Foundation Of Operations Research
Theory

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Abstract

Operations Research is the branch of applied mathematics dealing with quantitative methods to analyze and solve complex real-world decision-making problems.

The course covers some fundamental concepts and methods of Operations Research pertaining to graph optimization, linear programming and integer linear programming.

The emphasis is on optimization models and efficient algorithms with a wide range of important applications in engineering and management.

Contents

CHAPTER 1

Introduction

1.1 Definition

Definition (*Operations Research*)**.** Operations Research is the branch of mathematics in which mathematical models and quantitative methods are used to analyze complex decision-making problems and find near-optimal solutions.

This interdisciplinary discipline bridges the domains of applied mathematics, computer science, economics, and industrial engineering.

1.2 Decision-making problems

Definition (*Decision-making problems*)**.** Decision-making problems entail the challenge of selecting a viable solution from an array of alternatives, guided by one or multiple criteria.

For more intricate decision-making problems, a mathematical modeling approach is employed. These problems can be categorized as follows:

1. *Assignment problem*: given m jobs and m machines, suppose that each job can be executed by any machine and that t_{ij} is the execution time of job J_i one machine M_j. We want to decide which job assign to each machine to minimize the total execution time. Each job must be assigned to exactly one machine, and each machine to exactly one job. The number of feasible solution is equal to $m!$.

2. *Network design*: we want to decide how to connect n cities via a collection of possible links to minimize the total link cost. Given a graph $G = (N, E)$ with a node $i \in N$ for each city and an edge $\{i, j\} \in E$ of cost $c_i j$, select a subset of edges of minimum total cost, guaranteeing that all pairs of nodes are connected. The number of feasible solution is equal to $2^{|E|}$.

3. *Shortest path*: given a direct graph that represents a road network with distances (traveling times) for each arc, determine the shortest path between two points (nodes).

4. *Personnel scheduling*: determine the week schedule for the hospital personnel, to minimize the number of people involved while meeting the daily requirements.

5. *Service management*: determine how many desks to open at a given time of the day so that the average customer waiting time does not exceed a certain value.

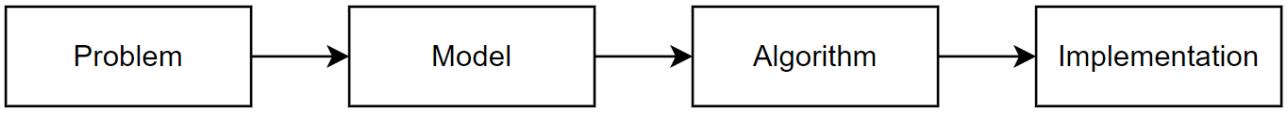
6. *Multi-criteria problem*: decide which laptop to buy considering the price, the weight and the performance.

7. *Maximum clique* (community detection in social networks): determine the complete subgraph of a graph, with the maximum number of vertices.

1.3 History

During World War II, groups of scientists were tasked with conducting research to determine the most effective methods for conducting military operations. In the years following the war, these techniques were declassified and began to be adopted on a broader scale, addressing various challenges in the realms of business, industry, and society. The post-war industrial expansion led to the growth of companies and organizations, which, in turn, gave rise to increasingly intricate decision-making dilemmas. This transformation was facilitated by the rapid advancements in Operations Research, numerical analysis methodologies, and the widespread adoption of computer technology.

1.4 Operations Research workflow



The fundamental stages involved in the examination of an Operations Research problem are as follows:

1. *Problem definition*: the initial step is to precisely articulate and understand the problem at hand.

2. *Model construction*: subsequently, a mathematical or computational model is constructed to encapsulate the problem's complexity.

3. *Algorithm selection or development*: to solve the model, an appropriate algorithm is chosen or developed, tailored to the specifics of the problem.

4. *Implementation*: the selected algorithm is either implemented or utilized within an existing software or program.

Upon completing this process, the results are thoroughly scrutinized, and feedback is integrated.

The resultant model, forged through this procedure, serves as a simplified representation of the real-world problem. To define it effectively, one must discern the fundamental components of the problem and delineate the principal relationships among them.

Example:
A company produces three types of electronic devices: D_1, D_2, D_3, which go through three main phases of the production process: assembly, refinement, and quality control. The time required for each phase and product is as follows:

	D_1	D_2	D_3
Assembly	80	70	120
Refinement	70	90	20
Quality control	40	30	20

The available resources within the planning horizon, in minutes, are:

Assembly	**Refinement**	**Quality control**
30 000	25 000	18 000

The unary product for each product is:

D_1	D_2	D_3
1600	1000	2000

The main assumption is that the company can sell whatever it produces.
The mathematical model that describes this problem is as follows:

- Decision variables: x_j represents the number of devices D_j produced for $j = 1, 2, 3$.
- Objective function: we need to maximize earnings, so we have:

$$\max [1.6x_1 + 1x_2 + 2x_3]$$

- Constraints: the constraints are based on the production limits of each phase:

$$80x_1 + 70x_2 + 120x_3 \leq 30000$$

$$70x_1 + 90x_2 + 20x_3 \leq 25000$$

$$40x_1 + 30x_2 + 20x_3 \leq 18000$$

- Variable type: the variables must be non-negative values, so we have $x_1, x_2, x_3 \geq 0$.

Example:
An insurance company must decide which investments to select out of a given set of possible assets. Here is the information about the investments:

Investments	**Area**	**Capital** (c_j)	**Return** (r_j)
A (automotive)	Germany	150000	11%
B (automotive)	Italy	150000	9%
C (ICT)	USA	60000	13%
D (ICT)	Italy	100000	10%
E (real estate)	Italy	125000	8%
F (real estate)	France	100000	7%
G (treasury bonds)	Italy	50000	3%
H (treasury bonds)	UK	80000	5%

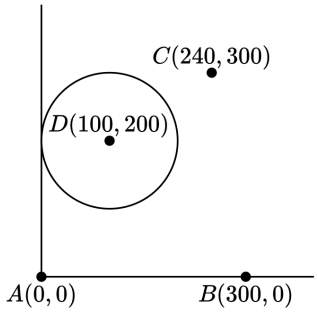
The available capital is 600,000 Euro. The company is required to make at most five different investments, with a maximum of three investments in Italy and a maximum of three abroad.

The mathematical model for this problem is as follows:

- Decision variables: binary variables x_j that indicate whether the j-th investment is selected ($x_j = 1$) or not ($x_j = 0$), for $j = 0, \ldots, 8$.
- Objective function: we need to maximize the expected return, so the objective function is:
$$\max \left[\sum_{j=1}^{8} c_j r_j x_j \right]$$
- Constraints: there is a constraint on the total capital invested, ensuring it doesn't exceed the available capital:
$$\sum_{j=1}^{8} c_j x_j \leq 800$$
There is a constraint on the maximum number of investments, which should be at most five:
$$\sum_{j=1}^{8} x_j \leq 5$$
Constraints on the maximum number of investments in Italy (at most three) and abroad (at most three):
$$x_2 + x_4 + x_5 + x_7 \leq 3$$
$$x_1 + x_3 + x_6 + x_8 \leq 3$$
- Variable type: the variables are binary integer defined as $x_j \in \{0, 1\}\ 1 \leq j \leq 8$.

The variant of the problem introduces an additional constraint. If any of the ICT investments (C or D) is selected, then at least one of the treasury bonds (G or H) must be selected. The constraint is as follows:
$$\frac{x_3 + x_4}{2} \leq x_7 + x_8$$
This constraint ensures that if both ICT investments are selected, at least one treasury bond must also be selected (not both).

Example:
There are three oil pits located at points $A = (0, 0)$, $B = (300, 0)$, and $C = (240, 300)$, and the goal is to connect them to a refinery with pipelines.



The cost of the pipelines is proportional to the square of their length. The refinery must be at least 100 km away from point $D = (100, 200)$. The objective is to find the location for the refinery that minimizes the total pipeline cost.

The mathematical model for this problem can be defined as follows:

- Decision variables: the coordinates of the refinery, denoted as x_1, x_2.
- Objective function: we need to minimize the total pipeline cost, which is the sum of the squared distances between the oil pits and the refinery:

$$\begin{aligned} \min z = & \left[(x_1 - 0)^2 + (x_2 - 0)^2\right] \\ & + \left[(x_1 - 300)^2 + (x_2 - 0)^2\right] \\ & + \left[(x_1 - 240)^2 + (x_2 - 300)^2\right] \end{aligned}$$

- Constraints: there is a constraint that ensures the refinery is at least 100 km away from point $D = (100, 200)$. The constraint can be represented as:

$$\sqrt{(x_1 - 100)^2 + (x_2 - 100)^2} \geq 100$$

- Variable type: $x_1, x_2 \in \mathbb{R}$

1.5 Mathematical programming problem

	Decisions	**Objective**	**Uncertainty**
Mathematical programming	single	one	no
Multi-objective programming	single	multiple	no
Stochastic programming	-	-	yes
Game theory	multiple	-	no

We will delve into the realm of mathematical programming problems, where the primary aim typically revolves around minimizing or maximizing a specified function. Notably, it's worth mentioning that the maximization of a function $f(x)$ is essentially equivalent to the minimization of $-f(x)$. These problems are defined by the following characteristics:

- *Decision variables* $x \in \mathbb{R}^n$: these are numerical variables that serve as identifiers for potential solutions.
- *Feasible region* $X \subseteq \mathbb{R}^n$: this is the set of admissible values for the decision variables.
- *Objective function* $f : X \to \mathbb{R}$: this function quantitatively expresses the value of each feasible solution.

The core objective in solving a mathematical programming problem is to uncover a feasible solution that is globally optimal. In some cases, the problem may prove to be infeasible, unbounded, possess a unique optimal solution, or offer a multitude of optimal solutions. When dealing with particularly challenging problems, it may be necessary to settle for a feasible solution that represents a local optimum.

Mathematical programming can be categorized into the following classes:

1. *Linear Programming.*
2. *Integer Linear Programming.*
3. *Nonlinear Programming.*

Multi-objective programming

Multi-objective programming can be approached in various ways. Suppose our goal is to minimize $f_1(x)$ while simultaneously maximizing $f_2(x)$. In this context, we can:

1. Combine the objectives into a single objective problem by representing both objectives in the same units. This involves minimizing a weighted combination of the objectives, as follows:
$$\min \lambda_1 f_1(x) - \lambda_2 f_2(x)$$
with appropriate scalar values λ_1 and λ_2.
2. Prioritize the primary objective and transform the other objective into a constraint. In this approach, the focus is on optimizing the primary objective function while ensuring that the secondary objective satisfies a specified constraint. This is achieved as follows:
$$\max_{x \in X} f_2(x) \quad f_1(x) \leq \varepsilon$$
with an appropriate constant value ε.

CHAPTER 2

Algorithms

2.1 Complexity

Definition (*Algorithm*)**.** An algorithm is a step-by-step sequence of instructions designed to solve any given instance of a problem.

Definition (*Instance*)**.** An instance, denoted as I, pertaining to a problem P, is a specific and unique case derived from the problem P.

The runtime of an algorithm is contingent on both the specific instance and the computer it runs on. To assess the algorithm's complexity while abstracting from hardware variations, we focus on evaluating its performance as a function of the instance's size, independently of the underlying hardware. For this purpose, we consider the count of elementary operations, presuming that each operation carries an equivalent cost. Since determining the precise count of elementary operations is often a formidable task, we resort to considering the asymptotic count of elementary operations in the worst-case scenario.

Definition (*Order of a function*)**.** A function f is denoted as being order of g and expressed as:

$$f(n) = O(g(n))$$

If there exists a positive constant c such that $f(n)$ is less than or equal to $c \cdot g(n)$ for sufficiently large values of n.

We can categorize algorithms into two primary classes based on their worst-case complexity:

- *Polynomial*: characterized by a complexity of $O(n^d)$, where d is a constant.
- *Exponential*: demonstrating a complexity of $O(2^n)$.

Algorithms with a high-order polynomial complexity are generally considered inefficient in practical applications.

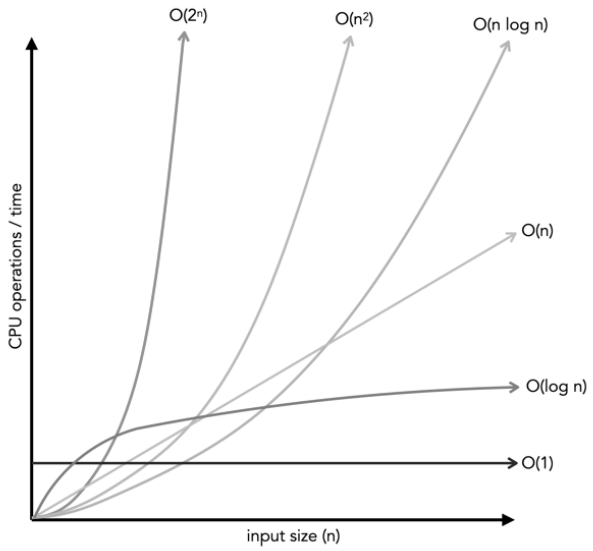


Figure 2.1: Plot of various algorithm's complexity

Definition (*Instance*)**.** The size of an instance denoted as $|I|$ represents the number of bits required to represent that instance.

Definition (*Polynomially solvable*)**.** A problem P is considered polynomially solvable if there exists a polynomial time algorithm that can provide an optimal solution for any given instance.

For many discrete optimization problems, the most efficient algorithm available today demands a number of elementary operations that, in the worst-case scenario, grows exponentially with the size of the instance.

Definition ($\mathcal{NP}$*-hard*)**.** $\mathcal{NP}$-hard computational problems are, at the very least, as challenging as a broad spectrum of exceptionally difficult problems for which no polynomial time algorithm has been identified to date.

The $\mathcal{NP}$-hardness of a problem is a very strong evidence that is inherently difficult. However, this doesn't imply that it cannot potentially be solved using a polynomial time algorithm.

2.2 Definitions

Definition (*Exact algorithm*)**.** An algorithm is considered exact when it is capable of delivering an optimal solution for every single instance.

Definition (*Heuristic algorithm*)**.** In contrast, an algorithm is deemed heuristic when it is not guaranteed to provide an optimal solution for all instances.

Definition (*Greedy algorithm*)**.** A greedy algorithm progressively builds a feasible solution by consistently making locally optimal choices at each step, without revisiting or reevaluating prior selections.

2.3 Dynamic programming

Dynamic programming, introduced by Richard Bellman in 1953, is a versatile method employed to find optimal solutions consisting of a sequence of elementary decisions. This is achieved by resolving a series of recursive equations.

Dynamic programming is well-suited for a wide range of sequential decision problems as long as they adhere to the optimality property.

In contemporary applications, dynamic programming finds utility across various domains, including optimal control, equipment maintenance and replacement, and the selection of inspection points along a production line.

CHAPTER 3

Network optimization models

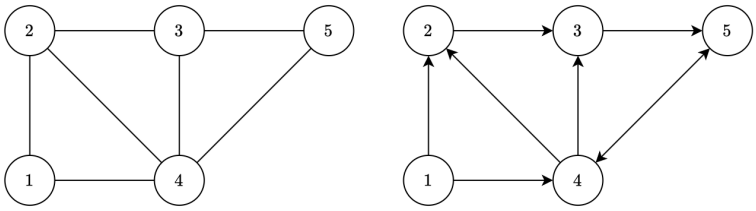
3.1 Introduction

Numerous decision-making problems find their representation in the realm of graphs.

Definition (*Graph*)**.** A graph is described as a pair $G = (N, E)$, comprising a set of nodes denoted as N and a set of edges or arcs E, where these edges link the nodes in pairs.

Definition (*Directed and undirected graphs*)**.** In the case of an undirected graph, an edge connecting nodes i and j is denoted as i, j, while in a directed graph, it is represented as (i, j).

Example:
Consider a road network connecting a total of "n" cities, which can be effectively represented by a graph. In this representation, each city corresponds to a node, and the connections between them are represented as edges.



The left graph is undirected and defined as follows:

- $N = \{1, 2, 3, 4, 5\}$
- $E = \{\{1, 2\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}, \{3, 5\}, \{4, 5\}\}$

The right graph is directed and defined as:

- $N = \{1, 2, 3, 4, 5\}$
- $E' = \{(1, 2), (1, 4), (2, 3), (2, 4), (3, 4), (3, 5), (4, 5)\}$

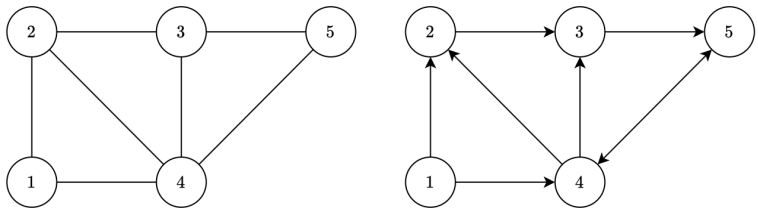
Definition (*Adjacent node*)**.** Nodes are considered adjacent when they are linked by an edge.

Definition (*Incident edge*)**.** An edge labeled as e is deemed incident to a node v when node v serves as one of the endpoints of the edge e.

Definition (*Degree of a node*)**.** In undirected graphs, the degree of a node signifies the count of edges incident to that particular node.

Definition (*In-degree and out-degree of a node*)**.** In directed graphs, the in-degree of a node represents the number of arcs that have it as their successor, while the out-degree of a node denotes the quantity of arcs that have it as their predecessor.

Example:
In the undirected graph, we observe that nodes 1 and 2 are adjacent, while node 1 and node 3 are not. The edge $\{1, 2\}$ is incident to nodes 1 and 2. Node 1 has a degree of 2, and node 4 has a degree of 4. In the directed graph, node 1 exhibits an in-degree of 0 and an out-degree of 2.



Definition (*Directed path*)**.** A directed path from $i \in N$ to $j \in N$ is a sequence $p = \langle\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{k-1}, v_k\}\rangle$ connecting nodes v_1 and v_k.

Definition (*Connected nodes*)**.** Nodes u and v are connected if there is a path connecting them. A graph (N, E) is connected if u, v are connected for any $u, v \in N$.

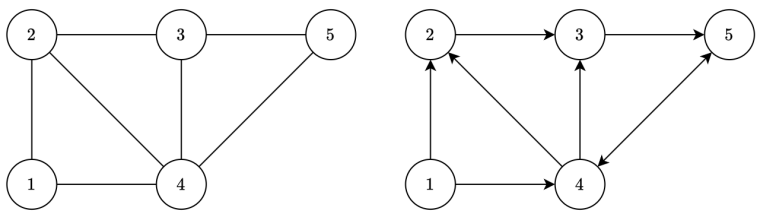
Definition (*Strongly connected nodes*)**.** A graph is *strongly connected* if u and v are connected by a directed path for any $u, v \in N$.

Definition (*Cycle*)**.** A cycle or circuit is a path with $v_1 = v_k$.

Example:
The undirected graph has a path $\langle\{2, 3\}, \{3, 4\}, \{4, 5\}\rangle$ connecting node 2 to node 5, thus indicating a connection between these nodes.

The directed graph has a directed path $\langle(3, 5), (5, 4), (4, 2), (2, 3), (3, 4)\rangle$ from node 3 to node 4. So we say those nodes are not strongly connected.

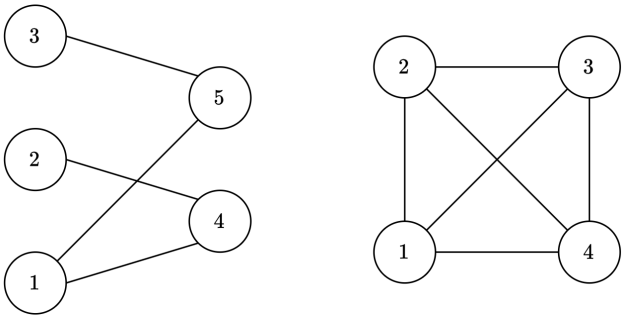
In the undirected graph, we observe a cycle $\langle\{2, 3\}, \{3, 5\}, \{5, 4\}, \{4, 2\}\rangle$. In the directed graph, a circuit is present, specifically $\langle(2, 3), (3, 4), (4, 2)\rangle$.



Definition (*Bipartite graph*)**.** A graph is bipartite if there is a partition $N = N_1 \cup N_2$ with $N_1 \cap N_1 = \varnothing$ such that no edge connects nodes in the same subset.

Definition (*Complete graph*)**.** A graph is complete if $E = \{\{v_i, v_j\} | v_i, v_j \in N \wedge i \leq j\}$.

Example:
The graphic on the left is bipartite because we can find two subsets of nodes such that $N = N_1 \cup N_2$ with $N_1 \cap N_1 = \varnothing$ that are: $N_1 = \{1, 2, 3\}$ and $N_2 = \{4, 5\}$. The graph on the right is a complete graph because all the nodes are connected with each other.



Definition (*Outgoing cut*)**.** Given a directed graph $G = (N, A)$ and $S \subset NM$, the *outgoing cut* induced by S is:

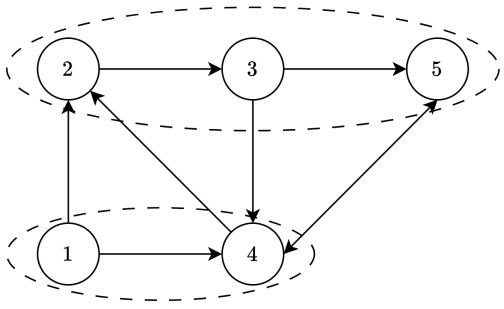
$$\delta^+(S) = \{(u, v) \in A | u \in S \wedge v \in N - S\}$$

Definition (*Incoming cut*)**.** Given a directed graph $G = (N, A)$ and $S \subset NM$, the *incoming cut* induced by S is:

$$\delta^-(S) = \{(u, v) \in A | v \in S \wedge u \in N - S\}$$

Example:
In the graph presented below, we can observe that:

- $\delta^+(\{1, 4\}) = \{(1, 2), (4, 2), (4, 5)\}$
- $\delta^-(\{1, 4\}) = \{(3, 4), (5, 4)\}$



An undirected graph with n nodes has at most $m = \dfrac{n(n-1)}{2}$ arcs. A directed graph with n nodes has at most $m = n(n-1)$ arcs.

Definition (*Dense graph*)**.** For a given graph with m representing the number of arcs or edges and n representing the number of nodes, we classify a graph as dense when:

$$m \approx n^2$$

Definition (*Sparse graph*)**.** For a given graph with m representing the number of arcs or edges and n representing the number of nodes, we classify a graph as sparse when:

$$m \ll n^2$$

Graph representation The most suitable method for representing a dense graph is by utilizing an $n \times n$ adjacency matrix, defined as follows:

$$\begin{cases} a_{ij} = 1 & \text{if } (i,j) \in A \\ a_{ij} = 0 & \text{otherwise} \end{cases}$$

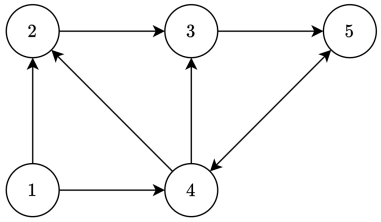
Conversely, for a sparse graph, the most effective approach is to use lists of successors for each node.

Example:
The adjacency matrix for the depicted graph is as follows:

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Furthermore, the list of successors is presented as follows:

$$S(1) = \{2,4\} \quad S(2) = \{3\} \quad S(3) = \{4,5\} \quad S(4) = \{2,5\} \quad S(5) = \{4\}$$



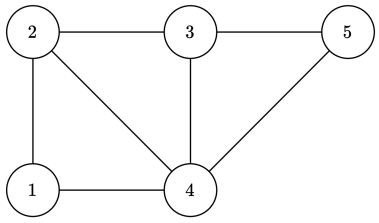
Definition (*Sub-graph*)**.** $G' = (N', E')$ is a sub-graph of $G = (N, E)$ if $N' \subseteq N$ and $E' \subseteq E$.

Definition (*Tree*)**.** A tree $G_T = (N', T)$ *of* G is a connected and acyclic sub-graph of G.

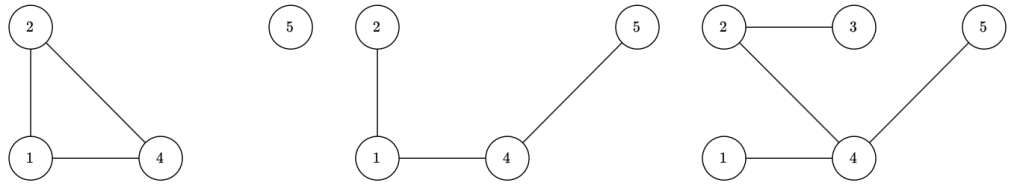
Definition (*Spanning tree*)**.** $G_T = (N', T)$ is a spanning tree of G if it contains all nodes in G.

Definition (*Leaves*)**.** The leaves of a tree are the nodes of degree one.

Example:
Considering the provided graph:



We can identify three different structures: a sub-graph, a tree, and a spanning tree. These structures are visually depicted in the modified graph below:



Property 3.1.1. Every tree with n nodes has $n - 1$ edges.

Proof. We will demonstrate this property with a proof by induction. For the base case we have that the claim holds for $n = 1$. For the inductive step we have to show that, if this is true for trees with n nodes, then it is also true for those with $n + 1$ nodes. Let T_1 be a tree with $n + 1$ nodes and recall that any tree with $n \geq 2$ nodes has at least two leaves. By deleting one of the leaf and its incident edge we obtain a tree T_2 with n nodes. By induction hypothesis, T_2 has $n - 1$ edges. Therefore, the tree T_1 has $n - 1 + 1 = n$ edges. □

Property 3.1.2. Any pair of nodes in a tree is connected via a unique path.

Property 3.1.3. By adding a new edge to a tree, we create a unique cycle.

Property 3.1.4. Let $G_T = (N, T)$ be a spanning tree of $G = (N, E)$. Consider an edge $e \notin T$ and the unique cycle C of $T \cup \{e\}$. For each edge $f \in C - \{e\}$, the sub-graph $T \cup \{e\} - \{f\}$ is also a spanning tree of G.

3.2 Graph reachability problem

In a directed graph $G = (N, A)$, given a specific node s, find all the nodes that can be reached starting from node s.

Algorithm 1 Graph reachability problem

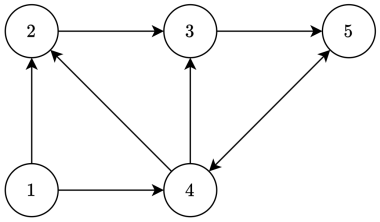
```
1: Q ← {s}
2: M ← {∅}
3: while Q ≠ 0 do
4:     u ← node in Q
5:     Q ← Q − {u}
6:     M ← M ∪ {u}
7:     for (u, v) ∈ δ+(u) do
8:         if v ∉ M and v ∉ Q then
9:             Q ← Q ∪ {v}
10:        end if
11:    end for
12: end while
```

The algorithm's worst-case time complexity is $O(n^2)$.

Example:
Given the graph and the starting node $s = 2$, the algorithm proceeds through the following steps:

1. $Q = \{2\}$ $M = \varnothing$
2. $Q = \{3\}$ $M = \{2\}$
3. $Q = \{4, 5\}$ $M = \{2, 3\}$
4. $Q = \{5\}$ $M = \{2, 3, 4\}$
5. $Q = \varnothing$ $M = \{2, 3, 4, 5\}$

As a result, nodes $\{2, 3, 4, 5\}$ are determined to be reachable from node two.



3.3 Minimum spanning tree problem

Given an undirected graph $G = (N, E)$ and a cost function, find a spanning tree $G_T = (N, T)$ of minimum total cost:

$$\min_{T \in X} \sum_{e \in T} c_e$$

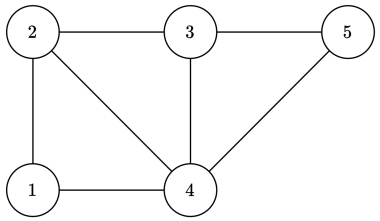
Here, X is the set of all spanning trees of G.

Theorem 3.3.1. *A complete graph with n nodes ($n \geq 1$) has $n^{(n-2)}$ spanning trees.*

To determine the spanning tree with the lowest total cost, we can employ an algorithm that incrementally constructs the spanning tree. The algorithm's fundamental concept is:

1. Start by choosing any node arbitrarily and establish a connection to the nearest distinct node.
2. Locate an unconnected node that is nearest to a connected node, and proceed to connect these two nodes. Continue this process until all nodes are connected.

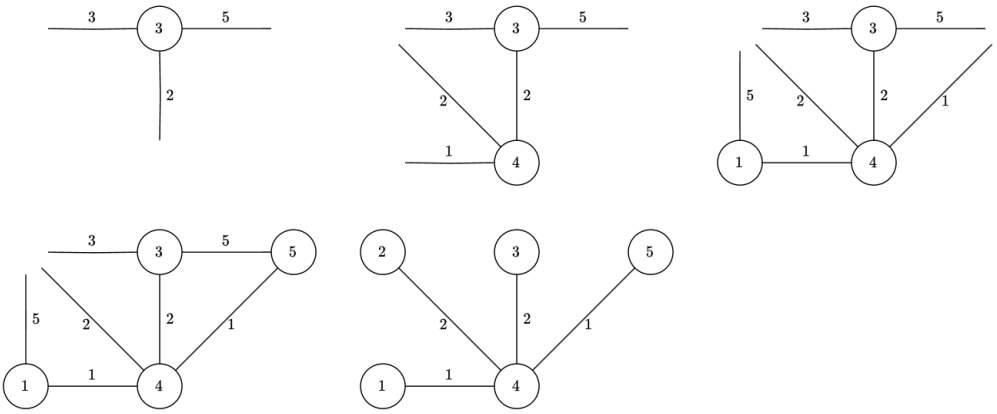
Example:
Let's apply Prim's algorithm to the following graph:



We start by selecting node 3 as our initial node. Therefore, we have two sets: $S = \{3\}$ and $T = \{\varnothing\}$. Next, we follow these steps:

- The edge with the minimum cost connects nodes 3 and 4. Now we have:
  - $S = \{3, 4\}$.
  - $T = \{\{3, 4\}\}$.
- The edge with the minimum cost connects nodes 1 and 4. Now we have:
  - $S = \{1, 3, 4\}$.
  - $T = \{\{3, 4\}, \{1, 4\}\}$.
- The edge with the minimum cost connects nodes 4 and 5. Now we have:
  - $S = \{1, 3, 4, 5\}$.
  - $T = \{\{3, 4\}, \{1, 4\}, \{4, 5\}\}$.
- The edge with the minimum cost connects nodes 4 and 5. Now we have:
  - $S = N$.
  - $T = \{\{3, 4\}, \{1, 4\}, \{4, 5\}, \{2, 4\}\}$.

In this case, the total cost is equal to $c(T) = 6$. Graphically, the resulting minimum spanning tree is as shown below:



Algorithm 2 Prim's algorithm for the minimum cost spanning tree problem

1: $S \leftarrow \{u\}$
2: $T \leftarrow \{\varnothing\}$
3: **while** $|T| < n - 1$ **do**
4: $\quad \{u, v\} \leftarrow$ edge in $\delta(S)$ of minimum cost
5: $\quad S \leftarrow S \cup \{v\}$
6: $\quad T \leftarrow T \cup \{u, v\}$
7: **end while**

Where $u \in S$ and $v \in N - S$. The worst-case complexity is $O(n^2)$.

Proposition. Prim's algorithm is exact.

The exactness does not depend on the choice of the first node nor on the selected edge of minimum cost in $\delta(S)$.

Property 3.3.1. Let F be a partial tree contained in an optimal tree of G. Consider $e\{u, v\} \in \delta(S)$ of minimum cost, then there exists a minimum cost spanning tree of G containing e.

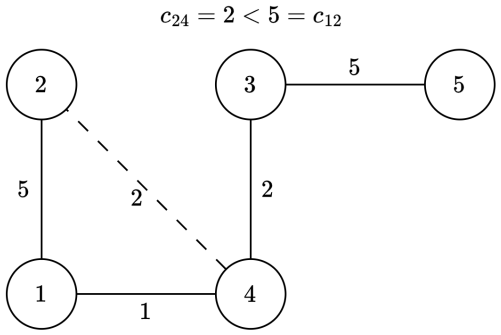
Proof. By contradiction, assume $T \subseteq E$ is a minimum cost spanning tree with $F \subseteq T$ and $e \notin T$. Adding edge e to T creates the cycle C. Let $f \in \delta(S) \cap C$. If $c_e = c_f$, then $T \cup \{e\} - \{f\}$ is also optimal since it has same cost of T. If $c_e < c_f$, then $c(T \cup \{e\} - \{f\}) < C(T)$, hence T^* is not optimal. □

Proposition. Prim's algorithm is greedy.

At each step a minimum cost edge is selected among those in the cut $\delta(S)$ induced by the current set of nodes S.

Definition (*Cost decreasing edge*)**.** Given a spanning tree T, an edge $e \notin T$ is cost decreasing if when e is added to T it creates a cycle C with $C \subseteq T \cup \{e\}$ and $\exists f \in C - \{e\}$ such that $c_e < c_f$.

Example:
In the context of the given graph:



Since we have the equation:

$$c(T \cup \{e\} - \{f\}) = c(T) + c_e - c_f$$

If edge e is cost decreasing, it implies that:

$$c(T \cup \{e\} - \{f\}) < c(T)$$

Theorem 3.3.2. *A tree T is of minimum total cost if and only if no cost decreasing edge exists.*

Proof. We prove that if the tree is of minimum total cost if no cost decreasing edge exists. If a cost-decreasing edge exists, then T is not of minimum total cost. □

Proof. We prove that if no cost decreasing edge exists the tree is of minimum total cost. When there are no cost-decreasing edges, it implies that the spanning tree T has the minimum total cost. Consider a minimum-cost spanning tree denoted as T^*, which is obtained through Prim's algorithm. It is possible to confirm that, by gradually exchanging one edge at a time, T^* can be transformed step by step into T without altering the overall cost. Consequently, we can conclude that T is also an optimal solution. □

The optimality condition provides a means to confirm the optimality of a spanning tree T. To determine if T is optimal, it is sufficient to examine whether each edge e in the set $E - T$ is not a cost-decreasing edge.

3.4 Graph shortest path problem

3.4.1 Dijkstra's algorithm

Given a directed graph $G = (N, A)$ with a cost $c_j \in R$ for each arc $(i, j) \in A$, and two nodes s and t, determine a minimum cost path from s to t.

Each value c_j represents the cost associated with arc $(i, j) \in A$. Here, node s is designated as the source or origin, while node t is specified as the destination or sink.

Definition (*Simple path*)**.** A path is considered simple when it ensures that each node is visited exactly once, without any repetitions.

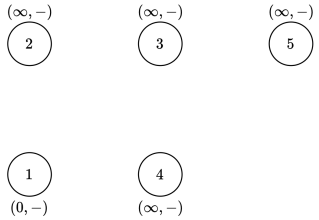
Property 3.4.1. If $c_j \geq 0$ for all $(i, j) \in A$, there is at least one shortest path which is simple.

Dijkstra's algorithm takes as input a graph $G = (N, A)$ with non-negative arc costs and a designated starting node s. The primary purpose of this algorithm is to determine the shortest paths from s to all other nodes within G. The algorithm's core principle involves systematically processing nodes based on their increasing order of the cost of the shortest path from s to each of them. For each node j belonging to N, we assign two labels:

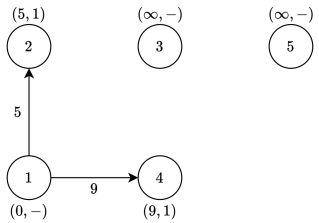
- L_j, which represents the cost of the minimum-cost path from s to j.
- $pred_j$, which identifies the predecessor of j along the shortest path from s to j.

It is important to note that Dijkstra's algorithm follows a greedy approach with respect to path from s to j.

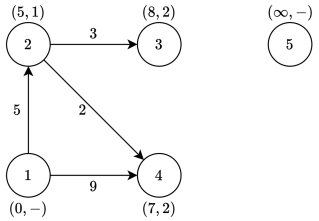
Example:
Let's apply Dijkstra's algorithm to a given graph, starting with node 1 as the initial node. We assign labels as follows:



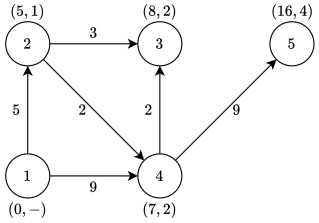
Next, we examine all arcs leading from the starting node to other nodes and update their labels:



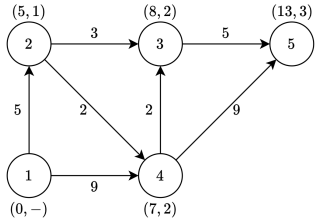
Moving to node 2, we explore reachable nodes. In this case, we find the shortest path from the initial node to 4, so we update the label for 4:



Now, we proceed to node 4, which is the closest node to the starting point. We check all arcs leading to other nodes:



We repeat this process for the remaining nodes, considering them in ascending order of cost from the start:



At this point, we can trace back the shortest path to any node in the graph using the predecessor label. For instance, the shortest path from s to 5 has a cost of 13, and the path is $(1, 2, 3, 5)$.

Proposition. Dijkstra's algorithm is exact.

Proof. At the k-th step: $S = \{s, i_2, \ldots, i_k\}$ and

$$L_j = \begin{cases} \text{cost of a minimum path from } s \text{ to } j,\ j \in S \\ \text{cost of a minimum path with all intermediate nodes in } S,\ j \notin S \end{cases}$$

We prove the proposition by induction on the number k of steps. For the base case it is easy to see that the statement holds for $k = 1$, since

$$S = \{s\},\ L_s = 0,\ L_j = +\infty,\ \forall j \neq S$$

For the inductive step we must prove that, if the statement holds at the k-th step, it must also hold for the $(k+1)$-th step. In the $(k+1)$-th step let $u \notin S$be the node that is inserted in S and $\varnothing$ the path from s to u such that:

$$L_v + c_{uv} \leq L_i + C_{ij},\ \forall (i, j) \in \delta^+(S)$$

Let us verify that every path π from s to u has $c(\pi) \geq c(\varnothing)$. There exist $i \in S$ and $j \notin S$ such that:

$$\pi = \pi_1 \cup \{(i,j)\} \cup \pi_2$$

Where (i,j) is the first arc in $\pi \cap \delta^+(S)$. Moreover:

$$c(\pi) = c(\pi_1) + c_{ij} + c(\pi_2) \geq L_i + c_{ij}$$

Because $c_{ij} \geq 0$, thus, $c(\pi_2) \geq 0$, and by the induction assumption, $c(\pi_1) \geq L_i$. Finally, by the choice of (v,u) we have:

$$L_i + c_{ij} \geq L_v + c_{vu} = c(\varnothing)$$

□

The inputs to the algorithm consist of a graph $G = (N, E)$ with non-negative costs assigned to its arcs, and a designated starting node s, which is a member of N.

Algorithm 3 Dijkstra's algorithm for the graph shortest path problem

```
1: S ← ∅
2: X ← {s}
3: for u ∈ N do
4:     L_u ← +∞
5: end for
6: L_s ← 0
7: while |S| ≠ n do
8:     u ← argmin{L_i | i ∈ X}
9:     X ← X − {u}
10:    S ← S ∪ {u}
11:    for (u, v) ∈ δ⁺(u) such that L_v > L_u + c_uv do
12:        L_v ← L_u + c_uv
13:        pred_v ← u
14:        X ← X ∪ {v}
15:    end for
16: end while
```

The worst-case time complexity of this algorithm is $O(n^3)$. We can observe the following points:

- A collection of the shortest paths from the starting node s to all nodes j can be obtained using the predecessor vector.
- The union of a set of the shortest paths from node s to all the other nodes of G is the shortest path trees rooted at s. These shortest path trees are distinct from minimum-cost spanning trees.
- Dijkstra's algorithm is not applicable when there are arcs with negative costs.

3.4.2 Floyd-Warshall's algorithm

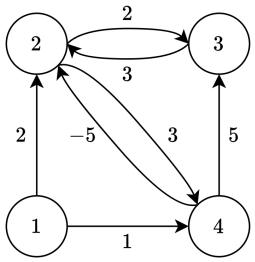
When the graph G contains a negative-cost cycle, the well-defined nature of the shortest path problem is compromised. Such cycles repeatedly lead to a decrease in cost, making it impossible to identify a finite shortest path from node s to t. To address this issue, the Floyd-Warshall algorithm is employed, as it is capable of detecting negative-cost cycles. The Floyd-Warshall algorithm provides a comprehensive set of shortest paths between all pairs of nodes, even in the presence of arcs with negative costs. This algorithm is founded on an iterative process known as a triangular operation. It employs $n \times n$ matrices, D and P, where their elements represent, upon completion of the algorithm:

- d_{ij}, denoting the cost of the shortest path from node i to j.
- p_{ij}, indicating the predecessor of node j in the shortest path from node i to j.

Definition (*Triangular operation*)**.** The triangular operation specifies that for every pair of nodes i and j, where i is not equal to u and j is not equal to u (including the case when i equals j), one should examine whether it is more advantageous to reach j from i via an intermediate node u. This determination is made by checking if the following relationship holds:

$$d_{iu} + d_{uj} < d_{ij}$$

Example:
Consider the following graph:



To initiate the algorithm, we initialize the matrices as follows:

$$D = \begin{bmatrix} 0 & 2 & \infty & 1 \\ \infty & 0 & 3 & 3 \\ \infty & 2 & 0 & \infty \\ \infty & -5 & 5 & 0 \end{bmatrix} \qquad P = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 \\ 4 & 4 & 4 & 4 \end{bmatrix}$$

In the first iteration with $u = 1$ (we focus on the first row and first column), the matrices remain unchanged because the triangular operation is consistently satisfied:

- $0 = d_{22} < d_{21} + d_{12} = \infty + 2$ (no changes).
- $3 = d_{23} < d_{21} + d_{13} = \infty + \infty$ (no changes).
- $3 = d_{24} < d_{21} + d_{14} = \infty + 1$ (no changes).
- $2 = d_{32} < d_{31} + d_{12} = \infty + 2$ (no changes).
- $0 = d_{33} < d_{31} + d_{13} = \infty + \infty$ (no changes).

- $\infty = d_{34} < d_{31} + d_{14} = \infty + 1$ (no changes).
- $-5 = d_{42} < d_{41} + d_{12} = \infty + 2$ (no changes).
- $5 = d_{43} < d_{41} + d_{13} = \infty + \infty$ (no changes).
- $0 = d_{44} < d_{41} + d_{14} = \infty + 1$ (no changes).

In the second iteration with $u = 2$ (focusing on the second row and second column), the matrices are modified, and the algorithm terminates due to the discovery of a negative-cost arc:

- $0 = d_{11} < d_{12} + d_{21} = 2 + \infty$ (no changes).
- $\infty = d_{13} < d_{12} + d_{23} = 2 + 3$ ($p_{13} \leftarrow p_{23}$).
- $1 = d_{14} < d_{12} + d_{24} = 2 + 3$ (no changes).
- $\infty = d_{31} < d_{32} + d_{21} = 2 + \infty$ (no changes).
- $0 = d_{33} < d_{32} + d_{23} = 2 + 3$ (no changes).
- $\infty = d_{34} < d_{32} + d_{24} = 2 + 3$ ($p_{34} \leftarrow p_{24}$).
- $\infty = d_{41} < d_{42} + d_{21} = 5 + \infty$ (no changes).
- $5 = d_{43} < d_{42} + d_{23} = -5 + 3$ ($p_{43} \leftarrow p_{23}$).
- $0 = d_{44} < d_{42} + d_{24} = -5 + 3$ (negative cost circuit found).

Given that the triangular operation is executed for all nodes u and for each pair of nodes i and j in the worst-case scenario, the overall time complexity of the Floyd-Warshall algorithm is $O(n^3)$.

Proposition. Floyd-Warshall's algorithm is exact.

Proof. Let's assume that the nodes of G are numbered from 1 to n. We can verify that when following the node index order, after the u-th cycle, the value of d_{ij} (for any i and j) corresponds to the cost of the shortest path from i to j with only intermediate nodes in the set $1, \ldots, u$. □

The algorithm takes as its inputs a directed graph $G = (N, A)$ and an $n \times n$ cost matrix, $C = [c_{ij}]$.

Algorithm 4 Floyd-Warshall's algorithm

for $i \in N$ **do**
  for $j \in N$ **do**
    $p_{ij} \leftarrow i$
    $d_{ij} \leftarrow \begin{cases} 0 & i = j \\ c_{ij} & i \neq j \wedge (i,j) \in A \\ +\infty & \text{otherwise} \end{cases}$
  end for
end for
for $u \in N$ **do**
  for $i \in N - \{u\}$ **do**
    for $j \in N - \{u\}$ **do**
      if $d_{iu} + d_{uj}\ d_{ij}$ **then**
        $p_{ij} \leftarrow p_{uj}$
        $d_{ij} \leftarrow d_{iu} + d_{uj}$
      end if
    end for
  end for
  for $i \in N$ **do**
    if $d_{ii} < 0$ **then**
      return
    end if
  end for
end for

3.4.3 Topological order algorithm

The topological order algorithm deals with a directed acyclic graph $G = (N, A)$, where each arc $(i, j) \in A$ has an associated cost $c_{ij} \in \mathbb{R}$, Given nodes s and t, the goal is to determine the shortest or longest path from s to t. Directed acyclic graphs possess a crucial property known as topological order, where the nodes in any such graph G can be indexed in a way that for each arc $(i, j) \in A$, we have $i < j$. This topological order property can be leveraged in an efficient dynamic programming algorithm to find shortest or longest paths in directed acyclic graphs. To apply the algorithm to a graph $G = (N, A)$, represented using lists of predecessors $\delta^-(v)$ and successors $\delta^+(v)$ for each node v, follow these steps:

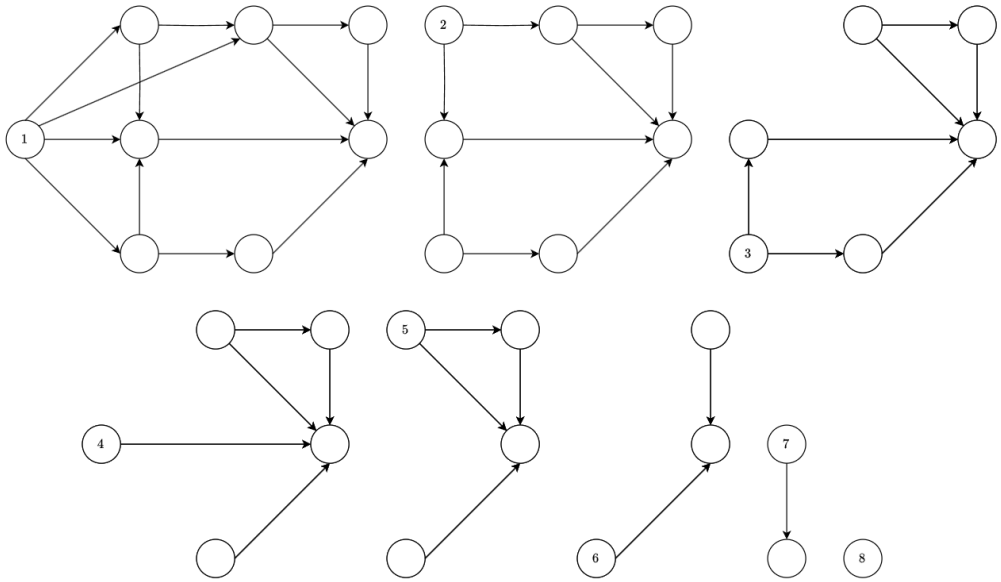
1. Assign the smallest positive integer that has not been assigned to a node $v \in N$ with $\delta^-(v) = \varnothing$.

2. Delete node v with all its incident arcs.

3. Repeat from step one until there are nodes remaining in the current sub-graph.

The complexity of this algorithm is $O(m)$, where m is the cardinality of A (the number of arcs), because each node and arc are considered at most once.

Example:
Here's a graphical illustration of the algorithm:



3.4.4 DAGs' dynamic programming algorithm

Any shortest path from 1 to π_t, with at least two arcs, can be subdivided into two parts: π_t and (i,t), where π_t is the shortest sub-path from s to i. For each node $i = 1, \ldots, t$, let L_i denote the cost of the shortest path from 1 to i. Therefore, we can express L_t as follows:

$$L_t = \min_{(i,t)\in\delta^-(t)} \{L_i + c_{it}\}$$

This minimum is taken over all possible predecessors i of t. If the graph G is directed, acyclic, and topologically ordered, the only possible predecessors of t in the shortest path π_t from 1 to t are those nodes with an index i less than t. Consequently:

$$L_t = \min_{i<t}\{L_i + c_{it}\}$$

However, in a graph with cycles, any node other than t can be a predecessor of t in π_t.

For Directed Acyclic Graphs (DAGs) whose nodes are topologically ordered, $L_{t-1}, \ldots, _L 1$ satisfy the same type of recursive relations:

$$L_{t-1} = \min_{i<t-1}\{L_i + c_{i,t-1}\}; \ldots; L_2 = \min_{i=1}\{L_i + c_i 2\} = L_1 + c_{12}; L_1 = 0$$

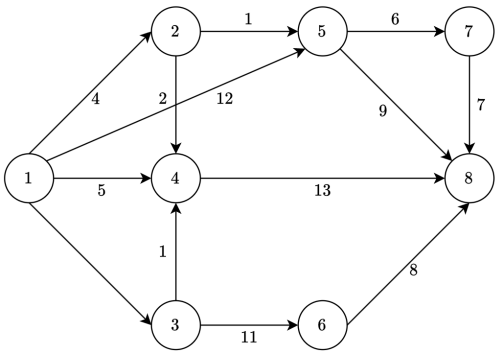
These relations can be solved in reverse order:

$$L_1 = 0; L_2 = L_1 + c_{12}; \ldots; L_t = \min_{i<t-1}\{L_i + c_t\}$$

Algorithm 5 Dynamic programming to find the shortest paths in DAGs

1: Sort the nodes of G topologically
2: $L_1 \leftarrow 0$
3: **for** $j = 2, \ldots, n$ **do**
4: $\quad L_j \leftarrow \min\{L_i + c_{ij} | (i,j) \in \delta^-(j) \wedge i < j\}$
5: $\quad pred_j \leftarrow v$ such that $(v,j) = \text{argmin}\{L_i + c_{ij} | (i,j) \in \delta^-(j) \wedge i < j\}$
6: **end for**

Example:
Consider the following graph:



To find the shortest paths in a Directed Acyclic Graph (DAG) using dynamic programming, follow this workflow:

- $L_1 = 0 \rightarrow \text{pred}_1 = 1$
- $L_2 = L_1 + c_{12} = 4 \rightarrow \text{pred}_2 = 1$
- $L_3 = L_1 + c_{13} = 2 \rightarrow \text{pred}_3 = 1$
- $L_4 = \min_{i=1,2,3}\{L_i + c_{i4}\} = \min 5, 6, 3 = 3 \rightarrow \text{pred}_4 = 3$
- $L_5 = \min_{i=1,2}\{L_i + c_{i5}\} = \min 12, 5 = 5 \rightarrow \text{pred}_5 = 2$
- $L_6 = L_3 + c_{36} = 13 \rightarrow \text{pred}_6 = 3$
- $L_7 = L_5 + c_{57} = 11 \rightarrow \text{pred}_7 = 5$
- $L_8 = \min_{i=4,5,6,7}\{L_i + c_{i8}\} = \min 16, 14, 21, 18 = 14 \rightarrow \text{pred}_8 = 5$

As the nodes are topologically ordered, each node and each arc is processed exactly once, resulting in a time complexity of $O(m)$, where m represents the cardinality of A (the number of arcs).

The same algorithm can also be applied to find the longest path using the following formula:

$$L_t = \max_{i<t}\{L_i + c_{it}\}, \ldots$$

Proposition. The dynamic programming algorithm for DAGs is exact.

Proof. This phenomenon can be attributed to the optimality principle. For any shortest (longest) path from node from 1 to t, π_t, there exists a pair of nodes i and j, where $i < j$, such that the path can be partitioned into two parts: π_i and (i, t), where π_i represents the minimum (maximum) length from s to i. □

3.4.5 Project planning algorithm

Definition (*Project*)**.** A project comprises a collection of m activities with an associated duration. Specifically, activity A_i has an estimated duration of $d_j \geq 0, i = 1, \ldots, m$.

Definition (*Precedence constraint*)**.** Certain pairs of activities are governed by precedence constraint. A notation such as $A_i \propto A_j$ signifies that the commencement of A_j is contingent upon the completion of A_i.

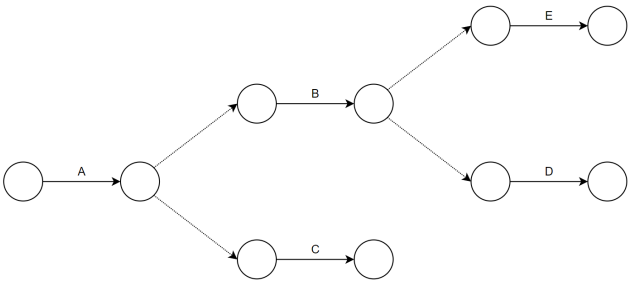
A project can be conveniently represented using a directed graph $G = (N, A)$, where each arc corresponds to an activity, and the length of the arc signifies the duration of the associated activity. In order to account for the precedence constraints, the arcs must be arranged in a manner such that the notation $A_i \propto A_j$ implies the existence of a directed path where the arc associated with A_i precedes the arc associated with A_j. This ensures the proper sequencing of activities. In this context, a node v marks an event corresponding to the end of all the activities $(i, v) \in \delta^-(v)$, consequently, it indicates the potential beginning of all activities $(v, j) \in \delta^+(v)$. To enhance the representation of graph G we use additional nodes that adheres to the following criteria:

- It includes a distinct initial node s, symbolizing the project's start event.
- It incorporates a unique final node t, representing the project's completion event.
- It maintains a structure where multiple arcs with the same endpoints are avoided, ensuring clarity and consistency in the representation.

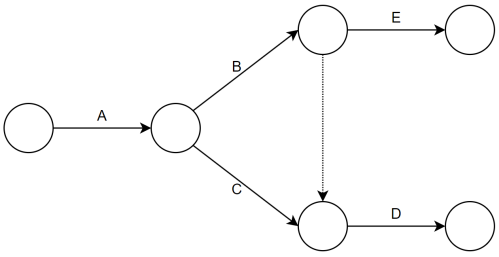
Property 3.4.2. The directed graph G representing any project is acyclic (it is a DAG).

Proof. By contradiction, if $A_{i1} \propto A_{12}, \ldots, A_{jk} \propto A_{ki}$ there would be a logical inconsistency. □

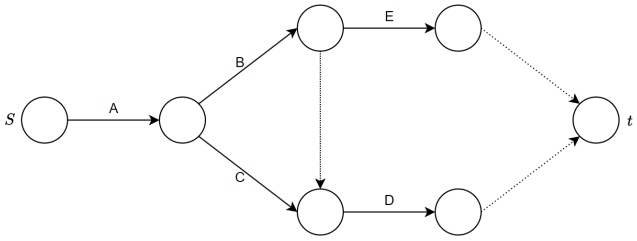
Example:
A project can be described as follows:



While it is possible to simplify this graph by contracting some arcs, care must be taken to avoid introducing unwanted precedence constraints. The correctly contracted graph appears as follows:



Additionally, we can introduce the initial and final nodes to create a more structured representation:



The problem to be addressed is as follows: given a project, the task is to schedule the activities in a manner that minimizes the total duration of the project.

Property 3.4.3. The minimum overall project duration is the length of the longest path from s to t.

Proof. Since any $s - t$ path represents a sequence of activities that must be executed in the specified order, its length provides a lower bound on the minimum overall project duration. □

To address this problem, we can employ the Critical Path Method (CPM), which accomplishes the following:

- Establishes a schedule to minimize the overall project duration.
- Determines the slack for each activity.

The algorithm takes as input the project's representation graph G and seeks to identify a topological order of the nodes. The steps involved are as follows:

1. Traverse the nodes in ascending order of indices. For each node $h \in N$, calculate the earliest time, denoted as T_{min_h}, at which the event associated with node h can occur (T_{min_n} corresponds to the minimum project duration).
2. Traverse the nodes in descending order of indices. For each node $h \in N$, compute the latest time, denoted as T_{max_h}, at which the event associated with node h can occur without causing a delay in the project's completion beyond T_{min_n}.
3. For each activity $(i, j) \in A$, determine the slack, denoted as σ_{ij}, using the formula:

$$\sigma_{ij} = T_{max_j} - T_{max_i} - d_{ij}$$

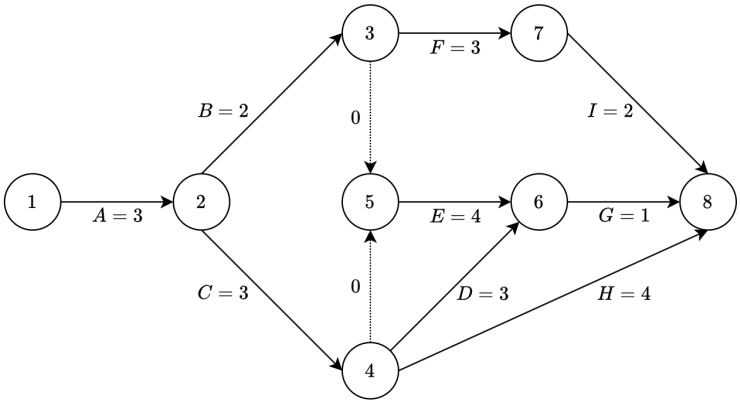
Example:
Consider the following project:

Activity	Duration	Predecessors
A	3	-
B	2	A
C	3	A
D	3	C
E	4	B, C
F	3	B
G	1	E, D
H	4	C
I	2	F

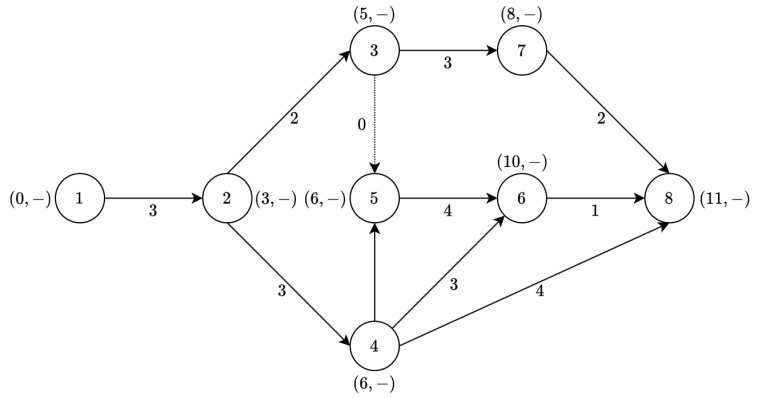
With the following precedence constraints:

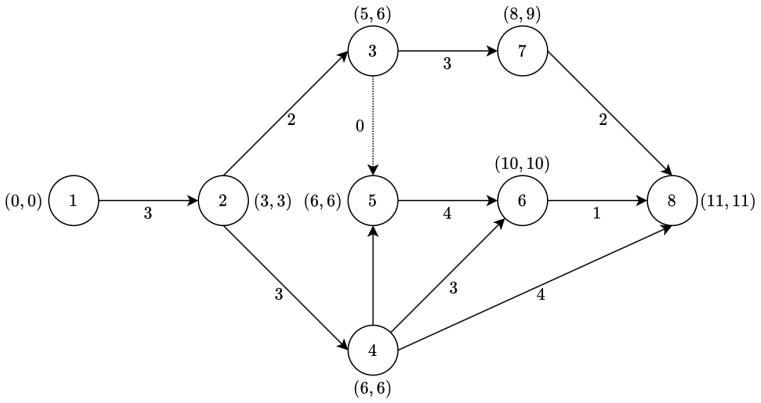
$$A \propto B, A \propto C, C \propto D, B \propto E, C \propto E, B \propto F, E \propto G, D \propto G, C \propto H, F \propto I$$

The objective is to determine the minimum overall duration of the project and calculate the slack for each activity. To do this, we start by creating a graph associated with the given problem:



The first two phases of the CPM algorithm result in the following graphs:





As a result, the longest path that is critical, and hence determines the minimum project duration, is $(1, 2, 4, 5, 6, 8)$.

The algorithm takes as inputs a graph $G = (N, A)$, where $n = |N|$, and the duration d_{ij} assigned to each $(i, j) \in A$.

Algorithm 6 Algorithm for the critical path method

1: Sort the nodes of G topologically
2: $T_{min_1} \leftarrow 0$
3: **for** $j = 2, \ldots, n$ **do**
4: $\quad T_{min_j} \leftarrow \max\{T_{min_i} + d_{ij} | (i, j) \in \delta^-(j)\}$
5: **end for**
6: $T_{max_n} \leftarrow T_{min_n}$
7: **for** $i = n - 1, \ldots, 1$ **do**
8: $\quad T_{max_j} \leftarrow \min\{T_{max_j} - d_{ij} | (i, j) \in \delta^+(j)\}$
9: **end for**

The algorithm's time complexity is $O(m)$, where $m = |A|$.

Definition (*Critical activity*)**.** An activity (i, j) with zero slack, which means that:

$$\sigma_{ij} = T_{max_j} - T_{min_i} - d_{ij} = 0$$

is considered critical.

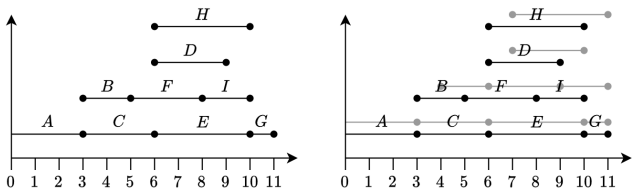
Definition (*Critical path*)**.** A critical path is defined an $s - t$ path only composed of critical activities (it always exists).

To determine the slack of a project, another approach is to use Gantt charts, which were initially introduced by Henry Gantt in 1910. These charts offer a visual representation of the project's timeline.

Example:
The outcome of the previous example is as follows:

(i, j)	d_{ij}	T_{min_i}	T_{max_j}
A	3	0	3
B	2	3	6
C	3	3	6
D	3	6	10
E	4	6	10
F	3	5	9
G	1	10	11
H	4	6	11
I	2	8	11

These results can be illustrated using a Gantt chart as shown below:



3.5 Minimum network flow problem

The network flow problem involves efficiently distributing a particular product from a set of sources to a set of users in order to optimize a predefined objective function.

Definition (*Network*)**.** A network is a directed and connected graph $G = (V, A)$ with a source $s \in V$ and a sink $t \in V$, with $s \neq t$, and a capacity $k_{ij} \geq 0$ for each arc $(i, j) \in A$.

Definition (*Feasible flow*)**.** A feasible flow x from s to t is a vector $x \in \mathbb{R}^m$ with a component x_{ij} for each arc $(i, j) \in A$ satisfying the capacity constraints:

$$0 \leq x_{ij} \leq k_{ij} \quad \forall (i, j) \in A$$

and the flow balance constraints at each intermediate node $u \in V (u \neq s, t)$:

$$\sum_{(i,u)\in\delta^-(u)} x_{iu} = \sum_{(u,j)\in\delta^+(u)} x_{uj} \quad \forall u \in N - \{s, t\}$$

Definition (*Value of flow*)**.** The value of flow x is:

$$\varphi = \sum_{(s,j)\in\delta^+(s)} x_{sj}$$

Definition (*Saturated arc*)**.** Given a network and a feasible flow x, an arc $(i, j) \in A$ is saturated if $x_{ij} = k_{ij}$.

Definition (*Empty arc*)**.** Given a network and a feasible flow x, an arc $(i, j) \in A$ is empty if $x_{ij} = 0$.

In the network flow problem, we are given a network $G = (V, A)$ with integer capacities k_{ij} for each arc $(i, j) \in A$, and two designated nodes $s, t \in V$. The goal is to determine a feasible flow from s to t with the maximum value.

In cases where there are multiple sources and sinks with a single type of product, dummy nodes s^* and t^* can be introduced. The linear programming model for this problem aims to maximize $\max \varphi$, subject to the following constraints:

$$\sum_{(u,j)\in\delta^+(u)} x_{uj} - \sum_{(i,u)\in\delta^-(u)} x_{iu} = \begin{cases} \varphi & u = s \\ -\varphi & u = t \\ 0 & \text{otherwise} \end{cases}$$

Here, φ represents the value of the feasible flow x, $0 \leq x_{ij} \leq k_{ij}$ with $\varphi, x_{ij} \in \mathbb{R}$, and $(i, j) \in A$.

Definition (*Cut*)**.** A cut separating s from t is $\delta(S)$ of G with $s \in S \subset V$ and $t \in V - S$. There are 2^{n-2} cuts separating s from t, where $n = |V|$.

Definition (*Capacity of the cut*)**.** The capacity of the cut $\delta(S)$ induced by S is equal to:

$$k(S) = \sum_{(i,j)\in\delta^+(S)} k_{ij}$$

Definition (*Value of the feasible flow*)**.** Given a feasible flow x from s to t and a cut $\delta(S)$ separating s from t, the value of the feasible flow x through the cut $\delta(S)$ is:

$$\varphi(S) = \sum_{(i,j)\in\delta^+(S)} x_{ij} - \sum_{(i,j)\in\delta^-(S)} x_{ij}$$

With this notation, the value of the flow x is represented as $\varphi = \varphi(\{s\})$.

Property 3.5.1. Given a feasible flow x from s to t, for each cut $\delta(S)$ separating s from t, we have $\varphi(S) = \varphi(\{s\})$.

Property 3.5.2. For every feasible flow x from s to t and every cut $\delta(S)$, with $S \subseteq V$, separating s from t, we have $\varphi(S) \leq k(S)$:

$$\varphi(S) \leq k(S)$$

This expression signifies that the flow's value is either less than or equal to the capacity of the cut.

Proof. According to the definition of the flow value through the cut $\delta(S)$, we express it as:

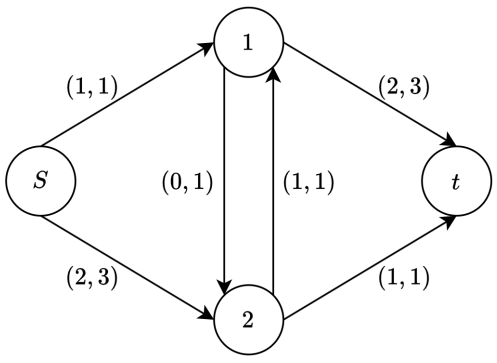
$$\varphi(S) = \sum_{(i,j)\in\delta^+(S)} x_{ij} - \sum_{(i,j)\in\delta^-(S)} x_{ij}$$

Furthermore, considering that $0 \leq x_{ij} \leq k_{ij}$ for any edge $(i,j) \in A$, we can establish:

$$\sum_{(i,j)\in\delta^+(S)} x_{ij} - \sum_{(i,j)\in\delta^-(S)} x_{ij} \leq \sum_{(i,j)\in\delta^+(S)} k_{ij} = k(S)$$

Hence, it follows that $\varphi(S) \leq k(S)$. □

Example:
The value of the feasible flow x passing through the cut $\delta(S)$ in the graph is illustrated below:



$$\varphi(\{s,1\}) = 2 + 0 + 2 - 1 = 3$$

The remaining values are as follows:

$$\delta(\{s,1\}) = \{(s,2),(1,2),(1,t)\}$$

$$k(S) = 7$$

If $\varphi(S) = k(S)$ for a subset $S \subseteq V$ containing the source $s \in S$ and excluding the destination $t \notin S$, then the flow x attains its maximum value, and the cut $\delta(S)$ represents the minimum capacity. The property $\varphi(S) \leq k(S)$ for any feasible flow x and any cut $\delta(S)$ that separates s from t signifies a weak duality connection between two problems:

- Determining a feasible flow of maximum value in a graph $G = (V, A)$ with integer capacities on the arcs and source-destination pair $s, t \in V$.
- Identifying a cut (one that separates s from t) of minimum capacity in a graph $G = (V, A)$ with integer arc capacities and source-destination pair $s, t \in V$.

The idea of the Ford-Fulkerson's algorithm to find network flows is as follows: start from a feasible flow x and try to iteratively increase its value φ by sending, at each iteration, an additional amount of product along an undirected path from s to t with a strictly positive residual capacity. If the arc (i, j) is not saturated, we can increase x_{ij}. If (i, j) is not empty, we can decrease x_{ij} while respecting $0 \leq x_{ij} \leq k_{ij}$.

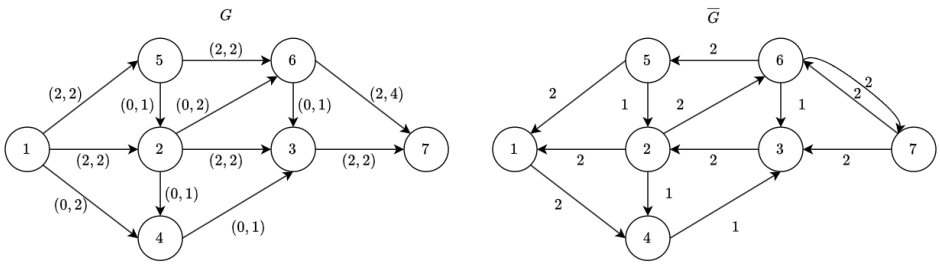
Definition (*Augmenting path*)**.** A path P from s to t is an augmenting path with respect to the current feasible flow x if $x_{ij} < k_{ij}$ for any forward arc $x_{ij} > 0$ for any backward arc.

Given a feasible flow x for $G = (V, A)$, we create the residual network denoted as $\overline{G} = (V, \overline{A})$ associated to x. This residual network encompasses all conceivable adjustments to the flow x:

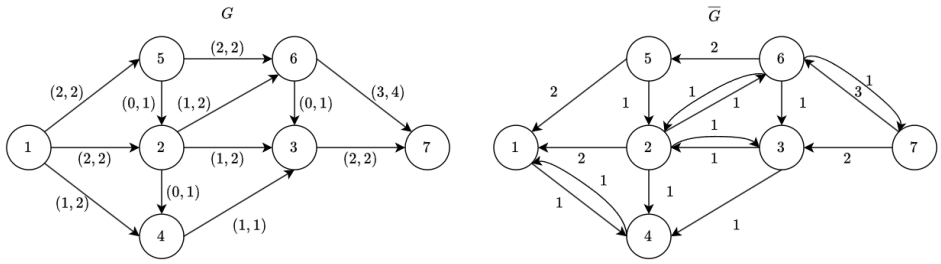
- If $(i, j) \in A$ is empty $(i, j) \in \overline{A}$ with $\overline{k}_{ij} = x_{ij} > 0$.
- If $(i, j) \in A$ is not empty $(i, j) \in \overline{A}$ with $\overline{k}_{ij} = k_{ij} - x_{ij} > 0$

Here, $\overline{k}_{ij}$ is termed residual capacity.

Example:
To determine the residual capacity of the graph depicted below, we follow this procedure:



Afterward, we update the flow and once more compute the residual capacity:



This process is repeated. However, since we are unable to identify additional paths, we conclude that the flow is now $\phi = 5$.

Proposition. Ford-Fulkerson's algorithm is exact.

Proof. A feasible flow denoted as x achieves its maximum value if and only if the destination node t cannot be reached from the source node s in the residual network associated with flow x. If an augmenting path exists, then flow x is not optimal for achieving the maximum value. When t is not reachable from s, it implies the presence of a cut in the residual network $\overline{G}$ such that $\delta^+_{\overline{G}}(S^*) = \varnothing$. By the definition of $\overline{G}$ every edge $(i,j) \in \delta^+_G(S^*)$ is saturated, and every edge in $\delta^-_G(S^*)$ is empty. Consequently,

$$\phi(S^*) = \sum_{(i,j)\in\delta^+_G(S^*)} x_{ij} - \sum_{(i,j)\in\delta^-_G(S^*)} x_{ij} = \sum_{(i,j)\in\delta^+_G(S^*)} k_{ij} = k(S^*)$$

According to weak duality, it follows that $\phi(S) < k(S)$ for all feasible flows x and $\forall S \subset V$ with $s \in S$, $t \notin S$. Therefore, the flow x has maximum value and the cut induced by S^* represents the minimum capacity. □

Theorem 3.5.1. *The value of a feasible flow of maximum value is equal to the capacity of a cut of minimum capacity.*

The algorithm takes as its inputs a graph denoted as $G = (V, A)$ with non-zero capacities $k_{ij} > 0$ for all edges $(i,j) \in A$, where $t \in N$.

Algorithm 7 Ford-Fulkerson's algorithm

```
1: x ← 0
2: φ ← 0
3: optimum ← false
4: while optimum = true do
5:     Build residual network G̅ associated to x
6:     P ← path from s to t in G̅
7:     if P is not defined then
8:         optimum ← true
9:     else
10:        δ ← min{k̅_ij | (i,j) ∈ P}
11:        φ ← φ + δ
12:        for (i,j) ∈ P do
13:            if (i,j) is a forward arc then
14:                x_ij ← x_ij + δ
15:            else
16:                x_ij ← x_ij − δ
17:            end if
18:        end for
19:    end if
20: end while
```

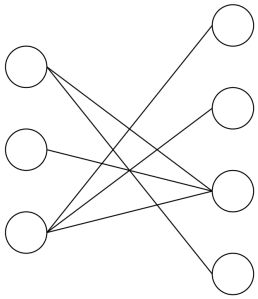
The algorithm's total complexity is $O(m^2 k_{max})$. The space complexity of the algorithm is $O(m \log k_{max})$. To render the algorithm polynomial, one can seek an augmenting path with the minimal number of arcs.

3.5.1 Minimum cost flow problem

In the context of a network with unit costs denoted as c_{ij} or each arc (i, j) and a specified value $\phi > 0$ the objective is to find a feasible flow from source s to destination t with a value of ϕ while minimizing the total cost. To address this problem, the approach involves initiating from a feasible flow x with a value of ϕ and dispatching an additional quantity of goods within the residual network through cycles characterized by negative costs.

3.5.2 Assignment problem

Definition (*Matching*)**.** Given an undirected bipartite graph $G = (V, E)$, a matching $M \subseteq E$ is a subset of non-adjacent edges.



Given a bipartite graph $G = (V, E)$, determine a matching with a maximum number of edges. This problem can be reduced to the problem of finding a feasible flow of maximum value from s to t in the following network. There is a correspondence between the feasible flow of value φ and the matching containing φ edges.

3.6 Traveling salesman problem

Given a directed graph $G = (N, A)$ with cost $c_{ij} \in \mathbb{Z}$ for each are $(i, j) \in A$, determine a circuit of minimum total cost visiting every node exactly once.

Definition (*Hamiltonian circuit*)**.** A Hamiltonian circuit C of G is a circuit that visits every node exactly once.

Therefore, by representing H as the set encompassing all Hamiltonian circuits within graph G, the problem can be formulated as:

$$\min_{C \in H} \sum_{(i,j) \in C} c_{ij}$$

It's important to note that this problem is classified as $\mathcal{NP}$-hard.

CHAPTER 4

Linear programming

4.1 Introduction

Definition (*Linear programming problem*)**.** A linear programming problem is an optimization problem defined as follows:

$$\begin{aligned} &\min \ f(x) \\ &\text{such that } \ x \in x \subseteq \mathbb{R}^n \to \mathbb{R} \end{aligned}$$

Here, the objective function, denoted as $f : x \to \mathbb{R}$, is linear, and the feasible region x is described as:

$$x = \{x \in \mathbb{R}^n | g_i(x) r_i 0 \wedge i \in (1, m)\}$$

Where r_i takes values from $\{=, \geq, \leq\}$, and $g_i : \mathbb{R}^n \to \mathbb{R}$ represents linear functions for $i = 1, \ldots, m$.

Definition (*Optimal solution*)**.** An optimal solution, denoted as $x^* \in \mathbb{R}^n$, of the linear program is defined as satisfying $f(x^*) \leq f(x)$ for all $x \in x$.

The general form of a linear programming problem can be expressed as follows:

$$\begin{aligned} \min z = \ & c^T x \\ \text{such that } \ & Ax \geq b \\ & x \geq 0 \end{aligned}$$

In this notation, x represents the vector of decision variables. It's important to note that the inequality for the matrix A can be transformed into an equality, and the constraints on the variable values can be relaxed in some instances. It's often helpful to convert all linear programming problems into the general form.

Transformation rules This transformation can be achieved using the following rules:

- To convert a maximization problem into a minimization problem, you can use:

$$\max (c^T x) = \min (-c^T x)$$

- For $a^T x \leq b$, you can introduce a slack variable s as:

$$a^T x \leq b \implies \begin{cases} a^T x + s = b \\ s \geq 0 \end{cases}$$

- For $a^T x \geq b$, you can introduce a surplus variable s as:

$$a^T x \geq b \implies \begin{cases} a^T x - s = b \\ s \geq 0 \end{cases}$$

- If a variable x_{ij} is unrestricted in sign, you can represent it as:

$$\begin{cases} x_j = x_j^+ - x_j^- \\ x_j = x_j^+, x_j^- \geq 0 \end{cases}$$

  After substituting x_j with $x_j^+ - x_j^-$, you can remove x_j from the problem.

- To change the direction of an inequality, you can use the following equivalences:

$$a^T x \leq b \text{ is equivalent to } -a^T x \geq -b$$

$$a^T x \geq b \text{ is equivalent to } -a^T x \leq -b$$

- For an equality constraint $a^T x = b$, you can represent it as:

$$\begin{cases} a^T x \geq b \\ a^T x \leq b \end{cases}$$

Example:
Given the following linear programming model, we'll rewrite it in the normal form:

$$\begin{aligned} \max f(x_1, x_2) &= 2x_1 - 3x_2 \\ \text{such that } & 4x_1 - 7x_2 \leq 5 \\ & 6x_1 - 2x_2 \geq 4 \\ & x_1 \geq 0, x_2 \in \mathbb{R} \end{aligned}$$

We can add two new variables, $x_2 = x_3 - x_4$, with $x_3, x_4 \geq 0$, and we obtain:

$$\begin{aligned} \max f(x_1, x_2) &= 2x_1 - 3x_2 \\ \text{such that } & 4x_1 - 7x_3 + 7x_4 \leq 5 \\ & 6x_1 - 2x_3 + 2x_4 \geq 4 \\ & x_1, x_2, x_4 \geq 0 \end{aligned}$$

Next, we introduce slack and surplus variables x_5 and x_6, resulting in:

$$\begin{aligned} \max f(x_1, x_2) &= 2x_1 - 3x_2 \\ \text{such that } & 4x_1 - 7x_3 + 7x_4 + x_5 = 5 \\ & 6x_1 - 2x_3 + 2x_4 - x_6 = 4 \\ & x_1, x_2, x_4, x_5, x_6 \geq 0 \end{aligned}$$

Finally, we need to change the sign of the objective function:

$$\begin{aligned} \min f(x_1, x_2) = & -2x_1 + 3x_2 \\ \text{such that } & 4x_1 - 7x_3 + 7x_4 + x_5 = 5 \\ & 6x_1 - 2x_3 + 2x_4 - x_6 = 4 \\ & x_1, x_2, x_4, x_5, x_6 \geq 0 \end{aligned}$$

The assumptions underlying linear programming models can be summarized as follows:

1. Linearity of the objective function and constraints.
2. Proportionality of the objective function and constraints: the contribution of each variable is scaled by a constant.
3. Additivity of the objective function and constraints: the overall contribution of all variables is the sum of their individual contributions.
4. Divisibility: variables are allowed to take rational values, and fractional values are acceptable.
5. Parameters are considered as constant which can be estimated with a sufficient degree of accuracy.

The linear programming sensitivity analysis is a technique used to assess how an optimal solution reacts to small alterations in parameter values.

4.2 Geometry of linear programming

Definition (*Level curve*)**.** A level curve for a value z of a function f is the collection of points in $\mathbb{R}^n$ where the function f remains constant and equals z.

Definition (*Hyperplane*)**.** A hyperplane is described as $H = \{x \in \mathbb{R}^n | a^T x = b\}$.

Definition (*Affine half-space*)**.** An affine half-space is represented as $H = \{x \in \mathbb{R}^n | a^T x \leq b\}$.

Every inequality constraint defines an affine half-space within the variable space.

Definition (*Polyhedron*)**.** The feasible region x in any linear programming problem forms a polyhedron, denoted as P.

Definition (*Convex*)**.** A subset $S \subseteq \mathbb{R}^n$ is considered convex if, for any pair $y^1, y^2 \in S$, the entire line segment connecting y^1 and y^2 is contained within S.

Definition (*Convex combinations*)**.** The line segment defined by y^1 and y^2 in S, encompassing all convex combinations of y^1 and y^2, is denoted as:

$$[y^1, y^2] = \{x \in \mathbb{R}^n | x = \alpha y^1 + (1 - \alpha) y^2 \wedge \alpha \in [0, 1]\}$$

Property 4.2.1. A polyhedron P is a convex set of $\mathbb{R}^n$.

This occurs because each half-space is inherently convex, and the intersection of a finite number of convex sets likewise forms a convex set.

Definition (*Vertex*)**.** A vertex of P is a point P that cannot be represented as a convex combination of two other distinct points of P.

In terms of algebraic expression, a vertex is defined as:

$$x = \alpha y^1 + (1-\alpha)y^2, \alpha \in [0,1], y^1, y^2 \in P \implies x = y^1 \vee x = y^2$$

Property 4.2.2. A non-empty polyhedron $P = \{x \in \mathbb{R}^n | Ax = b, x \geq 0\}$ (in standard form) or $P = \{x \in \mathbb{R}^n | Ax \geq b, x \geq 0\}$ (in canonical form) possesses a finite number ($n \geq 1$) of vertices.

Definition (*Unbounded feasible direction of P*)**.** In the context of a problem P, a vector $d \in \mathbb{R}^n$ where $d \neq 0$ is considered an unbounded feasible direction of P if, for any point $x_0 \in P$, the ray defined as $\{x \in \mathbb{R}^n | x = x_0 + \lambda d, \lambda \geq 0\}$ remains entirely within P.

Theorem 4.2.1 (Weyl-Minkoswki)**.** *Each point x within a polyhedron P can be represented as a convex combination of its vertices $x^1, \ldots, x^k$ and, if necessary, an unbounded feasible direction d of P:*

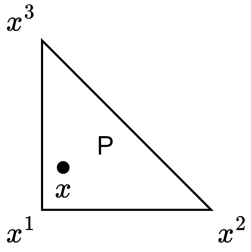
$$x = \alpha_1 x^1 + \cdots + \alpha_k x^k + d$$

Here, the coefficients $\alpha_i \geq 0$ fulfill the condition $\alpha_1 + \cdots + \alpha_k = 1$.

Definition (*Polytope*)**.** A polytope is a type of bounded polyhedron, which means it possesses only one unbounded feasible direction, and that direction is $d = 0$.

Each point x within a polytope P can be represented as a convex combination of its vertices.

Example:
The point $x = \alpha_1 x^1 + \alpha_2 x^2 + \alpha_3 x^3$ can also be expressed in the form $\alpha_1 + \alpha_2 + \alpha_3 = 1$ ($d = 0$).



Theorem 4.2.2 (Fundamental theorem of linear programming)**.** *Consider a linear programming problem $\min\{c^T x | x \in P\}$, where $P \subseteq \mathbb{R}^n$ represents a non-empty polyhedron of feasible solutions, one of the following scenarios must hold true. There exists at least one optimal vertex within P. The value of the objective function is unbounded from below over P.*

Proof. We prove that there exists at least one optimal vertex within P. If P contains an unbounded feasible direction d such that $c^T d < 0$, it signifies that P is unbounded, and the objective function values $z = c^T x$ decrease indefinitely along the direction d. □

Proof. We prove that the value of the objective function is unbounded from below over P. If P does not possess any unbounded feasible direction d such that $c^T d < 0$, meaning that for all directions d, we have $c^T d \geq 0$, then any point within P can be expressed as:

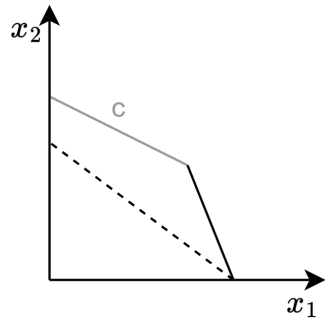
$$x = \sum_{i=1}^{k} \alpha_i x^i + d$$

Here, $x^1, \ldots, x^k$ are the vertices of P, $\alpha_i > 0$ with $\alpha_1 + \cdots + \alpha_k = 1$, and $d = 0$, or d represents an unbounded feasible direction. For any point $x \in P$, either $d = 0$ or $c^T d > 0$, which leads to the following inequality:

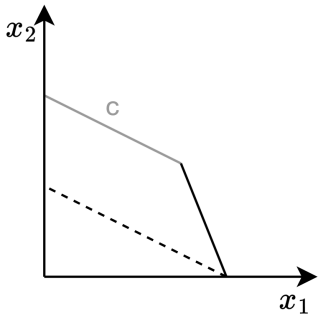
$$c^T x = c^T \left(\sum_{i=1}^{k} \alpha_i x^i + d \right) = \sum_{i=1}^{k} \alpha_i c^T x^i + c^T d \geq \min_{i=1,\ldots,n} c^T x^i$$

This is due to the fact that $\alpha_i > 0$ for any i, and $\alpha_1 + \cdots + \alpha_k = 1$. □

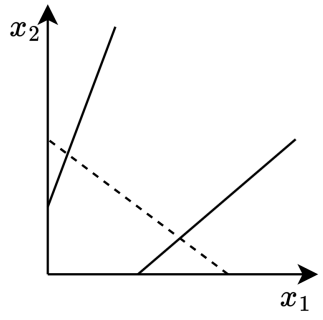
From this theorem, we can deduce that an interior point $x \in P$ cannot be an optimal solution. Furthermore, in an optimal vertex, all feasible directions lead to worse objective function values. This theorem suggests that even though the variables can take fractional values, linear programming can be approached as a combinatorial problem. Consequently, it is essential to investigate the vertices of the polyhedron of feasible solutions. However, the number of vertices is finite but can often grow exponentially, making graphical methods practical only when $n \leq 3$.



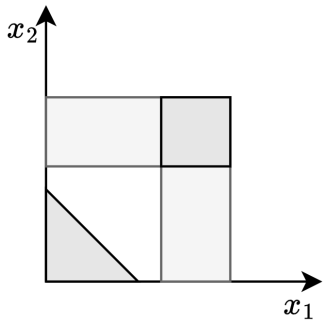
Unique optimal solution



Multiple optimal solutions



Unbounded linear program



Infeasible linear program

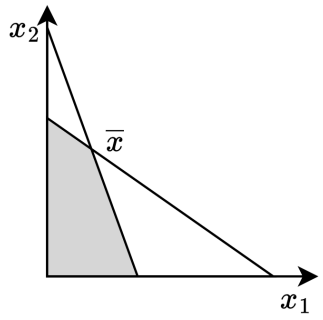
4.3 Basic feasible solutions

Thanks to the fundamental theorem of linear programming, solving any linear program involves examining the vertices of the polyhedron P containing feasible solutions. However, because the geometric definition of a vertex isn't suitable for algorithmic use, we require an algebraic characterization.

Example:
Consider the following linear program:

$$\begin{aligned} \min \quad & -x_1 - 3x_2 \\ \text{such that} \quad & x_1 + x_2 \leq 6 \\ & 2x_1 + x_2 \leq 8 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Its graphical representation is shown below:



From the graph, it's evident that a vertex corresponds to the intersection of the hyperplanes associated with n inequalities. In this example, the vertex $\overline{x}$ represents the intersection of the first two inequalities, which is the solution to the following system:

$$\begin{cases} x_1 + x_2 = 6 \\ 2x_1 + x_2 = 8 \end{cases}$$

To convert this problem into standard form, we formulate it as:

$$\begin{aligned} \min \quad & -x_1 - 3x_2 \\ \text{such that} \quad & x_1 + x_2 + s_1 = 6 \\ & 2x_1 + x_2 + s_2 = 8 \\ & x_1, x_2, s_1, s_2 \geq 0 \end{aligned}$$

To solve it, we set $s_1 = s_2 = 0$ and solve the system found before. Using this formulation, we can compute all the intersections with the axes:

1. $x_1 = 0, x_2 = 0 \rightarrow s_1 = 6, s_2 = 8$
2. $x_1 = 0, s_1 = 0 \rightarrow x_2 = 6, s_2 = 2$
3. $x_1 = 0, s_2 = 0 \rightarrow x_2 = 8, s_1 = -2$
4. $x_2 = 0, s_1 = 0 \rightarrow x_1 = 6, s_2 = -4$
5. $x_2 = 0, s_2 = 0 \rightarrow x_1 = 4, s_1 = 2$
6. $s_1 = 0, s_2 = 0 \rightarrow x_1 = 2, x_2 = 4$

Note that solutions three and four are infeasible since the values of s_1 and s_2 are negative.

Property 4.3.1. For any polyhedron $P = \{x \in \mathbb{R}^n | Ax = b, x \geq 0\}$ the facets (equivalent to edges in $\mathbb{R}^2$) are determined by setting one variable to 0.

Property 4.3.2. For any polyhedron $P = \{x \in \mathbb{R}^n | Ax = b, x \geq 0\}$ the vertices are determined by setting $n - m$ variables to 0.

When dealing with a matrix $A \in \mathbb{R}^{m \times n}$ where $m \leq n$, we can make the following observations:

- If $m = n$, there is a unique solution of $Ax = b$.
- If $m < n$, there are infinitely many solutions to $Ax = b$. In this case the system has $n - m$ degrees of freedom. As a result $n - m$ variables can be set arbitrarily.

Definition (*Basis of a matrix*)**.** A basis of a matrix $A \in \mathbb{R}^{m \times n}$ containing n variables and m coefficients consists of a subset of m columns from A. These columns are linearly independent and collectively form an $m \times m$ non-singular matrix denoted as B.

Using the earlier definition, we can rearrange the columns of matrix A and then split it to identify the basis as follows:

$$A = [B|N]$$

In this partition, B represents an $m \times m$ matrix, and N is a $(n - m) \times m$ matrix.

Let $x^T = \left[x_B^T | x_N^T\right]$, where x_B^T contains m components and x_N^T contains $m - n$ components. Then, any system $Ax = b$ can be expressed as:

$$Bx_B + Nx_N = b$$

For any given set of values for x_N, if matrix B is non-singular, we can compute x_B as:

$$x_B = B^{-1}b - B^{-1}Nx_N$$

Definition (*Basic solution*)**.** A basic solution is a solution obtained by setting $x_N = 0$ and, consequently, letting $x_B = B^{-1}b$.

Definition (*Basic feasible solution*)**.** A basic solution with $x_B \geq 0$ is a basic feasible solution.

Definition (*Basic variables*)**.** The variables in x_B are the basic variables.

Definition (*Non-basic variables*)**.** The variables in x_N are the non-basic variables.

Theorem 4.3.1. *An $x \in \mathbb{R}^n$ is a basic feasible solution if and only if x is a vertex of the polyhedron:*

$$P = \{x \in \mathbb{R}^n | Ax = b, x \geq 0\}$$

Example:
In the given linear program:

$$\begin{aligned} \min \quad & 2x_1 + x_2 + 5x_3 \\ \text{such that} \quad & x_1 + x_2 + x_3 + x_4 = 4 \\ & x_1 + x_5 = 2 \\ & x_3 + x_6 = 3 \\ & 3x_2 + x_3 + x_7 = 6 \\ & x_1, x_2, x_3, x_4, x_5, x_6, x_7 \geq 0 \end{aligned}$$

The matrices associated with the given constraints are:

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 3 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad b = \begin{bmatrix} 4 \\ 2 \\ 3 \\ 6 \end{bmatrix}$$

It can be observed that columns 4, 5, 6, and 7 are linearly independent. By selecting these columns, we obtain:

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I = B^{-1}$$

The solution in this case is $x_B = B^{-1}b = b$, which is a feasible solution. However, if we choose columns 2, 5, 6, and 7, the solution is $x_B = \begin{bmatrix} 4 & 2 & 3 & -6 \end{bmatrix}^T$, which is an infeasible solution.

The total number of feasible solutions can be calculated as:

$$\text{number of feasible solutions} = \binom{n}{m}$$

4.4 Simplex method

The idea of the simplex method is that, given a linear program in standard form:

$$\begin{aligned} \min \quad & z = c^T x \\ \text{such that} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

we will examine a sequence of basic feasible solutions with non-increasing objective function values until an optimal solution is reached, or the linear program is found to be unbounded.

4.4.1 Optimality test

Because a basic feasible solution is characterized by $x_B = B^{-1}b$ and $x_N = 0$, we can express the objective function in terms of only the non-basic variables as follows:

$$c^T x = c_B^T B^{-1} b + \left(c_N^T - c_B^T B^{-1} N\right) x_N$$

Definition (*Vector of reduced cost*)**.** The vector:

$$\overline{c}^T = c^T - C_B^T B^{-1} A = \left[c^T - C_B^T B^{-1} B, c^T - C_B^T B^{-1} N\right]$$

is the vector of reduced costs with respect to the basis B.

The vector $\overline{c}_j$ represents the change in the objective function value if non-basic x_j is increased from 0 to 1 while keeping all other non-basic variables at 0. The solution value changes by:

$$\Delta z = \theta \cdot \overline{c}_j$$

Consider a linear program $\min\{c^T x | Ax = b, x \geq 0\}$ and a feasible basis B.

Proposition. If $\overline{c}_N \geq 0$, then the basic feasible solution (x_B^T, x_N^T), where $x_B = B^{-1}b \geq 0$ and $x_N = 0$, with a cost of $c_B^T B^{-1} b$ is a global optimum.

Proof. If $\overline{c}_N \geq 0$, it implies that:

$$c^T x = c_B^T B^{-1} b + \overline{c}_N^T x_N \geq c_B^T B^{-1} b \ \forall x \geq 0, Ax = b$$

□

This optimality condition is sufficient, but in general, it's not necessary.

Example:
Consider the following linear program:

$$\begin{aligned} \min \quad & -x_1 - x_2 \\ \text{such that} \quad & x_1 - x_2 + s_1 = 1 \\ & x_1 + x_2 + s_2 = 3 \\ & x_1, x_2, s_1, s_2 \geq 0 \end{aligned}$$

The matrices associated with the given constraints are:

$$A = \begin{bmatrix} 1 & -1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix} \qquad b = \begin{bmatrix} -1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$$

Consider the solution with $x_B = (x_1, s_2) = (1, 2)$, $x_n = (x_2, s_1)$, and $z = -1$. We have:

$$B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \qquad N = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$$

Therefore, we have:

$$c_B^T = c_N^T = \begin{bmatrix} -1 & 0 \end{bmatrix} \qquad B^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

The reduced costs for x_2, s_1 are:

$$\overline{c}_N^T = c_N^T - c_B^T B^{-1} N = \begin{bmatrix} -2 & 0 \end{bmatrix}$$

Since $\overline{c}_2 = -2 < 0$, increasing x_2 to 1 will improve the solution by -2.

4.4.2 Vertex selection

Example:
Consider the following linear problem:

$$\begin{aligned} \min \quad & -x_1 - 3x_2 \\ \text{such that} \quad & x_1 + x_2 + s_1 = 6 \\ & 2x_1 + x_2 + s_2 = 8 \\ & x_1, x_2, s_1, s_2 \geq 0 \end{aligned}$$

Moving from vertex one to vertex five occurs as follows:

- In vertex one, $x_1 = 0, X_2 = 0 \implies s_1 = 6, s_2 = 8$ with $x_B = (s_1, s_2), x_N = (x_1, x_2)$.
- In vertex five, $x_2 = 0, s_2 = 0 \implies x_1 = 4, s_1 = 2$ with $x_B = (x_1, s_1), x_N = (x_2, s_2)$.

As a result, x_1 enters the basis B while s_2 exits the basis B.

By expressing the basic variables in terms of the non-basic variables, we obtain:

$$s_1 = 6 - x_1 - x_2$$

$$s_2 = 8 - 2x_1 - x_2$$

Now, we increase x_1 while keeping $x_2 = 0$. Since $s_1 = 6 - x_1 \geq 0$ implies $x_1 \leq 6$ and $s_2 = 8 - 2x_1 \geq 0$ implies $x_1 \leq 4$, the upper limit on the increase of x_1 is:

$$x_1 \leq \min\{6, 4\} = 4$$

We move from vertex one to vertex five by letting x_1 enter the basis and s_2 exit the basis ($s_1 = 2$ and $s_2 = 0$).

Note that when transitioning from the current vertex to an adjacent one, we substitute one column of the basis matrix B with one column of the non-basis matrix N.

Given a basis B, the system:

$$Ax = b \Leftrightarrow \sum_{j=1}^{m} a_{ij} x_i = b_j \quad \text{for } i = 1, \ldots, m$$

Can be expressed in canonical form:

$$x_B + B^{-1} N_{x_N} = B^{-1} b \Leftrightarrow x_B + \overline{N}_{x_N} = \overline{b}$$

This form emphasizes the basic feasible solution as $(x_B, x_N) = (B^{-1}b, 0)$. This involves pre-multiplying the system by B^{-1}:

$$B^{-1} B x_B + B^{-1} N x_N = B^{-1} b \implies I x_B + \overline{N} x_N = \overline{b}$$

In the canonical form:

$$x_{B_i} + \sum_{j=1}^{n-m} \overline{a}_{ij} x_{N_j} = \overline{b}_i \quad \text{for } i = 1, \ldots, m$$

The basic variables are expressed in terms of non-basic variables: $x_B = \overline{b} - \overline{N}_{x_N}$. If we increase the value of a non-basic variable x_s (starting from 0) while keeping all other non-basic variables

at 0, the system becomes:

$$x_{B_i} + \overline{a}_{is} x_s = \overline{b}_i \Leftrightarrow x_{B_i} = \overline{b}_i - \overline{a}_{is} x_s \quad \text{for } i = 1, \ldots, m$$

To ensure that $x_{B_i} \geq 0$ for each i, we need to satisfy:

$$\overline{b}_i - \overline{a}_{is} x_s \geq 0 \implies x_s \geq \frac{\overline{b}_i}{\overline{a}_{is}} \quad \text{for } \overline{a}_{is} \geq 0$$

The value of x_s can be increased up to:

$$\theta^* = \min_{i=1,\ldots,m} \left[\frac{\overline{b}_i}{\overline{a}_{is}} \text{ such that } \overline{a}_{is} \geq 0 \right]$$

The value of x_r where $r = \text{argmin}_{i=1,\ldots,m} \left[\frac{\overline{b}_i}{\overline{a}_{is}} \text{ such that } \overline{a}_{is} \geq 0 \right]$ decreases to 0 and exits the basis.

4.4.3 Change of basis

Consider a feasible basis B and a non-basic variable x_s (part of x_N) with a reduced cost $\overline{c}_{<}0$. To improve the objective function value, we aim to increase x_s as much as possible while keeping all other non-basic variables at zero. Among the basic variables x_r (in x_B) where $x_r > 0$, one imposes the strictest upper limit, denoted as θ^* on the increase of x_s. If $\theta^* > 0$, the resulting basic feasible solution yields a superior objective function value. This new basis differs from the previous one by just a single column, representing adjacent vertices in the solution space. Transitioning from the canonical form of the current basic feasible solution: $B^{-1}Bx_B + B^{-1}Nx_N = B^{-1}b$ to that of an adjacent basic feasible solution doesn't require computing B^{-1} from scratch. The inverse matrix B^{-1} of the new basis B can be obtained by applying a unique pivoting operation to the inverse of the previous basis, which differs by just one column.

The pivoting operation aligns with the Gaussian elimination method used to solve systems of linear equations. For a given system $Ax = b$, the steps are as follows:

1. Select a non-zero coefficient $\overline{a}_{rs}$ as the pivot.
2. Divide the r-th row by $\overline{a}_{rs}$.
3. For each row i where $i \neq r$ and $\overline{a}_{rs} \neq 0$, subtract the resulting r-th row multiplied by $\overline{a}_{rs}$.

It's important to note that this procedure preserves the feasibility of the solutions.

4.4.4 Tableau representation

A system with non-negativity constraints can be represented as a table in the following manner:

	$x_1 \ldots x_n$
0	c^T
b	A

If we consider a basis B and a partition $A = [B|N]$, with $0 = c^T x - z$, the representation becomes:

	$x_1 \dots x_n$	$x_{m+1} \dots x_n$	z
0	c_B^T	c_N^T	-1
b	A	N	$0 \\ \vdots \\ 0$

By performing pivoting operations, we transform the tableau into canonical form with respect to B:

		$x_1 \dots x_n$	$x_{m+1} \dots x_n$
$-z$	$-z_0$	$0 \dots 0$	$\overline{c}_N^T$
$X_{B[1]} \\ \vdots \\ X_{B[m]}$	$\overline{b}$	I	$\overline{N}$

Example:
For the linear program:

$$\begin{aligned} \min \quad & -x_1 - x_2 \\ \text{such that} \quad & 6x_1 + 4x_2 + x_3 = 24 \\ & 3x_1 - 2x_2 + x_4 = 6 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

The corresponding tableau is:

		x_1	x_2	x_3	x_4
$-z$	0	-1	-1	0	0
x_3	24	6	4	1	0
x_4	6	3	-2	0	1

Now we pivot the tableau with respect to 3 to derive an expression for x_1 from the pivot row and substituting it in all other rows. We have that x_1 enters the basis and x_4 exits the basis. The tableau after these changes looks like this:

		x_1	x_2	x_3	x_4
$-z$	2	0	-5/3	0	1/3
x_3	12	0	8	1	-2
x_1	2	1	-2/3	0	1/3

In this case, only x_2 can enter the basis, and x_3 is the only one that can exit. The optimal solution can be found in subsequent tableaux, and the final result is:

		x_1	x_2	x_3	x_4
$-z$	9/2	0	0	5/24	-1/12
x_2	3/2	0	1	1/8	-1/4
x_1	3	1	0	1/12	1/6

		x_1	x_2	x_3	x_4
$-z$	6	1/2	0	1/4	0
x_2	6	3/2	1	1/4	0
x_4	18	6	0	1/2	1

With all reduced costs greater than or equal to zero, the optimal basic solution is:

$$x^{*T} = (0, 6, 0, 18) \text{ with } z^* = -6$$

4.4.5 Algorithm

Algorithm 8 Simplex algorithm (LP with minimization)

Let $B[1], \ldots, B[m]$ be the column indices of the initial feasible basis B
Construct the initial tableau $\overline{A} = \{\overline{a}[i,j] | 0 \leq i \leq m, 0 \leq j \leq n\}$ in canonical form with respect to B
unbounded $\leftarrow$ false
optimal $\leftarrow$ false
while optimal=false and unbounded=false **do**
    if $\overline{a}[i,j] \geq 0 \, \forall j = 1, \ldots, m$ **then**
        optimal $\leftarrow$ true
    else
        Select a non-basic variable x_s with $\overline{a}[0,s] < 0$
        if $\overline{a}[i,s] \geq 0 \, \forall j = 1, \ldots, m$ **then**
            unbounded $\leftarrow$ true
        else
            $r \leftarrow \text{argmin} \left[\frac{\overline{a}[i,0]}{\overline{a}[i,s]} \text{ with } 1 \leq i \leq m \text{ and } \overline{a}[i,s] > 0 \right]$
            $pivot(r, s)$
            $B[r] \leftarrow s$
        end if
    end if
end while

4.4.6 Degenerate basic feasible solution and convergence

Definition (*Degenerate basic feasible solution*)**.** A basic feasible solution x is degenerate if it contains at least one basic variable $x_j = 0$.

This property implies that a solution x with more than $n - m$ zero values corresponds to multiple distinct bases. When degenerate basic feasible solutions are present, changing the basis may not necessarily result in a decrease in the objective function value. In such cases, it is possible to cycle through a sequence of degenerate bases associated with the same vertex. To manage these scenarios, various anti-cycling rules have been proposed to select the variables that enter and exit the bases (indices r and s). One such rule is Bland's rule, which dictates choosing the variable with the smallest index among all candidates to enter or exit the basis.

Proposition. The Simplex algorithm, when using Bland's rule, terminates after a maximum of $\binom{n}{m}$ iterations.

In some exceptional cases, the number of iterations may grow exponentially with respect to the values of n or m. However, in general, the Simplex algorithm is highly efficient, and extensive experimental studies have shown that the number of iterations increases linearly with respect to m and very slowly with respect to n.

4.4.7 Two-phase simplex method

The two-phase Simplex method differs from the standard simplex method in that it first addresses an auxiliary problem aiming to minimize the sum of artificial variables. Once this auxiliary problem is solved and the final tableau is reorganized, the second phase, which resembles the standard Simplex method, begins.

Consider a linear program (P):

$$\begin{aligned} \min \ & z = c^T \\ \text{such that } & Ax = b \\ & x \geq 0 \end{aligned}$$

where $b \geq 0$. An auxiliary linear program, equipped with artificial variables y_i, $i = 1, \ldots, m$, is formulated as (P_A):

$$\begin{aligned} \min \ & \sum_{i=1}^{m} y_i \\ \text{such that } & Ax + ly = b \\ & x \geq 0, x \geq 0 \end{aligned}$$

The initial basic feasible solution $y = b \geq 0$ and $x = 0$ is obvious. Two possibilities arise:

1. If $v^* > 0$, then (P) is infeasible.
2. If $v^* = 0$, where $t^* = 0$ and x^* represents a basic feasible solution of (P). In this case:
   - If y_i is non-basic for all i within $1 \leq i \leq m$, the corresponding columns and be removed to obtain a tableau in canonical form with respect to a basis, with the row for z determined by substitution.
   - In the case of a basic y_i (indicating a degenerate basic feasible solution), a pivot operation is performed, exchanging y_i with any non-basic variable x_j.

Example:
Consider the following linear program:

$$\begin{aligned} \min \ & x_1 + x_2 + 10x_3 \\ \text{such that } & x_2 + 4x_3 = 2 \\ & -2x_1 + x_2 - 6x_3 = 2 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

In standard form, it can be expressed as:

$$\begin{aligned} \min v &= y_1 + y_2 \\ \text{such that } & x_2 + 4x_3 + y_1 = 2 \\ & -2x_1 + x_2 - 6x_3 + y_2 = 2 \\ & x_1, x_2, x_3, y_1, y_2 \geq 0 \end{aligned}$$

To obtain the canonical form of $v = y_1 + y_2$, we can express y_1 and y_2 in terms of x_1, x_2, x_3.

		x_1	x_2	x_3	y_1	y_2
$-v$	-4	2	-2	2	0	0
y_1	2	0	1	4	1	0
y_2	2	-2	1	-6	0	1

		x_1	x_2	x_3	y_1	y_2
$-v$	0	2	0	10	2	0
x_2	2	0	1	4	1	0
y_2	0	-2	0	-10	-1	1

Following the pivot operations, we have identified the optimal solution for this auxiliary problem:

$$x^* = (0, 2, 0), y^* = (0, 0)$$

This optimal solution yields an objective value of $v^* = 0$.

To achieve this result, we selected the coefficient -2 from the row associated with y_2 as the pivot.

		x_1	x_2	x_3	y_1	y_2
$-v$	0	0	0	0	1	1
x_2	2	0	1	4	1	0
x_1	0	1	0	5	1/2	-1/2

Hence, this represents an optimal basic feasible solution for (P_A), and the corresponding values $x = (0, 2, 0)$ form a basic feasible solution for (P). In the context of the original linear program, the objective function is given as:

$$z = x_1 + x_2 + 10x_3$$

It's important to note that this expression is not in canonical form, primarily because x_3 is considered a non-basic variable. However, by performing the following substitution:

$$\begin{cases} x_2 = 2 - 4x_3 \\ x_1 = -5x_3 \end{cases} \implies z = 2 + x_3$$

Now, we derive the tableau that corresponds to the initial basic feasible solution for (P):

		x_1	x_2	x_3
$-z$	-2	0	0	1
x_2	2	0	1	4
x_1	0	1	0	5

Since the basic feasible solution is already optimal, there is no need for a phase two.

4.4.8 Polynomial-time algorithms for linear programming

Over the years, polynomial-time algorithms for linear programming have been developed, including:

- *Ellipsoid method* (Khachiyan, 1979), which holds significant theoretical importance.
- *Interior point methods* (Karmarkar, 1984), known for their highly efficient variants when dealing with certain types of problem instances.

4.5 Linear programming duality

A closely related maximization (minimization) linear program can be associated with any minimization (maximization) linear program, sharing the same parameters. Despite differing spaces and objective functions, the optimal values of the objective functions generally coincide. This duality is valuable because determining the best lower bound (maximum) is challenging, while finding the best upper bound is relatively simpler.

The general strategy involves linearly combining the constraints with non-negative multiplicative factors.

Example:
Consider the following original problem:

$$\begin{aligned} \max \ & 4x_1 + x_2 + 5x_3 + 3x_4 \\ \text{such that } & x_1 - x_2 - x_3 + 3x_4 \leq 1 \\ & 5x_1 + x_2 + 3x_3 + 8x_4 \leq 55 \\ & -x_1 + 2x_2 + 3x_3 - 5x_4 \leq 3 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

The dual problem can be constructed for easier solving:

$$\begin{aligned} \min \ & y_1 + 55y_2 + 3y_3 \\ \text{such that } & y_1 + 5y_2 - y_3 \geq 4 \\ & -y_1 + y_2 + 2y_3 \geq 1 \\ & -y_1 + 3y_2 + 3y_3 \geq 5 \\ & 3y_1 + 8y_2 - 5y_3 \geq 3 \\ & y_1, y_2, y_3 \geq 3 \end{aligned}$$

Property 4.5.1. The dual of the dual problem coincides with the primal problem.

Primal	*min*	*max*	**Dual**
Constraints	$\geq b_j$	$\geq b_j$	Variables
	$\leq b_j$	$\leq b_j$	
	$= b_j$	b_j free	
Variables	$\geq b_j$	$\leq b_j$	Constraints
	$\leq b_j$	$\geq b_j$	
	$= b_j$	b_j free	

Table 4.1: Rules to transform a problem into its dual

4.5.1 Weak duality theorem

Theorem 4.5.1 (Weak duality theorem)**.** *Given the primal problem:*

$$\begin{aligned} \min \ & z = c^T \\ \text{such that} \ & Ax \geq b \\ & x \geq 0 \end{aligned}$$

And its dual:

$$\begin{aligned} \max \ & w = b^T y \\ \text{such that} \ & A^t y \leq c \\ & y \geq 0 \end{aligned}$$

With $X = \{x | Ax \geq b, x \geq 0\} \neq \varnothing$ and $Y = \{y | A^T y \leq c, y \geq 0\} \neq \varnothing$. For every feasible solution $x \in X$ of the primal problem and every feasible solution $y \in Y$ of the dual problem, we have:

$$b^T y \leq c^T x$$

Proof. For every pair $x \in X$ and $y \in Y$, we have $Ax \geq b$, $x \geq O$ and $A^T y \leq c$, $y \geq 0$, which imply that:

$$b^T y \leq x^T A^T y \leq x^T c = c^T x$$

□

As a result, if x is a feasible solution of (P) $(x \in X)$, y is a feasible solution of (D) $(y \in Y)$, and the values of the respective objective functions coincide, $c^T x = b^T y$, then x is optimal for (P) and y is optimal for (D). The optimal solutions are denoted by x^* and y^*.

4.5.2 Strong duality theorem

Theorem 4.5.2 (Strong duality theorem)**.** *If $X = \{x | Ax \geq b, b, x \geq 0\}$ and $\min\{c^T x | x \in X\}$ is finite, there exist $x^* \in X$ and $y^* \in Y$ such that $c^T x^* = b^T y^*$.*

This is equivalent to saying:

$$\min\{c^T x | x \in X\} = \max\{b^T y | y \in Y\}$$

Proof. Derive an optimal solution of (D) from one of (P). Given the primal problem:

$$\begin{aligned} \min \ & z = c^T \\ \text{such that} \ & Ax = b \\ & x \geq 0 \end{aligned}$$

And its dual:

$$\begin{aligned} \max \ & w = b^T y \\ \text{such that} \ & y^t A \leq c^T \\ & y \in \mathbb{R}^m \end{aligned}$$

And:

$$x^* = \begin{bmatrix} x_B^* \\ x_N^* \end{bmatrix}$$

Here, $x_B^* = B^{-1}b$ and $x_N^* = 0$ an optimal feasible solution of (P). This solution is obtained after a finite number of iterations using the Simplex algorithm with Bland's rule.

Consider $\overline{y}^T = c_B^T B^{-1}$. Verify that $\overline{y}$ is a feasible solution of (D). For the non-basic variables:

$$\overline{c}_N^T = c_N^T - (c_B^T B^{-1})N = c_N^T - \overline{y}^T N \geq 0^T \implies \overline{y}^T N \geq c_N^T$$

For the basic variables:

$$\overline{c}_B^T = c_B^T - (c_B^T B^{-1})B = c_B^T - \overline{y}^T B \geq 0^T \implies \overline{y}^T B \geq c_B^T$$

According to weak duality, $\overline{y}$ is an optimal solution for (D):

$$\overline{y}^T b ? (c_B^T B^{-1})b = c_B^T (B^{-1}b) = c_B^T x_B^* = c^T x^*$$

Thus, $\overline{y} = y^*$ □

Consequences In linear programming problems, only one of the three cases can occur:

1. There exists an optimal solution.
2. The problem is unbounded: the optimal cost is $+\infty$ for maximization problems and $-\infty$ for minimization problems.
3. The problem is infeasible.

		Dual		
		Finite optimum	*Unbounded*	*Infeasible*
Primal	*Finite optimum*	✓	×	×
	Unbounded	×	×	✓
	Infeasible	×	✓	✓

4.5.3 Complementary slackness

Theorem 4.5.3 (Complementary slackness)**.** *Let $x \in X^*$ and $y \in Y^*$ be feasible solutions to the primal and dual problem, respectively. Then x and p are optimal for their respective problems if and only if:*

$$y_i^* \left(a_i^T x - b_i\right) = 0 \quad \forall, i \in \{i = 1, \ldots, m\}$$

$$\left(c_j - y^T A_j\right) x_j^* = 0 \quad \forall, j \in \{j = 1, \ldots, n\}$$

Where a_i denotes the i-th row of A and A_j the j-th column of A.

At optimality, the product of each variable with the corresponding slack variable in the constraint of the dual problem must be zero.

4.6 Sensitivity analysis

The sensitivity analysis of a linear programming problem involves examining the impact of minor modifications to the problem's data on the optimal solution. It assesses the sensitivity of an optimal solution concerning variations in the model parameters.

Definition (*Shadow price*)**.** The shadow price associated with the i-th resource is the maximum price the company is willing to acquire an additional unit of the i-th resource.

4.6.1 Algebraic form

Examine the sensitivity of an optimal solution concerning variations in the model parameters $\langle c_j, a_{ij}, b_i \rangle$. Given the linear programming problem:

$$\begin{aligned} \min \ & c^T \\ \text{such that } \ & Ax = b \\ & x \geq 0 \end{aligned}$$

And optimal basic solution x^*:

$$x_B^* = B^{-1} b \geq 0$$

$$x_N^* = 0$$

The objective is to assess the limits that maintain optimality for B under two conditions:

1. *Feasibility*: $B^{-1} b \geq 0$.
2. *Optimality*: $c_N^T = c_N^T - c_B^T B^{-1} N = 0^T$.

Changes in the requirement vector b Consider a change in a component b_i of the requirements vector b by a small amount δ. The updated requirements vector is given by:

$$b^T := b + \delta e_k$$

Here, e_k is the unitary vector with a value of 1 on the k-th position ($1 \leq k \leq n$) and 0 elsewhere. The goal is to determine the range of values for δ within which the current basis remains optimal.

Since the optimality conditions are unaffected by the change, only the feasibility condition needs verification. The basis B with the basic solution:

$$x^* = \begin{bmatrix} B^{-1}b^T \\ 0 \end{bmatrix} = \begin{bmatrix} B^{-1}\left(b + \delta e_k\right) \\ 0 \end{bmatrix}$$

remains optimal as long as the following relation holds:

$$B^{-1}\left(b + \delta e_k\right) \geq 0 \Rightarrow B^{-1}b \geq -\delta B^{-1}e_k$$

This results in m inequalities defining an interval of values for δ. If δ falls outside this allowed range, the current solution satisfies the optimality conditions but is primal infeasible. While the basis B remains optimal, the optimal basic feasible solution x^* changes. The objective function value shifts from $c_B^T B^{-1}b$ to $c_B^T B^{-1}\left(b + \delta e_k\right)$. Consequently, the optimal value of the dual variable is given by:

$$\Delta z^* = c_B^T B^{-1}\left(\delta e_k\right) = \delta y_k^* = \delta \frac{\partial z^*}{\partial b_k}$$

where Δz^* represents the shadow price of the k-th resource.

Changes in the cost vector c Consider a change in a cost coefficient c_j of the objective function c by a small amount δ. The updated cost vector is given by:

$$c^T := c + \delta e_j$$

Here, e_j is the unitary vector with a value of 1 on the j-th position ($1 \leq j \leq n$) and 0 elsewhere. The primal feasibility condition is unaffected, so the only condition to verify is the optimality condition:

$$c_B^T B^{-1} A \leq c^T$$

Two cases then arise:

1. c_j is the cost of a basic variable: the optimal solution remains unchanged.
2. c_j is the cost of a non-basic variable: the optimal solution changes.

However, the basic solution does not change:

$$x_B^* = B^{-1}b \ \wedge x_N^* = 0$$

Cost associated with a non-basic variable If c_j is the cost coefficient of a non-basic variable x, then c_B is not affected, and the only inequality impacted is the one for the reduced cost of x_j:

$$c_B^T B^{-1} A_j \leq c_j + \delta$$

This inequality establishes a lower bound for δ:

$$\delta \geq -c_j$$

If this condition is satisfied, the current basis is optimal. The reduced cost represents the maximum decrease of c_j for which B remains optimal.

Cost associated with a basic variable If c_j is the cost coefficient of the l-th basic variable, then c_B is affected by the change in c_j:

$$c_B^T := c_B + \delta e_l$$

All the optimality conditions are then affected by the change in c_j:

$$\left(c_B + \delta e_l\right)^T B^{-1} A \leq c_i \quad \forall i \neq j$$

The variable x_j is skipped as it is basic, and its reduced cost stays at zero. The condition is equal to:

$$\delta_{q_{li}} \leq c_i \quad \forall i \neq j$$

Here, q_{li} is the coefficient corresponding to the position of the l-th basic variable in the i-th row of A. These inequalities determine the range of δ for which the same basis remains optimal.

CHAPTER 5

Integer linear programming

5.1 Introduction

Definition (*Integer linear programming problem*)**.** An Integer linear programming (ILP) problem is an optimization problem of the form:

$$\begin{aligned} \min \ & c^T x \\ \text{such that } \ & Ax = b \\ & x \in \mathbb{Z}^n \end{aligned}$$

Additionally:

- If $x_j \in \{0, 1\}\ \forall, j$, the problem is called binary linear programming.
- If $\exists, i$ such that $x_i \notin \mathbb{Z}^n$, then the problem is called mixed integer linear programming.

Note that the integrality condition $x_i \in \mathbb{Z}$ is non-linear, as it can be expressed as $\sin(\pi x_j) = 0$.

Definition (*Linear relaxation*)**.** Let ILP be an ILP problem:

$$\begin{aligned} z_{ILP} := \min \ & c^T x \\ \text{such that } \ & Ax \leq b \\ & x \in \mathbb{Z}^n \\ & x \leq 0 \end{aligned}$$

Then the Linear Programming (LP) problem:

$$\begin{aligned} z_{LP} := \max \ & c^T x \\ \text{such that } \ & Ax \leq b \\ & x \leq 0 \end{aligned}$$

is the linear (or continuous) relaxation of ILP.

Property 5.1.1 (Bounds of ILP solutions)**.** For any ILP with max, the optimal solution is bounded by the optimal solution of the LP relaxation:

$$z_{ILP} \leq z_{LP}$$

For any ILP with min, the optimal solution is bounded by the optimal solution of the LP relaxation:

$$z_{ILP} \geq z_{LP}$$

The feasible region of any ILP is a lattice of points, either finite or infinite, depending on the type of problem. By removing the integrality constraint, the ILP problem becomes an LP problem, and the optimal solution of the ILP problem isn't always the optimal solution of the LP problem.

(a) Lattice of integer points (b) Relaxed lattice of integer points

Figure 5.1: Lattice of integer points

5.1.1 Solutions of the ILP problem

A viable approach to finding a solution to the ILP problem is to identify a solution to the LP problem and then round it to the nearest integer point. If an optimal solution of the LP problem is an integer, it is also an optimal solution of the ILP problem. However, often the rounded optimal solutions of the LP are either:

- *Infeasible* solutions for the ILP.
- *Useless* solutions for the ILP, as they are very different from an optimal solution of the ILP.

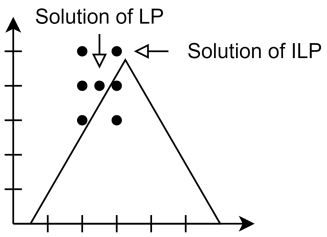


Figure 5.2: Solution of the LP relaxation of the ILP problem

5.1.2 Solution of assignment and transportation problems

Two crucial ILP problems, namely the assignment and transportation problems, have a solution that is the optimal solution of the LP relaxation.

Assignment problem Given:

- m machines, $i = 1, \ldots, m$
- n jobs, $j = 1, \ldots, n$, $n < m$
- c_{ij} cost of assigning job j to machine i

The goal is to determine an assignment of jobs to the machines to minimize the total cost. Each job must be assigned to exactly one machine, and each machine should have at least one job. The binary variables x_{ij} represent the assignment, where:

$$x_{ij} = \begin{cases} 1 & \text{if job } j \text{ is assigned to machine } i \\ 0 & \text{otherwise} \end{cases}$$

The assignment problem can be formulated as the following ILP problem:

$$\begin{aligned} \min \quad & \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} \\ \text{s.t.} \quad & \sum_{i=1}^{m} x_{ij} = 1 \quad \forall j \quad \text{at most one machine for each job} \\ & \sum_{j=1}^{n} x_{ij} = 1 \quad \forall i \quad \text{at least one job for each machine} \\ & x_{ij} \in \{0, 1\} \quad \forall i, j \end{aligned}$$

Transportation problem Given:

- m productions plant, $i = 1, \ldots, m$.
- n clients, $j = 1, \ldots, n$, $n > m$ by assumption.
- c_{ij} cost of shipping one unit from plant i to client j.
- p_i production capacity of plant i.
- d_j demand of client j.
- $q_{ij} \geq 0$ quantity shipped from plant i to client j.

The goal is to determine a transportation plan that minimizes the total costs while satisfying the production and demand constraints. The assumption is $\sum_{i=1}^{m} p_i \geq \sum_{j=1}^{n} d_j$. Variables x_{ij} represent the quantity shipped from plant i to client j.

Problems analysis In the assignment problem example, a forcing constraint is present, while the transportation problem introduces a constraint limiting the active number of variables.

Definition (*Forcing constraint*)**.** A constraint in the form:

$$x \leq y$$

is termed a forcing constraint if both x and y are binary variables.

Definition (*Constraint on binary variables*)**.** A constraint of the form:

$$\sum_{i=1}^{n} x_i \leq 1$$

where all x_i are binary variables, implies that at most one of the variables x_i can be one. Similarly, a constraint in the form:

$$\sum_{i=1}^{n} x_i = 1$$

where all x_i are binary variables, implies that exactly one variable x_i must be one.

The transportation problem illustrates that the optimal solution of the LP relaxation of the transportation problem is also the optimal solution of the ILP problem.

Theorem 5.1.1 (Solution of the transportation problem)**.** *If in a transportation problem p_i, d_{ij}, q_{ij} are all integers, all the basic feasible solutions (vertices) of its linear relaxation are integers.*

Proof. Let A be an integer constraint matrix of size $(mn + n + m) \times (mn)$, where $a_{ij} \in \{-1, 0, 1\}$. The right-hand side vector b is composed of integer elements. The optimal solution for the linear relaxation is:

$$x^* = \begin{bmatrix} B^{-1}b \\ 0 \end{bmatrix} \qquad B^{-1} = \frac{1}{|B|} \begin{bmatrix} \alpha_{11} & \dots & \alpha_{1n} \\ \dots & \dots & \dots \\ \alpha_{m1} & \dots & \alpha_{mn} \end{bmatrix}$$

where $\alpha_{ij} = (-1)^{i+j} \det(M_{ij})$, and M_{ij} is the square submatrix obtained from B by deleting the i-th row and the j-th column. Then:

- B is integer $\Rightarrow$ α_{ij} is an integer.
- $\det(B) = \pm 1 \Rightarrow B^{-1}$ is an integer $\Rightarrow x^*$ is an integer.
- It can be shown that A is totally unimodular, implying $\det(Q) = \{-1, 0, 1\}$ for any square submatrix Q of A.

□

Complexity of the ILP problem Most ILP problems are $\mathcal{NP}$-hard, meaning that there is no known algorithm capable of solving them and proving the solution's correctness in polynomial time. Various methods exist for finding optimal solutions, categorized as follows:

- *Implicit enumeration* methods: these aim to provide an exact solution, i.e., a global optimum. Examples of this category include branch and bound and dynamic programming methods.
- *Cutting planes* methods: these aim to provide an exact solution, i.e., a global optimum.
- *Heuristic* algorithms: these aim to provide an exact solution, i.e., a local optimum. Greedy and local search algorithms fall into this category.

5.2 Branch and bound method

The Branch and Bound method employs a divide and conquer strategy to address ILP problems by systematically exploring the set of feasible solutions. Let F represent the set of feasible solutions for the problem:

$$\begin{aligned} &\min\ c^T \\ &\text{such that } x \in F \end{aligned}$$

The method unfolds in two main phases:

1. *Branch*: the problem is partitioned into simpler subproblems.
2. *Bound*: the subproblems are solved, and the optimal solution is determined.

These two phases are applied recursively until a solution is discovered. If a problem remains unsolved during the bound phase, it is branched again until it becomes simple enough to find a solution. It's worth noting that this technique is applicable to a broad spectrum of problems, not exclusively limited to ILP problems.

Branching The set of solutions F is divided into k disjoint subsets:

$$F = F_1 \cup, \ldots, \cup F_k \quad F_i \cap F_j = \emptyset\ \forall, i \neq j$$

Let z_i be defined as:

$$z_i = \min\{c(x), |, x \in F_i\} \quad i = 1, , \ldots, , k$$

The solution z is then determined as:

$$z = \min\{c(x), |, c \in F\} = \min\{z_i, , \ldots, , z_k\}$$

Bounding For each subproblem $\min\{c(x), |, x \in F_i\}$, whose solution is found in the partition of F, the optimal solution z_i is determined by either:

1. Finding an optimal solution for $\min\{c(x), |, x \in F_i\}$.
2. Proving that $F_i = \emptyset$.
3. Proving that $z_i \geq z$, where z is the optimal solution of the original problem found so far.

If the subproblem is not resolved, it is necessary to generate a new subproblem by branching it again.

Generic branch and bound method If the lower bound $b(F_i)$ of the optimal solution for the subproblem $\min\{c(x), |, x \in F_i\}$ is greater than the optimal solution z found so far, the subproblem is discarded, as it cannot contain a better solution than the one found so far. At any point, the algorithm keeps in memory a set of active subproblems and the cost U of the best feasible solution found so far. Initially, U is either ∞ or the cost of the best feasible solution found by the heuristic method (if any). A typical step of the algorithm is:

1. Select an active subproblem F_i to be solved.

2. If the subproblem is infeasible, delete it; otherwise, compute $b(F_i)$ for the corresponding subproblem.
3. If $b(F_i) \geq U$, delete F_i.
4. If F_i is feasible and $b(F_i) < U$, either:
    - Obtain an optimal solution to the subproblem.
    - Break the corresponding subproblem into further subproblems, which are added to the set of active subproblems.

There are several parameters that can be arbitrarily chosen. There is no fixed rule for many of them, as the best choice depends on the problem to be solved:

1. There are multiple ways of choosing an active subproblem.
2. There are multiple ways of breaking a subproblem into further subproblems.
3. There are multiple ways of computing the lower bound $b(F_i)$.

5.2.1 Branching tree

The branching tree serves as a visual depiction of the branching progression within the Integer Linear Programming (ILP) problem. Each node in this tree corresponds to a subproblem derived from the larger ILP problem, and the edges symbolize the steps taken in the branching process. Notably, a branching tree may not encompass all conceivable nodes or all potential leaves, which amount to 2^d. A node in the tree is considered "fathomed" (having no child) under the following conditions:

1. The initial constraint, along with those on the arcs from the root to the node, renders the subproblem infeasible.
2. The optimal solution of the linear relaxation is an integer.
3. The value $C^T\overline{x}_{\text{LP}}$ of the optimal solution $\overline{x}_{\text{LP}}$ from the linear relaxation is inferior to the best feasible solution z discovered thus far.

This set of criteria, termed the branching criterion, enables the exclusion of numerous nodes (representing subproblems) that would not contribute to a valid solution, thereby curtailing the size of the tree.

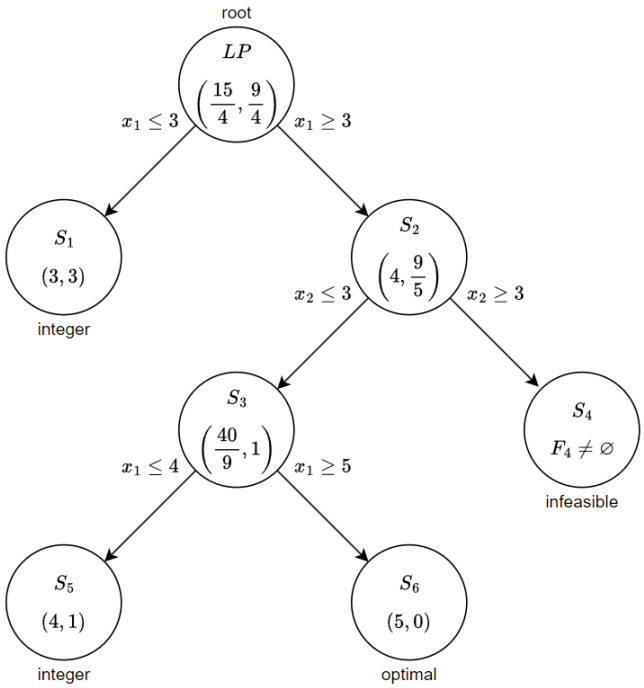


Figure 5.3: Branching tree

5.2.2 Branch and bound for the ILP problem

In the context of an Integer Linear Programming (ILP) problem given by:

$$\min\left\{c^T x, |, Ax = b,\ x \in \mathbb{N}\right\}$$

the Branch and Bound method proves to be an effective strategy for determining the optimal solution z.

Branching The first step involves partitioning the feasible region F. Let $\overline{x}$ represent an optimal solution of the linear relaxation of the ILP:

$$\min\left\{c^T x, |, Ax = b,\ x \geq 0,\ x \in \mathbb{R}\right\} \quad z_{\text{LP}} = c^T \overline{x}$$

If $\overline{x}$ is an integer, it serves as the optimal solution for the ILP problem. However, if $\overline{x}$ is non-integer, the ILP is divided into two subproblems:

$$\min\left\{c^T x, \big|, Ax = b, x \leq \lfloor \overline{x} \rfloor\right\}$$

$$\min\left\{c^T x, \big|, Ax = b, x \geq \lceil \overline{x} \rceil\right\}$$

Here, $\lfloor \overline{x} \rfloor$ and $\lceil \overline{x} \rceil$ represent the floor and ceiling of the vector $\overline{x}$. While choosing the variable x_h closest to 0.5 for branching can be a common approach, it may not always yield the best results. An alternative and more effective strategy is strong branching, wherein candidate variables (specifically, fractional basic ones) are considered. Evaluation of the corresponding objective function values helps determine the variable that, upon branching, produces the most substantial improvement in the objective function value.

Bounding The process of bounding involves determining a lower or upper bound on the optimal value z_i of a subproblem within the Integer Linear Programming (ILP). The nature of the bound depends on whether the ILP is a minimization or maximization problem. This is achieved by solving the linear relaxation of the subproblem. To decide which node (subproblem) to address next, the following criteria can be considered:

- *Deeper nodes first* (depth-first search strategy): this strategy selects the node with the greatest depth in the tree of subproblems. While simple to implement and optimize, it may result in a large tree of subproblems if the branching variable is chosen incorrectly.
- *More promising nodes first* (best-bound first strategy): this approach prioritizes the node with the best linear relaxation value $b(F_i)$. Although it generates a smaller tree of subproblems, the subproblems are less constrained, leading to longer times to find a feasible solution and improve it.

5.2.3 Remarks on the branch and bound method

Branch and Bound extends its utility to mixed Integer Linear Programming (ILP) problems. During the branching step, treating fractional variables as integers and vice versa facilitates its application.

Initiating the method with a well-crafted initial feasible solution from a heuristic algorithm can enhance performance. This contributes to obtaining more accurate lower or upper bounds in the context of maximization or minimization problems, respectively.

As a versatile approach, Branch and Bound can serve as a heuristic method for ILP problem resolution, halting when the best feasible solution discovered surpasses the best integer solution identified thus far.

Efficient solution of the linear relaxation Iteratively solving the linear relaxations of ILP subproblems from scratch at each branching step is unnecessary. The solution of the linear relaxation at the parent node can be utilized as a starting point for solving the linear relaxation at the child node.

Employing a single iteration of the Dual Simplex method (the simplex method applied to the dual problem) efficiently yields an optimal solution to the linear relaxation with an additional constraint. This optimization builds upon the optimal Branch and Bound outcome of the preceding node (linear relaxation of the parent subproblem).

Applicability of branch and bound approach The applicability of the Branch and Bound method extends beyond discrete optimization problems to encompass numerous non-linear optimization challenges. However, adaptations are required in two key procedures:

- The division of the feasible region into subregions (branching).
- The determination of a bound on the optimal value of the subproblem (bounding).

5.3 Cutting plane method and Gomory fractional cuts

Considering the ILP problem:

$$\begin{aligned} \min \ & c^T x \\ \text{such that } & Ax \geq b \\ & x \in \mathbb{N}, x \geq 0 \end{aligned}$$

with a feasible region:

$$X = \{x \in \mathbb{Z} \mid Ax \geq b, x \geq 0\}$$

Assuming that $a_{ij} \in N$, $b_i \in N$, $c_i \in N$, the feasible region can be expressed by different sets of constraints, varying in strength. Although all infinite formulations are equivalent, the optimal solution of the linear relaxations of the ILP (x^*_{LP}) may exhibit significant variation.

Definition (*Ideal formulation*)**.** The ideal formulation describes the convex hull of X, denoted as conv(X), representing the smallest convex subset containing X.

Given that all vertices have integer coordinates, for any $c \in \text{conv}(X)$, the following relation holds:

$$z^*_{LP} = z^*_{ILP}$$

This implies that the optimum of the linear programming relaxation is also the optimum of the ILP.

Theorem 5.3.1 (Ideal formulation)**.** *For any feasible region X whether bounded or unbounded, there exists an ideal formulation—a description of conv(X) involving a finite number of linear constraints.*

However, the number of constraints can be substantial, growing exponentially with the size of the original formulation.

This theorem suggests that the solution of any ILP can be reduced to that of a single LP. Nevertheless, determining the ideal formulation is often challenging, as it tends to be either very extensive or difficult to ascertain.

5.3.1 Cutting plane method

A detailed characterization of conv(X) isn't necessary; what's essential is a robust description of the vicinity surrounding the optimal solution.

Definition (*Cutting plane*)**.** A cutting plane is represented by the inequality $a^T x \leq b$.

This inequality is not met by x^*_{LP} but holds true for all feasible solutions within the ILP.

5.3.2 Cutting plane methods and Gomory fractional cuts

In the iterative process, cutting planes are successively added until the linear relaxation yields an optimal integer solution. Consider the optimal solution x^*_{LP} of the linear relaxation for the ILP problem formulated as:

$$\min \left\{c^T x \mid Ax = b, x \geq 0\right\}$$

Let x^*_{Br} denote the fractional basic variable. The corresponding row of the optimal tableau is expressed as:

$$x_{B[r]} + \sum_{j|x_j \in \mathbb{N}} \overline{a}_{rj} x_j = \overline{b}_r$$

Definition (*Gomory cut*)**.** The Gomory cut, concerning the fractional basic variable x_{Br}, takes the form of the inequality:

$$\sum_{j|x_j \in \mathbb{N}} \left(\overline{a}_{rj} - \lfloor \overline{a}_{rj} \rfloor\right) x_j \geq \overline{b}_r - \lfloor \overline{b}_r \rfloor$$

Property 5.3.1. Both the integer form:

$$x_{B[r]} + \sum_{j|x_j \in \mathbb{N}} \lfloor \overline{a}_{rj} \rfloor x_j \leq \lfloor \overline{b}_r \rfloor$$

and the fractional form:

$$x_{B[r]} + \sum_{j|x_j \in \mathbb{N}} \left(\overline{a}_{rj} - \lfloor \overline{a}_{rj} \rfloor\right) x_j \geq \overline{b}_r - \lfloor \overline{b}_r \rfloor$$

are equivalent.

5.3.3 Algorithm

Algorithm 9 Cutting plane method

1: solve the linear relaxation of the ILP problem $\min\left\{c^T x | Ax = b, x \geq 0\right\}$
2: let x^*_{LP} be the optimal solution of the linear relaxation
3: **while** x^*_{LP} is not integer **do**
4: select a basic variable $x_{B[r]}$ with fractional value
5: generate the corresponding Gomory cut
6: add constraint to the optimal tableau of the linear relaxation
7: perform one iteration of the dual simplex method
8: **end while**

Theorem 5.3.2. *If the ILP has a finite number of optimal solutions, the Cutting plane method with Gomory cuts is guaranteed to find an optimal solution.*

However, the algorithm's required number of iterations is unknown in advance and often proves to be very large.

5.3.4 Alternative techniques

Other cutting planes Various generic cutting planes, tailored for specific problems, are available beyond Gomory cuts. A comprehensive exploration of the combinatorial structure of different problems has resulted in:

- The characterization of entire classes of facets.
- The development of efficient procedures for generating these facets.

Branch-and-cut The approach of branch-and-cut integrates elements from both branch-and-bound and cutting plane methods, seeking to address the limitations of each. In the branch-and-cut method, for each subproblem, a set of cuts that are valid for that specific subproblem is generated and added to its formulation. When these cutting planes become less effective, the cut generation is halted, and a branching operation is executed. This process strengthens the formulation of various subproblems, interrupting long sequences of cuts without substantial improvements through branching operations.

CHAPTER 6

Python mip package

6.1 Linear program general structure

```
import mip
# declare the number of variables
N = 10
# declare I and J as a range
I = range(N)
J = range(N)
# declare d as a NxN matrix
d = [[0 for j in J] for i in I]
# declare the model
model = mip.model()
# problem:
#   x_i binary variables for i in I
#   y_ij >= 0 for i in I, j in J
#   minimize sum_i sum_j d_ij y_ij + 100 sum_i x_i
x = [ model.add_var(name = str(i), var_type = mip.BINARY) for i in I ]
y = [ model.add_var(name = str(i) + str(j), var_type = mip.CONTINUOUS, lb = 0)
    ↪ for i in I for j in J ]
# declare the objective function
model.objective = mip.minimize( mip.xsum(d[i][j] * y[i][j] for i in I for j in J)
    ↪ + 100 * mip.xsum(x[i] for i in I))
# declare constraints
for i in I:
  model.add_constr(mip.xsum(y[i][j] <= x[i] if j != i else 0))
# optimize the model
model.optimize()
```

6.2 Quick documentation

Model The model declaration is done with the following syntax:

```
model = mip.Model(name = "", sense = mip.MINIMIZE, solver_name = mip.CBC)
```

Here, `name` is the model name, `sense` is the model sense, and `solver_name` is the solver name. The possible senses are:

- *Minimize*: `mip.MINIMIZE` or `min`.
- *Maximize*: `mip.MAXIMIZE` or `max`.
- *Knapsack*: `mip.KNAPSACK` or `knap`.

Variables The variable declaration is done with the following syntax:

```
model.add_var(name = "", lb = 0.0, ub = inf, obj = 0.0, var_type = mip.CONTINUOUS)
```

Here, `name` is the variable name, `lb` is the lower bound, `ub` is the upper bound, `obj` is the objective coefficient, and `var_type` is the variable type. The possible types (also for constraints) are:

- *Binary variable*: `mip.BINARY` or `B`.
- *Integer variable*: `mip.INTEGER` or `I`.
- *Continuous variable*: `mip.CONTINUOUS` or `C`.

Constraints The constraint declaration is done with the following syntax:

```
model.add_constr(lin_expr, name = "", priority = None)
```

Here, `lin_expr` is the linear expression, `name` is the constraint name, and `priority` is the constraint priority. The possible types (also for constraints) are:

- *Binary constraint*: `mip.BINARY` or `B`.
- *Integer constraint*: `mip.INTEGER` or `I`.
- *Continuous constraint*: `mip.CONTINUOUS` or `C`.

Linear expressions The summation of linear expression is done with the following syntax:

```
mip.xsum(terms)
```

Here, `terms` is the list of terms. This function is used to create a linear expression from a summation of terms.