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Problem 1 (Real scalar field) **(20 points)**

Consider a real scalar quantum field $\phi(x)$.

a. Show that the time-ordered product $\mathrm{T}[\phi(x_1)\,\phi(x_2)]$ and the normal-ordered product $:\ \phi(x_1)\,\phi(x_2)\ :$ are both symmetric under the interchange $x_1 \leftrightarrow x_2$. 6 P

b. Show that the Feynman-propagator $\Delta_F(x_1 - x_2)$ shares this property as well. 4 P

c. Check Wick's theorem for the case of three scalar fields, 10 P

$$\begin{aligned}
\mathrm{T}[\phi(x_1)\,\phi(x_2)\,\phi(x_3)] \quad &= \quad :\ \phi(x_1)\,\phi(x_2)\,\phi(x_3)\ : + \phi(x_1)\, i\Delta_F(x_2 - x_3) \\
&\quad + \phi(x_2)\, i\Delta_F(x_1 - x_3) + \phi(x_3)\, i\Delta_F(x_1 - x_2)\,.
\end{aligned}$$

Hint: Note that the symmetry property under the interchange $x_1 \leftrightarrow x_2$ of the time-ordered product and the normal-ordered product in **a.** and **b.** holds for any number of fields. Hence, it is sufficient to check the formula above for one particular time ordering.

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Problem 2 (Dirac equation and Gordon identity) **(20 points)**

a. Starting from the Dirac equation in coordinate space, $(i\partial\!\!\!/ - m)\psi(x) = 0$, prove that the spinor ψ also satisfies the Klein-Gordon equation. 5 P

b. Let's consider the Dirac equation in momentum space for a particle spinor u,

$$(p\!\!\!/ - m)u(p) = 0 \quad \text{and} \quad \bar{u}(p')(p\!\!\!/' - m) = 0\,.$$

Use the Clifford-algebra $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$ to show the following identity, 15 P

$$\bar{u}(p')\,\gamma^\mu\,u(p) \quad = \quad \frac{1}{2m}\bar{u}(p')\,(p + p')^\mu\,u(p) + \frac{1}{m}\bar{u}(p')\,i\sigma^{\mu\nu}(p' - p)_\nu\,u(p)\,,$$

where $\sigma^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$.

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Problem 3 (Static external potential) **(20 points)**

A real scalar field $\phi(x)$ is described by the Lagrangian density,

$$\mathcal{L} = \tfrac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \tfrac{1}{2}\mu^2\,\phi^2(x) + \mu U(\boldsymbol{x})\,\phi^2(x)$$

where $U(\boldsymbol{x})$ is a static external classical potential.

a. Derive the equation of motion 5 P

$$(\partial^2 + \mu^2)\phi(x) = 2\mu U(\boldsymbol{x})\,\phi(x)\,.$$

b. Show that, in lowest order, the S-matrix element for an incoming boson, with four momentum $k_i^\mu = (\omega_i, \boldsymbol{k}_i)$, to be scattered to a state with momentum $k_f^\mu = (\omega_f, \boldsymbol{k}_f)$, is given by 15 P

$$\langle \boldsymbol{k}_f |\, S^{(1)}\, | \boldsymbol{k}_i \rangle = \frac{i2\pi\delta(\omega_f - \omega_i)}{\sqrt{2V\omega_i}\sqrt{2V\omega_f}} 2\mu\tilde{U}(\boldsymbol{k}_f - \boldsymbol{k}_i)\,,$$

where

$$\tilde{U}(\boldsymbol{q}) = \int d^3\boldsymbol{x}\, U(\boldsymbol{x})\, \mathrm{e}^{-i\boldsymbol{q}\cdot\boldsymbol{x}}\,.$$

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Problem 4 (Gauge-independence of Feynman amplitudes) (20 points)

In the lecture we introduced the polarization vector $\varepsilon_r^\mu(k)$ for plane light wave solutions of the Maxwell equations. Those polarization vectors are gauge dependent. For example, for a free photon, described in a Lorentz gauge by a plane wave $A^\mu(x) = \text{const}\,\varepsilon_r^\mu(k)\,\mathrm{e}^{\pm ik\cdot x}$, the gauge transformation $A^\mu(x) \to A^\mu(x) + \partial^\mu f(x)$, with $f(x) = \tilde{f}(k)\,\mathrm{e}^{\pm ik\cdot x}$, implies

$$\varepsilon_r^\mu(k)\,\mathrm{e}^{\pm ik\cdot x} \to [\varepsilon_r^\mu(k) \pm i\tilde{f}(k)\,k^\mu]\,\mathrm{e}^{\pm ik\cdot x}.$$

Consider the invariant amplitude $\mathcal{M}$ for the Compton scattering process of a photon and an electron, $\gamma(\boldsymbol{k}) + e^-(\boldsymbol{p}) \to \gamma(\boldsymbol{k}') + e^-(\boldsymbol{p}')$.

a. Draw all of the lowest order QED Feynman diagrams for this process and give explicit expressions for them using Feynman rules. 5 P

b. Show that the sum of the invariant amplitudes for lowest order Compton scattering is gauge invariant by considering the gauge transformations 15 P

$$\varepsilon_r^\mu(k) \to \varepsilon_r^\mu(k) + \lambda\,k^\mu \quad ; \quad \varepsilon_s^\mu(k') \to \varepsilon_s^\mu(k') + \lambda'\,k'^\mu.$$

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Problem 5 (Scalar QED) **(20 points)**

The Lagrangian density of scalar QED is given by

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + (D_\mu \phi)^* (D^\mu \phi) - m^2 \phi^* \phi \,,$$

with $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ and $D^\mu = \partial^\mu + ie A^\mu$.

a. Show that the scalar QED Lagrangian is invariant under the transformation $\phi \to e^{ie\alpha} \phi$, with $\alpha \in \mathbb{R}$. 5 P

b. Calculate the Noether current and charge corresponding to the symmetry transformation in **a.** 5 P

c. Expand the Lagrangian in terms of the interaction terms and state the Feynman rules for the vertices of scalar QED. 10 P