Image Analysis And Computer Vision
Theory

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Abstract

The course begins with an introduction to camera sensors, including their transduction, optics, geometry, and distortion characteristics. It then covers the basics of projective geometry, focusing on modeling fundamental primitives such as points, lines, planes, conic sections, and quadric surfaces, as well as understanding projective spatial transformations and projections.

The course continues with an exploration of camera geometry and single-view analysis, addressing topics like calibration, image rectification, and the localization of 3D models. This is followed by a study of multi-view analysis techniques, which includes 3D shape reconstruction, self-calibration, and 3D scene understanding.

Students will also learn about linear filters and convolutions, including space-invariant filters, the Fourier Transform, and issues related to sampling and aliasing. Nonlinear filters are discussed as well, with a focus on image morphology and operations such as dilation, erosion, opening, and closing, as well as median filters.

The course further explores edge detection and feature detection techniques, along with feature matching and tracking in image sequences. It addresses methods for inferring parametric models from noisy data and outliers, including contour segmentation, clustering, the Hough Transform, and RANSAC (random sample consensus).

Finally, the course applies these concepts to practical problems such as object tracking, recognition, and classification.

Contents

1 Optical sensors **1**
- 1.1 Camera . . . 1
- 1.2 Light rays deviation . . . 2
- 1.3 Focalization of parallel light rays . . . 3
- 1.4 Path of a light ray . . . 3
- 1.5 Pin-hole camera . . . 4
- 1.6 From real world to two-dimensional images . . . 4
- 1.7 Perspective and vanishing point . . . 5

2 Two-dimensional planar projective geometry **7**
- 2.1 Introduction . . . 7
- 2.2 Points . . . 7
- 2.3 Lines . . . 8
- 2.4 Conics . . . 12
- 2.5 Dual conics . . . 17
- 2.6 Transformations . . . 19

3 Two-dimensional reconstruction **27**
- 3.1 Introduction . . . 27
- 3.2 Affine reconstruction . . . 28
- 3.3 Shape reconstruction . . . 29
- 3.4 Accuracy issues . . . 32

4 Three-dimensional space projective geometry **34**
- 4.1 Introduction . . . 34
- 4.2 Points . . . 34
- 4.3 Planes . . . 34

CHAPTER 1

Optical sensors

1.1 Camera

Definition (*Camera*)**.** A camera is an optical sensor that generates data using electric transducers. It features an optical system designed to direct incoming light to its millions of photosensitive elements. Modern cameras are typically capable of recording 30 to 60 frames per second.

For simplicity, we will consider the optical system of a camera as a single lens with the following characteristics:

- *Spherical*: the lens is formed by the intersection of two spherical surfaces.
- *Thin*: the distance between the centers of the two spheres is nearly equal to the sum of their radii.
- *Small angles*: the light rays make only slight angles with respect to the optical axis.

These assumptions simplify the calculations involved in determining the path of a light ray as it passes through the lens. Specifically, the refraction of light at the boundary between two media is described by Snell's law:

$$\frac{\sin\theta_2}{\sin\theta_1} = \frac{n_1}{n_2}$$

Here:

- θ_1 and θ_2 are the angles between the normal at the surface and the direction of the light ray before and after crossing the boundary, respectively.
- n_1 and n_2 are the refractive indices of the two materials.

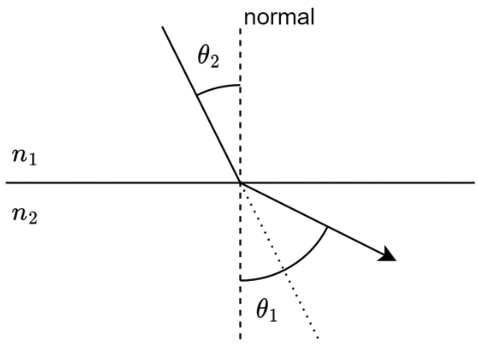
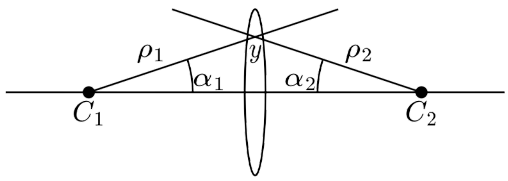


Figure 1.1: Snell's law

Definition (*Optical axis*)**.** The optical axis is the straight line that connects the centers of the two spheres that form the lens.

The angles of a ray passing through the centers of the spheres can be expressed as follows:

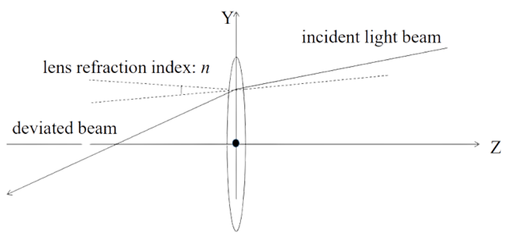
$$\alpha_1 = \frac{y_1}{\rho_1} \qquad \alpha_2 = -\frac{y_2}{\rho_2}$$



In this context, with the simplified lens, it is reasonable to assume:

$$y_1 = y_2 = y$$

1.2 Light rays deviation



For a lens with a refractive index n, the following equations apply:

$$\frac{\theta - \alpha_1}{\theta' - \alpha_1} \Rightarrow \frac{\sin(\theta - \alpha_1)}{\sin(\theta' - \alpha_1)} = n$$

$$\frac{\theta'' - \alpha_2}{\theta' - \alpha_2} \Rightarrow \frac{\sin(\theta'' - \alpha_2)}{\sin(\theta' - \alpha_2)} = n$$

Here:

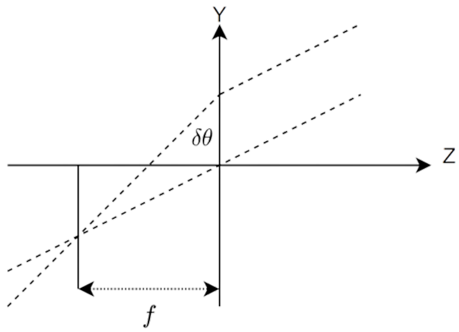
- θ is the angle of the incoming ray before entering the lens.
- θ' is the angle of the ray within the lens (not visible in the image).
- θ'' is the angle of the ray after exiting the lens.

By comparing these two equations, we can express the difference between the input angle θ and the output angle θ'' as:

$$\delta\theta = y(n-1)\left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right)$$

Here, the term $n-1$ reflects the influence of the lens material, while the term $\frac{1}{\rho_1} + \frac{1}{\rho_2}$ is determined by the curvature of the lens surfaces.

1.3 Focalization of parallel light rays



In the image, we observe two rays: one passing through the center of the lens and another that remains parallel to the first ray but passes through a different point. From this, we can make the following observations:

- When $Y = 0$, the ray experiences no deviation and continues straight without deflection.
- Using the relationship $Y = f \cdot \delta\theta$, we can express the focal length of the lens as follows:

$$f = \frac{1}{(n-1)\left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right)}$$

This indicates that all parallel rays converge at a common point known as the focal point, denoted as Z. The distance from the focal point to the y axis is given by:

$$Z = -f$$

1.4 Path of a light ray

To determine the trajectory of a light ray as it passes through a lens at any given position, you can follow these steps:

1. Draw a line parallel to the selected ray, passing through the center of the lens.

2. Identify the intersection point of this line with the focal plane.
3. The ray will travel from the point where it crosses the lens to the point on the focal plane.

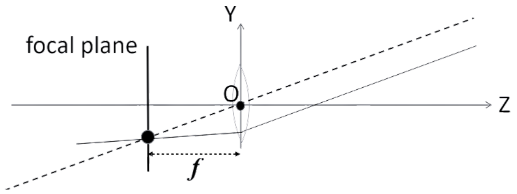


Figure 1.2: Path of a light ray through a lens

1.5 Pin-hole camera

To achieve a sharply focused image, it's crucial that each light ray converges precisely onto a single pixel on the camera's focal plane. To ensure this, the following conditions must be met:

- The distance between the lens and the light source, denoted as $Z(P)$, should be significantly greater than the lens aperture a ideally at least 1000 times larger.
- By positioning the screen at a distance Z from the lens, all rays can maintain parallel trajectories as they pass through the lens, resulting in a well-focused image.

The camera described is commonly known as a pin-hole camera, and it requires the following characteristics:

1. A thin spherical lens.
2. Utilization of small angles.
3. Ensuring that $Z(P) \gg a$.
4. Maintaining $Z = f$, where f represents the focal length.

1.6 From real world to two-dimensional images

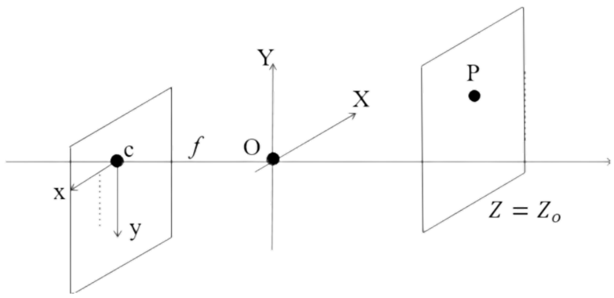
Images exist on a 2D plane, while the real world is three-dimensional, leading to a reduction of information compared to the original subject. This reduction is a result of perspective projection, which exhibits the following characteristics: nonlinearity, lack of shape preservation, and failure to maintain length ratios.

Using the triangle equality, we can express this perspective projection as:

$$x = f\frac{X}{Z} \qquad y = f\frac{Y}{Z}$$

To minimize information loss, one effective approach is to capture an image of a planar scene on a plane that is parallel to the image plane. This requires that:

$$Z = Z_0 = \text{constant}$$



In this scenario, the only difference between reality and the projection is a uniform downscaling, while other dimensions are preserved, yielding:

$$x = f\frac{X}{Z_0} = kX \qquad y = f\frac{Y}{Z_0} = kY$$

1.7 Perspective and vanishing point

When considering all lines parallel to the direction parameters $\begin{bmatrix}\alpha & \beta & 1\end{bmatrix}$, we can establish the following system of equations:

$$\begin{cases} X = X_0 + \alpha Z \\ Y = Y_0 + \beta Z \\ Z = 1 \cdot Z \end{cases}$$

To project these lines onto the 2D image using the triangle equality, we derive the following expressions:

$$x = f\frac{X}{Z} = f\frac{X_0 + \alpha Z}{Z} = f\alpha + \frac{X_0}{Z}$$
$$y = f\frac{Y}{Z} = f\frac{Y_0 + \beta Z}{Z} = f\beta + \frac{Y_0}{Z}$$

Next, we find the image of the point at infinity along these lines, which results in the point:

$$\begin{bmatrix}f\alpha & f\beta\end{bmatrix}$$

Remarkably, this image point is independent of the values of X_0 and Y_0. Therefore, all parallel lines share the same image of their points at infinity.

Definition (*Vanishing point*)**.** The image of the point at infinity of the lines is known as the vanishing point.

Consequently, we observe that all parallel lines in the real world project onto converging lines in the image.

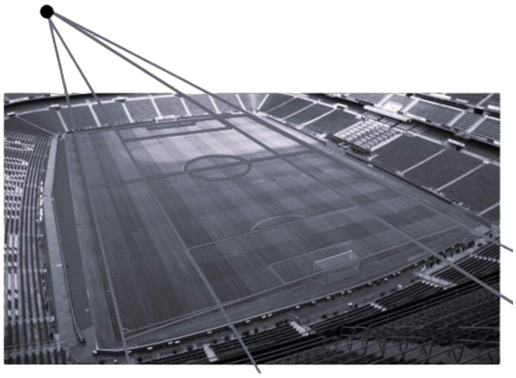


Figure 1.3: Vanishing point

CHAPTER 2

Two-dimensional planar projective geometry

2.1 Introduction

In the realm of planar geometry, the foundational elements consist of points, lines, conics, and dual conics. The transformations allowed within this geometry include projectivities, affinities, similarities, and isometries.

2.2 Points

To define points in Cartesian coordinates, we establish a Euclidean plane with a designated origin. Each point is uniquely represented by a pair of Cartesian coordinates, $\begin{bmatrix} x & y \end{bmatrix}$.

For image analysis, it is advantageous to use homogeneous coordinates. This involves constructing a 3D space with axes labeled $\begin{bmatrix} x & y & w \end{bmatrix}$. To represent a point, we assign three values, which allows for an infinite number of representations by varying the value of w. The relationship between Cartesian and homogeneous coordinates can be expressed as follows:

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ w \end{bmatrix} = w \begin{bmatrix} X \\ Y \\ 1 \end{bmatrix}$$

Consequently, a vector $\mathbf{x} = \begin{bmatrix} x & y & w \end{bmatrix}^T$ and all its nonzero multiples, including $\begin{bmatrix} \frac{x}{w} & \frac{y}{w} & 1 \end{bmatrix}^T$, represent the same point in Cartesian coordinates $\begin{bmatrix} X & Y \end{bmatrix}^T = \begin{bmatrix} \frac{x}{w} & \frac{y}{w} \end{bmatrix}^T$ on the Euclidean plane.

Property 2.2.1 (Homogeneity)**.** Any vector $\mathbf{x}$ is equivalent to all its nonzero multiples $\lambda\mathbf{x}$, where $\lambda \neq 0$, as they denote the same point.

The null vector does not represent any point.

Definition (*Projective plane*)**.** We define the projective plane as:

$$\mathbb{P}^2 = \left\{ \begin{bmatrix} x & y & w \end{bmatrix}^T \in \mathbb{R}^3 \right\} \setminus \left\{ \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T \right\}$$

Example:
The origin of the plane is defined as:

$$\begin{bmatrix}0 & 0 & 1\end{bmatrix}^T$$

A generic point in homogeneous coordinates can easily be transformed into Cartesian coordinates by simple division. For instance, the point:

$$\begin{bmatrix}0 & 8 & 4\end{bmatrix}^T$$

in Cartesian coordinates is:

$$\begin{bmatrix}\frac{x}{w} & \frac{y}{w}\end{bmatrix}^T = \begin{bmatrix}\frac{0}{4} & \frac{8}{4}\end{bmatrix} = \begin{bmatrix}0 & 4\end{bmatrix}$$

Consider a point $\mathbf{x} = \begin{bmatrix}x & y & w\end{bmatrix}^T$, and let w slowly decrease from $w = 1$. As w decreases, the point moves in a constant direction $\begin{bmatrix}x & y\end{bmatrix}$, distancing itself from the origin. As w approaches 0, the point tends toward infinity along the direction $\begin{bmatrix}x & y\end{bmatrix}$.

Definition (*Point at the infinity along the direction*)**.** We define the point at the infinity along the direction $\begin{bmatrix}x & y\end{bmatrix}$ as:

$$\mathbf{x} = \begin{bmatrix}x \\ y \\ w\end{bmatrix}$$

Points at infinity, representing directions, exist outside the Euclidean plane and are well-defined within the projective plane. Thus, the projective plane encompasses not only the Euclidean plane but also these points at infinity.

2.3 Lines

In the Euclidean plane, a line is typically defined by the equation:

$$aX + bY + c = 0$$

In the homogeneous plane, lines are represented as:

$$a\frac{x}{w} + b\frac{y}{w} + c = 0 \Longrightarrow ax + by + cw = 0$$

This equation can also be expressed using two vectors, denoted as $\mathbf{l}^T$ and $\mathbf{x}$, as follows:

$$\begin{bmatrix}a & b & c\end{bmatrix}\begin{bmatrix}x \\ y \\ w\end{bmatrix} = 0$$

Here, the vector $\mathbf{l} = \begin{bmatrix}a & b & c\end{bmatrix}^T$ epresents a line, with all its nonzero multiples also representing the same line.

Property 2.3.1 (Homogeneity)**.** Any vector $\mathbf{l}$ is equivalent to all its nonzero multiples, denoted as $\lambda\mathbf{l}$ (where $\lambda \neq 0$), since they denote the same line.

The coefficients a, b, and c are known as the homogeneous parameters of the line.

Similar to numbers, there are multiple equivalent representations for a single line, specifically all nonzero multiples of the unit normal vector. However, the null vector does not represent any lines.

Definition (*Projective dual plane*)**.** The projective dual plane is defined as:

$$\mathbb{P}^2 = \left\{ \begin{bmatrix} a & b & c \end{bmatrix}^T \in \mathbb{R}^3 \right\} \setminus \left\{ \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T \right\}$$

Property 2.3.2. If the third parameter is zero, denoted as $\mathbf{l} = \begin{bmatrix} a & b & 0 \end{bmatrix}^T$, then the line passes through the point $\begin{bmatrix} 0 & 0 \end{bmatrix}$.

Property 2.3.3. In the Euclidean plane, the direction $\begin{bmatrix} a & b \end{bmatrix}$ is perpendicular to the line represented by $\mathbf{l} = \begin{bmatrix} a & b & c \end{bmatrix}^T$.

Property 2.3.4. Two lines, $\mathbf{l} = \begin{bmatrix} a & b & c \end{bmatrix}^T$ and $\mathbf{l} = \begin{bmatrix} a & b & c' \end{bmatrix}^T$, are considered parallel if they share the same direction, represented by $\begin{bmatrix} b & -a \end{bmatrix}$.

> **Example:**
> The Cartesian axes are defined as:
>
> $$\mathbf{l}_x = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T$$
>
> $$\mathbf{l}_y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$$

In this context, the incidence relation of a line $\mathbf{l}^T\mathbf{x} = 0$ is defined when the point $\mathbf{x}$ lies on the line $\mathbf{l}$ or when the line $\mathbf{l}$ goes through the point $\mathbf{x}$.

Definition (*Line at the infinity*)**.** The line

$$\begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix} = w = 0$$

is called the line at the infinity, denoted as $\mathbf{l}_\infty = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$.

The principle of duality between points and lines states that the incidence relation is commutative, as the dot product is commutative.

To find the intersection of two lines l_1 and l_2, the following condition is imposed:

$$\begin{bmatrix} l_1^T \\ l_2^T \end{bmatrix} x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This equation leads to finding the right null space of the first column vector:

$$x = \text{RNS}\left(\begin{bmatrix} l_1^T \\ l_2^T \end{bmatrix} \right)$$

The system is under-determined, meaning there is only one intersection point between two lines, which can be represented in multiple ways in homogeneous coordinates. In 2D projective geometry, the vector x is orthogonal to both lines and can be found using the cross product:

$$x = l_1 \times l_2$$

Example:

Suppose we have two parallel lines, $l_1 = \begin{bmatrix} a & b & c_1 \end{bmatrix}^T$ and $l_2 = \begin{bmatrix} a & b & c_2 \end{bmatrix}^T$. The point that is common to both lines can be found using the system:

$$\begin{cases} ax + by + c_1 w = 0 \\ ax + by + c_2 w = 0 \end{cases}$$

The solution is $x = \begin{bmatrix} b & -a & 0 \end{bmatrix}^T$, which represents the point at infinity along the direction of both lines.

The line passing through two points can be determined using the dual of the previous problem, expressed as:

$$\begin{bmatrix} x_1^T \\ x_2^T \end{bmatrix} l = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

In 2D, this simplifies to:

$$l = x_1 \times x_2$$

Property 2.3.5. A point x obtained by a linear combination $x = \alpha x_1 + \beta x_2$ of two points x_1 and x_2 lies on the line l through x_1 and x_2.

Proof. The line l passing through both points satisfies $l^T x_1 = 0$ and $l^T x_2 = 0$. By adding α times the first equation to β times the second one, we obtain:

$$0 = l^T \left(\alpha x_1 + \beta x_2\right) = l^T x = 0$$

□

This establishes the duality between co-linear and concurrent.

Theorem 2.3.1. *For any true sentence containing the words: point, line, is on, goes through, co-linear and concurrent there exists a dual sentence (also true) obtained by replacing each occurrence of:*

- *Point* $\Leftrightarrow$ *line.*
- *Is on* $\Leftrightarrow$ *goes through.*
- *Co-linear* $\Leftrightarrow$ *concurrent.*

Within the euclidean plane, the direction normal to the line $l = \begin{bmatrix} a & b & c \end{bmatrix}^T$ is represented by $\begin{bmatrix} a & b \end{bmatrix}$. This relationship between lines can be explained by understanding that the angle between two lines is equal to the angle between their respective normal vectors. The formula for the angle between two vectors is:

$$\cos\theta = \frac{u \cdot v}{|u|\,|v|}$$

This applies to the angle between two lines $l_1 = \begin{bmatrix} a_1 & b_1 & c_1 \end{bmatrix}^T$ and $l_2 = \begin{bmatrix} a_2 & b_2 & c_2 \end{bmatrix}^T$. In this context, it is the angle between their respective normal directions $\begin{bmatrix} a_1 & b_1 \end{bmatrix}$ and $\begin{bmatrix} a_2 & b_2 \end{bmatrix}$, which can be calculated as follows:

$$\cos\theta = \frac{a_1 a_2 + b_1 b_2}{\sqrt{\left(a_1^2 + b_1^2\right)\left(a_2^2 + b_2^2\right)}}$$

Now, consider a line with four points related as follows:

$$X_1 = \alpha_1 Y + \beta_1 Z$$

$$X_2 = \alpha_2 Y + \beta_2 Z$$

The cross ratio is given by:

$$CR_{X_1,X_2,Y,Z} = \frac{c-a}{c-b} \Big/ \frac{d-a}{d-b} = \frac{\beta_1/\alpha_1}{\beta_2/\alpha_2}$$

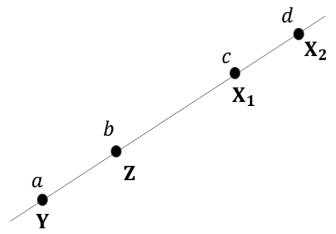
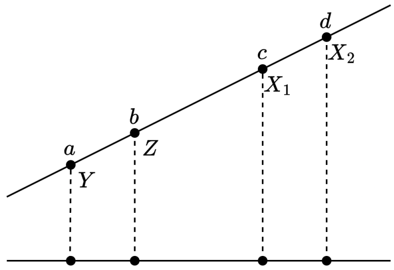


Figure 2.1: Line with the point of previous problem

Proof. Since the abscissae are proportional, the abscissae can be replaced by the X coordinate, as illustrated in the figure below:



The relation:

$$CR_{X_1,X_2,Y,Z} = \frac{c-a}{c-b} \Big/ \frac{d-a}{d-b}$$

still holds. If we consider $Y = \begin{bmatrix} y & * & v \end{bmatrix}^T$ and $Z = \begin{bmatrix} z & * & w \end{bmatrix}^T$, we can determine that:

$$X_1 = \begin{bmatrix} \alpha_1 y + \beta_1 z \\ * \\ \alpha_1 v + \beta_1 w \end{bmatrix} \qquad X_2 = \begin{bmatrix} \alpha_2 y + \beta_2 z \\ * \\ \alpha_2 v + \beta_2 w \end{bmatrix}$$

The difference between the X coordinates of X_1 and Y is calculated as:

$$c - a = \frac{\beta_1(zv - yw)}{(\alpha_1 y + \beta_1 z)v}$$

Similarly, the difference between the X coordinates of X_1 and Z is:

$$c - b = \frac{-\alpha_1(zv - yw)}{(\alpha_1 y + \beta_1 z)w}$$

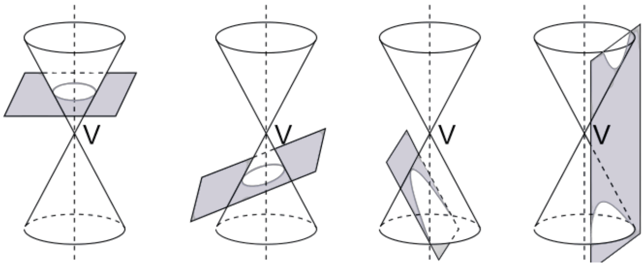
By substituting these expressions, we obtain:

$$\frac{c-a}{c-b} = -\frac{\beta_1 w}{\alpha_1 v} \qquad \frac{d-a}{d-b} = -\frac{\beta_2 w}{\alpha_2 v}$$

□

2.4 Conics

Conics are geometric shapes that result from the intersection of cones with planes. They include circles, ellipses, parabolas, and hyperbolas, as depicted in the following figure:



Definition. A point x is considered to be on a *conic* C if it satisfies a homogeneous quadratic equation of the form:

$$x^T C x = 0$$

Where C is a symmetric matrix is a symmetric matrix, which is a convention.

Conics are curves described by second-degree equations in the plane. In Euclidean coordinates, a conic can be expressed as:

$$aX^2 + bXY + cY^2 + dX + eY + f = 0$$

In homogeneous coordinates, it becomes:

$$ax^2 + bxy + cy^2 + dxw + eyw + fw^2 = 0$$

Alternatively, it can be represented in matrix form as:

$$x^T \begin{bmatrix} a & b/2 & d/2 \\ b/2 & c & e/2 \\ d/2 & e/2 & f \end{bmatrix} x = 0$$

Conics have five degrees of freedom, which means that five points are required to uniquely define a conic.

Example:
A circle can be expressed in Cartesian coordinates as:

$$(X - X_0)^2 + (Y - Y_0)^2 - r^2 = 0$$

In homogeneous coordinates, it is represented as:

$$\begin{bmatrix} x & y & w \end{bmatrix} \begin{bmatrix} 1 & 0 & -X_0 \\ 0 & 1 & -Y_0 \\ -X_0 & -Y_0 & X_0^2 + Y_0^2 - r^2 \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix} = 0$$

When you have a quadratic equation representing a conic and a linear equation for a line, their intersection results in a second-degree equation for the point x. Consequently, there will always be two intersection points between a line and a conic. These intersection points can fall into one of the following categories:

- Real and distinct: this occurs when the line and conic intersect at two separate, real points.
- Real and coincident: in this case, the line and conic intersect at a single real point, but it is a repeated or double root of the equation.
- Complex and distinct: the intersection points are two complex conjugate points.
- Complex and coincident: the line and conic intersect at a single complex point, and it is a repeated or double root.

This behavior is due to the fundamental theorem of algebra, which guarantees that a second-degree equation will have exactly two solutions when considering complex numbers.

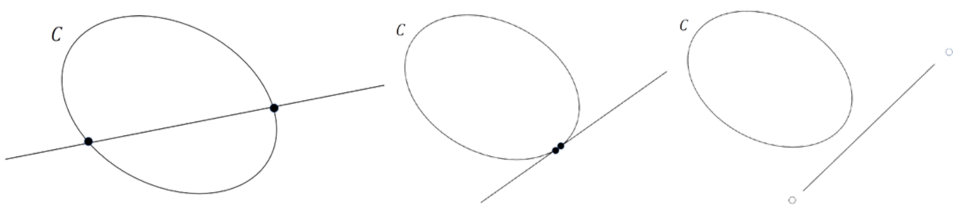


Figure 2.2: Intersection with two real roots, two coincident roots and two imaginary roots

The intersection between the line at infinity and a conic results in the following scenarios:

- Parabola: when there are two coincident solutions, indicating the point at infinity along the axis.
- Ellipse: when there are two complex-conjugate solutions, meaning there are no real solutions.
- Hyperbola: when there are two real and distinct solutions, representing lines that serve as the asymptotes.

Circular points

Example:
When we intersect a circumference and the line at infinity, we obtain the following system:

$$\begin{cases} x^2 - 2X_0w + X_0^2w^2 + y^2 - 2Y_0w + Y_0^2w^2 - r^2w^2 = 0 \\ w = 0 \end{cases}$$

This system simplifies to:

$$x^2 + y^2 = 0$$

It's evident that the parameters of the circumference (center and radius) have disappeared from the equation. Consequently, the two intersection points are the same for all circumferences.

Definition. The two intersection points remain the same for all circumferences when intersected with the line at infinity are referred to as the *circular points*. These points are defined as:

$$I = \begin{bmatrix} 1 \\ i \\ 0 \end{bmatrix} \qquad J = \begin{bmatrix} 1 \\ -i \\ 0 \end{bmatrix}$$

Polar line

Definition. Given a point y and a conic C in the plane, the line $l = Cy$ is called the *polar line* of point y with respect to the conic C.

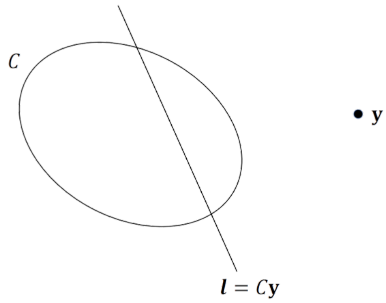


Figure 2.3: Example of polar line

Harmonic tuples

Definition. A 4-tuple of co-linear points A, B, C, D, whose cross ratio is = -1, is referred to as a *harmonic tuple*.

This specific value of the cross ratio is also shared by other 4-tuples of co-linear points. A notable example is:

$$(T, Z, \text{mid\_point}(Y, Z), P(\text{at the infinity}))$$

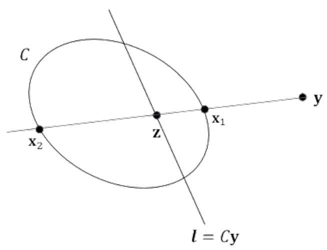
Furthermore, if (A, B, C, D) is a harmonic 4-tuple, then (C, D, A, B) is also harmonic.

Definition. In a harmonic tuple (A, B, C, D), points A and B are said to be *conjugate* of each other concerning points C and D.

Since the cross ratio of a harmonic tuple is negative, it follows that two conjugate points, A and B, concerning C and D, are positioned in such a way that one is located within the segment (C, D), while the other is situated outside this segment.

Polar line and harmonic tuples

Take any point z on the polar line $l = Cy$ and then consider the line passing through points y and z. Let's denote by x_1 and x_2 the points at which this line intersects the conic.



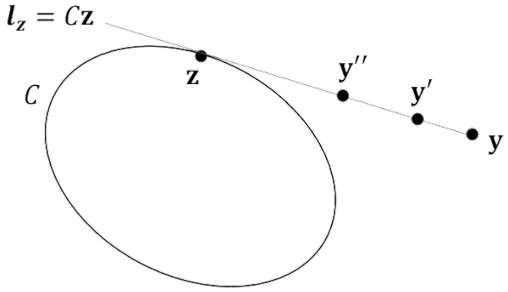
Theorem 2.4.1. *Let x_1 and x_2 represent the points at which the line passing through y and z intersects the conic C. In this case, y and z are conjugate with respect to x_1 and x_2.*

The polar line $l = Cy$ represents the set of points that are conjugate to y with respect to the conic C. More precisely, it includes points that are conjugate with respect to the intersection points of C with any line passing through y.

Polar line and tangency points

As the line through y approaches tangency with the conic C, the points x_1 and x_2 coincide with the points of tangency to C. Consequently, the conjugate point z, which remains within the interval (x_1, x_2), also coincides with the tangency point. This applies to any line that is tangent to C from the point y. Therefore, we have established that the polar line $l = Cy$ passes through the points of tangency from y to the conic C.

This leads us to the conclusion that if a point z lies on the conic C, then point y is one of its conjugates with respect to the same conic. The tangent line lz to the conic C passing through point z is the set of points that are conjugate to z. Therefore, we can assert that lz is the polar line of z with respect to conic C.



In the accompanying illustration, you can observe that the polar line $lz = Cz$ for a point z situated on the conic C corresponds to the tangent line to the conic C at the point z.

> **Example:**
> Consider a circumference with radius r centered in the origin of the plane and the point $y = \begin{bmatrix} X & 0 & 1 \end{bmatrix}^T$. The equation of the polar line is given by:
>
> $$l = Cy = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -r^2 \end{bmatrix} \begin{bmatrix} X \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} X \\ 0 \\ -r^2 \end{bmatrix}$$
>
> Therefore, the Cartesian equation of the polar line becomes:
>
> $$Xx - r^2 = 0 \rightarrow x = \frac{r^2}{X}$$
>
> This equation describes a vertical line.

From the previous example, we can conclude that the polar of a point P with respect to a circle is a line that is perpendicular to the line segment connecting the center of the circle to point P.

Example:
Consider a circumference with radius r centered in the origin of the plane and the point $y = \begin{bmatrix} x & 0 & 0 \end{bmatrix}^T$. The equation of the polar line is given by:

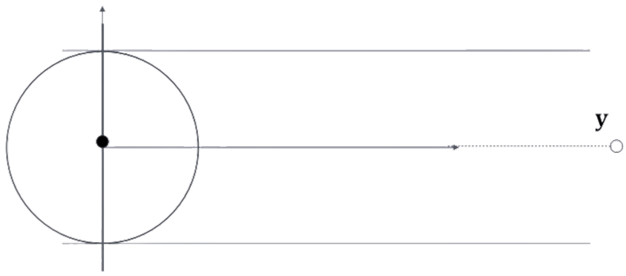
$$l = Cy = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -r^2 \end{bmatrix} \begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} X \\ 0 \\ 0 \end{bmatrix}$$

Therefore, the Cartesian equation of the polar line becomes:

$$Xx = 0 \rightarrow X = 0$$

This equation describes the diameter of the circumference perpendicular at the direction of the point y.

Tangent lines emerging from a point at infinity are always parallel. Consequently, the points of tangency lie along a diameter that is perpendicular to the direction of these parallel tangents.



Example:
Consider a circumference with radius r centered in the origin of the plane and the point $y = \begin{bmatrix} x & 0 & 0 \end{bmatrix}^T$. The equation of the polar line is given by:

$$l = Cy = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -r^2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -r^2 \end{bmatrix}$$

Therefore, the Cartesian equation of the polar line becomes:

$$-r^2 w = 0 \rightarrow X = 0$$

This equation describes the line at the infinity.

Here are the general properties of the polar lines.

Property 2.4.1. The polar line of any point at infinity is a diameter.

Property 2.4.2. Any diameter goes through the center of the circle.

Property 2.4.3. The center is conjugate to every point at infinity.

Property 2.4.4. All points at infinity are conjugate to the center.

Property 2.4.5. The polar of the center is the line that includes all the points at infinity.

Property 2.4.6. The polar line of the center is the line at infinity.

Degenerate conics

Definition. A *non-degenerate conic* is a conic where the matrix C is non-singular, indicating that:

$$\text{rank}(C) = 3$$

Conversely, a *degenerate conic* is a conic for which the matrix C is singular, characterized by:

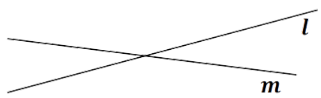
$$\text{rank}(C) < 3$$

There are two distinct scenarios to consider:

- When $\text{rank}(C) = 2$, any symmetric 3×3 matrix C can be expressed as:

$$C = lm^T + ml^T$$

  Here, l and m are column vectors. The conic corresponds to the set of points x that satisfy $x^T Cx = 0$. This equation is met when either $x^T l = 0$ or $m^T x = 0$. Consequently, x lies on the union of lines represented by l and m



- When $\text{rank}(C) = 1$, a symmetric 3×3 matrix C can be expressed as:

$$C = ll^T$$

  In this case, l is a column vector. The conic consists of the points x that satisfy $x^T Cx = 0$. This equation holds when $x^T l = 0$ is met (twice). Thus, x is on the repeated line represented by l.

2.5 Dual conics

Definition

Definition. A *dual conic* is a set of lines l that satisfy equation:

$$l^T C^* l = 0$$

where C^* is a 3×3 symmetric matrix.

A *non-degenerate dual conic* is a dual conic whose matrix C^* is non-singular:

$$\text{rank}(C^*) = 3$$

Consider a non-degenerate conic, denoted as C, and the collection of all lines l that are tangents to it. For each point c on the conic C, there exists a line l that is tangent to C. Since l is the polar line of x with respect to C, we can express it as $l = Cc$. Consequently, we can represent x as:

$$x = C^{-1}l$$

Moreover, given that C is a symmetric matrix, we have:

$$x^T = l^T l^{-T} = l^T C^{-1}$$

Now, considering that the point x lies on the conic C, we have:

$$x^T C x = 0$$

By substituting the previously derived expressions, we arrive at:

$$l^T C^{-1} l = 0$$

This equation represents a quadratic homogeneous equation on l. Therefore, we can conclude that for the dual conic holds $C^* = C^{-1}$. We can also note that a non-degenerate dual conic C^* is the collection of lines that are tangent to a non-degenerate conic C.

Degenerate dual conics

Definition. A *degenerate dual conic* is a conic where the matrix C^* is singular:

$$\text{rank}(C^*) < 3$$

There are two possible scenarios to consider:

- When $\text{rank}(C^*) = 2$, any symmetric 3×3 matrix C^* can be expressed as:

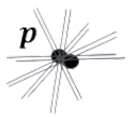
$$C^* = pq^T + qp^T$$

In this case, the conic represents the line l passing through point p or the line l passing through point q.

- When $\text{rank}(C^*) = 1$, any symmetric 3×3 matrix C^* can be expressed as:

$$C^* = pp^T$$

In this situation, the conic corresponds to the line l going through point p repeated twice.



Definition. The degenerate dual conic $C^* = pq^T + qp^T$ going through two circular point p and q is known as the *conic dual to the circular points*, and it can be expressed as:

$$C^*_\infty = IJ^T + JI^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

2.6 Transformations

Definition. A *projective mapping* between a projective plane $\mathbb{P}^2$ and another projective plane $\mathbb{P}^{'2}$ is an invertible mapping which preserves co-linearity:

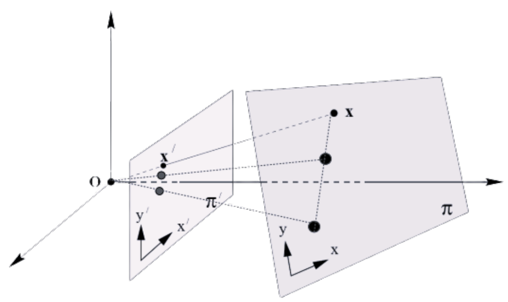
$$h : \mathbb{P}^2 \to \mathbb{P}^{'2}, x^{'} = h(x), x_1, x_2, x_3 \text{ are colinear}$$

$$\Leftrightarrow$$

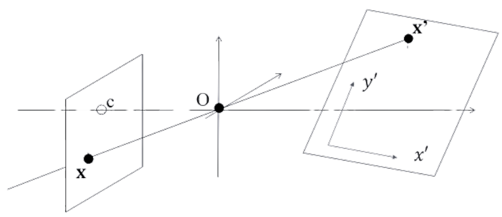
$$x_1^{'} = h(x_1), x_2^{'} = h(x_2), x_3^{'} = h(x_3) \text{ are colinear}$$

Projective mapping is also called projectivity or homography.

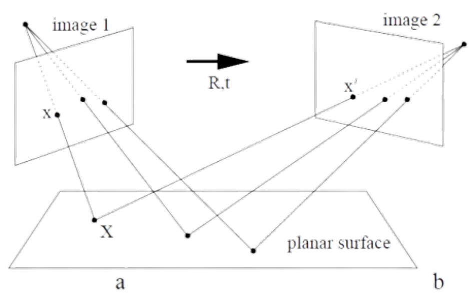
Example:
Mappings between two planes induced by central projection are projective, since they preserve co-linearity.



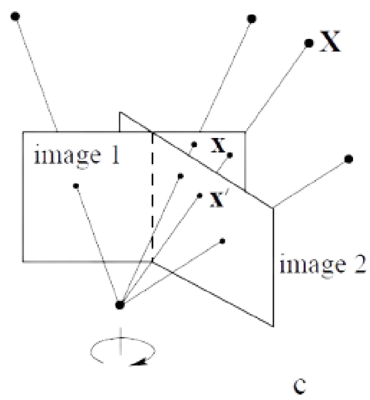
Mapping between a planar scene and its image is a homography, since it is induced by a central projection



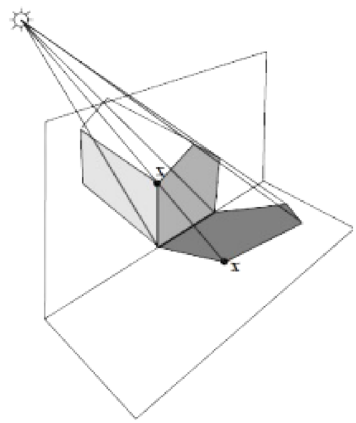
Mapping between two images of a planar scene is a homography.



Two images of a 3D scene, taken by a camera rotating around its center are related by a homography, since the second image is a central projection of the first image.



The shadow cast by a planar silhouette onto a ground plane is a projective transformation of the planar silhouette, since they are related by a central projection.



Theorem 2.6.1. *A mapping $h : \mathbb{P}^2 \to \mathbb{P}'^2$ is projective if and only if there exists an invertible 3×3 matrix H such that for any point in $\mathbb{P}^2$ represented by the vector x, is $h(x) = Hx$, where:*

$$H = \begin{bmatrix} h_{11} & h_{12} & h13 \\ h_{21} & h_{22} & h23 \\ h_{31} & h_{32} & h33 \end{bmatrix}$$

Projective mappings are linear when expressed in homogeneous coordinates, but they do not exhibit linearity when represented in Cartesian coordinates.

According to the theorem, if we have $h(x) = x^{'} = Hx$, then multiplying the matrix H by any nonzero scalar λ still satisfies the relation for the same points, giving us $x^{'} = \lambda Hx$. Therefore, any nonzero scalar multiple of the matrix H represents the same projective mapping as H. As a result, we can conclude that H is a homogeneous matrix. Despite having nine entries, it possesses only eight degrees of freedom, specifically the ratios between its elements. Consequently, we can estimate H using just four point correspondences. Each point correspondence, expressed as $x^{'} = Hx$, provides two independent equations in this estimation process.

Definition. A *homography* transforms various geometric entities as follows:

1. It maps a point x to a point $x^{'}$, where the transformation is expressed as:
$$x \rightarrow Hx = x^{'}$$
2. It maps a line l to a line $l^{'}$, and this transformation is represented as:
$$l \rightarrow H^{-T}l = l^{'}$$
3. It maps a conic C to a conic $C^{'}$, and the transformation is given by:
$$C \rightarrow H^{-T}CH^{-1} = C^{'}$$
4. It maps a dual conic C^{*} to a dual conic $C^{*'}$, with the transformation being:
$$C^{*} \rightarrow HC^{*}H^{T} = C^{*'}$$

of mapping two. To transform the equation of the line in terms of x, given by $l^{T}x = 0$, into a constraint on $x^{'} = Hx$, we combine the two equations, resulting in a linear equation on $x^{'}$:
$$l^{'T}x^{'} = 0$$
Here, $l^{'T} = l^{T}H^{-1}$. Thus, we have:
$$l^{'} = H^{-T}l$$
□

of mapping three. To transform the equation of the conic in terms of x, given by $x^{T}Cx = 0$, into a constraint on $x^{'} = Hx$, we have$x = H^{-1}x^{'}$ and $x^{T} = x^{iT}H^{-T}$. Combining these three equations, we obtain a linear equation on $x^{'}$:
$$x^{'T}C^{'}x^{'} = 0$$
Hence, we have:
$$C^{'} = H^{-T}CH^{-1}$$
□

of mapping four. or the transformation of a dual conic, we apply the same idea, yielding:
$$C^{*'} = HC^{*}H^{T}$$
□

The point-line incidence is preserved.

Proof. Let x be a point on the line l. This is expressed as $l^{T}x = 0$. When we apply the projective transformation H to both $x$and l, resulting in $Hx = x^{'}$ and $H^{-1}l = l^{'}$, they remain incident if $l^{'T}x^{'} = 0$:
$$l^{'T}x^{'} = l^{T}H^{-1}x^{'} = l^{T}H^{-1}Hx = l^{T}x = 0$$
□

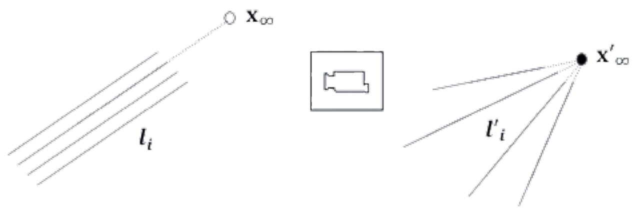
Vanishing points and vanishing line

The point that is common to both parallel lines $l_1 = \begin{bmatrix} a & b & c_1 \end{bmatrix}^T$ and $l_2 = \begin{bmatrix} a & b & c_2 \end{bmatrix}^T$ is the point $x = \begin{bmatrix} b & -a & 0 \end{bmatrix}^T$. This point is situated at infinity along the direction of both lines. When seeking the common point of the infinite lines l_i, we find that they all share the same point:

$$x_\infty = \begin{bmatrix} b & -a & 0 \end{bmatrix}^T$$

Hence, it becomes apparent that all these lines converge at $\begin{bmatrix} b & -a & 0 \end{bmatrix}^T$.

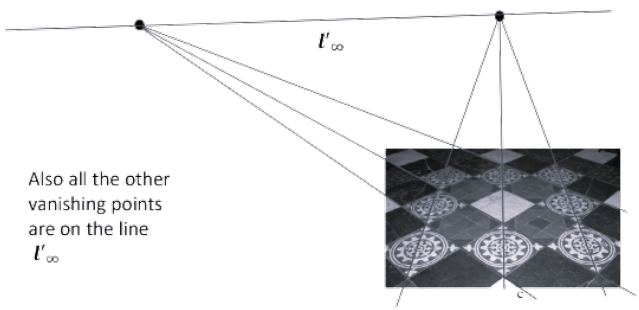
If we apply a projective transformation to all the aforementioned parallel lines l_i, we obtain the transformed lines $l_i^{'}$. The common point x_∞, shared by all lines l_i, is mapped to a point $x_\infty^{'}$ which belongs to each of the lines $l_i^{'}$.



Therefore, we can assert that all lines $l_i^{'}$ intersect at the point $x_\infty^{'} = Hx_\infty$, referred to as the vanishing point associated with the direction $(b, -a)$ of the parallel lines.

Theorem 2.6.2. *The image of a set of parallel lines l_i is a set of lines $l_i^{'}$ concurrent at a common point $x^{'}$ known as the vanishing point of the direction of lines l_i.*

By applying a projective transformation to the line at infinity l_∞, we obtain a line $l_\infty^{'}$. This line intersects the image all the points at the infinity x_∞ from the original plane. Consequently, the vanishing line $l_\infty^{'}$ can be determined from two vanishing points.



Polarity

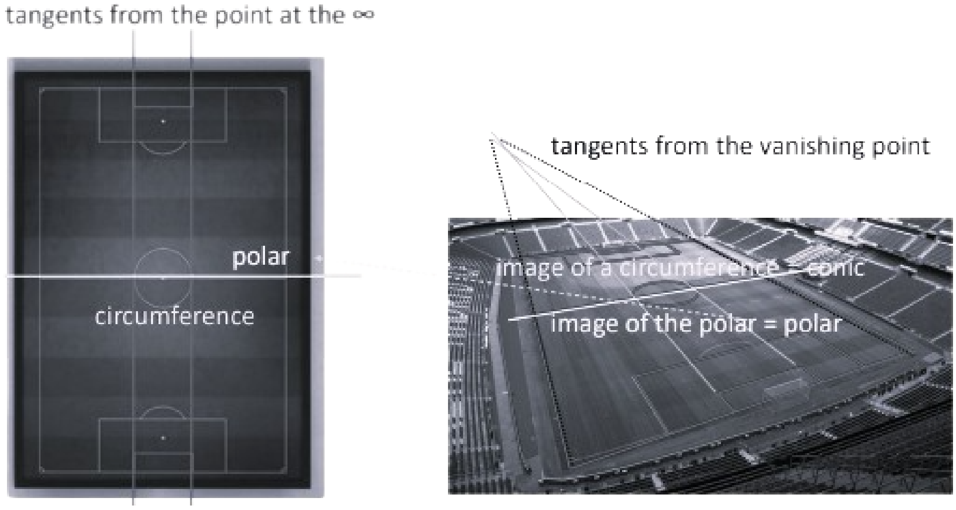
Polarity remains unaltered in the presence of projective mappings. The polar line $l = Cx$ corresponding to a point x with respect to a conic C gets mapped to the polar line of the transformed point $x^{'} = Hx$ with respect to the transformed conic:

$$C^{'} = H^{-T}CH^{-1}$$

Proof. This property holds because:

$$C^{'}x^{'} = H^{-T}CH^{-1}Hx = H^{-T}Cx = H^{-T}l = l^{'}$$

Therefore, the polar line of the transformed point aligns with the polar line of the original point. □



In conclusion, as polarity remains intact under projective mappings, conjugacy is similarly preserved, and the relationship $CR = -1$ is also upheld.

Cross ratio

Given a line defined by four points with the following relationships:

$$x_1 = \propto_1 Y + \beta_1 Z$$

$$x_2 = \propto_2 Y + \beta_2 Z$$

The cross ratio is expressed as:

$$CR_{X_1,X_2,Y,Z} = \frac{\beta_1/\alpha_1}{\beta_2/\alpha_2}$$

Upon applying a projective transformation H to these four points:

$$Y^{'} = HY \qquad Z^{'} = HZ$$

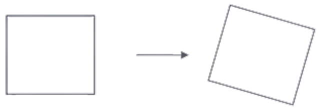
$$x_1^{'} = HX_1 = x_1 = \propto_1 Y^{'} + \beta_1 Z^{'} \qquad x_2^{'} = HX_2 = \propto_2 Y^{'} + \beta_2 Z^{'}$$

The coefficients of the linear combination remain the same. Hence, the cross ratio is conserved, maintaining its original value:

$$CR_{X_1',X_2',Y',Z'} = \frac{\beta_1/\alpha_1}{\beta_2/\alpha_2} = CR_{X_1,X_2,Y,Z}$$

Isometries

Isometries possess three degrees of freedom, which include translation denoted as t and the rotation angle represented by ϑ. Consequently, the invariants of this transformation encompass lengths, distances, and areas.



Definition. The *orthogonal matrix* $R_\perp$ is defined as follows:

$$R_\perp^{-1} = R_\perp^T$$

Hence, the matrix H_I for isometries takes the following form:

$$H_I = \begin{bmatrix} \cos\vartheta & -\sin\vartheta & t_x \\ \sin\vartheta & \cos\vartheta & t_y \\ 0 & 0 & 1 \end{bmatrix}$$

Here, $\begin{bmatrix} \cos\vartheta & -\sin\vartheta \\ \sin\vartheta & \cos\vartheta \end{bmatrix} = R_\perp$

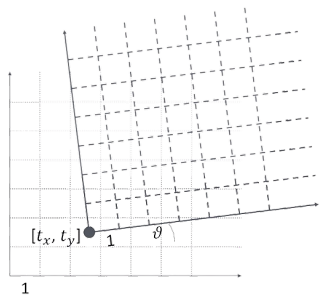
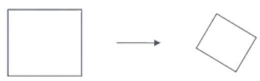


Figure 2.4: Isometry

Similarities

Similarities are characterized by four degrees of freedom, encompassing the translation, denoted as t; the scale, represented by s; and the rotation angle, expressed as ϑ. Consequently, the invariants of this transformation encompass the ratio of lengths and angles. Furthermore, the circular points I and J remain invariant throughout this transformation.



Hence, the matrix H_S for similarities is as follows:

$$H_I = \begin{bmatrix} s\cos\vartheta & -s\sin\vartheta & t_x \\ s\sin\vartheta & s\cos\vartheta & t_y \\ 0 & 0 & 1 \end{bmatrix}$$

Here, $\begin{bmatrix} s\cos\vartheta & -s\sin\vartheta \\ s\sin\vartheta & s\cos\vartheta \end{bmatrix} = sR_\perp$

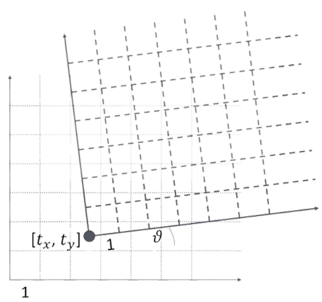
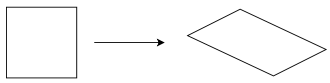


Figure 2.5: Similarity

Affinities

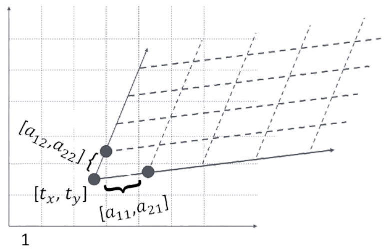
Affinities exhibit six degrees of freedom, consisting of the sub-matrix A and the translation component. As a result, the invariants of this transformation encompass parallelism, the ratio of parallel lengths, and the ratio of areas. The matrix A is defined as a 2×2 matrix with a rank of two. Additionally, the line at infinity, denoted as l_∞, remains invariant throughout the transformation.



Hence, the matrix H_A for affinities takes the following form:

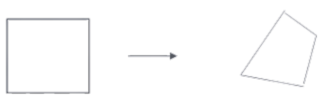
$$H_I = \begin{bmatrix} a_{11} & a_{21} & t_x \\ a_{12} & a_{21} & t_y \\ 0 & 0 & 1 \end{bmatrix}$$

Here, $\begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{21} \end{bmatrix} = A$



Projectivities

Projectivities possess eight degrees of freedom, encompassing the sub-matrix A, the vector v, and the translation component. Therefore, the invariants of this transformation include co-linearity, incidence, and the order of contact. The matrix A is defined as a 2×2 matrix with a rank of two. Furthermore, the cross ratio remains invariant throughout this transformation.



Hence, the matrix H_P for projectivities takes the following form:

$$H_I = \begin{bmatrix} a_{11} & a_{21} & t_x \\ a_{12} & a_{21} & t_y \\ v_1 & v_2 & 1 \end{bmatrix}$$

Here, $\begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{21} \end{bmatrix} = A$

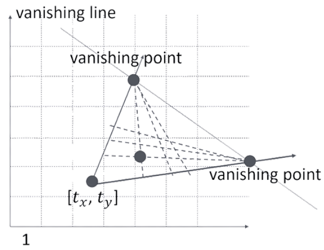


Figure 2.6: Affinity

CHAPTER 3

Two-dimensional reconstruction

3.1 Introduction

When we aim to recover a model of an unknown planar scene based on an image of the scene, which is a projective transformation denoted as $x_i^{'} = Hx_i$, we face a significant challenge. The key issue is that we know the values of x_i (the scene points), but we don't have direct knowledge of the transformation matrix H, which complicates a straightforward inversion of the mapping.

The general problem is inherently unsolvable due to the excessive number of unknown variables. To tackle this challenge, we can adopt two primary strategies:

1. Reduce unknowns: in many cases, it is unnecessary to precisely recover the original scene configuration. Instead, the objective is to retrieve the overall shape of the scene, known as shape reconstruction. By doing so, we can reduce the number of unknowns from eight to four, and the matrix H takes on the following form:

$$H = \begin{bmatrix} s\cos\vartheta & -s\sin\vartheta & t_x \\ s\sin\vartheta & s\cos\vartheta & t_y \\ 0 & 0 & 1 \end{bmatrix}$$

   Additionally, it's possible to perform similarity reconstruction, which reduces the unknowns by two, or affine reconstruction, which reduces the unknowns by six.

2. Add constraints: this strategy involves utilizing extra information to recover a model of the scene. The valuable information typically pertains to parameters that remain invariant under the desired class of mappings but are not invariant under more general classes.

The reconstruction can fall into one of two categories:

- Affine reconstruction: in this scenario, the reconstructed scene is an affine mapping of the original scene.
- Shape reconstruction: in this case, the reconstructed scene follows a similarity mapping of the original scene, aiming to capture the overall shape while simplifying the problem.

3.2 Affine reconstruction

Theorem 3.2.1. *A projective transformation H that maps the line at the infinity l_∞ onto itself implies that H is affine.*

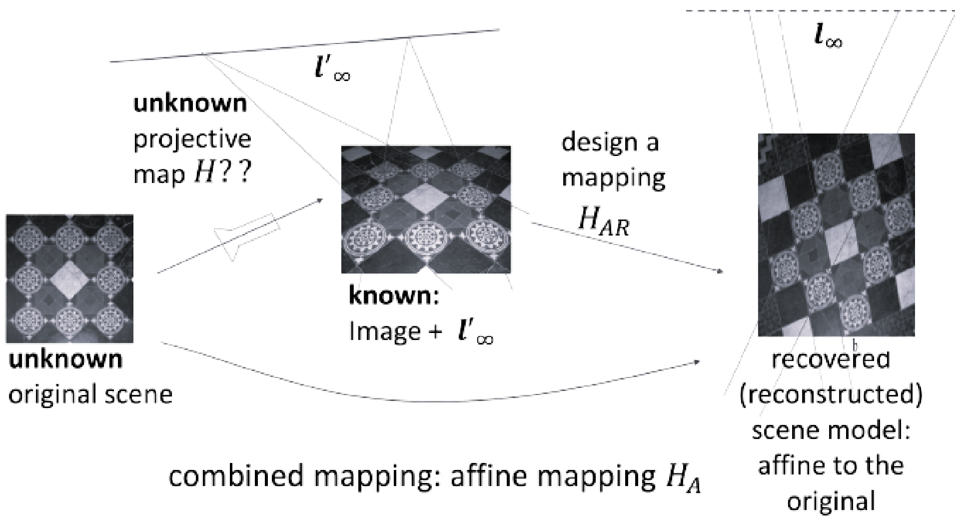
Proof. A point at the infinity $x_\infty = \begin{bmatrix} x & y & 0 \end{bmatrix}^T$ is mapped to another point $x' = Hx_\infty$ that remains at infinity only if the third coordinate of x' is zero. For all (x, y), this condition is expressed as:

$$\begin{bmatrix} v_1 & v_2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} = 0 \rightarrow \begin{bmatrix} v_1 & v_2 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$$

In other words, H is affine. □

The image provided results from a general projective mapping of the original scene. Consequently, the vanishing line l'_∞ in the image differs from the original l_∞. This observation leads to the possibility of using l'_∞ as additional information. By applying a new projective transformation H_{AR} to the image that restores l'_∞ to l_∞, a modified image is obtained. Notably, the image of the line at infinity l_∞ in this new model remains as l_∞.

Based on the theorem, the resulting model (i.e., the new image) is an affine mapping of the original scene. Therefore, the achieved model is an affine reconstruction of the scene.



The challenges associated with this approach include:

- Determine a projective transformation H_{AR} that restores l'_∞ to l_∞.
- Identify the vanishing line.

Determine a projective transformation

To find a projective mapping H_{AR} that restores l'_∞ to l_∞ the mapping should satisfy the condition of mapping any point $x' \in l'_\infty$ onto a set of point at infinity:

$$H_{AR}x' = \begin{bmatrix} * \\ * \\ 0 \end{bmatrix}$$

The mapping can be effectively represented as:

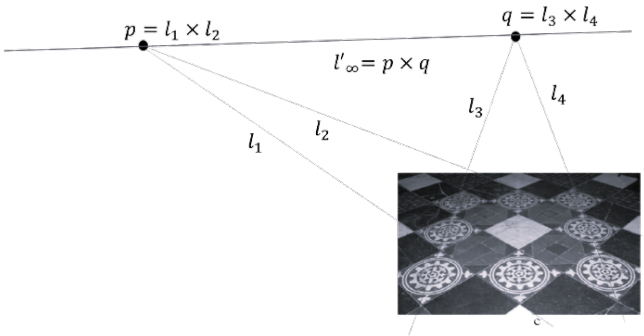
$$H_{AR} = \begin{bmatrix} * & * & * \\ * & * & * \\ & l_{\infty}^{\prime T} & \end{bmatrix}$$

In this matrix representation, we achieve the desired mapping:

$$H_{AR}x^{'} = \begin{bmatrix} * & * & * \\ * & * & * \\ & l_{\infty}^{\prime T} & \end{bmatrix} x^{'} = \begin{bmatrix} * \\ * \\ l_{\infty}^{\prime T}x^{\prime} \end{bmatrix}$$

Identify the vanishing line

To determine the vanishing line, additional information can be employed, such as the image of parallel lines.

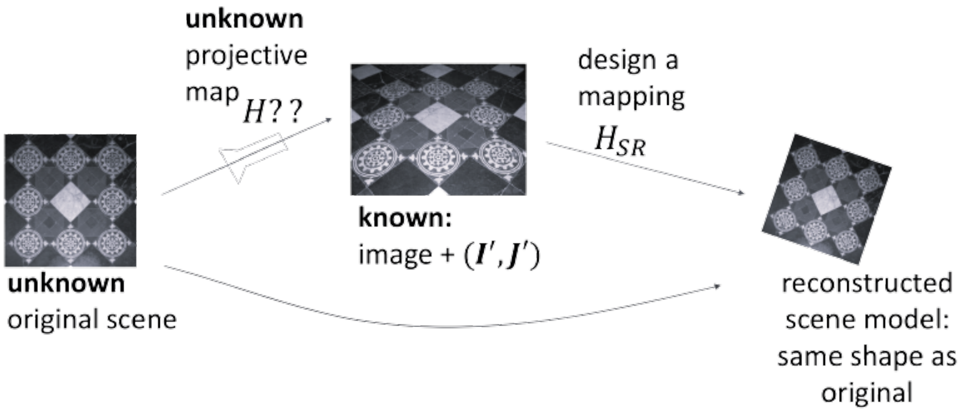


3.3 Shape reconstruction

Theorem 3.3.1. *A projective transformation H that maps the circular points I and J onto themselves implies that H is a similarity transformation.*

Proof. When we multiply a similarity matrix H_S by the circular point I, we obtain a multiple of I. Similarly, the same result is obtained for the other circular point, J. □

The given image represents a general projective mapping of the original scene. Consequently, the image of the circular points, denoted as $(I^{'}, J^{'})$, differs from I and J. To utilize $I^{'}, J^{'}$ as additional information, we apply a new projective mapping H_{SR} to map $I^{'}, J^{'}$ back to I, J. This process generates a new modified image where the circular points I, J are restored. As per the theorem, the obtained model (new image) is similar to the original scene, representing a shape reconstruction of the scene.



The method comes with certain challenges:

- Finding a projective mapping H_{SR} that restores $I^{'}, J^{'}$ to I, J.
- Determining the vanishing line.

Identify a projective transformation

Discovering a projective transformation H_{SR} that restores $I^{'}, J^{'}$ to I, J is equivalent to finding one of the ∞^4 matrices that satisfy:

$$\begin{cases} H_{SR} I^{'} = I \\ H_{SR} J^{'} = J \end{cases}$$

This task can be quite challenging.

Let's utilize alternative information: the degenerate conic dual to $I^{'}, J^{'}$, represented as $C_{\infty}^{'*} = I^{'} J^{'T} + J^{'} I^{'T}$, which corresponds to the image of the original conic dual to the circular points (I, J), expressed as:

$$C_{\infty}^{*} = I J^{T} + J I^{T}$$

Since $(I^{'}, J^{'})$ is the image of (I, J), then also $C_{\infty}^{'*}$ is the image of C_{∞}^{*}. As a result, any projective transformation H_{SR} that restores $(I^{'}, J^{'})$ to (I, J) also restores $C_{\infty}^{'*}$ to C_{∞}^{*}. By applying the transformation rule for dual conics under projective mappings, we get:

$$C_{\infty}^{*} = H_{SR} C_{\infty}^{'*} H_{SR}^{T}$$

Reversing this relationship, we find:

$$C_{\infty}^{'*} = H_{SR}^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} H_{SR}^{-T}$$

Applying singular value decomposition (SVD) to this equation, we find that the matrices H_{SR}^{-1} and H_{SR}^{-T} are orthogonal. Consequently, we have:

$$\text{SVD}(C_{\infty}^{'*}) = U_{\perp} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} U_{\perp}^{T}$$

Which provides a unique solution: $H_{SR} = U_{\perp}^{-1} = U_{\perp}^{T}$. To address the issues with image rectification, we need to modify the matrix H_{SR} as follows:

$$H_{SR} = \begin{bmatrix} \frac{1}{\sqrt{a}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{b}} & 0 \\ 0 & 0 & 1 \end{bmatrix} U^T$$

Determine the vanishing line

To determine the vanishing line, one can leverage additional information from the observed scene. This information can be used to establish the following constraints:

1. Known angles between lines: when the angles between lines in the scene are known, these angles can be used to constrain the vanishing line. The angle between two lines is related to the angle between their normal directions and is independent of parameters c_1 and c_2. Mathematically, this relationship is expressed as:

$$\cos\vartheta = \frac{a_1 a_2 + b_1 b_2}{\sqrt{(a_1^2 + b_1^2)(a_2^2 + b_2^2)}}$$

   Here, a_1, b_1, a_2, and b_2 are coefficients of the normal vectors of the lines. By rewriting the terms, this equation can be expressed as:

$$\cos\vartheta = \frac{l^T C_\infty^* m}{\sqrt{(l^T C_\infty^* l)(m^T C_\infty^* m)}}$$

   This equation can be further simplified by using the rules obtaining $C_\infty^* = H^{-1} C_\infty^{*\prime} H^{-T}$. Now, we can rewrite $l^T C_\infty^* m$ as $l^{\prime T} C_\infty^{*\prime} m^{\prime}$. With these transformations, the equation becomes:

$$\cos\vartheta = \frac{l^{\prime T} C_\infty^{*\prime} m^{\prime}}{(l^{\prime T} C_\infty^{*\prime} l^{\prime})(m^{\prime T} C_\infty^{*\prime} m^{\prime})}$$

   In this case, $m^{\prime}$ and $l^{\prime}$ are obtained from the image. Since the angle is known, this equation provides a linear constraint on $C_\infty^{*\prime}$, that is linear when the lines are perpendicular ($\cos\vartheta = 0$). The unknown matrix $C_\infty^{\prime}$ is symmetric, homogeneous, and singular, providing four independent constraints.

2. Known shape of objects: if the shape of objects in the scene is known, the reconstruction matrix H_{SR} can be determined. The transformation matrix is defined as:

$$H_{SR} = \begin{bmatrix} \frac{1}{\sqrt{a}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{b}} & 0 \\ 0 & 0 & 1 \end{bmatrix} U^T$$

   The Euclidean reconstructed image is calculated as $M_S = H_{SR} \cdot \text{image}$

3. Combinations of constraints: it is also possible to use a combination of known angles between lines and the shape of objects for additional constraints.

4. Observation of rigid planar motion: when observing rigid planar motion, which is a similarity transformation, the circular points remain invariant. The object has three degrees of freedom, and the center of rotation and the rotation angle can be determined. Given a matrix H, the eigenvectors of H correspond to fixed points, and the eigenvectors of H^{-T} correspond to fixed lines of the transformation. The eigenvectors can be used to extract important information:

- Eigenvectors $I^{'}, J^{'}$ correspond to complex eigenvalues.
- The phase of these eigenvectors is the rotation angle.
- Eigenvector $O^{'}$ correspond to real eigenvalues.

The three eigenvectors of H are proportional to three distinct values: 1, $e^{i\theta}$, and $-e^{i\theta}$. The eigenvector corresponding to the eigenvalue 1 represents the image of the center of rotation, denoted as O, and the angle θ corresponds to the rotation angle. The eigenvectors associated with the complex eigenvalues represent the images of the circular points $I^{'}, J^{'}$. Therefore, using the relationship $C_{\infty}^{'*} = I^{'}J^{'T} + J^{'}I^{'T}$, the singular value decomposition can be applied to obtain $\text{SVD}(C_{\infty}^{'*}) = UC_{\infty}^{*}U^{T}$, where U^{T} is the rectification matrix. Two methods can be used to address this:

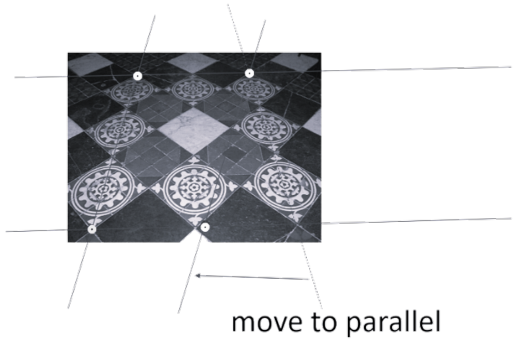
- Direct method:
  (a) Find $C_{\infty}^{*'}$.
  (b) Compute H_{rect} for rectification.
- Stratified method:
  (a) Perform affine reconstruction from projective to affine.
  (b) Perform shape reconstruction from affine to metric.

In some cases, the stratified method reduces numerical errors, providing a more accurate result.

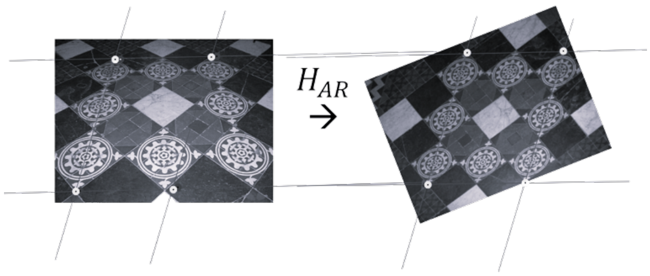
3.4 Accuracy issues

There are various accuracy issues when doing image rectification:

1. Noise and numerical errors: noise and numerical errors in the input data can affect the accuracy of the rectification process. It's essential to preprocess and filter the data to minimize these issues.
2. Little information: when choosing lines to identify vanishing points, it's crucial to select lines that are sufficiently far apart. Choosing lines that are too close to each other can lead to inaccuracies in vanishing point estimation and rectification.
3. Vanishing point near infinity: in cases where the vanishing point is nearly at infinity, it can be challenging to perform accurate affine rectification. To address this issue:
   - Draw two lines in the scene that are perpendicular to the given lines in the image. If the two new lines are not parallel, adjust one of the intersection points to make them parallel.



Finally, apply affine reconstruction to obtain accurate results.



- When dealing with sets of parallel lines, you can choose one line from each set and randomly select another pair of lines, making sure they are perpendicular. With these four lines, you can compute the matrix product of K and its transpose, denoted as KK^T, and then derive K through Cholesky factorization. Afterward, you can apply the rectifying transformation using the matrix:

$$H_{rect} = \begin{bmatrix} K & t \\ 0 & 1 \end{bmatrix}^{-1}$$

The accuracy of image rectification is crucial for various computer vision and image processing applications, and addressing these issues is essential for obtaining reliable results.

CHAPTER 4

Three-dimensional space projective geometry

4.1 Introduction

In space geometry, the fundamental elements required for defining the geometry include points, planes, quadrics, and dual quadrics. The allowable transformations within this type of geometry encompass projectivities, affinities, similarities, and isometries.

4.2 Points

Points in space geometry can be represented in Cartesian coordinates by defining a Euclidean space with its origin. This approach allows for the unambiguous definition of every point using three Cartesian coordinates (x, y, z).

However, when analyzing images, it is more convenient to employ homogeneous coordinates. The relationship between Cartesian and homogeneous coordinates is as follows:

$$X = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = w \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

This representation exhibits homogeneity, meaning any vector x is equivalent to all its non-zero multiples λx, where $\lambda \neq 0$, since they all represent the same point. The null vector, however, does not represent any point.

Definition. The *projective space* is defined as:

$$\mathbb{P}^3 = \{\begin{bmatrix} x & y & z & w \end{bmatrix}^T \in \mathbb{R}^4\} - \{\begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}^T\}$$

4.3 Planes

In the homogeneous coordinates, planes are defined by using the matrix:

$$\pi = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

Here, the direction normal to the plane is given by (a, b, c), and the distance from the origin to the plane is calculated as:

$$\text{distance} = -\frac{d}{\sqrt{a^2 + b^2 + c^2}}$$

Similar to homogeneous point coordinates, this representation of planes also exhibits the homogeneity property. Any vector π is equivalent to all its non-zero multiples $\lambda\pi$, where $\lambda \neq 0$, as they all represent the same plane. The parameters a, b, c, d are referred to as the homogeneous parameters of the plane. As with points, there are an infinite number of equivalent representations for a single plane, which includes all non-zero multiples of the unit normal vector. The null vector does not represent any plane. If $d = 0$, it signifies that the plane π passes through the origin of space.

To determine whether a point lies on a plane or if a plane passes through a point, you can solve the following system of equations:

$$\begin{cases} ax + by + cz + dw = 0 \\ \pi^T X = X^T \pi = 0 \end{cases}$$

Definition. The plane

$$\begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = w = 0$$

is called the *plane at the infinity* $\pi_\infty = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}^T$.

It's important to note that this plane has an undefined normal direction.