FOUR

The Maximum Principle and Economic Interpretations

Until this juncture, our development of the necessary and sufficient conditions of optimal control theory was essentially a reformulation of those from the classical calculus of variations. As a result, the power and reach of optimal control theory have not been fully exposed or exploited. The goal of this chapter, therefore, is to state and prove some necessary and sufficient conditions for a class of control problems that permit the full capability of optimal control theory to be realized. The theorems are not the most general we will encounter, but they do highlight the motivation for the name *Maximum Principle*. Moreover, our proof of the necessary conditions will employ some continuity assumptions that are, strictly speaking, not needed for a rigorous proof, but will nonetheless be employed so as to ease the technical burden and bring in some connections with the principle of optimality and dynamic programming. We will then use the necessary conditions to explicitly solve for the optimal paths of some examples, some of which will be devoid of any economic content so as to emphasize how to arrive at a solution in practice. The reader is encouraged to work through the proof of the necessary and sufficient conditions, as it provides the reader with a better understanding of how they differ from those presented earlier. Note that the theorems are stated and proven for a class of control problems with many state and control variables.

The optimal control problem under consideration is to find a piecewise continuous control vector function $\mathbf{u}(\cdot) \stackrel{\text{def}}{=} (u_1(\cdot), u_2(\cdot), \ldots, u_M(\cdot))$ and its associated piecewise smooth state vector function $\mathbf{x}(\cdot) \stackrel{\text{def}}{=} (x_1(\cdot), x_2(\cdot), \ldots, x_N(\cdot))$, defined on the fixed time interval $[t_0, t_1]$ that will solve the optimal control problem

$$V(t_0, \mathbf{x}_0, t_1, \mathbf{x}_1) \stackrel{\text{def}}{=} \left\{ \max_{\mathbf{u}(\cdot)} J[\mathbf{x}(\cdot), \mathbf{u}(\cdot)] \stackrel{\text{def}}{=} \int_{t_0}^{t_1} f(t, \mathbf{x}(t), \mathbf{u}(t))\, dt \right\}$$

$$\text{s.t.} \quad \dot{\mathbf{x}}(t) = \mathbf{g}(t, \mathbf{x}(t), \mathbf{u}(t)), \tag{1}$$

$$\mathbf{u}(t) \in U \subseteq \Re^M, \;\; \mathbf{x}(t_0) = \mathbf{x}_0, \;\; \mathbf{x}(t_1) = \mathbf{x}_1,$$

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where $\mathbf{g}(\cdot) \stackrel{\text{def}}{=} (g^1(\cdot), g^2(\cdot), \ldots, g^N(\cdot))$ and $\dot{\mathbf{x}}(\cdot) \stackrel{\text{def}}{=} (\dot{x}_1(\cdot), \dot{x}_2(\cdot), \ldots, \dot{x}_N(\cdot))$. Notice that the terminal boundary condition $\mathbf{x}(t_1) = \mathbf{x}_1$ specifies that the value of the state variable must be equal to some given (i.e., fixed) value at the terminal date of the planning horizon, seeing as $\mathbf{x}_1$ is not specified to be a decision variable in the problem statement. Thus problem (1) is a *fixed endpoints problem*, unlike that encountered in Chapters 2 and 3, in which $\mathbf{x}(t_1) = \mathbf{x}_1$ was a decision variable. Also recall that the control set U specifies the region in which the values of the control variable must lie, and that it is a fixed subset of ^{M} that is assumed to be *independent of the state variables*. We will relax this assumption in a later chapter.

In typical economic problems, $V(t_0, \mathbf{x}_0, t_1, \mathbf{x}_1)$ represents the maximum present discounted profit of a firm that begins its operations with the capital stock $\mathbf{x}_0$ at time t_0, and ends its operations with the capital stock $\mathbf{x}_1$ at time t_1. Alternatively, $V(t_0, \mathbf{x}_0, t_1, \mathbf{x}_1)$ is the maximum total value of the asset $\mathbf{x}_0$ at time t_0, and thus is the purchase price of the asset $\mathbf{x}_0$ at time t_0, because the maximum present value of the stream of net benefits from holding $\mathbf{x}_0$ at time t_0 and using the optimal plan is $V(t_0, \mathbf{x}_0, t_1, \mathbf{x}_1)$. The function $V(\cdot)$ is called the *optimal value function*.

Because we are allowing for a more general class of control variables than we did earlier, we must amend our definition of an admissible control. To this end, we have the following definition, which should be contrasted with that given earlier in Definition 2.1.

Definition 4.1: We call $(\mathbf{x}(t), \mathbf{u}(t))$ an *admissible pair* if $\mathbf{u}(\cdot)$ is any piecewise continuous control vector function such that $\mathbf{u}(t) \in U \;\forall\, t \in [t_0, t_1]$ and $\mathbf{x}(\cdot)$ is a piecewise smooth state vector function such that $\dot{\mathbf{x}}(t) = \mathbf{g}(t, \mathbf{x}(t), \mathbf{u}(t))$, $\mathbf{x}(t_0) = \mathbf{x}_0$, and $\mathbf{x}(t_1) = \mathbf{x}_1$.

In developing the necessary conditions in Chapter 2, we found it convenient to define a function called the Hamiltonian, and we will find it convenient to do so here too. In the present case there are N state variables and N ordinary differential equations describing their rates of changes with respect to time. Consequently, we associate with each of the differential equations a costate function, say $\lambda_n(\cdot)$, $n = 1, 2, \ldots, N$, and form the Hamiltonian for problem (1), namely,

$$
\begin{aligned}
H(t, \mathbf{x}, \mathbf{u}, \boldsymbol{\lambda}) &\stackrel{\text{def}}{=} f(t, \mathbf{x}, \mathbf{u}) + \sum_{n=1}^{N} \lambda_n g^n(t, \mathbf{x}, \mathbf{u}) \\
&\stackrel{\text{def}}{=} f(t, \mathbf{x}, \mathbf{u}) + \boldsymbol{\lambda}'\mathbf{g}(t, \mathbf{x}, \mathbf{u}), \qquad (2)
\end{aligned}
$$

where $\boldsymbol{\lambda} \stackrel{\text{def}}{=} (\lambda_1, \lambda_2, \ldots, \lambda_N)$ and $\boldsymbol{\lambda}'\mathbf{g}(t, \mathbf{x}, \mathbf{u}) \stackrel{\text{def}}{=} \sum_{n=1}^{N} \lambda_n g^n(t, \mathbf{x}, \mathbf{u})$ is the scalar or inner product of the vectors $\boldsymbol{\lambda}$ and $\mathbf{g}(t, \mathbf{x}, \mathbf{u})$.

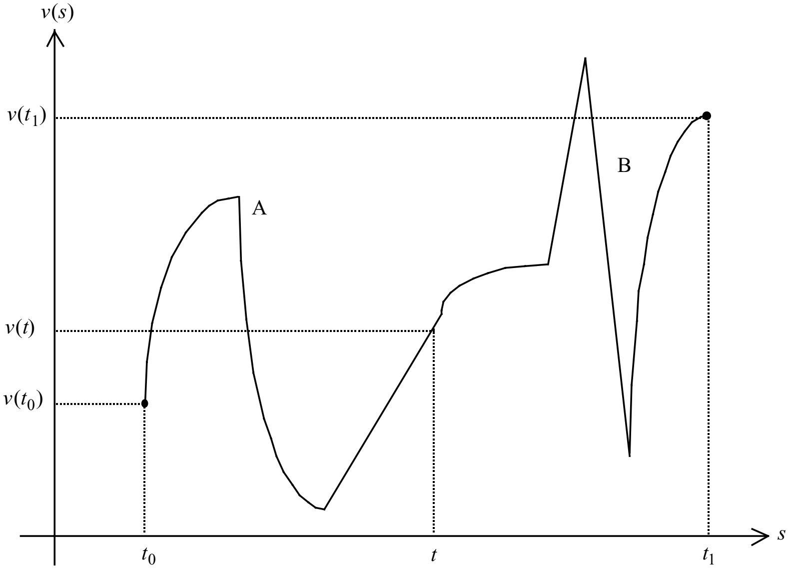

Figure 4.1

The ensuing proof of the necessary conditions for problem (1) will rely on the following three assumptions, in addition to the fundamental ones stated in Chapter 1:

(A.1) Let $S \in \Re^N$ be a nonempty set of points from which one can bring the system to the terminal point $\mathbf{x}_1$ at time t_1 by the use of an optimal control, where $\mathbf{x}_0 \in S$.

(A.2) S is an *open* set.

(A.3) $V(\cdot) \in C^{(2)}\ \forall\,(t, \mathbf{x}) \in (t_0, t_1) \times S$, where $V(\cdot)$ is the optimal value function for problem (1).

Assumption (A.1) means that there exists values of the state vector such that when we take the initial state as one of these values, we can arrive at the given terminal state by the terminal date of the planning horizon using an optimal control. Assumption (A.2) says that the boundary of the set S is not part of S, that is to say, S consists only of interior points. The $C^{(2)}$ nature of the optimal value function in assumption (A.3) is adopted so as to avoid getting caught up in technical details. It will be pointed out during the proof of the necessary conditions where these assumptions are used. The approach we take below to prove the necessary conditions of problem (1) is often referred to as the “dynamic programming proof” of the necessary conditions. As you will shortly see, this method of proof is quite different from the variational proof of the necessary conditions given in Chapter 2.

Before we state and prove the necessary conditions, it is preferable to introduce the key principle behind the foregoing proof, videlicet, the *principle of optimality*. The principle of optimality is illustrated in Figure 4.1 for an optimal control problem

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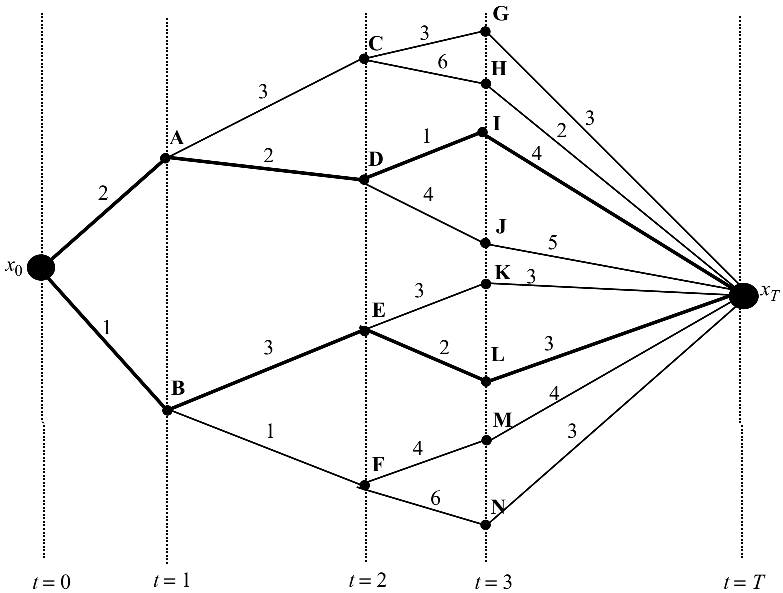

Figure 4.2

with one control variable. Let the curve $v(s)$ be the optimal path of the control variable for the entire planning horizon $[t_0, t_1]$, beginning with stock x_0 at time t_0 and ending with stock x_1 at time t_1. In Figure 4.1, this path is divided into two parts, A and B, with time $s = t$ being the break point in the time horizon. The principle of optimality asserts that trajectory B, defined for the interval $[t, t_1]$, must, in its own right, be the optimal path for this interval given that, at time $s = t$, $t \in (t_0, t_1)$, the initial state is taken as $z(t)$, the terminal state from the interval $[t_0, t]$. Notice that we have not drawn the optimal path of the control variable as a differentiable function of time so as to emphasize the more general character of the optimal control problem we are studying in this chapter relative to that studied in the prior two chapters.

An alternative view of the principle of optimality is given in Figure 4.2, where the problem is to find the least cost path from the initial state x_0 to the terminal state x_T. The numbers along each arc indicate the cost of moving between the adjacent points. First note that the optimal path is *not* unique and is given by the two thick paths. They represent the least cost paths of reaching the terminal state x_T from the given initial state x_0. In this context, the principle of optimality asserts that, roughly speaking, if you chop off the first arc (or any beginning set of arcs) from an optimal sequence of arcs, the remaining abridged sequence of arcs must still be optimal in its own right. In other words, the principle of optimality asserts that any portion of an optimal arc is optimal.

For example, in Figure 4.2, the path x_0ADIx_T is an optimal path from x_0 to x_T. Thus by the principle of optimality, the least cost path from A to x_T must be $ADIx_T$, which is precisely the remaining part of the overall optimal path when taking A as the initial state, as inspection of Figure 4.2 confirms. Conversely, because the path $ADIx_T$ is the least cost path from state A to state x_T, any longer optimal path, say from x_0 to x_T, that passes through A must use the path $ADIx_T$ to reach state x_T. Again, inspection of Figure 4.2 confirms this too. The same results would also hold if we took, say, state D as the initial state, as you should confirm. Likewise, similar results hold if we use the other optimal arc x_0BELx_T, as you should also confirm.

Finally, note that the *myopic* optimal path is given by x_0BFMx_T. This is the path chosen by an individual who looks only at the cost of moving from one state to the next, and chooses the path that minimizes the period-by-period cost of moving from x_0 to x_T. Observe that the total cost along the path x_0BFMx_T is greater than that along paths x_0ADIx_T and x_0BELx_T. This illustrates the general result that a myopic one-stage-at-a-time optimization procedure will *not*, in general, yield the least cost path in a dynamic environment.

The principle of optimality can be proven in a straightforward way by a contrapositive argument. It is important to prove this principle and to follow it carefully because, as noted above, the principle of optimality forms the basis of the ensuing proof of the necessary conditions of optimal control theory.

Theorem 4.1 (The Principle of Optimality): *An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision.*

Proof: Let $\mathbf{v}(s)$ be the optimal control path $\forall\, s \in [t_0, t_1]$, beginning at time t_0 in state $\mathbf{x}_0$ and ending at time t_1 in state $\mathbf{x}_1$, and let $\mathbf{z}(s)$ be the corresponding state path. A contrapositive proof will be employed. Assume, therefore, that $\mathbf{v}(s)$ is the optimal control path $\forall\, s \in [t_0, t]$, $t \in (t_0, t_1)$, beginning at time t_0 in state $\mathbf{x}_0$ and ending at time t with state $\mathbf{z}(t)$. Also assume, contrary to the conclusion of the theorem, that $\mathbf{u}(s) \neq \mathbf{v}(s)$ is the optimal control path $\forall\, s \in (t, t_1]$, beginning with state $\mathbf{z}(t)$ at time t and ending with state $\mathbf{x}_1$ at time t_1. Then $\mathbf{v}(s)$ could not be the optimal control path $\forall\, s \in [t_0, t_1]$ because one could use $\mathbf{v}(s)\, \forall\, s \in [t_0, t]$ and switch to $\mathbf{u}(s)\, \forall\, s \in (t, t_1]$, thereby giving the objective functional a larger value than using $\mathbf{v}(s)\, \forall\, s \in [t_0, t_1]$. This contradicts the assumed optimality (the hypothesis) of $\mathbf{v}(s)\, \forall\, s \in [t_0, t_1]$, thereby proving the principle of optimality. Q.E.D.

We are now ready to state and prove the necessary conditions for problem (1).

Theorem 4.2 (The Maximum Principle): *Let* $(\mathbf{z}(t), \mathbf{v}(t))$ *be an admissible pair for problem (1). Then if* $(\mathbf{z}(t), \mathbf{v}(t))$ *yields the absolute maximum of* $J[\cdot]$, *it is necessary that there exists a piecewise smooth vector-valued function*

$\boldsymbol{\lambda}(\cdot) \overset{\text{def}}{=} (\lambda_1(\cdot), \lambda_2(\cdot), \ldots, \lambda_N(\cdot))$ *such that for all* $t \in [t_0, t_1]$,

$$\mathbf{v}(t) = \arg\max_{\mathbf{u}\in U} H(t, \mathbf{z}(t), \mathbf{u}, \boldsymbol{\lambda}(t)),$$

that is, if

$$M(t, \mathbf{z}(t), \boldsymbol{\lambda}(t)) \overset{\text{def}}{=} \max_{\mathbf{u}\in U} H(t, \mathbf{z}(t), \mathbf{u}, \boldsymbol{\lambda}(t)),$$

then

$$M(t, \mathbf{z}(t), \boldsymbol{\lambda}(t)) \equiv H(t, \mathbf{z}(t), \mathbf{v}(t), \boldsymbol{\lambda}(t)),$$

or equivalently,

$$H(t, \mathbf{z}(t), \mathbf{v}(t), \boldsymbol{\lambda}(t)) \geq H(t, \mathbf{z}(t), \mathbf{u}, \boldsymbol{\lambda}(t))\,\forall\, \mathbf{u} \in U.$$

That is, for each $t \in [t_0, t_1]$, $H(\cdot)$ *attains at* $\mathbf{v}(t)$ *its maximum with respect to* $\mathbf{u}$, $\forall\, \mathbf{u} \in U$ *and* $(t, \mathbf{z}(t), \boldsymbol{\lambda}(t))$ *fixed. Furthermore, except for the points of discontinuities of* $\mathbf{v}(t)$,

$$\dot{z}_n(t) = \frac{\partial H}{\partial \lambda_n}(t, \mathbf{z}(t), \mathbf{v}(t), \boldsymbol{\lambda}(t)) = g^n(t, \mathbf{z}(t), \mathbf{v}(t)), \quad n = 1, 2, \ldots, N,$$

$$\dot{\lambda}_n(t) = -\frac{\partial H}{\partial x_n}(t, \mathbf{z}(t), \mathbf{v}(t), \boldsymbol{\lambda}(t)), \quad n = 1, 2, \ldots, N,$$

where the above notation means that the functions are first differentiated with respect to the particular variable and then evaluated at $(t, \mathbf{z}(t), \mathbf{v}(t), \boldsymbol{\lambda}(t))$.

Proof: To begin the proof of the necessary conditions, recall the definition of the optimal value function $V(\cdot)$ from problem (1):

$$\begin{aligned} V(t_0, \mathbf{x}_0) &\overset{\text{def}}{=} \max_{\mathbf{u}(\cdot)} \int_{t_0}^{t_1} f(s, \mathbf{x}(s), \mathbf{u}(s))\, ds \\ &\quad \text{s.t. } \dot{\mathbf{x}}(s) = \mathbf{g}(s, \mathbf{x}(s), \mathbf{u}(s)),\ \mathbf{u}(s) \in U, \mathbf{x}(t_0) = \mathbf{x}_0,\ \mathbf{x}(t_1) = \mathbf{x}_1. \end{aligned} \tag{3}$$

Note that we now use s as the dummy variable of integration in light of the fact that we intend to let t denote a particular date in the planning horizon. Because the optimal pair $(\mathbf{z}(\cdot), \mathbf{v}(\cdot))$ is assumed to exist, this definition is equivalent to

$$V(t_0, \mathbf{x}_0) \equiv \int_{t_0}^{t_1} f(s, \mathbf{z}(s), \mathbf{v}(s))\, ds. \tag{4}$$

Given that $(t_0, \mathbf{x}_0, t_1, \mathbf{x}_1)$ are the parameters of problem (1), the optimal value function $V(\cdot)$ depends on all four parameters in general, as well as any other

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parameter that might appear in problem (1). The notation in Eqs. (3) and (4), namely, $V(t_0, \mathbf{x}_0)$, is adopted to streamline the exposition, for the parameters $(t_1, \mathbf{x}_1)$ play no essential role in what follows.

By the principle of optimality, it follows that Eq. (4) can be written as

$$V(t_0, \mathbf{x}_0) \equiv \int_{t_0}^{t} f(s, \mathbf{z}(s), \mathbf{v}(s))\, ds + V(t, \mathbf{z}(t)), \tag{5}$$

which holds for every $t \in [t_0, t_1]$. From the definitions of $V(\cdot)$ in Eqs. (3) and (4), it should be clear that

$$V(t, \mathbf{z}(t)) \equiv \int_{t}^{t_1} f(s, \mathbf{z}(s), \mathbf{v}(s))\, ds, \tag{6}$$

in which case the correctness of Eq. (5) should now seem more evident given the additive property of integrals with respect to the limits of integration. Equation (6) asserts that $V(t, \mathbf{z}(t))$ is the maximum total value of the asset $\mathbf{z}(t)$ at time t, and thus is the purchase price of the asset $\mathbf{z}(t)$ at time t, because the maximum present value of the stream of net benefits from holding $\mathbf{z}(t)$ at time t and using the optimal plan is $V(t, \mathbf{z}(t))$. Equation (5) thus asserts that the total value of the asset $\mathbf{x}_0$ at time t_0 may be thought of as two separate additive components: (i) the return beginning with stock $\mathbf{x}_0$ at time t_0 and continuing the optimal plan until time t, and (ii) the return beginning with stock $\mathbf{z}(t)$ at time t and continuing on with the optimal plan, where $\mathbf{z}(t)$ is the stock at time t that resulted from the optimal plan in (i). This assertion is, in its essence, the principle of optimality.

Now differentiate Eq. (5) with respect to t using the chain rule and Leibniz's rule to get

$$0 \equiv \frac{d}{dt}\left[\int_{t_0}^{t} f(s, \mathbf{z}(s), \mathbf{v}(s))\, ds + V(t, \mathbf{z}(t))\right],$$

or

$$0 \equiv f(t, \mathbf{z}(t), \mathbf{v}(t)) + V_t(t, \mathbf{z}(t)) + \sum_{n=1}^{N} V_{x_n}(t, \mathbf{z}(t))\dot{z}_n(t), \tag{7}$$

which also holds for every $t \in [t_0, t_1]$. Because $(\mathbf{z}(\cdot), \mathbf{v}(\cdot))$ is the optimal pair in problem (1), it must satisfy the state equation identically, that is, $\dot{\mathbf{z}}(t) \equiv \mathbf{g}(t, \mathbf{z}(t), \mathbf{v}(t))$ for all $t \in [t_0, t_1]$. Substituting $\dot{\mathbf{z}}(t) \equiv \mathbf{g}(t, \mathbf{z}(t), \mathbf{v}(t))$ into Eq. (7) therefore yields

$$V_t(t, \mathbf{z}(t)) + \sum_{n=1}^{N} V_{x_n}(t, \mathbf{z}(t)) g^n(t, \mathbf{z}(t), \mathbf{v}(t)) + f(t, \mathbf{z}(t), \mathbf{v}(t)) \equiv 0\, \forall\, t \in [t_0, t_1]. \tag{8}$$

This is the fundamental partial differential equation obeyed by the optimal value function $V(\cdot)$, known as the *Hamilton-Jacobi-Bellman equation*. The Hamilton-Jacobi-Bellman equation is commonly written in the alternative form $-V_t(t, \mathbf{z}(t)) \equiv f(t, \mathbf{z}(t), \mathbf{v}(t)) + V_{\mathbf{x}}(t, \mathbf{z}(t))\, \mathbf{g}(t, \mathbf{z}(t), \mathbf{v}(t))$. We will return to this fundamental result in later chapters, when we discuss continuous time dynamic programming and intertemporal duality theory. Equation (8) is an important result in this proof. We will return to it shortly.

Now consider the control vector $\boldsymbol{\varepsilon} \in U$, and define the *perturbed control* as

$$\mathbf{u}(s) \stackrel{\text{def}}{=} \begin{cases} \boldsymbol{\varepsilon} & \forall\, s \in [t_0, t_0 + \Delta t) \\ \mathbf{u}^{\varepsilon}(s) & \forall\, s \in [t_0 + \Delta t, t_1], \end{cases}$$

where $\Delta t > 0$, $\mathbf{u}^{\varepsilon}(s)$ is the time path of the optimal control corresponding to the initial point $(t_0 + \Delta t, \mathbf{x}^{\varepsilon}(t_0 + \Delta t))$, and $\mathbf{x}^{\varepsilon}(s)$ is the time path of the state vector corresponding to the perturbed control path $\mathbf{u}(s)$. By assumption, $\mathbf{v}(\cdot)$ is the optimal control, thereby implying that $\mathbf{u}(\cdot)$ is a *suboptimal control*. As a result, it follows that

$$\begin{aligned} V(t_0, \mathbf{x}_0) &\equiv J[\mathbf{z}(\cdot), \mathbf{v}(\cdot)] \geq J[\mathbf{x}^{\varepsilon}(\cdot), \mathbf{u}(\cdot)] \\ &= \int_{t_0}^{t_0+\Delta t} f(s, \mathbf{x}^{\varepsilon}(s), \boldsymbol{\varepsilon})\, ds + V(t_0 + \Delta t, \mathbf{x}^{\varepsilon}(t_0 + \Delta t)). \end{aligned} \tag{9}$$

The pair $(\mathbf{x}^{\varepsilon}(\cdot), \mathbf{u}(\cdot))$ must be admissible if we are to compare the value of problem (1) using the controls $\mathbf{u}(\cdot)$ and $\mathbf{v}(\cdot)$. This implies that $\mathbf{x}^{\varepsilon}(t_0) = \mathbf{z}(t_0) = \mathbf{x}_0$ and $\mathbf{x}^{\varepsilon}(t_1) = \mathbf{z}(t_1) = \mathbf{x}_1$. Upon substituting $\mathbf{x}^{\varepsilon}(t_0) = \mathbf{x}_0$ in Eq. (9) and rearranging it, we get

$$V(t_0 + \Delta t, \mathbf{x}^{\varepsilon}(t_0 + \Delta t)) - V(t_0, \mathbf{x}^{\varepsilon}(t_0)) \leq - \int_{t_0}^{t_0+\Delta t} f(s, \mathbf{x}^{\varepsilon}(s), \boldsymbol{\varepsilon})\, ds. \tag{10}$$

Now divide Eq. (10) by $\Delta t > 0$, and let $\Delta t \to 0$ to get

$$\lim_{\Delta t \to 0} \frac{V(t_0 + \Delta t, \mathbf{x}^{\varepsilon}(t_0 + \Delta t)) - V(t_0, \mathbf{x}^{\varepsilon}(t_0))}{\Delta t} \leq - \lim_{\Delta t \to 0} \frac{\int_{t_0}^{t_0+\Delta t} f(s, \mathbf{x}^{\varepsilon}(s), \boldsymbol{\varepsilon})\, ds}{\Delta t}. \tag{11}$$

The inequality in Eq. (10) is preserved as $\Delta t \to 0$ because of the assumed differentiability of $V(\cdot)$ from assumption (A.3). Our goal now is to evaluate the limits in Eq. (11) so that it can be put in the form of Eq. (8).

We begin with the left-hand side of Eq. (11). To this end, define the function $G(\cdot)$ by the formula $G(\Delta t) \stackrel{\text{def}}{=} V(t_0 + \Delta t, \mathbf{x}^{\varepsilon}(t_0 + \Delta t))$, which implies that $G(0) = V(t_0, \mathbf{x}^{\varepsilon}(t_0))$. By the chain rule, we also have that $G'(\Delta t) = V_t(t_0 + \Delta t, \mathbf{x}^{\varepsilon}(t_0 + \Delta t)) + \sum_{n=1}^{N} V_{x_n}(t_0 + \Delta t, \mathbf{x}^{\varepsilon}(t_0 + \Delta t)) \quad \dot{x}_n^{\varepsilon}(t_0 + \Delta t)$. Using these results and the definition of the derivative, the left-hand side of Eq. (11)

becomes

$$\lim_{\Delta t\to 0}\frac{V(t_0+\Delta t,\mathbf{x}^{\varepsilon}(t_0+\Delta t))-V(t_0,\mathbf{x}^{\varepsilon}(t_0))}{\Delta t}=\lim_{\Delta t\to 0}\frac{G(0+\Delta t)-G(0)}{\Delta t}$$

$$\stackrel{\text{def}}{=} G'(\Delta t)\big|_{\Delta t=0}=G'(0)=V_t(t_0,\mathbf{x}^{\varepsilon}(t_0))+\sum_{n=1}^{N}V_{x_n}(t_0,\mathbf{x}^{\varepsilon}(t_0))\,\dot{x}_n^{\varepsilon}(t_0). \quad (12)$$

We will return to this result shortly.

To evaluate the right-hand side of Eq. (11), define the function $F(\cdot)$ by the formula

$$F(\Delta t)\stackrel{\text{def}}{=}\int_{t_0}^{t_0+\Delta t} f(s,\mathbf{x}^{\varepsilon}(s),\varepsilon)\,ds,$$

and then recognize that $F(0)=\int_{t_0}^{t_0} f(s,\mathbf{x}^{\varepsilon}(s),\varepsilon)\,ds=0$ and that $F'(\Delta t)=f(t_0+\Delta t,\mathbf{x}^{\varepsilon}(t_0+\Delta t),\varepsilon)$ by Leibniz's rule. Using these results and the definition of the derivative, the right-hand side of Eq. (11) becomes

$$\begin{aligned}
-\lim_{\Delta t\to 0}\frac{\int_{t_0}^{t_0+\Delta t} f(s,\mathbf{x}^{\varepsilon}(s),\varepsilon)\,ds}{\Delta t} &= -\lim_{\Delta t\to 0}\frac{F(\Delta t)}{\Delta t}=-\lim_{\Delta t\to 0}\frac{F(0+\Delta t)-F(0)}{\Delta t}\\
&= -\,F'(\Delta t)\big|_{\Delta t=0}\\
&= -F'(0)=-f\left(t_0+\Delta t,\mathbf{x}^{\varepsilon}(t_0+\Delta t),\varepsilon\right)\big|_{\Delta t=0}\\
&= -f(t_0,\mathbf{x}^{\varepsilon}(t_0),\varepsilon). \qquad (13)
\end{aligned}$$

Substituting Eqs. (12) and (13) in Eq. (11) gives

$$V_t(t_0,\mathbf{x}_0)+\sum_{n=1}^{N}V_{x_n}(t_0,\mathbf{x}_0)\dot{x}_n^{\varepsilon}(t_0)\le -f(t_0,\mathbf{x}_0,\varepsilon), \quad (14)$$

where $\mathbf{x}^{\varepsilon}(t_0)=\mathbf{x}_0$ was used because the pair $(\mathbf{x}^{\varepsilon}(\cdot),\mathbf{u}(\cdot))$ is admissible by assumption. Recall also that admissibility requires that $\mathbf{x}^{\varepsilon}(\cdot)$ satisfy the state equation; hence $\dot{\mathbf{x}}^{\varepsilon}(t_0)=\mathbf{g}(t_0,\mathbf{x}^{\varepsilon}(t_0),\varepsilon)$. Substituting $\dot{\mathbf{x}}^{\varepsilon}(t_0)=\mathbf{g}(t_0,\mathbf{x}^{\varepsilon}(t_0),\varepsilon)$ into Eq. (14) and again using $\mathbf{x}^{\varepsilon}(t_0)=\mathbf{x}_0$ gives

$$V_t(t_0,\mathbf{x}_0)+\sum_{n=1}^{N}V_{x_n}(t_0,\mathbf{x}_0)g^n(t_0,\mathbf{x}_0,\varepsilon)+f(t_0,\mathbf{x}_0,\varepsilon)\le 0. \quad (15)$$

Now replace the initial point $(t_0,\mathbf{x}_0)$ in Eq. (15) with an *arbitrary point* $(t,\mathbf{x})\in[t_0,t_1]\times S$ to get the inequality

$$V_t(t,\mathbf{x})+\sum_{n=1}^{N}V_{x_n}(t,\mathbf{x})g^n(t,\mathbf{x},\varepsilon)+f(t,\mathbf{x},\varepsilon)\le 0, \quad (16)$$

which is valid $\forall\,\mathbf{x}\in S$, $\forall\,t\in[t_0,t_1]$, and $\forall\,\varepsilon\in U$. This is where assumption (A.1) is used.

By defining the function $\phi(\cdot)$ as

$$\phi(t, \mathbf{x}, \varepsilon) \stackrel{\text{def}}{=} V_t(t, \mathbf{x}) + \sum_{n=1}^{N} V_{x_n}(t, \mathbf{x}) g^n(t, \mathbf{x}, \varepsilon) + f(t, \mathbf{x}, \varepsilon), \tag{17}$$

and then referring to Eq. (8) and Eq. (16), respectively, it should be clear that we have shown that

$$\phi(t, \mathbf{z}(t), \mathbf{v}(t)) \equiv 0 \, \forall \, t \in [t_0, t_1], \tag{18}$$

as well as

$$\phi(t, \mathbf{x}, \varepsilon) \leq 0 \, \forall \, \mathbf{x} \in S, \ \forall \, t \in [t_0, t_1], \quad \text{and} \quad \forall \, \varepsilon \in U. \tag{19}$$

The Maximum Principle and the costate equation follow from Eqs. (18) and (19), as we now proceed to demonstrate.

To see how the Maximum Principle follows from Eqs. (18) and (19), first set $\mathbf{x} = \mathbf{z}(t)$ and hold it fixed. Then Eqs. (18) and (19) imply that

$$\phi(t, \mathbf{z}(t), \mathbf{v}(t)) = \max_{\varepsilon \in U} \phi(t, \mathbf{z}(t), \varepsilon). \tag{20}$$

Next, define

$$\lambda_j(t) \stackrel{\text{def}}{=} V_{x_j}(t, \mathbf{z}(t)), \quad j = 1, 2, \ldots, N, \tag{21}$$

$$\lambda_0 \stackrel{\text{def}}{=} 1, \tag{22}$$

$$H(t, \mathbf{x}, \mathbf{u}, \boldsymbol{\lambda}) \stackrel{\text{def}}{=} \lambda_0 f(t, \mathbf{x}, \mathbf{u}) + \boldsymbol{\lambda}' \mathbf{g}(t, \mathbf{x}, \mathbf{u}) = \lambda_0 f(t, \mathbf{x}, \mathbf{u}) + \sum_{n=1}^{N} \lambda_n g^n(t, \mathbf{x}, \mathbf{u}). \tag{23}$$

Upon using Eq. (17), that is, the definition of the function $\phi(\cdot)$, Eq. (20) takes the form

$$V_t(t, \mathbf{z}(t)) + \sum_{n=1}^{N} V_{x_n}(t, \mathbf{z}(t)) g^n(t, \mathbf{z}(t), \mathbf{v}(t)) + f(t, \mathbf{z}(t), \mathbf{v}(t))$$
$$= \max_{\varepsilon \in U} \left\{ V_t(t, \mathbf{z}(t)) + \sum_{n=1}^{N} V_{x_n}(t, \mathbf{z}(t)) g^n(t, \mathbf{z}(t), \varepsilon) + f(t, \mathbf{z}(t), \varepsilon) \right\}.$$

Because $V_t(t, \mathbf{z}(t))$ is independent of ε, it can be canceled from both sides of the above equality. On using the definitions (21), (22), and (23), the above equation becomes

$$H(t, \mathbf{z}(t), \mathbf{v}(t), \boldsymbol{\lambda}(t)) = \max_{\varepsilon \in U} H(t, \mathbf{z}(t), \varepsilon, \boldsymbol{\lambda}(t)),$$

which is the Maximum Principle.

To derive the adjoint or costate equation, set $\varepsilon = \mathbf{v}(t)$ and hold it fixed in Eqs. (18) and (19). Then Eqs. (18) and (19) imply that $\phi(t, \mathbf{x}, \mathbf{v}(t))$ attains its maximum value

https://doi.org/10.1017/CBO9780511806827.006 Published online by Cambridge University Press

at $\mathbf{x} = \mathbf{z}(t)$. Because $\mathbf{x} \in S$ and S is an open set by assumption (A.2), the fact that $\phi(t, \mathbf{x}, \mathbf{v}(t))$ attains its maximum value at $\mathbf{x} = \mathbf{z}(t)$ implies the first-order necessary condition

$$\frac{\partial \phi}{\partial x_j}(t, \mathbf{x}, \mathbf{v}(t))\bigg|_{\mathbf{x}=\mathbf{z}(t)} = 0, \quad j = 1, 2, \ldots, N. \tag{24}$$

Using the definition of $\phi(\cdot)$ in Eq. (17) and carrying out the differentiation in Eq. (24) thus yields

$$\begin{aligned} &V_{tx_j}(t, \mathbf{z}(t)) + \sum_{n=1}^{N} V_{x_n}(t, \mathbf{z}(t))\, g^n_{x_j}(t, \mathbf{z}(t), \mathbf{v}(t)) \\ &\qquad + \sum_{n=1}^{N} V_{x_n x_j}(t, \mathbf{z}(t))\, g^n(t, \mathbf{z}(t), \mathbf{v}(t)) + f_{x_j}(t, \mathbf{z}(t), \mathbf{v}(t)) = 0, \end{aligned} \tag{25}$$

for $j = 1, 2, \ldots, N$. Note that assumption (A.3), namely, $V(\cdot) \in C^{(2)}\, \forall\, (t, \mathbf{x}) \in (t_0, t_1) \times S$, has been used in deriving Eq. (25). Now differentiate Eq. (21) with respect to t, and use the fact that $\dot{\mathbf{z}}(t) \equiv \mathbf{g}(t, \mathbf{z}(t), \mathbf{v}(t))$ to get

$$\dot{\lambda}_j(t) = V_{x_j t}(t, \mathbf{z}(t)) + \sum_{n=1}^{N} V_{x_j x_n}(t, \mathbf{z}(t))\, g^n(t, \mathbf{z}(t), \mathbf{v}(t)), \quad j = 1, 2, \ldots, N. \tag{26}$$

Using Young's theorem on Eq. (26) and then substituting Eq. (26) into Eq. (25) and rearranging the resulting expression yields

$$\dot{\lambda}_j(t) = -f_{x_j}(t, \mathbf{z}(t), \mathbf{v}(t)) - \sum_{n=1}^{N} V_{x_n}(t, \mathbf{z}(t))\, g^n_{x_j}(t, \mathbf{z}(t), \mathbf{v}(t)), \quad j = 1, 2, \ldots, N. \tag{27}$$

Finally, using the definitions (21), (22), and (23) in Eq. (27) gives

$$\dot{\lambda}_j(t) = -H_{x_j}(t, \mathbf{z}(t), \mathbf{v}(t), \boldsymbol{\lambda}(t)), \quad j = 1, 2, \ldots, N,$$

which is the costate or adjoint equation. This completes the proof of the necessary conditions. The reader should note that given our three assumptions, this proof is rigorous. Q.E.D.

Notice that in the definition of the Hamiltonian in Eq. (23), a *constant* costate variable λ_0 multiplies the integrand $f(t, \mathbf{x}, \mathbf{u})$. This is the correct way to write the Hamiltonian in general. In this case, the necessary conditions would then include the additional requirement that

$$(\lambda_0, \boldsymbol{\lambda}(t)') \neq \mathbf{0}'_{N+1}\, \forall\, t \in [t_0, t_1],$$

where $\mathbf{0}_{N+1}$ is the null (column) vector in $\Re^{N+1}$, and furthermore that

$$\lambda_0 = 0 \quad \text{or} \quad \lambda_0 = 1.$$

In most economic applications of optimal control theory, one can safely set $\lambda_0 = 1$. In essence, assuming that $\lambda_0 = 1$ means that the integrand matters in determining the optimal solution to the control problem. Alternatively, if $\lambda_0 = 0$, then the integrand does not matter for determining the optimal solution to the control problem in that the necessary conditions of Theorem 4.2 do not change if we replace the integrand function $f(\cdot)$ with *any* other function. The case $\lambda_0 = 0$ is thus appropriately referred to as *abnormal*. We will typically assume that $\lambda_0 = 1$ for the remainder of the book, even though, strictly speaking, it is possible for $\lambda_0 = 0$ to hold, as the ensuing example illustrates. The mental exercises present some other control problems for which $\lambda_0 = 0$ in an optimal solution.

Example 4.1: Consider the fixed endpoints optimal control problem

$$\max_{u(\cdot)} J[x(\cdot), u(\cdot)] \stackrel{\text{def}}{=} \int_0^T u(t)\, dt$$

$$\text{s.t.} \quad \dot{x}(t) = [u(t)]^2,$$

$$x(0) = 0, \; x(T) = 0.$$

Notice that the *only* admissible pair for this problem is $(x(t), u(t)) = (0, 0)$ for all $t \in [0, T]$, as inspection of the state equation and boundary conditions will verify. Defining the Hamiltonian by setting $\lambda_0 = 1$ yields $H(t, x, u, \lambda) \stackrel{\text{def}}{=} u + \lambda u^2$. Given that there are no constraints on the control variable, the necessary condition $\max_u H(t, x, u, \lambda)$ of Theorem 4.2 reduces to the familiar condition $H_u(t, x, u, \lambda) = 1 + 2\lambda u = 0$. Note, however, that the only admissible value of the control variable, namely, $u(t) = 0$, is not a solution to this necessary condition. If we instead define the Hamiltonian with a constant costate variable λ_0 attached to the integrand function and assume it is not equal to unity, then we have $\bar{H}(t, x, u, \lambda) \stackrel{\text{def}}{=} \lambda_0 u + \lambda u^2$. The necessary condition then reads $\bar{H}_u(t, x, u, \lambda) = \lambda_0 + 2\lambda u = 0$. If we choose $\lambda_0 = 0$, then $u(t) = 0$ does indeed satisfy this necessary condition.

Let's now provide an economic interpretation to the necessary conditions of Theorem 4.2 and the functions used in their derivation. First, consider a typical costate variable $\lambda_j(t)$, and recall that it is defined in Eq. (21) as $\lambda_j(t) \stackrel{\text{def}}{=} V_{x_j}(t, \mathbf{z}(t))$, $j = 1, 2, \ldots, N$. Also recall that $V(t, \mathbf{z}(t))$ is the *total value* of the stock $\mathbf{z}(t)$ in an optimal plan that begins at time t. This interpretation therefore implies that

$\lambda_j(t) \stackrel{\text{def}}{=} V_{x_j}(t, \mathbf{z}(t))$, $j = 1, 2, \ldots, N$, is the *marginal value* or *shadow value* of the jth stock $z_j(t)$ at time t in an optimal plan. That is, $\lambda_j(t)$ is the dollar amount by which the value of the optimal program $V(t, \mathbf{z}(t))$ would increase owing to a marginal increase in the jth stock at time t, the initial date of the optimal plan. Note that we will further refine and expand on the economic interpretation of $\boldsymbol{\lambda}(t)$ when we prove the dynamic envelope theorem. For now, this interpretation and intuition will suffice.

Next we turn to an economic interpretation of the Hamiltonian defined in Eq. (2), scilicet, $H(t, \mathbf{x}, \mathbf{u}, \boldsymbol{\lambda}) \stackrel{\text{def}}{=} f(t, \mathbf{x}, \mathbf{u}) + \boldsymbol{\lambda}'\mathbf{g}(t, \mathbf{x}, \mathbf{u})$. The integrand $f(t, \mathbf{x}, \mathbf{u})$ may be thought of as the flow of profits over a short time interval to the value of the objective functional $J[\mathbf{x}(\cdot), \mathbf{u}(\cdot)]$. The state equation $\dot{\mathbf{x}} = \mathbf{g}(t, \mathbf{x}, \mathbf{u})$ represents the flow of investment in the capital stock in physical terms over this time interval, since $\mathbf{x}$ is the capital stock vector and $\dot{\mathbf{x}}$ is the time rate of change of, or investment in, the vector of stocks. Taking the inner product of $\mathbf{g}(t, \mathbf{x}, \mathbf{u})$ with $\boldsymbol{\lambda}$ converts the physical flow $\mathbf{g}(t, \mathbf{x}, \mathbf{u})$ into dollar terms, since $\boldsymbol{\lambda}$ is the marginal value of the capital stock vector at time t. The sum $\boldsymbol{\lambda}'\mathbf{g}(t, \mathbf{x}, \mathbf{u})$ therefore represents the value of the capital accumulated in a short interval of time. Thus the Hamiltonian $H(t, \mathbf{x}, \mathbf{u}, \boldsymbol{\lambda}) \stackrel{\text{def}}{=} f(t, \mathbf{x}, \mathbf{u}) + \boldsymbol{\lambda}'\mathbf{g}(t, \mathbf{x}, \mathbf{u})$ measures the total contribution of the activities that take place in a short interval of time, which includes the immediate effect $f(t, \mathbf{x}, \mathbf{u})$ of profit flow, and the future effect $\boldsymbol{\lambda}'\mathbf{g}(t, \mathbf{x}, \mathbf{u})$ of the value of the capital accumulated. The Maximum Principle, namely, $H(t, \mathbf{z}(t), \mathbf{v}(t), \boldsymbol{\lambda}(t)) \geq H(t, \mathbf{z}(t), \mathbf{u}, \boldsymbol{\lambda}(t))\, \forall\, \mathbf{u} \in U$, thus directs us to maximize the sum of the current and future value flow at each point in time with our choice of the control variable $\mathbf{u}$. That is, it is simply an assertion that for every $t \in [t_0, t_1]$, the value $\mathbf{v}(t)$ of the optimal control vector must maximize the value of the Hamiltonian over all admissible values of the control vector $\mathbf{u}(t) \in U$.

Finally, we come to the adjoint or costate equation $-\dot{\lambda}_j = f_{x_j}(t, \mathbf{x}, \mathbf{u}) + \sum_{n=1}^{N} \lambda_n g^n_{x_j}(t, \mathbf{x}, \mathbf{u})$, $j = 1, 2, \ldots, N$. Because λ_j is the shadow value of the jth stock, $\dot{\lambda}_j$ may be thought of as the rate at which the shadow value of the jth stock is changing or appreciating through time. Thus, $-\dot{\lambda}_j$ is the rate at which the marginal value of the jth stock is depreciating through time. The costate equation therefore asserts that along the optimal path, the decrease in the marginal value of the jth capital stock at any point in time is the sum of jth capital stock's marginal contribution to profit, videlicet, $f_{x_j}(t, \mathbf{x}, \mathbf{u})$, and its marginal contribution to the enhancement of the value of all the capital stocks, to wit, $\sum_{n=1}^{N} \lambda_n g^n_{x_j}(t, \mathbf{x}, \mathbf{u})$.

With the economic interpretation of Theorem 4.2 now complete, let's consider a corollary to Theorem 4.2 that applies to a large class of optimal control problems that is often encountered in economic theory. Consider, therefore, a situation in which $\mathbf{v}(t) \in \operatorname{int} U\ \forall\, t \in [t_0, t_1]$, that is, the optimal solution for the control variables (assuming one exists) is in the interior of the control region for the entire planning horizon. Equivalently, assume that the constraints on the control variables are not binding on the optimal control for the entire planning horizon. Alternatively, one could more strongly assume that $U = \Re^M$, thereby implying that there are no constraints on the control variables. Under either of these two assumptions, the Maximum Principle of Theorem 4.2, namely, $H(t, \mathbf{z}(t), \mathbf{v}(t), \boldsymbol{\lambda}(t)) \geq H(t, \mathbf{z}(t), \mathbf{u}, \boldsymbol{\lambda}(t)) \, \forall \, \mathbf{u} \in U$, implies the first-order necessary condition

$$H_{\mathbf{u}}(t, \mathbf{z}(t), \mathbf{v}(t), \boldsymbol{\lambda}(t))' = \mathbf{0}_M \, \forall \, t \in [t_0, t_1],$$

as well as the second-order necessary condition

$$\mathbf{h}' H_{\mathbf{uu}}(t, \mathbf{z}(t), \mathbf{v}(t), \boldsymbol{\lambda}(t))\mathbf{h} \leq 0 \, \forall \, t \in [t_0, t_1] \quad \text{and} \quad \forall \, \mathbf{h} \in \Re^M,$$

assuming, of course, that $H(\cdot) \in C^{(2)}$ in $\mathbf{u}$. The proof of these necessary conditions follows immediately from the first-order necessary and second-order necessary conditions from static optimization theory, since (i) we are choosing $\mathbf{u} \in U$ at each $t \in [t_0, t_1]$ to maximize $H(\cdot)$, and (ii) the constraints are either not binding on the optimal control or simply do not exist under the above two scenarios. We have therefore proven the following economically important corollary to Theorem 4.2.

Corollary 4.2 (Simplified Maximum Principle): *Let* $(\mathbf{z}(t), \mathbf{v}(t))$ *be an admissible pair for problem (1). Then if* $\mathbf{v}(t) \in \operatorname{int} U \; \forall t \; \in [t_0, t_1]$, $(\mathbf{z}(t), \mathbf{v}(t))$ *yields the absolute maximum of* $J[\cdot]$, *and* $H(\cdot) \in C^{(1)}$ *in* $\mathbf{u}$, *it is necessary that there exists a piecewise smooth vector-valued function* $\boldsymbol{\lambda}(\cdot) \stackrel{\text{def}}{=} (\lambda_1(\cdot), \lambda_2(\cdot), \ldots, \lambda_N(\cdot))$ *such that for all* $t \in [t_0, t_1]$,

$$H_{u_m}(t, \mathbf{z}(t), \mathbf{v}(t), \boldsymbol{\lambda}(t)) = 0, \quad m = 1, 2, \ldots, M,$$

and

$$\sum_{\ell=1}^{M} \sum_{m=1}^{M} H_{u_\ell u_m}(t, \mathbf{z}(t), \mathbf{v}(t), \boldsymbol{\lambda}(t)) h_\ell h_m \leq 0 \, \forall \, \mathbf{h} \in \Re^M.$$

Furthermore, except for the points of discontinuities of $\mathbf{v}(t)$,

$$\dot{z}_n(t) = \frac{\partial H}{\partial \lambda_n}(t, \mathbf{z}(t), \mathbf{v}(t), \boldsymbol{\lambda}(t)) = g^n(t, \mathbf{z}(t), \mathbf{v}(t)), \quad n = 1, 2, \ldots, N,$$

$$\dot{\lambda}_n(t) = -\frac{\partial H}{\partial x_n}(t, \mathbf{z}(t), \mathbf{v}(t), \boldsymbol{\lambda}(t)), \quad n = 1, 2, \ldots, N.$$

We now pause to illustrate how to use Corollary 4.2 with a simple mathematical example.

Example 4.2: Consider the optimal control problem

$$\max_{u(\cdot)} \int_0^1 \left[-x(t) - \frac{1}{2}\alpha[u(t)]^2 \right] dt$$

$$\text{s.t.} \quad \dot{x}(t) = u(t), \; x(0) = x_0, \; x(1) = x_1,$$

where $\alpha > 0$ is a parameter. Define the Hamiltonian as $H(x, u, \lambda) \stackrel{\text{def}}{=} -x - \frac{1}{2}\alpha u^2 + \lambda u$. Given that there are no constraints on the control variable, the necessary conditions are given by Corollary 4.2; hence

$$H_u(x, u, \lambda) = -\alpha u + \lambda = 0,\ H_{uu}(x, u, \lambda) = -\alpha \le 0.$$

Because $\alpha > 0$, it follows that $H_{uu}(x, u, \lambda) = -\alpha < 0$. Solving $H_u(x, u, \lambda) = -\alpha u + \lambda = 0$ for the control variable gives $u = \alpha^{-1}\lambda$. The other necessary conditions are thus

$$\dot{x} = H_\lambda(x, u, \lambda) = u,\ x(0) = x_0,\ x(1) = x_1,$$
$$\dot{\lambda} = -H_x(x, u, \lambda) = 1.$$

Substituting $u = \alpha^{-1}\lambda$ into these differential equations yields

$$\dot{x} = \alpha^{-1}\lambda,\ x(0) = x_0,\ x(1) = x_1,$$
$$\dot{\lambda} = 1.$$

Separating the variables in $\dot{\lambda} = 1$ and integrating gives the general solution $\lambda(t) = t + c_1$, where c_1 is a constant of integration. Substituting $\lambda(t) = t + c_1$ into $\dot{x} = \alpha^{-1}\lambda$ gives $\dot{x} = \alpha^{-1}[t + c_1]$, and separating the variables and integrating yields the general solution $x(t) = \frac{1}{2}\alpha^{-1}t^2 + \alpha^{-1}c_1 t + c_2$, where c_2 is another constant of integration. The constants of integration c_1 and c_2 are determined by using the initial and terminal conditions $x(0) = x_0$ and $x(1) = x_1$:

$$x(0) = x_0 \Rightarrow c_2 = x_0,$$
$$x(1) = x_1 \Rightarrow c_1 = \alpha[x_1 - x_0] - \frac{1}{2}.$$

These constants of integration are then substituted into the general solution for the state and costate variables found above to get the specific solutions

$$z(t; \alpha, x_0, x_1) = \frac{1}{2}\alpha^{-1}t^2 + \left[x_1 - x_0 - \frac{1}{2}\alpha^{-1}\right]t + x_0,$$
$$\lambda(t; \alpha, x_0, x_1) = t + \alpha[x_1 - x_0] - \frac{1}{2}.$$

Finally, substitute $\lambda(t; \alpha, x_0, x_1) = t + \alpha[x_1 - x_0] - \frac{1}{2}$ into $u = \alpha^{-1}\lambda$ to find the path of the control variable that satisfies the necessary conditions:

$$v(t; \alpha, x_0, x_1) = \alpha^{-1}\left[t - \frac{1}{2}\right] + x_1 - x_0.$$

Notice that the solution to the necessary conditions depends explicitly on the parameters (α, x_0, x_1) of the control problem. The above solution to the necessary conditions should be compared to that of Example 2.1, since the control problems are identical in the two examples, save for a fixed terminal state in the present example.

The ensuing example has a closed and bounded (i.e., compact) control region. As a result, we must use Theorem 4.2 in order to derive the correct necessary conditions. Moreover, the optimal control problem is linear in the control variable and thus must be handled differently if one is to solve the necessary conditions.

Example 4.3: Consider the following optimal control problem:

$$\max_{u(\cdot),x_1} J[x(\cdot), u(\cdot)] \stackrel{\text{def}}{=} \int_0^1 x(t)\,dt$$

$$\text{s.t.} \quad \dot{x}(t) = x(t) + u(t),\ x(0) = 0,\ x(1) = x_1,$$

$$u(t) \in U \stackrel{\text{def}}{=} \{u(\cdot) : -1 \leq u(t) \leq 1\}.$$

Here we have an explicit representation of the control set U, a *closed interval* on the real line. The fact that U is a closed set (i.e., it includes its boundary as part of the set) is important, for this means the optimal value of the control may be at the boundary, and thus the derivative form of the Maximum Principle, namely, $H_u(t, x, u, \lambda) = 0$, is not in general valid. Hence, by Theorem 4.2, the necessary condition is to choose u to maximize the Hamiltonian subject to $u \in U$.

The Hamiltonian is defined as $H(x, u, \lambda) \stackrel{\text{def}}{=} x + \lambda[x + u] = x[1 + \lambda] + \lambda u$. The optimal path of the costate variable must necessarily satisfy $\dot{\lambda} = -H_x(x, u, \lambda) = -1 - \lambda$ and $\lambda(1) = 0$, the latter of which results from the fact that x_1 is a decision variable. The integrating factor for this linear ordinary differential equation with constant coefficients is $\mu(t) \stackrel{\text{def}}{=} \exp[\int^t ds] = e^t$, so upon multiplying both sides of $\dot{\lambda} + \lambda = -1$ by e^t, it follows that the resulting differential equation is equivalent to $\frac{d}{dt}[\lambda e^t] = -e^t$. Upon integrating, we find that $\lambda(t) = ce^{-t} - 1$, where c is a constant of integration. The free boundary condition $\lambda(1) = 0$ implies that $c = e$. Thus the specific solution to the costate equation is $\lambda(t) = e^{1-t} - 1$, so the costate path is fully determined. Finally, because $\dot{\lambda}(t) = -e^{1-t} < 0\ \forall\, t \in [0, 1]$ and $\lambda(1) = 0$, we conclude that $\lambda(t) > 0\ \forall\, t \in [0, 1)$. This fact is important in what follows.

Now let's examine the Hamiltonian $H(x, u, \lambda) \stackrel{\text{def}}{=} x[1 + \lambda] + \lambda u$. Seeing as the term λu in the Hamiltonian is the only one that depends on u, maximizing the Hamiltonian with respect to u subject to $u \in [-1, 1]$ is equivalent to maximizing λu with respect to u subject to $u \in [-1, 1]$. Because $\lambda(t) > 0\ \forall\, t \in [0, 1)$, the maximum of λu (and thus $H(x, u, \lambda)$) with respect to u subject to $u \in [-1, 1]$ is attained at $u = v(t) = 1$. From the transversality condition $\lambda(1) = 0$, it follows that the Hamiltonian is independent of the control variable at $t = 1$, thereby implying that maximizing $H(\cdot)$ with respect to u subject to $u \in [-1, 1]$ does not determine the value of the control at $t = 1$. As a result, the choice of the control at $t = 1$ is arbitrary. We therefore set $v(1) = 1$ so that the control function $v(\cdot)$ is continuous on $[-1, 1]$. In sum, the control that maximizes the Hamiltonian is $v(t) = 1\, \forall\, t \in [0, 1]$.

Given this control, the state variable must satisfy the differential equation $\dot{x} - x = 1$ and the initial condition $x(0) = 0$. The integrating factor of this linear first-order differential equation is $\gamma(t) \stackrel{\text{def}}{=} \exp[-\int^t ds] = e^{-t}$, so that upon multiplying both sides of $\dot{x} - x = 1$ by e^{-t}, it follows that the resulting differential equation is equivalent to $\frac{d}{dt}[xe^{-t}] = e^{-t}$. Upon integrating, we have the general solution $x(t) = ke^t - 1$, where k is a constant of integration. The initial condition $x(0) = 0$ implies that $k = 1$; hence $z(t) = e^t - 1$ is the specific solution of the state equation. Note also that the value of the objective functional is

$$J[z(\cdot), v(\cdot)] = \int_0^1 [e^t - 1]\, dt = [e^t - t]\Big|_{t=0}^{t=1} = e - 2 > 0.$$

Because we don't have as of yet any sufficiency theorems for this class of control problems, we will have to wait to conclude that our solution of the necessary conditions actually is the solution of the optimal control problem.

It is natural at this juncture to complement the necessary conditions of Theorem 4.2 with a set of sufficient conditions that also apply directly to optimal control problem (1). The proof of the following sufficient conditions differs from our earlier proof of sufficient conditions in Chapter 3, however, because now we have to be careful to properly account for the presence of the control region U in our analysis, just as we did in the proof of Theorem 4.2. We begin, therefore, with a preliminary technical result that is of value in the proof of the sufficiency theorem.

Lemma 4.1: *Let $F(\cdot) : A \to \Re$, where $A \subseteq \Re^N$, and let $F(\cdot) \in C^{(1)}$ on A. Also, let $S \in \operatorname{int} A$ be a convex set and $\bar{\mathbf{x}} \in S$.*

(i) $F(\bar{\mathbf{x}}) \geq F(\mathbf{x})\, \forall\, \mathbf{x} \in S \Rightarrow F_{\mathbf{x}}(\bar{\mathbf{x}})\, [\mathbf{x} - \bar{\mathbf{x}}] \leq 0\, \forall\, \mathbf{x} \in S$.

(ii) If $F(\cdot)$ is concave $\forall\, \mathbf{x} \in S$, then

$$F_{\mathbf{x}}(\bar{\mathbf{x}})\, [\mathbf{x} - \bar{\mathbf{x}}] \leq 0\, \forall\, \mathbf{x} \in S \Rightarrow F(\bar{\mathbf{x}}) \geq F(\mathbf{x})\, \forall\, \mathbf{x} \in S.$$

Proof: To prove part (i), first define $G(\alpha) \stackrel{\text{def}}{=} F(\bar{\mathbf{x}}) - F(\alpha\mathbf{x} + (1-\alpha)\bar{\mathbf{x}})$ for $\alpha \in [0, 1]$. Given that S is a convex set, $\alpha\mathbf{x} + (1-\alpha)\bar{\mathbf{x}} \in S\, \forall\, \mathbf{x} \in S$ and $\alpha \in [0, 1]$. Thus $F(\bar{\mathbf{x}}) \geq F(\mathbf{x})\, \forall\, \mathbf{x} \in S$ is equivalent to the statement $G(\alpha) \stackrel{\text{def}}{=} F(\bar{\mathbf{x}}) - F(\alpha\mathbf{x} + (1-\alpha)\bar{\mathbf{x}}) \geq 0$ for $\alpha \in [0, 1]$. Because $G(\alpha) \geq 0$ for all $\alpha \in [0, 1]$ and $G(0) = F(\bar{\mathbf{x}}) - F(\bar{\mathbf{x}}) = 0$, it follows that $G'(0) \geq 0$ must hold; hence

$$G'(0) = -F_{\mathbf{x}}(\bar{\mathbf{x}})\, [\mathbf{x} - \bar{\mathbf{x}}] \geq 0.$$

Multiplying through by minus unity completes the proof of part (i).

https://doi.org/10.1017/CBO9780511806827.006 Published online by Cambridge University Press

To prove part (ii), first note that by Theorem 21.3 of Simon and Blume (1994), we have the familiar inequality fundamental to concave functions, namely,

$$F(\mathbf{x}) \le F(\bar{\mathbf{x}}) + F_{\mathbf{x}}(\bar{\mathbf{x}})\,[\mathbf{x} - \bar{\mathbf{x}}]\,\forall\,\mathbf{x} \in S.$$

Because $F_{\mathbf{x}}(\bar{\mathbf{x}})\,[\mathbf{x} - \bar{\mathbf{x}}] \le 0\,\forall\,\mathbf{x} \in S$ by hypothesis, the above inequality immediately gives the desired conclusion, to wit, $F(\bar{\mathbf{x}}) \ge F(\mathbf{x})\,\forall\,\mathbf{x} \in S$. Q.E.D.

With Lemma 4.1 in hand, we may proceed to prove the ensuing sufficiency theorem for the class of control problems we have been considering in this chapter.

Theorem 4.3 (Mangasarian Sufficient Conditions): *Let* $(\mathbf{z}(t), \mathbf{v}(t))$ *be an admissible pair for problem (1). Assume that* U *is a convex set and that* $\partial f(\cdot)/\partial\mathbf{u}$ *and* $\partial\mathbf{g}(\cdot)/\partial\mathbf{u}$ *exist and are continuous, in addition to our basic assumptions on* $f(\cdot)$ *and* $\mathbf{g}(\cdot)$*. Suppose that* $(\mathbf{z}(t), \mathbf{v}(t))$ *satisfies the necessary conditions of Theorem 4.2 for problem (1) with costate vector* $\boldsymbol{\lambda}(t)$*, and let* $H(t, \mathbf{x}, \mathbf{u}, \boldsymbol{\lambda}) \overset{\text{def}}{=} f(t, \mathbf{x}, \mathbf{u}) + \boldsymbol{\lambda}'\mathbf{g}(t, \mathbf{x}, \mathbf{u})$ *be the value of the Hamiltonian function. If* $H(\cdot)$ *is a concave function of* $(\mathbf{x}, \mathbf{u})\,\forall\,t \in [t_0, t_1]$ *over an open convex set containing all the admissible values of* $(\mathbf{x}(\cdot), \mathbf{u}(\cdot))$ *when the costate vector is* $\boldsymbol{\lambda}(t)$*, then* $\mathbf{v}(t)$ *is an optimal control and* $(\mathbf{z}(t), \mathbf{v}(t))$ *yields the global maximum of* $J[\cdot]$*. If* $H(\cdot)$ *is a strictly concave function under the same conditions, then* $(\mathbf{z}(t), \mathbf{v}(t))$ *yields the unique global maximum of* $J[\cdot]$*.*

Proof: Let $(\mathbf{x}(t), \mathbf{u}(t))$ be any admissible pair. By hypothesis, $H(\cdot)$ is a $C^{(1)}$ concave function of $(\mathbf{x}, \mathbf{u})\,\forall\,t \in [t_0, t_1]$. It therefore follows from Theorem 21.3 of Simon and Blume (1994) that

$$\begin{aligned} H(t, \mathbf{x}(t), \mathbf{u}(t), \boldsymbol{\lambda}(t)) &\le H(t, \mathbf{z}(t), \mathbf{v}(t), \boldsymbol{\lambda}(t)) + H_{\mathbf{x}}(t, \mathbf{z}(t), \mathbf{v}(t), \boldsymbol{\lambda}(t))\,[\mathbf{x}(t) - \mathbf{z}(t)] \\ &\quad + H_{\mathbf{u}}(t, \mathbf{z}(t), \mathbf{v}(t), \boldsymbol{\lambda}(t))\,[\mathbf{u}(t) - \mathbf{v}(t)], \end{aligned} \tag{28}$$

for every $t \in [t_0, t_1]$. Integrating both sides of the above inequality over the interval $[t_0, t_1]$ and using the definitions of $H(\cdot)$ and $J[\cdot]$ yields

$$\begin{aligned} J[\mathbf{x}(\cdot), \mathbf{u}(\cdot)] \le J[\mathbf{z}(\cdot), \mathbf{v}(\cdot)] &+ \int_{t_0}^{t_1} \boldsymbol{\lambda}(t)'[\mathbf{g}(t, \mathbf{z}(t), \mathbf{v}(t)) - \mathbf{g}(t, \mathbf{x}(t), \mathbf{u}(t))]\,dt \\ &+ \int_{t_0}^{t_1} H_{\mathbf{x}}(t, \mathbf{z}(t), \mathbf{v}(t), \boldsymbol{\lambda}(t))\,[\mathbf{x}(t) - \mathbf{z}(t)]\,dt \\ &+ \int_{t_0}^{t_1} H_{\mathbf{u}}(t, \mathbf{z}(t), \mathbf{v}(t), \boldsymbol{\lambda}(t))\,[\mathbf{u}(t) - \mathbf{v}(t)]\,dt. \end{aligned} \tag{29}$$

By admissibility, $\dot{\mathbf{z}}(t) \equiv \mathbf{g}(t, \mathbf{z}(t), \mathbf{v}(t))$ and $\dot{\mathbf{x}}(t) \equiv \mathbf{g}(t, \mathbf{x}(t), \mathbf{u}(t))$ for every $t \in [t_0, t_1]$, whereas Theorem 4.2 implies that $\dot{\boldsymbol{\lambda}}(t)' \equiv -H_{\mathbf{x}}(t, \mathbf{z}(t), \mathbf{v}(t), \boldsymbol{\lambda}(t))$ for every $t \in [t_0, t_1]$. Substituting these three results in Eq. (29) gives

$$\begin{aligned} J[\mathbf{x}(\cdot), \mathbf{u}(\cdot)] \leq J[\mathbf{z}(\cdot), \mathbf{v}(\cdot)] &+ \int_{t_0}^{t_1} [\boldsymbol{\lambda}(t)'[\dot{\mathbf{z}}(t) - \dot{\mathbf{x}}(t)] + \dot{\boldsymbol{\lambda}}(t)'[\mathbf{z}(t) - \mathbf{x}(t)]]\, dt \\ &+ \int_{t_0}^{t_1} H_{\mathbf{u}}(t, \mathbf{z}(t), \mathbf{v}(t), \boldsymbol{\lambda}(t))\, [\mathbf{u}(t) - \mathbf{v}(t)]\, dt. \end{aligned} \tag{30}$$

Because $H(\cdot)$ is concave in $(\mathbf{x}, \mathbf{u})\,\forall\, t \in [t_0, t_1]$, it is also concave in $\mathbf{u}$ alone for each $t \in [t_0, t_1]$. Thus, by Lemma 4.1, we have that

$$\begin{aligned} &H_{\mathbf{u}}(t, \mathbf{z}(t), \mathbf{v}(t), \boldsymbol{\lambda}(t))\, [\mathbf{u}(t) - \mathbf{v}(t)] \leq 0\, \forall\, \mathbf{u} \in U \\ &\quad \Leftrightarrow H(t, \mathbf{z}(t), \mathbf{v}(t), \boldsymbol{\lambda}(t)) \geq H(t, \mathbf{z}(t), \mathbf{u}, \boldsymbol{\lambda}(t))\, \forall\, \mathbf{u} \in U. \end{aligned}$$

Using this fact in Eq. (30) permits the conclusion

$$J[\mathbf{x}(\cdot), \mathbf{u}(\cdot)] \leq J[\mathbf{z}(\cdot), \mathbf{v}(\cdot)] + \int_{t_0}^{t_1} [\boldsymbol{\lambda}(t)'[\dot{\mathbf{z}}(t) - \dot{\mathbf{x}}(t)] + \dot{\boldsymbol{\lambda}}(t)'[\mathbf{z}(t) - \mathbf{x}(t)]]\, dt. \tag{31}$$

To wrap up the proof, simply note that

$$\begin{aligned} \frac{d}{dt}[\boldsymbol{\lambda}(t)'[\mathbf{z}(t) - \mathbf{x}(t)]] &= \boldsymbol{\lambda}(t)'[\dot{\mathbf{z}}(t) - \dot{\mathbf{x}}(t)] + [\mathbf{z}(t) - \mathbf{x}(t)]'\dot{\boldsymbol{\lambda}}(t) \\ &= \boldsymbol{\lambda}(t)'[\dot{\mathbf{z}}(t) - \dot{\mathbf{x}}(t)] + \dot{\boldsymbol{\lambda}}(t)'[\mathbf{z}(t) - \mathbf{x}(t)], \end{aligned}$$

and substitute this result into Eq. (31) to get

$$\begin{aligned} J[\mathbf{x}(\cdot), \mathbf{u}(\cdot)] &\leq J[\mathbf{z}(\cdot), \mathbf{v}(\cdot)] + \int_{t_0}^{t_1} \frac{d}{dt}[\boldsymbol{\lambda}(t)'[\mathbf{z}(t) - \mathbf{x}(t)]]\, dt \\ &= J[\mathbf{z}(\cdot), \mathbf{v}(\cdot)] + \boldsymbol{\lambda}(t)'\,[\mathbf{z}(t) - \mathbf{x}(t)]\Big|_{t=t_0}^{t=t_1} \\ &= J[\mathbf{z}(\cdot), \mathbf{v}(\cdot)], \end{aligned}$$

since by admissibility, we have $\mathbf{x}(t_0) = \mathbf{x}_0$, $\mathbf{z}(t_0) = \mathbf{x}_0$, $\mathbf{x}(t_1) = \mathbf{x}_1$, and $\mathbf{z}(t_1) = \mathbf{x}_1$. We have thus shown that $J[\mathbf{x}(\cdot), \mathbf{u}(\cdot)] \leq J[\mathbf{z}(\cdot), \mathbf{v}(\cdot)]$ for all admissible functions $(\mathbf{x}(\cdot), \mathbf{u}(\cdot))$, just as we wished to. If $H(\cdot)$ is strictly concave in $(\mathbf{x}, \mathbf{u})\,\forall\, t \in [t_0, t_1]$, then the inequality in Eq. (28) becomes strict if either $\mathbf{x}(t) \neq \mathbf{z}(t)$ or $\mathbf{u}(t) \neq \mathbf{v}(t)$ for some $t \in [t_0, t_1]$. Then $J[\mathbf{x}(\cdot), \mathbf{u}(\cdot)] < J[\mathbf{z}(\cdot), \mathbf{v}(\cdot)]$ follows. This shows that any admissible pair of functions $(\mathbf{x}(\cdot), \mathbf{u}(\cdot))$ that are not identically equal to $(\mathbf{z}(\cdot), \mathbf{v}(\cdot))$ are suboptimal. Q.E.D.

Let's pause briefly so we can see this theorem in action.

Example 4.4: We begin by examining Example 4.2. Recall that this control problem does not have any constraints on the control variable, that is, $U = \Re$, a convex set. Also recall that

$$z(t; \alpha, x_0, x_1) = \frac{1}{2}\alpha^{-1}t^2 + \left[x_1 - x_0 - \frac{1}{2}\alpha^{-1}\right]t + x_0,$$

$$v(t; \alpha, x_0, x_1) = \alpha^{-1}\left[t - \frac{1}{2}\right] + x_1 - x_0,$$

$$\lambda(t; \alpha, x_0, x_1) = t + \alpha[x_1 - x_0] - \frac{1}{2}$$

is the *only* solution of the necessary conditions. We will now show that the Hamiltonian for this control problem, scilicet, $H(x, u, \lambda) \stackrel{\text{def}}{=} -x - \frac{1}{2}\alpha u^2 + \lambda u$, is a concave function of (x, u) for all $t \in [0, 1]$. To check concavity, we compute the Hessian matrix of $H(\cdot)$:

$$\begin{bmatrix} H_{uu} & H_{ux} \\ H_{xu} & H_{xx} \end{bmatrix} = \begin{bmatrix} -\alpha & 0 \\ 0 & 0 \end{bmatrix}.$$

Because the eigenvalues are $-\alpha < 0$ and 0, the Hessian matrix is negative semidefinite and therefore $H(\cdot)$ is concave in (x, u) for all $t \in [0, 1]$. Thus the above triplet is the unique solution to the posed control problem. Note that since $H(\cdot)$ is a polynomial, the continuity conditions of Theorem 4.3 are satisfied.

Now turn to Example 4.3. In this example, the control region $U \stackrel{\text{def}}{=} [-1, 1]$ is a closed and bounded convex set. Given that the Hamiltonian $H(x, u, \lambda) \stackrel{\text{def}}{=} x + \lambda[x + u]$ is linear in (x, u), it is concave in (x, u) for all $t \in [0, 1]$. The continuity conditions of Theorem 4.3 are satisfied because of the aforementioned linearity. Seeing as $v(t) = 1$, $z(t) = e^t - 1$, and $\lambda(t) = e^{1-t} - 1$ satisfy all the necessary conditions of Theorem 4.2 and are unique except at $t = 1$ when any admissible value of the control variable is optimal, Theorem 4.3 implies that this triplet is indeed the unique solution of the control problem.

With the mechanics of Theorems 4.2 and 4.3 in place, let's now analyze the inventory accumulation problem presented in Example 1.4 using these theorems. If your memory of this optimal control model is a bit fuzzy, you should return to Example 1.4 right now and refresh it.

https://doi.org/10.1017/CBO9780511806827.006 Published online by Cambridge University Press

Example 4.5: Referring to Example 1.4, the optimal control problem that is to be solved to determine the optimal inventory accumulation policy is given by

$$\min_{u(\cdot)} J[x(\cdot), u(\cdot)] \stackrel{\text{def}}{=} \int_0^T \left[c_1[u(t)]^2 + c_2 x(t)\right] dt$$

$$\text{s.t.} \quad \dot{x}(t) = u(t),$$

$$x(0) = 0,\ x(T) = x_T,$$

where $x(t)$ is the stock of inventory at time t, $u(t)$ is the production rate at time t, and $x_T > 0$ is the contracted (and thus fixed) amount of the good that must be delivered by the terminal date of the planning horizon $T > 0$. Recall that $c_1 > 0$ is a constant that shifts the average and marginal cost of production and $c_2 > 0$ is the constant unit cost of holding inventory. The fundamental economic trade-off in this control problem is between the cost of producing the good and the cost of storing it. For example, if the firm's manager produces a relatively large quantity of the good early in the planning horizon, then those goods must be held in inventory for a relatively long period of time, thereby driving up the inventory holding cost. On the other hand, if the manager waited until later in the planning horizon to produce a large quantity of the good, then inventory cost would be relatively low but production cost would be relatively high because of the "rush" at the end of the planning period to produce the contracted amount $x_T > 0$. The optimal production plan balances these two costs. Note that the production rate is not constrained to be nonnegative in the above statement of the control problem. We thus assume that it is not binding in the optimal plan. Later in this example, we will reexamine this assumption. Furthermore, after we introduce two additional theorems on necessary and sufficient conditions, we will explicitly account for the nonnegativity constraint on the production rate and examine how it affects the optimal solution.

The Hamiltonian for the control problem is defined as $H(x, u, \lambda) \stackrel{\text{def}}{=} c_1 u^2 + c_2 x + \lambda u$, where λ is the shadow *cost* of the inventory, for we are dealing with a cost minimization problem. Because we have assumed that the nonnegativity constraint on the production rate is not binding, the necessary conditions are given by Corollary 4.2, and therefore read

$$H_u(x, u, \lambda) = 2c_1 u + \lambda = 0, \tag{32}$$

$$H_{uu}(x, u, \lambda) = 2c_1 \geq 0, \tag{33}$$

$$\dot{\lambda} = -H_x(x, u, \lambda) = -c_2, \tag{34}$$

$$\dot{x} = H_\lambda(x, u, \lambda) = u,\ x(0) = 0,\ x(T) = x_T. \tag{35}$$

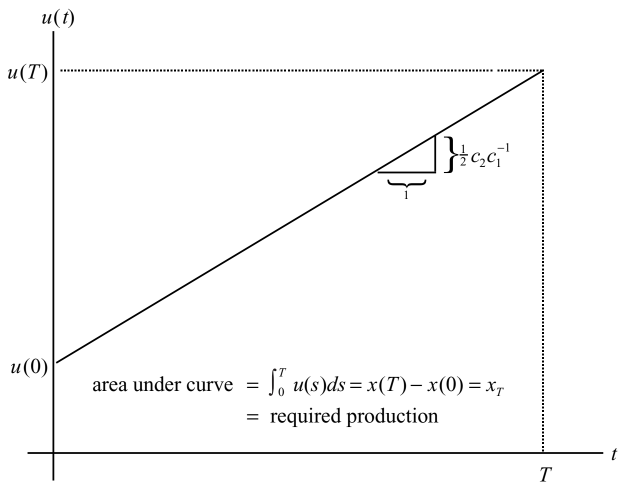

Figure 4.3

Because $c_2 > 0$, it follows that Eq. (33) is satisfied with a strict inequality, that is to say, $H_{uu}(x, u, \lambda) = 2c_1 > 0$ (remember that we are dealing with a minimization problem here).

Without even solving the necessary conditions, we can glean some economic information about the pattern of production. To see this, first solve Eq. (32) for the production rate to get $u = -\lambda/(2c_1)$, and then differentiate this expression with respect to t and use Eq. (34) to arrive at $\dot{u} = c_2/(2c_1) > 0$. This shows that the production rate rises over the planning horizon. Thus the firm produces the fewest units of the good per unit of time at the beginning of the planning horizon, and the most units of the good per unit of time at the end of the planning horizon. Figure 4.3 displays this qualitative conclusion graphically. Note that the slope of the curve in Figure 4.3 is given by the right-hand side of $\dot{u} = c_2/(2c_1)$, which is a positive constant.

Examination of Eq. (34) similarly reveals qualitative information. In particular, we see that the shadow cost of inventory falls over the planning horizon. This is not at all surprising, for an additional unit of stock received at the beginning of the planning period must be held in inventory longer (and is thus more costly) than one received near the end of the planning horizon. The declining shadow cost of the stock is essentially dual to the rising production rate, for one can interpret the former as driving the latter, or the latter as driving the former. Also noteworthy is the fact that because we assumed that the nonnegativity constraint on the production rate is not binding, the shadow cost of inventory is *negative*. That this is true can be seen from Eq. (32), or equivalently, from $u = -\lambda/(2c_1)$. This observation means that one more unit of stock in inventory lowers the total cost of production, which is not at all surprising seeing as one more unit of the good in inventory implies that one less unit of the good must be produced in order to reach the terminal stock of x_T.

Let's now solve the necessary conditions. Separating the variables and integrating the costate equation (34) gives the general solution $\lambda(t) = k_1 - c_2 t$, where k_1 is a constant of integration. Substituting this expression in Eq. (32) yields $u = (c_2 t - k_1)/(2c_1)$, which when substituted in the state equation (35), produces $\dot{x} = (c_2 t - k_1)/(2c_1)$. Separating the variables and integrating the state equation gives the general solution for the inventory stock $x(t) = \frac{1}{4}c_2 c_1^{-1} t^2 - \frac{1}{2}c_1^{-1} k_1 t + k_2$, where k_2 is another constant of integration. Now use the two boundary conditions to find the two constants of integration:

$$x(0) = 0 \;\Rightarrow\; k_2 = 0,$$

$$x(T) = x_T \;\Rightarrow\; k_1 = \frac{1}{2}c_2 T - 2c_1 x_T T^{-1}.$$

Next substitute the values of k_1 and k_2 into the general solution for the inventory stock to find the specific or definite solution that satisfies the boundary conditions:

$$z(t; c_1, c_2, T, x_T) = \frac{1}{4}c_2 c_1^{-1} t[t - T] + x_T T^{-1} t. \tag{36}$$

The corresponding production rate may be found by using $u = (c_2 t - k_1)/(2c_1)$ or differentiating Eq. (26) with respect to t, since $\dot{x} = u$:

$$v(t; c_1, c_2, T, x_T) = \frac{1}{4}c_2 c_1^{-1}[2t - T] + x_T T^{-1}. \tag{37}$$

Finally, the time path of the shadow cost of the inventory stock is found by substituting the value of k_1 in $\lambda(t) = k_1 - c_2 t$ to arrive at

$$\lambda(t; c_1, c_2, T, x_T) = \frac{1}{2}c_2[T - 2t] - 2c_1 x_T T^{-1}. \tag{38}$$

It is important to emphasize that the solution to the necessary conditions depends explicitly on the parameters $\boldsymbol{\alpha} \stackrel{\text{def}}{=} (c_1, c_2, T, x_T)$ of the inventory accumulation problem, just as the solution to a static optimization problem does. Unlike the solution to a static optimization problem, however, the solution to the necessary conditions in general depends on the independent variable time as well, thus indicating that even if the parameters are constant over the planning horizon, the inventory level and the production rate will in general vary over the planning period.

Before studying the comparative dynamics properties of the above solution, let's confirm that it is in fact the solution to the posed optimal control problem. Because we are dealing with a minimization problem and we have assumed that the nonnegativity constraint on the production rate is not binding, all we have to do is verify that the Hamiltonian is a convex function of the state and control variables. Inspection of the optimal control problem shows that the integrand function is convex in (x, u) and the transition function is linear in (x, u); hence by the analogue of Theorem 3.2 for convex functions, we may conclude that the Hamiltonian is a convex function of (x, u). As a result, by the analogue of Theorem 4.3 for minimization problems, we know that the above solution to the necessary conditions is a solution of the optimal control problem. Actually, we can claim a bit more. In view of the fact that the above solution of the necessary conditions is the only one, it is therefore the unique solution of the control problem. Recall, however, that this conclusion is true only if $v(t; \boldsymbol{\alpha}) \geq 0 \, \forall \, t \in [0, T]$.

Turning to the comparative dynamics, observe that because we have an explicit solution to the inventory accumulation problem, we may proceed to differentiate it with respect to the parameters of interest to conduct the comparative dynamics analysis. Let's therefore proceed to uncover and economically interpret some of the comparative dynamics properties of the unique solution, leaving the rest of them for a mental exercises.

The production rate $v(t; \boldsymbol{\alpha})$ and the shadow cost of the inventory stock $\lambda(t; \boldsymbol{\alpha})$ are the more interesting economic variables, so we will focus our attention on them rather than on the inventory stock. To begin the comparative dynamics analysis, partially differentiate the production rate given in Eq. (37), and the shadow cost of the inventory given in Eq. (38), with respect to the unit cost of holding inventory c_2 to find that

$$\frac{\partial v}{\partial c_2}(t; \boldsymbol{\alpha}) = \frac{1}{4} c_1^{-1} [2t - T] \begin{cases} < 0 \, \forall \, t \in [0, \frac{1}{2}T) \\ = 0 \text{ for } t = \frac{1}{2}T \\ > 0 \, \forall \, t \in (\frac{1}{2}T, T] \end{cases}, \tag{39}$$

$$\frac{\partial \lambda}{\partial c_2}(t; \boldsymbol{\alpha}) = \frac{1}{2} [T - 2t] \begin{cases} > 0 \, \forall \, t \in [0, \frac{1}{2}T) \\ = 0 \text{ for } t = \frac{1}{2}T \\ < 0 \, \forall \, t \in (\frac{1}{2}T, T] \end{cases}. \tag{40}$$

The first thing to notice about Eqs. (39) and (40) is that unlike a comparative statics result, a comparative dynamics result can change sign with the passage of time. The economic interpretation of Eq. (39) is straightforward and makes economic sense. Because inventory is now more costly to hold, the production rate is decreased in the first half of the planning horizon so that less inventory accumulates in this period. Because the firm must still have $x_T > 0$ units in inventory at the terminal time $t = T$, and T is fixed (i.e., the planning horizon is fixed), the production rate must increase in the second half of the planning horizon in order to make up for the lower production rate in the first half of the planning horizon. In this way, fewer units are held in inventory for a longer period of time, thereby partially offsetting the higher cost of holding inventory. Figure 4.4 depicts the effect of the increase in the inventory holding cost on the production rate. Notice that the production rate path pivots counterclockwise about the point $t = \frac{1}{2}T$ as a result of the higher inventory holding costs. Observe, however, that because the firm is required to always produce the same number of units of the good over the same period of time, the area under each time path, which is the required production, is the same in each case. In other words, the area between the two curves in Figure 4.4 must be the same, implying that Area 1 equals Area 2.

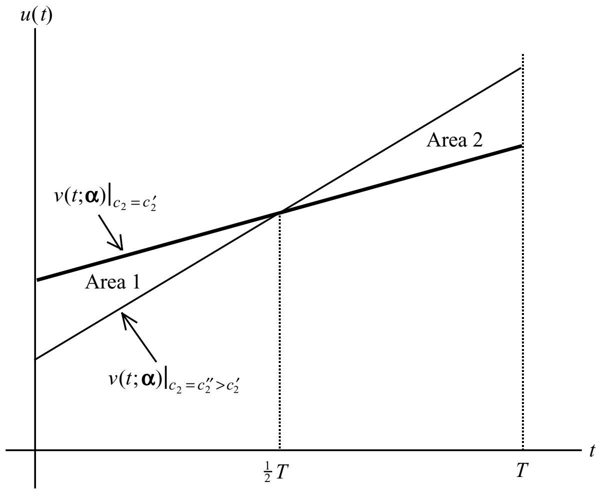

Figure 4.4

The effect of the increase in inventory holding costs on the shadow cost of inventory, given in Eq. (40), also has a straightforward economic interpretation. In this case, we see that an increase in the cost of holding inventory increases the shadow cost of inventory in the first half of the planning horizon and decreases it in the second half, just the opposite of its effect on the production rate. The economic explanation for this feature is essentially the same as that behind the declining value of the shadow cost of inventory over the planning horizon, namely, an additional unit of stock received at the beginning of the planning period must be held in inventory longer (and is thus more costly) than one received near the end of the planning horizon.

Now let's reexamine our assumption that the nonnegativity constraint on the production rate is not binding. We begin by checking to see if in fact the production rate $v(t;\boldsymbol{\alpha})$ satisfies the constraint $u(t) \geq 0 \,\forall\, t \in [0, T]$. Given that $\dot{v}(t;\boldsymbol{\alpha}) = \frac{1}{2}c_2c_1^{-1} > 0$, the production rate $v(t;\boldsymbol{\alpha})$ is an increasing function of time. Therefore, if the production rate at the initial date of the planning period is nonnegative, then the production rate is nonnegative throughout the planning period because it is a strictly increasing function of time. Conversely, if the production rate is nonnegative throughout the planning horizon, then it is surely nonnegative at the initial date of the planning period. Stated more formally, we've shown that $v(0;\boldsymbol{\alpha}) \geq 0$ if and only if $v(t;\boldsymbol{\alpha}) \geq 0 \,\forall\, t \in [0, T]$. Therefore, the nonnegativity constraint $u(t) \geq 0$ will be satisfied for all $t \in [0, T]$ if and only if $v(0;\boldsymbol{\alpha}) = x_T T^{-1} - \frac{1}{4}c_2c_1^{-1}T \geq 0$, or equivalently, if and only if

$$x_T \geq \frac{1}{4}c_2c_1^{-1}T^2. \tag{41}$$

Thus the pair of curves $(z(t;\boldsymbol{\alpha}), v(t;\boldsymbol{\alpha}))$ is the solution of the inventory accumulation problem if and only if Eq. (41) holds. The economic interpretation of the inequality in Eq. (41) is that the total production x_T is large relative to the delivery date T, and the unit storage cost c_2 is sufficiently small relative to the unit production cost coefficient c_1, if and only if the production rate is nonnegative throughout the planning period. If Eq. (41) does not hold, then the start of production is postponed in the optimal plan. We will investigate this situation in the subsequent example, in which we will explicitly take the nonnegativity constraint into account.

Recall that Theorem 4.2 was developed under the assumption that U is a *fixed* set, that is, U is *not* a function of the state variables or time. We will treat the case in which U is a function of the state variables in a later chapter. It turns out, however, that Theorem 4.2 actually covers the case in which U is a function of t, say, $U(t)$, provided a constraint qualification (to be detailed below) is satisfied. Let's endeavor to see why this is so.

At one level, it is easy to see why Theorem 4.2 applies when $U(t)$ is the control region: nothing in the proof of Theorem 4.2 depended on the fact that the control region U was independent of t. In actual applications of optimal control theory to economic problems, however, the set $U(t)$ is almost always defined by a system of inequality constraints, that is,

$$U(t) \stackrel{\text{def}}{=} \{\mathbf{u}(\cdot) : h^k(t, \mathbf{u}(t)) \geq 0, \quad k = 1, 2, \ldots, K\}. \tag{42}$$

For instance, in Example 4.3, we had that $u(t) \in U \stackrel{\text{def}}{=} \{u(t) : -1 \leq u(t) \leq 1\}$. This constraint can be rewritten to conform to Eq. (42) as follows:

$$h^1(t, u(t)) \stackrel{\text{def}}{=} u(t) + 1 \geq 0,$$
$$h^2(t, u(t)) \stackrel{\text{def}}{=} 1 - u(t) \geq 0.$$

For our purposes, therefore, there is no loss in generality in defining the control region by a system of inequality constraints. We will thus do so for the remainder of the textbook when constraints are considered. Because of this convention, we must therefore introduce a constraint qualification on the constraint functions $h^k(\cdot)$, $k = 1, 2, \ldots, K$, just as is done when studying static optimization problems with constraints.

The optimal control problem for which we now seek to develop necessary and sufficient conditions can be stated as follows:

$$\max_{\mathbf{u}(\cdot)} J[\mathbf{x}(\cdot), \mathbf{u}(\cdot)] \stackrel{\text{def}}{=} \int_{t_0}^{t_1} f(t, \mathbf{x}(t), \mathbf{u}(t))\, dt$$
$$\text{s.t.} \quad \dot{\mathbf{x}}(t) = \mathbf{g}(t, \mathbf{x}(t), \mathbf{u}(t)), \ \mathbf{x}(t_0) = \mathbf{x}_0, \ \mathbf{x}(t_1) = \mathbf{x}_1, \tag{43}$$
$$\mathbf{h}(t, \mathbf{u}(t)) \geq \mathbf{0}_K,$$

where $\mathbf{u}(\cdot) \stackrel{\text{def}}{=} (u_1(\cdot), u_2(\cdot), \ldots, u_M(\cdot))$, $\mathbf{g}(\cdot) \stackrel{\text{def}}{=} (g^1(\cdot), g^2(\cdot), \ldots, g^N(\cdot))$, $\dot{\mathbf{x}}(\cdot) \stackrel{\text{def}}{=} (\dot{x}_1(\cdot), \dot{x}_2(\cdot), \ldots, \dot{x}_N(\cdot))$, $\mathbf{h}(\cdot) \stackrel{\text{def}}{=} (h^1(\cdot), h^2(\cdot), \ldots, h^K(\cdot))$, and $\mathbf{0}_K$ is the K-element null vector in $\Re^K$. It is important to observe that the vector of constraints $\mathbf{h}(t, \mathbf{u}(t)) \geq \mathbf{0}_K$ in problem (43) is equivalent to the requirement $\mathbf{u}(t) \in U(t) \stackrel{\text{def}}{=} \{\mathbf{u}(\cdot) : h^k(t, \mathbf{u}(t)) \geq 0,\ k = 1, 2, \ldots, K\}$ from Eq. (42). Before stating the theorems, we first introduce a definition that is a prerequisite for the constraint qualification we shall adopt.

Definition 4.2: Let X be a set. Then card(X), the *cardinal number* of X, is the number of elements in X.

For finite sets, the only type that we will employ when using Definition 4.2, this is an elementary and straightforward concept. As usual, we let $(\mathbf{z}(\cdot), \mathbf{v}(\cdot))$ be an optimal pair of functions for problem (43). We may now state the constraint qualification of interest to us.

Rank Constraint Qualification: Define $\iota(t, \mathbf{v}(t)) \stackrel{\text{def}}{=} \{k : h^k(t, \mathbf{v}(t)) = 0,\ k = 1, 2, \ldots, K\}$ as the index set of the binding constraints along the optimal path. For all $t \in [t_0, t_1]$, if $\iota(t, \mathbf{v}(t)) \neq \emptyset$, that is, $\iota(t, \mathbf{v}(t))$ is nonempty, then the card$(\iota(t, \mathbf{v}(t))) \times M$ Jacobian matrix

$$\left[\frac{\partial h^k}{\partial u_m}(t, \mathbf{v}(t)) \right]_{\substack{k \in \iota(t, \mathbf{v}(t)) \\ m = 1, 2, \ldots, M}} \tag{44}$$

has a rank equal to card$(\iota(t, \mathbf{v}(t)))$. That is, the rank of the above Jacobian matrix is equal to the number of its rows.

For example, if k' of the inequality constraints bind at a given point t in the planning horizon, $0 < k' < K$, then the above Jacobian matrix will be of order $k' \times M$ and the constraint qualification will be satisfied if the rank of the Jacobian equals k'. This rank condition on the Jacobian has two important consequences. First, it implies that at least one control variable must be present in each of the binding constraints. To see this, assume that the constraint qualification holds but that one of the k' binding constraints, say, the first (without loss of generality), does not have any control variables in it. In this case, the first row of the above Jacobian would be identically zero, thereby implying that the rows of the Jacobian are linearly dependent. This, in turn, implies that the Jacobian is of rank less than k', thus violating the constraint qualification. Second, the constraint qualification implies that the number of binding constraints k' cannot be greater than the number of control variables M. To see this, assume that the constraint qualification holds but that $k' > M$. Then there are more rows (k') in the Jacobian than there are columns (M). But the rank of a matrix cannot exceed the minimum of the number of its rows or columns. This, in turn, implies that the rank of the Jacobian is $\min(k', M) = M$, which is less than k', thereby violating the constraint qualification.

With the constraint qualification now dealt with, we can turn to the necessary conditions for problem (43). For ease of exposition, we first remind the reader of the definition of the Hamiltonian:

$$H(t,\mathbf{x},\mathbf{u},\boldsymbol{\lambda}) \stackrel{\text{def}}{=} f(t,\mathbf{x},\mathbf{u}) + \boldsymbol{\lambda}'\mathbf{g}(t,\mathbf{x},\mathbf{u}) = f(t,\mathbf{x},\mathbf{u}) + \sum_{n=1}^{N}\lambda_n g^n(t,\mathbf{x},\mathbf{u}).$$

We should also note that in addition to the standard assumptions on the functions $f(\cdot)$ and $\mathbf{g}(\cdot)$ given in Chapter 1, we now make the following *additional* ones:

(A.1) $\partial f(\cdot)/\partial u_m \in C^{(0)}$ with respect to the $1+N+M$ variables $(t,\mathbf{x},\mathbf{u})$ for $m = 1,2,\ldots,M$.
(A.2) $\partial g^n(\cdot)/\partial u_m \in C^{(0)}$ with respect to the $1+N+M$ variables $(t,\mathbf{x},\mathbf{u})$ for $m = 1,2,\ldots,M$ and $n = 1,2,\ldots,N$.
(A.3) $h^k(\cdot) \in C^{(0)}$ with respect to the $1+M$ variables $(t,\mathbf{u})$ for $k = 1,2,\ldots,K$.
(A.4) $\partial h^k(\cdot)/\partial u_m \in C^{(0)}$ with respect to the $1+M$ variables $(t,\mathbf{u})$ for $k = 1, 2,\ldots,K$ and $m = 1,2,\ldots,M$.

We may now state the necessary conditions pertaining to problem (43).

Theorem 4.4 (Necessary Conditions, Inequality Constraints): *Let* $(\mathbf{z}(t),\mathbf{v}(t))$ *be an admissible pair for problem* (43)*, and assume that the rank constraint qualification is satisfied. Then if* $(\mathbf{z}(t),\mathbf{v}(t))$ *yields the absolute maximum of* $J[\cdot]$*, it is necessary that there exist a piecewise smooth vector-valued function* $\boldsymbol{\lambda}(\cdot) \stackrel{\text{def}}{=} (\lambda_1(\cdot),\lambda_2(\cdot),\ldots,\lambda_N(\cdot))$ *and a piecewise continuous vector-valued Lagrange multiplier function* $\boldsymbol{\mu}(\cdot) \stackrel{\text{def}}{=} (\mu_1(\cdot),\mu_2(\cdot),\ldots,\mu_K(\cdot))$*, such that for all* $t \in [t_0,t_1]$,

$$\mathbf{v}(t) = \arg\max_{\mathbf{u}}\{H(t,\mathbf{z}(t),\mathbf{u},\boldsymbol{\lambda}(t)) \ \text{ s.t. } \ \mathbf{h}(t,\mathbf{u}(t)) \geq \mathbf{0}_K\};$$

that is, if

$$M(t,\mathbf{z}(t),\boldsymbol{\lambda}(t)) \stackrel{\text{def}}{=} \max_{\mathbf{u}}\{H(t,\mathbf{z}(t),\mathbf{u},\boldsymbol{\lambda}(t)) \ \text{ s.t. } \ \mathbf{h}(t,\mathbf{u}(t)) \geq \mathbf{0}_K\},$$

then

$$M(t,\mathbf{z}(t),\boldsymbol{\lambda}(t)) \equiv H(t,\mathbf{z}(t),\mathbf{v}(t),\boldsymbol{\lambda}(t)),$$

or equivalently

$$H(t,\mathbf{z}(t),\mathbf{v}(t),\boldsymbol{\lambda}(t)) \geq H(t,\mathbf{z}(t),\mathbf{u},\boldsymbol{\lambda}(t))\forall\, \mathbf{u} \in U(t),$$

where $U(t) \stackrel{\text{def}}{=} \{\mathbf{u}(\cdot) : h^k(t,\mathbf{u}(t)) \geq 0,\ k = 1,2,\ldots,K\}$*. Because the rank constraint qualification is assumed to hold, the above necessary condition implies that*

$$\frac{\partial L}{\partial u_m}(t,\mathbf{z}(t),\mathbf{v}(t),\boldsymbol{\lambda}(t),\boldsymbol{\mu}(t)) = 0, \quad m = 1,2,\ldots,M,$$

$$\frac{\partial L}{\partial \mu_k}(t,\mathbf{z}(t),\mathbf{v}(t),\boldsymbol{\lambda}(t),\boldsymbol{\mu}(t)) \geq 0,$$

$$\mu_k(t) \geq 0,\ \mu_k(t)\frac{\partial L}{\partial \mu_k}(t, \mathbf{z}(t), \mathbf{v}(t), \boldsymbol{\lambda}(t), \boldsymbol{\mu}(t)) = 0, \quad k = 1, 2, \ldots, K,$$

where

$$\begin{aligned} L(t, \mathbf{x}, \mathbf{u}, \boldsymbol{\lambda}, \boldsymbol{\mu}) &\stackrel{\text{def}}{=} H(t, \mathbf{x}, \mathbf{u}, \boldsymbol{\lambda}) + \boldsymbol{\mu}'\mathbf{h}(t, \mathbf{u}) \\ &= f(t, \mathbf{x}, \mathbf{u}) + \sum_{n=1}^{N} \lambda_n g^n(t, \mathbf{x}, \mathbf{u}) + \sum_{k=1}^{K} \mu_k h^k(t, \mathbf{u}) \end{aligned}$$

is the Lagrangian function. Furthermore, except for the points of discontinuities of $\mathbf{v}(t)$*,*

$$\dot{z}_n(t) = \frac{\partial H}{\partial \lambda_n}(t, \mathbf{z}(t), \mathbf{v}(t), \boldsymbol{\lambda}(t)) = g^n(t, \mathbf{z}(t), \mathbf{v}(t)), \quad n = 1, 2, \ldots, N,$$

$$\dot{\lambda}_n(t) = -\frac{\partial H}{\partial x_n}(t, \mathbf{z}(t), \mathbf{v}(t), \boldsymbol{\lambda}(t)), \quad n = 1, 2, \ldots, N,$$

where the above notation means that the functions are first differentiated with respect to the particular variable and then evaluated at $(t, \mathbf{z}(t), \mathbf{v}(t), \boldsymbol{\lambda}(t), \boldsymbol{\mu}(t))$.

The proof of this theorem is a straightforward consequence of Theorem 4.2 and the Karush-Kuhn-Tucker theorem, stated as Theorem 18.4 in Simon and Blume (1994). It is therefore left for a mental exercise. Before stating a corresponding sufficiency theorem, we note several important consequences of Theorem 4.4. First, in general, the Lagrange multiplier function $\boldsymbol{\mu}(\cdot) \stackrel{\text{def}}{=} (\mu_1(\cdot), \mu_2(\cdot), \ldots, \mu_K(\cdot))$ is piecewise continuous on $[t_0, t_1]$. It turns out, however, that $\boldsymbol{\mu}(\cdot)$ is continuous whenever the optimal control function $\mathbf{v}(\cdot)$ is continuous. Thus the discontinuities in the Lagrange multipliers can only occur at points where the optimal control is discontinuous. Second, the Lagrangian function evaluated along the optimal solution is a continuous function of t, that is, $L(t, \mathbf{z}(t), \mathbf{v}(t), \boldsymbol{\lambda}(t), \boldsymbol{\mu}(t))$ is a continuous function of t. Third, the *total derivative* of $L(t, \mathbf{z}(t), \mathbf{v}(t), \boldsymbol{\lambda}(t), \boldsymbol{\mu}(t))$ with respect to t, scilicet,

$$\dot{L}(t, \mathbf{z}(t), \mathbf{v}(t), \boldsymbol{\lambda}(t), \boldsymbol{\mu}(t)) \stackrel{\text{def}}{=} \frac{d}{dt}L(t, \mathbf{z}(t), \mathbf{v}(t), \boldsymbol{\lambda}(t), \boldsymbol{\mu}(t)),$$

is equal to the *partial derivative* of $L(t, \mathbf{z}(t), \mathbf{v}(t), \boldsymbol{\lambda}(t), \boldsymbol{\mu}(t))$ with respect to t, namely,

$$L_t(t, \mathbf{z}(t), \mathbf{v}(t), \boldsymbol{\lambda}(t), \boldsymbol{\mu}(t)) \stackrel{\text{def}}{=} \frac{\partial}{\partial t}L(t, \mathbf{z}(t), \mathbf{v}(t), \boldsymbol{\lambda}(t), \boldsymbol{\mu}(t)),$$

at all continuity points of the optimal control function $\mathbf{v}(\cdot)$, assuming that $f(\cdot)$, $\mathbf{g}(\cdot)$, and $\mathbf{h}(\cdot)$ are $C^{(1)}$ functions of $(t, \mathbf{x}, \mathbf{u})$. Other results similar to those derived in Mental Exercise 2.31 also hold, as you will discover in a mental exercise in this chapter.

It is also worthwhile at this juncture to provide a rather thorough set of remarks on the role that the rank constraint qualification plays in solving an optimal control problem. Following Seierstad and Sydsæter (1987, pp. 278–279), the crucial observation is that the set of candidates for optimality are contained in the *union* of the following two sets:

$$A \stackrel{\text{def}}{=} \{\text{admissible pairssatisfying the necessary conditions}\},$$

$$B \stackrel{\text{def}}{=} \{\text{admissible pairs that fail to satisfy the rank constraintqualification}\}.$$

In other words, the set $A \cup B$ contains all the candidates for optimality. Because we have the identity $A \cup B \equiv B \cup (A - B)$, a valid two-step approach for obtaining all of the candidates for optimality is as follows:

(i) First find all the admissible pairs that fail to satisfy the rank constraint qualification for some $t \in [t_0, t_1]$.
(ii) Then find all the admissible pairs that satisfy the necessary conditions *and* the rank constraint qualification.

This is the approach applied in the book to find the candidates for optimality, and it is also the procedure recommended because it is often the most successful. Note, however, that the identity $A \cup B \equiv A \cup (B - A)$ is equally valid and suggests a different two-step procedure for obtaining all of the candidates for optimality, to wit:

(i′) First find all the admissible pairs that satisfy the necessary conditions, *disregarding* the rank constraint qualification.
(ii′) Then find all the admissible pairs that fail to satisfy the rank constraint, *excluding* those obtained in (i′).

Remember that either two-step procedure will yield all the candidates for optimality.

The following sufficiency theorem is not the most general known, but it will suffice for our present purposes. Its proof is similar to that of Theorem 4.3 and so is left for a mental exercise. It is a Mangasarian-type sufficiency result.

Theorem 4.5 (Sufficient Conditions, Inequality Constraints): *Let* $(\mathbf{z}(t), \mathbf{v}(t))$ *be an admissible pair for problem (43). Suppose that* $(\mathbf{z}(t), \mathbf{v}(t))$ *satisfies the necessary conditions of Theorem 4.4 for problem (43) with costate vector* $\boldsymbol{\lambda}(t)$ *and the Lagrange multiplier vector* $\boldsymbol{\mu}(t)$*, and let* $L(t, \mathbf{x}, \mathbf{u}, \boldsymbol{\lambda}, \boldsymbol{\mu}) \stackrel{\text{def}}{=} f(t, \mathbf{x}, \mathbf{u}) + \boldsymbol{\lambda}'\mathbf{g}(t, \mathbf{x}, \mathbf{u}) + \boldsymbol{\mu}'\mathbf{h}(t, \mathbf{u})$ *be the value of the Lagrangian function. If* $L(\cdot)$ *is a concave function of* $(\mathbf{x}, \mathbf{u})\, \forall\, t \in [t_0, t_1]$ *over an open convex set containing all the admissible values of* $(\mathbf{x}(\cdot), \mathbf{u}(\cdot))$ *when the costate vector is* $\boldsymbol{\lambda}(t)$ *and the Lagrange multiplier vector is* $\boldsymbol{\mu}(t)$*, then* $\mathbf{v}(t)$ *is an optimal control and* $(\mathbf{z}(t), \mathbf{v}(t))$ *yields the global maximum of* $J[\cdot]$*. If* $L(\cdot)$ *is a strictly concave function under the same conditions, then* $(\mathbf{z}(t), \mathbf{v}(t))$ *yields the unique global maximum of* $J[\cdot]$.

A very important observation about this sufficiency theorem is that the rank constraint qualification is *not* required, in sharp contrast to the necessary conditions of

Theorem 4.4. It is also noteworthy that this theorem may be strengthened to give an Arrow-type sufficiency result. We will pursue this in a later chapter, however, when we study a more general class of optimal control problems. Let's now demonstrate Theorems 4.4 and 4.5 by returning to the inventory accumulation problem of Example 4.5, but this time, we explicitly take into account the nonnegativity restriction on the production rate.

Example 4.6: Recall the inventory accumulation problem from Example 4.5:

$$\max_{u(\cdot)} J[x(\cdot), u(\cdot)] \stackrel{\text{def}}{=} \int_0^T -\left[c_1[u(t)]^2 + c_2 x(t)\right] dt$$

$$\text{s.t.} \quad \dot{x}(t) = u(t),\ x(0) = 0,\ x(T) = x_T,$$

$$h(t, u(t)) \stackrel{\text{def}}{=} u(t) \geq 0.$$

Notice that in this instance, we have multiplied the objective functional by minus unity in order to convert it into a maximization problem, the form required by Theorems 4.4 and 4.5. Recall that in Example 4.5, we showed that the production rate constraint $u(t) \geq 0\, \forall\, t \in [0, T]$ is satisfied if and only if $x_T \geq \frac{1}{4}c_2 c_1^{-1} T^2$ [recall Eq. (41)]. In the present situation, let's therefore assume that $x_T < \frac{1}{4}c_2 c_1^{-1} T^2$, so that the nonnegativity constraint on the production rate will be binding for some period of time in the planning horizon. This is clearly the only case in which the nonnegativity constraint is interesting.

To solve this version of the problem, we begin by defining the Hamiltonian and Lagrangian functions:

$$H(x, u, \lambda) \stackrel{\text{def}}{=} -c_1 u^2 - c_2 x + \lambda u,$$

$$L(x, u, \lambda, \mu) \stackrel{\text{def}}{=} -c_1 u^2 - c_2 x + \lambda u + \mu u.$$

Because $h_u(t, u) = 1$, the rank constraint qualification is satisfied. Thus by Theorem 4.4, the necessary conditions are

$$L_u(x, u, \lambda, \mu) = -2c_1 u + \lambda + \mu = 0, \tag{45}$$

$$L_\mu(x, u, \lambda, \mu) = u \geq 0,\ \mu \geq 0,\ \mu u = 0, \tag{46}$$

$$\dot{\lambda} = -H_x(x, u, \lambda) = c_2, \tag{47}$$

$$\dot{x} = H_\lambda(x, u, \lambda) = u,\ x(0) = 0,\ x(T) = x_T. \tag{48}$$

The Hessian matrix of the Lagrangian function is given by

$$\begin{bmatrix} L_{uu}(x, u, \lambda, \mu) & L_{ux}(x, u, \lambda, \mu) \\ L_{xu}(x, u, \lambda, \mu) & L_{xx}(x, u, \lambda, \mu) \end{bmatrix} = \begin{bmatrix} -2c_1 & 0 \\ 0 & 0 \end{bmatrix},$$

which has eigenvalues $-2c_1 < 0$ and zero. Thus the Hessian matrix of the Lagrangian function is negative semidefinite, which, as you know by now, is equivalent to the concavity of $L(\cdot)$ in (x, u). By Theorem 4.5, therefore, any solution of the necessary conditions is a solution of the optimal control problem. Note, however, that because $L(\cdot)$ is not strictly concave in (x, u), we cannot appeal to Theorem 4.5 to claim uniqueness.

Given that we have assumed that $x_T < \frac{1}{4}c_2c_1^{-1}T^2$, we know that the length of the planning horizon T is long relative to the amount of the product required to be produced x_T. Moreover, we also know that inventory is costly to hold, that is, $c_2 > 0$. In light of these two observations, a reasonable hypothesis as to the structure of the solution for the production rate is that there is an initial period of the planning horizon, say, $0 \le t \le s$, $s \in (0, T)$, in which the production rate is zero, that is, $u(t) = 0$ for all $t \in [0, s]$, after which the production rate is positive, that is, $u(t) > 0$ for all $t \in (s, T]$. The time after which the production rate turns positive, namely, $s \in (0, T)$, is called the *switching time*. It is an endogenous variable, determined by the condition that the cumulative amount produced by the end of the planning horizon equals the amount required by the contract. We thus seek a solution of the necessary conditions with these characteristics. Such a solution will be the solution of the control problem because of the aforementioned concavity of $L(\cdot)$ in (x, u).

To begin, first solve Eq. (45) for the control variable to get

$$u = \frac{\lambda + \mu}{2c_1}. \tag{49}$$

Our conjecture is that $u(t) = 0$ for all $t \in [0, s]$, $s \in (0, T)$. Clearly, this is equivalent to $\lambda(t) + \mu(t) = 0$ for all $t \in [0, s]$. Seeing as $\dot{\lambda} = c_2 > 0$ by Eq. (47) (the costate equation), $\lambda(t) \neq 0$ for all $t \in [0, T]$, with the possible exception of an instant. This implies that the Lagrange multiplier $\mu(t) \neq 0$ for all $t \in [0, s]$, with the possible exception of an instant, since $\lambda(t) + \mu(t) = 0$ for all $t \in [0, s]$. Integrating $\dot{\lambda} = c_2 > 0$ yields $\lambda(t) = c_2 t + a_1$ for all $t \in [0, T]$, where a_1 is a constant of integration. Because $\lambda(t) + \mu(t) = 0$ for all $t \in [0, s]$, $\mu(t) = -c_2 t - a_1$ for all $t \in [0, s]$. To find the value of the constant of integration, recall our conjecture that $u(t) > 0$ for all $t \in (s, T]$, which implies, by way of Eq. (46), that $\mu(t) = 0$ for all $t \in (s, T]$. Because the control function is a continuous (but not a differentiable) function of t under our hypothesis, the Lagrange multiplier function is too, confirming a remark made after Theorem 4.4. We thus use the fact that $\mu(\cdot)$ is continuous at $t = s$ to get $\mu(s) = -c_2 s - a_1 = 0$, or $a_1 = -c_2 s$. Using this value of the constant of integration yields the specific solution for the costate variable and Lagrange multiplier:

$$\lambda(t; s, c_2) = c_2[t - s]\,\forall\, t \in [0, T], \tag{50}$$

$$\mu(t; s, c_2) = \begin{cases} c_2[s - t] & \forall t \in [0, s], \\ 0 & \forall\, t \in (s, T]. \end{cases} \tag{51}$$

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Note that on the closed interval $[0, s]$, $\lambda(t; s, c_2)$ and $\mu(t; s, c_2)$ are only equal to zero at $t = s$, an instant, just as claimed above. Substituting Eqs. (50) and (51) into Eq. (49) yields our specific solution for the production rate:

$$v(t; s, c_1, c_2) = \begin{cases} 0 & \forall\, t \in [0, s], \\ \dfrac{c_2[t-s]}{2c_1} & \forall\, t \in (s, T]. \end{cases} \tag{52}$$

Observe that $\mu(\cdot)$ and $v(\cdot)$ are continuous functions of t, just as asserted above. More precisely, $\mu(\cdot)$ and $v(\cdot)$ are piecewise smooth functions of t.

To finish up the solution, we must find the time path of the stock of inventory and the switching time s. Using Eq. (52), the state equation $\dot{x} = u$ becomes

$$\dot{x} = \begin{cases} 0 & \forall\, t \in [0, s], \\ \dfrac{c_2[t-s]}{2c_1} & \forall\, t \in (s, T]. \end{cases}$$

Integrating this differential equation separately over each time interval yields the general solution

$$x(t) = \begin{cases} a_2 & \forall\, t \in [0, s], \\ \dfrac{c_2 t^2}{4c_1} - \dfrac{c_2 s t}{2c_1} + a_3 & \forall\, t \in (s, T], \end{cases} \tag{53}$$

where a_2 and a_3 are constants of integration. Using the initial condition $x(0) = 0$ gives $a_2 = 0$. Because $x(\cdot)$ is a continuous function of t by Theorem 4.4, we can solve for a_3 by equating both portions of the solution in Eq. (53) at $t = s$ using $a_2 = 0$. This yields the equation

$$\frac{c_2 s^2}{4c_1} - \frac{c_2 s^2}{2c_1} + a_3 = 0,$$

which, when solved for a_3, gives

$$a_3 = \frac{c_2 s^2}{4c_1}. \tag{54}$$

Substituting Eq. (54) into Eq. (53) yields the specific solution of the state equation

$$z(t; s, c_1, c_2) = \begin{cases} 0 & \forall\, t \in [0, s], \\ \dfrac{c_2}{4c_1}[t-s]^2 & \forall\, t \in (s, T]. \end{cases} \tag{55}$$

As expected, $z(\cdot)$ is a piecewise smooth function of t.

Finally, we seek to determine the switching time s. Because the only unused boundary condition is the terminal one, namely, $x(T) = x_T$, it can be used to

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determine s. Using this boundary condition in Eq. (55) gives $\frac{c_2}{4c_1}[T - s]^2 = x_T$, which yields

$$s = T - 2\sqrt{\frac{c_1 x_T}{c_2}} \tag{56}$$

as the solution for the switching time, as you are asked to show in a mental exercise. In sum, Eqs. (50), (51), (52), (53), and (56) represent the unique solution to the problem, since they are the only solution to the necessary and sufficient conditions.

An interesting feature of the solution for the switching time is that the length of the subinterval in which production occurs, videlicet,

$$T - s = 2\sqrt{\frac{c_1 x_T}{c_2}},$$

is identical to the length of the planning horizon the firm would choose if T was a decision variable, as we will see in Chapter 8. This observation means that the length of the subinterval in which production occurs is unaffected by how distant the terminal period T is, as long as our original assumption on the length of the planning horizon holds, to wit, $x_T < \frac{1}{4}c_2 c_1^{-1} T^2$. This, in turn, implies that the production rate in the interval $[s, T]$ in also unaffected by an increase in T. As can be seen from Eq. (56), $\partial s/\partial T = 1$; thus an increase in T lengthens the initial period when no production occurs, but leaves the length of the period in which positive production occurs unchanged, thereby implying that the positive production rate in the interval $[s, T]$ is unaffected too. We leave the remaining comparative dynamics results for a mental exercise.

In the next chapter, we take up the study of optimal control problems that often exhibit piecewise continuous optimal control functions. In order to solve such problems, we often must approach them somewhat differently because the differential calculus has a more limited role in the characterization of the optimal control. To convince you that this generality is not simply of mathematical interest, we will present and solve three economic problems that exhibit piecewise continuous optimal control functions.

MENTAL EXERCISES

4.1 Consider the fixed endpoints optimal control problem

$$\min_{u(\cdot)} J[x(\cdot), u(\cdot)] \stackrel{\text{def}}{=} \int_0^1 [u(t)]^2 \, dt$$

$$\text{s.t.} \quad \dot{x}(t) = x(t) + u(t),\ x(0) = 1,\ x(1) = 0.$$

(a) Find the solution to the necessary conditions.
(b) Prove that the solution found in (a) solves the optimal control problem.

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4.2 Consider the fixed endpoints optimal control problem

$$\max_{u(\cdot)} J[x(\cdot), u(\cdot)] \stackrel{\text{def}}{=} \int_0^1 [\alpha t u(t) - \frac{1}{2}[u(t)]^2]\, dt$$

$$\text{s.t.} \quad \dot{x}(t) = u(t) - x(t),\ x(0) = x_0,\ x(1) = 0.$$

(a) Find the solution to the necessary conditions.
(b) Prove that the solution found in (a) solves the optimal control problem.

4.3 The following questions pertain to Example 4.3.
(a) Show that $u_1(t) = \sin t$ is an admissible control.
(b) Show that $u_2(t) = \frac{1}{2}$ is an admissible control.
(c) Compute $J[x_1(\cdot), u_1(\cdot)]$ and$J[x_2(\cdot), u_2(\cdot)]$.
(d) Compare $J[x_1(\cdot), u_1(\cdot)]$ and $J[x_2(\cdot), u_2(\cdot)]$ with the value of $J[\cdot]$ computed in the example.

4.4 Consider the optimal control problem

$$\max_{u(\cdot), x_1} J[x(\cdot), u(\cdot)] \stackrel{\text{def}}{=} \int_0^1 [x(t) + u(t)]\, dt$$

$$\text{s.t.} \quad \dot{x}(t) = -x(t) + u(t) + t,\ x(0) = 1,\ x(1) = x_1,$$

$$u(t) \in U \stackrel{\text{def}}{=} \{u(\cdot) : 0 \le u(t) \le 1\}.$$

(a) Write down all the necessary conditions for this problem.
(b) Find the unique solution of the costate equation.
(c) Find the decision rule governing the optimal control, and find $v(t)$.
(d) Determine the associated path of the state variable $z(t)$.
(e) Prove that the solution to the necessary conditions is the optimal solution of the control problem.
(f) Solve the control problem directly, that is, without the aid of the Maximum Principle. **Hint:** Solve the state equation.

4.5 Consider the fixed endpoints optimal control problem

$$\max_{u(\cdot)} J[x(\cdot), u(\cdot)] \stackrel{\text{def}}{=} \int_0^1 u(t)\, dt$$

$$\text{s.t.} \quad \dot{x}(t) = x(t) + [u(t)]^2,\ x(0) = 1,\ x(1) = \alpha,$$

where α is some parameter.
(a) Show that there is no solution to the optimal control problem if $\alpha \le 1$.
(b) Would your answer to part (a) change if "max" were replaced by "min"? Why or why not?

4.6 Consider the optimal control problem

$$\max_{u(\cdot)} \int_0^1 u(t)\,dt$$

$$\text{s.t.}\quad \dot{x}(t) = [u(t) - (u(t))^2]^2,\ x(0) = 0,\ x(1) = 0,$$

$$u(t) \in [0, 2].$$

Prove that $v(t) = 1\,\forall\, t \in [0, 1]$ is the optimal control and satisfies the Maximum Principle only for $\lambda_0 = 0$.

4.7 Consider the two state variable optimal control problem

$$\max_{u(\cdot),x_1} \int_0^1 \left[t - \frac{1}{2}\right] u(t)\,dt$$

$$\text{s.t.}\quad \dot{x}(t) = u(t),\ x(0) = 0,\ x(1) = x_1,$$

$$\dot{y}(t) = [x(t) - tu(t)]^2,\ y(0) = 0,\ y(1) = 0.$$

Prove that any constant control is optimal and satisfies the Maximum Principle only for $\lambda_0 = 0$. **Hint:** Note that $x(t) - tu(t) \equiv 0 \Rightarrow \dot{x}(t) - t\dot{u}(t) - u(t) \equiv -t\dot{u}(t) \equiv 0$.

4.8 Consider the two control variable optimal control problem

$$\max_{u_1(\cdot),u_2(\cdot)} \int_0^1 [u_1(t) - 2u_2(t)]\,dt$$

$$\text{s.t.}\quad \dot{x}(t) = [u_1(t) - u_2(t)]^2,\ x(0) = 0,\ x(1) = 0,$$

$$u_1(t) \in [-1, 1],\ u_2(t) \in [-1, 1].$$

Prove that $v_1(t) = 1\,\forall\, t \in [0, 1]$ and $v_2(t) = 1\,\forall\, t \in [0, 1]$ are optimal and satisfy the Maximum Principle only for $\lambda_0 = 0$.

4.9 Find necessary conditions for a solution of the fixed endpoints optimal control problem with a bounded control variable, namely,

$$\max_{u(\cdot)} \int_{t_0}^{t_1} f(t, x(t), u(t))\,dt$$

$$\text{s.t.}\quad \dot{x}(t) = g(t, x(t), u(t)),\ x(t_0) = x_0,\ x(t_1) = x_1,$$

$$a(t) \le u(t) \le b(t).$$

4.10 Consider the optimal control problem

$$\max_{u(\cdot),x_2} J[x(\cdot), u(\cdot)] \stackrel{\text{def}}{=} \int_0^2 [2x(t) - 3u(t) - [u(t)]^2]\, dt$$

$$\text{s.t.} \quad \dot{x}(t) = x(t) + u(t),\ x(0) = 4,\ x(2) = x_2,$$

$$u(t) \in U \stackrel{\text{def}}{=} \{u(\cdot) : 0 \leq u(t) \leq 2\}.$$

Note that this problem is probably best tackled *after* Example 5.1 is understood.

(a) Find the solution to the necessary conditions.

(b) Prove that the solution found in (a) solves the optimal control problem.

4.11 Consider the optimal control problem

$$\max_{u(\cdot),x_4} J[x(\cdot), u(\cdot)] \stackrel{\text{def}}{=} \int_0^4 3x(t)\, dt$$

$$\text{s.t.} \quad \dot{x}(t) = x(t) + u(t),\ x(0) = 5,\ x(4) = x_4,$$

$$u(t) \in U \stackrel{\text{def}}{=} \{u(\cdot) : 0 \leq u(t) \leq 2\}.$$

(a) Find the solution to the necessary conditions.

(b) Prove that the solution found in (a) solves the optimal control problem.

4.12 Prove Theorem 4.4.

4.13 Consider the optimal control problem (43):

$$\max_{\mathbf{u}(\cdot)} J[\mathbf{x}(\cdot), \mathbf{u}(\cdot)] \stackrel{\text{def}}{=} \int_{t_0}^{t_1} f(t, \mathbf{x}(t), \mathbf{u}(t))\, dt$$

$$\text{s.t.} \quad \dot{\mathbf{x}}(t) = \mathbf{g}(t, \mathbf{x}(t), \mathbf{u}(t)),\ \mathbf{x}(t_0) = \mathbf{x}_0,\ \mathbf{x}(t_1) = \mathbf{x}_1,$$

$$\mathbf{h}(t, \mathbf{u}(t)) \geq \mathbf{0}_K.$$

Let $(\mathbf{z}(t), \mathbf{v}(t))$ be the optimal pair, $\boldsymbol{\lambda}(t)$ the corresponding time path of the costate vector, and $\boldsymbol{\mu}(t)$ the time path of the corresponding Lagrange multiplier vector. Define the Hamiltonian as

$$H(t, \mathbf{x}, \mathbf{u}, \boldsymbol{\lambda}) \stackrel{\text{def}}{=} f(t, \mathbf{x}, \mathbf{u}) + \boldsymbol{\lambda}'\mathbf{g}(t, \mathbf{x}, \mathbf{u}) = f(t, \mathbf{x}, \mathbf{u}) + \sum_{n=1}^{N} \lambda_n g^n(t, \mathbf{x}, \mathbf{u})$$

and the Lagrangian as

$$\begin{aligned} L(t, \mathbf{x}, \mathbf{u}, \boldsymbol{\lambda}, \boldsymbol{\mu}) &\stackrel{\text{def}}{=} H(t, \mathbf{x}, \mathbf{u}, \boldsymbol{\lambda}) + \boldsymbol{\mu}'\mathbf{h}(t, \mathbf{u}) \\ &= f(t, \mathbf{x}, \mathbf{u}) + \sum_{n=1}^{N} \lambda_n g^n(t, \mathbf{x}, \mathbf{u}) + \sum_{k=1}^{K} \mu_k h^k(t, \mathbf{u}). \end{aligned}$$

Assume that $f(\cdot) \in C^{(1)}$, $\mathbf{g}(\cdot) \in C^{(1)}$, and $\mathbf{h}(\cdot) \in C^{(1)}$ in $(t, \mathbf{x}, \mathbf{u})$.

(a) Prove that

$$\frac{d}{dt} L(t, \mathbf{z}(t), \mathbf{v}(t), \boldsymbol{\lambda}(t), \boldsymbol{\mu}(t)) = \frac{\partial}{\partial t} L(t, \mathbf{z}(t), \mathbf{v}(t), \boldsymbol{\lambda}(t), \boldsymbol{\mu}(t)).$$

(b) Prove that if the optimal control problem is autonomous, that is, the independent variable t doesn't enter the functions $f(\cdot)$, $\mathbf{g}(\cdot)$, or $\mathbf{h}(\cdot)$ explicitly, that is, $f_t(t, \mathbf{x}, \mathbf{u}) \equiv 0$, $\mathbf{g}_t(t, \mathbf{x}, \mathbf{u}) \equiv \mathbf{0}_N$, and $\mathbf{h}_t(t, \mathbf{u}) \equiv \mathbf{0}_K$, then $L(\cdot)$ is constant along the optimal path.

(c) Prove that

$$\frac{d}{dt} H(t, \mathbf{z}(t), \mathbf{v}(t), \boldsymbol{\lambda}(t)) = \frac{\partial}{\partial t} H(t, \mathbf{z}(t), \mathbf{v}(t), \boldsymbol{\lambda}(t)).$$

Note that this result is *not* the same as that in Mental Exercise 2.31.

(d) Prove that if the optimal control problem is autonomous, then $H(\cdot)$ is constant along the optimal path too.

4.14 Prove Theorem 4.5.

4.15 This mental exercise asks you to compute and economically interpret the remaining comparative dynamics of the optimal inventory accumulation problem in Example 4.5.

(a) Derive the comparative dynamics result for an increase in total required production x_T on the production rate and the shadow cost of inventory. Provide an economic interpretation.

(b) Derive the comparative dynamics result for an increase in the unit cost of production coefficient c_1 on the production rate and the shadow cost of inventory. Provide an economic interpretation.

(c) Derive the comparative dynamics result for an increase in the length of the production run T on the production rate and the shadow cost of inventory. Provide an economic interpretation.

4.16 With reference to Example 4.6, the inventory accumulation problem with a nonnegativity constraint on the production rate:

(a) Explain why $s \in (0, T)$ must hold.

(b) Provide an economic interpretation of $\lambda(s; s, c_2) = 0$.

(c) Derive Eq. (56) for the switching time and explain your steps fully and carefully.

(d) Derive the comparative dynamics for the switching time and production rate for an increase in c_1. Provide an economic interpretation.

(e) Derive the comparative dynamics for the switching time and production rate for an increase in c_2. Provide an economic interpretation.

(f) Derive the comparative dynamics for the switching time and production rate for an increase in x_T. Provide an economic interpretation.

4.17 Consider the optimal control problem

$$\min_{u(\cdot)} J[x(\cdot), u(\cdot)] \stackrel{\text{def}}{=} \int_0^1 [tu(t) + [u(t)]^2]\, dt$$

$$\text{s.t.} \quad \dot{x}(t) = u(t),\ x(0) = x_0,\ x(1) = x_1.$$

(a) Find the solution of the necessary conditions, and denote the solution triplet by $(z(t; x_0, x_1), v(t; x_0, x_1), \lambda(t; x_0, x_1))$.
(b) Prove that the solution found in part (a) is the unique solution of the optimal control problem.
(c) Define the optimal value function by

$$V(x_0, x_1) \stackrel{\text{def}}{=} \int_0^1 \left[tv(t; x_0, x_1) + [v(t; x_0, x_1)]^2\right] dt.$$

The optimal value function is thus the value of the objective functional $J[x(\cdot), u(\cdot)]$ evaluated along the solution curve. Find an explicit expression for $V(x_0, x_1)$.
(d) Compute $\frac{\partial}{\partial x_0} V(x_0, x_1)$ and $\frac{\partial}{\partial x_1} V(x_0, x_1)$.
(e) Show that

$$\frac{\partial}{\partial x_0} V(x_0, x_1) = \lambda(0; x_0, x_1)$$

and

$$\frac{\partial}{\partial x_1} V(x_0, x_1) = -\lambda(1; x_0, x_1).$$

This is a special case of the *dynamic envelope theorem*, which will be covered in a general fashion in Chapter 9. Note also that the first result is equivalent to the definition given in Eq. (21).

4.18 Consider a simple generalization of the optimal inventory accumulation problem in Example 4.5. Assume now that the firm discounts its costs at the rate $r > 0$, but the problem is otherwise unchanged. Thus the firm is asserted to solve

$$\min_{u(\cdot)} J[x(\cdot), u(\cdot)] \stackrel{\text{def}}{=} \int_0^T \left[c_1[u(t)]^2 + c_2 x(t)\right] e^{-rt}\, dt$$

$$\text{s.t.} \quad \dot{x}(t) = u(t),\ x(0) = 0,\ x(T) = x_T.$$

Assume that the nonnegativity constraint $u(t) \geq 0\, \forall\, t \in [0, T]$ is *not* binding. This assumption will be addressed later in this question.

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(a) Derive the necessary conditions. Show that

$$2e^{-rt}c_1u(t) + \int_t^{t+\varepsilon} e^{-rs}c_2\,ds = 2e^{-r[t+\varepsilon]}c_1u(t+\varepsilon)$$

along an optimal path, where $\varepsilon > 0$ is sufficiently small. Provide an economic interpretation of this integral equation.

(b) Find the solution of the necessary conditions, and denote the solution triplet by $(z(t;\gamma), v(t;\gamma), \lambda(t;\gamma))$, where $\gamma \stackrel{\text{def}}{=} (c_1, c_2, r, T, x_T)$.

(c) Prove that the solution of the necessary conditions is the unique solution of the optimal control problem.

(d) Show that $\dot{v}(t;\gamma) > 0$ holds in the optimal plan. Provide an economic interpretation of this result.

(e) Derive a necessary and sufficient condition for the nonnegativity constraint $u(t) \geq 0\,\forall\, t \in [0, T]$ to be nonbinding in an optimal production plan.

(f) Show that $\partial\lambda(t;\gamma)/\partial x_T < 0\,\forall\, t \in [0, T]$ and $\partial\dot{z}(t;\gamma)/\partial x_T > 0\,\forall\, t \in [0, T]$. Provide an economic interpretation of these comparative dynamics results. Note that there are no simple refutable comparative dynamics results for the shadow cost of the inventory or the production rate for the remaining parameters, in sharp contrast to the basic inventory accumulation model studied in Example 4.5 and in Mental Exercise 4.15, which had the discount rate set equal to zero.

4.19 Consider another generalization of the optimal inventory accumulation problem in Example 4.5, in which now the production cost function is given by $g(\cdot)$ and is assumed to have the following properties:

$$g(\cdot) \in C^{(2)},\ g(0) = 0,\ g'(u) \geq 0, \quad \text{and} \quad g''(u) > 0 \text{ for } u \geq 0.$$

Thus production costs are a nondecreasing, strongly convex function of the production rate. Also assume that costs are discounted at the rate $r > 0$. Under these assumptions, the optimal inventory accumulation problem is given by

$$\min_{u(\cdot)} J[x(\cdot), u(\cdot)] \stackrel{\text{def}}{=} \int_0^T e^{-rt}\,[g(u(t)) + c_2x]\,dt$$

$$\text{s.t.} \quad \dot{x}(t) = u(t),\ x(0) = 0,\ x(T) = x_T.$$

(a) Derive the necessary conditions.

(b) Show that $\dot{u}(t) > 0$ holds in an optimal plan. Provide an economic interpretation of this result.

(c) Show that

$$e^{-rt}g'(u(t)) + \int_t^{t+\varepsilon} e^{-rs}c_2\,ds = e^{-r[t+\varepsilon]}g'(u(t+\varepsilon))$$

in an optimal plan, where $\varepsilon > 0$ is sufficiently small. Provide an economic interpretation of this integral equation.

(d) Now set the discount rate to zero, that is, set $r \equiv 0$, and derive the necessary conditions.

(e) Show that for a given $u(t)$, $\dot{u}(t)$ is larger for $r > 0$ than for $r \equiv 0$. Explain, in economic terms, why this makes sense. Draw a graph in tu-space to show your reasoning geometrically.

(f) For the $r > 0$ and $r \equiv 0$ cases, show that for a given $u(t)$, $\dot{u}(t)$ increases with an increase in the per-unit holding cost of the inventory c_2. Explain, in economic terms, why this makes sense. Draw a graph in tu-space to show your reasoning geometrically.

4.20 Recall the intertemporal utility maximization problem developed in Example 1.3:

$$\max_{c(\cdot)} J[k(\cdot), c(\cdot)] \stackrel{\text{def}}{=} \int_0^T e^{-rt} U(c(t))\, dt$$

$$\text{s.t.} \quad \dot{k}(t) = w + ik(t) - c(t),\ k(0) = k_0,\ k(T) = k_T.$$

Please refer back to Example 1.3 for the economic interpretation of the model and its variables and parameters if your memory of it is vague. Also recall that $U(\cdot) \in C^{(2)}$ and that $U'(c) > 0$ and $U''(c) < 0$.

(a) Derive the necessary conditions and provide an economic interpretation of the Maximum Principle equation.

(b) Prove that a solution of the necessary conditions, assuming that one exists, is a solution of the intertemporal utility maximization problem.

(c) Prove that

$$e^{-rt} U'(c(t)) = i \int_t^{t+\varepsilon} e^{-rs} U'(c(s))\, ds + e^{-r(t+\varepsilon)} U'(c(t+\varepsilon))$$

along an optimal path, where $\varepsilon > 0$ is sufficiently small. Provide an economic interpretation of this integral equation.

(d) Prove that

$$\dot{c} = \left[\frac{-U'(c)}{U''(c)}\right][i - r]$$

along an optimal path. Provide an economic interpretation.

In order to tease out some more qualitative results and come up with a closed form solution of the problem, assume that $U(c) \stackrel{\text{def}}{=} \ln c$, $k_T = 0$, and $w = 0$.

(e) Derive the unique solution of the control problem under the above simplifying assumptions. Denote it by $(k^*(t;\boldsymbol{\beta}), c^*(t;\boldsymbol{\beta}), \lambda(t;\boldsymbol{\beta}))$, where $\boldsymbol{\beta} \stackrel{\text{def}}{=} (i, r, k_0, T)$.

(f) Obtain the comparative dynamics of the optimal triplet with respect to i, and provide an economic interpretation.
(g) Obtain the comparative dynamics of the optimal triplet with respect to r, and provide an economic interpretation.
(h) Obtain the comparative dynamics of the optimal triplet with respect to k_0, and provide an economic interpretation.
(i) Obtain the comparative dynamics of the optimal triplet with respect to T, and provide an economic interpretation.

4.21 Suppose that at time $t = 0$ (the present), a mine contains an amount $x_0 > 0$ of a nonrenewable resource stock, say, for example, coal. The profit flow that results from extracting and selling the resource at the rate $q(t)$ is given by $\pi(q) \stackrel{\text{def}}{=} \ln q$. The firm lives over the closed interval $[0, T]$ and discounts its profit at the rate $r > 0$. Moreover, the firm is required to extract all the asset by time $t = T$, implying that cumulative extraction over the planning horizon $[0, T]$ must equal the initial size of the deposit $x_0 > 0$. Stated in mathematical terms, this constraint is given by

$$\int_0^T q(s)\,ds = x_0.$$

The firm is asserted to maximize its present discounted value of profit. The optimal control problem can therefore be posed as

$$\max_{q(\cdot)} \int_0^T \pi(q(t))e^{-rt}\,dt$$

$$\text{s.t.} \quad \int_0^T q(t)\,dt = x_0.$$

Unfortunately, at this juncture, we are not able to solve this *isoperimetric* control problem. We will have to wait until Chapter 7 to solve it in this form. We can, however, transform it to a form readily solvable by our current methods. To this end, define $x(t)$ as the cumulative amount of the asset extracted by time t, that is,

$$x(t) \stackrel{\text{def}}{=} \int_0^t q(s)\,ds.$$

Applying Leibniz's rule to this definition and using the integral constraint on the extraction rate yields the associated differential equation and boundary conditions, namely,

$$\dot{x}(t) = q(t),\ x(0) = 0,\ x(T) = x_0.$$

https://doi.org/10.1017/CBO9780511806827.006 Published online by Cambridge University Press

Hence, the equivalent optimal control problem in a form we can handle via the theorems developed so far is given by

$$\max_{q(\cdot)} J[x(\cdot), q(\cdot)] \stackrel{\text{def}}{=} \int_0^T [\ln q(t)]\, e^{-rt}\, dt$$

$$\text{s.t.} \quad \dot{x}(t) = q(t),\ x(0) = 0,\ x(T) = x_0.$$

(a) Derive the solution of the necessary conditions and denote the resulting triplet by $(x^*(t; r, x_0, T), q^*(t; r, x_0, T), \lambda(t; r, x_0, T))$.

(b) Prove that the solution of the necessary conditions is the unique solution of the control problem.

(c) Compute the comparative dynamics of an increase in the initial stock on the present value shadow price of the stock and the extraction rate. That is, derive

$$\frac{\partial \lambda}{\partial x_0}(t; r, x_0, T) \text{ and } \frac{\partial q^*}{\partial x_0}(t; r, x_0, T).$$

Explain, in economic jargon, your results.

(d) Compute the comparative dynamics of an increase in the discount rate on the present value shadow price of the stock and the extraction rate. Because you will not be able to sign this result $\forall\, t \in [0, T]$, evaluate your comparative dynamics results at $t = 0$ and $t = T$. Explain, in economic jargon, your results. **Hint:** Recall that the Taylor series representation for e^y about the point $y = 0$ is given by $e^y = \sum_{k=0}^{+\infty} \frac{y^k}{k!}$.

(e) Compute the comparative dynamics of an increase in the planning horizon on the cumulative stock of the resource and the extraction rate. Explain, in economic jargon, your results.

Now let the profit flow be a general $C^{(2)}$ function of the extraction rate, say, $\pi(q)$, with $\pi'(q) > 0$ and $\pi''(q) < 0$ for $q \geq 0$. Answer the remaining three parts of the exercise based on the general profit function.

(f) Prove that the present value of marginal profit is constant over the entire horizon.

(g) Prove that marginal profit grows exponentially at the discount rate r.

(h) Prove that the optimal extraction rate declines through time.

4.22 Consider the so-called simplest problem in the calculus of variations, defined by

$$\max_{x(\cdot)} J[x(\cdot)] \stackrel{\text{def}}{=} \int_{t_0}^{t_1} f(t, x(t), \dot{x}(t))\, dt$$

$$\text{s.t.} \quad x(t_0) = x_0,\ x(t_1) = x_1.$$

https://doi.org/10.1017/CBO9780511806827.006 Published online by Cambridge University Press

Observe that there are no constraints in the problem and that both of the endpoints are fixed. Assume that $f(\cdot) \in C^{(2)}$ over an open set and that admissible functions are $C^{(2)}$. It is well known that a necessary condition obeyed by the optimizing curve $z(t)$ is given by the differential equation

$$\frac{d}{dt} f_{\dot{x}}(t, x(t), \dot{x}(t)) = f_x(t, x(t), \dot{x}(t)),$$

known as the *Euler equation*. This question asks you to prove that the necessary conditions of the optimal control problem corresponding to the simplest problem in the calculus of variations are equivalent to the Euler equation. One benefit of answering this question is that it provides some background material for Chapter 7.

(a) Transform the above calculus of variations problem into an equivalent optimal control problem by letting $\dot{x}(t) \stackrel{\text{def}}{=} u(t)$ be the state equation.

(b) Write down the necessary conditions for the optimal control problem in (a).

(c) Show that if $(z(\cdot), v(\cdot)) = (z(\cdot), \dot{z}(\cdot))$ is a solution of the necessary conditions of the optimal control problem in (a), then it is also a solution of the Euler equation.

(d) Show that if $(z(\cdot), \dot{z}(\cdot))$ is a solution of the Euler equation, then it is also a solution of the necessary conditions of the optimal control problem in (a). **Hint:** Define the variable $\lambda(t) \stackrel{\text{def}}{=} -f_{\dot{x}}(t, z(t), \dot{z}(t))$.

FURTHER READING

The original statement of the principle of optimality is contained in Bellman (1957). Bellman and Dreyfus (1962) is another useful reference on the principle. Both of these books still provide readable and informative introductions to dynamic programming. Our use of the principle of optimality in this chapter was very specific to the purpose at hand, namely, to prove the necessary conditions for a rather general optimal control problem. As we will see in later chapters, the principle of optimality will lead to the derivation of a useful partial differential equation called the Hamilton-Jacobi-Bellman equation. This equation will become the centerpiece of our foray into intertemporal duality theory. Chiang (1992), Léonard and Van Long (1992), and Kamien and Schwartz (1991) all provide at least some discussion of the principle of optimality and dynamic programming, with the latter providing a bit more than the first two.

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