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Exercise sheet 5
WS 2022/23
November 22, 2022

Quantum Field Theory

Exercise 16:

Show that the Lagrangian density starting from

$$\mathcal{L} \; = \; -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

and adding the term $-\frac{1}{2}\left(\partial_\mu A^\mu(x)\right)\left(\partial_\nu A^\nu(x)\right)$ is equivalent to the Lagrangian density suggested by Fermi with (4P)

$$\mathcal{L} \; = \; -\frac{1}{2}\left(\partial_\nu A_\mu(x)\right)\left(\partial^\nu A^\mu(x)\right) \; .$$

Exercise 17:

The Feynman-propagator $D_F^{\mu\nu}$ of the photon is defined as:

$$\langle 0|T\{A^\mu(x_1)A^\nu(x_2)\}|0\rangle \; = \; iD_F^{\mu\nu}(x_1 - x_2) \; .$$

Show that $D_F^{\mu\nu}$ is given as: (4P)

$$D_F^{\mu\nu}(x) \; = \; \lim_{m\to 0}\left(-g^{\mu\nu}\Delta_F(x)\right)$$

and

$$D_F^{\mu\nu}(x) \; = \; \frac{-g^{\mu\nu}}{(2\pi)^4}\int d^4k \; \frac{e^{-ikx}}{k^2 + i\epsilon} \quad .$$

$\Delta_F(x)$ is the Feynman-propagator of the real Klein-Gordon field as derived in the lectures.

Exercise 18:

The most general state representing the physical vacuum, i.e. the state in which there are no transverse photons present, but which contains the most general allowed admixture of scalar and longitudinal photons, is given by

$$|\Psi_{SL}>=\sum_{n=0}^{\infty}|\Psi_n> \quad \text{with} \quad |\Psi_n>=\sum_{\vec{k}_1}\dots\sum_{\vec{k}_n} f_n(\vec{k}_1,\dots,\vec{k}_n)\prod_{i=1}^{n}\left(a_{\vec{k}_i}^{3\,\dagger}-a_{\vec{k}_i}^{0\,\dagger}\right)|0>$$

where $f_n(\vec{k}_1,\dots,\vec{k}_n)$ are arbitrary functions. The state $|\Psi_0>=|0>$ is the vacuum state in which there are no photons of any kind present. The $a_{\vec{k}}^{r\,\dagger}$ $(r=0,3)$ are the creation operators for a scalar $(r=0)$ and a longitudinal $(r=3)$ photon.

(a) Show that $|\Psi_{SL}>$ represents a physical state which fulfills the Gupta-Bleuler condition. (2P)

(b) Determine the norm of the state $<\Psi_{SL}|\Psi_{SL}>$. What is the most general state in which there are a definite number of transverse photons, with definite momenta and polarization vectors, present? (2P)

Exercise 19:

The state $|\Psi_T\rangle$ contains transverse photons only. Now we consider the state

$$|\Psi_T'\rangle \;=\; \left\{1+c\left[a_{\vec{k}}^{3\,\dagger}-a_{\vec{k}}^{0\,\dagger}\right]\right\}|\Psi_T\rangle$$

where c is a complex constant and $a_{\vec{k}}^{r\,\dagger}$ $(r=0,3)$ are the creation operators for a scalar $(r=0)$ and a longitudinal $(r=3)$ photon. Show that the replacement $|\Psi_T\rangle \;\rightarrow\; |\Psi_T'\rangle$ corresponds to a gauge transformation (8P), i.e.

$$\langle\Psi_T'|A^\mu(x)|\Psi_T'\rangle \;=\; \langle\Psi_T|A^\mu(x)+\partial^\mu\Lambda(x)|\Psi_T\rangle,$$

where

$$\Lambda(x) \;=\; \left(\frac{2}{V\omega_{\vec{k}}^3}\right)^{1/2} Re(ic\, e^{-ikx})\ .$$

Worked-out solutions to the homework problems should be handed in at the beginning of the lecture of Tuesday, Nov. 29.