Nonlinear Optimization

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Abstract

The course aims to present the main methods for non-linear optimization, both continuous and discrete. Topics include: optimality conditions for both unconstrained and constrained problems, Lagrangian functions, and duality; gradient-based methods, Newton's methods, and step-size reduction techniques; recursive quadratic programming; and methods using penalty functions.

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CHAPTER 1

Optimization

1.1 Introduction

Optimization is a powerful and widely used field of applied mathematics, playing a crucial role in solving real-world problems across various domains.

Given a set $X \subseteq \mathbb{R}^n$ and a function $f : X \to \mathbb{R}^n$ hat we aim to minimize, the goal is to find an optimal solution $x^* \in X$ such that:

$$f(\underline{x}^*) \leq f(\underline{x}) \qquad \forall \underline{x} \in X$$

Many decision-making problems cannot be effectively modeled using linear approaches due to their inherent nonlinearity.

Example:
We are given:

- m warehouses, indexed by $i = 1, \ldots, m$, each with a capacity p_i and a location constraint within an area $A_i \subseteq \mathbb{R}^2$.
- n clients, indexed by $j = 1, \ldots, n$, each located at coordinates (a_j, b_j) with a demand d_j.

Our goal is to determine the optimal warehouse locations and how to distribute the product to clients while minimizing transportation costs, ensuring that warehouse capacities and client demands are met. We assume that the total available supply is sufficient: $\sum_{i=1}^{m} p_i \geq \sum_{j=1}^{n} d_j$.

The decision variables for this problem are:

- (x_i, y_i): the coordinates of of warehouse i (for $1 \leq i \leq m$).
- w_{ij}: the quantity of product transported from warehouse i to client j (for $1 \leq i \leq m$, $1 \leq j \leq n$).
- t_{ij}: the distance between warehouse i and client j, given by:

$$t_{ij} = \sqrt{(x_i - a_j)^2 + (y_i - b_j)^2}$$

We aim to minimize the total transportation cost:

$$\min \sum_{i=1}^{m} \sum_{j=1}^{n} t_{ij} w_{ij}$$

Subject to the following constraints:

1. Warehouse capacity constraints:

$$\sum_{j=1}^{n} w_{ij} \leq p_i \qquad \forall i$$

2. Client demand satisfaction:

$$\sum_{i=1}^{m} w_{ij} \geq d_j \qquad \forall j$$

3. Warehouse location constraints:

$$(x_i, y_i) \in A_i \subseteq \mathbb{R}^2 \qquad \forall i$$

4. Non-negativity constraints:

$$w_{ij} \geq 0, t_{ij} \geq 0 \qquad \forall i, j$$

This formulation ensures that all client demands are met while keeping transportation costs minimal and adhering to warehouse capacities and location constraints.

Example:
In computerized tomography, we analyze a 3D volume $V \subseteq \mathbb{R}^3$ that is subdivided into n small cubes, called voxels V_j. We assume that the matter density is constant within each voxel.

Our goal is to reconstruct a 2D slice of V, meaning we need to determine the density x_j for each voxel V_j based on measurements from m X-ray beams. For the i-th beam:

- a_{ij} represents the path length of the beam within voxel V_j.
- I_0 is the initial X-ray intensity at the source.
- I_i is the intensity after passing through the volume.

According to the Beer-Lambert law, the total log-attenuation of the beam is linearly related to the density:

$$\sum_{j=1}^{n} a_{ij} x_j = b_i = \log \frac{I_0}{I_i} \qquad i = 1, \ldots, m$$

We can formulate this as a linear system:

$$Ax = b, \quad x_j \geq 0 \qquad \forall j = 1, \ldots, n$$

However, this system is often infeasible due to measurement errors, non-uniformity of voxels, and other practical factors.

To handle inconsistencies, we use a least squares formulation to minimize the reconstruction error:

$$\min \sum_{i=1}^{m} \left(b_i - \sum_{j=1}^{n} a_{ij} x_j \right)^2$$

Subject to:

$$x_j \geq 0 \qquad j = 1, \dots, n$$

Since $n \gg m$ (many voxels, fewer beams), the problem may have multiple optimal solutions. To improve stability, we introduce a regularization term and minimize:

$$f(x) = \sum_{i=1}^{m} \left(b_i - \sum_{j=1}^{n} a_{ij} x_j \right)^2 + \delta \sum_{j=1}^{n} x_j$$

Here, $\delta > 0$ controls the strength of regularization.

The function $f(x)$ can include nonlinear terms to better account for material properties, image characteristics, or a stochastic model of attenuation. The number and directions of beams can also be optimized to improve reconstruction quality. In dynamic imaging, we can extend this to four dimensions to account for respiratory motion over time.

Example:
In this auction setting, bidders can place bids on combinations of discrete items, rather than bidding on individual items separately. Given:

- A set N of n bidders.
- A set M of m distinct items,
- For each subset $S \subseteq M$, bidder $j \in N$ is willing to pay $b_j(S)$ for that specific bundle S.

We assume that bidders may place higher bids for bundles than for individual items separately, i.e., there may be synergies in bundling items.

Our goal is to determine the allocation of items to bidders to maximize total revenue while ensuring that no item is allocated more than once. The decision variables will be:

- $b(S) = \max_{j \in N} b_j(S)$: the highest bid received for bundle.
- x_S: a binary variable indicating whether the highest bid for S is accepted:

$$x_S = \begin{cases} 1 & \text{if the highest bid on } S \text{ is accepted} \\ 0 & \text{otherwise} \end{cases}$$

The formulation is the following:

$$\max \sum_{S \subseteq M} b(S) x_S$$

Subject to:

1. Each item can be allocated to at most one winning bundle:

$$\sum_{S \subseteq M | i \in S} x_S \leq 1 \qquad \forall i \in M$$

2. Each bundle is either selected or not:

$$x_S \in \{0, 1\} \qquad \forall S \subseteq M$$

In essence, when $x_S = 1$, bundle S is awarded to a bidder willing to pay the highest price, ensuring that the total revenue is maximized while maintaining a valid allocation of items.

This formulation has $2^{|M|}$ variables, making it computationally challenging for large sets of items. Efficient algorithms, such as branch-and-bound, linear programming relaxations, or heuristic methods, may be necessary for practical problem sizes.

1.2 Optimization problem

The general optimization problem is formulated as follows:

$$\begin{aligned} \min \ & f(\underline{x}) \\ \text{such that } & g_i(\underline{x}) \leq 0 \qquad i \leq i \leq m \\ & \underline{x} \in S \subseteq \mathbb{R}^n \end{aligned}$$

Definition (*Feasible region*)**.** The feasible region consists of all points that satisfy both the set constraints and the algebraic constraints:

$$X = S \cap \{\underline{x} \in \mathbb{R}^n \mid g_i(\underline{x}) \leq 0, 1 \leq i \leq m\}$$

Here, each constraint function $g_i : S \to \mathbb{R}$ defines a restriction on the feasible set.

Definition (*Objective function*)**.** The function to be minimized, known as the objective function, is given by:

$$f : X \to \mathbb{R}$$

It assigns a numerical value to each feasible solution $\underline{x}$, which we seek to minimize.

Without loss of generality, we assume:

- The problem is a minimization problem, since maximization can be rewritten as:

$$\max_{\underline{x} \in X} f(\underline{x}) = \min_{\underline{x} \in X} -f(\underline{x})$$

- All algebraic constraints are inequality constraints, since equality constraints can be rewritten as two inequalities:

$$g(\underline{x}) = 0 \equiv \begin{cases} g(\underline{x}) \geq 0 \\ g(\underline{x}) \leq 0 \end{cases}$$

Definition (*Global optimum*)**.** A feasible solution $\underline{x}^* \in X$ is a global optimum if:

$$f(\underline{x}^*) \leq f(\underline{x}) \qquad \forall \underline{x} \in X$$

Definition (*Local optimum*)**.** A feasible solution $\overline{\underline{x}} \in X$ is a local optimum if there exists $\epsilon > 0$ such that:

$$f(\overline{\underline{x}}) \leq f(\underline{x}) \qquad \forall \underline{x} \in X \cap \mathcal{N}_\epsilon(\overline{\underline{x}})$$

Here, $\mathcal{N}_\epsilon(\overline{\underline{x}}) = \{\underline{x} \in X \mid \|\underline{x} - \overline{\underline{x}} \leq \epsilon\|\}$ is an epsilon-neighborhood around $\overline{\underline{x}}$.

For complex optimization problems, finding a global optimum is often computationally infeasible. Instead, we focus on obtaining good local optima within a reasonable computation time.