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Exercise sheet 7
WS 2022/23
December 6, 2022

Quantum Field Theory

Exercise 24:

Space inversion, or parity operation, corresponds to the improper Lorentz transformation

$$\Lambda^{\mu}{}_{\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

where the Dirac spinor Ψ and the adjoint $\bar{\Psi}$ transform according to

$$\Psi'(x') = S_P(\Lambda)\Psi(x),\ \ \bar{\Psi}'(x') = \bar{\Psi}(x)S_P^{-1}(\Lambda)$$

with

$$S_P = S_P^{-1} = \gamma^0.$$

In the lecture we also introduced the 4x4 Dirac matrix

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3.$$

(a) Determine how the bilinear covariants $\bar{\Psi}\Psi$, $\bar{\Psi}\gamma^{\mu}\Psi$, $\bar{\Psi}\gamma_5\Psi$ and $\bar{\Psi}\gamma^{\mu}\gamma_5\Psi$ transform under the parity transformation. (3P)

(b) Further show that $\bar{\Psi}\Psi$, $\bar{\Psi}\gamma^{\mu}\Psi$, $\bar{\Psi}\gamma_5\Psi$ and $\bar{\Psi}\gamma^{\mu}\gamma_5\Psi$ transform under proper Lorentz-transformations with $\Psi'(x') = S(\Lambda)\Psi(x)$ either as Lorentz-scalars or 4-vectors. In a first step it might be helpful to show that $\gamma_5 S(\Lambda) = S(\Lambda)\gamma_5$. (3P)

Exercise 25:

For a Dirac field, the transformations

$$\psi(x) \to \psi'(x) = \exp(i\alpha\gamma_5)\psi(x), \quad \psi^{\dagger}(x) \to \psi^{\dagger\prime}(x) = \psi^{\dagger}(x)\exp(-i\alpha\gamma_5)\,,$$

where α is an arbitrary parameter, are called chiral phase transformations.

(a) Show that the Lagrangian $\mathcal{L}= \bar{\psi}(i\partial\!\!\!/ - m)\psi$ is invariant under chiral phase transformations for a vanishing mass term $m = 0$, and that in this case the corresponding conserved current can be expressed as $j^{\mu}_A(x) \equiv \bar{\psi}(x)\gamma^{\mu}\gamma_5\psi(x)$, the so-called axial current. (3P)

(b) Derive the equations of motion for the fields

$$\psi_L(x) \equiv \frac{1}{2}(1-\gamma_5)\psi(x), \quad \psi_R(x) \equiv \frac{1}{2}(1+\gamma_5)\psi(x)\,,$$

for a non-vanishing mass $m \neq 0$, and show that they decouple in the limit $m = 0$. (2P)

Exercise 26:

In the expansion of the real Klein-Gordon field $\phi(x)$ with

$$\phi(x) = \frac{1}{\sqrt{V}} \sum_{\vec{k}} \frac{1}{\sqrt{2\omega_k}} \left(a_{\vec{k}}\, e^{-ik\cdot x} + a^\dagger_{\vec{k}}\, e^{ik\cdot x} \right)$$

we now impose the anticommutation relations

$$\{a_{\vec{k}}, a^\dagger_{\vec{p}}\} = \delta_{\vec{k},\vec{p}}, \quad \{a_{\vec{k}}, a_{\vec{p}}\} = \{a^\dagger_{\vec{k}}, a^\dagger_{\vec{p}}\} = 0\,.$$

Show that the commutator and anticommutator of the field operators for a space-like separation $(x-y)^2 < 0$ do not vanish, i.e., (4P)

$$[\phi(x), \phi(y)] \neq 0 \quad \text{und} \quad \{\phi(x), \phi(y)\} \neq 0 \quad .$$

Exercise 27:

The Dirac field operators Ψ_i with spinor component i satisfy the equal-time anticommutation relations

$$\{\Psi_a(x), \Psi_b(y)\}_{x0=y0} = \{\Psi^\dagger_a(x), \Psi^\dagger_b(y)\}_{x0=y0} = 0\,,$$

and

$$\{\Psi_a(x), \Psi^\dagger_b(y)\}_{x0=y0} = \delta_{a,b}\delta^{(3)}(\vec{x}-\vec{y})\,.$$

Based on these anticommutations relations show that the charge-current density operator

$$j^\mu(x) = -e\bar{\Psi}(x)\gamma^\mu\Psi(x)$$

satisfies the relation (5P)

$$[j^\mu(x), j^\nu(y)] = 0, \text{ for } (\mathrm{x}-\mathrm{y})^2 < 0\,.$$

Worked-out solutions to the homework problems should be handed in at the beginning of the lecture of Tuesday, Dec. 13.