In matrices, we have:

$$\begin{cases} \min_{\mathbf{x},v} v : \\ P^T \mathbf{x} \geq v \mathbf{1}_m \\ \mathbf{x} \geq 0 \qquad \begin{pmatrix} 1 & \mathbf{x} \end{pmatrix} = 1 \end{cases}$$

Player two Player 2 must choose a probability distribution $\mathbf{y} = \begin{pmatrix} y_1 & \cdots & y_m \end{pmatrix} \in \sum_m$ in order to maximize w with the following constraints:

$$\begin{cases} (x, p_{1,\cdot}) = x_1 p_{11} + \cdots + x_m p_{1m} \leq w \\ \cdots \\ (x, p_{i,\cdot}) = x_1 p_{i1} + \cdots + x_m p_{im} \leq w \\ \cdots \\ (x, p_{n,\cdot}) = x_1 p_{n1} + \cdots + x_m p_{nm} \leq w \end{cases}$$

It is a linear maximization problem where we need to find the value w and we do not know the vector $\mathbf{y}$. In matrices, we have:

$$\begin{cases} \min_{\mathbf{y},w} w : \\ P \mathbf{y} \leq w \mathbf{1}_n \\ \mathbf{y} \geq 0 \qquad \begin{pmatrix} 1 & \mathbf{y} \end{pmatrix} = 1 \end{cases}$$

Here, $\mathbf{1}$ is a vector of right dimensions whose componenst are all 1's. Ideally, the maximum value for v is equal to the minimal value for w, so as to yield the value of the game.

3.4 Equivalent formulation

Consider a zero sum game described by a payoff matrix P. We can assume, without loss of generality, that $p_{ij} > 0$ for all i, j. This implies $v > 0$.

If $\alpha_j = \frac{x_i}{v}$, then $\sum x_i = 1$ becomes $\sum \alpha_i = \frac{1}{v}$ and maximizing v is equivalent to minimizing $\sum \alpha_i$. Likewise, if we set $\beta_j = \frac{y_j}{v}$ we can do the same. Consider the two problems in duality:

$$\begin{cases} \min(c, \alpha) \\ A\alpha \geq b \\ \alpha \geq 0 \end{cases} \qquad \begin{cases} \max(b, \beta) \\ A^T \beta \leq c \\ \beta \geq 0 \end{cases}$$

Here, $A = P^T$. Denote by v the common value of the two problems. Then we have:

- x is the optimal strategy for Player 1 if and only if $x = v\alpha$ for some α optimal solution of the primal problem.
- y is the optimal strategy for Player 2 if and only if $y = v\beta$ for some β optimal solution of the dual problem.

Consider again the complementarity conditions for the above problems, with x and y being strategies for the two players:

$$\begin{cases} \forall i \bar{x}_i > 0 \implies \sum_{k=1}^{m} p_{ik}\bar{y}_k = c_i \\ \forall i \bar{y}_i > 0 \implies \sum_{k=1}^{n} p_{kj}\bar{x}_k = b_i \end{cases}$$

Since $\bar{y}$ is optimal for Player 2, one has $\sum_{j=1}^{m} p_{ij}\bar{y}_j$ for all i, and hence $x_i > 0$ implies that the row i is optimal for Player 1. And conversely the same holds for Player 2.

3.5 Symmetric games

Definition (*Antisymmetric*)**.** A $n \times n$ matrix P with elements (p_{ij}) is said to be antisymmetric provided that $p_{ij} = -p_{ji}$ for all $i, j = 1, \cdots, n$.

Definition (*Fair game*)**.** A finite zero sum game is fair if the associated matrix is antisymmetric.

In fair games there is no favorite plater: in fact, their role can be exchanged.

Proposition. If $P = (p_{ij})$ is antisymmetric the conservative value $v = 0$ and $\bar{x}$ is an optimal strategy for Player 1 if and only if it is optimal for Player 2.

Proof. Recall that $P^T = -P$ if P is an antisymmetric matrix. Then, since:

$$(Px.x) = (x, P^T x) = -(x, Px) = -(Px, x)$$

one has $f(x, x) = 0$ for all x. This implies $v_1 \leq 0, v_2 \geq 0$ If $\bar{x}$ is optimal for the first player, then:

$$(\bar{x}, Py) \geq 0 \qquad \forall y \in \Sigma_n$$

so that $(P^T\bar{x}, y) \geq 0$, which by the fact that P is antisymmetric becomes:

$$(P\bar{x}, y) \leq 0 \qquad \forall y \in \Sigma_n$$

Therefore $\bar{x}$ is optimal also for the second player, and conversely. □

3.5.1 Optimal strategies in fair games

In order to find optimal strategies in fair games, we need to solve the system of inequalities:

$$\begin{cases} x_1 p_{11} + \cdots + x_n p_{n1} \geq 0 \\ \cdots \\ x_1 p_{1j} + \cdots + x_n p_{nj} \geq 0 \\ \cdots \\ x_1 p_{1n} + \cdots + x_n p_{nn} \geq 0 \end{cases}$$

With extra conditions $x_i \geq 0$ and $\sum_{i=1}^{n} x_i = 1$.

CHAPTER 4

Nash model

4.1 Introduction

Definition (*Non cooperative strategic game*)**.** A two player non cooperative game in strategic form is $(X, Y, f : X \times Y \to \mathbb{R}, g : X \times Y \to \mathbb{R})$. Here, X and Y are the strategy sets of the two players, f is the utility function of Player 1, and g is the utility function of Player 2.

4.2 Nash equilibrium

Definition (*Nash equilibrium*)**.** A Nash equilibrium profile for $(X, Y, f : X \times Y \to \mathbb{R}, g : X \times Y \to \mathbb{R})$ is a pair $(\bar{x}, \bar{y}) \in X \times Y$ such that $f(\bar{x}, \bar{y}) \geq f(x, \bar{y})$ for all $x \in X$ and $f(\bar{x}, \bar{y}) \geq f(\bar{x}, y)$ for all $y \in Y$.

A Nash equilibrium profile is a joint combination of strategies which is stable with respect to unilateral deviations of any individual player. At equilibrium, neither player can improve their utilities by changing strategy. In fact, it is not even convenient for the players to change, given that each one takes for granted that the other one will play the selected strategy.

The main ideas of the Nash model can be seen with two player: having more players does not add complexity to the concept (except for the notation). Let us consider a n-player game with strategy sets X_i for each player and payoffs $u_i : X \to \mathbb{R}$ with $X = \prod_{i=1}^{n} X_i$. Let $x = (x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_n)$ be a strategic profile x_{-i} denotes the vector $x_{-i} = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$ and write also $x = (x_i, x_{-i})$ to emphasize the role of x_i. Then, $\bar{x} = (\bar{x}_i)_{i=1}^{n}$ is a Nash equilibrium profile if for every i, for every $x \in X_i$:

$$u_i(\bar{x}) \geq u_i(x, \bar{x}_{-i})$$

The notion of Nash equilibrium provides a new definition of rationality. We have to see the connection with dominant strategies, backward induction, and optimal strategies in zero sum games.

4.2.1 Dominant strategies

Suppose $\bar{x}$ is a weakly dominant strategy for Player 1:

$$f(\bar{x}, y) \geq f(x, y) \qquad \forall x, y$$

If $\bar{y}$ maximizes the function $y \rightarrowtail g(\bar{x}, y)$ for Player 2, then $(\bar{x}, \bar{y})$ is a Nash equilibrium profile. In fact, for $\bar{x}$ weakly dominant it is true, in particular, that $f(\bar{x}, \bar{y}) \geq f(x, \bar{y})$ for all $x \in X$, thereby satisfying Nash condition on the utility function f of player 1. Then, the maximization requirement that $\bar{y}$ be such that $g(\bar{x}, \bar{y}) \geq g(\bar{x}, y)$ for all $y \in Y$ naturally satisfies the Nash condition on the utility function g of Player 2.

Non uniqueness Let us suppose $\bar{y}$ maximizes the function $y \rightarrowtail g(\bar{x}, y)$:

- If $\bar{x}$ is a weakly dominant strategy for Player 1, then other Nash equilibria beyond $(\bar{x}, \bar{y})$ can exists.
- If $\bar{x}$ is a strictly dominant strategy for Player 1, then no other Nash equilibria exists different from the above ones.

Proof. Assume that there us another Nash equilibrium (x_i, y_i) different than $(\bar{x}, \bar{y})$. By definition, it implies the fact that $f(x_i, y_i) \geq f(\bar{x}, y_j)$. However:

- If $\bar{x}$ is weakly dominant, since $f(\bar{x}, y) \geq f(x, y)$ for all x and all y it follows in particular that such inequality holds for y_j, that is $f(\bar{x}, y_i) \geq f(x_i, y_i)$, which is consistent with the above fact. Hence, (x_i, y_i) can be a Nash equilibrium.
- If $\bar{x}$ is strictly dominant, since $f(\bar{x}, y) > f(x, y)$ for all x and all y it follows in particular that such inequality holds for y_j, that is $f(\bar{x}, y_i) > f(x_i, y_i)$, but that is in contradiction with the above fact. Hence, no pair (x_i, y_i) other than $(\bar{x}, \bar{y})$ can be a Nash equilibrium.

$\square$

4.2.2 Backward induction

Backward induction provides a Nash equilibrium for a game of perfect information, since players systematically make an optimal choice in every part of the tree of the game. It is possible that in games of perfect information there are more equilibria than the ones provided by backward induction.

4.2.3 Zero sum games

Theorem 4.2.1. *Let X, Y be nonempty sets and $f : X \times Y \to \mathbb{R}$ a function (so that in zero sum games $g(x, y) = -f(x, y)$). Then, the following are equivalent:*

1. *The pair $(\bar{x}, \bar{y})$ fulfills:*

$$f(x, \bar{y}) \geq f(\bar{x}, \bar{y}) \geq f(\bar{x}, y) \qquad \forall x, y$$

2. *The following conditions are satisfied:*

$$\begin{aligned}
&\inf_y \sup_x f(x, y) = \sup_x \inf_y f(x, y) \\
&\inf_y f(\bar{x}, y) = \sup_x \inf_y f(x, y) \\
&\sup_x f(\bar{x}, y) = \inf_y \sup_x f(x, y)
\end{aligned}$$

By the first condition, the equilibrium $(\bar{x}, \bar{y})$ yields conservative values for both players. By the second condition, the conservative value agree; moreover the players must solve independent problems: $\bar{x}$ maximizes $f(\cdot, y)$ and $\bar{y}$ minimizes $f(x, \cdot)$.

Proof. Starting from the first we have:

$$v_2 = \inf_y \sup_x f(x,y) \le \sup_x f(x,\bar{y}) = f(\bar{x},\bar{y}) = \inf_y f(\bar{x},y) \le \sup_x \inf_y f(x,y) = v_1$$

Since $v_1 \le v_2$ always holds, all above inequalities are equalities.

Suppose now that the second condition holds, we have that:

$$\inf_y \sup_x f(x,y) = \sup_x f(x,\bar{y}) \ge f(\bar{x},\bar{y}) \ge \inf_y f(\bar{x},y) = \sup_x \inf_y f(x,y)$$

Because of the first consequence of the second condition, all inequalities are equalities. □

As a consequence, given a general zero sum game $X, Y, f : X \times Y \to \mathbb{R}$:

- Any Nash equilibrium $(\bar{x}, \bar{y})$ provides optimal strategies for the players; moreover $f(\bar{x}, \bar{y}) = v$ is the value of the game.
- Any pair of optimal strategies $\bar{x}$ for the Player 1 ant $\bar{y}$ for Player 2 are such that $(\bar{x}, \bar{y})$ is a Nash equilibrium profile of the game and $f(\bar{x}, \bar{y}) = v$.

4.3 Nash equilibrium existence

$(\bar{x}, \bar{y})$ is a Nash equilibrium for the game if and only if

$$(\bar{x}, \bar{y}) \in BR(\bar{x}, \bar{y})$$

Thus, the existence of a Nash equilibrium can be proved by using a fixed point.

Theorem 4.3.1. *Let Z be a compact convex subset of an Euclidean space, let $F : Z \to 2^Z$ be such that $F(z)$ is a nonempty closed convex set for all z. Suppose also F has a closed graph. Then, F has a fixed point: there is $\bar{z} \in Z$ such that $\bar{z} \in F(\bar{z})$*

Theorem 4.3.2 (Nash theorem)**.** *Given the game $(X, Y, f : X \times Y \to \mathbb{R}, g : X \times Y \to \mathbb{R})$, suppose:*

- *X and Y are compact convex subsets of some Euclidean space.*
- *f, g continuous.*
- *$x \mapsto f(x, y)$ is quasi concave for all $y \in Y$.*
- *$y \mapsto g(x, y)$ is quasi concave for all $x \in X$.*

Then, the game has an equilibrium.

Proof. $BR_1(y)$ and $BR_2(x)$ are nonempty (X and Y are compact), closed (f and g are continuous), and convex valued (f and g are quasi concave).

BR has closed graph: suppose $(u_n, v_n) \in BR(x_n, y_n)$ for all n and $(u_n, v_n) \to (u, v)$, $(x_n, y_n) \to (x, y)$. We want to prove that $(u, v) \in BR(x, y)$. We have:

$$f(u_n, y_n) \geq f(z, y_n) \qquad g(x_n, v_n) \geq g(x_n, t) \qquad \forall z \in X, t \in Y$$

Taking limits:

$$f(u, y) \geq f(z, y) \qquad g(x, v) \geq g(x, t) \qquad \forall z \in X, t \in Y$$

□

Corollary 4.3.2.1. *A finite game with utilities functions (A, B) always admits a Nash equilibrium profile in mixed strategies.*

In this case X and Y are simplexes, thus the assumptions of the theorem are fulfilled.

4.4 Nash equilibrium search

To find the Nash equilibrium we can use a brute force algorithm:

1. Guess the support of the equilibria $\text{spt}(\bar{x})$ and $\text{spt}(\bar{y})$.
2. Ignore the sub-optimal strategies and find x, y, u, w by solving the linear system, of $n + m + 2$ equations:

$$\begin{cases} \sum_{i=1}^{n} x_i = 1 \\ \sum_{j=1}^{m} a_{ij} y_j = v \quad \forall i \in \text{spt}(\bar{x}) \\ x_i = 0 \qquad \forall i \notin \text{spt}(\bar{x}) \end{cases} \qquad \begin{cases} \sum_{i=1}^{m} x_i = 1 \\ \sum_{j=1}^{n} b_{ij} x_i = w \quad \forall j \in \text{spt}(\bar{y}) \\ y_i = 0 \qquad \forall j \notin \text{spt}(\bar{x}) \end{cases}$$

3. Check wether the ignored inequalities are satisfied. If $x_i \geq 0$, $y_j \geq 0$, $\sum_{j=1}^{m} a_{ij} y_j \leq v$ and $\sum_{i=1}^{n} b_{ij} x_i \leq w$ then stop since we have found a mixed equilibrium profile. Otherwise, go back to step 1 and try another guess of the support.

Enumerating all the possible supports in the brute force algorithm quickly becomes computationally prohibitive; there are potentially $(2^n - 1)(2^m - 1)$ options. For $n \times n$ games the number of combinations grows very quickly.

Lemke-Howson proposed a more efficient algorithm, though still with exponential running time in the worst case.

4.5 Potential games

Consider a finite game with strategy sets X_i and suppose that all the players have the same payoff $p : Z \to \mathbb{R}$, that is for all i, the utility function are:

$$u_i(x_1, \ldots, x_n) = p(x_1, \ldots, x_n)$$

If $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_n) \in X$ is a strategy profile such that $p(\bar{x}) \geq p(x)$ for all strategy profiles $x \in X$, then $\bar{x}$ is a Nash equilibrium in pure strategies. Note that there might be other Nash equilibria in pure o mixed strategies. However, $\bar{x}$ is the best strategy for all players.

Consider the following payoff-improving procedure:

1. Start from an arbitrary strategy profile $(x_1, \dots, x_n) \in X$.
2. Ask if any player has a better strategy x'_i that strictly increases her payoff. If yes, replace x_i with x'_i and repeat. Otherwise stop, ew have found a pure Nash equilibrium profile.

Each iteration strictly increases the value $p(x)$, so that no strategy profile $x \in X$ can be visited twice. Since X is a finite set, the procedure must reach a pure Nash equilibrium after at most $|X|$ steps. Therefore, this procedure guarantees to reach the global minimum $\bar{x}$.

Consider now an arbitrary finite game with payoffs $u_i : X \to \mathbb{R}$. We can add a constant c_i to the payoff of player i:

$$\tilde{u}_i(x_1, \dots, x_n) = u_i(x_1, \dots, x_n) + c_i$$

If, instead, c_i depends only on x_{-i} and not on x_i, the best responses and equilibria remain the same.

Definition. The payoffs $\tilde{u}_i$ and u_i are said diff-equivalent for player i, if the difference:

$$\tilde{u}_i(x_1, \dots, x_n) - u_i(x_1, \dots, x_n) = c_i(x_{-i})$$

does not depend on her decision x_i but only on the strategies of the other players.

Theorem 4.5.1. *Finite games with diff-equivalent payoffs have the same pure Nash equilibria.*

Proof. The best reaction multi-function, for every player i, is the same when considering two diff-equivalent payoffs u_i and $\tilde{u}_i$, no matter how different from each other the latter functions are. □

Definition (*Potential game*)**.** A finite game with strategy set X_i and payoffs $u_i : X \to \mathbb{R}$ is called a potential game, if it is diff-equivalent to a game with common payoffs.

That is, there exists a potential function $p : XX \to \mathbb{R}$ such that for each i, for every $x_{-i} \in X_{-i}$, and all $x'_i, x_i \in X_i$ we have:

$$\Delta u_i(x'_i, x_i, x_{-i}) = \Delta p(x'_i, x_i, x_{-i})$$

here, $\Delta p(x'_i, x_i, x_{-i}) = p(x'_i, x_{-i}) - (x_i, x_{-i})$

Corollary 4.5.1.1. *Every finite potential game has at least one pure Nash equilibrium.*

Corollary 4.5.1.2. *In a finite potential game every best response iteration reaches a pure Nash equilibrium in finitely many steps.*

4.5.1 Potential search

A potential $p : X \to \mathbb{R}$ is characterized by:

$$\Delta u_i(x'_i, x_i, x_{-i}) = \Delta p(x'_i, x_i, x_{-i})$$

Adding a constant to $p(\cdot)$ provides a new potential. Now, the potential $p(\cdot)$ is determined uniquely:

$$p(x_1, \dots, x_n) = \sum_{i=1}^{n} [u_i(\bar{x}_1, \dots \bar{x}_{i-1}, x_i, \dots, x_n) u_i(\bar{x}_1, \dots \bar{x}_{i-1}, \bar{x}_i, \dots, x_n)]$$

Existence If a game admits a potential the sum on the right hand side of the previous equation is independent of the particular order used. The converse is also true. However, checking that all these orders yield the same answer is impractical for more than two or three players.

4.6 Cost and efficiency

Definition (*Pareto efficient*)**.** An equilibrium is Pareto efficient if it is not possible to increase the utility of a player without decreasing the utility of some other player.

Nash equilibria need to be Pareto efficient and in fact they can be bad for all the players. To quantify how bad an outcome is, lut us consider costs, rather than utilities.

The quality of a strategy profile $x = (x_1, \dots, x_n)$ is measured by a social cost function $x \mapsto C(x)$ from X to $\mathbb{R}_+$. The smaller $C(x)$ the better the outcome x.

Definition (*Benchmark*)**.** The benchmark is the minimal value that a benevolent social planner could achieve:

$$\text{opt} = \min_{x \in X} C(x)$$

For $x \in X$ the ration $\frac{C(x)}{\text{opt}}$ measures how far outcome x is from being optimal. So, a large value implies a big loss in terms of social welfare, whereas a quotient close to 1 implies that x is almost as efficient as an optimal solution.

Definition (*Price of anarchy*)**.** Let $NE \subseteq X$ be the set of pure Nash equilibria of a cost game. The price of anarchy is defined as:

$$\text{PoS} = \max_{\bar{x} \in NE} \frac{C(\bar{x})}{\text{opt}}$$

Definition (*Price of stability*)**.** Let $NE \subseteq X$ be the set of pure Nash equilibria of a cost game. The price of stability is defined as:

$$\text{PoS} = \min_{\bar{x} \in NE} \frac{C(\bar{x})}{\text{opt}}$$

Note that $1 \leq \text{PoS} \leq \text{PoA}$:

- $\text{PoA} \leq \alpha$ means that in every possible pure equilibrium the social cost $C(\bar{x})$ is no worse than α opt.
- $\text{PoS} \leq \alpha$ means that there exists some equilibrium with social cost at most α opt.

Proposition. Consider a cost minimization finite potential game with potential $p : X \to \mathbb{R}$, and suppose that there exists $\alpha, \beta \geq 0$ such that:

$$\frac{1}{\alpha} C(x) \leq p(x) \leq \beta C(x) \qquad \forall x \in X$$

Then, $\text{PoS} \leq \alpha\beta$

Proof. Let $\bar{x}$ be a minimum of $p(\cdot)$ so that $\bar{x}$ is a Nash equilibrium. For all $x \in X$:

$$\frac{1}{\alpha}C(\bar{x}) \leq p(\bar{x}) \leq p(x) \leq \beta C(x) \qquad \forall x \in X$$

Since this is true for all x, we can choose the strategy x yielding the optimal outcome opt $= \min_{x \in X} C(x)$, and hence it follows that $C(\bar{x}) \leq \alpha\beta\text{opt}$. □

In case a game deals with utilities rather than costs, one defined:

$$\text{opt} = \max_{x \in X} U(x)$$

Here, $U(x)$ is some fixed utility function.

4.7 Repeated game

When a game is repeated many times, collaboration between players, eben if dominated in the one shot game, can be based on rationality. The common strategy of the Nash equilibria profile has a weakness: it is based on mutual threat of the players, which is not completely credible since by pushing the player who deviates from the agreement the other will also damage himself. In general, the number of Nash equilibria profile in the repetition of the game is very large.

Definition (*Stage game*)**.** A stage game is played with infinite horizon by the players.

We need to define the strategy and the payoff.

4.7.0.1 Strategy

Assume that at each stage τ, the player know which outcome has been selected at stage $\tau - 1$. Thus the strategy for a player is:

$$s = s(\tau), \tau = 0, \ldots$$

Here, for each τ, $s(\tau)$ is a specification of the moves of the stage game, which is in general a function of the past choices of the players.

4.7.0.2 Payoff

In general, it is not possible to sum payoffs obtained at each stage since the sum will be infinite for $\tau = \infty$. There are different possible choices to construct the payoff function. One standard choice is to use a discount factor δ, where $0 < \delta < 1$. So, the utility function becomes:

$$u_i(s,t) = (1-\delta)\sum_{\tau=0}^{\infty} \delta^{\tau} u_i(s(\tau), t(\tau))$$

Here, $u_i(s(\tau), t(\tau))$ is the stage-game payoff of the player i at time τ given strategy profile $(s(\tau), t(\tau))$.

Definition (*Threat value*)**.** For the bi-matrix game (A, B) representing the stage game:

$$\underline{\mathrm{v}}_1 = \min_j \max_i a_{ij} \qquad \underline{\mathrm{v}}_2 = \min_i \max_j b_{ij}$$

Are called threat values of Player 1 and Player 2, respectively.

Note that v_1 and v_2 are not the conservative values of the two players.

Theorem 4.7.1. *For every feasible payoff vector $v = (v_1, v_2) = (a_{\bar{i}\bar{j}}, b_{\bar{i}\bar{j}})$ such that $v_i > \underline{v}_i$ where $i = 1, 2$, there exists $\bar{\delta} < 1$ such that for all $\delta > \bar{\delta}$ there is a Nash equilibrium of the repeated game with discounting factor δ, which yields payoffs v.*

Proof. $v = (v_1, v_2) = (a_{\bar{i}\bar{j}}, b_{\bar{i}\bar{j}})$ such that $v_i > \underline{\text{v}}_i$. Define the following strategy s: play the strategy yielding v at any stage, unless the opponents deviates at time t. In the latter case play the threat strategy form the stage $t+1$ onwards. We need to prove that s provides utility vector v and s is a Nash equilibrium for all δ close to 1.

At time $\tau = t$ player i could gain at most $\max_{i,j} a_{ij}$. Denote by s_t the strategy of deviating at time t. So, if the Player 1 deviates, after t he will gain at most $\underline{\text{v}}_1$. Hence, the payoff is such that:

$$u_1(s_t) \leq (1-\delta)\left(\sum_{\tau=0}^{t-1} \delta^\tau v_1 + \delta^t \max_{i,j} a_{ij} + \sum_{\tau=t+1}^{\infty} \delta^\tau \underline{\text{v}}_1\right) \quad (1-\delta^t)v_1 + (1-\delta)\delta^\tau \max_{i,j} a_{ij} + (\delta^{t+1})\underline{\text{v}}_1$$

Instead with strategy s the payoff is:

$$u_1(s) = (1-\delta)\sum_{\tau=0}^{\infty} \delta^\tau v_1 = v_1$$

Then:

$$u_1(s) = v_1 \geq u_1(s_t) = (1-\delta^t)v_1 + (1-\delta)\delta^\tau \max_{i,j} a_{ij} + (\delta^{t+1})\underline{\text{v}}_1$$

If and only if:

$$\begin{aligned}
(1-\delta^t)v_1 + (1-\delta)\delta^\tau \max_{i,j} a_{ij} + (\delta^{t+1})\underline{\text{v}}_1 &\leq v_1 \\
(1-\delta)\delta^\tau \max_{i,j} a_{ij} + (\delta^{t+1})\underline{\text{v}}_1 &\leq \delta^t v_1 \\
(1-\delta)\max_{i,j} a_{ij} + \delta\underline{\text{v}}_1 &\leq v_1 \\
\delta(\max_{i,j} a_{ij} - \underline{\text{v}}_1) &\geq \max_{i,j} a_{ij} - v_1
\end{aligned}$$

By properly setting $\boldsymbol{\delta}_i = \frac{\max_{i,j} a_{ij} - v_i}{\max_{i,j} a_{ij} - \underline{\text{V}}_i} < 1$ we have:

$$\boldsymbol{\delta} = \max_{i=1,2} \boldsymbol{\delta}_i$$

□

4.7.1 Correlated equilibrium

Given a game (A, B) with n strategies for Player 1 and m strategies for Player 2. Let $I = \{1, \ldots, n\}$, $J = \{1, \ldots, m\}$, and $X = I \times J$.

Definition (*Correlated equilibrium*)**.** A correlated equilibrium is a probability distribution $P = (p_{ij})$ on X such that for all $\bar{i} \in I$:

$$\sum_{j=1}^{m} p_{\bar{i}j} a_{\bar{i}j} \geq \sum_{j=1}^{m} p_{\bar{i}j} a_{ij} \qquad \forall i \in I$$

And for all $\bar{j} \in J$:

$$\sum_{j=1}^{m} p_{i\bar{j}} b_{i\bar{j}} \geq \sum_{j=1}^{m} p_{i\bar{j}} b_{ij} \qquad \forall j \in J$$

4.7.1.1 Existence

The set of correlated equilibria of a finite game is nonempty.

Theorem 4.7.2. *A Nash equilibrium profile generates a correlated equilibrium.*

Given the Nash equilibrium profile $(\bar{x}, \bar{y})$, the probability distribution of the outcome matrix is p, where each element is such that $p_{ij} = \bar{x}_i \bar{y}_j$

Proof. We have to prove that:

$$\sum_{j=1}^{m} \bar{x}_{\bar{i}} \bar{y}_j a_{\bar{i}j} \geq \sum_{j=1}^{m} \bar{x}_{\bar{i}} \bar{y}_j a_{ij} \forall i \in I$$

That is obvious for $\bar{x}_{\bar{i}} = 0$. If $\bar{x}_{\bar{i}} > 0$ we need to show that:

$$\sum_{j=1}^{m} \bar{y}_j a_{\bar{i}j} \geq \sum_{j=1}^{m} \bar{y}_j a_{ij} \qquad \forall i \in I$$

The left (right) hand side is the expected utility of Player 1 is he chooses row $\bar{i}$ (i) given that Player 2 chooses his equilibrium strategy $\bar{y}$. The inequality holds since the pure strategy $\bar{i}$ is played with positive probability, hence $\bar{i}$ must be (one of) the best reaction(s) to $\bar{y}$. □

Theorem 4.7.3. *The set of the correlated equilibria of a finite game is a nonempty convex polytope.*

Proof. Remember that a convex polytope is the smallest convex set containing a finite number of points. The set of the correlated equilibria is the solution set of a system $n^2 + m^2$ linear inequalities called incentive constraints, plus the condition of being a probability distribution. □

Proposition. If a row $\bar{i}$ is strictly dominated, then $P_{\bar{i}j}$ for every j.

Proof. Suppose $\bar{i}$ is strictly dominated by i. Since:

$$\sum_{j=1}^{m} p_{\bar{i}j}(a_{\bar{i}j} - a_{ij}) \geq 0$$

It must be $p_{\bar{i}j}$ for every j. □

The most important conclusion we can draw is that there is essentially a unique rationality paradigm in the whole theory: the idea of best reaction.