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Exercise sheet 10
WS 2022/23
January 17, 2023

Quantum Field Theory

Exercise 37:

In the expression for the differential cross section $d\sigma = \frac{|\mathcal{M}|^2}{F} dQ$ of a collision between A and B we introduced the flux factor with $F = 4E_A E_B |\vec{v}_A - \vec{v}_b|$ involving the modulus of the relative velocity $|\vec{v}_A - \vec{v}_b|$. Show that for a general collinear collision between A and B the flux factor can be written in the manifestly invariant form

$$F = 4\left\{(p_A \cdot p_B)^2 - m_A^2 m_B^2\right\}^{1/2} \ .$$

The quantities $m_{A(B)}$, $p_{A(B)}$, $E_{A(B)}$ and $\vec{v}_{A(B)}$ refer to mass, 4-momentum, energy and velocity of particle A (B) in the initial state. (4P)

Exercise 38:

The γ-matrices fulfill the anticommutation relations

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu}$$

where an identity matrix $\mathbb{1}_{4\times 4}$ is implicit on the right-hand side.

(a) Show the following trace identities (4P),

$$\begin{aligned}
\mathrm{Tr}[\gamma^\mu \gamma^\nu] &= 4g^{\mu\nu} \, , \\
\mathrm{Tr}[\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma] &= 4(g^{\mu\nu} g^{\rho\sigma} + g^{\mu\sigma} g^{\nu\rho} - g^{\mu\rho} g^{\nu\sigma}) \, , \\
\mathrm{Tr}[\gamma^\mu \gamma^\nu \gamma_5] &= 0 \, , \\
\mathrm{Tr}[\gamma^{\mu_1} ... \gamma^{\mu_n}] &= 0 \, \mathrm{for\, n\, odd.}
\end{aligned} \tag{1}$$

(b) Prove the following contraction identities (4P)

$$\begin{aligned}
\gamma^\mu \gamma_\mu &= 4, \\
\gamma_\mu \gamma^\alpha \gamma^\mu &= -\, 2\gamma^\alpha, \\
\gamma_\mu \gamma^\alpha \gamma^\beta \gamma^\mu &= 4g^{\alpha\beta}, \\
\gamma_\mu \gamma^\alpha \gamma^\beta \gamma^\gamma \gamma^\mu &= -\, 2\, \gamma^\gamma \gamma^\beta \gamma^\alpha \, .
\end{aligned}$$

Exercise 39:

We consider the scattering process

$$A(p_A) + B(p_B) \to C(p_C) + D(p_D)$$

with corresponding 4-momenta $p_A, \ldots, p_D$. In the following we evaluate the generic expression for the differential cross section in the centre-of-momentum (CM) frame defined by

$$\vec{p}_A + \vec{p}_B = \vec{p}_C + \vec{p}_D = 0 \ .$$

The 4-momenta of the particles in the inital and final state can therefore be written as $p_A = (E_A, \vec{p}_i)$, $p_B = (E_B, -\vec{p}_i)$, $p_C = (E_C, \vec{p}_f)$ und $p_D = (E_D, -\vec{p}_f)$.
In the CM frame for the process $AB \to CD$ show that the flux factor F and the Lorentz invariant phase space factor dQ can be expressed as:

$$F = 4p_i\sqrt{s} \quad , \quad dQ = \frac{1}{(4\pi)^2}\,\frac{p_f}{\sqrt{s}}\,d\Omega$$

where $d\Omega$ is the element of the solid angle about $\vec{p}_C$, $s = (E_A + E_B)^2$, $|\vec{p}_A| = |\vec{p}_B| = p_i$ and $|\vec{p}_C| = |\vec{p}_D| = p_f$. Hence the differential cross section is (8P)

$$\frac{d\sigma}{d\Omega}_{CM} = \frac{1}{64\pi^2 s}\,\frac{p_f}{p_i}|\mathcal{M}|^2 \quad .$$

Hints:

i) the flux factor is defined as $F = 4\left\{(p_A \cdot p_B)^2 - m_A^2 m_B^2\right\}^{1/2}$,

ii) the Lorentz-invariant phase space factor is defined as

$$dQ = \frac{1}{(2\pi)^2}\delta^{(4)}(p_A + p_B - p_C - p_D)\,\frac{d^3p_C}{2E_C}\,\frac{d^3p_D}{2E_D} \ ,$$

iii) integrate with respect to $\vec{p}_D$ and to $|\vec{p}_C|$.

Worked-out solutions to the homework problems should be handed in latest at the beginning of the lecture of January 24, 2023.