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Exercise sheet 8
WS 2022/23
December 13, 2022

Quantum Field Theory

Exercise 28:

We split the Hamiltonian up as $H = H_0 + H_{int}$ where H_0 is the Hamiltonian for a free field theory. In the interaction or Dirac picture the interaction part of the Hamiltonian is $H_{int,I}(t) = e^{iH_0(0)t}(H(0) - H_0(0))e^{-iH_0(0)t}$. Show in the following that the time evolution operator

$$U(t, t_0) = T\left[exp\left(-i\int_{t_0}^{t} dt' H_{int,I}(t')\right)\right]$$

is a solution to the equation (8P)

$$i\frac{\partial}{\partial t}U(t, t_0) = H_{int,I}(t)U(t, t_0)\ ,$$

with the initial condition $U(t_0, t_0) = \mathbb{1}$. Follow the procedure outlined in the lecture.

Exercise 29:

Show that the contractions of Dirac fields ψ_a $(a = 1, 2, 3, 4)$ are given as (6P)

$$\begin{aligned} \psi_a(x)\bar{\psi}_b(x') &= -\bar{\psi}_b(x')\psi_a(x) = iS_{Fab}(x - x')\ , \\ \psi_a(x)\psi_b(x') &= \bar{\psi}_a(x)\bar{\psi}_b(x') = 0\ . \end{aligned}$$

$S_F(x - x')$ is the Feynman propagator of the Dirac field. The Dyson-Wick contraction of two field operators is defined as

$$A(x)B(x') \equiv \mathcal{T}[A(x)B(x')] -\ :\hat{A}(x)\hat{B}(x'):\ .$$

Exercise 30:

Use Wick's theorem to calculate the following vacuum expectation values of time-ordered products (8P):

(a) $\langle 0|\mathcal{T}\left[\phi^4(x)\phi^4(y)\right]|0\rangle$ for a real scalar field ϕ (also give a graphical representation of the result),

(b) $\langle 0|\mathcal{T}\left[:\phi^4(x)::\phi^4(y):\right]|0\rangle$ for a real scalar field ϕ,

(c) and $\langle 0|\,\mathcal{T}\left[\psi_a(x_1)\psi_b(x_2)\psi_c(x_3)\bar{\psi}_d(x_4)\bar{\psi}_e(x_5)\bar{\psi}_f(x_6)\right]|0\rangle$ for a Dirac field ψ .

Exercise 31:

In ϕ^3 theory evaluate the particular amplitude (6P)

$$\frac{1}{2}\left(\frac{-i\lambda}{3!}\right)^2 \int d^4y_1 d^4y_2 \langle 0|\mathcal{T}\left[\phi(x_1)\phi(x_2)\phi^3(y_1)\phi^3(y_2)\right]|0\rangle.$$

Note that the integrals should not be evaluated explicitly. Give a graphical representation of the result.
Above term is part of the perturbative series of a so-called 2-point correlation function, a quantity briefly mentioned in the lecture.

Exercise 32:

A real scalar field $\phi(x)$, associated with a spin-zero boson B, is described by the Lagrangian density

$$\mathcal{L}(x) \;=\; \mathcal{L}_0(x) \;+\; \mathcal{L}_I(x)\;.$$

$\mathcal{L}_0$ is the free-field density with

$$\mathcal{L}_0(x) \;=\; \frac{1}{2}\partial_\mu\phi(x)\partial^\mu\phi(x) \;-\; \frac{1}{2}m^2\phi^2(x) \quad ; \quad \mathcal{L}_I(x) \;=\; \frac{g}{4!}\,\phi^4(x)\,.$$

$\mathcal{L}_I$ represents an interaction of the field with itself. The parameter g is a real coupling constant. Normal ordering of operators is assumed throughout.
Write down the S-matrix expansion, and pick out the normal ordered term that gives rise to the BB scattering process

$$B(\vec{k}_1) \;+\; B(\vec{k}_2) \;\to\; B(\vec{k}_3) \;+\; B(\vec{k}_4)$$

in first-order perturbation theory. Draw the Feynman diagram representing this term, and show that the corresponding S-matrix element is given by

$$< k_3, k_4|S^{(1)}|k_1, k_2 >= (2\pi)^4\delta^{(4)}(k_3 + k_4 - k_1 - k_2)\prod_i\left(\frac{1}{2V\omega_i}\right)^{1/2}\mathcal{M}$$

with the Feynman amplitude $\mathcal{M} = ig$. Note that $\mathcal{M}$ is independent of the boson 4-momenta $k_i^\mu = (\omega_i, \vec{k}_i)$. (6P)

Exercise 33:

A reminder of classical Electrodynamics:
The Hamiltonian of a non-relativistic particle of mass m and charge q, moving in an electromagnetic field, is given by

$$H = \frac{1}{2m}\left(\vec{p} - q\vec{A}\right)^2 + q\phi \, ,$$

where $\vec{A} = \vec{A}(\vec{x}, t)$ and $\phi = \phi(\vec{x}, t)$ are the vector and scalar potentials of the electromagnetic field at position $\vec{x}$ of the particle at time t. Show that the resulting Hamilton equations lead to

$$m\frac{d}{dt}\dot{\vec{x}} = q\left(\vec{E} + \dot{\vec{x}} \times \vec{B}\right)$$

where $\vec{E}$ and $\vec{B}$ are the electric and magnetic fields at the instantaneous position of the charge (6P).

Worked-out solutions to the homework problems should be handed in latest at the beginning of the lecture of January 10, 2023.

We wish you Happy Holidays and a Happy New Year 2023!