Exercise 3 (online: 08.05.2023. Return by: **Mo 15.05.2023 10:00**) 13P

1. **Fang-Howard wavefunctions** 6P

   Consider a triangular quantum well as discussed for the formation of a 2-dimensional electron gas (2DEG) at the interface of a heterostructure, with

$$V(z) = \begin{cases} eFz, & z > 0; \\ \infty, & z < 0. \end{cases}, \tag{1}$$

   where e is the electron charge and F is the strength of effective electric field. The groundstate can be approximately described by the so-called Fang-Howard wavefunction

$$\Phi(z) = \begin{cases} cz \exp\left(-bz/2\right), & \text{for } z > 0; \\ 0, & \text{for } z < 0, \end{cases} \tag{2}$$

   where the amplitude $c > 0$ and decay length $b > 0$.

   (a) Calculate the normalization constant c for the well (1). (1P)

   (b) Calculate the groundstate energy

$$\epsilon_0 = \langle \Phi^* \hat{H} \Phi \rangle \equiv \int_{-\infty}^{+\infty} \Phi^* \hat{H} \Phi \, dz, \text{ with } \hat{H} = \frac{-\hbar^2}{2m} \frac{\partial^2}{\partial z^2} + V(z)$$

   as a function of the variational parameter b.(2P)

   (c) Minimize the ground state energy by varying the parameter b and find its optimal value b_0. Calculate the corresponding energy ϵ_0 and compare it with the exact result (calculated from the exact solution $\Phi(z)$ given by the Airy function)

$$E_o = \left(\frac{9\pi}{8}\right)^{2/3} \left(\frac{\hbar^2}{2m^*}\right)^{1/3} (eF)^{2/3}. \tag{3}$$

   Calculate E_o of a 2DEG according to eq. (3) for the case of GaAs with $F = 5\,\mathrm{MV/m}$ and the effective mass $m^*(\text{GaAs}) = 0.067 m_e$. (3P)

   (d) Occasionally an approximate description of the first excited state is also required:

$$\Phi_1(z) = \begin{cases} z(d - bz) \exp\left(-bz/2\right), & \text{for } z > 0; \\ 0, & \text{for } z < 0, \end{cases} \tag{4}$$

   Determine the constant d from the orthogonality condition of the wavefunctions. (1P, Optional)

2. **1D-harmonic oscillator in a magnetic field** $\boxed{7P}$

A 1D wire aligned along the y-direction may be described by the confining potential $V(x) = \frac{1}{2}m\omega_o^2x^2$. In the z-direction the system is assumed to be confined to a 2D plane and the resulting energy quantization may be neglected in the following.

(a) Show explicitly, step-by-step, that in the presence of a perpendicular magnetic field $\boldsymbol{B} = (0, 0, B)$ the assumption of the usual Landau gauge, *i.e.*, $\boldsymbol{A} = (0, Bx, 0)$, results in a 1D Schrödinger equation with an additional parabolic potential:

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega_c^2\left(x - \frac{\hbar k_y}{m\omega_c}\right)^2 + \frac{1}{2}m\omega_o^2x^2\right]\Phi(x) = \varepsilon\Phi(x), \tag{5}$$

where we used the following ansatz for a wavefunction $\psi(x, y) = \Phi(x) \cdot e^{ik_y y}$ and the cyclotron frequency $\omega_c \equiv eB/m$. Hint: Use $\boldsymbol{p} - e\boldsymbol{A}$ instead of $\boldsymbol{p}$ in the Hamiltonian for the Schrödinger Equation $\left[\frac{1}{2m}\boldsymbol{p}^2 + V(x)\right]\psi = \epsilon\psi$ to account for the Lorentz force. (3P)

(b) Show that the eigenenergies are given by

$$\varepsilon_n(k_y) = \left(n + \frac{1}{2}\right)\hbar\omega_*(B) + \frac{\hbar^2k_y^2}{2m(B)} \tag{6}$$

and determine the values of $\omega_*(B)$ und $m(B)$. Hint: rewrite the terms corresponding to the potential energy in Eq. (5) in the form $C(x - A)^2 + D$.(3P)

(c) Based on the result (6), describe the changes in the structure of energy levels and dispersion as B increases from small fields ($B \to 0$) to intermediate fields and further to high fields ($\omega_c \gg \omega_o$). How is the energy level structure at high fields called? (1P)