Game Theory
Theory

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Abstract

The theory begins by examining the main assumptions that distinguish decision theory from interactive decision theory. While decision theory focuses on individual decision-making in isolation, interactive decision theory explores how multiple decision-makers interact, considering each other's potential actions.

In the context of non-cooperative games, the discussion extends to games represented in extensive form, where players make decisions at various points, and games with perfect information, where all players are fully informed of prior moves. The technique of backward induction is key in solving such games. Additionally, combinatorial games are explored, emphasizing their strategic complexity.

Zero-sum games are analyzed in terms of conservative values, where each player seeks to minimize potential losses. The concept of equilibrium in pure strategies is introduced, and this is extended to mixed strategies in finite games, invoking von Neumann's theorem. Finding optimal strategies and determining the value of finite games is achieved through the use of linear programming techniques.

The Nash non-cooperative model plays a central role in understanding strategic interactions. Nash equilibrium is discussed, focusing on the existence of equilibria in both pure and mixed strategies within finite games. Examples of potential games are provided, along with methods for identifying potential functions. Notable examples include congestion games, routing games, network games, and location games. Concepts such as the price of stability, price of anarchy, and correlated equilibria are explored to analyze the efficiency and stability of these systems.

Finally, the discussion shifts to cooperative games, defining key concepts such as the core, nucleolus, Shapley value, and power indices. Examples of cooperative scenarios illustrate how these concepts help to determine fair outcomes and power distribution among players.

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CHAPTER 1

Introduction

1.1 Games

Games provide valuable models for simulating a variety of real-world situations.

Definition (*Game*)**.** A game is a structured process that includes the following components:

- A group of participants, referred to as players, with at least two members.
- An initial state or starting condition.
- A set of rules that define how players can act.
- A range of possible outcomes or end states.
- The preferences of each player concerning these potential outcomes.

1.2 Game Theory Assumptions

Game theory operates under the following key assumptions about the players involved:

1. *Self-interested.*
2. *Rational.*

1.2.1 Self-interest

Players are assumed to focus solely on their own preferences concerning the outcomes of the game. This is a mathematical assumption, not an ethical judgment. In fact, it is essential for defining what constitutes a rational choice within the framework of game theory.

1.2.2 Rationality

Definition (*Preference relation*)**.** Let X be a set. A preference relation on X is a binary relation $\preceq$ that satisfies the following properties for all $x, y, z \in X$:

- *Reflexive*: $x \preceq x$ (every element is at least as preferred as itself).
- *Complete*: $x \preceq y$ or $y \preceq x$ (any two elements can be compared).
- *Transitive*: if $x \preceq y$ and $y \preceq z$, then $x \preceq z$ (preferences are consistent across comparisons).

The transitive property ensures that preferences can be consistently ranked.

Definition (*Utility function*)**.** Given a preference relation $\preceq$ over a set X, a utility function representing $\preceq$ is a function $u : X \to \mathbb{R}$ such that:

$$u(x) \geq u(y) \Leftrightarrow x \preceq y$$

While a utility function may not always exist in specific cases, it does exist in general settings, particularly when X is finite. If a utility function does exist, there are infinitely many such functions, differing by any strictly increasing transformation of the original function.

Each player i is assigned a set X_i, representing all the choices available to them. Therefore, the set $X = xX_i$ over which the utility function u is defined represents the combined choices of all players.

Rationality assumptions The following assumptions define the rational behavior of players:

1. *Consistent preferences*: players can establish a preference relation over the game's outcomes, and this ordering is consistent.
2. *Utility representation*: players can define a utility function that represents their preference relations when needed.
3. *Consistent use of probability*: players apply the laws of probability consistently, including computing expected utilities and updating probabilities according to Bayes' rule.
4. *Understanding consequences*: players comprehend the outcomes of their actions, the impact on other players, and the resulting chain of consequences.
5. *Application of decision theory*: players use decision theory to maximize their utility. Given a set of alternatives X and a utility function u, each player seeks $\bar{x} \in X$ such that:

$$u(\bar{x}) \geq u(x) \qquad \forall x \in X$$

One significant consequence of these axioms is the principle of eliminating strictly dominated strategies: a player will not choose an action a if there exists another action b that yields a strictly better outcome, regardless of the actions of other players.

Example:
Consider the following games:

Gain	**Probability**
2500	33%
2400	66%
0	1%

Table 1.1: Game A

Gain	Probability
2500	0%
2400	100%
0	0%

Table 1.2: Game B

In a sample of 72 participants, 82% chose to play Game B, indicating a preference for certainty—characteristic of risk-averse individuals. According to expected utility theory, this decision is rational if:

$$u(2400) > \frac{33}{100}u(2500) + \frac{66}{100}u(2400)$$

This simplifies to:

$$\frac{34}{100}u(2400) > \frac{33}{100}u(2500)$$

Now consider the following alternatives:

Gain	Probability
2500	33%
0	67%

Table 1.3: Game C

Gain	Probability
2400	34%
0	66%

Table 1.4: Game D

In this new setup, 83% of participants preferred Game C, reflecting a preference for a larger gain even with a lower probability of success. Rationality in this scenario requires:

$$\frac{34}{100}u(2400) < \frac{33}{100}u(2500)$$

This contradicts the earlier experiment, where the opposite preference was observed. Such behavior violates the independence axiom in expected utility theory, which states that consistent preferences should hold under similar probabilistic transformations.

This contradiction is known as the Allais Paradox, demonstrating that individuals do not always act as fully rational decision-makers.

Example:
A group of players is asked to choose an integer between 1 and 100. The mean of all chosen numbers, M is then calculated. The objective of the game is to select the number closest to qM, where $0 < q < 1$.

A purely rational player would conclude that the optimal number to choose is 1, regardless of the value of q. However, this player is likely to lose.

For example, let $q = \frac{1}{2}$. Since $M \leq 100$, in the first step, it seems irrational to choose a number greater than $\frac{1}{2} \cdot 100$, as this is the initial target value based on the game's rules.

However, in the second step, assuming all players are rational and recognize that others are also rational, each player would realize that others will also choose numbers below 50. Therefore, the new logical step would be to pick a number less than $\left(\frac{1}{2}\right)^2 \cdot 100$.

This reasoning continues iteratively: at step n, it becomes irrational to choose a number greater than $\left(\frac{1}{2}\right)^n \cdot 100$. Ultimately, after enough steps, the only rational choice would appear to be selecting the smallest possible number (1).

Despite this reasoning, experiments show that the actual winning number is far higher than 1. In fact, the winning number tends to increase as the value of q increases, revealing that real-life behavior often deviates from purely rational game theory predictions.

1.3 Bimatrices

Conventionally, Player 1 selects a row, while Player 2 selects a column. This results in a pair of values that represent the utilities for Player 1 and Player 2, respectively. These options can be conveniently displayed in a bimatrix.

Example:
Consider the following bimatrix:

$$\begin{pmatrix} \begin{pmatrix} 8 & 8 \end{pmatrix} & \begin{pmatrix} 2 & 7 \end{pmatrix} \\ \begin{pmatrix} 7 & 2 \end{pmatrix} & \begin{pmatrix} 0 & 0 \end{pmatrix} \end{pmatrix}$$

In this example, Player 1's utilities are given by:

$$\begin{pmatrix} 8 & 2 \\ 7 & 0 \end{pmatrix}$$

Since the second row is strictly dominated by the first (i.e., Player 1's utility in the first row is higher for any choice by Player 2), Player 1 will rationally choose the first row. Similarly, Player 2 will select the first column, as it strictly dominates the second column.

While the principle of eliminating strictly dominated strategies may seem simplistic, it can lead to surprisingly powerful insights and outcomes.

Example:
Consider the following two games:

$$\begin{pmatrix} \begin{pmatrix} 10 & 10 \end{pmatrix} & \begin{pmatrix} 3 & 15 \end{pmatrix} \\ \begin{pmatrix} 15 & 3 \end{pmatrix} & \begin{pmatrix} 5 & 5 \end{pmatrix} \end{pmatrix}$$

$$\begin{pmatrix} \begin{pmatrix} 8 & 8 \end{pmatrix} & \begin{pmatrix} 2 & 7 \end{pmatrix} \\ \begin{pmatrix} 7 & 2 \end{pmatrix} & \begin{pmatrix} 0 & 0 \end{pmatrix} \end{pmatrix}$$

Note that in the first game, players have outcomes like $\begin{pmatrix} 10 & 10 \end{pmatrix}$ and $\begin{pmatrix} 15 & 3 \end{pmatrix}$, which individually seem to offer higher utilities than most outcomes in the second game. However, applying rational decision-making principles leads to a surprising result.

According to the principle of elimination of dominated strategies, players will end up choosing the outcome pair $\begin{pmatrix} 8 & 8 \end{pmatrix}$ in the second game because it dominates other available outcomes. This leads them to prefer the second game over the first game, despite the fact that the first game contains outcomes with higher individual utilities, like $\begin{pmatrix}10 & 10\end{pmatrix}$ and $\begin{pmatrix}15 & 3\end{pmatrix}$.

Now, consider the expanded form of the first game, which contains even more outcomes:

$$\begin{pmatrix} \begin{pmatrix}1 & 1\end{pmatrix} & \begin{pmatrix}11 & 0\end{pmatrix} & \begin{pmatrix}4 & 0\end{pmatrix} \\ \begin{pmatrix}0 & 11\end{pmatrix} & \begin{pmatrix}8 & 8\end{pmatrix} & \begin{pmatrix}2 & 7\end{pmatrix} \\ \begin{pmatrix}0 & 4\end{pmatrix} & \begin{pmatrix}7 & 2\end{pmatrix} & \begin{pmatrix}0 & 0\end{pmatrix} \end{pmatrix}$$

This expanded version of the first game includes all the outcomes from the second game, plus some additional options. However, rationality axioms suggest that in the first game, players should choose the outcome $\begin{pmatrix}10 & 10\end{pmatrix}$, which dominates the other possibilities.

Interestingly, in the second game, where fewer options are available, the players end up selecting $\begin{pmatrix}8 & 8\end{pmatrix}$. This leads to a paradoxical outcome: having fewer available actions can actually make players better off by simplifying the decision-making process and avoiding suboptimal choices.

Example:
Consider the rational outcomes of the following game.

$$\begin{pmatrix} \begin{pmatrix}0 & 0\end{pmatrix} & \begin{pmatrix}1 & 1\end{pmatrix} \\ \begin{pmatrix}1 & 1\end{pmatrix} & \begin{pmatrix}0 & 0\end{pmatrix} \end{pmatrix}$$

While we may not know the rational outcomes formally, it is clear that the preferred outcome for both players is $\begin{pmatrix}1 & 1\end{pmatrix}$. However, this leads to a coordination problem.

Both pairs of actions result in the same outcome $\begin{pmatrix}1 & 1\end{pmatrix}$, but there is no clear way for the players to distinguish between these two strategies. As a result, while the rational outcome is obvious, the players face difficulty coordinating on which specific actions to take to achieve it.

Example:
Consider a voting game with three players, each having the following preferences:

1. Player 1: $A \precneqq B \precneqq C$
2. Player 2: $B \precneqq C \precneqq A$
3. Player 3: $C \precneqq A \precneqq B$

Here, the notation $A \precneqq B$ indicates that Player 1 prefers $B \preceq A$, but not vice versa. The winner is determined by the alternative that receives the most votes. However, if there is a tie among three different votes, the alternative chosen by Player 1 will win.

Let's now analyse the rational outcome of the game through the elimination of dominated actions:

- Alternative A is a weakly dominant strategy for Player 1.
- Players 2 and 3 have their least preferred choice as a weakly dominated strategy.

To avoid their worst outcome, Player 2 retains options B and C (ordered in rows), while Player 3 keeps C and A (ordered in columns). Player 1 will consistently choose A. This simplifies the game to a 2×2 table with the following outcomes:

	C	**A**
B	A	A
C	C	A

Since $C \precneqq A$ for both Players 2 and 3, they will choose the outcome in the second row and first column, leading to the final result being C, which is the worst outcome for Player 1.

Example:
Consider the classic Chicken Game. Two cars are driving toward each other on a narrow road, and there isn't enough room for both cars to pass. If both cars keep going straight, they will crash, but if at least one deviates, the crash is avoided. The payoff bimatrix for the game is:

$$\begin{pmatrix} \begin{pmatrix} -1 & -1 \end{pmatrix} & \begin{pmatrix} 1 & 10 \end{pmatrix} \\ \begin{pmatrix} 10 & 1 \end{pmatrix} & \begin{pmatrix} -10 & -10 \end{pmatrix} \end{pmatrix}$$

The payoffs are as follows:

- Both players receive a utility of -1 if they both deviate.
- A player earns 10 for going straight when the other deviates, while the deviating player receives 1.
- If both go straight and crash, they both receive -10.

The best outcomes are either $\begin{pmatrix} 10 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 10 \end{pmatrix}$. However, there is no way to decisively determine which of these two outcomes will occur, as both are equally viable equilibria.

CHAPTER 2

Extensive form games

2.1 Games tree

Example:
Three politicians are tasked with deciding whether to raise their salaries. The voting is public and happens sequentially. Each politician prefers a salary increase but also wants to vote against it to maintain public support.

The optimal outcome for each politician is to receive a salary increase while voting against it. The game has the following characteristics:

1. The voting process is sequential, with politicians voting one after another.
2. Every possible situation is fully known to all players: they are aware of the entire history and all possible future actions.
3. The final outcome is determined by the majority vote.

To represent such a game we can use a tree, where each branch represents a player's vote: a YES vote goes left, and a NO vote goes right. The utilities depend on each player's vote and the final outcome:

1. YES, but no raise.
2. NO, and no raise.
3. YES, and a raise.
4. NO, and a raise.

The corresponding decision tree looks like this:

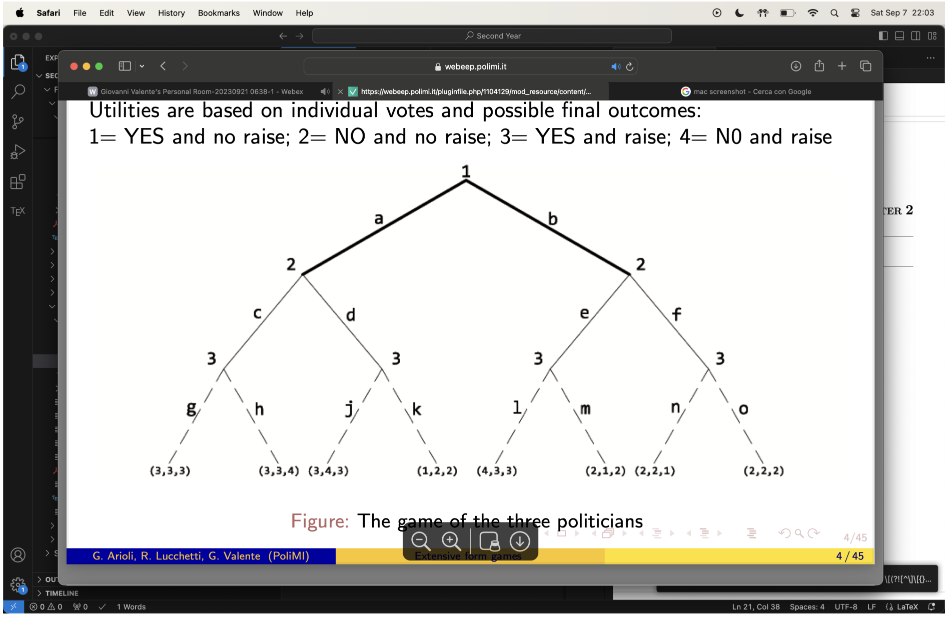


Figure 2.1: Voting game tree

This is an example of a game with perfect information, where each player is fully aware of all prior events.

Example:
Two players, Player 1 and Player 2, must decide sequentially whether to participate in a game. If both choose to play, a coin is flipped (random component R): Player 1 wins if it lands heads, and Player 2 wins if it lands tails. The corresponding decision tree is as follows:

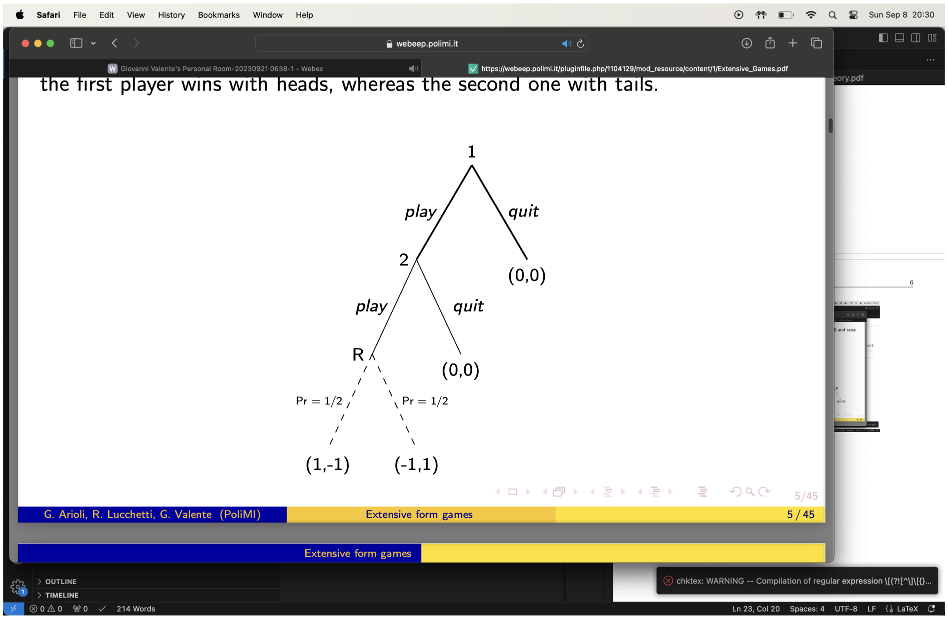


Figure 2.2: Chance game tree

Definition (*Finite directed graph*)**.** A finite directed graph is a pair (V, E) where:

- V is a finite set, called the set of vertices.
- $E \subset V \times V$ is a set of ordered pairs of vertices, called the directed edges.

Definition (*Path*)**.** A path from a vertex v_1 to a vertex v_{k+1} is a finite sequence of vertices and edges $v_1, e_1, v_2, \ldots, v_k, e_k, v_{k+1}$ such that $e_i \neq e_j$ if $i \neq j$ and $e_j = (v_j, v_{j+1})$.

The number k is called the length of the path.

Definition (*Oriented graph*)**.** An oriented graph is a finite directed graph with no bidirectional edges. That is, for all vertices v_j and v_k, at most one of (v_j, v_k) and (v_k, v_j) can be an edge in the graph.

Definition (*Tree*)**.** A tree is a triple (V, E, x_0) where (V, E) is an oriented graph, and x_0 is a vertex in V such that there is a unique path from x_0 to any other vertex $x \in V$.

Definition (*Child*)**.** A child of a vertex v is any vertex x such that $(v, x) \in E$.

Definition (*Leaf*)**.** A vertex is called a leaf if it has no children.

We say that the vertex x follows the vertex v if there is a path from v to x.

2.2 Extensive games

Definition (*Extensive form game with perfect information*)**.** An extensive form game with perfect information consists of:

1. A finite set $N = \{1, \ldots, n\}$ of players.
2. A game tree (V, E, x_0).
3. A partition of the non-leaf vertices into sets $P1, P2, \ldots, Pn + 1$.
4. A probability distribution for each vertex in P_{n+1}, defined on the edges from that vertex to its children.

The game includes the following:

1. The set P_i, for $i \leq n$, consists of nodes where Player i must choose a child of v, representing a possible move by that player.
2. P_{n+1} is the set of nodes where chance plays a role (i.e., random events). Here, $n+1$ refers to the players plus the random component. P_{n+1} can be empty, indicating no random events in the game.
3. When P_{n+1} is empty, the n players only have preferences over the leaves, meaning no random component influences the outcome, so a utility function isn't required.

2.2.1 Extensive game solution

To determine the optimal outcome, we apply the axioms of rationality.

> **Example:**
> For the voting game described earlier:

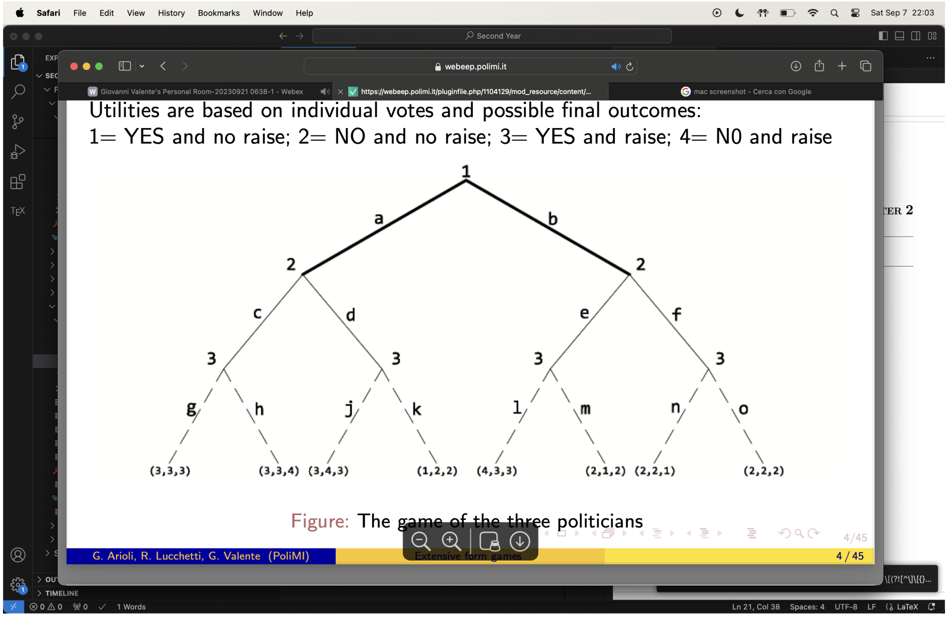


Figure 2.3: Voting game tree

We can identify the preferences as follows:

- For Player 3:
  - $h = \begin{pmatrix} 3 & 3 & 4 \end{pmatrix}$ over $g = \begin{pmatrix} 3 & 3 & 3 \end{pmatrix}$.
  - $j = \begin{pmatrix} 3 & 4 & 3 \end{pmatrix}$ over $k = \begin{pmatrix} 1 & 2 & 2 \end{pmatrix}$.
  - $l = \begin{pmatrix} 4 & 3 & 3 \end{pmatrix}$ over $m = \begin{pmatrix} 2 & 1 & 2 \end{pmatrix}$.
  - $o = \begin{pmatrix} 2 & 2 & 2 \end{pmatrix}$ over $n = \begin{pmatrix} 2 & 2 & 1 \end{pmatrix}$.
- For Player 2:
  - $d = \begin{pmatrix} 3 & 4 & 3 \end{pmatrix}$ over $c = \begin{pmatrix} 3 & 3 & 4 \end{pmatrix}$.
  - $e = \begin{pmatrix} 4 & 3 & 3 \end{pmatrix}$ over $f = \begin{pmatrix} 2 & 2 & 2 \end{pmatrix}$.
- For Player 1:
  - $b = \begin{pmatrix} 4 & 3 & 3 \end{pmatrix}$ over $a = \begin{pmatrix} 3 & 4 & 3 \end{pmatrix}$.

Thus, the optimal outcome is found.

Definition (*Length*)**.** The length of a game is defined as the length of the longest path in the game tree.

Using decision theory and the assumption of rationality:

- Rationality assumption 5 allows us to solve games of length 1.
- Rationality assumption 4 allows us to solve games of length $i + 1$ if all games of length at most i have already been solved.

This iterative process is called backward induction, where we work backwards from the leaves of the tree to the root to determine the optimal sequence of actions.

Theorem 2.2.1 (First rationality theorem)**.** *The rational outcomes of a finite game with perfect information are those determined by the backward induction procedure.*

Backward induction can be applied because every vertex v in the game is the root of a subgame consisting of all the vertices that follow v. These subgames are derived from the original game.

Example:
For the chance game:

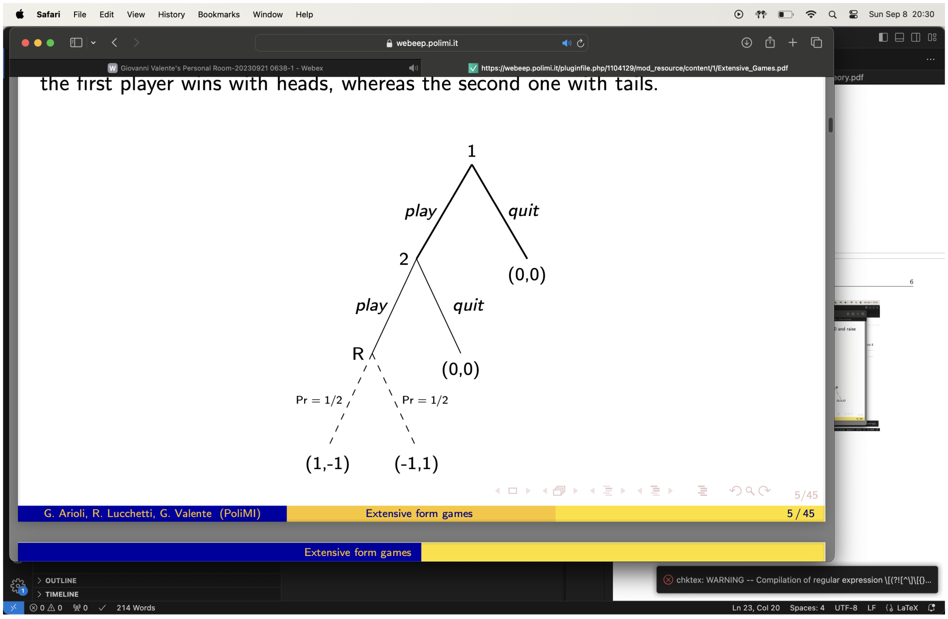


Figure 2.4: Chance game tree

The outcomes obtained through backward induction are $\begin{pmatrix}4 & 3\end{pmatrix}$ and $\begin{pmatrix}3 & 4\end{pmatrix}$. Player 2 has no strict preference between $\begin{pmatrix}4 & 3\end{pmatrix}$ and $\begin{pmatrix}0 & 3\end{pmatrix}$, indicating that in general, the solutions may not be unique.

2.3 Chess theorem

Theorem 2.3.1 (Von Neumann)**.** *In the game of chess, one and only one of the following alternatives holds:*

1. *The White has a way to win, no matter what the Black does.*
2. *White has a strategy to guarantee a win, regardless of what Black does.*
3. *Both White and Black can force at least a draw, regardless of the opponent's actions.*

Proof. Assume the game has a finite length of $2K$ moves, where each player makes K moves. Let a_i represent White's move at their i-th stage, and b_i represent Black's corresponding move.

The first possibility in the theorem can be formulated as follows:

$$\exists a_1 : \forall b_1 \exists a_2 : \forall b_2 \dots \exists a_K : \forall b_K \implies \text{white wins}$$

Now, suppose this is false. Then the negation is:

$$\forall a_1 \exists b_1 : \forall a_2 : \exists b_2 : \ldots \forall a_K : \exists b_K \implies \text{white does not win}$$

This means Black has the possibility to prevent White from winning, ensuring at least a draw.

If White does not have a winning strategy, then Black can secure at least a draw. Similarly, if Black does not have a winning strategy, then White can secure at least a draw. Therefore, if neither of the first two possibilities holds, the third one must be true. □

2.3.1 Extension

Von Neumann's theorem can be extended to any finite game of perfect information where the possible outcomes are either a win for one player or a tie.

Corollary 2.3.1.1. *Consider a finite, perfect information game with two players, where the only possible outcomes are a win for one of the players or a tie. Then, exactly one of the following holds:*

1. *Player 1 has a winning strategy, no matter what the second player does.*

2. *Player 2 has a winning strategy, no matter what the second player does.*

The possible solutions for a game are classified as follows:

- *Very weak solution*: the game has a rational outcome, but it is not accessible in practice, as with chess.

- *Weak solution*: the outcome of the game is known, but the method to achieve it is not generally understood.

- *Solution*: there exists an algorithm that can determine the outcome.

Example:
In the game of Chomp, players take turns removing tiles from a rectangular grid, where removing a tile also removes all tiles to its right and above. The game ends when a player is unable to make a move, and the last player to play wins.

In this scenario, we can distinguish between two types of solutions: a definite solution occurs when the grid is square, while a weak solution arises with a rectangular grid. Let's consider a specific configuration, illustrated in the provided figure.

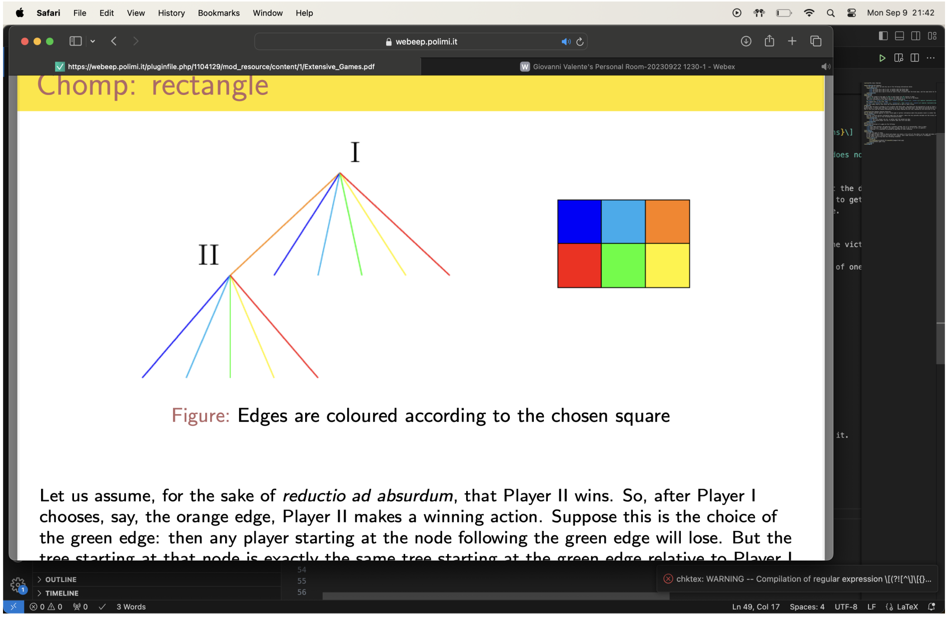


Figure 2.5: Rectangular chomp

To explore the outcome of the game, let's assume for contradiction that Player 2 is the winner. Suppose Player 1 chooses a particular edge (represented in orange), and then Player 2 makes a winning move by selecting another edge (the green edge). If we examine the game state following this move, any player starting from the node that follows the green edge is destined to lose.

However, this situation mirrors the tree of outcomes that begins at the green edge from Player 1's perspective. This means that Player 1 has a move that guarantees a victory against Player 2's position. Hence, the assumption that Player 2 can win leads to a contradiction, implying that Player 1 must be the one who wins the game.

Thus, we conclude that Player 1 has a winning strategy in this instance of the game of Chomp.

2.4 Impartial combinatorial games

Definition (*Impartial combinatorial game*)**.** An impartial combinatorial game is defined by the following characteristics:

1. There are two players who alternate turns.
2. The game consists of a finite number of positions.
3. Both players adhere to the same set of rules.
4. The game concludes when no further moves can be made.
5. The outcome of the game is not influenced by chance.
6. In the classical version of the game, the winner is the player who leaves the opponent with no available moves; in the misère version, the objective is reversed.

Example:
Several examples illustrate impartial combinatorial games:

- k piles of cards: each player, on their turn, can take any number of cards (at least one) from a single pile.
- k piles of cards with restrictions: each player can take any number of cards (at least one) from no more than $j < k$ piles during their turn.
- k cards in a row: players can take j_l cards on their turn.

In all these variations, the player who is left without cards loses. In the first two examples, the positions can be represented as $(n_1, \cdots, n_k)$, where each n_i is a non-negative integer corresponding to the number of cards in each pile. In the third example, the positions are characterized by all non-negative integers less than or equal to k.

To solve impartial combinatorial games, we begin by partitioning the set of all possible positions (which are finite in number) into two distinct categories:

1. P-positions: these are positions where the previous player has a winning strategy, meaning they are losing positions for the player who is about to move.
2. N-positions: these are positions where the next player has a winning strategy, indicating they are winning positions for the player who is about to move.

It is important to note that the current state of the game is what matters, rather than which player is designated to move.

Partition rules The rules for the partitions are:

- The terminal position $(0, 0, \cdots, 0)$ is classified as a P-position. This is a losing position because the player has no cards left to play.
- From any P-position, only N-positions can be reached. This means that the next player is guaranteed to have a winning strategy.
- From any N-position it is possible (but not obligatory) to move to a P-position. The player in an N-position can make a move that leads their opponent to a losing position.

Therefore, the player who starts from an N-position is assured of a victory, given that they play optimally.

2.4.1 Nim game

The Nim game is characterized by a tuple $(n_1, \cdots, n_k)$, where each n_i is a positive integer. During their turn, each player must choose one pile n_i and replace it with $\hat{n}_i$, ensuring that $\hat{n}_i < n_i$. The player who reduces the position to $(0, \ldots, 0)$ wins.

Therefore, each player's action involves removing cards from a single pile with the objective of clearing the entire table.

Theorem 2.4.1 (Bouton)**.** *A position $(n_1, n_2, \ldots, n_k)$ in the Nim game is a P-position if and only if:*

$$n_1 \oplus n_2 \oplus \cdots \oplus n_k = 0$$

Proof. The terminal position $(0, 0, \dots, 0)$ is a P-position corresponding to a Nim-sum of zero.

If the Nim-sum $n_1 \oplus n_2 \oplus \dots \oplus n_k = 0$, any subsequent position will have a non-zero Nim-sum. Assume the next position is $(\hat{n}_1, n_2, \dots, n_k)$ such that $\hat{n}_1 \oplus n_2 \oplus \dots \oplus n_k = 0$. Then we would have:

$$n_1 \oplus n_2 \oplus \dots \oplus n_k = 0$$

Which, by the cancellation law, implies $\hat{n}_1 = n_1$. This is a contradiction, as the game rules stipulate that $\hat{n}_1 < n_1$.

Conversely, if $n_1 \oplus n_2 \oplus \dots \oplus n_k \neq 0$, it is possible to move to a position with a zero Nim-sum. Let $z = n_1 \oplus n_2 \oplus \dots \oplus n_k \neq 0$. Identify a pile where the binary representation of z has a 1 in its leftmost column. Change that digit to 0 and adjust the digits to the right, leaving unchanged the digits that correspond to 0. This operation produces a new number that is smaller than the original. □

Example:
Consider the configuration with a non-zero Nim-sum: $4, 6, 5$. By removing a card from the first pile, we can reach the configuration $3, 6, 5$, which has a zero Nim-sum. Thus, there are three initial advantageous moves, one available for each pile.

2.4.2 Conclusions

Games with perfect information can typically be resolved through backward induction. However, this method is primarily effective for relatively simple games due to the constraints of limited rationality. Depending on the specifics of the game, we may arrive at varying degrees of solutions.

2.5 Strategies

In backward induction, a specific move must be identified at every node. Let P_i denote the set of all nodes at which player i is required to make a decision.

Definition (*Pure strategy*)**.** A pure strategy for player i is defined as a function on the set P_i, which associates each node v in P_i with a child node x, or equivalently, an edge (v, x).

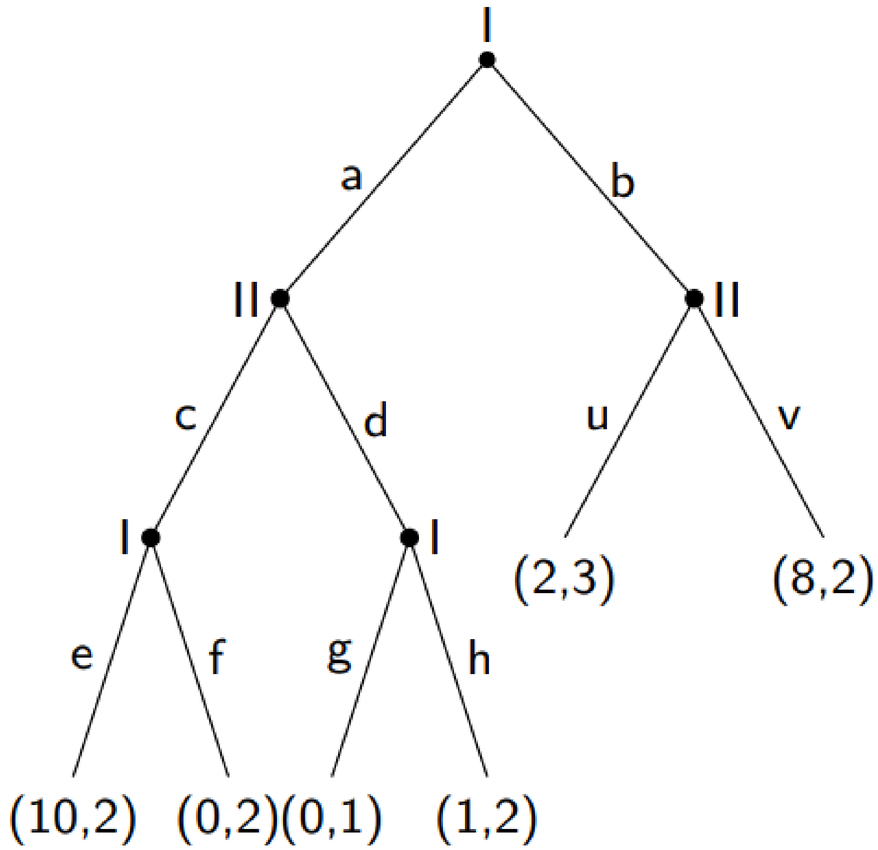
Definition (*Mixed strategy*)**.** A mixed strategy refers to a probability distribution over the set of pure strategies.

When a player possesses n pure strategies, the collection of their mixed strategies is represented as:

$$\sum_n = \left\{ p = (p_1, \dots, p_n) | p_i \geq 0 \text{ and } \sum p_i = 1 \right\}$$

Here, $\sum_n$ forms the fundamental simplex in n-dimensional space.

Example:
Consider the following tree:



The strategies depicted in the tree are shown below:

	cu	cv	du	dv
aeg	(10,2)	(10,2)	(0,1)	(0,1)
aeh	(10,2)	(10,2)	(1,2)	(1,2)
afg	(0,2)	(0,2)	(0,1)	(0,1)
afh	(0,2)	(0,2)	(1,2)	(1,2)
beg	(2,3)	(8,2)	(2,3)	(8,2)
beh	(2,3)	(8,2)	(2,3)	(8,2)
bfg	(2,3)	(8,2)	(2,3)	(8,2)
bfh	(2,3)	(8,2)	(2,3)	(8,2)

Note that Player 1's strategies are listed in the rows, while Player 2's are in the columns. All combinations are included, even if they are equivalent (e.g., strategies b– for Player 2). The table may contain repeated pairs, as different strategies can lead to the same outcomes.

- *Extensive form*: the various moves of the players are presented sequentially.

- *Strategic form*: all players' strategies are presented simultaneously.

Theorem 2.5.1 (Von Neumann on strategies)**.** *In the game of chess, one of the following scenarios must hold:*

1. *White has a winning strategy.*
2. *Black has a winning strategy.*
3. *Both players possess a strategy that guarantees at least a tie.*

The first outcome occurs when there exists a row containing all winning elements. The second outcome arises when there is a column consisting of all winning elements. The third outcome features mixed results, including ties, but does not encompass all three outcomes in a single row or column.

If $P_i = \{v_1, \ldots, v_k\}$ and v_j has n_j children, then the total number of strategies available to Player i is $n_1 \cdot n_2 \cdot \cdots \cdot n_k$. This illustrates that the number of strategies, even in short games, is typically quite substantial.

Example:
In the game of Tic-Tac-Toe, if the game is halted after three moves, the first player has $9 \cdot 7^{(8\times 9)}$ strategies available (not accounting for symmetrical configurations).

2.6 Games with imperfect information

In certain scenarios, players must make their moves simultaneously, which prevents them from having complete knowledge of each other's actions. This situation can still be represented using a game tree.

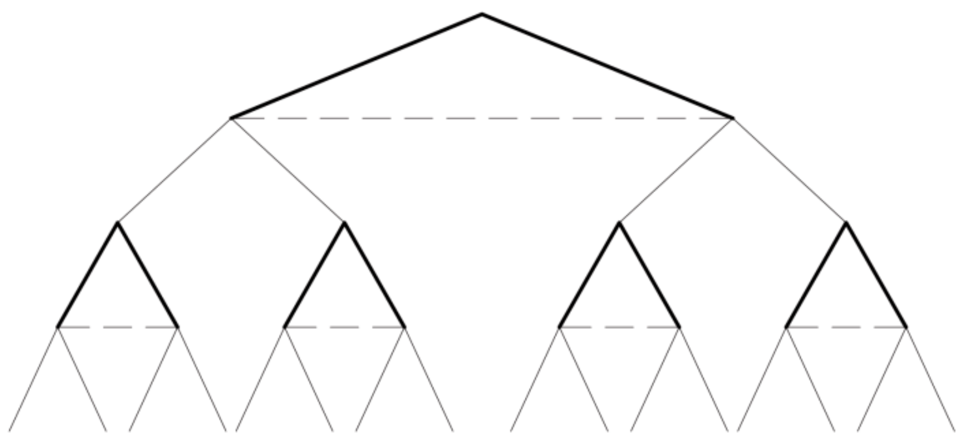


Figure 2.6: Tree with imperfect information

The dashed lines in the figure indicate that a player does not know exactly which vertex they occupy.

Definition (*Information set*)**.** An information set for Player i is a pair $(U_i, A(U_i))$ satisfying the following conditions:

1. $U_i \subset P_i$ is a non-empty set of vertices $v_1, \cdots, v_k$.

2. Each vertex $v_j \in U_i$ has the same number of children.

3. $A_i(U_i)$ is a partition of the children of $v_1 \cup \cdots \cup v_k$ such that each element of the partition contains exactly one child from each vertex v_j.

Thus, Player i knows they are in U_i but cannot determine the exact vertex. The partition defines the choice function, indicating that each set in $A_i(U_i)$ corresponds to an available move for the player (graphically, this represents the same choice, or edge, emanating from different vertices).

Definition (*Extensive form game with imperfect information*)**.** An extensive form game with imperfect information is characterized by the following components:

1. A finite set $N = \{1, \ldots, n\}$ of players.

2. A game tree (V, E, x_0).

3. A partition comprising sets $P_1, P_2, \ldots, P_{n+1}$ of the non-leaf vertices.

4. A partition $(U_i^j), j = 1, \ldots, ki$ of the set P_i, for all i, with (U_i^j, A_i^j) being the information set for all players i at all vertices j (having the same number of children).

5. A probability distribution defined for each vertex in P_{n+1} on the edges leading to its children.

6. An n-dimensional vector assigned to each leaf.

It is important to note that if the partition consists of only a single vertex, then a game with imperfect information effectively becomes a game with perfect information.

Definition (*Pure strategy*)**.** A pure strategy for player i in an imperfect information game is a function defined over the collection $\mathcal{U}$ of their information sets, assigning to each $U_i \in \mathcal{U}$ an element from the partition $A(U_i)$.

Definition (*Mixed strategy*)**.** A mixed strategy is defined as a probability distribution over the pure strategies.

A game of perfect information is a specific type of imperfect information game where all information sets for all players are singletons.

CHAPTER 3

Zero sum games

3.1 Introduction

Definition (*Zero sum game*)**.** A two player zero sum game in strategic form is the triplet $(X, Y, f : X \times Y \to \mathbb{R})$. Here:

- X is the strategy space of Player 1.
- Y the strategy space of Player 2.
- $f(x, y)$ is what Player 1 gets from Player 2, when they play x, y respectively.

Given that f is the utility function of Player 1, by definition of zero sum games the utility function g of Player 2 must be:

$$g = -f$$

In the finite case $X = \{1, 2, \ldots, n\}$, $Y = \{1, 2, \ldots, m\}$ the game is described by a payoff matrix P, wherein Player 1 selects row i while Player 2 selects column j:

$$\begin{pmatrix} p_{11} & \cdots & p_{1m} \\ \cdots & p_{ij} & \cdots \\ p_{n1} & \cdots & p_{nm} \end{pmatrix}$$

Here, p_{ij} is the payment of Player 2 to Player 1 when they play i, j respectively. In order to choose the optimal strategy, each player can reason as follows:

- Player 1 can guarantee herself to get at least $v_1 = \max_i \min_j p_{ij}$.
- Player 2 can guarantee himself to pay at most $v_2 = \min_j max_i p_{ij}$.

v_1 and v_2 are said to be the conservative values of Player 1 and Player 2, respectively.

Example:
Consider the game:

$$\begin{pmatrix} 4 & 3 & 1 \\ 7 & 5 & 8 \\ 8 & 2 & 0 \end{pmatrix}$$

For the player 1 we pick the minimum for each row, that is : $\begin{pmatrix} 1 & 5 & 0 \end{pmatrix}$ and then we choose the maximum between them. Thus, the conservative value for the player 1 is $v_1 = 5$.

For the player 2 we pick the maximum for each column, that is : $\begin{pmatrix} 8 & 5 & 8 \end{pmatrix}$ and then we choose the mimimum between them. Thus, the conservative value for the player 2 is $v_2 = 5$.

Accordingly, the rational outcome is 5 and the rational behavior is $(\bar{i} = 2, \bar{j} = 2)$.

3.1.1 Rationality in zero sum games

Let us suppose the following:

- $v_1 = v_2 := v$,
- $\bar{i}$ the row such that $p_{\bar{i}\bar{j}} \geq v_1 = v$ for all j
- $\bar{j}$ the column such that $p_{\bar{i}\bar{j}} \leq v_2 = v$ for all i

Then $p_{\bar{i}\bar{j}} = v$ and $p_{\bar{i}\bar{j}} = v$ is the rational outcome of the game

$\bar{i}$ is an optimal strategy for Pl1, because she cannot get more than v_2, since v_2 is the conservative value of the second player $\bar{j}$ is an optimal strategy for Pl2, because he cannot pay less than v_1, since v_1 is the conservative value of the first player $\bar{i}$ maximizes the function $\alpha(i) = minjp_{ij}$; $\bar{j}$ minimizes the function $\beta(j) = \max_i p_{ij}$.

3.1.2 Extension to arbitrary games

Let the triplet $(X, Y, f : X \times Y \to \mathbb{R})$ represent a game between two players, wherein their respective strategy spaces X and Y may not be finite sets. For the sake of rational behaviour, the players can guarantee to themselves the following outcomes:

- Player 1: $v_1 = \sup_x \inf_y f(x, y)$
- Player 2: $v_2 = \inf_y \sup_x f(x, y)$

The outcomes v_1, v_2 are the conservative values of the players. If $v_1 = v_2$, we set $v = v_1 = v_2$ and we say that the game has value v.

Optimality Let X and Y be arbitrary sets. Suppose:

1. $v_1 = v_2 := v$
2. there exists strategy $\bar{x}$ such that $f(\bar{x}, y) \geq v$ for all $y \in Y$
3. there exists strategy $\bar{y}$ such that $f(x, \bar{y}) \leq v$ for all $x \in X$

(the last two conditions are needed if the sets are infinite and not compact). Then:

- v is the rational outcome of the game.
- $\bar{x}$ is an optimal strategy for Pl1.
- $\bar{y}$ is an optimal strategy for Pl2.

Observe $\bar{x}$ is optimal for Pl1 since it maximizes the function $\alpha(x) = \inf_y f(x, y)$ $\bar{y}$ is optimal for Pl2 since it minimizes the function $\beta(y) = \sup_x f(x, y)$ where $\alpha(x)$ is the value of the optimal choice of Pl2 if he knows that Pl1 plays x, and $\beta(y)$ is the value of the optimal choice of Pl1 if she knows that Pl2 plays y.

Proposition. Let X, Y be nonempty sets and let $f : X \times Y \to \mathbb{R}$ be an arbitrary real valued function. Then:

$$v_1 = \sup_x \inf_y f(x,y) \leq \inf_y \sup_x f(x,y) = v_2$$

Proof. By definition, for all x, y:

$$\inf_y f(x,y) \leq f(x,y) \leq \sup_x f(x,y)$$

Thus:

$$\alpha(x) = \inf_y f(x,y) \leq \sup_x f(x,y) = \beta(y)$$

Since for all $x \in X$ and $y \in Y$ it holds that

$$\alpha(x) \leq \beta(y)$$

it follows that

$$\sup_x \alpha(x) \leq \inf_y \beta(y)$$

□

As a consequence, in every game $v_1 \leq v_2$.

Example:
Consider the rock, scissors, and paper game:

$$\begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$$

The conservative values are not the same: in fact, $v_1 = -1$ and $v_2 = 1$.

There is no winning strategy since each player always plays randomly. But if the game is repeated many times, the rational solution for both players is to play each option one-third of the times, so that in the long run their expected utility is zero. By extending the game with mixed strategies, both conservative values become 0.

Case $v_1 < v_2$ Consider the case of mixed strategies with a gmae with an $n \times m$ utility matrix P. In this case the strategy spaces are defined as:

$$\sum_k = \left\{ x = (x_1, \dots, x_k) | x_i \geq 0 \text{ and } \sum_{i=1}^{k} x_i = 1 \right\}$$

with $k = n$ for Pl1 and $k = m$ for Pl2. The utility function is:

$$f(x,y) = \sum_{i=1,\dots,n, j=i,\dots,m} x_i y_j p_{ij} = (x, Py)$$

Here p_{ij} is an element of P corresponding to the utility of Pl1 when she plays row i and Pl2 plays column j (of course the utility of Pl2 is just the opposite). Thus, the mixed extension of the initial game is

$$\left(\sum_n, \sum_m, f(x,y) = (x, Py) \right)$$

CHAPTER 4

Mathematical concepts

4.1 Binary sum

We define a binary operation $\oplus$ on the set of natural numbers $\mathbb{N} = \{0, 1, 2, \ldots, n, \ldots\}$ as follows. For any two natural numbers $n_1, n_2 \in \mathbb{N}$:

1. Convert n_1 and $_n2$ into their binary representations, denoted as $[n_1]_2$ and $[n_2]_2$.
2. Perform the binary addition of $[n_1]_2$ and $[n_2]_2$ using the standard addition method, but without carrying over. This means if the addition of two bits results in 2 (i.e., $1 + 1$), it should be represented as 0 in that position with no carry to the next higher bit.
3. The result of the operation $\oplus$ is then represented in binary form, which corresponds to the sum computed in step 2.

Example:
Let's apply the $\oplus$ operation to the numbers $1, 2, 4$, and 1:

1. Convert the numbers to binary:
   - 1 in binary: $[1]_2 = 001$
   - 2 in binary: $[2]_2 = 010$
   - 4 in binary: $[4]_2 = 100$
   - 1 in binary: $[1]_2 = 001$
2. Perform the $\oplus$ operation:

$$\begin{array}{c} [1]_2 = 001 + \\ [2]_2 = 010 + \\ [4]_2 = 100 + \\ [1]_2 = 001 = \\ \hline [6]_2 = 110 \end{array}$$

3. The result of the operation $1 \oplus 2 \oplus 4 \oplus 1$ in decimal form is 6.

4.2 Group

Definition (*Group*)**.** A group is defined as a nonempty set A equipped with a binary operation $\cdot$ such that the following conditions hold:

1. *Closure*: for any elements $a, b \in A$, the result of the operation $a \cdot b$ is also an element of A.
2. *Associativity*: the operation $\cdot$ is associative, meaning that for all $a, b, c \in A$, it holds that $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.
3. *Identity element*: there exists a unique element e known as the identity element, such that for every $a \in A$, the following holds: $a \cdot e = e \cdot a = a$.
4. *Inverse element*: for every element $a \in A$, there exists a unique element $b \in A$ (denoted as a^{-1}) such that $a \cdot b = b \cdot a = e$. This element b is called the inverse of a.

Definition (*Abelian group*)**.** A group A is termed an abelian group (or commutative group) if the operation is commutative; that is, for all $a, b \in A$, the equation $\cdot\ a \cdot b = b \cdot a$ holds true.

Example:
Examples of abelian groups:

1. *The integers* $\mathbb{Z}$: the set of integers, equipped with the usual addition operation $(+)$, forms an abelian group. This group satisfies all the group properties:
   - *Closure*: the sum of any two integers is an integer.
   - *Associativity*: addition is associative, i.e., $(a + b) + c = a + (b + c)$.
   - *Identity*: the identity element is 0 since $a + 0 = a$ for any integer a.
   - *Inverses*: for every integer a, the inverse is $-a$ because $a + (-a) = 0$.
2. *The non-zero real numbers* $\mathbb{R}^*$: the set of all real numbers except 0, equipped with the usual multiplication operation $(\times)$, is an abelian group. It fulfills the following criteria:
   - *Closure*: the product of any two non-zero real numbers is also a non-zero real number.
   - *Associativity*: multiplication is associative, i.e., $(a \times b) \times c = a \times (b \times c)$.
   - *Identity*: the identity element is 1 because $a \times 1 = a$ for any non-zero real number a.
   - *Inverses*: for every non-zero real number a, the inverse is $\frac{1}{a}$ since $a \times \frac{1}{a} = 1$.

Examples of non abelian groups:

1. *The group of $n \times n$ matrices with non-zero determinant*: the set of all $n \times n$ matrices with a non-zero determinant, equipped with the usual matrix multiplication, is a non-abelian group (often denoted as $\mathrm{GL}(n, \mathbb{R})$). This group satisfies the group properties as follows:
   - *Closure*: the product of two invertible matrices is invertible, thus remaining in the group.

- *Associativity*: matrix multiplication is associative, i.e., $(A \cdot B) \cdot C = A \cdot (B \cdot C)$.
- *Identity*: the identity matrix serves as the identity element.
- *Inverses*: each invertible matrix has an inverse that is also an invertible matrix.
- *Non-abelian*: for matrices A and B, it is generally true that $A \cdot B \neq B \cdot A$.

Proposition. Let $(A, \cdot)$ be a group. Then the cancellation law holds:

$$a \cdot b = a \cdot c \implies b = c$$

Proof. To demonstrate the cancellation law, we start with the equation $a \cdot b = a \cdot c$.

By multiplying both sides of this equation by the inverse of a, denoted as a^{-1}, we obtain:

$$a^{-1}a \cdot b = a^{-1}a \cdot c$$

Utilizing the property of inverses, this simplifies to:

$$e \cdot b = e \cdot c$$

Here, e is the identity element of the group. By the definition of the identity, we have $b = c$. □

Proposition. The set of natural numbers with the operation $\oplus$ forms an abelian group.

Proof. We verify the group properties. The identity element is 0, since $n \oplus 0 = n$ for any natural number n. For any natural number n, the inverse with respect to $\oplus$ is n itself. However, since we consider the natural numbers as a set starting from 0, the formal definition of inverses in this context might not strictly apply, but 0 serves as an absorbing element for addition. The operation $\oplus$ is associative, as $(n_1 \oplus n_2) \oplus n_3 = n_1 \oplus (n_2 \oplus n_3)$ holds for all natural numbers n_1, n_2, n_3. The operation $\oplus$ is commutative since $n_1 \oplus n_2 = n_2 \oplus n_1$ for any natural numbers n_1 and n_2.

Therefore, since all group properties are satisfied, $(\mathbb{N}, \oplus)$ is an abelian group. Consequently, the cancellation law holds:

$$n_1 \oplus n_2 = n_1 \oplus n_3 \implies n_2 = n_3$$

□

4.3 Convexity

Definition (*Convex set*)**.** A set $C \subset \mathbb{R}^n$ is convex just in case for any $x, y \in C$, provided $\lambda \in [0, 1]$, one has:

$$\lambda x + (1 - \lambda)y \in C$$

The intersection of an arbitrary family of convex sets is convex. A closed convex set with nonempty interior coincides with the closure of its internal points.

Definition (*Convex combination*)**.** We shall call a convex combination of elements $x_1, \ldots, x_n$ any vector x of the form

$$x = \lambda_1 x_1 + \cdots + \lambda_n x_n$$

with $\lambda_1 \geq 0, \ldots, \lambda_n \geq 0$ and $\sum_{i=1}^{n} \lambda_i = 1$

Proposition. A set C is convex if and only if for every $\lambda_1 \geq 0, \ldots, \lambda_n \geq 0$ such that $\sum_{i=1}^{n} \lambda_i = 1$, for every $c_1, \ldots, c_n \in C$, for all n, then $\sum_{i=1}^{n} \lambda_i c_i \in C$.

If C is not convex, then there is a smallest convex set containing C: it is the intersection of all convex sets containing C.

Definition (*Convex hull*)**.** The convex hull of a set C, denoted by co C, is:

$$\text{co } C = \bigcap_{A \in \mathcal{C}} A$$

where $\mathcal{C} = \{A | C \subset A \text{ and } A \text{ is convex}\}$.

Proposition. Given a set C, then

$$\text{co } C = \left\{ \sum_{i=1}^{n} \lambda_i c_i | \lambda_i \geq 0, \sum_{i=1}^{n} \lambda_i = 1, c_i \in C \quad \forall i, n \in N \right\}$$

The convex hull of any set C is the set of all convex combinations of points in C. When C is a finite collection of points, its convex hull co C is called a polytope. E.g. if C contains three points, co C is the triangle (i.e. a polygone) with such points at its angles, which includes also all the points inside

Theorem 4.3.1. *Given a closed convex set C and a point x outside C, there is a unique element $p \in C$ such that, for all $c \in C$:*

$$\|p - x\| \leq \|c - x\|$$

The projection p is characterized by the following properties:

1. $p \in C$
2. $(x - p, c - p) \leq 0$ *for all* $c \in C$

That is, p is the closest point to x belonging to the set C, and by the last property it forms an obtuse angle between x and any $c \in C$.

Theorem 4.3.2. *Let C be a convex proper subset of the Euclidean space $\mathbb{R}^l$ and assume $\bar{x} \in$ cl C^c. Then there is an element $0 \neq x^* \in \mathbb{R}^l$ such that, $\forall c \in C$:*

$$(x^*, c) \geq (x^*, \bar{x})$$

This theorem means that a criterion to tell apart an external point from any internal point.

Proof. Suppose $\bar{x} \notin$ cl C and call p its projection on cl C. Based on the previous theorem, in follows that $(\bar{x} - p, c - p) \leq 0$ for all $c \in C$. By setting $x^* := p - \bar{x} \neq 0$, the inequality becomes $(-x^*, c - \bar{x} - x^*) = (-x^*, -x^*) + (-x^*, c - \bar{x}) \leq 0$, that is $(x^*, c - \bar{x}) \geq \|x^*\|^2$. Then, since $\|x^*\|^2 > 0$, by linearity one obtains

$$(x^*, c) \geq (x^*, \bar{x}) \quad \forall c \in C$$

As x^* appears in both sides, by renormalization we can choose $\|x^*\| = 1$. If $\bar{x} \in$ cl $C \setminus C$, take a sequence $\{x_n\} \subset C^c$ such that $x_n \to \bar{x}$. From the first step of the proof, we can find some norm one x_n^* such that

$$(x_n^*, c) \geq (x_n^*, x_n) \quad \forall c \in C$$

So, given that for some sub-sequence one has $x_n^* \to x^*$, taking the limit of the above inequality yields

$$(x^*, c) \geq (x^*, \bar{x}) \quad \forall c \in C$$

□

Corollary 4.3.2.1. *Let C be a closed convex set in a Euclidean space, let x be on the boundary of C. Then there is a hyperplane containing x and leaving all of C in one of the halfspaces determined by the hyperplane.*

Such an hyperplane is said to be an hyperplane supporting C at x

Corollary 4.3.2.2. *Let C be a closed convex set in a Euclidean space. Then C is the intersection of all half-spaces containing it.*

Theorem 4.3.3. *Let A, C be closed convex subsets of $\mathbb{R}^l$ such that $intA$ is nonempty and $intA \cap C = \varnothing$. Then there are $0 \neq x^*$ and $b \in \mathbb{R}$ such that, $\forall a \in A$, $\forall c \in C$:*

$$(x^*, a) \geq b \geq (x^*, c)$$

This means a criterion to determine whether a point is in A or in C.

Proof. Since $\bar{x} = 0 \in (\text{int}A - C)^c$, from the previous separation theorem with $\bar{x} = 0$ there is $x^* \neq 0$ such that

$$(x^*, x) \geq 0 \quad \forall x \in \text{int}A - C$$

Thus, for $x = a - c$ by linearity we obtain

$$(x^*, a) \geq (x^*, c) \quad \forall a \in \text{int}A, \forall c \in C$$

By extension this implies

$$(x^*, a) \geq (x^*, c) \quad \forall a \in \text{cl int}A = A, \forall c \in C$$

□

$H = \{x : (x^*, x) = b\}$ is called the separating hyperplane: A and C are contained in the two different half-spaces generated by H.