Foundation Of Operations Research
Exercises

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Abstract

Operations Research is the branch of applied mathematics dealing with quantitative methods to analyze and solve complex real-world decision-making problems.

The course covers some fundamental concepts and methods of Operations Research pertaining to graph optimization, linear programming and integer linear programming.

The emphasis is on optimization models and efficient algorithms with a wide range of important applications in engineering and management.

Contents

Chapter 1

Exercise session I

1.1 Linear programming modeling

A bank has a capital of C billions of Euro and two available stocks:

1. With an annual revenue of 15% and risk factor of $\frac{1}{3}$.
2. With an annual revenue of 25% and risk factor of 1.

The risk factor represents the maximum fraction of the stock value that can be lost. A risk factor of 25% implies that, if stocks are bought for 100 euro up to 25 euro can be lost. It is required that at least half of C is risk-free. The amount of money used to buy stocks of two must not be larger than two times that used to buy stocks of one. At least $\frac{1}{6}$ of C must be invested into one.

Give a Linear Programming formulation for the problem of determining an optimal portfolio for which the profit is maximized. Solve the problem graphically.

Solution

- The parameters are:
  - The quantity of available capital C.
- The decision variables are:
  - The amount of money invested in stock of type one x_1.
  - The amount of money invested in stock of type two x_2.

- The objective function is:

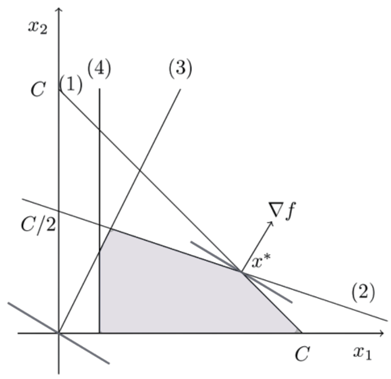
$$\max\left[0.15x_1 + 0.25x_2\right]$$

- The constraints are:
  - Maximum capital:
  $$x_1 + x_2 \leq C$$
  - Half of the invested capital is risk-free:
  $$\frac{1}{3}x_1 + x_2 \leq \frac{C}{2}$$
  - The amount of money used to buy stocks of two must not be larger than two times that used to buy stocks of one:
  $$x_2 \leq 2x_1$$
  - At least $\frac{1}{6}$ of C must be invested into one:
  $$x_1 \geq \frac{1}{6}C$$
  - Constraint on the variables:
  $$x_1, x_2 \geq 0$$

To solve the problem graphically, we must identify the feasible region in $\mathbb{R}^2$ that satisfies the constraints. To draw a constraint, it suffices to find any two points that satisfy it with equality (as an equation). The border of the constraint is then represented by the only line containing such points. There are two possible ways to identify which of the two half planes is the feasible one:

- In the first one, it suffices to pick a random point and checking whether it satisfies the constraint. If it does, the half space to which the point belongs is the feasible one, otherwise the other half space is.
- Alternatively, we can consider the gradient of the constraint and compare it to the direction of the inequality.

The region found with the constraints is the following:



The feasible region is as shown in the picture. To find the feasible point where the objective function attains its maximal value, we can draw the level curves $f(x_1, x_2) = 0.15x_1 + 0.25x_2 = z$ where each level curve is the set of points whose objective function value is equal to z, for any constant z.

Since f is linear, the level curve $f(x_1, x_2) = z$ is a line, orthogonal to its gradient, and parametric in z. When z is increased, we obtain parallel level lines that move towards the direction of the gradient $\nabla f(x_1, x_2)$.

Note that, by starting with $z = 0$ and by increasing it in a continuous way, the level lines of f will first intersect the feasible region at $\left(\frac{C}{6}, 0\right)$, and then, increasing z, at any other point, until the intersection is empty. The last feasible point having a nonempty intersection is the maximizer of f over the feasible set. In this problem there is a single maximizer. The maximizer, denoted by x^*, can be found as the solution to the following linear system:

$$\begin{cases} x_1 + x_2 = C \\ \frac{1}{3}x_1 + x_2 = \frac{C}{2} \end{cases}$$

which yields $x^* = \left(\frac{3C}{4}, \frac{C}{4}\right)$, where $f(x^*) = \frac{7C}{40}$.

1.2 Linear programming modeling

A refinery produces two types of gasoline, mixing three basic oils according to the following gasoline mixture rules:

	Oil 1	**Oil 2**	**Oil 3**	**Revenue**
Gasoline A	$\leq 30\%$	$\geq 40\%$	-	5.5
Gasoline B	$\leq 40\%$	$\geq 10\%$	-	4.5

The last column of the previous table indicates the profit (euro/barrel). The availability of each type of oil (in barrel) and the cost (euro/barrel) are as follows:

Oil	**Availability**	**Cost**
1	3 000	3
2	2 000	6
3	4 000	4

Give a Linear Programming formulation for the problem of determining a mixture that maximizes the profit (difference between revenues and costs).

Solution

- The decision variables are:
  - The amount of the i-th oil used to produce the j-th gasoline, $i \in \{1, 2, 3\}$ and $j \in \{A, B\}$ x_{ij}.
  - The amount of gasoline of type j-th that is produced, $j \in \{A, B\}$ y_j.
- The objective function is:

$$\max 5.5y_A + 4.5y_B + 3(x_{1A} + x_{1B}) - 6(x_{2A} + x_{2B}) - 4(x_{3A} + x_{3B})$$

- The constraints are:
  - Availability of 1:

$$x_{1A} + x_{1B} \leq 3000$$

  - Availability of 2:

$$x_{2A} + x_{2B} \leq 2000$$

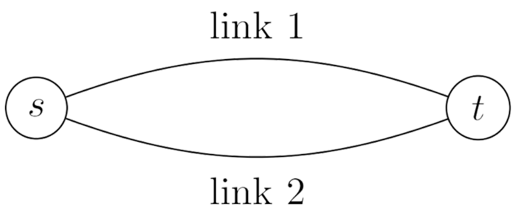
- Availability of 3:
$$x_{3A} + x_{3B} \leq 4000$$
- Conservation of A:
$$y_A = x_{1A} + x_{2A} + x_{3A}$$
- Conservation of B:
$$y_B = x_{1B} + x_{2B} + x_{3B}$$
- Minimum quantity of A:
$$x_{1A} \leq 0.3y_A$$
- Minimum quantity of B:
$$x_{1B} \leq 0.5y_B$$
- Maximum quantity of A:
$$x_{2A} \geq 0.4y_A$$
- Maximum quantity of B:
$$x_{2B} \geq 0.1y_B$$
- The variable must be non-negative:
$$x_{1A}, x_{2A}, x_{3A}, x_{1B}, x_{2B}, x_{3B}, y_A, y_B \geq 0$$

Chapter 2

Exercise session II

2.1 Linear programming modeling

Assume that n packets of data must be routed from node s to node t, along one of two available links, with capacity (bandwidth) $k_1 = 1$ Mbps and $k_2 = 2$ Mbps.



The cost per unit of capacity of link 2 is 30% larger than that of link 1. The following table indicates the quantity of capacity consumed by each packet i, $i \in l, \ldots, n$, and the cost to route it on link 1.

Packet	**Consumed capacity**	**Cost on link one**
1	0.3	200
2	0.2	200
3	0.4	250
4	0.1	150
5	0.2	200
6	0.2	200
7	0.5	700
8	0.1	150
9	0.1	150
10	0.6	900

Give an integer linear programming formulation for the problem of minimizing the total cost of routing all the packets. Give also an integer linear programming formulation for the more general case where m links are available.

Solution

The 2-link case can be formulated as the following integer linear program.

- The sets are:
  - The set of packets $I = \{1, \dots, n\}$.
  - The set of links $J = \{1, \dots, m\}$.
- The parameters are:
  - The capacity consumed by packet i, for $i \in I$ a_i.
  - The routing cost for packet i on link j, for $i \in I, j \in J$ c_{ij}.
  - The capacity for link j and $j \in J$ k_j.
- The decision variables are:
  - $x_i j$: 1 if packet i is routed on link j, or 0 otherwise, for $i \in I, j \in J$
- The objective function is:

$$\min \sum_{i \in I} c_{ij} x_{ij}$$

- The constraints are:
  - The assignment:

$$\sum_{j \in J} x_{ij} = 1$$

  - The capacity:

$$\sum_{j \in J} a_i x_{ij} \leq k_j$$

  - The variables must be binary:

$$x_{ij} \in \{0, 1\} \quad i \in I, j \in J$$

The m-link formulation requires a new set of binary variables, one for each packet and link. The packet-to-link assignment is also to be explicitly introduced.

2.2 Linear programming modeling

A company A, which produces one type of high-precision measuring instrument, has to plan the production for the next 3 months. Each month, A can produce at most 110 units, at a unit cost of 300 Euro. Moreover, each month, up to 60 additional units produced by another company B can be bought at a unit cost of 330 Euro. Unsold units can be stored. The inventory cost is of 10 Euro per unit of product, per month. Sales forecasts indicate a demand of 100, 130, and 150 units of product for the next 3 months.

1. Give a linear programming formulation for the problem of determining a production plan (direct or indirect) which minimizes the total costs, while satisfying the monthly demands.

2. Give a mixed integer linear programming formulation for the variant of the problem where production lots have a minimum size. In particular, if any strictly positive quantity is produced in a given month, this quantity cannot be smaller than 15 units.

Solution

1.
   - The sets are:
     - The set of months $T = \{1, 2, 3\}$.
   - The parameters are:
     - The production capacity of A b.
     - The production capacity of B $b^{'}$.
     - The unit production cost for A c.
     - The unit production cost for B $c^{'}$.
     - The inventory cost per unit and month m.
     - The sales forecast for month t, for $t \in T$ d_t.
   - The decision variables are:
     - The units produced by A in month t, $t \in T$ x_t.
     - The units bought from B in month t, for $t \in T$ x_t $x_t^{'}$.
     - The units in inventory at the end of month t, for $t \in T \cup \{0\}$ z_t.
   - The objective function is:

$$\min \sum_{t \in T} cx_t + c^{'} x_t^{'} + mz_t$$

- The constraints are:
  - The capacity of A:
$$x_t \leq b$$
  - The capacity of A:
$$x_t^{'} \leq b^{'}$$
  - The demand:
$$x_{t-1} + x_t + x_t^{'} \geq d_t$$
  - The inventory balance:
$$x_{t-1} + x_t + x_t^{'} - d_t = z_t t$$
  - The starting condition:
$$z_0 = 0$$
  - The non-negative variables:
$$x_t, x_t^{'}, z_t \geq 0$$

2. To take into account the minimum lot size, we add the binary variables y_t, that is 1 if production is active at month t, or 0 otherwise, for $t \in T$ and the constraints:

   - The minimum lot size:
$$x_t \geq l_{y_t}$$
   - The activation:
$$x_t \leq M_{y_t}$$

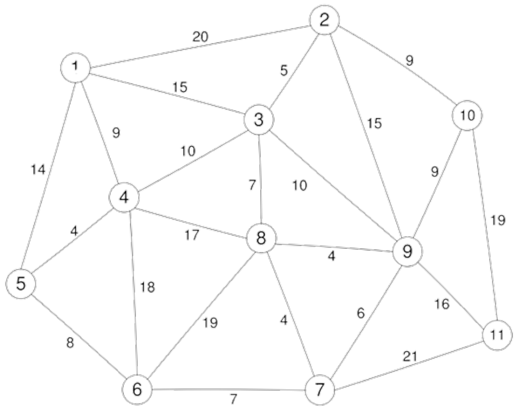
   where $l = 15$ is the minimum lot size, and M is a large enough value, such that constraint $x_t \leq M_{y_t}$ is redundant when $y_t = 1$. For instance, we can choose $M = 110$. Such constraints are usually called big-M constraints.

Chapter 3

Exercise session III

3.1 Minimum cost spanning tree

Find the minimum-cost spanning tree in the graph given in the figure by using Prim's algorithm, starting from the node three.

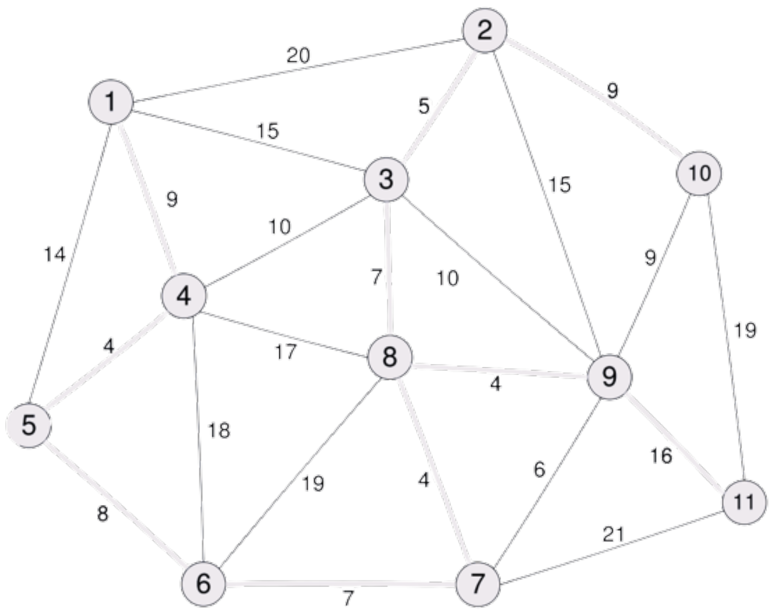


Solution

We apply Prim's algorithm, starting from node three and in the end we need to do the 10 iterations since we have 11 nodes. The iterations are summarized in the following table:

Reached nodes	Added edge	Edge cost	Iteration
3	$\{2,3\}$	5	1
2,3	$\{3,8\}$	7	2
2,3,8	$\{7,8\}$	4	3
2,3,7,8	$\{8,9\}$	4	4
2,3,7,8,9	$\{6,7\}$	7	5
2,3,6,7,8,9	$\{5,6\}$	8	6
2,3,5,6,7,8,9	$\{4,5\}$	4	7
2,3,4,5,6,7,8,9	$\{2,10\}$	9	8
2,3,4,5,6,7,8,9,10	$\{1,4\}$	9	9
1,2,3,4,5,6,7,8,9,10	$\{9,11\}$	16	10

The minimum cost spanning tree that has been found has total cost 73, and it is shown in the following figure.



3.2 Kruskal's algorithm

In 1956 Joseph Kruskal proposed the following greedy algorithm to find a minimum cost spanning tree in an arbitrary connected undirected graph $G = (N, E)$ with a cost c_e attached to each edge $e \in E$.

Algorithm 1 Kruskal's algorithm

sort the edges of E as $\{e_1, \ldots, e_m\}$ where $c_{e_1} \le c_{e_2} \le \cdots \le c_{e_m}$
$i \leftarrow 1$
initialize the sub graph $G' = (N, E)$ of G with $F = \varnothing$
while $|F| < n - 1$ **do**
    if the two endpoints of the edge e_i belong to different connected components of the current sub graph $G^{'}$ **then**
        $F \leftarrow F \cup \{e_i\}$
        merge the two connected components
    end if
    $i \leftarrow i + 1$
end while
return the spanning tree $G^{'} = (N, E)$

In other words, we order the edges by increasing (non-decreasing) cost, we consider the edges in that order and, at each step, we select the current edge (which is one of the cheapest edges still available) only if it does not create a cycle with the previously selected edges. The algorithm terminates when $n - 1$ edges have been selected.

1. Describe an efficient way to identify/keep track of the connected components of the sub graph $G^{'}$ and to check that a new edge is creating a cycle with the previously selected edges (is connecting two distinct connected components of $G^{''}$). Determine the overall computational complexity of this implementation of Kruskal's algorithm.

2. By invoking the optimality condition for minimum-cost spanning trees, verify that Kruskal's algorithm is exact.

3. Find the maximum-cost spanning tree in the graph of the previous exercise by using a straightforward adaptation of Kruskal's algorithm.

4. Apply the Kruskal's algorithm on the graph of the previous exercise to find the minimum spanning tree.

Solution

1. To identify/keep track of the connected components (subtrees) of the sub graph G, we use a vector v with as many components as vertices in the graph, where v_i indicates the index of the connected component containing node i. At the beginning of the algorithm, we start with $v_i = i$, for $i = 1, \ldots, n$. When an edge $e = i, j$ is considered for addition to G'', we compare the values v_i and v_j. If $v_i \neq v_j$, then we can add the edge e to G' because it does not create a cycle. Since the two connected components of indices v_i and v_j are merged, the indices are updated as follows: in the vector v we substitute each occurrence of the index of node i with that of node j. If $v_i = v_j$, edge e is skipped because it would create a cycle.

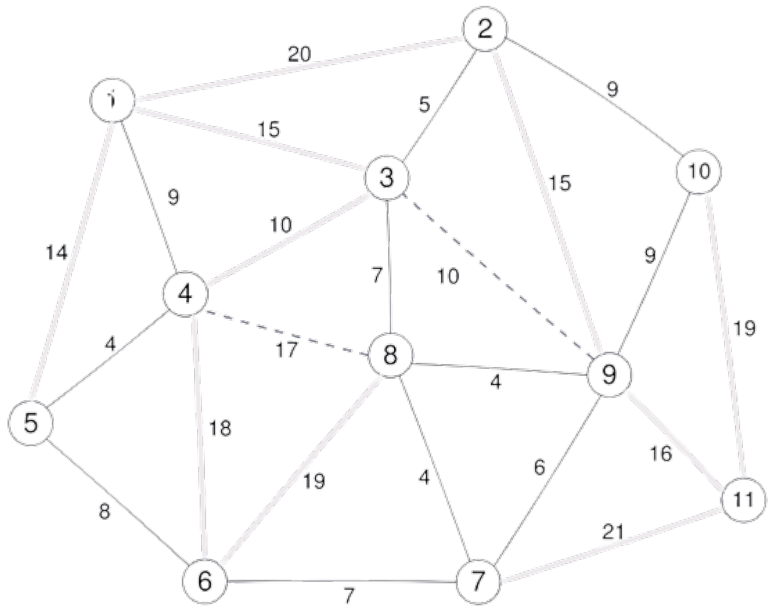
2. The m edges can be ordered by non-decreasing cost in $O(m \log m)$, which is $O(m \log n)$ since $m \log m \leq m \log n^2 = 2m \log n$. At most m edges are considered for addition to the current sub graph G'. At each iteration, an edge $e = i, j$ is considered and the vector v is updated (in $O(n)$) only if $v_i \neq v_j$. Since a merging operation occurs exactly$n - 1$ times (a spanning tree contains $n - 1$ edges), the overall complexity is $O(m \log n + m + n^2) = O(mlogn + n^2)$.

3. To verify that Kruskal's algorithm is exact, we just need to recall that the edges are considered in order of non-decreasing cost and to invoke the optimality condition for minimum cost spanning tree. Since each edge e that has been discarded (not added to F) has a cost c_e which is at least as large as the cost of all the previously selected edges, it is not a cost-decreasing edge. According to the optimality condition for minimum cost spanning trees, the resulting spanning tree is of minimum total cost because no cost-decreasing edge exists.

4. The algorithm needs 10 iteration to find the maximum spanning tree. The iterations are summarized in the following table (remember that we order the edges, and we add them only if they do not create any cycle):

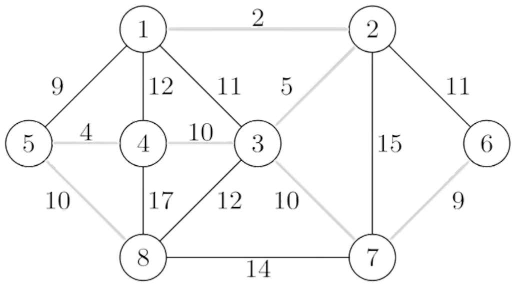
Connected components	Edge	Cost	Iteration
∅	{7,11}	21	1
{7,11}	{1,2}	20	2
{1,2}, {7,11}	{6,8}	19	3
{1,2}, {7,11}, {6,8}	{10,11}	19	4
{1,2}, {7,10,11}, {6,8}	{4,6}	18	5
{1,2}, {7,10,11}, {4,6,8}	{4,8}	NO	6
{1,2}, {7,10,11}, {4,6,8}	{9,11}	16	7
{1,2}, {7,9,10,11}, {46,8}	{1,3}	15	8
{1,2,3}, {7,9,10,11}, {4,6,8}	{2,9}	15	9
{1,2,3,7,9,10,11}, {4,6,8}	{1,5}	14	10
{1,2,3,5,7,9,10,11}, {4,6,8}	{3,9}	No	11
{1,2,3,4,5,6,7,8,9,10,11}	{3,4}	10	12

The spanning tree of maximum cost, of value 167, is shown in the figure.



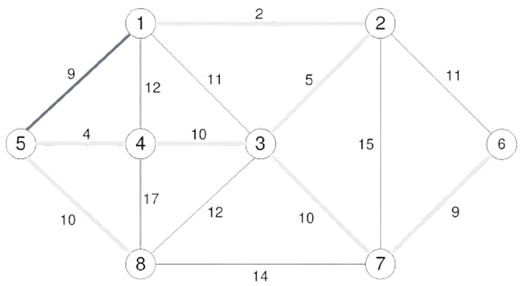
3.3 Optimality check

Without applying any one of Prim's and Kruskal's algorithms, verify whether the following spanning tree is of minimum cost.

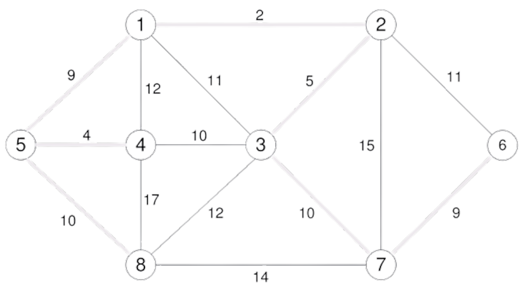


Solution

It suffices to verify that there exists a cost decreasing edge. By inspection, we observe that, by adding edge $\{1, 5\}$ to the tree, the cycle $\langle 1, 5, 4, 3, 2, 1\rangle$ is introduced. In such cycle, edge $\{4, 3\}$ has a strictly larger cost than $\{1, 5\}$.



Therefore, by removing edge $\{4, 3\}$ and adding edge $\{1, 5\}$, a spanning tree of strictly smaller total cost is obtained.



3.4 Compact storage of similar sequences

Consider the problem of storing a large set of strings. We assume that the strings have many similar entries (they differ only in a small number of positions) and we wish to store them in a compact way. This problem arises in several contexts such as when storing DNA sequences, where the characters correspond to the four DNA bases. In this exercise, we consider the simplified version of the problem with only two characters. Given a set of k sequences of M bits, we compute for each pair i, j, with $1 < i, j < k$, the Hamming distance between the sequences i and j, i.e., the number of bits that need to be flipped in sequence i to obtain sequence j. This function clearly satisfies the three usual properties of a distance: non-negativity, symmetry and triangle inequality. Consider the following set of 6 sequences and the corresponding matrix $D = d_{ij}$ of Hamming distances:

1. 011100011101
2. 101101011001
3. 110100111001
4. 101001111101
5. 100100111101
6. 010101011100

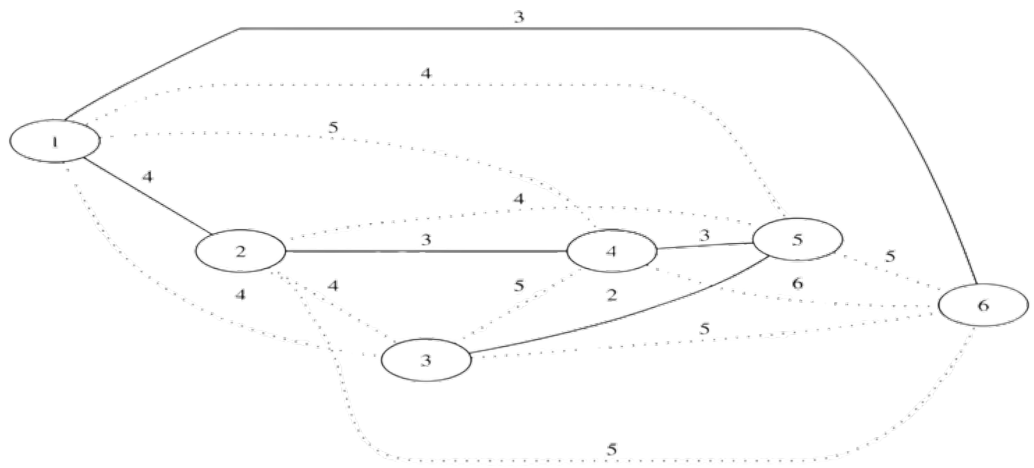
	1	2	3	4	5	6
1	0	4	4	5	4	3
2		0	4	3	4	5
3			0	5	2	5
4				0	3	6
5					0	5
6						0

Where, due to symmetry, only the upper triangle of the matrix is shown. In order to exploit redundancies between sequences and to save memory, we can store: one of the sequences, called the reference sequence, completely and for every other sequence, only the set of bit flips that allow us to retrieve it either directly from the reference sequence or from another sequence.

Show how the problem of deciding which differences to memorize, to minimize the total number of bits used for storage, can be reduced to the problem of finding a minimum-cost spanning tree in an appropriate graph. Solve the problem for the given instance.

Solution

We construct a complete graph G with a node for each sequence and an edge for each pair of sequences. Moreover, each edge $\{i, j\}$ is assigned a cost $d_{ij} \log_2(M)$, which corresponds to the number of bits needed to store the indices of the positions in which the i-th and j-th sequences differ. The problem is then to look for a sub graph G' of G of minimum total cost. Since only one sequence is completely stored, G' must be connected to be able to retrieve any sequence. Since a sub graph of minimal cost is sought, G' will be acyclic. Therefore, a minimum cost spanning tree in G provides an optimal solution to the problem under consideration. The following minimum cost spanning tree has been found using Prim's algorithm.

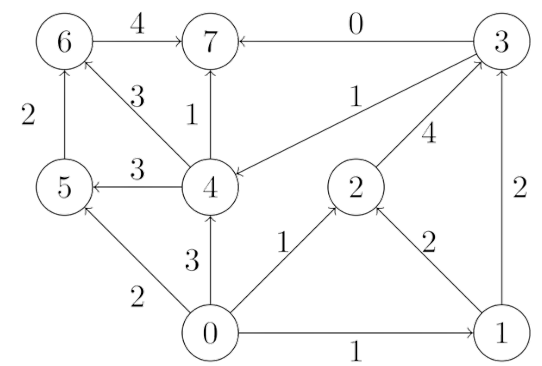


Chapter 4

Exercise session IV

4.1 Shortest paths with non-negative costs

Given the following directed graph, find a set of the shortest paths from node 0 to all the other nodes, using Dijkstra's algorithm. Can we solve the problem with Dynamic Programming? If yes, do so.

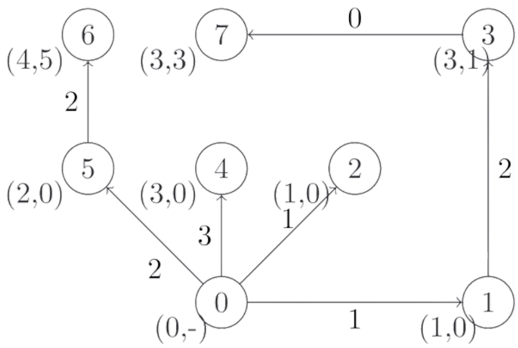


Solution

Dijkstra's algorithm determines a set of shortest paths from a given node s to all the other nodes in the graph. It can be applied to any (directed) graph with non-negative arc costs. The iteration is summarized in the following table.

Iteration	0		1		2		3		4		5		6		7		8	
Node	L	p	L	p	L	p	L	p	L	p	L	p	L	p	L	p	L	p
0	0	-	0	-	0	-	0	-	0	-	0	-	0	-	0	-	0	-
1	∞	-	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0
2	∞	-	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0
3	∞	-	∞	-	3	1	3	1	3	1	3	1	3	1	3	1	3	1
4	∞	-	3	0	3	0	3	0	3	0	3	0	3	0	3	0	3	0
5	∞	-	2	0	2	0	2	0	2	0	2	0	2	0	2	0	2	0
6	∞	-	∞	-	∞	-	∞	-	4	5	4	5	4	5	4	5	4	5
7	∞	-	∞	-	∞	-	∞	-	∞	-	3	3	3	3	3	3	3	3

Graphically, we have that the final graph is the following.



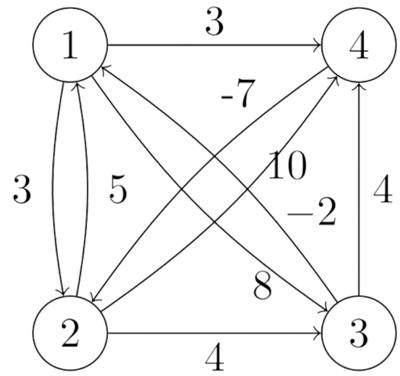
A topological order can be obtained as follows. At iteration i, for $i = 1, \dots, n$, pick a node with no incoming arcs, label it as node i, remove it from the graph, iterate until all nodes are removed, or there is no node with no incident arcs. The graph we are dealing with is acyclic, and its node indices already correspond to a topological order. The Dynamic Programming technique can therefore be applied. The steps are the following:

- $L(0) = 0, p(0) = 0$
- $L(1) = L(0) + c_{01} = 1, p(1) = 0$
- $L(2) = \min\{L(0) + c_{02}, L(1) + c_{12}\} = \min\{0 + 1, 1 + 2\} = 1, p(2) = 0$
- $L(3) = \min\{L(1) + c_{13}, L(2) + c_{23}\} = \min\{1 + 2, 1 + 4\} = 3, p(3) = 1$
- $L(4) = \min\{L(0) + c_{04}, L(3) + c_{34}\} = \min\{0 + 3, 3 + 1\} = 3, p(4) = 0$
- $L(5) = \min\{L(0) + c_{05}, L(4) + c_{45}\} = \min\{0 + 2, 3 + 3\} = 3, p(5) = 0$
- $L(6) = \min\{L(4) + c_{46}, L(5) + c_{56}\} = \min\{3 + 3, 2 + 2\} = 3, p(6) = 5$

- $L(7) = \min\{L(3)+c_{37}, L(4)+c_{47}, L(6)+c_{67}\} = \min\{3+0, 3+1, 4+4\} = 3, p(7) = 3$

4.2 Shortest paths with negative costs

Given the following directed graph, find the shortest paths between all pairs of nodes, or show that the problem is ill-posed by exhibiting a circuit of total negative cost.



Solution

Since the graph contains negative arc costs, we use Floyd-Warshall's algorithm that finds the shortest path between each pair of nodes, or establishes that the problem is ill-posed by exhibiting a circuit of negative cost. The graph can contain cycles, but such cycles must be of non-negative cost. The steps are the following:

1. The initial configuration is the following.

D	1	2	3	4
1	0	3	8	3
2	5	0	4	10
3	-2	∞	0	4
4	∞	-7	∞	0

P	1	2	3	4
1	1	1	1	1
2	2	2	2	2
3	3	3	3	3
4	4	4	4	4

2. The first iteration is the following.

   (a) $d_{21} + d_{12} = 8 > d_{22} = 0$

   (b) $d_{21} + d_{13} = 13 > d_{23} = 4$

   (c) $d_{21} + d_{14} = 8 < d_{24} = 10 \Rightarrow$ update d_{24}, p_{24}

   (d) $d_{31} + d_{12} = 1 < d_{32} = \infty \Rightarrow$ update d_{32}, p_{32}

   (e) $d_{31} + d_{13} = 6 > d_{33} = 0$

(f) $d_{31} + d_{14} = 1 < d_{34} = 4 \Rightarrow$ update d_{34}, p_{34}

(g) $d_{41} + d_{ij} = \infty (\forall i, j)$

The matrices become:

D	1	2	3	4
1	0	3	8	3
2	5	0	4	8
3	-2	1	0	1
4	∞	-7	∞	0

P	1	2	3	4
1	1	1	1	1
2	2	2	2	1
3	3	1	3	1
4	4	4	4	4

3. The second iteration is the following.

(a) $d_{12} + d_{21} = 8 > d_{11} = 0$

(b) $d_{12} + d_{23} = 7 < d_{13} = 8 \Rightarrow$ update d_{13}, p_{13}

(c) $d_{12} + d_{24} = 11 > d_{24} = 3$

(d) $d_{32} + d_{21} = 6 > d_{31} = -2$

(e) $d_{32} + d_{23} = 5 > d_{33} = 0$

(f) $d_{32} + d_{24} = 9 > d_{34} = 1$

(g) $d_{42} + d_{21} = -2 < d_{41} = \infty \Rightarrow$ update d_{41}, p_{41}

(h) $d_{42} + d_{23} = -3 < d_{43} = \infty \Rightarrow$ update d_{43}, p_{43}

(i) $d_{42} + d_{24} = 1 > d_{44} = 0$

The matrices become:

D	1	2	3	4
1	0	3	7	3
2	5	0	4	8
3	-2	1	0	1
4	-2	-7	-3	0

P	1	2	3	4
1	1	1	2	1
2	2	2	2	1
3	3	1	3	1
4	2	4	2	4

4. The third iteration is the following.

(a) $d_{13} + d_{31} = 5 > d_{11} = 0$

(b) $d_{13} + d_{32} = 8 > d_{12} = 3$

(c) $d_{13} + d_{34} = 8 > d_{14} = 3$

(d) $d_{23} + d_{31} = 2 < d_{21} = 5 \Rightarrow$ update d_{21}, p_{21}

(e) $d_{23} + d_{32} = 5 > d_{22} = 0$

(f) $d_{23} + d_{34} = 5 < d_{24} = 8 \Rightarrow$ update d_{24}, p_{24}

(g) $d_{43} + d_{31} = -5 < d_{41} = -2 \Rightarrow$ update d_{41}, p_{41}

(h) $d_{43} + d_{32} = -2 > d_{42} = -7$

(i) $d_{43} + d_{34} = -2 < d_{44} = 0 \Rightarrow$ update d_{44}, p_{44}

The matrices become:

D	1	2	3	4
1	0	3	7	3
2	2	0	4	5
3	-2	1	0	1
4	-5	-7	-3	-2

P	1	2	3	4
1	1	1	2	1
2	3	2	2	1
3	3	1	3	1
4	3	4	2	1

We obtain $d_{44} = -2 < 0$ and the algorithm halts: we found a circuit with a total negative cost of -2, $\langle(4,2),(2,3),(3,1),(1,4)\rangle$.

4.3 Dynamic Programming

A company must buy a new machine and then determine a renewal (maintenance-replacement) plan for the next 5 years, making sure that, at any point in time, the available machine works properly. At the beginning of each year of the planning horizon, the company must decide whether to keep the old machine or to substitute it with a new machine. The maintenance costs and the expected revenue (when the machine is sold) depend on how old the machine. They are indicated in the following table.

Years	**Maintenance (k)**	**Revenue when sold (k)**
0	2	-
1	4	7
2	5	6
3	9	2
4	12	1

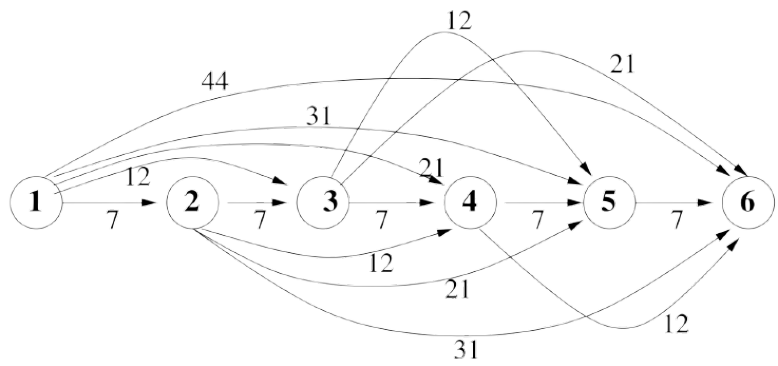
To avoid high maintenance costs of an old machine, the machine can be sold at the beginning of the second, third, fourth, and fifth year, and a new one can be bought. For the sake of simplicity, we suppose that a new machine always costs 12k Euro. Show that the problem of determining a machine renewal plan of minimum total net cost (total cost for buying/rebuying the machine + maintenance costs - total revenue) can be solved via Dynamic Programming by finding the shortest path in an ad hoc directed acyclic graph. Find an optimal renewal plan. Is it unique?

Solution

Consider a directed graph with six nodes: nodes 1 to 5 are associated to the beginning of each year, while node 6 corresponds to the end of the 5 years time horizon. For each pair i, j, with $i, j = 1, \ldots, 5$ and $i < j$, the arc (i, j) that represents the choice of buying a machine at the beginning of year i and selling it at the beginning of year j. The cost c_{ij} of arc (i, j) is defined as the net cost of the corresponding partial renewal plan from the beginning of year i to the beginning of year j, namely:

$$c_{ij} = c_b + \left(\sum_{k=0}^{j-i-1} m_k \right) - r_{j-i}$$

where c_b is the buying cost of 12000 Euro, m_k is the annual maintenance cost for a machine that is k years old, and r_k is the selling price of a machine which is k years old. We obtain the following directed acyclic graph:



Any path from node 1 to node 6 corresponds to a (complete) renewal plan whose total net cost is equivalent to the total cost of the path. To look for the shortest path from node 1 to node 6, we apply the Dynamic Programming algorithm, obtaining:

1. $L(1) = 0$
2. $L(2) = 7, p(2) = 1$
3. $L(3) = 12, p(3) = 1$
4. $L(4) = 19, p(4) = 3$
5. $L(5) = 24, p(5) = 3$
6. $L(6) = 31, p(6) = 5$

The shortest path (of cost 31) is $\langle (1,3),(3,5),(5,6) \rangle$. It amounts to buy a new machine every 2 years. Note that there are two other optimal solutions: path $\langle (1,2),(2,4)(4,6) \rangle$ and path $\langle (1,3),(3,4),(4,6) \rangle$.

4.4 Project planning

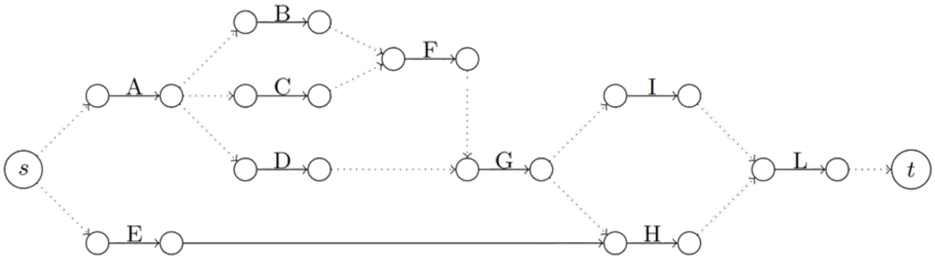
The preparation of the apple pie has long been a tradition at Rossi's family. First the weight of the ingredients has to be determined: flour, sugar, butter, eggs, apples, cream. The butter must then be melted down, and added to a mixture of flour, sugar, and eggs. Apples must be added to this new mixture, once they have been peeled and cut into thin slices. The mixture can then be cooked, in the already heated oven. It is advisable to whip the cream only after the apple slices have been added to the mixture. Once the cake is cooked, the cream is used to garnish it. The following table reports the time needed for each activity.

	Activity	**Times (minutes)**
A	Weight the ingredients	5
B	Melt the butter	3
C	Mix flour, eggs and sugar	5
D	Peel the apples and cut them into slices	10
E	Heat the oven	20
F	Add butter to the mixture	8
G	Add apples to the mixture	4
H	Cook the mixture in the oven	40
I	Whip the cream	10
L	Garnish	5

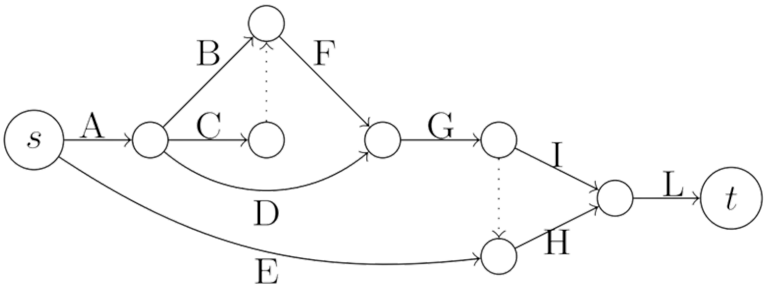
Draw the graph (with activities associated to arcs) which represents the project precedence relations. Determine the minimum total completion time of the project as well as the earliest times and the latest times associated to each node. Identify the critical activities and draw Gantt's chart at earliest.

Solution

We derive the directed graph representing the precedence relations as follows. For each activity, we introduce an arc whose cost is equivalent to the duration of the activity (its two nodes represent the beginning and the end of the activity). For each precedence relation $A_i < A_j$, a fictitious arc (i, j) of duration 0 is introduced (dashed line) between the ending node of the arc associated to Ai and the beginning node of the arc associated to A_j. We include a node s and, for each activity without predecessors associated to arc (v, w), we add the arc (s, v) of cost 0. Similarly, we include a node t and, for each activity without successors associated to arc (v, w), we add the arc (w, t) of cost 0.



By deleting some fictitious arcs while paying attention not to create any unwanted precedence relations, we obtain the following more compact directed acyclic graph representing the project.

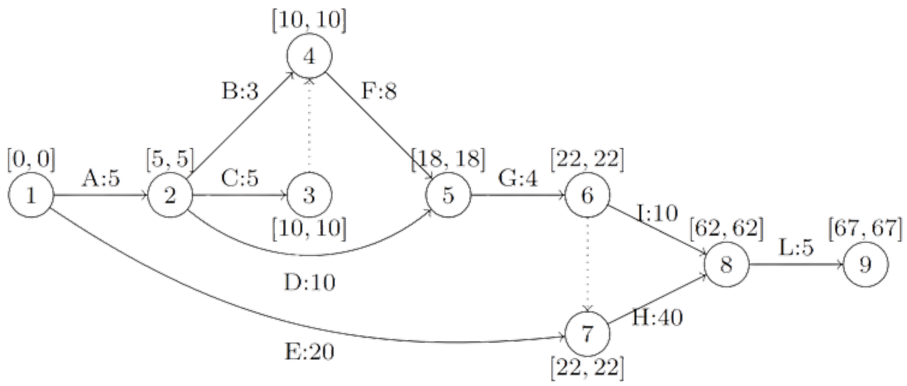


We use the Critical Path Method to determine, for each node v of the graph, the at earliest and at latest times, denoted by T_{min_v} and respectively T_{max_v}. The algorithm exploits the topological order of the nodes and consists of two phases where Dynamic Programming is applied considering the n nodes in the increasing/decreasing order of the indices. Here is the pseudocode of the algorithm:

Algorithm 2 Critical Path Method algorithm

sort the nodes topologically
$T_{min_1} \leftarrow 0$
for $j = 2, \dots, n$ **do**
  $T_{min_j} \leftarrow \max\{T_{min_i} + d_{ij} | (i, j) \in \delta^-(j)\}$
end for
$T_{max_n} \leftarrow T_{min_n}$
for $i = n - 1, \dots, 1$ **do**
  $T_{max_i} \leftarrow \min\{T_{mmax_j} - d_{ij} | (i, j) \in \delta^+(i)\}$
end for

After ordering topologically the nodes, we obtain the following earliest times and latest times:



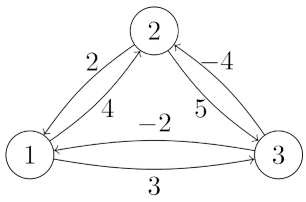
The slacks for the activities are:

- $\sigma(A) = T_{max_2} - T_{min_1} - d_{12} = 5 - 0 - 5 = 0$
- $\sigma(B) = T_{max_4} - T_{min_2} - d_{24} = 10 - 5 - 3 = 2$
- $\sigma(C) = T_{max_3} - T_{min_2} - d_{23} = 10 - 5 - 5 = 0$
- $\sigma(D) = T_{max_5} - T_{min_2} - d_{25} = 18 - 5 - 10 = 3$
- $\sigma(E) = T_{max_6} - T_{min_1} - d_{16} = 22 - 0 - 20 = 2$
- $\sigma(F) = T_{max_5} - T_{min_4} - d_{45} = 18 - 10 - 8 = 0$
- $\sigma(G) = T_{max_6} - T_{min_5} - d_{56} = 22 - 18 - 4 = 0$
- $\sigma(H) = T_{max_8} - T_{min_7} - d_{78} = 62 - 22 - 40 = 0$
- $\sigma(I) = T_{max_8} - T_{min_6} - d_{68} = 62 - 22 - 10 = 30$
- $\sigma(L) = T_{max_9} - T_{min_8} - d_{89} = 67 - 62 - 7 = 0$

The critical activities are A, C, F, G, H, L.

4.5 Shortest paths with negative costs

Given the following directed graph, find the shortest paths between all pairs of nodes, or show that the problem is ill-posed, by exhibiting a circuit of total negative cost.



Solution

The steps of the Warshall-Floyd's algorithm are the following:

1. The initial configuration is the following.

D	1	2	3
1	0	4	3
2	2	0	5
3	-2	-4	0

P	1	2	3
1	1	1	1
2	2	2	2
3	3	3	3

2. The first iteration is the following.

   (a) $d_{21} + d_{12} = 2 + 4 = 6 > d_{22} = 0$

   (b) $d_{21} + d_{13} = 2 + 3 = 5 = d_{23} = 5$

   (c) $d_{31} + d_{13} = -2 + 3 = 1 > d_{33} = 0$

   (d) $d_{31} + d_{12} = -2 + 4 = 2 > d_{32} = -4$

   The matrices become:

D	1	2	3
1	0	4	3
2	2	0	5
3	-2	-4	0

P	1	2	3
1	1	1	1
2	2	2	2
3	3	3	3

3. The second iteration is the following.

   (a) $d_{12} + d_{21} = 4 + 2 = 6 > d_{11} = 0$

(b) $d_{12} + d_{23} = 4 + 5 = 9 > d_{13} = 3$

(c) $d_{32} + d_{23} = -4 + 5 = 1 > d_{33} = 0$

(d) $d_{32} + d_{21} = -4 + 2 = -2 = d_{31} = -2$

The matrices become:

D	1	2	3
1	0	4	3
2	2	0	5
3	-2	-4	0

P	1	2	3
1	1	1	1
2	2	2	2
3	3	3	3

4. The third iteration is the following.

   (a) $d_{13} + d_{31} = 3 - 2 = 1 > d_{11} = 0$

   (b) $d_{13} + d_{32} = 3 - 4 = -1 < d_{12} = 4 \Rightarrow$ update d_{12}, p_{12}

   (c) $d_{23} + d_{32} = 5 - 4 = 1 > d_{22} = 0$

   (d) $d_{23} + d_{31} = 5 - 2 = 3 > d_{21} = 2$

   The matrices become:

D	1	-1	3
1	0	4	3
2	2	0	5
3	-2	-4	0

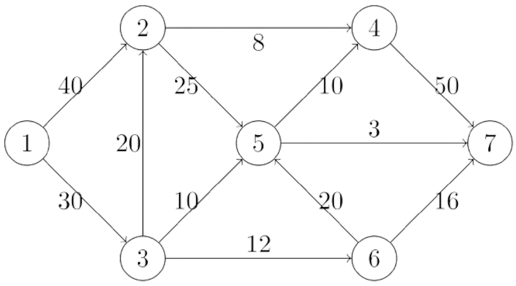
P	1	2	3
1	1	3	1
2	2	2	2
3	3	3	3

Chapter 5

Exercise session V

5.1 Maximum flow

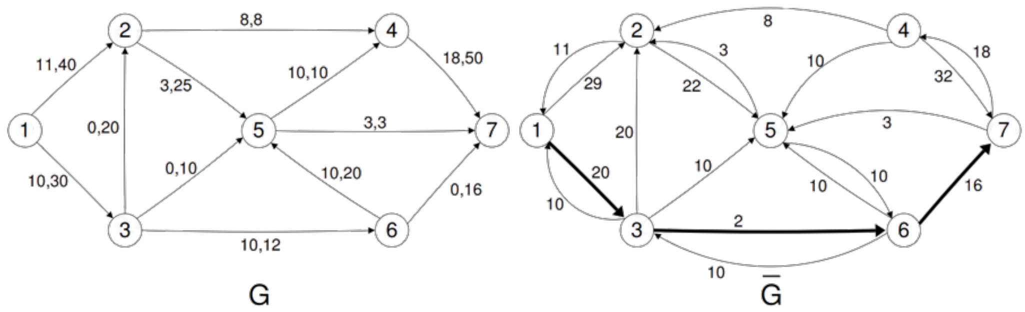
Given the following network with capacities on the arcs:



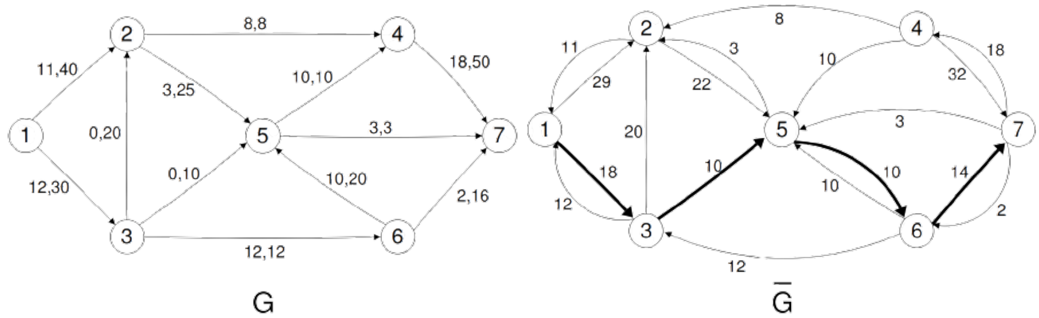
find a maximum flow from node 1 to node 7, starting from the feasible flow of value 21 in which $x_{12} = 11$, $x_{13} = x_{36} = x_{54} = x_{65} = 10$, $x_{24} = 8$, $x_{25} = x_{57} = 3$, $x_{47} = 18$, and $x_{ij} = 0$ for the remaining arcs. Determine a corresponding minimum cut.

Solution

The initial feasible flow $\underline{x_0}$ of value $\varphi_0 = 21$ and the associated incremental (residual) network are as follows:

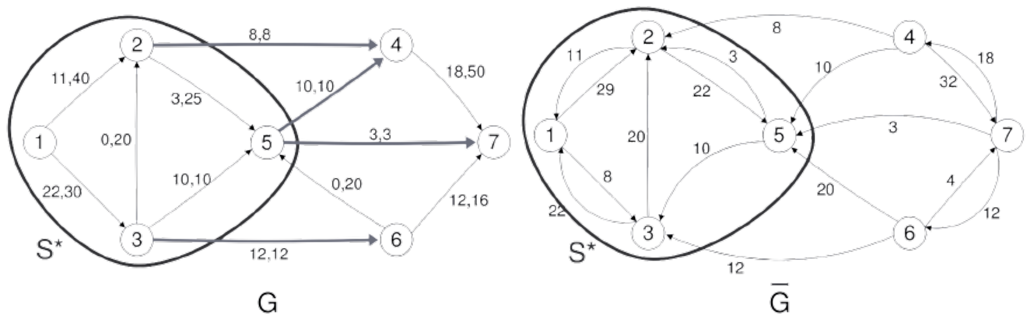


By sending $\delta = 2$ additional units along the path $\langle(1,3),(3,6),(6,7)\rangle$, we obtain the following feasible flow x_3 of value $\varphi_3 = 23$ and the associated residual network:



Observe that all augmenting paths for this network use the backward arc $(5,6)$. From the flow point of view, this amounts to unload arc $(6,5)$, decreasing the amount of product flowing through it.

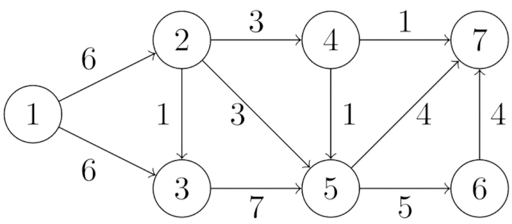
By sending $\delta = 10$ units along the path $\langle(1,3),(3,5),(5,6),(6,7)\rangle$, we obtain the following feasible flow x_4 of value $\varphi_4 = 33$ and the associated residual network.



No other augmenting path exists. The set $S^* = \{1, 2, 3, 5\}$, highlighted in blue, contains all the nodes that can be reached in $\overline{G}$ from the source one. S^* induces in the network G the cut $\delta_G(S^*)$ of minimum total capacity $k(S^*) = 33$ which is highlighted in red. Note that, according to strong duality, the value $\varphi_4 = 9$ of the feasible flow x_4 is equal to the total capacity $k(S^*)$ of by the cut $\delta_G(S^*) = \{(2,4),(3,6),(5,4),(5,7)\}$ induced by S^*.

5.2 Maximum flow and minimum cut

Given the following network with capacities on the arcs:

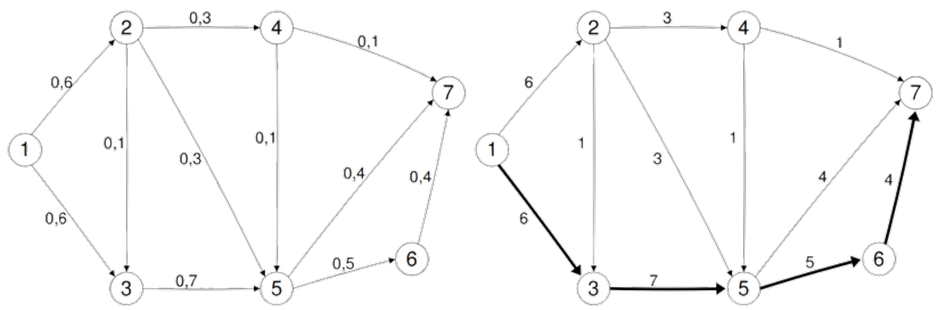


find a maximum (feasible) flow from node 1 to node 7, and determine a corresponding minimum (capacity) cut.

Solution

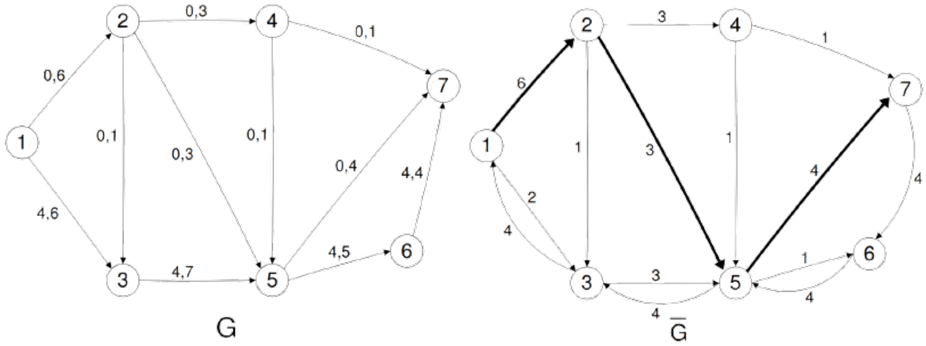
We apply Ford-Fulkerson's algorithm. In the following figures, on the left we report the current feasible flow in the network G (with on each arc the quantity of product x_{ij} flowing through it and its capacity k_{ij}) and on the right the incremental (residual) network $\overline{G}$ associated to the current feasible flow.

We start from the null feasible flow $\underline{x}_0 = 0$ of value $\varphi_0 = 0$. Since all arcs are empty, $\overline{G}$ is equivalent to G.



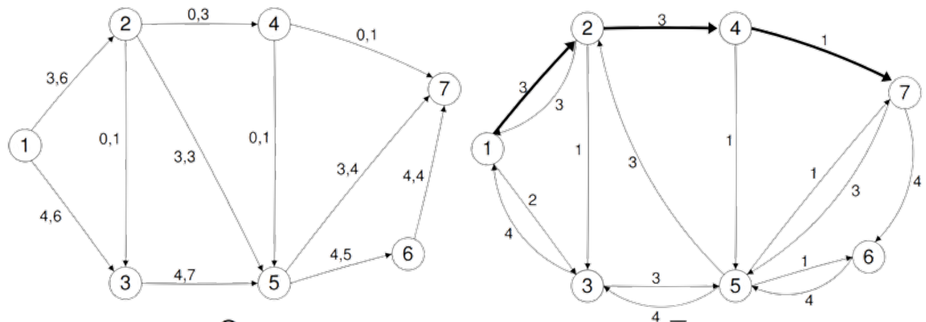
G $\qquad\qquad$ $\overline{G}$

Along the augmenting path $\langle(1,3),(3,5),(5,6),(6,7)\rangle$ we can send up to $\delta = 4$ additional units of product. δ is given by arc $(6,7)$ which has the smallest capacity k_{ij} of all arcs on the path. Adding these $\delta = 4$ units of product to the feasible flow $\underline{x}_0$, we obtain the following feasible flow $\underline{x}_1$ (on the left) of value $\varphi_1 = 0 + 4 = 4$. The associated incremental (residual) network is reported on the right.



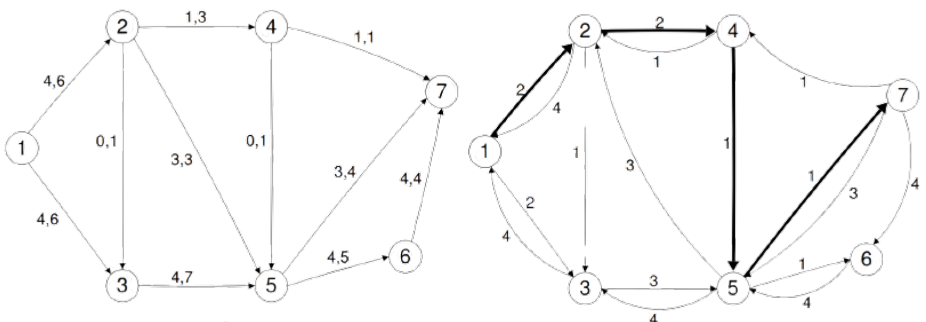
G $\overline{\text{G}}$

Along the augmenting path $\langle(1,2),(2,5),(5,7)\rangle$ we can send up to $\delta = 3$ additional units of product. δ is given by arc $(2,5)$ which has the smallest residual capacity $\underline{k}_{ij}$ of all arcs on the path. Adding these $\delta = 3$ units of product to the feasible flow $\underline{x}_1$, we obtain the following feasible $\underline{x}_2$ of value $\varphi_2 = 4 + 3 = 7$. The associated incremental (residual) network is reported on the right.



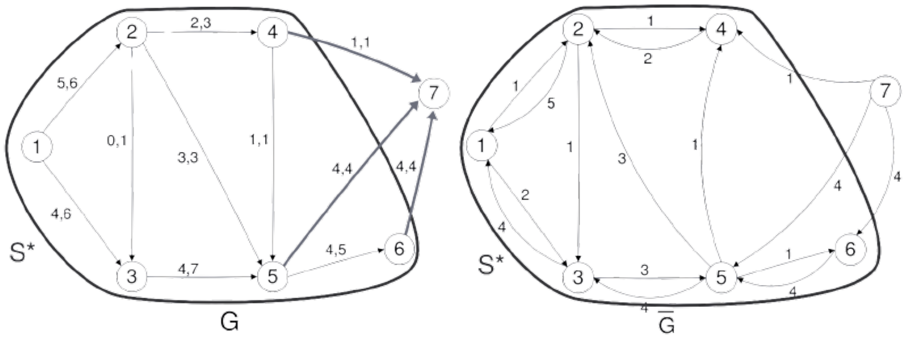
G $\overline{\text{G}}$

Along the augmenting path $\langle(1,2),(2,4),(4,7)\rangle$ we can send up to $\delta = 1$ additional units of product, where δ is given by arc $(4,7)$. We obtain the following feasible flow $\underline{x}_3$ (on the left) of value $\varphi_3 = 7+1 = 8$. The associated incremental (residual) network is reported on the right.



G $\overline{\text{G}}$

Along the augmenting path $\langle(1,2),(2,4),(4,5),(5,7)\rangle$ we can send up to $\delta = 1$ additional units of product, where δ is given by arc $(4,5)$ (or $(5,7)$). We obtain the following feasible flow $\underline{x}_4$ (on the left) of value $\varphi_4 = 8+1 = 9$. The associated incremental (residual) network is reported on the right.



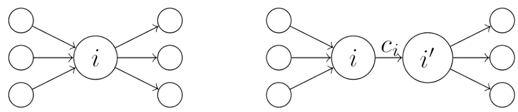
Since no other augmenting path exists (no path from node 1 to node 7 in the incremental/residual network $\overline{G}$), the algorithm halts. The set $S^* = \{1,2,3,4,5,6\}$, highlighted in blue, contains all the nodes that can be reached from the source 1 in $\overline{G}$. S^* induces the cut $\delta(S^*)$ of minimum total capacity $k(S^*) = 9$ which is highlighted in red. Note that, according to strong duality, the value $\varphi_4 = 9$ of the feasible flow $\underline{x}_4$ is equal to the total capacity $k(S^*)$ of the cut $\delta(S^*) = \{(4,7),(5,7),(6,7)\}$ induced by S^*.

5.3 Maximum flow with node capacities

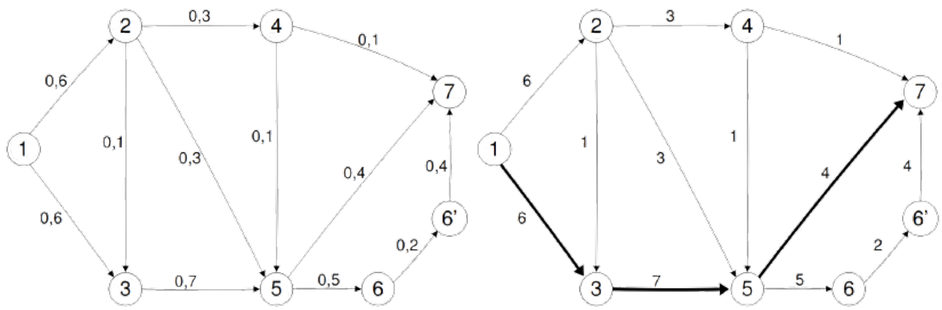
In maximum flow problems, how can we deal with capacities on both nodes and arcs? Find a maximum flow from node 1 to node 7 in the network of the previous exercise, with a node capacity of 2 on node 6.

Solution

The capacities on nodes can be easily reduced to capacities on arcs. Indeed, each node i with a capacity c_i can be substituted with two auxiliary nodes, which are connected with an arc whose capacity is equal to c_i and where all entering arcs entering the left node and all exiting arcs exit from the right node.



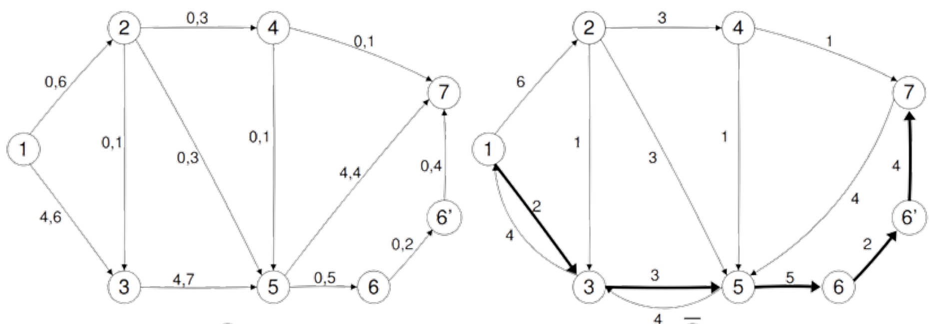
The network is modified as follows.



G $\overline{\text{G}}$

We send $\delta = 4$ additional units along the path

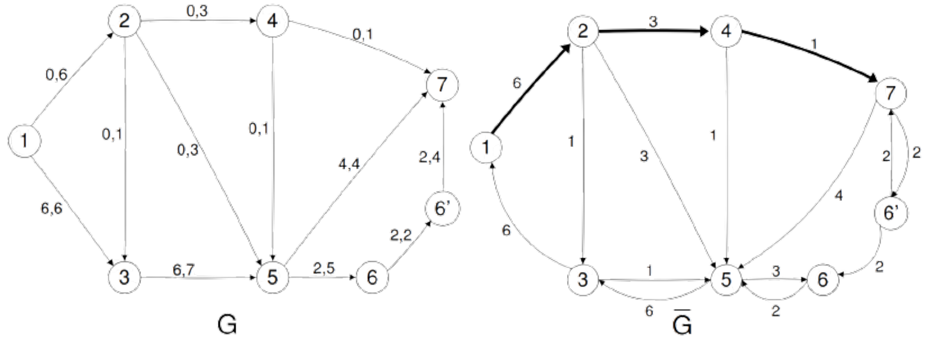
$$\langle (1,3), (3,5), (5,7) \rangle$$



G $\overline{\text{G}}$

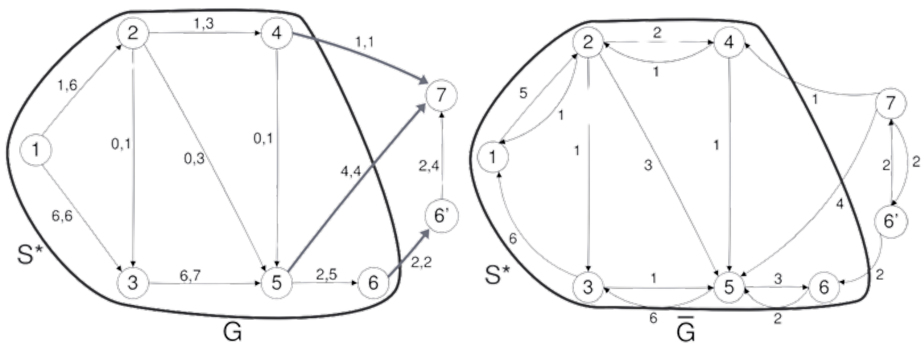
We send $\delta = 2$ additional units along the augmenting path

$$\Big\langle (1,3), (3,5), (5,6), (6,6'), (6',7) \Big\rangle$$



We send $\delta = 1$ additional units along the augmenting path

$$\langle (1,2), (2,4), (4,7) \rangle$$



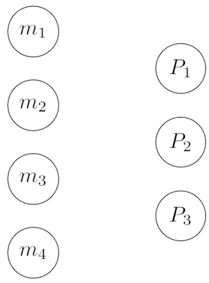
No other augmenting path exists. The minimum cut induced by $S^* = \{1,2,3,4,5,6\}$ is highlighted in red. It has a total capacity $k(S^*) = \varphi = 7$.

5.4 Indirect application of maximum flows

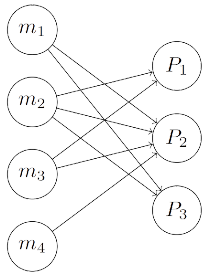
A software house has to handle 3 projects, P_1, P_2, P_3, over the next 4 months. The projects require 8, 10, and 12 man/months, respectively. P_1 can only begin after month 1, and must be completed at latest by the end of month 3. P_2 and P_3 can begin from month 1, and must be completed by the end of month 4 and 2, respectively. Each month, 8 engineers are available. Due to the internal structure of the company, no more than 6 engineers can work, at the same time, on the same project. Determine whether it is possible to complete the three projects within the time constraints and, if it is possible, find a feasible workforce plan. Describe how this problem can be reduced to the problem of finding a maximum flow in an appropriate network.

Solution

We build a network with nodes m_1, m_2, m_3, m_4 associated to the four months and nodes P_1, P_2, P_3 associated to the three projects.



For each pair of month-node mi and project-node P_j, the arc (m_i, P_j) is included in the network if man/hours of month i can be allocated to project P_j. For instance, since project P_1 can only begin after month 1 and must be completed within month 3, there are only two arcs entering in P_1, namely (m_2, P_1) and (m_3, P_1). Thus, we obtain the following bipartite oriented graph.



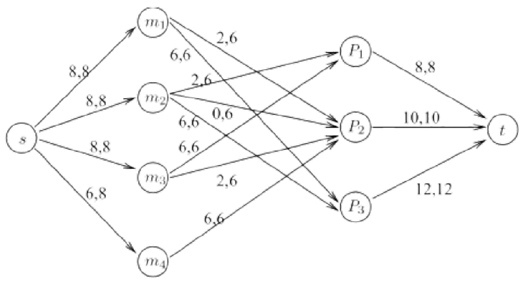
All arcs outgoing from s have capacity 8, since there are 8 engineers available. All arcs connecting month-nodes to project-nodes have capacity 6, as no more than 6 engineers can work on the same project at the same time. All arcs entering in t have a capacity that is equivalent to the number of man/months needed to complete the corresponding project P_j. All arcs outgoing from s have capacity 8, since there are 8 engineers available. All arcs connecting month/nodes to project-nodes have capacity 6, as no more than 6 engineers can work on the same project at the same time. All arcs entering in t have a capacity that is equivalent to the number of man/months needed to complete the corresponding project P_j.

Since all capacities are integer, the maximum flow will be integer as well. To check whether all projects can be completed within the given time limits, it suffices to check whether the network admits a feasible flow of value $8 + 10 + 12 = 30$.

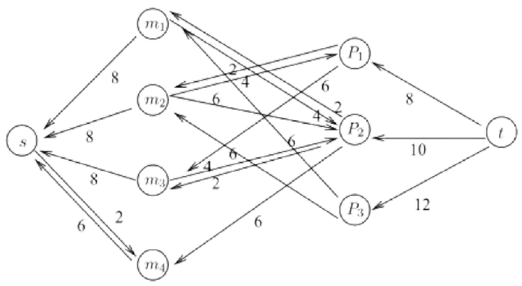
A feasible flow of maximum value 30 can be found by applying Ford-Fulkerson algorithm. We can start from the feasible flow $\underline{x}_0 = 0$ of value $\varphi_0 = 0$ and use the following augmenting paths:

- $\langle(s, m_3), (m_3, P_1), (P_1, t)\rangle$ with $\delta = 6$, yielding a feasible flow of value $\varphi_1 = 6$.
- $\langle(s, m_2), (m_2, P_1), (P_1, t)\rangle$ with $\delta = 2$, yielding a feasible flow of value $\varphi_2 = 8$.
- $\langle(s, m_1), (m_1, P_3), (P_3, t)\rangle$ with $\delta = 6$, yielding a feasible flow of value $\varphi_3 = 14$.
- $\langle(s, m_2), (m_2, P_3), (P_3, t)\rangle$ with $\delta = 6$, yielding a feasible flow of value $\varphi_4 = 20$.
- $\langle(s, m_4), (m_4, P_2), (P_2, t)\rangle$ with $\delta = 6$, yielding a feasible flow of value $\varphi_5 = 26$.
- $\langle(s, m_3), (m_1, P_2), (P_2, t)\rangle$ with $\delta = 2$, yielding a feasible flow of value $\varphi_6 = 28$.
- $\langle(s, m_3), (m_3, P_2), (P_2, t)\rangle$ with $\delta = 2$, yielding a feasible flow of value $\varphi_7 = 30$.

The resulting feasible flow $\underline{x}_7$ of value $\varphi_7 = 30$ is as follows.



And the associated residual network $\overline{G}_7$ is the following.



Note that in $\overline{G}_7$ only node m_4 can be reached from node s, hence $S^* = \{s, m_4\}$.

Since the cut $\delta G(S^*) = \{(s, m_1), (s, m_2), (s, m_3), (m_4, P_2)\}$ has a total capacity of $k(S^*) = 8 + 8 + 8 + 6 = 30$ which is equal to the value $\varphi_7 = 30$ of the feasible flow $\underline{x}_7$, weak duality implies that the feasible flow $\underline{x}_7$ is of maximum value and the cut $\delta G(S^*)$ is of minimum total capacity (among all the cuts separating the source s from the sink t).

Chapter 6

Laboratory session I

Linear programming modeling

A canteen has to plan the composition of the meals that it provides. A meal can be composed of the types of food indicated in the following table. Costs, in Euro per hg, and availabilities, in hg, are also indicated.

Food	**Cost**	**Availability**
Bread	0.1	4
Milk	0.5	3
Eggs	0.12	1
Meat	0.9	2
Cake	1.3	2

A meal must contain at least the following amount of each nutrient:

Nutrient	Minimal quantity
Calories	600 cal
Proteins	50 g
Calcium	0.7 g

Each hg of each type of food contains to following amount of nutrients:

Food	**Calories**	**Proteins**	**Calcium**
Bread	30 cal	15 g	0.02 g
Milk	50 cal	15 g	0.15 g
Eggs	150 cal	30 g	0.05 g
Meat	180 cal	90 g	0.08 g
Cake	400 cal	70 g	0.01 g

Give a linear programming formulation for the problem of finding a meal of minimum total cost which satisfies the minimum nutrient requirements.

Solution

```
# Import the package mip
!pip install mip
import mip
# Food
I = {'Bread', 'Milk', 'Eggs', 'Meat', 'Cake'}
# Nutrients
J = {'Calories', 'Proteins', 'Calcium'}
# Cost in Euro per hg of food
c = {'Bread':0.1, 'Milk':0.5, 'Eggs':0.12, 'Meat':0.9, 'Cake':1.3}
# Availability per hg of food
q = {'Bread':4, 'Milk':3, 'Eggs':1, 'Meat':2, 'Cake':2}
# minum nutrients
b = {'Calories':600, 'Proteins':50, 'Calcium':0.7}
# Nutrients per hf of food
a = {  ('Bread','Calories'):30,
       ('Milk','Calories'):50,
       ('Eggs','Calories'):150,
       ('Meat','Calories'):180,
       ('Cake','Calories'):400,
       ('Bread','Proteins'):5,
       ('Milk','Proteins'):15,
       ('Eggs','Proteins'):30,
       ('Meat','Proteins'):90,
       ('Cake','Proteins'):70,
       ('Bread','Calcium'):0.02,
       ('Milk','Calcium'):0.15,
       ('Eggs','Calcium'):0.05,
       ('Meat','Calcium'):0.08,
       ('Cake','Calcium'):0.01}
# Define a empty model
model = mip.Model()
# Define variables
x = [model.add_var(name = i,lb=0) for i in I]
# Define the objective function
model.objective = mip.minimize(mip.xsum())
# Availability constraint
for i,food in enumerate(I):
```

```
model.add_constr()
# Minum nutrients constraint
for j in J:
model.add_constr(mip.xsum()>=)
# Optimizing command
model.optimize()
# Optimal objective function value
model.objective.x
# Printing the variables values
for i in model.vars:
print(i.name)
print(i.x)
```