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The Nematic Phase of a System of Long Hard Rods

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Dedicated to the 70^{th} *birthday of Giovanni Gallavotti*

Abstract: We consider a two-dimensional lattice model for liquid crystals consisting of long rods interacting via purely hard core interactions, with two allowed orientations defined by the underlying lattice. We rigorously prove the existence of a nematic phase, i.e., we show that at intermediate densities the system exhibits orientational order, either horizontal or vertical, but no positional order. The proof is based on a two-scales cluster expansion: we first coarse grain the system on a scale comparable with the rods' length; then we express the resulting effective theory as a contour's model, which can be treated by Pirogov-Sinai methods.

1. Introduction

In 1949, L. Onsager [34] proposed a statistical theory for a system of elongated molecules interacting via repulsive short-range forces, based on an explicit computation of the first few Mayer's coefficients for the pressure. Onsager's theory predicted the existence at intermediate densities of a nematic liquid crystal phase, that is a phase in which the distribution of orientations of the particles is anisotropic, while the distribution of the particles in space is homogeneous and does not exhibit the periodic variation of densities that characterizes solid crystals (periodicity in all space dimensions) or smectic liquid crystals (periodicity in one dimension).

From a microscopic point of view, the most natural lattice model describing elongated molecules with short-range repulsive forces is a system of rods of length k and thickness 1 at fixed density ρ (here ρ = average number of rods per unit volume), arranged on a cubic lattice, say a large squared box portion of $\mathbb{Z}^2$, and interacting via a purely hard core potential. Even though very natural, this model is not easy to treat and its phase diagram in the plane (ρ, k) is still not understood in many physically relevant parameters' ranges. Of course, for all k's, at very small density there is a unique isotropic Gibbs state, invariant under translations and under discrete rotations of 90^o; this can be proved by standard cluster expansion methods. If $k = 2$, it is known [19] that the state is analytic and, therefore, there is no phase transition, for all densities but, possibly, at the close packing density, i.e., at the maximal possible density $\rho_{max} = 1/k$. If k is sufficiently large ($k \geq 7$ should be enough [15]) there is numerical evidence [15,30] for *two* phase transitions as ρ is increased from zero to the maximal density. The first, isotropic to nematic, seems to take place at a $\rho_c^{(1)} \simeq C_1/k^2$, while the second, nematic to isotropic, seems to take place at $\rho_c^{(2)} \simeq \rho_{max} - C_2/k^3$. These findings renovated the interest of the condensed matter community in the phase diagram of long hard rod systems and stimulated more systematic numerical studies of the nature of the critical points at $\rho_c^{(1)}$ and $\rho_c^{(2)}$ [8,10,28,29,31,32]. From a mathematical point of view there is no rigorous proof of any of these behaviors yet, with the exception of the "trivial" case of very low densities: namely, there is neither a proof of nematic order at intermediate densities, nor a proof of the absence of orientational order at very high densities, nor a rigorous understanding of the nature of the transitions.

In this work we give a rigorous proof of some of the conjectures stated above on the nature of the phase diagram of long hard rods systems. More precisely, we show that well inside the interval $(\rho_c^{(1)}, \rho_c^{(2)})$, the system is in a nematic phase, i.e., in a phase characterized by two distinguished Gibbs states, with different orientational order, either horizontal or vertical, but with no positional order. To the best of our knowledge, this is the first proof of the existence of a nematic phase in a microscopic model with molecules of fixed finite length and finite thickness, interacting via a purely repulsive potential. In this respect, our result is a strong confirmation of Onsager's proposal that orientational ordering can be explained as an excluded volume effect.

Our proof is based on a two-scales cluster expansion method, in which we first coarse grain the system on scale k; we next realize that the resulting effective model can be expressed as a contour model, reminiscent of the contour theory for the Ising model at low temperatures. However, contrary to the Ising case, the contour theory we have to deal with here is not invariant under a $\mathbb{Z}^2$ symmetry: therefore, we cannot apply the Peierls' argument and we need to make use of a Pirogov-Sinai method.

Of course, our proof leaves many questions about the phase diagram of long hard rod systems open, the most urgent being, we believe, the question about the nature of the densely packed phase at $\rho \geq \rho_c^{(2)}$: can one prove the absence of orientational order, at least at close packing? Is the densely packed phase characterized by some "hidden" (striped-like) order? Progress on these problems would be important for the understanding of the emergence of hidden order in more complicated systems than elongated molecules with purely hard core interactions, in which short range repulsion competes with attractive forces acting on much longer length scales.

Previous results. There is a limited number of papers where important previous results on the existence of orientational order in lattice or continuum models for liquid crystals were obtained, related to the ones found in this work.

A first class of liquid crystal models that has been considered in the literature describe long rods with purely repulsive interactions and discrete orientations, like ours; of course, the case of continuous orientations would be of great interest, but its treatment appears to be beyond the current state of the art. In [18,21,25], the existence of orientational order for different variants of lattice gases of anisotropic molecules with repulsive interactions was proved, by using Peierls-like estimates and cluster expansion; however, in all these cases, orientational comes together with translational order, which is not the case in a nematic phase. A continuum version of the model in [25], i.e., a continuum system of infinitely thin rods with two allowed orientations and hard core interactions, was later proved to have a phase transition from an isotropic to a nematic phase [5,38], by using improved estimates on the contours' probabilities and a Pirogov-Sinai method. More recently, the existence of an isotropic to nematic transition in an integrable model of polydisperse long rods in $\mathbb{Z}^2$ with hard core interactions was proved [22], by mapping the partition function of the polydisperse hard rods gas into that of the nearest neighbor two-dimensional Ising model.

A second class of liquid crystal models studied in the literature assumes the existence of attractive forces favoring the alignment of the molecular axes: in fact, in some cases, the attraction is expected to originate from the inter-molecular Coulomb interaction [33] and to play a more prominent role than the Onsager's excluded volume effect. The emergence of a nematic phase in such models was first understood at the mean field level [3,7,33,36]. Later, it was understood that in the presence of attractive forces, even the monomer-dimer system can exhibit an oriented phase at low temperatures, as proved in [20] by reflection positivity methods; the absence of positional order for the same model, known as the Heilmann-Lieb's model [20], was then proved on the basis of cluster expansion methods [27]. Remarkably, if attractive forces favoring the alignment of the molecules' axes are allowed, there are models displaying a full $O(m)$ orientational symmetry, $m \geq 2$, for which it is possible to rigorously prove the existence of nematic order (or quasi-long range order, depending on the dimensionality). In particular, in [1,2,40] certain d-dimensional lattice-gas models describing particles with an internal ("spin") continuous orientational degree of freedom were introduced; the existence of orientational order was proved both in $d > 2$ for short-ranged interactions and in $d = 1, 2$ with sufficiently long-ranged interactions, via a combination of infrared bounds and chessboard estimates [11]. In [17], a proof of the existence of orientational quasi-long range order à la Kosterlitz-Thouless was given for a similar system in $d = 2$ with short ranged interactions, by using a combination of the Gruber-Griffiths method [16], originally applied to the study of an orientational phase transition in a continuum system of particles with internal Ising-like degrees of freedom, and of the Fröhlich-Spencer method [13], originally applied to the study of the Kosterlitz-Thouless transition in the classical two-dimensional XY model.

Summary. The rest of the paper is organized as follows. In Sect. 2 we "informally" introduce the model, state the main results and explain the key ideas involved in the proof. In Sect. 3 we define the model and state the main theorem (Theorem 1 below) in a mathematically precise form. In the following sections we prove Theorem 1: in Sect. 4 we rewrite the partition function with q boundary conditions in terms of a sum over contours' configurations, where the contours are defined in a way suitable for later application of a Pirogov-Sinai argument. In Sect. 6 we prove the convergence of the cluster expansion for the pressure, under the assumption that the activity of the contours is small and decays sufficiently fast in the contour's size. In Sect. 7 we complete the proof of convergence of the cluster expansion for the pressure, by inductively proving the desired bound on the activity of the contours. Finally, in Sect. 8 we adapt our expansion to the computation of correlation functions and we prove Theorem 1, while in Sect. 9 we draw the conclusions and discuss possible future generalizations of our main result.

2. The Model

We consider a finite square box $\Lambda \subset \mathbb{Z}^2$ of side L, to be eventually sent to infinity. We fix k and the average density $\rho \in (0, 1/k)$. The finite volume Gibbs measure at activity z gives weight z^n to every allowed configuration of n rods: we say that a configuration is allowed if no pair of rods overlaps. Of course, one also needs to specify boundary conditions: we consider, say, periodic boundary conditions, open boundary conditions, horizontal or vertical boundary conditions, the latter meaning that all the rods within a distance $\sim k$ from the boundary of Λ are horizontal or vertical – see below for a more precise definition. The grand canonical partition function is:

$$Z_\Lambda(z) = \sum_{n\geq 0} z^n w_n^\Lambda, \tag{2.1}$$

where w_n^Λ is the number of allowed configurations of n rods in the box Λ, in the presence of the prescribed boundary conditions. Note that $w_n^\Lambda = 0$ for all $n \geq |\Lambda|/k$, which shows that $Z_\Lambda(z)$ is a finite (and, therefore, well defined) sum for all finite Λ's. The activity z is fixed in such a way that

$$\lim_{|\Lambda|\to\infty} \frac{\langle n\rangle_\Lambda}{|\Lambda|} = \lim_{|\Lambda|\to\infty} \frac{1}{|\Lambda|} \frac{\sum_{n\geq 0} n z^n w_n^\Lambda}{Z_\Lambda(z)} = \rho. \tag{2.2}$$

The goal is to understand the properties of the partition function and of the associated Gibbs state in the limit $|\Lambda| \to \infty$ at fixed ρ. An informal statement of our main result is the following.

Main Result. *For k large enough, if $k^{-2} \ll \rho \ll k^{-1}$, the system admits two distinct infinite volume Gibbs states, characterized by long range orientational order (either horizontal or vertical) and no translational order, selected by the boundary conditions.*

Sketch of the proof. The idea is to coarse grain Λ in squares of side $\ell \simeq k/2$. Each square is large, since in average it contains many ($\sim \rho k^2 \gg 1$) rods. On the other hand, its side ℓ is small enough to ensure that only rods of the same orientation are allowed to have centers in the same square. This means that the partition function restricted to a single square contains only sums over vertical or horizontal configurations. Let us consider the case where the rods are all horizontal (vertical is treated in the same way). A typical horizontal configuration consists of many ($\sim \rho k^2$) horizontal rods with centers distributed approximately uniformly (Poisson-like) in the square, since their interaction, once we prescribe their direction, is very weak: they "just" have a hard core repulsion that prevents two rods to occupy the same row, an event that is very rare, since the density of occupied rows ($\sim \rho k^2/k$) is very small, thanks to the condition that $\rho \ll 1/k$. Because of this small density of occupied rows, we are able to quantify via cluster expansion methods how close to Poissonian is the distribution of the centers in the given square (once we condition with respect to a prescribed orientation of the rods).

To control the interaction between different squares we use a Pirogov-Sinai argument. Each square can be of three types: (i) either it is of type +1, if it contains only horizontal rods, (ii) or it is of type -1, if it contains only vertical rods, (iii) or it is of type 0, if it is empty. The values $-1, 0, +1$ associated to each square play the role of spin values associated to the coarse grained system. The interaction between the spins is only finite range and squares with vertical (+1) and horizontal (-1) spin have a strong repulsive interaction, due to the hard core constraint. On the other hand, the vacuum configurations (the spins equal to 0) are very unlikely, since the probability of having a large deviation event such that a square of side ℓ is empty is expected to be exponentially small $\sim \exp\{-c\rho k^2\}$, for a suitable constant c.

Therefore the typical spin configurations consist of big connected clusters of "uniformly magnetized spins", either of type +1 or of type −1 separated by boundary layers (the contours), which contain zeros or pairs of neighboring opposite spins. These contours can be shown to satisfy a Peierls' condition; i.e., if we consider the contour partition sum with the constraint that a given contour Γ appears, then its ratio to the unconstrained partition sum is exponentially small in the geometric size of Γ. The contour theory is not symmetric under spin flip and, therefore, we are forced to study it by the (non-trivial although standard) methods first introduced by Pirogov and Sinai [37].

A more detailed summary of the main steps of the proof, including some comments about the specific way in which Pirogov-Sinai theory is implemented in our context, is postponed to the beginning of Sect. 4. Before that, let us state our main results in a mathematically more sound form.

3. Main Results

Definitions. For any region $X \subseteq \mathbb{Z}^2$ we call Ω_X the set of rod configurations $R = \{r_1, \ldots, r_n\}$ where all the rods belong to the region X. A rod r "*belongs to*" a region X if the center of the rod is inside the region, in which case we write $r \in X$. Here each rod is identified with a sequence of k adjacent sites of $\mathbb{Z}^2$ in the horizontal or vertical direction. If k is odd, the center of the rod belongs to the lattice $\mathbb{Z}^2$ itself and, therefore, the notion of "rod belonging to X" is unambiguously defined. On the contrary, if k is even, the geometrical center of the rod does not belong to the original lattice $\mathbb{Z}^2$; however, for what follows, it is convenient to pick one of the sites belonging to r and elect it to the role of "center of the rod": if r is horizontal (vertical), we decide that the "center of r" is the site of r that is closest to its geometrical center from the left (bottom). We shall also say that: a rod r "*touches*" a region X, if $r \cap X \neq \emptyset$; a rod r "*is contained in*" a region X, if $r \cap X^c = \emptyset$, in which case we write $r \subseteq X$.

The rod configurations in Ω_X can contain overlapping and even coinciding rods; we denote by $R(r)$ the multiplicity of r in $R \in \Omega_X$. The grand canonical partition function in X with open boundary conditions is

$$Z_0(X) = \sum_{R \in \Omega_X} z^{|R|} \varphi(R), \tag{3.1}$$

where $|R| := \sum_r R(r)$ and $\varphi(R)$ implements the hard core interaction:

$$\varphi(R) = \prod_{r,r' \in R} \varphi(r, r'), \qquad \varphi(r, r') = \begin{cases} 1 & \text{if } r \cap r' = \emptyset \\ 0 & \text{if } r \cap r' \neq \emptyset. \end{cases} \tag{3.2}$$

Let $\ell := \lceil k/2 \rceil$ and assume that $\Lambda \subseteq \mathbb{Z}^2$ is a square box of side divisible by 4ℓ. We pave Λ by squares of side ℓ, called "*tiles*", and by squares of side 4ℓ, called "*smoothing squares*". The lattice of the tiles' centers is a coarse grained lattice of mesh ℓ, called Λ'; similarly, the lattice of the smoothing squares' centers is a coarse grained lattice of mesh 4ℓ, called Λ''. Given $\xi \in \Lambda'$, the tile centered at ξ is denoted by Δ_ξ; given $a \in \Lambda''$, the smoothing square centered at a is denoted by $\mathcal{S}_a$. Given two sets $X, Y \subseteq \Lambda$, we indicate their euclidean distance by $\mathrm{dist}(X, Y) = \min_{x \in X, y \in Y} |x - y|$. If X and

Y are union of tiles, we shall also indicate by $X', Y' \subset \Lambda'$ the coarse versions of X and Y, i.e., the sets of sites in Λ' such that $X = \cup_{\xi \in X'} \Delta_\xi$ and $Y = \cup_{\xi \in Y'} \Delta_\xi$. The distance between X' and Y' is denoted by $\mathrm{dist}(X', Y')$ and their rescaled distance by $\mathrm{dist}'(X', Y') := \ell^{-1}\mathrm{dist}(X', Y')$; with these conventions, if ξ and η are nearest neighbor sites on Λ', then $\mathrm{dist}(\xi, \eta) = |\xi - \eta| = \ell$ and $\mathrm{dist}'(\xi, \eta) = 1$. The complement of Λ is denoted by $\Lambda_c := \mathbb{Z}^2 \backslash \Lambda$ and its coarse version by Λ'_c, with obvious meaning.

The size of the tiles is small enough to ensure that if one vertical (horizontal) rod belongs to a given tile, then all other rods belonging to the same tile and respecting the hard core repulsion condition must be vertical (horizontal). If a tile is empty, i.e., no rod belongs to it, then we assign it an extra fictitious label, which can take three possible values, either 0 or + or −. A rod configuration $R \in \Omega_\Lambda$ (combined with an assignment of these extra fictitious labels) induces a spin configuration $\sigma = \{\sigma_\xi\}_{\xi \in \Lambda'}$ on Λ', $\sigma_\xi \in \{-1, 0, +1\}$, via the following rules:

– $\sigma_\xi = +1$, if all rods belonging to Δ_ξ are horizontal or if the tile is empty with the extra label equal to +,
– $\sigma_\xi = -1$, if all rods belonging to Δ_ξ are vertical or if the tile is empty with the extra label equal to −,
– $\sigma_\xi = 0$, if Δ_ξ is empty with the extra label equal to 0.

The corresponding set of rod configurations in the tile Δ_ξ is denoted by $\Omega^{\sigma_\xi}_{\Delta_\xi}$: $\Omega^{+}_{\Delta_\xi}$ ($\Omega^{-}_{\Delta_\xi}$) is the set of rod configurations in Δ_ξ consisting either of horizontal (vertical) rods or of the empty configuration; similarly, $\Omega^{0}_{\Delta_\xi}$ consists only of the empty configuration.

Note that the grand canonical partition function in Λ with open boundary conditions can be rewritten as

$$Z_0(\Lambda) = \sum_{\sigma \in \Theta_{\Lambda'}} \sum_{R \in \Omega_\Lambda(\sigma)} \bar{\varphi}(R), \tag{3.3}$$

where $\Theta_{\Lambda'} := \{-1, 0, +1\}^{\Lambda'}$ and $\Omega_\Lambda(\sigma) := \cup_{\xi \in \Lambda'} \Omega^{\sigma_\xi}_{\Delta_\xi}$. Moreover,

$$\bar{\varphi}(R) := \Big[\prod_{\xi \in \Lambda'} \zeta(\xi)\Big] \varphi(R), \tag{3.4}$$

where the activity of a tile is defined as

$$\zeta(\xi) = \begin{cases} z^{|R_\xi|} & \text{if } \sigma_\xi = \pm 1 \\ -1 & \text{if } \sigma_\xi = 0. \end{cases} \tag{3.5}$$

The sign -1 is necessary to avoid over-counting of the empty configurations. Note that $\bar{\varphi}(R)$ depends both on σ and on R; however, in order not to overwhelm the notation, we shall drop the label σ.

The partition function with q boundary conditions, $q = \pm$, denoted by $Z(\Lambda|q)$, can be defined in a similar fashion:

$$Z(\Lambda|q) = \sum_{\sigma \in \Theta^q_{\Lambda'}} \sum_{R \in \Omega_\Lambda(\sigma)} \bar{\varphi}(R), \tag{3.6}$$

where $\Theta^q_{\Lambda'} \subset \Theta_{\Lambda'}$ is the set of spin configurations such that $\mathrm{dist}'(\xi, \Lambda'_c) \le 5 \Rightarrow \sigma_\xi = q$. The number 5 appearing here is related to the choice of smoothing squares of side 4ℓ: in fact, the condition that all the spins σ_ξ with $\mathrm{dist}'(\xi, \Lambda'_c) \le 5$ are equal to q guarantees that all the smoothing squares adjacent to the boundary of Λ are uniformly "magnetized" with magnetization q and that, moreover, all such smoothing squares are surrounded by a 1-tile-thick peel of spins equal to q. These two conditions are convenient for an explicit construction of a contour representation for $Z(\Lambda|q)$, as we will show below.

Correspondingly, the ensemble $\langle\cdot\rangle_\Lambda^q$ with q boundary conditions is defined by

$$\langle A_X\rangle_\Lambda^q = \frac{1}{Z(\Lambda|q)} \sum_{\sigma\in\Theta^q_{\Lambda'}} \sum_{R\in\Omega_\Lambda(\sigma)} \bar{\varphi}(R)\, A_X(R), \tag{3.7}$$

where A_X is a local observable, depending only on the restriction R_X of the rod configuration R to a given finite subset $X \subset \Lambda$. The infinite volume states $\langle\cdot\rangle^q$ with q boundary conditions are defined by

$$\langle A_X\rangle^q = \lim_{|\Lambda|\to\infty} \langle A_X\rangle_\Lambda^q, \tag{3.8}$$

if the limit exists for all local observables A_X, $X \subset \mathbb{Z}^2$. Our main results can be stated as follows.

Theorem 1. *If zk and $(zk^2)^{-1}$ are small enough, then the two infinite volume states $\langle\cdot\rangle^q$, $q = \pm$, exist. They are translationally invariant and are different among each other. In particular, if $\chi^\sigma_{\xi_0}$ is the projection onto the rod configurations such that $R_{\xi_0} \in \Omega^\sigma_{\Delta_{\xi_0}}$, then*

$$\langle\chi^{-q}_{\xi_0}\rangle^q \le e^{-czk^2}, \tag{3.9}$$

for a suitable constant c. Moreover, let n_{x_0} be the indicator function that is equal to 1 if a rod has a center in $x_0 \in \mathbb{Z}^2$ and 0 otherwise, then

$$\rho = \langle n_{x_0}\rangle^+ = \langle n_{x_0}\rangle^- = z(1 + O(\max\{zk, e^{-c'zk^2}\})) \tag{3.10}$$

and

$$\rho(x-y) = \langle n_x n_y\rangle^+ = \langle n_x n_y\rangle^- = \rho^2\Big(1 + O(e^{-c''|x-y|/k})\Big), \tag{3.11}$$

for suitable constants $c', c'' > 0$.

Equation (3.9) proves the existence of orientational order in the system. Equations (3.10)–(3.11) prove the absence of translational symmetry breaking. These two behaviors together prove that the system is in a nematic liquid crystal phase, as announced in the Introduction. The rest of the paper is devoted to the proof of Theorem 1, which is based on a two-scales cluster expansion. As it will be clear from the discussion in the next sections, our construction proves much more than what is explicitly stated in Theorem 1, namely it allows us to compute the averages of all the local observables in terms of an explicit exponentially convergent series.

4. The Contour Theory

The proof of Theorem 1 will be split in several steps. We start by developing a representation of the partition function $Z(\Lambda|q)$ with q boundary conditions and of its logarithm in terms of a set of interacting contours. The description and proof of convergence of this expansion will be presented in this and in the following three sections (Sects. 4-5-6-7). Later (Sect. 8), we will adapt the contour expansion to the computation of the correlations. In the following we will try to be as self-consistent as possible and to keep things simple, by avoiding as much as we can general and abstract settings. Still, the discussion will be unavoidably technical. Therefore, we find it useful to remind the reader the main steps of the proof and the way in which their presentation is organized.

1. As a preliminary step, we make the definition of contours precise, by introducing the notion of "good" and "bad" regions (roughly corresponding to the uniformly magnetized regions and their complements): the contours are the maximal connected components of the bad region and they naturally come with boundary conditions, i.e., the value of the magnetization of the tiles immediately outside or inside the contours themselves. This is explained in Sect. 4.1.
2. Next, we explain how to perform cluster expansion within a good region, that is within a region where the magnetization is constant or, in other words, the rods are all constrained to have the same orientation. This is effectively a one-dimensional problem, since horizontal rods belonging to different rows do not interact, and similarly for vertical rods. Moreover, the a-dimensional density of the effective one-dimensional system is $\sim zk \ll 1$, and it is then easy to prove convergence of the expansion of the thermodynamic and correlation functions. There are several, sharp methods to deal with the low density one-dimensional system, and one of them is reviewed in Sect. 4.2.
3. A key step of the construction is the derivation of a contour representation where only contours with fixed (say positive - that is horizontal) external boundary conditions are present. This is the core of Pirogov-Sinai (PS) theory; see [37] for the original version of this method and [4,23,41,42] for several alternative simplified versions of it. The key statement is contained in Lemma 1, which can be found, together with its proof, in Sect. 4.3. It is worth stressing that our model offers some simplifications as compared to the general PS theory. In particular, while the PS method allows one to construct the coexistence line in macroscopically a-symmetric models, in our case it is a priori clear that the "horizontal phase" in the big squared box Λ is equivalent to the "vertical phase" in the same box; in other words, we are from the beginning on the coexistence line between the two phases. This allows us to avoid the auxiliary construction of "metastable" models, or "contour models with parameters" [4,23,41,42], and the corresponding part of PS theory dealing with the adjustment of the auxiliary parameters of the metastable model is naturally missing here.
4. The contour representation contained in Lemma 1 involves many-body contour interactions, which are usually very hard to deal with. Fortunately, in our case the many-body contour interaction is quasi-one dimensional. In fact, it is induced by the rods with fixed orientation corresponding to the external boundary condition, say horizontal. This allows us to reorganize the contour expansion as a polymer expansion (this is explained in Sect. 5), where the polymers consist of unions of contours and horizontal rods that interact among themselves via a two-body hardcore interaction, see Fig. 4 below.

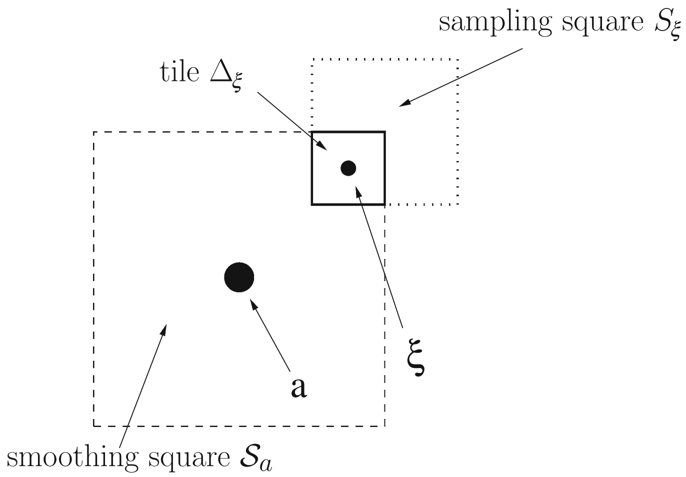

Fig. 1. An example of tile, sampling square and smoothing square

5. In Sect. 6, we prove the convergence of the polymer expansion, assuming that the activity of the contours is exponentially small. There is an additional difficulty, due to the fact that the activity of a contour depends on the partition function restricted to its interior, which is the very object we are trying to compute. To circumvent this problem, in Sect. 7 we set up an inductive scheme that is standard in the context of PS theory. The idea is to first prove the bound on the activity of the contours that are so small that they have no interior, and then move to larger and larger contours. At each induction step we know how to compute the partition function restricted to the interior of a contour (and, therefore, the activity itself), simply because it involves contours that are smaller than the considered one. Finally, in Sect. 8 we adapt the whole discussion to the computation of the correlation functions.

We are now finally ready to start the technical discussion.

4.1. Goodness, badness and contours. Let us collect here some useful definitions on bad and good regions, sampling squares and contours.

Definition 1. Sampling squares. *Given a spin configuration $\sigma \in \Theta^q_{\Lambda'}$, this induces a partition of Λ' into regions where the spins are "uniformly magnetized up or down" (i.e., regions where the spins are constantly equal to* $+1$ *or to* -1*) and boundary regions separating the "uniformly magnetized regions" among each other, which can possibly contain spins equal to zero. To make this more precise we introduce the notion of "sampling squares", defined as follows: given $\xi \in \Lambda'$, the sampling square associated to ξ is defined as $S_\xi = \cup_{\eta\in\Lambda' :\, 0\le\eta_i-\xi_i\le\ell}\,\Delta_\eta$, where η_i and η_i, $i = 1, 2$, are the coordinates of $\xi, \eta \in \Lambda'$. Note that if* $\mathrm{dist}'(\xi, \Lambda'_c) > 1$, *then S_ξ contains exactly* **4 tiles**. *See Fig. 1 for an example. We say that a sampling square is*

- **good** *if the spins inside S_ξ are all equal either to* $+1$ *or to* -1. *Each good sampling square comes with a* **magnetization** $m = \pm 1$.
- **bad** *otherwise; note that each bad sampling square is such that either it contains at least one spin equal to zero, or it contains at least one pair of neighboring spins with opposite values,* $+1$ *and* -1.

Definition 2. Connectedness, good and bad regions. *Given a configuration $\sigma \in \Theta_{\Lambda'}$, we call*

$$B(\sigma) = \cup_{\substack{\xi\in\Lambda':\\ S_\xi \text{ is bad}}} S_\xi \tag{4.1}$$

the union of all bad sampling squares. The "smoothening" of $B(\sigma)$ on scale 4ℓ is defined as:

$$\overline{B}(\sigma) = \cup_{\substack{a\in\Lambda'':\\ \mathcal{S}_a\cap B(\sigma)\neq\emptyset}} \mathcal{S}_a, \tag{4.2}$$

where the lattice Λ'' and the smoothing squares $\mathcal{S}_a$ were defined in the paragraph following Equation (3.2).

Let $X \subseteq \Lambda$ be a union of tiles: we say that X is connected if, given any pair of points $x, y \in X$, there exists a sequence $(x_0 = x, x_1, \ldots, x_{n-1}, x_n = y)$ such that $x_i \in X$ and $|x_i - x_{i-1}| = 1$, for all $i = 1, \ldots, n$. We also say that X is D-connected (with the prefix "D" meaning "diagonal") if, given any pair of points $x, y \in X$, there exists a sequence $(x_0 = x, x_1, \ldots, x_{n-1}, x_n = y)$ such that $x_i \in X$ and $|x_i - x_{i-1}| \leq \sqrt{2}$, for all $i = 1, \ldots, n$ (here $|x - y|$ is the euclidean distance between x and y).

The maximal D-connected components of $\overline{B}(\sigma)$ are denoted by Γ_j and are the geometric supports of the contours that we will introduce below. The complement of the bad region,

$$G(\sigma) := \Lambda\backslash\overline{B}(\sigma), \tag{4.3}$$

can be split into uniformly magnetized disconnected regions, each of which is a union of tiles; these are denoted by Y_j and m_j and are the corresponding magnetizations.

Remarks. 1. Note that distinct D-disconnected bad regions in $\overline{B}(\sigma)$, $\Gamma_j(\sigma)$, $\Gamma_{j'}(\sigma)$ with $j \neq j'$, do not interact directly; i.e., $\varphi(R_\xi, R_\eta) = 1$ for all $\xi \in \Gamma_j$, $\eta \in \Gamma_{j'}$. This is because $\Gamma_j(\sigma)$ and $\Gamma_{j'}(\sigma)$ are separated by at least one smoothing square (hence 4 tiles). Similarly, distinct uniformly magnetized disconnected regions, $Y_j(\sigma), Y_{j'}(\sigma) \in G(\sigma)$ with $j \neq j'$ and magnetizations $m_j, m_{j'}$, do not interact directly; i.e., $\varphi(R_\xi, R_\eta) = 1$ for all $\xi \in Y_j$, $\eta \in Y_{j'}$ and for all $R_\xi \in \Omega_\xi^{m_j}$, $R_\eta \in \Omega_\eta^{m_{j'}}$. In fact, note that R_ξ and R_η can interact only in one of the following two cases: ξ and η are on the same row (column) and $|\xi - \eta| \leq 2\ell$, or $|\xi_1 - \eta_1| = |\xi_2 - \eta_2| = \ell$. If $\xi \in Y_j$ and $\eta \in Y_{j'}$ with $j \neq j'$, then the first case can occur only if $|\xi - \eta| \geq 5\ell$ (in the horizontal or vertical directions, Y_j and $Y_{j'}$ are separated by at least one smoothing square), in which case R_ξ and R_η certainly do not interact, whatever is the alignment of the rods. In the second case necessarily $m_j = m_{j'}$, otherwise the sampling square containing both ξ and η would be bad and both tiles would belong to $B(\sigma)$ instead of $G(\sigma)$. Now, if $m_j = m_{j'}$ the rods in R_ξ have the same orientation as those in R_η, while their centers belong to different rows and columns and, therefore, do not interact.

2. In terms of the definitions above, the set $\Theta_{\Lambda'}^q \subset \Theta_{\Lambda'}$ of spin configurations with q boundary conditions can be thought as the set of spin configurations such that all the contours' supports $\Gamma_j \subset \overline{B}(\sigma)$ are D-disconnected from Λ^c and separated from it by at least one smoothing square.

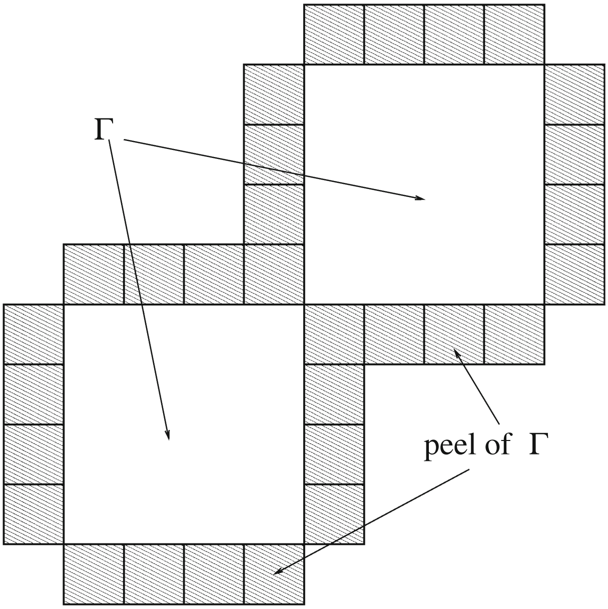

Fig. 2. The peel of a bad region Γ

Definition 3. Contours. *Given a spin configuration with q boundary conditions* $\sigma \in \Theta^q_{\Lambda'}$*, and a rod configuration* $R \in \Omega_\Lambda$ *compatible with it, let* Γ *be one of the maximal connected components of* $\overline{B}(\sigma)$*. By construction, the complement of* Γ, $\Lambda \backslash \Gamma$*, consists of one or more connected components: one of these components is adjacent to (i.e., it is at a distance* 1 *from)* Λ^c *and is naturally identified as the exterior of* Γ*; it is denoted by* $\text{Ext}\,\Gamma$*. If* Γ *is simply connected this is the only connected component of* $\Lambda \backslash \Gamma$*; if not, i.e., if* Γ *has* $h_\Gamma \geq 1$ *holes, then there are other connected components of* $\Lambda \backslash \Gamma$*, to be called the interiors of* Γ *and denoted by* $\text{Int}_j \Gamma$, $j = 1, \dots, h_\Gamma$*. The interior of* Γ *is then* $\text{Int}\Gamma = \cup_j \text{Int}_j \Gamma$*. For what follows, it is also convenient to introduce the* **1-tile-thick peel** *of* Γ *(see Fig. 2):*

$$P_\Gamma = \cup_{\substack{\xi \in \Lambda': \\ \text{dist}'(\xi, \Gamma') = 1}} \Delta_\xi . \tag{4.4}$$

Note that, since distinct D-disconnected regions are separated by at least one smoothing square (i.e., 4 tiles), then also the peels associated to distinct Γ*'s are mutually D-disconnected.*

The **contour** γ *associated to the support* $\Gamma = \text{supp}(\gamma)$ *is defined as the collection:*

$$\gamma = (\Gamma, \sigma_\gamma, R_\gamma, m_{ext}, \underline{m}_{int}), \tag{4.5}$$

where

- σ_γ *is the restriction of the spin configuration* σ *to* Γ*;*
- R_γ *is the restriction of the rod configuration* R *to* Γ*;*
- m_{ext} *is the magnetization of* $P^{ext}_\Gamma := \text{Ext}\,\Gamma \cap P_\Gamma$*;*

- *$\underline{m}_{int} = \{m^1_{int}, \ldots, m^{h_\Gamma}_{int}\}$, with m^j_{int} the magnetization of* $\mathrm{Int}_j\Gamma \cap P_\Gamma$*; if $h_\Gamma = 0$, then $\underline{m}_{int}$ is the empty set. In the following we shall also denote by $P^{int}_\Gamma :=$* $\mathrm{Int}\Gamma \cap P_\Gamma$ *the internal peel of* Γ.

If $m_{ext} = q$, then we say that γ is a q-contour.

Remark. The set γ must satisfy a number of constraints. In particular, given m_{ext} and $\underline{m}_{int}, \sigma_\gamma$ must be compatible with the conditions that: (i) all the sampling squares having non-zero intersection with P_Γ are good (otherwise the contour would also contain these squares); (ii) each smoothing square contained in Γ has non-zero intersection with at least one bad sampling square. Moreover, R_γ must be compatible with σ_γ itself.

In the following we want to write an expression for $Z(\Lambda|q)$ purely in terms of contours. Roughly speaking, given a contour configuration contributing to the r.h.s. of Eq. (3.6), we first want to freeze the rods inside the supports of the contours, next sum over all the rod configurations in the good regions and show that the resulting effective theory is a contour theory treatable by the Pirogov-Sinai method. The resummation of the configurations within the good regions can be performed by standard cluster expansion methods, as explained in the following digression.

4.2. Partition function restricted to a good region. Given a set $X \subseteq \Lambda$ consisting of a union of tiles, let $\Omega^q_X = \cup_{\xi\in X'}\Omega^q_{\Delta_\xi}$, $q = \pm$. The restricted theory of the "uniformly q-magnetized" region X (with open boundary conditions) is associated with the partition function:

$$Z^q(X) = \sum_{R\in\Omega^q_X} z^{|R|}\varphi(R), \tag{4.6}$$

which can be easily computed by standard cluster expansion methods, some aspects of which are briefly reviewed here (for extensive reviews, see, e.g., [6] and [14, Chap. 7]). The logarithm of Eq. (4.6) can be expressed in terms of a convergent series as:

$$\log Z^q(X) = \sum_{R\in\Omega^q_X} z^{|R|}\varphi^T(R) = z|X|(1 + O(zk)), \tag{4.7}$$

where φ^T are the *Mayer's coefficients*, which admit the following explicit representation. Given the rod configuration $R = \{r_1, \ldots, r_n\}$, consider the graph $\mathcal{G}$ with n nodes, labelled by $1, \ldots, n$, with edges connecting all pairs i, j such that $r_i \cap r_j \neq \emptyset$ ($\mathcal{G}$ is sometimes called the connectivity graph of R). Then one has $\varphi^T(\emptyset) = 0$, $\varphi^T(r) = 1$ and, for $|R| > 1$:

$$\varphi^T(R) = \frac{1}{R!}\sum^{*}_{C\subseteq\mathcal{G}}(-1)^{\text{number of edges in } C}, \tag{4.8}$$

where $R! = \prod_r R(r)!$ and the sum runs over all the connected subgraphs C of G *that visit all the n points* $1, \ldots, n$. In particular, if $|R| > 1$, then $\varphi^T(R) = 0$ unless R is connected.

The sum in the r.h.s. of Eq. (4.7) is exponentially convergent for $zk \ll 1$; in particular, if $x_0 \in \Lambda$, then for a suitable constant $C > 0$,

$$\sum_{\substack{R \in \Omega^q_\Lambda: \\ R \ni x_0,\ |R| \geq m}} |z|^{|R|} |\varphi^T(R)| \leq Cz(Czk)^{m-1}, \tag{4.9}$$

uniformly in Λ, where $R \ni x_0$ means that R contains at least one rod with center in x_0. Moreover, the sum $\sum_{R \in \Omega^q_\Lambda : R \ni x_0} z^{|R|} \varphi^T(R)$ is analytic in zk, uniformly in Λ, for zk small enough and its limit as $\Lambda \nearrow \mathbb{Z}^2$ is analytic, too. A useful corollary of Eq. (4.9) is the following: if $V(R)$ is the union of the centers of the rods in R, supp(R) is the support of the union of rods $r \in R$ (thought of as a subset of Λ) and diam(supp(R)) is its diameter, then for any finite region $X \subset \Lambda$:

$$\begin{aligned}
\sum_{\substack{R \in \Omega^q_\Lambda: \\ V(R) \cap X \neq \emptyset, \\ \mathrm{diam}(\mathrm{supp}(R)) \geq d}} |z|^{|R|} |\varphi^T(R)| &\leq \sum_{x_0 \in X} \sum_{m \geq \lceil \frac{d}{k-1} \rceil} \sum_{\substack{R \in \Omega^q_\Lambda: \\ R \ni x_0,\ |R| \geq m}} |z|^{|R|} |\varphi^T(R)| \\
&\leq 2Cz|X|(Czk)^{\frac{d}{k-1}-1},
\end{aligned} \tag{4.10}$$

uniformly in Λ; here, in the first inequality, we used the fact that in order for supp(R) to have diameter d, the configuration R needs to have at least $\lceil d/(k-1) \rceil$ rods, while in the second inequality we used Eq. (4.9). In a similar fashion, all the correlation functions can be computed in terms of convergent series, as long as zk is small enough. These results are classical, see [39] or, e.g., [6, 14]. The restricted theory is applied to the computation of the sums over the rod configurations in the good regions, as described in the following.

4.3. Contour representation of the partition function. Given a contour γ, let $Z_\gamma(\mathrm{Int}_j \Gamma | m^j_{int})$ be the partition function on the j^{th} interior of Γ with the boundary conditions created by the presence of the "frozen" rods R_γ. Moreover, if $\xi \in P'_\Gamma$, let

$$A_\gamma(\Delta_\xi) = \cup_{\eta \in a_\gamma(\xi)} \Delta_\eta, \qquad C_\gamma(\Delta_\xi) = \cup_{\substack{\eta \in \Gamma': \\ \mathrm{dist}'(\eta,\xi) \leq 2}} \Delta_\eta, \tag{4.11}$$

where

$$a_\gamma(\xi) := \{\xi\} \cup \{\eta \in \Lambda' : \mathrm{dist}'(\eta, \xi) = 1, \mathrm{dist}'_1(\eta, \Gamma') = 2, \eta_{j(-q)} = \xi_{j(-q)}\}, \tag{4.12}$$

with $\mathrm{dist}'_1(\cdot, \cdot)$ the rescaled ("coarse") L_1 distance on Λ' and $j(+) = 1$, $j(-) = 2$. Finally, given $\Delta \subseteq P_\Gamma$, let f_Δ and g_Δ be the following characteristic functions:

$$f_\Delta(R) = \begin{cases} 1 & \text{if } R \text{ has at least one rod belonging to} A_\gamma(\Delta) \\ & \text{and one belonging to } C_\gamma(\Delta), \\ 0 & \text{otherwise,} \end{cases} \tag{4.13}$$

$$g_\Delta(R) = \begin{cases} 1 & \text{if } R \cap R_\gamma \neq \emptyset,\ R \text{ has at least one rod belonging to } A_\gamma(\Delta) \\ & \text{and } R_\gamma \text{has at least one rod belonging } C_\gamma(\Delta), \\ 0 & \text{otherwise.} \end{cases} \tag{4.14}$$

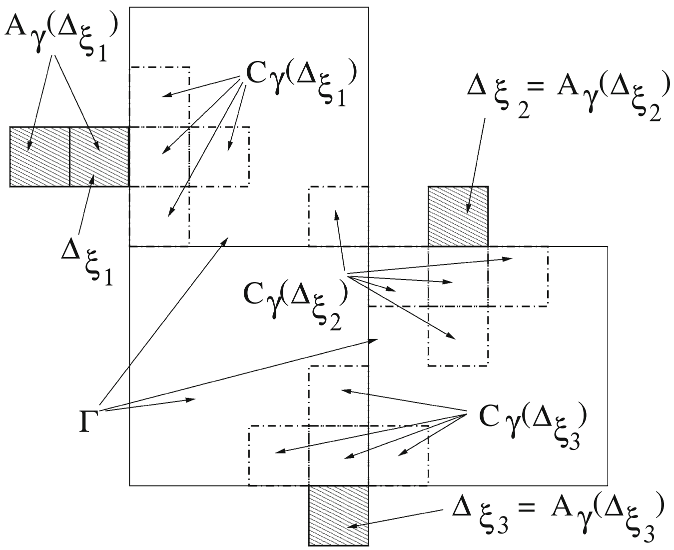

Fig. 3. The two sets $A_\gamma(\Delta_\xi)$ and $C_\gamma(\Delta_\xi)$ in the case that $q = +$

Pictorially speaking, f_Δ is the characteristic function of the event "R crosses the boundary of Γ at Δ", while g_Δ is the characteristic function of the event "R intersects R_γ across Δ". Note that, by construction, given two distinct tiles, $\Delta_1 \subseteq P_{\Gamma_1}$ and $\Delta_2 \subseteq P_{\Gamma_2}$ such that $\Delta_1 \cap \Delta_2 = \emptyset$, then $A_{\gamma_1}(\Delta_1) \cap A_{\gamma_2}(\Delta_2) = \emptyset$, even in the case that $\Gamma_1 \equiv \Gamma_2$.

In terms of these definitions, the following contours' representation for $Z(\Lambda|q)$ is valid.

Lemma 1. *The conditioned partition function* $Z(\Lambda|q)$, $q = \pm 1$, *can be written as*

$$Z(\Lambda|q) = Z^q(\Lambda) \sum_{\partial \in \mathcal{C}(\Lambda,q)} \Big[\prod_{\gamma\in\partial} \zeta_q(\gamma)\Big] e^{-W(\partial)}, \tag{4.15}$$

where:

- $\mathcal{C}(\Lambda, q)$ *is the set of all the well D-disconnected* q*-contour configurations in* Λ *(here we say that* $\{\gamma_1, \dots, \gamma_n\}$ *is well D-disconnected if the supports* $\Gamma_1, \dots, \Gamma_n$ *are separated among each other and from* Λ^c *by at least one smoothing square);*
- $\zeta_q(\gamma)$ *is the activity of* γ*:*

$$\zeta_q(\gamma) = \zeta_q^0(\gamma) \exp\Big\{ -\sum_{R\in\Omega_\Lambda^q} \varphi^T(R) z^{|R|} \sum_{\Delta\subseteq P_\Gamma} F_\Delta(R)\Big\}, \tag{4.16}$$

where

$$\zeta_q^0(\gamma) = \frac{\bar\varphi(R_\gamma)}{Z^q(\Gamma)} \prod_{j=1}^{h_\Gamma} \frac{Z_\gamma(\mathrm{Int}_j\Gamma|m_{int}^j)}{Z(\mathrm{Int}_j\Gamma|q)} \tag{4.17}$$

and $F_\Delta = f_\Delta$ *if* $\Delta \subseteq P_\Gamma^{int}$ *while* $F_\Delta = f_\Delta + g_\Delta(1 - f_\Delta)$ *if* $\Delta \subseteq P_\Gamma^{ext}$.

- *$W(\partial)$ is the interaction between the contours in ∂:*

$$W(\partial) = \sum_{R \in \Omega^q_\Lambda} \varphi^T(R) z^{|R|} \sum_{n \geq 2} (-1)^{n+1} \sum^{*}_{\Delta_1 < \cdots < \Delta_n} F_{\Delta_1}(R) \cdots F_{\Delta_n}(R), \quad (4.18)$$

where the $*$ on the sum indicates the constraint that $\Delta_1, \ldots, \Delta_n$ are all contained in the peel of some contour of ∂ and their centers $\xi_1, \ldots \xi_n$ all belong to the same row (if $q = +$) or column (if $q = -$) of Λ', namely $\xi_{1,j(-q)} = \cdots = \xi_{n,j(-q)}$. Moreover, by writing $\Delta_1 < \cdots < \Delta_n$, we mean that $\xi_{1,j(q)} < \cdots < \xi_{n,j(q)}$. Finally, $F_\Delta = f_\Delta$ if Δ is contained in the internal peel of some contour in ∂ or $F_\Delta = f_\Delta + g_\Delta(1 - f_\Delta)$ if Δ is contained in the external peel of some contour in ∂.

Remarks. 1. The contour configurations $\{\gamma_1, \ldots, \gamma_n\} \in \mathcal{C}(\Lambda, q)$ consist of n-ples of well D-disconnected q-contours, which means that the geometric supports $\Gamma_1, \ldots, \Gamma_n$ are separated among each other and from Λ^c by at least one smoothing square. Note, however, that their external and internal magnetizations are not necessarily compatible among each other: for instance, Γ_1 may have one hole surrounding Γ_2, and the internal magnetization of Γ_1 may be different from the external magnetization of Γ_2 (which is q). It is actually an important point of the representation Eq. (4.15) that we can forget about the compatibility conditions among the internal and external magnetizations of different contours. There exist different (and even more straightforward) contour representation of $Z(\Lambda|q)$, where the internal and external contours' magnetizations satisfy natural but non-trivial constraints (e.g., in the example above, the natural constraint is that the internal magnetization of Γ_1 is the same as the external magnetization of Γ_2). However, the magnetization constraints are not suitable to apply cluster expansion methods to the resulting contour theory. Therefore, it is convenient to eliminate such constraints, at the price of adding the extra factors $Z_\gamma(\mathrm{Int}_j\Gamma|m^j_{int})/Z(\mathrm{Int}_j\Gamma|q)$ in the definition of the contours' activities, see Eq. (4.17).

2. The interest of the representation Eq. (4.15) is that the contour activities and the multi-contour interaction satisfy suitable bounds, allowing us to study the r.h.s. of Eq. (4.15) by cluster expansion methods. In particular, $\sup^*_{\sigma_\gamma} \sum_{R_\gamma \in \Omega_\Gamma(\sigma_\gamma)} |\zeta_q(\gamma)| \leq \exp\{-(\text{const.}) z k^2 |\Gamma'|\}$, where the $*$ on the sup reminds the constraint that all the smoothing squares in Γ must have a non-zero intersection with at least one bad sampling square. Moreover, $W(\partial)$ is a quasi-one-dimensional potential, exponentially decaying to zero in the mutual distance between the supports of the contours in ∂. The proofs of these claims will be postponed to the next sections.

Proof of Lemma 1. Given $\sigma \in \Theta^q_{\Lambda'}$ a spin configuration with q boundary conditions consider the corresponding set of contours $\{\gamma_1, \ldots, \gamma_n\}$. Some of them are *external*, in the sense that they are not surrounded by any other contour in $\{\gamma_1, \ldots, \gamma_n\}$. By construction, these external contours are all q-contours. We denote by $\mathcal{C}_{ext}(\Lambda, q)$ the set of external q-contour configurations. Given $\partial \in \mathcal{C}_{ext}(\Lambda, q)$, there is a common external region to all the contours in ∂, which we denote by $\mathrm{Ext}(\partial)$. Besides this, there are several internal regions within each contour $\gamma \in \partial$. For each external contour $\gamma \in \partial$, we freeze the corresponding rod configuration R_γ and sum over the rod configurations inside all the internal regions $\mathrm{Int}_j\Gamma$, $j = 1, \ldots, h_\Gamma$. In this way, for each such interior, we reconstruct the partition function $Z_\gamma(\mathrm{Int}_j\Gamma|m^j_{int})$. On the other hand, by construction all rods inside $\mathrm{Ext}(\partial)$ are either horizontal or vertical, according to the value of q. Therefore, if we sum over all the allowed rod configurations inside this region we get the restricted partition function $Z^q_\partial(\mathrm{Ext}(\partial))$, where the subscript ∂ reminds the fact that the rods $R_\partial = \cup_{\gamma\in\partial}$ create an excluded volume for the rods in $\mathrm{Ext}(\partial)$. Using these definitions, we can rewrite

$$Z(\Lambda|q) = \sum_{\partial\in\mathcal{C}_{ext}(\Lambda,q)} Z^q_\partial(\mathrm{Ext}(\partial)) \prod_{\gamma\in\partial}\Big[\bar{\varphi}(R_\gamma)\prod_{j=1}^{h_\Gamma} Z_\gamma(\mathrm{Int}_j\Gamma|m^j_{int})\Big]. \tag{4.19}$$

Note that here we used the fact that the exterior and the interior(s) of ∂ do not interact directly (i.e., they only interact through R_γ). Using the definition of $\zeta^0_q(\gamma)$, Eq. (4.17), we can rewrite $Z(\Lambda|q)$ as

$$\frac{Z(\Lambda|q)}{Z^q(\Lambda)} = \sum_{\partial\in\mathcal{C}_{ext}(\Lambda,q)} \prod_{\gamma\in\partial}\Big[\zeta^0_q(\gamma)\prod_{j=1}^{h_\Gamma}\frac{Z(\mathrm{Int}_j\Gamma|q)}{Z^q(\mathrm{Int}_j\Gamma)}\Big] e^{-W_0^{ext}(\partial)}, \tag{4.20}$$

where

$$e^{-W_0^{ext}(\partial)} = \frac{Z^q_\partial(\mathrm{Ext}(\partial))\prod_{\gamma\in\partial}\big[Z^q(\Gamma)\prod_{j=1}^{h_\Gamma} Z^q(\mathrm{Int}_j\Gamma)\big]}{Z^q(\Lambda)}. \tag{4.21}$$

The factors $\frac{Z(\mathrm{Int}_j\Gamma|q)}{Z^q(\mathrm{Int}_j\Gamma)}$ have the same form as the l.h.s. of Eq. (4.20) itself, with Λ replaced by $\mathrm{Int}_j\Gamma$: therefore, the equation can be iterated until the interior of all the contours is so small that it cannot contain other contours. The result of the iteration is

$$\frac{Z(\Lambda|q)}{Z^q(\Lambda)} = \sum_{\partial\in\mathcal{C}(\Lambda,q)}\Big[\prod_{\gamma\in\partial}\zeta^0_q(\gamma)\Big] e^{-W_0(\partial)}, \tag{4.22}$$

where

$$e^{-W_0(\partial)} = \frac{Z^q_\partial(\Lambda(\partial))\prod_{\gamma\in\partial} Z^q(\Gamma)}{Z^q(\Lambda)}, \tag{4.23}$$

$\Lambda(\partial) = \Lambda\setminus\cup_{\gamma\in\partial}\Gamma$ is the complement of the contours' supports and $Z^q_\partial(\Lambda(\partial))$ is the restricted partition function with magnetization q in the volume $\Lambda(\partial)$ and in the presence of the hard rod constraint generated by the frozen rods R_γ in the region $\cup_{\gamma\in\partial}\cup_{\Delta\subseteq P^{ext}_\Gamma} A_\gamma(\Delta)$.

We now use Eq. (4.7) and the analogous expression for $Z^q_\partial(\Lambda(\partial))$, i.e.,

$$\log Z^q_\partial(\Lambda(\partial)) = \sum_{\substack{R\overset{ext}{\cap}R_\partial=\emptyset \\ R\in\Omega^q_{\Lambda(\partial)}}} z^{|R|}\varphi^T(R), \tag{4.24}$$

where $R\overset{ext}{\cap}R_\partial=\emptyset$ means that R does not intersect R_∂ from the outside, namely:

$$R\overset{ext}{\cap}R_\partial=\emptyset \overset{def}{\Leftrightarrow} \prod_{\gamma\in\partial}\prod_{\Delta\subseteq P^{ext}_\Gamma}(1-g_\Delta(R)) = 1, \tag{4.25}$$

where g_Δ was defined in Eq. (4.14). Then we can rewrite:

$$e^{-W_0(\partial)} = \frac{Z^q(\Lambda(\partial))\prod_{\gamma\in\partial} Z^q(\Gamma)}{Z^q(\Lambda)} \cdot \frac{Z^q_\partial(\Lambda(\partial))}{Z^q(\Lambda(\partial))},$$

$$= \exp\Big\{-\sum_{R\in\Omega^q_\Lambda}^{R\overset{\partial}{\rightsquigarrow}2} z^{|R|}\varphi^T(R)\Big\} \cdot \exp\Big\{-\sum_{R\in\Omega^q_{\Lambda(\partial)}}^{R\cap R^{ext}_\partial\neq\emptyset} z^{|R|}\varphi^T(R)\Big\}, \quad (4.26)$$

where $R\overset{\partial}{\rightsquigarrow}2$ means that R must contain two rods r_1, r_2 belonging, respectively, to two distinct elements of the partition $\mathcal{P}(\partial)$ of Λ induced by the contours in ∂; i.e., either $r_1, r_2 \in R$ belong, respectively, to two disconnected components of $\Lambda(\partial)$, or they belong to two different contours' supports, or r_1 belongs to one contour's support and r_2 to one of the components of $\Lambda(\partial)$. Using the definitions of the characteristic functions f_Δ and g_Δ defined in Eqs. (4.13)–(4.14), the two exponential in the r.h.s. of Eq. (4.26) can be written as

$$\sum_{R\in\Omega^q(\Lambda)}^{R\overset{\partial}{\rightsquigarrow}2} z^{|R|}\varphi^T(R) = \sum_{R\in\Omega^q_\Lambda} z^{|R|}\varphi^T(R)\Big[1-\prod_{\gamma\in\partial}\prod_{\Delta\subseteq P_\Gamma}(1-f_\Delta(R))\Big], \quad (4.27)$$

$$\sum_{R\in\Omega^q_{\Lambda(\partial)}}^{R\cap R^{ext}_\partial\neq\emptyset} z^{|R|}\varphi^T(R) = \sum_{R\in\Omega^q_\Lambda} z^{|R|}\varphi^T(R)\Big[1-\prod_{\gamma\in\partial}\prod_{\Delta\subseteq P^{ext}_\Gamma}(1-g_\Delta(R))\Big]$$
$$\cdot\Big[\prod_{\gamma\in\partial}\prod_{\Delta\subseteq P_\Gamma}(1-f_\Delta(R))\Big]. \quad (4.28)$$

Using the representations Eqs. (4.26), (4.27), (4.28) into Eq. (4.22), we find

$$\frac{Z(\Lambda|q)}{Z^q(\Lambda)} = \sum_{\partial\in\mathcal{C}(\Lambda,q)}\Bigg[\prod_{\gamma\in\partial}\zeta^0_q(\gamma)\Bigg]\exp\Big\{-\sum_{R\in\Omega^q_\Lambda} z^{|R|}\varphi^T(R)$$
$$\cdot\Big[1-\Big(\prod_{\gamma\in\partial}\prod_{\Delta\subseteq P^{ext}_\Gamma}(1-g_\Delta(R))\Big)\cdot\Big(\prod_{\gamma\in\partial}\prod_{\Delta\subseteq P_\Gamma}(1-f_\Delta(R))\Big)\Big]\Big\}. \quad (4.29)$$

Note that the expression in square brackets in the second line can be conveniently rewritten as

$$1-\prod_{\gamma\in\partial}\Big(\prod_{\Delta\subseteq P^{ext}_\Gamma}(1-g_\Delta(R))(1-f_\Delta(R))\Big)\cdot\Big(\prod_{\Delta\subseteq P^{int}_\Gamma}(1-f_\Delta(R))\Big)$$
$$\equiv 1-\prod_{\gamma\in\partial}\prod_{\Delta\subseteq P_\Gamma}(1-F_\Delta), \quad (4.30)$$

where F_Δ was defined in the statement of Lemma 1. Plugging Eq. (4.30) into Eq. (4.29) gives

$$\frac{Z(\Lambda|q)}{Z^q(\Lambda)} = \sum_{\partial\in\mathcal{C}(\Lambda,q)}\Big[\prod_{\gamma\in\partial}\zeta^0_q(\gamma)\Big]\exp\Big\{-\sum_{R\in\Omega^q_\Lambda} z^{|R|}\varphi^T(R)$$
$$\cdot\Big[\sum_{\Delta\subseteq P_\partial}F_\Delta(R)+\sum_{n\geq 2}(-1)^{n+1}\sum_{\{\Delta_1,\ldots,\Delta_n\}}F_{\Delta_1}(R)\cdots F_{\Delta_n}(R)\Big]\Big\}, \quad (4.31)$$

where the sum $\sum_{\{\Delta_1,\ldots,\Delta_n\}}$ runs over collections of distinct tiles $\Delta_i \subseteq P_\partial$, with $P_\partial := \cup_{\gamma\in\partial} P_\Gamma$. Finally, using the fact that $\varphi^T(R)$ forces R to be connected and, therefore, to live on a single row or column, depending on whether q is + or −, we find that the only non-vanishing contributions in the latter sum come from n-ples of tiles all living on the same row or column. This proves the desired result. □

5. Reorganizing the Contour Expansion

Standard cluster expansion methods are more easily implemented in the case of two-body interactions. Our contour interaction Eq. (4.18) is many-body but it can be reduced to the two-body case by a slight reorganization of the expansion.

Lemma 2. *The contour representation (4.15) for the conditioned partition function $Z(\Lambda|q)$, $q = \pm 1$, can be reorganized as follows:*

$$\frac{Z(\Lambda|q)}{Z^q(\Lambda)} = 1 + \sum_{m\geq 1}\sum_{\{X_1,\ldots,X_m\}} K_q^{(\Lambda)}(X_1)\cdots K_q^{(\Lambda)}(X_m)\,\phi(\{X_1, X_2\ldots, X_m\}), \tag{5.1}$$

where:

- *each* polymer X_i *is a D-connected union of tiles in* Λ*;*
- ϕ *implements the hard core interaction, i.e.,*

$$\phi(\{X_1, X_2\ldots, X_m\}) = \prod_{i<j}\phi(X_i, X_j), \tag{5.2}$$

$$\phi(X_i, X_j) = \begin{cases} 1 & \text{if } X_i \text{ D-disconnected from } X_j \\ 0 & \text{otherwise.}\end{cases}$$

- $K_q^{(\Lambda)}(X)$ *is a suitable function of X, called the polymer's activity, which is defined by Eq. (5.3) below.*

Remark. The definition Eq. (5.2) of the polymer interaction is the analogue of Eq. (3.2) with the rods replaced by polymers and the notion of intersection replaced by D-connectedness.

Definition of the polymer's activity. Given $\partial = \{\gamma_1, \ldots \gamma_n\} \in \mathcal{C}(\Lambda, q)$ and $X_\partial = \cup_{i=1}^n \Gamma_i$, let $Y = \{\Delta_1, \ldots \Delta_m\}$ be a collection of $m \geq 2$ distinct tiles, all contained in the peel of X_∂, i.e., $\Delta_i \in \cup_{j=1}^n P_{\Gamma_j}$, and all belonging to the same row (if $q = +$) or column (if $q = -$). Since the tiles are all on the same row (column), we can order them from left to right (bottom to top), $\Delta_1 < \Delta_2 < \cdots < \Delta_m$. We denote by $\Upsilon^q_{X_\partial}$ the set of all such collections. Moreover, for each $Y = \{\Delta_1, \ldots \Delta_m\} \in \Upsilon^q_{X_\partial}$ with $\Delta_1 < \cdots < \Delta_m$, we define $\overline{Y}$ to be the union of all the tiles between Δ_1 and Δ_m. With these definitions, the activity of the polymer X is given by

$$K_q^{(\Lambda)}(X) = \sum_{n\geq 1,\, p\geq 0}\ \sum_{\substack{\partial\in\mathcal{C}(\Lambda,q):\ |\partial|=n \\ \{Y_1,\ldots,Y_p\}:\ Y_i\in\Upsilon^q_{X_\partial} \\ X_\partial\cup\{\cup_j \overline{Y}_j\}=X}} \Big[\prod_{\gamma\in\partial}\zeta_q(\gamma)\Big]\Big[\prod_{i=1}^p (e^{\mathcal{F}(Y_i)} - 1)\Big], \tag{5.3}$$

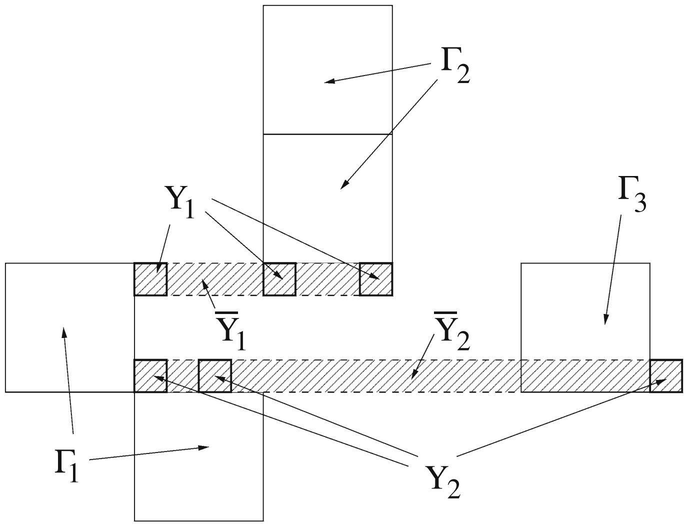

Fig. 4. An example of polymer X and of a possible way of realizing it as a union of three contours' supports $\Gamma_1, \Gamma_2, \Gamma_3$ and of two sets Y_1, Y_2

where $\zeta_q(\gamma)$ was introduced in Eq. (4.16) and, if $Y = \{\Delta_1, \ldots, \Delta_m\} \in \Upsilon^q_{X_\partial}$,

$$\mathcal{F}(Y) := (-1)^n \sum_{R \in \Omega^q_\Lambda} z^{|R|} \varphi^T(R) F_{\Delta_1}(R) \cdots F_{\Delta_n}(R). \tag{5.4}$$

An example of a polymer X with non-vanishing activity and of a possible way of realizing it as a union of sets Γ_i and $\overline{Y}_j$ is given in Fig. 4.

Proof of Lemma 2. Using the definition of $\mathcal{F}(Y)$, we can rewrite Eqs. (4.15)–(4.18) as:

$$\frac{Z(\Lambda|q)}{Z^q(\Lambda)} = \sum_{\partial \in \mathcal{C}(\Lambda,q)} \Big[\prod_{\gamma \in \partial} \zeta_q(\gamma)\Big]\Big[\prod_{Y \in \Upsilon^q_{X_\partial}} e^{\mathcal{F}(Y)}\Big]. \tag{5.5}$$

Let us now add and subtract 1 to each of the factors $e^{\mathcal{F}(Y)}$. In this way we turn each factor into a binomial $1 + (e^{\mathcal{F}(Y)} - 1)$. If $Y \in \Upsilon^q_{X_\partial}$, we associate the quantity $(e^{\mathcal{F}(Y)} - 1)$ with the region $\overline{Y}$; similarly, we associate the activity $\zeta(\gamma)$ with the region Γ. In this way, every factor of the form $\prod_{i=1}^n \zeta(\gamma_i) \prod_{j=1}^p (e^{\mathcal{F}(Y_j)} - 1)$ is geometrically associated with the region $X = \{\cup_{i=1}^n \Gamma_i\} \cup \{\cup_{j=1}^p \overline{Y}_j\}$. We develop the binomials $1 + (e^{\mathcal{F}(Y)} - 1)$ and collect together the contribution corresponding to the maximally D-connected regions, obtained as unions of Γ_i's and $\overline{Y}_j$'s. The result is Eq. (5.1). □

6. Convergence of the Contours' Expansion

In this and in the next section we prove the convergence of the cluster expansion for the logarithm of the partition function with q boundary conditions, starting from Eq. (4.15). The proof will be split in two main steps: first, in this section, we prove convergence under the assumption that the activities $\zeta_q(\gamma)$ satisfy suitable decay bounds in the size of $|\Gamma|$. Then, in the next section, we prove the validity of such a decay bound via an induction in the size of $|\Gamma|$. From now on, $C, C', \dots$ and $c, c', \dots$ indicate universal positive constants (to be thought of as "big" and "small", respectively), whose specific values may change from line to line.

Lemma 3. *Suppose that, for zk and $(zk^2)^{-1}$ small enough,*

$$\sup_{\sigma_\gamma}{}^{*} \sum_{R_\gamma \in \Omega_\Gamma(\sigma_\gamma)} |\zeta_q(\gamma)| \leq e^{-c_0\, zk^2 |\Gamma'|}, \tag{6.1}$$

where the $*$ on the sup recalls the constraint that all the smoothing squares in Γ must have a non-zero intersection with at least one bad sampling square, and $c_0 = 5 * 10^{-4}$. Then the logarithm of the partition function admits a convergent cluster expansion

$$\log Z(\Lambda|q) = \sum_{R \in \Omega^q_\Lambda} z^{|R|} \varphi^T(R) + \sum_{\mathcal{X} \subseteq \Lambda} \Big[\prod_{X \in \mathcal{X}} K_q^{(\Lambda)}(X) \Big] \phi^T(\mathcal{X}), \tag{6.2}$$

where $\mathcal{X} = \{X_1, \dots, X_n\}$ is a polymers' configuration (possibly, some of the X_i's may coincide), each polymer X being a D-connected subset of Λ consisting of a union of tiles.

Remarks. 1. The constant $c_0 = 5 * 10^{-4}$ is a possible explicit constant for which the result of the lemma holds (certainly, it is not the sharp one). Its specific value is motivated by Lemma 4 and by its proof, see next section.
2. The function $\phi^T(\mathcal{X})$ in Eq. (6.2) is the Mayer's coefficient of $\mathcal{X}$, defined as in Eq. (4.8), with R replaced by $\mathcal{X}$, r_i by X_i, and the notion "$r_i \cap r_j \neq \emptyset$" replaced by "$X_i$ is D-connected to X_j".

Proof. By Lemma 2, Eqs. (4.15)–(4.18) can be equivalently rewritten as

$$\frac{Z(\Lambda|q)}{Z^q(\Lambda)} = 1 + \sum_{m \geq 1} \sum_{\{X_1, \dots, X_m\}} K_q^{(\Lambda)}(X_1) \cdots K_q^{(\Lambda)}(X_m)\, \phi(\{X_1, \dots X_m\}). \tag{6.3}$$

It is well-known [6, 14, 39], that if the activities $K_q^{(\Lambda)}(X)$ are sufficiently small and decay fast enough with the size of X, then one can apply standard cluster expansion methods (analogous to those sketched above, after Eq. (4.6)) for computing the logarithm of Eq. (5.1) and put it in the form of the exponentially convergent sum. More specifically, a sufficient condition for the application of the standard cluster expansion is, see e.g. [14, Prop. 7.1.1],

$$|K_q^{(\Lambda)}(X)| \leq C \varepsilon_0^{|X'|} e^{-\kappa_0 \delta'(X')}, \tag{6.4}$$

for some $\kappa_0 > 0$ and ε_0 small enough (here $\delta'(X')$ is the rescaled tree length of the coarse set $X' \subset \Lambda'$, i.e., it is the number of nearest neighbor edges of the smallest tree on Λ' that covers X'). In the following, we will prove that under the assumption of the lemma, the polymers' activities $K_q^{(\Lambda)}(X)$ satisfy

$$|K_q^{(\Lambda)}(X)| \leq \varepsilon_1 \varepsilon^{|X'|-1}, \quad \varepsilon_1 := e^{-\frac{c_0}{6} zk^2}, \quad \varepsilon_2 := (zk)^{\frac{1}{32}}\,, \quad \varepsilon := \max\{\varepsilon_1, \varepsilon_2\}, \tag{6.5}$$

where c_0 is the same constant as in Eq. (6.1). Using the fact that X is D-connected, we see that Eq. (6.5) implies Eq. (6.4) with $\varepsilon_0 = e^{-\kappa_0} = \varepsilon^{1/2}$. Therefore, by [14, Prop. 7.1.1], we get

$$\log \frac{Z(\Lambda|q)}{Z^q(\Lambda)} = \sum_{\mathcal{X}\subseteq\Lambda} \Big[\prod_{X\in\mathcal{X}} K_q^{(\Lambda)}(X)\Big]\phi^T(\mathcal{X}). \tag{6.6}$$

Combining this equation with Eq. (4.7) gives Eq. (6.2).

The rest of this section is devoted to the proof of Eq. (6.5). The polymer's activity Eq. (5.3) can be rewritten as

$$\begin{aligned} K_q^{(\Lambda)}(X) = &\sum_{\substack{X_0,X_1\subseteq X:\\ X_0\cup X_1=X}} \sum_{n\geq 1} \sum_{\substack{\partial=\{\gamma_1,\ldots,\gamma_n\}\in\mathcal{C}(\Lambda,q):\\ \cup_{i=1}^n \Gamma_i = X_0}} \Big[\prod_{j=1}^n \zeta_q(\gamma_j)\Big] \\ &\cdot \sum_{Q\subseteq X_1} \sum_{p\geq 0} \sum_{\substack{\{Y_1,\ldots,Y_p\}:\ Y_i\in\Upsilon^q_{X_0}\\ \cup_i \mathrm{supp}(Y_i)=Q\\ \cup_i \overline{Y}_i = X_1}} \Bigg[\prod_{i=1}^p (e^{\mathcal{F}(Y_j)}-1)\Bigg]. \end{aligned} \tag{6.7}$$

Note that since all the tiles in Q belong to the peel of some contour then $Q\cap X_0 = \emptyset$. On the other hand, the sets X_0 and X_1 *may very well overlap*, i.e., $X_0\cap X_1 \neq \emptyset$ in general (see Fig. 4 for an example). Moreover, once X_0 is fixed, the supports of the contours are automatically fixed too, since they must be the D-connected components of X_0. Then we can rewrite the sum as

$$\begin{aligned} K_q^{(\Lambda)}(X) = &\sum_{\emptyset\neq X_0\subset X} \sum_{\substack{\{\gamma_1,\ldots,\gamma_n\}\in\mathcal{C}(\Lambda,q):\\ \mathrm{supp}(\gamma_i)=\Gamma_i(X_0)}} \Big[\prod_{j=1}^n \zeta_q(\gamma_j)\Big] \\ &\cdot \sum_{\substack{X_1\subseteq X:\\ X_0\cup X_1 = X}} \sum_{Q\subseteq X_1} \sum_{p\geq 0} \sum_{\substack{\{Y_1,\ldots,Y_p\}:\ Y_i\in\Upsilon^q_{X_0}\\ \cup_i \mathrm{supp}(Y_i)=Q\\ \cup_i \overline{Y}_i = X_1}} \Big[\prod_{i=1}^p (e^{\mathcal{F}(Y_j)}-1)\Big], \end{aligned} \tag{6.8}$$

where in the second sum $\Gamma_i(X_0)$ are the maximally D-connected components of X_0, which must be well D-disconnected (otherwise the corresponding contribution to the activity is zero).

Now, note that $\mathcal{F}(Y)$ given in Eq. (5.4) is at least of order n (with $n\geq 2$) in z, by the very definition of the characteristic function F_Δ. In fact, $F_\Delta(R)$ is either equal to f_Δ or to $f_\Delta + g_\Delta(1-f_\Delta)$; therefore, using the definitions of f_Δ and g_Δ, Eqs. (4.13)–(4.14), we see that $F_\Delta(R)$ is different from zero only if R contains a rod belonging to $A_{\gamma(\Delta)}(\Delta)$. Now recall that, as already observed after Eq. (4.14), distinct tiles $\Delta_1\neq\Delta_2$ correspond to distinct sets $A_{\gamma(\Delta_1)}(\Delta_1)$ and $A_{\gamma(\Delta_2)}(\Delta_2)$, such that $A_{\gamma(\Delta_1)}(\Delta_1)\cap A_{\gamma(\Delta_2)}(\Delta_2)=\emptyset$ (here $\gamma(\Delta_i)$ is the contour whose peel Δ_i belongs to, $\Delta_i\in P_{\Gamma_i}$: since the peels of different contours are disconnected, the contour $\gamma(\Delta_i)$ is unique). Therefore, the r.h.s. of Eq. (5.4) is non zero only if R contains at least n distinct rods. Using Eq. (4.9), we find that, if $Y = \{\Delta_{\xi_1}, \dots, \Delta_{\xi_m}\}$ with $\Delta_{\xi_1} < \dots < \Delta_{\xi_n}$,

$$|\mathcal{F}(Y)| \le \sum_{\substack{R\in\Omega^q_\Lambda:\ |R|\ge 2\\ V(R)\cap\Delta_1\neq\emptyset,\\ \mathrm{diam}(\mathrm{supp}(R))\ge\mathrm{diam}(Y)}} |z|^{|R|}|\varphi^T(R)| \le 2Cz\ell^2(Czk)^{\max\{\frac{\mathrm{diam}(Y)}{k-1}-1,1\}}$$
$$\le C'zk^2(zk)^{\alpha\cdot\mathrm{diam}'(Y)}, \tag{6.9}$$

where $\mathrm{diam}'(Y) = |\xi_n - \xi_1|/\ell$ is the rescaled diameter of the set $\cup_{\Delta\in Y}\Delta$, and α can be chosen to be $\alpha = 1/4$. Using this bound and the fact that $|e^x - 1| \le |x|e^{|x|}$, we find:

$$\sum_{\substack{\{Y_1,\dots,Y_p\}:\ Y_i\in\Upsilon^q_{X_0}\\ \cup_i\mathrm{supp}(Y_i)=Q\\ \cup_i\overline{Y}_i=X_1}} \prod_{i=1}^p \left|e^{\mathcal{F}(Y_j)} - 1\right|$$
$$\le \sum_{\substack{\{Y_1,\dots,Y_p\}:\ Y_i\in\Upsilon^q_{X_0}\\ \cup_i\mathrm{supp}(Y_i)=Q\\ \cup_i\overline{Y}_i=X_1}} \prod_{j=1}^p \left\{C'zk^2(zk)^{\alpha\cdot\mathrm{diam}'(Y_j)}e^{\left[C'zk^2(zk)^{\alpha\cdot\mathrm{diam}'(Y_j)}\right]}\right\} \tag{6.10}$$

Now, since the choice of Y only depends on the union of the contours' supports X_0, in Eq. (6.8) we can start by performing the sums over the contours' spin attributions, rod configurations and internal colors. Using the bound Eq. (6.1) on the contours activities, we get

$$\sum_{\substack{\{\gamma_1,\dots,\gamma_n\}\in\mathcal{C}(\Lambda,q):\\ \mathrm{supp}(\gamma_i)=\Gamma_i(X_0)}} \left[\prod_{j=1}^n \zeta_q(\gamma_j)\right] \le \prod_{j=1}^n \left[\sum_{\substack{\gamma_j:\\ \mathrm{supp}(\gamma_j)=\Gamma_j(X_0)}} |\zeta_q(\gamma_j)|\right]$$
$$\le \prod_{j=1}^n 6^{|\Gamma'_j|}e^{-c_0\cdot zk^2|\Gamma'_j|} = 6^{|X'_0|}e^{-c_0\cdot zk^2|X'_0|}, \tag{6.11}$$

where the factor $6^{|\Gamma'|}$ bounds the sums over σ_γ and $\underline{m}_{int}$ at Γ fixed. Putting these results together into Eq. (6.8), we find

$$|K_q^{(\Lambda)}(X)| \le \sum_{\emptyset\neq X_0\subset X} 6^{|X'_0|}e^{-c_0\cdot zk^2|X'_0|} \sum_{\substack{X_1\subseteq X:\\ X_0\cup X_1=X}} \sum_{Q\subseteq X_1} \sum_{p\ge 0} \sum_{\substack{\{Y_1,\dots,Y_p\}:\ Y_i\in\Upsilon^q_{X_0}\\ \cup_i\mathrm{supp}(Y_i)=Q\\ \cup_i\overline{Y}_i=X_1}}$$
$$\cdot\prod_{j=1}^p \left\{C'zk^2(zk)^{\alpha\cdot\mathrm{diam}'(Y_j)}e^{\left[C'zk^2(zk)^{\alpha\cdot\mathrm{diam}'(Y_j)}\right]}\right\}. \tag{6.12}$$

Now, note that: (i) $\sum_{j=1}^p \mathrm{diam}'(Y_j) \ge |X'_1| - 1 \ge |X'_1|/2$; (ii) $|X'_0| \ge |Q'|$, because every tile in $Q = \cup_i\mathrm{supp}(Y_i)$ belongs to the peel of X_0;

$$\text{(iii)}\quad \sum_{j=1}^p (zk)^{\alpha\cdot\mathrm{diam}'(Y_j)} \le \sum_{\xi\in Q'}\sum_{\overline{Y}'\ni\xi}(zk)^{\alpha|\overline{Y}'|} \le C''|Q'|(zk)^\alpha.$$

Plugging these estimates into Eq. (6.12) we find

$$
\begin{aligned}
|K_q^{(\Lambda)}(X)| \le & \sum_{\emptyset \neq X_0 \subseteq X} (6e^{-\frac{c_0}{2}\cdot zk^2})^{|X_0'|} \sum_{\substack{X_1 \subseteq X: \\ X_0 \cup X_1 = X}} (zk)^{\frac{\alpha}{4}|X_1'|} \\
& \cdot \sum_{Q \subseteq X_1} e^{-zk^2|Q'|(\frac{c_0}{2} - C'C''(zk)^{\alpha})} \sum_{p \ge 0} \sum_{\substack{\{Y_1,\dots,Y_p\}:\ Y_i \in \Upsilon^q_{X_0} \\ \cup_i \mathrm{supp}(Y_i) = Q \\ \cup_i \overline{Y}_i = X_1}} \prod_{j=1}^{p} \Big[C' zk^2 (zk)^{\frac{\alpha}{2}\mathrm{diam}'(Y_j)} \Big],
\end{aligned}
\tag{6.13}
$$

which can be further bounded by:

$$
\begin{aligned}
|K_q^{(\Lambda)}(X)| \le & \sum_{\emptyset \neq X_0 \subseteq X} e^{-\frac{c_0}{3} zk^2 |X_0'|} \sum_{\substack{X_1 \subseteq X: \\ X_0 \cup X_1 = X}} (zk)^{\frac{\alpha}{4}|X_1'|} \\
& \cdot \sum_{Q \subseteq X_1} e^{-\frac{c_0}{3} zk^2 |Q'|} \sum_{p \ge 0} \frac{1}{p!} \Big[C' zk^2 \sum_{\substack{A \cap Q \neq \emptyset \\ |A'| \ge 2}} (zk)^{\frac{\alpha}{2}\delta'(A')} \Big]^p,
\end{aligned}
\tag{6.14}
$$

where in the last sum A is a generic subset of Λ consisting of a union of tiles, and $\delta'(A')$ is its rescaled tree length. The expression in square brackets in the second line is bounded above by $C'' zk^2 |Q'| (zk)^{\frac{\alpha}{2}}$, so that

$$
\begin{aligned}
|K_q^{(\Lambda)}(X)| & \le \sum_{\emptyset \neq X_0 \subseteq X} e^{-\frac{c_0}{3} zk^2 |X_0'|} \sum_{\substack{X_1 \subseteq X: \\ X_0 \cup X_1 = X}} (zk)^{\frac{\alpha}{4}|X_1'|} \sum_{Q \subseteq X_1} e^{-zk^2|Q'|(\frac{c_0}{3} - C''(zk)^{\frac{\alpha}{2}})} \\
& \le \sum_{\emptyset \neq X_0 \subseteq X} e^{-\frac{c_0}{3} zk^2 |X_0'|} \sum_{\substack{X_1 \subseteq X: \\ X_0 \cup X_1 = X}} (zk)^{\frac{\alpha}{4}|X_1'|} \sum_{Q \subseteq X_1} e^{-\frac{c_0}{4} zk^2 |Q'|}.
\end{aligned}
\tag{6.15}
$$

The last sum can be rewritten as $\sum_{Q \subseteq X_1} e^{-\frac{c_0}{4} zk^2 |Q'|} = (1 + e^{-\frac{c_0}{4} zk^2})^{|X_1'|}$, so that, defining $\tilde{\varepsilon}_1 := e^{-\frac{c_0}{3} zk^2}$, $\tilde{\varepsilon}_2 := (zk)^{\frac{\alpha}{4}}$ and $\tilde{\varepsilon} := \max\{\tilde{\varepsilon}_1, \tilde{\varepsilon}_2\}$:

$$
\begin{aligned}
|K_q^{(\Lambda)}(X)| & \le \sum_{\emptyset \neq X_0 \subseteq X} \tilde{\varepsilon}_1^{|X_0'|} \sum_{\substack{X_1 \subseteq X: \\ X_0 \cup X_1 = X}} \Big((1 + \tilde{\varepsilon}_1^{\frac{3}{4}}) \tilde{\varepsilon}_2 \Big)^{|X_1'|} \\
& \le \sum_{\emptyset \neq X_0 \subseteq X} \tilde{\varepsilon}_1^{|X_0'|} \sum_{\substack{X_1 \subseteq X: \\ X_0 \cup X_1 = X}} (2\tilde{\varepsilon}_2)^{|X_1'|} = (\tilde{\varepsilon}_1 + 2\tilde{\varepsilon}_1\tilde{\varepsilon}_2 + 2\tilde{\varepsilon}_2)^{|X'|} - (2\tilde{\varepsilon}_2)^{|X'|} \\
& \le |X'| \tilde{\varepsilon}_1 (1 + 2\tilde{\varepsilon}_2)\, (\tilde{\varepsilon}_1 + \tilde{\varepsilon}_1 2\tilde{\varepsilon}_2 + 2\tilde{\varepsilon}_2)^{|X'|-1} \le \tilde{\varepsilon}_1 (\sqrt{\tilde{\varepsilon}})^{|X'|-1},
\end{aligned}
\tag{6.16}
$$

where $\tilde{\varepsilon} = \max\{\tilde{\varepsilon}_1, \tilde{\varepsilon}_2\}$. Setting $\varepsilon_1 = \sqrt{\tilde{\varepsilon}_1}$, $\varepsilon_2 = \sqrt{\tilde{\varepsilon}_2}$ and recalling that $\alpha = \frac{1}{4}$, we obtain the desired estimate on $K_q^{(\Lambda)}(X)$. This concludes the proof of the lemma. $\square$

Remark. The dependence of the activities $K_q^{(\Lambda)}(X)$ on Λ is inherited from the constraint that X must be separated from Λ^c by at least one smoothing square, and by the fact that the quantities $\zeta(\gamma)$ and $\mathcal{F}(Y)$ themselves are Λ-dependent, simply because their definitions involve sums over rods collections in Ω_Λ^q. However, this dependence is very weak: in fact, if $K_q(X)$ is the infinite volume limit of $K_q^{(\Lambda)}(X)$, we have:

$$\left|K_q^{(\Lambda)}(X) - K_q(X)\right| \leq \left(\varepsilon_1 \varepsilon^{|X'|-1}\right)^{1/2} \varepsilon^{c' \cdot \mathrm{dist}'(X', \Lambda_c')}, \tag{6.17}$$

for some $c' > 0$. The proof of Eq. (6.17) proceeds along the same lines used to prove Eq. (6.5) and, therefore, we will not belabor the details of this computation.

7. The Activity of the Contours

In this section we prove the assumption Eq. (6.1) used in the proof of Lemma 3. Let us first recall, for the reader's convenience, the definition of $\zeta_q(\gamma)$:

$$\zeta_q(\gamma) = \zeta_q^0(\gamma) \exp\Big\{ - \sum_{R \in \Omega_\Lambda^q} \varphi^T(R) z^{|R|} \sum_{\Delta \subseteq P_\Gamma} F_\Delta(R) \Big\}, \tag{7.1}$$

where

$$\zeta_q^0(\gamma) = \frac{\bar{\varphi}(R_\gamma)}{Z^q(\Gamma)} \prod_{j=1}^{h_\Gamma} \frac{Z_\gamma(\mathrm{Int}_j \Gamma | m_{int}^j)}{Z(\mathrm{Int}_j \Gamma | q)}. \tag{7.2}$$

By using the same considerations used to get the bound Eq. (6.9), we see that the expression in braces in the r.h.s. of Eq. (7.1) is equal to a contribution of order one in z plus a rest, which is bounded in absolute value by $Czk^2|\Gamma'|(zk)^\alpha$. On the other hand, the contribution of order one in z is equal to $-z \sum_{R \in \Omega_\Lambda^q:\, |R|=1} \sum_{\Delta \subseteq P_\Gamma} F_\Delta(R)$, which is *negative*, simply because $F_\Delta \geq 0$. Therefore,

$$|\zeta_q(\gamma)| \leq |\zeta_q^0(\gamma)| e^{Czk^2|\Gamma'|(zk)^\alpha}, \tag{7.3}$$

which makes it apparent that, in order to prove Eq. (6.1), we need to prove an analogous bound for $\zeta_q^0(\gamma)$. By definition, $Z_\gamma(X|m) \leq Z(X|m)$, so that

$$|\zeta_q^0(\gamma)| \leq |\bar{\zeta}_q^0(\gamma)| \prod_{j=1}^{h_\Gamma} \max\Big\{1, \frac{Z(\mathrm{Int}_j \Gamma | -q)}{Z(\mathrm{Int}_j \Gamma | q)}\Big\}, \qquad \bar{\zeta}_q^0(\gamma) := \frac{\bar{\varphi}(R_\gamma)}{Z^q(\Gamma)}. \tag{7.4}$$

The estimate that we need on the quantities $\bar{\zeta}_q^0(\gamma)$ and $\frac{Z(\mathrm{Int}_j \Gamma | -q)}{Z(\mathrm{Int}_j \Gamma | q)}$ is summarized in the following two lemmas.

Lemma 4. *Let zk and $(zk^2)^{-1}$ be small enough. Then*

$$\sup_{\sigma_\gamma}{}^{*} \sum_{R_\gamma \in \Omega_\Gamma(\sigma_\gamma)} |\bar{\zeta}_q^0(\gamma)| \leq e^{-2c_0\, zk^2 |\Gamma'|}, \tag{7.5}$$

where the $*$ on the sup recalls the constraint that all the smoothing squares in Γ must have a non-zero intersection with at least one bad sampling square, and $c_0 = 5 * 10^{-4}$.

Remark. The specific choice of c_0 in the lemma comes from Eq. (7.11) below. It is related to the size of the smoothing squares, to the number of zero spins and to the number of pairs of neighboring spins with opposite sign that can appear in a contour (as explained below, it comes from the remark that every smoothing square - which contains 64 tiles - in a contour must intersect at least one bad sampling square - of size $\ell^2 \geq k^2/4$).

Lemma 5. *Let zk and $(zk^2)^{-1}$ be small enough. Then there exist two positive constants $C, c_1 > 0$ such that, for any simply connected region $X \subset \mathbb{Z}^2$ consisting of a union of smoothing squares,*

$$e^{-|P'_X|(Czk^2(zk)+\varepsilon^{c_1})} \leq \frac{Z(X|+)}{Z(X|-)} \leq e^{|P'_X|(Czk^2(zk)+\varepsilon^{c_1})}, \tag{7.6}$$

where ε was defined in Eq. (6.5) and P_X is the 1-tile-thick peel of X.

These two estimates combined with Eq. (7.3) give

$$\begin{aligned}
\sup^*_{\sigma_\gamma} \sum_{R_\gamma \in \Omega_\Gamma(\sigma_\gamma)} |\zeta_q(\gamma)| &\leq e^{-2c_0\, zk^2|\Gamma'|} e^{C'zk^2|\Gamma'|(zk)^\alpha} \prod_{j=1}^{h_\Gamma} e^{|P'_{\mathrm{Int}_j\Gamma}|(Czk^2(zk)+\varepsilon^{c_1})} \\
&\leq e^{|\Gamma'|(Czk^2(zk)+\varepsilon^{c_1})} e^{C'zk^2|\Gamma'|(zk)^\alpha} e^{-2c_0\, zk^2|\Gamma'|} \\
&= e^{-zk^2|\Gamma'|\left(2c_0 - C'(zk)^\alpha - C(zk) - \frac{\varepsilon^{c_1}}{zk^2}\right)} \leq\; e^{-c_0 zk^2|\Gamma'|}
\end{aligned} \tag{7.7}$$

under the only assumptions that zk and $(zk^2)^{-1}$ are small enough. Therefore, these two lemmas imply the convergence of the cluster expansion Eq. (6.2), which completes the computation of the partition function of our hard rod system with q boundary conditions. A computation of the correlation functions based on a similar expansion will be discussed in the next section. The rest of this section is devoted to the proofs of Lemma 4 and 5.

Proof of Lemma 4. Let σ_γ be a spin configuration compatible with the fact that γ is a contour. In particular, let us recall that every smoothing square contained in Γ has a non zero intersection with at least one bad sampling square; moreover, by its very definition, each such bad square must contain either one tile with magnetization equal to 0, or one pair of neighboring tiles with magnetizations + and −, respectively. Therefore, given σ_γ, it is possible to exhibit a partition $\mathcal{P}$ of Γ such that: (i) all the elements of the partition consist either of a single tile or of a pair of neighboring tiles with opposite magnetizations + and − (we shall call such pairs "domino tiles"); (ii) if $\mathcal{N}_0$ is the number of single tiles in $\mathcal{P}$ with magnetization equal to 0 and $\mathcal{N}_d$ is the number of domino tiles in $\mathcal{P}$, then $\mathcal{N}_0 + \mathcal{N}_d \geq |\Gamma'|/64$. The factor 64 comes from the consideration that in Γ, by definition, we have at least one bad square every four smoothing squares, and by the fact that four smoothing squares contain 64 tiles.

By the definition of $\bar{\varphi}(R_\gamma)$, we have: $\bar{\varphi}(R_\gamma) \leq \prod_{P\in\mathcal{P}} \bar{\varphi}(R_P)$. Moreover, using the standard cluster expansion described after Eq. (4.6), we find that $Z^q(\Gamma) \geq \prod_{P\in\mathcal{P}} Z^q(P) e^{-Czk^2(zk)|\Gamma'|}$. By combining these two bounds we get

$$\sum_{R_\gamma \in \Omega_\Gamma(\sigma_\gamma)} |\bar{\zeta}^0_q(\gamma)| \leq e^{Czk^2(zk)|\Gamma'|} \prod_{P\in\mathcal{P}} \Big| \sum_{R_P} \frac{\bar{\varphi}(R_P)}{Z^q(P)} \Big|, \tag{7.8}$$

where the sum over R_P runs over rods configurations in $\Omega_P(\cup_{\xi\in P'}\sigma_\xi)$. Now, if P is a single tile with magnetization either $+$ or $-$, then $\sum_{R_P}\frac{\bar\varphi(R_P)}{Z^q(P)}=1$. Moreover, if P is a single tile with magnetization equal to 0, then $\sum_{R_P}\frac{\bar\varphi(R_P)}{Z^q(P)}=-\frac{1}{Z^q(P)}=-e^{-z\ell^2(1+O(zk))}$.

Finally, let us consider the case that P is a domino tile. We assume without loss of generality that $P=\{\Delta_{\xi_1},\Delta_{\xi_2}\}$, with $\xi_2-\xi_1=(\ell,0)$, and $\sigma_{\xi_1}=-\sigma_{\xi_2}=+$. Since the rods interact via a hard core, $\bar\varphi(R_{\xi_1},R_{\xi_2})$ is different from zero only if at least one of the two rod configurations R_{ξ_1} and R_{ξ_2} is *untypical*: here we say that R_{ξ_1} is untypical if it does not contain any rod in the right half of Δ_{ξ_1} and, similarly, that R_{ξ_2} is untypical if it does not contain any rod in the left half of Δ_{ξ_2}. Therefore,

$$\sum_{R_P}\frac{\bar\varphi(R_P)}{Z^q(P)}\le e^{Czk^2(zk)}\Big[\sum_{\substack{R_{\xi_1}\in\Omega^+_{\Delta_{\xi_1}}:\\ R_{\xi_1}\ \text{untypical}}}\frac{\bar\varphi(R_{\xi_1})}{Z^+(\Delta_{\xi_1})}+\sum_{\substack{R_{\xi_2}\in\Omega^-_{\Delta_{\xi_2}}:\\ R_{\xi_2}\ \text{untypical}}}\frac{\bar\varphi(R_{\xi_2})}{Z^-(\Delta_{\xi_2})}\Big],\tag{7.9}$$

where we used that $\sum_{R\in\Omega^q_\Delta}\bar\varphi(R)=Z^q(\Delta)$. Equation (7.9) can be rewritten and estimated (defining $\Delta^L_{\xi_1}$ to be the left half of Δ_{ξ_1}) as

$$\sum_{R_P}\frac{\bar\varphi(R_P)}{Z^q(P)}\le 2e^{Czk^2(zk)}\frac{Z^+(\Delta^L_{\xi_1})}{Z^+(\Delta_{\xi_1})}\le 2e^{C'zk^2(zk)}e^{-z\ell^2/2}.\tag{7.10}$$

Plugging the bounds on $\sum_{R_P}\frac{\bar\varphi(R_P)}{Z^q(P)}$ into Eq. (7.8) gives:

$$\begin{aligned}\sum_{R_\gamma\in\Omega_\Gamma(\sigma_\gamma)}|\bar\zeta^0_q(\gamma)|&\le e^{Czk^2(zk)|\Gamma'|}e^{-z\ell^2(1-Czk)(\mathcal{N}_0+\frac12\mathcal{N}_d)}\\&\le e^{-z\ell^2(1-C'zk)|\Gamma'|/128},\end{aligned}\tag{7.11}$$

where in the last line we used the bound $\mathcal{N}_0+\mathcal{N}_d\ge|\Gamma'|/64$. Using $\ell\ge k/2$ we obtain Eq. (7.5) so the proof of the lemma is complete. □

Proof of Lemma 5. We proceed by induction on the size of X. If X is so small that it cannot contain contours D-disconnected from X^c, then

$$\frac{Z(X|+)}{Z(X|-)}=\frac{Z^+(X)}{Z^-(X)}=\exp\Big\{\sum_{R\in\Omega^+_X}\varphi^T(R)z^{|R|}-\sum_{R\in\Omega^-_X}\varphi^T(R)z^{|R|}\Big\}.\tag{7.12}$$

Let $V(R)$ be the union of the centers of the rods in R and let $R\in\Omega^q_X$. Since the orientation of all rods in R is fixed, $V(R)$ identifies uniquely the rod configuration. Then

$$\begin{aligned}\sum_{R\in\Omega^q_X}\varphi^T(R)z^{|R|}&=\sum_{R\in\Omega^q_X}\sum_{x\in V(R)}\frac{\varphi^T(R)z^{|R|}}{|V(R)|}=\sum_{x\in X}\sum_{\substack{R\in\Omega^q_X\\ V(R)\ni x}}\frac{\varphi^T(R)z^{|R|}}{|V(R)|}\\&=\sum_{x\in X}\sum_{\substack{R\in\Omega^q_{\mathbb{Z}^2}\\ V(R)\ni x}}\frac{\varphi^T(R)z^{|R|}}{|V(R)|}-\sum_{x\in X}\sum_{\substack{R\in\Omega^q_{\mathbb{Z}^2}\setminus\Omega^q_X\\ V(R)\ni x}}\frac{\varphi^T(R)z^{|R|}}{|V(R)|}.\end{aligned}\tag{7.13}$$

The first sum in the second line is equal to

$$\sum_{x\in X}\sum_{\substack{R\in\Omega^q_{\mathbb{Z}^2}\\ V(R)\ni x}}\frac{\varphi^T(R)z^{|R|}}{|V(R)|}=|X|s(z),\tag{7.14}$$

where

$$s(z):=\sum_{\substack{R\in\Omega^q_{\mathbb{Z}^2}\\ V(R)\ni x}}\frac{\varphi^T(R)z^{|R|}}{|V(R)|}\tag{7.15}$$

is an analytic function of z, of the form $s(z)=z(1+O(zk))$, *independent of* q *and* x. The second sum in the second line of Eq. (7.13) involves rod configurations containing at least one rod belonging to X and one belonging to X^c. Therefore, it is of order at least 2 in z and scales like the boundary of X:

$$\Big|\sum_{x\in X}\sum_{\substack{R\in\Omega^q_{\mathbb{Z}^2}\setminus\Omega^q_X\\ V(R)\ni x}}\frac{\varphi^T(R)z^{|R|}}{|V(R)|}\Big|\le C_1zk^2(zk)|P'_X|,\tag{7.16}$$

for a suitable constant $C_1>0$, independent of q. Plugging Eqs. (7.13)–(7.16) into Eq. (7.12) gives:

$$\frac{Z(X|+)}{Z(X|-)}=\exp\Big\{-\sum_{x\in X}\sum_{\substack{R\in\Omega^+_{\mathbb{Z}^2}\setminus\Omega^+_X\\ V(R)\ni x}}\frac{\varphi^T(R)z^{|R|}}{|V(R)|}+\sum_{x\in X}\sum_{\substack{R\in\Omega^-_{\mathbb{Z}^2}\setminus\Omega^-_X\\ V(R)\ni x}}\frac{\varphi^T(R)z^{|R|}}{|V(R)|}\Big\},\tag{7.17}$$

which is bounded from above and below by $e^{2C_1zk^2(zk)|P'_X|}$ and $e^{-2C_1zk^2(zk)|P'_X|}$, respectively. Setting $C\ge 2C_1$, this proves the inductive hypothesis Eq. (7.6) at the first step, i.e., for regions X small enough.

Let us now assume the validity of Eq. (7.6) for all the regions of size strictly smaller than Λ_0, and let us prove it for Λ_0. As explained in Sect. 6, $Z(\Lambda_0|q)$ admits the cluster expansion Eq. (6.2) involving polymers X that are D-disconnected from Λ_0^c, whose activities are defined in Eq. (5.3). In particular, the cluster expansion is convergent provided that $\zeta_q(\gamma)$ is bounded as in Eq. (6.1). Now, note that the interiors of the contours γ_i involved in the cluster expansion for $Z(X_0|q)$ via Eqs. (6.2) and (5.3) have all sizes strictly smaller than Λ_0. Therefore, using the inductive hypothesis, the product $\max\Big\{1,\frac{Z(\mathrm{Int}_j\Gamma|-q)}{Z(\mathrm{Int}_j\Gamma|q)}\Big\}$ in Eq. (7.4) can be bounded from above by $e^{|\Gamma'|(Czk^2(zk)+\varepsilon^c)}$ that, if combined with Eqs. (7.3), (7.5), implies Eq. (6.1) for all the the contours γ_i involved in the cluster expansion for $Z(X_0|q)$. We can then write:

$$\frac{Z(\Lambda_0|+)}{Z(\Lambda_0|-)}=\frac{Z^+(\Lambda_0)}{Z^-(\Lambda_0)}\exp\Big\{\sum_{\mathcal{X}\subseteq\Lambda_0}\Big[K_+^{(\Lambda_0)}(\mathcal{X})-K_-^{(\Lambda_0)}(\mathcal{X})\Big]\phi^T(\mathcal{X})\Big\},\tag{7.18}$$

where $K_q^{(\Lambda_0)}(\mathcal{X})=\prod_{X\in\mathcal{X}}K_q^{(\Lambda_0)}(X)$ and $K_q^{(\Lambda_0)}(X)$ admits the bound Eq. (6.5). The first factor in the r.h.s. of Eq. (7.18) is rewritten as in Eq. (7.17) and is bounded from above and below by $e^{2C_1 zk^2(zk)|P'_{\Lambda_0}|}$ and $e^{-2C_1 zk^2(zk)|P'_{\Lambda_0}|}$, respectively, exactly in the same way as Eq. (7.18) itself.

The second factor in the r.h.s. of Eq. (7.18) can be bounded as follows. We rewrite

$$\begin{aligned}
&\exp\Big\{\sum_{\mathcal{X}\subseteq\Lambda_0}\Big[K_+^{(\Lambda_0)}(\mathcal{X})-K_-^{(\Lambda_0)}(\mathcal{X})\Big]\phi^T(\mathcal{X})\Big\}\\
&\quad=\exp\Big\{\sum_{\substack{\mathcal{X}\subseteq\Lambda_0\\ q=\pm}} qK_q(\mathcal{X})\phi^T(\mathcal{X})\Big\}\cdot\exp\Big\{\sum_{\substack{\mathcal{X}\subseteq\Lambda_0\\ q=\pm}} q\Big[K_q^{(\Lambda_0)}(\mathcal{X})-K_q(\mathcal{X})\Big]\phi^T(\mathcal{X})\Big\},
\end{aligned} \tag{7.19}$$

where $K_q(\mathcal{X})=\prod_{X\in\mathcal{X}}K_q(X)$. Now

$$\begin{aligned}
\prod_{j=1}^{n}K_q^{(\Lambda_0)}(X_j)-\prod_{j=1}^{n}K_q(X_j)=\sum_{m=1}^{n}\prod_{j=1}^{m-1}K_q(X_j)\,[K_q^{(\Lambda_0)}(X_m)\\
-K_q(X_m)]\prod_{j=m+1}^{n}K_q^{(\Lambda_0)}(X_j),
\end{aligned} \tag{7.20}$$

then using Eq. (6.17) and (6.5) , we have

$$\begin{aligned}
|K_q^{(\Lambda_0)}(\mathcal{X})-K_q(\mathcal{X})| &\le \sum_{m=1}^{n}(\sqrt{\varepsilon})^{|X'_m|}\varepsilon^{c'\mathrm{dist}'(X'_m,\Lambda'_{0,c})}\prod_{j\neq m}\varepsilon^{|X'_j|}\\
&\le \varepsilon^{c'\mathrm{dist}'(\mathcal{X},\Lambda'_{0,c})}\prod_{X\in\mathcal{X}}\varepsilon^{\frac{|X'|}{2}}.
\end{aligned} \tag{7.21}$$

Therefore, the second factor in the second line of Eq. (7.19) can be bounded from above and below by $e^{\varepsilon^{c_2}|P'_{\Lambda_0}|}$ and $e^{-\varepsilon^{c_2}|P'_{\Lambda_0}|}$, respectively for a suitable constant c_2. We are left with the first factor in the second line of Eq. (7.19), which involves the partition sum

$$\sum_{\mathcal{X}\subseteq\Lambda_0}K_q(\mathcal{X})\phi^T(\mathcal{X})=\sum_{\xi\in\Lambda'_0}\sum_{\substack{\mathcal{X}\supseteq\Delta_\xi\\ \mathcal{X}\subseteq\Lambda_0}}\frac{K_q(\mathcal{X})\phi^T(\mathcal{X})}{|\mathcal{X}'|}, \tag{7.22}$$

where $|\mathcal{X}'|$ is number of tiles in $\cup_{X\in\mathcal{X}}X$. Equation (7.22) can be further rewritten as

$$\sum_{\mathcal{X}\subseteq\Lambda_0}K_q(\mathcal{X})\phi^T(\mathcal{X})=|\Lambda'_0|\mathcal{S}+\sum_{\xi\in\Lambda'_0}\sum_{\substack{\mathcal{X}\supseteq\Delta_\xi\\ \mathcal{X}\cap\Lambda_0^c\neq\emptyset}}\frac{K_q(\mathcal{X})\phi^T(\mathcal{X})}{|\mathcal{X}'|}, \tag{7.23}$$

where

$$\mathcal{S}:=\sum_{\substack{\mathcal{X}\supseteq\Delta_\xi\\ \mathcal{X}\subseteq\mathbb{Z}^2}}\frac{K_q(\mathcal{X})\phi^T(\mathcal{X})}{|\mathcal{X}'|}, \tag{7.24}$$

is independent of q and ξ. The second term in the r.h.s. of Eq. (7.23) is bounded in absolute value from above by $|P'_{\Lambda_0}|\varepsilon^{c_3}$ for a suitable $c_3>0$; therefore,

$$\exp\Big\{\sum_{\substack{\mathcal{X}\subseteq\Lambda_0\\ q=\pm}} qK_q(\mathcal{X})\phi^T(\mathcal{X})\Big\} = \exp\Big\{\sum_{\substack{\xi\in\Lambda_0'\\ q=\pm}}\ \sum_{\substack{\mathcal{X}\supseteq\Delta_\xi\\ \mathcal{X}\cap\Lambda_0^c\neq\emptyset}} q\frac{K_q(\mathcal{X})\phi^T(\mathcal{X})}{|\mathcal{X}'|}\Big\} \le e^{2|P_{\Lambda_0'}|\varepsilon^{c_3}} \tag{7.25}$$

and is bounded from below by $e^{-2|P_{\Lambda_0'}|\varepsilon^{c_3}}$. Choosing c_1 such that $\varepsilon^{c_1} \ge \varepsilon^{c_2} + 2\varepsilon^{c_3}$ this completes the inductive proof of Eq. (7.6). □

8. Existence of Nematic Order

In this section we prove Theorem 1. We start by proving Eq. (3.9). The probability that the tile centered at ξ_0 has magnetization $-q$ in the presence of boundary conditions q can be written as

$$\langle\chi_{\xi_0}^{-q}\rangle_\Lambda^q = \frac{\partial}{\partial z_0}\log Z_{z_0}(\Lambda|q)\Big|_{z_0=1}, \tag{8.1}$$

where $Z_{z_0}(\Lambda|q)$ is defined in a way completely analogous to Eqs. (3.4)–(3.6), with the only difference that the activity $\zeta(\xi)$ in Eq. (3.4) is replaced by $\tilde\zeta(\xi)$, where $\tilde\zeta(\xi) = \zeta(\xi)$ if $\xi \neq \xi_0$, while

$$\tilde\zeta(\xi_0) = \begin{cases} z^{|R_\xi|} & \text{if } \sigma_\xi = q\\ z_0 z^{|R_\xi|} & \text{if } \sigma_\xi = -q\\ -1 & \text{if } \sigma_\xi = 0.\end{cases} \tag{8.2}$$

The change of $\zeta(\xi)$ into $\tilde\zeta(\xi)$ induces a corresponding change of $\zeta_q(\gamma)$ and $K_q^{(\Lambda)}(X)$ into $\tilde\zeta_q(\gamma)$ and $\tilde K_q^{(\Lambda)}(X)$, respectively. The activity $\tilde K_q^{(\Lambda)}(X)$ admits the same bound Eq. (6.5) (possibly with a slightly different constant c''), uniformly in z_0 for z_0 close to 1, and it depends explicitly on z_0 only if $X \supseteq \Delta_{\xi_0}$. In such a case, the derivative of $\tilde K_q^{(\Lambda)}(X)$ with respect to z_0 is bounded by $\sqrt{\varepsilon_1\varepsilon^{|X'|-1}}$, uniformly in z_0 for z_0 close to 1.

The logarithm of the modified partition function admits a convergent cluster expansion analogous to Eq. (6.2):

$$\log\frac{Z_{z_0}(\Lambda|q)}{Z^q(\Lambda)} = \sum_{\mathcal{X}\subseteq\Lambda}\tilde K_q^{(\Lambda)}(\mathcal{X})\phi^T(\mathcal{X}), \tag{8.3}$$

so that

$$\langle\chi_{\xi_0}^{-q}\rangle_\Lambda^q = \sum_{\mathcal{X}\subseteq\Lambda}\partial_{z_0}\tilde K_q^{(\Lambda)}(\mathcal{X})\phi^T(\mathcal{X})\Big|_{z_0=1}. \tag{8.4}$$

The sum in the r.h.s. of Eq. (8.4) is exponentially convergent for ε small enough, and it only involves polymer configurations containing Δ_{ξ_0}, simply because $\tilde K_q^{(\Lambda)}(X)$ is independent of z_0 whenever $\Delta_{\xi_0}\cap X = \emptyset$. Therefore,

$$\begin{aligned}\langle\chi_{\xi_0}^{-q}\rangle_\Lambda^q &\le \sum_{\mathcal{X}\subseteq\Lambda}|\phi^T(\mathcal{X})|\cdot|\partial_{z_0}\tilde K_q^{(\Lambda)}(\mathcal{X})|_{z_0=1} \le (\varepsilon_1)^{\frac14}\sum_{\mathcal{X}\supseteq\Delta_{\xi_0}}|\phi^T(\mathcal{X})|\prod_{X\in\mathcal{X}}\varepsilon^{\frac14|X'|}\\ &\le (\text{const.})(\varepsilon_1)^{\frac14},\end{aligned} \tag{8.5}$$

which proves Eq. (3.9).

In order to compute the density-density correlation functions we proceed in a similar fashion. We replace the activity z of a rod r centered at x by z_x and we define $\tilde{Z}_{\mathbf{z}}(\Lambda|q)$ to be the modified partition function with boundary conditions q and variable rod activities $\mathbf{z} = \{z_x\}_{x\in\Lambda}$. Correspondingly, we rewrite:

$$\begin{aligned} \langle n_x\rangle_\Lambda^q &= z\partial_{z_x} \log \tilde{Z}_{\mathbf{z}}(\Lambda|q)\Big|_{\mathbf{z}=z}, \\ \langle n_x n_y\rangle_\Lambda^q - \langle n_x\rangle_\Lambda^q \langle n_y\rangle_\Lambda^q &= z^2\partial_{z_x}\partial_{z_y} \log \tilde{Z}_{\mathbf{z}}(\Lambda|q)\Big|_{\mathbf{z}=z}, \end{aligned} \tag{8.6}$$

where $\mathbf{z} = z$ means that $z_x = z$, $\forall x \in \Lambda$; the higher order density correlation functions have a similar representation. Once again, $\log \tilde{Z}_{\mathbf{z}}(\Lambda|q)$ admits a cluster expansion completely analogous to $\log Z(\Lambda|q)$:

$$\log \tilde{Z}_{\mathbf{z}}(\Lambda|q) = \sum_{R\in\Omega_\Lambda^q} \Big[\prod_{r\in R} z_{x(r)}\Big]\varphi^T(R) + \sum_{\mathcal{X}\subseteq\Lambda} \tilde{K}_{q,\mathbf{z}}^{(\Lambda)}(\mathcal{X})\phi^T(\mathcal{X})\Big|_{\mathbf{z}=z}, \tag{8.7}$$

where $x(r)$ is the center of r. Moreover, $\tilde{K}_{q,\mathbf{z}}^{(\Lambda)}(X)$, together with its derivatives with respect to z_x and/or z_y, admit the same bound Eq. (6.5), possibly with a different constant c''; the derivative of $\tilde{K}_{q,\mathbf{z}}^{(\Lambda)}(X)$ with respect to z_x and/or z_y is different from zero only if $X \ni x$ and/or $X \ni y$. Therefore,

$$\begin{aligned} \langle n_x\rangle_\Lambda^q &= \sum_{R\in\Omega_\Lambda^q} z^{|R|} R(r(x))\varphi^T(R) + \sum_{\mathcal{X}\subseteq\Lambda} \partial_{z_x}\tilde{K}_{q,\mathbf{z}}^{(\Lambda)}(\mathcal{X})\phi^T(\mathcal{X})\Big|_{\mathbf{z}=z}, \\ &\langle n_x n_y\rangle_\Lambda^q - \langle n_x\rangle_\Lambda^q\langle n_y\rangle_\Lambda^q \\ &\quad= \sum_{R\in\Omega_\Lambda^q} z^{|R|} R(r(x))R(r(y))\varphi^T(R) + \sum_{\mathcal{X}\subseteq\Lambda} \partial^2_{z_x z_y}\tilde{K}_{q,\mathbf{z}}^{(\Lambda)}(\mathcal{X})\phi^T(\mathcal{X})\Big|_{\mathbf{z}=z}, \end{aligned} \tag{8.8}$$

where $R(r)$ is the multiplicity of r in R. The sums in the first line involve connected rod or polymer configurations containing at least one rod centered at x; similarly, the sums in the second line involve connected rod or polymer configurations containing at least one rod centered at x and one rod centered at y. All the sums are exponentially convergent and their evaluation finally leads to the finite volume analogues of Eqs. (3.10)–(3.11). The infinite volume counterparts are obtained simply by replacing all the finite volume activities with their infinite volume counterparts and by dropping the constraints that the polymers should be contained in Λ. The infinite volume limit is reached exponentially fast and all the observables share the same invariance properties as the infinite volume activities themselves. In particular, the infinite volume Gibbs measures $\langle\cdot\rangle^q$ are translation invariant, and the averages $\langle\chi_{\xi_0}^{-q}\rangle^q$ and $\langle\prod_j n_{x_j}\rangle^q$ are all independent of q. We will not belabor the proofs of these claims, since they are all straightforward consequences of the cluster expansion described in the previous sections, in the same sense as the representations for $\langle\chi_{\xi_0}^{-q}\rangle^q$, $\langle n_x\rangle^q$ and $\langle n_x n_y\rangle^q$ and the proof of their convergence, discussed in this section, are a consequence of the bounds of sects. 4 and 6. This concludes the proof of the main theorem. □

9. Conclusions and Outlook

We considered a simple two-dimensional lattice model for liquid crystals, known as the hard-core $k$mer-monomer system. We rigorously proved the validity of the "Onsager's excluded volume effect" underlying the emergence of a spontaneously oriented phase at intermediate densities. The oriented phase comes without any translational order: therefore, it can be considered as a lattice analogue of the nematic phase in liquid crystals. Our proof is based on a rigorous coarse graining operation, which maps the original rod system into a contour model for effective Ising spins. The resulting contour model is not invariant under "spin flip" and has long range many-body interaction. In order to control its thermodynamic and correlation functions we use cluster expansion methods in the form originally proposed by Pirogov and Sinai.

Our methods can be naturally extended to more general situations than the one that we explicitly considered here. For instance, while it is important that the number of allowed orientations of the molecules is finite, we do not really use the fact that the centers are constrained to belong to a lattice. Therefore, we expect our result to be valid also for systems of finite size hard rods with centers in $\mathbb{R}^2$ and $2 \leq N < \infty$ distinct allowed orientations. Similarly, we expect the same results to be generalizable to the three-dimensional case, that is to hard rods with centers in $\mathbb{R}^3$ and $2 \leq N < \infty$ distinct allowed orientations. The nice thing is that in three dimensions we can also play with the shape of the "rods": for instance, we can imagine to consider flat anisotropic $1 \times k^\alpha \times k$ molecules, with k a large parameter and $\alpha \in [0, 1]$. Depending on the shape of these molecules, that is on the value of α, there may be competition between a nematic-like and a smectic-like phase (by "smectic" we mean a phase where orientational symmetry is completely broken, while translational symmetry is only partially broken: e.g., a phase where the molecules organize themselves into planes that are displaced periodically in the transverse direction, and on each such plane the molecules' centers are disordered). It would be interesting to establish whether this possibility realizes or not and, in case, to construct the phase diagram at intermediate densities as a function of α and k. We plan to come back to this question in a future publication.

Another natural generalization of our work is to consider models in which the phases with different orientations are not macroscopically equivalent among each other: if we added a small perturbing interaction non-symmetric with respect to 90^o rotations, then in order to determine the coexistence line between the two phases we would be forced to introduce an external field, balancing the effect of the perturbation. This would force us to enter the usual discussion of Pirogov-Sinai theory about "metastable phases" and "auxiliary contour models with parameters" that we were able to avoid from the beginning in our case. See Comment 3 at the beginning of Sect. 4.

Another interesting question, which is non trivial already at the level of k-mers in $\mathbb{Z}^2$ is whether by increasing density towards close packing we encounter a second phase transition to a "dense phase" or not. The conjecture, based on a simple and appealing variational computation [15] and some more recent simulations [24], is that there should be a high density phase where orientational order is lost, but maybe a hidden striped order could survive. The variational computation underlying this conjecture gives a bound on the entropy of the close packed system in the form: $S_{cp}/\text{Volume} \geq C_s k^{-2} \log k$ for an explicitly known constant C_s; the "minimal" thing to prove, in order to give some rigorous support to the conjecture about the structure of the dense phase, would be an upper bound of the same asymptotic form: $S_{cp}/\text{Volume} \sim k^{-2} \log k$, asymptotically as $k \to \infty$. Some attempts in this direction indicate that the estimate is not trivial at all.

The problem may be of interest also for the community that studies random packings and glassy behavior in hard core particles, see [35] and references therein.

Of course, the most important and difficult open problem is to prove the existence of orientational order in a model with genuine rotational invariance. In this context, the understanding of the Onsager's excluded volume effect is completely open. The real obstacle is our lack of understanding of continuous symmetry breaking phenomena. A good example is provided by the three-dimensional Heisenberg antiferromagnet, where the emergence of spontaneous magnetization has been proved by reflection positivity methods [9,12]. Unfortunately, the proof is very fragile under apparently harmless changes in the Hamiltonian and it would be very important to adapt it to cases where, e.g., a weak next-to-nearest-neighbor interaction is added to the dominant nearest neighbor exchange. It is very likely that future progress on the theory of the Heisenberg model will also help us in understanding the validity of the Onsager's excluded volume effect in continuous three dimensional systems. It may also be of help to try to attack the issue of continuous rotational symmetry by first looking at mean-field like models, possibly in the spirit of [26].

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