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Exercise sheet 6
WS 2022/23
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Quantum Field Theory

Exercise 20:

Show in the following that the total angular momentum $\vec{J} = \vec{L} + \vec{S}$ of a Dirac-particle with the Hamilton-operator $\hat{H} = \vec{\alpha} \cdot \hat{\vec{p}} + \beta m$ is a conserved quantity.

(a) First show that $[\hat{H}, \hat{\vec{L}}] = -i\vec{\alpha} \times \hat{\vec{p}}$, where $\hat{\vec{L}} = \hat{\vec{r}} \times \hat{\vec{p}}$ is the orbital angular momentum operator. (2P)

(b) In addition you should prove that $[\hat{H}, \frac{1}{2}\vec{\Sigma}] = i\vec{\alpha} \times \hat{\vec{p}}$, where $\frac{1}{2}\vec{\Sigma} = \hat{\vec{S}} = \frac{1}{2}\begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}$. (2P)

Exercise 21:

(a) Operate on the Dirac equation $(i\gamma^\mu \partial_\mu - m)\,\Psi = 0$ with $\gamma^\nu \partial_\nu$ and show that each of the four components ψ_i of the Dirac spinor satisfies the Klein-Gordon equation (2P)

$$(\Box + m^2)\psi_i = 0 \, .$$

(b) Solutions of the Dirac equation for a free particle of mass m and energy $E = \sqrt{\vec{p}^{\,2} + m^2} > 0$ can be expressed in the form

$$\Psi_{s=1,2}(x) = \sqrt{E+m} \begin{pmatrix} \chi_s \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\chi_s \end{pmatrix} e^{-ip\cdot x} \, .$$

Show that the 4-current $\bar{\Psi}\gamma^\mu\Psi$ with $\bar{\Psi} = \Psi^\dagger\gamma^0$ satisfies the relation (4P)

$$\bar{\Psi}(x)\gamma^\mu\Psi(x) = 2p^\mu = 2(E, \vec{p}) \, .$$

Exercise 22:

The free particle and antiparticle solutions of the Dirac equation have been given in momentum space as

$$u_s(p) \equiv u(p,s) = \sqrt{E+m}\begin{pmatrix} \chi_s \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\chi_s \end{pmatrix} \text{ and } v_s(p) \equiv v(p,s) = \sqrt{E+m}\begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\chi_s \\ \chi_s \end{pmatrix}$$

where $p^0 = E > 0$ (both for u and v). Both solutions satisfy the Dirac equations

$$(\not{p} - m)u(p,s) = 0, \quad (\not{p} + m)v(p,s) = 0 \, .$$

(a) Show that the corresponding equations for the adjoint spinors are (2P)

$$\bar{u}(p,s)(\not{p} - m) = 0 \, , \quad \bar{v}(p,s)(\not{p} + m) = 0 \, .$$

(b) Derive the following completeness relations (4P)

$$\sum_{s=1,2} u(p,s)\bar{u}(p,s) = \not{p} + m \, ,$$

$$\sum_{s=1,2} v(p,s)\bar{v}(p,s) = \not{p} - m \, .$$

Note that these are 4×4 matrix relations which are often used in the evaluation of Feynman diagrams.

Exercise 23:

(a) Show that the matrix $\sigma^{\mu\nu} \equiv \frac{i}{2}[\gamma^\mu, \gamma^\nu]$ is traceless and that the following relation is valid for an arbitrary antisymmetric tensor $\omega_{\mu\nu}$, (2P)

$$-\frac{i}{4}\omega_{\mu\nu}[\gamma^\rho, \sigma^{\mu\nu}] = \omega^\rho{}_\sigma \gamma^\sigma \, .$$

(b) Another important Dirac matrix is the matrix $\gamma_5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3$. Show that it anti-commutes with all the other γ-matrices, i.e., $\{\gamma^\mu, \gamma_5\} = 0$. Also, show that $\gamma_5^2 = 1$ and that γ_5 is hermitian. Express γ_5 explicitly in the Dirac-Pauli representation. (2P)

Worked-out solutions to the homework problems should be handed in at the beginning of the lecture of Tuesday, Dec. 6.