Model Identification And Data Analysis I
Exercises

Christian Rossi

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Abstract

The course topics are:

- Basic concepts of stochastic processes.
- ARMA and ARMAX classes of parametric models for time series and for Input/Output systems.
- Parameter identification of ARMA and ARMAX models.
- Analysis of identification methods.
- Model validation and pre-processing.

Contents

CHAPTER 1

Exercise session I

1.1 Exercise one

Consider an MA (2) process defined by the function:

$$y(t) = e(t) + \frac{1}{2}e(t-1) - e(t-2) \qquad e(t) \sim WN(0,1)$$

1. Find the transfer function for this system.
2. Calculate the expected value of the process $y(t)$.
3. Compute the covariance function of the process $y(t)$.

Solution

1. Utilizing the Z-transform, we express the MA (2) process as:

   $$y(t) = e(t) + \frac{1}{2}e(t)z^{-1} - e(t)z^{-2}$$

   Grouping the $e(t)$ factor, we obtain:

   $$y(t) = e(t)\left(1 + \frac{1}{2}z^{-1} - z^{-2}\right)$$

   This yields the polynomial:

   $$W(z) = 1 + \frac{1}{2}z^{-1} - z^{-2}$$

   In normal form, $W(z)$ becomes:

   $$W(z) = \frac{z^2 + \frac{1}{2}z - 1}{z^2}$$

2. The expected value is computed as follows:

$$\begin{aligned}
\mathbb{E}\left[y(t)\right] &= \mathbb{E}\left[e(t) + \frac{1}{2}e(t-1) - e(t-2)\right] \\
&= \mathbb{E}\left[e(t)\right] + \mathbb{E}\left[\frac{1}{2}e(t-1)\right] - \mathbb{E}\left[e(t-2)\right] \\
&= \underbrace{\mathbb{E}\left[e(t)\right]}_{0} + \frac{1}{2}\underbrace{\mathbb{E}\left[e(t-1)\right]}_{0} - \underbrace{\mathbb{E}\left[e(t-2)\right]}_{0} \\
&= 0
\end{aligned}$$

3. For the covariance:

$$\begin{aligned}
\gamma_y(0) &= \mathbb{E}\left[y(t)^2\right] \\
&= \mathbb{E}\left[\left(e(t) + \frac{1}{2}e(t-1) - e(t-2)\right)^2\right] \\
&= \mathbb{E}\left[e(t)^2 + \frac{1}{2}e(t-1)^2 + e(t-2)^2 + \text{cross products}\right] \\
&= \underbrace{\mathbb{E}\left[e(t)^2\right]}_{1} + \frac{1}{4}\underbrace{\mathbb{E}\left[e(t-1)^2\right]}_{1} + \underbrace{\mathbb{E}\left[e(t-2)^2\right]}_{1} + \underbrace{\mathbb{E}\left[\text{cross products}\right]}_{0} \\
&= 1 + \frac{1}{4} + 1 \\
&= \frac{9}{4}
\end{aligned}$$

The covariance at lag one is:

$$\gamma_y(1) = 0$$

We need to compute another time lag since we have two correlated time instants in the formula (square of the same time instant). The covariance of two is as follows:

$$\gamma_y(2) = -1$$

There is another correlation of the time instant $t-2$, but it is the only one, so for time instants after two, we have a null covariance. The final result is:

$$\begin{cases}
\gamma_y(0) = \frac{9}{4} \\
\gamma_y(1) = 0 \\
\gamma_y(2) = -1 \\
\gamma_y(\tau) = 0 \qquad \forall\,|\tau| \geq 3
\end{cases}$$

1.2 Exercise two

Consider a process with the following covariance:

$$\gamma(0) = \frac{5}{2} \qquad \gamma(1) = 1 \qquad \gamma(\tau) = 0 \quad |\tau| > 1$$

1. Examine the process.
2. Determine the expression of the process.

Solution

- The process adheres to an MA (1) model.
- Utilizing the general system, we have:

$$y(t) = c_0 e(t) + c_1 e(t-1) \qquad e \sim WN(0, \lambda^2)$$

The coefficients can be found using the following system of equations:

$$\begin{cases} (c_0^2 + c_1^2)\,\lambda^2 = \frac{5}{2} \\ (c_0 c_1)\,\lambda^2 = 1 \end{cases}$$

To simplify, we set $c_0 = 1$ and solve the system:

$$\begin{cases} (1 + c_1^2)\,\lambda^2 = \frac{5}{2} \\ (1 c_1)\,\lambda^2 = 1 \end{cases}$$

Solving the system yields:

$$\begin{cases} c_{1,2} = 2, \frac{1}{2} \\ \lambda_{1,2} = \frac{1}{2}, 2 \end{cases}$$

1.3 Exercise three

Consider an AR (2) process described by the following equation:

$$y(t) = \frac{1}{2} y(t-1) - \frac{1}{4} y(t-2) + e(t)$$

Here, $e(t) \sim WN(0, 1)$.

1. Determine the transfer function of the given system.
2. Calculate the expected value.
3. Compute the covariance.

Solution

1. Using the Z-transform, we get:

$$y(t) = \frac{1}{2} y(t) z^{-1} - \frac{1}{4} y(t) z^{-2} + e(t)$$

This yields:

$$y(t) = \frac{1}{1 - \frac{1}{2} z^{-1} + \frac{1}{4} z^{-2}} e(t)$$

2. The expected value is determined as follows:

$$
\begin{aligned}
\mathbb{E}\left[y(t)\right] &= \mathbb{E}\left[\frac{1}{2}y(t-1) - \frac{1}{4}y(t-2) + e(t)\right] \\
&= \frac{1}{2}\mathbb{E}\left[y(t-1)\right] + \frac{1}{4}\mathbb{E}\left[y(t-2)\right] - \underbrace{\mathbb{E}\left[e(t)\right]}_{0} \\
&= \frac{1}{2}\mathbb{E}\left[y(t-1)\right] + \frac{1}{4}\mathbb{E}\left[y(t-2)\right]
\end{aligned}
$$

Now, $y(t)$ is a stationary stochastic process because $e(t)$ is an SSP and $W(z)$ is asymptotically stable, we have $\mathbb{E}\left[y(t)\right] = m$ for all instants. Thus, rewriting the previous formula:

$$
m = \frac{1}{2}m + \frac{1}{4}m \rightarrow m = 0
$$

This value coincides with the expected value.

To confirm the hypothesis, we need to check if the input process is a stationary stochastic process (white noise is a stationary stochastic process) and if the transfer function is stable:

$$
W(x) = \frac{z^2}{z^2 - \frac{1}{2}z + \frac{1}{4}}
$$

Stability requires that all the modules of the poles are inside the unit circle:

$$
z^2 - \frac{1}{2}z + \frac{1}{4} = 0
$$

The solutions to this equation are:

$$
z_{1,2} = \frac{1}{4} \pm i\frac{\sqrt{3}}{4}
$$

From which the modules are:

$$
|z_{1,2}| = \frac{1}{2}
$$

Thus, the system is stable, confirming the hypothesis.

3. The covariance at lag zero is calculated as follows:

$$
\gamma_y(0) = \mathbb{E}\left[\frac{1}{2}y(t-1) - \frac{1}{4}y(t-2) + e(t)\right]
$$

From this we have:

$$
\begin{aligned}
\gamma_y(0) = \frac{1}{4}\underbrace{\mathbb{E}\left[y(t-1)^2\right]}_{\gamma_y(0)} + \frac{1}{16}\underbrace{\mathbb{E}\left[y(t-2)^2\right]}_{\gamma_y(0)} + \underbrace{\mathbb{E}\left[e(t^2)\right]}_{1} + \frac{1}{4}\underbrace{\mathbb{E}\left[y(t-1)y(t-2)\right]}_{\gamma_y(1)} + \\
+ \underbrace{\mathbb{E}\left[y(t-1)e(t)\right]}_{0} + \frac{1}{2}\underbrace{\mathbb{E}\left[y(t-2)e(t)\right]}_{0}
\end{aligned}
$$

The resulting equation is:

$$
\frac{11}{16}\gamma_y(0) + \frac{1}{4}\gamma_y(1) = 1
$$

To determine the covariance at lag one, we compute:

$$\begin{aligned}\gamma_y(1) &= \mathbb{E}\left[\left(\frac{1}{2}y(t-1) - \frac{1}{4}y(t-2) + e(t)\right) y(t-1)\right] \\ &= \frac{1}{2}\underbrace{\mathbb{E}\left[y(t-1)^2\right]}_{\gamma_y(0)} - \frac{1}{4}\underbrace{\mathbb{E}\left[y(t-2)y(t-1)\right]}_{\gamma_y(1)} + \underbrace{\mathbb{E}\left[e(t)y(t-1)\right]}_{0} \\ &= \frac{1}{2}\gamma_y(0) - \frac{1}{4}\gamma_y(1)\end{aligned}$$

This leads to the equation:

$$\gamma_y(1) = \frac{1}{2}\gamma_y(0) - \frac{1}{4}\gamma_y(1)$$

The resulting system of equations is:

$$\begin{cases} \dfrac{11}{16}\gamma_y(0) + \frac{1}{4}\gamma_y(1) = 1 \\ -\dfrac{1}{2}\gamma_y(0) + \frac{5}{4}\gamma_y(1) = 0 \end{cases}$$

Solving this system yields:

$$\begin{cases} \gamma_y(0) = \frac{80}{63} \\ \gamma_y(1) = \frac{32}{63} \end{cases}$$

Now, we can compute the covariance at lag two:

$$\begin{aligned}\gamma_y(2) &= \mathbb{E}\left[\left(\frac{1}{2}y(t-1) - \frac{1}{4}y(t-2) + e(t)\right) y(t-2)\right] \\ &= \frac{1}{2}\underbrace{\mathbb{E}\left[y(t-1)y(t-2)\right]}_{\gamma_y(1)} - \frac{1}{4}\underbrace{\mathbb{E}\left[y(t-2)^2\right]}_{\gamma_y(0)} + \underbrace{\mathbb{E}\left[e(t)y(t-2)\right]}_{0} \\ &= \frac{1}{2}\gamma_y(1) - \frac{1}{4}\gamma_y(0) \\ &= -\frac{4}{63}\end{aligned}$$

The final result is:

$$\begin{cases} \gamma_y(0) = \frac{80}{63} \\ \gamma_y(1) = \frac{32}{63} \\ \gamma_y(\tau) = \frac{1}{2}\gamma_y(\tau-1) - \frac{1}{4}\gamma_y(\tau-2) \qquad \forall\,|\tau| \geq 2 \end{cases}$$

CHAPTER 2

Exercise session II

2.1 Exercise one

Consider the AR (1) process:

$$y(t) = \frac{1}{3}y(t-1) + e(t) + 2 \qquad e(t) \sim WN(1,1)$$

1. Determine the transfer function of the system and confirm its stationary stochastic nature.
2. Calculate the expected value.
3. Compute the covariance.

Solution

1. Applying the input delay operator yields:

$$y(t) = \frac{1}{3}z^{-1}y(t) + e(t) + 2$$

Rearranging terms, we get:

$$y(t) = \left[\frac{z}{z-\frac{1}{3}}\right](e(t)+2)$$

As the input is a stationary stochastic process, the poles of the transfer function are:

$$z - \frac{1}{3} = 0 \rightarrow z = \frac{1}{3}$$

Since the pole is inside the unity circle, the process is stationary and stochastic.

2. The expected value is:

$$\mathbb{E}\left[y(t)\right] = \mathbb{E}\left[\frac{1}{3}y(t-1) + e(t) + 2\right] = \frac{1}{3}\mathbb{E}\left[y(t-1)\right] + 1 + 2$$

Given that we have a stationary stochastic process, the mean is constant:

$$m_y = \frac{1}{3}m_y + 3 \rightarrow m_y = \frac{9}{2}$$

3. We define the unbiased process:

$$\begin{cases} \tilde{y}(t) = y(t) - m_y \\ \tilde{e}(t) = e(t) - m_e \end{cases}$$

In our case, this yields:

$$\tilde{y}(t) + \frac{9}{2} = \frac{1}{3}\left(\tilde{y}(t-1) + \frac{9}{2}\right) + \tilde{e}(t) + 1 + 2 \rightarrow \tilde{y}(t) = \frac{1}{3}\tilde{y}(t-1) + \tilde{e}(t)$$

Finally, we compute the covariance function as:

$$\gamma_y(\tau) = \mathbb{E}\left[\tilde{y}(t)\tilde{y}(t-\tau)\right]$$

Beginning with the covariance at $\tau = 0$:

$$\gamma_{\tilde{y}}(0) = \mathbb{E}\left[\tilde{y}(t)^2\right] = \mathbb{E}\left[\left(\frac{1}{3}\tilde{y}(t-1) + \tilde{e}(t)\right)^2\right] = \frac{1}{9}\gamma_{\tilde{y}}(0) + 1 \rightarrow \gamma_{\tilde{y}}(0) = \frac{9}{8}$$

Next, we compute the covariance at $\tau = 1$:

$$\gamma_{\tilde{y}}(1) = \mathbb{E}\left[\tilde{y}(t)\tilde{y}(t-1)\right] = \mathbb{E}\left[\left(\frac{1}{3}\tilde{y}(t-1) + \tilde{e}(t)\right)\tilde{y}(t-1)\right] = \frac{1}{3}\gamma_{\tilde{y}}(0) \rightarrow \gamma_{\tilde{y}}(1) = \frac{3}{8}$$

For a generic τ:

$$\gamma_{\tilde{y}}(\tau) = \frac{1}{3}\gamma_{\tilde{y}}(\tau - 1) \qquad |\tau| \geq 1$$

2.2 Exercise two

Consider the ARMA $(1,1)$ process described by the expression:

$$y(t) = \frac{1}{2}y(t-1) + e(t) - e(t-1) \qquad e(t) \sim WN(1,9)$$

1. Determine the transfer function and verify if it is a stationary stochastic process.
2. Calculate the expected value.
3. Compute the covariance function.

Solution

1. We express the formula in operatorial representation:

$$y(t) = \frac{1}{2}y(t)z^{-1} + e(t) - e(t)z^{-1} \rightarrow y(t) = \frac{z-1}{z-\frac{1}{2}}e(t)$$

The system exhibits a zero at $z = 1$ and a pole in $z = \frac{1}{2}$, indicating asymptotic stability. As the input, White Noise, is a stationary stochastic process, $y(t)$ is also a stationary stochastic process.

2. The expected value is computed as:

$$\mathbb{E}\left[y(t)\right] = \mathbb{E}\left[\frac{1}{2}y(t-1) + e(t) - e(t-1)\right] = \frac{1}{2}\mathbb{E}\left[y(t-1)\right] + 1 - 1$$

Since $y(t)$ is a stationary stochastic process, its mean is constant:

$$m_y = \frac{1}{2}m_y \rightarrow m_y = 0$$

Alternatively, it can be computed using the theorem:

$$\mathbb{E}\left[y(t)\right] = W(1) \cdot \mathbb{E}\left[e(t)\right] = 0 \cdot 1 = 0$$

3. Define the unbiased process as:

$$\begin{cases} \tilde{y}(t) = y(t) - m_y \\ \tilde{e}(t) = e(t) - m_e \end{cases}$$

In this case, we obtain:

$$\tilde{y}(t) + m_y = \frac{1}{2}\left(\tilde{y}(t-1) + m_y\right) + \tilde{e}(t) + m_e - \left(\tilde{e}(t-1) + m_e\right)$$

Simplifying, we have:

$$\tilde{y}(t) = \frac{1}{2}\tilde{y}(t-1) + \tilde{e}(t) + 1 - \tilde{e}(t-1) - 1 \rightarrow \tilde{y}(t) = \frac{1}{2}\tilde{y}(t-1) + \tilde{e}(t) - \tilde{e}(t-1)$$

Starting with the covariance at $\tau = 0$:

$$\gamma_{\tilde{y}}(0) = \mathbb{E}\left[\tilde{y}(t)^2\right] = \mathbb{E}\left[\left(\frac{1}{2}\tilde{y}(t-1) + \tilde{e}(t) - \tilde{e}(t-1)\right)^2\right] = \frac{1}{4}\gamma_{\tilde{y}}(0) + 9 - 9 - 9 \rightarrow \gamma_{\tilde{y}}(0) = 12$$

Next, compute the covariance at $\tau = 1$:

$$\gamma_{\tilde{y}}(1) = \mathbb{E}\left[\tilde{y}(t)\tilde{y}(t-1)\right] = \mathbb{E}\left[\left(\frac{1}{2}\tilde{y}(t-1) + \tilde{e}(t) - \tilde{e}(t-1)\right)y(t-1)\right] \rightarrow \gamma_{\tilde{y}}(1) = -3$$

Then, compute the covariance at $\tau = 2$:

$$\gamma_{\tilde{y}}(2) = \mathbb{E}\left[\tilde{y}(t)\tilde{y}(t-2)\right] = \mathbb{E}\left[\left(\frac{1}{2}\tilde{y}(t-1) + \tilde{e}(t) - \tilde{e}(t-1)\right)y(t-2)\right] \rightarrow \gamma_{\tilde{y}}(1) = -\frac{3}{2}$$

For a generic τ:

$$\gamma_{\tilde{y}}(\tau) = \frac{1}{2}\gamma_{\tilde{y}}(\tau - 1) \qquad |\tau| \geq 2$$

2.3 Exercise three

Consider the MA (2) process generated by the expression:

$$y(t) = e(t) + 0.5e(t-1) + 0.5e(t-2) \qquad e(t) \sim WN(2, 1)$$

1. Determine the transfer function and verify if it is a stationary stochastic process.
2. Calculate the expected value.
3. Compute the covariance function.

Solution

1. We express the formula in operatorial representation:

$$y(t) = e(t) + 0.5e(t)z^{-1} + 0.5e(t)z^{-2} \rightarrow y(t) = \frac{z^2 + 0.5z + 0.5}{z^2} e(t)$$

   The system has two zeros at $z_{1,2} = -\frac{1}{4} \pm \frac{\sqrt{7}}{4} i$ and a pole at $z = 0$, indicating asymptotic stability. As the input, White Noise, is a stationary stochastic process, $y(t)$ is also a stationary stochastic process.

2. The expected value is computed as:

$$\mathbb{E}\left[y(t)\right] = \mathbb{E}\left[e(t) + 0.5e(t-1) + 0.5e(t-2)\right] = 2 + 1 + 1 = 4$$

   Alternatively, it can be computed using the theorem:

$$\mathbb{E}\left[y(t)\right] = W(1) \cdot \mathbb{E}\left[e(t)\right] = 2 \cdot 2 = 4$$

3. Define the unbiased process as:

$$\begin{cases} \tilde{y}(t) = y(t) - m_y \\ \tilde{e}(t) = e(t) - m_e \end{cases}$$

   In this case, we have:

$$\tilde{y}(t) + m_y = \left(\tilde{e}(t) + m_e\right) + 0.5\left(\tilde{e}(t-1) + m_e\right) + 0.5\left(\tilde{e}(t-2) + m_e\right)$$

   Simplifying, we obtain:

$$\tilde{y}(t) = \tilde{e}(t) + 0.5\tilde{e}(t-1) + 0.5\tilde{e}(t-2)$$

   Since it is a Moving Average process, we can directly find the covariance as:

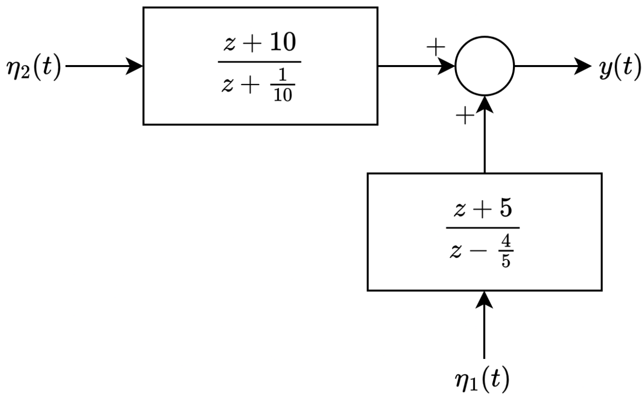
$$\begin{cases} \left(c_0^2 + c_1^2 + c_2^2\right)\lambda^2 & \tau = 0 \\ \left(c_0c_1 + c_1c_2\right)\lambda^2 & |\tau| = 1 \\ \left(c_0c_2\right)\lambda^2 & |\tau| = 2 \\ 0 & |\tau| \geq 3 \end{cases} \rightarrow \begin{cases} \frac{3}{2} & \tau = 0 \\ \frac{3}{4} & |\tau| = 1 \\ \frac{1}{2} & |\tau| = 2 \\ 0 & |\tau| \geq 3 \end{cases}$$

CHAPTER 3

Exercise session III

3.1 Exercise one

Consider the stochastic process defined by the following diagram:



Here, $\eta_1 \sim WN(1,1)$ and $\eta_2 \sim WN(0,1)$ are uncorrelated.
Find the characteristic values of the given process $y(t)$.

Solution

Remember that for an ARMA (n_a, n_b) process:

- If $n_a > n_b$, the covariance becomes recursive for $\tau = n_a$.
- If $n_a \leq n_b$, the covariance becomes recursive for $\tau = n_b + 1$

The output process is composed of two uncorrelated processes because the White Noise sources are uncorrelated:

$$y(t) = y_1(t) + y_2(t)$$

Since both $y_1(t)$ and $y_2(t)$ are stationary, $y(t)$ is also stationary.
The mean is:

$$m_y = \mathbb{E}\left[y(t)\right] = \mathbb{E}\left[y_1(t) + y_2(t)\right] = W_1(1)\mathbb{E}\left[\eta_1(t)\right] + W_2(1)\mathbb{E}\left[\eta_2(t)\right] = \frac{15}{2}$$

The covariance can be computed as the sum of the covariances of $y_1(t)$ and $y_2(t)$ (since they are uncorrelated):

$$\gamma_y(\tau) = \gamma_{y_1}(\tau) + \gamma_{y_2}(\tau)$$

For the stochastic process $y_1(t)$ in the time domain:

$$y_1(t) = \frac{1}{5} y_1(t-1) + \eta_1(t) + 5\eta_1(t-1)$$

Define the unbiased process by:

$$\begin{cases} \tilde{y}_1(t) = y_1(t) - m_{y_1} \\ \tilde{\eta}_1(t) = \eta_1(t) - m_{\eta_1} \end{cases}$$

Then, the process becomes:

$$\tilde{y}_1(t) = \frac{1}{5} \tilde{y}_1(t-1) + \tilde{\eta}_1(t) + 5\tilde{\eta}_1(t-1)$$

The covariance at different time lags is:

$$\gamma_{y_1}(\tau) = \begin{cases} \frac{175}{6} & \tau = 0 \\ \frac{65}{6} & |\tau| = 1 \\ \frac{13}{6} & |\tau| = 2 \\ \frac{1}{5}\gamma_{y_1}(\tau-1) & |\tau| \geq 3 \end{cases}$$

For the stochastic process $y_2(t)$:

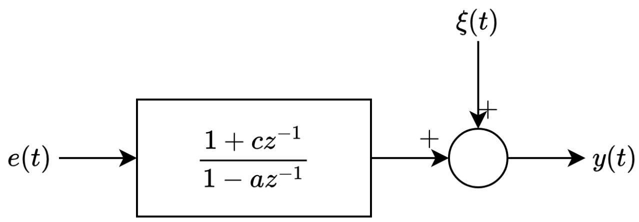
$$y_2(t) = -\frac{1}{10} y_2(t-1) + \eta_2(t) + 10\eta_2(t-1)$$

The covariance function is:

$$\gamma_{y_2}(\tau) = \begin{cases} 100 & \tau = 0 \\ 0 & |\tau| \geq 1 \end{cases}$$

3.2 Exercise two

Consider the stochastic process defined by the following diagram:



Here, $e(t) \sim WN(1, 1)$ and $\xi(t) \sim WN(0, 1)$ are uncorrelated.

1. Determine when the process is stationary.
2. Given $\gamma_y(0) = 6$, $\gamma_y(1) = -2$, and $\gamma_y(\tau) = 0$ for $\tau \geq 2$, compute the values of a and c.

Solution

1. The process $y(t)$ is stationary when both $\xi(t)$ and $y_1(t)$ are stationary. Since $\xi(t)$ is a White Noise process, it is stationary by definition. The process $y_1(t)$ is stationary when $|a| < 1$.

2. Since $\gamma_y(\tau) = 0$ for $\tau \geq 2$, this implies that $y(t)$ is a Moving Average Process of order one. Hence, $a = 0$.

   The process in the time domain is:

$$y(t) = -ay(t-1) + e(t) + ce(t-1) + \xi(t)$$

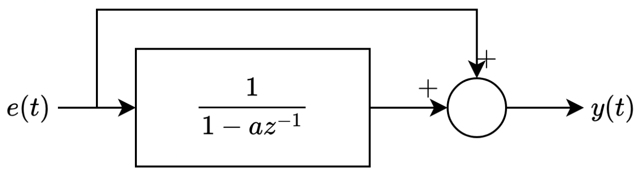
   We can compute the covariance at $\tau = 0$:

$$\gamma_y(0) = \mathbb{E}\left[y(t)^2\right] = 0$$

   From this, we obtain $c = \pm 2$.

3.3 Exercise three

Consider the stochastic process defined by the following diagram:



Here, $e(t) \sim WN(0, \lambda^2)$, and $|a| < 1$.
Find the characteristic values of the given process $y(t)$.

Solution

To begin, let's compute the expected value of $y(t)$:

$$m_y = \mathbb{E}\left[y(t)\right] = \mathbb{E}\left[ay(t-1) + 2e(t)\right] = a\mathbb{E}\left[y(t-1)\right] \rightarrow m_y = 0$$

The covariance function at $\tau = 0$ is given by:

$$\gamma_y(0) = \mathbb{E}\left[y(t)^2\right] = \mathbb{E}\left[(y_1(t) + y_2(t))^2\right] = \frac{4 - 3a^2}{1 - a^2}\lambda^2$$

The covariance function at $\tau = 1$ is given by:

$$\gamma_y(1) = \mathbb{E}\left[y(t)y(t-1)\right] = \frac{a\lambda^2(2 - a^2)}{1 - a^2}$$

Alternatively, noting that we have two processes in parallel with a transfer function equal to:

$$y(t) = \frac{1}{1 - az^{-1}}e(t) + e(t) = \frac{2 - az^{-1}}{1 - az^{-1}}e(t)$$

The canonical form becomes:

$$y(t) = \frac{1 - \frac{a}{2}z^{-1}}{1 - az^{-1}} e_1(t)$$

Here, $e_1(t) = 2e(t)$, implying that $e(t) \sim WN(0, 2^2\lambda^2)$. We can now find the time-domain representation, which is:

$$y(t) = ay(t-1) + \eta_1(t) - \frac{a}{2}\eta_1(t-1)$$

From this, we can compute the covariance in a more straightforward manner.

CHAPTER 4

Exercise session IV

4.1 Exercise one

Consider the process described by the expression:

$$y(t) = \frac{1}{2}y(t-1) + e(t) - e(t-1) \qquad e(t) \sim WN(0, 9)$$

Determine the spectral density function of the provided process.

Solution

For a stationary stochastic process, the following formula holds:

$$\Gamma_y(\omega) = \left|W(e^{j\omega})\right|^2 \Gamma_u(\omega) = \left|W(e^{j\omega})\right|^2 \lambda^2$$

We start by computing the transfer function:

$$y(t) = \frac{z-1}{z-\frac{1}{2}}$$

Since the pole is inside the unit circle and $e(t)$ is a stationary stochastic process (White Noise), $y(t)$ is also a stationary stochastic process. We can then use the fundamental theorem of spectral analysis:

$$\Gamma_y(\omega) = \left|\frac{e^{j\omega}-1}{e^{j\omega}-\frac{1}{2}}\right|^2 9$$

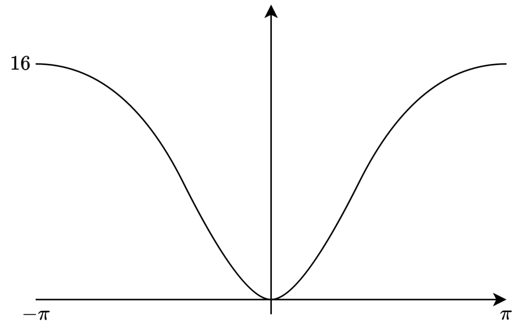
We compute the squares as follows:

- $\left|e^{j\omega}-1\right|^2 = (e^{j\omega}-1)(e^{-j\omega}-1) = 2(1-\cos\omega)$
- $\left|e^{j\omega}-\frac{1}{2}\right|^2 = \left(e^{j\omega}-\frac{1}{2}\right)\left(e^{-j\omega}-\frac{1}{2}\right) = \frac{5}{4} - \cos\omega$

Thus, the spectral density function is:

$$\Gamma_y(\omega) = \frac{1-\cos\omega}{\frac{5}{4}-\cos\omega}18$$

This allows us to generate the graph:



4.2 Exercise two

Consider the process generated by the following expression:

$$y(t) = \left(1 - z^{-1} + z^{-2}\right)\left(1 + \frac{3}{2}z^{-1}\right)e(t) \qquad e(t) \sim N(0,1)$$

Find the spectral density function of the given process.

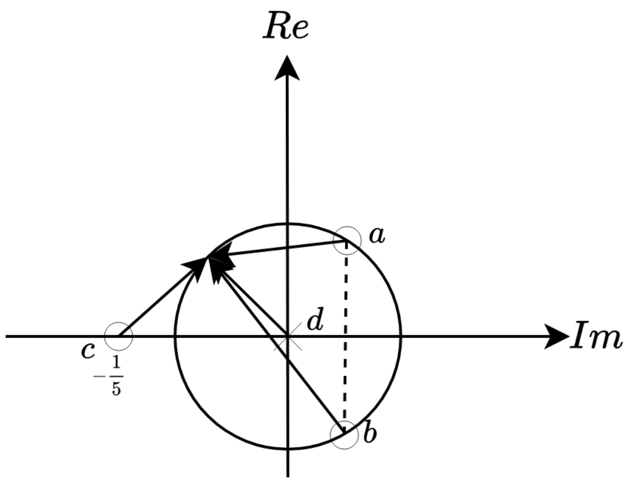
Solution

This can be rewritten as:

$$y(t) = \frac{\left(z^2 - z + 1\right)\left(z + \frac{3}{2}\right)}{z^2}e(t)$$

The poles are at $z = 0$, and the zeros are at $z_{1,2,3} = -\frac{3}{2}, \frac{1}{2} \pm j\frac{\sqrt{3}}{2}$

The simplest way to compute the spectral density function is by using the vectors that connect a generic point $e^{j\omega}$ to the poles (d) and the zeros (a, b, c):



In this case, the spectral density function is computed as:

$$\Gamma_y(\omega) = \frac{|a|^2\,|b|^2\,|c|^2}{|d|^2}\lambda^2$$

For e^{j0}:

- $|a|^2 = 1$
- $|b|^2 = 1$
- $|c|^2 = \frac{25}{4}$
- $|d|^2 = 1$

Thus, $\Gamma_y(0) = \frac{25}{4}$.

For $e^{j\frac{\pi}{2}}$:

- $|a|^2 = 2 - \sqrt{3}$
- $|b|^2 = 2 + \sqrt{3}$
- $|c|^2 = \frac{13}{4}$
- $|d|^2 = 1$

Therefore, $\Gamma_y\left(\frac{\pi}{2}\right) = \frac{13}{4}$.

For $e^{j\pi}$:

- $|a|^2 = 3$
- $|b|^2 = 3$
- $|c|^2 = \frac{1}{4}$
- $|d|^2 = 1$

Hence, $\Gamma_y(\pi) = \frac{9}{4}$.

Note that $\Gamma_y\left(\frac{\pi}{3}\right) = 0$.

4.3 Exercise three

Consider the process described by the function:

$$y(t) = \frac{z^4}{\left(z - \frac{1}{2} - j\frac{1}{2}\right)\left(z - \frac{1}{2} + j\frac{1}{2}\right)\left(z + \frac{1}{2} - j\frac{1}{2}\right)\left(z + \frac{1}{2} + j\frac{1}{2}\right)} e(t)$$

Here, $e(t) \sim WN(0, 1)$. Find the spectral density function of the given process.

Solution

In this case, the spectral density function is computed as:

$$\Gamma_y(\omega) = \frac{1}{|a|^2 |b|^2 |c|^2 |d|^2} \lambda^2$$

Starting at e^{j0}, we have:

- $|a|^2 = \frac{1}{2}$

- $|b|^2 = \frac{5}{2}$
- $|c|^2 = \frac{5}{2}$
- $|d|^2 = \frac{1}{2}$

Thus, $\Gamma_y(0) = \dfrac{16}{25}$.

For $e^{j\frac{\pi}{2}}$ and $e^{j\pi}$, we have the same result.

Using the fundamental theorem of spectral analysis, we have:

$$\Gamma_y(\omega) = \left|W(e^{j\omega})\right|^2$$

This can be rewritten as:

$$y(t) = \frac{z^4}{z^4 + \frac{1}{4}} e(t)$$

Thus,

$$\Gamma_y(\omega) = \left|\frac{e^{j\omega 4}}{e^{j\omega 4} + \frac{1}{4}}\right|^2 \cdot 1 = \frac{16}{17 + 8\cos(4\omega)}$$

4.4 Exercise four

Consider the following process:

$$y(t) = \frac{1}{2} y(t-1) + e(t) - e(t-1) \qquad e(t) \sim WN(0, 9)$$

We have that $\gamma_y(0) = 12$, $\gamma_y(\pm 1) = -3$, and $\gamma_y(\pm\tau) = \frac{1}{2}\gamma_y(\tau - 1)$ with $|\tau| \geq 2$. Find the spectral density function of the given process.

Solution

The spectrum is the sum of all covariances:

$$\begin{aligned}
\Gamma_y(\omega) &= \sum_{\tau=-\infty}^{+\infty} \gamma_y(\tau) e^{-j\omega\tau} \\
&= 12e^{-j\omega 0} - 3e^{-j\omega} - 3e^{j\omega} - \frac{3}{2}e^{-j\omega 2} - \frac{3}{2}e^{j\omega 2} + \dots \\
&= 12 - 6\left[\frac{1}{2}e^{j\omega} + \frac{1}{2}e^{-j\omega} + \frac{1}{4}e^{j\omega} + \frac{1}{4}e^{-j\omega} + \dots\right] \\
&= 12 - 6\left[-1 + 1 + \frac{1}{2}e^{-j\omega} + \frac{1}{4}e^{-j\omega} + \dots - 1 + 1 + \frac{1}{2}e^{j\omega} + \frac{1}{4}e^{j\omega}\right] \\
&= 24 - 6\left[\sum_{i=0}^{+\infty}\left(\frac{1}{2}e^{-j\omega}\right)^i + \sum_{i=0}^{+\infty}\left(\frac{1}{2}e^{j\omega}\right)^i\right] \\
&= 24 - 6\left[\frac{1}{1 - \frac{1}{2}e^{-j\omega}} + \frac{1}{1 - \frac{1}{2}e^{j\omega}}\right] \\
&= \frac{1 - \cos(\omega)}{\frac{5}{4} - \cos(\omega)} 18
\end{aligned}$$

CHAPTER 5

Exercise session V

5.1 Exercise one

Consider the given process:

$$y(t) = \frac{1}{2}y(t-2) + \eta(t) + 4\eta(t-1) \qquad \eta(t) \sim WN(0,1)$$

1. Determine the transfer function.
2. Calculate $\hat{y}(t+1|t)$.
3. Validate the obtained predictor.
4. Find $\hat{y}(t+2|t)$.

Solution

1. The transfer function is given by:

$$y(t) = \left[\frac{z(z+4)}{z^2 - \frac{1}{2}}\right]\eta(t)$$

2. To find $\hat{y}(t+1|t)$, we need to follow these steps:

   - Check if the process is in canonical form:
     (a) Numerator and denominator are monic (coefficient of the highest power equal one): both are equal to one.
     (b) Numerator and denominator have the same degree: both are of second degree.
     (c) Numerator and denominator are co-prime: they have no common roots.
     (d) The singularities must be inside the unit circle: not satisfied. Therefore, redefine:

$$\left[\frac{z(z+4)}{z^2 - \frac{1}{2}}\right]\left[\frac{z+\frac{1}{2}}{z+4}\right] = \frac{z\left(z-\frac{1}{4}\right)}{z^2 - \frac{1}{2}}$$

     And we must redefine the White Noise as $e(t) \sim WN(0, 4^2 \cdot 1)$

- Compute the predictor via long division:

$$\hat{y}(t|t-1) = E(z)e(t) + \frac{F(z)}{A(z)}e(t-1) = e(t) + \frac{\frac{1}{4} - \frac{1}{2}z^{-1}}{1 - \frac{1}{2}z^{-2}}e(t-1)$$

But we don't know the value of $e(t)$ because we have given only past samples until $t-1$. So the predictor is:

$$\hat{y}(t|t-1) = \frac{\frac{1}{4}z^{-1} - \frac{1}{2}z^{-2}}{1 - \frac{1}{2}z^{-2}} \frac{1 - \frac{1}{2}z^{-1}}{1 + \frac{1}{4}z^{-1}}y(t) = \frac{\frac{1}{4} - \frac{1}{2}z^{-1}}{1 + \frac{1}{4}z^{-1}}y(t-1)$$

Equivalently:

$$\hat{y}(t+1|t) = \frac{\frac{1}{4} - \frac{1}{2}z^{-1}}{1 + \frac{1}{4}z^{-1}}y(t)$$

In time domain, this becomes:

$$\hat{y}(t+1|t) = -\frac{1}{4}\hat{y}(t|t-1) + \frac{1}{4}y(t) + \frac{1}{2}y(t-1)$$

Note that predictors from noise can also be computed as:

$$\hat{y}(t|t-1) = \frac{C(z) - A(z)}{A(z)}e(t)$$

Predictors from data can be computed as:

$$\hat{y}(t|t-1) = \frac{C(z) - A(z)}{C(z)}y(t)$$

3. The prediction error is given by:

$$\varepsilon(t|t-1) = y(t) - \hat{y}(t|t-1) = E(z)e(t)$$

The variance is:

$$\text{Var}\left[\varepsilon(t|t-1)\right] = \mathbb{E}\left[\varepsilon(t|t-1)^2\right] = \mathbb{E}\left[1 \cdot e(t)^2\right] = 16$$

Since the variance of the process is approximately 23, the predictor is optimal but not very good because the variance and covariance are similar.

4. The two-step ahead predictor can only be found via long division. After performing two steps in the division, we obtain:

$$\hat{y}(t|t-2) = \frac{F(z)}{A(z)}e(t) = \frac{\frac{1}{2}z^{-2} + \frac{1}{8}z^{-3}}{1 - \frac{1}{2}z^{-2}}e(t)$$

The predictor from data can be found knowing that:

$$e(t) = \frac{1 - \frac{1}{2}z^{-1}}{1 + \frac{1}{4}z^{-1}}y(t)$$

By substitution, we obtain:

$$\hat{y}(t|t-2) = \frac{\frac{1}{2} + \frac{1}{8}z^{-1}}{1 + \frac{1}{4}z^{-1}}y(t-2)$$

The prediction error is:

$$\varepsilon(t|t-2) = y(t) - \hat{y}(t|t-2) = E(z)e(t)$$

The variance is:

$$\mathrm{Var}\left[\varepsilon(t|t-2)\right] = \mathbb{E}\left[\varepsilon(t|t-2)^2\right] = \mathbb{E}\left[\left(e(t) + \frac{1}{4}e(t-1)\right)^2\right] = 17$$

Since the variance of the process is approximately 23, the predictor is optimal but not very good because the variance and covariance are similar.

CHAPTER 6

Exercise session VI

6.1 Exercise one

Consider the following process described by the expression:

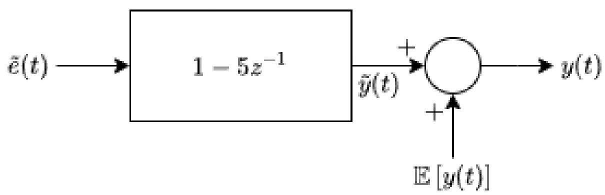
$$y(t) = e(t) + 5e(t-1) \qquad e(t) \sim WN(1,1)$$

The expected value of the process $y(t)$ is 6.

1. Determine the unbiased process.
2. Find the predictor $\hat{y}(t|t-1)$.

Solution

1. The given system can be represented as:



In the block diagram, we define:

$$\begin{cases} \tilde{y}(t) = y(t) - m_y \\ \tilde{e}(t) = e(t) - m_e \end{cases} \rightarrow \begin{cases} \tilde{y}(t) = y(t) - 6 \\ \tilde{e}(t) = e(t) - 1 \end{cases}$$

The process $y(t)$ is composed of:

$$y(t) = \tilde{y}(t) + 6 = \tilde{e}(t)\left(1 + 5z^{-1}\right) + 6 = (e(t)-1)\left(1+5z^{-1}\right) + 6 = e(t) + 5e(t-1)$$

The unbiased process is:

$$\tilde{y}(t) = \tilde{e}(t) + 5\tilde{e}(t-1)$$

Since the unbiased process is not in canonical form, an all-pass filter must be used:

$$\tilde{y}(t) = \frac{1+\frac{1}{5}z^{-1}}{1}\frac{1+5z^{-1}}{1+\frac{1}{5}z^{-1}1}\eta(t)$$

Here, $\eta(t) \sim WN(0, 25)$.

In the time domain, this becomes:

$$\tilde{y}(t) = \eta(t) + \frac{1}{5}\eta(t-1)$$

2. The predictor from noise is:

$$\hat{\tilde{y}}(t|t-1) = \frac{1}{5}\eta(t-1)$$

The predictor from data is:

$$\hat{\tilde{y}}(t|t-1) = \frac{1}{5}z^{-1}\frac{1}{1+\frac{1}{5}z^{-1}}\tilde{y}(t) = -\frac{1}{5}\tilde{y}(t-1|t-2) + \frac{1}{5}\tilde{y}(t-1)$$

To find the predictor of the original process by substitution, as the prediction is linear, we have:

$$\begin{aligned}
&\hat{\tilde{y}}(t+1|t) = -\frac{1}{5}\tilde{y}(t|t-1) + \frac{1}{5}\tilde{y}(t) \rightarrow \\
&\hat{y}(t+1|t) - 6 = -\frac{1}{5}\left(y(t|t-1) - 6\right) + \frac{1}{5}\left(y(t) - 6\right) \rightarrow \\
&\hat{y}(t+1|t) - 6 = -\frac{1}{5}y(t|t-1) + \frac{6}{5} + \frac{1}{5}y(t) - \frac{6}{5} \rightarrow \\
&\hat{y}(t+1|t) = -\frac{1}{5}y(t|t-1) + \frac{1}{5}y(t) + 6
\end{aligned}$$

6.2 Exercise two

Consider the given process:

$$\frac{1}{2}y(t) = -\frac{1}{3}y(t-1) - \frac{1}{18}y(t-2) + 3e(t-2) - 8e(t-3) - 3e(t-4)$$

Here, $e(t) \sim WN(0, 1)$. Let's compute the one-step ahead predictor.

Solution

The transfer function is:

$$y(t) = \frac{3z^{-2} - 8z^{-3} - 3z^{-4}}{\frac{1}{2} + \frac{1}{3}z^{-1} + \frac{1}{18}z^{-2}}e(t)$$

We need to rewrite this function in canonical form:

$$y(t) = \frac{z^2 - \frac{8}{3}z - 1}{z^2 + \frac{2}{3}z + \frac{1}{9}}\frac{3}{\frac{1}{2}}e(t)$$

To ensure the same degree, it becomes:

$$y(t) = \frac{z^2 - \frac{8}{3}z - 1}{z^2 + \frac{2}{3}z^1 + \frac{1}{9}} \frac{3}{\frac{1}{2}} z^{-2} e(t)$$

Now, define the new White Noise as:

$$\eta(t) = \frac{3}{\frac{1}{2}} z^{-2} e(t) \rightarrow \eta(t) \sim WN\left(0, 36\right)$$

Thus, we have:

$$y(t) = \frac{1 - \frac{8}{3}z^{-1} - z^{-2}}{1 + \frac{2}{3}z^{-1} + \frac{1}{9}z^{-2}} \eta(t) \qquad \eta(t) \sim WN\left(0, 36\right)$$

The poles are at $z_{1,2} = -\frac{1}{3}$, and the zeros are at $z_{1,2} = -\frac{1}{3}, 3$. We have a zero that is not inside the unit circle.

Next, factorize the numerator and denominator:

$$y(t) = \frac{\left(1 + \frac{1}{3}z^{-1}\right)\left(1 - 3z^{-1}\right)}{\left(1 + \frac{1}{3}z^{-1}\right)\left(1 + \frac{1}{3}z^{-1}\right)} \eta(t) = \frac{1 - 3z^{-1}}{1 + \frac{1}{3}z^{-1}} \eta(t)$$

Use an all-pass filter to remove the zero at three:

$$y(t) = \frac{1 - 3z^{-1}}{1 + \frac{1}{3}z^{-1}} \frac{1 - \frac{1}{3}z^{-1}}{1 + \frac{1}{3}z^{-1}} \eta(t)$$

Redefined the White Noise as:

$$\xi(t) = \frac{1 - 3z^{-1}}{1 + \frac{1}{3}z^{-1}} \eta(t) \rightarrow \xi(t) \sim WN(0, 324)$$

The canonical form is:

$$y(t) = \frac{1 - \frac{1}{3}z^{-1}}{1 + \frac{1}{3}z^{-1}} \xi(t)$$

Now, with the canonical representation, compute the one-step ahead predictor as:

$$\hat{y}(t|t-1) = \frac{C(z) - A(z)}{C(z)} y(t) = \frac{1 + \frac{1}{3}z^{-1} - \left(1 - \frac{1}{3}z^{-1}\right)}{1 + \frac{1}{3}z^{-1}} y(t) = \frac{-\frac{2}{3}z^{-1}}{1 - \frac{1}{3}z^{-1}} y(t)$$

6.3 Exercise three

Consider the ARMAX process described by the expression:

$$y(t) = \frac{1}{3} y(t-1) + u(t-1) + 3e(t-1) + e(t-2) \qquad e(t) \sim WN(0, 1)$$

Let's compute the one-step ahead predictor.

Solution

The ARMAX process can be rewritten as:

$$y(t) = \frac{C(z)}{A(z)}e(t) + \frac{B(z)}{A(z)}u(t-1) = \frac{3z^{-1} + z^{-2}}{1 - \frac{1}{3}z^{-1}}e(t) + \frac{1}{1 - \frac{1}{3}z^{-1}}u(t-1)$$

The transfer function we consider is the one multiplied by the noise $e(t)$:

$$W(z) = \frac{3z^{-1} + z^{-2}}{1 - \frac{1}{3}z^{-1}}e(t) = \frac{3z + 1}{z^2 - \frac{1}{3}z}e(t)$$

By collecting $3z^{-1}$ at the numerator, we get:

$$W(z) = \frac{1 + \frac{1}{3}z^{-1}}{1 - \frac{1}{3}z^{-1}}\eta(t) \qquad \eta(t) \sim WN(0, 9)$$

The canonical form of the full ARMAX is:

$$y(t) = \frac{1 + \frac{1}{3}z^{-1}}{1 - \frac{1}{3}z^{-1}}\eta(t) + \frac{1}{1 - \frac{1}{3}z^{-1}}u(t-1)$$

The one-step ahead predictor for an ARMAX is:

$$\hat{y}(t|t-1) = \frac{F(z)}{C(z)}y(t) + \frac{B(z)E(z)}{C(z)}u(t-1) = \frac{\frac{2}{3}z^{-1}}{1 + \frac{1}{3}z^{-1}}y(t) + \frac{1}{1 + \frac{1}{3}z^{-1}}u(t-1)$$

6.4 Exercise four

Consider the process:

$$y(t) = 3 + v(t)$$

Let's find the predictor $\hat{y}(t|t-k)$ for all k when:

1. $v(t) \sim WN(0, 1)$
2. $v(t) = e(t) + \frac{1}{2}e(t-2) \quad e(t) \sim WN(0, 1)$

Solution

1. In this case, the process becomes:

$$y(t) = 3 + v(t) \qquad v(t) \sim WN(0, 1)$$

The only predictable part at any time different from zero is the constant, so:

$$\hat{y}(t|t-k) = 3$$

2. Here, the process becomes:

$$y(t) = 3 + e(t) + \frac{1}{2}e(t-2) \quad e(t) \sim WN(0, 1)$$

The expected value of the process is three, so we consider the unbiased process:

$$\tilde{y}(t) = y(t) - 3$$

Thus,

$$\tilde{y}(t) = e(t) + \frac{1}{2}e(t-2) = \frac{1 + \frac{1}{2}z^{-2}}{1}e(t)$$

The process is in canonical form. With long division, we get $F_1(z) = \frac{1}{2}z^{-2}$, $F_2(z) = \frac{1}{2}z^{-2}$, and $F_{3\to\infty} = 0$.

For the one-step ahead predictor:

$$\hat{\tilde{y}}(t|t-1) = \frac{\frac{1}{2}z^{-2}}{1+\frac{1}{2}z^{-2}}\tilde{y}(t) = -\frac{1}{2}\tilde{y}(t-2|t-3) + \frac{1}{2}\tilde{y}(t-2)$$

Thus,

$$\hat{y}(t|t-1) = -\frac{1}{2}y(t-2|t-3) + \frac{1}{2}y(t-2) + 3$$

For the two-step predictor:

$$\hat{y}(t|t-2) = -\frac{1}{2}y(t-2|t-4) + \frac{1}{2}y(t-2) + 3$$

For the general k:

$$\hat{y}(t|t-k) = 3$$

6.5 Exercise five

Consider the process:

$$y(t) = \frac{1}{4}y(t-2) + \eta(t-2) + \frac{1}{3}\eta(t-3) \qquad \eta(t) \sim WN(0,1)$$

Let's compute the predictor $\hat{y}(t|t-2)$.

Solution

The transfer function of the expression is:

$$y(t) = \frac{z^{-1} + \frac{1}{3}z^{-3}}{1 - \frac{1}{4}z^{-2}}\eta(t) = \frac{z^3 + \frac{1}{3}z}{z^3 - \frac{1}{4}z}\eta(t)$$

In canonical form it becomes:

$$y(t) = \frac{1 + \frac{1}{3}z^{-2}}{1 - \frac{1}{4}z^{-2}}(t) \qquad e(t) \sim WN(0,1)$$

All the poles and zeros are inside the unit circle, so the transfer function is stable.

By performing the long division for two steps, we get $F_2(z) = \frac{7}{12}z^{-2}$ and $E(z) = 1$. The predictor is:

$$\hat{y}(t|t-2) = e(t) + \frac{\frac{7}{12}}{1 - \frac{1}{4}z^{-2}}e(t-2) = \frac{\frac{7}{12}z^{-2}}{1 - \frac{1}{4}z^{-2}}e(t)$$

In terms of data, it becomes:

$$\hat{y}(t|t-2) = \frac{\frac{7}{12}z^{-2}}{1 - 1\frac{1}{4}z^{-2}} \frac{1 - \frac{1}{4}z^{-2}}{1 + \frac{1}{3}z^{-2}} y(t) = \frac{\frac{7}{12}z^{-2}}{1 + \frac{1}{3}z^{-2}} y(t)$$

In the time domain:

$$\hat{y}(t|t-2) = \frac{7}{12} y(t-2) - \frac{1}{3} z^{-2} \hat{y}(t-2|t-4)$$

CHAPTER 7

Exercise session VII

7.1 Exercise one

Consider the system:

$$\mathcal{S} : y(t) = e(t) + \frac{1}{2}e(t-1) \qquad e(t) \sim WN(0,1)$$

And the model:

$$\mathcal{M} : y(t) = ay(t-1) + \xi(t) \qquad \xi(t) \sim WN(0,\lambda^2)$$

Compute the value of a^* and λ^{*2}.

Solution

The steps are:

1. Compute the predictor:
$$\hat{y}(t|t-1) = ay(t-1)$$
2. Compute the prediction error (by substituting the real system to $y(t)$):
$$\varepsilon(t|t-1) = y(t) - ay(t-1) = \left(1 - az^{-1}\right) y(t) = \left(1 - az^{-1}\right)\left(1 + \frac{1}{2}z^{-1}\right) e(t)$$
That is:
$$\varepsilon(t|t-1) = e(t) + \left(\frac{1}{2} - a\right) e(t-1) - \frac{1}{2}ae(t-2)$$
3. Compute the variance of the prediction error:
$$\bar{J}(a^*) = \mathrm{Var}\left[\varepsilon\right] = \mathbb{E}\left[\varepsilon^2\right] = \frac{5}{4} + \frac{5}{4}a^2 - a$$
4. Derive with respect to the variable a^*:
$$\frac{d\bar{J}(a^*)}{da^*} = \frac{5}{2}a^* - 1$$
We want a minimum, so we set this derivative to zero:
$$\frac{5}{2}a^* - 1 \rightarrow a^* = \frac{2}{5}$$

5. The value of λ^{*2} can be computed by substituting the value of a^* into the variance function:
$$\lambda^{*2} = \frac{5}{4} + \frac{5}{4}\left(\frac{2}{5}\right)^2 - \frac{2}{5} = \frac{21}{20}$$
The prediction is good since it is similar to the variance of the White Noise.

The model is stable since the poles are inside the unit circle.

7.2 Exercise two

Consider the system:

$$\mathcal{S} : y(t) = e(t) + \frac{1}{2}e(t-1) \qquad e(t) \sim WN(0,1)$$

And the model:
$$\mathcal{M} : y(t) = \eta(t) + b\eta(t-1) \qquad \eta(t) \sim WN(0,\lambda^2)$$

Find the value of b^* and λ^{*2}.

Solution

Since both the model and the system are of the same type (Moving Average of order one), we can conclude that:

- $b^* = \frac{1}{2}$.
- $\lambda^{*2} = 1$.

Thus, we obtain the same formulation for the system. The model is stable since the poles are inside the unit circle.

7.3 Exercise three

Consider the system:

$$\mathcal{S} : y(t) = e(t) + \frac{1}{2}e(t-1) \qquad e(t) \sim WN(0,1)$$

And the model:

$$\mathcal{M} : y(t) = \frac{1}{1 + az^{-1} + bz^{-2}}\eta(t) \qquad \eta(t) \sim WN(0,\lambda^2)$$

Find the value of $\theta^* = \begin{bmatrix} a^* & b^* \end{bmatrix}$ and λ^{*2}.

Solution

The steps are:

1. Compute the predictor:
$$\hat{y}(t|t-1) = \frac{C(z) - A(z)}{C(z)} y(t)$$

2. Compute the prediction error (by substituting the real system to $y(t)$):
$$\varepsilon(t|t-1) = \frac{A(z)}{C(z)} y(t) = \frac{1 + az^{-1} + bz^{-2}}{1} \left(e(t) + \frac{1}{2} e(t-1) \right)$$

3. Compute the variance of the prediction error:
$$\bar{J}(\theta^*) = \mathrm{Var}\left[\varepsilon(t|t-1)\right] = \mathbb{E}\left[\varepsilon(t|t-1)^2\right] = \frac{5}{4}a^2 + \frac{5}{4}b^2 + a + ab + \frac{5}{4}$$

4. Derive with respect to the variable θ^*:
$$\begin{cases} \dfrac{\partial \theta^*}{\partial a^*} = \dfrac{5}{2}a^* + 1 + b^* \\ \dfrac{\partial \theta^*}{\partial b^*} = \dfrac{5}{2}b^* + a^* \end{cases}$$
We want a minimum, so we set those derivatives to zero:
$$\begin{cases} \dfrac{5}{2}a^* + 1 + b^* = 0 \\ \dfrac{5}{2}b^* + a^* = 0 \end{cases} \rightarrow \begin{cases} a = -\dfrac{10}{21} \\ b = \dfrac{4}{21} \end{cases}$$

5. The value of λ^{*2} can be computed by substituting the value of θ^* into the variance function:
$$\lambda^{*2} = 1.011$$
The prediction is good since it is similar to the variance of the White Noise.

The model is stable since the poles are inside the unit circle.

7.4 Exercise four

Consider the system:

$$\mathcal{S} : y(t) = 3e(t) + 9e(t-1) \qquad e(t) \sim WN(0,1)$$

And the model:
$$\mathcal{M} : y(t) = \eta(t) + b\eta(t-1) \qquad \eta(t) \sim WN(0,\lambda^2)$$
Find the value of b^* and λ^{*2}.

Solution

The system is not written in canonical form, so rewrite it as:

$$\begin{aligned}\mathcal{S}: y(t) &= 3\left(e(t) + 3e(t-1)\right) \\ &= 3\left(1 + 3z^{-1}\right)e(t) \\ &= 3\frac{\left(1 + 3z^{-1}\right)e(t)}{1 + \frac{1}{3}z^{-1}}\left(1 + \frac{1}{3}z^{-1}\right)e(t)\end{aligned}$$

Now we obtain:

$$\xi(t) = \frac{1 + 3z^{-1}}{1 + \frac{1}{3}z^{-1}}e(t) \qquad \xi(t) \sim WN(0, 81)$$

And the system expression becomes:

$$\mathcal{S}: y(t) = \xi(t) + \frac{1}{3}\xi(t)(t-1)$$

Now, with the same expression, we find:

- $b^* = \frac{1}{3}$.
- $\lambda^{*2} = 81$.

The model is stable since the poles are inside the unit circle.

CHAPTER 8

Exercise session VIII

8.1 Exercise one

Consider the system:

$$\mathcal{S} : y(t) = e(t) + \frac{1}{3}e(t-1) \qquad e(t) \sim WN(0,1)$$

And the model is:

$$\mathcal{M} : y(t) = -ay(t-1) + \eta(t) \qquad \eta(t) \sim WN(0,\lambda^2)$$

Find the value of a^* and λ^{*2}.

Solution

The steps are:

1. Compute the predictor:
$$\hat{y}(t|t-1) = -ay(t-1)$$

2. Compute the prediction error:
$$\varepsilon(t|t-1) = y(t) - \hat{y}(t|t-1) = (1+az^{-1})y(t) = y(t) + ay(t-1)$$

3. Compute the variance of the prediction error:
$$\bar{J}(a^*) = \text{Var}\left[\varepsilon(t)\right] = \mathbb{E}\left[(y(t)+ay(t-1))^2\right] = \gamma_y(0) + a^2\gamma_y(0) + 2a\gamma_y(1)$$

4. Derive with respect to the variable a^*:
$$\frac{d\bar{J}(a^*)}{da^*} = 2a^*\gamma_y(0) + 2\gamma_y(1)$$

   We want a minimum, so set this derivative to zero:
$$2a^*\gamma_y(0) + 2\gamma_y(1) = 0 \rightarrow a^* = -\frac{\gamma_y(1)}{\gamma_y(0)}$$

5. Find the value of the covariance from the system $\mathcal{S}$:

$$\gamma_y(0) = \mathbb{E}\left[\left(e(t) + \frac{1}{3}e(t-1)\right)^2\right] = \frac{10}{9}$$

$$\gamma_y(1) = \mathbb{E}\left[\left(e(t) + \frac{1}{3}e(t-1)\right)\left(e(t-1) + \frac{1}{3}e(t)\right)\right] = \frac{1}{3}$$

Thus,

$$a^* = -\frac{\gamma_y(1)}{\gamma_y(0)} = -\frac{3}{10}$$

6. The value of λ^{*2} can be computed by substituting the value of a^* into the variance function:

$$\lambda^{*2} = \gamma_y(0) + a^{*2}\gamma_y(0) + 2a^*\gamma_y(1) = \frac{10}{9} + \left(-\frac{3}{10}\right)^2\frac{10}{9} + 2\left(-\frac{3}{10}\right)\frac{1}{3} = \frac{91}{90}$$

This is similar to the variance of the White Noise, indicating good identification.

Since that absolute value of a is less than one, the system is in canonical form.

8.2 Exercise two

Consider the system:

$$\mathcal{S} : y(t) = \frac{1}{3}y(t-1) + u(t-1) + \eta(t) + \frac{1}{2}\eta(t-1)$$

Here $\eta(t) \sim WN(0, 1)$, $u(t) \sim WN(0, 1)$ are two independent White Noises And the model:

$$\mathcal{M} : y(t) = -ay(t-1) + bu(t-1) + e(t) \qquad e(t) \sim WN(0, \lambda^2)$$

Find the value of θ^* and λ^{*2}.

Solution

The steps are:

1. Compute the predictor:

$$\hat{y}(t|t-1) = \frac{F(z)}{C(z)}y(t) + \frac{B(z)E(z)}{C(z)}u(t) = -\frac{a}{1}y(t-1) + \frac{bu(t-1)}{1}$$

2. Compute the prediction error:

$$\varepsilon(t|t-1) = y(t) - \hat{y}(t|t-1) = y(t) + ay(t-1) - bu(t-1)$$

3. Compute the variance of the prediction error:

$$\begin{aligned}\bar{J}(\theta^*) &= \text{Var}\left[\varepsilon(t)\right] \\ &= \mathbb{E}\left[\left(y(t) + ay(t-1) - bu(t-1)\right)^2\right] \\ &= \left(1 + a^{*2}\right)\gamma_y(0) + b^{*2}\gamma_u(0) + 2a^*\gamma_y(1) - 2b^*\mathbb{E}\left[y(t)u(t-1)\right]\end{aligned}$$

4. Derive with respect to the variables a^* and b^*:

$$\frac{\partial \bar{J}(\theta^*)}{\partial a^*} = 2a^*\gamma_y(0) + 2\gamma_y(1)$$

$$\frac{\partial \bar{J}(\theta^*)}{\partial b^*} = 2b^*\gamma_y(0) + 2\mathbb{E}\left[u(t-1)y(t)\right]$$

We want a minimum, so we impose those derivatives to be null:

$$2a^*\gamma_y(0) + 2\gamma_y(1) = 0 \rightarrow a^* = -\frac{\gamma_y(1)}{\gamma_y(0)}$$

$$2b^*\gamma_u(0) + 2\mathbb{E}\left[u(t-1)y(t)\right] = 0 \rightarrow b^* = \frac{\mathbb{E}\left[u(t-1)y(t)\right]}{\gamma_u(0)}$$

5. We may now find the value of the covariance from the system $\mathcal{S}$:

$$\gamma_y(0) = \frac{69}{32}$$

$$\gamma_y(1) = \frac{7}{32}$$

As a result:

$$a^* = -\frac{\gamma_y(1)}{\gamma_y(0)} = -\frac{7}{69}$$

$$b^* = \frac{\mathbb{E}\left[u(t-1)y(t)\right]}{\gamma_y(0)} = -\frac{\gamma_y(1)}{\gamma_y(0)} = 1$$

6. The value of λ^{*2} can be computed by substituting the value of a^* and b^* into the variance function:

$$\lambda^{*2} = \left(1 + a^{*2}\right)\gamma_y(0) + b^{*2}\gamma_u(0) + 2a^*\gamma_y(1) - 2b^*\mathbb{E}\left[y(t)u(t-1)\right] = 1.134$$

That is similar to the variance of the White Noise, so the identification is good.

8.3 Exercise three

Consider the system:

$$\mathcal{S} : y(t) = -\frac{1}{2}y(t-1) + e(t) \qquad e(t) \sim WN(0,1)$$

And the model is:

$$\mathcal{M} : y(t) = -ay(t-2) + \eta(t) \qquad \eta(t) \sim WN(0,\lambda^2)$$

Find the value of a^* and λ^{*2}.

Solution

The steps are:

1. Compute the predictor:
$$\hat{y}(t|t-1) = \frac{C(z) - A(z)}{C(z)} y(t)$$
2. Compute the prediction error:
$$\varepsilon(t|t-1) = y(t) - \hat{y}(t|t-1) = y(t) - \frac{C(z) - A(z)}{C(z)} y(t) = y(t) + ay(t-2)$$
3. Compute the variance of the prediction error:
$$\bar{J}(a^*) = \text{Var}\left[\varepsilon(t)\right] = \gamma_y(0) + a^{*2}\gamma_y(0) + 2a^*\gamma_y(2)$$
4. Derive with respect to the variable a^*:
$$\frac{d\bar{J}(a^*)}{da^*} = 2a^*\gamma_y(0) + 2\gamma_y(2)$$
We want a minimum, so we impose those derivatives to be null:
$$2a^*\gamma_y(0) + 2\gamma_y(2) = 0 \rightarrow a^* = -\frac{\gamma_y(2)}{\gamma_y(0)}$$
5. We may now find the value of the covariance from the system $\mathcal{S}$:
$$\gamma_y(0) = \frac{4}{3}$$
$$\gamma_y(2) = \frac{1}{3}$$
As a result:
$$a^* = -\frac{\gamma_y(2)}{\gamma_y(0)} = -\frac{1}{4}$$
The system is stable since $|a^*| < 1$.
6. The value of λ^{*2} can be computed by substituting the value of a^* into the variance function:
$$\lambda^{*2} = \frac{5}{4}$$
That is similar to the variance of the White Noise, so the identification is good.

8.4 Exercise four

Consider the system:

$$\mathcal{S} : y(t) = 3e(t) + e(t-2) \qquad e(t) \sim WN(0, 1)$$

And the model:

$$\mathcal{M} : y(t) = a_1 y(t-1) + a_2 y(t-2) + \eta(t) \qquad \eta(t) \sim WN(0, \lambda^2)$$

Find the value of θ^* and λ^{*2}.

Solution

The steps are:

1. Compute the predictor:
$$\hat{y}(t|t-1) = \frac{C(z) - A(z)}{C(z)} y(t)$$

2. Compute the prediction error:
$$\varepsilon(t|t-1) = y(t) - \hat{y}(t|t-1) = y(t) - \frac{C(z) - A(z)}{C(z)} y(t) = y(t) - a_1 y(t-1) - a_2 y(t-2)$$

3. Compute the variance of the prediction error:
$$\begin{aligned}\bar{J}(\theta^*) &= \mathrm{Var}\left[\varepsilon(t)\right] \\ &= \mathbb{E}\left[\left(y(t) - a_1^* y(t-1) - a_2^* y(t-2)\right)^2\right] \\ &= \left(1 + a_1^{*2} + a_2^{*2}\right)\gamma_y(0) + 2a_1^*\left(a_2^* - 1\right)\gamma_y(1) - 2a_2^*\gamma_y(2)\end{aligned}$$

4. Derive with respect to the variables a_1^* and a_2^*:
$$\frac{\partial \bar{J}(\theta^*)}{\partial a_1^*} = 2a_1^*\gamma_y(0) + 2(a_2^* - 1)\gamma_y(1)$$
$$\frac{\partial \bar{J}(\theta^*)}{\partial a_2^*} = 2a_2^*\gamma_y(0) + 2a_1^*\gamma_y(1) - 2\gamma_y(2)$$
We want a minimum, so we impose those derivatives to be null:
$$\begin{cases} 2a_1^*\gamma_y(0) + 2(a_2^* - 1)\gamma_y(1) = 0 \\ 2a_2^*\gamma_y(0) + 2a_1^*\gamma_y(1) - 2\gamma_y(2) = 0 \end{cases}$$

5. We may now find the value of the covariance from the system $\mathcal{S}$:
$$\gamma_y(0) = 10$$
$$\gamma_y(1) = 0$$
$$\gamma_y(1) = 3$$
As a result:
$$a_1^* = 0$$
$$a_2^* = -\frac{3}{10}$$

6. The value of λ^{*2} can be computed by substituting the value of a_1^* and a_2^* into the variance function:
$$\lambda^{*2} = 9.1$$
That is similar to the variance of the White Noise (remember to consider the system in canonical form), so the identification is good.

CHAPTER 9

Exercise session IX

9.1 Exercise one

Consider a stationary process $y(t)$ of which we know:

$$y(1) = 1 \qquad y(2) = 0 \qquad y(3) = -1$$

And the model:

$$\mathcal{M} : y(t) = ay(t-1) + \xi(t) + a\xi(t-1) \qquad \xi(t) \sim WN(0, \lambda^2)$$

Let's compute the parameter a.

Solution

1. Check if the mean of the given samples is zero.
2. Compute the predictor of the model:

$$\hat{y}(t|t-1) = -a\hat{y}(t-1|t-2) + 2ay(t-1)$$

3. Compute the predictions on the given data applying the heuristic at time zero:

t	$y(t)$	$\hat{y}(t\|t-1)$
0	0	$\hat{y}(0\|-1) = 0$
1	1	$\hat{y}(1\|0) = 0$
2	0	$\hat{y}(2\|1) = 2a$
3	-1	$\hat{y}(3\|2) = -2a^2$

4. Compute the cost function:

$$\hat{J}_3 = \frac{1}{3}\sum_{i=1}^{3}\left(y(i) - \hat{y}(i|i-1)\right)^2 = \frac{1}{3}\left[(1-0)^2 + (1-2a)^2\left(-1+2a^2\right)^2\right]$$

Thus,

$$\hat{J}_3 = \frac{1}{3}\left(2 + 4a^2\right)$$

5. Find the derivative and equal to zero:
$$\frac{8}{3}\hat{a} = 0 \rightarrow \hat{a} = 0$$

9.2 Exercise two

Consider an input defined as:
$$u(t) = 1$$
With the following output:
$$y(t) = \begin{cases} 1 & t \text{ is odd} \\ -1 & t \text{ is even} \end{cases}$$
We have the data from $t = 0$ to $t = 15$. And the model:
$$\mathcal{M} : y(t) = ay(t-1) + bu(t-1) + \xi(t)$$
Here $\xi(t) \sin WN(0, \lambda^2)$

Let's identify the parameter θ^*.

Solution

1. Compute the predictor:
$$\hat{y}(t|t-1) = ay(t-1) + bu(t-1)$$
2. The cost function is:
$$\hat{J}_a = \frac{1}{15}\sum_{i=1}^{15} (y(i) - \hat{y}(i|i-1))^2$$
3. Apply the Least Squares formula. We can rewrite the model as:
$$\mathcal{M} : y(t) = \theta^T\varphi(t) + \xi(t) \qquad \varphi(t) = \begin{bmatrix} y(t-1) \\ u(t-1) \end{bmatrix}$$
In this way, the predictor becomes:
$$\hat{y}(t|t-1) = \theta^T\varphi(t)$$
At this point, we have:
$$\hat{\theta}_{15} = \left[\sum_{i=1}^{15} \varphi(i)\varphi(i)^T\right]^{-1} + \left[\sum_{i=1}^{15} \varphi(i)y(i)\right]$$
4. Compute the formula:
$$\hat{\theta}_{15} = \left[\sum_{i=1}^{15} \begin{bmatrix} y(i-1) \\ u(i-1) \end{bmatrix} \begin{bmatrix} y(i-1) & u(i-1) \end{bmatrix}\right]^{-1} + \left[\sum_{i=1}^{15} \begin{bmatrix} y(i-1) \\ u(i-1) \end{bmatrix} y(i)\right]$$
We have:
$$\hat{\theta}_{15} = \begin{bmatrix} \sum_{i=1}^{15} y(i-1)^2 = 15 & \sum_{i=1}^{15} y(i-1)u(i-1) \\ \sum_{i=1}^{15} u(i-1)y(i-1) & \sum_{i=1}^{15} u(t-1)^2 \end{bmatrix}^{-1} \begin{bmatrix} \sum_{i=1}^{15} y(i-1)y(i) \\ \sum_{i=1}^{15} u(i-1)y(i) \end{bmatrix}$$
$$= \begin{bmatrix} 15 & 1 \\ 1 & 15 \end{bmatrix}^{-1} \begin{bmatrix} -15 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

9.3 Exercise three

Consider a model:

$$\mathcal{M}: y(t) = \frac{1}{4}y(t-1) + \frac{1}{a}y(t-2) + e(t) \qquad e(t) \sim WN(0, \lambda^2)$$

We are given:

$$y(0) = 2 \qquad y(1) = 0 \qquad y(2) = -1$$

We also know that $y(t) = 0$ for all $t < 0$.

Let's find the parameter $\hat{a}$

Solution

1. Check if the mean of the given samples is zero. In this case, we don't need normalization since we have an infinite number of samples with a value of zero.

2. Compute the predictor of the model:

$$\hat{y}(t|t-1) = \frac{1}{4}y(t-1) + \frac{1}{a}y(t-2)$$

3. Compute the predictions on the given data applying the heuristic at time zero:

t	$y(t)$	$\hat{y}(t\|t-1)$
0	2	$\hat{y}(0\|-1) = 0$
1	0	$\hat{y}(1\|0) = \frac{1}{2}$
2	-1	$\hat{y}(2\|1) = \frac{2}{a}$

4. Compute the cost function:

$$\hat{J}_3 = \frac{1}{3}\sum_{i=1}^{3}(y(i) - \hat{y}(i|i-1))^2 = \frac{1}{3}\left[(2-0)^2 + \left(0 - \frac{1}{2}\right)^2 \left(-1 + \frac{2}{a}\right)^2\right]$$

Thus,

$$\hat{J}_3 = \frac{7}{4} + \frac{4}{3a} + \frac{1}{a^2}$$

5. Find the derivative and equal to zero:

$$\frac{4}{3} - \frac{a^2 - 2a}{a^4} = 0 \to \hat{a} = -2$$

Roots and poles are inside the unit circle, so the system is stable.