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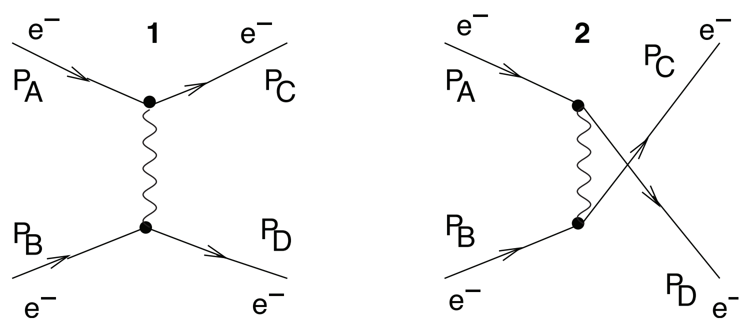
Exercise sheet 11
WS 2022/23
January 24, 2023

Quantum Field Theory

Exercise 40:

We consider Møller scattering which is the QED process $e^- \, e^- \to e^- \, e^-$. In lowest order of perturbative QED this scattering process is described by following Feynman diagrams with the 4-momenta p_i (i=A,B,C,D):



(a) Write down an expression for the invariant Feynman amplitude $\mathcal{M} = \mathcal{M}_1 + \mathcal{M}_2$ following the Feynman rules which includes the contributions $\mathcal{M}_{i=1,2}$ of both diagrams. (4P)

(b) Show that the spin-summed modulus squared of the Feynman amplitude in the high-energy limit (the mass m_e of the electron can be neglected) is given by

$$\overline{|\mathcal{M}|^2} \; = \; \frac{1}{4} \sum_{\textit{spin projections}} |\mathcal{M}_1 + \mathcal{M}_2|^2 \; = \; 2e^4 \left\{ \frac{s^2+u^2}{t^2} + \frac{2s^2}{tu} + \frac{s^2+t^2}{u^2} \right\} \, .$$

The quantities s, u and t are the invariant Mandelstam variables which are defined with respect to the p_i (i=A,B,C,D) as

$$\begin{aligned} s &\equiv (p_A + p_B)^2 \approx 2p_A \cdot p_B \approx 2p_C \cdot p_D \\ t &\equiv (p_A - p_C)^2 \approx -2p_A \cdot p_C \approx -2p_B \cdot p_D \\ u &\equiv (p_A - p_D)^2 \approx -2p_A \cdot p_D \approx -2p_C \cdot p_B \end{aligned}$$

and where the last identities correspond to the high-energy limit. (8P)

Hint: to evaluate the spin-summed modulus squared of $\mathcal{M}_1$ and $\mathcal{M}_2$ you can use results already derived in the context of $e^+e^- \to \mu^+\mu^-$.

(c) Show that the differential cross-section in the center-of-momentum (CM) system for electron-electron scattering in the high-energy limit ($E >> m_e$) is given by (8P)

$$\left(\frac{d\sigma}{d\Omega}\right)_{CM} = \frac{\alpha^2}{8E^2}\left[\frac{1+\cos^4(\theta/2)}{\sin^4(\theta/2)} + \frac{2}{\sin^2(\theta/2)\cos^2(\theta/2)} + \frac{1+\sin^4(\theta/2)}{\cos^4(\theta/2)}\right]$$

where θ is the scattering angle and E is the energy of either electron in the CM frame. The result for the differential cross section is the high-energy limit of the Møller formula already derived in C. Møller, *Ann. Phys.*, **14**, 531 (1932).

Exercise 41 (bonus points):

In the lecture we introduced the polarization vector $\varepsilon_r^\mu(k)$ in the plane wave solutions of the photon. Those polarization vectors are gauge dependent. For example, for a free photon, described in a Lorentz gauge by a plane wave $A^\mu(x) = \text{const}\,\varepsilon_r^\mu(k)\,\mathrm{e}^{\pm ik\cdot x}$, the gauge transformation $A^\mu(x) \to A^\mu(x) + \partial^\mu f(x)$, with $f(x) = \tilde{f}(k)\,\mathrm{e}^{\pm ik\cdot x}$, implies

$$\varepsilon_r^\mu(k)\,\mathrm{e}^{\pm ik\cdot x} \to [\varepsilon_r^\mu(k) \pm i\tilde{f}(k)\,k^\mu]\,\mathrm{e}^{\pm ik\cdot x}.$$

Consider the invariant amplitude $\mathcal{M}$ for the Compton scattering process of a photon and an electron, $\gamma(\boldsymbol{k}) + e^-(\boldsymbol{p}) \to \gamma(\boldsymbol{k}') + e^-(\boldsymbol{p}')$.

(a) Draw all of the lowest order QED Feynman diagrams for this process and give explicit expressions for them using Feynman rules. (2P)

(b) Show that the sum of the lowest order Compton scattering Feynman diagrams is gauge invariant, although the individual contributions of the diagrams are not, by considering the gauge transformations (8P)

$$\varepsilon_r^\mu(k) \to \varepsilon_r^\mu(k) + \lambda\,k^\mu \quad ; \quad \varepsilon_s'^\mu(k') \to \varepsilon_s'^\mu(k') + \lambda'\,k'^\mu.$$

Worked-out solutions to the homework problems should be handed in latest at the beginning of the lecture of January 31, 2023.