Foundation Of Operations Research
Theory

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Abstract

Operations Research is the branch of applied mathematics dealing with quantitative methods to analyze and solve complex real-world decision-making problems.

The course covers some fundamental concepts and methods of Operations Research pertaining to graph optimization, linear programming and integer linear programming.

The emphasis is on optimization models and efficient algorithms with a wide range of important applications in engineering and management.

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Chapter 1

Introduction

1.1 Definition

Definition

Operations Research is the branch of mathematics in which mathematical models and quantitative methods are used to analyze complex decision-making problems and find near-optimal solutions.

It is an interdisciplinary field at the interface of applied mathematics, computer science, economics and industrial engineering.

1.2 Decision-making problems

Definition

The *decision-making problems* are problems in which we must choose a feasible solution among many alternatives based on one or several criteria.

The more complex decision-making problems are tackled via a mathematical modelling approach (mathematical models, algorithms and computer implementations). Those problems can be classified in the following categories:

1. Assignment problem: given m jobs and m machines, suppose that each job can be executed by any machine and that t_{ij} is the execution time of job J_i one machine M_j. We want to decide which job assign to each machine to minimize the total execution time. Each job must be assigned to exactly one machine, and each machine to exactly one job. The number of feasible solution is equal to $m!$.

2. Network design: we want to decide how to connect n cities via a collection of possible links to minimize the total link cost. Given a graph

$G = (N, E)$ with a node $i \in N$ for each city and an edge $\{i, j\} \in E$ of cost $c_i j$, select a subset of edges of minimum total cost, guaranteeing that all pairs of nodes are connected. The number of feasible solution is equal to $2^{|E|}$.

3. Shortest path: given a direct graph that represents a road network with distances (traveling times) for each arc, determine the shortest path between two points (nodes).

4. Personnel scheduling: determine the week schedule for the hospital personnel, to minimize the number of people involved while meeting the daily requirements.

5. Service management: determine how many desks to open at a given time of the day so that the average customer waiting time does not exceed a certain value.

6. Multi-criteria problem: decide which laptop to buy considering the price, the weight and the performance.

7. Maximum clique (community detection in social networks): determine the complete sub-graph of a graph, with the maximum number of vertices.

1.3 History

In the World War II, teams of scientists were asked to do research on the most efficient way to conduct the operations. In the decades after the war, the techniques became public and began to be applied more widely to problems in business, industry and society. During the industrial boom, the substantial increase in the size of the companies and organizations gave rise to more complex decision-making problems. The favorable circumstances that permitted this were:

- Fast progress in Operations Research and in numerical analysis methodologies.

- Advent and diffusion of computers (more computing power and widespread software).

1.4 Operations Research workflow

$\boxed{\textsf{Problem}} \rightarrow \boxed{\textsf{Model}} \rightarrow \boxed{\textsf{Algorithm}} \rightarrow \boxed{\textsf{Implementation}}$

The main steps in studying an Operations Research problem are:

1. Define the problem.
2. Build the model.
3. Select or develop an appropriate algorithm.
4. Implement it or use an existing program.

After all this process we need to analyze the results with feedbacks (and eventually modify some previous step).

The model obtained with this process is a simplified representation of a real-world problem. To define it we must identify the fundamental elements of the problem and the main relationships among them.

Example : A company produces three types of electronic devices: D_1, D_2, D_3, going through three main phases of the production process: assembly, refinement and quality control. The time required for each phase and product is:

	D_1	D_2	D_3
Assembly	80	70	120
Refinement	70	90	20
Quality control	40	30	20

The available resources within the planning horizon in minutes are:

Assembly	**Refinement**	**Quality control**
30 000	25 000	18 000

The unary product for each product in:

D_1	D_2	D_3
1600	1000	2000

The main assumption is that the company can sell whatever it produces.

The mathematical model that describes the problem given before is the following:

- Decision variables: x_j is the number of devices D_j produced for $j = 1, 2, 3$.
- Objective function: we need to maximize the earning, so we have:

$$\max z = 1.6x_1 + 1x_2 + 2x_3$$

- Constraints: they are on the production limit of each phase, that are:

$$80x_1 + 70x_2 + 120x_3 \leq 30000$$

$$70x_1 + 90x_2 + 20x_3 \leq 25000$$

$$40x_1 + 30x_2 + 20x_3 \leq 18000$$

- Variable type: the variables must be non-negative values, so we have $x_1, x_2, x_3 \geq 0$.

Example : An insurance company must decide which investments to select out of a given set of possible assets.

Investments	**Area**	**Capital** (c_j)	**Return** (r_j)
A (automotive)	Germany	150000	11%
B (automotive)	Italy	150000	9%
C (ICT)	USA	60000	13%
D (ICT)	Italy	100000	10%
E (real estate)	Italy	125000	8%
F (real estate)	France	100000	7%
G (treasury bonds)	Italy	50000	3%
H (treasury bonds)	UK	80000	5%

The available capital is 600000 euro. It is required to take at most five different investments. It is also required to take at maximum three investments in Italy and maximum three abroad.

The mathematical model that describes the problem given before is the following:

- Decision variables: boolean value to communicate if the investment is selected or not: $x_j = 1$ if the j-th investment is selected and $x_j = 0$ otherwise, for $j = 0, \ldots, 8$.
- Objective function: we need to maximize the expected return, so we have:

$$\max z = \sum_{j=1}^{8} c_j r_j x_j$$

- Constraints: there is a constraint on the capital that insurance

$$\sum_{j=1}^{8} c_j x_j \leq 800$$

There is a constraint also on the max number of general investment and on the region they are coming from formalized asked

$$\sum_{j=1}^{8} x_j \leq 5$$

$$x_2 + x_4 + x_5 + x_7 \leq 3$$

$$x_1 + x_3 + x_6 + x_8 \leq 3$$

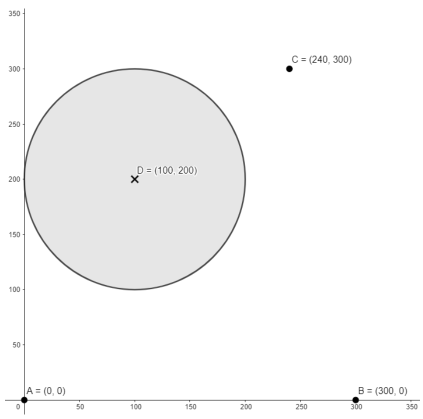
- Variable type: the variables are binary integer defined as $x_j \in \{0, 1\}$ $1 \leq j \leq 8$.

The variant requires that if any of the ICT investment is selected, then at least one of the treasury bond must be select. This requires one new constraint that is:

$$\frac{x_3 + x_4}{2} \leq x_7 + x_8$$

It is divided by two because if both ICT are selected at least one treasury bound must be selected and not two.

Example: Consider three oil pits, located in positions $A = (0, 0)$, $B = (300, 0)$, and $C = (240, 300)$, from which oil is extracted.



Connect them to a refinery with pipelines whose cost is proportional to the square of their length. The refinery must be at least 100 km away from point $D = (100, 200)$, but the oil pipelines can cross the corresponding forbidden zone. Give a mathematical model to decide where to locate the refinery to minimize the total pipeline cost.

- Decision variables: the coordinates of the refinery x_1, x_2.
- Objective function: we need to minimize the cost, so we have:

$$\begin{aligned}\min z = &\left[(x_1-0)^2+(x_2-0)^2\right]\\ &+\left[(x_1-300)^2+(x_2-0)^2\right]\\ &+\left[(x_1-240)^2+(x_2-300)^2\right]\end{aligned}$$

- Constraints: there is a constraint on the location that is

$$\sqrt{(x_1-100)^2+(x_2-100)^2} \geq 100$$

- Variable type: $x_1, x_2 \in \mathbb{R}$

1.5 Mathematical programming problem

The decision-making problems can be classified as:

- Mathematical programming: one decision maker and one objective.
- Multi-objective programming: one decision maker and several objectives.
- Stochastic programming: uncertainty level greater than zero.
- Game theory: several decision makers.

We will analyze the mathematical programming problems, so we have a single decision maker and a single objective. The optimization usually requires to minimizing or maximizing a given function. Note that maximizing $f(x)$ is the same problem of minimizing $-f(x)$. The problems are characterized by:

- Decision variables $x \in \mathbb{R}^n$: numerical variables whose values identify a solution of the problem.
- Feasible region $X \subseteq \mathbb{R}^n$: distinguish between feasible and infeasible solutions.
- Objective function $f : X \to \mathbb{R}$: expresses in quantitative terms the value of each feasible solution.

Solving a mathematical programming problem consists in finding a feasible solution which is globally optimum. It may happen that: the problem is infeasible, unbounded, has a single optimal solution or has numerous optimal solutions. When the problem is very hard we must settle for a feasible solution that is a local optimum. AN optimization problem can have many local optima. Mathematical programming can be classified in three main categories:

1. Linear Programming.
2. Integer Linear Programming.
3. Nonlinear Programming.

Multi-objective programming can be taken into account in different ways. Suppose we wish to minimize $f_1(x)$ and maximize $f_2(x)$, we can:

1. Turn it into a single objective problem by expressing the two objectives in terms of the same unit:

   $$\min \lambda_1 f_1(x) - \lambda_2 f_2(x)$$

   for appropriate scalars λ_1 and λ_2.

2. Optimize the primary objective function and turn the other objective into a constraint:

   $$\max_{x \in X} f_2(x) \quad f_1(x) \leq \epsilon$$

   for an appropriate constant ϵ.

Chapter 2

Network optimization models

2.1 Definitions

Many decision-making problems can be formulated in terms of graphs and networks.

Definition

A *graph* is a pair $G = (N, E)$, with a set of nodes N and a set of edges or arcs $E \subseteq N \times N$ connecting them pairwise. An edge connecting the nodes i and j is represented by $\{i, j\}$ or (i, j) if the graph is *undirected* or *directed* respectively.

Example : A road network which connects n cities can be modelled, by a graph where a city corresponds to a node, and a connection corresponds to an edge.

The graph on the left is undirected and defined as:

- $N = \{1, 2, 3, 4, 5\}$
- $E = \{\{1, 2\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}, \{3, 5\}, \{4, 5\}\}$

The graph on the right is directed and defined as:

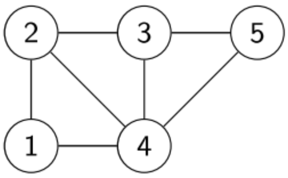
- $N = \{1, 2, 3, 4, 5\}$
- $E' = \{(1, 2), (1, 4), (2, 3), (2, 4), (3, 4), (3, 5), (4, 5)\}$

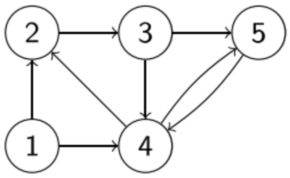
Definition

Two nodes are *adjacent* if they are connected by an edge. An edge e is *incident* in a node v if v is an endpoint of e. In undirected graphs, the *degree* of a node is the number of incident edges in a given node. In directed graph, the *id-degree* (*out-degree*) of a node is the number of arcs that ave it as successor (predecessor).

Example : In the undirected graph, we have that nodes 1 and 2 are adjacent and 1 and 3 are not. The edge $\{1, 2\}$ is incident in nodes 1 and 2. Node 1 has a degree 2, and node 4 has a degree of 4.

In the directed graph, the node 1 has an in-degree equal to 0, and an out-degree equal to 2.





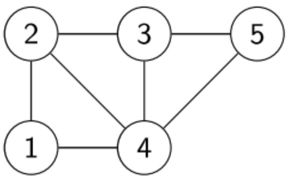
Definition

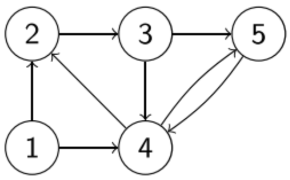
A *directed path from* $i \in N$ *to* $j \in N$ is a sequence of arcs $p = \langle \{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{k-1}, v_k\} \rangle$ connecting nodes v_1 and v_k. Nodes u and v are *connected* if there is a path connecting them. A graph (N, E) is connected if u, v are connected for any $u, v \in N$. A graph is *strongly connected* if u, v are connected by a directed path for any $u, v \in N$. A *cycle* or circuit is a path with $v_1 = v_k$.

Example : The undirected graph has a path $\langle \{2, 3\}, \{3, 4\}, \{4, 5\} \rangle$ from node 2 to node 5. So we say those nodes are connected.

The directed graph has a directed path $\langle (3, 5), (5, 4), (4, 2), (2, 3), (3, 4) \rangle$ from node 3 to node 4. So we say those nodes are not strongly connected.

In the undirected graph $\langle \{2, 3\}, \{3, 5\}, \{5, 4\}, \{4, 2\} \rangle$ is a cycle. In the directed graph $\langle (2, 3), (3, 4), (4, 2) \rangle$ is a circuit.

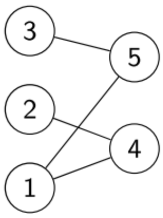


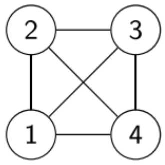


Definition

A graph is *bipartite* if there is a partition $N = N_1 \cup N_2$ with $N_1 \cap N_1 = \varnothing$ such that no edge connects nodes in the same subset. A graph is *complete* if $E = \{\{v_i, v_j\} | v_i, v_j \in N \wedge i \leq j\}$

Example: The graphic on the left is bipartite because we can find two subsets of nodes such that $N = N_1 \cup N_2$ with $N_1 \cap N_1 = \varnothing$ that are: $N_1 = \{1, 2, 3\}$ and $N_2 = \{4, 5\}$. The graph on the right is a complete graph because all the nodes are connected with each other.





Definition

Given a directed graph $G = (N, A)$ and $S \subset NM$, the *outgoing cut* induced by S is:

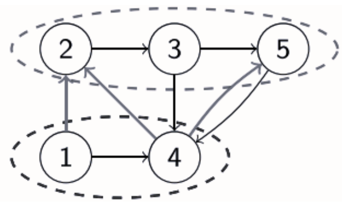
$$\delta^+(S) = \{(u, v) \in A | u \in S \wedge v \in N - S\}$$

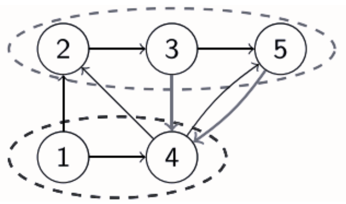
while the *incoming cut* induced by S is:

$$\delta^-(S) = \{(u, v) \in A | v \in S \wedge u \in N - S\}$$

Example: In the following graph we can note that:

- $\delta^+(\{1, 4\}) = \{(1, 2), (4, 2), (4, 5)\}$
- $\delta^-(\{1, 4\}) = \{(3, 4), (5, 4)\}$





An undirected graph with n nodes has at most $m = n(n-1)$ arcs. A directed graph with n nodes has at most $m = \dfrac{n(n-1)}{2}$ arcs.

Definition

A graph is said to be *dense* if:

$$m \approx n^2$$

and *sparse* if:

$$m \ll n^2$$

The best way to represent a dense graph is by using an $n \times n$ adjacency matrix, that is defined in the following way:

$$\begin{cases} a_{ij} = 1 & \text{if } (i,j) \in A \\ a_{ij} = 0 & \text{otherwise} \end{cases}$$

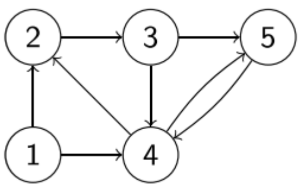
The best way to represent a sparse graph is by using lists of successors for each node.

Example : The adjacency matrix for the following graph is:

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

And the list of successor is:

$$S(1) = \{2,4\} \quad S(2) = \{3\} \quad S(3) = \{4,5\} \quad S(4) = \{2,5\} \quad S(5) = \{4\}$$



2.2 Graph reachability problem

Given the directed graph $G = (N, A)$ and a node s, determine all the nodes that are reachable from s.

Algorithm 1 Graph reachability problem

$Q \leftarrow \{s\}$
$M \leftarrow \{\varnothing\}$
while $Q \neq 0$ **do**
  $u \leftarrow$ node in Q
  $Q \leftarrow Q - \{u\}$
  $M \leftarrow M \cup \{u\}$
  for $(u, v) \in \delta^+(u)$ **do**
    if $v \notin M$ and $v \notin Q$ **then**
      $Q \leftarrow Q \cup \{v\}$
    end if
  end for
end while

The worst case complexity of the previous algorithm is $O(n^2)$.

Example : Given the following graph and $s = 2$ the algorithm makes the following steps:

1. $Q = \{2\}$ $M = \varnothing$
2. $Q = \{3\}$ $M = \{2\}$
3. $Q = \{4, 5\}$ $M = \{2, 3\}$
4. $Q = \{5\}$ $M = \{2, 3, 4\}$
5. $Q = \varnothing$ $M = \{2, 3, 4, 5\}$

So the nodes $\{2, 3, 4, 5\}$ are reachable from node 2.