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Exercise sheet 3
WS 2022/23
November 7, 2022

Quantum Field Theory

Exercise 9:

The real Klein-Gordon field is described by the Hamiltonian density

$$\mathcal{H} \;=\; \frac{1}{2}\left[\pi^2(x) + (\vec{\nabla}\phi(x))^2 + m^2\,\phi^2(x)\right]\,.$$

Use the commutation relations

$$[\phi(\vec{x},t),\pi(\vec{y},t)] \;=\; i\,\delta^{(3)}(\vec{x}-\vec{y})\,,$$

$$[\phi(\vec{x},t),\phi(\vec{y},t)] \;=\; [\pi(\vec{x},t),\pi(\vec{y},t)] = 0\,,$$

to show that

$$[H,\phi(x)] = -i\,\pi(x) \quad , \quad [H,\pi(x)] = i(m^2-\vec{\nabla}^2)\phi(x)\,,$$

where $H = \int d^3x\,\mathcal{H}$ is the Hamilton operator. Use Heisenberg's equations of motion for the operators $\phi(x)$ and $\pi(x)$ to show that (6P)

$$\dot{\phi}(x) = \pi(x) \quad , \quad (\partial^2 + m^2)\phi(x) = 0\,.$$

Exercise 10:

From the expansion of the free real Klein-Gordon field $\phi(x)$ with

$$\phi(x) \;=\; \frac{1}{\sqrt{V}}\sum_{\vec{k}}\frac{1}{\sqrt{2\omega_{\vec{k}}}}\left(a_{\vec{k}}\,e^{-ik\cdot x} + a^\dagger_{\vec{k}}\,e^{ik\cdot x}\right)\;,$$

where $\omega_{\vec{k}} = \sqrt{(\vec{k})^2 + m^2}$, derive the following expression for the annihilation operator $a_{\vec{k}}$:

$$a_{\vec{k}} \;=\; \frac{1}{\sqrt{2V\omega_{\vec{k}}}}\int d^3x\,e^{ik\cdot x}\left(i\,\dot{\phi}(x) + \omega_{\vec{k}}\,\phi(x)\right)\,.$$

Hence derive the commutation relations

$$[a_{\vec{k}}, a^\dagger_{\vec{k}'}] = \delta_{\vec{k},\vec{k}'} \quad , \quad [a_{\vec{k}}, a_{\vec{k}'}] = [a^\dagger_{\vec{k}}, a^\dagger_{\vec{k}'}] = 0$$

from the commutation relations for the fields $\phi(x)$ and $\pi(x)$ (cf. exercise 9). (6P)

Exercise 11:

The normal-ordered Hamilton operator of the free real Klein-Gordon field is given by

$$H = \sum_{\vec{p}} \omega_{\vec{p}}\, a^\dagger_{\vec{p}} a_{\vec{p}} \equiv \sum_{\vec{p}} \omega_{\vec{p}} \mathcal{N}(\vec{p})\,,$$

with $\omega_{\vec{p}} = \sqrt{\vec{p}^2 + m^2}$. H is written either in terms of the usual creation and annihilation operators $a^\dagger_{\vec{p}}$, $a_{\vec{p}}$ or the occupation number operator

$$\mathcal{N}(\vec{p}) \equiv a^\dagger_{\vec{p}} a_{\vec{p}}\,.$$

We further have $N \equiv \sum_{\vec{p}} \mathcal{N}(\vec{p})$.

(a) Use the following commutator relations

$$[a_{\vec{p}}, a^\dagger_{\vec{p}'}] = \delta_{\vec{p},\vec{p}'} \quad , \quad [a_{\vec{p}}, a_{\vec{p}'}] = [a^\dagger_{\vec{p}}, a^\dagger_{\vec{p}'}] = 0 \tag{1}$$

to show that (4P)

$$\begin{array}{rl}
(i) & [\mathcal{N}(\vec{p}), a_{\vec{p}'}] = -\delta_{\vec{p},\vec{p}'} a_{\vec{p}}\,, \\
(ii) & [\mathcal{N}(\vec{p}), a^\dagger_{\vec{p}'}] = \delta_{\vec{p},\vec{p}'} a^\dagger_{\vec{p}}\,, \\
(iii) & [\mathcal{N}(\vec{p}), \mathcal{N}(\vec{p}')] = 0\,, \\
(iv) & [\mathcal{N}(\vec{p}), H] = 0\,, \\
(v) & [N, a_{\vec{p}}] = -a_{\vec{p}}\,, \\
(vi) & [N, a^\dagger_{\vec{p}}] = a^\dagger_{\vec{p}}\,, \\
(vii) & [N, H] = 0\,.
\end{array}$$

(b) We define the multi-particle state $|n\rangle$ by

$$|n\rangle \equiv \sum_{\vec{p}_1} ... \sum_{\vec{p}_n} \varphi(\vec{p}_1, ..., \vec{p}_n)\; a^\dagger_{\vec{p}_1} ... a^\dagger_{\vec{p}_n}\, |0\rangle$$

with a suitable function $\varphi(\vec{p}_1, ..., \vec{p}_n)$. Show that the state $|n\rangle$ contains n particles by proving that (2P)

$$N|n\rangle = n\,|n\rangle\,.$$

(c) Is there a change in the relation $(i) - (vii)$ if the commutators (1) are replaced by anticommutators? (2P)

Worked-out solutions to the homework problems should be handed in at the beginning of the lecture of Tuesday, Nov. 15.