Quantum Field Theory[1]

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[1]This preliminary version of the script is intended only for use in parallel with attendance of the lecture. The study of the script cannot replace the attendance of the lecture! If you find any errors, please let us know!

Contents

Chapter 1

Introduction and Preliminaries

Quantum field theory (QFT) can be divided into two branches, one is relativistic quantum field theory, which is relevant for particle physics. The other branch is non-relativistic quantum field theory, which is relevant in condensed matter physics or statistical physics. In this lecture we focus on relativistic QFT, it is a theory of infinite degrees of freedom that unifies special relativity (SR) and quantum mechanis (QM).

1.1 Natural units

The usual units that are used in physics are the S.I. units, that is, kg, m, s and so on. Two fundamental constants that often appear in QFT are *Planck's constant*

$$\hbar = \frac{h}{2\pi} = 1.055 \cdot 10^{-35}\,\mathrm{Js} \tag{1.1.1}$$

and the *speed of light*

$$c = 2.99 \cdot 10^{8}\,\frac{\mathrm{m}}{\mathrm{s}}. \tag{1.1.2}$$

A clever way to avoid these in equations is to use a different choice of units, the n*atural units*. Here we set

$$\hbar = c = 1. \tag{1.1.3}$$

As a consequence all physical quantities are given in powers of energy (or equivalently mass)

$$[\text{energy}]^{\mu} = \mathrm{eV}^{\mu},\ \mathrm{MeV}^{\mu},\ \mathrm{GeV}^{\mu}, \ldots, \tag{1.1.4}$$

where $\mu \in \mathbb{Z}$ and $1\,\mathrm{eV} = 1.6021 \cdot 10^{-19}$ J. Moreover, since $c = 1$ we have

$$[c] = 1 = \frac{[L]}{[T]} \quad \Longleftrightarrow \quad [L] = [T], \tag{1.1.5}$$

meaning that both length and time have the same dimension. From the relativistic energy-momentum relation

$$E^2 = \vec{p}^2c^2 + m^2c^4 = \vec{p}^2 + m^2, \tag{1.1.6}$$

we obtain that $[E] = [|\vec{p}|] = [m]$ =MeV or GeV. Furthermore, since we set $\hbar = 1$ we get

$$[\hbar] = 1 = \left[\int dtL\right] = [T][L] = [T][H]. \tag{1.1.7}$$

Hence, $[L] = [T] = \frac{1}{\text{MeV}} = \text{MeV}^{-1}$. A useful relation is

$$\hbar c = 1 = 200\,\text{MeV fm}, \tag{1.1.8}$$

where $1\,\text{fm} = 10^{-15}\,\text{m}$.

1.2 Special relativity

A fundamental observation in special relativity is that the speed of light c is universal in all reference frames. As a consequence the time t becomes a coordinate

$$c = \frac{|\vec{x}|}{t} = \frac{|\vec{x}'|}{t'} \Leftrightarrow c^2t^2 - x^2 - y^2 - z^2 = c^2t'^2 - x'^2 - y'^2 - z'^2 = 0. \tag{1.2.1}$$

Thus, two frames S and S' are related through a *Lorentz transformationation* (LT)[1]

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}, \tag{1.2.2}$$

where $\beta = v/c$, $\gamma = \frac{1}{\sqrt{1-\beta^2}} = \cosh(\omega)$ and $\beta\gamma = \sinh(\omega)$ with ω the rapidity. The above transformation matrix is called the *Lorentz-matrix* Λ.

1.2.1 Relativistic notation

In this notation we write $c = 1$, furthermore, the contravariant 4-vector is given as

$$x \equiv (x^\mu) = (t, \vec{x}) = (x^0, x^1, x^2, x^3). \tag{1.2.3}$$

[1] Boost in x-direction.

The metric tensor is

$$g_{\mu\nu} = g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{1.2.4}$$

One defines the "dot product" of the two vectors x^μ, y^μ as

$$\begin{aligned} x \cdot y &= x^\mu y^\nu g_{\mu\nu} \\ &= x^0 y^0 - x^1 y^1 - x^2 y^2 - x^3 y^3 \\ &= x^0 y^0 - \vec{x} \cdot \vec{y}. \end{aligned} \tag{1.2.5}$$

Note that the last equality is an implicit sum over repeated indices called the *Einstein summation convention*. This is not a scalar product in the mathematical sense, becuase it is not positive definit. For the case that $x = y$ one obtains

$$x \cdot x = x^2 = (x^0)^2 - \vec{x}^2 = t^2 - \vec{x}^2. \tag{1.2.6}$$

A four dimensional space of 4-vectors equipped with such a dot product is called a *Minkowski space*. A covariant vector has subscript indices and is formally written as

$$x_{\mu=0,1,2,3} = (t, -\vec{x}) = g_{\mu\nu} x^\nu. \tag{1.2.7}$$

Analogously, one can write

$$x^\mu = g^{\mu\nu} x_\nu. \tag{1.2.8}$$

Thus, the metric tensor lowers or raises indices. Additionally, we may write

$$x \cdot y = x^\mu y^\mu g_{\mu\nu} = x^\mu y_\mu. \tag{1.2.9}$$

An important identity is

$$g^{\nu\mu} g_{\mu\alpha} = \delta^\nu_\alpha = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{1.2.10}$$

The Lorentz transformation in this notation is written as $\Lambda^\mu{}_\nu$ and

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu. \tag{1.2.11}$$

The Lorentz transformation shall leave the dot product invariant, that is, $x' \cdot y' = x \cdot y$. This yields the condition

$$g_{\rho\sigma} = \Lambda^{\mu}{}_{\sigma} g_{\mu\nu} \Lambda^{\nu}{}_{\rho}. \tag{1.2.12}$$

Any matrix Λ that satisfies this equation is a Lorentz transformation. The product of matrices can thus be written as

$$g = \Lambda^T g \Lambda \Longleftrightarrow g_{\rho\sigma} = \Lambda^{\mu}{}_{\rho} g_{\mu\nu} \Lambda^{\nu}{}_{\sigma}. \tag{1.2.13}$$

We can also calculate the determinant of matrices in this notation

$$\det g = -1 = \det(\Lambda^T g \Lambda) = \det(g) \cdot (\det \Lambda)^2 = -(\det \Lambda)^2 \Rightarrow \det \Lambda = \pm 1. \tag{1.2.14}$$

Transformations with $\det \Lambda = +1$ describe boosts, rotations etc. and are called *proper* Lorentz transformations. Transformations with $\det \Lambda = -1$ are called *discrete* Lorentz transformationations, examples are space inversion and time reversal. If the determinant is different from zero, then there exists an inverse LT

$$\Lambda^{-1} = g \Lambda^T g \Longleftrightarrow (\Lambda^{-1})^{\mu}{}_{\nu} = \Lambda_{\nu}{}^{\mu}. \tag{1.2.15}$$

Covariant 4-vectors transform with the inverse LT in the following way

$$x'_{\mu} = \Lambda_{\mu}{}^{\nu} x_{\nu}. \tag{1.2.16}$$

Definition: Contravariant 4-tensors

An arbitrary multi-index quantity $T^{\mu_1 \cdots \mu_M}$ is called a *contravariant 4-tensor of rank M* if it transforms like

$$(T^{\mu_1 \cdots \mu_M})' = \Lambda^{\mu_1}_{\nu_1} \ldots \Lambda^{\mu_M}_{\nu_M} T^{\nu_1 \ldots \nu_M}. \tag{1.2.17}$$

Example: 4-vector

A general 4-vector a^{μ} is a 4-tensor of rank 1 where

$$(a^{\mu})' = \Lambda^{\mu}{}_{\nu} a^{\nu}. \tag{1.2.18}$$

Definition: Time-, space- and lightlike vectors

The dot-product of a^{μ} is not positive definite, thus a^2 can be positive or negative or 0 ($a^{\mu} = 0$). We call the different cases

- $a^2 > 0$: *time-like* vector

- $a^2 < 0$: *space-like* vector
- $a^2 = 0$: *light-like* vector

A time-like vector connects two events that are causally connected, that is, the second event is within the light cone of the first event. This means that they can cause each other to happen. It is not possible to swap the order of those events. A space-like vector connects two events that are causally disconnected, that is the second event is outside the light cone of the first event. It is possible to change the temporal order of those events when changing the reference frame.

Principle of relativity

All physical laws must have the same form in all reference frames. Meaning, all equations should be form invariant under LT.

Example: Different types of 4-vectors

- **4-gradient** The LT matrix can be expressed as

$$\frac{\partial x^\rho}{\partial x'^\nu} = \Lambda_\nu{}^\rho \tag{1.2.19}$$

or

$$\frac{\partial x'^\rho}{\partial x^\nu} = \Lambda^\rho{}_\nu = (\Lambda^{-1})_\nu{}^\rho, \tag{1.2.20}$$

where the atter one is an alternative representation for Λ^{-1}. We write for the 4-gradient

$$\partial'_\nu \equiv \frac{\partial}{\partial x'^\nu} = \Lambda_\nu{}^\mu \frac{\partial}{\partial x^\mu} \equiv \Lambda_\nu{}^\mu \partial_\mu. \tag{1.2.21}$$

As we see it transforms like a covariant 4-vector. The covariant 4-gradient is

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu} = (\partial_0, \partial_1, \partial_2, \partial_3) = \left(\frac{\partial}{\partial t}, \vec{\nabla}\right) \qquad \text{covariant 4-gradient,} \tag{1.2.22}$$

$$\partial^\mu \equiv \frac{\partial}{\partial x_\mu} = (\partial^0, \partial^1, \partial^2, \partial^3) = \left(\frac{\partial}{\partial t}, -\vec{\nabla}\right) \qquad \text{contravariant 4-gradient.} \tag{1.2.23}$$

The 4-divergence of 4-vector field $A^\mu(x)$ is

$$\partial A = \partial_\mu A^\mu(x) = \frac{\partial A^0}{\partial t}(x) + \vec{\nabla} \cdot \vec{A}(x) \tag{1.2.24}$$

and is Lorentz invariant. The D'Alembert operator can be written as

$$\partial_\mu \partial^\mu = \partial^2 = \Box = \frac{\partial^2}{\partial t^2} - \Delta, \tag{1.2.25}$$

which is again Lorentz invariant.

- **4-velocity** We first introduce the Lorentz invariant distance

$$ds^2 \equiv c^2 dt^2 - (d\vec{x})^2 \tag{1.2.26}$$

and the *proper time*, which is also Lorentz invariant

$$d\tau = \frac{ds}{c} = \frac{dt}{\gamma}. \tag{1.2.27}$$

The physical interpretation of the proper time is the that it is the time of the clock, which moves along the trajectory with a particle. With this we can now define the 4-velocity

$$v^\mu = \frac{dx^\mu}{d\tau} = \frac{dx^\mu}{dt}\frac{dt}{d\tau} = \gamma \frac{d}{dt}\begin{pmatrix} ct \\ \vec{x} \end{pmatrix} = \gamma \begin{pmatrix} c \\ \vec{v} \end{pmatrix}. \tag{1.2.28}$$

The dot product of the 4-velocity with itself yields

$$v \cdot v = v^2 = \gamma^2 (c^2 - \vec{v}^2) = c^2 \frac{1 - \frac{\vec{v}^2}{c^2}}{1 - \frac{\vec{v}^2}{c^2}} = c^2 = v^2. \tag{1.2.29}$$

- **4-momentum** (of a point-like mass m) Note that m is the mass of a particle at rest. Then the 4-momentum is given as

$$P^\mu = m v^\mu = m \frac{dx^\mu}{d\tau}. \tag{1.2.30}$$

As we can see P^μ is a 4-vector, hence we can write it as $P^\mu = m\gamma(c, \vec{v})$. Taking the dot product with itself we obtain

$$P_\mu P^\mu = P^2 = m^2 v^2 = m^2 c^2 = P^2. \tag{1.2.31}$$

For $P^\mu = (P^0, \vec{p}) \equiv (\frac{E}{c}, \vec{p})$ we have E as the relativistic energy and $\vec{p}$ as the 3-momentum. Then the relativisitc energy-momentum relation is given as

$$E = \sqrt{\vec{p}^2 c^2 + m^2 c^4}. \tag{1.2.32}$$

Chapter 2

Elements of Classical Field Theory

2.1 Lagrangian Formulation of Field Theory

Consider N point-like masses $m_{i=1,...,M}$ and suppose we have f degrees of freedom. In analytic mechanics we would have f generalized coordinates $q_{i=1,...,f}$. We first write a coordinate vector $\vec{r}_i$ of each mass m_i as a function of the generalized coordinates, that is, $\vec{r}_i = \vec{r}_i(\vec{q})$. We then construct the kinetic energy, which is given as

$$T = \sum_{i=1}^{N} \frac{1}{2} m_i \dot{\vec{r}}_i(\vec{q})^2 \tag{2.1.1}$$

and the potential $V(\vec{r}_i; \vec{q})$. The *Lagrangian function* is given as $L = T - V = L(\vec{q}, \dot{\vec{q}})$. We can also look at other quantities such as the *action*

$$S[\vec{q}] = \int_{t_1}^{t_2} dt L(\vec{q}, \dot{\vec{q}}), \tag{2.1.2}$$

which is a central quantity. The *Principle of least action* is

$$\delta S[\vec{q}] = 0. \tag{2.1.3}$$

The action $S[\vec{q}]$ is minimal, if the pointlike *Euler-Lagrange equations* or equation of motions

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} = 0, \tag{2.1.4}$$

are satisfied. These are a set of f differential equations in classical mechanics. For the continous case in three dimensions one has

$$\frac{\partial}{\partial t} \frac{\partial \mathcal{L}}{\partial(\partial_t \phi)} - \frac{\partial \mathcal{L}}{\partial \phi} + \vec{\nabla} \cdot \frac{\partial \mathcal{L}}{\partial(\vec{\nabla}\phi)} = 0, \tag{2.1.5}$$

where the last term is the so called *field theoretic term* and $\mathcal{L} = \mathcal{L}(\phi, \partial_t\phi, \vec{\nabla}\phi)$ is the *Lagrangian density* or often just *Lagrangian*. The power of the Lagrangian field theory is that we can construct it such that we maintain manifest Lorentz-invariance at all times! Furthermore, we can construct a relativistic field theory.

Remark.

- $\int_{t_1}^{t_2} dt \int_\Omega d^3x \to \int_{\mathbb{R}} dt \int_{\mathbb{R}^3} d^3x = \int_{\mathbb{R}^4} d^4x$ and the measure d^4x is Lorentz invariant.
- $\mathcal{L}$ *is assumed to be a Lorentz-scalar!* The guiding principle is to construct the Lagrangian.
- We want $\mathcal{L} = \mathcal{L}(\phi, \partial_t\phi, \vec{\nabla}\phi) \overset{!}{=} \mathcal{L}(\phi, \partial_\mu\phi)$, this is the case in QED, QCD and in the standard model. Note that a Lagrangian can contain higher derivatives, that is, $\mathcal{L}(\phi, \partial_\mu\phi, \partial_\mu\partial_\nu\phi, \ldots)$. These yield *effective field theories*, which are "non-renormalizable".
- We will study *local* Lagrangians where all fields depend on the same coordinate x^μ. A "non-local Lagrangian" would be

$$\mathcal{L} = \int d^4y \phi(x)\phi(y-x). \tag{2.1.6}$$

We will also study Lagrangians of the form

$$\mathcal{L} = \mathcal{L}(\phi, \partial_\mu\phi) \to S[\phi] = \int d^4x \mathcal{L}(\phi, \partial_\mu\phi). \tag{2.1.7}$$

Applying the principle of least action, $\phi \to \phi + \delta\phi$, gives

$$\begin{aligned}
\delta S[\phi] &= \int d^4x \delta\mathcal{L}(\phi + \delta\phi, \partial_\mu\phi + \partial_\mu\delta\phi) \\
&= \int d^4x \left[\frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\mu(\delta\phi)\right] && (2.1.8) \\
&= \int d^4x \left[\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right]\delta\phi && (2.1.9) \\
&= 0 \\
\Leftrightarrow \frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} &= 0, && (2.1.10)
\end{aligned}$$

where we used partial integration to go from Eq. (2.1.8) to Eq. (2.1.9).

Note that

$$\delta\mathcal{L}(\phi+\delta\phi,\partial_\mu\phi+\partial_\mu(\delta\phi))=\mathcal{L}(\phi+\delta\phi,\partial_\mu\phi+\partial_\mu(\delta\phi))-\mathcal{L}(\phi,\partial_\mu\phi). \tag{2.1.11}$$

Recall that the *momentary virtual displacement* (momentary here means $\delta t=0$) for an arbitrary and sufficiently smooth function $\vec{f}=\vec{f}(q_1,\ldots,q_N,t)$ is defined as

$$\delta\vec{f}=\sum_{i=1}^{N}\frac{\partial\vec{f}}{\partial q_i}\delta q_i. \tag{2.1.12}$$

2.2 Hamilton Formulation of Field Theory

In point mechanics we have $L=L(q_k,\dot{q}_k)$ and the *canonical momenta*

$$p_k=\frac{\partial L}{\partial\dot{q}_k}, \tag{2.2.1}$$

which we can solve for $\dot{q}_k$ and obtain $\dot{q}_k=f(p_k,q_k)$. From this we can construct the *Hamiltonian function*

$$H=H(p_k,q_k)=\sum_k p_k\dot{q}_k-L, \tag{2.2.2}$$

which is defined as the *Legendre transformation* of the Lagrangian function. Recall

$$\dot{q}_k=\frac{\partial H}{\partial p_k},\qquad \dot{p}_k=-\frac{\partial H}{\partial q_k}. \tag{2.2.3}$$

We can do the same in the field theory. In field theory though we have the *canoncial momentum field*

$$\pi(x)\equiv\frac{\partial\mathcal{L}}{\partial(\partial_t\phi)}, \tag{2.2.4}$$

which we can solve for $\partial_t\phi$ and obtain $(\partial_t\phi)=f(\pi,\vec{\nabla}\phi,\phi)$. The *Hamiltonian density* can be defined by the Legendre transofrmation as follows

$$\mathcal{H}=\pi\partial_t\phi-\mathcal{L}, \tag{2.2.5}$$

with $\mathcal{L}=\mathcal{L}(\phi,\partial_\mu\phi)$. The Hamiltonian then just is

$$H=\int d^3x\mathcal{H}. \tag{2.2.6}$$

As for the *canonical equations of motions*, we have

$$\partial_t\phi=\frac{\partial\mathcal{H}}{\partial\pi},\qquad \partial_t\pi=-\frac{\partial\mathcal{H}}{\partial\phi}+\vec{\nabla}\cdot\frac{\partial\mathcal{H}}{\partial(\vec{\nabla}\phi)}. \tag{2.2.7}$$

2.3 Lagrange Formalism for Fields with N components

A function $\phi_a(x,t) = \phi_a(x)$ with $a = 1,\dots,N$ components, describes a field. The Lagrange density is

$$\mathcal{L} = \mathcal{L}(\phi_a(x), \partial_\mu \phi_a(x)) \quad \text{such that} \quad L = \int d^3x \mathcal{L}. \tag{2.3.1}$$

For the Euler-Lagrange equations of motion we find as usual

$$\frac{\partial \mathcal{L}}{\partial \phi_a} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} = 0 \quad \text{with} \quad a = 1,\dots,N. \tag{2.3.2}$$

The momentum density is conjugate to ϕ_a and is given by

$$\pi_a = \frac{\partial \mathcal{L}}{\partial \dot{\phi}_a} = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi_a)} = \frac{\partial \mathcal{L}}{\partial(\partial_t \phi a)}. \tag{2.3.3}$$

As before the Hamiltonian H is

$$H = \int \mathcal{H} d^3x = \int (\pi_a \dot{\phi}_a - \mathcal{L}) d^3x. \tag{2.3.4}$$

2.4 Symmetries and Conservation Laws

If the Lagrange density $\mathcal{L}$ is invariant under *continous symmetry transformation*, one obtains *conserved currents*. This will lead to the so called *Noether theorem*, the general idea is the following: Change of $\mathcal{L}$ while changing ϕ_a and $\partial_\mu \phi_a$,

$$\phi_a \to \phi'_a = \phi_a + \delta\phi_a, \tag{2.4.1}$$

$$\partial_\mu \phi_a \to (\partial_\mu \phi_a)' = \partial_\mu \phi_a + \delta(\partial_\mu \phi_a). \tag{2.4.2}$$

We find that the change of the Lagrange density is

$$\delta\mathcal{L} = \frac{\partial \mathcal{L}}{\partial \phi_a} \delta\phi_a + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \partial_\mu(\delta\phi_a). \tag{2.4.3}$$

Furthermore, we have Euler-Lagrange

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \right) - \frac{\partial \mathcal{L}}{\partial \phi_a} = 0 \Leftrightarrow \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \right) = \frac{\partial \mathcal{L}}{\partial \phi_a}. \tag{2.4.4}$$

The change of the Lagrange density can then be written as

$$\delta\mathcal{L} \overset{(2.4.4)}{=} \partial_\mu \left\{ \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \delta\phi_a \right\}, \tag{2.4.5}$$

where we used the product rule to pull out the derivative ∂_μ. We now want to take a look at *geometrical space-time symmetries.* The translation in 4-dimensional space is given by an infinitesimal change

$$x_\nu \to x'_\nu = x_\nu + \epsilon_\nu. \tag{2.4.6}$$

Note that ϵ_ν does not depend on x_ν. The Lagrange density is invariant under such a translation, because it does not depend on x_ν that is $\mathcal{L} \neq \mathcal{L}(x_\nu)$. The change of $\mathcal{L}$ becomes

$$\delta\mathcal{L} = \epsilon_\nu \frac{\partial \mathcal{L}}{\partial x_\nu} = \epsilon_\nu \partial^\nu \mathcal{L}(\phi_a, \partial_\mu \phi_a) \tag{2.4.7}$$

and the change of a field ϕ_a under such a translation is given by

$$\delta\phi_a = \epsilon_\nu \frac{\partial \phi_a}{\partial x_\nu} = \epsilon_\nu \partial^\nu \phi_a. \tag{2.4.8}$$

Using Eq. (2.4.5) we obtain

$$\delta\mathcal{L} = \epsilon_\nu \partial_\mu \left\{ \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \right\} \tag{2.4.9}$$

Comparing the last equation to expressions of the change of the Lagrange density we can deduce that for arbitrary ϵ_ν we can write

$$\partial_\mu \left\{ \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \partial^\mu \phi_a - g^{\mu\nu} \mathcal{L} \right\} = 0. \tag{2.4.10}$$

We now define the *Energy-Momentum Tensor* $T_{\mu\nu}$ by

$$T^{\mu\nu} \equiv \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \partial^\mu \phi_a - g^{\mu\nu} \mathcal{L}. \tag{2.4.11}$$

We thus obtain

$$\partial_\mu T^{\mu\nu} = 0, \tag{2.4.12}$$

which is just a continuity equation if we compare with the previous result for a 4-current j^μ where we had $\partial_\mu j^\mu = 0$. Integrating the continuity equation yields

$$\int_V d^3x \partial_\mu T^{\mu\nu} = 0, \tag{2.4.13}$$

$$\frac{d}{dt} \left\{ \int_V d^3x T^{0\nu}(x) \right\} = - \int_V \frac{\partial}{\partial x^i} d^3x \overset{\text{Gauss}}{=} - \oint_{O(V)} \vec{S}^{(\nu)} \cdot d\vec{f}, \tag{2.4.14}$$

with $\vec{S}^{(\nu)} = (T^{1\nu}, T^{2\nu}, T^{3\nu})$ the 3-momentum and energy flux. For localized fields $V \to \infty$, $\phi_a \to 0$, $\oint = 0$ we can define the *4-momentum of the field* as

$$P^\nu \equiv \int_V d^3x T^{0\nu}. \tag{2.4.15}$$

Now, let us take a look at the 0th component or time-component and see what we get

$$\begin{aligned} P^0 &= \int d^3x T^{00} \\ &= \int d^3x \left\{ \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi_a)} \partial^0 \phi_a - \mathcal{L} \right\} \\ &= \int d^3x \left\{ \pi_a \dot{\phi}_a - \mathcal{L} \right\} \\ &= H. \end{aligned} \tag{2.4.16}$$

Therefore, P^0 stands for the energy of the field, which is conserved. For the spatial components we get

$$P^{k=1,2,3} = \int d^3x T^{0k} = \int d^3x \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi_a)} \partial^k \phi_a, \tag{2.4.17}$$

the *field momentum* and the conservation of this quantity. We can, thus, conclude that translational invariance in terms of space and time leads to

1. energy and momentum conservation and
2. the definition of these quantities for the fields.

Another aspect is, that we could have looked at invariance under rotations in space. This leads to angular momentum conservation and the definition of these quantities.

2.5 Internal Symmetries

Symmetries of internal degrees of freedom are, for example, charge and isospin. We illustrate this as follows: imagine we have a field $\phi_a(x) \to \phi'_a(x) = e^{-iq_a\epsilon}\phi_a(x)$ with $q_a \in \mathbb{C}$ such that $q_a \neq q_a(x)$ and with ϵ a fixed parameter independent of $a = 1, \ldots, N$. Note that $\mathcal{L}$ depends on the product of fields ϕ_a and $\partial_\mu \phi_a$, meaning

$$\phi_1(x)\phi_2(x)\ldots\phi_n(x) \to e^{-i(q_1+q_2+\ldots+q_n)\epsilon}\phi_1(x)\phi_2(x)\ldots\phi_n(x) \tag{2.5.1}$$

or

$$\partial_\nu\phi_1(x)\partial_\lambda\phi_2(x)\ldots\partial_\xi\phi_n(x) \to e^{-i(q_1+q_2+\ldots+q_n)\epsilon}\partial_\nu\phi_1(x)\partial_\lambda\phi_2(x)\ldots\partial_\xi\phi_n(x). \tag{2.5.2}$$

We then have invariance of $\mathcal{L}$ under $\phi_a \to \phi'_a$ with $q_1 + q_2 + \ldots + q_n = 0$. One could take for example the electric charge q_a. In that case the total charge of the system of fields vanishes for all times. More generally, consider mixing of different, independant fields. A more general expression is

$$\phi_a(x) \to \phi'_a = e^{-i\epsilon q_{ab}}\phi_b, \tag{2.5.3}$$

where we implicitly sum over b. Again ϵ is a fixed parameter and $q_{ab} \in \mathbb{C}$. In the infinitesimal version one has

$$\phi_a(x) \to \phi'_a(x) = \phi_a(x) \underbrace{-i\epsilon q_{ab}\phi_b}_{=\delta\phi_a(x)}. \tag{2.5.4}$$

We now state the following: $\mathcal{L}$ is invariant under this variation $\delta\phi_a$, that is,

$$\delta\mathcal{L} = \partial_\mu \left\{ \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\delta\phi_a \right\} \stackrel{!}{=} 0. \tag{2.5.5}$$

This is a continuouity equation $\partial_\mu j^\mu = 0$ with the 4-vector

$$j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\delta\phi_a \cdot \frac{1}{\epsilon} = -i\frac{\mathcal{L}}{\partial(\partial_\mu\phi_a)}q_{ab}\phi_b, \tag{2.5.6}$$

which we call *Noether current* j^μ. Note that we are free to pick ϵ and since

$$\partial_\mu j^\mu = \frac{\partial}{\partial_t}j^0 + \vec{\nabla}\cdot\vec{j} = 0, \tag{2.5.7}$$

we obtain that the charge is given by $Q = \int d^3x j^0(x)$.

2.6 Classification of Fields

Transformation properties under Lorentz transformations are

$$x^\mu = \Lambda^\mu{}_\nu x^\nu, \tag{2.6.1}$$

with $\det(\Lambda) = +1$, which we call *proper* LT. If $\det(\Lambda) = -1$, then we call it an *inproper* LT. We deal with different types of fields

- **Scalar fields** $\phi(x)$: These transform as $\phi'(x') = \phi(x)$. They describe spin 0 particles with $J^P = 0^+$ where P is the parity and J the total spin. An example is the Higgs Boson.

- **Pseudoscalar fields** $\phi(x)$: These transform as $\phi'(x') = \det(\Lambda)\pi(x)$. They describe spin 0 particles with $J^P = 0^-$. Examples are pions, that is, $\pi^{\pm}, \pi^0$.

- **Vector fields** $A^\mu(x)$: These transform as $A'^\mu(x') = \Lambda^\mu{}_\nu A^\nu(x)$. They describe spin 1 particles with $J = 1$. Examples are the photon γ, W-bosons $W^\pm$ or the Z-boson Z^0.

- **Spinor fields or Fermion fields** $\psi(x)$: They are solutions to the Dirac equation. These transform as $\psi'(x') = S(\Lambda)\psi(x)$ and they describe spin 1/2 particles such as quarks and leptons.

Chapter 3

Quantisation of Free Fields

3.1 Canonical Quantization of Fields

The usual path in quantum mechanics is the following: first we consider point particle mechanics with the Lagrange function $L = m\dot{q}^2/2 - V(q)$. We can then construct the Hamilton function $H(p,q) = p\dot{q} - L(q,\dot{q})$ by introducing the conjugate momentum $p \equiv \partial L/\partial\dot{q}$. Imagine we have a quantity $A(q,p)$ and we look at this quantity and how it changes with time. We can write down the equation of motion

$$\dot{A}(q,p) = \Big\{A, H\Big\}_{PB}, \tag{3.1.1}$$

with the *Poisson brackets* $\{\cdot\}_{PB}$ defined as

$$\{A,B\}_{PB} = \left(\frac{\partial A}{\partial q}\frac{\partial B}{\partial p} - \frac{\partial A}{\partial p}\frac{\partial B}{\partial q}\right). \tag{3.1.2}$$

Then, for example, we obtain the Hamilton equations of motion

$$\dot{p} = \{p,H\} = -\frac{\partial H}{\partial q}; \qquad \dot{q} = \{q,H\} = \frac{\partial H}{\partial p}. \tag{3.1.3}$$

If the Poisson brackets $\{A,B\} = C$, then especially $\{x,p\} = 1$ with the replacement $q \to x$. Quantization means, that the functions A, B and C become operators, as can be seen in the following "mapping" (note that we just substitute, there is no one to one mapping)

$$\{A,B\} = C \longrightarrow \frac{1}{i\hbar}[\hat{A},\hat{B}] = \hat{C}. \tag{3.1.4}$$

with $[\hat{A},\hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$ the *commutator*. In particular, we obtain $[\hat{x},\hat{p}] = i$ with $\hat{x},\hat{p}$ hermitian operators for position and momentum respectively. For $i = 1,\dots N$ degrees

of freedom we have

$$[\hat{q}_i, \hat{p}_j] = i\delta_{i,j}, \qquad [\hat{q}_i, \hat{q}_j] = [\hat{p}_i, \hat{p}_j] = 0. \tag{3.1.5}$$

In ordinary quantum mechanics we usually work in the so-called *Schrödinger picture*. In this picture the Schrödinger equation is given by

$$\hat{H}(\hat{p}, \hat{q}) \left|\psi, t\right\rangle_S = i\frac{\partial}{\partial t} \left|\psi, t\right\rangle_S , \tag{3.1.6}$$

where $\hat{p}, \hat{q}$ are time dependent. We call $\left|\psi, t\right\rangle_S$ a state vector. The time dependence of the Schrödingers equation is at least formally solved by

$$\left|\psi, t\right\rangle_S = U \left|\psi, t_0\right\rangle_S , \tag{3.1.7}$$

with t_0 a fixed time and $U = U(t, t_0) = e^{-i\hat{H}(t-t_0)/\hbar}$ the *time evolution operator*. U actually defines a unitary transformation from the Schrödinger picture (S) to the so-called *Heisenberg picture* (H) of states and operators. A state in the Heisenberg picture is defined by

$$\left|\psi, t\right\rangle_H \equiv U^\dagger \left|\psi, t\right\rangle_S = \left|\psi, t_0\right\rangle_S , \tag{3.1.8}$$

where U unitary time evolution operator, that is, $U^\dagger U = UU^\dagger = 1$ and

$$\hat{O}_H(t) = e^{i\hat{H}t} \hat{O}_S e^{-i\hat{H}t}, \tag{3.1.9}$$

where $t_0 = 0$. To summarize, in the Heisenberg picture

- state vectors are constant in time, operators $\hat{O}_H$ are time dependent, but $H_S = H_H = H \neq H(t)$.
- U is unitary (i.e. $UU^\dagger = U^\dagger U = 1$). This is important because

  $$_S\langle\psi, t|\hat{O}_S|\phi, T\rangle_S = {}_H\langle\psi, t|\hat{O}_H(t)|\phi, t\rangle_H \tag{3.1.10}$$

  where we inserted $U^\dagger U$ and $UU^\dagger$. Furthermore,

  $$[\hat{A}_S, \hat{B}_S] = \hat{C}_S \to [\hat{A}_H(t), \hat{B}_H(t)] = \hat{C}_H(t). \tag{3.1.11}$$

  Differentiating the equation of transformation yields

  $$\frac{d}{dt}\hat{O}_H(t) = \frac{1}{i}[\hat{O}_H, \hat{H}], \tag{3.1.12}$$

  which is called the *Heisenberg equation of motion*. We have a clear correspondance to classical physics, meaning operators follow certain equations of motions we already know from classical mechanics.

- in QFT we first work in the Heisenberg picture

We have to think about fundamental commutation relations, or what we call quantization conditions according to (note that from now on we drop the index H since we work in the Heisenberg picture)

$$[\hat{q}_i(t), \hat{p}_j(t)] = i\delta_{i,j} \tag{3.1.13}$$

$$[\hat{p}_i(t), \hat{p}_j(t)] = [\hat{q}_i(t), \hat{q}_j(t)] = 0 \tag{3.1.14}$$

at equal time t. We now generalize this to a continuous system with an infinte number of degrees of freedom

$$q_i \to \phi(\vec{x}) \tag{3.1.15}$$

$$i \to \vec{x} \tag{3.1.16}$$

and

$$\pi = \frac{\partial \mathcal{L}}{\partial \phi} = \text{"field momentum density"}. \tag{3.1.17}$$

Instead of a q_i we now have a field ϕ. We, thus, obtain

$$[\hat{\phi}(\vec{x}, t), \hat{\pi}(\vec{x}\,', t)] = i\delta^3(\vec{x}' - \vec{x}) \qquad \text{(density)}, \tag{3.1.18}$$

$$[\hat{\pi}(\vec{x}, t), \hat{\pi}(\vec{x}\,', t)] = 0, \tag{3.1.19}$$

$$[\hat{\phi}(\vec{x}, t), \hat{\phi}(\vec{x}\,', t)] = 0, \tag{3.1.20}$$

the *canonical quantization of field operators*.

3.2 Quantization of the real Klein-Gordon Field

In this chapter we quantize the known Klein-Gordon equation of motion. We will first derive the equation starting from the classical energy-momentum relation $E = \vec{p}^{\,2}/2m$ with the known substitution $E \to i\partial_t$, $\vec{p} \to -i\vec{\nabla}$. Recall, that $\hbar = c = 1$. This results in

$$i\partial_t \psi(\vec{x}, t) = -\frac{\vec{\nabla}^2}{2m}\psi(\vec{x}, t), \tag{3.2.1}$$

where ψ is a wavefunction. In the relativistic case consider the squared the energy-momentum relation

$$E^2 = \vec{p}^{\,2} + m^2 \quad \overset{\text{replacement}}{\longrightarrow} \quad -\frac{\partial^2}{\partial t^2}\phi(\vec{x}, t) = (-\vec{\nabla}^2 + m^2)\phi(\vec{x}, t). \tag{3.2.2}$$

Together with the d'Alembert operator

$$\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2 = \partial_\mu \partial^\mu \equiv \Box, \tag{3.2.3}$$

one obtains the so-called *Klein-Gordon equation*

$$(\Box + m^2)\phi(\vec{x}, t) = 0. \tag{3.2.4}$$

The problem with this equation is, that it results in "negative energy" solutions and "negative" probability densities as discussed in the Advanced Quantum Mechanics lecture. Let us ignore these problems for now and apply it to the fields.

3.2.1 Klein-Gordon-Field

If one sets the Lagrange density to $\mathcal{L} = \frac{1}{2}\{(\partial_\mu \phi(x))^2 - m^2\phi^2(x)\}$, from the Euler-Langrange equations we obtain $(\partial_\mu \partial^\mu + m^2)\phi = 0$. The canonical field momentum then reads $\pi = \partial\mathcal{L}/\partial\dot{\phi} = \dot{\phi}(x)$. We can also write down the Hamilton density

$$\mathcal{H} = \pi\dot{\phi} - \mathcal{L} = \pi^2 - \frac{1}{2}\{\pi^2 - (\vec{\nabla}\phi)^2 - m^2\phi^2\} = \frac{1}{2}(\pi^2 + (\vec{\nabla}\phi)^2 + m^2\phi^2). \tag{3.2.5}$$

Note that in the Lagrange density there is a negative sign in front of the m^2 whereas in the Klein-Gordon equation there is a plus sign. Quantization means that we have the following two commutation relations

$$[\phi(\vec{x}, t), \pi(\vec{x}, t)] = i\delta^{(3)}(\vec{x} - \vec{x}'), \tag{3.2.6}$$

$$[\hat{\pi}(\vec{x}, t), \hat{\pi}(\vec{x}, t)] = [\hat{\phi}(\vec{x}, t), \hat{\phi}(\vec{x}', t)] = 0. \tag{3.2.7}$$

In the following we will leave out the hats, as it is clear that we work with operators. The field operator ϕ fulfills the KG-equation $(\partial_\mu \partial^\mu + m^2)\phi = 0$. The idea now is that we expand the operator ϕ into solutions of the classical KG equation (i.e.Solution of a free particle $\rightarrow$ plane waves $\rightarrow$ Fourier expansion). Let us first look at such solutions of the KG equation. We use a plane wace ansatz

$$\varphi(\vec{x}) = e^{-(k^0 t - \vec{k}\cdot\vec{x})} = e^{ik^\mu x_\mu} = e^{ik\cdot x}. \tag{3.2.8}$$

Inserting in the KG equation yields

$$(\partial_t^2 - \vec{\nabla}^2 + m^2)\varphi = (-(k^0)^2 + \vec{k}^2 + m^2)\varphi = 0. \tag{3.2.9}$$

The next step is to say that $\vec{k}$ is arbitrary and by the above condition we find $k^0 = \pm\sqrt{m^2+\vec{k}^2}$. What one gets are two types of solutions

$$\varphi_+(x) = N_k e^{i(\omega_k t - \vec{k}\cdot\vec{x})} \tag{3.2.10}$$

$$\varphi_-(x) = N_k e^{-i(\omega_k t - \vec{k}\cdot\vec{x})} \tag{3.2.11}$$

with $\omega_k = \sqrt{m^2+\vec{k}^2}$. These solutions have the following properties:

- They are normalized to a finite volume $V = L^3$, that is, a box.
- They have periodic boundary conditions $\psi(x+L,y,z) = \psi(x,y,z)$. One then gets discrete values $\vec{k} = \frac{2\pi}{L}\vec{\ell}$ with $\vec{\ell} = (\ell_x, \ell_y, \ell_z)$ and $\ell_i = 0, \pm 1, \pm 2, \ldots$
- Choose the normalization constant to be $N_k = \frac{1}{\sqrt{V}\sqrt{2\omega_k}}$.
- They yield the completeness relation

$$\frac{1}{\sqrt{V}}\sum_{\vec{k}} e^{i\vec{k}\cdot(\vec{x}-\vec{x}')} = \delta^{(3)}(\vec{x}-\vec{x}') \tag{3.2.12}$$

- and they fulfill orthogonality

$$\frac{1}{V}\int e^{i(\vec{k}-\vec{k}')\cdot\vec{x}} d^3x = \delta_{\vec{k},\vec{k}'}. \tag{3.2.13}$$

  In some literature a continuous basis is used and one writes

$$\frac{1}{\sqrt{V}}\sum_{\vec{k}} \longrightarrow \frac{1}{(2\pi)^3}\int d^3x e^{i(\vec{k}-\vec{k}')\cdot\vec{x}} = \delta(\vec{k}-\vec{k}'). \tag{3.2.14}$$

  as well as

$$\delta_{\vec{k},\vec{k}'} \longrightarrow \delta^{(3)}(\vec{k}-\vec{k}') \tag{3.2.15}$$

In total we have

$$\phi(x) = \sum_{\vec{k}} N_k (e^{ik\cdot x}\alpha^*(\vec{k}) + e^{-ik\cdot x}\alpha(\vec{k})) \in \mathbb{R}, \tag{3.2.16}$$

with $k^\mu = (\omega_k, \vec{k}), k\cdot x = k^\mu x_\mu = \omega_k t - \vec{k}\cdot\vec{x}$ and $\alpha(\vec{k})$ is an arbitrary complex function. We now go over to so-called *field operators*. For a field operator it should hold that $\phi(x) = \phi^\dagger(x)$, since we want to obtain real values. For the Fourier expansion of $\phi(x)$ instead of α, α^* we from now on write creation and annihilation operators $a^\dagger_{\vec{k}}, a_{\vec{k}}$. The field operator then reads

$$\phi(x) = \frac{1}{\sqrt{V}}\sum_{\vec{k}} \frac{1}{\sqrt{2\omega_k}}(a_{\vec{k}} e^{-ik\cdot x} + a^\dagger_{\vec{k}} e^{ik\cdot x}). \tag{3.2.17}$$

We also need the conjugate momentum $\pi = \dot{\phi}$ which in this case is

$$\pi(x) = \frac{1}{\sqrt{V}} \sum_{\vec{k}} \frac{-i\omega_k}{\sqrt{2\omega_k}} (a_{\vec{k}} e^{-ik\cdot x} - a_{\vec{k}}^\dagger e^{ik\cdot x}). \tag{3.2.18}$$

Inverting the expressions for ϕ and π yields an expression vor the annihilation operator

$$a_{\vec{k}} = \frac{1}{\sqrt{V}\sqrt{2\omega_k}} \int d^3x e^{ikx} (i\dot{\phi}(x) + \omega_k \phi(x)). \tag{3.2.19}$$

This is derived in the exercises. Inserting these back gives commutations relations for $a_{\vec{k}}$ and $a_{\vec{k}}^\dagger$

$$[a_{\vec{k}}, a_{\vec{k}'}^\dagger] = \delta_{\vec{k},\vec{k}'}, \tag{3.2.20}$$

$$[a_{\vec{k}}, a_{\vec{k}'}] = [a_{\vec{k}}^\dagger, a_{\vec{k}}^\dagger] = 0. \tag{3.2.21}$$

The idea is now that all physical quantities can be expressed via creation and annihilation operators $a^\dagger, a$. The Hamiltonian is given as

$$H = \frac{1}{2} \int d^3x (\pi^2 + (\nabla\phi)^2 + m^2\phi^2). \tag{3.2.22}$$

Remark. For a function of the form

$$f_{\vec{k}}(x) = \frac{1}{\sqrt{V 2\omega_k}} e^{-ikx}, \tag{3.2.23}$$

the orthogonality relations are

$$\int d^3x f_{\vec{k}}^*(x) f_{\vec{k}}(x) = \frac{1}{2\omega_k} \delta_{\vec{k},\vec{k}'} \tag{3.2.24}$$

$$\int d^3x f_{\vec{k}'}(x) f_{\vec{k}}(x') = \frac{1}{2\omega_k} \delta(\vec{k} - \vec{k}') e^{-i2\omega_k t}. \tag{3.2.25}$$

Using the above remark and after some calculation one obtains for the Hamiltonian

$$H = \frac{1}{2} \sum_{\vec{k}} \omega_{\vec{k}} \big(a_{\vec{k}}^\dagger a_{\vec{k}} + a_{\vec{k}} a_{\vec{k}}^\dagger \big). \tag{3.2.26}$$

Using the commutation relation $[a_{\vec{k}}, a_{\vec{k}}^\dagger] = \delta_{\vec{k},\vec{k}'}$ we can write it as

$$H = \sum_{\vec{k}} \omega_{\vec{k}} \left(a_{\vec{k}}^\dagger a_{\vec{k}} + \frac{1}{2} \right). \tag{3.2.27}$$

Analogous, one obtains for the field momentum

$$P^{k=1,2,3} = \int d^3x T^{0k}(x) = \int d^3x \frac{\partial \mathcal{L}}{\partial(\partial_0\phi)}\partial^k\phi = \int d^3x \pi(x)\partial^k\phi \tag{3.2.28}$$

or

$$\vec{P} = -\int d^3x \pi(x)(\vec{\nabla}\phi) = \frac{1}{2}\sum_{\vec{k}} \vec{k}(a^\dagger_{\vec{k}}a_{\vec{k}} + a_{\vec{k}}a^\dagger_{\vec{k}}). \tag{3.2.29}$$

As a final result we have

$$P = \sum_{\vec{k}} \vec{k}\left(a^\dagger_{\vec{k}}a_{\vec{k}} + \frac{1}{2}\right). \tag{3.2.30}$$

We still introduce the quantum states and its interpretation as particles. Again, the Hamiltonian is

$$H = \frac{1}{2}\int d^3x(\pi^2 + (\nabla\phi)^2 + m^2\phi^2). \tag{3.2.31}$$

- **Classically:** Energetically the lowest state is the state with $\phi(x) = 0$, it is called the *vacuum state.*
- **Quantum mechanically:** $H = \sum_{\vec{k}} \omega_{\vec{k}}(a^\dagger_{\vec{k}}a_{\vec{k}} + \frac{1}{2})$ for any $\vec{k}$ it looks like a harmonic oscillator with frequency $\omega_{\vec{k}} = \sqrt{\vec{k}^2 + m^2}$

Reminder. The *simple harmonic oszillator* in non-relativistic quantum mechanics is given by the Hamiltonian

$$\hat{H} = \frac{\vec{p}^2}{2} + \frac{1}{2}\omega^2\hat{q}^2, \tag{3.2.32}$$

where the mass m is set to one and with $[\hat{q}, \hat{p}] = i$. We can define the so-called *creation* and *annihiliation operators*

$$\hat{a} = \sqrt{\frac{\omega}{2}}\hat{q} + \frac{i}{\sqrt{2\omega}}\hat{p}, \qquad \hat{a}^\dagger = \sqrt{\frac{\omega}{2}}\hat{q} - \frac{i}{\sqrt{2\omega}}\hat{p}. \tag{3.2.33}$$

They fulfill the commutation relation $[\hat{a}, \hat{a}^\dagger] = \mathbb{1}$ and the Hamiltonian can be rewritten as

$$\hat{H} = \frac{\omega}{2}(\hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a}) = \omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right). \tag{3.2.34}$$

It is easy to show that

$$[\hat{H}, \hat{a}^\dagger] = \omega\hat{a}^\dagger \qquad \text{and} \qquad [\hat{H}, \hat{a}] = -\omega\hat{a}. \tag{3.2.35}$$

From this one obtains that $\hat{a}$ and $\hat{a}^\dagger$ connect different eigenstates. To see this imagine $|E\rangle$ is an eigenstate with energy E, that is, $\hat{H}|E\rangle = E|E\rangle$. Construction of further eigenstates with $\hat{a}$ and $\hat{a}^\dagger$ is done by the following procedure

$$\hat{H}\hat{a}^\dagger|E\rangle \overset{(3.2.35)}{=} \hat{a}^\dagger\hat{H}|E\rangle + \omega\hat{a}^\dagger|E\rangle = (E+\omega)\hat{a}^\dagger|E\rangle \tag{3.2.36}$$

and similarly

$$\hat{H}\hat{a}\,|E\rangle = (E-\omega)\hat{a}\,|E\rangle\,. \tag{3.2.37}$$

Doing this multiple times we find the following property: The system has a ladder of states with energy $\ldots, E-2\omega, E-\omega, E, E+\omega, E+2\omega, \ldots$ All states with such energies are of course again eigenstates of the Hamiltonian. When the energy is bounded below we call the lowest energy state the *ground state*. In this lecture it is denoted by $|0\rangle$ with $\hat{a}\,|0\rangle = 0$ and $\hat{H}\,|0\rangle = \frac{1}{2}\omega\,|0\rangle$. We call $\omega/2$ the *zero point energy.* Excited energy states can be generated by applying $\hat{a}^\dagger$ multiple times

$$|n\rangle = (\hat{a}^\dagger)^n\,|0\rangle \qquad \text{with} \qquad \hat{H}\,|n\rangle = \left(n+\frac{1}{2}\right)|n\rangle\,. \tag{3.2.38}$$

We can now define the general ground state as the vacuum state $|0\rangle$ such that $a_{\vec{k}}\,|0\rangle = 0$ for all $\vec{k}$. The expectation value of the field in this state is $\langle 0|\phi|0\rangle = 0$, which is classically consistent. We now take the Hamilton operator and act on the corresponding ground state

$$H\,|0\rangle = \sum_{\vec{k}} \frac{\omega_{\vec{k}}}{2}\,|0\rangle = \text{“}\infty\cdot|0\rangle\,\text{”}, \tag{3.2.39}$$

with $E_0 = \frac{\omega_{\vec{k}}}{2}$ the zero point energy. Note that we have a divergence, that is, $E_0 \to \infty$ for $\vec{k} \to \infty$, which is called the *ultra-violet divergence* ($k = \omega/c = 2\pi/\lambda$ and $\omega = 2\pi/f$). We make the assumption that the theory is still valid for arbitrarily high energies. The practical approach now follows from the fact that in physics we are only interested in energy differences relative to the vacuum because we can only measure relative to the vacuum. Therefore, we redefine or *renormailze* the Hamiltonian meaning that instead of H we use a new quantity

$$:H: \equiv H - \langle 0|H|0\rangle = \sum_{\vec{k}} \omega_{\vec{k}} a^\dagger_{\vec{k}} a_{\vec{k}}, \tag{3.2.40}$$

with $\langle 0|\,:H:\,|0\rangle = 0$. One calls $:H:$ a *normal ordered operator* in terms of $a_{\vec{k}}$ and $a^\dagger_{\vec{k}}$. The *normal order* is the state in which all creation operators are on the left of the annihilation operators. In the *anti-normal order* all the annihilation operators are on the left of all creation operators. The normal ordering $:O:$ of an arbitrary operator O can also be written as $\mathcal{N}(O)$. For example

$$:a^\dagger_{\vec{k}} a_{\vec{k}} + a_{\vec{k}} a^\dagger_{\vec{k}}: \equiv a^\dagger_{\vec{k}} a_{\vec{k}} + a^\dagger_{\vec{k}} a_{\vec{k}} = 2a^\dagger_{\vec{k}} a_{\vec{k}} \tag{3.2.41}$$

or

$$:H: = :\int d^3x \mathcal{H}: = \sum_{\vec{k}} \omega_{\vec{k}} a^\dagger_{\vec{k}} a_{\vec{k}}. \tag{3.2.42}$$

The n*ormal ordering of the 3-momentum operator* is defined as

$$:\vec{P}:=\frac{1}{2}\sum_{\vec{k}}\vec{k}:(a^{\dagger}_{\vec{k}}a_{\vec{k}}+a_{\vec{k}}a^{\dagger}_{\vec{k}}):=\sum_{\vec{k}}\vec{k}a^{\dagger}_{\vec{k}}a_{\vec{k}} \tag{3.2.43}$$

and $\vec{P}\,|0\rangle=0$.

3.3 Particles and Fock Space

Until now we have discussed the vacuum and ground state, but now we want to be able to describe excitations. This is done by using the so called *Fock space*. In order to introduce the Fock space we have to set up our algebra for the creation and annihilation operators $\hat{a}^{\dagger},\hat{a}$. To do so we start from the commutation relation

$$[:H:,a^{\dagger}_{\vec{q}}]=\left[\sum_{\vec{k}}\omega_{\vec{k}}a^{\dagger}_{\vec{k}}a_{\vec{k}},a^{\dagger}_{\vec{q}}\right]=\sum_{\vec{k}}a^{\dagger}_{\vec{k}}\underbrace{[a_{\vec{k}},a_{\vec{q}}]}_{\delta_{\vec{k},\vec{q}}}=\omega_{\vec{q}}a^{\dagger}_{\vec{q}} \tag{3.3.1}$$

as well as $[:H:,a_{\vec{q}}]=-\omega_{\vec{q}}a_{\vec{q}}$. Now construct energy eigenstates

$$|\vec{q}\rangle=a^{\dagger}_{\vec{q}}|0\rangle\qquad\text{with}\qquad :H:|\,|q\rangle=:H:a^{\dagger}_{\vec{q}}|0\rangle \tag{3.3.2}$$

and we have an energy eigenstate $:H:|\,|q\rangle=\omega_{\vec{q}}|\,|q\rangle$ with $\omega_{\vec{q}}=\sqrt{\vec{q}^2+m^2}$. Note that $|\vec{q}\rangle$ is also a 3-momentum eigenstate with

$$\vec{P}\,|\vec{q}\rangle=\vec{q}\,|\,|q\rangle\,, \tag{3.3.3}$$

where we used $[\vec{P},a^{\dagger}_{\vec{q}}]=\vec{q}a^{\dagger}_{\vec{q}}$. This makes clear that $|\vec{q}\rangle$ is an eigenstate (energy and 3-momentum) of a particle with mass m, 3-momentum $\vec{q}$, energy $E_{\vec{q}}=\sqrt{\vec{q}^2+m^2}$ and spin $s=0$. It is clear now that $a_{\vec{q}}$ creates field quanta or particles and $a_{\vec{q}}$ is the corresponding annihilation operator that destroys particles. We could also have multi-particle states, which are denoted as

$$|\vec{q}_1,\ldots,\vec{q}_k\rangle\equiv a^{\dagger}_{\vec{q}_1}\ldots a^{\dagger}_{\vec{q}_k}|0\rangle\,, \tag{3.3.4}$$

with $[a^{\dagger}_{\vec{q}},a_{\vec{q}}]=0$. Such a state is symmetric under exchange of two particles. For example, we have $|\vec{p},\vec{q}\rangle=|\vec{q},\vec{p}\rangle$. Note that for Bosons the spin quantum number s is 0. The energy of $|\vec{p},\vec{q}\rangle$ is given by

$$:H:\;|\vec{p},\vec{q}\rangle=:H:\;a^{\dagger}_{\vec{p}}a^{\dagger}_{\vec{q}}|0\rangle=\ldots=(\omega_{\vec{q}_1}+\omega_{\vec{q}_2})\,|\vec{p},\vec{q}\rangle\,. \tag{3.3.5}$$

The physical space of the theory is given by

$$|0\rangle\,,\;a^{\dagger}_{\vec{p}}|0\rangle\,,\;a^{\dagger}_{\vec{p}_1}a^{\dagger}_{\vec{p}_2}|0\rangle\,,\ldots \tag{3.3.6}$$

and is also known as *Fock space.*

3.4 Causality and the "Scalar" Propagator

In the beginning we started with the Lagrangian $\mathcal{L} = \frac{1}{2}(\partial_\mu\phi\partial^\mu\phi - m^2\phi^2)$ of a scalar field ϕ with $\phi = \phi^*$. Then, we introduced the conjugate momentum field $\pi = \dot{\phi}$. To quantize our field we introduced the equal-time commutation relations

$$[\phi(\vec{x},t), \dot{\phi}(\vec{y},t)] = i\delta^{(3)}(\vec{x}-\vec{y}), \tag{3.4.1}$$
$$[\phi(\vec{x},t), \phi(\vec{y},t)] = 0 = [\dot{\phi}(\vec{x},t), \dot{\phi}(\vec{y},t)], \tag{3.4.2}$$

where we replaced $\phi \to \hat{\phi},\ \phi^* \to \hat{\phi}^\dagger$. Next, the quantum field shall satisfy the Klein-Gordon equation $(\partial^2 + m^2)\phi = 0$. From here we can motivate the second quantisation, that is, writing operators in terms of creation and annihilation operators, which until now we have only done for the Hamiltonian and the momentum operator. These create and annihilate field quanta, these we interpret as particles. For the field operator we can write

$$\hat{\phi}(x) = \frac{1}{\sqrt{V}}\sum_{\vec{k}} \frac{1}{\sqrt{2\omega_{\vec{k}}}}[\hat{a}(\vec{k})e^{-ik\cdot x} + \hat{a}^\dagger(\vec{k})e^{ik\cdot x}] \equiv \phi^+(x) + \phi^-(x), \tag{3.4.3}$$

with $k^0 = \omega_{\vec{k}} = \sqrt{k^2+m^2}$ and $[a(\vec{k}), a^\dagger(\vec{k}')] = \delta_{\vec{k},\vec{k}'},\ [a(\vec{k}), a(\vec{k}')] = [a^\dagger(\vec{k}), a^\dagger(\vec{k}')] = 0$. These commutation relations yield the spectrum of free particle states. *But what about covariant commutation relations?* We first consider the commutator

$$\begin{aligned}
[\phi(x), \phi(y)] &= [\phi^+(x) + \phi^-(x), \phi^+(y) + \phi^-(y)] \\
&= [\phi^+(x), \phi^-(y)] + [\phi^-(x), \phi^+(y)] + \underbrace{[\phi^+(x), \phi^+(y)]}_{[a,a]=0} + \underbrace{[\phi^-(x), \phi^-(y)]}_{[a^\dagger,a^\dagger]=0} \\
&= [\phi^+(x), \phi^-(y)] + [\phi^-(x), \phi^+(y)]
\end{aligned} \tag{3.4.4}$$

for $x^0 \neq y^0$. We start by calculating the first term in Eq. (3.4.4)

$$\begin{aligned}
[\phi^+(x), \phi^-(y)] &= \frac{1}{2V}\sum_{\vec{k},\vec{k}'} \frac{1}{\sqrt{\omega_{\vec{k}}\omega_{\vec{k}'}}} e^{-ik\cdot x}e^{ik'\cdot x} \underbrace{[a(\vec{k}), a^\dagger(\vec{k}')]}_{\delta_{\vec{k},\vec{k}'}} \\
&= \frac{1}{2V}\sum_{\vec{k}} \frac{1}{\omega_{\vec{k}}} e^{-ik\cdot(x-y)} \\
&\overset{V\to\infty}{\longrightarrow} \int \frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_{\vec{k}}} e^{-ik\cdot(x-y)},
\end{aligned} \tag{3.4.5}$$

where in the limit $V \to \infty$ we replaced the sum by an integral. The second term yields the same result but with x and y interchanged. In total, the commutator is

$$\begin{aligned}[\phi(x), \phi(y)] &= \int \frac{d^3k}{(2\pi)^2} \frac{1}{2\omega_{\vec{k}}} \left(e^{-ik\cdot(x-y)} - e^{-ik\cdot(y-x)} \right) \\ &= -i \int \frac{d^3k}{(2\pi)^2} \frac{1}{\omega_{\vec{k}}} \sin(k \cdot (x-y)) \\ &\equiv i\Delta(x-y) \in \mathbb{C}. \end{aligned} \tag{3.4.6}$$

The question remains: "Is this covariant?" In order to investigate that, we claim the following:

Claim. We can rewrite the just found operator as

$$\begin{aligned}\Delta(x) &= -\int \frac{d^3k}{(2\pi)^2} \frac{1}{2\omega_{\vec{k}}} \frac{1}{2i} \left(e^{ik\cdot x} - e^{-ik\cdot x} \right) \Big|_{k^0=\omega_k=\sqrt{k^2+m^2}} \\ &\stackrel{!}{=} -i \int \frac{d^4k}{(2\pi)^3} \delta(k^2 - m^2) \mathrm{sign}(k^0) e^{-ik\cdot x}. \end{aligned} \tag{3.4.7}$$

Proof. To show equality we split the integral into its spatial and temporal part

$$I = -\frac{i}{(2\pi)^3} \int_{\mathbb{R}^3} d^3\vec{k} \int_{-\infty}^{\infty} dk^0 \delta((k^0)^2 - (\vec{k}^2 + m^2)) \mathrm{sign}(k^0) e^{-ik\cdot x} \tag{3.4.8}$$

and then just calculate it. To do so we use a property of the δ-distribution, which states that the δ-distribution of a function can be rewritten as the sum $\delta(f(x)) = \sum_{x_i} \frac{1}{f'(x_i)} \delta(x - x_i)$ where x_i are the zeros of f such that $f'(x_i) \neq 0$. Then $\delta((k^0)^2 - (\vec{k}^2 + m^2)) = f(k^0) = (k^0)^2 - (\vec{k} + m^2)$ and $f'(k^0) = 2k^0$, where $k^0_\pm = \pm\sqrt{\vec{k} + m^2}$. Inserting this into Eq. (3.4.8) gives us

$$\begin{aligned} I &= -\frac{i}{(2\pi)^3} \int d^3\vec{k} \int_{-\infty}^{\infty} dk^0 \frac{1}{2\sqrt{\vec{k}^2 + m^2}} (\delta(k^0 + \omega_k)) + \delta(k^0 - \omega_k) \mathrm{sign}(k^0) e^{-ik\cdot x} \\ &= -\frac{1}{(2\pi)^3} \int d^3k \frac{1}{2\omega_k} \Bigg[-\int_{-\infty}^{0} dk^0 (\delta(k^0 + \omega_k) + \underbrace{\delta(k^0 - \omega_k)}_{=0}) e^{-ik\cdot x} \\ &\qquad + \int_0^{\infty} dk^0 (\underbrace{\delta(k^0 + \omega_k)}_{=0} + \delta(k^0 - \omega_k)) e^{-ik\cdot x} \Bigg] \\ &= -\frac{i}{(2\pi)^3} \int \frac{d^3k}{2\omega_k} (e^{-i\omega_k x^0 + i\vec{k}\cdot\vec{x}} - \underbrace{e^{i\omega_k x^0 + i\vec{k}\cdot\vec{x}}}_{\vec{k}\to-\vec{k}}) \\ &= -\int \frac{d^3k}{(2\pi)^3} \frac{1}{\omega_k} \frac{1}{2i} (e^{ikx} - e^{-ik\cdot x}) \Big|_{k^0=\omega_k} \\ &= \Delta(x). \end{aligned} \tag{3.4.9}$$

Remarks. We want to list a few important remarks regarding what we just did

- The obtained operator $\Delta(x) = -\int \frac{d^3k}{(2\pi)^3}\delta(k^2 - m^2)\text{sign}(k^0)e^{-ikx}$ has components d^3k, $\delta(k^2 - m^2)$, e^{-ikx} that are all Lorentz invariant. Note that $k^2 = m^2 > 0$, so k is a time-like vector. A proper LT does not change the sign of k^0 for time-like k^0.

- Equal-time commutation relations $[\phi(\vec{x}, t), \phi(\vec{y}, t)] = i\Delta(0, \vec{x} - \vec{y}) = 0$ for $x^0 = y^0 = t$ are fulfilled.

- $\Delta(x - y) \neq 0$ if $(x - y)^2 > 0$ e.g. $\vec{x} \cdot \vec{y} = 0$, but $x^0 \neq y^0$. Then $\Delta(x^0 - y^0) = -\int \frac{d^3k}{(2\pi)^3}\frac{1}{\omega_k}\sin(\omega_k(x^0 - y^0)) \neq 0$. Note that the LT can never turn a space-like vector into a time-like vector and vice versa. Hence, $[\phi(x), \phi(y)] = 0$ for $(x - y)^2 > 0$, that is, space-like. This is a manifestation of *causality.* Consider the commutator of two observables O_1, O_2

$$[O_1(\vec{x}, t_1), O_2(\vec{y}, t_2)] = 0 \qquad \text{for} \qquad c^2(t_1 - t_2)^2 - (\vec{x} - \vec{y})^2 < 0 \tag{3.4.10}$$

  and implies that the field $\phi(x)$ satisfies (micro-)causality! That is very important for conceptual consistency! For interacting theories: causality is often a postulate.

The physical interpretation of the operator Δ now is the following: Let $\langle 0|0\rangle = 1$, that is, $|0\rangle$ is a normalized vacuum state, then

$$\begin{aligned} i\Delta(x - y) &= [\phi(x), \phi(y)] \\ &= \langle 0|[\phi(x), \phi(y)]|0\rangle \\ &= \langle 0|\phi(x)\phi(y)|0\rangle - \langle 0|\phi(y)\phi(x)|0\rangle \\ &= \langle 0|\phi^+(x)\phi^-(y)|0\rangle - \langle 0|\phi^+(y)\phi^-(x)|0\rangle. \end{aligned} \tag{3.4.11}$$

The first term is the amplitude for the creation of a particle at position y and annihilation of a particle at position x. So, this represents the propagation of the particle from x to y. The second term is the same process in reversed order. What about time-ordering? Our expectation is, that the particle should be created first and only then should it be annihilated. For $x^0 > y^0$: $\langle 0||\phi(x)\phi(y)|0\rangle$ is okey, but is $\langle 0|\phi(y)\phi(x)|0\rangle$?[1] If $y^0 > x^0$: the order $\langle 0|\phi(y)\phi(x)|0\rangle$ is okey, but is $\langle 0|\phi(x)\phi(y)|0\rangle$? As we see the physical

[1]Note that $x^0 > y^0$ means that the first event happens at position y and the second event happens at x. This means that the first order of the field operators represents this correctly, since we first act with the operator $\phi(y)$ and *then* with the operator $\phi(x)$. The second order of operators would mean the event at later time happens before the event at an earlier time.

interpretation is flawed. We define a new object called the *Feynman propagator*. For that we need a *time-ordering product*

$$T\{O_1(x)O_2(y)\} = \begin{cases} O_1(x)O_2(y), & \text{if } x^0 > y^0 \\ O_2(y)O_1(x), & \text{if } y^0 > x^0 \end{cases}, \tag{3.4.12}$$

which always gives us the order of operators such that the earlier operator acts first. Another way to write it is $T\{(O_1(x)O_2(y))\} = \Theta(x^0-y^0)O_1(x)O_2(y)+\Theta(y^0-x^0)O_2(y)O_1(x)$. Using this we can write our derived operator $\Delta(x)$ as

$$\begin{aligned} i\Delta_F(x-y) &= \langle 0|T\{\phi(x)\phi(y)\}|0\rangle \\ &= \langle 0|\Theta(x^0-y^0)\phi(x)\phi(y) + \Theta(y^0-x^0)\phi(y)\phi(x)|0\rangle \\ &= \Theta(x^0-y^0)\langle 0|\phi^+(x)\phi^-(y)|0\rangle + \Theta(y^0-x^0)\langle 0|\phi^+(y)\phi^-(x)|0\rangle \\ &= \Theta(x^0-y^0)\int \frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_k}e^{-ik\cdot(x-y)}\Big|_{k^0=\omega_k} \qquad (3.4.13) \\ &\quad + \Theta(y^0-x^0)\int \frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_k}e^{-ik\cdot(y-x)}\Big|_{k^0=\omega_k}. \qquad (3.4.14) \end{aligned}$$

Claim. We can write the Feynman propagator $\Delta_F(x-y)$ in manifestly covariant form

$$i\Delta_F(x-y) = \int \frac{d^4k}{(2\pi)^4}\frac{ie^{ik\cdot(x-y)}}{k^2-m^2+i\delta}. \tag{3.4.15}$$

Note that δ is infinitesimally small and we want to take the limit $\delta \to 0$ in the end. Note also that in the integral we usually have $k^2 \neq m^2$. In particular, k^0 can be any value. Also k^μ does not satisfy the energy momentum relation! Recall that k^μ is "off the mass shell" or short "off-shell". We say a particle that travels from x to y carries *off-shell momentum* k^μ.

Proof. We start with the right-hand side of the equality calculate the integral

$$\begin{aligned} I &= \int \frac{d^4k}{(2\pi)^4}\frac{ie^{-ik\cdot(x-y)}}{k^2-m^2+i\delta} \\ &= \int \frac{d^3\vec{k}}{(2\pi)^3}e^{i\vec{k}\cdot(\vec{x}-\vec{y})}\underbrace{\int_{-\infty}^{\infty}\frac{dk^0}{2\pi}\frac{ie^{-ik^0(x^0-y^0)}}{(k^0)^2-(\underbrace{\vec{k}^2+m^2}_{=\omega_k^2}+i\delta)}}_{\equiv A}. \end{aligned} \tag{3.4.16}$$

First note that we can write

$$\begin{aligned}(k^0)^2-(\omega_k^2-i\delta) &\cong (k^0-\omega_k+i\delta)(k^0+\omega_k-i\delta)\\ &= (k^0)^2-\omega_k^2+i\underbrace{2\omega_k\delta}_{=\tilde{\delta}}+\delta^2\\ &\overset{\delta\to 0}{\approx} (k^0)^2-\omega_k^2+i\tilde{\delta}\end{aligned} \tag{3.4.17}$$

and similarly

$$\begin{aligned}\frac{1}{(k^0)^2-(\omega_k^2-i\delta)} &\cong \frac{1}{(k^0-\omega_k+i\delta)(k^0+\omega_k-i\delta)}\\ &= \frac{1}{2(\omega_k-i\delta)}\left[\frac{1}{k^0-\omega_k+i\delta}-\frac{1}{k^0+\omega_k-i\delta}\right].\end{aligned} \tag{3.4.18}$$

Using these two equations the integral A can now be written as

$$A=\frac{1}{2\omega_k}\left[\int_{-\infty}^{\infty}\frac{dk^0}{2\pi}\frac{ie^{-ik^0(x^0-y^0)}}{k^0-(\omega_k-i\delta)}-\int_{-\infty}^{\infty}\frac{dk^0}{2\pi}\frac{ie^{-ik^0(x^0-y^0)}}{k^0-(-\omega_k+i\delta)}\right]. \tag{3.4.19}$$

Now that A has this form we can use the *theorem of residues*[2]. For $x^0>y^0$ we choose the closed contour in the lower half plane at $|k^0|\to\infty$. For $x^0<y^0$ we close the contour in the upper half plane. Thus, we obtain

$$\begin{aligned}A &= \Theta(x^0-y^0)\frac{1}{2\omega_k}\left(\frac{-2\pi i}{2\pi}ie^{-i\omega_k(x^0-y^0)}\right)+\Theta(y^0-x^0)\frac{1}{2\omega_k}\left(\frac{(+2\pi i)}{2\pi}(-i)e^{i\omega_k(x^0-y^0)}\right)\\ &= \Theta(x^0-y^0)\int\frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_k}e^{-ik\cdot(x-y)}\Big|_{k^0=\omega_k}+\Theta(y^0-x^0)\underbrace{\int\frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_k}e^{-ik\cdot(y-x)}}_{\vec{k}\to-\vec{k},\text{ in last step}}\Big|_{k^0=\omega_k}\\ &= i\Delta_F(x-y).\end{aligned} \tag{3.4.20}$$

3.5 Complex Klein-Gordon Field

We start from the Lagragian $\mathcal{L}=(\partial_\mu\phi)^*(\partial^\mu\phi)-m^2\phi^*\phi$ with two independent fields ϕ^*,ϕ which respectively yield a conjugate field $\pi_\phi=\frac{\partial\mathcal{L}}{\partial(\partial_0\phi)}=\partial_0\phi^*\equiv\pi$ and $\pi_{\phi^*}=\frac{\partial\mathcal{L}}{\partial(\partial_0\phi^*)}=\partial_0\phi\equiv\pi^*$. Again replace $\phi\to\hat{\phi}(x)$, $\phi^*(x)\to\hat{\phi}^*(x)$, $\pi(x)\to\hat{\pi}(x)$, $\pi^*(x)\to\hat{\pi}^*(x)$ with $\hat{\phi}(x)\neq\hat{\phi}^*(x)$ and $\hat{\pi}(x)\neq\hat{\pi}^*(x)$. Then the commutation relations read

$$[\hat{\phi}(\vec{x},t),\hat{\pi}(\vec{x},t)]=i\delta^{(3)}(\vec{x}-\vec{y}), \tag{3.5.1}$$

$$[\hat{\phi}^\dagger(\vec{x},t),\hat{\pi}^\dagger(\vec{x},t)]=i\delta^{(3)}(\vec{x}-\vec{y}). \tag{3.5.2}$$

[2] $\oint dz\frac{f(z)}{z-z_0}=2\pi i f(z_0)$.

Note that all other commutators such as $[\phi(\vec{x},t)\phi^\dagger(\vec{y},t)]=0$ vanish as before. The field operators $\hat{\phi},\hat{\phi}^\dagger$ shall satisfy the Klein-Gordon equation, that is, $(\partial^2+m^2)\hat{\phi}(x)=0$ and $(\partial^2+m^2)\hat{\phi}^\dagger(x)=0$. It is easy to see that plain wave solutions at $k^0=\omega_{\vec{k}}=\sqrt{\vec{k}^2+m^2}$ are solutions because $(\partial_x^2+m^2)e^{\pm ik\cdot x}=(k^2-m^2)=0$. We can write the field operators as superpositions of such plane waves

$$\hat{\phi}=\hat{\phi}^+ + \hat{\phi}^- = \frac{1}{\sqrt{2V}}\sum_{\vec{k}}\frac{1}{\sqrt{\omega_{\vec{k}}}}\big(\hat{a}(\vec{k})e^{-ik\cdot x}+\hat{b}^\dagger(\vec{k})e^{ik\cdot x}\big)|_{k^0=\omega_{\vec{k}}} \tag{3.5.3}$$

and

$$\hat{\phi}^\dagger=(\hat{\phi}^+)^\dagger+(\hat{\phi}^-)^\dagger = \frac{1}{\sqrt{2V}}\sum_{\vec{k}}\frac{1}{\sqrt{\omega_{\vec{k}}}}\big(\hat{a}^\dagger(\vec{k})e^{ik\cdot x}+\hat{b}(\vec{k})e^{-ik\cdot x}\big)|_{k^0=\omega_{\vec{k}}}. \tag{3.5.4}$$

The further analysis is completley analogous to the real scalar field, we therefore only point out the differences. The difference to the previous analysis is that we have two sets of creation and annihilation operators. One takes inverted expansions of the field operators and then plugs them into the equal time relations. This gives the commutation relations

$$[\hat{a}(\vec{k}),\hat{a}^\dagger(\vec{k}')]=\delta_{\vec{k},\vec{k}'} \tag{3.5.5}$$

$$[\hat{b}(\vec{k}),\hat{b}^\dagger(\vec{k}')]=\delta_{k,k'} \tag{3.5.6}$$

$$[\hat{a}(\vec{k}),\hat{a}(\vec{k}')]=[\hat{b}(\vec{k}),\hat{b}(\vec{k}')]=[\hat{a}(\vec{k}),\hat{b}^\dagger(\vec{k}')]=[\hat{a}(\vec{k}),\hat{b}(\vec{k}')]=\ldots=0. \tag{3.5.7}$$

We have two sets of creation of annihilation operators which create / annihilate two different types of particles, which for now we call a- and b-particles. We construct multi-particle states analogously to the case of real fields by creating them from the vacuum

$$\hat{a}^\dagger(\vec{k}_1)\ldots\hat{a}^\dagger(k_m)\hat{b}^\dagger(p_1)\ldots\hat{b}^\dagger(p_n)\,|0\rangle=|a,k_1;\ldots;a,k_m;b,p_1;\ldots;b,p_n\rangle. \tag{3.5.8}$$

From this we are now able to construct a Hamiltonian

$$\hat{H}=\int d^3x\,\underbrace{:(\pi_\phi\partial_0\phi+\pi_{\phi^+}\partial_0\phi^+-\mathcal{L}):}_{\mathcal{H}\to\hat{\mathcal{H}}}. \tag{3.5.9}$$

We can insert the expansions of $\phi,\phi^\dagger$ in terms of annihilation and creation operators to obtain (cf. homework ex. 12)

$$\hat{H}=\sum_{\vec{k}}E_{\vec{k}}\big(\hat{N}_a(\vec{k})+\hat{N}_b(\vec{k})\big), \tag{3.5.10}$$

where $\hat{N}_a(\vec{k}) \equiv \hat{a}^\dagger(k)a(k)$ and $\hat{N}_b(\vec{k}) \equiv \hat{b}^\dagger(k)b(k)$ are the *occupation number operators* for a- and b-particles. Also, $E_{\vec{k}} = \omega_{\vec{k}} = \sqrt{\vec{k}^2 + m^2}$. The Hamiltonian counts a- and b-particles and as we see the Hamiltonian in this case is symmetric under a- and b-particles (meaning they can be exchanged without changing the Hamiltonian due to the plus sign between the occupation number operators). This means that the eigenvalues of the Hamiltonian are *degenerate*! For example $E[a, \vec{k}_1; a, \vec{k}_2; b, \vec{p}_1] = E[a, \vec{k}_1; b, \vec{k}_2; b, \vec{p}_1]$. Next, we calculate the momentum operator which we can write as

$$\hat{\vec{P}} = -\int d^3x :(\pi_\phi \vec{\nabla}\phi + \pi_{\phi^\dagger}\vec{\nabla}\phi^\dagger):= \ldots = \sum_{\vec{k}} \vec{k}(N_a(\vec{k}) + N_b(\vec{k})). \tag{3.5.11}$$

This operator is also symmetric under exchange of a and b. We now want to know how particles a and b differ and if there is a difference at all. Yes! This has to do with the fact that we have a complex field. Note that the Lagragian is invariant under a global phase transformation. A global phase transformation[3] is for example $\phi \to e^{iq\alpha}\hat{\phi}$ which means that $\phi^\dagger \to e^{iq\alpha}\hat{\phi}^\dagger$ and for $\mathcal{L} = \partial_\mu\phi^*\partial^\mu\phi - m^2\phi^*\phi$ nothing changes. Noethers theorem (cf. homework ex. 9) tells us that the symmetry gives a conserved current

$$j^\mu(x) = iq :(\phi^\dagger\partial^\mu\phi - \phi\partial^\mu\phi^\dagger):, \tag{3.5.12}$$

with $\partial_\mu j^\mu(x) = 0 = \partial_0 j^0 + \vec{\nabla}\cdot\vec{j} = 0$. Taking the volume integral and applying Gauss law we obtain a conserved quantity we call a *conserved charge*

$$\begin{aligned}
\hat{Q} &= \int d^3x j^0(x) \\
&= iq\int d^3x :\phi^\dagger\partial^0\phi - \phi\partial^0\phi^\dagger: \\
&= \ldots \\
&= q\sum \vec{k}\big(\hat{N}_a(\vec{k}) - \hat{N}_b(\vec{k})\big) \\
&= q(\hat{N}_a - \hat{N}_b),
\end{aligned} \tag{3.5.13}$$

with $N_i = \sum_{\vec{k}} N_i(\vec{k})$. Note that compared to the Hamilton we now have a minus sign. This means that the conserved charge is not a symmetric operator under exchange of a and b. This means a-, b-particles are equal in mass, energy and momentum but opposite in charge. We can therefore interpret the b-particle as the *anti-particles* of a!

[3] A *global transformation* does not depend on the position x e.g. $\phi \to e^{-iq\alpha}\phi$ where $\alpha \neq \alpha(x)$. For a *local transformation* we would have $\alpha = \alpha(x)$ which would yield additional terms because we can not pull out the exponential as in the global case. These extra terms that appear represent interactions.

Remarks.

- In the real scalar field theory we had $\phi = \phi^\dagger$ and thus $\hat{b}^\dagger \to \hat{a}^\dagger$ and if this is the case we have

$$:\hat{Q}:= q(\hat{N}_a - \hat{N}_a) = 0 \tag{3.5.14}$$

  This means that "uncharged" particles are their own anti-particles!

- One can show that $[\hat{H}, \hat{Q}] = 0$, this means that the Hamiltonian $\hat{H}$ and the conserved charge $\hat{Q}$ are compatible observables meaning they have a common set of eigenstates, which are the multi-particle states

$$|(\vec{k}, +q), (\vec{k}_2, +q), \ldots, (\vec{k}_n, +q); (\vec{p}_1, -q), \ldots, (\vec{p}_m, -q)\rangle \tag{3.5.15}$$

  and we can label them with momenta k and charge q.

- What about causality? We know that causality is a property that is satisfied by a real field theory. As a reminder, causality means that for two observables $[\hat{Q}_i(x), \hat{Q}_j(y)] = 0$ for space-like distances we have $(x-y)^2 < 0$. We can now analyse this property for the case of a complex scalar field. For example, we can consider the covariant commutation relation. This is

$$\begin{aligned}
\mathbb{C} \ni i\Delta(x-y) &= [\hat{\phi}(x), \hat{\phi}^\dagger(y)] \\
&= \langle 0|[\hat{\phi}(x), \hat{\phi}^\dagger(y)]|0\rangle \\
&= \langle 0|\underbrace{\phi(x)\phi^\dagger(y)}_{\sim \hat{a}\hat{a}^\dagger}|0\rangle - \langle 0|\underbrace{\phi^\dagger(y)\phi(x)}_{\sim \hat{b}\hat{b}^\dagger}|0\rangle \overset{!}{=} 0,
\end{aligned} \tag{3.5.16}$$

  for space like distances $(x-y)^2 < 0$. Again, we could interprete these matrix elements as amplitude for, e.g., creation of a particle a at position y and annihilation of the particle at position x. This means that the particle propagates from y to x. The situation is analogous for the b-particle, but in the reverse direction, meaning it moves backwards in time. We could change the reference frame via a Lorentz transformation that reverses the ordering, which is possible for space-like distances. In summary, an a-particle moves backwards in time, whereas a b-particle moves forward in time. Both amplitudes $\langle 0|\phi(x)\phi^\dagger(y)|0\rangle$ and $\langle 0|\phi^\dagger(y)\phi(x)|0\rangle$ are Lorentz invariant and hold in each frame. In order to have causality, that is, Eq. (3.5.16) holds, we *need* the anti-particles. Anti-particles are, thus, essential for ensuring causality.

3.6 Quantization of the Electromagnetic Field

In the tutorials (and also in the Classical Field Theory lecutre) we have already introduced the electromagnetic field strength tensor

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}, \tag{3.6.1}$$

with A being the 4-vector potential. By definition the field strengh tensor $F^{\mu\nu}$ satisfies the *Bianchi identity*

$$\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0. \tag{3.6.2}$$

From it the homogeneous Maxwell equations $\vec{\nabla} \cdot \vec{B} = 0$, $\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$ follow. From the Lagrangian of the electromagnetic field one gets

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \overset{\text{ELG}}{\Longrightarrow} \partial_\mu F^{\mu\nu} = 0 \Leftrightarrow \Box A^\nu - \partial^\nu (\partial_\mu A^\mu) = 0, \tag{3.6.3}$$

which yields the inhomogeneous Maxwell equations for the case of no source. The Maxwell equations are invariant under a transformation

$$A^\mu \to A'^\mu = A^\mu - \partial^\mu \chi(x), \tag{3.6.4}$$

which is called a *gauge transformation.* This is due to the fact the the four vector potential is not uniquely defined and such a transformation does not change the electric and magnetic field components, they stay the same. Additionally, the Maxwell equations are invariant under such a gauge transformation. There are many different such transformations, two of such we discuss now.

- **Coulomb gauge**: For the case of $\rho = 0$, $\vec{j} = 0$ we can choose a function χ in such a fashion that the conditions $A^0 = 0$ and $\vec{\nabla} \cdot \vec{A} = 0$ are fulfilled. This is a choice to restrain the four vector potential A[4]. But we have no *Lorentz covariance*,

[4]One can always choose $\vec{A}$ such that $\vec{\nabla} \cdot \vec{A} = 0$, because if $\vec{\nabla} \cdot \vec{A} = f(\vec{x}, t) \neq 0$, then one can pick a gauge transformation $A \to A'$ in such a way that $\vec{\nabla} \cdot \vec{A}' \overset{!}{=} 0 = \vec{\nabla} \cdot \vec{A} + \Delta\chi$ where the function χ is chosen such that $\Delta\chi = -f$. This illustrates that one can always find a four vector potential which fulfills the condition. Then we have $\Box A^0 - \partial^0(\partial_\mu A^\mu) = 0$ which is just $\partial_t^2 A^0 - \vec{\nabla}^2 A^0 - \partial^0(\partial_0 A^0) = -\Delta A^0 = 0$. In electrostatics on the right hand side we usually have the charge density, which then gives the famous *Poisson equation.* But since $\rho = 0$ or in other words the fields vanish at infinity, a solution of this equation is $A^0 = 0$.

the above conditions are only valid in one specific frame. This measn that after Lorentz transformation these conditions are not fulfilled anymore, they are not *form invariant.*

- **Lorenz gauge**: The gauge we will use in the lecture is $\partial_\mu A^\mu = 0$. We then get a more simplified form of the inhomogeneous Maxwell's equation $\Box A^\nu = 0$. This is the equation of motion of the 4-vector potential A^μ for this choice. Note that it looks like the Klein-Gordon equation without the mass term in every component. This choice does not fix the vector potential absolutely, it is not uniquely determined. We therefore have a *residual gauge freedom.* We could, for example, make a transformation $A^\mu \to A'^\mu$ with $\Box\chi = 0$. So, for this choice we find $\partial_\mu A^\mu = \partial_\mu A'^\mu = 0$.

We now develop a covariant formalism with the four potential and we want to do quantization in the Lorenz gauge. We start with the Lagrange density $\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ and the real components $A_\mu(x)$ are the degrees of freedom. The field momentum component can be computed to be

$$\pi^\mu(x) = \frac{\partial\mathcal{L}}{\partial\dot{A}_\mu} = -\frac{1}{2}\frac{\partial}{\partial\dot{A}_\mu}(\partial_\alpha A_\beta\partial^\alpha A^\beta - \partial_\alpha A_\beta\partial^\beta A^\alpha) = \partial^\mu A^0 - \partial^0 A^\mu = F^{\mu 0} \tag{3.6.5}$$

and $\pi^{k=1,2,3} = F^{k0} = E^k$, where $\vec{E}$ is the electric field. But $\pi^0 = 0$ and $[A^0, \pi^0] = 0$ and not 1 or $\delta(\vec{x} - \vec{x}')$. This commutator is important and we do not want to discard it. The next idea is to change $\mathcal{L}$ such that the equation of motion $\Box A^\mu = 0$ results directly and to then use the condition $\partial_\mu A^\mu = 0$. To achieve this we add an additional term to the Lagrange density, that is,

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}(\partial_\mu A^\mu)^2. \tag{3.6.6}$$

This does the job and is called the *Feynman gauge.* The extra term is called *gauge fixing term* and is chosen in such a way that the equation of motion is solved. Another ansatz is

$$\mathcal{L}_{\text{Fermi}} = -\frac{1}{2}(\partial_\nu A_\mu)(\partial^\nu A^\mu). \tag{3.6.7}$$

This Lagrangian, the so-called *Fermi-Lagrange*, is equivalent to the one before, however, the two are not identical. By that we mean that the two Lagrangian are completely equivalent up to a 4-divergence. The previous Lagrangian though fixed the gauge. We use the Fermi-Lagrange and the Euler-Lagrange equations of motion are

$$\partial_\alpha\frac{\partial\mathcal{L}}{\partial(\partial_\alpha A_\mu)} - \underbrace{\frac{\partial\mathcal{L}}{\partial A_\beta}}_{=0} = 0, \tag{3.6.8}$$

with

$$\partial_\nu \partial^\nu A^\mu = 0. \tag{3.6.9}$$

The momentum field is

$$\pi^\mu(x) = \frac{\partial \mathcal{L}}{\partial(\partial_0 A_\mu)} = -\partial^0 A^\mu = -\dot{A}^\mu \tag{3.6.10}$$

for all $\mu \in \{0,1,2,3\}$. Now all components have a non-vanishing momentum, but note that this is *not* the electric field, it depends on the Lagrangian! We now quantize and implement $\partial_\mu A^\mu = 0$. First, we expand the free electromagnetic field in terms of a 4-vector potential. Recall that every component fulfills the massless Klein-Gordon equation. We know those solutions, they are plain waves and we can write down classically the real four vector potential as

$$A^\mu(x) = \sum_{r=0,1,2,3} \sum_{\vec{k}} \sqrt{\frac{1}{2V\omega_{\vec{k}}}} \big(\varepsilon_r^\mu(\vec{k}) a_{\vec{k}}^r e^{-ik\cdot x} + \varepsilon_r^\mu(\vec{k})(a_{\vec{k}}^r)^\dagger e^{ik\cdot x}\big), \tag{3.6.11}$$

with $k^0 = \omega_{\vec{k}} = |\vec{k}|$ and $m = 0$. The sum over r means that we have 4 linearly independent polarization states of A^μ for each $\vec{k}$. The $\varepsilon_r^\mu(\vec{k})$ for $r \in \{0,1,2,3\}$ are called *polarization vectors* and they are chosen to be real. They are also orthonormal, that is,

$$\varepsilon_r(\vec{k}) \cdot \varepsilon_s(\vec{k}) = -\zeta_r \delta_{r,s} \tag{3.6.12}$$

with $\zeta_0 = -1$ (time-like) and $\zeta_{1,2,3} = 1$ (space-like). Additionally, they fulfill the completeness relation

$$\sum_r \zeta_r \varepsilon_r^\mu(\vec{k}) \varepsilon_r^\nu(\vec{k}) = -g^{\mu\nu}. \tag{3.6.13}$$

To be explicit, in a given frame we might have

$$\varepsilon_0^\mu(\vec{k}) = (1,0,0,0). \tag{3.6.14}$$

In this case we have a so called *space* or *timelike polarization*. All others are

$$\varepsilon_r^\mu = (0, \vec{\varepsilon}_r(\vec{k})) \tag{3.6.15}$$

for $r \in \{1,2,3\}$ and $\vec{\varepsilon}_r(\vec{k})$ are mutually orthogonal and they are unit vectors with the longitudinal polarization

$$\vec{\varepsilon}_3(\vec{k}) = \frac{\vec{k}}{||\vec{k}||} \tag{3.6.16}$$

and the transverse polarization being

$$\vec{k} \cdot \vec{\varepsilon}_r(\vec{k}) = 0 \tag{3.6.17}$$

for $r \in \{1, 2\}$. Furthermore, we choose them such that

$$\vec{\varepsilon}_r(\vec{k}) \cdot \vec{\varepsilon}_s(\vec{k}) = \delta_{r,s} \tag{3.6.18}$$

for $r \in \{1, 2, 3\}$. This choice fulfills the orthogonality and completeness relation. If we pick a particular $\vec{k} = k\hat{e}_z$, then $\vec{\varepsilon}_3 = (0, 0, 1)$, $\vec{\varepsilon}_1 = (1, 0, 0)$, $\vec{\varepsilon}_2 = (0, 1, 0)$. The point is that we have

- 2 independent physical polarizations $\vec{\varepsilon}_1$ and $\vec{\varepsilon}_2$, the transversal ones. This we know from the Coulomb gauge.
- $\vec{E}$ and $\vec{B}$ are transversal to the 3-momentum $\vec{k}$, that is, $\vec{B} = \vec{\nabla} \times \vec{A} \propto \vec{k} \times \vec{\varepsilon}_{1,2}$ and $\vec{E} = -\vec{\nabla} A^0 - \frac{\partial \vec{A}}{\partial t} = -\frac{\partial \vec{A}}{\partial t} \propto \vec{\varepsilon}_{1,2}$.

We now talk about the covariant quantization and equal-time commutation relations. We have four fields A_μ and they are connected to the conjugate field momenta

$$A_\mu \leftrightarrow \pi^\mu = \frac{\partial \mathcal{L}}{\partial \dot{A}_\mu} = -\dot{A}^\mu. \tag{3.6.19}$$

In general, we have the basic commutator in the quantization

$$[\phi_r(\vec{x}, t), \pi_s(\vec{x}', t)] = -\delta_{r,s}\delta(\vec{x} - \vec{x}'), \tag{3.6.20}$$

but here we have

$$[A_\sigma(\vec{x}, t), \pi^\nu(\vec{x}', t)] = i\delta^\nu_\sigma \delta(\vec{x} - \vec{x}'). \tag{3.6.21}$$

We now multiply from the left with $g^{\mu\sigma}$ to obtain

$$[A^\mu, \dot{A}^\mu] = -ig^{\mu\nu}\delta(\vec{x} - \vec{x}'). \tag{3.6.22}$$

Putting everything together yields the two commutation relations

$$[A^\mu(\vec{x}, t), \dot{A}^\nu(\vec{x}, t)] = -ig^{\mu\nu}\delta(\vec{x} - \vec{x}'), \tag{3.6.23}$$

$$[A^\mu(\vec{x}, t), A^\nu(\vec{x}', t)] = [\dot{A}^\mu(\vec{x}, t), \dot{A}^\nu(\vec{x}', t)] = 0. \tag{3.6.24}$$

Inserting the previous expansion of the field operators Eq. (3.6.11) will give the commutation relations[5]

$$[a^r_{\vec{k}}, a^s_{\vec{k}'}] = [(a^r_{\vec{k}})^\dagger, (a^s_{\vec{k}'})^\dagger] = 0, \tag{3.6.25}$$

$$[a^r_{\vec{k}}, (a^s_{\vec{k}'})^\dagger] = \zeta_r \delta_{r,s}\delta_{\vec{k},\vec{k}'}, \tag{3.6.26}$$

[5]Remember for the scalar field we had $[\phi, \dot{\phi}] = i\delta(\vec{x} - \vec{x}')$.

with $\zeta_r = -1$ for $r = 0$ and $\zeta_r = 1$ for $r \in \{1, 2, 3\}$. So, for $\zeta_r = 1$ and $r \in \{1, 2, 3\}$ we have the usual commutation relations of bosons. This refers to the so called *transversal and longitudinal photons*[6]. For $\zeta_0 = -1$ the role of a and $a^\dagger$ is interchanged, this refers to *scalar photons*. These are the quanta to the corresponding polarization.

Gupta-Bleuler (1950): The $a^r_{\vec{k}}$ for $r \in \{0, 1, 2, 3\}$ are annihilation operator in the sense that

$$a^r_{\vec{k}} |0\rangle = 0, \tag{3.6.27}$$

with $|0\rangle$ the vacuum state. We can then already produce the eigenstates, the so called *one-photon states*

$$|1_{\vec{k},r}\rangle = (a^r_{\vec{k}})^\dagger |0\rangle , \tag{3.6.28}$$

where r stand for a transversal, longitudinal or scalar photon with 3-momentum $\vec{k}$. We now look at the Hamilton operator for further justification of this interpretation. In general it is

$$H = \int d^3x : \{\pi_r(x)\dot{\phi}_r(x) - \mathcal{L}\} :, \tag{3.6.29}$$

here it is

$$H = \int d^3x : \{\pi^\mu \dot{A}_\mu - \mathcal{L}\} : . \tag{3.6.30}$$

Inserting the Fermi-Lagrange density as well as the the field expansion Eq. (3.6.11) and the conjugate momentum in terms of creation and annihilation operators gives

$$H = \sum_{r,\vec{k}} \omega_k \zeta_k (a^r_{\vec{k}})^\dagger a^r_{\vec{k}} \tag{3.6.31}$$

for $\omega_{\vec{k}} = |\vec{k}|$. For $r = 0$ (scalar polarization) the energy is positive definite

$$H |1_{\vec{k},r}\rangle = \sum_{s,\vec{q}} \omega_q \zeta_s (a^s_{\vec{q}})^\dagger a^s_{\vec{q}} (a^r_{\vec{q}})^\dagger |0\rangle = \omega_k \zeta_r^2 |1_{\vec{k},r}\rangle , \tag{3.6.32}$$

where we used that $a^s_{\vec{q}}(a^r_{\vec{q}})^\dagger |0\rangle = ([a, a^\dagger] + a^\dagger) |0\rangle = \zeta_r \delta_{r,s} \delta_{\vec{k},\vec{q}} |0\rangle$. The normalization $\langle 1_{\vec{k},r}|1_{\vec{k},r}\rangle$ of one photon states gives a negative norm. Let's discuss normalization, we have $\langle 1_{\vec{k},r}|1_{\vec{k},r}\rangle = \langle 0|a^r_{\vec{k}}(a^r_{\vec{k}})^\dagger|0\rangle = \langle 0|\zeta_r + (a^r_{\vec{k}})^\dagger a^r_{\vec{k}}|0\rangle = \zeta_r \langle 0|0\rangle = \zeta_r$. Now we find that the norm for time-like polarization $r = 0$ is negative! This is a contradiction to the probabilistic nature of qunatum mechanics. This affects the time-like photon, which is not physical. So, how do we solve the problem? We still have to implement the Lorenz

[6] In the quantized version quanta of the electromagnetic field are created.

condition $\partial_\mu A^\mu = 0$, which we did not do until now. The first attempt would be that $\partial_\mu A^\mu = 0$ for the field operator A^μ. The KG equation results in $[\phi(x), \phi(y)] = i\Delta(x-y)$. We can generalize this to

$$[A^\nu(x), A^\nu(x')] = -ig^{\mu\nu} \lim_{m\to 0} \Delta(x-x') = iD^{\mu\nu}(x-x'), \tag{3.6.33}$$

but

$$[\partial_\mu A^\mu(x), A^\mu(x')] = i\partial_\mu^x D^{\mu\nu}(x-x') \neq 0 \tag{3.6.34}$$

in general. The next attempt is to use it as a condition on the Hilbert space, then $\partial_\mu A^\mu |\psi\rangle = 0$ for a "good" physical state $|\psi\rangle$ (no longitudinal or time-like photons). We write

$$A_\mu(x) = A_\mu^+(x) + A_\mu^-(x), \tag{3.6.35}$$

where

$$A_\mu^+ = \sum_{\vec{k},r} \sqrt{\frac{1}{2V\omega_k}} \varepsilon_r^\mu(\vec{k}) e^{-ik\cdot x} a_{\vec{k}}^r \tag{3.6.36}$$

and similarly for A^- as for the scalar fields. If one takes A_μ^+ and acts on the ground state we obtain $A_\mu^+(x)|0\rangle = 0$, but $A_\mu^-|0\rangle \neq 0$ and not even $\partial^\mu A_\mu^-|0\rangle \neq 0$. Thus, the ground state is no physical state, because it does not fulfill the necessary condition. Therefore, this approach does not work either, since we want to keep $|0\rangle$ as a physical state. The next attempt puts an even weaker condition on Hilbert space, this procedure is called the *Gupta-Bleuler (GB) condition*. It goes as follows: We now have $\partial_\mu (A^\mu)^+(x)|\psi\rangle = 0$. This means that there is a restriction on the physical states $|\psi\rangle$. We can now use the adjoint state and then $\langle\psi| \partial_\mu (A^\mu)^-(x) = 0$. Adding both of these conditions we obtain

$$\langle\psi| \partial_\mu (A^\mu)^+(x) + \partial_\mu (A^\mu)^-(x) |\psi\rangle = \langle\psi|\partial_\mu A^\mu(x)|\psi\rangle = 0. \tag{3.6.37}$$

Hence, the expectation value fulfills the Lorenz condition and Maxwell's equations hold in the classical limit of the GB theory. We now have to identify these states $|\psi\rangle$ and what this means for the Hilbert space. Consider

$$\partial_\mu (A^\mu)^+(x)|\psi\rangle = 0. \tag{3.6.38}$$

For a fixed $\vec{k}$ we get, if we act with the four derivative on the $(A^\mu)^+$

$$\sum_r k_\mu \varepsilon_r^\mu(\vec{k}) a_{\vec{k}}^r |\psi\rangle . \tag{3.6.39}$$

We either have $r \in \{1,2\}$ and $\vec{k}\cdot\vec{\varepsilon}_r = 0$ or $(k_0 a_{\vec{k}}^0 - |\vec{k}| a_{\vec{k}}^3)|\psi\rangle$. We have a photon and in this case this means that with $k^0 = |\vec{k}|$ or $(a_{\vec{k}}^3 - a_{\vec{k}}^0)|\psi\rangle = 0$ for all $\vec{k}$. This means

that physical states can contain combinations of time-like and longitudinal photons, which means if we find a longitudinal photon with our annihilator, then we also find a time-like photon with the same 3-momentum $\vec{k}$. In general, there is no restriction on pairs of time-like and longitudinal photons. It means that a physical state $|\psi\rangle = |\psi_T\rangle$ contains only transversal photons. But we are allowed to multiply with a component $|\phi\rangle = \sum_{k=0}^{\infty} c_k |\phi_n\rangle$ (n pairs of timelike and longitudinal photons such that

$$|\psi\rangle = |\psi_T\rangle |\phi\rangle \tag{3.6.40}$$

What is the difference? We change the admixture, which corresponds to a gauge transforrmation between 4-vector potential in Lorenz gauge. This does not change the physics. We will now further evaluate this and check if operators, energy etc. are influenced. Consider

$$a^3_{\vec{k}} |\psi\rangle = a^0_{\vec{k}} |\psi\rangle \Rightarrow \langle\psi|(a^3_{\vec{k}})^\dagger a^3_{\vec{k}}|\psi\rangle \tag{3.6.41}$$

and similarly we get

$$\langle\psi| (a^3_{\vec{k}})^\dagger = \langle\psi| (a^0_{\vec{k}})^\dagger \Rightarrow \langle\psi|(a^0_{\vec{k}})^\dagger a^0_{\vec{k}}|\psi\rangle. \tag{3.6.42}$$

The expecation value of the Hamilton becomes

$$\langle\psi|H|\psi\rangle = \langle\psi|\sum_{r,\vec{k}} \omega_k \zeta_r (a^r_{\vec{k}})^\dagger a^r_{\vec{k}}|\psi\rangle = \langle\psi|\sum_{\vec{k}} \sum_{r=1}^{2} \omega_k (a^r_{\vec{k}})^\dagger a^r_{\vec{k}}|\psi\rangle. \tag{3.6.43}$$

As we see only transversal photons contribute. *This result is valid for all expectation values of gauge invariant operators. This is independent of the c_n.* In Lorenz gauge (for our choice of $\mathcal{L}$) we find for the *photon propagator* reads

$$\begin{aligned} \langle 0|T(A^\mu(x)A^\nu(x'))|0\rangle &= i \lim_{m\to 0}(-g^{\mu\nu}\Delta_F(x-x')) \\ &= -ig^{\mu\nu}\frac{1}{(2\pi)^4}\int d^4k \frac{e^{ik\cdot(x-x')}}{k^2+i\varepsilon}, \end{aligned} \tag{3.6.44}$$

with Δ_F the Feynman propagator.

Chapter 4

Dirac Equation and Quantization of Dirac Fields

4.1 Dirac Equation as a Wave equation

We make the transition from classical mechanics to quantum mechanics via the *correspondence principle*

$$\vec{p} \to -i\vec{\nabla}, \qquad E \to i\frac{\partial}{\partial t}, \quad \text{or} \quad p_\mu \to i\partial_\mu. \tag{4.1.1}$$

In non-relativistic mechanics the energy momentum relation becomes

$$E = \frac{\vec{p}^2}{2m} + V(\vec{r}) \longrightarrow i\frac{\partial}{\partial t}\psi(\vec{x},t) = \left(-\frac{\Delta}{2m} + V(\vec{x}) \right)\psi(\vec{x},t), \tag{4.1.2}$$

resulting in the Schrödinger equation with ψ the wave function. For the relativistic energy momentum relation

$$E = \sqrt{\vec{p}^2 + m^2} \tag{4.1.3}$$

applying the correspondence prinicple we find

$$i\frac{\partial}{\partial t}\psi = \sqrt{-\Delta + m^2}\psi. \tag{4.1.4}$$

This is a rather unpractical equation, because expanding the square root would yield spatial partial derivatives of arbitrary order. We would thus treat space and time differently, which is against the spirit of special relativity. We therefore try the squared equation $E^2 = \vec{p}^2 + m^2$. This gives

$$-\frac{\partial^2}{\partial t^2}\psi = (-\Delta + m^2)\psi \tag{4.1.5}$$

or equivalently

$$(\partial_t^2 - \Delta + m^2)\psi = (\partial_\mu \partial^\mu + m^2)\psi = (\partial^2 + m^2)\psi = 0, \tag{4.1.6}$$

which is the known *Klein-Gordon equation*. The problem is that the wavefunction cannot be interpeted as a probability amplitude as for the Schrödinger equation, because the solutions will be such that $|\psi|^2 < 0$. Thus, we have no probabilistic interpretation[1]. In order to construct a relativistic wave equation we can use *Dirac's trick*: Factorize the relativistic energy momentum relation

$$E^2 - \vec{p}^2 - m^2 = 0 = p_\mu p^\mu - m^2. \tag{4.1.7}$$

Suppose we are in the rest frame of the particle, that is, $\vec{p} = 0$, then

$$E^2 - m^2 = (E - m)(E + m) = 0. \tag{4.1.8}$$

However, if $\vec{p} \neq 0$, we make an ansatz

$$p_\mu p^\mu - m^2 \stackrel{!}{=} (\beta^\rho p_\rho + m)(\gamma^\sigma p_\sigma - m) = 0 \tag{4.1.9}$$

and determine the coefficients β, γ using

$$(\beta^\rho p_\rho + m)(\gamma^\sigma p_\sigma - m) = \beta^\rho \gamma^\sigma p_\rho p_\sigma + m(\gamma^\sigma - \beta^\sigma)p_\sigma - m^2. \stackrel{!}{=} p_\rho p^\rho - m^2 \tag{4.1.10}$$

Comparison yields that $\gamma^\rho = \beta^\rho$ such that the mixed term cancels. Comparing the terms $\beta^\rho \gamma^\sigma p_\rho p_\sigma$ and $p_\rho p^\rho = (p^0)^2 - (p^i)^2$ one can see that for example for $\beta^0 \beta^0 p^0 p^0$ we need that $(\beta^0)^2 = (\gamma^0)^2 = 1$. For the mixed terms we have $+\beta^0 \gamma^i p^0 p^i$ and $-\beta^i \gamma^0 p^i p^0$, but on the right hand side there appear no such mixed terms, thus

$$(\beta^0 \gamma^i + \beta^i \gamma^0) p^0 p^i = 0 \Rightarrow (\beta^0 \gamma^i + \beta^i \gamma^0) = 0. \tag{4.1.11}$$

In the same way one can show that also $\beta^i \gamma^j + \beta^j \gamma^i = 0$. Lastly, for terms where $i = j$ we obtain that $(\gamma^i)^2 = -1$. To summarize, we have four conditions

1. $(\beta^0)^2 = (\gamma^0)^2 = 1$
2. $(\gamma^i)^2 = -1$

[1] One might wonder why we still use it then in quantum field theory, but note that there we do not interprete ψ as a wavefunction but as a field operator. While the wavefunction in standard quantum mechanics describes a state of the system, in quantum field theory this is an operator in terms of creation and annihilation operators. Thus, we do not care about positivity of ψ there.

3. $(\beta^0\gamma^i + \beta^i\gamma^0) = 0$

4. $(\beta^i\gamma^j + \beta^j\gamma^i) = 0$

and in short we can write them as

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = \{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}. \tag{4.1.12}$$

This relation is important as it defines the γ's. In mathematics one calls this, the γ-matrices with Eq. (4.1.12) a so-called *Clifford algebra*. No ordinary number can satisfy Eq. (4.1.12), in order to satisfy this algebra the γ's have to be at least 4×4-matrices. Any set of γ's that satisfies the Clifford algebra will work. The most popular choice in our context is the *Pauli-Dirac representation*, where one chooses

$$\gamma^0 = \begin{pmatrix} \mathbb{1}_{2\times2} & 0 \\ 0 & -\mathbb{1}_{2\times2} \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \tag{4.1.13}$$

with $\mathbb{1}_{2\times2}$ the identity of the two by two matrices and σ^i the known *Pauli matrices*

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{4.1.14}$$

One can check that

$$(\gamma^0)^\dagger = \gamma^0, \qquad (\gamma^i)^\dagger = -\gamma^i. \tag{4.1.15}$$

The other representation is the so called *chiral or Weyl representation*. Here the γ^i are the same but γ^0 is different,

$$\gamma^0 = \begin{pmatrix} 0 & 1_{2\times2} \\ 1_{2\times2} & 0 \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}. \tag{4.1.16}$$

We will use the Pauli-Dirac representation in the following. As a consequence we find

$$p_\mu p^\mu - m^2 = (\gamma^\mu p_\mu + m)(\gamma^\mu p_\mu - m) \stackrel{!}{=} 0. \tag{4.1.17}$$

This can only be the case if either

$$(\gamma^\mu p_\mu + m) = 0, \qquad \text{or} \qquad (\gamma^\mu p_\mu - m) = 0. \tag{4.1.18}$$

We introduce now a notation for an often appearing expression, we write $\gamma^\mu p_\mu = \not{p}$. This is called the *Feynman slash*, which for general vectors a_μ is written as $\gamma^\mu a_\mu = \not{a}$. Transition to quantum mechanics using the correspondence principle yields

$$(i\gamma^\mu\partial_\mu - m)\psi(x) = 0, \tag{4.1.19}$$

or in the slash notation

$$(i\partial\!\!\!/ - m)\psi(x) = 0. \tag{4.1.20}$$

This is called the *free Dirac equation*. Note that when we write m, what we really mean here is $m \cdot \mathbb{1}_{4\times4}$, but usually one omits the unity matrix. The wavefunction ψ now becomes a 4-dimensional vector with entries ψ_i and is called *spinor* or *Dirac spinor*

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}. \tag{4.1.21}$$

It is important to keep in mind that while the Minkowski index runs from 0 to 3, the Dirac index here runs from 1 to 4. One might wonder why we choose the factor with a minus sign, that is because the adjoint equation contains the same information. For the *adjoint spinor* we write

$$\bar{\psi}_j(x) \equiv \psi_i^\dagger(x)\gamma^0_{ij}. \tag{4.1.22}$$

We can take the Dirac equation and take the hermitian adjoint of it to obtain

$$[(i\partial\!\!\!/ - m)\psi]^\dagger = \psi^\dagger[-i\overleftarrow{\partial_\mu}(\gamma^\mu)^\dagger - m] = 0. \tag{4.1.23}$$

With $(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0$ the equation becomes

$$\psi^\dagger[-i\overleftarrow{\partial_\mu}\gamma^0\gamma^\mu\gamma^0 - m\gamma^0\gamma^0] = -\bar{\psi}[i\partial\!\!\!/ + m]\gamma^0 = 0. \tag{4.1.24}$$

Multiplication with γ^0 from the right gives

$$\bar{\psi}(x)[i\overleftarrow{\partial\!\!\!/} + m] = 0. \tag{4.1.25}$$

This leads us to the *Dirac Hamiltonian*. We can introduce a slightly different set of matrices called β- and α-matrices with $\beta = \gamma^0$, $\beta\alpha^i = \gamma^i$, $\alpha^i = \gamma^0\gamma^i$ and in matrix form

$$\beta = \begin{pmatrix} 1_{2\times2} & 0 \\ 0 & -1_{2\times2} \end{pmatrix}, \quad \alpha^i = \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix}. \tag{4.1.26}$$

They are hermitian, that is, $\beta^\dagger = \beta$ and $(\alpha^i)^\dagger = \alpha^i$. Furthermore, we have $\beta^2 = 1 = (\alpha^i)^2$, $\{\alpha^i, \beta\} = 0$, $\{\alpha^i, \alpha^i\} = 2\delta^{ij}$. With these matrices the Dirac equation reads

$$i\gamma^\mu\partial_\mu\psi - m\psi = i\beta\partial_t\psi + i\beta\alpha^i\nabla^i\psi - m\psi = 0. \tag{4.1.27}$$

Multiplication with β from the right finally gives us the following expression

$$i\frac{\partial}{\partial t}\psi(x) = [-i\vec{\alpha}\cdot\vec{\nabla} + \beta m]\psi \equiv H_D\psi(x). \tag{4.1.28}$$

This is the Dirac equation in Schrödinger form with the *Dirac Hamiltonian*

$$H_D = -i\vec{\alpha}\cdot\vec{\nabla} + \beta m, \tag{4.1.29}$$

which now is a 4×4 - matrix. The continutiy equation for some density ρ and some current $\vec{j}$ is

$$\partial_t \rho + \vec{\nabla}\cdot\vec{j} = 0 \tag{4.1.30}$$

and it describes that for a volume V the influx is equal to the outflux. We now want to derive an equation like that, also called the *Dirac field* for the Dirac spinors. We can then identify ρ and $\vec{j}$. The conjugate of the Dirac equation (4.1.28) is

$$-i\frac{\partial}{\partial t}\psi^\dagger(x) = \psi^\dagger(x)[i(\vec{\alpha})^\dagger\cdot\overleftarrow{\nabla} + m(\beta)^\dagger]. \tag{4.1.31}$$

Multiplying from the left with ψ and subtracting the adjoint Dirac equation from the Dirac equation gives

$$\begin{aligned} i\left(\psi^\dagger\frac{\partial}{\partial t}\psi + \left(\frac{\partial}{\partial t}\psi^\dagger\right)\psi\right) &= \frac{\partial}{\partial t}(\psi^\dagger\psi) \\ &= -i\left(\psi^\dagger[(\vec{\alpha}\cdot\vec{\nabla})\psi] + (\vec{\nabla}\psi^\dagger)\cdot\vec{\alpha}\psi\right) \\ &= -i\vec{\nabla}\cdot(\psi^\dagger\vec{\alpha}\psi), \end{aligned} \tag{4.1.32}$$

or equivalently

$$\frac{\partial}{\partial t}(\psi^\dagger\psi) + i\vec{\nabla}\cdot(\psi^\dagger\vec{\alpha}\psi) = 0. \tag{4.1.33}$$

Comparison with the continuity equation gives the following identification

$$\rho = \psi^\dagger\psi = |\psi_1|^2 + |\psi_2|^2 + |\psi_3|^2|\psi_4|^\dagger > 0 \quad \text{and} \quad \vec{j} = \psi^\dagger\vec{\alpha}\psi, \tag{4.1.34}$$

where ρ is a probability density and $\vec{j}$ a current density. The 4-current is going to be

$$j^\mu = (\rho, \vec{j}) = (\psi^\dagger\beta^2\psi, \psi^\dagger\beta^2\vec{\alpha}\psi) = (\bar{\psi}\gamma^0\psi, \bar{\psi}\vec{\gamma}^i\psi) = \bar{\psi}(x)\gamma^\mu\psi(x) \tag{4.1.35}$$

with $\beta = \gamma^0$, $\gamma^i = \beta\alpha^i$, $\bar{\psi} = \psi^\dagger\gamma^0 = \psi^\dagger\beta$. Then the short form of the continuity equation is

$$\partial_\mu j^\mu = \partial_t j^0 + \vec{\nabla}\cdot\vec{j} = 0 \tag{4.1.36}$$

and the probability current or *Diract 4-current* $j^\mu(x) = \bar{\psi}(x)\gamma^\mu\psi(x)$ transforms like a 4-vector!

4.1.1 Solutions of the Free Dirac Equation

Let us now solve the free Dirac equation (4.1.28). To do so we need to find the spinor ψ. We know it depends on the position $\vec{x}$ and time t. We first suppose that there is no $\vec{x}$ dependence but only time dependence of the spinor, that is, $\psi = \psi(t)$, so it is constant in space. That means in particular that

$$-i\vec{\nabla}\psi(t) = 0 \mathrel{\widehat{=}} \hat{p}\psi = 0. \tag{4.1.37}$$

Naively speaking, we can interprete the solution as a particle at rest. Then the Dirac equation is simple

$$i\partial_t\psi(t) = m\beta\psi(t) \quad \text{with} \quad \beta = \begin{pmatrix} 1_{2\times 2} & 0 \\ 0 & -1_{2\times 2} \end{pmatrix}. \tag{4.1.38}$$

This is easy to solve if we also write the spinor ψ as

$$\psi = \begin{pmatrix} \chi \\ \varphi \end{pmatrix}, \quad \chi = \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix}, \quad \varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}. \tag{4.1.39}$$

We get two equations

$$i\partial_t\chi(t) = m\chi, \tag{4.1.40}$$

$$i\partial\varphi(t) = -m\phi. \tag{4.1.41}$$

Because χ and ϕ span a 2-dimensional vector space we get two independent solutions for each of the above equations. The solutions of Eq. (4.1.40) are called *positive energy solutions*. We write them as

$$\psi^{(+)}_{s=1,2}(t) = e^{-imt}\begin{pmatrix} \chi_s \\ 0 \end{pmatrix} \quad \text{with} \quad \chi_{s=1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \chi_{s=2}\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{4.1.42}$$

and the rest energy $E = mc^2 = m$ of the particle. The solutions of Eq. (4.1.41)

$$\psi^{(-)}_{s=1,2}(t) = e^{-imt}\begin{pmatrix} 0 \\ \chi_s \end{pmatrix} \tag{4.1.43}$$

are called the *negative energy solutions*. The question though is how to interprete the negative energy solutions. If we take the Dirac equation as wave equation, then we can use the so called *hole theory*[2]. To do so consider the vacuum, see Fig. 4.1b. A

[2]First done in 1930

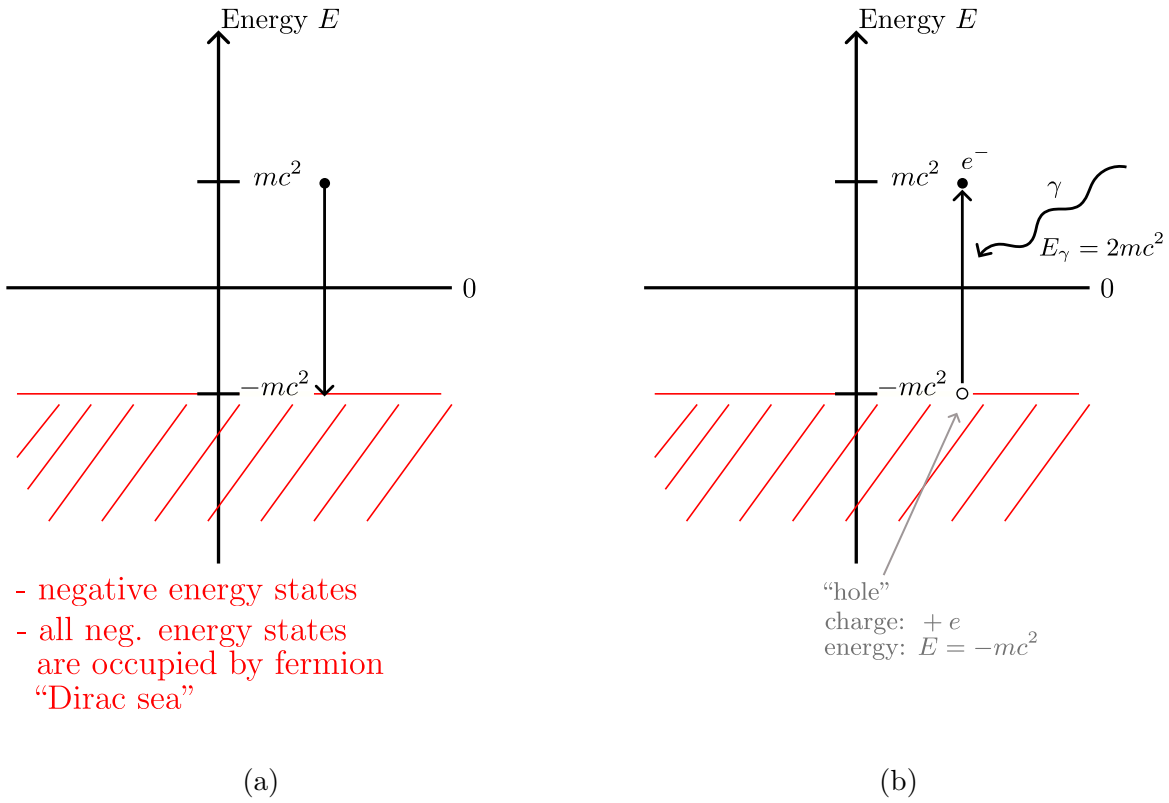


Fig. 4.1: Vacuum, that is, there is no potential present. (a) A particle at rest has energy mc^2, this is the postive energy solution of the Dirac equation, but we also have the negative energy $-mc^2$ solution. Below this threshold there is a whole continuum of states, the *Dirac sea*. This continuum is filled with particles called fermions. Therefore, the Pauli principle prohibits the de-excitation of a particle. (b) Excitation of a particle from the Dirac sea leaving a hole (empty state) in the vacuum.

particle at rest has energy mc^2, this is a positive solution, but there are also negative solutions with $-mc^2$. Below that threshold there exists a continuum of states called the *Dirac sea*. The Dirac sea is filled with fermions. So as we see the vacuum is not really empty, it has infinitely many particles but they are all at negative energy. Now suppose we have an electron with energy mc^2, if all the negative energy states were not filled, then the electron would occupy a state in the Dirac sea, because it would always go to the lowest energy. But it cannot, because all states are filled and the Pauli prinicple prevents this de-excitation. Now consider the excitation of a fermion via a photon from the Dirac sea, see Fig. 4.1a, to the positive energy solution mc^2. The photon needs to have at least an energy of $E_\gamma = 2mc^2$. The fermion at mc^2 is then interpreted as

an electron at rest with energy $E = mc^2$ and charge $-e$. In the continuum of the negative energy state there now seems to be a *hole*, because there is a state missing. The hole behaves like another particle in this continuum, we could interprete this as a quasi-particle with charge $+e$ and an energy $E = -mc^2$. Thus, we can interprete holes as anti-particles. In the case of an electron that would be a *positron*[3]. In quantum field theory the concept of holes and the Dirac sea is not needed because we only deal with positive energies. There antiparticles appear naturally when considering complex fields. Though, the terminology is still used e.g. for the *"sea quarks"* in the nucleon. It is also used in solid state physics. There the holes are interpreted as quasi-particles.

As we see the Dirac equation describes two particles with vectors that have two components. Naturally, the question arises what the two components mean. Simply put, it already includes spin 1/2, meaning we can interprete the two vectors χ_s as *spin vectors*. One can introduce a *generalized Dirac spin operator*

$$\vec{S} = \frac{1}{2}\vec{\Sigma} = \frac{1}{2}\begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}, \tag{4.1.44}$$

with $\vec{\sigma}$ the usual Pauli matrices. Because in the spinor the upper part χ describes particles and the lower part ϕ describes antiparticles we deduce that the upper $\vec{\sigma}$ measures the spin of positive energy solutions (particles) and the lower $\vec{\sigma}$ measures the spin of negative energy solutions (antiparticles). The action of the z-component of the spin operator on a solution is

$$S_z \psi^{(\pm)}_{s=1,2} = \frac{1}{2}\begin{pmatrix} \sigma_z & 0 \\ 0 & \sigma_z \end{pmatrix} \psi^{(\pm)}_{s=1,2} = \pm\frac{1}{2}\psi^{(\pm)}_{s=1,2}. \tag{4.1.45}$$

Thus, the $\psi^{(\pm)}_s$ are eigenstates of the z-component of the *Dirac spin operator*. That means we interprete $\psi^{+}_{s=1}$ as an electron with *spin up*[4] and $\psi^{+}_{s=2}$ as an electron with *spin down*[5]. Analogously the negative solutions are interpreted as positrons with either spin up or down. We will now talk about solutions with $\vec{p} \neq 0$, for that we will try the plane wave ansatz

$$\psi^{(+)}_{s=1,2}(x) = u_{s=1,2}(p)e^{-ip\cdot x} \quad \text{with} \quad u_i(p) = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}. \tag{4.1.46}$$

[3]Discovered in 1932 by C. Anderson
[4]Denoted as $(e^-)^{\uparrow}$
[5]Denoted as $(e^-)^{\downarrow}$

The momentum has to be on-shell, that is, $p^2 = m^2$. In the limit $\vec{p} \to 0$ this gives again the positive solution $u_{s=1,2}(m, \vec{0})e^{-imt}$ from before. Inserting this ansatz into the Dirac equation gives

$$\begin{aligned}(i\partial\!\!\!/ - m)\psi^{(+)}_{s=1,2}(x) &= \left(i\gamma^\mu_{ij} u_j(p) \Big(\underbrace{\frac{\partial}{\partial x^\mu} e^{-ip_\mu x^\mu}}_{-ip_\mu e^{-ip\cdot x}} \Big) - m u_i(p) e^{-ip\cdot x} \right) \\ &= [(p\!\!\!/ - m)u(p)]e^{-ip\cdot x} \\ &= 0, \end{aligned} \tag{4.1.47}$$

and we obtain the *Dirac equation in momentum space*

$$(p\!\!\!/ - m)u_s(p) = 0. \tag{4.1.48}$$

A trick in general is to write

$$(p\!\!\!/ - m)(p\!\!\!/ + m) = p\!\!\!/p\!\!\!/ - m(p\!\!\!/ - p\!\!\!/) - m^2 = p^2 - m^2 = 0, \tag{4.1.49}$$

where we used that $p\!\!\!/p\!\!\!/ = p_\mu p_\nu \gamma^\mu \gamma^\nu = p_\mu p_\nu \frac{1}{2}(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu) = p_\mu p_\nu \frac{1}{2}\{\gamma^\mu, \gamma^\nu\} = g^{\mu\nu} p_\mu p_\nu = p^2$ and that p is chosen to be on-shell. Similarly, we also obtain that

$$(p\!\!\!/ + m)(p\!\!\!/ - m) = 0. \tag{4.1.50}$$

The general solution of $(p\!\!\!/ - m)u_s(p) = 0$ is

$$u_s(p) = N(p\!\!\!/ + m) \underbrace{\begin{pmatrix} \chi_s \\ 0 \end{pmatrix}}_{\text{spinor at rest}}, \tag{4.1.51}$$

with N being a normalization constant. Using the Dirac-Pauli representation of γ^0, γ^i from Eq. (4.1.13) we get

$$(p\!\!\!/ + m) = (p^0\gamma^0 - p^1\gamma^1 - p^2\gamma^2 - p^3\gamma^3 + m) = \begin{pmatrix} E+m & 0 & -p^3 & -p^1 + ip^2 \\ 0 & E+m & -p^1 - ip^2 & p^3 \\ p^3 & p^1 - ip^2 & -E+m & 0 \\ p^1 + ip^2 & -p^3 & 0 & -E+m \end{pmatrix}, \tag{4.1.52}$$

where $p^0 = E$. We can use this matrix, which then acts on the spin-vectors

$$\begin{pmatrix} \chi_1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \chi_2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}. \tag{4.1.53}$$

The solutions become

$$u_{s=1,2}(p) = N(E+m)\begin{pmatrix} \chi_s \\ \\ \dfrac{\vec{\sigma}\cdot\vec{p}}{E+m}\chi_s \end{pmatrix}, \tag{4.1.54}$$

with $\chi_1 = \begin{pmatrix} 1 & 0 \end{pmatrix}^T$, $\chi_2 = \begin{pmatrix} 0 & 1 \end{pmatrix}^T$ and $E = \sqrt{\vec{p}^2 + m^2}$. The normalization constant is chosen to be

$$N = \frac{1}{\sqrt{E+m}}. \tag{4.1.55}$$

Other choices such as $N = \frac{1}{\sqrt{2m(E+m)}}$ are possible[6]. Thus, the total positive energy solution of the free Dirac equation becomes

$$\psi^{(+)}_{s=1,2} = \sqrt{E+m}\begin{pmatrix} \chi_s \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\chi_s \end{pmatrix} e^{-ip\cdot x}. \tag{4.1.56}$$

The first component in the spinor is called the *large component*, while the second component is called the *small component* due to the fact that the small component goes to zero if we go to the rest frame. In this case the second component is negligible. For the negative energy solutions we also make a plane wave ansatz

$$\psi^{(-)}_{s=1,2} = v_s(p)e^{ip\cdot x} \xrightarrow{\vec{p}\to 0} v_s(m,\vec{0})e^{imt}. \tag{4.1.57}$$

As before we insert this ansatz into the free Dirac equation

$$(i\not{\partial} - m)\psi^{(-)}_s(x) = -(\not{p} + m)v_s(p)e^{ip\cdot x} = 0 \tag{4.1.58}$$

and this means that

$$(\not{p} + m)v_s(p) = 0. \tag{4.1.59}$$

This is the Dirac equation for the v spinors. Next we make an ansatz

$$v_s(p) = N(\not{p} - m)\begin{pmatrix} 0 \\ \chi_s \end{pmatrix}, \tag{4.1.60}$$

since $(\not{p} + m)(\not{p} - m) = 0$. The only thing that changes in Eq. (4.1.52) is the sign of the mass in the diagonal. The Pauli-Dirac representation implies that we can write

$$v_s(p) = \sqrt{E+m}\begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\chi_s \\ \chi_s, \end{pmatrix} \tag{4.1.61}$$

[6]Mandl & Shaw

with the normalization constant $N = -\frac{1}{\sqrt{E+m}}$ and finally

$$\psi^{(-)}_{s=1,2} = \begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\chi_s \\ \chi_s \end{pmatrix} e^{ip\cdot x}. \tag{4.1.62}$$

Hence, we have constructed four linearly independent solutions of the free Dirac equation. There are a few interesting relations we want to discuss next, but first let us introduce again the adjoint spinors now in momentum space

$$\bar{u}_s(p) = u^\dagger_s(p)\gamma^0, \quad \bar{v}_s(p) = v^\dagger_s(p)\gamma^0. \tag{4.1.63}$$

These spinors satisfy an orthogonality relation

$$(\bar{u}_r)_i(p)(u_s)_i(p) = 2m\delta_{r,s}, \tag{4.1.64}$$

$$(\bar{v}_r)_i(p)(v_s)_i(p) = -2m\delta_{r,s}, \tag{4.1.65}$$

$$\bar{u}_r(p)v_s(p) = 0, \tag{4.1.66}$$

$$\bar{v}_r(p)u_s(p) = 0, \tag{4.1.67}$$

as well as a completeness relation, also called *spin sums*

$$\sum_{s=1,2} [u_s(p)]_i[\bar{u}_s(p)]_j = (\not{p} + m)_{ij}, \tag{4.1.68}$$

$$\sum_{s=1,2} [v_s(p)]_i[\bar{v}_s(p)]_j = (\not{p} - m)_{ij}, \tag{4.1.69}$$

where r, s are the polarization indices. These relations will also be relevant for the evaluation of Feynman diagrams.

4.2 Lorentz- (Form)Invariance of the Dirac Equation

Suppose we have a Lorentz transformation and a translation, this is called a *Poincaré transformation*

$$x' = \Lambda x + a. \tag{4.2.1}$$

In order to derive how a spinor $\psi_i(x)$ transforms we need that the Dirac equation (4.1.20) shall have the same form in all reference frames. If the Dirac equation in the reference frame S is given by

$$(i\not{\partial}_x - m)\psi(x) = 0, \tag{4.2.2}$$

then in a reference frame S' it needs to be of the same form, that is

$$(i\partial\!\!\!/_{x'} - m)\psi'(x') = 0, \tag{4.2.3}$$

in order to have *form invariance*. The goal will now be to connect the spinor in S with the transformed spinor in S'. This means we want to construct a map F such that

$$\psi(x) = \psi'(x') = F(\psi(x)). \tag{4.2.4}$$

It is important to note that since the Dirac equation (4.1.20) is linear and ψ, ψ' are solutions of that this implies that also F needs to be a linear map. We make the following ansatz

$$\psi'_i(x') = S_{ij}(\Lambda)\psi_j(x) = S_{ij}(\Lambda)\psi_j(\Lambda^{-1}(x' - a)). \tag{4.2.5}$$

Moreover, we expect that the determinent of S is non-vanishing (i.e. $\det S \neq 0$), that therefore the inverse of S should exist and $S^{-1}(\Lambda)\psi'(x') = \psi(x)$. Naturally, $S^{-1}(\Lambda) = S(\Lambda^{-1})$ should hold. In order to determine the transformation matrix S we insert our ansatz into the Dirac equation (4.1.20)

$$S(\Lambda)(i\partial\!\!\!/ - m)\psi(x) = 0. \tag{4.2.6}$$

What we then get is

$$S(\Lambda)\left(i\gamma^\mu\left(\Lambda^\nu{}_\mu \frac{\partial}{\partial x'^\nu}\right) - m\right)S^{-1}(\Lambda)\psi'(x') = 0. \tag{4.2.7}$$

Reordering gives

$$(i[S(\Lambda)\gamma^\mu S^{-1}(\Lambda)\Lambda^\nu{}_\mu]\partial'_\nu - m)\psi'(x') \stackrel{!}{=} (i\gamma^\nu\partial'_\nu - m)\psi'(x') = 0, \tag{4.2.8}$$

where we already imposed form invariance of the Dirac equation. By comparison of the coefficients we find

$$S(\Lambda)\gamma^\mu S^{-1}(\Lambda)\Lambda^\nu{}_\mu \stackrel{!}{=} \gamma^\nu. \tag{4.2.9}$$

Multiplying with the inverse of S from the left and with S from the right gives

$$S^{-1}(\Lambda)\gamma^\nu S(\Lambda) = \Lambda^\nu{}_\mu \gamma^\mu. \tag{4.2.10}$$

This equation determines $S(\Lambda)$ meaning that we can construct the matrix S from this equation. Note that γ^μ is the same in *all* forms. We will now use Eq. (4.2.10) to determine S by taking an *infinitesimal* and *proper* Lorentz transformation. Remember

that a proper Lorentz transform has $\det\Lambda = +1$ and since it is infinitesimal we can write

$$\Lambda^{\nu}{}_{\mu} = g^{\nu}{}_{\mu} + \omega^{\nu}{}_{\mu} = \delta^{\nu}_{\mu} + \omega^{\nu}{}_{\mu}, \tag{4.2.11}$$

with ω being infinitesimal. By raising the index $\omega^{\nu}{}_{\mu} \to \omega^{\nu\mu} = -\omega^{\mu\nu}$ we see that ω is antisymmetric in ν, μ, but note that $\omega^{\nu}{}_{\mu}$ is not![7] Now we have

$$S_{ij}(\Lambda) = S_{ij}(\mathbb{1} + \omega) = \underbrace{S_{ij}(\mathbb{1})}_{=\mathbb{1}_{ij}} + \omega_{\mu\nu}\Gamma^{\mu\nu}_{ij} + o(\omega^2), \tag{4.2.12}$$

where $\Gamma^{\mu\nu}$ is also a Dirac matrix and a priori we assume it to be antisymmetric in μ and ν, that is, $\Gamma^{\mu\nu} = -\Gamma^{\nu\mu}$. Inserting the Taylor expansion of S Eq. (4.2.12) into Eq. (4.2.10) gives

$$\begin{aligned} S^{-1}(\Lambda)\gamma^{\nu}S(\Lambda) &= (\mathbb{1} - \omega_{\rho\sigma}\Gamma^{\rho\sigma})\gamma^{\nu}(\mathbb{1} + \omega_{\alpha\beta}\Gamma^{\alpha\beta}) \\ &= \gamma^{\nu} - \omega_{\rho\sigma}\Gamma^{\rho\sigma}\gamma^{\nu} + \omega_{\rho\sigma}\gamma^{\nu}\Gamma^{\rho\sigma} + o(\omega^2) \\ &\overset{(4.2.10)}{=} \Lambda^{\nu}{}_{\mu}\gamma^{\mu} \\ &= (g^{\nu}{}_{\mu} + \omega^{\nu}{}_{\mu})\gamma^{\mu} \\ &= \gamma^{\nu} + \omega^{\nu}{}_{\mu}\gamma^{\mu} \end{aligned} \tag{4.2.13}$$

and thus we obtain

$$\omega_{\rho\sigma}[\gamma^{\nu}, \Gamma^{\rho\sigma}] = \omega^{\nu}{}_{\mu}\gamma^{\mu}. \tag{4.2.14}$$

This is not all, we have one more condition, we still need to normalize S, because that is not fixed by Eq. (4.2.10). We choose $\det S = +1$, so

$$\begin{aligned} \det S(\mathbb{1} + \omega) &= \det(\mathbb{1}_{ij} + \omega_{\mu\nu}\Gamma^{\mu\nu}_{ij}) \\ &= \det \begin{pmatrix} (1 + \omega_{\mu\nu}\Gamma^{\mu\nu}_{11}) & \omega_{\mu\nu}\Gamma^{\mu\nu}_{12} & \dots & \dots \\ \omega_{\mu\nu}\Gamma^{\mu\nu}_{21} & (1 + \omega_{\mu\nu}\Gamma^{\mu\nu}_{22}) & \dots & \dots \\ \vdots & \vdots & \ddots & \dots \\ \vdots & \vdots & \vdots & (1 + \omega_{\mu\nu}\Gamma^{\mu\nu}_{44}) \end{pmatrix} \\ &= (1 + \omega_{\mu\nu}\Gamma^{\mu\nu}_{11}) \dots (1 + \omega_{\mu\nu}\Gamma^{\mu\nu}_{44}) + o(\omega^2) \\ &= 1 + \omega_{\mu\nu}(\Gamma^{\mu\nu}_{11} + \Gamma^{\mu\nu}_{22} + \Gamma^{\mu\nu}_{33} + \Gamma^{\mu\nu}_{44}) + o(\omega^2) \\ &= 1 + \omega_{\mu\nu}\underbrace{\mathrm{Tr}[\Gamma^{\mu\nu}]}_{=0} \overset{!}{=} 1 \end{aligned} \tag{4.2.15}$$

[7] Check by inserting Eq. (4.2.11) into $\Lambda^{\mu}{}_{\rho}g^{\rho\sigma}\Lambda^{\nu}{}_{\sigma} = g^{\mu\nu}$ and expand to first order in ω.

and note that in the above calculation we only considered terms of order $o(\omega^2)$. From this we get another condition that Γ must fulfill. So in total we have two conditions now, namely

$$\mathrm{Tr}[\Gamma^{\mu\nu}] = 0 \quad \text{and} \quad \omega_{\rho\sigma}[\gamma^\nu, \Gamma^{\rho\sigma}] = \omega^\nu{}_\mu \gamma^\mu. \tag{4.2.16}$$

The matrix that satisfies these conditions is

$$\Gamma^{\mu\nu} = -\frac{i}{4}\sigma^{\mu\nu} \quad \text{with} \quad \sigma^{\mu\nu} \equiv \frac{i}{2}[\gamma^\mu, \gamma^\nu]. \tag{4.2.17}$$

The proof that this matrix fulfills the conditions in Eq. (4.2.16) is a homework assignment. In total we can write S as

$$S(\Lambda) = \mathbb{1} - \omega_{\mu\nu}\frac{i}{4}\sigma^{\mu\nu} + o(\omega^2) \tag{4.2.18}$$

for an infinitesimal Lorentz transformation. What about finite Lorentz transformations? To that end we now use the group property of the Lorentz transformation. We can construct a finite Λ by taking infinitely many infinitesimal steps N with ω/N the stepsize. Then we can write

$$\begin{aligned}
\Lambda^\mu{}_\nu &= \lim_{N\to\infty}\left(g^\mu{}_\alpha + \frac{1}{N}\omega^\mu{}_\alpha\right)\left(g^\alpha{}_\beta + \frac{1}{N}\omega^\alpha{}_\beta\right)\ldots\left(g^\chi{}_\nu + \frac{1}{N}\omega^\chi{}_\nu\right) \\
&= \lim_{N\to\infty}\left(g^\mu{}_\nu + \frac{1}{N}\omega^\mu{}_\nu\right)^N \\
&= \lim_{N\to\infty}\left(\mathbb{1} + \frac{1}{N}\omega\right)^N \\
&= (e^\omega)^\mu{}_\nu
\end{aligned} \tag{4.2.19}$$

and thus

$$\begin{aligned}
S_{ij}(\Lambda) &= \lim_{N\to\infty}\underbrace{S_{ij}\left(g + \frac{1}{N}\omega\right)\ldots S_{ij}\left(g + \frac{1}{N}\omega\right)}_{\text{N-times}} \\
&= \lim_{N\to\infty}\left(\mathbb{1} - \omega_{\mu\nu}\frac{i}{4}\sigma^{\mu\nu}\right)^N_{ij} \\
&= \left(e^{-\frac{i}{4}\sigma^{\mu\nu}\omega_{\mu\nu}}\right)_{ij}
\end{aligned} \tag{4.2.20}$$

for $\Lambda = e^\omega$.

Example: Rotations

Rotation around z-axis with angle ω in $\mathbb{R}^4$,

$$\Lambda^{\mu}{}_{\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\omega & \sin\omega & 0 \\ 0 & -\sin\omega & \cos\omega & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \overset{\omega \ll 1}{=} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \omega & 0 \\ 0 & -\omega & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + o(\omega^2) = g^{\mu}{}_{\nu} + \omega^{\mu}{}_{\nu}, \tag{4.2.21}$$

with $\omega^1{}_2 = -\omega^{12} = \omega$ and all other $\omega^{\mu}{}_{\nu} = 0$. Then

$$S(\Lambda) = 1 + \omega_{12}\Gamma^{12} + \omega_{21}\Gamma^{21} = 1 + \frac{i}{2}\sigma^{12}\omega + o(\omega^2), \tag{4.2.22}$$

where

$$\sigma^{12} = \frac{i}{2}(\gamma^1\gamma^2 - \gamma^2\gamma^1) = i\gamma^1\gamma^2 \overset{(4.1.13)}{=} \begin{pmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{pmatrix} = \Sigma^3. \tag{4.2.23}$$

So we can write

$$S(\Lambda) = \mathbb{1} + \frac{i}{2}\Sigma^3\omega. \tag{4.2.24}$$

For a finite rotation where we now have a finite ω we can also do N rotations with angle ω/N. Then we can write

$$S(\omega) = \lim_{N\to\infty}\left[S\left(\frac{\omega}{N}\right)\right]^N = \lim_{N\to\infty}\left(\mathbb{1} + \frac{i}{2}\Sigma^3\frac{\omega}{N}\right)^N = e^{\frac{i}{2}\Sigma^3\omega} \tag{4.2.25}$$

and a spinor transforms as $\psi'(x') = e^{\frac{i}{2}\Sigma^3\omega}\psi(x)$.

We now know how a spinor transforms, that is, $\psi'(x') = e^{-\frac{i}{2}\sigma^{\mu\nu}\omega_{\mu\nu}}\psi(x)$. The next question is how the adjoint spinor $\bar{\psi}(x) = \psi^\dagger(x)\gamma^0$ transforms. We take the hermitian conjugate of Eq. (4.2.5)

$$(\psi')^\dagger(x') = \psi^\dagger(x')S^\dagger(\Lambda) = \psi^\dagger(x')\underbrace{\gamma^0\gamma^0}_{=\mathbb{1}}S^\dagger(\Lambda). \tag{4.2.26}$$

Multiplication with γ^0 gives

$$(\bar{\psi}'(x')) = \bar{\psi}(x)(\gamma^0 S^\dagger(\Lambda)\gamma^0). \tag{4.2.27}$$

Hence, the adjoint spinor is transformed by the matrix $\gamma^0 S^\dagger(\Lambda)\gamma^0$! For a proper Lorentz

transformation Λ ($\det\Lambda = +1$) and ${\Lambda^0}_0 > 0$ one can show that

$$\gamma^0 S^\dagger(\Lambda)\gamma^0 = S^{-1}(\Lambda). \tag{4.2.28}$$

The proof of this identity can be found in the book of *Schwabl*. It is somewhat technical, but it is easy to see that it holds for infinitesimal $S(\Lambda)$. Eq. (4.2.28) implies

$$\bar\psi'(x') = \bar\psi(x)S^{-1}(\Lambda). \tag{4.2.29}$$

Although the γ^μ's stay the same in all frames, the covariant bilinar $\bar\psi(x)\Gamma\psi(x)$ transform like scalar, vector or tensor for some arbitrary matrix Γ. For example

$$(\bar\psi'(x'))\gamma^\mu\psi'(x') = \bar\psi(x)S^{-1}(\Lambda)\gamma^\mu S(\Lambda)\psi(x) \overset{(4.2.10)}{=} {\Lambda^\mu}_\nu(\bar\psi(x)\gamma^\nu\psi(x)) \tag{4.2.30}$$

transforms like a vector. In order to see what other kind of covariant bilinears exist we introduce another Dirac matrix

$$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3. \tag{4.2.31}$$

It has the following properties

1. $\{\gamma^\mu, \gamma_5\} = 0$,
2. $(\gamma_5)^2 = \mathbb{1}$,
3. $\gamma_5 = (\gamma_5)^\dagger$.

The proof of these properties is a homework assignment. It is a relevant quantity in the context of the electroweak interaction. We have a set of 16 Dirac matrices

$$\{\gamma^\mu, \gamma^\mu\gamma_5, \sigma^{\mu\nu}, \gamma^\nu, \gamma_5, \mathbb{1}\}, \tag{4.2.32}$$

where the γ^μ and $\gamma^\mu\gamma_5$ are four matrices, the $\sigma^{\mu\nu}$ give six linearly independent matrices and γ_5, $\mathbb{1}$ are two additional matrices. These 16 linearly independent matrices form a basis of Dirac matrices, which is 16-dimensional. We can therefore represent any Dirac matrix M in terms of these matrices. Such a linear combination would look like

$$M = v_\mu\gamma^\mu + a_\mu\gamma^\mu\gamma_5 + t_{\mu\nu}\sigma^{\mu\nu} + p\gamma_5 + s\mathbb{1}, \tag{4.2.33}$$

where $v_\mu, a_\mu, t_{\mu\nu}, p, s$ are coefficients. If we want to study bilinear maps such as $\bar\psi(x)M\psi(x)$ then it is interesting to know how the basis matrices transform. We can build bilinear

maps from the 16 basis matrices

$\bar{\psi}(x)\gamma^\mu\psi(x)$	transforms like vector,
$\bar{\psi}(x)\gamma^\mu\gamma_5\psi(x)$	transforms like pseudo-vector,
$\bar{\psi}(x)\sigma^{\mu\nu}\psi(x)$	transforms like tensor (anti-symmetric),
$\bar{\psi}(x)\gamma_5\psi(x)$	transforms like pseudo-scalar,
$\bar{\psi}(x)\mathbb{1}\psi(x)$	transforms like scalar.

While the Dirac matrices alone do not transform under Lorentz transformation, the covariant bilinears do transform as expected! This will be our guiding principle in constructing covariant Lagrangians. The difference between a vector and a pseudo-vector is that they transform differently under *parity* $\vec{x} \to -\vec{x},\ t \to t$. The corresponding Lorentz-matrix is not infinitesimal and not a proper Lorentz transformation

$$(\Lambda_P)^\mu{}_\nu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{4.2.34}$$

with $\psi'(x') = S(\Lambda_P)\psi(x) \equiv P\psi(x)$. One can easily show that if we choose $P = \gamma^0$, then this choice satisfies $P^{-1}\gamma^\nu P = (\Lambda_p)^\nu{}_\mu\gamma^\mu$ and $\det P = +1$. The momentum spinors $u_s(p),\ v_s(p)$ for a particle at rest, that is, $\vec{p} = 0,\ p^\mu = (m, \vec{0})$, are eigenstates of the parity. It means that u describes fermions, while v describes antifermions and due to

$$Pu_s(\vec{p} = 0, m) = \gamma^0 \begin{pmatrix} \chi_s \\ 0 \end{pmatrix} = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} \begin{pmatrix} \chi_s \\ 0 \end{pmatrix} = + \begin{pmatrix} \chi_s \\ 0 \end{pmatrix} \tag{4.2.35}$$

$$Pv_s(\vec{p} = 0, m) = \gamma^0 \begin{pmatrix} 0 \\ \chi_s \end{pmatrix} = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} \begin{pmatrix} 0 \\ \chi_s \end{pmatrix} = - \begin{pmatrix} 0 \\ \chi_s \end{pmatrix} \tag{4.2.36}$$

fermions have *even intrinsic parity* while antifermions have *odd intrinisc parity*. For the covariant bilinears we want to know what the difference between a vector and a pseudo-vector is. The bilinear map

$$\begin{aligned} \bar{\psi}'(x')\gamma^\mu\psi(x) &= \bar{\psi}(x)P^{-1}\gamma^\mu P\psi(x) \\ &= \bar{\psi}(x)\gamma^0\gamma^\mu\gamma^0\psi(x) \\ &= \bar{\psi}(x)(\gamma^\mu)^\dagger\psi(x) \\ &= \begin{pmatrix} \bar{\psi}\gamma^0\psi \\ -\bar{\psi}\gamma^i\psi \end{pmatrix} \end{aligned} \tag{4.2.37}$$

under parity transforms like a vector, where the time-component stays the same and the spatial component changes its sign. If we would do the same with

$$\bar{\psi}'(x')\gamma^\mu\gamma_5\psi(x) = \bar{\psi}(x)\gamma^0\gamma^\mu\gamma_5\gamma^0\psi(x) = -\bar{\psi}(x)(\gamma^\mu)^\dagger\gamma_5\psi(x) = -\begin{pmatrix}\bar{\psi}\gamma^0\psi\\ -\bar{\psi}\gamma^i\psi\end{pmatrix} \tag{4.2.38}$$

we see that this transforms like a pseudo-vector. As we see, the sign of a pseudo-vector changes, but for a vector it does not.

4.3 Quantization of the Dirac Field

4.3.1 Lagrange- and Hamilton Formalism

We consider the classical, complex Dirac field $\psi(x)$ with Lagrange density

$$\mathcal{L} = \bar{\psi}(x)(i\gamma_\mu\partial^\mu - m)\psi(x) \tag{4.3.1}$$

and $\bar{\psi} = \psi^\dagger\gamma^0$. Let us check if this Lagrange density makes sense. To do so we check the Euler-Lagrange equation of motion. The independent degrees of freedom are either ψ_a or $\bar{\psi}_b$ where the label a, b refers to the spinor component. Equivalently, we could take Reψ and Imψ. Therefore,

$$\partial_\nu\frac{\partial\mathcal{L}}{\partial(\partial_\nu\bar{\psi}_a)} - \frac{\partial\mathcal{L}}{\partial\bar{\psi}_a} = 0. \tag{4.3.2}$$

Written out with spinor components the Lagrange density is $\mathcal{L} = \bar{\psi}_c(i\gamma_\mu\partial^\mu - m)_{cd}\psi_d(x)$. The first term vanishes, because no derivative with respect to $\bar{\psi}$ exists, hence only the second term remains and we obtain

$$(i\gamma^\mu\partial_\mu - m)\psi(x) = 0. \tag{4.3.3}$$

As we see the Lagrange density is constructed such that the Dirac equation is retrieved. Moreover,

$$\partial_\nu\frac{\partial\mathcal{L}}{\partial(\partial_\nu\psi_a)} - \frac{\partial\mathcal{L}}{\partial\psi_a} = 0 \tag{4.3.4}$$

and we obtain

$$i\partial_\nu\bar{\psi}\gamma^\nu + \bar{\psi}m = 0. \tag{4.3.5}$$

This is nothing else but the adjoint Dirac equation. For the momentum we get

$$\pi_a = \frac{\partial\mathcal{L}}{\partial\dot{\psi}_a} = (i\bar{\psi}\gamma^0)_a = i\psi_a^\dagger, \tag{4.3.6}$$

where we use that $\bar{\psi} = \psi^\dagger \gamma^0$ and $(\gamma^0)^2 = 1$. The adjoint momentum operator is

$$\bar{\pi}_a = \frac{\partial \mathcal{L}}{\partial \dot{\bar{\psi}}_a} = 0. \tag{4.3.7}$$

In our analysis we need the Hamiltonian, its density is

$$\begin{aligned}
\mathcal{H} &= \pi_a \dot{\psi}_a - \mathcal{L} \\
&= i\psi^\dagger \dot{\psi} - \bar{\psi}(i\gamma^0 \partial_0 + i\gamma^i \partial_i - m)\psi \\
&= -\bar{\psi}(i\gamma^i \partial_i - m)\psi \\
&\overset{(4.1.20)}{=} i\bar{\psi}\gamma^0 \partial_0 \psi \\
&= i\psi^\dagger \partial_0 \psi
\end{aligned} \tag{4.3.8}$$

and thus

$$\mathcal{H} = i\psi^\dagger \partial_0 \psi. \tag{4.3.9}$$

We will now derive the Noether current. Invariance of $\mathcal{L}$ under a transformation $\psi \to \psi' = e^{-i\varepsilon q}\psi$ and similarly $\bar{\psi} \to \bar{\psi}' = e^{i\varepsilon q}\bar{\psi}$ for small ε or in the infinitesimal version $\psi \to \psi' = \psi - i\varepsilon q \psi = \psi + \delta\psi$ implies (cf. Chapter 2.5)

$$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi_a)} \delta\psi_a \frac{1}{\varepsilon} = (i\bar{\psi}\gamma^0)_a (-iq\psi_a) = q\bar{\psi}\gamma^\mu \psi. \tag{4.3.10}$$

As a consequence we get a 4-current

$$j^\mu(x) = q\bar{\psi}\gamma^\mu \psi \tag{4.3.11}$$

that is conserved as it satisfies $\partial_\mu j^\mu(x) = 0$. Up to now we used a classical formalism, we have now a classical field that depends on space, time, is complex and has four components. It fulfills the Dirac equation of motion and we can associate a Lagrange density, a Hamiltonian, momenta and a current that is conserved.

4.3.2 Canonical Commutation Relations

The title may alternatively be "How not to quantize the Dirac field" because as we will see, it does not work. In complete analogy let us look at the commutation relations

$$[\psi_a(\vec{x}, t), \pi_b(\vec{y}, t)] = i\delta_{a,b}\delta^{(3)}(\vec{x} - \vec{y}), \tag{4.3.12}$$

with $\pi_b = i\psi_b^\dagger$ and

$$[\psi_a(\vec{x}, t), \psi_b^\dagger(\vec{y}, t)] = \delta_{a,b}\delta^{(3)}(\vec{x} - \vec{y}), \tag{4.3.13}$$

$$[\psi_a(\vec{x}, t), \psi_b(\vec{y}, t)] = [\psi_a^\dagger(\vec{x}, t), \psi_b^\dagger(\vec{y}, t)] = 0. \tag{4.3.14}$$

Next, expand ψ in plain waves (cf. Chapter 3.5)

$$\psi(\vec{x},t) = \frac{1}{\sqrt{V}}\sum_{\vec{p}}\sum_{s=1,2}\frac{1}{\sqrt{2E_p}}\left(b^s_{\vec{p}}u(\vec{p},s)e^{-ip\cdot x} + (c^s_{\vec{p}})^\dagger v(\vec{p},s)e^{ip\cdot x}\right), \tag{4.3.15}$$

with $p^0 = E_p = \sqrt{m^2+p^2} > 0$. We expect that the operatos $b^{(\dagger)}, c^{(\dagger)}$ are annihilation and creation operators of particles and respectively antiparticles (e.g. electron and positron). Because we already did the computation in a previous chapter we make the following ansatz

$$[b^s_{\vec{p}}, (b^s_{\vec{p}'})^\dagger] = B\delta_{r,s}\delta_{\vec{p},\vec{p}'}, \tag{4.3.16}$$
$$[c^s_{\vec{p}}, (c^s_{\vec{p}'})^\dagger] = C\delta_{r,s}\delta_{\vec{p},\vec{p}'}. \tag{4.3.17}$$

All other commutators are vanishing. Now plug this into the commutator of the spinor and its adjoint, we find

$$\begin{aligned}
&[\psi(\vec{x},t),\psi^\dagger(\vec{y},t)]\\
&= \sum_{r,s}\sum_{\vec{p},\vec{p}'}\frac{1}{V}\frac{1}{\sqrt{2E_p}\sqrt{2E_{p'}}}\left\{\begin{array}{l}[b^r_{\vec{p}},(b^s_{\vec{p}'})^\dagger]u(\vec{p},r)\bar{u}(\vec{p}',s)\gamma^0 e^{ip'\cdot y - ip\cdot x}\\ +[c^r_{\vec{p}},(c^s_{\vec{p}'})^\dagger]v(\vec{p},r)\bar{v}(\vec{p}',s)\gamma^0 e^{-ip'\cdot y + ip\cdot x}\end{array}\right\}\\
&\overset{(4.3.16)}{\underset{(4.3.17)}{=}} \frac{1}{V}\sum_{\vec{p}}\frac{1}{2E_p}\{B\cdot(\not{p}+m)\gamma^0 e^{-i\vec{p}\cdot(\vec{y}-\vec{x})} - \underbrace{C\cdot(\not{p}-m)\gamma^0 e^{i\vec{p}\cdot(\vec{y}-\vec{x})}}_{\vec{p}\to-\vec{p}}\}\\
&= \frac{1}{V}\sum_{\vec{p}}\frac{1}{2E_p}e^{-i\vec{p}\cdot(\vec{y}-\vec{x})}\{B(p^0\gamma^0 + p_i\gamma^i + m)\gamma^0 - C(p_0\gamma^0 - p_i\gamma^i - m)\gamma^0\},
\end{aligned} \tag{4.3.18}$$

where we used the completeness relation $\sum_r u(\vec{p},r)\bar{u}(\vec{p},r) = \not{p} + m = p_\mu\gamma^\mu + m$ and analogously $\sum_r v(\vec{p},r)\bar{v}(\vec{p},r) = \not{p} - m$. Also note that the time components of x and y are equal and cancel. Setting $B = -C = 1$ gives

$$[\psi(\vec{x},t),\psi^\dagger(\vec{y},t)] = \frac{1}{V}\sum_{\vec{p}} e^{-i\vec{p}\cdot(\vec{x}-\vec{y})} = \delta^{(3)}(\vec{x}-\vec{y}). \tag{4.3.19}$$

We get the desired result, the commutator in Eq. (4.3.16) is as expected but not the one of in Eq. (4.3.17) where we need a minus sign. Let us now consider other quantities such as the Hamiltonian, that is, the energy of the system

$$H = \int d^3x\mathcal{H} = \int d^3x i\psi^\dagger\partial_0\psi = \ldots = \sum_r\sum_{\vec{p}} E_p((b^r_{\vec{p}})^\dagger b^r_{\vec{p}} - c^r_{\vec{p}}(c^r_{\vec{p}})^\dagger), \tag{4.3.20}$$

where $E_p = \sqrt{\vec{p}^2 + m^2}$ and we get a minus sign between the terms due to the derivative ∂_0 acting on the exponential of the spinor ψ. Finally, using the commutators from Eq. (4.3.16) and (4.3.17) (where $B = 1$ and $C = -1$) we obtain

$$H = \sum_r \sum_{\vec{p}} E_p((b^r_{\vec{p}})^\dagger b^r_{\vec{p}} - (c^r_{\vec{p}})^\dagger c^r_{\vec{p}} + 1). \tag{4.3.21}$$

Note that the term with a 1, which after performing the sum is an infinity, we can get rid off by normal ordering. The negative sign, however, is problematic. The operator $b^\dagger$ creates a positive energy state because

$$[H, (b^s_{\vec{p}})^\dagger] = E_p (b^s_{\vec{p}})^\dagger, \tag{4.3.22}$$

with $H\,|E\rangle = E\,|E\rangle$ we get

$$H(b^s_{\vec{p}})^\dagger\,|E\rangle = E_p(b^s_{\vec{p}})^\dagger\,|E\rangle + (b^s_{\vec{p}})^\dagger H\,|E\rangle = (E + E_p)\,|E\rangle\,. \tag{4.3.23}$$

As we see, the energy is raised by one quanta of energy $E_{\vec{p}}$. This supports the above interpretation (recall harmonic oscillator). Similarly,

$$[H, (c^s_{\vec{p}})^\dagger] = E_{\vec{p}} (c^s_{\vec{p}})^\dagger, \tag{4.3.24}$$

$$[H, c^s_{\vec{p}}] = -E_{\vec{p}} c^s_{\vec{p}}. \tag{4.3.25}$$

So far everything seems fine as the energy can be raised and lowered. Hence, $c^\dagger, c$ are creation and annihilation operators, but note that $[c^s_{\vec{p}}, (c^s_{\vec{p}'})^\dagger] = -\delta_{r,s}\delta_{\vec{p},\vec{p}}$. This means that we cannot define $c\,|0\rangle = 0$, because $c^\dagger_p\,|0\rangle = |\vec{p}, r\rangle$, $\langle\vec{p}, r| = \langle 0|\,c^r_p$ and now we can calculate the normalization $\langle\vec{p}, r|\vec{p}, r\rangle = \langle 0|cc^\dagger|0\rangle = -1$. Thus, we have a negative norm and we cannot interprete such a state! Instead, we could say, we interchange the role of c and $c^\dagger$. Then c would be a creation operator and $c^\dagger$ an annihilation operator with $[H, c] = -E_{\vec{p}}$. On the one hand our norm would be positive, but on the other hand acting with the creation operator would decrease the energy of the system instead of raising it, which we would expect to happen. The consequence is that the energy would go to minus infinity, which implies that there would be no stability in the system, that is, there exists no stable ground state! If we recall, a similar problem arised when we first introduced the Dirac equation and calculated its solutions. There we also had negative energy solutions and interpreted them as the Dirac sea. It is important to say, that this only worked because we talked about fermions meaning that at every level of this negative energy only two fermions of different spin can occupy the same state. If we would talk about bosons, the whole interpretation would fall apart, as it did with the Klein-Gorodon equation, because we could fill a state with as many bosonsas as we wish. In summary, in this formalism there is a problem which we cannot solve, when keeping the fundamental commutation relation.

4.3.3 Canonical Anticommutation Relations

The Dirac field represents spin 1/2 particles, which are fermions. As we know, fermions have to obey the *Fermi-Dirac-statistics*. It states that a quantum state gets a minus sign upon interchange of two particles. This is embedded in the structure of quantum field theory, meaning that it is consistent with microcausality. We demand the following, instead of commutation relations, the operators b and c from Chapter 4.3.2 should fulfill anticommutation relations

$$\{b^r_{\vec{p}}, (b^s_{\vec{p}'})^\dagger\} = b^r_{\vec{p}}(b^s_{\vec{p}'})^\dagger + (b^s_{\vec{p}'})^\dagger b^r_{\vec{p}} = \delta_{r,s}\delta_{\vec{p},\vec{p}'}, \tag{4.3.26}$$

$$\{c^r_{\vec{p}}, (c^s_{\vec{p}'})^\dagger\} = c^r_{\vec{p}}(c^s_{\vec{p}'})^\dagger + (c^s_{\vec{p}'})^\dagger c^r_{\vec{p}} = \delta_{r,s}\delta_{\vec{p},\vec{p}'}. \tag{4.3.27}$$

All other anticommutators vanish. At this point we will not do all the calculations in detail again, we rather sketch the procedure. These anticommutation relations, Eq. (4.3.26) and (4.3.27), correspond to

$$\{\psi_a(\vec{x},t), \psi^\dagger_b(\vec{y},t)\} = \delta_{a,b}\delta^{(3)}(\vec{x}-\vec{y}), \tag{4.3.28}$$

$$\{\psi_a, \psi_b\} = \{\psi^\dagger_a, \psi^\dagger_b\} = 0. \tag{4.3.29}$$

Now we have for the Hamiltonian

$$H = \sum_r \sum_{\vec{p}} E_p((b^r_{\vec{p}})^\dagger b^r_{\vec{p}} - c^r_{\vec{p}}(c^r_{\vec{p}})^\dagger) \overset{(4.3.27)}{=} \sum_r \sum_{\vec{p}} E_p((b^r_{\vec{p}})^\dagger b^r_{\vec{p}} + (c^r_{\vec{p}})^\dagger c^r_{\vec{p}} - 1) \tag{4.3.30}$$

and the constant term can again be eliminated by normal ordering. Normal ordering for fermion fields has the following structure

$$\begin{aligned} :\psi_a\psi_b: &= \mathcal{N}\{\psi_a\psi_b\} \\ &=: \psi^+_a + \psi^-_a(\psi^+_b + \psi^-_b): \\ &= \psi^+_a\psi^+_b - \psi^-_b\psi^+_a + \psi^-_a\psi^+_b + \psi^-_a\psi^-_b, \end{aligned} \tag{4.3.31}$$

where $\psi^+_{a,b} \sim b$, $\psi^-_{a,b} \sim c^\dagger$. Note that again $\mathcal{N}$ is just a different way of writing the normal ordering. From this calculation we see that the exchange of two fermionic operators gives a minus sign (as if all anticommutators $\{\cdot,\cdot\}$ are zero). We can redefine the Hamiltonian (energy operator) including normal ordering

$$H = \sum_r \sum_{\vec{p}} E_p((b^r_{\vec{p}})^\dagger b^r_{\vec{p}} + (c^r_{\vec{p}})^\dagger c^r_{\vec{p}}). \tag{4.3.32}$$

Similarly, if we consider the momentum operator $\vec{P} =: -i\int d^3x \psi^\dagger(x)\vec{\nabla}\psi(x):$ we obtain

$$\vec{P} = \sum_{r,\vec{p}} \vec{p}((b^r_{\vec{p}})^\dagger b^r_{\vec{p}} + (c^r_{\vec{p}})^\dagger c^r_{\vec{p}}). \tag{4.3.33}$$

Moreover, the charge operator based on the corresponding Noether current is

$$Q =: q \int d^3x \psi^\dagger(x)\psi :, \tag{4.3.34}$$

with $\bar{\psi}\gamma^0\psi = \psi^\dagger\gamma^0\gamma^0\psi = \psi^\dagger\psi$. Applying the normal ordering gives

$$Q = \sum_{r,\vec{p}} q((b^r_{\vec{p}})^\dagger b^r_{\vec{p}} - (c^r_{\vec{p}})^\dagger c^r_{\vec{p}}). \tag{4.3.35}$$

Again we find two types of particles [b and c or particle (electron e^-) and antiparticle (positron e^+)]. We will now look at the particle states and the consequences of the Fermi-Dirac statistic. The vacuum state $|0\rangle$ satisfies $b^s_{\vec{p}}|0\rangle = c^s_{\vec{p}}|0\rangle = 0$. As a check $[H, b^r_{\vec{p}}] = -E_p b^r_{\vec{p}}$ (analogously for the c operators) and $[H, (b^r_{\vec{p}})^\dagger] = E_p (b^r_{\vec{p}})^\dagger$ (analogously for the $c^\dagger$ operators). For completeness, the one particle states are

$$|\vec{p}, r\rangle_b = (b^r_{\vec{p}})^\dagger |0\rangle , \quad |\vec{p}, r\rangle_c = (c^r_{\vec{p}})^\dagger |0\rangle , \tag{4.3.36}$$

with norm $\langle \vec{p}', s|\vec{p}, r\rangle = \langle 0|b^s_{\vec{p}'}(b^r_{\vec{p}})^\dagger|0\rangle \overset{(4.3.26)}{=} \delta_{r,s}\delta_{\vec{p},\vec{p}'} > 0$. We can go a little further and find a two particle state

$$|\vec{p}_1, r; \vec{p}_2, s\rangle = (b^r_{\vec{p}_1})^\dagger (b^r_{\vec{p}_2})^\dagger |0\rangle = -(b^s_{\vec{p}_2})^\dagger (b^r_{\vec{p}_1})^\dagger |0\rangle = -|\vec{p}_2, s; \vec{p}_1, r\rangle , \tag{4.3.37}$$

where we assumed that $\vec{p}_1 \neq \vec{p}_2$ and $r \neq s$. Atleast one of these two inequalities[8] has to be fulfilled to obtain the above result, that is, states change sign upon interchange of two particles (Fermi-Dirac statistics) or in quantum mechanics it means that the wavefunction is antisymmetric. This is a consequence of the anticommutators. We could use the same quantum states for the same particle and when interchanging we get $|\vec{p}_1, r; \vec{p}_2, r\rangle = 0$. Thus, two fermions cannot reside in the same quantum state (Pauli exclusion principle).

4.3.4 Microcausality

We consider again the field operator

$$\begin{aligned}\psi(\vec{x}, t) &= \frac{1}{\sqrt{V}} \sum_{\vec{p}} \sum_s \frac{1}{\sqrt{2E_p}} (b^s_{\vec{p}}\, u(\vec{p}, s) e^{-ip\cdot x} + (c^s_{\vec{p}})^\dagger v(\vec{p}, s) e^{ip\cdot x}) \\ &= \psi^+(x) + \psi^-(x).\end{aligned} \tag{4.3.38}$$

[8] $\vec{p}_1 \neq \vec{p}_2$ and $r \neq s$

with $\psi^+ \sim b$ and $\psi^- \sim c^\dagger$. Analogously, $\bar{\psi} = \bar{\psi}^+(x) + \bar{\psi}^-(x)$ with $\bar{\psi}^+ \sim c$ and $\bar{\psi}^- \sim c^\dagger$. The usual exercise is now to compute

$$\{\psi_a(x), \bar{\psi}_b(y)\} = \{\psi_a^+(x), \bar{\psi}_b^-(y)\} + \{\psi_a^-(x), \bar{\psi}_b^+(y)\}, \tag{4.3.39}$$

where in general $x^0 \neq y^0$. The reason why we look at this combination $\{\psi_a, \bar{\psi}_b\}$ is because as we saw at the end of Chapter 4.2 we can construct so called bilinears, such as scalars, vectors and tensors. These are the quantities, which later on will call observables (i.e. charge, energy and so on). So we always have this bilinear structure and in these ψ, $\bar{\psi}$ are always contained. Therefore, this anticommutator is especially interesting as we will see later. We get

$$\{\psi^+(x), \bar{\psi}^-(y)\} = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} (\not{p} + m) e^{-ip\cdot(x-y)} \equiv i(i\not{\partial}_x + m)\Delta^+(x-y), \tag{4.3.40}$$

with $\frac{1}{V}\sum_{\vec{p}} \to \frac{1}{(2\pi)^3}\int d^3p$ and $\Delta^+(x-y) = \frac{-i}{2(2\pi)^3}\int \frac{d^3p}{\omega_p} e^{-ip\cdot(x-y)}$ where $\omega_p = E_p$. Similarly, we have

$$\{\psi^-(x), \bar{\psi}^+(y)\} = i(i\not{\partial}_x + m)\{-\Delta^+(y-x)\}. \tag{4.3.41}$$

Adding both anticommutator gives

$$\{\psi(x), \bar{\psi}(y)\} = i(i\not{\partial}_x + m)\{\Delta^+(x-y) - \Delta^+(y-x)\} \equiv i(i\not{\partial}_x + m)\Delta(x-y), \tag{4.3.42}$$

with $\Delta(x) = -\frac{1}{(2\pi)^3}\int \frac{d^3p}{\omega_p}\sin(p\cdot x)$. For space-like seperation, that is, $(x-y)^2 < 0$, we have $\{\psi(x), \bar{\psi}(y)\} = 0$. The difference to the Klein-Gordon case is that we have an anticommutator and a matrix structure, but there are also certain similarities. Other commutation relations such as

$$\{\psi(x), \psi(y)\} = \{\bar{\psi}(x), \bar{\psi}(y)\} = 0, \tag{4.3.43}$$

are vanishing. Consider now *microcausality* for the obeservables $A(x)$ and $B(y)$ with $[A(x), B(y)] = 0$ for $(x-y)^2 < 0$. For Dirac fields, the commutator of bilinear forms results in

$$\begin{aligned}
&[\bar{\psi}_a(x)\psi_b(x), \bar{\psi}_c(y)\psi_d(y)] \\
&= \bar{\psi}_c[\bar{\psi}_a\psi_b, \psi_d] + [\bar{\psi}_a\psi_b, \bar{\psi}_c]\psi_d \\
&= \bar{\psi}_c\bar{\psi}_a\{\psi_b, \psi_d\} - \bar{\psi}_c\{\bar{\psi}_a, \psi_d\}\psi_b + \bar{\psi}_a\{\psi_b, \bar{\psi}_c\}\psi_d - \{\bar{\psi}_a, \bar{\psi}_c\}\psi_b\psi_d \\
&= 0
\end{aligned} \tag{4.3.44}$$

for $(x-y)^2 < 0$, where we used $[C, AB] = A[C, B] + [C, A]B$ and $[AB, C] = A\{B, C\} - \{A, C\}B$. *Thus, microcausality is fulfilled for bilinear forms.* This is due to the fact that we used anticommutation relations for the fermionic field. Remember that when we used commutation relations for the fermionic field, that is, using Bose-Einstein statisitcs, we ran into problems, since we had no lower bound for the energy. But using Fermi-Dirac statistics, that is, anticommutation relations, for the fermionic field results in a lower bound for the energy and microcausality is fulfilled. Therefore, particles with integral spin must be quantized with Bose-Einstein statistics whereas particles with half-integer spin must be quantized according to Fermi-Dirac statistics. We conclude that the wrong spin-statistics connection leads to difficulties such as violation of microcausality.

4.3.5 Fermion Propagator

We now write the *Fermion propagator* as $i(S_F)_{ab}(x-x') = \langle 0|T\{\psi_a(x)\bar{\psi}_b(x')\}|0\rangle$. Note that we have a $\bar{\psi}$ because it was the only commutator that has a contribution. For fermions

$$T\{\psi_a(x)\bar{\psi}_b(x')\} = \Theta(t-t')\psi_a(x)\bar{\psi}_b(x') - \Theta(t'-t)\bar{\psi}_b(x')\psi_a(x). \tag{4.3.45}$$

The minus sign is due to the anticommutation relations for fermionic field operators and Lorentz covariance. As previously we have

$$\begin{aligned}\langle 0|\psi_a(x)\bar{\psi}_b(x')|0\rangle &= \langle 0|\psi_a^+(x)\bar{\psi}_b^-(x')|0\rangle \\ &\overset{(4.3.26)}{=} \langle 0|\{\psi_a^+, \bar{\psi}_b^-(x')\}|0\rangle \\ &\overset{(4.3.40)}{=} i(i\partial\!\!\!/_x + m)_{ab}\Delta^+(x-x')\end{aligned} \tag{4.3.46}$$

and analogously

$$\langle 0|\bar{\psi}_b(x')\psi_a(x)|0\rangle = i(i\partial_x + m)_{ab}(-\Delta^+(x'-x)). \tag{4.3.47}$$

Summing these two contributions gives

$$\begin{aligned}iS_F(x-x') &= i(i\partial\!\!\!/_x + m)\{\Theta(t-t')\underbrace{\Delta^+(x-x')}_{\text{b-particles}} - \Theta(t'-t)\underbrace{(-\Delta^+(x'-x))}_{\text{c-particles}}\} \\ &= i(i\partial\!\!\!/_x + m)\Delta_F(x-x'),\end{aligned} \tag{4.3.48}$$

where Δ_F is the Feynman-Propagator for the Klein-Gordon field (see Eq. (3.4.15)). Acting with $i(i\partial\!\!\!/_x + m)$ on Δ_F finally gives

$$S_F(x-x') = \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot(x-x')} \frac{p\!\!\!/ + m}{p^2 - m^2 + i\varepsilon}. \tag{4.3.49}$$

This is called the *Feynman propagator for spin* $1/2$ *particles.*

Chapter 5

Interactions & Perturbation Theory

Up to know we only talked about free quantum theory where particles do not interact with each other. All equations of motion we considered so far[1] are linear and we used those to construct plain wave solutions of the general form

$$\Phi(x) = \frac{1}{\sqrt{V}} \sum_{\vec{p}} \frac{1}{\sqrt{2E_p}} (\phi_1(\vec{p})\hat{a}(\vec{p})e^{-ip\cdot x} + \phi_2(\vec{p})\hat{b}^\dagger(\vec{p})e^{ip\cdot x}). \tag{5.0.1}$$

Let us now add more complicated interaction terms to the free Lagrangians[2]. These interaction terms contain more than two fields and the guiding principle is that these are supposed to be local as well as covariant.

Example: Real scalar field Lagrangian and interactions

The Lagrangian in the real scalar field theory with some interaction terms is given by

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m^2\phi^2 s \underbrace{- \sum_{n\geq 3} \frac{\lambda_n}{n!}\phi^n(x)}_{\text{interaction terms}}. \tag{5.0.2}$$

To obtain the equations of motion we insert the Lagrangian into the Euler-Lagrange equation

$$\frac{\partial}{\partial x^\mu}\frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)} - \frac{\partial \mathcal{L}}{\partial \phi} = 0, \tag{5.0.3}$$

[1] For real scalar fields $(\partial^2 + m^2)\phi(x) = 0$, for the Dirac field $(i\partial\!\!\!/ - m)\psi(x) = 0$ and for the photon field in Lorenz gauge $\Box A^\mu(x) = 0$.

[2] For the real scalar field $\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m^2\phi^2$, for the Dirac field $\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi$ and for the free photon field $\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$.

which yields

$$(\partial^2 - m^2)\phi(x) = -\frac{\lambda_3}{2!}\phi^2(x) - \frac{\lambda_4}{3!}\phi^3(x) - \dots \tag{5.0.4}$$

We call the coefficients λ_n *coupling constants*. As we see, we end up with non-linear terms on the right hand side. Solutions to such non-linear equations are typically unknown. In fact, even if we were able to construct solutions, because of these non-linear terms a superposition of solutions would not again be a solution to the equation, as it is the case for linear differential equations. We therefore continue with an approximate method called *perturbation theory*. There we take the unperturbed theory, which is the free theory, and we assume that these interactions are small. We can then make a Taylor expansion in these interaction. Thus, the question we have to ask is "When are these interactions small?". Naïvely, one would say that this is the case, if the coupling constants λ_n are small. If this is the case we can treat interactions as small perturbations of the free theory.

We now make a *(mass) dimensional analysis* of the coupling constants, that is, we want to find out what mass dimension the coupling constants have. Since we work in natural units, all physical quantities can be described as having a dimension which is some power of the mass.

- In natural units $[\hbar] = 1$, meaning that the action has no mass dimension, that is, $[S] = [\int d^4x\mathcal{L}] = 0$ with $[x] = [dx] = -1$ and $[d^4x] = -4$ (cf. Chapter 1.1). Hence, the mass dimension of the Lagrangian must be $[\mathcal{L}] = 4$. This holds for every Lagrangian!

- Also, $[m] = [\frac{\partial}{\partial x^\mu}] = [\partial_\mu] = 1$ as they are given in GeV.

- For a scalar field we had the Lagrangian $\mathcal{L} = \frac{1}{2}(\partial_\mu\phi\partial^\mu\phi - m^2\phi^2)$, $[\mathcal{L}] = 4 = [m^2] + [\phi^2] = 2 + [\phi^2]$ and therefore $[\phi] = 1$. For the Dirac field $\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi$ we have $[\mathcal{L}] = 4 = [m] + 2[\psi] = [\partial\!\!\!/ + 2[\psi]]$ and thus $[\psi] = 3/2$. For the photonic field the Lagrangian is $\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ and we have $[\mathcal{L}] = 4 = [\partial] + [\partial] + [A] + [A]$, where $[\partial] = 1$. So, $[A^\mu] = 1$.

- The *interaction Lagrangian* is $\mathcal{L}_{\text{int}} = -\sum_{n\geq 3}\frac{\lambda_n}{n!}\phi^n(x)$ and $[\mathcal{L}] = 4 = [\lambda_n] + [\phi^n]$, thus the coupling constants have mass dimension $[\lambda_n] = 4 - n$. For $n = 3$ the dimension of the coupling constant is $[\lambda] = 1$: the interaction is characterized by a number $\tilde{\lambda}_3 = (\lambda/E)$ where E is some typical energy or mass scale of the interaction. From this one can conclude that $\tilde{\lambda}_3$ is important at small energies,

but negligable at high energies. Such interactions are called *super-renormalizable* and for those we can apply perturbation theory only at high energies. For $n = 4$ the dimension of the coupling constant is $[\lambda_4] = 0$, that is, it is just a number: The perturbation will be small if $\lambda_4 \ll 1$. This is then called *renormalizable*[3]. For $n > 4$ the dimension of the coupling constant is $[\lambda_n] < 0$ and $\tilde{\lambda}_n = (\lambda_n E^{n-4})$. The interaction in this case is small at low energies and large at high energies. We call this *non-renormalizable*. The most prominent example of a non-renormalizable theory is Fermi's 4-fermion theory for the weak interactions with the interaction Lagrangian $G_F \cdot (\bar{\psi}\psi)(\bar{\psi}\psi)$ and the *Fermi constant* $G_F = 1.16 \cdot 10^{-5} \frac{1}{\mathrm{GeV}^2}$.

For the moment let us focus on renormalizable interactions with mass dimension 4. Examples of such interactions are

- Higgs potential $\phi^4(x)$.
- Interactions for QED, QCD $(-e\bar{\psi}\gamma^\mu\psi)A_\mu$ with mass dimension 4.
- An interaction that can be constructed from A itself is $A_\mu A_\nu \partial^\mu A^\nu$, $(A_\mu A^\mu)(A_\nu A^\nu)$, it has mass dimension 4. This appears in QCD.
- Numerous other fields appear in the electroweak theory $|\phi|^2 A_\mu A^\nu$, $A^\mu \phi \partial_\mu \phi^\nu$, $\bar{\psi}\psi\phi$ (Yukawa - coupling).

The standard model of particle physics contains (almost) all possible types of renormalizable, local and covariant interaction terms of scalar, Dirac and vector fields.

5.1 Interaction or Dirac Picture

Suppose we have a free, time independent Hamiltonian $\hat{H}_0$ and in free quantum field theory we write $H_0 = \sum_k E_{\vec{k}} a^\dagger_{\vec{k}} a_{\vec{k}}$. In the Schrödinger picture we deal with operators that are time-independent but the states are time dependent and the evolution of the states is given by the Schrödinger equation

$$i\frac{\partial}{\partial t}\left|\psi_s(t)\right\rangle = \hat{H}_0\left|\psi_s(t)\right\rangle . \tag{5.1.1}$$

In the Heisenberg picture the states are independent of the time, but operators are time dependent. In particular, also the fields are time dependent. The transformation of an

[3]Examples are QED, QCD, gauge theory and Higgs interactions.

operator in the Schrödinger picture to an operator in the Heisenberg picture is given by an unitary transformation

$$\hat{O}_H(t) = e^{i\hat{H}_0 t}\hat{O}_S e^{-i\hat{H}_0 t}, \tag{5.1.2}$$

with $\hat{U}_0(t) = e^{-i\hat{H}_0 t}$. States on the other hand become time-independant, they can be transformed as follows

$$|\psi_H\rangle = e^{i\hat{H}_0 t}\,|\psi_S(t)\rangle = |\psi_H(0)\rangle\,. \tag{5.1.3}$$

The time evolution of operator is given by the Heisenberg equation

$$i\frac{\partial}{\partial t}\hat{O}_H(t) = [\hat{O}_H(t), \hat{H}_0]. \tag{5.1.4}$$

Now we also include interactions, that is, we add an additional term $\hat{H}_{\text{int}}$ to the free Hamilton. We then have $\hat{H} = \hat{H}_0 + \hat{H}_{\text{int}}$ with $[\hat{H}_0, \hat{H}_{\text{int}}] \neq 0$. The problem with $\hat{H}_0$ can be solved exactly, but if we add an interaction term this is no longer the case. We thus want to do perturbation theory and impose the *interaction picture*, which is particularly designed for perturbation theory. To get to the interaction picture we again need an unitary transformation. An operator in the interaction picture is given by

$$\hat{O}_I(t) \equiv e^{i\hat{H}_0 t}\hat{O}_S e^{-i\hat{H}_0 t}, \tag{5.1.5}$$

in particular a field in the interaction picture is given by

$$\Phi(\vec{x}, t) = e^{i\hat{H}_0 t}\hat{\Phi}(\vec{x}, 0)e^{-i\hat{H}_0 t} \tag{5.1.6}$$

The time evolution of (field) operators is governed by the free Hamiltonian $\hat{H}_0$

$$i\frac{\partial}{\partial t}\Phi_I(\vec{x}, t) = [\Phi_I(\vec{x}, 0), \hat{H}_0]. \tag{5.1.7}$$

That means we can treat fields in the interaction picture as free fields. We can therefore apply our free-field formalism and work again with creation and annihilation operators. Note that the interaction does not vanish, it appears in the evolution of the physical states. Again, in the Schrödinger picture a state evolves via

$$i\frac{\partial}{\partial t}\,|\psi_S(t)\rangle = (\hat{H}_0 + \hat{H}_{\text{int}})\,|\psi_S(t)\rangle\,. \tag{5.1.8}$$

The unitary transformation to the interaction picture is

$$|\psi_I(t)\rangle = e^{i\hat{H}_0 t}\,|\psi_s(t)\rangle\,. \tag{5.1.9}$$

Equivalently, this means that $\langle\psi_S(t)|\hat{O}_S|\phi_S(t)\rangle = \langle\psi_I(t)|\hat{O}_I(t)|\phi_I(t)\rangle$ by construction. Let us now take a state in the interaction picture and see how it evolves with time

$$\begin{aligned} i\frac{\partial}{\partial t}|\psi_I(t)\rangle &= i\frac{\partial}{\partial t}(e^{i\hat{H}_0 t}|\psi_S(t)\rangle) \\ &= (-\hat{H}_0 e^{-i\hat{H}_0 t}|\psi_S(t)\rangle + e^{i\hat{H}_0 t} i\frac{\partial}{\partial t}|\psi_S(t)\rangle) \\ &\overset{(5.1.8)}{=} (-i\hat{H}_0 e^{iH_0 t}|\psi_S(t)\rangle + e^{i\hat{H}_0 t}(\hat{H}_0 + \hat{H}_{\text{int}})e^{-i\hat{H}_0 t}e^{i\hat{H}_0 t}|\psi_S(t)\rangle) \\ &= \hat{H}_{\text{int, I}}(t)|\psi_I(t)\rangle \end{aligned} \tag{5.1.10}$$

In total we have a Schrödinger-type equation

$$i\frac{\partial}{\partial t}|\psi_I(t)\rangle = \hat{H}_{\text{int,I}}(t)|\psi_I(t)\rangle\,, \tag{5.1.11}$$

but the operator that governs the time-dependence is the interaction Hamiltonian $\hat{H}_{\text{int,I}}(t)$. Hence, states in the interaction picture evolve with the interaction Hamiltonian $\hat{H}_{\text{int,I}}(t)$. We will now discuss the time-evolution operator in the interaction picture, which will be particularly important when we talk about scattering-amplitudes, scattering experiments and so on. First, we want to connect the states in the interaction picture at different times, to do so consider

$$|\psi_I(t)\rangle = \hat{U}(t,t_0)|\psi_I(t_0)\rangle\,, \tag{5.1.12}$$

where $\hat{U}(t,t_0)$ is the *time-evolution operator*. Note that $\hat{U}(t_0,t_0) = \mathbb{1}$. Inserting this into a Schrödinger-type equation yields

$$i\frac{\partial}{\partial t}|\psi_S(t)\rangle = i\frac{\partial}{\partial t}[\hat{U}(t,t_0)|\psi_I(t_0)\rangle] = \hat{H}_{\text{int},I}|\psi_I(t)\rangle = H_{\text{int},I}\hat{U}(t,t_0)|\psi_I(t_0)\rangle\,. \tag{5.1.13}$$

Note that $|\psi_I(t_0)\rangle$ does not depend on t, so we can write

$$i\frac{\partial}{\partial t}[\hat{U}(t,t_0)|\psi_I(t_0)\rangle] = i\left(\frac{\partial}{\partial t}\hat{U}(t,t_0)\right)|\psi_I(t_0)\rangle\,. \tag{5.1.14}$$

Eq. (5.1.13) thus gives an operator equation

$$i\frac{\partial}{\partial t}\hat{U}(t,t_0) = \hat{H}_{\text{int},I}(t)\hat{U}(t,t_0) \tag{5.1.15}$$

for the time-evolution operator with initial condition $\hat{U}(t_0,t_0) = \mathbb{1}$. If $\hat{H}_{\text{int},I}$ would be time-independent, then the solution would be simple, because we can just write

$$\hat{U}(t,t_0) = e^{-i\hat{H}_{\text{int},I}(t-t_0)}. \tag{5.1.16}$$

In general though, this is not the case, since $[\hat{H}_{\text{int},I},(t_1),H_{\text{int, I}}(t_2)]\neq 0$ for $t_1\neq t_2$. The formal solution is

$$\hat{U}(t,t_0)=Te^{-i\int_{t_0}^{t}dt' H_{\text{int},I}(t')}, \tag{5.1.17}$$

where T is the *time ordering operator* or *time-ordering product*. It orders the operators in such a way that operators with a smaller time argument act first. If we expand the exponential into a power series this gives

$$\begin{aligned}\hat{U}(t,t_0)=\mathbb{1}&-i\int_{t_0}^{t}dt_1 H_{\text{int},I}(t_1)\\&+\frac{(-i)^2}{2!}\int_{t_0}^{t}dt_1\int_{t_0}^{t}dt_2 T[H_{\text{int},I}(t_1)H_{\text{int},I}(t_2)]\\&+\frac{(-i)^3}{3!}\int_{t_0}^{t}dt_1\int_{t_0}^{t}dt_2\int_{t_0}^{t}dt_3 T[H_{\text{int},I}(t_1)H_{\text{int},I}(t_2)H_{\text{int},I}(t_3)]\\&+\dots\end{aligned} \tag{5.1.18}$$

A sketch of the proof is given here, the rest of it is a homework assignment.

proof. We want to show that

$$U(t,t_0)\overset{!}{=}Te^{-i\int_{t_0}^{t}dt' H_{\text{int},I}(t')} \tag{5.1.19}$$

by showing

$$i\frac{\partial}{\partial t}U(t,t_0)=H_{\text{int},I}U(t,t_0). \tag{5.1.20}$$

Inserting the expansion into Eq. (5.1.15) yields

$$\begin{aligned}&i\frac{\partial}{\partial t}\Big[\mathbb{1}-i\int_{t_0}^{t}dt' H_{\text{int},I}(t')+\sum_{n=2}^{\infty}\frac{(-i)^n}{n!}\int_{t_0}^{t}dt_1\dots\int_{t_0}^{t}dt_n T[H_{\text{int},I}(t_1)\dots H_{\text{int},I}(t_n)]\Big]\\&=H_{\text{int},I}(t)+\sum_{n=2}^{\infty}\frac{(-i)^{n-1}}{n!}\frac{\partial}{\partial t}\int_{t_0}^{t}dt_1\underbrace{\Big[\int_{t_0}^{t}dt_2\dots\int_{t_0}^{t}dt_n T[H_{\text{int},I}(t_1)\dots H_{\text{int},I}(t_n)]\Big]}_{=:f(t_1,t)}.\end{aligned} \tag{5.1.21}$$

Using the forumla

$$\frac{\partial}{\partial t}\int_{t_0}^{t}dt' f(t',t)=f(t,t)+\int_{t_0}^{t}dt'\frac{\partial f}{\partial t}(t',t) \tag{5.1.22}$$

the first term $f(t,t)$ will just be another Hamiltonian, which means that it can be pulled out. What remains is somwhat the same expression but with a reduced number of integrals. Iterating this procedure n-times one has in the last step $\frac{\partial}{\partial t}T[\dots]=0$, this then gives the desired result.

5.1.1 Summary of Interaction Picture

The full Hamiltonian can be written as $H = H_0 + H_{\text{int}}$, that is, it can be split into an exactly solvable problem with H_0 and an additional interaction term, which makes calculations more complex. The interaction picture is constructed such that the fields evolve according to the *free Hamiltonian* H_0

$$\phi_j(\vec{x}, t) = \hat{U}_0^\dagger(t, t_0)\phi_I(\vec{x}, t_0)\hat{U}(t, t_0), \tag{5.1.23}$$

with $\hat{U}(t, t_0) = e^{-i\hat{H}_0(t-t_0)}$. The states evolve according to the *interaction Hamiltonian* H_{int}

$$|\psi_I(t)\rangle = \hat{U}(t, t_0)\,|\psi_I(t_0)\rangle\,, \tag{5.1.24}$$

with $\hat{U}(t, t_0) = Te^{-i\int_{t_0}^{t} dt' H_{\text{int},I}(t')}$.

5.2 S-Matrix for Scattering Processes

If we include interactions, we have an interacting system $|\psi(t)\rangle$, which involves many-body particle states, which is can be rather complicated. The goal of this chapter is to take the introduced *interaction picture* and adjust it to the scattering situation. In this context when we talk about scattering we imagine two particles which are far apart in the initial state, that is, they are seperated in space and time. They come together and interact at a certain point, where as a result a bunch of particles are produced. After the interaction they move apart and are detected by some detector. Suppose we have two localized wavepackages which do not overlap at $t = -\infty$. Then they move with a certain momentum towards each other. At $t = 0$ they collide in a complicated way. At $t = +\infty$ the particles created in the process are spatially seperated and are detected. We describe the inital system by the Hilbert space vector $|\psi(-\infty)\rangle = |i\rangle$ and the final state $\psi(\infty) = |f\rangle$ (see Fig. 5.1). What we want to do is to connect these two states by the so called *S-Matrix*. It does so in the sense that

$$|\psi(\infty)\rangle = \hat{S}\,|\psi(-\infty)\rangle\,. \tag{5.2.1}$$

In principle, the state $|\psi(\infty)\rangle$ contains all possible outcomes of the scattering process, we call those *scattering channels*, that is,

$$e^- + \gamma \to \begin{cases} e^- + \gamma \quad \text{Compton scattering} \\ e^- + \gamma + (e^+e^-) \\ \vdots \\ \text{all channels in } |\psi(\infty)\rangle \end{cases} \tag{5.2.2}$$

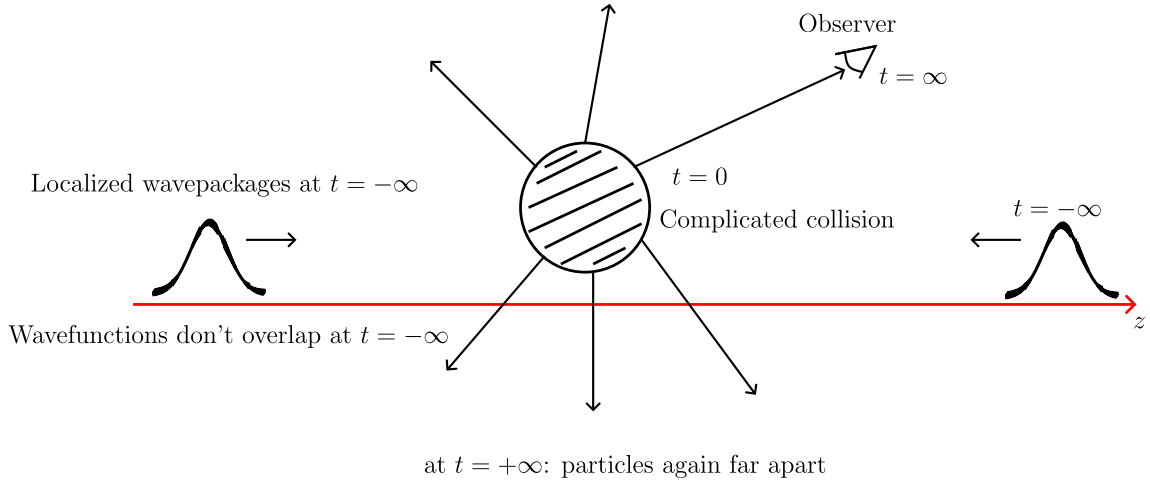


Fig. 5.1: Two localized wavepackages at $t = -\infty$ collide at $t = 0$ and are far apart at time $t = +\infty$. The z-axis along which the particles travel is more generally called the *beam axis*.

Suppose we require a definite channel $|f\rangle$ (one of the possible outcomes). The *transition probability* $|i\rangle \to |f\rangle$ is $|\langle f|\psi(+\infty)\rangle|^2$. We could also write this as $\langle f|\psi(+\infty)\rangle = \langle f|\hat{S}|i\rangle \equiv S_{fi}$ and call it the *S-matrix element*. In the interaction picutre we can write

$$|\psi(t = +\infty)\rangle = \hat{S}_I \, |\psi_I(t = -\infty)\rangle = \hat{U}(t = +\infty, t_0 = -\infty) \, |\psi_I(t = -\infty)\rangle \, . \tag{5.2.3}$$

Hence,

$$\hat{S}_I = T e^{-i \int_{-\infty}^{\infty} dt H_{\text{int},I}(t)} \tag{5.2.4}$$

and with $H_{\text{int},I} = \int d^3x \mathcal{H}_{\text{int},I}(t, \vec{x})$ we can write

$$\hat{S}_I = T e^{-i \int d^4x \mathcal{H}_{\text{int},I}(x)}. \tag{5.2.5}$$

In the case that $\frac{\partial \mathcal{L}_{\text{int}}}{\partial \dot{\phi}} = 0$ we have $\mathcal{L}_{\text{int}} = -\mathcal{H}_{\text{int}}$ and $\hat{S}_I$ becomes

$$\hat{S}_I = T e^{+i \int d^4x \mathcal{L}_{\text{int},I}(x)}. \tag{5.2.6}$$

In particular, $\mathcal{L}_{\text{int},I}$ is constructed from fields in the interaction picture and for those fields we take the free fields. From now on we only work in the interaction picture and therefore drop the subscript I. Next, we want to find out the initial conditions, that is, we want to find $|\psi(-\infty)\rangle = |i\rangle$ and $|f\rangle$. This is a subtle question in an interacting theory. Ideally, we may assume that $|i\rangle$ and $|f\rangle$ are eigenstates of the free Hamiltonian. Note that this is something that needs a rigerious justification! At this point though this is difficult, but it will be done in the follow-up lecture Quantum field theory II. Questions that arise are

- Why are we allowed to switch off interactions at $t = \pm\infty$?
- What exactly are vacuum & isolated particles in interacting theories?
- Are there even isolated particles in interacting theories?

We could illustrate this in the scattering situation (adiabatic hypothesis). Let us suppose we have the initial state at $t = -\infty$, the final state at $t = \infty$ and slowly adiabatically turn on the interaction represented by a function $f(t)$ (see Fig. 5.2).

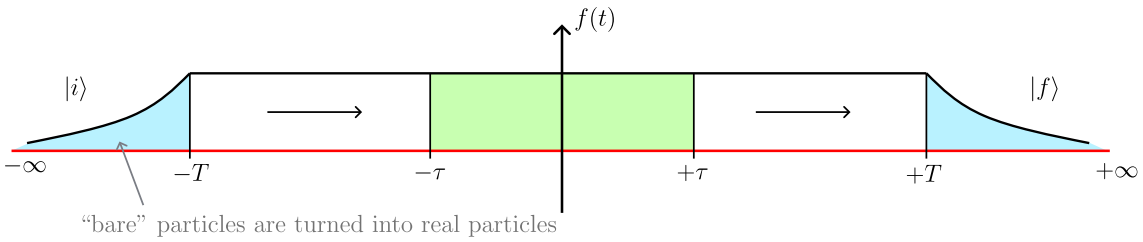


Fig. 5.2: In the intervals $[-\infty, T]$ and $[T, +\infty]$ we slowly turn on the interaction, which then turns the the *bare particle* into a *real particle* in the sense that it becomes surrounded by the photon cloud. In the region $[-\tau, \tau]$ the scattering occurs. The argument why we can use the bare states is that the interaction should not depend on the behaviour of the interaction $f(t)$ at its tails.

$$H_I(t) \to H_I(t) f(t) \tag{5.2.7}$$

5.3 Wick-Theorem

Let us now evaluate the S-matrix elements $S_{fi} = \langle f|\hat{S}|i\rangle$ where

$$|i\rangle = a^\dagger_{\vec{k}_1} a^\dagger_{\vec{k}_2} |0\rangle \tag{5.3.1}$$

is a two-particle state for a scattering situation. Or we could have $|i\rangle = a^\dagger_{\vec{k}}|0\rangle$ for the situation that a heavy particle decays. The n-particle final state can be written as

$$|f\rangle = a^\dagger_{\vec{k}_1} \dots a^\dagger_{\vec{k}_n} |0\rangle \,. \tag{5.3.2}$$

Then the matrix element reads

$$\begin{aligned}\langle f|\hat{S}|i\rangle &= \sum_{n=0}^{\infty}\frac{i^n}{n!}\int d^4x_1\ldots\int d^4x_n\langle f|T[:\mathcal{L}_{\text{int}}(x_1):\ldots:\mathcal{L}_{\text{int}}(x_n):]|i\rangle \\ &= \sum_{n=0}^{\infty} S_{fi}^{(n)}\end{aligned} \tag{5.3.3}$$

and $\mathcal{L}_{\text{int}}$ consist of a product of fields, for example $\mathcal{L}_{\text{int}} = \frac{\lambda}{4!}:\phi^4(x):$ or $\mathcal{L}_{\text{int}} = e:\bar{\psi}(x)\not{A}(x):\psi(x)$. Fields in the interaction picture evolve like free fields

$$\begin{aligned}\Phi(x) &= \frac{1}{\sqrt{V}}\sum_{\vec{k}}\frac{1}{\sqrt{2E_{\vec{k}}}}[\phi_1(\vec{k})\hat{a}(\vec{k})e^{-ik\cdot x}] + \frac{1}{\sqrt{V}}\sum_{\vec{k}}\frac{1}{\sqrt{2E_{\vec{k}}}}[\phi_2(\vec{k})\hat{b}^\dagger(\vec{k})e^{ik\cdot x}] \\ &\equiv \Phi^+(x) + \Phi^-(x),\end{aligned} \tag{5.3.4}$$

with $\Phi^+ \sim \hat{a}(\vec{k})$ and $\Phi^- \sim \hat{b}^\dagger(\vec{k})$. What we want to calculate are S-matrix elements of the form

$$\langle f|T[\mathcal{L}_{\text{int}}(x_1)\ldots\mathcal{L}_{\text{int}}(x_n)]|i\rangle = \langle 0|a_{\vec{p_1}_n}\ldots a_{\vec{p}_n}T[\mathcal{L}_{\text{int}}(x_1)\ldots\mathcal{L}_{\text{int}}(x_n)]a^\dagger_{\vec{k}_1}a^\dagger_{\vec{k}_2}|0\rangle. \tag{5.3.5}$$

In principle we could commute or anticommute all annihilation (creation) operators to the right (left), but this would be inefficient. A better way to proceed with is to calculate the matrix element through *Wick's theorem*, it gives us an efficient way to calculate S-matrix elements.

Definition: Normal ordering

The *normal ordering* for the case that Φ_i is either Φ_i^+ or Φ_i^- is defined as

$$:\Phi_1(x_1)\ldots\Phi_n(x_n): \equiv \text{sgn}(P)\cdot\Phi_{P(1)}(x_{P(1)})\ldots\Phi_{P(n)}(P(x_n)), \tag{5.3.6}$$

where P is a *permutation*, such that the annihilation parts Φ_i^+ are permuted to the right and the creation operator parts Φ_i^- to the left. The *signum* $\text{sgn}(P)$ of the permutation P is $+1$ for an even number of *transpositions* and -1 for an odd number of transpositions. One often also writes ε_P for the signum.

Example: Normal ordering of fermionic and bosonic fields

The normal ordering for a bosonic field is

$$\begin{aligned}:\phi(x_1)\phi(x_2):&=:(\phi^+(x_1)+\phi^-(x_2)(\phi^+(x_2)+\phi^-(x_2)):\\&=\phi^+(x_1)\phi^+(x_2)+\phi^-(x_1)\phi^-(x_2)+\phi^-(x_1)\phi^+(x_2)+\underbrace{:\phi^+(x_1)\phi^-(x_2):}_{=\phi^-(x_2)\phi^+(x_1)}\end{aligned} \tag{5.3.7}$$

and for a fermionic field it is

$$\begin{aligned}:\psi_i(x_1)\bar{\psi}_j(x_2):&=:(\psi^+(x_1)+\psi^-(x_1))(\bar{\psi}^+(x_2)+\bar{\psi}^-(x_2)):\\&=\psi^+(x_1)\bar{\psi}^+(x_2)+\psi^-(x_1)\bar{\psi}^-(x_2)+\psi^-(x_1)\bar{\psi}^+(x_2)\\&\quad+\underbrace{:\psi^+(x_1)\bar{\psi}^-(x_2):}_{=-\bar{\psi}^-(x_2)\psi^+(x_1)}\end{aligned} \tag{5.3.8}$$

Definition: Time ordering

We define the *time ordering* of generic fields via permutations as

$$T(\Phi_1(x_1)\dots\Phi_n(x_n))\equiv \mathrm{sgn}(P)\Phi_{P(1)}(x_{P(1)})\dots\Phi_{P(n)}(x_{P(n)}), \tag{5.3.9}$$

where P is a permutation such that $x^0_{P(1)}>x^0_{P(2)}>\ldots>x^0_{P(n)}$ and the signum of the permutation is defined as in the definition of the normal order.

Example: Time ordering of bosonic fields

The time ordering for two bosonic fields is

$$T(\phi(x_1)\phi(x_n))=\phi(x_1)\phi(x_2)\Theta(x_1^0-x_2^0)+\phi(x_2)\phi(x_1)\Theta(x_2^0-x_1^0) \tag{5.3.10}$$

and for fermionic fields

$$T(\psi_i(x_1)\bar{\psi}_j(x_2))=\psi_i(x_1)\bar{\psi}_j(x_2)\Theta(x_1^0-x_2^0)-\bar{\psi}_j(x_2)\psi_i(x_1)\Theta(x_2^0-x_1^0). \tag{5.3.11}$$

Definition: Dyson-Wick-contraction

The *Wick contraction* of two field operators $\hat{A},\hat{B}$ is defined as[4]

$$\underbrace{\hat{A}(x)\hat{B}}(x')\equiv T[\hat{A}(x)\hat{B}(x')]-:\hat{A}(x)\hat{B}(x'):. \tag{5.3.12}$$

For real fields with $x_1^0 > x_2^0$ the Wick contraction is

$$\begin{aligned}
\underbrace{\phi(x_1)\phi}(x_2) &= (\phi^+(x_1)+\phi^-(x_1))(\phi^+(x_2)+\phi^-(x_2)) \\
&\quad - (\phi^+(x_1)\phi^+(x_2)+\phi^+(x_1)\phi^-(x_2)+\phi^-(x_1)\phi^+(x_2)+\phi^-(x_2)\phi^+(x_1)) \\
&= [\phi^+(x_1),\phi^-(x_2)] \\
&= \frac{1}{2V}\sum_{\vec{k}}\frac{1}{\omega_{\vec{k}}}e^{-ik\cdot(x_1-x_2)} \\
&\to \int \frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_{\vec{k}}}e^{-ik\cdot(x_1-x_2)},
\end{aligned} \tag{5.3.14}$$

where we used our previous result for the commutator of scalar fields, see for example Eq. (3.4.5). For $x_2^0 > x_1^0$ it is

$$\underbrace{\phi(x_1)\phi}(x_2) = [\phi^+(x_2),\phi^-(x_1)] \to \int \frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_{\vec{k}}}e^{-ik\cdot(x_2-x_1)} \tag{5.3.15}$$

and in total we have

$$\begin{aligned}
\underbrace{\phi(x_1)\phi}(x_2) &= \Theta(x_1^0-x_2^0)\int \frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_k}e^{-ik\cdot(x_1-x_2)} + \Theta(x_2^0-x_1^0)\int \frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_k}e^{-ik\cdot(x_2-x_1)} \\
&= i\Delta_F(x_1-x_2) \\
&= \int \frac{d^4k}{(2\pi)^4}\frac{ie^{-ik\cdot(x_1-x_2)}}{k^2-m^2+i\delta}.
\end{aligned} \tag{5.3.16}$$

For fermion fields (cf. Homework Assignment) we find

$$\begin{aligned}
\underbrace{\psi_i(x_1)\bar{\psi}_j}(x_2) &= T(\ldots) - :\ldots: \\
&= iS_{ij}^F(x_1-x_2) \\
&= \int \frac{d^4p}{(2\pi)^4}\frac{i(\not{p}+m)_{ij}e^{-ip\cdot(x_2-x_1)}}{p^2-m^2+i\delta} \\
&= -\underbrace{\bar{\psi}_j(x_2)\psi}_i(x_1).
\end{aligned} \tag{5.3.17}$$

Also note that the following is true

$$\underbrace{\psi_i(x_1)\psi}_j(x_2) = 0 = \underbrace{\bar{\psi}_i(x_1)\bar{\psi}}_j(x_2). \tag{5.3.18}$$

[4]For two field operators Φ_1, Φ_2 we may also write

$$\underbrace{\Phi_1(x_1)\Phi}_2(x_2) \equiv T\{\Phi_1(x_1)\Phi_2(x_2)\} - :\Phi_1(x_1)\Phi_2(x_2): \tag{5.3.13}$$

For photons we have

$$\underbrace{A^\mu(x_1)A^\nu}(x_2) = iD_F^{\mu\nu}(x_1 - x_2) = \int \frac{d^4k}{(2\pi)^4} \frac{(-ig^{\mu\nu})e^{-ik\cdot(x_1-x_2)}}{k^2 + i\delta}. \tag{5.3.19}$$

As we see, the Wick contraction of fields recovers the corresponding propagator.

Definition: Normal ordering with contractions

Suppose we have $n > m$ arbitrary fields $\Phi_i(x_i)$ at positions x_i, then the *normal ordering with contractions* is

$$\begin{aligned}
&:\underbrace{\Phi_1(x_1)\Phi_2(x_2)\Phi_3(x_3)\Phi_4(x_4)\ldots\Phi_{m-1}(x_{m-1})\Phi_m(x_m)}\Phi_{m+1}(x_{m+1})\ldots\Phi_n(x_n):\\
&= (-1)^P \underbrace{\Phi_1(x_1)\Phi_{m-1}(x_{m-1})}\underbrace{\Phi_2(x_2)}\Phi_3(x_3)\underbrace{\Phi_4(x_4)}\Phi_{m+1}(x_{m+1})\ldots\\
&\quad :\Phi_5(x_5)\ldots\Phi_{m-2}(x_{m-2})\Phi_m(x_m)\Phi_{m+2}(x_{m+2})\ldots\Phi_n(x_n):,
\end{aligned} \tag{5.3.20}$$

where P is the number of transpositions of neighboring fermionic operators.

Example: Normal ordering with contractions

Normal ordered product of Dirac fields ψ and a photonic field A^μ with contraction

$$:\psi_i(x_1)\underbrace{\psi_j(x_2)A^\mu(x_3)\bar\psi_k(x_4)}\bar\psi_l(x_5):= (-1)\underbrace{\psi_j(x_2)}\bar\psi_l(x_5)\,:\psi_i(x_1)A^\mu(x_3)\bar\psi_k(x_4):$$

(5.3.21)

We have to switch $\bar\psi_l(x_5)$ with $\bar\psi_k(x_4)$ and since fields of different types, for example here $\bar\psi_l$ and A^μ, commute aynyway, we get only get one minus sign.

Theorem: Wick theorem for time-ordered product of fields

Let Φ_i be n free fields, then the time ordered product is given by

$$T[\Phi_1(x_1)\ldots\Phi_n(x_n)] =: \Phi_1(x_1)\ldots\Phi_n(x_n): + \sum :\text{all possible contractions}:, \tag{5.3.22}$$

with unequal times $x_1^0 \neq x_2^0 \neq \ldots \neq x_n^0$. ◇

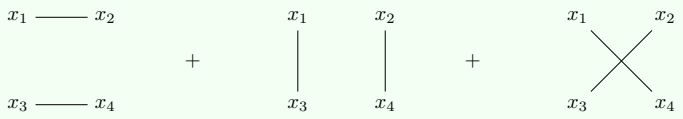
Example: Wick's theorem for real fields

Consider real fields $\phi(x_i) \equiv \phi_i$, then for the case of free fields, the time-ordered product is by Wick's theorem given as

$$\begin{aligned} T[\phi_1\phi_2\phi_3\phi_4] =& :\phi_1\phi_2\phi_3\phi_4: \\ &+ :\underbrace{\phi_1\phi_2}\phi_3\phi_4: + :\phi_1\underbrace{\phi_2\phi_3}\phi_4: + :\phi_1\phi_2\underbrace{\phi_3\phi_4}: \\ &+ :\underbrace{\phi_1\phi_2\phi_3}\phi_4: + :\phi_1\underbrace{\phi_2\phi_3\phi_4}: + :\underbrace{\phi_1\phi_2}\underbrace{\phi_3\phi_4}: \\ &+ :\underbrace{\phi_1\phi_2}\underbrace{\phi_3\phi_4}: + :\underbrace{\phi_1\phi_2\phi_3\phi_4}: + :\underbrace{\phi_1\phi_2}\underbrace{\phi_3\phi_4}: . \end{aligned} \tag{5.3.23}$$

This is very useful for vacuum expectation values, because if we sandwich the time-ordered product between vacuum states, that is $\langle 0|:\ldots:|0\rangle$, only the terms with maximal contractions survive. Sandwiching Eq. (5.3.23) from the previous example for the time-ordered product of fields we obtain

$$\begin{aligned} \langle 0|T[\phi_1\phi_2\phi_3\phi_4]|0\rangle =& (i\Delta_F(x_1-x_2))(i\Delta_F(x_3-x_2)) \\ &+ (i\Delta_F(x_1-x_3))(i\Delta_F(x_2-x_4)) \\ &+ (i\Delta_F(x_1-x_4))(i\Delta(x_2-x_3)) \end{aligned}$$



(5.3.24)

These are our first *Feynman diagrams* in coordinate space.

This, however, is not really what we need. What we are really after are interaction Lagrangians at equal times

$$T[:\mathcal{L}_{\text{int}}(x_1):\ldots:\mathcal{L}_{\text{int}}(x_m):] = T[:(\Phi_1\ldots\Phi_m)(x_1):\ldots:(\Phi_1\ldots\Phi_m)(x_m):]. \tag{5.3.25}$$

One can modify Wick's theorem to this situation, because one can show that

$$\begin{aligned} &T[:(\Phi_1\ldots\Phi_m)(x_1):\ldots:(\Phi_1\ldots\Phi_m)(x_m):] \\ &= T[(\Phi_1\ldots\Phi_m)(x_1)\ldots(\Phi_1\ldots\Phi_m)(x_m)]_{\text{no equal-time contraction}}, \end{aligned} \tag{5.3.26}$$

where we do not take into account equal-time interaction. A short argumentation about this can be found in Mandl & Shaw.

Example: Modified Wick's theorem for real fields

Modified Wick theorem for squared fields ϕ at different times

$$\begin{aligned} T[:\phi^2(x_1)::\phi^2(x_2):] &\overset{(5.3.26)}{=} T[\phi^2(x_1)\phi^2(x_2)]_{\text{no equal time contraction}} \\ &\overset{(5.3.22)}{=} :\phi^2(x_1)\phi^2(x_2): \\ &\quad + :\underbrace{\phi(x_1)\phi(x_1)\phi}(x_2)\phi(x_2): + :\phi(x_1)\underbrace{\phi(x_1)\phi(x_2)\phi}(x_2): \\ &\quad + :\phi(x_1)\underbrace{\phi(x_1)\phi}(x_2)\phi(x_2): + :\phi(x_1)\underbrace{\phi(x_1)\phi(x_2)\phi}(x_2): \\ &\quad + :\underbrace{\phi(x_1)\underbrace{\phi(x_1)\phi}(x_2)\phi}(x_2): + :\phi(x_1)\underbrace{\phi(x_1)\underbrace{\phi(x_2)\phi}(x_2)}:, \end{aligned} \tag{5.3.27}$$

where in the second step we applied Wick's theorem. Then for the expectation value in the vacuum state $|0\rangle$ we obtain

$$\begin{aligned} \langle 0|T[:\phi^2(x_1)::\phi^2(x_2):]|0\rangle &= 2\cdot(i\Delta_F(x_1-x_2))^2 \\ &= 2\cdot\ x_1 \quad \text{  } \quad x_2 \end{aligned} \tag{5.3.28}$$

Wick's theorem now tells us how to calculate operators $T[:\mathcal{L}_{\text{int}}:(x_1)\ldots:\mathcal{L}_{\text{int}}:(x_n)]$, but what we want to calculate are matrix elements of these operators between an initial state and a final state

$$\langle f|T[:\mathcal{L}_{\text{int}}(x_1):\ldots:\mathcal{L}_{\text{int}}:]|i\rangle, \tag{5.3.29}$$

where

$$|i\rangle = |\psi_2(-\infty)\rangle\,, \qquad |f\rangle \ \text{ at } \ +\infty, \tag{5.3.30}$$

$$|i\rangle = a^\dagger(p_1)\ldots a^\dagger(p_m)\,|0\rangle\,, \quad |f\rangle = a^\dagger(k_1)\ldots a^\dagger(k_m), \quad \langle f| = \langle 0|\,a(k_m)\ldots a(k_1). \tag{5.3.31}$$

As we see, we again made the assumption that both initial and final state are free momentum and energy eigenstates. Then

$$\langle \vec{k}_1,\ldots,\vec{k}_n|T[\ldots]|\vec{p}_1,\ldots,\vec{p}_n\rangle, \tag{5.3.32}$$

where $T[\ldots]$ is after Wick's theorem a sum of normal-ordered fields. One of the terms

that remains, for the case $m = 1$ is

$$\begin{aligned}
&\langle \vec{k}_1| :\phi_1(x_1)\phi_2(x_2): |\vec{p}_1\rangle \\
&= \langle \vec{k}_1|(\phi^-(x_1)\phi^-(x_2) + \phi^+(x_1)\phi^+(x_2) + \phi^-(x_1)\phi^+(x_2) \pm \phi^-(x_2)\phi^+(x_1))|\vec{p}_1\rangle \\
&= \langle \vec{k}_1|(\phi^-(x_1)\phi^+(x_2) \pm \phi^-(x_2)\phi^-(x_1))|\vec{p}_1\rangle, \qquad (5.3.33)
\end{aligned}$$

where the sign depends on whether ϕ fermionic or bosonic. It is useful to know that for $\Phi^+(x)\,|\vec{k}\rangle$ we have

$$\begin{aligned}
\Phi^+(x)\,|\vec{k}\rangle &= \frac{1}{\sqrt{V}} \sum_{\vec{p}} \frac{1}{2E_p} \phi_1(\vec{p}) e^{-ip\cdot x} \underbrace{\hat{a}(\vec{p})\hat{a}^\dagger(\vec{k})}_{\substack{=\delta_{\vec{k},\vec{p}}|0\rangle \pm \hat{a}^\dagger(\vec{k})a(\vec{p})|0\rangle \\ =\delta_{\vec{k},\vec{p}}|0\rangle}} |0\rangle \\
&= \frac{1}{\sqrt{V}} \frac{1}{\sqrt{2E_p}} \phi_1(\vec{k}) e^{-ik\cdot x}\,|0\rangle\,, \qquad (5.3.34)
\end{aligned}$$

or for the adjoint

$$\langle \vec{k}|\,\Phi^-(x) = \frac{1}{\sqrt{V}} \frac{1}{\sqrt{2E_k}} \Phi_1^*(\vec{k}) e^{ik\cdot x}\,\langle 0|\,. \qquad (5.3.35)$$

Now, in principle, we can calculate any S-matrix element $\langle f|\hat{S}|i\rangle$!

Chapter 6

Quantum Electrodynamics & Feynman Diagram

The theory of quantum electrodynmics was mainly developed in the late 40s of the 21st century, with important contributions from Schwinger, Feynman and Dyson. It is one of the best theories available in theoretical physics and a prime example is the calculation and experimental detection of the magnetic moment of the muon. In this lecture we already discussed classical electrodynamics, now we will look at the quantum version of it. In the following we will look at the $\hat{S}$-matrix and we will evaluate the $\hat{S}$-matrix using Wick's theorem order by order, at least for the lowest ones. Moreover, we will formulate diagrammatic representations for mathematical expressions of transition amplitudes, called *Feynman diagrams*. Furthermore, we will see that there is in fact a one-to-one correspondence between the explicit mathematical expressions of the amplitudes and the corresponding diagram, called *Feynman rules*. Once we did this for the first terms of the expansion, we will look at elementary processes in quantum electrodynamics, where one can also compare obtained cross-sections to real data from experiments in order to convince you, that the theory works rather well even in lower orders.

6.1 QED Lagrange Density for Interaction of Light with Matter

Quantum electrodynamics is a theory for the interaction of light and matter, where by matter we mean charged fermion fields (e.g. electron e^- and positron e^+) and by light we mean the massless vectorfield (photon γ). We are coupling the four-potential A_μ to

matter fields in the following way

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - j_\mu A^\mu, \tag{6.1.1}$$

where the first term is the *kinetic term for the electromagnetic field* expressed in terms of the four-potential A^μ and the second term is the *interaction term*, where a four-current j_μ for the matter fields and the four-potential enters. Applying the Euler-Lagrange equations of motion one obtains the inhomogeneous Maxwell equation and a continuity equation

$$\partial_\mu F^{\mu\nu} = j^\nu, \quad \partial_\nu j^\nu = 0. \tag{6.1.2}$$

Remember, when quantizing the electromagnetic fields in terms of the four-potential we had problems with gauge freedom, choice of gauge, arbitraryness as well as unphysical degrees of freedom of the photons as discussed in Chapter 3.6. In order to circumvent these problems we replace the first term of this Lagrangian by

$$\mathcal{L}_{\text{Fermi}} = -\frac{1}{2}(\partial_\nu A_\mu)(\partial^\nu A^\mu), \tag{6.1.3}$$

which we had in Eq. (3.6.7). The coupling to matter fields, which are fermions, goes as follows. First note that the fermions are described by the Dirac Lagrangian for free fermions

$$\mathcal{L}_{\text{Dirac}} = \bar{\psi}(i\partial\!\!\!/ - m)\psi, \tag{6.1.4}$$

where m is the mass of the electron or positron. From the fermions we already know that they fulfill the Dirac equation (4.1.20). In order to use what we know from classical electrodynamics we have to generate a four-current, the electromagnetic current. We can generate such a four-current by using a global phase transformation

$$\psi(x) \to e^{-i\alpha}\psi(x) \tag{6.1.5}$$

and obtain

$$j^\mu = q\bar{\psi}\gamma^\mu\psi, \tag{6.1.6}$$

where q is the electric charge ($q = -e$ for an electron). This current fulfills the continuity equation $\partial_\mu j^\mu = 0$ by Noethers theorem. Finally, we get the *Lagrange density for QED*

$$\mathcal{L}_{\text{QED}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\partial\!\!\!/ - m)\psi - q\bar{\psi}\gamma^\mu A_\mu\psi. \tag{6.1.7}$$

Again the ψ stands for electron, positron and A^μ for photons. The first term of this Lagrangian describes free photon fields, the second term describes free fermion fields

and the last term describes their interaction. Equivalently, one can write the Lagrangian in a more compact form

$$\mathcal{L}_{\text{QED}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\not{D} - m)\psi \tag{6.1.8}$$

using the *covariant derivative* $D_\mu\psi \equiv \partial_\mu\psi + iqA_\mu\psi$. One also calls this *minimal substitution*, recall that this was also done back in non-relativistic quantum mechanics in order to introduce electromagnetic fields into the theory of free particles. The point is that the Lagrangian $\mathcal{L}$ from Eqs. (6.1.7), (6.1.8) is invariant under *local gauge transformation*, by this we mean the following. Recall that from classical electrodynmaics we know that we can redefine the four-potential in such a way that

$$A_\mu \to A_\mu + \partial_\mu\lambda(x), \tag{6.1.9}$$

where λ is an arbitrary function. In the quantum mechanical version we find

$$\psi \to e^{-iq\lambda(x)}\psi. \tag{6.1.10}$$

Here we multiply the spinors with a phase, that depends on space-time. Moreover, note that the function λ is the same as in the gauge transformation of the four-potential. Furthermore, this time-dependence does not give us any new conservation laws, though charge conservation and the continuity equation are still given. The QED Lagrangian (6.1.7) is invariant under such a local gauge transformation since $D_\mu\psi = \partial_\mu\psi + iqA_\mu$ transforms as

$$\begin{aligned}
&\partial_\mu(e^{-iq\lambda(x)}\psi) + iq(A_\mu + \partial_\mu\lambda(x))(e^{-iq\lambda}\psi) \\
&= e^{iq\lambda(x)}(-iq\partial_\mu\lambda(x))\psi + iq(\partial_\mu\lambda(x))e^{-iq\lambda(x)}\psi + e^{-iq\lambda(x)}\partial\psi + e^{-iq\lambda(x)}iqA_\mu\psi \\
&= e^{-iq\lambda(x)}D_\mu\psi,
\end{aligned} \tag{6.1.11}$$

where the first two terms cancel. If also the $\bar{\psi}$ is transformed, that is,

$$\bar{\psi} \to e^{+iq\lambda(x)}\bar{\psi}, \tag{6.1.12}$$

one sees that in fact $\mathcal{L}_{\text{QED}}$ is invariant under local guage transformations such as in Eqs. (6.1.10) and (6.1.9). For $q = -e$ and with the *finestructure constant*

$$\alpha = \frac{e^2}{4\pi\underbrace{\hbar c}_{=1}} \approx \frac{1}{137} \tag{6.1.13}$$

we can solve for the electric charge

$$e = \sqrt{4\pi\alpha} \approx 0.3. \tag{6.1.14}$$

At first our $\hat{S}$ matrix approach is justified in the sense that we have an expansion parameter which is small. Finally, what we need is the interaction Hamiltonian

$$\mathcal{H}_{\text{int}} = -\mathcal{L}_{\text{int}} = -e :\bar{\psi}(x)\gamma^\mu A_\mu(x)\psi(x): \tag{6.1.15}$$

6.2 Feynman Diagrams & Feynman Rules

We discuss the S-matrix

$$\begin{aligned} S &= T\exp\left\{ -i\int d^4x \mathcal{H}_{\text{int}}^{\text{QED}}(x)\right\} \\ &= \sum_{n=0}^{\infty}\frac{(-i)^n}{n!}\int d^4x_1 \ldots \int d^4x_n T\{\mathcal{H}_{\text{int}}(x_1)\ldots\mathcal{H}_{\text{int}}(x_n)\} \\ &= \sum_{n=0}^{\infty} S^{(n)}, \end{aligned} \tag{6.2.1}$$

with $\mathcal{H}_{\text{int}}^{\text{QED}} = -e :\bar{\psi}(x)\gamma^\mu A_\mu(x)\psi(x):$. In the following we will only discuss $S^{(1)}, S^{(2)}$, which is sufficient. We will then introduce a graphical notation, called *Feynman diagrams*.

6.2.1 First Order S-Matrix Expansion

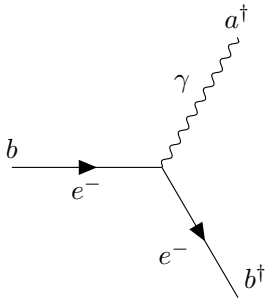
We begin with order $n = 1$, by Wick's theorem and its modified version (cf. Eq. (5.3.22) and Eq. (5.3.26) resepctively) the only contribution we have is

$$S^{(1)} = ie\int d^4x :\bar{\psi}(x)\gamma^\mu A_\mu(x)\psi(x):, \tag{6.2.2}$$

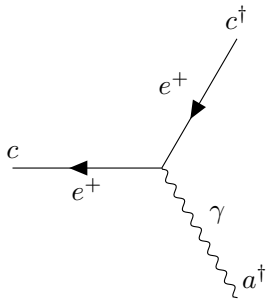
since the fields are at equal time x^0. Here, no contraction is possible. This means that

$$S^{(1)} = ie\int d^4x :(\bar{\psi}^+(x)+\bar{\psi}^-(x))\gamma^\mu(A_\mu^+(x)+A_\mu^-(x))(\psi^+(x)+\psi^-(x)):, \tag{6.2.3}$$

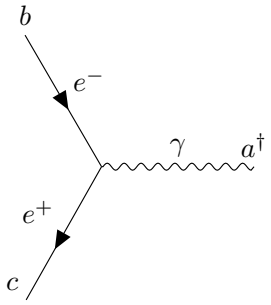
where again $\bar{\psi}^+ \sim c$ (annihilation of a positron e^+), $\bar{\psi}^- \sim b^\dagger$ (creation of an electron e^-), $A_\mu^+ \sim a$ (annihilation of a photon γ), $A_\mu^+ \sim a^\dagger$ (creation of a photon γ), $\psi^+ \sim b$ (annihilation of an electron e^-) and $\psi^- \sim c^\dagger$ (creation of a positron e^+). This describes 8 processes, some of them are diagrammtically depicted in Fig. 6.1.



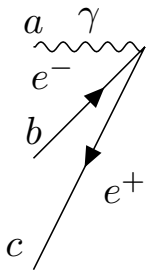
(a) $b^\dagger a^\dagger b$



(b) $a^\dagger cb$



(c) $c^\dagger ac$



(d) cba

Fig. 6.1: Four out of the 8 fundamental Feynman diagrams.

These processes are called the *fundamental vertices* of QED, because all other processes or diagrams are combinations of these fundamental vertices. However, for free fields (particles) all these eight processes are kinematically forbidden! Consider now a particular process Fig. 6.2 in order to learn more about the structure[1]. The initial state

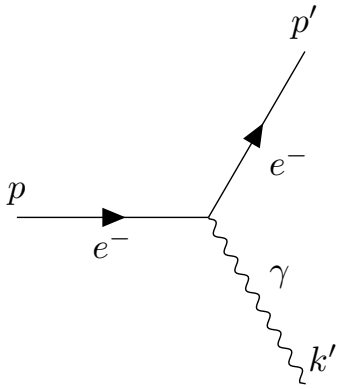


Fig. 6.2: Example for an electron e^- with four-momentum p and energy $p^0 = \sqrt{m^2 + \vec{p}^2}$, that emits a photon γ with four-momentum k' and thus has a final momentum $p' = p - k'$.

is a single particle state $|i\rangle = b^\dagger_{\vec{p}}|0\rangle = |e^-, \vec{p}\rangle$ and the final state is two-particle state $|f\rangle = b^\dagger_{\vec{p}} b^\dagger_{\vec{k}'}|0\rangle = \ |e^-, \vec{p}\,'; \gamma, \vec{k}\,'\rangle$. The S-matrix element is then given as

$$\begin{aligned}\langle f|S^{(1)}|i\rangle &= \langle f|ie\int d^4x \bar{\psi}^-(x)\gamma^\mu A^-_\mu(x)\psi^+(x)|i\rangle \\ &= ie\int d^4x \left\{\frac{1}{\sqrt{2VE_{p'}}}\bar{u}(\vec{p}\,')e^{ip'\cdot x}\right\}\gamma_\mu\left\{\frac{1}{\sqrt{2V\omega_{k'}}}\varepsilon^\mu_{\vec{k}'}e^{ik'\cdot x}\right\}\left\{\frac{1}{\sqrt{2VE_p}}u(\vec{p})e^{-ip\cdot x}\right\},\end{aligned} \tag{6.2.4}$$

with

$$\langle f|b^\dagger_{\vec{p}_3}a^\dagger_{\vec{p}_2}b_{\vec{p}_1}|i\rangle = \langle 0|a_{\vec{k}'}b_{\vec{p}'}b^\dagger_{\vec{p}_3}a^\dagger_{\vec{p}_2}\underbrace{b_{\vec{p}_1}b^\dagger_{\vec{p}}}_{=-b^\dagger_{\vec{p}}b_{\vec{p}_1}+\delta_{\vec{p},\vec{p}_1}}|0\rangle = \delta_{\vec{k}',\vec{p}_2}\delta_{\vec{p}',\vec{p}_3}\delta_{\vec{p},\vec{p}_1}. \tag{6.2.5}$$

Integration of the matrix element in Eq. (6.2.4) is now trivial, because the x-dependence is only contained in the exponentials, it is

$$\int \mathrm{d}^4x \exp\{ix\cdot(p'+k'-p)\} = (2\pi)^4\cdot\delta^{(4)}(p'+k'-p) \tag{6.2.6}$$

[1]In the following, spin and polarization indices are suppressed, we write $\varepsilon^p_\mu \to \varepsilon_\mu$, $v(\vec{p},s) \to v(\vec{p})$.

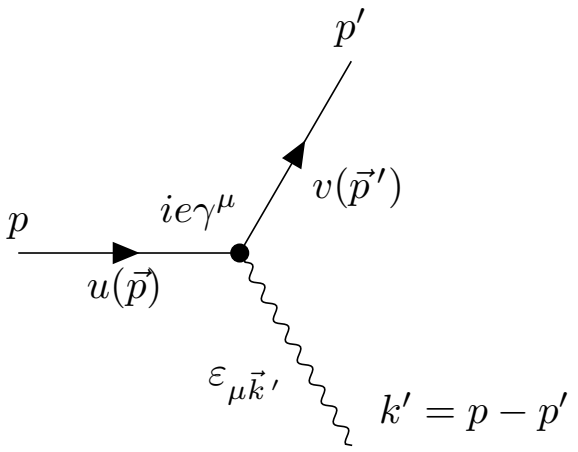


Fig. 6.3: Diagrammatic representation of the Feynman amplitude from Eq. (6.2.8).

by definition of the Dirac-Delta distribution. What we obtain is

$$\langle f|S^{(1)}|i\rangle = (2\pi)^4\delta^{(4)}(p'+k'-p)\left(\frac{1}{\sqrt{2VE_{p'}}}\right)^{1/2}\left(\frac{1}{\sqrt{2V\omega_{k'}}}\right)^{1/2}\left(\frac{1}{\sqrt{2VE_p}}\right)^{1/2}\mathcal{M}, \tag{6.2.7}$$

where the Dirac-Delta distribution $\delta^{(4)}(p'+k'-p)$ is synonymous to energy and momentum conservation $p = p' + k'$. The *Feynman amplitude* is denoted by $\mathcal{M}$, it reads

$$\mathcal{M} = ie\bar{u}(\vec{p}\,')\gamma^\mu\varepsilon_{\mu,\vec{k}\,'=\vec{p}-\vec{p}\,'}u(\vec{p}) \tag{6.2.8}$$

in momentum space. Now, $\vec{k}\,' = \vec{p} - \vec{p}\,'$ is synonymous to the conservation of 3-momentum. The corresponding Feynman diagram is depicted in Fig. 6.3. This is actually not a physical process as it is kinematically not allowed. To see this set $\vec{p} = 0$ and $\vec{p}\,' = -\vec{k}\,'$, then use $p = p' + k'$ and square it $p^2 = (p'+k')^2$. We then have

$$m_e^2 = p'^2 + k'^2 + 2p'\cdot k' = m_e^2 + 0 + 2p'\cdot k' \tag{6.2.9}$$

and thus $p'\cdot k' = 0$, otherwise it does not work. This means that $E_{\vec{p}\,'}k'^0 - \vec{p}\,'\cdot\vec{k}\,' = 0 = E_{\vec{p}\,'}k'^0 + \vec{k}'^2$ and with $k'^0 = |\vec{k}\,'|$ we obtain

$$E_{\vec{p}\,'} + k'^0 = 0. \tag{6.2.10}$$

It is not possible to fulfill this equality. Hence, the process where a single free electron radiates off a photon is not physical, because energy conservation cannot be fulfilled. We could also consider the process depicted in Fig. 6.4. The difference to the previous process is that we replace

$$\bar{\psi}^-(x) \to \bar{\psi}^+(x) = (\ldots)^{1/2}c\bar{v}(\vec{p}\,')e^{-ip'\cdot x}, \tag{6.2.11}$$

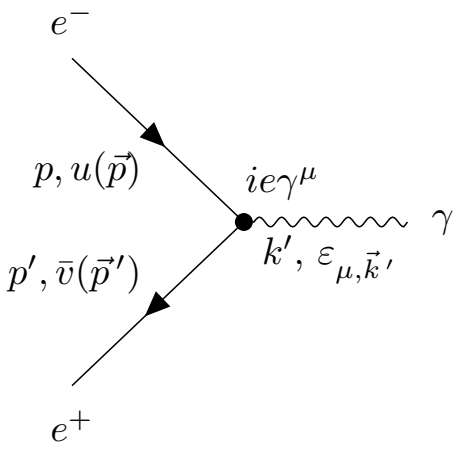


Fig. 6.4: Electron e^- with momentum p and positron e^+ with momentum p' annihilate and absorb a photon γ with momentum k'.

which yields

$$\langle f|S^{(1)}|i\rangle = (2\pi)^4\delta^{(4)}(\vec{k}' - p - p')(\ldots)^{1/2}(\ldots)^{1/2}(\ldots)^{1/2}\mathcal{M}, \tag{6.2.12}$$

with the Feynman amplitude

$$\mathcal{M} = ie\bar{v}(\vec{p}')\not{\varepsilon}_{\vec{k}'=\vec{p}+\vec{p}'}u(\vec{p}). \tag{6.2.13}$$

This amplitude can be easily obtained from the diagram Fig. 6.4 if one follows the fermion line in direction of the arrows and writes *spinorial factors* from right to left.

6.2.2 Second Order S-Matrix Expansion

We now consider the second order $n = 2$, the only contribution is

$$S^{(2)} = \frac{-e^2}{2!}\int d^4x_1 \int d^4x_2 T\{:\bar{\psi}\not{A}\psi(x_1)::\bar{\psi}\not{A}\psi(x_2):\}. \tag{6.2.14}$$

We now can use Wick's theorem to rewrite the time ordered product into a term with no contractions

$$:\bar{\psi}(1)\not{A}(1)\psi(1)\bar{\psi}(2)\not{A}(2)\psi(2):, \tag{6.2.15}$$

where $x_1 \equiv 1$. One of the corresponding Feynman diagrams is given in Fig. 6.5. These are $2S^{(1)}$ processes independent from each other and again unphysical. The next terms are those with one contraction, where the only contractions that are non-vanishing are the ones between ψ and $\bar{\psi}$, so we have the following two terms

$$:\bar{\psi}(1)\not{A}(1)\underbrace{\psi(1)\bar{\psi}}(2)\not{A}(2)\psi(2): + :\underbrace{\bar{\psi}(1)\not{A}(1)\psi(1)\bar{\psi}(2)\not{A}(2)\psi}(2): . \tag{6.2.16}$$

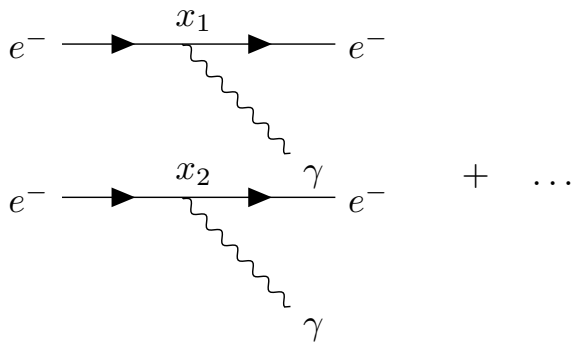


Fig. 6.5: Feynman diagram of the second order S-matrix term without any contractions.

In fact, both of them yield identical contributions, which can be seen when interchanging the labels 1 and 2, pulling the contractions out of the normal ordering without applying the commutator or anti-commutator and integrating. This means that the the second order term with one contraction is

$$S_B^{(2)} = -e^2 \int d^4x_1 \int d^4x_2 \, :(\bar{\psi} \not{A} \psi)_{x_1} (\bar{\psi} \not{A} \psi)_{x_2} :, \tag{6.2.17}$$

where B denotes the process. Recall that ψ contains both an annihilation operator b of an electron and a creation operator c of a positrons. The $\not{A}$ contains an annihilation and creation operator of a photon, denoted with $a, a^\dagger$ resepctively. The $\bar{\psi}$ contains both an annihilation operator c of a positron and a creation operator $b^\dagger$ of an electron. Hidden in this expression are thus $(2 \times 2) \times (2 \times 2) = 16$ terms, one of them being Fig. 6.6a.

Remember, that the contraction of two fields gives a Feynman propagator. For the contraction of two fermion fields we get the propagation of a fermion from space-time point x_1 to x_2, that is, $\psi_a(1)\bar{\psi}_b(2) = iS_{Fab}(x_1 - x_2)$. Diagrammatically this means that we have a photon and an electron which annihilate at x_2, this is where the interaction takes place, then we have a propagator relating x_2 with x_1. At x_1 we create a corresponding electron and a photon. The intermediate line in Fig. 6.6a stands for the *fermion propagator*, relating the two points x_2 and x_1. Note that *all* time-orderings are possible, that is, there is no preferred time ordering for which one of the two interaction vetices happens earlier. The second term out of the 16 terms is shown in Fig. 6.6b. These two contributions are called *Compton scattering*, which refers to the process $e^-\gamma \to e^-\gamma$. Another possibility of a physical process is $e^-e^+ \to 2\gamma$, which can be obtained from Fig. 6.6a by flipping the electron line on the right side, to the left and similarly we flip the photon line from the left side to the right side. By doing the reverse one obtains $\gamma\gamma \to e^+e^-$. Physical processes are the ones where $|i\rangle$ and $|f\rangle$ contain 2 particles each.

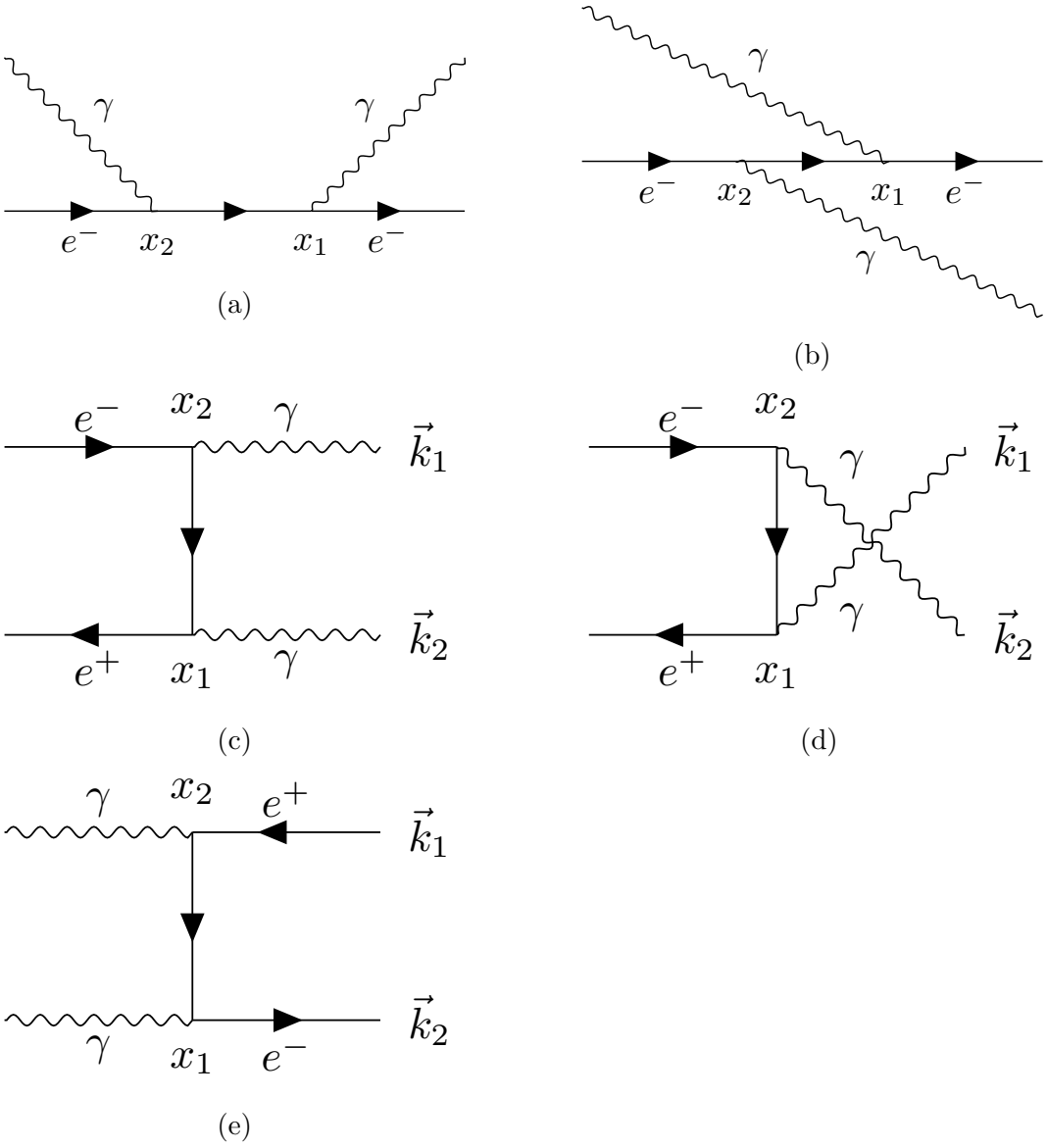


Fig. 6.6: Feynman diagrams of second order S-matrix term with one contraction. The first two Feynman diagrams 6.6a and 6.6b correspond to the process $e^-\gamma \to e^-\gamma$, called *Compton scattering*. In the second row (see diagrams 6.6c and 6.6d) the process $e^-e^+ \to 2\gamma$ is depicted, it is to say that in the two pictures only the momenta of the photons in the final state is interchanged. The process can also be reversed in the sense that we have $\gamma\gamma \to e^+e^-$, this is shown in diagram 6.6e.

By looking at one contraction of the fermions we found three processes. We now want to take a closer look at the diagrams Fig.6.6a and 6.6b, which represent the Compton scattering process $e^-\gamma \to e^-\gamma$. We use

$$\underbrace{\psi(x_1)\bar{\psi}}(x_2) = iS_F(x_1 - x_2) = \frac{1}{(2\pi)^4}\int d^4q e^{-iq(x_1-x_2)} \underbrace{\frac{i(\not{q}+m)}{q^2 - m^2 + i\varepsilon}}_{=iS_F(q)} \tag{6.2.18}$$

in order to obtain

$$S_a = -e^2 \int d^4x_1 \int d^4x_2 \bar{\psi}^-(x_1)\gamma^\alpha iS_F(x_1 - x_2)\gamma^\beta A^-_\alpha(x_1)A^+_\beta(x_2)\psi^+(x_2), \tag{6.2.19}$$

$$S_b = e^2 \int d^4x_1 \int d^4x_2 \bar{\psi}^-(x_1)\gamma^\alpha iS_F(x_1 - x_2)\gamma^\beta A^-_\beta(x_2)A^+_\alpha(x_1)\psi^+(x_2), \tag{6.2.20}$$

from Eq. (6.2.17). The initial state of the Compton process is $|i\rangle = b^\dagger_{\vec{p}} a^\dagger_{\vec{k}} |0\rangle \, |e^-, \vec{p}; \gamma, \vec{k}\rangle$ and the final state is $|f\rangle = b^\dagger_{\vec{p}'} a^\dagger_{\vec{k}'} |0\rangle = |e^- \vec{p}'; \gamma, \vec{k}\,'\rangle$. Note again that we still suppress spin and polarization indices. We are now interested in the transition matrix $\langle f|S_a|i\rangle$. Proceeding as in Chapter 6.2.1 yields

$$\begin{aligned}\langle f|S_a|i\rangle = &\left(\frac{1}{2VE_{\vec{p}'}}\right)^{1/2} \cdot \ldots \cdot \left(\ldots\right)^{1/2} \cdot (-e^2) \cdot \int d^4x_1 \int d^4x_2 \bar{u}(\vec{p}')e^{ip'\cdot x_1} \not{\epsilon}_{\vec{k}'} e^{ik'\cdot x_1} \\ &\cdot \frac{1}{(2\pi)^4}\int d^4q e^{-iq\cdot(x_1-x_2)} \frac{i\not{q}+m}{q^2 - m^2 + i\varepsilon} \not{\epsilon}_{\vec{k}} e^{-ik\cdot x_2} u(\vec{p}) e^{-ip\cdot x_2}.\end{aligned} \tag{6.2.21}$$

Again, the dependence on the 4-vertex x_i only appears in the exponentials, so the integrals result in

$$\begin{aligned}\int d^4x_1 \int d^4x_2 e^{ix_1(p'+k'-q)} e^{ix_1\cdot(q-p-k)} &= (2\pi)^4\delta^{(4)}(p' + k' - q)(2\pi)^4\delta^{(4)}(q - p - k) \\ &= (2\pi)^4\delta^{(4)}(p' + k' - p - k)(2\pi)^4\delta^{(4)}(q - p + k),\end{aligned} \tag{6.2.22}$$

where q is the 4-momentum of the intermediate electron / positron state with $q = p+k = p'+k'$. Moreover, the two δ-distributions in the first line relate to the energy-momentum conservation at each vertex. In the second line on the other hand the δ-distributions are rewritten so that $\delta^{(4)}(p' + k' - p - k)$ represents the momentum conservation of the initial and final state, whereas $\delta^{(4)}(q - p + k)$ is related to the 4-momentum of the intermediate state. Our final result thus is

$$\langle f|S_a|i\rangle = (2\pi)^4\delta^{(4)}(p' + k' - p - k) \cdot (\ldots)^{1/2} \cdot \ldots \cdot (\ldots)^{1/2} \cdot \mathcal{M}_a, \tag{6.2.23}$$

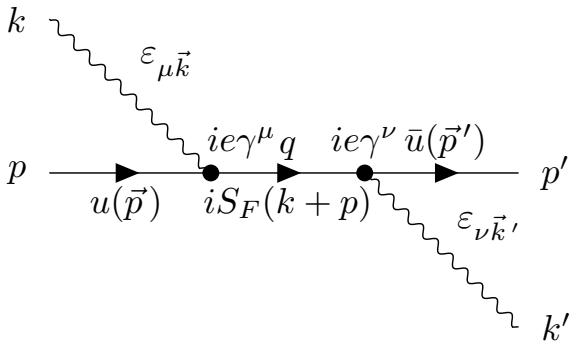


Fig. 6.7: Feynman diagram of second order S-matrix term with one contraction of the fermion fields. This is a diagramatic representation of the expression we derived for the Feynman amplitude $\mathcal{M}_a$.

with

$$\mathcal{M}_a = -e^2 \bar{u}(\vec{p}') \not{\varepsilon}_{\vec{k}'} i S_F(q = p + k) \not{\varepsilon}_{\vec{k}'} u(\vec{p}). \tag{6.2.24}$$

This expression can now be used to apply the diagrammatic representation, see Fig. 6.7. We have 4-momentum conservation at each vertex and $q^2 \neq m^2$, where q is the momentum of the intermediate state and m the mass of the incoming electron. Such a particle representing an intermediate state is called *virtual particle*. These particles are *off-shell* and thus do not fulfill Einstein's relativistic energy momentum relation for free particles $E^2 = \vec{p}^2 + m^2$. More importantly, they are *not* physical particles which can be experimentally detected. The second important message here is that one follows the direction of the fermion line, which means that we write the spinorial factors from the right side to the left side. The same procedure can be done for Fig. 6.6b, it is the same just with interchanged indices. We obtain

$$\mathcal{M}_b = -e^2 \bar{u}(\vec{p}') \not{\varepsilon}_{\vec{k}'} i S_F(q = p - k') \not{\varepsilon}_{\vec{k}'} u(\vec{p}) \tag{6.2.25}$$

and diagrammatically this corresponds to Fig. 6.8. If we can do Compton scattering of electrons, then we can also do Compton scattering of positrons. The situation is the following, we have the same graph, but the direction of the fermion line is reversed, see Fig. 6.9. The Feynman amplitude reads

$$M'_a = e^2 \bar{v}(\vec{p}) \not{\varepsilon}_{\vec{k}} i S_F(q = -p - k) \not{\varepsilon}_{\vec{k}'} v(\vec{p}'). \tag{6.2.26}$$

Again, one follows the fermion line, but note that since we are now dealing with antiparticles the final state corresponds to the right side of the mathematical expression.

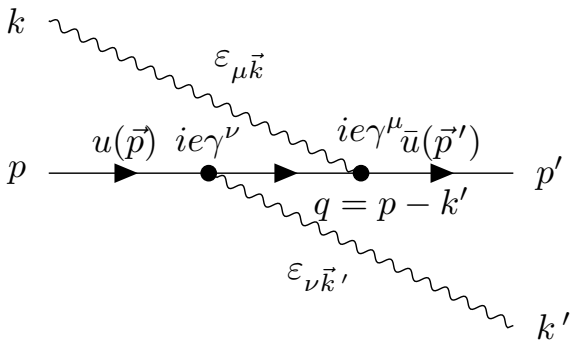


Fig. 6.8: Feynman diagram of the second order S-matrix term with one contraction of the fermion fields. This is a diagrammatic representation of the expression we derived for the Feynman amplitude $\mathcal{M}_b$.

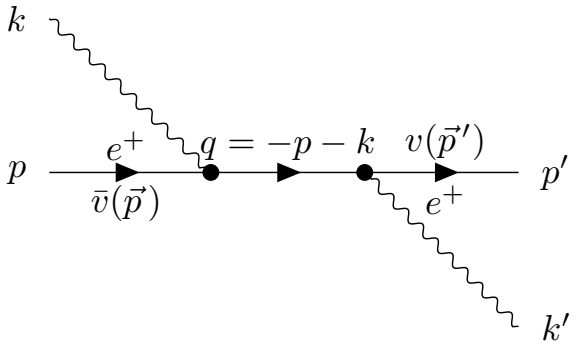


Fig. 6.9: Feynman diagram of the second order S-matrix term with one contraction of the fermion fields. This is a diagrammatic representation of the expression we derived for the Feynman amplitude $\mathcal{M}'_a$.

Before the final state of the process was on the left hand side of the mathematical expression. A momentum p of an external line stands for the physical 4-momentum. The flow of 4-momentum of electrons is in the same direction as that of the arrows of the corresponding external line. For positrons on the other hand the flow of 4-momentum is in the opposite direction as that of the arrows of the corresponding external line. For internal lines the flow of 4-momentum is always in the direction of the arrow, because the propagator depends on how we choose the sign of q, since it is in the numerator of

$$S_I(q) = \frac{\not{q} + m}{q^2 - m^2 + i\varepsilon}. \tag{6.2.27}$$

Furthermore, the 4-momentum at a vertex, which is the sum of all momenta of lines at that vertex, must add up to 0.

We can also consider the following term in second order of the expansion with

$$:(\bar{\psi}\not{A}\psi)_{x_1}(\bar{\psi}\not{A}\psi)_{x_2}: . \tag{6.2.28}$$

One of the 16 possible processes is depicted in the diagrammatic language in Fig. 6.10. The propagator we need is the one we first had in Eq. (3.6.44) or later in Eq. (5.3.19)

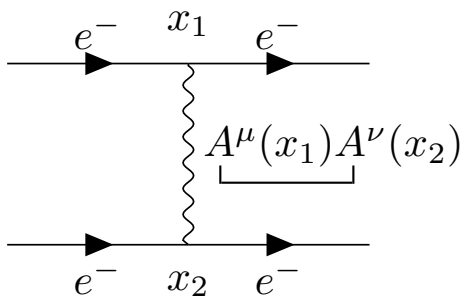


Fig. 6.10: Feynman diagram of the second order S-matrix term with one contraction of the photon field. This is a diagramatic representation of Mϕller scattering for electrons, that is, $e^-e^- \to e^-e^-$.

with the Wick contraction. It reads

$$\underbrace{A^\mu(x_1)A^\nu}(x_2) = iD_F^{\mu\nu}(x_1 - x_2) = \frac{i}{(2\pi)^4}\int d^4k e^{-ik\cdot(x_1-x_2)}\frac{-g^{\mu\nu}}{k^2 + i\varepsilon} = D_F^{\mu\nu}(k). \tag{6.2.29}$$

The process $e^-e^- \to e^-e^-$ is called *Møller scattering*. We could also consider the processes depicted in Fig. 6.11. The process $e^-e^+ \to e^-e^+$ and its two contributions

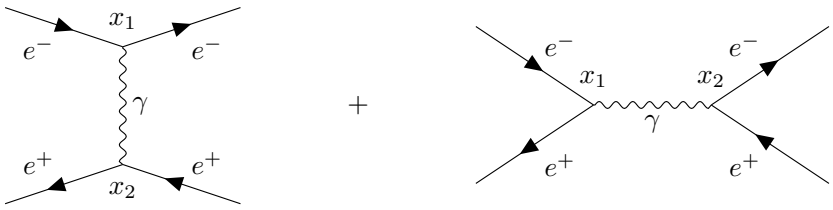


Fig. 6.11: Feynman diagrams of the second order S-matrix term with one contraction of the photon field. This is a diagramatic representation of the two contributions for Bhaba scattering $e^-e^+ \to e^-e^+$.

are called *Bhabha scattering*. Another process is $e^+e^+ \to e^+e^+$, which is exactly the same as for the Mϕller process except that we have positrons. There are again multiple contributions, but all other than the one depicted in Fig. 6.12 are kinematically not allowed.

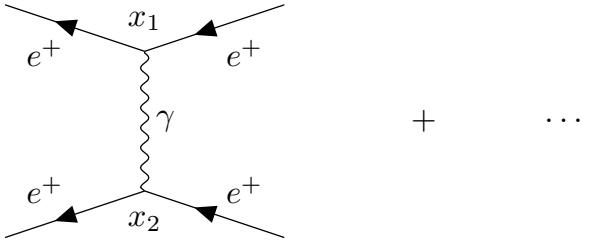


Fig. 6.12: Feynman diagram of second order S-matrix term with one contraction of the photon field. This is a diagramatic representation of the expression for the process $e^+e^+ \to e^+e^+$.

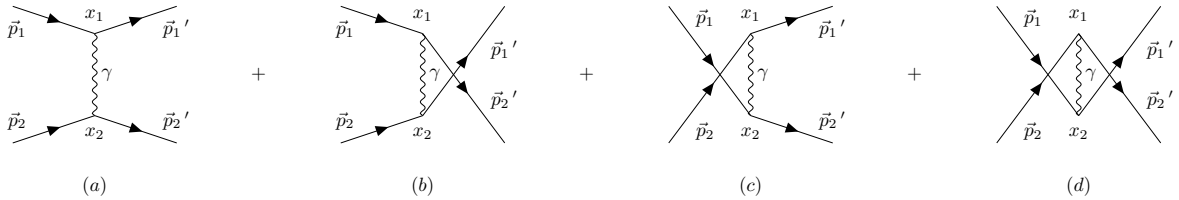


Fig. 6.13: The four-contributions of the Mϕller scattering where (c) and (d) are the same as (a) and (b).

Let us take a closer look at Mϕller scattering, in this case the initial and final state are

$$|i\rangle = |e^-, \vec{p}_1; e^-, \vec{p}_2\rangle\,, \qquad |f\rangle = |e^-, \vec{p}_1\,'; e^-, \vec{p}_2\,'\rangle\,, \tag{6.2.30}$$

and the S-matrix element is

$$\langle f|S^{(2)}|i\rangle = -\frac{e^2}{2}\langle f|\int d^4x_1 \int d^4x_2 :(\bar{\psi}^-\gamma^\alpha\psi^+)_{x_1}(\bar{\psi}^-\gamma^\beta\psi^+)_{x_2}: \cdot\, iD_{F\alpha\beta}(x_1-x_2)|i\rangle. \tag{6.2.31}$$

The four-contributions can graphically be written as in Fig. 6.13. We can see that the last two contributions in Fig. 6.13 are topologically equivalent to the first two, as only the labels are interchanged. Thus, one only has to write down topologically inequivalent Feynman diagrams. This is why we obtain an additional factor 2, which cancels with the factor 2! from the S-matrix expansion.

Creation and annihilation operators should be in the correct order. For example consider the following order of operators $b^\dagger_{\vec{p}_2\,'}, b^\dagger_{\vec{p}_1\,'}, b_{\vec{p}_1} b_{\vec{p}_2}$, then the operators for the two contributions (a) and (b) in Fig. 6.13 are

$$-\bar{\psi}^-_{\vec{p}_1\,'}(x_1)\bar{\psi}^-_{\vec{p}_2\,'}(x_2)\psi^+_{\vec{p}_1}(x_1)\psi^+_{\vec{p}_2}(x_2) \tag{6.2.32}$$

and

$$+\bar{\psi}^-_{\vec{p}_1\,'}(x_2)\bar{\psi}^-_{\vec{p}_2\,'}(x_1)\psi^+_{\vec{p}_1}(x_1)\psi^+_{\vec{p}_2}(x_2). \tag{6.2.33}$$

Note that the additional minus sign in the second expression comes from interchanging two fermionic field operators. Now as we see both contributions (a) and (b) have a relative minus sign and thus are an "*antisymmetric wave function*" of the final state.

Example: Feynman amplitude of $e^-e^- \to e^-e^-$

Consider now the Feynman diagram shown in Fig. 6.14. One could give the propagator a direction, in fact one can give it a direction and one should! But the orientation does not matter, because the propagator does not depend on it, that is, $D_{F\alpha\beta(k)} = D_{F\alpha\beta}(-k)$ with $k = p_2 - p_2'$ from 4-momentum conservation at the lower vertex. For the first contribution (a) the transition amplitude is

$$\langle f|S^{(2)}_{e^-e^-\to e^-e^-}|i\rangle = (2\pi)^4\delta^{(4)}(p_1' + p_2' - p_1 - p_2)\cdot\Pi(\ldots)^{1/2}[\mathcal{M}_a + \mathcal{M}_b], \quad (6.2.34)$$

with the Feynman amplitude

$$\mathcal{M}_a = -e^2\bar{u}(\vec{p}_1\,')\gamma^\alpha u(\vec{p}_1)iD_{F\alpha\beta}(k = p_2 - p_2')\bar{u}(\vec{p}_2\,')(\gamma^\beta)u(\vec{p}_2). \quad (6.2.35)$$

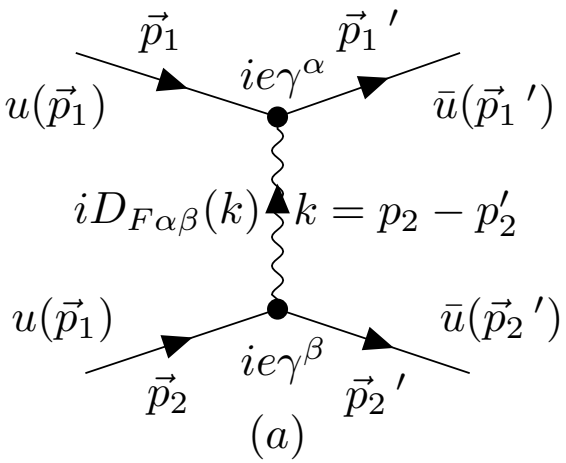


Fig. 6.14: First contribution of Møller scattering where the photon propagator has a direction given and from which one can conclude 4-momentum conservation at the vertices. The direction is irrelevant, because it does not matter if we plug in k or $-k$ into the mathematical expression Eq. (6.2.29).

We now consider non-vanishing terms with two contractions where we contract two photon fields and two fermion fields, that is,

$$:(\bar{\psi}\not{A}\psi)_{x_1}(\bar{\psi}\not{A}\psi)_{x_2}: + :(\bar{\psi}\not{A}\psi)_{x_1}(\bar{\psi}\not{A}\psi)_{x_2}: . \quad (6.2.36)$$

Again, both terms give identical contributions when we include the integrations and interchange x_1 and x_2 thus yielding a factor of 2. The diagram Fig. 6.15 is a modification

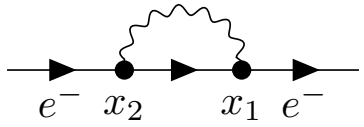


Fig. 6.15: Feynman diagram for Eq. (6.2.36) in coordinate space. It is the first diagram we discuss which contains a loop. Such a diagram is therefore called *loop diagram*.

of the *bare* electron, which yields the *physical electron*. We see that an electron can emit a photon and re-absorbs it. It creates its own electromagnetic field! As we see an electron can self-interact. We conclude that the real physical electron is surrounded by a photon cloud. Note that this process is in fact kinematically possible! This process has an influence on the energy or equivalently on the mass of the particle. When one considers such a process, the mass of the particle will change. The difference between the initial mass of the particle and the final mass is called *self-energy* of the particle (see later when we discuss radiative corrections). We will skip the calculation for deriving the Feynman amplitude of this process, we will only sketch it. When one evaluates Eq. (6.2.36) one has two integrations involving x_1 and x_2. The integration of these gives two delta-distributions. One is used for overall momentum conservation, the other one is used to fix the momentum of the internal line. Remember, that every propagator in coordinate space comes with an extra integration, since we move to momentum space where we have our simple expressions for the propagator. In total we have four-integrations, the integrals of x_1 and x_2 can be performed easily, because only the exponentials are position dependent. As mentioned before they yield two delta-distributions. One is needed and the other is used in order to evaluate one of the two remaining integrations, therefore one integral is left. The transition amplitude is given by

$$\langle f|S^{(2)}_{e^-\to e^-}|i\rangle = (2\pi)^4\delta^{(4)}(p-p')(\ldots)^{1/2}(\ldots)^{1/2}\mathcal{M} \tag{6.2.37}$$

and the Feynman amplitude is

$$\mathcal{M} = \frac{e^2}{(2\pi)^4}\int d^4k i D_{F\alpha\beta}(k)\bar{u}(\vec{p})\gamma^\beta i S_F(p-k)\gamma^\alpha u(\vec{p}), \tag{6.2.38}$$

where

$$D_{F\alpha\beta}(k) \sim \frac{-g_{\alpha\beta}}{k^2+i\varepsilon}, \qquad S_F(p-k) \sim \frac{\not{p}-\not{k}+m}{(p-k)^2-m^2+i\varepsilon}. \tag{6.2.39}$$

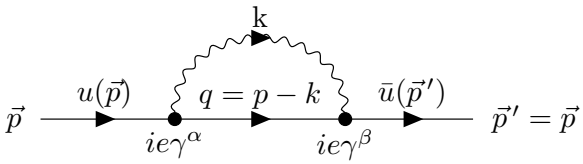


Fig. 6.16: Feynman diagram for Eq. (6.2.36) in momentum space.

The corresponding diagram in momentum space is depicted in Fig. 6.16. We have 4-momentum conservation at both vertices as well as an additional integral $\frac{1}{(2\pi)^4}\int d^4k$ for the internal momentum k. A diagram of this form is called *loop diagram*.

Next let us consider the non-vanishing terms where we only contract the fermion fields but not the photon field

$$:(\bar{\psi}\not{A}\psi)_{x_1}(\bar{\psi}\not{A}\psi)_{x_2}: . \tag{6.2.40}$$

This corresponds to the diagram Fig. 6.17. This process is called *photon self-energy* or

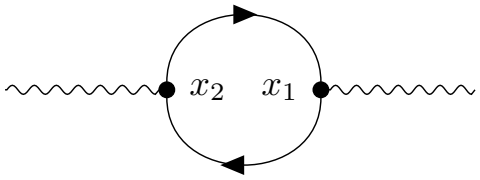


Fig. 6.17: Feynman diagram for Eq. (6.2.40) in coordinate space. We have a electron-positron pair in this loop that decays back into a photon. The creation and annihilation of this virtual pair is a quantum fluctuation.

vacuum polarisation. The physical interpretation is that a photon decays and creates a positron-electron pair which annihilates again and emits a photon, a *quantum fluctuation*. In a sense, the photon is able to excite virtual quantum fluctuations in terms of charged particles and antiparticles. The new feature here is that what we have is a *fermion-loop* and not a photon-loop as previously. The same diagram but in momentum space is shown in Fig. 6.18. For the Feynman amplitude of the fermion loop we get

$$\mathcal{M} = -\frac{e^2}{(2\pi)^4}\int d^4p\ \mathrm{tr}\{\not{\epsilon}_k S_F(p+k)\not{\epsilon}_k S_F(p)\}. \tag{6.2.41}$$

What we learn here is that if we have a fermion loop, we take the trace and multiply with a factor of (-1).

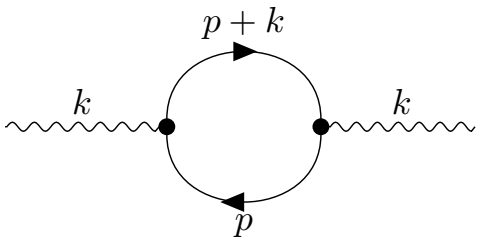


Fig. 6.18: Feynman diagram for Eq. (6.2.40) in momentum space.

Next we consider terms with three contractions

$$:(\bar{\psi}\not{A}\psi)_{x_1}(\bar{\psi}\not{A}\psi)_{x_2}: . \tag{6.2.42}$$

For this term we have the the Feynman diagram depicted in Fig. 6.19. We call such

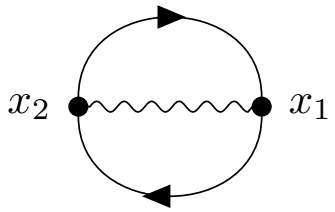


Fig. 6.19: Feynman diagram for Eq. (6.2.42) in coordinate space.

processes *vacuum fluctation*. These terms yield no contribution to the relevant S-matrix elements, because to the inital and final vertex we cannot add any physical particles. There are no new properties concering the 1-1-correspondence between the Feynman amplitudes and the corresponding diagrams. The same holds for higher orders of the expansion. We will not find any new rules. Therefore we will now summarize all the rules we know.

6.2.3 Feynman Rules of QED

As we have seen all the calculations leading to the final Feynman amplitudes using Wick's theorem are very tedious. We therefore draw Feynman digrams in order to represent the expansion of $\langle f|S|i\rangle$. Yet, one should be careful when interpreting these, since they do not reflect nature perfectly but only help us avoid writing out all the terms. What we do is, we

- draw Feynman diagrams

- draw external lines[2] for each particle in $|i\rangle$ and $|f\rangle$, draw internal lines for each propagator [3], associate momenta (spin projection, polarization etc.) to each line, add arrows to fermion lines to denote the charge[4], join external lines together with the interaction vertices

- associate numbers (or integrals) with the diagrams.

How we exactly proceed is determined by the so called *Feynman rules*. Consider a transition $|i\rangle \to |f\rangle$, the S-matrix element is

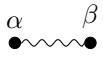
$$\langle f|S|i\rangle = \delta_{f,i} + \left\{(2\pi)^4\delta^{(4)}(P_f - P_i)\prod_{\text{ext.}}\left(\frac{1}{2VE_{\text{ext.}}}\right)^{1/2}\right\}\cdot\mathcal{M}, \tag{6.2.43}$$

where $P_f = \sum_j p_j^f$ and $P_i = \sum_j p_j^i$. The Feynman amplitude is given by

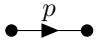
$$\mathcal{M} = \sum_{n=1}^{\infty}\mathcal{M}^{(n)}, \tag{6.2.44}$$

where $\mathcal{M}^n$ is the Feynman amplitude of n-th order in perturbation theory, it includes all *topologically different*, *connected* Feynman diagrams with n vertices and the correct number of external lines. By topologically different we mean that two diagrams A and B are not equivalent in the sense that we cannot rearrange lines without cutting any connection in order to obtain B from A. By connected we mean that every part of the diagram is connected to an external line. We now summarize all the rules we have for the diagrammatic elements:

1. each vertex: $ie\gamma^\alpha$

2. internal photon line: 

$iD_{F\alpha\beta}(k) = i\frac{-g_{\alpha\beta}}{k^2+i\varepsilon}$

3. internal fermion line: 

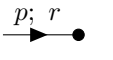
$iS_F(p) = i\frac{\not{p}+m}{p^2-m^2+i\varepsilon}$

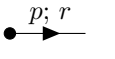
[2]Lines where we attach spinors or polarization vectors.

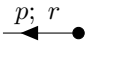
[3]In the internal line all timer orderings are included, so one could have the propagation of a particle *or* an antiparticle.

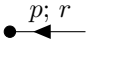
[4]In direction of time for particle and in opposite direction for antiparticle

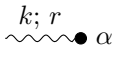
4. external lines:

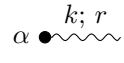
e^- in $|i\rangle$:  $u(\vec{p},r)$

e^- in $|f\rangle$:  $\bar{u}(\vec{p},r)$

e^+ in $|i\rangle$:  $\bar{v}(\vec{p},r)$

e^+ in $|f\rangle$:  $v(\vec{p},r)$

γ in $|i\rangle$:  $\varepsilon_{r\alpha}(\vec{k})$

γ in $|f\rangle$:  $\varepsilon^{(*)}_{r\alpha}(\vec{k})$

5. follow the direction of arrows in fermion lines: ordering of spinor factors $(\gamma, S_F, u, v, \ldots)$ from right to left,

6. fermion loop: take trace Tr and multiply the whole amplitude by a factor of -1,

7. 4-momentum conservation at each vertex. For each internal q (not fixed by momentum conservation), factor $\frac{1}{(2\pi)^4}\int d^4q$,

8. multiplication by $S_p = \pm 1$ (phase factor) for even (odd) interchange of neighboring boson (fermion) operators (see $e^-e^- \to e^-e^-$ or $e^+e^- \to e^+e^-$).

6.3 QED and Leptons

What we have discussed up to now is the interaction between three types of particles e^-, e^+ and γ. However, in the context of QED there are also other charged particles such as hadrons. These are composite particles and examples of such are protons or neutrons. The proton carries one elementary charge e, whereas the neutron is neutral, it carries no electric charge. Nonetheless the neutron has a magnetic moment, *ananomalous magnetic dipol moment*, and can therefore interact with a magnetic field. As hadrons are composite particles, they are not point particles.

More interesting are actually point particles, here we consider *charged leptons*. Besides the electron there are also *muons* $\mu^\pm$ and *tauons* $\tau^\pm$. They all have half-integer spin and charge $\pm e$. They differ in their mass

$$m_{e^-} \approx 0.5\,\mathrm{MeV} < m_\mu \approx 106\,\mathrm{MeV} < m_\tau \approx 1784\,\mathrm{MeV}. \tag{6.3.1}$$

allowed allowed allowed forbidden

Fig. 6.20: Fundamental vertices for leptons. The last Feynman diagram is an example for a forbidden process because both the electron and muon number are not conserved.

The electron is a stable particle, muons and tauons decay weakly[5]. At this point we want to focus on electromagnetic interactions. The message when including these particles in QED is that we have $e-\mu-\tau$-*universality*. That is, since leptons are spin 1/2 particles we associate them with a spinor field. So, if we write down the Dirac Lagrange density without any interactions we have

$$\mathcal{L}_0 = \sum_{l=e,\mu,\tau} \bar{\psi}_l (i\partial\!\!\!/ - m_e)\psi_l \tag{6.3.2}$$

and in order to include interaction we use minimal substitution. Then the interaction Hamiltonian is

$$\mathcal{H}_{\text{int}}(x) = -\mathcal{L}_{\text{int}}(x) = -e \sum_{\ell=e,\mu,\tau} :\bar{\psi}_\ell(x) A\!\!\!/(x)\psi_\ell(x): \,. \tag{6.3.3}$$

It is

1. only valid for point particles and
2. each term in the interaction Hamiltonian $\mathcal{H}_{\text{int}}$ contains only one type of leptons, the fundamental vertices are depicted in Fig. 6.20.

The *electron number* $N(e)$ is conserved in QED, that is,

$$N(e) = N(e^-) - N(e^+) \tag{6.3.4}$$

is the same before and after an interaction process. The same is true for the *muon number* $N(\mu)$ and *tauon number* $N(\tau)$. Processes like

$$e^- + \mu^+ \to e^+ + \mu^- \tag{6.3.5}$$

are forbidden. Now we have new particles and new processes like

$$e^+ + e^- \to \mu^+ + \mu^- \tag{6.3.6}$$

are possible.

[5] A weak interaction is responsible for the fact that muons and tauons do not live long.

6.4 QED Observables

Observables are objects which in principle can be measured in an experiment. They allow us to make predictions and connect them to experimental data. In particle physics there are two types of observables that we can measure in experiments

1. Decay of particles: One particle in the rest frame decays into N particles via, for example, weak interactions, strong interaction but not in QED

2. Cross sections: They can be analyzed in scattering experiments. A priori, we have two particles that collide and may create N particles, see Fig. 6.21. Examples are

$$e^+e^- \to \gamma^* \to \mu^+\mu^-, \tag{6.4.1}$$

   where γ^* is a virtual photon and

$$\gamma e^- \to \gamma e^-. \tag{6.4.2}$$

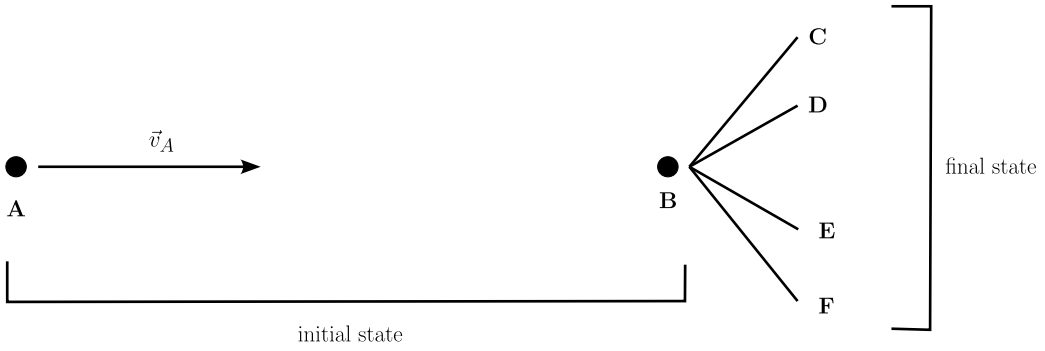


Fig. 6.21: Two particles A and B, where the latter is at rest while the other is moving towards B with velocity $\vec{v}_A$. They collide and produce further particles C, D, E, F.

So far we have learned how to calculate the S-matrix elements $S_{fi} = \langle f|\hat{S}|i\rangle$. Recall that the S-matrix is the time evolution operator in the interaction picture and thus its matrix elements are the transition amplitudes for a transition from $|i\rangle \to |f\rangle$. They are given by

$$S_{fi} = \delta_{fi} + (2\pi)^4\delta^{(4)}(P_f - P_i)\prod_i\left(\frac{1}{\sqrt{2VE_i}}\right)\prod_f\frac{1}{\sqrt{2VE_f}}\mathcal{M}, \tag{6.4.3}$$

where $\delta_{fi} = \langle f|\mathbb{1}|i\rangle$, $P_f = \sum_f p_f$, $P_i = \sum_i p_i$ and $\mathcal{M}$ the Feynman-amplitude which represents the true scattering. In QED we can calculate $\mathcal{M}$ perturbatively using Feynman rules. In order to make the connection to observables we need the *transition probability* for $|i\rangle \to |f\rangle$ where $|i\rangle \neq |f\rangle$. To do so we need to calculate

$$P_{fi} = |\langle f|\hat{S} - \mathbb{1}|i\rangle|^2. \tag{6.4.4}$$

Often one defines the so-called T-matrix $\hat{T} = \hat{S} - \mathbb{1}$. This, however, means that we have to square a δ-distribution $\delta^{(4)}(P_f - P_i)$ and one might wonder if that is problematic. In principle there are two ways to deal with this problem:

1. Work carefully with wavepackets $|k_A\rangle \to \int \frac{d^3p_A}{(2\pi)^3\sqrt{2E_p}}\phi(p_A)\,|\vec{p}_A\rangle$. This procedure is rather tedious and shall not be discussed here in detail. It can be found in Peskin & Schroeder in Chapter 4.5.

2. Another way is to work in a finite volume V and a finite time interval T. This procedure is a bit more hand waving but results in the same final formula. It can be found in Mandl & Shaw in Chapter 8.1.

We will follow the latter approach and begin by regularization of the δ-distribution

$$(2\pi)^4\delta^{(4)}(P_f - P_i) = \lim_{\substack{V\to\infty \\ T\to\infty}} \int_{-T/2}^{T/2} dt \int_V d^3x e^{ix\cdot(P_f - P_i)} \equiv \delta_{TV}(P_f - P_i), \tag{6.4.5}$$

where V is the volume of the box and T the "time box", that is, the time scale where the interaction happens. Let us now consider the squared expression

$$\begin{aligned}(2\pi)^8[\delta^{(4)}(P_f - P_i)]^2 &= (2\pi)^8\delta^{(4)}(P_f - P_i)\delta^{(4)}(0) \\ &\overset{\text{regularize}}{\longrightarrow} \delta_{TV}(P_f - P_i) \\ &\cong \delta_{TV}(P_i - P_f)\cdot TV.\end{aligned} \tag{6.4.6}$$

Hence, the probability grows linearly with the interaction time T and we can write $P_{fi} = w_{fi}\cdot T$ with w_{fi} being the probability rate. We introduce this rate as

$$w_{fi} = \frac{P_{fi}}{T}. \tag{6.4.7}$$

From now on we will work with its regularized version

$$w_{fi}^{TV} = \delta_{TV}(P_f - P_i)\cdot V \cdot \prod_i \frac{1}{2VE_i}\prod_f \frac{1}{2VE_f}|\mathcal{M}|^2. \tag{6.4.8}$$

Note that the time T cancelled in the last expression. We now want to take the limit $V \to \infty$. In order to do so recall that we assumed that the final state $|f\rangle = |\vec{p}_1, \ldots, \vec{p}_f\rangle$ is a momentum eigenstate. In an experiment, however, this is not realistic, because we never measure a particle with a definite momentum. We will always measure the momentum in some sort of range of values, we call these *bins*. For example in the x-direction it might be that the measured momentum of a particle is

$$p_{i,x} \in [p_{i,x,\min}, p_{i,x,\max}] = \text{"bin"}. \tag{6.4.9}$$

In order to accomodate for that we need to sum over bins. We therefore modify the transition rate again

$$w_{fi}^{TV} = \sum_{\vec{p}_i,\ldots,\vec{p}_f \in \text{bins}} \frac{P_{fi}}{T} = \frac{1}{V}\sum_{\vec{p}_i} \frac{1}{2E_i} \cdots \frac{1}{V}\sum_{\vec{p}_f} \frac{1}{2E_f} \delta_{TV}(P_f - P_i)) \cdot V \cdot \prod_i \frac{1}{2VE_i} |\mathcal{M}|^2. \tag{6.4.10}$$

6.4.1 Decay Rate

We now derive first the *decay rate* of a particle with mass m. The initial state $|i\rangle$ is a one-particle state in the rest frame denoted by $|k\rangle$ with $k^\mu = (m, \vec{0})$. In that case the transition amplitude is

$$\begin{aligned} w_{fi}^{TV} &= \Gamma_{fi}^{TV} \\ &= \frac{1}{V}\sum_{\vec{p}_f} \frac{1}{2E_i} \cdots \frac{1}{V}\sum_{\vec{p}_f} \frac{1}{2E_f} \cdot \delta_{TV}(P_i - P_f) \frac{1}{2m} \cdot |\mathcal{M}|^2 \\ &\to \int \frac{d^3p_i}{(2\pi)^3}\frac{1}{2E_i} \cdots \int \frac{d^3p_f}{(2\pi)^3}\frac{1}{2E_f} \cdot (2\pi)^4 \delta^{(4)}(P_i - P_f) \frac{1}{2m} \cdot |\mathcal{M}|^2, \end{aligned} \tag{6.4.11}$$

where we used that $\prod_i \frac{1}{2VE_i} = \frac{1}{2VE} = \frac{1}{2mV}$. Γ_{fi}^{TV} now has a well-defined limit for $V \to \infty$ and we can relax this regularization. We can finally write the formula for the decay rate

$$\begin{aligned} \Gamma_{fi}^{1\to f} &= \frac{1}{2m} \underbrace{\left[\prod_{m=2}^{f} \int_{\text{bins}} \frac{d^3p_m}{(2\pi)^3}\frac{1}{2E_m}(2\pi)^4\delta^{(4)}(P_f - P_i)\right]_{P_i=k}}_{\text{Lorentz-invariant f-particle phase space (LIPS)}} \cdot |\mathcal{M}|^2(p_1, \ldots, p_f) \\ &= \frac{1}{2m}\prod_{m=2}^{f} \int_{\text{bins}} \frac{d^3p_m}{(2\pi)^3} \delta(p_m^2 - m_m^2)\Theta(p_m^0)(2\pi)^4\delta^{(4)}(P_f - k) \cdot |\mathcal{M}|^2(p_1, \ldots, p_f) \\ &= \int_{\text{bins}} dPS_f. \end{aligned} \tag{6.4.12}$$

If we integrate over the full space we would allow particles to decay with all associate momenta, in that case we would call it the *total decay rate*. If, however, the bin is rather restricted or the bins are rather small then this will correspond to a *differential decay rate*.

6.4.2 Cross Section

Let us now derive the *cross section*, we consider $2 \to f$ scattering or to be more precise, a process

$$A(k_1) + B(k_2) \to C_1(p_1) + \ldots + C_f(p_f). \tag{6.4.13}$$

We consider the same situation as shown in Fig. 6.21 with the addition of a detector that measures and identifies all of the created particles or only one of the particles. By detection we mean the following, we have a beam of particles of type A, which hit a target of many particles of type B and the detector counts the number of particles of, for example C_1. The experimental cross section is defined as

$$\sigma_{fi} = \frac{\#\text{ particles counted in detector per time unit}}{(\text{incoming particle current density}) \cdot (\#\text{ target particles } B)}, \tag{6.4.14}$$

where # stands for 'number of'. In theory we consider an idealization where one particle of type A hits one particle B. Let us consider now the incoming current density of particles A

$$\vec{j} = \rho\vec{v} = \frac{1}{V}|\vec{v}|, \tag{6.4.15}$$

where the number of target particles in our idealization is just 1. Then

$$\begin{aligned}\sigma_{fi}^{TV} &= \frac{w_{fi}^{TV}}{\frac{|\vec{v}|}{V} \cdot 1} \\ &\overset{(6.4.10)}{=} \frac{V}{|\vec{v}|}\frac{1}{2VE_A}\frac{1}{2VE_B}\left(\frac{1}{V}\sum_{p_i}\frac{1}{2E_i}\right)\cdots\left(\frac{1}{V}\sum_{p_f}\frac{1}{2E_f}\right)\cdot\delta_{TV}(P_i - P_f)\cdot|\mathcal{M}|^2.\end{aligned} \tag{6.4.16}$$

The limit for $V \to \infty$ is again well defined and we obtain the *cross section for a fixed target B at rest*

$$\sigma_{fi} = \frac{1}{2E_A 2m_B|\vec{v}_A|}\int dPS_f|\mathcal{M}_{fi}|^2. \tag{6.4.17}$$

In general, however, what happens at particle colliders is that both beams are moving. In that case we have to change the *flux factor*

$$2E_A 2m_B|\vec{v}_A| \longrightarrow 2E_A 2E_B \underbrace{|\vec{v}_A - \vec{v}_B|}_{\text{relative velocity}}. \tag{6.4.18}$$

One may ask if Eq. (6.4.16) is Lorentz invariant and in fact $\int dPS_f |M_{fi}|^2$ is exactly that. Also the prefactor can be written covariantly, the expression for that is called *Møller flux factor*

$$F = 4\sqrt{(k_1 \cdot k_2)^2 - m_1^2 m_2^2}. \tag{6.4.19}$$

In the center of mass (CM) frame it reduces to

$$F = 2E_1 2E_2 |\vec{v}_1 - \vec{v}_2| \tag{6.4.20}$$

and the proof of that is a homework assignment. Using this expression, the cross section can be written as

$$\sigma_{fi} = \frac{1}{F} \int_{\text{bins}} dPS_f |\mathcal{M}_{fi}|^2. \tag{6.4.21}$$

Integrating over the whole space yields the *total cross section*, if the momentum regions are restricted, that is, if we have bins, then we do not have a total cross section. This yields us the *differential cross sections*, they are cross sections with infinitesimal bins $p_i^{x,y,z} \in [p_i^{x,y,z}, p_i^{x,y,z} + dp_i^{x,y,z}]$. Note that these cannot be measured in an experiment, it is only a theoretical construction. Nethertheless it is a useful construction as we will see. We can write the differential cross section as

$$d\sigma_{fi} = \frac{|\mathcal{M}_{fi}|^2(p_1, \ldots, p_f)}{4\sqrt{(k_1 - k_2)^2 - m_1^2 m_2^2}} \prod_{m=1}^{f} \frac{d^4 p_m}{(2\pi)^3} \delta^+(p_m^2 - m_m^2)(2\pi)^4 \delta^{(4)}(P_f - P_i). \tag{6.4.22}$$

This formula can be found in any book and can be seen as a *master formula for the cross section* for a $2 \to f$ process.

6.5 Two-Body Scattering Process and its Cross Section

Consider a two-body final state with the same mass m and a 4-momentum

$$k_1 + k_2 \to p_1 + p_2 \quad \text{with} \quad p_1^2 = p_2^2 = m^2, \tag{6.5.1}$$

for example Møller scattering $e^- e^- \to e^- e^-$ (see assignments). A very useful quantity is the *center of mass energy*, which is defined as

$$s = (k_1 + k_2)^2 = (p_1 + p_2)^2. \tag{6.5.2}$$

Let us now work out the master formula for this example, the differential cross section is

$$d\sigma = \frac{1}{F}|\mathcal{M}|^2(p_1,p_2)(2\pi)^4 \underbrace{\delta^{(4)}(k_1+k_2-p_1-p_2)}_{\substack{p_2 \text{ restricted by} \\ \text{4-momentum conservation}}} \frac{d^4p_1}{(2\pi)^3}\frac{d^4p_2}{(2\pi)^3}\delta^+(p_1^2-m^2)\delta^+(p_2^2-m^2). \tag{6.5.3}$$

Since it is impossible to measure the δ-distribution in an experiment we have to get rid of all appearences in our expression. To do so we integrate over p_2

$$\int d^4p_2 d\sigma = \frac{|\mathcal{M}|^2(p_1,p_2=k_1+k_2-p_1)}{(2\pi)^2\cdot F}\delta^+(p_1^2-m^2)\delta^+((k_1+k_2-p_1)^2-m^2)d^4p_1. \tag{6.5.4}$$

Because the expression is covariant we further on can work in a specific frame that suits us. A particular suitable frame to work out the cross section is the center-of-mass CM frame, that is, we choose the frame such that

$$\vec{k}_1+\vec{k}_2=0=\vec{p}_1+\vec{p}_2 \Rightarrow \vec{k}_1=-\vec{k}_2 \text{ and } \vec{p}_1=-\vec{p}_2. \tag{6.5.5}$$

We can then write the 4-momentum as $p_1^\mu=(p_1^0,\vec{p}_1)=(p_1^0,|\vec{p}_1|\vec{e})$ and we express the unit vector $\vec{e}=(\sin\theta\cos\phi,\sin\theta\sin\phi,\cos\theta)$ in spherical coordinates. The angle θ is called the *scattering angle*. The situation is show in Fig. 6.22. Also in this frame we

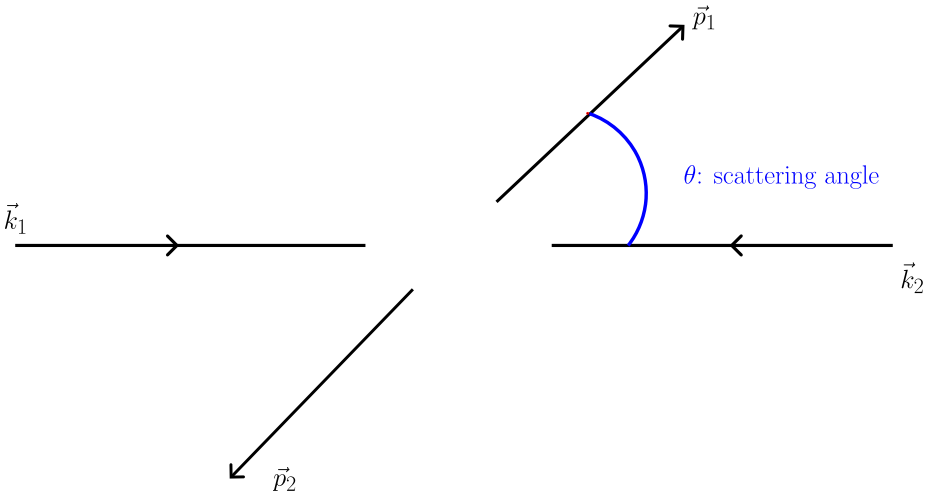


Fig. 6.22: 2-body scattering experiment.

have $s=(k_1+k_2)^2=(k_1^0+k_2^0)^2$ and thus the *CM energy*, which is the total energy in this process, is given by $\sqrt{s}=k_1^0+k_2^0$. Now the two δ-distributions in Eq. (6.5.4) can

be easily evaluated in this frame, since the sum of the 3-momenta is vanishing. So, for the first δ-distribution we have

$$\begin{aligned}\delta^+(p_1^2-m^2) &= \Theta(p_1^0)\delta((p_1^0)^2-(\vec{p}_1^{\,2}+m^2)) \\ &= \Theta(p_1^0)\frac{1}{2|p_1^0|}\left(\delta\big(p_1^0-\sqrt{\vec{p}_1^{\,2}+m^2}\big)+\delta\big(p_1^0+\sqrt{\vec{p}_1^{\,2}+m^2}\big)\right) \\ &= \Theta(p_1^0)\frac{1}{2|p_1^0|}\delta\big(p_1^0-\sqrt{\vec{p}_1^{\,2}+m^2}\big), \end{aligned} \tag{6.5.6}$$

where we used

$$\delta(f(x)) = \frac{1}{|(f'(x_i))|}\delta(x-x_i), \tag{6.5.7}$$

with x_i the zeroes of f. For the other one we have

$$\begin{aligned}\delta^+((k_1+k_2-p_1)^2-m^2) &= \Theta(k_1^0+k_2^0-p_1^0)\delta((k_1+k_2)^2+p_1^2-m^2-2p_1\cdot(k_1+k_2)) \\ &= \Theta(\sqrt{s}-p_1^0)\frac{1}{2\sqrt{s}}\delta(p_1^0-\sqrt{s}/2). \end{aligned} \tag{6.5.8}$$

Inserting these two expressions back into Eq. (6.5.4) yields

$$\begin{aligned}&\int d^4p_2 d\sigma \\ &= \frac{|\mathcal{M}|^2(p_1)}{(2\pi)^2\cdot F}\Theta(p_1^0)\Theta(\sqrt{s}-p_1^0)\frac{1}{2\sqrt{s}}\delta\left(p_1^0-\frac{\sqrt{s}}{2}\right)\cdot\frac{1}{2p_1^0}\delta\big(p_1^0-\sqrt{\vec{p}_1^{\,2}+m^2}\big)d^4p_1 \\ &= \frac{|\mathcal{M}|^2}{(2\pi)^2\cdot F}\Theta(p_1^0)\Theta\big(\sqrt{s}-p_1^0\big)\frac{1}{8}\sqrt{\frac{s-4m^2}{s}}\delta\left(p_1^0-\frac{\sqrt{s}}{2}\right) \\ &\quad\times\delta\left(|\vec{p}_1|-\frac{\sqrt{s-4m^2}}{2}\right)dp_1^0 d|\vec{p}_1|d\Omega, \end{aligned} \tag{6.5.9}$$

where we used

$$d^4p_1 = dp_1^0 d^3\vec{p}_1 = dp_1^0 d|\vec{p}_1||\vec{p}_1|^2 d\Omega, \tag{6.5.10}$$

with $d\Omega = d\theta d\phi\sin\theta$ the *solid angle* and

$$\begin{aligned}\delta\big(\sqrt{s}/2-\sqrt{\vec{p}_1^{\,2}+m^2}\big) &= \frac{\sqrt{|\vec{p}_1|^2+m^2}}{|\vec{p}_1|}\delta\big(|\vec{p}_1|-\sqrt{s-4m^2}/2\big) \\ &= \sqrt{\frac{s}{s-4m^2}}\delta\big(|\vec{p}_1|-\sqrt{s-4m^2}\big). \end{aligned} \tag{6.5.11}$$

Note that here in the first step we rewrote the argument as follows

$$\frac{\sqrt{s}}{2}-\sqrt{\vec{p}_1^{\,2}+m^2}=0 \Leftrightarrow \frac{s}{4}\vec{p}_1^{\,2}+m^2 \Leftrightarrow |\vec{p}_1| = \frac{1}{2}\sqrt{s-4m}. \tag{6.5.12}$$

Now we can safely integrate out the energy and the momentum. Finally, we got rid of all of the δ-distributions and we are only left with the differential of the solid angle.

$$\int dp_1^0 \int d|\vec{p}_1| \int d^4p_2 d\sigma = d\sigma(\Omega) = \frac{|\mathcal{M}|^2(\Omega)}{(2\pi)^2 \cdot F} \frac{1}{8} \sqrt{\frac{s-4m^2}{s}} d\Omega, \tag{6.5.13}$$

where the Feynman amplitude first depended on p_1 and p_2, that is, $\mathcal{M}(p_1, p_2) = \mathcal{M}(p_1, p_2 = k_1 + k_2 - p_1)$, but now since

$$p_{1,2} = \left(\frac{\sqrt{s}}{2}, \frac{\sqrt{s-4m^2}}{2} \vec{e}(\Omega) \right), \tag{6.5.14}$$

final results are determined by Ω. Finally, the differential cross section in the CM frame is

$$\left(\frac{d\sigma}{d\Omega} \right)_{\text{CM}} = \frac{1}{8} \sqrt{\frac{s-4m^2}{s}} \frac{|\mathcal{M}|^2(\Omega)}{(2\pi)^2 \cdot F}, \tag{6.5.15}$$

integrating over all angles gives the total cross section

$$\sigma(s) = \int d\Omega \left(\frac{d\sigma}{d\Omega} \right) = \frac{1}{8} \sqrt{\frac{s-4m^2}{s}} \frac{1}{(2\pi)^2 \cdot F} \int d\Omega |\mathcal{M}^2|(\Omega). \tag{6.5.16}$$

The simplest process we can consider for the two-body scattering is the *lepton pair annihilation / production*, that is, a process

$$e^+e^- \to \mu^+\mu^- \text{ or } \tau^+\tau^-. \tag{6.5.17}$$

The corresponding Feynman diagram is shown in Fig. 6.23 and has an amplitude

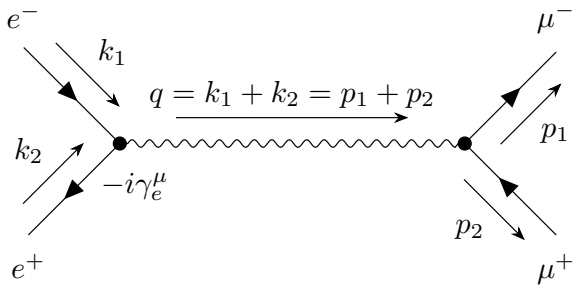


Fig. 6.23: Feynman diagram of lepton pair annihilation or production.

$$\begin{aligned} \mathcal{M} &= \bar{v}(k_2, \sigma_2)(-ie\gamma^\mu)u(k_1, \sigma_1) \frac{-ig_{\mu\nu}}{q^2 + i\delta} \bar{u}(p_1, s_1)(-ie\gamma^\nu)v(p_2, s_2) \\ &\overset{q^2=s}{=} i \underbrace{e^2}_{=4\pi\alpha} \frac{1}{s} \bar{v}(k_2, \sigma_2)\gamma^\mu u(k_1, \sigma_1) \bar{u}(p_1, s_1)\gamma_\mu v(p_2, s_2). \end{aligned} \tag{6.5.18}$$

To calculate the cross section one needs $|\mathcal{M}|^2$. We already had the Feynman amplitude $\mathcal{M}$ in Eq. (6.5.18). Thus, we only need its complex conjugate $\mathcal{M}^*$, which is easy to obtain. We find

$$|\mathcal{M}|^2 = \frac{(4\pi)^2\alpha_{\text{em}}^2}{s^2}(\bar{v}(k_2,\sigma_2)\gamma^\mu u(k_1,\sigma_1))(\bar{u}(p_1,s_1))\gamma_\mu v(p_2,s_2)) \\ \underbrace{(\bar{v}(k_2,\sigma_2)\gamma^\nu u(k_1,\sigma_1))^\dagger}_{=\bar{u}(k_1,\sigma_1)\gamma^\nu v(k_2,\sigma_2)} \underbrace{(\bar{u}(p_1,s_1)\gamma_\nu v(p_2,s_2))^\dagger}_{=\bar{v}(p_2,s_2)\gamma_\nu u(p_1,s_1)}. \tag{6.5.19}$$

Recall that $\bar{v} = v^\dagger\gamma^0$, so the complex conjuagtion gives

$$(\bar{v}\gamma^\nu u)^\dagger = u^\dagger(\gamma^\nu)^\dagger(\gamma^0)^\dagger v = u^\dagger\gamma^0\gamma^0\gamma^0\gamma^\nu\gamma^0\gamma^0 v = \bar{u}(k_1,\sigma_1)\gamma^\nu v(k_2,\sigma_2) \tag{6.5.20}$$

and similarly for the second factor. The Feynman diagram of $|\mathcal{M}|^2$ can be drawn as a diagram of $\mathcal{M}$ and a mirrored version for $\mathcal{M}^*$, see Fig. 6.24. In order to evaluate

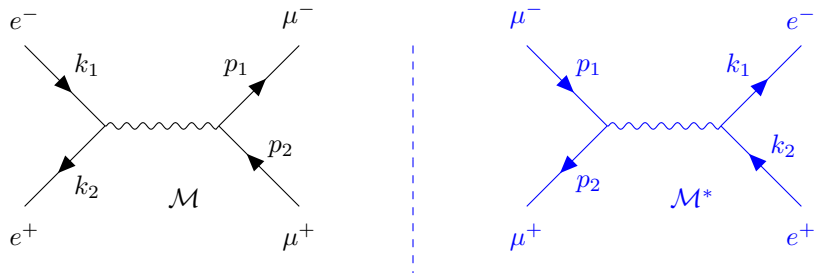


Fig. 6.24: Feynman diagram for $|\mathcal{M}|^2 = \mathcal{M}\mathcal{M}^*$, the black diagram represents $\mathcal{M}$ whereas the red diagram represents $\mathcal{M}^*$. The dashed line (or sometimes a $\int$ symbol) indicates a phase space integration.

the products in Eq. (6.5.19), note that each factor is of the form vector-matrix-vector. We therefore write these explicitly in components and we rearrange the spinors in the following way

$$\frac{1}{4}\sum_{\sigma_1,\sigma_2}\sum_{s_1,s_2}|\mathcal{M}|^2 = \frac{(4\pi)^2\alpha_{\text{em}}^2}{s^2}\sum_{\sigma_2}\Big[v_m(k_2,\sigma_2)\bar{v}_i(k_2,\sigma_2)\Big]\gamma^\mu_{ij}\sum_{\sigma_1}\Big[u_j(k_1,\sigma_1)\bar{u}_m(k_1,\sigma_1)\Big]\gamma^\nu_{mn} \\ \sum_{s_1}\Big[u_s(p_1,s_1)\bar{u}_k(p_1,s_1)\Big]\gamma^\mu_{kl}\sum_{s_2}\Big[v_l(p_2,s_2)\bar{v}_r(p_2,s_2)\Big]\gamma^\nu_{rs}. \tag{6.5.21}$$

In this expression the spin-information, that is the $\sigma_{1,2}, s_{1,2}$, is still included. Recall that for each spinor there are two components indicating the spin direction of spin-$\frac{1}{2}$. In the following we assume that the detectors are insensitive to the spin of those particles we

measure, that is, the detector does not care about whether the particle is spin-up or spin-down, it measures both possibilities. This translates to the fact that we sum over the final states, $\sum_{s_1=1}^{2}\sum_{s_2}^{2}$. Moreover, we consider unpolarized initial particle beams, that means that we consider an ensemble of spins with *arbitrary* spin polarization. We thus average out the spin information, $\frac{1}{2}\sum_{\sigma_1=1}^{2}\frac{1}{2}\sum_{\sigma_2=1}^{2}$. Now with that we can use the polarization sums (completeness relations)

$$\sum_s u_i(k,s)\bar{u}_j(k,s) = (\not{k}+m)_{ij}, \tag{6.5.22}$$

$$\sum_s v_i(k,s)\bar{v}_j(k,s) = (\not{k}-m)_{ij}, \tag{6.5.23}$$

to find the unpolarized cross section

$$\frac{1}{4}\sum_{s_1,s_2}\sum_{\sigma_1,\sigma_2}|\mathcal{M}|^2 = \frac{(4\pi)^2\alpha_{\text{em}}^2}{4s^2}\cdot\text{tr}[(\not{k}_2-m_e)\gamma^\mu(\not{k}_1+m_e)\gamma^\nu]\cdot\text{tr}[(\not{p}_1+m_\mu)\gamma_\mu(\not{p}_2-m_\mu)\gamma_\nu]. \tag{6.5.24}$$

In the next step we have to calculate those traces and for that we already proofed some useful formulas (homework sheet 6, ex. 21 c))

$$\text{tr}[\gamma^\mu\gamma^\nu)] = 4g^{\mu\nu}, \tag{6.5.25}$$

$$\text{tr}[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma] = 4(g^{\mu\nu}g^{\rho\sigma}+g^{\mu\sigma}g^{\nu\rho}-g^{\mu\rho}g^{\nu\sigma}), \tag{6.5.26}$$

$$\text{tr}[\text{odd }\#\gamma\text{'s}] = 0. \tag{6.5.27}$$

Let us now calculate the first trace in Eq. (6.5.21) using these formulas

$$\begin{aligned}\text{tr}[(\not{k}_2-m_e)\gamma^\mu(\not{k}_1+m_e)\gamma^\nu] &= \text{tr}[\not{k}_2\gamma^\mu\not{k}_1\gamma^\nu] - m_e^2\text{tr}[\gamma^\mu\gamma^\nu] \\ &= 4(k_2^\mu k_1^\nu + k_2^\nu k_1^\mu - \underbrace{(k_1\cdot k_2)g^{\mu\nu} - m_e^2 g^{\mu\nu}}_{=-((k_1\cdot k_2)+m_e^2)g^{\mu\nu}}),\end{aligned} \tag{6.5.28}$$

similarly we obtain for the second trace

$$\text{tr}[(\not{p}_1+m_\mu)\gamma_\mu(\not{p}_2-m_\mu)\gamma_\nu] = 4(p_{1\mu}p_{2\nu}+p_{1\nu}p_{2\mu}-(p_1\cdot p_2)g_{\mu\nu}-m_\mu^2 g_{\mu\nu}). \tag{6.5.29}$$

In the next step we have to contract the traces, the result of that is

$$\begin{aligned}&\text{tr}[(\not{k}_2-m_e)\gamma^\mu(\not{k}_1+m_e)\gamma^\nu]\cdot\text{tr}[(\not{p}_1+m_\mu)\gamma_\mu(\not{p}_2-m_\mu)\gamma_\nu] \\ &= 16\cdot\{2(k_2\cdot p_1)(k_1\cdot p_2)+2(k_2\cdot p_2)(k_1\cdot p_1) \\ &\quad -2(k_1\cdot k_2)((p_1\cdot p_2)+m_\mu^2)-2((k_1\cdot k_2)+m_e^2)(p_1\cdot p_2) \\ &\quad +[(k_1\cdot k_2)(p_1\cdot p_2)\underbrace{g^{\mu\nu}g_{\mu\nu}}_{=4}m_\mu^2(k_1\cdot k_2)\underbrace{g^{\mu\nu}g_{\mu\nu}}_{=4}+m_e^2(p_1\cdot p_2)\underbrace{g_{\mu\nu}g^{\mu\nu}}_{=4}+m_e^2m_\mu^2\underbrace{g^{\mu\nu}g_{\mu\nu}}_{=4}]\} \\ &= 32\cdot[(k_1\cdot p_1)(k_2\cdot p_2)+(k_1\cdot p_2)(k_2\cdot p_1)+m_\mu^2(k_1\cdot k_2)+m_e^2(p_1\cdot p_2)+2m_e^2m_\mu^2].\end{aligned} \tag{6.5.30}$$

Now what we have to do is to evaluate these dot products which we will again do in the CM frame. Before we did not choose any particular axis, but we will choose one now, namely we choose the *beam axis* as the z-axis. Moreover, we choose $\vec{k}_1$ in the positive z-direction. In the CM frame we have $\vec{k}_1 + \vec{k}_2 = \vec{p}_1 + \vec{p}_2 = \vec{0}$ and

$$k_1^\mu = (k_1^0, 0, 0, k_1^z), \tag{6.5.31}$$
$$k_2^\mu = (k_2^0, 0, 0, k_2^z = -k_1^z), \tag{6.5.32}$$
$$k_1^2 = k_2^2 = m_e^2 \Rightarrow (k_1^0)^2 = (k_1^z)^2 + m_e^2 = (k_2^0)^2, \tag{6.5.33}$$
$$s = (k_1 + k_2)^2 = (k_1^0 + k_2^0)^2 = 4(k_1^0)^2 = 4(k_2^0)^2. \tag{6.5.34}$$

Solving for the energies we obtain

$$k_1^0 = k_2^0 = \frac{\sqrt{s}}{2}, \tag{6.5.35}$$
$$k_1^z = \frac{1}{2}\sqrt{s - 4m_e^2} = -k_2^z, \tag{6.5.36}$$

which we can insert into our previously derived expression Eq. (6.5.14) in order to get

$$p_{1,2} = \left(\frac{\sqrt{s}}{2}, \pm\frac{1}{2}\sqrt{s - 4m_\mu^2}\vec{e}\right). \tag{6.5.37}$$

This can now be used to evaluate the dot products in Eq. (6.5.30)

- $k_1 \cdot p_1 = \frac{s}{4} - \frac{1}{4}\sqrt{(s - 4m_e^2)(s - 4m_\mu^2)}\cos\theta = k_2 \cdot p_2$,
- $k_1 \cdot p_2 = \frac{s}{4} + \frac{1}{4}\sqrt{(s - 5m_e^2)(s - 4m_\mu^2)}\cos\theta = k_2 \cdot p_1$,
- $k_1 \cdot k_2 = \frac{s}{4} + \frac{s}{4} - m_e^2 = \frac{1}{2}s - m_e^2$,
- $p_1 \cdot p_2 = \frac{s}{2} - m_\mu^2$,

and finally gives

$$\begin{aligned}
&\mathrm{tr}[(\not{k}_2 - m_e)\gamma^\mu(\not{k}_1 + m_e)\gamma^\nu] \cdot \mathrm{tr}[(\not{p}_1 + m_\mu)\gamma_\mu(\not{p}_2 - m_\mu)\gamma_\nu] \\
&= 32 \cdot \left(2\Big(\frac{s^2}{16} + \frac{1}{16}(s - 4m_e^2)(s - 4m_\mu^2)\cos^2\theta\Big) + m_\mu^2\Big(\frac{s}{2} - m_e^2\Big) + m_e^2\Big(\frac{s}{2} - m_\mu^2\Big) + 2m_e^2 m_\mu^2\right) \\
&= 4s^2\Big[\Big(1 + 4\frac{m_\mu^2 + m_e^2}{s}\Big) + \Big(1 - \frac{4m_e^2}{s}\Big)\Big(1 - \frac{4m_\mu^2}{s}\cos^2\theta\Big)\Big].
\end{aligned} \tag{6.5.38}$$

The unpolarized differential cross section in the CM frame becomes

$$\left(\frac{d\sigma_{\text{unp.}}}{d\Omega}\right)_{\text{CM}} = \sqrt{\frac{s - 4m_\mu^2}{s}}\frac{\frac{1}{4}\sum_{s_1,s_2}\sum_{\sigma_1,\sigma_2}|\mathcal{M}|^2}{2 \cdot (4\pi)^2 \cdot F}, \tag{6.5.39}$$

where $F = 4\sqrt{(k_1 \cdot k_2)^2 - 4m_e^4} = 2\sqrt{s(s-4m_e^2)}$. For the lepton pair annihilation / production this is then

$$\begin{aligned}\left(\frac{d\sigma_{\text{unp.}}}{d\Omega}\right)_{\text{CM}} &= \frac{\alpha_{\text{em}^2}}{4s}\sqrt{\frac{1-4\frac{m_\mu^2}{s}}{1-4\frac{m_e^2}{s}}}\left[\left(1+4\frac{m_\mu^2+m_e^2}{s}\right)\right] + \left(1-\frac{4m_e^2}{s}\right)\left(1-4\frac{m_\mu^2}{s}\cos^2\theta\right) \\ &\overset{m_e\approx 0}{\approx} \frac{\alpha_{\text{em}}^2}{4s}\sqrt{1-4\frac{m_\mu^2}{s}}\left[\left(1+4\frac{m_\mu^2}{s}\right)+\left(1-4\frac{m_\mu^2}{s}\right)\right] \\ &\overset{s\gg m_\mu^2}{=} \frac{\alpha_{\text{em}}^2}{4s}[1+\cos^2\theta], \end{aligned} \tag{6.5.40}$$

which is a QED prediction and in principle this formula can be experimentally tested. Note that everything in Eq. (6.5.40) is covariant except for the factor $\cos\theta$, which has an interpretation only in the CM frame. The total cross section, which again is just integration of the differential cross section over the full solid angle $\sigma(s) = \int d\Omega (d\sigma/d\Omega)_{\text{CM}}$ with $\int d\Omega = 4\pi, \int d\Omega \cos^2\theta = 4\pi/3$, will only depend on invariant variables, basically only on s and is given by

$$\begin{aligned}\sigma_{\text{tot.}}^{\text{unp.}}(s, m_e^2, m_\mu^2) &= \frac{\pi\alpha_{\text{em}}^2}{3}\sqrt{\frac{1-4\frac{m_\mu^2}{s}}{1-4\frac{m_e^2}{s}}}\left[\left(1+4\frac{m_\mu^2+m_e^2}{s}\right)+\frac{1}{3}\left(1-4\frac{m_e^2}{s}\right)\left(1-4\frac{m_\mu^2}{s}\right)\right] \\ &\overset{s\gg m_e^2, m_\mu^2}{\longrightarrow} \frac{4\pi\alpha_{\text{em}}^2}{3s} \\ &= \frac{\pi\alpha_{\text{em}}^2}{3E^2}. \end{aligned} \tag{6.5.41}$$

In the last step we set the center of mass energy s equal to $4E$ with E the beam energy of e^+ and e^-. In particular,

$$E^2\sigma \overset{E\gg m_e^2, m_e^2}{\longrightarrow} \frac{\pi\alpha_{\text{em}}^2}{3}. \tag{6.5.42}$$

We just demonstrated how all of these calculations work, in principle. We just follow this recipe: 1) Calculate the polarization sums 2) calculate the expressions for the traces 3) contract the traces and 4) inserting $|\mathcal{M}|^2$.

We now want to consider the more complicated process of Bhabha scattering $e^+e^- \to e^+e^-$. The difference to the previous case is that instead of only one diagram for the amplitude, we now have 2 leading order diagrams, see Fig. 6.25. In a previous homework

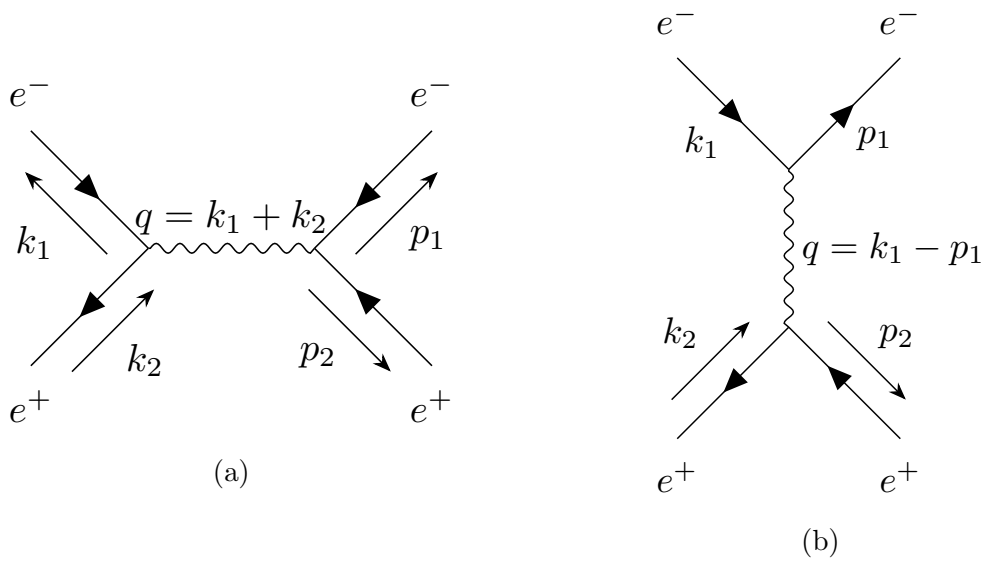


Fig. 6.25: Leading order Feynman diagrams of Bhabha scattering, (a) s-channel and (b) t-channel.

exercise we derived the two corresponding Feynman amplitudes

$$\mathcal{M}_a = i\frac{4\pi\alpha_{\text{em}}}{s}\bar{v}(k_2,\sigma_2)\gamma^\mu u(k_1,\sigma_1)\bar{u}(p_1,s_1)\gamma_\mu v(p_2,s_2), \tag{6.5.43}$$

$$\mathcal{M}_b = -i\frac{4\pi\alpha_{\text{em}}}{t}\bar{u}(p_1,s_1)\gamma^\mu u(k_1,\sigma_1)\bar{v}(k_2,\sigma_2)\gamma_\mu v(p_2,s_2). \tag{6.5.44}$$

Note that $\mathcal{M}_b$ contains an additional minus sign. We call $\mathcal{M}_a$ the s-channel and $\mathcal{M}_b$ the t-channel. The appearing s and t actually refer to the so called *Mandelstam variables*. They are defined for an arbitrary $2 \to 2$ process (cf. Fig. 6.26), and not only for the process we are talking about, by

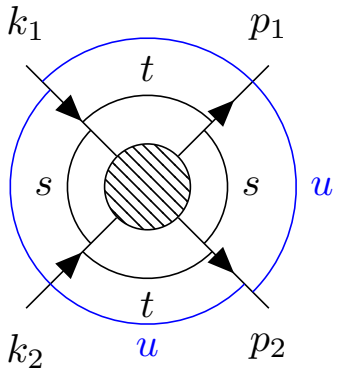


Fig. 6.26: Diagram of an arbitrary $2 \to 2$ process with Mandelstam variables.

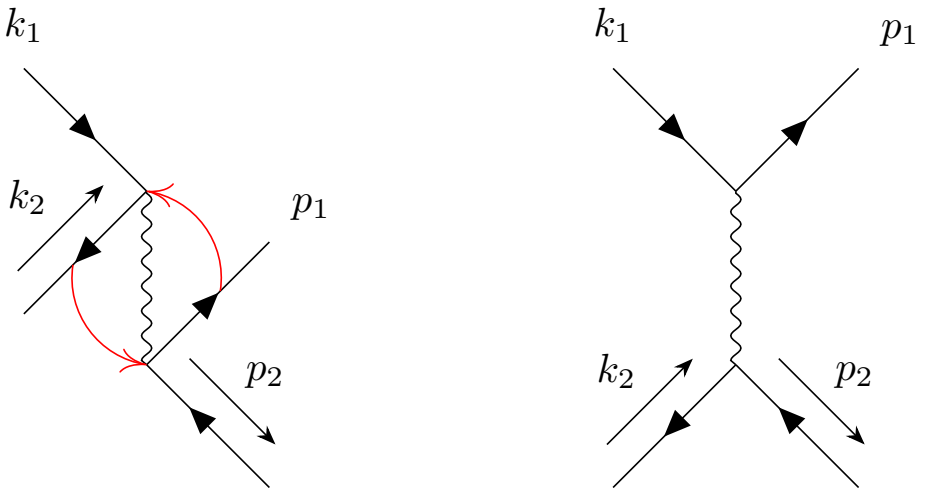


Fig. 6.27: Crossing symmetry: Obtaining (b) $\mathcal{M}_b$ from (a) $\mathcal{M}_a$ by interchanging the fermion lines as indicated by the red arrows.

$$s = (k_1 + k_2)^2 = (p_1 + p_2)^2, \tag{6.5.45}$$

$$t = (k_1 - p_1)^2 = (k_2 - p_2)^2, \tag{6.5.46}$$

$$u = (k_1 - p_2)^2 = (k_2 - p_1)^2. \tag{6.5.47}$$

These variables fulfill the following relation

$$s + t + u = \sum_i m_i^2, \tag{6.5.48}$$

where m_i are the external masses. It is possible to construct diagram b) in Fig. 6.25 from diagram a) by replacing k_2 by $-p_1$, p_1 by $-k_2$ and interchanging the order of the spinors $\bar{u}$ and $\bar{v}$. This feature is called *crossing symmetry*, see Fig. 6.27. Let us now calculate the unpolarized cross section for this process. That will give us 4 terms, since

$$\frac{1}{4}\sum_{\text{spins}} |\mathcal{M}|^2 = \frac{1}{4}\sum_{\text{spins}} |\mathcal{M}_a + \mathcal{M}_b|^2 = \frac{1}{4}\sum_{\text{spins}} [|\mathcal{M}_a|^2 + |\mathcal{M}_b|^2 + \underbrace{\mathcal{M}_a\mathcal{M}_b^* + \mathcal{M}_b\mathcal{M}_a^*}_{\text{interference terms}}]. \tag{6.5.49}$$

The first term we already know, it is

$$\frac{1}{4}\sum_{\text{spins}} |\mathcal{M}_a|^2 = \frac{(4\pi)^2\alpha_{\text{em}}^2}{4s^2} \cdot 32 \cdot [(k_1 \cdot p_1)(k_2 \cdot p_2) + (k_1 \cdot p_2)(k_2 \cdot p_1) + \text{ mass terms}] \tag{6.5.50}$$

and the second term we obtain from crossing symmetry

$$\frac{1}{4}\sum_{\text{spins}} |\mathcal{M}_b|^2 = \frac{(4\pi)^2\alpha_{\text{em}}^2}{4t^2} \cdot 32 \cdot [(k_1 \cdot k_2)(p_1 \cdot p_2) + (k_1 \cdot p_2)(k_2 \cdot p_1) + \text{ mass terms }]. \tag{6.5.51}$$

What is left to do now, is to calculate the interference terms

$$\begin{aligned}\frac{1}{4}\sum_{\text{spins}}\mathcal{M}_a\mathcal{M}_b^* &= -\frac{(4\pi)^2\alpha_{\text{em}}^2}{st}\cdot\frac{1}{4}\cdot\sum_{\text{spins}}\bar{v}(k_2,\sigma_2)\gamma^\mu u(k_1,\sigma_1)\bar{u}(p_1,s_1)\gamma_\mu v(p_2,s_2)\\ &\qquad\qquad\cdot[\bar{u}(k_1,\sigma_1)\gamma^\nu u(p_1,s_1)\bar{v}(p_2,s_2)\gamma_\nu v(k_2,\sigma_2)]\\ &= -\frac{(4\pi)^2\alpha_{\text{em}}^2}{st}\cdot\frac{1}{4}\cdot\text{tr}[(\not{k}_2-m)\gamma^\mu(\not{k}_1+m)\gamma^\nu(\not{p}_1+m)\gamma_\mu(\not{p}_2-m)\gamma_\nu]\\ &= -\frac{(4\pi)^2\alpha_{\text{em}}^2}{st}\cdot\frac{1}{4}\cdot(\text{tr}[\not{k}_2(\gamma^\mu\not{k}_1\gamma^\nu\not{p}_1\gamma_\mu)\not{p}_2\gamma_\nu]+\text{ mass terms}),\end{aligned}\tag{6.5.52}$$

where in the second step we rearanged the spinors and applied the spin sums, see Eqs. (6.5.22), (6.5.23). So now we have to calculate a trace with 8 γ-matrices. This is quite difficult, but we can use a few tricks. We see for example that the Minkowski index μ is contracted, similarly for ν. Furthermore, there are a few *contraction identites*, that were derived in a homework assignment

$$\gamma^\mu\gamma_\mu = 4, \tag{6.5.53}$$

$$\gamma^\mu\not{a}\gamma_\mu = -2\not{a}, \tag{6.5.54}$$

$$\gamma^\mu\not{a}\not{b}\gamma_\mu = 4ab, \tag{6.5.55}$$

$$\gamma^\mu\not{a}\not{b}\not{c}\gamma_\mu = -2\not{c}\not{b}\not{a}. \tag{6.5.56}$$

These can be proven using the Clifford algebra $\{\gamma^\mu,\gamma^\nu\}=2g^{\mu\nu}$. Using these identities we obtain

$$\begin{aligned}\frac{1}{4}\text{tr}[\not{k}_2\underbrace{(\gamma^\mu\not{k}_1\gamma^\nu\not{p}_1\gamma_\mu)}_{=-2\not{p}_1\gamma^\nu\not{k}_1}\not{p}_2\gamma_\nu] &= -\frac{1}{2}\text{tr}[\not{k}_2\not{p}_1\underbrace{\gamma^\nu\not{k}_1\not{p}_2\gamma_\nu}_{=4k_1\cdot p_2}]\\ &= -2\underbrace{\text{tr}[\not{k}_2\not{p}_1]}_{=4k_2\cdot p_1}(k_1\cdot p_2)\\ &= -8(k_1\cdot p_2)(k_2\cdot p_1).\end{aligned}\tag{6.5.57}$$

Since this is just a real number its complex conjugate is identica and we get

$$\frac{1}{4}\sum_{\text{spins}}(\mathcal{M}_a\mathcal{M}_b^*+\mathcal{M}_a^*\mathcal{M}_b) = \frac{(4\pi)^2\alpha_{em}^2}{4st}\cdot 16\cdot(k_1\cdot p_2)(k_2\cdot p_1)\tag{6.5.58}$$

for the interference terms. Next thing that we need to do is to evaluate the dot products in the CM frame and express them through the CM angle θ (in general we can have an additional dependence on φ)

$$k_1\cdot p_1 = k_2\cdot p_2 = \frac{s}{4}-\frac{1}{4}(s-4m_e^2)\cos\theta = \frac{s}{4}(1-\cos\theta)+\text{ mass terms},\tag{6.5.59}$$

$$k_1\cdot p_2 = k_2\cdot p_1 = \frac{s}{4}+\frac{1}{4}(s-4m_e^2)\cos\theta = \frac{s}{4}(1+\cos\theta)+\text{ mass terms}.\tag{6.5.60}$$

We also need the Mandelstam variable t, which is something we have not calculated yet,

$$t = (k_1 - p_1)^2 - 2k_1 \cdot p_1 = -\frac{s}{2}(1 - \cos\theta) < 0. \tag{6.5.61}$$

Finally, by inserting both Eq. (6.5.59) and (6.5.60) into Eq. (6.5.58) we get the unpolarized differential cross section for Bhabha scattering

$$\begin{aligned}\left(\frac{d\sigma_{\text{unp.}}}{d\Omega}\right)_{\text{CM}}^{e^+e^-\to e^+e^-} &\overset{s\gg m_e^2}{=} \frac{1}{8(2\pi)^2 F}\left(\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2\right) \\ &= \frac{\alpha_{\text{em}}^2}{s}\left[\frac{1}{4}(1+\cos^2\theta) + 2\frac{1+\frac{1}{4}(1+\cos^2\theta)^2}{(1-\cos\theta)^2} - \frac{\frac{1}{4}(1+\cos^2\theta)^2}{(1+\cos\theta)}\right] \\ &= \frac{\alpha_{\text{em}}^2}{s}\Bigg[\underbrace{\frac{1}{2}(1+\cos^2\theta)}_{s\text{-channel}} + \underbrace{\frac{1+\cos^4\frac{\theta}{2}}{\sin^4\frac{\theta}{2}}}_{t\text{-channel}} - \underbrace{\frac{2\cos^4\frac{\theta}{2}}{\sin^2\frac{\theta}{2}}}_{\text{interference terms}}\Bigg],\end{aligned} \tag{6.5.62}$$

where we used $\sin\frac{\theta}{2} = \sqrt{\frac{1}{2}(1-\cos\theta)}$ and $\cos\frac{\theta}{2} = \sqrt{\frac{1}{2}(1+\cos\theta)}$ in the last step.

6.6 Scattering off an External Classical Field

In QED an electromagnetic field $A^\mu(x)$ is usually quantized and as a consequence, we obtain photons. Though, sometimes it is useful to keep $A^\mu(x)$ classical, for example, for an external magnetic field or a Coulomb field of a heavy nucleus. Therefore, we split the field as

$$A^\mu(x) \to A_q^\mu(x) + A_{\text{ext.}}^\mu(x), \tag{6.6.1}$$

where $A_\mu^q(x)$ is the quantized field and $A_{\text{ext.}}^\mu(x)$ the classical field. One might ask "When is a field classical?", the answer to that question is that we do not have a coupling to the system which creates this 4-potential such as a heavy nucleus. A heavy nucleus carries charge and the charge is responsible for the electrostatic field. Consider now the interaction of an electron with that nucleus, we scatter it off, but the heavy nucleus does not move, since there is no recoil, that is, there is no backcoupling. Alternativley, we can consider an external magnetic field and investigate the behaviour of an electron inside that field. For example we have a magnetic field, which in leading order couples to the magnetic dipole moment of the electron. The idea is that in the quantum system when one has an external magnetic field, could we for example study the observable magnetic dipole moment of the electron? The answer is yes.

In QED the interaction Hamiltonian involving the free electromagnetic field is given as

$$\mathcal{H}_{\text{int}} = -e :\bar{\psi}(x)(\not{A}_q(x) + \not{A}_{\text{ext}}(x))\psi(x): \tag{6.6.2}$$

and the S-matrix is

$$\begin{aligned}\hat{S} &= Te^{-i\int d^4x \mathcal{H}_{\text{int}}(x)} \\ &= \sum_{n=0}^{\infty} \frac{(ie)^n}{n!} \int d^4x_1 \ldots \int d^4x_n T\{:\bar{\psi}(x_1)(\not{A}_q + \not{A}_{\text{ext.}})(x_1): \ldots :\bar{\psi}(x_1)(\not{A}_q + \not{A}_{\text{ext.}})(x_n):\}.\end{aligned} \tag{6.6.3}$$

That external field could for example be a static external field (which does not depend on time)

$$A^{\mu}_{\text{ext.}}(x) = A^{\mu}_{\text{ext}}(0,\vec{x}) = \frac{1}{(2\pi)^3}\int d^3q e^{-i\vec{q}\cdot\vec{x}} \tilde{A}^{\mu}_{\text{ext.}}(\vec{q}). \tag{6.6.4}$$

This, however, requires a specific frame, because in another frame the field might not be static. The first order term of the S-matrix is

$$\begin{aligned}\hat{S}^{(1)}_{\text{ext.}} &= ie\int d^4x :\bar{\psi}(x)\not{A}_{\text{ext.}}(x)\psi(x): \\ &= ie\int d^4x \underbrace{\bar{\psi}^-(x)}_{\sim c^\dagger}\not{A}_{\text{ext.}}(x)\underbrace{\psi^+(x)}_{\sim c} - ie\int d^4x \underbrace{\psi^-_j(x)}_{\sim d^\dagger}\underbrace{\bar{\psi}^+_i(x)}_{\sim d}(\not{A}_{\text{ext.}})_{ij}(x) + \ldots.\end{aligned} \tag{6.6.5}$$

$\hat{S}$ is an operator, but we are interested in its matrix elements. We take for simplicity an electron both in the inital / final state, that is,

$$|i\rangle = |e^-(p)\rangle = c^\dagger(p)\,|0\rangle\,, \qquad |f\rangle = |e^-(p')\rangle = c^\dagger(p')\,|0\rangle\,. \tag{6.6.6}$$

Before we calculate the matrix elements S_{fi} let us first look at the effect of the field operators on these states

$$\psi^+(x)c^\dagger(p)\,|0\rangle = \frac{1}{\sqrt{2VE_p}}u(p)e^{-ip\cdot x}\,|0\rangle = \{\psi^+(x), c^\dagger(p)\}\,|0\rangle\,, \tag{6.6.7}$$

$$\langle 0|\,c(p')\bar{\psi}^-(x) = \frac{1}{\sqrt{2VE_{p'}}}\bar{u}(p')e^{ip'\cdot x}\,\langle 0| = \{c(p'), \bar{\psi}^-(x')\}\,\langle 0|\,. \tag{6.6.8}$$

With that we are now able to calculate the first order $\hat{S}$-matrix elements

$$\begin{aligned}\langle f|\hat{S}^1|i\rangle &= ie\int \frac{d^3\vec{q}}{(2\pi)^3}\frac{1}{\sqrt{2VE_p}}\frac{1}{\sqrt{2VE_{p'}}}\bar{u}(p')\tilde{\not{A}}_{\text{ext.}}(\vec{q})u(p)\cdot\int d^4x e^{-ix\cdot(p'-p)}e^{i\vec{q}\cdot\vec{x}} \\ &= (2\pi)\delta(E'_p - E_p)\frac{1}{\sqrt{2VE_p}}\frac{1}{\sqrt{2VE_{p'}}}\mathcal{M},\end{aligned} \tag{6.6.9}$$

with the Feynman amplitude $\mathcal{M}$. In this case we can simply read off the latter as

$$\mathcal{M} = :e\bar{u}(\vec{p}\,')\tilde{\not{A}}_{\text{ext.}}(\vec{q}=\vec{p}\,'-\vec{p})u(p):, \tag{6.6.10}$$

where $E_{p'} = E_p$, which is equivalent to $|\vec{p}\,'| = |\vec{p}\,|$. We use that no energy is transformed in the scattering process. Thus, we add an additional Feynman rule for external (static) classical fields. We have an additional graphic notation

(6.6.11)

which indicates the classical field and we add a factor

$$\tilde{A}^{\mu}_{\text{ext.}}(\vec{q}) = \int d^3x e^{-iq\cdot x}. \tag{6.6.12}$$

As a next step we can calculate the transition rate, which was defined as

$$w^T_{fi} = \frac{|\langle f|\hat{S}-\mathbb{1}|i\rangle|^2}{T} = 2\pi\delta_T(E_{p'}-E_p)\frac{1}{(2VE_p)^2}|\mathcal{M}|^2. \tag{6.6.13}$$

For the cross section, for which we have to divide the transition rate by the flux factor $\rho|\vec{v}| = \frac{1}{V}|\vec{v}| = \frac{1}{V}\frac{|\vec{p}|}{E_p}$ of the incoming electron beam[6], we have

$$\sigma^{TV} = \frac{1}{\frac{|\vec{v}|}{V}}\sum_{\vec{p}\,'\in\text{bins}} w^{TV}_{fi} = \frac{1}{\frac{|\vec{p}|}{E_pV}2VE_p}\frac{1}{V}\sum_{p'}\frac{(2\pi)\delta_T(E_{p'}-E_p)}{2E_{p'}}|\mathcal{M}|^2. \tag{6.6.14}$$

Taking the continuum limit $V, T \to \infty$ gives

$$\sigma = \frac{1}{2|\vec{p}|}\int_{\text{bins}}\frac{d^3p'}{(2\pi)^3 2E_{p'}}(2\pi)\delta(E_{p'}-E_p)|\mathcal{M}|^2(p'). \tag{6.6.15}$$

In the same way we are now able to calculate the differential cross section, for which we take infinitesimal small bins, that is, bin $= [\vec{p}\,', \vec{p}\,' + d^3\vec{p}\,']$,

$$d\sigma = \frac{1}{2|\vec{p}|(2\pi)^2}\delta(E_{p'}-E_p)|\mathcal{M}|^2(\vec{p}\,')\frac{1}{2E_{p'}}d^3p'. \tag{6.6.16}$$

This expression is still too "differential" due to the appearing δ-distribution. In order to deal with that experimentally, that is, in order to compare with experiments, we have to get rid of that δ-distribution. For that we use spherical coordinates and write

[6] Recall that the 4-vector $v^\mu = \gamma(c=1, \vec{v})$, that $p^\mu = mv^\mu = m\gamma(1,\vec{v}) = (E_p, \vec{p})$ and thus $|\vec{v}| = |\vec{p}|/E_p$.

$d^3p' = |\vec{p}'|^2 d|\vec{p}'|d\Omega$. What we have to do in order to integrate over p' is to write the δ-distribution in a more suitable way

$$\delta(E_{p'} - E_p) = \delta(\sqrt{\vec{p}'^2 + m_e^2} - \sqrt{\vec{p}^2 + m_e^2}) = \frac{\sqrt{(\vec{p}\,p)^2 + m_e^2}}{|\vec{p}'|}\delta(|\vec{p}' - |\vec{p}||), \tag{6.6.17}$$

where we used the known Eq. (6.5.7). Integration over the differential cross section then yields

$$d\sigma(\Omega) = \int d|\vec{p}'|d\sigma = \frac{1}{4(2\pi)^2}|\mathcal{M}|^2(\Omega)|_{|\vec{p}|=|\vec{p}'|}d\Omega. \tag{6.6.18}$$

Inserting the Feynman amplitude for the classical electromagnetic field (cf. Eq. (6.6.10)) then gives us, for example, the differential cross section for a scattering off a Coulomb field with

$$\left(\frac{d\sigma(\Omega)}{d\Omega}\right) = \frac{1}{(4\pi)^2}|\mathcal{M}|^2(\Omega). \tag{6.6.19}$$

Example: Form factor of static charge distribution

Consider a static charge distribution that is smeared out, that is, $j^{\mu=0}(\vec{x}) = Ze\rho(\vec{x})$, where the total charge is Ze and $\int d^3x\rho(\vec{x}) = 1$. The 4-potential of this configuration is

$$A^{\mu}_{\text{ext.}}(\vec{x}\,) = (\phi, \vec{0}) \tag{6.6.20}$$

with the electrostatic potential $\phi(\vec{r}) = \frac{Ze}{4\pi}\int d^3r' \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|}$ which is the standard solution to the Poisson equation $\nabla^2\phi(\vec{r}) = -Ze\rho(\vec{r})$. This helps us to make a connection between ϕ and ρ, which we need in Eq. (6.6.20). Next we need the Fourier transformation $\mathcal{F}$ of the electrostatic potential ϕ

$$\begin{aligned}
\int d^3r e^{-i\vec{q}\cdot\vec{r}}\phi(\vec{r}) &= \frac{Ze}{4\pi}\int d^3r \int d^3r' e^{-i\vec{q}\cdot\vec{r}}\frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|} \\
&= \frac{Ze}{4\pi}\left(-\frac{1}{\vec{q}^2}\right)\int d^3r \int d^3r' e^{-i\vec{q}\cdot\vec{r}}\rho(\vec{r}')\vec{\nabla}^2\frac{1}{|\vec{r}-\vec{r}'|} \\
&= \frac{Ze}{4\pi}\frac{4\pi}{\vec{q}^2}\int d^3r e^{-i\vec{q}\cdot\vec{x}}\rho(\vec{r}') \\
&= \frac{Ze}{\vec{q}^2}\mathcal{F}(\vec{q}),
\end{aligned} \tag{6.6.21}$$

where we use $e^{-i\vec{q}\cdot\vec{r}} = -\frac{1}{\vec{q}^2}\vec{\nabla}^2 e^{-i\vec{q}\cdot\vec{r}}$ and $\vec{\nabla}^2\frac{1}{|\vec{r}-\vec{r}'|} = 4\pi\delta^{(3)}(\vec{r}-\vec{r}')$. The Fourier transform of the charge distribution $\rho(\vec{r}\,)$ is denoted as $\mathcal{F}(\vec{q}\,) \equiv \int d^3r e^{-i\vec{q}\cdot\vec{x}}\rho(\vec{r}')$. Then the differential cross section is

$$\frac{d\sigma}{d\Omega} = \frac{1}{(4\pi)^2}\frac{Z^2e^4}{\vec{q}^4}|\mathcal{F}(\vec{q}\,)|^2 \cdot \frac{1}{2}\sum_{\text{spins}}|\bar{v}(\vec{p}\,')\gamma^0 v(\vec{p}\,)|^2 \tag{6.6.22}$$

and thus finally we obtain

$$\frac{d\sigma}{d\Omega} = \left(\frac{d\sigma}{d\Omega}\right)_{\text{Mott}} \cdot |\mathcal{F}(\vec{q})|^2, \tag{6.6.23}$$

where

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{Mott}} = \frac{(\alpha Z)^2(1 - v^2\sin^2(\theta/2))}{4E^2v^4\sin^4(\theta/2)}, \tag{6.6.24}$$

with $\alpha = e^2/4\pi$ and $v = |\vec{p}|/E$ for relativistic electrons. To show the identity of Eq. (6.6.22) with Eq. (6.6.23) remains as an exercise for the reader. Note that θ is the angle between $\vec{p}\,'$ and $\vec{p}$. The form factor

$$\mathcal{F}(\vec{q}) = \int d^3r\rho(\vec{r})e^{i\vec{q}\cdot\vec{r}}, \qquad \vec{q} = \vec{p} - \vec{p}\,' \longrightarrow \vec{q}^2 = 4p^2\sin^2(\theta/2) \tag{6.6.25}$$

for a point particle in the origin with $\rho(\vec{r}) = \delta(\vec{r})$ is

$$\mathcal{F}(\vec{q}) = 1 \tag{6.6.26}$$

and for a smooth distribution $\rho(\vec{r}) = \frac{m^3}{8\pi}e^{-mr}$, the form factor is

$$\mathcal{F}(\vec{q}) = \left(1 + \frac{\vec{q}^2}{m^2}\right)^{-2}. \tag{6.6.27}$$

The last form factor has a so-called *dipole form*.

An overview of different charge distributions and their corresponding form factors can be seen in Fig. 6.28. Using electron scattering on a charge distribution one is able

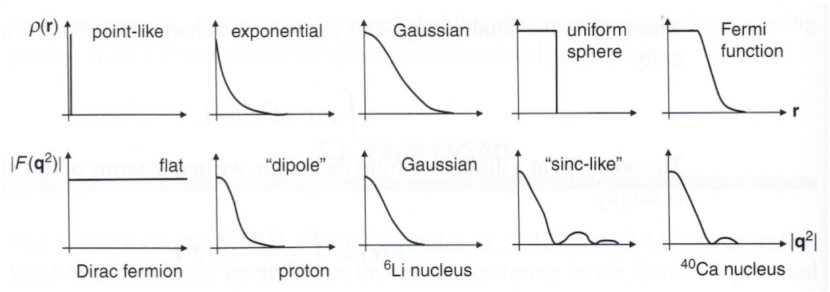


Fig. 6.28: Overview of different charge distributions and their corresponding formfactors. Taken from Thomson, M.: Modern Particle Physics, Cambridge University Press, 2013.

to relate this property to the form factor known for all values of $\vec{q}^2$. Hence, we are able

to determine, whether a particle is fundamental or not.