Sheet 3:

Exercise 1:

Potential
$$V(z) = \begin{cases} eFz & z > 0 \\ \infty & z < 0 \end{cases}$$

Fang-Howard-Wavefunctions:

$$\phi(z) = \begin{cases} cz \exp(-bz/2) & z > 0 \\ 0 & z < 0 \end{cases}$$

a) $\int_0^\infty \phi^*(z)\phi(z)\,dz = \int_0^\infty |c|^2 z^2 e^{-bz}\,dz = \left[-\frac{|c|^2}{b} z^2 e^{-bz}\right]_0^\infty$

$+ \int_0^\infty 2\frac{|c|^2 z}{b} e^{-bz}\,dz = \left[-\frac{|c|^2}{b^2} 2z e^{-bz}\right]_0^\infty + \int_0^\infty 2\frac{|c|^2}{b^2} e^{-bz}$

$= \left[-2\frac{|c|^2}{b^3} e^{-bz}\right]_0^\infty = \frac{2|c|^2}{b^3}$

$\Rightarrow c = \sqrt{b^3/2}$

b) $E_0 := \int_{-\infty}^{\infty} \phi^* \left(-\frac{\hbar^2}{2m}\partial_z^2 + V(z)\right)\phi\,dz$

$= \int_0^\infty \left\{-\frac{\hbar^2}{2m} cz\left(b - \frac{b^2}{2}z\right)e^{-bz} + eFz\, b\, z\, e^{-bz}\right\} dz$

$= \int_0^\infty \left\{-\frac{\hbar^2}{2m}\frac{b^3}{2} z\left(1 - \frac{b}{2}z\right)e^{-bz} + eFz^3\frac{b^3}{2}e^{-bz}\right\} dz$

$= -\frac{\hbar}{2m}\left\{\left[-\frac{b^2}{2} z\left(1 - \frac{b}{2}z\right)e^{-bz}\right]_0^\infty + \int_0^\infty \frac{b^2}{2}(1 - bz)e^{-bz}\,dz\right\}$

$+ eF\left(\left[-\frac{b^2}{2} z^3 e^{-bz}\right]_0^\infty + \int_0^\infty \frac{3b^2}{2} z^2 e^{-bz}\right.$

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$$\cancel{= -\frac{\hbar^2}{2m}\left\{ \left[-\frac{b}{2}(1-bz)e^{-bz}\right]_0^\infty + \int_0^\infty \frac{b^2}{2} e^{-bz} \right.}$$

$$\cancel{+3\frac{eF}{b} = -\frac{\hbar^2}{2m}\left(\frac{b}{2}-\frac{b}{2}\right) + \frac{3eF}{b} = \frac{3eF}{b}}$$

$$\partial_z^2 \phi(z) = \partial_z\, c\left(1-\frac{b}{2}z\right)e^{-bz/2} = c\left(-\frac{b}{2}-\frac{b}{2}+\frac{b^2}{4}z\right)e^{-bz/2}$$

$$\Rightarrow \varepsilon_0 = \int_0^\infty \left\{ -\frac{\hbar^2}{2m}\frac{b^3}{2} z \left(\frac{b^2}{4}z - b\right) e^{-bz} + \underbrace{eFz^3 \frac{b^3}{2} e^{-bz}}_{= \left[-eFz^3 \frac{b^2}{2} e^{-bz}\right]_0^\infty + \int_0^\infty eFz^2 \frac{3}{2}b^2 e^{-bz}\,dz} \right\} dz$$

$$= \int_0^\infty \frac{\hbar^2}{2m}\frac{b^4}{2} z e^{-bz}\,dz + \int_0^\infty \left(-\frac{\hbar^2}{2m}\frac{b^5}{8} + eF\frac{3}{2}b^2\right) z^2 e^{-bz}\,dz$$

$$= \int_0^\infty \frac{\hbar}{2m}\frac{b^3}{2} e^{-bz}\,dz + \left(\frac{eF}{2}3b^2 - \frac{\hbar^2}{2m}\frac{b^5}{8}\right)\frac{2}{b^3}$$

$$= \frac{\hbar}{2m}\frac{b^2}{2} + \left(\frac{3eF}{b} - \frac{\hbar^2}{2m}\frac{b^2}{4}\right) = \frac{3eF}{b} + \frac{\hbar^2}{2m}\frac{b^2}{4}$$

c) minimize ε_0

$$0 \overset{!}{=} \frac{\partial \varepsilon_0}{\partial b} = \frac{\hbar^2}{2m}\frac{b}{2} - \frac{3eF}{b^2} \Rightarrow b^3 = \frac{12\,emF}{\hbar^2} \rightsquigarrow b_0 = \left(\frac{12\,meF}{\hbar^2}\right)^{1/3}$$

$$\varepsilon_0(b_0) = 3\left(\frac{\hbar^2}{12m}\right)^{1/3}(eF)^{2/3} + \frac{1}{2}\left(\frac{\hbar^2}{m}\right)^{1/3}(12\,eF)^{2/3}$$

$$= \left(\left(\frac{9}{2}\right)^{1/3} + 6^{2/3}\right)\left(\frac{\hbar^2}{2m}\right)^{1/3}(eF)^{2/3} = (40{,}5)^{1/3}\left(\frac{\hbar^2}{2m}\right)^{1/3}(eF)^{2/3}$$

compared to $\left(\frac{9\pi}{8}\right)^{2/3}\left(\frac{\hbar^2}{2m}\right)^{1/3}(eF)^{2/3}$ with $\left(\frac{9\pi}{8}\right)^{2/3} = (12{,}5)^{1/3}$

We see that only the ~~terms~~ prefactor is different to the exact result, ~~which~~ which may be absorbed by the effective mass.

For the given values from the exercise $E_0 = 0{,}056\,eV$

d) $\int_{-\infty}^{\infty} \phi_1^* \phi_0 \, dz \overset{!}{=}$

$$= \int_0^{\infty} c\, z^2 (d - bz) \exp(-bz)\, dz = c \cdot d \frac{2}{b^3}$$

$$+ \left[c\, z^3 e^{-bz} \right]_0^{\infty} - \int_0^{\infty} 3c\, z^2 e^{-bz}\, dz = (cd - 3c) \frac{2}{b^3}$$

$\Rightarrow d = 3$

Exercise 2:

We look at a 2 dimensional problem with a potential

$V(x,y) = \frac{1}{2} m \omega_a^2 x^2$ ~~and~~ in a magnetic ~~Fi~~ Field $(0, 0, B)$

the corresponding vector potential $\vec{A} = (0, B_z x, 0)$ produces the correct B as $\vec{\nabla} \times \vec{A} = (0, 0, \partial_x B_z x) = \vec{B}$

The stationary Schrödinger equation reads

$$E\, \Psi(x,y) = \left(\frac{1}{2m} (\hat{\vec{p}} - e\vec{A})^2 + \hat{V}(x,y) \right) \Psi(x,y)$$

$$= \left(\frac{\hbar^2}{2m} (\partial_x^2 + \partial_y^2) + \frac{e^2 B^2 x^2}{2m} + i\hbar \frac{eB}{m} x\, \partial_y + \frac{1}{2} m \omega_a^2 x^2 \right) \Psi(x,y)$$

with the approach $\psi(x,y) = \phi(x)\, e^{i k_y y}$
we rewrite

$$\varepsilon\phi(x) = \left[-\frac{\hbar^2}{2m}\left(\partial_x^2 - k_y^2\right) + \frac{1}{2} m\omega_c^2 x^2 - \omega_c x k_y \hbar + \frac{1}{2} m \omega_a x^2\right]\phi(x)$$

$$= \left(-\frac{\hbar^2}{2m}\partial_x^2 + \underbrace{\frac{1}{2}m\left(\cancel{m}\omega_c x - \frac{\hbar}{m}k_y\right)^2}_{= \frac{1}{2} m\omega_c^2 \left(x - \frac{\hbar k_y}{m\omega_c}\right)^2} + \frac{1}{2} m\omega_a^2 x^2\right)\phi(x)$$

b) in order to derive the eigenenergies ε_n we rewrite the equation again.

$$\varepsilon\phi(x) = \left(-\frac{\hbar^2}{2m}\partial_x^2 + \frac{1}{2} m(\omega_c^2+\omega_a^2)x^2 - \omega_c \hbar k_y x + \frac{\hbar^2}{2m}k_y^2\right)\phi(x)$$

$$= \left[-\frac{\hbar^2}{2m}\partial_x^2 + \frac{1}{2} m(\omega_c^2+\omega_a^2)\left(x - \frac{\hbar k_y \omega_c}{m(\omega_c^2+\omega_a^2)}\right)^2 + \frac{\hbar^2 k_y^2}{2m}\Big(1 - \underbrace{\frac{\omega_c^2}{\omega_c^2+\omega_a^2}}_{= \frac{\omega_a^2}{\omega_c^2+\omega_a^2}}\Big)\right]\phi(x) = \varepsilon(x)$$

$$\Rightarrow \left(-\frac{\hbar^2}{2m}\partial_x^2 + \frac{1}{2} m(\omega_c^2+\omega_a^2)\left(x - \frac{\hbar k_y \omega_c}{m(\omega_c^2+\omega_a^2)}\right)^2\right)\phi(x)$$

$$\underbrace{\left(\varepsilon - \frac{\hbar^2 k_y^2}{2m}\frac{\omega_a^2}{\omega_c^2+\omega_a^2}\right)}_{=: \tilde{\varepsilon}}\phi(x)$$

This the equation of a harmonic oscillator moved in space an energy. It has eigenenergies of the known form

$$\tilde{\varepsilon}_n = \hbar\tilde{\omega}\left(n+\frac{1}{2}\right) \quad \text{where } \tilde{\omega} = \sqrt{\omega_c^2+\omega_a^2}$$

and so therefore

$$\varepsilon_n = \tilde{\varepsilon}_n + \frac{\hbar^2 k_y^2}{2m}\frac{\omega_a^2}{\omega_a^2+\omega_c^2} =: \tilde{\varepsilon}_n + \frac{\hbar^2 k_y^2}{2m(B)}$$

c) as $\omega_c \sim B$, we get for very small fields our normal Landau-levels

$$\varepsilon \approx \hbar\omega_a\left(n+\frac{1}{2}\right)+\frac{\hbar^2 k_y^2}{2m}$$

when $\omega_a \approx \omega_c$ $\quad \varepsilon \approx \sqrt{2}\,\hbar\omega_a\left(n+\frac{1}{2}\right)+\frac{1}{2}\frac{\hbar^2 k_y^2}{2m}$

the ~~correction~~ landau lebel correction $\frac{\hbar^2 k_y^2}{2m}$ of the harmonic ~~oscillator~~ energies get less important, until the energie converge for $\omega_c \gg \omega_a$ ~~th~~ to the normal harmonic oscillator of the cyclotron frequency. $\varepsilon \approx \hbar\omega_c\left(n+\frac{1}{2}\right)$

(it's an harmonic energylevel structure)