Game Theory

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Abstract

The theory begins by examining the main assumptions that distinguish decision theory from interactive decision theory. While decision theory focuses on individual decision-making in isolation, interactive decision theory explores how multiple decision-makers interact, considering each other's potential actions.

In the context of non-cooperative games, the discussion extends to games represented in extensive form, where players make decisions at various points, and games with perfect information, where all players are fully informed of prior moves. The technique of backward induction is key in solving such games. Additionally, combinatorial games are explored, emphasizing their strategic complexity.

Zero-sum games are analyzed in terms of conservative values, where each player seeks to minimize potential losses. The concept of equilibrium in pure strategies is introduced, and this is extended to mixed strategies in finite games, invoking von Neumann's theorem. Finding optimal strategies and determining the value of finite games is achieved through the use of linear programming techniques.

The Nash non-cooperative model plays a central role in understanding strategic interactions. Nash equilibrium is discussed, focusing on the existence of equilibria in both pure and mixed strategies within finite games. Examples of potential games are provided, along with methods for identifying potential functions. Notable examples include congestion games, routing games, network games, and location games. Concepts such as the price of stability, price of anarchy, and correlated equilibria are explored to analyze the efficiency and stability of these systems.

Finally, the discussion shifts to cooperative games, defining key concepts such as the core, nucleolus, Shapley value, and power indices. Examples of cooperative scenarios illustrate how these concepts help to determine fair outcomes and power distribution among players.

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CHAPTER 1

Introduction

1.1 Games

Game theory extends decision theory to settings involving multiple decision-makers, making it inherently more complex than scenarios with a single agent. It provides a powerful framework for modeling a wide range of real-world situations where the outcomes depend not only on an individual's choices but also on the actions of others.

Definition (*Game*)**.** A game is a process composed of the following elements:

- *Players*: the decision-makers involved (at least two).
- *Initial situation*: the starting conditions or state of the game.
- *Rules*: the set of actions available to each player and how the game is played.
- *Outcomes*: all possible end states that can result from players' actions.
- *Preferences*: each player's ranking of the possible outcomes, reflecting their individual goals or interests.

1.2 Players

In game theory, players are assumed to be both selfish and rational. These assumptions form the foundation for analyzing strategic interactions.

1.2.1 Selfish player

A selfish player is one who is solely concerned with the outcomes that affect their own preferences. That is, they evaluate the game purely based on personal benefit, without regard for the preferences or well-being of others.

1.2.2 Rational player

Definition (*Preference relation*). A preference relation on a set X is a binary relation $\succeq$ satisfying the following properties for all $x, y, z \in X$:

- *Reflexivity*: $x \succeq x$.
- *Completeness*: either $x \succeq y$ or $y \succeq x$.
- *Transitivity*: if $x \succeq y$ and $y \succeq z$, then $x \succeq z$.

Definition (*Utility function*). Given a preference relation $\succeq$ over a set X, a utility function is a function $u : X \to \mathbb{R}$ that represents $\succeq$ such that:

$$u(x) \geq u(y) \Leftrightarrow x \succeq y$$

While a utility function may not exist for all possible preference structures, it is guaranteed to exist in many common cases (particularly when the outcome set X is finite). Moreover, if a utility function exists, there are infinitely many such functions, each differing by a strictly increasing transformation of the original.

Each player i is associated with a choice set X_i representing all available actions. The global outcome space consists of the joint actions of all players. Utility functions are then defined over this combined set.

1.2.2.1 Rationality assumptions

Rational behavior in game theory is guided by the following assumptions:

1. Players can define a consistent preference relation over the possible outcomes.
2. Players can represent their preferences using a utility function when needed.
3. Players apply probability theory consistently when dealing with uncertainty.
4. Players can comprehend the consequences of all possible actions.
5. Players apply principles of decision theory whenever applicable.

Given a set of alternatives X and a utility function u, a rational player chooses an alternative $\bar{x} \in X$ such that:

$$u(\bar{x}) \geq u(x) \qquad \forall x \in X$$

One direct implication of these assumptions is the elimination of strictly dominated strategies: a rational player will never choose an action that yields a strictly worse outcome than another, regardless of the other players' choices.

1.2.3 Actions

The available actions and corresponding outcomes can be represented as pairs of utilities (one for each player). In two-player games, this data is often organized in a bi-matrix, where Player 1 selects a row and Player 2 selects a column. Each cell in the matrix represents a pair of payoffs corresponding to the selected strategies of both players.

CHAPTER 2

Extensive game

2.1 Introduction

An extensive game is a type of game in which the order of players' actions is explicitly modeled. It is commonly represented using a tree structure, known as the game tree, where:

- Each node represents a point at which a player makes a decision.
- Edges correspond to the available actions from that decision point.
- Terminal nodes represent final outcomes of the game, typically associated with payoffs for each player.

2.2 Extensive game with perfect information

An extensive game with perfect information is a formal model used to represent sequential interactions between players, where each player is fully aware of all previous actions taken.

Definition (*Extensive Game with Perfect Information*)**.** An extensive game with perfect information consists of the following components:

1. A finite set $N = \{1, \ldots, n\}$ of players.
2. A game tree (V, E, x_0).
3. A partition of the non-leaf vertices into sets $P_1, P_2, \ldots, P_{n+1}$.
4. A probability distribution for each vertex in P_{n+1}, defined on the edges from that vertex to its children.
5. A n-dimensional vector attached to each leaf (list of possible outcomes).

If P_{n+1} is empty, the game contains no random events and is entirely deterministic.

2.2.1 Solution

To determine the rational outcomes of an extensive game, we apply the axioms of rationality recursively to the structure of the game tree.

Definition (*Length*)**.** The length of a game is the length of the longest path from the root to a leaf in the game tree.

The fifth rationality assumption allows players to solve games of length 1. The fourth assumption ensures that if players can solve all games of length at most i, they can also solve games of length $i+1$. This recursive process leads to the concept of backward induction.

Theorem 2.2.1 (First rationality theorem)**.** *The rational outcomes of a finite extensive game with perfect information are exactly those determined by the backward induction procedure.*

Backward induction is applicable because each node v in the tree defines a sub-game rooted at v, which can be solved independently using the same principles.

2.2.2 Possible outcomes

Extensive games may admit multiple possible outcomes depending on the strategies chosen by the players.

Theorem 2.2.2 (Von Neumann)**.** *In the game of chess, exactly one of the following holds:*

1. *White has a winning strategy, regardless of Black's moves.*
2. *Black has a winning strategy, regardless of White's moves.*
3. *Both players can force at least a draw, no matter what the opponent does.*

Proof. Assume the game has a finite maximum length of $2k$ moves. Let a_i denote White's i-th move and b_i denote Black's i-th move.

The first possibility can be expressed formally as:

$$\exists a_1 \mid \forall b_1 \exists a_2 \mid \forall b_2 \dots \exists a_k \mid \forall b_k \implies \text{white wins}$$

If this statement is false, its negation must be true:

$$\forall a_1 \exists b_1 \mid \forall a_2 \mid \exists b_2 \mid \dots \forall a_k \mid \exists b_k \implies \text{white does not win}$$

Which implies that Black has a strategy to prevent White from winning, ensuring at least a draw. The same reasoning applies symmetrically to the second possibility. □

Corollary 2.2.2.1. *In a finite, perfect-information, two-player game where outcomes are limited to a win for one player or a tie, exactly one of the following must be true:*

1. *Player 1 has a winning strategy, no matter what Player 2 does.*
2. *Player 2 has a winning strategy, no matter what Player 1 does.*

This leads to a classification of solution types in game theory:

- *Very weak solution*: the game has a rational outcome, but it is inaccessible.
- *Weak solution*: the outcome is known to exist, but the procedure to obtain it is unknown.
- *Solution*: there exists an explicit, constructive algorithm to determine the outcome.

2.3 Combinatorial game

Definition (*Impartial combinatorial game*)**.** An impartial combinatorial game is characterized by the following properties:

1. Two players take turns alternately.
2. The game has a finite number of distinct positions.
3. The set of allowed moves from a position is the same for both players.
4. The game ends when no legal moves remain.
5. There is no element of chance.
6. In the classical version, the player who makes the last move wins.

To analyze such games, we categorize all possible positions into two types winning (P) and losing (N). Importantly, whether a position is a P- or N-position depends solely on the configuration of the game, not on which player is about to move.

The terminal position $(0, 0, \cdots, 0)$ is a P-position, as no moves are available. Any move from a P-position must lead to an N-position. From any N-position, there exists at least one move that leads to a P-position.

2.3.1 Nim game

The classical game of Nim is represented by a tuple $(n_1, \cdots, n_k)$, where each $n_i \in \mathbb{N}$ represents the size of pile i. On each turn, a player selects a pile $n_i > 0$ and reduces it to a smaller value $\hat{n}_i < n_i$. The objective is to force the opponent into the terminal position $(0, \ldots, 0)$, thereby winning the game.

Theorem 2.3.1 (Bouton)**.** *A position $(n_1, n_2, \ldots, n_k)$ in Nim game is a P-position if and only if:*

$$n_1 \oplus n_2 \oplus \cdots \oplus n_k = 0$$

Proof. Consider the position $(0, 0, \ldots, 0)$ with a Nim-sum of 0 (P-position). Suppose a player chooses to modify pile n_i to $\hat{n}_i < n_i$. Then the new Nim-sum becomes:

$$\hat{n}_i \oplus \hat{n}_1 \oplus n_2 \oplus \cdots \oplus n_k$$

Then we would have:

$$\hat{n}_i \oplus 0 \neq 0$$

The resulting position is an N-position.

Conversely, suppose the current position has Nim-sum $n_1 \oplus n_2 \oplus \cdots \oplus n_k \neq 0$. Then there exists some pile n_j such that changing it to $\hat{n}_j$ yields:

$$\hat{n}_i \oplus \hat{n}_1 \oplus n_2 \oplus \cdots \oplus n_k = 0$$

The resulting position is an P-position. Thus, from any position with non-zero Nim-sum, the player can always move to a position with zero Nim-sum. □

2.4 Strategy game

In the context of extensive-form games, backward induction requires that a specific action be determined at every decision point for each player. Let P_i denote the set of all decision nodes where Player i must choose an action.

Definition (*Pure strategy*)**.** A pure strategy for Player i is a function $s_i : P_i \to V$, where for each node $v \in P_i$, the function selects a child node x of v.

Definition (*Mixed strategy*)**.** A mixed strategy for Player i is a probability distribution over the set of all pure strategies available to Player i.

If Player i has n distinct pure strategies, the set of all mixed strategies corresponds to the standard simplex in $\mathbb{R}^n$

$$\sum_n = \left\{ p = (p_1, \dots, p_n) | p_i \geq 0 \text{ and } \sum p_i = 1 \right\}$$

This simplex represents all convex combinations of the pure strategies (each point in Σ_n encodes a possible mixed strategy).

Theorem 2.4.1 (Von Neumann)**.** *In the game of chess, exactly one of the following outcomes holds:*

1. *White has a strategy that guarantees a win, regardless of Black's moves.*
2. *Black has a strategy that guarantees a win, regardless of White's moves.*
3. *Both players have strategies that ensure at least a draw, regardless of the opponent's play.*

The outcome of the game depends on the structure of the payoff matrix or tree:

1. A row of all wins for White implies a winning strategy for White.
2. A column of all wins for Black implies a winning strategy for Black.
3. If neither of these occurs, the best both players can secure is a draw.

2.5 Extensive game with imperfect information

n many strategic settings, players are required to make decisions without full knowledge of the prior actions taken by others. This lack of information—such as when players move simultaneously. Despite this, such scenarios can still be modeled using a game tree.

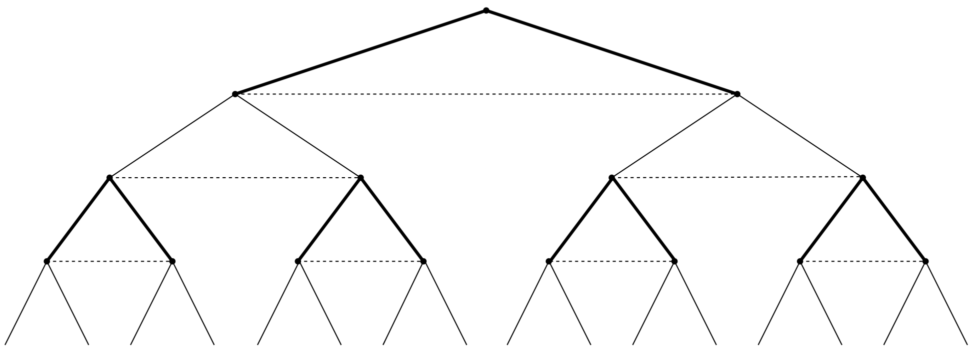

Figure 2.1: Extensive game with imperfect information

In the figure, dashed lines connect nodes that are indistinguishable to a player. That is, the player knows they are at one of the connected nodes, but cannot tell which one specifically.

Definition (*Information set*)**.** An information set for Player i is a pair $(U_i, A(U_i))$ satisfying the following conditions:

1. $U_i \subset P_i$ is a non-empty set of vertices $v_1, \cdots, v_k$.
2. Each vertex $v_j \in U_i$ has the same number of children.
3. $A_i(U_i)$ is a partition of the children of $v_1 \cup \cdots \cup v_k$ such that each element of the partition contains exactly one child from each vertex v_j.

Intuitively, an information set represents the player's uncertainty about their exact position in the game tree. The associated partition $A(U_i)$ defines the available actions.

Definition (*Extensive game with imperfect information*)**.** An extensive-form game with imperfect information consists of the following elements:

1. A finite set of players $N = \{1, \dots, n\}$.
2. A game tree (V, E, x_0).
3. A partition of the non-terminal (non-leaf) nodes into sets $P_1, P_2, \dots, P_{n+1}$.
4. A partition $(U_i^j), j = 1, \dots, ki$ of the set P_i, for all i, with (U_i^j, A_i^j) being the information set for all players i at all vertices j (having the same number of children).
5. A probability distribution defined for each vertex in P_{n+1} on the edges leading to its children.
6. An n-dimensional vector assigned to each leaf.

Note that when all information sets contain singletons, the game reduces to one with perfect information.

Definition (*Pure strategy*)**.** A pure strategy for Player i in an imperfect information game is a function defined over the collection $\mathcal{U}$ of their information sets, assigning to each $U_i \in \mathcal{U}$ an element from the partition $A(U_i)$.

Definition (*Mixed strategy*)**.** A mixed strategy is defined as a probability distribution over the pure strategies.

CHAPTER 3

Zero sum games

3.1 Introduction

Definition (*Zero sum game*)**.** A two-player zero-sum game in strategic form can be described as a triplet $(X, Y, f : X \times Y \to \mathbb{R})$, where:

- X is the strategy space of Player 1.
- Y is the strategy space of Player 2.
- $f(x, y)$ represents the payoff Player 1 receives from Player 2 when they play strategies x and y, respectively.

Since this is a zero-sum game, Player 2's utility function g is defined as the negative of Player 1's utility function:

$$g(x, y) = -f(x, y)$$

In the case where the strategy spaces are finite the game can be represented by a payoff matrix P. In this matrix, Player 1 chooses a row i, and Player 2 chooses a column j:

$$\begin{pmatrix} p_{11} & \cdots & p_{1m} \\ \cdots & p_{ij} & \cdots \\ p_{n1} & \cdots & p_{nm} \end{pmatrix}$$

Here, p_{ij} denotes the payment Player 2 makes to Player 1 when they select strategies i and j, respectively.

To determine the optimal strategy, both players can employ conservative reasoning:

- Player 1 can ensure a minimum payoff of $v_1 = \max_i \min_j p_{ij}$.
- Player 2 can limit their losses to at most $v_2 = \min_j \max_i p_{ij}$.

These values, v_1 and v_2, are known as the conservative values for Player 1 and Player 2, respectively.

In more general cases where the strategy spaces X and Y are not finite, a similar reasoning applies. Let $(X, Y, f : X \times Y \to \mathbb{R})$ describe the game, where X and Y are arbitrary strategy sets. The conservative values can be defined as follows:

- Player 1: $v_1 = \sup_x \inf_y f(x, y)$.
- Player 2: $v_2 = \inf_y \sup_x f(x, y)$.

These values, v_1 and v_2, are known as the conservative values for Player 1 and Player 2, respectively.

3.2 Rationality

Now, suppose the following holds:

- $v_1 = v_2 = v$.
- There exists a row $\bar{i}$ such that $p_{\bar{i}\bar{j}} \geq v_1 = v$ for all j.
- There exists a column $\bar{j}$ such that $p_{\bar{i}\bar{j}} \leq v_2 = v$ for all i.

In this case, $p_{\bar{i}\bar{j}} = v$, and this value represents the rational outcome of the game. Thus, $\bar{i}$ maximizes the function $\alpha(i) = \min_j p_{ij}$, and $\bar{j}$ minimizes the function $\beta(j) = \max_i p_{ij}$.

3.2.1 Existence of a rational outcome

To demonstrate the existence of a rational outcome in a zero-sum game, we need to establish the following:

1. *Equality of conservative values*: the conservative values of both players agree, i.e., $v_1 = v_2$.
2. *Existence of an optimal strategy for Player 1*: there exists a strategy $\bar{x}$ such that:

$$v_1 = \inf_y f(\bar{x}, y)$$

   This ensures that $\bar{x}$ is an optimal strategy for Player 1.

3. *Existence of an optimal strategy for Player 2*: there exists a strategy $\bar{y}$ such that:

$$v_2 = \sup_x f(x, \bar{y})$$

   This ensures that $\bar{y}$ is an optimal strategy for Player 2.

In the case where the strategy spaces are finite, such optimal strategies $\bar{x}$ and $\bar{y}$ always exist. Therefore, proving the existence of a rational outcome is equivalent to demonstrating the equality of the conservative values, i.e., $v_1 = v_2$.

Theorem 3.2.1 (Von Neumann)**.** *There always exists a rational outcome for a finite two-player zero-sum game, as described by its payoff matrix P.*

This fundamental result, known as the Minimax theorem, guarantees that in every finite zero-sum game, the conservative values for both players coincide, and optimal strategies exist for both players, leading to a rational outcome.

Proof. Suppose, without loss of generality, that all p_{ij} in the matrix P are positive. Consider the column vectors $p_1, \dots, p_m \in \mathbb{R}^n$, and let C denote their convex hull. Define the set

$$Q_t = \{x \in \mathbb{R}^n : x_i \le t\}$$

and

$$v = \sup\{t \ge 0 : Q_t \cap C = \emptyset\}$$

Since $\operatorname{int} Q_v \cap C = \emptyset$, the sets Q_v and C can be separated by a hyperplane. Hence, there exist coefficients $\bar{x}_1, \dots, \bar{x}_n$, with some $\bar{x}_i \ne 0$, and $b \in \mathbb{R}$ such that:

$$(\bar{x}, u) = \sum_{i=1}^{n} \bar{x}_i u_i \le b \le \sum_{i=1}^{n} \bar{x}_i w_i = (\bar{x}, w)$$

for all $u = (u_1, \dots, u_n) \in Q_v$ and $w = (w_1, \dots, w_n) \in C$.

Since all $\bar{x}_i$'s must be non-negative, we can assume $\sum \bar{x}_i = 1$. Additionally, $b = v$, since $\bar{v} := (v, \dots, v) \in Q_v$, and

$$(\bar{x}, \bar{v}) = \sum_i \bar{x}_i v = v \sum_i \bar{x}_i = v$$

Therefore, $b \ge v$. If $b > v$, by choosing a small $a > 0$ such that $b \ge v + a$, we would have:

$$\sup\left\{\sum_{i=1}^{n} \bar{x}_i u_i : u \in Q_{v+a}\right\} < b$$

which would imply $Q_{v+a} \cap C = \emptyset$, contradicting the definition of v.

Next, since $Q_v \cap C \ne \emptyset$, let $\bar{w} = \sum_{j=1}^{m} \bar{y}_j p_j$ (as C is convex) for some $\bar{y} = (\bar{y}_1, \dots, \bar{y}_m) \in \Sigma_m$. Since $\bar{w} \in Q_v$, we have $\bar{w}_i \le v$ for all i.

We now show that $\bar{x}$ is optimal for Player 1, $\bar{y}$ is optimal for Player 2, and v is the value of the game.

For Player 1, since $(\bar{x}, w) \ge v$ for every $w \in C$ by the separation result, and since each column $p_{\cdot j} \in C$, we have:

$$(\bar{x}, p_{\cdot j}) \ge v, \quad \text{for all } j$$

For Player 2, consider $w = \sum_{j=1}^{m} \bar{y}_j p_j \in Q_v \cap C$ as before. Then, $w_i = \bar{y} p_{i\cdot}$, and since $w \in Q_v$, it follows that $w_i \le v$ for every i. Hence, we have:

$$v \ge w_i = \bar{y} p_{i\cdot}$$

□

Von Neumann's theorem guarantees that even when a finite zero-sum game has no solutions in pure strategies, the following holds:

- For Player 1, there exists a mixed strategy, represented as a probability distribution $\mathbf{x} = \begin{pmatrix} x_1 & \dots & x_n \end{pmatrix}$, over her pure strategies. For every column j:

$$(x, p_{\cdot j}) = \sum_{i=1}^{n} x_i p_{ij} = x_1 p_{1j} + x_2 p_{2j} + \dots + x_n p_{nj} \ge v$$

- For Player 2, there exists a mixed strategy, represented as a probability distribution $\mathbf{y} = \begin{pmatrix} y_1 & \dots & y_n \end{pmatrix}$, over her pure strategies. For every row i:

$$(y, p_{i\cdot}) = \sum_{j=1}^{n} y_j p_{ij} = y_j p_{i1} + y_2 p_{i2} + \dots + y_m p_{im} \le v$$

The constant v is the value of the game under mixed strategies. Player 1 aims to maximize v, while Player 2 seeks to minimize it.

3.3 Optimality

Let X and Y be arbitrary sets. Suppose:

1. $v_1 = v_2 := v$.
2. There exists a strategy $\bar{x}$ such that $f(\bar{x}, y) \geq v$ for all $y \in Y$.
3. There exists a strategy $\bar{y}$ such that $f(x, \bar{y}) \leq v$ for all $x \in X$.

Then:

- v is the rational outcome of the game.
- $\bar{x}$ is an optimal strategy for Player 1.
- $\bar{y}$ is an optimal strategy for Player 2.

It follows that $\bar{x}$ is optimal for Player 1 since it maximizes $\alpha(x) = \inf_y f(x, y)$, while $\bar{y}$ is optimal for Player 2 since it minimizes $\beta(y) = \sup_x f(x, y)$. The values $\alpha(x)$ and $\beta(y)$ represent the best responses for the players if they knew the opponent's strategy.

3.3.1 Conservative values different or equal

Proposition. Let X and Y be nonempty sets, and let $f : X \times Y \to \mathbb{R}$ be an arbitrary real-valued function. Then:

$$v_1 = \sup_x \inf_y f(x, y) \leq \inf_y \sup_x f(x, y) = v_2$$

Proof. By definition, for all $x \in X$ and $y \in Y$:

$$\inf_y f(x, y) \leq f(x, y) \leq \sup_x f(x, y)$$

Thus, for all x and y, it holds that:

$$\alpha(x) = \inf_y f(x, y) \leq \sup_x f(x, y) = \beta(y)$$

Taking the supremum over x and the infimum over y, we conclude:

$$\sup_x \alpha(x) \leq \inf_y \beta(y)$$

□

As a result, it follows that for any game, $v_1 \leq v_2$.

3.3.2 Conservative values not equal

When the conservative values differ, mixed strategies must be considered. In this case, the strategy spaces for both players are probability distributions:

$$\sum_k = \left\{ x = (x_1, \dots, x_k) | x_i \geq 0 \text{ and } \sum_{i=1}^{k} x_i = 1 \right\}$$

Here, $k = n$ for Player 1 and $k = m$ for Player 2. The utility function is extended to:

$$f(x, y) = \sum_{i=1,\dots,n, j=i,\dots,m} x_i y_j p_{ij} = (x, Py)$$

Thus, the mixed extension of the original game is given by:

$$\left(\sum_n \sum_m f(x, y) = (x, Py) \right)$$

3.3.3 Pure strategies optimality

Theorem 3.3.1. *If a player knows the strategy being used by the opposing player, they can always adopt a pure strategy to achieve the best possible outcome.*

This means that once one player's choice is fixed, the optimization problem reduces to a linear problem over a simplex, given that the utility function in such a game is bilinear.

Proof. Consider Player 2, who knows that Player 1 is using a mixed strategy $\bar{x}$. Player 2's task is then to minimize the function:

$$f(\bar{x}, y) = (\bar{x}, Py)$$

over the simplex $\sum_m$ (the set of mixed strategies for Player 2). The optimal value will be attained at one of the vertices e_j of the simplex, which corresponds to a pure strategy. Thus, Player 2 can use a pure strategy to achieve the optimal outcome. □

Given a payoff matrix P, let the column vector corresponding to the j-th pure strategy be denoted as $p_{\cdot j}$, and the row vector corresponding to the i-th pure strategy as $p_{i\cdot}$, respectively. The payoff of the first player in the mixed extension of the game is given by:

$$f(x, y) = (x, Py)$$

The previous theorem implies that, to verify the existence of a rational outcome for the game, we need to show the existence of mixed strategies $\bar{x}$ and $\bar{y}$, as well as a value v, such that:

- $(\bar{x}, P_{e_j}) = (\bar{x}, p_{\cdot j})$ for every column j.
- $(e_i, p_{i\cdot}\bar{y}) \leq v$ for every row i.

Here, e_j is the j-th strategy of Player 2, and e_i is the i-th strategy of Player 1.

3.3.4 General case optimality

Von Neumann proof can be efficiently used to find rational outcome of payoff matrices that can be reduced to matrices where ine player has only two strategies. However, in higher dimensions this procedure becomes more complicated, since it is not clear when and where the set Q_t meets C. Therefore, we need to use Linear Programming.

Player one Player 1 must choose a probability distribution $\mathbf{x} = \begin{pmatrix} x_1 & \cdots & x_n \end{pmatrix} \in \sum_n$ in order to maximize v with the following constraints:

$$\begin{cases} (x, p_{\cdot,1}) = x_1 p_{11} + \cdots + x_n p_{n1} \geq v \\ \cdots \\ (x, p_{\cdot,j}) = x_1 p_{1j} + \cdots + x_n p_{nj} \geq v \\ \cdots \\ (x, p_{\cdot,m}) = x_1 p_{1m} + \cdots + x_n p_{nm} \geq v \end{cases}$$

It is a linear maximization problem where we need to find the value v and we do not know the vector $\mathbf{x}$. In matrices, we have:

$$\begin{cases} \min_{\mathbf{x},v} v : \\ P^T \mathbf{x} \geq v \mathbf{1}_m \\ \mathbf{x} \geq 0 \qquad \begin{pmatrix} 1 & \mathbf{x} \end{pmatrix} = 1 \end{cases}$$

Player two Player 2 must choose a probability distribution $\mathbf{y} = \begin{pmatrix} y_1 & \cdots & y_m \end{pmatrix} \in \sum_m$ in order to maximize w with the following constraints:

$$\begin{cases} (x, p_{1,\cdot}) = x_1 p_{11} + \cdots + x_m p_{1m} \leq w \\ \cdots \\ (x, p_{i,\cdot}) = x_1 p_{i1} + \cdots + x_m p_{im} \leq w \\ \cdots \\ (x, p_{n,\cdot}) = x_1 p_{n1} + \cdots + x_m p_{nm} \leq w \end{cases}$$

It is a linear maximization problem where we need to find the value w and we do not know the vector $\mathbf{y}$. In matrices, we have:

$$\begin{cases} \min_{\mathbf{y},w} w : \\ P \mathbf{y} \leq w \mathbf{1}_n \\ \mathbf{y} \geq 0 \qquad \begin{pmatrix} 1 & \mathbf{y} \end{pmatrix} = 1 \end{cases}$$

Here, $\mathbf{1}$ is a vector of right dimensions whose componenst are all 1's. Ideally, the maximum value for v is equal to the minimal value for w, so as to yield the value of the game.

3.4 Equivalent formulation

Consider a zero sum game described by a payoff matrix P. We can assume, without loss of generality, that $p_{ij} > 0$ for all i, j. This implies $v > 0$.

If $\alpha_j = \frac{x_i}{v}$, then $\sum x_i = 1$ becomes $\sum \alpha_i = \frac{1}{v}$ and maximizing v is equivalent to minimizing $\sum \alpha_i$. Likewise, if we set $\beta_j = \frac{y_j}{v}$ we can do the same. Consider the two problems in duality:

$$\begin{cases} \min(c, \alpha) \\ A\alpha \geq b \\ \alpha \geq 0 \end{cases} \qquad \begin{cases} \max(b, \beta) \\ A^T \beta \leq c \\ \beta \geq 0 \end{cases}$$

Here, $A = P^T$. Denote by v the common value of the two problems. Then we have:

- x is the optimal strategy for Player 1 if and only if $x = v\alpha$ for some α optimal solution of the primal problem.
- y is the optimal strategy for Player 2 if and only if $y = v\beta$ for some β optimal solution of the dual problem.

Consider again the complementarity conditions for the above problems, with x and y being strategies for the two players:

$$\begin{cases} \forall i \bar{x}_i > 0 \implies \sum_{k=1}^{m} p_{ik}\bar{y}_k = c_i \\ \forall i \bar{y}_i > 0 \implies \sum_{k=1}^{n} p_{kj}\bar{x}_k = b_i \end{cases}$$

Since $\bar{y}$ is optimal for Player 2, one has $\sum_{j=1}^{m} p_{ij}\bar{y}_j$ for all i, and hence $x_i > 0$ implies that the row i is optimal for Player 1. And conversely the same holds for Player 2.

3.5 Symmetric games

Definition (*Antisymmetric*). A $n \times n$ matrix P with elements (p_{ij}) is said to be antisymmetric provided that $p_{ij} = -p_{ji}$ for all $i, j = 1, \cdots, n$.

Definition (*Fair game*). A finite zero sum game is fair if the associated matrix is antisymmetric.

In fair games there is no favorite plater: in fact, their role can be exchanged.

Proposition. If $P = (p_{ij})$ is antisymmetric the conservative value $v = 0$ and $\bar{x}$ is an optimal strategy for Player 1 if and only if it is optimal for Player 2.

Proof. Recall that $P^T = -P$ if P is an antisymmetric matrix. Then, since:

$$(Px.x) = (x, P^T x) = -(x, Px) = -(Px, x)$$

one has $f(x, x) = 0$ for all x. This implies $v_1 \leq 0, v_2 \geq 0$ If $\bar{x}$ is optimal for the first player, then:

$$(\bar{x}, Py) \geq 0 \qquad \forall y \in \Sigma_n$$

so that $(P^T\bar{x}, y) \geq 0$, which by the fact that P is antisymmetric becomes:

$$(P\bar{x}, y) \leq 0 \qquad \forall y \in \Sigma_n$$

Therefore $\bar{x}$ is optimal also for the second player, and conversely. □

3.5.1 Optimal strategies in fair games

In order to find optimal strategies in fair games, we need to solve the system of inequalities:

$$\begin{cases} x_1 p_{11} + \cdots + x_n p_{n1} \geq 0 \\ \cdots \\ x_1 p_{1j} + \cdots + x_n p_{nj} \geq 0 \\ \cdots \\ x_1 p_{1n} + \cdots + x_n p_{nn} \geq 0 \end{cases}$$

With extra conditions $x_i \geq 0$ and $\sum_{i=1}^{n} x_i = 1$.

CHAPTER 4

Nash model

4.1 Introduction

Definition (*Non cooperative strategic game*)**.** A two player non cooperative game in strategic form is $(X, Y, f : X \times Y \to \mathbb{R}, g : X \times Y \to \mathbb{R})$. Here, X and Y are the strategy sets of the two players, f is the utility function of Player 1, and g is the utility function of Player 2.

4.2 Nash equilibrium

Definition (*Nash equilibrium*)**.** A Nash equilibrium profile for $(X, Y, f : X \times Y \to \mathbb{R}, g : X \times Y \to \mathbb{R})$ is a pair $(\bar{x}, \bar{y}) \in X \times Y$ such that $f(\bar{x}, \bar{y}) \geq f(x, \bar{y})$ for all $x \in X$ and $f(\bar{x}, \bar{y}) \geq f(\bar{x}, y)$ for all $y \in Y$.

A Nash equilibrium profile is a joint combination of strategies which is stable with respect to unilateral deviations of any individual player. At equilibrium, neither player can improve their utilities by changing strategy. In fact, it is not even convenient for the players to change, given that each one takes for granted that the other one will play the selected strategy.

The main ideas of the Nash model can be seen with two player: having more players does not add complexity to the concept (except for the notation). Let us consider a n-player game with strategy sets X_i for each player and payoffs $u_i : X \to \mathbb{R}$ with $X = \prod_{i=1}^n X_i$. Let $x = (x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_n)$ be a strategic profile x_{-i} denotes the vector $x_{-i} = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$ and write also $x = (x_i, x_{-i})$ to emphasize the role of x_i. Then, $\bar{x} = (\bar{x}_i)_{i=1}^n$ is a Nash equilibrium profile if for every i, for every $x \in X_i$:

$$u_i(\bar{x}) \geq u_i(x, \bar{x}_{-i})$$

The notion of Nash equilibrium provides a new definition of rationality. We have to see the connection with dominant strategies, backward induction, and optimal strategies in zero sum games.

4.2.1 Dominant strategies

Suppose $\bar{x}$ is a weakly dominant strategy for Player 1:

$$f(\bar{x}, y) \geq f(x, y) \qquad \forall x, y$$

If $\bar{y}$ maximizes the function $y \rightarrowtail g(\bar{x}, y)$ for Player 2, then $(\bar{x}, \bar{y})$ is a Nash equilibrium profile. In fact, for $\bar{x}$ weakly dominant it is true, in particular, that $f(\bar{x}, \bar{y}) \geq f(x, \bar{y})$ for all $x \in X$, thereby satisfying Nash condition on the utility function f of player 1. Then, the maximization requirement that $\bar{y}$ be such that $g(\bar{x}, \bar{y}) \geq g(\bar{x}, y)$ for all $y \in Y$ naturally satisfies the Nash condition on the utility function g of Player 2.

Non uniqueness Let us suppose $\bar{y}$ maximizes the function $y \rightarrowtail g(\bar{x}, y)$:

- If $\bar{x}$ is a weakly dominant strategy for Player 1, then other Nash equilibria beyond $(\bar{x}, \bar{y})$ can exists.
- If $\bar{x}$ is a strictly dominant strategy for Player 1, then no other Nash equilibria exists different from the above ones.

Proof. Assume that there us another Nash equilibrium (x_i, y_i) different than $(\bar{x}, \bar{y})$. By definition, it implies the fact that $f(x_i, y_i) \geq f(\bar{x}, y_j)$. However:

- If $\bar{x}$ is weakly dominant, since $f(\bar{x}, y) \geq f(x, y)$ for all x and all y it follows in particular that such inequality holds for y_j, that is $f(\bar{x}, y_i) \geq f(x_i, y_i)$, which is consistent with the above fact. Hence, (x_i, y_i) can be a Nash equilibrium.
- If $\bar{x}$ is strictly dominant, since $f(\bar{x}, y) > f(x, y)$ for all x and all y it follows in particular that such inequality holds for y_j, that is $f(\bar{x}, y_i) > f(x_i, y_i)$, but that is in contradiction with the above fact. Hence, no pair (x_i, y_i) other than $(\bar{x}, \bar{y})$ can be a Nash equilibrium.

□

4.2.2 Backward induction

Backward induction provides a Nash equilibrium for a game of perfect information, since players systematically make an optimal choice in every part of the tree of the game. It is possible that in games of perfect information there are more equilibria than the ones provided by backward induction.

4.2.3 Zero sum games

Theorem 4.2.1. *Let X, Y be nonempty sets and $f : X \times Y \to \mathbb{R}$ a function (so that in zero sum games $g(x, y) = -f(x, y)$). Then, the following are equivalent:*

1. *The pair $(\bar{x}, \bar{y})$ fulfills:*

$$f(x, \bar{y}) \geq f(\bar{x}, \bar{y}) \geq f(\bar{x}, y) \qquad \forall x, y$$

2. *The following conditions are satisfied:*

$$\begin{aligned}
&\inf_y \sup_x f(x, y) = \sup_x \inf_y f(x, y) \\
&\inf_y f(\bar{x}, y) = \sup_x \inf_y f(x, y) \\
&\sup_x f(\bar{x}, y) = \inf_y \sup_x f(x, y)
\end{aligned}$$

By the first condition, the equilibrium $(\bar{x}, \bar{y})$ yields conservative values for both players. By the second condition, the conservative value agree; moreover the players must solve independent problems: $\bar{x}$ maximizes $f(\cdot, y)$ and $\bar{y}$ minimizes $f(x, \cdot)$.

Proof. Starting from the first we have:

$$v_2 = \inf_y \sup_x f(x,y) \leq \sup_x f(x, \bar{y}) = f(\bar{x}, \bar{y}) = \inf_y f(\bar{x}, y) \leq \sup_x \inf_y f(x,y) = v_1$$

Since $v_1 \leq v_2$ always holds, all above inequalities are equalities.

Suppose now that the second condition holds, we have that:

$$\inf_y \sup_x f(x,y) = \sup_x f(x, \bar{y}) \geq f(\bar{x}, \bar{y}) \geq \inf_y f(\bar{x}, y) = \sup_x \inf_y f(x,y)$$

Because of the first consequence of the second condition, all inequalities are equalities. □

As a consequence, given a general zero sum game $X, Y, f : X \times Y \to \mathbb{R}$:

- Any Nash equilibrium $(\bar{x}, \bar{y})$ provides optimal strategies for the players. $f(\bar{x}, \bar{y}) = v$ is the value of the game.
- Any pair of optimal strategies $\bar{x}$ for the Player 1 ant $\bar{y}$ for Player 2 are such that $(\bar{x}, \bar{y})$ is a Nash equilibrium profile of the game and $f(\bar{x}, \bar{y}) = v$.

4.3 Nash equilibrium existence

$(\bar{x}, \bar{y})$ is a Nash equilibrium for the game if and only if

$$(\bar{x}, \bar{y}) \in BR(\bar{x}, \bar{y})$$

Thus, the existence of a Nash equilibrium can be proved by using a fixed point.

Theorem 4.3.1. *Let Z be a compact convex subset of an Euclidean space, let $F : Z \to 2^Z$ be such that $F(z)$ is a nonempty closed convex set for all z. Suppose also F has a closed graph. Then, F has a fixed point: there is $\bar{z} \in Z$ such that $\bar{z} \in F(\bar{z})$*

Theorem 4.3.2 (Nash theorem)**.** *Given the game $(X, Y, f : X \times Y \to \mathbb{R}, g : X \times Y \to \mathbb{R})$, suppose:*

- *X and Y are compact convex subsets of some Euclidean space.*
- *f, g continuous.*
- *$x \mapsto f(x,y)$ is quasi concave for all $y \in Y$.*
- *$y \mapsto g(x,y)$ is quasi concave for all $x \in X$.*

Then, the game has an equilibrium.

Proof. $BR_1(y)$ and $BR_2(x)$ are nonempty (X and Y are compact), closed (f and g are continuous), and convex valued (f and g are quasi concave).

BR has closed graph: suppose $(u_n, v_n) \in BR(x_n, y_n)$ for all n and $(u_n, v_n) \to (u, v)$, $(x_n, y_n) \to (x, y)$. We want to prove that $(u, v) \in BR(x, y)$. We have:

$$f(u_n, y_n) \geq f(z, y_n) \qquad g(x_n, v_n) \geq g(x_n, t) \qquad \forall z \in X, t \in Y$$

Taking limits:

$$f(u, y) \geq f(z, y) \qquad g(x, v) \geq g(x, t) \qquad \forall z \in X, t \in Y$$

□

Corollary 4.3.2.1. *A finite game with utilities functions (A, B) always admits a Nash equilibrium profile in mixed strategies.*

In this case X and Y are simplexes, thus the assumptions of the theorem are fulfilled.

4.4 Nash equilibrium search

To find the Nash equilibrium we can use a brute force algorithm:

1. Guess the support of the equilibria $\mathrm{spt}(\bar{x})$ and $\mathrm{spt}(\bar{y})$.
2. Ignore the sub-optimal strategies and find x, y, u, w by solving the linear system, of $n + m + 2$ equations:

$$\begin{cases} \sum_{i=1}^{n} x_i = 1 \\ \sum_{j=1}^{m} a_{ij} y_j = v \quad \forall i \in \mathrm{spt}(\bar{x}) \\ x_i = 0 \qquad \forall i \notin \mathrm{spt}(\bar{x}) \end{cases} \qquad \begin{cases} \sum_{i=1}^{m} x_i = 1 \\ \sum_{j=1}^{n} b_{ij} x_i = w \quad \forall j \in \mathrm{spt}(\bar{y}) \\ y_i = 0 \qquad \forall j \notin \mathrm{spt}(\bar{x}) \end{cases}$$

3. Check wether the ignored inequalities are satisfied. If $x_i \geq 0$, $y_j \geq 0$, $\sum_{j=1}^{m} a_{ij} y_j \leq v$ and $\sum_{i=1}^{n} b_{ij} x_i \leq w$ then stop since we have found a mixed equilibrium profile. Otherwise, go back to step 1 and try another guess of the support.

Enumerating all the possible supports in the brute force algorithm quickly becomes computationally prohibitive; there are potentially $(2^n - 1)(2^m - 1)$ options. For $n \times n$ games the number of combinations grows very quickly.

Lemke-Howson proposed a more efficient algorithm, though still with exponential running time in the worst case.

4.5 Potential games

Consider a finite game with strategy sets X_i and suppose that all the players have the same payoff $p : Z \to \mathbb{R}$, that is for all i, the utility function are:

$$u_i(x_1, \ldots, x_n) = p(x_1, \ldots, x_n)$$

If $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_n) \in X$ is a strategy profile such that $p(\bar{x}) \geq p(x)$ for all strategy profiles $x \in X$, then $\bar{x}$ is a Nash equilibrium in pure strategies. Note that there might be other Nash equilibria in pure o mixed strategies. However, $\bar{x}$ is the best strategy for all players.

Consider the following payoff-improving procedure:

1. Start from an arbitrary strategy profile $(x_1, \ldots, x_n) \in X$.
2. Ask if any player has a better strategy x'_i that strictly increases her payoff. If yes, replace x_i with x'_i and repeat. Otherwise stop, ew have found a pure Nash equilibrium profile.

Each iteration strictly increases the value $p(x)$, so that no strategy profile $x \in X$ can be visited twice. Since X is a finite set, the procedure must reach a pure Nash equilibrium after at most $|X|$ steps. Therefore, this procedure guarantees to reach the global minimum $\bar{x}$.

Consider now an arbitrary finite game with payoffs $u_i : X \to \mathbb{R}$. We can add a constant c_i to the payoff of player i:

$$\tilde{u}_i(x_1, \ldots, x_n) = u_i(x_1, \ldots, x_n) + c_i$$

If, instead, c_i depends only on x_{-i} and not on x_i, the best responses and equilibria remain the same.

Definition. The payoffs $\tilde{u}_i$ and u_i are said diff-equivalent for player i, if the difference:

$$\tilde{u}_i(x_1, \ldots, x_n) - u_i(x_1, \ldots, x_n) = c_i(x_{-i})$$

does not depend on her decision x_i but only on the strategies of the other players.

Theorem 4.5.1. *Finite games with diff-equivalent payoffs have the same pure Nash equilibria.*

Proof. The best reaction multi-function, for every player i, is the same when considering two diff-equivalent payoffs u_i and $\tilde{u}_i$, no matter how different from each other the latter functions are. $\square$

Definition (*Potential game*)**.** A finite game with strategy set X_i and payoffs $u_i : X \to \mathbb{R}$ is called a potential game, if it is diff-equivalent to a game with common payoffs.

That is, there exists a potential function $p : XX \to \mathbb{R}$ such that for each i, for every $x_{-i} \in X_{-i}$, and all $x'_i, x_i \in X_i$ we have:

$$\Delta u_i(x'_i, x_i, x_{-i}) = \Delta p(x'_i, x_i, x_{-i})$$

here, $\Delta p(x'_i, x_i, x_{-i}) = p(x'_i, x_{-i}) - (x_i, x_{-i})$

Corollary 4.5.1.1. *Every finite potential game has at least one pure Nash equilibrium.*

Corollary 4.5.1.2. *In a finite potential game every best response iteration reaches a pure Nash equilibrium in finitely many steps.*

4.5.1 Potential search

A potential $p : X \to \mathbb{R}$ is characterized by:

$$\Delta u_i(x'_i, x_i, x_{-i}) = \Delta p(x'_i, x_i, x_{-i})$$

Adding a constant to $p(\cdot)$ provides a new potential. Now, the potential $p(\cdot)$ is determined uniquely:

$$p(x_1, \ldots, x_n) = \sum_{i=1}^{n} [u_i(\bar{x}_1, \ldots \bar{x}_{i-1}, x_i, \ldots, x_n) u_i(\bar{x}_1, \ldots \bar{x}_{i-1}, \bar{x}_i, \ldots, x_n)]$$

Existence If a game admits a potential the sum on the right hand side of the previous equation is independent of the particular order used. The converse is also true. However, checking that all these orders yield the same answer is impractical for more than two or three players.

4.6 Cost and efficiency

Definition (*Pareto efficient*)**.** An equilibrium is Pareto efficient if it is not possible to increase the utility of a player without decreasing the utility of some other player.

Nash equilibria need to be Pareto efficient and in fact they can be bad for all the players. To quantify how bad an outcome is, lut us consider costs, rather than utilities.

The quality of a strategy profile $x = (x_1, \dots, x_n)$ is measured by a social cost function $x \mapsto C(x)$ from X to $\mathbb{R}_+$. The smaller $C(x)$ the better the outcome x.

Definition (*Benchmark*)**.** The benchmark is the minimal value that a benevolent social planner could achieve:

$$\text{opt} = \min_{x \in X} C(x)$$

For $x \in X$ the ration $\frac{C(x)}{\text{opt}}$ measures how far outcome x is from being optimal. So, a large value implies a big loss in terms of social welfare, whereas a quotient close to 1 implies that x is almost as efficient as an optimal solution.

Definition (*Price of anarchy*)**.** Let $NE \subseteq X$ be the set of pure Nash equilibria of a cost game. The price of anarchy is defined as:

$$\text{PoS} = \max_{\bar{x} \in NE} \frac{C(\bar{x})}{\text{opt}}$$

Definition (*Price of stability*)**.** Let $NE \subseteq X$ be the set of pure Nash equilibria of a cost game. The price of stability is defined as:

$$\text{PoS} = \min_{\bar{x} \in NE} \frac{C(\bar{x})}{\text{opt}}$$

Note that $1 \leq \text{PoS} \leq \text{PoA}$:

- $\text{PoA} \leq \alpha$ means that in every possible pure equilibrium the social cost $C(\bar{x})$ is no worse than α opt.
- $\text{PoS} \leq \alpha$ means that there exists some equilibrium with social cost at most α opt.

Proposition. Consider a cost minimization finite potential game with potential $p : X \to \mathbb{R}$, and suppose that there exists $\alpha, \beta \geq 0$ such that:

$$\frac{1}{\alpha} C(x) \leq p(x) \leq \beta C(x) \qquad \forall x \in X$$

Then, $\text{PoS} \leq \alpha\beta$

Proof. Let $\bar{x}$ be a minimum of $p(\cdot)$ so that $\bar{x}$ is a Nash equilibrium. For all $x \in X$:

$$\frac{1}{\alpha} C(\bar{x}) \leq p(\bar{x}) \leq p(x) \leq \beta C(x) \qquad \forall x \in X$$

Since this is true for all x, we can choose the strategy x yielding the optimal outcome $\text{opt} = \min_{x \in X} C(x)$, and hence it follows that $C(\bar{x}) \leq \alpha\beta\text{opt}$. □

In case a game deals with utilities rather than costs, one defined:

$$\text{opt} = \max_{x \in X} U(x)$$

Here, $U(x)$ is some fixed utility function.

4.7 Repeated game

When a game is repeated many times, collaboration between players, eben if dominated in the one shot game, can be based on rationality. The common strategy of the Nash equilibria profile has a weakness: it is based on mutual threat of the players, which is not completely credible since by pushing the player who deviates from the agreement the other will also damage himself. In general, the number of Nash equilibria profile in the repetition of the game is very large.

Definition (*Stage game*)**.** A stage game is played with infinite horizon by the players.

We need to define the strategy and the payoff.

4.7.0.1 Strategy

Assume that at each stage τ, the player know which outcome has been selected at stage $\tau - 1$. Thus the strategy for a player is:

$$s = s(\tau), \tau = 0, \ldots$$

Here, for each τ, $s(\tau)$ is a specification of the moves of the stage game, which is in general a function of the past choices of the players.

4.7.0.2 Payoff

In general, it is not possible to sum payoffs obtained at each stage since the sum will be infinite for $\tau = \infty$. There are different possible choices to construct the payoff function. One standard choice is to use a discount factor δ, where $0 < \delta < 1$. So, the utility function becomes:

$$u_i(s, t) = (1 - \delta) \sum_{\tau=0}^{\infty} \delta^\tau u_i(s(\tau), t(\tau))$$

Here, $u_i(s(\tau), t(\tau))$ is the stage-game payoff of the player i at time τ given strategy profile $(s(\tau), t(\tau))$.

Definition (*Threat value*)**.** For the bi-matrix game (A, B) representing the stage game:

$$\underline{\mathrm{v}}_1 = \min_j \max_i a_{ij} \qquad \underline{\mathrm{v}}_2 = \min_i \max_j b_{ij}$$

Are called threat values of Player 1 and Player 2, respectively.

Note that v_1 and v_2 are not the conservative values of the two players.

Theorem 4.7.1. *For every feasible payoff vector $v = (v_1, v_2) = (a_{\bar{i}\bar{j}}, b_{\bar{i}\bar{j}})$ such that $v_i > \underline{v}_i$ where $i = 1, 2$, there exists $\bar{\delta} < 1$ such that for all $\delta > \bar{\delta}$ there is a Nash equilibrium of the repeated game with discounting factor δ, which yields payoffs v.*

Proof. $v = (v_1, v_2) = (a_{\bar{i}\bar{j}}, b_{\bar{i}\bar{j}})$ such that $v_i > \underline{v}_i$. Define the following strategy s: play the strategy yielding v at any stage, unless the opponents deviates at time t. In the latter case play the threat strategy form the stage $t+1$ onwards. We need to prove that s provides utility vector v and s is a Nash equilibrium for all δ close to 1.

At time $\tau = t$ player i could gain at most $\max_{i,j} a_{ij}$. Denote by s_t the strategy of deviating at time t. So, if the Player 1 deviates, after t he will gain at most $\underline{v}_1$. Hence, the payoff is such that:

$$u_1(s_t) \leq (1-\delta)\left(\sum_{\tau=0}^{t-1}\delta^\tau v_1 + \delta^t \max_{i,j} a_{ij} + \sum_{\tau=t+1}^{\infty}\delta^\tau \underline{v}_1\right) \quad (1-\delta^t)v_1 + (1-\delta)\delta^\tau \max_{i,j} a_{ij} + (\delta^{t+1})\underline{v}_1$$

Instead with strategy s the payoff is:

$$u_1(s) = (1-\delta)\sum_{\tau=0}^{\infty}\delta^\tau v_1 = v_1$$

Then:

$$u_1(s) = v_1 \geq u_1(s_t) = (1-\delta^t)v_1 + (1-\delta)\delta^\tau \max_{i,j} a_{ij} + (\delta^{t+1})\underline{v}_1$$

If and only if:

$$\begin{aligned}
(1-\delta^t)v_1 + (1-\delta)\delta^\tau \max_{i,j} a_{ij} + (\delta^{t+1})\underline{v}_1 &\leq v_1 \\
(1-\delta)\delta^\tau \max_{i,j} a_{ij} + (\delta^{t+1})\underline{v}_1 &\leq \delta^t v_1 \\
(1-\delta)\max_{i,j} a_{ij} + \delta\underline{v}_1 &\leq v_1 \\
\delta(\max_{i,j} a_{ij} - \underline{v}_1) &\geq \max_{i,j} a_{ij} - v_1
\end{aligned}$$

By properly setting $\boldsymbol{\delta}_i = \frac{\max_{i,j} a_{ij} - v_i}{\max_{i,j} a_{ij} - \underline{V}_i} < 1$ we have:

$$\boldsymbol{\delta} = \max_{i=1,2} \boldsymbol{\delta}_i$$

□

4.7.1 Correlated equilibrium

Given a game (A, B) with n strategies for Player 1 and m strategies for Player 2. Let $I = \{1, \ldots, n\}$, $J = \{1, \ldots, m\}$, and $X = I \times J$.

Definition (*Correlated equilibrium*)**.** A correlated equilibrium is a probability distribution $P = (p_{ij})$ on X such that for all $\bar{i} \in I$:

$$\sum_{j=1}^{m} p_{\bar{i}j} a_{\bar{i}j} \geq \sum_{j=1}^{m} p_{\bar{i}j} a_{ij} \qquad \forall i \in I$$

And for all $\bar{j} \in J$:

$$\sum_{j=1}^{m} p_{i\bar{j}} b_{i\bar{j}} \geq \sum_{j=1}^{m} p_{i\bar{j}} b_{ij} \qquad \forall j \in J$$

4.7.1.1 Existence

The set of correlated equilibria of a finite game is nonempty.

Theorem 4.7.2. *A Nash equilibrium profile generates a correlated equilibrium.*

Given the Nash equilibrium profile $(\bar{x}, \bar{y})$, the probability distribution of the outcome matrix is p, where each element is such that $p_{ij} = \bar{x}_i \bar{y}_j$

Proof. We have to prove that:

$$\sum_{j=1}^{m} \bar{x}_{\bar{i}} \bar{y}_j a_{\bar{i}j} \geq \sum_{j=1}^{m} \bar{x}_{\bar{i}} \bar{y}_j a_{ij} \forall i \in I$$

That is obvious for $\bar{x}_{\bar{i}} = 0$. If $\bar{x}_{\bar{i}} > 0$ we need to show that:

$$\sum_{j=1}^{m} \bar{y}_j a_{\bar{i}j} \geq \sum_{j=1}^{m} \bar{y}_j a_{ij} \qquad \forall i \in I$$

The left (right) hand side is the expected utility of Player 1 is he chooses row $\bar{i}$ (i) given that Player 2 chooses his equilibrium strategy $\bar{y}$. The inequality holds since the pure strategy $\bar{i}$ is played with positive probability, hence $\bar{i}$ must be (one of) the best reaction(s) to $\bar{y}$. □

Theorem 4.7.3. *The set of the correlated equilibria of a finite game is a nonempty convex polytope.*

Proof. Remember that a convex polytope is the smallest convex set containing a finite number of points. The set of the correlated equilibria is the solution set of a system $n^2 + m^2$ linear inequalities called incentive constraints, plus the condition of being a probability distribution. □

Proposition. If a row $\bar{i}$ is strictly dominated, then $P_{\bar{i}j}$ for every j.

Proof. Suppose $\bar{i}$ is strictly dominated by i. Since:

$$\sum_{j=1}^{m} p_{\bar{i}j}(a_{\bar{i}j} - a_{ij}) \geq 0$$

It must be $p_{\bar{i}j}$ for every j. □

The most important conclusion we can draw is that there is essentially a unique rationality paradigm in the whole theory: the idea of best reaction.

CHAPTER 5

Cooperative games

5.1 Introduction

Definition (*Cooperative game*)**.** A cooperative game is a pair (N, V) where N is the set of all players and $V : 2^N \to \mathbb{R}^n$ is a utility multi-function such that $V(A) \subseteq \mathbb{R}^{|A|}$ for any coalition $A \in \mathcal{P}(N)$ formed by a subset of all the players.

2^N is the collection of all subsets of the finite set N, such that $|N| = n$. $V(A)$ is the set of the aggregate utilities of the player in coalition A. If $V(A)$ gives cots rather than utilities, all inequalities must be reversed.

Definition (*Transferable utility game*)**.** A transferable utility game is a function $V : 2^N \to \mathbb{R}$ such that $v(\varnothing) = 0$.

Any TU game is also a cooperative game.

Definition (*Peer game*)**.** The peer game is a game v such that:

$$v(S) = \sum_{i \in N | S(i) \subseteq S} v_i$$

5.1.1 Cooperative games taxonomy

Let $\mathcal{G}(N)$ be the set of all cooperative games having N as the set of players. Fix a list $S_1, \dots, S_{2^n-1}$ of coalition. Then, a vector $(v_1, \dots, v_{2^n-1})$ represents a game, setting $v_i = v(S_i)$.

Proposition. The set $\mathcal{G}(N)$ is isomorphic to $\mathbb{R}^{2n-1}$.

Proposition. Given the set $\{u_A \mid A \subseteq N\}$ of the unanimity games

$$u_a(T) = \begin{cases} 1 & \text{if } A \subseteq T \\ 0 & \text{otherwise} \end{cases}$$

Is a basis for the space $\mathcal{G}(N)$.

Definition (*Additive game*)**.** A game is additive if:

$$v(A \cup B) = v(A) + v(B) \qquad \forall A \cap B = \varnothing$$

Definition (*Superadditive game*)**.** A game is superadditive if:

$$v(A \cup B) \geq v(A) + v(B) \qquad \forall A \cap B = \varnothing$$

The set of additive game is a vector space of dimension n.

Definition. A game $v \in \mathcal{G}$ is called simple if:

- $v(S) \in \{0, 1\}$ for nonempty coalition S.
- $A \subseteq C$ implies $v(A) \leq v(C)$.
- $v(N) = 1$.

$v(A) = 1$ means that the coalition A wins and $v(A) = 0$ means that the coalition A loses.

Definition. A coalition A in the simple game v is called minimal winning coalition if $v(A) = 1$ and $B \subsetneq A \implies v(B) = 0$.

5.2 Solution

Definition (*Solution vector*)**.** A solution vector for the game $v \in \mathcal{G}(N)$ is a vector $(x_1, \dots, x_n)$.

Definition (*Solution concept*)**.** A solution concept for the game $v \in \mathcal{G}(N)$ is a multifunction:

$$S : \mathcal{G}(N) \to \mathbb{R}^n$$

The solution vector assign utility or cost to each player. A solution assigns a set of solution vectors, possibly empty, to every game.

5.3 Imputation

Definition (*Imputation*)**.** The solution $I : \mathcal{G}(N) \to \mathbb{R}^n$ such that $x \in I(v)$ if:

1. $x_i \geq v(\{i\})$ for all i.
2. $\sum_{i=1}^{n} x_i = v(N)$.

is called imputation.

The first condition states that player i will not paretcipate if the solution assigns him a value x_i less than what he would be able to earn by himself. The second condition can be split in two inequalities:

- *Feasibility*: $\sum_{i=1}^{n} x_i \leq v(N)$ assures that if the grand coalition is formed the amount available to the player is $v(N)$.
- *Efficiency*: $\sum_{i=1}^{n} x_i \geq v(N)$ says that the overall amount will be effectively distributed among all the players.

If a game satisfies $v(N) \geq \sum_i v(\{i\})$, then the imputation is nonempty. If v is additive, them $I(v) = \{(v(1, \dots, v(n)))\}$.

Proposition. The imputation set $I(v)$ is a polytope.

Efficiency is a mandatory requirements in cooperative games. The imputation set is non empty if the game is superadditive and it reduces to a singleton if it is additive. The imputation set is the intersection of the half spaces defined by efficiency and feasibility constraints.

5.4 Core

Definition (*Core*)**.** The core is the solution $C : \mathcal{G}(N) \to \mathbb{R}^n$ of a game v such that:

$$C(v) = \left\{ x \in \mathbb{R}^n \mid \sum_{i=1}^{n} x_i = v(N) \wedge \sum_{i \in S} x_i \geq v(S) \quad \forall S \subseteq N \right\}$$

The core is a subset of the imputations. Imputations are efficient distributions of utilities accepted by all individual players, while core vectors are efficient distributions of utilities accepted by all coalitions.

Proposition. The core $C(v)$ is a polytope.

The core reduces to the singleton $(v(\{1\}), \ldots, v(\{n\}))$ if v is additive. But superadditive games can have an empty core.

Definition (*Veto player*)**.** In a game v, a player i is a veto player if $v(A) = 0$ for all A such that $i \notin A$.

Theorem 5.4.1. *Let v be a simple game. Then $C(v) \neq \varnothing$ if and only if there is at least one veto player. When a veto player exists, the core is the closed convex polytope with the vectors $(0, \ldots, 1, \ldots, 0)$ as extreme points, where 1 corresponds to the veto player.*

Proof. If there is no veto player, then for every i there is A_i such that $i \notin A_i$ and $v(A_i) = 1$. Suppose that $\mathbf{x} \in C(v)$, then it follows that:

$$\sum_{j \neq i} x_j \geq \sum_{j \in A_i} = 1 \qquad \forall i$$

However, by summing up the above inequalities from 1 to n one obtains:

$$(n-1) \sum_{j=1}^{n} x_j = n$$

Which yields a contradiction with $\sum_{j=1}^{n} x_j = 1$.

Conversely, and imputation assigning zero to the non-veto players must be in the core. $\square$

Theorem 5.4.2. *The Linear programming problem:*

$$\min \sum_{i=1}^{n} x_i$$
$$\textit{such that} \sum_{i \in S} x_i \geq v(S) \quad \forall S \subseteq N$$

Has always a nonempty set of solutions C. The core $C(v)$ is nonempty and $C(v) = C$ if and only if the optimal value of the problem is $v(N)$.

The value V of the Linear Programming problem is $V \geq v(N)$, due to the constraint $\sum_i x_i \geq v(N)$. Thus, for every x fulfilling the constraint one has $\sum_{i=1}^{n} x_i \geq v(N)$.

Theorem 5.4.3. *$C(v) \neq \varnothing$ if and only if every vector $(\lambda_S)_{S \subset N}$ such that:*

$$\lambda_S \geq 0 \qquad \forall S \subseteq N$$

$$\sum_{S|i \in S \subseteq N} \lambda_S = 1 \qquad \forall i$$

Satisfies also the following inequality:

$$\sum_{S \subseteq N} \lambda_S v(S) \leq v(N)$$

Note that the coefficients λ_S can be interpreted as indicating how much a given coalition S represents the players. Therefore, the theorem suggests that, no matter what quota the players contribute to the coalition, the weighted values must not exceed the overall amount of utility.

Proof. The first Linear Programming problem has the following matrix form:

$$\begin{gathered} \min \ \langle c, x \rangle \\ \text{such that } Ax \geq b \end{gathered}$$

Here $c = 1_n, b = (v(\{1\}), \ldots, v(N))$ and A si a $(2^n - 1) \times n$ matrix, whose rows are coalitions and columns are players: $A_{ij} = 1$ if player j is in the coalition i, $A_{ij} = 0$ otherwise. The dual of the problem takes the form:

$$\begin{aligned} \max & \sum_{S \subseteq N} \lambda_S v(S) \\ \text{such that} & \sum_{S|i \in S \subseteq N} \lambda_S = 1 \qquad \forall i \\ & \lambda_S \geq 0 \end{aligned}$$

Since the primal has at least one finite solution, the fundamental duality theorem states that this is true also for the dual and there is no duality gap. So, the core $C(v)$ is nonempty if and only if the value V of the dual problem is such that $V \leq v(N)$. □

5.4.1 Balanced coalitions

Definition (*Balanced family*)**.** A family $(S_1, \ldots, S_m)$ of coalitions is called balanced in case there exists $\lambda = (\lambda_1, \ldots, \lambda_m)$ such that $\lambda_j > 0$ for all $j \in [1, m]$ and for every $i \in N$:

$$\sum_{k|i \in S_k} \lambda_k = 1$$

λ is called balancing vector.

The set of λ_S is a convex polytope with a finite number of extreme points.

Definition (*Minimal balancing family*)**.** A minimal balancing family is a balancing family for which there is no sub-family that is balanced.

Lemma 5.4.4. *A balanced family is minimal if and only if its balancing vector is unique.*

Theorem 5.4.5. *The positive coefficient of the extreme points of the constraint set of the dual problem are the balancing vectors of the minimal balanced coalitions.*

To find the extreme points of the dual constraint set it is enough to find balanced families with unique balancing vector. The partitions of N are minimal balanced families. The relevant condition:

$$\sum_{S\subseteq N} \lambda_S v(S) \leq v(N)$$

is automatically fulfilled if the game is superadditive.

5.5 Nucleolus

Definition. The excess of a coalition A over the imputation x is:

$$e(A,x) = v(A) - \sum_{i\in A} x_i$$

$e(A,x)$ is a measure of the dissatisfaction of the coalition A with respect to the assignment of the imputation x. An imputation x of the game v belongs to $C(v)$ if and only if $e(A,x) \leq 0$ for all A.

Definition (*Lexicographic vector*)**.** The lexicographic vector attached to the imputation x is the $(2^n-1)^{\text{th}}$ dimensional vector $\theta(x)$ such that:

1. $\theta_i(x) = e(A,x)$ for some $A \subseteq N$.
2. $\theta_1(x) \geq \cdots \geq \theta_{2^n-1}(x)$

It arranges the excess of coalition over the imputation x in decreasing order.

Definition (*Nucleolus*)**.** The nucleolus solution is the solution $\nu : \mathcal{G}(N) \to \mathbb{R}^n$ such that $\nu(v)$ is the set of imputations x such that $\theta(x) \leq_L \theta(y)$, for all imputations y of the game v.

$x \leq_L y$ if $x = y$ or there exists j such that $x_i = y_i$ for all $i < j$, and $x_i < y_j$. $x \leq_L y$ defines a total order in any Euclidean space. That is, the nucleolus minimizes the excess.

Theorem 5.5.1. *For every transferable utility game v with nonempty imputation set, the nucleolus $\nu(v)$ is a singleton.*

Proposition. Suppose v is such that $C(v) \neq \varnothing$. Then $\nu(v) \in C(v)$.

Proof. For all $x \in C(v)$, by definition od core $\theta_1(x) \leq 0$. Since the nucleolus minimizes the excess, we have $\theta_i(\nu(v)) \leq 0$. Then, $\nu(v)$ is in the core. $\square$

5.6 Shapley value

Let $\phi : \mathcal{G}(N) \to \mathbb{R}^n$ be a one point solution.

Property 5.6.1. Efficiency: $\sum_{i\in N} \phi_i(v) = v(N)$ for every $v \in \mathcal{G}(N)$.

Property 5.6.2. Symmetry: symmetric players must get the same value. If $v \in \mathcal{G}(N)$ is a game such that $v(A\cup\{i\}) = v(A\cup\{j\})$ for every A not containing i and j, then $\phi_i(v) = \phi_j(v)$.

Property 5.6.3. Null player property: a player contributing nothing to any coalition cannot get anything. If $v \in \mathcal{G}(N)$ and $i \in N$ are such that $v(A) = v(A \cup \{i\})$ for all A, then $\phi_i(v) = 0$.

Property 5.6.4. Additivity: $\phi(v + w) = \phi(v) + \phi(w)$ for every $v, w \in \mathcal{G}(N)$.

Theorem 5.6.1. *Let $\sigma : \mathcal{G}(N) \to \mathbb{R}^n$ be defined by:*

$$\sigma_i(v) = \sum_{S \in 2^{N \setminus \{i\}}} \frac{s!(n-s-1)!}{n!} \left[v(S \cup \{i\}) - v(S)\right]$$

Then σ satisfies the properties of efficiency, symmetry, null player and additivity. Conversely, if $\tilde{\sigma}$ is a one point solution satisfying the property of efficiency, symmetry, null player and additivity, then $\tilde{\sigma} = \sigma$.

In other words, there exits one point solution satisfying the properties of efficiency, symmetry, null player and additivity. We call it the Shapley value. The term $m_i(v, S) = v(S \cup \{i\}) - v(S)$ is called the marginal contribution of player i to coalition $S \cup \{i\}$.

Efficiency proof. Consider a generic term $v(S \cup \{i\}) - v(S)$. The term $v(N)$ appears n times, once for every player, when $S = N \setminus \{i\}$. As $s = n - 1$ its coefficient is $\frac{(n-1)!(n-n)!}{n!} = \frac{1}{n}$. Consider now any other coalition $T \neq N$. The term $v(T)$ appears both with positive and negative coefficients;

- The positive coefficient $\frac{(t-1)!(n-t)!}{n!}$ appears t times, once for every player $i \in S$ when $S = T \setminus \{i\}$: hence the distribution is $\frac{t!(n-t)!}{n!}$.
- The negative coefficient $-\frac{(t-1)!(n-t)!}{n!}$ appears $n-t$ times, once for every player $i \notin T$ when $S = T$: hence the distribution is $-\frac{t!(n-t)!}{n!}$.

Thus in the sum:

$$\sum_{i=1}^{n} \sum_{S \in 2^{N \setminus \{i\}}} \frac{s!(n-s-1)!}{n!} \left[v(S \cup \{i\}) - v(S)\right]$$

$v(N)$ appears with coefficient one and every $A \neq N$ appears with null coefficient. □

Symmetry proof. We have that:

$$\begin{aligned}\sigma_i(v) = &\sum_{S \in 2^{N \setminus \{i \cup j\}}} \frac{s!(n-s-1)!}{n!} \left[v(S \cup \{i\}) - v(S)\right] + \\ &\sum_{S \in 2^{N \setminus \{i \cup j\}}} \frac{(s+1)!(n-s-2)!}{n!} \left[v(S \cup \{i \cup j\}) - v(S \cup \{j\})\right]\end{aligned}$$

$$\begin{aligned}\sigma_j(v) = &\sum_{S \in 2^{N \setminus \{i \cup j\}}} \frac{s!(n-s-1)!}{n!} \left[v(S \cup \{j\}) - v(S)\right] + \\ &\sum_{S \in 2^{N \setminus \{i \cup j\}}} \frac{(s+1)!(n-s-2)!}{n!} \left[v(S \cup \{i \cup j\}) - v(S \cup \{i\})\right]\end{aligned}$$

The terms in the sums are thus equal for symmetric players. □

Uniqueness proof. Consider a unanimity game u_A:

- Players not belonging to A are null players: thus σ assigns zero to them.
- Players belonging to A are symmetric, and so σ must assign the same value to both.
- σ is efficient.
- Then $\sigma_i(u_A) = \frac{1}{|A|}$ if $i \in A$, $\sigma_i(u_A) = 0$ otherwise.

Then ϕ is uniquely determined by the basis of $\mathcal{G}(N)$ give in terms of the unanimity games. The same argument applies to the game $c \cdot u_A$, for $c \in \mathbb{R}$. Because of the additivity axiom, at most once function satisfies the properties. □

5.6.1 Simple games

In the case of simple games, the Shapley value becomes:

$$\sigma_i = \sum_{A \in \mathcal{A}_i} \frac{a!(n-a-1)!}{n!}$$

Here, $\mathcal{A}_i$ is the set of the coalitions A such that $i \notin A$, A is not winning, and $A \cup \{i\}$ is winning.

Alternatively, it can be written:

$$\sigma_i = \sum_{A \in \mathcal{W}_i} \frac{(a-1)!(n-a)!}{n!}$$

Here, $\mathcal{W}_i$ is the set of the coalitions A such that $i \in A$, A is winning, and $A \setminus \{i\}$ is not winning.

In simple games the Shapley value measures the fraction of power of every player. In order to measure the relative power of the players in a simple game, the requirement of efficiency is not mandatory, hence coalitions could even form in a different way from the case of the Shapley value.

Definition (*Probabilistic power index*)**.** A probabilistic power index ψ one the set of simple games is:

$$\psi_i(v) = \sum_{S \in 2^{N \setminus \{i\}}} \Pr_i(S) m_i(v, S)$$

Here, $\Pr_i$ is a probability measure on $2^{N \setminus \{i\}}$.

Definition (*Semi-value*)**.** A probabilistic power index ψ one the set of simple games is a semi-value if there exists a vector $(\Pr_0, \ldots, \Pr_{n-1})$ such that:

$$\psi_i(v) = \sum_{S \in 2^{N \setminus \{i\}}} \Pr_s m_i(v, S)$$

Since the index is probabilistic, the two conditions must hold: $\Pr_s \geq 0$ and $\sum_{n=0}^{n-1} \binom{n-1}{s} \Pr_s = 1$.

Definition (*Regular semi-value*)**.** If $\Pr_s > 0$ for all s, the semi-value is called regular.

APPENDIX A

Additional concepts

A.1 Binary sum

We define a binary operation $\oplus$ on the set of natural numbers $\mathbb{N} = \{0, 1, 2, \ldots, n, \ldots\}$ as follows. For any two natural numbers $n_1, n_2 \in \mathbb{N}$:

1. Convert n_1 and ${}_n2$ into their binary representations, denoted as $[n_1]_2$ and $[n_2]_2$.
2. Perform the binary addition of $[n_1]_2$ and $[n_2]_2$ using the standard addition method, but without carrying over. This means if the addition of two bits results in 2 (i.e., $1 + 1$), it should be represented as 0 in that position with no carry to the next higher bit.
3. The result of the operation $\oplus$ is then represented in binary form, which corresponds to the sum computed in step 2.

A.2 Group

Definition (*Group*)**.** A group is defined as a nonempty set A equipped with a binary operation $\cdot$ such that the following conditions hold:

1. *Closure*: for any elements $a, b \in A$, the result of the operation $a \cdot b$ is also an element of A.
2. *Associativity*: the operation $\cdot$ is associative, meaning that for all $a, b, c \in A$, it holds that $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.
3. *Identity element*: there exists a unique element e known as the identity element, such that for every $a \in A$, the following holds: $a \cdot e = e \cdot a = a$.
4. *Inverse element*: for every element $a \in A$, there exists a unique element $b \in A$ (denoted as a^{-1}) such that $a \cdot b = b \cdot a = e$. This element b is called the inverse of a.

Definition (*Abelian group*)**.** A group A is termed an Abelian group (or commutative group) if the operation is commutative; that is, for all $a, b \in A$, the equation $\cdot\ a \cdot b = b \cdot a$ holds true.

Proposition. Let $(A, \cdot)$ be a group, then the cancellation law holds:

$$a \cdot b = a \cdot c \implies b = c$$

Proposition. The set of natural numbers with the operation $\oplus$ forms an Abelian group.

A.3 Convexity

Definition (*Convex set*)**.** A set $C \subset \mathbb{R}^n$ is called convex if for any points $x, y \in C$ and for any $\lambda \in [0, 1]$ the point $\lambda x + (1 - \lambda)y \in C$.

This means that the line segment connecting any two points in C is entirely contained within C. The properties of a convex set are:

- The intersection of an arbitrary family of convex sets is convex.
- A closed convex set with a nonempty interior coincides with the closure of its internal points.

Definition (*Convex combination*)**.** A convex combination of elements $x_1, \ldots, x_n$ is any vector x of the form:

$$x = \lambda_1 x_1 + \cdots + \lambda_n x_n$$

where $\lambda_1 \geq 0, \ldots, \lambda_n \geq 0$ and $\sum_{i=1}^{n} \lambda_i = 1$.

Proposition. A set C is convex if and only if for every $\lambda_1 \geq 0, \ldots, \lambda_n \geq 0$ such that $\sum_{i=1}^{n} \lambda_i = 1$, and for every $c_1, \ldots, c_n \in C$, we have

$$\sum_{i=1}^{n} \lambda_i c_i \in C$$

Definition (*Convex hull*)**.** The convex hull of a set C, denoted by co C, is the smallest convex set containing C. It is defined as:

$$\text{co } C = \bigcap_{A \in \mathcal{C}} A$$

where $\mathcal{C} = \{A | C \subset A \text{ and } A \text{ is convex}\}$.

Proposition. The convex hull of a set C can be expressed as:

$$\text{co } C = \left\{ \sum_{i=1}^{n} \lambda_i c_i | \lambda_i \geq 0, \sum_{i=1}^{n} \lambda_i = 1, c_i \in C \quad \forall i, n \in \mathbb{N} \right\}$$

The convex hull of a set C consists of all convex combinations of points in C. When C is a finite set, the convex hull is called a polytope.

Theorem A.3.1. *Given a closed convex set C and a point x outside C, there exists a unique point $p \in C$ such that for all $c \in C$:*

$$\|p - x\| \leq \|c - x\|$$

The projection p is the point in C closest to x and satisfies the following:

1. $p \in C$.
2. $(x - p, c - p) \leq 0$ for all $c \in C$.

Theorem A.3.2. *Let C be a convex subset of $\mathbb{R}^l$, and assume $\bar{x} \in cl\, C^c$ (the closure of the complement of C). Then, there exists a nonzero $x^* \in \mathbb{R}^l$ such that for all $c \in C$:*

$$(x^*, c) \geq (x^*, \bar{x})$$

This result provides a criterion to distinguish points outside of C from those inside.

Corollary A.3.2.1. *For any closed convex set C in Euclidean space, and any point x on the boundary of C, there exists a hyperplane that contains x and leaves all points in C on one side of the hyperplane.*

This hyperplane is called a supporting hyperplane for C at x.

Corollary A.3.2.2. *Any closed convex set C in Euclidean space can be represented as the intersection of all half-spaces that contain it.*

Theorem A.3.3. *Let A and C be closed convex subsets of $\mathbb{R}^l$, with int $A \neq \varnothing$ and int $A \cap C = \varnothing$. Then, there exists a nonzero vector x^* and a scalar $b \in \mathbb{R}$ such that for all $a \in A$ and $c \in C$:*

$$(x^*, a) \geq b \geq (x^*, c)$$

This provides a criterion to determine whether a point lies in A or C. The hyperplane $H = \{x : (x^*, x) = b\}$ is the separating hyperplane, with A and C located in different half-spaces defined by H.

Definition (*Quasi concavity*)**.** Quasi concavity for a real valued function h means that the sets:

$$h_a = \{z \mid h(z) \geq a\}$$

are convex for all $a \in \mathbb{R}$.

A.4 Linear programming

Definition (*Duality first form*)**.** The following two linear programs are said to be in duality:

$$\begin{cases} \min(\mathbf{c}, \mathbf{x}) \\ \mathbf{A}\mathbf{x} \geq \mathbf{b} \\ \mathbf{x} \geq 0 \end{cases} \qquad \begin{cases} \max(\mathbf{b}, \mathbf{y}) \\ \mathbf{A}^T\mathbf{y} \leq \mathbf{c} \\ \mathbf{y} \geq 0 \end{cases}$$

Here the matrix $A \in \mathbb{R}^{\gtrdot \times \ltimes}$ and the vectors $\mathbf{c}, \mathbf{x} \in \mathbb{R}^n$, $\mathbf{b}, \mathbf{y} \in \mathbb{R}^m$.

The minimization problem is called primal problem and the maximization is called dual problem.

Definition (*Duality second form*)**.** The following two linear programs are said to be in duality:

$$\begin{cases} \min(\mathbf{c}, \mathbf{x}) \\ \mathbf{A}\mathbf{x} \geq \mathbf{b} \end{cases} \qquad \begin{cases} \max(\mathbf{b}, \mathbf{y}) \\ \mathbf{A}^T\mathbf{y} \leq \mathbf{c} \\ \mathbf{y} \geq 0 \end{cases}$$

The minimization problem in the second form can be written in an equivalent way in the first form; dualizing this shows that the dual is equivalent to the dual of the second form, in the sense that the solution is the same.

Given two problems in duality, there are three options:

1. Both can be feasible.
2. Only one can be feasible.
3. They can both be infeasible.

A.4.1 Duality theorems

Theorem A.4.1 (*Weak duality*)**.** *Let v be the value of the primal minimization problem and V the value of the dual maximization problem. Then:*

$$v \geq V$$

Theorem A.4.2 (*Strong duality*)**.** *If the primal and the dual problems are feasible, then both problems have optimal solutions $\bar{\mathbf{x}}$ and $\bar{\mathbf{y}}$ and the optimal values coincide, that is:*

$$v = (\mathbf{c}, \bar{\mathbf{x}}) = (\mathbf{b}, \bar{\mathbf{y}}) = V$$

If the primal is feasible and the dual is infeasible, then $v = V = -\infty$.
If the primal is infeasible and the dual is feasible, then $v = V = +\infty$.
If both the primal and the dual are infeasible, then $v = +\infty > V = -\infty$.

Corollary A.4.2.1. *If one problem is feasible and has an optimal solution, then also the dual problem is feasible and has solution. Moreover, there is no duality gap.*

A.4.2 Complementarity

Theorem A.4.3 (Complementarity condition first form)**.** *Let $\bar{\mathbf{x}}$ and $\bar{\mathbf{y}}$ be primal and dual feasible. Then $\bar{\mathbf{x}}$ and $\bar{\mathbf{y}}$ are simultaneously optimal if and only if:*

$$\begin{cases} \forall i \bar{x}_i > 0 \implies \sum_{k=1}^{m} a_{ik}\bar{y}_k = c_i \\ \forall i \bar{y}_i > 0 \implies \sum_{k=1}^{n} a_{kj}\bar{x}_k = b_i \end{cases}$$

A.5 Multifunction and best response

Definition (*Multi-function*)**.** Given two sets A, B, a function $f : A \to 2^B$, where 2^B denotes the power set of B, is called a multifunction.

Definition (*Best response*)**.** Denote by BR_1 and BR_2, the following multi-functions:

$$BR_1 : Y \to 2^X : BR_1(y) = \text{argmax}\,\{f(\cdot, y)\}$$
$$BR_2 : X \to 2^Y : BR_2(x) = \text{argmax}\,\{f(x, \cdot)\}$$

We call best response multi-function the following:

$$BR : X \times Y \to 2^X \times 2^Y : BR(x, y) = (BR_1(y), BR_2(x))$$

A.6 Graph theory

Definition (*Finite directed graph*)**.** A finite directed graph is a pair (V, E) where:

- V is a set of vertices.
- $E \subset V \times V$ is a set of ordered pairs of directed edges.

Definition (*Path*)**.** A path from a vertex v_1 to a vertex v_{k+1} is a finite sequence of vertices and edges $v_1, e_1, v_2, \ldots, v_k, e_k, v_{k+1}$ such that $e_i \neq e_j$ if $i \neq j$ and $e_j = (v_j, v_{j+1})$.

The number k is called the length of the path.

Definition (*Oriented graph*)**.** An oriented graph is a finite directed graph where for all vertices v_j and v_k, at most one of (v_j, v_k) and (v_k, v_j) can be an edge in the graph.

Definition (*Tree*)**.** A tree is a triple (V, E, x_0) where (V, E) is an oriented graph, and x_0 is a vertex in V such that there is a unique path from x_0 to any other vertex $x \in V$.

Definition (*Child*)**.** A child of a vertex v is any vertex x such that $(v, x) \in E$.

Definition (*Leaf*)**.** A vertex is called a leaf if it has no children.