Institut für Theoretische Physik
Universität Tübingen
T. Gutsche

Exercise sheet 2
WS 2022/23
October 28, 2022

Quantum Field Theory

Exercise 5:

Show that the Lagrange density $\mathcal{L} = \mathcal{L}(\phi, \partial_\mu \phi)$ of a field ϕ can be replaced by

$$\mathcal{L}' \;=\; \mathcal{L} \;+\; \partial_\mu \Lambda^\mu(x)$$

without changing the equations of motion. Thereby the $\Lambda^\mu(x)$ with $\mu = 0, .., 3$ are arbitrary functions of the field $\phi(x)$. (4P)

Exercise 6:

Consider the Lagrange density of real vector fields $A^\mu(x)$,

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j^\mu A_\mu \quad \text{with} \quad F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu$$

and show that the Euler-Lagrange equations yield the Maxwell equations (4P)

$$\partial_\mu F^{\mu\nu} \;=\; j^\nu \;.$$

Exercise 7:

Classical electrodynamics (without external sources) follows from the Lagrange density

$$\mathcal{L} \;=\; -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \;.$$

Use the energy-momentum tensor $T^{\mu\nu}$ to show that the energy of the electromagnetic field can be written as (6P)

$$P^0 = \int d^3x \; T^{00} = \int d^3x \left\{ \frac{\partial \mathcal{L}}{\partial(\partial_0 A_\alpha)} \partial^0 A_\alpha - \mathcal{L} \right\} = \int d^3x \frac{1}{2} (\vec{E}^2 + \vec{B}^2) \;.$$

Exercise 8:

We start with the Lagrange density of a complex scalar field ϕ with

$$\mathcal{L} = (\partial_\mu \phi)^* (\partial^\mu \phi) - m^2 \phi^* \phi \, .$$

(a) Derive the resulting equations of motion using $\mathrm{Re}(\phi)$, $\mathrm{Im}(\phi)$ or ϕ, ϕ^* as independent components of the complex field ϕ. (3P)

(b) Show that the invariance of $\mathcal{L}$ under the phase transformation

$$\phi(x) \to \mathrm{e}^{-\mathrm{i}e\epsilon} \phi(\mathrm{x}) \quad ; \quad \phi^*(x) \to \mathrm{e}^{+\mathrm{i}e\epsilon} \phi^*(\mathrm{x})$$

leads to a continuity equation with the conserved current (3P)

$$j^\mu \quad = \quad -ie(\phi^* \partial^\mu \phi - \phi \partial^\mu \phi^*) \, .$$

Worked-out solutions to the homework problems should be handed in at the beginning of the lecture of Tuesday, Nov. 8. Please note again that the first exercise classes will start on Nov. 3.