Institut für Theoretische Physik
Universität Tübingen
T. Gutsche

Exercise sheet 4
WS 2022/23
November 15, 2022

Quantum Field Theory

Exercise 12:

We consider the Lagrangian density of the complex Klein-Gordon field

$$\mathcal{L} = (\partial_\mu \phi^\dagger)(\partial^\mu \phi) - m^2 \phi^\dagger \phi$$

with the Fourier expansion of the field operator

$$\phi(x) = \frac{1}{\sqrt{V}} \sum_{\vec{k}} \frac{1}{\sqrt{2\omega_k}} \left(a_{\vec{k}}\, e^{-ik\cdot x} + b^\dagger_{\vec{k}}\, e^{ik\cdot x} \right)$$

and the field conjugate to ϕ with

$$\pi = \frac{\partial \phi^\dagger}{\partial t} \, .$$

Imposing the usual commutation relations on the ϕ and π operators results in the non-vanishing commutator relations for the creation and annihilation operators

$$[a_{\vec{k}}, a^\dagger_{\vec{k}'}] = [b_{\vec{k}}, b^\dagger_{\vec{k}'}] = \delta_{\vec{k},\vec{k}'} \, .$$

(a) Show that the normal-ordered Hamiltonian of the complex scalar field can be written as

$$\begin{aligned} H &= \int d^3x : \left[\pi(x)\pi^\dagger(x) + \vec{\nabla}\phi(x) \cdot \vec{\nabla}\phi^\dagger(x) + m^2 \phi(x)\phi^\dagger(x) \right] : \\ &= \sum_{\vec{k}} \omega_{\vec{k}} \, (a^\dagger_{\vec{k}} a_{\vec{k}} + b^\dagger_{\vec{k}} b_{\vec{k}}) \end{aligned}$$

where $\omega_{\vec{k}} = \sqrt{\vec{k}^2 + m^2}$. (4P)

(b) Further show that the normal-ordered charge operator with

$$Q = iq \int d^3x : (\phi^\dagger \frac{\partial \phi}{\partial t} - \phi \frac{\partial \phi^\dagger}{\partial t}) :$$

can be expressed as

$$Q = q \sum_{\vec{k}} (a^\dagger_{\vec{k}} a_{\vec{k}} - b^\dagger_{\vec{k}} b_{\vec{k}}) \, .$$

Interpret the results for H and Q. (3P)

Exercise 13:

The Feynman-Propagator of a real Klein-Gordon field with mass m has been derived as

$$\Delta_F(x) \;=\; \frac{1}{(2\pi)^4}\int \frac{d^4k\ e^{-ikx}}{k^2-m^2+i\epsilon} \quad .$$

Show that Δ_F satisfies the inhomogeneous Klein-Gordon equation (2P)

$$(\Box_{x'}+m^2)\Delta_F(x'-x) \;=\; -\,\delta^{(4)}(x'-x) \ , \quad \Box=\partial_\mu\partial^\mu \, .$$

Exercise 14:

Derive the Feynman-propagator Δ'_F for the complex scalar Klein-Gordon field with mass m. The propagator of a charged meson is defined as

$$<0|T\{\phi(x)\phi^\dagger(x')\}|0> \;=\; i\Delta'_F(x-x') \ .$$

Interpret the propagator in terms of emission and absorption of particles and antiparticles. (4P)

Exercise 15:

Charge conjugation for the complex Klein-Gordon field $\phi(x)$ is defined by

$$\phi(x) \longrightarrow \mathcal{C}\phi(x)\mathcal{C}^{-1}=\eta_c\,\phi^\dagger(x) \ \ (*)$$

where $\mathcal{C}$ is a unitary operator which leaves the vacuum invariant (i.e., $\mathcal{C}|0>=|0>$), and η_c is a phase factor.

(a) Show that under the transformation $(*)$ the Lagrangian density (given in Exercise 12) is invariant and the charge-current density (see Exercise 8, $j^\mu=-ie(\phi^\dagger\partial^\mu\phi-\phi\partial^\mu\phi^\dagger)$) changes sign. (4P)

(b) Derive

$$\mathcal{C}a_{\vec{k}}\mathcal{C}^{-1}=\eta_c b_{\vec{k}},\ \mathcal{C}b_{\vec{k}}\mathcal{C}^{-1}=\eta_c^* a_{\vec{k}}$$

for the annihilation operators, and hence show that

$$\mathcal{C}|a,\vec{k}>=\eta_c^*|b,\vec{k}>,\ \mathcal{C}|b,\vec{k}>=\eta_c|a,\vec{k}>$$

where $|a,\vec{k}>$ denotes the state with a single a-particle of momentum $\vec{k}$ present, etc. (3P)

($\mathcal{C}$ is called the charge conjugation operator. It interchanges particles and antiparticles: $a\leftrightarrow b$.)

Worked-out solutions to the homework problems should be handed in at the beginning of the lecture of Tuesday, Nov. 22.