THREE

Concavity and Sufficiency in Optimal Control Problems

In unconstrained static optimization problems, we know that if the objective function is a $C^{(1)}$ concave function of the decision variables, then the solution of the FONCs is the solution of the optimization problem under consideration. That is, the FONCs are both necessary *and* sufficient conditions for optimality if the objective function is a $C^{(1)}$ concave function of the decision variables. The goal of this chapter is to demonstrate that essentially an analogous result holds for optimal control problems. More precisely, we seek to derive an economically useful set of conditions under which a solution of the necessary conditions of Theorems 2.2 or 2.3 is a solution of the posed optimal control problem. We will also prove a more general sufficiency theorem due to Arrow.

To begin, recall that the optimal control problem under consideration is given by

$$\begin{aligned}\max_{u(\cdot),x_1} J[x(\cdot),u(\cdot)] &\stackrel{\text{def}}{=} \int_{t_0}^{t_1} f(t,x(t),u(t))\,dt\\ \text{s.t.}\quad \dot{x}(t) &= g(t,x(t),u(t)),\\ x(t_0) &= x_0,\ \ x(t_1)=x_1,\end{aligned} \tag{1}$$

with the Hamiltonian defined by $H(t,x,u,\lambda) \stackrel{\text{def}}{=} f(t,x,u)+\lambda g(t,x,u)$. The necessary conditions are given by Theorem 2.2 and read

$$H_u(t,x,u,\lambda) = f_u(t,x,u)+\lambda g_u(t,x,u) = 0,$$

$$\dot{\lambda} = -H_x(t,x,u,\lambda) = -f_x(t,x,u)-\lambda g_x(t,x,u),\ \ \lambda(t_1)=0,$$

$$\dot{x} = H_\lambda(t,x,u,\lambda) = g(t,x,u),\ \ x(t_0)=x_0.$$

Given these preliminaries, we now state and prove the following sufficiency theorem for the optimal control problem (1).

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Theorem 3.1 (Mangasarian Sufficient Conditions): *Let $(z(t), v(t))$ be an admissible pair for problem (1). Suppose that $(z(t), v(t))$ satisfy the necessary conditions of Theorem 2.2 for problem (1) with costate variable $\lambda(t)$, and let $H(t, x, u, \lambda) \stackrel{\text{def}}{=} f(t, x, u) + \lambda g(t, x, u)$ be the value of the Hamiltonian function. If $H(\cdot)$ is a concave function of $(x, u)\,\forall\, t \in [t_0, t_1]$ over an open convex set containing all the admissible values of $(x(\cdot), u(\cdot))$ when the costate variable is $\lambda(t)$, then $v(t)$ is an optimal control and $(z(t), v(t))$ yields the global maximum of $J[\cdot]$. If $H(\cdot)$ is a strictly concave function under the same conditions, then $(z(t), v(t))$ yields the unique global maximum of $J[\cdot]$.*

Proof: Let $(x(t), u(t))$ be an admissible pair of curves. By hypothesis, $H(\cdot)$ is concave in $(x, u)\,\forall\, t \in [t_0, t_1]$. Moreover, because $f(\cdot)$ and $g(\cdot)$ are $C^{(1)}$ functions by assumption, so is $H(\cdot)$. Therefore, by Theorem 21.3 in Simon and Blume (1994), we have that

$$\begin{aligned} H(t, x(t), u(t), \lambda(t)) \leq H(t, z(t), v(t), \lambda(t)) + H_x(t, z(t),& \\ v(t), \lambda(t))[x(t) - z(t)] + H_u(t, z(t), v(t), \lambda(t))[u(t) - v(t)].& \end{aligned} \tag{2}$$

Now recall that $(z(t), v(t))$ satisfy the necessary conditions of Theorem 2.2 by hypothesis; hence $H_u(t, z(t), v(t), \lambda(t)) \equiv 0\,\forall\, t \in [t_0, t_1]$. This implies that the last term on the right-hand side of Eq. (2) is identically zero. Moreover, the inequality in Eq. (2) holds for all $t \in [t_0, t_1]$, so we can integrate it over $[t_0, t_1]$ and the inequality is preserved. Doing just that yields

$$\begin{aligned} \int_{t_0}^{t_1} H(t, x(t), u(t), \lambda(t))\, dt \leq \int_{t_0}^{t_1} [H(t, z(t), v(t), \lambda(t))& \\ + H_x(t, z(t), v(t), \lambda(t))[x(t) - z(t)]]\, dt.& \end{aligned} \tag{3}$$

Now use the costate equation $\dot{\lambda}(t) \equiv -H_x(t, z(t), v(t), \lambda(t))$, the definition of the Hamiltonian function $H(t, x, u, \lambda) \stackrel{\text{def}}{=} f(t, x, u) + \lambda g(t, x, u)$ evaluated along the curves $(z(t), v(t))$ and $(x(t), u(t))$, and the definition of functional $J[x(\cdot), u(\cdot)] \stackrel{\text{def}}{=} \int_{t_0}^{t_1} f(t, x(t), u(t))\, dt$ evaluated along the curves $(z(t), v(t))$ and $(x(t), u(t))$, to rewrite Eq. (3) in the form

$$\begin{aligned} J[x(\cdot), u(\cdot)] \leq J[z(\cdot), v(\cdot)] + \int_{t_0}^{t_1} [\lambda(t)[g(t, z(t), v(t)) - g(t, x(t), u(t))]& \\ -\dot{\lambda}(t)[x(t) - z(t)]]\, dt.& \end{aligned} \tag{4}$$

Next, define $h(t) \stackrel{\text{def}}{=} x(t) - z(t)$ and integrate the term $\int_{t_0}^{t_1} \dot{\lambda}(t)h(t)\, dt$ of Eq. (4) by

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parts to get

$$\int_{t_0}^{t_1} \dot{\lambda}(t)h(t)\,dt = \lambda(t_1)h(t_1) - \lambda(t_0)h(t_0) - \int_{t_0}^{t_1} \lambda(t)\dot{h}(t)\,dt. \tag{5}$$

Because $x(t)$ satisfies the state equation and initial condition by virtue of it being admissible, as does $z(t)$ by virtue of it being a solution of the necessary conditions, it therefore follows that $h(t_0) \overset{\text{def}}{=} x(t_0) - z(t_0) = x_0 - x_0 \equiv 0$. Moreover, $\lambda(t_1) = 0$ by the necessary transversality condition. Hence, using Eq. (5) and these two results in Eq. (4) yields

$$J[x(\cdot), u(\cdot)] \leq J[z(\cdot), v(\cdot)] + \int_{t_0}^{t_1} \lambda(t)[[g(t, z(t), v(t)) - \dot{z}(t)]$$
$$+[\dot{x}(t) - g(t, x(t), u(t))]]\,dt. \tag{6}$$

Because the curves $(x(t), u(t))$ are admissible, they must satisfy the state equation identically, that is, $\dot{x}(t) \equiv g(t, x(t), u(t))$. Similarly, seeing as the curves $(z(t), v(t))$ are a solution of the necessary conditions, they must satisfy the state equation identically too, that is, $\dot{z}(t) \equiv g(t, z(t), v(t))$. Thus the integrand of Eq. (6) is identically zero, thereby implying that $J[x(\cdot), u(\cdot)] \leq J[z(\cdot), v(\cdot)]$. If $H(\cdot)$ is strictly concave in $(x, u)\,\forall\, t \in [t_0, t_1]$, then the inequality in Eq. (2) becomes strict if either $x(t) \neq z(t)$ or $u(t) \neq v(t)$ for some $t \in [t_0, t_1]$. Continuity would then imply that there exists an interval over that $x(t) \neq z(t)$ or $u(t) \neq v(t)$ held. Carrying the strict inequality through the proof leads to the conclusion that $J[x(\cdot), u(\cdot)] < J[z(\cdot), v(\cdot)]$. This argument thus shows that any admissible pair $(x(t), u(t))$ that is not identically equal to $(z(t), v(t))$ is nonoptimal. Q.E.D.

Now the question becomes: Under what conditions is the Hamiltonian $H(\cdot)$ a concave function of $(x, u)\,\forall\, t \in [t_0, t_1]$? The following theorem, which you are asked to prove in the mental exercises, provides some necessary technical input for applying Theorem 3.1 in many optimal control problems in economics.

Theorem 3.2: *A nonnegative linear combination of concave functions is also a concave function. That is, if $f^i(\cdot) : \mathrm{X} \to \Re$, $i = 1, 2, \ldots, m$, are concave functions on a convex subset $\mathrm{X} \subset \Re^n$, then $f(\mathbf{x}) \overset{\text{def}}{=} \sum_{i=1}^{m} \alpha_i f^i(\mathbf{x})$, where $\alpha_i \in \Re_+$, $i = 1, 2, \ldots, m$, is also a concave function on $\mathrm{X} \subset \Re^n$.*

To see what Theorem 3.2 implies for Theorem 3.1, first recall the definition of the Hamiltonian, namely, $H(t, x, u, \lambda) \overset{\text{def}}{=} f(t, x, u) + \lambda g(t, x, u)$. Therefore, if $f(\cdot)$ and $g(\cdot)$ are concave functions of $(x, u)\,\forall\, t \in [t_0, t_1]$, and if $\lambda(t) \geq 0\,\forall\, t \in [t_0, t_1]$, then $H(\cdot)$ is a concave function of $(x, u)\,\forall\, t \in [t_0, t_1]$ by Theorem 3.2, because the Hamiltonian is a nonnegative linear combination of concave

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functions in this instance. Similarly, if $f(\cdot)$ is concave in $(x, u)\,\forall\, t \in [t_0, t_1]$, $g(\cdot)$ is convex in $(x, u)\,\forall\, t \in [t_0, t_1]$, and $\lambda(t) \le 0\,\forall\, t \in [t_0, t_1]$, then $H(\cdot)$ is concave in $(x, u)\,\forall\, t \in [t_0, t_1]$. To see this, first define $\mu(t) \stackrel{\text{def}}{=} -\lambda(t) \ge 0$. This definition allows us to rewrite the Hamiltonian in the form $H(t, x, u, \lambda) \stackrel{\text{def}}{=} f(t, x, u) + \lambda g(t, x, u) = f(t, x, u) + \mu[-1 \cdot g(t, x, u)]$. Given that $g(\cdot)$ is convex in $(x, u)\,\forall\, t \in [t_0, t_1]$, $-g(\cdot)$ is concave in $(x, u)\,\forall\, t \in [t_0, t_1]$ by definition. Moreover, because $\mu(t) \ge 0\,\forall\, t \in [t_0, t_1]$, $H(\cdot)$ is concave in $(x, u)\,\forall\, t \in [t_0, t_1]$ by Theorem 3.2 in view of the fact that it is a nonnegative linear combination of concave functions. Thus, in either case, $H(\cdot)$ is concave in $(x, u)\,\forall\, t \in [t_0, t_1]$, and therefore by Theorem 3.1, a solution of the necessary conditions of Theorem 2.2 is a solution to the optimal control problem (1).

Finally, if $g(\cdot)$ is linear in $(x, u)\,\forall\, t \in [t_0, t_1]$, then $\lambda(t)$ may be of any sign and $H(\cdot)$ will be concave in $(x, u)\,\forall\, t \in [t_0, t_1]$ if $f(\cdot)$ is concave in $(x, u)\,\forall\, t \in [t_0, t_1]$. This should be clear because if $g(\cdot)$ is linear in $(x, u)\,\forall\, t \in [t_0, t_1]$, then it is *both* concave and convex in $(x, u)\,\forall\, t \in [t_0, t_1]$, and therefore $\lambda(t)g(\cdot)$ is both concave and convex in $(x, u)\,\forall\, t \in [t_0, t_1]$ regardless of the sign of $\lambda(t)$. Hence, if $g(\cdot)$ is linear in $(x, u)\,\forall\, t \in [t_0, t_1]$ and $f(\cdot)$ is concave in $(x, u)\,\forall\, t \in [t_0, t_1]$, then $H(\cdot)$ is concave in $(x, u)\,\forall\, t \in [t_0, t_1]$ as it is a nonnegative linear combination of concave functions. In this instance, we may also conclude that a solution of the necessary conditions of Theorem 2.2 is a solution to the optimal control problem (1) by Theorem 3.1. In the mental exercises, you are asked to provide a different proof for the concavity of $H(\cdot)$, in this case using Theorem 21.5 of Simon and Blume (1994), which states that a function is concave in certain variables if and only if its Hessian matrix with respect to the said variables is negative semidefinite. We summarize our results of this and the preceding paragraph in the following corollary.

Corollary 3.1: *For problem (1), if $f(\cdot)$ is concave in $(x, u)\,\forall\, t \in [t_0, t_1]$, and any one of the following three additional conditions hold:*

(i) $g(\cdot)$ is concave in $(x, u)\,\forall\, t \in [t_0, t_1]$ and $\lambda(t) \ge 0\,\forall\, t \in [t_0, t_1]$, or
(ii) $g(\cdot)$ is convex in $(x, u)\,\forall\, t \in [t_0, t_1]$ and $\lambda(t) \le 0\,\forall\, t \in [t_0, t_1]$, or
(iii) $g(\cdot)$ is linear in $(x, u)\,\forall\, t \in [t_0, t_1]$,

then $H(\cdot)$ is concave in $(x, u)\,\forall\, t \in [t_0, t_1]$ and the solution to the necessary conditions of Theorem 2.2 is a solution to the optimal control problem (1).

One point that may cross your mind as you read Corollary 3.1 is that we must know the sign of the costate variable over the entire planning horizon to use it, unless, of course, the transition function $g(\cdot)$ is linear in $(x, u)\,\forall\, t \in [t_0, t_1]$, a case that is fairly common but certainly not universal in economic problems. It would be nice, therefore, if a simple sufficient condition could be found that would guarantee the uniform sign of the costate variable over the planning horizon. The following lemma provides such a simple sufficient condition. The proof is a technical one,

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but it introduces a clever way to get an economic interpretation of the costate variable and demonstrate the forward-looking nature of it, and so is included in the text.

Lemma 3.1: *Let* $(z(t), v(t))$ *be the optimal solution to problem (1), with corresponding costate variable* $\lambda(t)$.

(i) If $f_x(t, z(t), v(t)) > 0\, \forall\, t \in [t_0, t_1]$, *then* $\lambda(t) > 0\, \forall\, t \in [t_0, t_1)$;
(ii) If $f_x(t, z(t), v(t)) < 0\, \forall\, t \in [t_0, t_1]$, *then* $\lambda(t) < 0\, \forall\, t \in [t_0, t_1)$;
(iii) If $f_x(t, z(t), v(t)) = 0\, \forall\, t \in [t_0, t_1]$, *then* $\lambda(t) = 0\, \forall\, t \in [t_0, t_1]$.

Proof: The proof is not hard but involves a clever twist in solving the costate equation. Moreover, the twist imparts intuition to the economic interpretation of the costate variable. To begin, recall the costate equation and transversality condition evaluated along the optimal solution, and write out the individual functions rather than just the Hamiltonian, that is,

$$\dot{\lambda}(t) = -f_x(t, z(t), v(t)) - \lambda(t) g_x(t, z(t), v(t)), \lambda(t_1) = 0.$$

Defining $f_x^*(t) \stackrel{\text{def}}{=} f_x(t, z(t), v(t))$ and $g_x^*(t) \stackrel{\text{def}}{=} g_x(t, z(t), v(t))$, the costate equation can be rewritten more compactly as $\dot{\lambda} + g_x^*(t)\lambda = -f_x^*(t)$. The integrating factor for this first-order linear ordinary differential equation is found in the standard manner, scilicet, $\omega(t) \stackrel{\text{def}}{=} \exp[\int_{t_0}^{t} g_x^*(s) ds]$, where s is a dummy variable of integration. Note that the integrating factor is positive by definition. Next, multiply the costate equation $\dot{\lambda} + g_x^*(t)\lambda = -f_x^*(t)$ through by the integrating factor $\omega(t)$ to get

$$\omega(t)\dot{\lambda} + \omega(t) g_x^*(t)\lambda = -\omega(t) f_x^*(t). \tag{7}$$

Because $\dot{\omega}(t) = g_x^*(t)\omega(t)$ by Leibniz's rule, the left-hand side of Eq. (7) can be rewritten as $\omega(t)\dot{\lambda} + \omega(t) g_x^*(t)\lambda = \frac{d}{dt}[\omega(t)\lambda]$, thereby permitting Eq. (7) to be rewritten as

$$\frac{d}{dt}[\omega(t)\lambda] = -\omega(t) f_x^*(t). \tag{8}$$

Integrating both sides of Eq. (8) with respect to t yields the general solution of the costate equation, namely,

$$\lambda(t) = -[\omega(t)]^{-1} \int_{t_0}^{t} \omega(\tau) f_x^*(\tau)\, d\tau + [\omega(t)]^{-1} c_1, \tag{9}$$

where c_1 is a constant of integration and τ is another dummy variable of integration. That this is a general solution can be verified by differentiating it with respect to t using Leibniz's rule. The specific (or definite) solution can be found by applying the transversality condition to determine the value of c_1. In the mental exercises, you are asked to show that the resulting definite solution is identical to that derived below.

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We thus stop the present derivation so that we can proceed with the aforementioned clever solution method.

The clever idea in solving the differential equation (8) is to integrate it *forward* from date t, that is, from $t < t_1$ to t_1, rather than backward from date t as we did above, that is, from t_0 to $t > t_0$. Moreover, when integrating Eq. (8) forward from $t < t_1$ to t_1 we must insert a minus sign on the right-hand side to reflect the fact that the index t now occurs as a lower limit of integration rather than as the upper limit of integration, just as Leibniz's rule dictates. To be consistent with this approach to solving the costate equation, we therefore define a new integrating factor by the formula $\varpi(t) \overset{\text{def}}{=} \exp[-\int_t^{t_1} g_x^*(s)\,ds]$. This integrating factor is also positive by definition, and an application of Leibniz's rule shows it satisfies the differential equation $\dot{\varpi}(t) = g_x^*(t)\varpi(t)$, in accord with the previous integrating factor.

Proceeding as above, we multiply the costate equation $\dot{\lambda} + g_x^*(t)\lambda = -f_x^*(t)$ through by the integrating factor $\varpi(t)$ to get

$$\varpi(t)\dot{\lambda} + \varpi(t)g_x^*(t)\lambda = -\varpi(t)f_x^*(t). \tag{10}$$

In view of the fact that $\dot{\varpi}(t) = g_x^*(t)\varpi(t)$, as noted above, the left-hand side of Eq. (10) can be rewritten as $\varpi(t)\dot{\lambda} + \varpi(t)g_x^*(t)\lambda = \frac{d}{dt}[\varpi(t)\lambda]$, thereby permitting Eq. (10) to be rewritten as

$$\frac{d}{dt}[\varpi(t)\lambda] = -\varpi(t)f_x^*(t). \tag{11}$$

Integrating Eq. (11) from $t < t_1$ to t_1 therefore yields the general solution

$$\lambda(t) = [\varpi(t)]^{-1}\int_t^{t_1} \varpi(\tau)f_x^*(\tau)d\tau + [\varpi(t)]^{-1}c_2, \tag{12}$$

where c_2 is a constant of integration that is different from c_1, a fact that you will verify in the aforementioned mental exercise. That this is a general solution can be verified by differentiating it with respect to t using Leibniz's rule. Now apply the transversality condition $\lambda(t_1) = 0$ to Eq. (12) to find the value of the constant of integration c_2:

$$\lambda(t_1) = [\varpi(t_1)]^{-1}\int_{t_1}^{t_1} \varpi(\tau)f_x^*(\tau)\,d\tau + [\varpi(t_1)]^{-1}c_2 = [\varpi(t_1)]^{-1}c_2 = 0 \Leftrightarrow c_2 = 0,$$

because $\varpi(t_1) \overset{\text{def}}{=} \exp[-\int_{t_1}^{t_1} g_x(s, z(s), v(s))\,ds] = 1$. The specific solution to the costate equation is thus

$$\lambda(t) = [\varpi(t)]^{-1}\int_t^{t_1} \varpi(\tau)f_x(\tau, z(\tau), v(\tau))\,d\tau. \tag{13}$$

Inspection of Eq. (13) and the three parts of the lemma completes the proof. Q.E.D.

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The economic interpretation of Lemma 3.1 is straightforward. Take, for example, part (i), and interpret $f(t, x, u)$ as the net benefit associated with the capital stock x and control u at time t. Part (i) asserts that if the marginal net benefit of the capital stock is positive in an optimal plan, then the shadow value of the capital stock is likewise positive in an optimal plan, that is, the capital stock has a positive marginal value to the decision maker in an optimal plan. Part (ii) has an analogous interpretation and thus need not be explicitly discussed.

Part (iii) asserts that if the marginal net benefit of the capital stock is zero in an optimal plan, then the shadow value of the capital stock is similarly zero in an optimal plan, that is, the capital stock has no marginal value to the decision maker in an optimal plan. This result sheds light on Example 2.2, in which we concluded that the posed optimal control problem was not really a dynamic optimization problem. In Example 2.2, recall that the integrand function is independent of the state variable, thereby implying that $f_x(t, x, u) \equiv 0$. By Lemma 3.1(iii), this implies that $\lambda(t) = 0\,\forall\, t \in [t_0, t_1]$ in an optimal plan. But with $\lambda(t) = 0\,\forall\, t \in [t_0, t_1]$, the Hamiltonian $H(\cdot)$ is identical to the integrand function $f(\cdot)$, which implies that the value of the control variable that satisfies $H_u(t, x, u, \lambda) = 0$ is identically equal to the value of the control variable that satisfies $f_u(t, x, u) = 0$. This, then, is the reason that Example 2.2 is not really a dynamic optimization problem. In other words, with $\lambda(t) = 0\,\forall\, t \in [t_0, t_1]$, the state equation is essentially eliminated as a factor in determining the solution of the control problem. But it is precisely the state equation that makes the optimal control problem a dynamic problem, a feature we discussed in Chapter 1, as you may recall. Hence, with $\lambda(t) = 0\,\forall\, t \in [t_0, t_1]$, the state equation is irrelevant and the optimal control problem essentially reduces to a static optimization problem.

In each of the three cases in Lemma 3.1, you may have noticed that the sign of the derivative $g_x(t, z(t), v(t))$ plays absolutely no role in determining the sign of $\lambda(t)$. This may appear strange at first glance. Inspection of the costate equation in the form $\dot{\lambda} + g_x^*(t)\lambda = -f_x^*(t)$, however, shows that the role of $g_x(t, z(t), v(t))$ is to dictate the rate of growth of the shadow value of the stock, not its level or value.

As remarked at the beginning of the proof of Lemma 3.1, the solution for the costate variable points to a nice economic interpretation of it. To tease out the economic interpretation, let's again interpret $f(t, x, u)$ as the net benefit associated with the capital stock x and control u at time t. Because the integrating factor for the forward integration solution of the costate equation is $\varpi(t) \stackrel{\text{def}}{=} \exp[-\int_t^{t_1} g_x^*(s)\,ds]$, we have that

$$
\begin{aligned}
\frac{\varpi(\tau)}{\varpi(t)} &= \frac{\exp\left[-\int_\tau^{t_1} g_x^*(s)\,ds\right]}{\exp\left[-\int_t^{t_1} g_x^*(s)\,ds\right]} = \exp\left[-\int_\tau^{t_1} g_x^*(s)\,ds\right]\exp\left[\int_t^{t_1} g_x^*(s)\,ds\right] \\
&= \exp\left[\int_t^{t_1} g_x^*(s)\,ds - \int_\tau^{t_1} g_x^*(s)\,ds\right] = \exp\left[\int_t^{\tau} g_x^*(s)\,ds\right],
\end{aligned}
$$

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as $\tau \geq t$. Given the exponential nature of the ratio $\varpi(\tau)/\varpi(t)$, we may therefore interpret it as a *time-varying* discount factor. Using this observation, we can rewrite Eq. (13) as

$$\lambda(t) = \int_t^{t_1} \frac{\varpi(\tau)}{\varpi(t)} f_x(\tau, z(\tau), v(\tau))\, d\tau.$$

This form of the solution is easy to impart an economic interpretation to in light of the above development. In particular, the costate variable at time t, namely, $\lambda(t)$, may be interpreted as the discounted value of the net marginal benefits of the capital stock x from the present moment t until the terminal period t_1, in an optimal plan. This thus reinforces our earlier assertion that $\lambda(t)$ is the shadow value of the capital stock at time t.

Another aspect of Eq. (13) deserves to be reinforced, to wit, the *forward-looking* nature of the solution. That is, the optimal value of the costate variable at time t reflects the discounted value of the net marginal benefits of the capital stock x from the present moment t until the terminal period t_1. It is in this sense that the costate variable "looks ahead" to determine its optimal value. This is another reason why when $\lambda(t) = 0\ \forall\, t \in [t_0, t_1]$, the optimal control problem is essentially a static optimization problem.

Finally, observe that the conditions of Lemma 3.1 are straightforward and relatively easy to check. For example, in many optimal control problems in economics, the signs of the partial derivatives of the functions $f(\cdot)$ and $g(\cdot)$ are specified by the researcher as part of the problem statement. This means that the sign of $f_x(t, z(t), v(t))$ is known without ever solving the optimal control problem under consideration, and thus so is the sign $\lambda(t)$ by Lemma 3.1. We now return to our general discussion of sufficient conditions for problem (1).

Theorem 3.1 is due to Mangasarian and is straightforward (if at times tedious) to check in many optimal control problems in economics. Some dynamic economic models may not satisfy the strong concavity restrictions it requires, however. A generalization of Theorem 3.1 due to Arrow applies to a larger class of problems, but in practice, it can be more difficult to check than Mangasarian's theorem. Before we state and prove the Arrow sufficiency theorem, we need to define a new function and restate Theorem 2.2, the necessary conditions for problem (1), using this newly defined function. To this end, we have the following definition:

Definition 3.1: For the control problem (1), the *maximized Hamiltonian* $M(\cdot)$ is defined as

$$M(t, x, \lambda) \stackrel{\text{def}}{=} \max_u H(t, x, u, \lambda), \tag{14}$$

where $H(t, x, u, \lambda) \stackrel{\text{def}}{=} f(t, x, u) + \lambda g(t, x, u)$ is the Hamiltonian for problem (1).

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Seeing as no constraints are placed on the control variable u in problem (1), a necessary condition that must be obeyed by the control that solves the maximization problem in Eq. (14) is the familiar vanishing derivative condition, namely, $H_u(t, x, u, \lambda) = f_u(t, x, u) + \lambda g_u(t, x, u) = 0$, which must hold for every $t \in [t_0, t_1]$. Now if $H_{uu}(t, x, u, \lambda) \neq 0$ along the optimal path, then by the implicit function theorem, we can in principle solve $H_u(t, x, u, \lambda) = 0$ for u in terms of (t, x, λ), say, $u = \hat{u}(t, x, \lambda)$. That is, the necessary condition $H_u(t, x, u, \lambda) = 0$ implicitly defines u in terms of (t, x, λ). We acknowledge that this is the maximizing value of the control variable by using the notation $\hat{u}(t, x, \lambda) \overset{\text{def}}{=} \arg\max_u H(t, x, u, \lambda)$. Substituting $u = \hat{u}(t, x, \lambda)$ into the Hamiltonian $H(\cdot)$ yields the value of the maximized Hamiltonian $M(\cdot)$, that is to say

$$M(t, x, \lambda) \equiv H(t, x, \hat{u}(t, x, \lambda), \lambda) = f(t, x, \hat{u}(t, x, \lambda)) + \lambda g(t, x, \hat{u}(t, x, \lambda)). \tag{15}$$

Equation (15) demonstrates how one would go about constructing the maximized Hamiltonian in practice. Given Definition 3.1, we are now in a position to restate Theorem 2.2 in terms of the maximized Hamiltonian $M(\cdot)$.

Theorem 3.3 (Necessary Conditions): *Suppose $v(\cdot) \in C^{(0)}$ and $v(t) \in \operatorname{int} U \ \forall\, t \in [t_0, t_1]$, and let $z(\cdot) \in C^{(1)} \ \forall\, t \in [t_0, t_1]$ be the corresponding state function that satisfies the state equation $\dot{x}(t) = g(t, x(t), u(t))$ and initial condition $x(t_0) = x_0$, so that $(z(t), v(t))$ is an admissible pair. Then if $(z(t), v(t))$ yields the global maximum of $J[\cdot]$ when $x(t_1) = x_1$ is a decision variable, then it is necessary that there exists a function $\lambda(\cdot) \in C^{(1)} \ \forall\, t \in [t_0, t_1]$ such that*

$$H_u(t, z(t), v(t), \lambda(t)) = 0 \ \forall\, t \in [t_0, t_1],$$

$$\dot{\lambda}(t) = -M_x(t, z(t), \lambda(t)), \ \ \lambda(t_1) = 0,$$

$$\dot{z}(t) = M_\lambda(t, z(t), \lambda(t)), \ \ z(t_0) = x_0,$$

where $M(t, x, \lambda) \overset{\text{def}}{=} \max_u H(t, x, u, \lambda)$ is the maximized Hamiltonian for problem (1) and $v(t) \overset{\text{def}}{=} \hat{u}(t, z(t), \lambda(t))$.

Theorem 3.3 is equivalent to Theorem 2.2, so the two may be used interchangeably. The proof of their equivalence is short and relatively simple because it makes use of the envelope theorem, and so is left for the mental exercises. The sufficiency counterpart to Theorem 3.3 can now be stated. You are asked to complete the proof of it in the mental exercises.

Theorem 3.4 (Arrow Sufficiency Theorem): *Let $(z(t), v(t))$ be an admissible pair for problem (1). Suppose that $(z(t), v(t))$ satisfy the necessary conditions of Theorem 3.3 for problem (1) with costate variable $\lambda(t)$, and let*

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$M(t, x, \lambda) \overset{\text{def}}{=} \max_u H(t, x, u, \lambda)$ be the value of the maximized Hamiltonian function. If $M(\cdot)$ is a concave function of x for all $t \in [t_0, t_1]$ over an open convex set containing all the admissible values of $x(\cdot)$ when the costate variable is $\lambda(t)$, then the pair $(z(t), v(t))$ yields the global maximum of $J[\cdot]$. If $M(\cdot)$ is a strictly concave function under the same conditions, then $J[z(\cdot), v(\cdot)] > J[x(\cdot), u(\cdot)]$ and $z(t)$ is unique, but $v(t)$ is not necessarily unique.

Proof: Let $(x(t), u(t))$ be an admissible pair, where $u(t) \overset{\text{def}}{=} \hat{u}(t, x(t), \lambda(t))$. You are asked to show that $M(\cdot) \in C^{(1)}$ in x in completing this proof. Because $M(\cdot)$ is concave in x $\forall\, t \in [t_0, t_1]$ given $\lambda(t)$ by hypothesis, it follows from Theorem 21.3 in Simon and Blume (1994) that

$$M(t, x(t), \lambda(t)) \leq M(t, z(t), \lambda(t)) + M_x(t, z(t), \lambda(t))[x(t) - z(t)].$$

Given that this inequality holds $\forall\, t \in [t_0, t_1]$, we can integrate it over $[t_0, t_1]$ and the inequality is preserved, thereby yielding

$$\int_{t_0}^{t_1} M(t, x(t), \lambda(t))\, dt \leq \int_{t_0}^{t_1} [M(t, z(t), \lambda(t)) + M_x(t, z(t), \lambda(t))[x(t) - z(t)]]\, dt.$$

The remainder of the proof is similar to that of Theorem 3.1 and is therefore left for you to complete in a mental exercise. Q.E.D.

The Arrow sufficiency theorem replaces the assumption of concavity of the Hamiltonian $H(\cdot)$ in (x, u) of the Mangasarian theorem, with the assumption that the maximized Hamiltonian $M(\cdot)$ is concave in x. Note, however, that checking the curvature properties of a derived function like $M(\cdot)$ can be more difficult than checking the curvature properties of $H(\cdot)$.

The following theorem demonstrates that the Mangasarian sufficiency theorem is a *special case* of Arrow's sufficiency theorem by establishing that if $H(\cdot)$ is concave in (x, u), then $M(\cdot)$ is concave in x. It is also important to note that even if $H(\cdot)$ is not concave in (x, u), it is still possible that $M(\cdot)$ is concave in x, so that Arrow's theorem applies to a larger class of problems than does Mangasarian's. The ensuing theorem is stated for the vector case in view of the fact that we will make use of that form in a later chapter.

Theorem 3.5: *If $F(\cdot) : \mathrm{X} \times \mathrm{U} \to \Re$ is concave over the convex sets $\mathrm{X} \subset \Re^n$ and $\mathrm{U} \subset \Re^m$, then*

$$\phi(\mathbf{x}) \overset{\text{def}}{=} \max_{\mathbf{u}} F(\mathbf{x}, \mathbf{u})$$

is a concave function of $\mathbf{x}$.

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Proof: For given $\mathbf{x}_i$, let $\mathbf{u}_i = \mathbf{u}^*(\mathbf{x}_i) \stackrel{\text{def}}{=} \arg\max_{\mathbf{u}} F(\mathbf{x}_i, \mathbf{u})$, $i = 1, 2$, and let $\alpha \in [0, 1]$. Then we have the following string of equalities and inequalities, which we will explain below:

$$\begin{aligned}
\alpha\phi(\mathbf{x}_1) + (1-\alpha)\phi(\mathbf{x}_2) &\stackrel{\text{def}}{=} \alpha \max_{\mathbf{u}} F(\mathbf{x}_1, \mathbf{u}) + (1-\alpha) \max_{\mathbf{u}} F(\mathbf{x}_2, \mathbf{u}) \\
&= \alpha F(\mathbf{x}_1, \mathbf{u}_1) + (1-\alpha) F(\mathbf{x}_2, \mathbf{u}_2) \\
&\le F(\alpha\mathbf{x}_1 + (1-\alpha)\mathbf{x}_2, \alpha\mathbf{u}_1 + (1-\alpha)\mathbf{u}_2,) \\
&\le \max_{\mathbf{u}} F(\alpha\mathbf{x}_1 + (1-\alpha)\mathbf{x}_2, \mathbf{u}) \\
&\stackrel{\text{def}}{=} \phi(\alpha\mathbf{x}_1 + (1-\alpha)\mathbf{x}_2).
\end{aligned}$$

The first two equalities follow from the definitions of $\phi(\cdot)$ and $\mathbf{u}_i = \mathbf{u}^*(\mathbf{x}_i)$, respectively, and by noting that $\phi(\mathbf{x}_i) \equiv F(\mathbf{x}_i, \mathbf{u}^*(\mathbf{x}_i)), i = 1, 2$. The first inequality follows from the hypothesis that $F(\cdot)$ is concave $\forall\,(\mathbf{x}, \mathbf{u}) \in \mathrm{X} \times \mathrm{U}$, that is, we have used the defining property of concavity. The next inequality follows from the definition of a maximum, that is, $\alpha\mathbf{u}_1 + (1-\alpha)\mathbf{u}_2$ is a feasible choice for the given value of $\alpha\mathbf{x}_1 + (1-\alpha)\mathbf{x}_2$, but it is not necessarily the optimal value of $\mathbf{u}$. The last equality follows from the definition of $\phi(\cdot)$. Q.E.D.

Before stating our final sufficiency theorem of the chapter, it will be beneficial to examine five examples so that we can demonstrate how the theorems developed in this chapter are used. We begin with the two purely mathematical problems we first examined in Chapter 2. Next we proceed to a case in which the Mangasarian sufficiency theorem does not hold but the Arrow sufficiency theorem does. Then we present a more general problem with economic content and point out a crucial feature of the sufficiency theorems. Finally, we examine an intertemporal model of a nonrenewable resource–extracting firm.

Example 3.1: Consider the optimal control problem from Example 2.1:

$$\max_{u(\cdot), x_1} J[x(\cdot), u(\cdot)] \stackrel{\text{def}}{=} \int_0^1 \left[-x(t) - \frac{1}{2}\alpha[u(t)]^2\right] dt$$

$$\text{s.t.} \quad \dot{x}(t) = u(t),$$

$$x(0) = x_0, \; x(1) = x_1,$$

where $\alpha > 0$ is a parameter. The Hamiltonian is defined as $H(t, x, u, \lambda; \alpha) \stackrel{\text{def}}{=} -x - \frac{1}{2}\alpha u^2 + \lambda u$.

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One way to determine if $H(\cdot)$ is a concave function of (x, u), and thus conclude that a solution to the necessary conditions is a solution of the control problem, is to simply inspect the Hamiltonian to see if it is a concave function of (x, u). This approach is viable when $f(\cdot)$ and $g(\cdot)$ are made up of elementary functions, as in this example. Inspection of the transition function $g(\cdot)$ reveals that it is independent of the state variable and linear in the control variable. Similarly, the integrand function $f(\cdot)$ is jointly concave in the state and control variables. Hence, by Corollary 3.1(iii), a solution of the necessary conditions is also a solution of the optimal control problem.

Alternatively and equivalently, we can determine if $H(\cdot)$ is concave in (x, u) by seeing if the Hessian matrix of the Hamiltonian with respect to (x, u) is negative semidefinite, seeing as a function is concave with respect to a set of variables in its domain if and only if its Hessian matrix with respect to those variables is negative semidefinite by Theorem 21.5 of Simon and Blume (1994). Accordingly, we are led to compute the following three principal minors of the Hessian matrix of $H(\cdot)$ with respect to (x, u):

$$H_{uu} = -\alpha < 0,\, H_{xx} = 0,\, \begin{vmatrix} H_{uu} & H_{ux} \\ H_{xu} & H_{xx} \end{vmatrix} = \begin{vmatrix} -\alpha & 0 \\ 0 & 0 \end{vmatrix} = 0.$$

In view of the fact that all the first-order principal minors are nonpositive and the determinant is nonnegative, Theorem 16.2 of Simon and Blume (1994) implies that $H(\cdot)$ is a concave function of (x, u). Thus, by Theorem 3.1, the solution of the necessary conditions is also a solution of the optimal control problem. Note, however, that because $H(\cdot)$ is linear in x, the Hamiltonian is not strictly concave in the state and control variables. This means that we are not permitted to use Theorem 3.1 to conclude that the solution of the control problem is unique. In spite of this observation, we still may in fact conclude that the solution of the control problem is unique seeing as we found only one solution of the necessary conditions in Example 2.1.

A third way to check the concavity of the Hamiltonian in the state and control variables is to compute the eigenvalues of the Hessian matrix of $H(\cdot)$ with respect to (x, u). Because the Hessian matrix of $H(\cdot)$ with respect to (x, u) is diagonal, the eigenvalues are the diagonal elements themselves by Theorem 23.1 of Simon and Blume (1994). Inspection of the above Hessian matrix reveals that one eigenvalue is negative and the other is zero. By Theorems 23.17 and 21.5 of Simon and Blume (1994), we may again conclude that $H(\cdot)$ is concave in (x, u). Thus, by Theorem 3.1, the solution of the necessary conditions is a solution of the optimal control problem.

Though we know that Arrow's sufficiency theorem will hold (why?), let's verify that explicitly. Because $H_{uu}(t, x, u, \lambda; \alpha) = -\alpha < 0$, the solution $u = \hat{u}(t, x, \lambda; \alpha) \stackrel{\text{def}}{=} \alpha^{-1}\lambda$ of the necessary condition $H_u(t, x, u, \lambda; \alpha) = -\alpha u + \lambda = 0$

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is in fact the unique value of the control variable that maximizes the Hamiltonian. Hence the value of the maximized Hamiltonian $M(\cdot)$ is given by

$$\begin{aligned} M(t, x, \lambda; \alpha) &\equiv H(t, x, \hat{u}(t, x, \lambda; \alpha), \lambda; \alpha) \\ &= -x - \frac{1}{2}\alpha(\alpha^{-1}\lambda)^2 + \lambda\alpha^{-1}\lambda \\ &= -x + \frac{1}{2}\alpha^{-1}\lambda^2, \end{aligned}$$

which is linear and thus concave in the state variable. Theorem 3.4 thus implies that the solution of the necessary conditions is a solution of the optimal control problem. In a mental exercise, you are asked to verify that the necessary conditions of Theorem 3.3 for the above control problem are identical to those of Theorem 2.2 given in Example 2.1. Finally, note that the solution for the costate variable $\lambda(t) = t - 1$ is negative $\forall\, t \in [0, 1)$, just as one would expect based on Lemma 3.1(ii), given that $f_x(t, x, u) = -1 < 0$ in the control problem.

Example 3.2: Consider the optimal control problem from Example 2.2:

$$\max_{u(\cdot), x_1} J[x(\cdot), u(\cdot)] \stackrel{\text{def}}{=} \int_0^1 \left[\alpha t u(t) - \frac{1}{2}[u(t)]^2\right] dt$$

$$\text{s.t.} \quad \dot{x}(t) = u(t) - x(t),$$

$$x(0) = x_0, \; x(1) = x_1.$$

The Hamiltonian is given by $H(t, x, u, \lambda; \alpha) \stackrel{\text{def}}{=} \alpha t u - \frac{1}{2}u^2 + \lambda[u - x]$. Because this example is not that different from Example 3.1, we will be more terse in our exposition, thereby leaving some details to a mental exercise.

The integrand is a concave function of the state and control variables, and the state equation is linear in the state and control variables. As a result, we may again appeal to Corollary 3.1(iii) to conclude that a solution of the necessary conditions is a solution of the control problem. Uniqueness then follows from the same argument we employed in Example 3.1.

By Theorem 3.5, we know that Arrow's sufficiency theorem will hold. Nonetheless, let's verify explicitly that it indeed does for practice. Since $H_{uu}(t, x, u, \lambda; \alpha) = -1 < 0$, the solution $u = \hat{u}(t, x, \lambda; \alpha) \stackrel{\text{def}}{=} \alpha t + \lambda$ of the necessary condition $H_u(t, x, u, \lambda; \alpha) = \alpha t - u + \lambda = 0$ is the unique value of the control variable that maximizes the Hamiltonian. Hence the maximized Hamiltonian is given by

$$\begin{aligned} M(t, x, \lambda; \alpha) &\equiv H(t, x, \hat{u}(t, x, \lambda; \alpha), \lambda; \alpha) \\ &= \alpha t(\alpha t + \lambda) - \frac{1}{2}(\alpha t + \lambda)^2 + \lambda(\alpha t + \lambda - x) \end{aligned}$$

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$$= \alpha^2 t^2 + \alpha t\lambda - \frac{1}{2}\alpha^2 t^2 - \alpha t\lambda - \frac{1}{2}\lambda^2 + \alpha t\lambda + \lambda^2 - \lambda x$$
$$= \frac{1}{2}\alpha^2 t^2 + \frac{1}{2}\lambda^2 + \alpha t\lambda - \lambda x,$$

which is linear and thus concave in the state variable. Theorem 3.4 thus implies that a solution of the necessary conditions is a solution of the optimal control problem. In passing, note that the solution for the costate variable, namely $\lambda(t) = 0\,\forall\, t \in [0, 1]$, is as expected based on Lemma 3.1(iii), seeing as $f_x(t, x, u) \equiv 0$ in this control problem.

In the third example, we encounter a case in which the Mangasarian sufficiency theorem does not hold but the Arrow sufficiency theorem does, thereby reinforcing our earlier conclusion of the more general nature of the latter.

Example 3.3: Now consider the ensuing optimal control problem:

$$\max_{u(\cdot),x_T} J[x(\cdot), u(\cdot)] \stackrel{\text{def}}{=} \int_0^T \left[-\frac{1}{2}[u(t)]^2 - x(t)\right] dt$$
$$\text{s.t.}\quad \dot{x}(t) = [u(t)]^2 x(t),$$
$$x(0) = x_0 > 0,\; x(T) = x_T.$$

First, observe that the integrand is a strictly decreasing function of the state variable, so that by Lemma 3.1(ii), we may conclude that $\lambda(t) < 0\,\forall\, t \in [0, T)$, and furthermore that $\lambda(T) = 0$ by the necessary transversality condition. Next, notice that because $x(0) = x_0 > 0$, the state equation implies that *all* admissible values of the state variable are positive.

The Hamiltonian for this problem is defined as $H(t, x, u, \lambda) \stackrel{\text{def}}{=} -\frac{1}{2}u^2 - x + \lambda u^2 x$. To determine if $H(\cdot)$ is concave in (x, u), we compute the Hessian matrix of the Hamiltonian with respect to (x, u) and check to see if it is negative semidefinite. This leads to the results

$$H_{uu} = -1 + 2\lambda x < 0,\; H_{xx} = 0,$$
$$\begin{vmatrix} H_{uu} & H_{ux} \\ H_{xu} & H_{xx} \end{vmatrix} = \begin{vmatrix} -1 + 2\lambda x & 2\lambda u \\ 2\lambda u & 0 \end{vmatrix} = -4\lambda^2 u^2 < 0\,\forall\, u \neq 0.$$

Because the determinant of the Hessian matrix is negative, it is not negative semidefinite, and consequently, $H(\cdot)$ is not concave in (x, u). Thus Theorem 3.1 (Mangasarian's theorem) does not apply. Let's now see if Theorem 3.4 (Arrow's theorem) does.

The necessary condition $H_u(t, x, u, \lambda) = -u + 2\lambda ux = 0$ has the solution $u = \hat{u}(t, x, \lambda) \stackrel{\text{def}}{=} 0$ *or* $2\lambda x = 1$. The latter equation cannot be satisfied by any admissible solution, however, because it implies that λ and x have the same sign, which we know

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they cannot because all admissible $x(t) > 0\,\forall\, t \in [0, T]$ and $\lambda(t) \leq 0\,\forall\, t \in [0, T]$, as noted above. We may therefore conclude that $u = \hat{u}(t, x, \lambda) \stackrel{\text{def}}{=} 0$ is the solution of the necessary condition $H_u(t, x, u, \lambda) = -u + 2\lambda u x = 0$. Given that $H_{uu}(t, x, u, \lambda) = -1 < 0$, the Hamiltonian is a strictly concave function of the control variable. Thus $u = \hat{u}(t, x, \lambda) \stackrel{\text{def}}{=} 0$ yields the unique global maximum of the Hamiltonian. The maximized Hamiltonian is then given by

$$M(t, x, \lambda) \equiv H(t, x, \hat{u}(t, x, \lambda), \lambda) = -x,$$

which is linear and thus concave in x. Therefore, by Theorem 3.4 (Arrow's theorem), we may conclude that the solution to the necessary conditions, namely, $v(t) = 0$, $z(t) = x_0$, and $\lambda(t) = t - T$, is also a solution to the optimal control problem. In the mental exercises, you are asked to argue directly, that is, without using the sufficiency theorems, that this solution to the necessary conditions is the solution to the control problem.

We now return to the intertemporal utility maximization problem and point out a particular feature of the sufficiency theorems that will put you on alert when you go to apply them in your own work.

Example 3.4: As you may recall, the intertemporal utility maximization problem is given by

$$\max_{c(\cdot), k_T} J[k(\cdot), c(\cdot)] \stackrel{\text{def}}{=} \int_0^T U(c(t))\, e^{-rt}\, dt$$

$$\text{s.t.} \quad \dot{k}(t) = w + ik(t) - c(t),\ k(0) = k_0,\ k(T) = k_T.$$

The Hamiltonian is defined as $H(t, k, c, \lambda) \stackrel{\text{def}}{=} U(c)\, e^{-rt} + \lambda[w + ik - c]$. Also recall that we have assumed that $U'(c) > 0$ and $U''(c) < 0$.

First, we check to see if $H(\cdot)$ is concave in (k, c) for a given λ by computing the Hessian matrix of the Hamiltonian with respect to (k, c) and seeing if it is negative semidefinite:

$$H_{cc} = U''(c)\, e^{-rt} < 0,\ H_{kk} \equiv 0,\ H_{cc}H_{kk} - [H_{ck}]^2 = U''(c)\, e^{-rt} \cdot 0 - 0 = 0.$$

These calculations show that $H(\cdot)$ is concave in (k, c) for a given λ, so a solution (if it exists) to the necessary conditions is a solution to the control problem by Mangasarian's theorem (Theorem 3.1).

To check whether Arrow's theorem can be applied to reach the same conclusion, we need the maximized Hamiltonian $M(\cdot)$. Because $H_{cc} = U''(c)\, e^{-rt} < 0$, the solution (if one exists) to the necessary condition $H_c = U'(c)\, e^{-rt} - \lambda = 0$ does indeed maximize the Hamiltonian. Moreover, because $H_{cc} = U''(c)\, e^{-rt} < 0$, the implicit

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function theorem implies that if the solution to $H_c = U'(c)\, e^{-rt} - \lambda = 0$ exists, one could in principle find it in the form $c = \hat{c}(t, \lambda; r)$. Hence, the maximized Hamiltonian is given by

$$M(t, k, \lambda) \equiv H(t, k, \hat{c}(t, \lambda; r), \lambda) = U(\hat{c}(t, \lambda; r))\, e^{-rt} + \lambda[ik - \hat{c}(t, \lambda; r)].$$

Because $M(\cdot)$ is a linear function of k, it is therefore is a concave function of the capital stock, that is, the state variable. Hence by Arrow's theorem (Theorem 3.4), a solution of the necessary conditions, if one exists, is a solution of the optimal control problem.

Notice that all throughout this example, we have continually said "if a solution exists," because the sufficiency theorems presume the existence of a solution to the necessary conditions. In fact, no solution exists to the necessary conditions for this form of the problem under the stated assumptions. To see this, first observe that the integrand is independent of the capital stock, thereby implying, via Lemma 3.1(iii), that the costate variable is zero throughout the planning horizon. This, in turn, implies that the necessary condition $H_c = U'(c)\, e^{-rt} - \lambda = 0$ reduces to $U'(c)\, e^{-rt} = 0$. Seeing as $e^{-rt} \neq 0$, the necessary condition becomes $U'(c) = 0$, which *does not* have a solution because it was assumed that $U'(c) > 0$. Hence no solution to the necessary conditions exists for the intertemporal utility maximization problem under the stated assumptions. The way out of this situation is to fix the terminal capital stock or add a salvage value term to the problem. The important point to take away from this example is that you must be sure that a solution exists to the necessary conditions to employ the sufficiency theorems.

Let's now examine, in some detail, a classical economic model of the firm first introduced into the economics literature by Hotelling (1931). It will afford us the opportunity to use both the necessary and sufficient conditions developed so far in a model with economic content, as well as give us a taste of how the literature attempts to extract economic information from a dynamic economic model specified in general qualitative terms.

Example 3.5: Consider a firm that wishes to maximize the present discounted value of extracting a known and finite stock of a nonrenewable resource from the ground and selling it over a fixed and finite period of time. Define $x(t)$ as the stock of the nonrenewable resource in the ground at time t, $q(t)$ as the extraction rate of the nonrenewable resource at time t, $x_0 > 0$ as the given initial amount of the nonrenewable resource in the ground at the initial date of the planning horizon $t = 0$, and $T > 0$ as the terminal date of the planning horizon. As a result, the stock of the nonrenewable resource remaining in the ground at time t by definition equals the initial stock less the cumulative amount of it extracted up to time t,

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that is,

$$x(t) \stackrel{\text{def}}{=} x_0 - \int_0^t q(s)\, ds.$$

By Leibniz's rule, it therefore follows that

$$\dot{x}(t) = -q(t),\ x(0) = x_0.$$

We assume that the firm is free to choose how much of the nonrenewable resource to leave unextracted, that is, in the ground, when the planning horizon comes to a close. In other words, $x(T) = x_T$ is a choice or decision variable for the firm. We also assume that the firm's instantaneous profit function, say $\pi(\cdot) \in C^{(2)}$, depends not only on the rate of extraction $q(t)$, but also on the stock of the nonrenewable resource in the ground $x(t)$, and that $\pi_q(q, x) > 0$ and $\pi_x(q, x) > 0$. The former implies that profit increases with the extraction rate for a given stock, and the latter implies that for a given extraction rate, a larger stock of the nonrenewable resource in the ground is cheaper to extract, thus resulting in higher profit. The latter effect, videlicet, $\pi_x(q, x) > 0$, is called the *stock effect*, and can be motivated by imagining that in order to extract increasing amounts of the nonrenewable resource, the firm must dig deeper and deeper in the ground. This results in higher extraction costs, and thus lower profit, compared with the extraction of the initial units of the stock, which lie closer to the surface. Finally, we assume that the marginal profit of extraction is a decreasing function of the extraction rate, that is, $\pi_{qq}(q, x) < 0$.

The firm is asserted to maximize the present discounted value of profit using the discount rate $r > 0$, subject to the above differential equation governing the movement of the nonrenewable resource over time. Hence the optimal control problem facing the firm is given by

$$\max_{q(\cdot), x_T} \int_0^T \pi(q(t), x(t))\, e^{-rt}\, dt$$

$$\text{s.t.} \quad \dot{x}(t) = -q(t),\ x(0) = x_0,\ x(T) = x_T.$$

To simplify matters at this stage, we assume that both the extraction rate and resource stock are nonnegative through the planning horizon, and that a solution to the necessary conditions exists.

The Hamiltonian for this control problem is given by $H(t, x, q, \lambda) \stackrel{\text{def}}{=} \pi(q, x)\, e^{-rt} - \lambda q$. By Theorem 2.2, the necessary conditions are thus

$$H_q(t, x, q, \lambda) = \pi_q(q, x)\, e^{-rt} - \lambda = 0, \tag{16}$$

$$\dot{\lambda} = -\pi_x(q, x)\, e^{-rt},\ \lambda(T) = 0, \tag{17}$$

$$\dot{x}(t) = -q(t),\ x(0) = x_0. \tag{18}$$

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As remarked in Chapter 2 and discussed in the context of Lemma 3.1, the costate variable at any time t has the interpretation of the shadow value or price of the state variable at time t. In this model, therefore, $\lambda(t)$ is the *present value* shadow price of the nonrenewable resource stock in the ground at time t. The noun *present value* is employed because the objective functional is the present discounted value of profit. With this in mind, Eq. (16) asserts that in an optimal extraction plan, the firm will equate the present value marginal profit of extraction at time t with the present value shadow price of the nonrenewable resource stock at time t. That is, an optimal extraction rate is one that at every date of the planning horizon leaves the firm indifferent to extracting another unit of the nonrenewable resource stock and increasing the present value of marginal profit by $\pi_q(q(t), x(t))\, e^{-rt}$, and leaving the unit in the ground to extract at a later date, which has the present value of $\lambda(t)$ to the firm.

Because the costate equation in Eq. (17) holds for all $t \in [0, T]$, we may integrate it over the interval $[t, t+\varepsilon]$ for any $t \in [0, T-\varepsilon]$ and it stills holds. Doing just that and using s as the dummy variable of integration yields

$$\int_t^{t+\varepsilon} \dot{\lambda}(s)\, ds = -\int_t^{t+\varepsilon} \pi_x(q(s), x(s))\, e^{-rs}\, ds.$$

Integrating the left-hand side and rearranging the result yields

$$\lambda(t) = \int_t^{t+\varepsilon} \pi_x(q(s), x(s))\, e^{-rs} ds + \lambda(t+\varepsilon).$$

Substituting $\lambda(t) = \pi_q(q(t), x(t))\, e^{-rt}$ from Eq. (16) into the previous equation gives the expression of interest, namely,

$$\pi_q(q(t), x(t))\, e^{-rt} = \int_t^{t+\varepsilon} \pi_x(q(s), x(s))\, e^{-rs} ds + \pi_q(q(t+\varepsilon), x(t+\varepsilon))\, e^{-r(t+\varepsilon)}. \tag{19}$$

This equation asserts that in an optimal extraction plan, the firm is indifferent to extracting "today" and earning the present value marginal profit $\pi_q(q(t), x(t))\, e^{-rt}$, and extracting "tomorrow" and earning the present value marginal profit $\pi_q(q(t+\varepsilon), x(t+\varepsilon))\, e^{-r(t+\varepsilon)}$, *plus* the present value of the sum of the marginal profit earned, that is, costs saved, over the interval $[t, t+\varepsilon]$ because of the higher stock that results from delaying the extraction, to wit, $\int_t^{t+\varepsilon} \pi_x(q(s), x(s))\, e^{-rs} ds$.

Now define $\mu(t) \stackrel{\text{def}}{=} \lambda(t)\, e^{rt}$ as the *current value* shadow price of the nonrenewable resource stock. Differentiating this definition with respect to t yields $\dot{\mu}(t) = r\lambda(t)\, e^{rt} + \dot{\lambda}(t)\, e^{rt}$. Substituting $\dot{\lambda}(t) = -\pi_x(q(t), x(t))\, e^{-rt}$ from Eq. (17) and $\mu(t) \stackrel{\text{def}}{=} \lambda(t)\, e^{rt}$ in $\dot{\mu}(t) = r\lambda(t)\, e^{rt} + \dot{\lambda}(t)\, e^{rt}$ gives $\dot{\mu}(t) = r\mu(t) - \pi_x(q(t), x(t))$,

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which can be rearranged to read

$$\frac{\dot{\mu}(t)}{\mu(t)} = r - \frac{\pi_x(q(t), x(t))}{\mu(t)} < r. \tag{20}$$

This is the generalized version of *Hotelling's rule*, given that the instantaneous profit of the firm depends on the extraction rate as well as the amount of the nonrenewable resource stock in the ground. Equation (1) shows that in general, the current value shadow price of the nonrenewable resource stock grows at a rate less than the discount rate because $\mu(t) = \pi_q(q(t), x(t)) > 0$ from Eq. (16) and $\pi_x(q, x) > 0$. In fact, if $\pi_x(q, x)/\mu > r$, then the current value shadow price of the nonrenewable resource stock would be decreasing over some interval in the planning horizon. Thus, under the rather mild assumptions we are invoking, the current value shadow price of the nonrenewable resource stock may increase or decrease over the planning horizon, that is, its movement over time is not necessarily monotonic.

We now show that in general, the optimal extraction rate is not necessarily a decreasing function of time when stock effects are present. This is the analogue to the prior conclusion concerning the nonrenewable resource stock. Thus, periods of increasing extraction, declining extraction, or even constant extraction are entirely consistent with the nonrenewable resource–extracting model of the firm in the presence of stock effects. To see this mathematically, first differentiate $\mu(t) = \pi_q(q(t), x(t))$ with respect to t using the chain rule to get the equation $\dot{\mu}(t) = \pi_{qq}(q(t), x(t))\dot{q}(t) + \pi_{qx}(q(t), x(t))\dot{x}(t)$. Next, solve this equation for $\dot{q}(t)$ and substitute the current value version of the costate equation $\dot{\mu}(t) = r\mu(t) - \pi_x(q(t), x(t))$ and the state equation $\dot{x}(t) = -q(t)$ in it to get

$$\dot{q}(t) = \frac{\pi_{qx}(q(t), x(t))q(t) + r\pi_q(q(t), x(t)) - \pi_x(q(t), x(t))}{\pi_{qq}(q(t), x(t))} \gtreqless 0. \tag{21}$$

Seeing as $\pi_q(q, x) > 0$ and $\pi_x(q, x) > 0$, it is clear from inspection of Eq. (21) that the optimal extraction rate may be increasing, decreasing, or constant over time, or may display all three characteristics over the planning horizon, regardless of the sign of the cross-partial derivative $\pi_{qx}(q, x)$. Typically, it is assumed that $\pi_{qx}(q, x) > 0$, which means that the marginal profit of extraction is higher when the stock of the asset in the ground is larger, which may be termed the *marginal stock effect*. This is entirely consistent with our prior assumption $\pi_x(q, x) > 0$ reflecting the stock effect.

In the special case in which profit (or equivalently, extraction cost) doesn't depend on the remaining stock in the ground, that is, $\pi_x(q, x) \equiv 0$, the archetype Hotelling rule follows immediately from Eq. (20), namely, $\dot{\mu}(t)/\mu(t) = r$. This implies that the current value shadow price of the nonrenewable resource stock rises at the discount rate. Moreover, in view of the fact that $\pi_x(q, x) \equiv 0$ implies that $\pi_{qx}(q, x) \equiv 0$, Eq. (21) reduces to

$$\dot{q}(t) = \frac{r\pi_q(q(t), x(t))}{\pi_{qq}(q(t), x(t))} < 0,$$

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because $\pi_q(q, x) > 0$ and $\pi_{qq}(q, x) < 0$. Thus, in the absence of stock effects, the optimal extraction rate falls over the planning period. This is not an unexpected result, for the effect of positive discounting is to make the firm prefer current profit over future profit, thereby providing the incentive to extract at a higher rate in the beginning of the planning horizon.

For the special case in which the firm doesn't discount future profits at a positive rate, that is, $r \equiv 0$, it follows from Eq. (20) that the current value shadow price of the nonrenewable resource grows at a negative rate given by $\dot{\mu}(t)/\mu(t) = -\pi_x(q(t), x(t))/\mu(t) < 0$ because $\pi_x(q, x) > 0$. Similarly, it follows from Eq. (21) that

$$\dot{q}(t) = \frac{\pi_{qx}(q(t), x(t))q(t) - \pi_x(q(t), x(t))}{\pi_{qq}(q(t), x(t))}. \tag{22}$$

Thus, under the archetype marginal stock effect assumption $\pi_{qx}(q, x) > 0$, the optimal extraction rate may be increasing, decreasing, or constant over the planning horizon, or, more generally, display all three of these characteristics. If, in addition, we assume that $\pi_{qx}(q, x) \equiv 0$ but continue to maintain that $\pi_x(q, x) > 0$, then it follows from Eq. (22) that the optimal extraction rate is increasing over the planning horizon. If we furthermore assume that $\pi_x(q, x) \equiv 0$, then it follows from Eq. (22) that the optimal extraction rate is constant over the planning horizon.

To finish up the examination of the necessary conditions, we now consider the transversality condition associated with the choice of the stock of the nonrenewable resource left in the ground at the terminal time. From Eq. (17), we know that the terminal value of the present value shadow price of the resource stock is zero at the ending date, that is, $\lambda(T) = 0$. Moreover, from Eq. (16), we have that $\pi_q(q(T), x(T))\, e^{-rT} = \lambda(T)$. Seeing as $e^{-rT} \neq 0$, the prior two observations allow us to write the transversality condition associated with the choice of the terminal stock of the nonrenewable resource as

$$\pi_q(q(T), x(T))\, e^{-rT} = 0.$$

In the last period, therefore, with the terminal amount of the nonrenewable resource stock remaining in the ground an unrestricted choice variable, the firm behaves just like a static profit maximizing firm, setting marginal profit equal to zero. But this is surely not surprising, for if the firm is in the last period of its planning horizon, there is no future for which to plan. Thus, all the firm has to consider when making a decision in the last period is the consequences of its decision in that period, just like a static profit maximizing firm.

To finish up the analysis of this model, we seek to determine if the above necessary conditions given in Eqs. (16) through (18) are sufficient. We thus begin by computing the Hessian matrix of the Hamiltonian function with respect to the resource stock and extraction rate, that is,

$$\begin{bmatrix} H_{xx} & H_{xq} \\ H_{qx} & H_{qq} \end{bmatrix} = \begin{bmatrix} \pi_{xx}(q, x)\, e^{-rt} & \pi_{xq}(q, x)\, e^{-rt} \\ \pi_{qx}(q, x)\, e^{-rt} & \pi_{qq}(q, x)\, e^{-rt} \end{bmatrix}.$$

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From this we see that concavity of the $H(\cdot)$ in (x, q) is equivalent to concavity of the $\pi(\cdot)$ in (x, q), given that $e^{-rt} > 0$. As a result, by Theorem 3.1, a solution of the necessary conditions given in Eqs. (16) through (18) is a solution of the optimal control problem if $\pi(\cdot)$ is concave in (x, q).

The last theorem of the chapter gives a Mangasarian-type set of sufficient conditions for the following *scrap value* or *salvage value* optimal control problem:

$$\max_{u(\cdot),x_1} J_S[x(\cdot), u(\cdot)] \stackrel{\text{def}}{=} \int_{t_0}^{t_1} f(t, x(t), u(t))\, dt + S(x_1)$$
$$\text{s.t.} \quad \dot{x}(t) = g(t, x(t), u(t)), \tag{23}$$
$$x(t_0) = x_0,\ x(t_1) = x_1.$$

The proof of the ensuing theorem follows that of Theorem 3.1 and is therefore left for a mental exercise.

Theorem 3.6 (Mangasarian Sufficient Conditions, Scrap Value): *Let $(z(t), v(t))$ be an admissible pair for problem (23). Suppose, moreover, that the pair $(z(t), v(t))$ satisfies the necessary conditions of Theorem 2.4 for problem (23) with the costate variable $\lambda(t)$, and let $H(t, x, u, \lambda) \stackrel{\text{def}}{=} f(t, x, u) + \lambda g(t, x, u)$ be the Hamiltonian that corresponds to the costate variable $\lambda(t)$. If $H(\cdot)$ is a concave function of $(x, u)\, \forall\, t \in [t_0, t_1]$ over an open convex set containing all the admissible values of $(x(\cdot), u(\cdot))$ and $S(\cdot)$ is a concave function of x over an open convex set containing all the admissible values of $x(\cdot)$, then $v(t)$ is an optimal control and $(z(t), v(t))$ yields the global maximum of $J_S[\cdot]$. If $H(\cdot)$ and $S(\cdot)$ are strictly concave functions under the same conditions, then $(z(t), v(t))$ yields the unique global maximum of $J_S[\cdot]$.*

We finish up this chapter by applying Theorem 3.6 to our modified inventory accumulation problem in Example 2.4.

Example 3.6: Recall the version of the inventory accumulation problem with a salvage value function from Example 2.4:

$$\max_{u(\cdot),x_T} J_S[x(\cdot), u(\cdot)] \stackrel{\text{def}}{=} px(T) - \int_0^T [c_1[u(t)]^2 + c_2 x(t)]\, dt$$
$$\text{s.t.} \quad \dot{x}(t) = u(t),\ x(0) = 0,\ x(T) = x_T.$$

The Hamiltonian, defined as $H(x, u, \lambda; c_1, c_2) \stackrel{\text{def}}{=} -c_1 u^2 - c_2 x + \lambda u$, is a concave function of $(x, u)\, \forall\, t \in [0, T]$ because its Hessian matrix with respect to (x, u) is

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negative semidefinite, that is,

$$H_{uu} = -2c_1 < 0,\ H_{xx} = 0, \begin{vmatrix} H_{uu} & H_{ux} \\ H_{xu} & H_{xx} \end{vmatrix} = \begin{vmatrix} -2c_1 & 0 \\ 0 & 0 \end{vmatrix} = 0.$$

Given that the salvage function is linear in the state variable, it is concave in it too. Hence by Theorem 3.6, the solution to the necessary conditions given in Example 2.4 is also a solution to the optimal control problem. Uniqueness follows from the observation that there is only one solution of the necessary conditions.

Thus far in our study of optimal control theory, we have limited our attention to problems with one state variable, one control variable, a terminal value of the state variable that is an unrestricted decision variable, and a salvage value function. Moreover, we have developed necessary and sufficient conditions for this class of control problems in which the optimal control is a continuous function of time, and the corresponding state and costate functions are continuously differentiable with respect to time. In the next chapter, we broaden the class of optimal control problems we examine to a considerable degree. Specifically, we extend the theorems of this and the previous chapter by examining optimal control problems with many state and control variables, constraints on the control variables, optimal controls that are piecewise continuous functions of time, and states and costates that are piecewise smooth functions of time. This will permit us to tackle many more dynamic problems of interest to economists, as we will see in the chapters to come.

MENTAL EXERCISES

3.1 For Mental Exercises 2.15–2.19, determine whether the solution of the necessary conditions solves the optimal control problem under consideration. Is the solution unique? Be sure to show your work.

3.2 For Mental Exercises 2.20–2.23, determine whether the solution of the necessary conditions solves the optimal control problem under consideration. Is the solution unique? Be sure to show your work.

3.3 Prove that if the Hamiltonian $H(t, x, u, \lambda) \stackrel{\text{def}}{=} f(t, x, u) + \lambda g(t, x, u)$ for the optimal control problem

$$\max_{u(\cdot), x_1} J[x(\cdot), u(\cdot)] \stackrel{\text{def}}{=} \int_{t_0}^{t_1} f(t, x(t), u(t))\, dt$$

$$\text{s.t.} \quad \dot{x}(t) = g(t, x(t), u(t)),\ x(t_0) = x_0,\ x(t_1) = x_1,$$

is concave in $(x, u)\, \forall\, t \in [t_0, t_1]$, and the triplet $(z(t), v(t), \lambda(t))$ satisfies the necessary conditions of Theorem 3.2, then $v(t)$ maximizes $H(t, z(t), u, \lambda(t))$ for each $t \in [t_0, t_1]$ with respect to u.

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3.4 Prove Theorem 3.2.

3.5 Prove part (iii) of Corollary 3.1 using the fact that a function is concave in a subset of variables if and only if its Hessian matrix with respect to the subset of variables is negative semidefinite.

3.6 With respect to the derivation in Lemma 3.1, show that the specific solution for the costate variable is identical whether you integrate the costate equation backward from t_0 to $t > t_0$ or forward from $t < t_1$ to t_1. Show further that the constant of integration c_1 differs from c_2.

3.7 Prove Theorem 3.3.

3.8 Complete the proof of Theorem 3.4.

3.9 In Example 3.1, show that the necessary conditions of Theorem 3.3 are identical to those of Theorem 2.2 given in Example 2.1.

3.10 In Example 3.2, demonstrate that:

(a) $H(\cdot)$ is concave in (x, u) by showing that the Hessian matrix of $H(\cdot)$ with respect to (x, u) is negative semidefinite.

(b) $H(\cdot)$ is concave in (x, u) by showing that the eigenvalues of the Hessian matrix of $H(\cdot)$ with respect to (x, u) are nonpositive.

(c) The solution of the necessary conditions is the unique solution of the control problem.

(d) The necessary conditions of Theorem 3.3 are identical to those of Theorem 2.2 given in Example 2.2.

11. In Example 3.3, demonstrate that:

(a) The solution to the necessary conditions is given by $v(t) = 0$, $z(t) = x_0$, and $\lambda(t) = t - T$.

(b) The solution to the necessary conditions is the solution to the control problem *without* using the sufficiency theorems.

12. Prove Theorem 3.6.

13. Show that for Mental Exercise 2.30, the solution of the necessary conditions does indeed solve the optimal control problem. Is the solution unique? Show your work.

14. State and prove a Mangasarian-type sufficiency theorem for the optimal control problem

$$\max_{u(\cdot),x_0,x_1} J[x(\cdot), u(\cdot)] \stackrel{\text{def}}{=} \int_{t_0}^{t_1} f(t, x(t), u(t))\,dt$$

$$\text{s.t.} \quad \dot{x}(t) = g(t, x(t), u(t)),\ x(t_0) = x_0,\ x(t_1) = x_1.$$

15. Show that for Mental Exercise 2.28, the solution of the necessary conditions does indeed solve the optimal control problem. Is the solution unique? Show your work.

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16. In Example 3.4, show and explain why the integrand is not strictly concave in the state and control variables even though $U''(c) < 0$. Explain carefully how fixing the terminal capital stock or adding a salvage function can get one out of the problem of nonexistence.
17. **True, False, Uncertain:** If the Hamiltonian is not a concave function of the state and control variables and the maximized Hamiltonian is not a concave function of the state variable, then one *cannot* determine which of several solutions of the necessary conditions is optimal, assuming that an optimal solution exists.

FURTHER READING

An excellent book, reference and otherwise, for the prerequisite mathematics, is Simon and Blume (1994). If you do not have a copy of this book, then it would most likely behoove you to buy a copy of it as soon as possible. As is clear from the chapter proper, we will rely on many of the theorems developed in that book during the course of our exposition of optimal control theory. A proof of the Arrow sufficiency theorem, under more general conditions than stated here, may be found in Arrow and Kurz (1970). Kamien and Schwartz (1971) offer another proof of Arrow's theorem. Lemma 3.1 is a special case of a more general result established by Léonard (1981). Chiang (1992), Kamien and Schwartz (1991), Léonard and Van Long (1992), and Seierstad and Sydsaeter (1987) contain sufficiency theorems akin to those given in this chapter. As should be obvious, Theorem 3.1 is due to Mangasarian (1966). In the next chapter, we will introduce more general sufficiency theorems than those examined in the present chapter.

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