

Image Analysis And Computer Vision
Theory

Christian Rossi

Academic Year 2023-2024

Abstract

The topics of the course are:

- Introduction.
- Camera sensors: transduction, optics, geometry, distortion
- Basics on Projective geometry: modelling basic primitives (points, lines, planes, conic sections, quadric surfaces) and projective spatial transformations and projections.
- Camera geometry, and single view analysis: calibration, image rectification, localization of 3D models.
- Multi-view analysis: 3D shape reconstruction, self-calibration, 3D scene understanding.
- Linear filters and convolutions, space-invariant filters, Fourier Transform, sampling and aliasing.
- Nonlinear filters: image morphology and morphology operators (dilate, erode, open, close), median filters.
- Edge detection and feature detection techniques. Feature matching and feature tracking along image sequences.
- Inferring parametric models from noisy data (including outliers), contour segmentation, clustering, Hough Transform, Ransac (random sample consensus).
- Applications: object tracking, object recognition, classification.

Contents

1	Optical Sensors	2
1.1	Photocamera	2
1.2	Light rays deviation	3
1.3	Focalization of parallel light rays	4
1.4	Path of a light ray	5
1.5	Pin-hole camera	5
1.6	From real world to 2D images	5
1.7	Perspective and vanishing point	6
2	Planar Projective Geometry	8
2.1	Introduction	8
2.2	Points	8
2.3	Lines	9
2.4	Conics	13
2.4.1	Circular points	14
2.4.2	Polar line	14
2.4.3	Harmonic tuples	15
2.4.4	Polar line and harmonic tuples	15
2.4.5	Polar line and tangency points	16
2.4.6	Degenerate conics	18
2.5	Dual conics	19
2.5.1	Definition	19
2.5.2	Degenerate dual conics	20
2.6	Transformations	20
2.6.1	Vanishing points and vanishing line	23
2.6.2	Polarity	24
2.6.3	Cross ratio	25
2.6.4	Hierarchy of projective transformations	26
2.6.5	Isometries	26
2.6.6	Similarities	27
2.6.7	Affinities	28
2.6.8	Projectivities	29
3	Two-dimensional reconstruction	30
3.1	Introduction	30
3.2	Reconstruction	30
3.2.1	Affine reconstruction	31

3.2.2	Shape reconstruction	32
3.3	Accuracy issues	35

CHAPTER 1

Optical Sensors

1.1 Photocamera

Definition

The *photocamera* is an optical sensor; this means that produces data using electric transducers. It uses an optical system that select the direction of the incoming light at each element of its screen made with millions of photosensitive elements. Most of the actual cameras can capture up to 30-60 frames per second.

For simplicity, we suppose that the optical system of a photocamera is a single lens that is:

- Spherical: the lens is obtained by intersecting two spheres.
- Thin: the distance between the center of the two spheres is almost identical to the sum of the diameter of them.
- Small angles: the light forms small inclination angles with the optical axis.

This simplifies the computation of the path of the ray crossing the lens. In fact, the refraction of the light when crossing a border between two media is given by the Snell's law:

$$\frac{\sin \theta_2}{\sin \theta_1} = \frac{n_1}{n_2}$$

where:

- θ_1 and θ_2 are the angles between the normal at the surface and the direction of the light ray, respectively before and after crossing the border.
- n_1 and n_2 are the refraction indexes of the two materials.

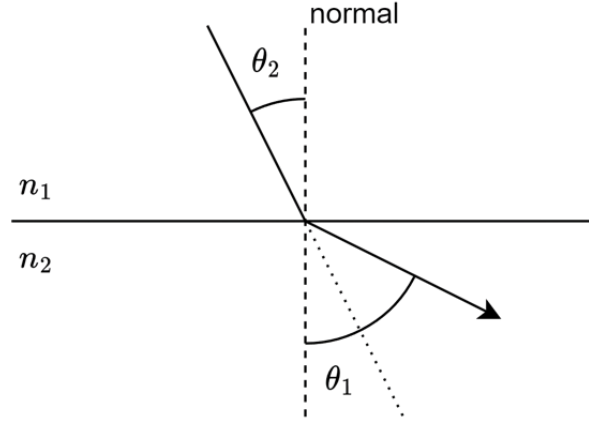


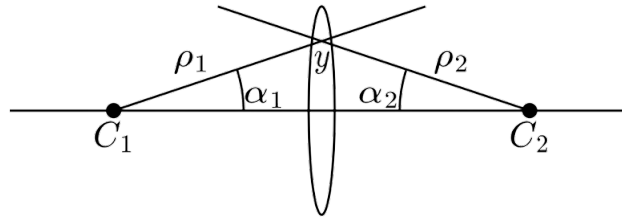
Figure 1.1: Snell's law

Definition

The *optical axis* is the straight line that connects the center of the two spheres that are used to form the lens.

The angles of a ray passing through the centers of the spheres can be calculated as follows:

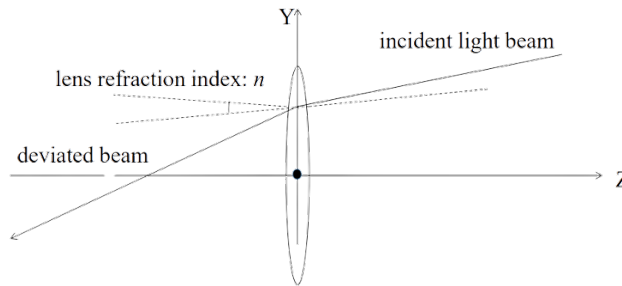
$$\alpha_1 = \frac{y_1}{\rho_1} \quad \alpha_2 = -\frac{y_2}{\rho_2}$$



Since we have a simplified lens, it is possible to say that:

$$y_1 = y_2 = y$$

1.2 Light rays deviation



Given a lens with refraction n we have that the following equations are valid:

$$\frac{\theta - \alpha_1}{\theta' - \alpha_1} \Rightarrow \frac{\sin(\theta - \alpha_1)}{\sin(\theta' - \alpha_1)} = n$$

$$\frac{\theta'' - \alpha_2}{\theta' - \alpha_2} \Rightarrow \frac{\sin(\theta'' - \alpha_2)}{\sin(\theta' - \alpha_2)} = n$$

where:

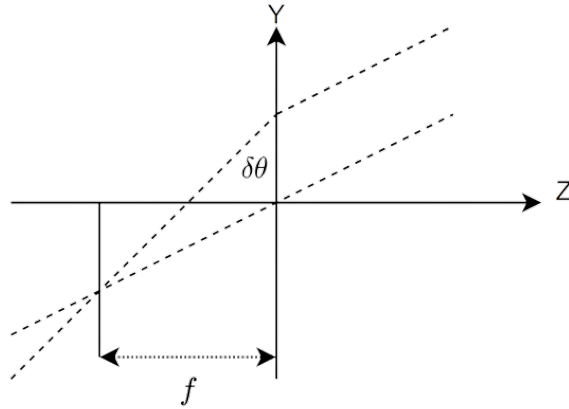
- θ is the angle before the lens.
- θ' is the angle in the lens (not visible in the image).
- θ'' is the angle after the lens.

Comparing the two equations it is possible to find the difference between the input angle θ and the output angle θ'' that is:

$$\delta\theta = y(n - 1) \left(\frac{1}{\rho_1} + \frac{1}{\rho_2} \right)$$

It is possible to see that the first term $(n - 1)$ is due to the matter of the lens and the second $\left(\frac{1}{\rho_1} + \frac{1}{\rho_2} \right)$ depend on the curvature of the surface.

1.3 Focalization of parallel light rays



In the image we have one ray that passes through the center of the lens and the other that passes in another point, but it is parallel to the first one. So we have that:

- $Y = 0$: the deviation of the ray is null and proceed without being deviated.
- $Y = f \cdot \delta\theta \Rightarrow f = \frac{1}{(n - 1) \left(\frac{1}{\rho_1} + \frac{1}{\rho_2} \right)}$.

This means that all the rays that proceed in parallel meets in one common point on the focal point Z , with a distance from the Y axis equal to:

$$Z = -f$$

1.4 Path of a light ray

To find the projection of a light ray crossing a lens in any position we need to:

1. Draw a line parallel to the selected ray and that passes through the center of the lens.
2. Intersect the line with the focal plane.
3. The ray will go from the point in which it crosses the lens to the point found on the focal plane.

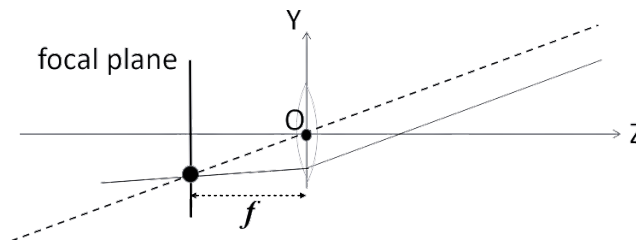


Figure 1.2: Path of a light ray through a lens

1.5 Pin-hole camera

To have a focussed image we need that every ray hits the focal plane of the camera in exactly one pixel. To obtain this we need that the distance between the lens and the source of the ray $Z(P)$ must be much greater than the lens aperture a (at least $1000\times$). In this way we can place the screen at distance Z from the lens. If all the rays respect this constraint they will be all parallel for the lens and the image will be on focus. The camera that we have defined so far is called pin-hole camera and needs:

1. Thin spherical lens.
2. Small angles.
3. $Z(P) \gg a$.
4. $Z = f$.

1.6 From real world to 2D images

The images are in a 2D plane while the real world is in three dimensions. So, a picture have less information than the original subject in the real world. In fact, the space projection of space in a 2D image is a perspective projection, that is:

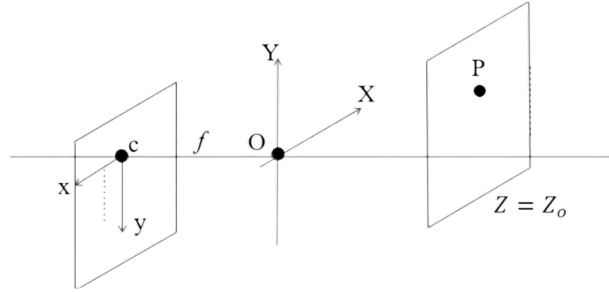
- Nonlinear.
- Not shape-preserving.
- Not length-ratio preserving.

Due to the triangle equality we have that:

$$x = f \frac{X}{Z} \quad y = f \frac{Y}{Z}$$

The only way to avoid this information loss is to take a picture of a planar scene on plane parallel to image plane, so we need to have

$$Z = Z_0 = \text{constant}$$



In this case the only difference between reality and projection will be the down scaling, while the other dimensions are preserved:

$$x = f \frac{X}{Z_0} = kX \quad y = f \frac{Y}{Z_0} = kY$$

1.7 Perspective and vanishing point

If we decide to choose all the lines parallel to the direction parameters $[\alpha \ \beta \ 1]$ we obtain the following system:

$$\begin{cases} X = X_0 + \alpha Z \\ Y = Y_0 + \beta Z \\ Z = 1 \cdot Z \end{cases}$$

All these lines can be projected in the 2D image with the triangle equality in the following way:

$$x = f \frac{X}{Z} = f \frac{X_0 + \alpha Z}{Z} = f\alpha + \frac{X_0}{Z}$$

$$y = f \frac{Y}{Z} = f \frac{Y_0 + \beta Z}{Z} = f\beta + \frac{Y_0}{Z}$$

We now want to find the image of the point at the infinity of these lines, so we obtain the point:

$$[f\alpha \ f\beta]$$

that is independent of X_0 and Y_0 . So, we have that all parallel lines have the same image if their points at the infinity.

Definition

The image of the point at the infinity of the lines is called *vanishing point*.

So, we have that all the lines that are parallel in the real world will project onto concurrent lines in the image.

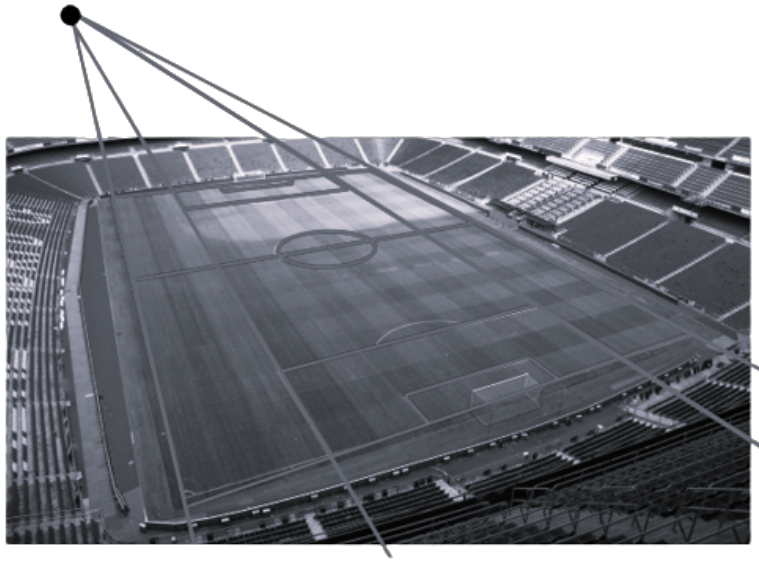


Figure 1.3: Example of vanishing point

CHAPTER 2

Planar Projective Geometry

2.1 Introduction

The elements needed to define a planar geometry are: point, lines, conics and dual conics. The possible transformations in this type of geometry are: projectivities, affinities, similarities and isometries.

2.2 Points

The points can be defined in Cartesian coordinates by defining a euclidean plane with his origin. In this way every point is defined unambiguously with two Cartesian coordinates (x, y) .

To analyze images it is better to use homogeneous coordinates. To define this type of coordinates we need to construct a 3D space with x, y, w axis. So, now to define a point we have to assign three values. This means that every number can be represented in infinite ways by changing the value of w . The correlation between Cartesian and homogeneous coordinates is the following:

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ w \end{bmatrix} = w \begin{bmatrix} X \\ Y \\ 1 \end{bmatrix}$$

So we have that a vector $\mathbf{x} = \begin{bmatrix} x \\ y \\ w \end{bmatrix}$ and all its nonzero multiples $\lambda \begin{bmatrix} x \\ y \\ w \end{bmatrix}$, including $\begin{bmatrix} x/w \\ y/w \\ 1 \end{bmatrix}$,

represent the point of Cartesian coordinates $\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} x/w \\ y/w \end{bmatrix}$ on the euclidean plane. This type of representation has the homogeneity property: any vector \mathbf{x} is equivalent to all its nonzero multiples $\lambda \mathbf{x}, \lambda \neq 0$, since they represent the same point. The null vector does not represent any point.

Definition

The *projective plane* is defined as:

$$\mathbb{P}^2 = \{[x \ y \ w]^T \in \mathbb{R}^3\} - \{[0 \ 0 \ 0]^T\}$$

Example: The origin of the plane is defined as:

$$\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$$

A generic point in heterogeneous coordinates can be transformed into a couple of Cartesian coordinate with a simple division. The point:

$$\begin{bmatrix} 0 & 8 & 4 \end{bmatrix}^T$$

in Cartesian coordinate is equal to:

$$\begin{bmatrix} x/w & y/w \end{bmatrix}^T = \begin{bmatrix} 0/4 & 8/4 \end{bmatrix} = \begin{bmatrix} 0 & 4 \end{bmatrix}$$

Consider the point $\mathbf{x} = \begin{bmatrix} x \\ y \\ w \end{bmatrix}$ and let w slowly drop to 0 starting from $w = 1$. As w decreases, the point will move along a constant direction $\begin{bmatrix} x & y \end{bmatrix}$, with increasing distance from the origin. As w tends to 0, this points tends to the infinity along the direction $\begin{bmatrix} x & y \end{bmatrix}$.

Definition

The point

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ w \end{bmatrix}$$

is called the *point at the infinity along the direction* $\begin{bmatrix} x & y \end{bmatrix}$.

Points at the infinity, who represent directions, are not part of the euclidean plane: they are extra points, well-defined within the projective plane. So, in general we have that the projective plane is the euclidean plane with also the points at the infinity.

2.3 Lines

In the euclidean plane a line is defined as:

$$aX + bY + c = 0$$

In the heterogeneous plane the lines are defined as:

$$a\frac{x}{w} + b\frac{y}{w} + c = 0 \implies ax + by + cw = 0$$

This equation can be also represented using two vectors called respectively \mathbf{l}^T and \mathbf{x} :

$$\begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix} = 0$$

where the vector $\mathbf{l} = \begin{bmatrix} a & b & c \end{bmatrix}^T$ and all its nonzero multiples represent a line. This type of representation has the homogeneity property: any vector \mathbf{l} is equivalent to all its nonzero multiples $\lambda\mathbf{l}$, $\lambda \neq 0$, since they represent the same line. a, b, c are called homogeneous parameters of the line.

As for numbers, there is an infinite number of equivalent representations for a single line, namely all nonzero multiples of the unit normal vector. The null vector does not represent any lines.

Definition

The *projective dual plane* is defined as:

$$\mathbb{P}^2 = \{[a \ b \ c]^T \in \mathbb{R}^3\} - \{[0 \ 0 \ 0]^T\}$$

There are three important remarks:

1. If the third parameter is null, $\mathbf{l} = [a \ b \ 0]^T$, then the line goes through point $[0 \ 0]$.
2. Within the euclidean plane, direction $[a \ b]$ is normal to the line $\mathbf{l} = [a \ b \ c]^T$.
3. Two lines $\mathbf{l} = [a \ b \ c]^T$ and $\mathbf{l}' = [a \ b \ c']^T$ are parallel: their common direction is $[b, -a]$.

Example: The Cartesian axes are defined as:

$$\mathbf{l}_x = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \mathbf{l}_y = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

The incidence relation of a line $\mathbf{l}^T \mathbf{x} = 0$ is defined when the point \mathbf{x} is on the line \mathbf{l} or the line \mathbf{l} goes through the point \mathbf{x} .

Definition

The line

$$[0 \ 0 \ 1] \begin{bmatrix} x \\ y \\ w \end{bmatrix} = \mathbf{w} = 0$$

is called the *line at the infinity* $\mathbf{l}_\infty = [0 \ 0 \ 1]^T$.

Since the dot product is commutative, also the incidence relation is commutative. This is called the principle of duality between points and lines. Given the lines \mathbf{l}_1^T and \mathbf{l}_2^T , the intersection can be found by imposing:

$$\begin{bmatrix} \mathbf{l}_1^T \\ \mathbf{l}_2^T \end{bmatrix} \mathbf{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

that is equal to find the right null space of the first column vector.

$$\mathbf{x} = \mathbf{RNS} \left(\begin{bmatrix} \mathbf{l}_1^T \\ \mathbf{l}_2^T \end{bmatrix} \right)$$

The system found is under determined, so there is only one intersection point between two lines (that can be represented in infinite ways in homogeneous coordinates). Note that the vector \mathbf{x} is orthogonal to both lines, so in the 2D projective geometry can be found with the cross product:

$$\mathbf{x} = \mathbf{l}_1 \times \mathbf{l}_2$$

Example: Suppose we have two parallel lines $\mathbf{l}_1 = [a \ b \ c_1]^T$ and $\mathbf{l}_2 = [a \ b \ c_2]^T$. The point that is common to both points can be found with the system:

$$\begin{cases} ax + by + c_1w = 0 \\ ax + by + c_2w = 0 \end{cases}$$

So the solution will be $\mathbf{x} = [b \ -a \ 0]^T$ that is the point at the infinity along the direction of both lines.

The line through two points can be found with the dual of the previous problem, that is:

$$\begin{bmatrix} \mathbf{x}_1^T \\ \mathbf{x}_2^T \end{bmatrix} \mathbf{l} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

that in 2D become:

$$\mathbf{l} = \mathbf{x}_1 \times \mathbf{x}_2$$

Property : The point \mathbf{x} given by the linear combination $\mathbf{x} = \alpha \mathbf{x}_1 + \beta \mathbf{x}_2$ of two points \mathbf{x}_1 and \mathbf{x}_2 is on the line \mathbf{l} through \mathbf{x}_1 and \mathbf{x}_2 .

Proof: The line \mathbf{l} through both points satisfies $\mathbf{l}^T \mathbf{x}_1 = 0$ and $\mathbf{l}^T \mathbf{x}_2 = 0$. By adding α times the first equation to β times the second one, we obtain:

$$0 = \mathbf{l}^T (\alpha \mathbf{x}_1 + \beta \mathbf{x}_2) = \mathbf{l}^T \mathbf{x} = 0$$

■

This proves that there is also a duality between colinear and concurrent.

Theorem

[duality principle] For any true sentence containing the words: point, line, is on and goes through there is a dual sentence (also true) obtained by substituting, in the previous one, each occurrence of:

- Point \Leftrightarrow line.
- Is on \Leftrightarrow goes through.
- Colinear \Leftrightarrow concurrent.

Within the euclidean plane, $[a \ b]$ is the direction normal to the line $\mathbf{l} = [a \ b \ c]^T$. Since we have that the angle between two lines is the angle between their normal and that the angle between two vectors is equal to:

$$\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}| |\mathbf{v}|}$$

we have that the angle between two lines $\mathbf{l}_1 = [a_1 \ b_1 \ c_1]^T$ and $\mathbf{l}_2 = [a_2 \ b_2 \ c_2]^T$ is the angle between their normal directions $[a_1 \ b_1]$ and $[a_2 \ b_2]$, namely:

$$\cos \theta = \frac{a_1 a_2 + b_1 b_2}{\sqrt{(a_1^2 + b_1^2)(a_2^2 + b_2^2)}}$$

Given a line with four points that have the following relations:

$$\mathbf{X}_1 = \alpha_1 \mathbf{Y} + \beta_1 \mathbf{Z}$$

$$\mathbf{X}_2 = \alpha_2 \mathbf{Y} + \beta_2 \mathbf{Z}$$

We have that the cross ratio is the following:

$$CR_{X_1, X_2, Y, Z} = \frac{c-a}{c-b} / \frac{d-a}{d-b} = \frac{\beta_1/\alpha_1}{\beta_2/\alpha_2}$$

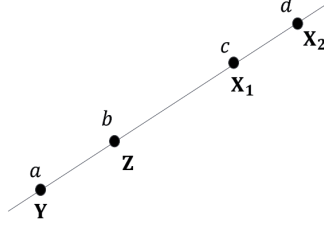
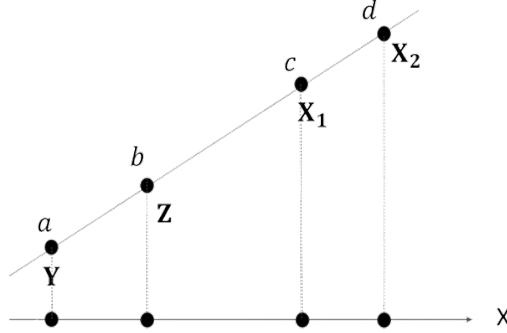


Figure 2.1: Line with the point of previous problem

Proof: Since the abscissae are proportional we can replace the abscissae by the X coordinate like in the figure.



And the relation

$$CR_{X_1, X_2, Y, Z} = \frac{c-a}{c-b} / \frac{d-a}{d-b}$$

still holds. Let $\mathbf{Y} = \begin{bmatrix} y \\ * \\ v \end{bmatrix}$ and $\mathbf{Z} = \begin{bmatrix} z \\ * \\ w \end{bmatrix}$, then:

$$\mathbf{X}_1 = \begin{bmatrix} \alpha_1 y + \beta_1 z \\ * \\ \alpha_1 v + \beta_1 w \end{bmatrix} \quad \mathbf{X}_2 = \begin{bmatrix} \alpha_2 y + \beta_2 z \\ * \\ \alpha_2 v + \beta_2 w \end{bmatrix}$$

The difference between the X coordinates of \mathbf{X}_1 and \mathbf{Y} is:

$$c - a = \frac{\beta_1(zv - yw)}{(\alpha_1 y + \beta_1 z)v}$$

The difference between the X coordinates of \mathbf{X}_1 and \mathbf{Z} is:

$$c - b = \frac{-\alpha_1(zv - yw)}{(\alpha_1 y + \beta_1 z)w}$$

By substitution we obtain:

$$\frac{c-a}{c-b} = -\frac{\beta_1 w}{\alpha_1 v} \quad \frac{d-a}{d-b} = -\frac{\beta_2 w}{\alpha_2 v}$$

■

2.4 Conics

The conics derives from the intersection of cones with planes.

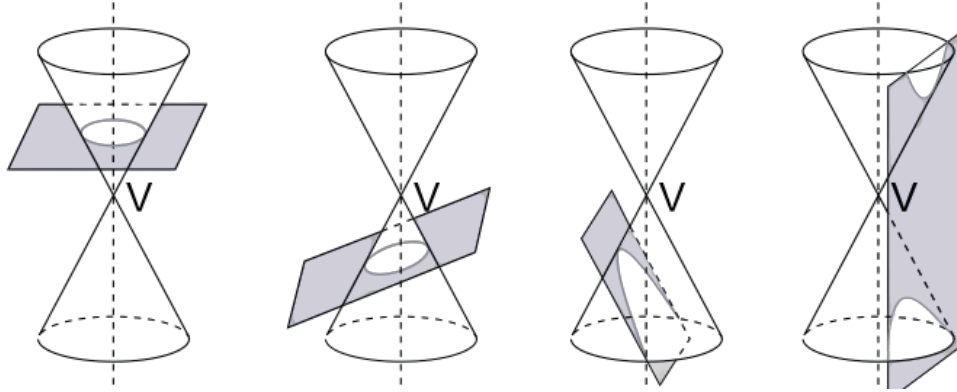


Figure 2.2: A circumference, an ellipse, a parabola and a hyperbole

Definition

A point \mathbf{x} is on a *conic* \mathbf{C} if it satisfies a homogeneous quadratic equation, namely

$$\mathbf{x}^T \mathbf{C} \mathbf{x} = 0$$

where \mathbf{C} is a symmetric matrix (this is a convention).

A conic is a curve described by a second-degree equation in the plane. In Euclidean coordinates a conic becomes:

$$aX^2 + bXY + cY^2 + dX + eY + f = 0$$

and in homogeneous coordinates becomes:

$$ax^2 + bxy + cy^2 + dxw + eyw + fw^2 = 0$$

or in matrix form:

$$\mathbf{x}^T \begin{bmatrix} a & b/2 & d/2 \\ b/2 & c & e/2 \\ d/2 & e/2 & f \end{bmatrix} \mathbf{x} = 0$$

The conics have five degrees of freedom, so we need five points to define a unique conic.

Example : The circumference can be expressed in Cartesian coordinates:

$$(X - X_0)^2 + (Y - Y_0)^2 - r^2 = 0$$

or in homogeneous coordinates:

$$\begin{bmatrix} x & y & w \end{bmatrix} \begin{bmatrix} 1 & 0 & -X_0 \\ 0 & 1 & -Y_0 \\ -X_0 & -Y_0 & X_0^2 + Y_0^2 - r^2 \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix} = 0$$

Given a quadratic equation of a conic and a linear equation of a line, their intersection leads to a two degree equation on \mathbf{x} . This means that there are always two intersection points between a line and a conic, and they can be: real or complex conjugate and distinct or coincident. This result is due to the fundamental theorem of algebra.

Theorem

[Fundamental theorem of algebra] The fundamental theorem of algebra states that every non-constant single-variable polynomial with complex coefficients has at least one complex root.

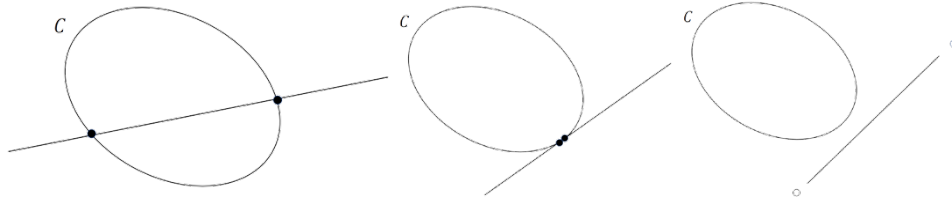


Figure 2.3: Intersection with two real roots, two coincident roots and two imaginary roots

The intersection between the line at the infinity and a conic is:

- A parabola if there are two coincident solutions, that is the point at the infinity along axis.
- An ellipse if there are two complex-conjugate, that is there are no real solution.
- A hyperbole if there are two real distinct solutions, that is the lines are the asymptotes.

2.4.1 Circular points

Example : Intersecting a circumference and the line at the infinity we obtain the following system:

$$\begin{cases} x^2 - 2X_0w + X_0^2w^2 + y^2 - 2Y_0w + Y_0^2w^2 - r^2w^2 = 0 \\ w = 0 \end{cases}$$

We obtain:

$$x^2 + y^2 = 0$$

As we can see, the circumference parameters (center and radius) disappear from the equation. So, we have that the two intersection points are the same for all circumferences.

Definition

The two intersection points are the same for all circumferences with the line at the infinity are the points:

$$\mathbf{I} = \begin{bmatrix} 1 \\ i \\ 0 \end{bmatrix} \quad \mathbf{J} = \begin{bmatrix} 1 \\ -i \\ 0 \end{bmatrix}$$

called *circular points*.

2.4.2 Polar line

Definition

Given a point \mathbf{y} and a conic \mathbf{C} in the plane, the line $\mathbf{l} = C\mathbf{y}$ is called the *polar line* of point \mathbf{y} with respect to the conic \mathbf{C} .

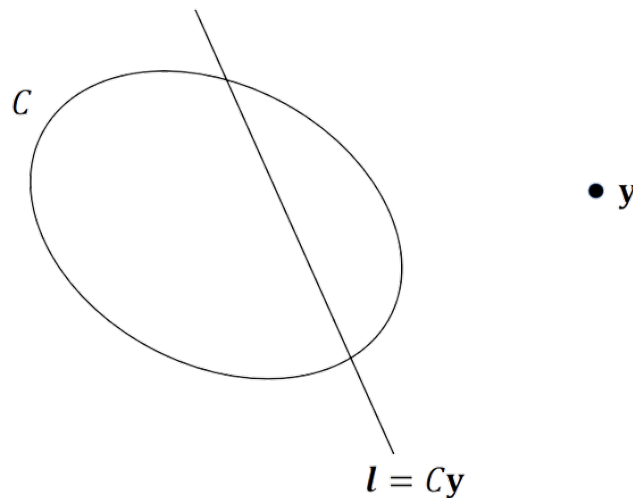


Figure 2.4: Example of polar line

2.4.3 Harmonic tuples

Definition

A 4-tuple of colinear points A, B, C, D , whose cross ratio is -1 , is said to be a *harmonic tuple*.

This value, of course, is also assumed by the cross ratio of other 4-tuples of colinear points. A notable example is:

$$(T, Z, \text{mid_point}(Y, Z), P(\text{at the infinity}))$$

If (A, B, C, D) is a harmonic 4-tuple, also (C, D, A, B) is harmonic.

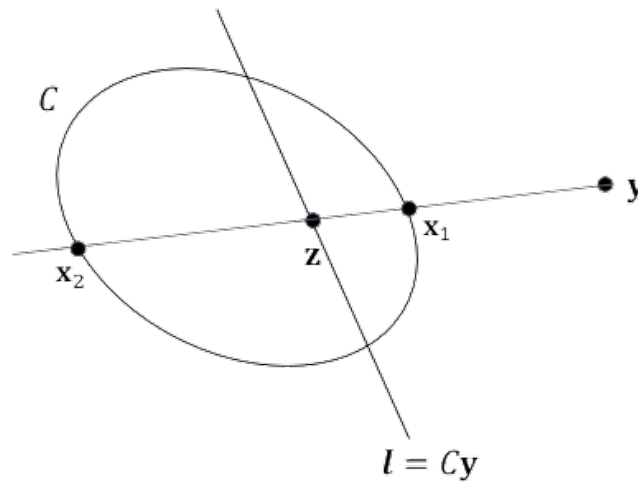
Definition

In a harmonic tuple (A, B, C, D) , A and B are said to be *conjugate* of each other with respect to C and D .

Since the cross ratio of a harmonic tuple is negative, then two conjugate points A, B with respect to C, D are one inside segment (C, D) and one outside (C, D) .

2.4.4 Polar line and harmonic tuples

Consider any point \mathbf{z} on the polar line $\mathbf{l} = C\mathbf{y}$, and consider the line through \mathbf{y} and \mathbf{z} : call \mathbf{x}_1 and \mathbf{x}_2 the points where this line crosses the conic.



Theorem

Let x_1 and x_2 be the points where the line through y and z crosses C : y and z are conjugate with respect to x_1 and x_2 .

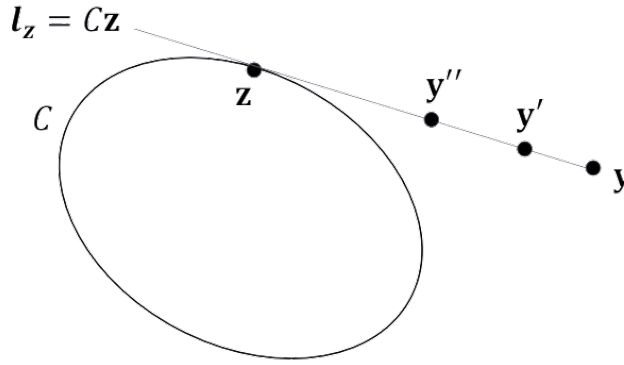
The polar line $l = Cy$ is the locus of the points conjugate of y with respect to conic C (more precisely, conjugate with respect to the intersection points of C with any line through y).

2.4.5 Polar line and tangency points

Since y is outside segment (x_1, x_2) then its conjugate z is inside (x_1, x_2) . This holds for any line through y . As the line through y tends to be tangent to C , the points x_1 and x_2 coincide with the tangency point to C : therefore the conjugate point z , which remains inside (x_1, x_2) , is also coincident with the tangency point. The conjugate point z , which remains inside (x_1, x_2) , also coincides with the tangency point. Analogous result holds for the other line tangent to C from y .

So, we have obtained that the polar line $l = Cy$ goes through the tangency points from y to C .

We have that points z and y are conjugate with respect to the conic C . So, if point z is on the conic C , then point y is one of its conjugates. We have that the tangent line l_z to C through z is the locus of points conjugate to z . So, we have that l_z is the polar line of z with respect to C .



We can see that the polar line $l_z = Cz$ of a point z on the conic C is the tangent line to C through z .

Example : Consider a circumference with radius r centered in the origin of the plane and the point $y = [X \ 0 \ 1]^T$, its equation is:

$$l = Cy = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -r^2 \end{bmatrix} \begin{bmatrix} X \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} X \\ 0 \\ -r^2 \end{bmatrix}$$

So we obtained that the polar line has the following Cartesian equation:

$$Xx - r^2 = 0 \rightarrow x = \frac{r^2}{X}$$

that is a vertical line.

With the previous example we obtained that the polar of a point P with respect to a circumference is a line perpendicular to the segment (center, P).

Example : Consider a circumference with radius r centered in the origin of the plane and the point $y = [x \ 0 \ 0]^T$, its equation is:

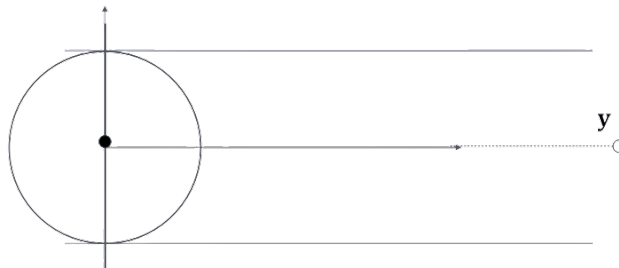
$$l = Cy = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -r^2 \end{bmatrix} \begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} X \\ 0 \\ 0 \end{bmatrix}$$

So we obtained that the polar line has the following Cartesian equation:

$$Xx = 0 \rightarrow X = 0$$

that is at the diameter of the circumference perpendicular at the direction of the point y .

Tangent lines from a point at the infinity are all parallel, so we have that tangency points are along a diameter perpendicular to the common direction of the parallel tangents.



Example : Consider a circumference with radius r centered in the origin of the plane and the point $\mathbf{y} = [x \ 0 \ 0]^T$, its equation is:

$$\mathbf{l} = C\mathbf{y} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -r^2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -r^2 \end{bmatrix}$$

So we obtained that the polar line has the following Cartesian equation:

$$-r^2 w = 0 \rightarrow X = 0$$

that is at the line at the infinity.

In general, we have the following properties:

1. The polar line of any point at the infinity is a diameter.
2. Any diameter goes through the center of the circumference.
3. The center is conjugate to each point at the infinity.
4. All the points at the infinity are conjugate to the center.
5. The polar of the center is the line containing all the points at the infinity.
6. The polar line of the center is the line at the infinity.

2.4.6 Degenerate conics

Definition

A *non-degenerate conic* is a conic where the matrix \mathbf{C} is non-singular:

$$\text{rank}(\mathbf{C}) = 3$$

A *degenerate conic* is a conic where the matrix \mathbf{C} is singular:

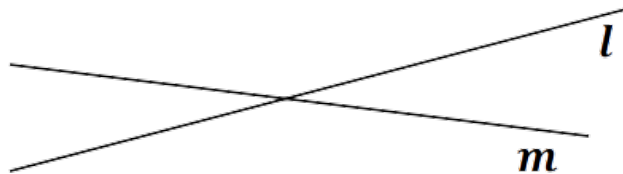
$$\text{rank}(\mathbf{C}) < 3$$

There are two possible cases:

- $\text{rank}(\mathbf{C}) = 2$: in this case any symmetric 3×3 matrix \mathbf{C} can be written as:

$$\mathbf{C} = \mathbf{l}\mathbf{m}^T + \mathbf{m}\mathbf{l}^T$$

where \mathbf{l} and \mathbf{m} are column vectors. This conic is the set of points \mathbf{x} satisfying $\mathbf{x}^T \mathbf{C} \mathbf{x} = 0$. This equation is satisfied if $\mathbf{x}^T \mathbf{l} = 0$ or $\mathbf{m}^T \mathbf{x} = 0$ are satisfied. So, we have that \mathbf{x} is on the set union of lines \mathbf{l} and \mathbf{m} .



- $\text{rank}(\mathbf{C}) = 1$: in this case any symmetric 3×3 matrix \mathbf{C} can be written as:

$$\mathbf{C} = \mathbf{l}\mathbf{l}^T$$

where \mathbf{l} is a column vector. This conic is the set of points \mathbf{x} satisfying $\mathbf{x}^T \mathbf{C} \mathbf{x} = 0$. This equation is satisfied if $\mathbf{x}^T \mathbf{l} = 0$ is satisfied (counted two times). So, we have that \mathbf{x} is on the repeated line \mathbf{l} .



2.5 Dual conics

2.5.1 Definition

Definition

A *dual conic* is a set of lines \mathbf{l} that satisfy equation

$$\mathbf{l}^T \mathbf{C}^* \mathbf{l} = 0$$

where \mathbf{C}^* is a 3×3 symmetric matrix.

A *non-degenerate dual conic* is a dual conic whose matrix \mathbf{C}^* is non-singular:

$$\text{rank}(\mathbf{C}^*) = 3$$

Consider a non-degenerate conic \mathbf{C} and the set of all lines \mathbf{l} that are tangent to it: through each point \mathbf{c} on \mathbf{C} there is a line \mathbf{l} tangent to \mathbf{C} . Since \mathbf{l} is the polar of \mathbf{x} with respect to \mathbf{C} , is $\mathbf{l} = \mathbf{C}\mathbf{c}$. Therefore, is

$$\mathbf{x} = \mathbf{C}^{-1}\mathbf{l}$$

and hence, since \mathbf{C} is symmetric we have

$$\mathbf{x}^T = \mathbf{l}^T \mathbf{l}^{-T} = \mathbf{l}^T \mathbf{C}^{-1}$$

Now, since the point \mathbf{x} is on \mathbf{C} ,

$$\mathbf{x}^T \mathbf{C} \mathbf{x} = 0$$

Substituting the values found previously we obtain:

$$\mathbf{l}^T \mathbf{C}^{-1} \mathbf{l} = 0$$

that is the quadratic homogeneous equation on \mathbf{l} . So we can say that for the dual conic $\mathbf{C}^* = \mathbf{C}^{-1}$

We can also say that a non-degenerate dual conic \mathbf{C}^* is the set of lines tangent to a non-degenerate conic \mathbf{C} .

2.5.2 Degenerate dual conics

Definition

A *degenerate dual conic* is a conic where the matrix \mathbf{C}^* is singular:

$$\text{rank}(\mathbf{C}^*) < 3$$

There are two possible cases:

- $\text{rank}(\mathbf{C}^*) = 2$: in this case any symmetric 3×3 matrix \mathbf{C}^* can be written as:

$$\mathbf{C}^* = \mathbf{p}\mathbf{q}^T + \mathbf{q}\mathbf{p}^T$$

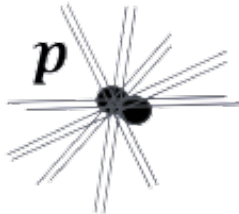
This conic is the line \mathbf{l} going through point \mathbf{p} or the line \mathbf{l} going through point \mathbf{q} .



- $\text{rank}(\mathbf{C}^*) = 1$: in this case any symmetric 3×3 matrix \mathbf{C}^* can be written as:

$$\mathbf{C}^* = \mathbf{p}\mathbf{p}^T$$

This conic is the line \mathbf{l} going through point \mathbf{p} repeated twice.



Definition

The degenerate dual conic $\mathbf{C}^* = \mathbf{p}\mathbf{q}^T + \mathbf{q}\mathbf{p}^T$ going through two circular point \mathbf{p} and \mathbf{q} is called the conic dual to the circular points, and it is equal to:

$$\mathbf{C}_{\infty}^* = \mathbf{I}\mathbf{J}^T + \mathbf{J}\mathbf{I}^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

2.6 Transformations

Definition

A *projective mapping* between a projective plane \mathbb{P}^2 and another projective plane \mathbb{P}'^2 is

an invertible mapping which preserves colinearity:

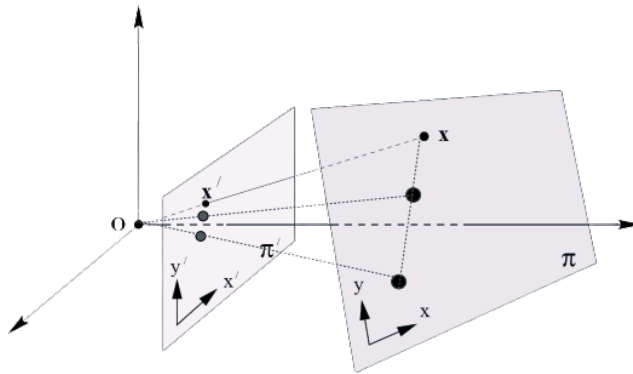
$$h : \mathbb{P}^2 \rightarrow \mathbb{P}^2, \mathbf{x}' = h(\mathbf{x}), \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \text{ are colinear}$$

$$\Leftrightarrow$$

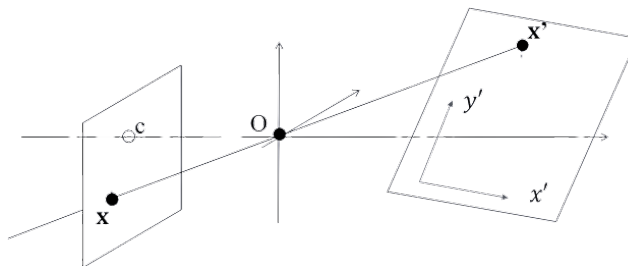
$$\mathbf{x}'_1 = h(\mathbf{x}_1), \mathbf{x}'_2 = h(\mathbf{x}_2), \mathbf{x}'_3 = h(\mathbf{x}_3) \text{ are colinear}$$

Projective mapping is also called projectivity or homography.

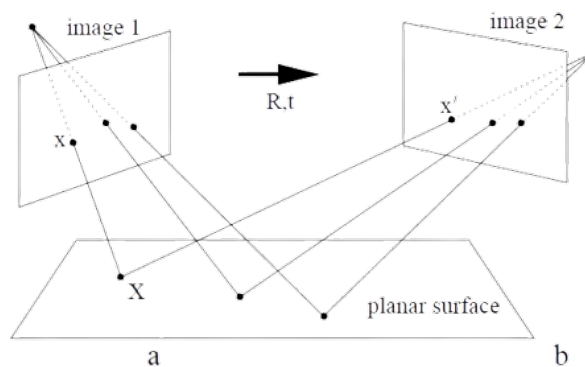
Example: Mappings between two planes induced by central projection are projective, since they preserve colinearity.



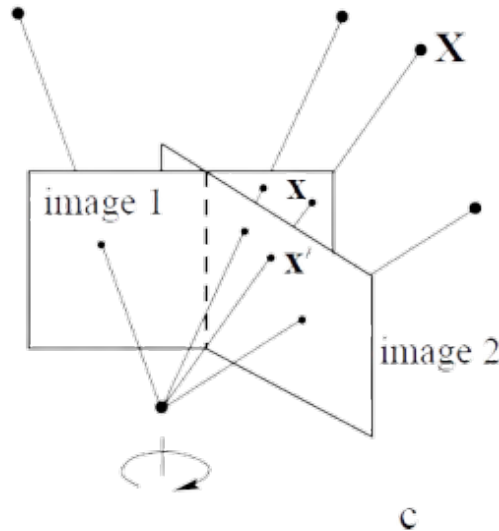
Mapping between a planar scene and its image is a homography, since it is induced by a central projection



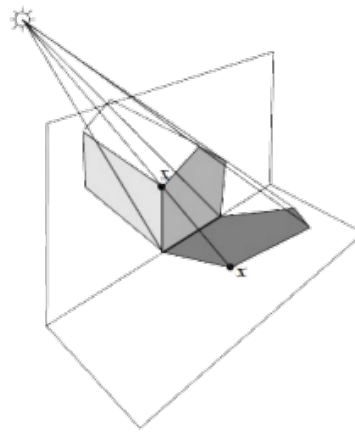
Mapping between two images of a planar scene is a homography.



Two images of a 3D scene, taken by a camera rotating around its center are related by a homography, since the second image is a central projection of the first image.



The shadow cast by a planar silhouette onto a ground plane is a projective transformation of the planar silhouette, since they are related by a central projection.



Theorem

[Fundamental theorem of projective geometry] A mapping $h : \mathbb{P}^2 \rightarrow \mathbb{P}'^2$ is projective if and only if there exists an invertible 3×3 matrix H such that for any point in \mathbb{P}^2 represented by the vector \mathbf{x} , is $h(\mathbf{x}) = H\mathbf{x}$, where:

$$H = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}$$

Projective mappings are linear in the homogeneous coordinates, but they are not in Cartesian coordinates.

From the theorem we have that $h(\mathbf{x}) = \mathbf{x}' = H\mathbf{x}$. Therefore, if we multiply the matrix H by any nonzero scalar λ , the relation is satisfied by the same points $\mathbf{x}' = \lambda H\mathbf{x}$. Thus, any nonzero multiple of the matrix H represents the same projective mapping as H . Hence, H is a homogeneous matrix: in spite of its 9 entries, H has only 8 degrees of freedom, namely the

ratios between its elements. Therefore, it can be estimated by just four point correspondences, since each point correspondence $\mathbf{x}' = H\mathbf{x}$ yields two independent equations.

Definition

A homography transform each:

1. Point \mathbf{x} into a point \mathbf{x}' such that: $\mathbf{x} \rightarrow H\mathbf{x} = \mathbf{x}'$
2. Line \mathbf{l} into a line \mathbf{l}' such that: $\mathbf{l} \rightarrow H^{-T}\mathbf{l} = \mathbf{l}'$
3. Conic C into a conic C' such that: $C \rightarrow H^{-T}CH^{-1} = C'$
4. Dual conic C^* into a dual conic $C^{*'}$ such that: $C^* \rightarrow HC^*H^T = C^{*'}$

Proof property two: The equation of the line on \mathbf{x} is $\mathbf{l}^T\mathbf{x} = 0$ has to be converted into a constraint on $\mathbf{x}' = H\mathbf{x}$. Combining the two equations we obtain a linear equation on \mathbf{x}' :

$$\mathbf{l}'^T\mathbf{x}' = 0$$

With $\mathbf{l}'^T = \mathbf{l}^T H^{-1}$ and so we have:

$$\mathbf{l}' = H^{-T}\mathbf{l}$$

Proof property three: The equation of the conic on \mathbf{x} is $\mathbf{x}^T C \mathbf{x} = 0$ has to be converted into a constraint on $\mathbf{x}' = H\mathbf{x}$, that is equal to:

$$\mathbf{x} = H^{-1}\mathbf{x}' \text{ and } \mathbf{x}^T = \mathbf{x}'^T H^{-T}$$

Combining the three equations we obtain a linear equation on \mathbf{x}' :

$$\mathbf{x}'^T C' \mathbf{x}' = 0$$

And so we have:

$$C' = H^{-T}CH^{-1}$$

Proof property four: It is possible to apply the same idea to prove the transformation of a dual conic, obtaining:

$$C^{*'} = HC^*H^T$$

The point line incidence is preserved.

Proof: Suppose that point \mathbf{x} is on the line \mathbf{l} , that is $\mathbf{l}^T\mathbf{x} = 0$. Apply the same projective transformation H both to \mathbf{x} and \mathbf{l} : $H\mathbf{x} = \mathbf{x}'$ and $H^{-1}\mathbf{l} = \mathbf{l}'$. They are still incident if $\mathbf{l}'^T\mathbf{x}' = 0$:

$$\mathbf{l}'^T\mathbf{x}' = \mathbf{l}^T H^{-1}\mathbf{x}' = \mathbf{l}^T H^{-1}H\mathbf{x} = \mathbf{l}^T\mathbf{x} = 0$$

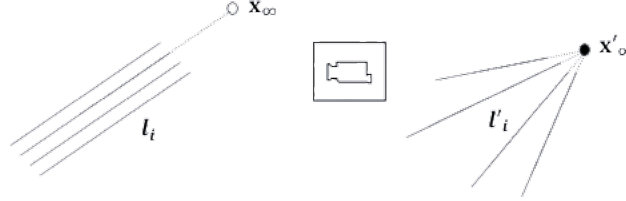
2.6.1 Vanishing points and vanishing line

The point common to both parallel lines $\mathbf{l}_1 = [a \ b \ c_1]^T$ and $\mathbf{l}_2 = [a \ b \ c_2]^T$ is the point $\mathbf{x} = [b \ -a \ 0]^T$, that is the point at the infinity along the direction of both lines. If we search for the common point of infinite lines \mathbf{l}_i we obtain that they all have in common the point:

$$\mathbf{x}_\infty = [b \ -a \ 0]^T$$

So, we obtained that all lines are concurrent at $[b \ -a \ 0]^T$.

Applying a projective transformation to all the above parallel lines l_i we obtain lines l'_i . The common point x_∞ to all lines l_i is mapped onto a point x'_∞ which belongs to each of the lines l'_i .

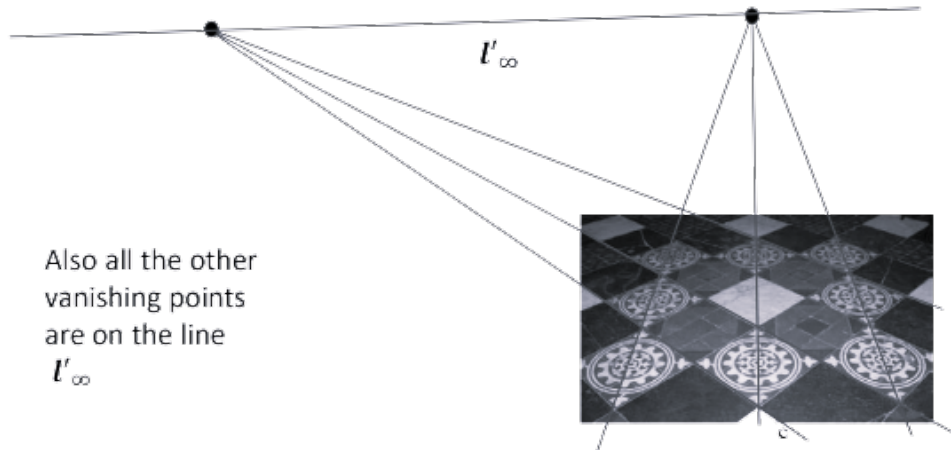


So, we have that all lines l'_i are concurrent at the point $x'_\infty = Hx_\infty$ called the vanishing point associated to the direction $(b, -a)$, of the parallel lines.

Theorem

The image of a set of parallel lines l_i is a set of lines l'_i concurrent at a common point x' called the vanishing point of the direction of lines l_i .

Applying a projective transformation to line at the infinity l_∞ we obtain a line l'_∞ . This line goes through the image all the points at the infinity x_∞ of the original plane. So, we have that the vanishing line l'_∞ can be determined from two vanishing points.



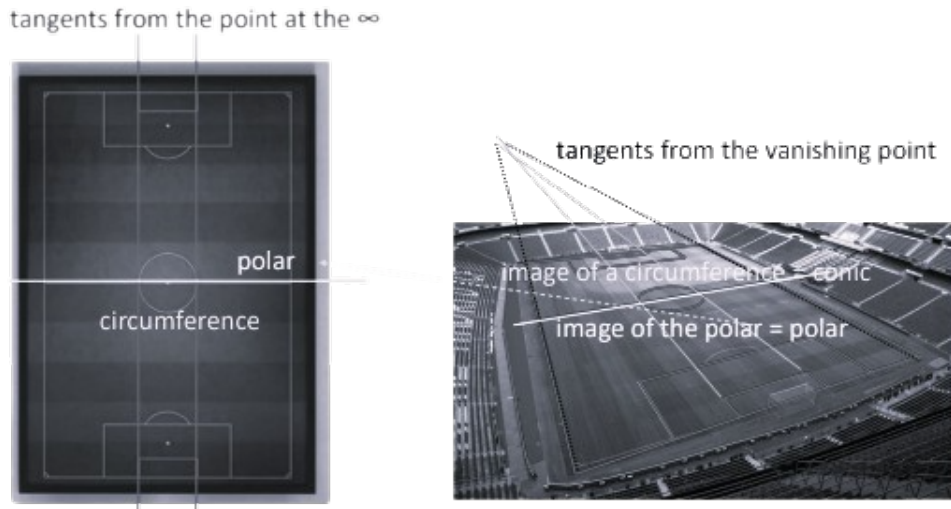
2.6.2 Polarity

Polarity is preserved under projective mappings. The polar line $l = Cx$ of a point x with respect to a conic C is mapped onto the polar line of the transformed point $x' = Hx$ with respect to the transformed conic:

$$C' = H^{-T}CH^{-1}$$

Proof: This property is satisfied because we have:

$$C'x' = H^{-T}CH^{-1}Hx = H^{-T}Cx = H^{-T}l = l'$$



In the end we have that since polarity is preserved, also conjugacy is preserved under projective mappings and also $CR = -1$ is preserved.

2.6.3 Cross ratio

Given a line with four points that have the following relations:

$$\mathbf{X}_1 = \alpha_1 \mathbf{Y} + \beta_1 \mathbf{Z}$$

$$\mathbf{X}_2 = \alpha_2 \mathbf{Y} + \beta_2 \mathbf{Z}$$

We have that the cross ratio is the following:

$$CR_{X_1, X_2, Y, Z} = \frac{\beta_1 / \alpha_1}{\beta_2 / \alpha_2}$$

If we apply a projective transformation H to the four points we will have:





$$\mathbf{Y}' = H\mathbf{Y} \quad \mathbf{Z}' = H\mathbf{Z}$$

$$\mathbf{X}'_1 = H\mathbf{X}_1 = \mathbf{X}_1 = \alpha_1 \mathbf{Y}' + \beta_1 \mathbf{Z}' \quad \mathbf{X}'_2 = H\mathbf{X}_2 = \alpha_2 \mathbf{Y}' + \beta_2 \mathbf{Z}'$$

The linear combination coefficients are the same. Therefore their cross ratio is preserved and it is again:

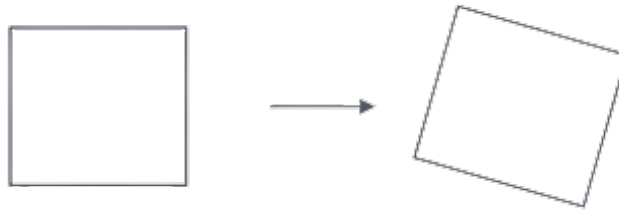
$$CR_{X'_1, X'_2, Y', Z'} = \frac{\beta_1 / \alpha_1}{\beta_2 / \alpha_2} = CR_{X_1, X_2, Y, Z}$$

2.6.4 Hierarchy of projective transformations

Projective 8 dof	$\begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}$		Views of a plane from different viewpoints, any view of a scene from the same viewpoint.
Affine 6 dof	$\begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$		Images of a "far away" object under any rotation
Similarity 4 dof	$\begin{bmatrix} sr_{11} & sr_{12} & t_x \\ sr_{21} & sr_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$		Camera looking at an assembly line w/ zoom.
Euclidean 3 dof	$\begin{bmatrix} r_{11} & r_{12} & t_x \\ r_{21} & r_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$		Camera looking at an assembly line.

2.6.5 Isometries

The isometries have three degree of freedom, which are the translation \mathbf{t} and the rotation angle ϑ . So, the invariants of this transformation are: lengths, distances, and areas.



Definition

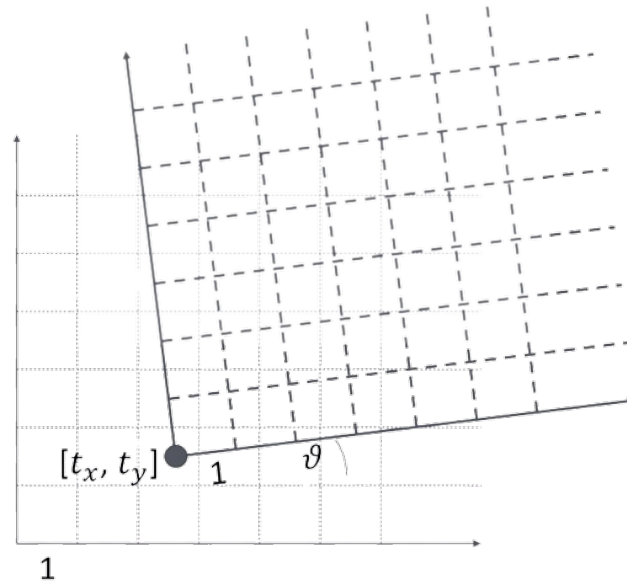
The orthogonal matrix R_{\perp} is defined as follows:

$$R_{\perp}^{-1} = R_{\perp}^T$$

So, we have that the matrix H_I for the isometries is the follow:

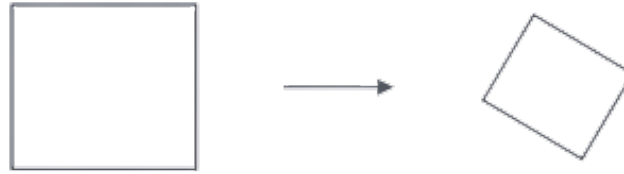
$$H_I = \begin{bmatrix} \cos \vartheta & -\sin \vartheta & t_x \\ \sin \vartheta & \cos \vartheta & t_y \\ 0 & 0 & 1 \end{bmatrix}$$

Where $\begin{bmatrix} \cos \vartheta & -\sin \vartheta \\ \sin \vartheta & \cos \vartheta \end{bmatrix} = R_{\perp}$



2.6.6 Similarities

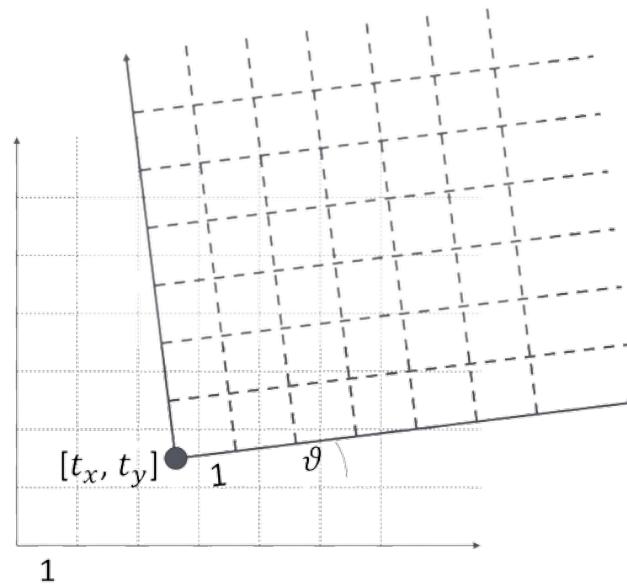
The similarities have four degree of freedom, which are the translation \mathbf{t} , the scale s , and the rotation angle ϑ . So, the invariants of this transformation are: ratio of lengths, and angles. Also, the circular points \mathbf{I} and \mathbf{J} are invariant of the transformation.



So, we have that the matrix H_S for the similarities is the follow:

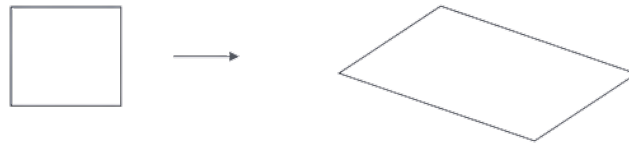
$$H_I = \begin{bmatrix} s \cos \vartheta & -s \sin \vartheta & t_x \\ s \sin \vartheta & s \cos \vartheta & t_y \\ 0 & 0 & 1 \end{bmatrix}$$

Where $\begin{bmatrix} s \cos \vartheta & -s \sin \vartheta \\ s \sin \vartheta & s \cos \vartheta \end{bmatrix} = sR_{\perp}$



2.6.7 Affinities

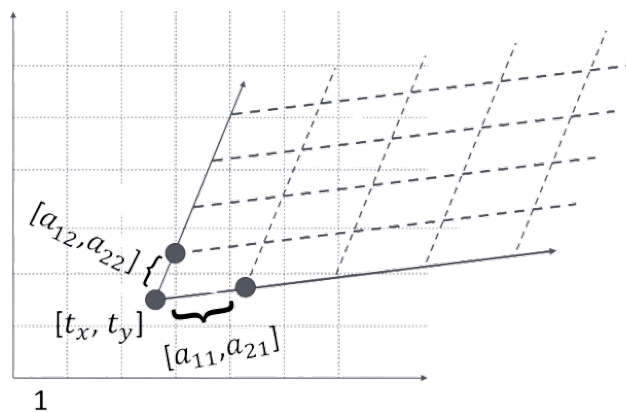
The affinities have six degree of freedom, which are the submatrix A and the translation. So, the invariants of this transformation are: parallelism, ratio of parallel lengths, ratio of areas. A is any 2×2 (rank two) matrix. Also the line at the infinity l_∞ is invariant of the transformation.



So, we have that the matrix H_A for the affinities is the follow:

$$H_I = \begin{bmatrix} a_{11} & a_{21} & t_x \\ a_{12} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$$

Where $\begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix} = A$



2.6.8 Projectivities

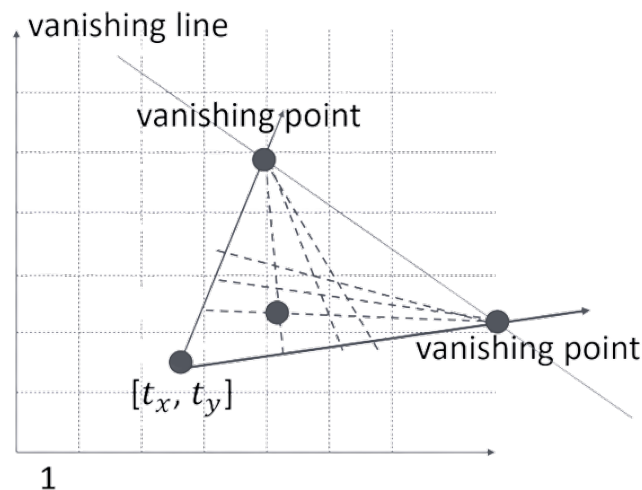
The projectivities have eight degree of freedom, which are the sub-matrix A , the vector v , and the translation. So, the invariants of this transformation are: colinearity, incidence, order of contact. A is any 2×2 (rank two) matrix. Also, the cross ratio is invariant of the transformation.



So, we have that the matrix H_A for the projectivities is the follow:

$$H_I = \begin{bmatrix} a_{11} & a_{21} & t_x \\ a_{12} & a_{21} & t_y \\ v_1 & v_2 & 1 \end{bmatrix}$$

Where $\begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{21} \end{bmatrix} = A$



CHAPTER 3

Two-dimensional reconstruction

3.1 Introduction

We want to recover a model of the scene, given an image of an unknown planar scene. The scene is a set of unknown points x_i on a plane. The image is a projective transformation of the scene, that is $\mathbf{x}'_i = H\mathbf{x}_i$. In this equation we know only the \mathbf{x}_i and not the matrix H . So, the main problem is that we cannot simply invert the mapping H .

The general problem is unsolvable because we have too many unknowns. So, to find the projection matrix we can combine two strategies:

1. Reduce unknowns: it is often not necessary to recover the original scene as it was, but it is simpler to recover just the shape of the scene. In this case the goal is to recover a model similar to the scene, that is called shape reconstruction. In this case, we can reduce the unknowns from eight to four and the matrix H becomes:

$$H = \begin{bmatrix} s \cos \vartheta & -s \sin \vartheta & t_x \\ s \sin \vartheta & s \cos \vartheta & t_y \\ 0 & 0 & 1 \end{bmatrix}$$

We can also make a similarity instead of the shape, but this will reduce the unknowns only by two. The affinity will reduce the unknowns by six.

2. Add constraints: we use additional information to recover a model of the scene. The useful information is the values of parameters, that are invariant under the desired class of mappings, but are not invariant under more general classes.

3.2 Reconstruction

The reconstruction can be of two types:

- Affine reconstruction: reconstructed scene is an affine mapping of the original one.
- Shape reconstruction: reconstructed scene is a similarity mapping of the original one.

3.2.1 Affine reconstruction

Theorem

A projective transformation H maps the line at the infinity \mathbf{l}_∞ onto itself implies that H is affine.

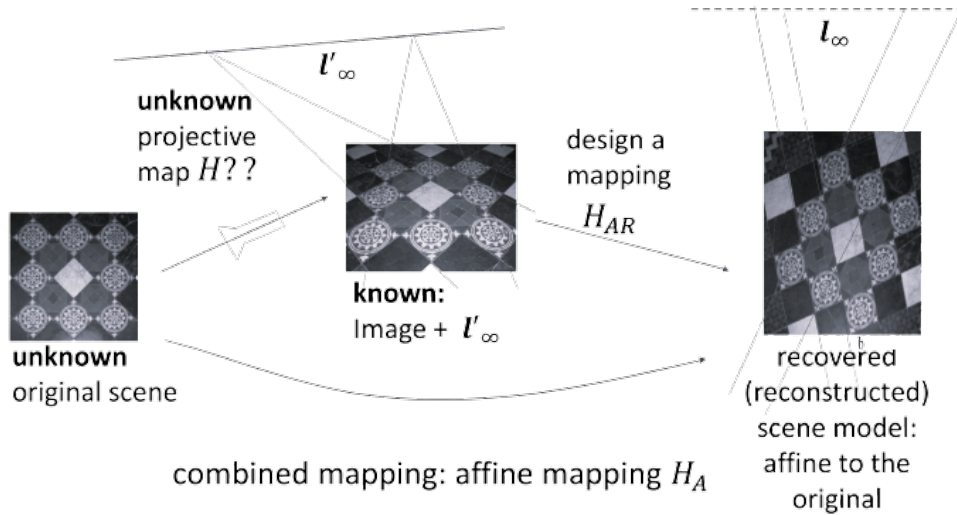
Proof: Any point at the infinity $\mathbf{x}_\infty = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$ mapped onto a point $\mathbf{x}' = H\mathbf{x}_\infty$ also at the infinity if and only if the third coordinate of \mathbf{x}' is zero. For all (x, y) we have that

$$\begin{bmatrix} v_1 & v_2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} = 0 \rightarrow \begin{bmatrix} v_1 & v_2 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$$

Namely, H is affine. ■

The given image is a general projective mapping of the original scene. This implies that the vanishing line \mathbf{l}'_∞ is in general different from \mathbf{l}_∞ . So, it is possible to use \mathbf{l}'_∞ as additional information: if we apply to the image a new projective mapping H_{AR} which sends \mathbf{l}'_∞ back to \mathbf{l}_∞ , we obtain a new, modified image. The (new) image of the line at the infinity \mathbf{l}_∞ is again \mathbf{l}_∞ .

From the theorem, the obtained model (i.e. the new image) is an affine mapping of the original scene. So, the obtained model is an affine reconstruction of the scene.



The issues with this method are:

- Find a projective mapping H_{AR} which sends \mathbf{l}'_∞ back to \mathbf{l}_∞ : this mapping must map any point $\mathbf{x}' \in \mathbf{l}'_\infty$ onto a set of point at the infinity:

$$H_{AR}\mathbf{x}' = \begin{bmatrix} * \\ * \\ 0 \end{bmatrix}$$

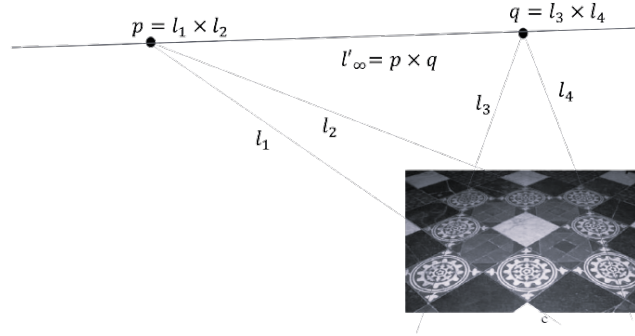
Everything works if we let

$$H_{AR} = \begin{bmatrix} * & * & * \\ * & * & * \\ & \mathbf{l}'_T & \end{bmatrix}$$

In fact we have:

$$H_{AR}\mathbf{x}' = \begin{bmatrix} * & * & * \\ * & * & * \\ & \mathbf{l}'_T & \end{bmatrix} \mathbf{x}' = \begin{bmatrix} * \\ * \\ \mathbf{l}'_T \mathbf{x}' \end{bmatrix}$$

- Find the vanishing line: it is possible to use additional knowledge items, such as the image of parallel lines.



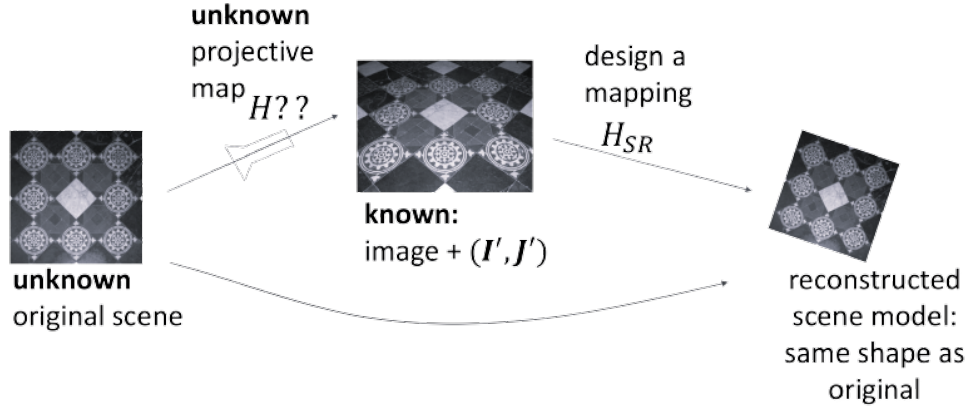
3.2.2 Shape reconstruction

Theorem

A projective transformation H maps the circular points \mathbf{I} and \mathbf{J} onto themselves implies that and implies from H is a similarity.

Proof: Multiplying a similarity matrix H_S by circular point \mathbf{I} , a multiple of \mathbf{I} is obtained. Analogous result is obtained for the other circular point \mathbf{J} . ■

The given image is a general projective mapping of the original scene. So, the image $(\mathbf{I}', \mathbf{J}')$ of circular points (\mathbf{I}, \mathbf{J}) of the scene plane is in general different from \mathbf{I}, \mathbf{J} . We can use \mathbf{I}', \mathbf{J}' as additional information: if we apply to the image a new projective mapping H_{SR} which maps \mathbf{I}', \mathbf{J}' back to \mathbf{I}, \mathbf{J} , we obtain a new, modified image. The new image of the circular points \mathbf{I}, \mathbf{J} is again \mathbf{I}, \mathbf{J} . So, from the theorem, the obtained model (new image) is similar to the original scene. We have that the obtained model is a shape reconstruction of the scene.



The issues with this method are:

- Find a projective mapping H_{SR} which maps \mathbf{I}', \mathbf{J}' back to \mathbf{I}, \mathbf{J} : finding one of the ∞^4 matrices such that: $\begin{cases} H_{SR}\mathbf{I}' = \mathbf{I} \\ H_{SR}\mathbf{J}' = \mathbf{J} \end{cases}$. Let us use equivalent information instead: the degenerate conic dual to \mathbf{I}', \mathbf{J}' , that is $C_\infty'^* = \mathbf{I}'\mathbf{J}'^T + \mathbf{J}'\mathbf{I}'^T$. This dual conic is the image of the original conic dual to the circular points (\mathbf{I}, \mathbf{J}) :

$$C_\infty^* = \mathbf{I}\mathbf{J}^T + \mathbf{J}\mathbf{I}^T$$

in fact, since $(\mathbf{I}', \mathbf{J}')$ is the image of (\mathbf{I}, \mathbf{J}) , then also $C_\infty'^*$ is the image of C_∞^* . So, any projectivity H_{SR} which maps $(\mathbf{I}', \mathbf{J}')$ back to (\mathbf{I}, \mathbf{J}) also maps $C_\infty'^*$ back to C_∞^* . Using the transformation rule for dual conics under projective mappings we get:

$$C_\infty^* = H_{SR}C_\infty'^*H_{SR}^T$$

Inverting this relationship yields:

$$C_\infty'^* = H_{SR}^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} H_{SR}^{-T}$$

If we apply the singular value decomposition to this equation we obtain that the matrices H_{SR}^{-1} and H_{SR}^{-T} are perpendicular, so:

$$SVC(C_\infty'^*) = U_\perp \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} U_\perp^T$$

This provides exactly one solution: $H_{SR} = U_\perp^{-1} = U_\perp^T$. To fix the problems on image rectification we have to change the matrix H_{SR} in this way:

$$H_{SR} = \begin{bmatrix} \frac{1}{\sqrt{a}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{b}} & 0 \\ 0 & 0 & 1 \end{bmatrix} U^T$$

- Find the vanishing line: it is possible to use additional information on the observed scene. It is possible to derive these constraints:

1. Known angles between lines: the angle between two lines is the angle between their normal direction, so they are independent of parameters c_1 and c_2 :

$$\cos \vartheta = \frac{a_1 a_2 + b_1 b_2}{\sqrt{(a_1^2 + b_1^2)(a_2^2 + b_2^2)}}$$

But since the term $a_1 a_2 + b_1 b_2$ can be written as:

$$\begin{bmatrix} a_1 & b_1 & c_1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix} = \mathbf{l}^T C_\infty^* \mathbf{m}$$

We have that the previous formula is equivalent to:

$$\cos \vartheta = \frac{\mathbf{l}^T C_\infty^* \mathbf{m}}{\sqrt{(\mathbf{l}^T C_\infty^* \mathbf{l})(\mathbf{m}^T C_\infty^* \mathbf{m})}}$$

But for the transformation rules we have that $C_\infty^* = H^{-1} C_\infty'^* H^{-T}$, it is possible to rewrite $\mathbf{l}^T C_\infty^* \mathbf{m}$ as $\mathbf{l}'^T C_\infty'^* \mathbf{m}'$, and the previous formula:

$$\cos \vartheta = \frac{\mathbf{l}'^T C_\infty'^* \mathbf{m}'}{(\mathbf{l}'^T C_\infty'^* \mathbf{l}')(\mathbf{m}'^T C_\infty'^* \mathbf{m}')}$$

In this case it is possible to extract \mathbf{m}' and \mathbf{l}' from the image. So, since we know the angle we have an equation on C_∞^* , that is linear when the lines are perpendicular ($\cos \vartheta = 0$). Since the unknown matrix is symmetric, homogeneous, and singular we have four independent constraints.

2. Known shape of objects, where we have that:

$$H_{SR} = \begin{bmatrix} \frac{1}{\sqrt{a}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{b}} & 0 \\ 0 & 0 & 1 \end{bmatrix} U^T$$

And the Euclidean reconstructed image is $M_S = H_{SR} \cdot \text{image}$

3. Combinations of one and two.
4. Observation of rigid planar motion: it is a similarity, so the circular points are invariants. The object has three degree of freedom since it is a pure rotation, and we can find the center of rotation, and the rotation angle. So, given a matrix H we have that the eigenvectors of H are the fixed points, and that the eigenvectors of H^{-T} are fixed lines of the transformation. It can be shown that:

- Eigenvectors \mathbf{I}', \mathbf{J}' correspond to complex eigenvalues.
- The phase of these eigenvectors is the rotation angle.
- Eigenvector \mathbf{O}' correspond to the real eigenvalues.

The three eigenvector associated to H are proportional respectively to $1, e^{i\theta}$, and $-e^{i\theta}$. The eigenvector associated to 1 is the image of \mathbf{O} , and the angle θ is the rotation angle. The eigenvectors associated to the complex values are the images of the circular points \mathbf{I}', \mathbf{J}' , but since we have

$$C_\infty'^* = \mathbf{I}' \mathbf{J}'^T + \mathbf{J}' \mathbf{I}'^T$$

If we apply the singular value decomposition to this matrix we have $\text{svd}(C_\infty'^*) = U C_\infty'^* U^T$, where U^T is the rectification matrix. It is possible to use two methods:

- Direct method, where the steps are:
 - (a) Find C_{∞}^{*} .
 - (b) Compute H_{rect} .
- Stratified method, where the steps are:
 - (a) Affine reconstruction from projective to affine.
 - (b) Shape reconstruction from affine to metric

Sometimes, the second method reduces numerical errors.

3.3 Accuracy issues