Game Theory
Theory

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Contents

CHAPTER 1

Introduction

1.1 Games

Games are efficient models for an enormous amount of everyday life situations.

Definition (*Game*)**.** A game is a process consisting in:

- A set of players (at least two).
- An initial situation.
- Rules that the player must follow.
- All possible final situations.
- The preferences of all players on the set of the final situations.

1.2 Game theory assumptions

Game theory assumes that the players are supposed to be:

1. *Selfish.*
2. *Rational.*

1.2.1 Selfishness

The players only care about their own preferences with respect to the outcomes of the game. This is not an ethical problem, but a mathematical assumption. In fact, we need it to define what is the meaning of a ratinal choice.

1.2.2 Rationality

Definition (*Preference relation*)**.** Let X be a set. A preference relation on X is a binary relation $\preceq$ such that for all $x, y, z \in X$:

- Reflexive: $x \preceq x$.
- Complete: $x \preceq y$ or $y \preceq x$ or both.
- Transitive: if $x \preceq y$ and $y \preceq z$, then $x \preceq z$.

The transitive property is useful to have a consistent ranking.

First rationality assumption The first rationality assumption is that the players are able to provide a preference relation over the outcomes of the game, and the order must be consistent.

Definition (*Utility function*)**.** Let $\preceq$ be a preference relation over X. A utility function representing $\preceq$ is a function $u : X \to \mathbb{R}$ such that:

$$u(x) \geq u(y) \Leftrightarrow x \preceq y$$

A utility function may not exists in particular cases, however it exists in the general setting, specifically when X is a finite set. If a utility function exists, then there exist infinitely many utility functions, given by any strictly increasing transformation of the former.

To player i there is assigned a set X_i, repesenting all the choices available to her. Hence, the set $X = xX_i$ over which u is defined comprises the possible choices of all players.

Second rationality assumption The second rationality assumption states that the agents are able to provide a utility function representing their preference relations, whenever it is necessary.

Third rationality assumption The third rationality assumption states that the players use consistently the laws of probability. In particular, they are consistent with the computation of the expected utilities, they are able to update probabilities according to Bayes rule.

Fourth rationality assumption The fourth rationality assumption states that the players are able to understand the consequences of all their actions, the consequences of this information on any other player, the consequences of the consequences and so on.

Fifth rationality assumption The fifth rationality assumption states that the players are able to use decision theory, whenever it is possible.

That is, given a set of alternatives X, and a utility function u on X, each player seeks a $\bar{x} \in X$ such that:

$$u(\bar{x}) \geq u(x) \qquad \forall x \in X$$

An important consequence of the previous axioms is the principle of elimination of strictly dominated strategies. . A player does not take an action a it she has available an action b providing her a strictly better result, no matter what the other players do.

1.3 Bimatrices

Conventionally, Player 1 chooses a row, while Player 2 chooses a column. This results in a pair of numbers, corresponding to the utilities of Player 1 and 2, respectively. The options can be summarized in a bimatrix.

Example:
Consider the following bimatrix:

$$\begin{pmatrix} \begin{pmatrix} 8 & 8 \end{pmatrix} & \begin{pmatrix} 2 & 7 \end{pmatrix} \\ \begin{pmatrix} 7 & 2 \end{pmatrix} & \begin{pmatrix} 0 & 0 \end{pmatrix} \end{pmatrix}$$

In this example the utilities of player 1 are:

$$\begin{pmatrix} 8 & 2 \\ 7 & 0 \end{pmatrix}$$

Since the second row is strictly dominated by the first, Player 1 selects the latter. Likewise, Player 2 selects the first column, which strictly dominates the second one

Even if the Principle of elimination of strictly dominated actions may not be very informative, it has rather surprising consequences.

Example:
Consider the following two games:

$$\begin{pmatrix} \begin{pmatrix} 10 & 10 \end{pmatrix} & \begin{pmatrix} 3 & 15 \end{pmatrix} \\ \begin{pmatrix} 15 & 3 \end{pmatrix} & \begin{pmatrix} 5 & 5 \end{pmatrix} \end{pmatrix}$$

$$\begin{pmatrix} \begin{pmatrix} 8 & 8 \end{pmatrix} & \begin{pmatrix} 2 & 7 \end{pmatrix} \\ \begin{pmatrix} 7 & 2 \end{pmatrix} & \begin{pmatrix} 0 & 0 \end{pmatrix} \end{pmatrix}$$

Observe: relative to any single outcome (i.e they always have greater utilities) in Nevertheless, according to the principle, interactive situation to play the second game rather than the first! (for, the outcome pair (8, 8) is greater than (5, 5))

The first game:

$$\begin{pmatrix} \begin{pmatrix} 8 & 8 \end{pmatrix} & \begin{pmatrix} 2 & 7 \end{pmatrix} \\ \begin{pmatrix} 7 & 2 \end{pmatrix} & \begin{pmatrix} 0 & 0 \end{pmatrix} \end{pmatrix}$$

The second game contains all outcomes of the first, plus some further outcomes: The first game:

$$\begin{pmatrix} \begin{pmatrix} 1 & 1 \end{pmatrix} & \begin{pmatrix} 11 & 0 \end{pmatrix} & \begin{pmatrix} 4 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 11 \end{pmatrix} & \begin{pmatrix} 8 & 8 \end{pmatrix} & \begin{pmatrix} 2 & 7 \end{pmatrix} \\ \begin{pmatrix} 0 & 4 \end{pmatrix} & \begin{pmatrix} 7 & 2 \end{pmatrix} & \begin{pmatrix} 0 & 0 \end{pmatrix} \end{pmatrix}$$

The second game contains all outcomes of the first, plus some further outcomes:

Here, the rationality axioms imply that in the first game the players should choose the outcome pair (10, 10), whereas in the second game they should choose the outcome pair (1, 1) in the first row and first column. Therefore, just having less available actions can make the players better off!

Example:
What are the rational outcomes of the following game?

$$\begin{pmatrix} \begin{pmatrix} 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 1 \end{pmatrix} \\ \begin{pmatrix} 1 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 0 \end{pmatrix} \end{pmatrix}$$

We formally do not know but it is obvious that the rational outcomes will be (1, 1) However, the actions prescribed by (first row, second column) and (second row, first column) yield the same preferred outcomes but they cannot be distinguished, thereby creating a coordination problem between the players!

Example:
Consider a voting game with three players with the following preferences:

1. $A \precneqq B \precneqq C$
2. $B \precneqq C \precneqq A$
3. $C \precneqq A \precneqq B$

The symbol $A \precneqq B$ means that $A \preceq B$ and not $B \preceq A$. The rule is that the alternative that receives most votes will win. Yet, in case of three different votes, the alternative selected by Player 1 will win. What can we expect to be the rational outcome of the game? Let us try with elimination of dominated actions:

- Alternative A is a weakly dominant strategy for Player 1.
- Players 2 and 3 have their worst choice as weakly dominated strategy.

In order to avoid their worst outcome, Pl2 keeps B and C (ordered in rows) and Pl3 keeps C and A (ordered in columns), while Pl1 always plays A. Thus, the game reduces to a 2×2 table with the following outcomes:

	C	**A**
B	A	A
C	C	A

Since $C \precneqq A$ for both Pl2 and Pl3, they will choose the outcome in the second row and first column. Therefore the final result is C, which is the worst one for Pl1!

CHAPTER 2

Extensive form games

2.1 Tree representation

2.1.1 Voting gmae

Three politicians are supposed to decide whether to raise their salaries or not. The vote is public and in sequence. They would prefer to receive a salary increase, yet they would also like to vote against it so as not to lose public support. Optimal result for each player: having salary increase while voting against it! Main features of the game: 1 The moves take place in sequence: the politicians vote one after the other 2 Every possible situation is known to the players: at any time they know the whole past history, as well as the possible developments 3 The final outcome is determined by the majority of votes This is an example of what is called a game with perfect information: each player has knowledge about all the events that have previously occurred. How can we represent such a game? And how can we solve it? We may use a tree in which Each player's vote is represented at a branch: YES on the left and NO on the right. Utilities are based on individual votes and possible final outcomes: 1= YES and no raise; 2= NO and no raise; 3= YES and raise; 4= N0 and raise The tree in this case will become:

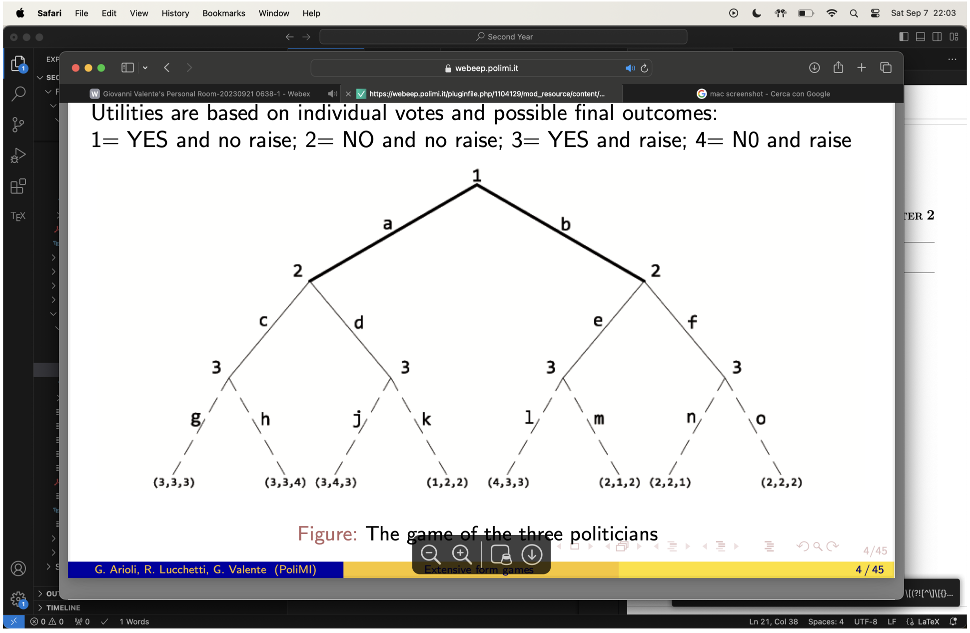

Figure 2.1: Voting game tree

2.1.2 A game with chance

Two players 1 and 2 must decide in sequence whether to play or not. If both of them decide to play, then a coin is tossed (random component R): the first player wins with heads, whereas the second one with tails.

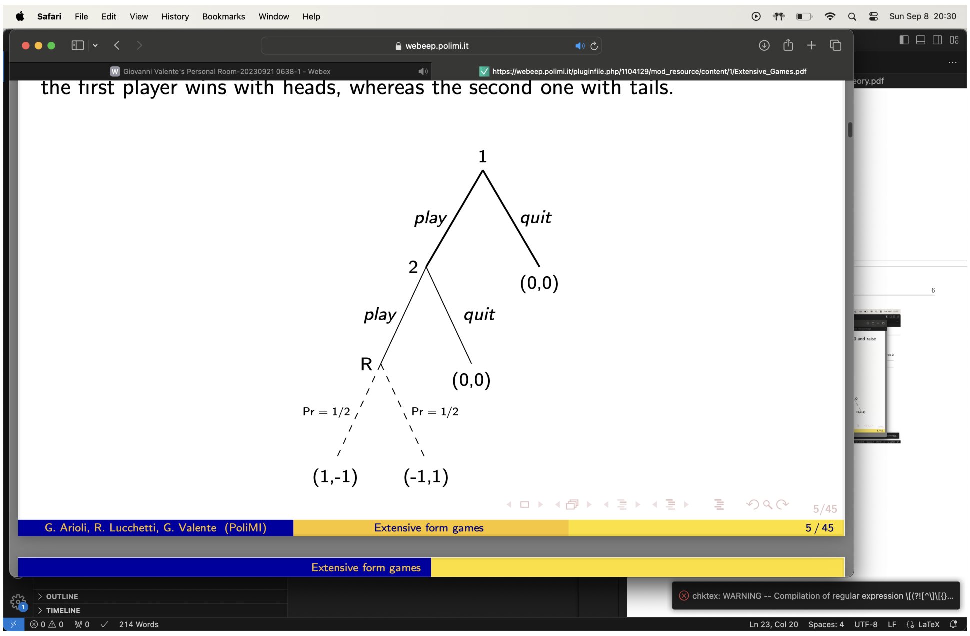

Figure 2.2: Chance game tree

2.1.3 Definitions

Definition (*Finite directed graph*)**.** A finite directed graph is a pair (V, E) where:

- V is a finite set, called the set of vertices.

- $E \subset V \times V$ is a set of ordered pairs of vertices called the set of the (directed) edges.

Definition (*Path*)**.** A path from a vertex v_1 to a vertex v_{k+1} is a finite sequence of vertices-edges $v_1, e_1, v_2, \ldots, v_k, e_k, v_{k+1}$ such that $e_i \neq e_j$ if $i \neq j$ and $e_j = (v_j, v_{j+1})$. k is called the length of the path.

Definition (*Oriented graph*)**.** An oriented graph is finite directed graph having no bidirected edges, that is for all j, k at most one between (v_j, v_k) and (v_k, v_j) may be arrows of the graph.

Definition (*Tree*)**.** A tree is a triple (V, E, x_0) where (V, E) is an oriented graph and $_x0$ is a vertex in V such that there is a unique path from x_0 to x, where x is any vertex in V.

Definition (*Child*)**.** A child of a vertex v is any vertex x such that $(v, x) \in E$. A vertex is called a leaf if it has no children. We say that the vertex x follows the vertex v if there is a path from v to x.

2.2 Extensive game

Definition (*Extensive form game with perfect information*)**.** An extensive form game with perfect information consists of:

1. A finite set $N = \{1, \ldots, n\}$ of players.
2. A game tree (V, E, x_0).
3. A partition of the vertices that are not leaves into sets $P1, P2, \ldots, Pn + 1$.
4. A probability distribution for each vertex in P_{n+1}, defined on the edges from the vertex to its children.

We have the following:

1. The set P_i , for $i \leq n$, is the set of the nodes v where Player i must choose a child of v, representing a possible move from him at v.
2. P_{n+1} is the set of the nodes where a chance move is present: that is $n + 1$ is the number of players plus the random component. P_{n+1} can be empty, meaning that the game does not admit any chance.
3. When P_{n+1} is empty, the n players have only preferences on the leaves: a utility function is not required.

2.2.1 Solving an extensive game

In order to find the optimal outcome, we employ the rationality axioms.

> **Example:**
> For the voting game described before:

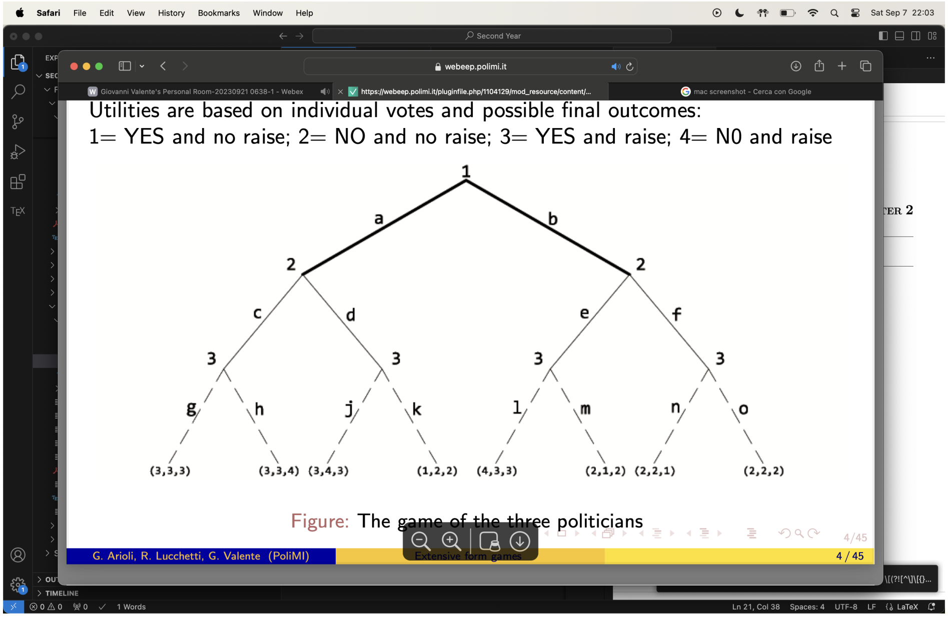

Figure 2.3: Voting game tree

We have the following:

- Player3: h = (3, 3, 4) over g = (3, 3, 3) ; j = (3, 4, 3) over k = (1, 2, 2); l = (4, 3, 3) over m= (2, 1, 2) ; o = (2, 2, 2) over n= (2, 2, 1)
- Player2: d =(3,4,3)overc =(3,3,4);e =(4,3,3)overf =(2,2,2)
- Player1: b = (4, 3, 3) over a = (3, 4, 3), thereby finding the optimal outcome

Definition (*Length*)**.** Define Length of the game as the length of the longest path in the game.

Decision theory, i.e. rationality assumption 5, enables us to solve games of length 1. Rationality assumption 4 allows us to solve a game of length $i+1$if the games of length at most i are solved. Thus, by repeated applications, we can solve games of any finite length. This method takes the name of backward induction: it is the process of reasoning backwards in time (that is from the leaves of the tree up to the root), so as to determine a sequence of actions leading one to the optimal outcome.

Theorem 2.2.1 (First rationality theorem)**.** *The rational outcomes of a finite, perfect information game are those given by the procedure of backward induction.*

The method of backwards induction can be applied since every vertex v of the game is the root of a new game, made by all followers of v in the initial game. Such a game is called a subgame of the original one.

Example:
For the second game:

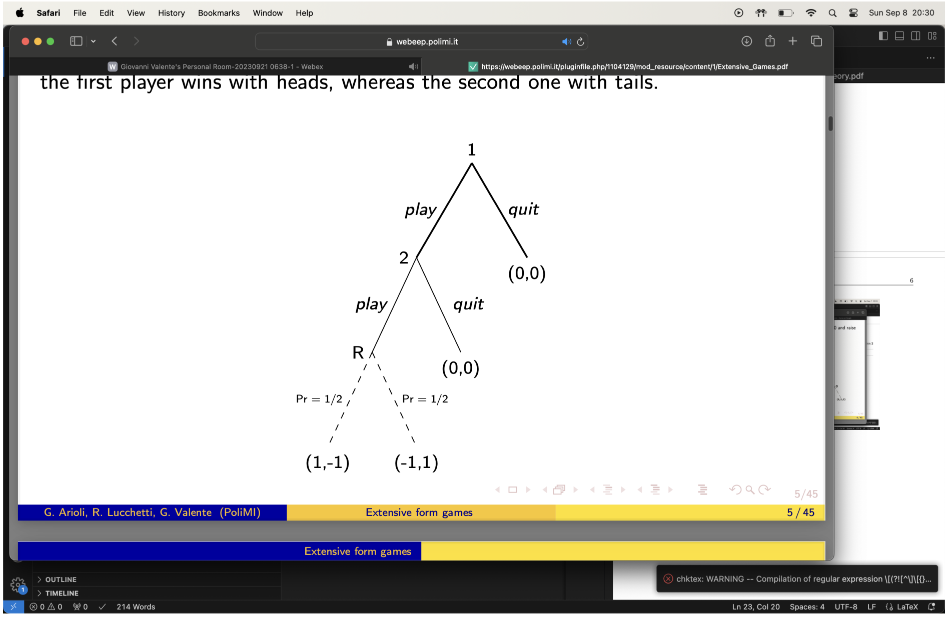

Figure 2.4: Chance game tree

The outcomes obtained by backward induction are: (4, 3) and (3, 4). In fact, Pl2 does not have any preference between (4, 3) and (0, 3)! Therefore, in general, uniqueness of solutions is not guaranteed.

2.3 The chess theorem

Theorem 2.3.1 (Von Neumann). *In the game of chess one and only one of the following alternatives holds:*

1. *The white has a way to win, no matter what the black does.*
2. *The black has a way to win, no matter what the white does.*
3. *The white has a way to force at least a draw, no matter what the black does, and the same holds for the black.*

Proof. Suppose the length of the game is $2K$ so each player has K choices to make. Call a_i the move of the White at her i-th stage and b_i the one of the Black. The first alternative in the chess theorem can be expressed as

$$\exists a_1 : \forall b_1 \exists a_2 : \forall b_2 \dots \exists a_K : \forall b_K \implies \text{white wins}$$

Now suppose this is not true. Then

$$\forall a_1 \exists b_1 : \forall a_2 : \exists b_2 : \dots \forall a_K : \exists b_K \implies \text{white does not win}$$

But this means exactly that Black has the possibility to get at least a draw. $\square$

If White does not have a strategy to win no matter what Black does, then Black has the possibility to get at least the draw. Symmetrically, if Black does not have a strategy to win no matter what White does, then White has the possibility to get at least the draw Thus if the first and the second alternatives in the chess theorem are not true, necessarily the third one is true.

Von Neumann theorem extension

The von Neumann theorem applies to every finite game of perfect information where the possible result is either the victory of one player or a tie. Thus the following corollary holds:

Corollary 2.3.1.1. *Consider a finite perfect information game with two players, where the only possible outcomes are the victory of one or the other player. Then one and only one of the following alternative holds:*

1. *The first player can win, no matter what the second one does.*
2. *The second player can win, no matter what the first one does.*

The possible solutions of a game are the follwing:

- Very weak solution: the game has a rational outcome, but it is inaccessible, like in chess.
- Weak solution: the outcome of the game is known, but how to get to it is not (in general).
- Solution: it is possible to provide an algorithm to find a solution.

Example:
Consider the game of chomp. In this game we have a grid in which each player can remove a tile with all the others on the right and above it. In this game we have a solution if the grid is a square, and a weak solution if the grid is rectangular. In the latter case we may have the following situation:

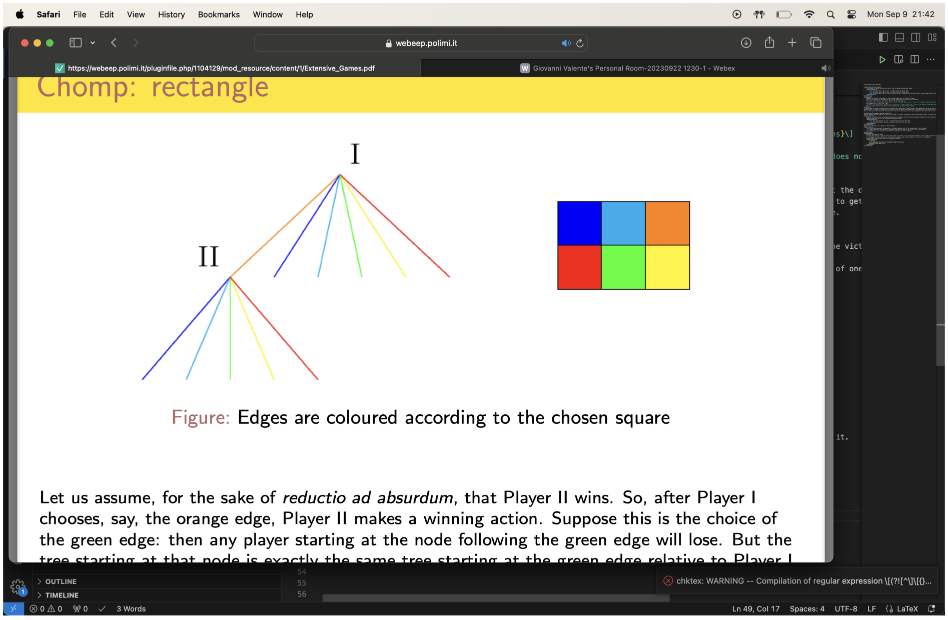

Figure 2.5: Rectangular chomp

Let us assume, for the sake of reductio ad absurdum, that Player II wins. So, after Player I chooses, say, the orange edge, Player II makes a winning action. Suppose this is the choice of the green edge: then any player starting at the node following the green edge will lose. But the tree starting at that node is exactly the same tree starting at the green edge relative to Player I. Thus Player I has a move available that guarantees victory, whereby Player II would have to lose. Since we derived a contradiction, the original assumption must be false: it is Player I that wins!