Necessary Conditions for a Simplified Control Problem

The goal of this chapter is to derive the necessary conditions that an optimal solution *path* or solution *curve* must obey, or equivalently, that an optimal *function* must obey, for the simplest problem in optimal control theory. The necessary conditions to be derived below are known collectively as the Pontryagin necessary conditions. They are the dynamic analogue of the FONCs in unconstrained optimization problems, the latter of which you will recall require that all the first-order partial derivatives of the objective function vanish at the optimal solution.

The simplest problem in optimal control requires that the planning horizon and initial value of the state variable be fixed. The terminal value of the state variable, however, is freely chosen, that is, it is a decision variable. To simplify matters even further in this chapter, we also assume that $M = 1$ and $N = 1$, thereby implying that we are dealing with a single control variable and a single state variable. Given these assumptions, the optimal control problem we will be analyzing in this chapter is given by

$$\begin{gathered}\max_{u(\cdot),x_1} J[x(\cdot), u(\cdot)] \stackrel{\text{def}}{=} \int_{t_0}^{t_1} f(t, x(t), u(t))\, dt \\ \text{s.t.} \quad \dot{x}(t) = g(t, x(t), u(t)), \\ u(t) \in U,\ x(t_0) = x_0,\ x(t_1) = x_1,\end{gathered} \tag{1}$$

where $U \subset \Re$ is a fixed control set that does not depend on the state or control variables. Recall, for example, that U may be a nonnegativity constraint on the control variable or a fixed upper bound on the control variable, among other constraints. Take note of the fact that the terminal value of the state variable is a decision variable, just as the notation of problem (1) conveys.

In order to make the ensuing proofs and presentation as clear as possible, and to avoid getting caught up in somewhat tangential mathematical details (for our purposes), we adopt an overly strong set of assumptions. The adopted assumptions, however, have the distinct advantage of allowing us to derive the necessary

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conditions for problem (1) from the so-called variational point of view. Therefore, to this end, we adopt the following assumptions in addition to the basic ones given in Chapter 1:

(A.1) $f(\cdot) \in C^{(1)}$ and $g(\cdot) \in C^{(1)}$ over an open set.
(A.2) $\exists$ a $C^{(0)}$ optimal control function $v(\cdot)$ defined on $[t_0, t_1]$ that solves problem (1), along with its $C^{(1)}$ associated state function $z(\cdot)$.
(A.3) $v(t) \in \text{int } U \; \forall \, t \in [t_0, t_1]$.

It is important to observe that the assumed $C^{(1)}$ nature of $f(\cdot)$ and $g(\cdot)$ can be weakened. At this point in the development of optimal control theory, however, we will require these stronger assumptions because of the variational-based proof of the necessary conditions that we shall adopt. More generally, all we really need to assume is that $f(\cdot)$ and $g(\cdot)$ are $C^{(0)}$ and that $f_x(\cdot)$ and $g_x(\cdot)$ are likewise $C^{(0)}$, as noted in Chapter 1. The assumed continuity of the optimal control function $v(\cdot)$ and the assumed $C^{(1)}$ nature of the corresponding state function $z(\cdot)$ are similarly overly strong. For now, the strong assumptions are fine given our pedagogical inclination, but we will certainly relax them later to show the full power and reach of optimal control theory. Assumption (A.3) asserts that the optimal control path lies in the *interior* of the control set for the entire planning horizon. This assumption therefore means that the constraints on the control variable are not binding in an optimal plan and can henceforth be ignored for our present development of the necessary conditions. More formally, assumption (A.3) implies that for each $t \in [t_0, t_1]$, $v(t) \in U$ and $B(v(t); \delta) \stackrel{\text{def}}{=} \{u(\cdot) : |u(t) - v(t)| < \delta\} \in U$ for some $\delta > 0$, that is, for each period of the planning horizon, the value of the optimal control belongs to the control set and is the center of an open ball that is contained in the control set for some positive radius defining the open ball. Finally, the assumption that $f(\cdot) \in C^{(1)}$ and $g(\cdot) \in C^{(1)}$ over an *open set* in (A.1) eliminates the need to consider one-sided derivatives in the foregoing analysis seeing as an open set consists only of interior points by its very definition.

Rather than denote the optimal pair of curves by, say, $(x^*(t), u^*(t))$, or the optimal pair of functions by $(x^*(\cdot), u^*(\cdot))$, we prefer to use the notation $(z(t), v(t))$ and $(z(\cdot), v(\cdot))$ for the optimal pair of curves and functions, respectively. As we will see, this has the advantage of reducing notational clutter as we proceed to more general optimal control problems. Proceeding as we would in static optimization theory, we now turn to the definition of admissibility for an optimal control problem (1).

Definition 2.1: If $u(\cdot) \in C^{(0)}$ and $u(t) \in \text{int } U \; \forall \, t \in [t_0, t_1]$, and $x(\cdot) \in C^{(1)} \; \forall \, t \in [t_0, t_1]$ is the corresponding state function that satisfies $\dot{x}(t) = g(t, x(t), u(t))$ and $x(t_0) = x_0$, then $(x(t), u(t))$ is called an *admissible pair*.

Recall that once we specify the path of the control variable $u(t) \; \forall \, t \in [t_0, t_1]$, the path of the state variable is completely determined by the solution of the state equation and initial condition. This follows from the fact that if we substitute a given control path $u(t)$ into the state equation, we can in principle integrate the state equation and use the initial condition to solve for the corresponding path of the state variable. Said differently, the state variable path is completely determined once the solution for the control variable is found. It is natural, therefore, to think of the state and control variables as coming in *pairs*, so we will use this term repeatedly throughout the text. Moreover, we may speak of finding the control function because a corresponding state function is implied. Given that the selection of the control determines the state, it determines the value of the functional $J[\cdot]$ as well.

The domain of the objective functional $J[\cdot]$ is taken to be the set of all admissible function pairs $(x(\cdot), u(\cdot))$. Now recall that $f(\cdot) \in C^{(1)}$ by assumption (A.1), and $u(\cdot) \in C^{(0)}\,\forall\, t \in [t_0, t_1]$ and $x(\cdot) \in C^{(1)}\,\forall\, t \in [t_0, t_1]$ by the definition of admissibility. Therefore, if a specific admissible function pair $(x(\cdot), u(\cdot))$ is substituted in $f(\cdot)$, then when we consider $f(\cdot)$ as a function of the single independent variable t, it must be $C^{(0)}$. Because the definite integral of a $C^{(0)}$ function exists by a fundamental theorem of integral calculus (see, e.g., Taylor and Mann, Chapter 18, Theorem III), the integral (or functional) $J[x(\cdot), u(\cdot)]$ exists. That is, $J[x(\cdot), u(\cdot)]$ is a finite value that depends on the admissible function pair $(x(\cdot), u(\cdot))$. In other words, the objective functional $J[\cdot]$ exists for all admissible function pairs $(x(\cdot), u(\cdot))$. The fundamental problem of optimal control, therefore, is that of finding an admissible function pair $(x(\cdot), u(\cdot))$ that extremizes the functional $J[\cdot]$.

In order to derive the necessary conditions in as simple and straightforward a manner as is possible, we take a *local* view of the situation. The reasoning behind this strategy is as follows. We know that if a function yields a global maximum to the objective functional $J[\cdot]$, it also yields a local maximum. Consequently, a condition that is necessary for a local maximum is also necessary for a lobal maximum. Thus, by deriving the necessary conditions for a local maximum, we simultaneously derive the necessary conditions for a global maximum. Moreover, seeing as an admissible pair $(z(\cdot), v(\cdot))$ that maximizes $J[\cdot]$ also minimizes $-J[\cdot]$, we need only consider optimal control problems in which a maximum is sought when deriving necessary conditions. As a consequence of these observations, we intend to construct an entire family of control curves that are close to, or live in a neighborhood of, the optimal control curve $v(t)$, with the property that for a particular value of some parameter, we can recover the optimal control curve $v(t)$ from the said family. Such a family of comparison curves is known as a weak variation of the optimal control function $v(\cdot)$.

To this end, we therefore consider a one-parameter family of *comparison* or *varied control curves*, that is, *weak variations* of the optimal control curve $v(t)$, defined by

$$u(t;\varepsilon) \stackrel{\text{def}}{=} v(t) + \varepsilon\eta(t), \tag{2}$$

where $v(\cdot) \in C^{(0)}$ and $v(t) \in \operatorname{int} U\,\forall\, t \in [t_0, t_1]$ is the optimal control by assumptions (A.2) and (A.3), respectively, $\eta(\cdot) \in C^{(0)}$ is an *arbitrary* fixed function, and ε is a

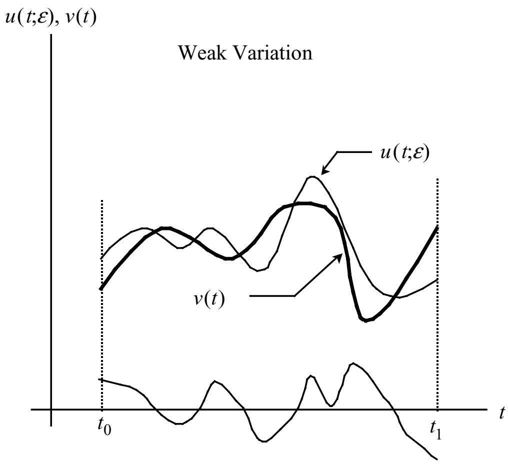

Figure 2.1

parameter. A typical graph of a weak variation of the optimal control curve $v(t)$ is given in Figure 2.1. Recall that because $v(\cdot) \in C^{(0)}$ and $u(\cdot;\varepsilon) \in C^{(0)}$, the two curves displayed in Figure 2.1 may have kinks in them, but not jumps or breaks. It is important to emphasize that in Eq. (2) the *entire* optimal path $v(t)$ has undergone a variation by the term $\varepsilon\eta(t)$, rather than just a particular value of (or point on) the optimal path. Note that this observation also holds in Figure 2.1, for the entire curve $v(t)$ has been altered to produce the neighboring curve $u(t;\varepsilon)$, entirely consistent with the construction in Eq. (2). In other words, the neighboring curve $u(t;\varepsilon)$ in Figure 2.1 was *not* produced from the curve $v(t)$ by changing only a few isolated points of it. This is directly related to the idea behind a functional introduced in Chapter 1, namely, that a functional is a map from curves to the real line. Hence, in considering how a functional varies, we must vary the *entire curve* or *function* and not just a particular point on the curve or a particular value of the function.

The idea behind the above definition of a weak variation of an optimal control curve is rather simple. In particular, given that the optimal control function $v(\cdot)$ is known to exist, we can construct an entire family of functions that have values close to those of the optimal control function by multiplying an arbitrary perturbing function $\eta(\cdot) \in C^{(0)}$ by a sufficiently small scalar ε, and then adding the resulting product $\varepsilon\eta(\cdot)$ to $v(\cdot)$. The smallness of ε ensures that the value of the product $\varepsilon\eta(t)$ is sufficiently small too. To see this, simply observe that because $\eta(\cdot) \in C^{(0)}\ \forall\, t \in [t_0, t_1]$, $\eta(t)$ is bounded due to a basic property of continuous functions defined on closed and bounded sets (see, e.g., Taylor and Mann, Chapter 17, Theorem III). In other words, because the arbitrary function $\eta(\cdot) \in C^{(0)}\ \forall\, t \in [t_0, t_1]$, there exists a number β such that $|\eta(t)| \leq \beta$ for all $t \in [t_0, t_1]$. Consequently, there exists a sufficiently small value of ε, say, $|\varepsilon| < \varepsilon_0$, such that the product $\varepsilon\eta(t)$ is sufficiently small too. Moreover, such an $\varepsilon_0 > 0$ can be found for every arbitrary perturbing function $\eta(\cdot) \in C^{(0)}$. As a result of these observations, it follows that the comparison control curves $u(t;\varepsilon)$ defined in Eq. (2) can be made to lie arbitrarily close to the optimal control curve $v(t)$

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for all $t \in [t_0, t_1]$, a fact you are asked to verify in a mental exercise. What's more, because the optimal control curve $v(t) \in \text{int}\, U\ \forall\, t \in [t_0, t_1]$ by assumption (A.3), it follows that $u(t;\varepsilon) \in \text{int}\, U\ \forall\, t \in [t_0, t_1]$ as well, another fact left for a mental exercise. Note that no restrictions are placed on the value of the function $\eta(\cdot)$ at $t = t_0$ or $t = t_1$ in view of the fact that the control path is not required to meet any initial or terminal boundary conditions, unlike the state variable path.

Let us now examine what the above construction of a weak variation of the optimal control curve $v(t)$ implies for the resulting state variable time path. To begin, define $x(t;\varepsilon)$ as the time path of the state variable that results from substituting the comparison control time path $u(t;\varepsilon) \stackrel{\text{def}}{=} v(t) + \varepsilon\eta(t)$ into the state equation $\dot{x}(t) = g(t, x(t), u(t;\varepsilon))$, and integrating it using the initial condition $x(t_0) = x_0$. That is, $x(t;\varepsilon)$ is the comparison curve of the state variable that is the companion to, or paired with, $u(t;\varepsilon) \stackrel{\text{def}}{=} v(t) + \varepsilon\eta(t)$. By their very construction, therefore, it follows that $(x(t;\varepsilon), u(t;\varepsilon))$ is an admissible pair of curves for all $|\varepsilon| < \varepsilon_0$, or equivalently, $(x(\cdot;\varepsilon), u(\cdot;\varepsilon))$ is an admissible pair of functions for all $|\varepsilon| < \varepsilon_0$. In line with assumption (A.2), we now assume that $x(\cdot) \in C^{(1)}$ in both of its arguments, where ε enters parametrically. Finally, note that given a general functional form for the transition function $g(\cdot)$, it is, in general, impossible to get an explicit solution for the comparison curves $x(t;\varepsilon)$ of the state variable.

Next, observe that by definition (or construction), $u(t;0) = v(t)$ is the optimal path of the control variable, and thus $x(t;0) = z(t)$ is the corresponding optimal path of the state variable. It is important to emphasize that admissibility requires that *all* comparison paths of the state variable satisfy the initial condition $x(t_0) = x_0$. In other words, we have the following identity for the initial value of the comparison paths of the state variable:

$$x(t_0;\varepsilon) \equiv x_0. \tag{3}$$

This identity must hold for all $|\varepsilon| < \varepsilon_0$ in order for the comparison curves $x(t;\varepsilon)$ of the state variable to be admissible. We will return to this identity in due course.

Now *fix* the functions $\eta(\cdot) \in C^{(0)}$, $z(\cdot) \in C^{(1)}$, and $v(\cdot) \in C^{(0)}$ for all $t \in [t_0, t_1]$, and then evaluate the objective functional $J[\cdot]$ along the comparison functions $u(\cdot;\varepsilon) \stackrel{\text{def}}{=} v(\cdot) + \varepsilon\eta(\cdot)$ and $x(\cdot;\varepsilon)$. This implies that $J[\cdot]$ is a *function* of the parameter ε rather than a functional, for we have fixed the functions in its domain, and therefore the only thing left to vary is the parameter ε. Hence, we can define the function $\Phi(\cdot)$ of the single parameter ε by

$$\Phi(\varepsilon) \stackrel{\text{def}}{=} \int_{t_0}^{t_1} f(t, x(t;\varepsilon), u(t;\varepsilon))\, dt. \tag{4}$$

Recalling that $(x(t;0), u(t;0)) = (z(t), v(t))$ is the optimal pair by construction, it follows that $\Phi(0) = J[z(\cdot), v(\cdot)]$, which is the maximum value of the objective functional $J[\cdot]$ by assumption (A.2). Thus, by construction, $\Phi(\varepsilon) \le \Phi(0)$ for all $|\varepsilon| < \varepsilon_0$, thereby implying that the function $\Phi(\cdot)$ of one real variable ε has a relative maximum at $\varepsilon = 0$ by construction. It is important to remember, however, that the optimal and comparison paths of the state and control variables must obey the state equation in order for them to be admissible. Consequently, the optimal value of ε, videlicet, $\varepsilon = 0$, cannot be chosen without taking into account how the state equation impinges on it. For that reason, in finding the necessary conditions that a solution to problem (1) must obey, we follow a procedure reminiscent of solving constrained optimization problems by way of the method of Lagrange multipliers.

More formally, because $(x(t; \varepsilon), u(t; \varepsilon))$ is an admissible pair for all sufficiently small ε by construction, it must satisfy the state equation as an identity, that is,

$$g(t, x(t; \varepsilon), u(t; \varepsilon)) - \dot{x}(t; \varepsilon) \equiv 0 \; \forall \, t \in [t_0, t_1], \tag{5}$$

and for all $|\varepsilon| < \varepsilon_0$. Seeing as Eq. (5) is identically zero for all $t \in [t_0, t_1]$, we may multiply it by some as of yet unknown *function* $\lambda(\cdot) \in C^{(1)} \; \forall \, t \in [t_0, t_1]$, and the resulting expression will still be identically equal to zero for all $t \in [t_0, t_1]$. It is worthwhile to emphasize that because Eq. (5) is a constraint that holds for each $t \in [t_0, t_1]$, we must multiply it by a function of t rather than a single multiplier value as it is not a single constraint in the ordinary sense. Doing just that and then integrating the resulting expression over the closed interval $[t_0, t_1]$ gives

$$\int_{t_0}^{t_1} \lambda(t)[g(t, x(t; \varepsilon), u(t; \varepsilon)) - \dot{x}(t; \varepsilon)] \, dt \equiv 0 \; \forall \, t \in [t_0, t_1]. \tag{6}$$

Observe that Eq. (6) is identically zero for each $t \in [t_0, t_1]$ and for all $|\varepsilon| < \varepsilon_0$ because the integrand is identically zero for each $t \in [t_0, t_1]$ and for all $|\varepsilon| < \varepsilon_0$ by Eq. (5). The function $\lambda(\cdot)$ is called the *costate* or *adjoint function*. Given that the left-hand side of Eq. (6) is identically zero, we are permitted to add it to $\Phi(\varepsilon)$ in Eq. (4) without affecting the latter's value. We then use the fact that the sum of integrals is the integral of the sum, provided the integrands are continuous, which they are here under our assumptions. The result of these two operations is a new form for $\Phi(\varepsilon)$, namely,

$$\Phi(\varepsilon) \stackrel{\text{def}}{=} \int_{t_0}^{t_1} [f(t, x(t; \varepsilon), u(t; \varepsilon)) + \lambda(t)[g(t, x(t; \varepsilon), u(t; \varepsilon)) - \dot{x}(t; \varepsilon)]] \, dt, \tag{7}$$

which incorporates the state equation into the analysis. We are now in a position to proceed with the derivation of the necessary conditions.

To commence with the said derivation, we integrate the term

$$\int_{t_0}^{t_1} \lambda(t) \dot{x}(t; \varepsilon) \, dt$$

by parts. This is the expression to integrate by parts, for any term in the integrand that has been differentiated with respect to the independent variable is a prime candidate

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for an integration by parts operation. Thus,

$$\text{let:} \quad \left.\begin{matrix} p = \lambda(t) & dq = \dot{x}(t;\varepsilon)\,dt \\ dp = \dot{\lambda}(t)\,dt & q = x(t;\varepsilon) \end{matrix}\right\} \Rightarrow$$

$$\int_{t_0}^{t_1} \lambda(t)\dot{x}(t;\varepsilon)\,dt = \lambda(t_1)x(t_1;\varepsilon) - \lambda(t_0)x(t_0;\varepsilon) - \int_{t_0}^{t_1} \dot{\lambda}(t)x(t;\varepsilon)\,dt. \tag{8}$$

By substituting Eq. (8) into Eq. (7), we can write $\Phi(\varepsilon)$ in the following convenient form:

$$\begin{aligned} \Phi(\varepsilon) \overset{\text{def}}{=} \int_{t_0}^{t_1} & [f(t, x(t;\varepsilon), u(t;\varepsilon)) + \lambda(t)g(t, x(t;\varepsilon), u(t;\varepsilon)) + \dot{\lambda}(t)x(t;\varepsilon)]\,dt \\ & - \lambda(t_1)x(t_1;\varepsilon) + \lambda(t_0)x(t_0;\varepsilon). \end{aligned} \tag{9}$$

Because $\Phi(\varepsilon)$ attains its maximum value at $\varepsilon = 0$ by construction, as noted above, we know from static optimization theory that a necessary condition that the optimal value of ε must obey is $\Phi'(\varepsilon)|_{\varepsilon=0} = 0$ in view of the fact that we have already accounted for the effects of the state equation on the choice of ε. Hence, differentiating $\Phi(\varepsilon)$ using Leibniz's rule (Theorem A.2.1 in the appendix to this chapter), evaluating the resulting derivative at $\varepsilon = 0$, and then collecting terms, the necessary condition $\Phi'(0) \overset{\text{def}}{=} \Phi'(\varepsilon)|_{\varepsilon=0} = 0$ takes the form

$$\begin{aligned} \Phi'(0) = & \int_{t_0}^{t_1} [f_x(t, z(t), v(t)) + \lambda(t)g_x(t, z(t), v(t)) + \dot{\lambda}(t)]\, x_\varepsilon(t;0)\,dt \\ & + \int_{t_0}^{t_1} [f_u(t, z(t), v(t)) + \lambda(t)g_u(t, z(t), v(t))]\eta(t)\,dt \\ & - \lambda(t_1)x_\varepsilon(t_1;0) + \lambda(t_0)x_\varepsilon(t_0;0) = 0, \end{aligned} \tag{10}$$

where we have used $(x(t;0), u(t;0)) = (z(t), v(t))$ and $u_\varepsilon(t;0) = \eta(t)$ to simplify the resulting expression. This is the equation we want to work with in order to determine the necessary conditions of optimal control problem (1).

The first thing to notice about Eq. (10) is that the last term is identically zero due to the fixed initial condition $x(t_0;\varepsilon) \equiv x_0$, as given in Eq. (3). Recall that this identity must hold for all admissible state paths. Noting that x_0 is given (or fixed) and is therefore independent of ε, we can differentiate the identity $x(t_0;\varepsilon) \equiv x_0$ with respect to ε and evaluate the resulting derivative at $\varepsilon = 0$ to get $x_\varepsilon(t_0;0) \equiv 0$. This proves that the last term in Eq. (10) vanishes identically, as claimed. Using this result, we may rewrite Eq. (10) as

$$\Phi'(0) = \int_{t_0}^{t_1} [f_x(t, z(t), v(t)) + \lambda(t) g_x(t, z(t), v(t)) + \dot{\lambda}(t)]\, x_\varepsilon(t; 0)\, dt$$
$$+ \int_{t_0}^{t_1} [f_u(t, z(t), v(t)) + \lambda(t) g_u(t, z(t), v(t))]\, \eta(t)\, dt - \lambda(t_1) x_\varepsilon(t_1; 0) = 0, \tag{11}$$

which is the form with which we will henceforth use.

Now recall that the perturbing function $\eta(\cdot) \in C^{(0)}$ was held fixed in the above development, *but was otherwise arbitrary*. Moreover, the function $x(\cdot; \varepsilon)$ and the value $x_\varepsilon(t_1; 0)$ are arbitrary too, for the curve $u(t; \varepsilon)$ depends on the arbitrarily chosen function $\eta(\cdot)$ and thus so does the curve $x(t; \varepsilon)$, as it is the companion to $u(t; \varepsilon)$. Given these two observations, the only way to guarantee that Eq. (11) holds for *any* continuous function $\eta(\cdot)$ is for the coefficients of the two arbitrary functions $\eta(\cdot)$ and $x_\varepsilon(\cdot; 0)$ to vanish for all $t \in [t_0, t_1]$ and for the coefficient of the arbitrary value $x_\varepsilon(t_1; 0)$ to vanish at $t = t_t$. This fact therefore implies that the time path of the costate variable $\lambda(t)$ satisfies the ensuing linear ordinary differential equation and terminal boundary condition

$$\dot{\lambda}(t) = -[f_x(t, z(t), v(t)) + \lambda(t) g_x(t, z(t), v(t))],\ \lambda(t_1) = 0, \tag{12}$$

and that the following stationary condition holds:

$$f_u(t, z(t), v(t)) + \lambda(t) g_u(t, z(t), v(t)) = 0\ \forall\, t \in [t_0, t_1]. \tag{13}$$

Remember that $z(\cdot)$ and $v(\cdot)$ are *given* functions of t, namely, the optimal ones in problem (1). Also recall that we required that the state equation and initial condition be satisfied in the course of the above development, for they were required to hold by the definition of admissibility. Hence, we have proven the following basic theorem.

Theorem 2.1 (Necessary Conditions): *Suppose $v(\cdot) \in C^{(0)}$ and $v(t) \in \mathrm{int} U\ \forall\, t \in [t_0, t_1]$, and let $z(\cdot) \in C^{(1)}\ \forall\, t \in [t_0, t_1]$ be the corresponding state function that satisfies the state equation $\dot{x}(t) = g(t, x(t), u(t))$ and initial condition $x(t_0) = x_0$, so that $(z(t), v(t))$ is an admissible pair. Then if $(z(t), v(t))$ yields the absolute (or global) maximum of $J[x(\cdot), u(\cdot)]$ when $x(t_1) = x_1$ is a decision variable, it is necessary that there exist a function $\lambda(\cdot) \in C^{(1)}\ \forall\, t \in [t_0, t_1]$ such that*

$$f_u(t, z(t), v(t)) + \lambda(t) g_u(t, z(t), v(t)) = 0\ \forall\, t \in [t_0, t_1], \tag{14}$$
$$\dot{\lambda}(t) = -[f_x(t, z(t), v(t)) + \lambda(t) g_x(t, z(t), v(t))],\ \lambda(t_1) = 0, \tag{15}$$
$$\dot{z}(t) = g(t, z(t), v(t)),\ z(t_0) = x_0. \tag{16}$$

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Equation (14) may be called the *simplified Maximum Principle*, the adjective simplified arising from our assumption that $v(t) \in \operatorname{int} U \ \forall\, t \in [t_0, t_1]$. The pair of differential equations (15) and (16) are frequently called the *canonical equations*. Individually, Eq. (15) is referred to as the *costate equation*, whereas Eq. (16) is called the *state equation*. The free terminal boundary condition $\lambda(t_1) = 0$ is often called a *transversality condition*.

Recall that in constrained static optimization problems, the necessary and sufficient conditions are best stated and remembered by introducing an auxiliary function known as the Lagrangian. We can employ a similar function for optimal control problems, called the *Hamiltonian* $H(\cdot)$. It is defined as

$$H(t, x, u, \lambda) \stackrel{\text{def}}{=} f(t, x, u) + \lambda g(t, x, u), \tag{17}$$

from which it follows that

$$\frac{\partial H}{\partial u}(t, x, u, \lambda) = f_u(t, x, u) + \lambda g_u(t, x, u),$$

$$\frac{\partial H}{\partial x}(t, x, u, \lambda) = f_x(t, x, u) + \lambda g_x(t, x, u),$$

$$\frac{\partial H}{\partial \lambda}(t, x, u, \lambda) = g(t, x, u).$$

Given these results, we can restate Theorem 2.1 in the following form, which is far easier to remember and use.

Theorem 2.2 (Necessary Conditions): *Suppose $v(\cdot) \in C^{(0)}$ and $v(t) \in \operatorname{int} U \ \forall\, t \in [t_0, t_1]$, and let $z(\cdot) \in C^{(1)} \ \forall\, t \in [t_0, t_1]$ be the corresponding state function that satisfies the state equation $\dot{x}(t) = g(t, x(t), u(t))$ and initial condition $x(t_0) = x_0$, so that $(z(t), v(t))$ is an admissible pair. Define the Hamiltonian as $H(t, x, u, \lambda) \stackrel{def}{=} f(t, x, u) + \lambda g(t, x, u)$. Then if $(z(t), v(t))$ yields the absolute (or global) maximum of $J[x(\cdot), u(\cdot)]$ when $x(t_1) = x_1$ is a decision variable, it is necessary that there exist a function $\lambda(\cdot) \in C^{(1)} \ \forall\, t \in [t_0, t_1]$ such that*

$$H_u(t, z(t), v(t), \lambda(t)) = 0 \ \forall\, t \in [t_0, t_1], \tag{18}$$

$$\dot{\lambda}(t) = -H_x(t, z(t), v(t), \lambda(t)), \ \ \lambda(t_1) = 0, \tag{19}$$

$$\dot{z}(t) = H_\lambda(t, z(t), v(t), \lambda(t)), \ \ z(t_0) = x_0. \tag{20}$$

The form of the necessary conditions given in Theorem 2.2 is that which you should remember. Moreover, it is important to understand the notation employed in the necessary condition $H_u(t, z(t), v(t), \lambda(t)) = 0 \ \forall\, t \in [t_0, t_1]$. It means that the Hamiltonian function $H(\cdot)$ is first partially differentiated with respect to the control variable, and *then* evaluated along the optimal solution path. That is, more explicitly,

$$H_u(t, z(t), v(t), \lambda(t)) = \left. \frac{\partial H}{\partial u}(t, x, u, \lambda) \right|_{(z(t), v(t), \lambda(t))} = 0 \ \forall\, t \in [t_0, t_1].$$

Notice that by formulating the necessary conditions in terms of the Hamiltonian, it becomes clear that the condition $\partial H/\partial u = 0$ means that for each $t \in [t_0, t_1]$, we choose the control variable so that the Hamiltonian function is stationary for given values of the state and costate variables. In fact, as we will show in Chapter 4, when we present the necessary conditions for a more general class of control problems, the control variable will be chosen so as to maximize the Hamiltonian at each $t \in [t_0, t_1]$ for given values of the state and costate variables.

In principle, the optimal control problem can be solved in the following way to arrive at the paths of the optimal solution triplet $(z(t), v(t), \lambda(t))$, assuming, of course, that such an optimal solution exists. First, notice that we have three equations in Theorem 2.2 in which to solve for the three unknown paths $(z(t), v(t), \lambda(t))$, so in principle, we have the correct number of equations. Moreover, the solution of the canonical equations yields two constants of integration, which can be determined from the initial condition and transversality condition.

Next, notice that the necessary condition $\partial H/\partial u = 0$ is just like a first-order necessary condition from static optimization theory. Written in full it is

$$\frac{\partial H}{\partial u}(t, x, u, \lambda) = 0. \tag{21}$$

Hence if $H_{uu} \neq 0$ along the optimal path for all $t \in [t_0, t_1]$, then by the implicit function theorem we can in principle solve Eq. (21) for u in terms of (t, x, λ), say, $u = \hat{u}(t, x, \lambda)$. Then we can substitute $u = \hat{u}(t, x, \lambda)$ into the canonical differential equations to get

$$\dot{x} = H_\lambda(t, x, \hat{u}(t, x, \lambda), \lambda),\ x(t_0) = x_0, \tag{22}$$

$$\dot{\lambda} = -H_x(t, x, \hat{u}(t, x, \lambda), \lambda),\ \lambda(t_1) = 0. \tag{23}$$

Notice that the control variable u no longer appears in Eqs. (22) and (23) since we have used the implicit function theorem to solve $\partial H/\partial u = 0$ for u in terms of (t, x, λ). Hence, the canonical differential equations (22) and (23) depend only on (t, x, λ). They form a two-point boundary value problem because the state variable is fixed at the initial date of the planning horizon and the costate variable is fixed at zero at the terminal date. Solution of the canonical differential equations along with the boundary conditions yields the optimal solution path for the state and costate variables, scilicet, $(z(t), \lambda(t))$. Substituting these optimal paths into $u = \hat{u}(t, x, \lambda)$ yields the optimal path of the control variable, namely, $v(t) \stackrel{\text{def}}{=} \hat{u}(t, z(t), \lambda(t))$. The optimal path of the control variable $v(t)$ is termed the *open-loop* solution because the control variable is expressed as a function of the independent variable t (and the problem's parameters) but not the value of the state variable at time t.

It is beneficial at this juncture to introduce the economic interpretation of the costate variable $\lambda(t)$. To this end, let's interpret the state variable $x(t)$ as the capital stock of a firm at time t. The objective of the firm is the maximization of its present discounted value of profit, that is, its wealth. Such a wealth-maximizing firm implicitly places a monetary value on the capital stock it owns, and thus similarly places a monetary value on increments to its capital stock at each date of its planning horizon. The costate variable $\lambda(t)$ is precisely the value to the firm of an additional unit of the capital stock. That is, $\lambda(t)$ is the firm's *internal* valuation of another unit of capital. In other words, $\lambda(t)$ measures the increase in the firm's wealth if it had one more unit of capital on hand at time t. Thus $\lambda(t)$ is the most the firm would pay for another unit of capital at time t. In economics, such internal incremental valuations go by the name of *shadow price, shadow value*, or *marginal value*. We may therefore interpret $\lambda(t)$ as the shadow value of the firm's capital stock at time t. Formal justification for this economic interpretation will have to wait for Chapter 9, but we will see some mathematical justification for it in Chapters 3 and 4.

Let's now examine two simple mathematical examples to get some practice at solving the necessary conditions of Theorem 2.2 under our set of simplifying assumptions. After this, we will solve an optimal control problem of some importance in dynamic economic theory, to wit, the adjustment cost model of the firm, and then investigate some of its qualitative properties.

Example 2.1: Consider the optimal control problem

$$\max_{u(\cdot),x_1} J[x(\cdot), u(\cdot)] \stackrel{\text{def}}{=} \int_0^1 \left[-x(t) - \frac{1}{2}\alpha[u(t)]^2\right] dt$$

$$\text{s.t.} \quad \dot{x}(t) = u(t),$$

$$x(0) = x_0,\ x(1) = x_1,$$

where $\alpha > 0$ is a parameter. The Hamiltonian is defined as $H(t, x, u, \lambda; \alpha) \stackrel{\text{def}}{=} -x - \frac{1}{2}\alpha u^2 + \lambda u$. By Theorem 2.2, the necessary conditions are

$$H_u(t, x, u, \lambda; \alpha) = -\alpha u + \lambda = 0,$$

$$\dot{\lambda} = -H_x(t, x, u, \lambda; \alpha) = 1,\ \lambda(1) = 0,$$

$$\dot{x} = H_\lambda(t, x, u, \lambda; \alpha) = u,\ x(0) = x_0.$$

Solving the necessary condition $H_u(t, x, u, \lambda; \alpha) = -\alpha u + \lambda = 0$, we get $u = \alpha^{-1}\lambda$. Substituting this into the canonical differential equations gives

$$\dot{\lambda} = 1,\ \lambda(1) = 0,$$

$$\dot{x} = \alpha^{-1}\lambda,\ x(0) = x_0.$$

Notice that we have followed the recipe given earlier for solving the necessary conditions of optimal control problems.

Now integrate the costate equation $\dot{\lambda} = 1$ to get the general solution $\lambda(t) = t + c_1$, where c_1 is an arbitrary constant of integration. Using the transversality condition $\lambda(1) = 0$ implies that $c_1 = -1$, so the specific solution of the costate equation is $\lambda(t) = t - 1$. Substituting this into the state equation yields $\dot{x} = \alpha^{-1}(t - 1)$, and a routine integration results in the general solution $x(t) = (2\alpha)^{-1}t^2 - \alpha^{-1}t + c_2$, where c_2 is another constant of integration. Using the initial condition $x(0) = x_0$ implies that $c_2 = x_0$, so the specific solution of the state equation is $z(t; \alpha, x_0) = (2\alpha)^{-1}t^2 - \alpha^{-1}t + x_0$. Finally, substitute $\lambda(t) = t - 1$ into $u = \alpha^{-1}\lambda$ to find the corresponding path of the control variable, namely, $v(t; \alpha) = \alpha^{-1}(t - 1)$. Hence the solution to the necessary conditions is given by

$$v(t; \alpha) = \alpha^{-1}(t - 1),$$
$$z(t; \alpha, x_0) = (2\alpha)^{-1}t^2 - \alpha^{-1}t + x_0,$$
$$\lambda(t) = t - 1.$$

Notice the paths for the state and control variables depend on the parameters α and x_0, just like solutions to static optimization problems. Hence, we could differentiate $v(\cdot)$ and $z(\cdot)$ with respect to α to compute the comparative dynamics of the problem. We will delay such considerations until Example 2.3, however, when a meaningful economic problem is developed and solved.

Example 2.2: Consider the control problem

$$\max_{u(\cdot), x_1} J[x(\cdot), u(\cdot)] \stackrel{\text{def}}{=} \int_0^1 \left[\alpha t u(t) - \frac{1}{2}[u(t)]^2\right] dt$$
$$\text{s.t.} \quad \dot{x}(t) = u(t) - x(t),$$
$$x(0) = x_0, \; x(1) = x_1.$$

The Hamiltonian is given by $H(t, x, u, \lambda; \alpha) \stackrel{\text{def}}{=} \alpha t u - \frac{1}{2}u^2 + \lambda[u - x]$, so the necessary conditions from Theorem 2.2 are given by

$$H_u(t, x, u, \lambda; \alpha) = \alpha t - u + \lambda = 0,$$
$$\dot{\lambda} = -H_x(t, x, u, \lambda; \alpha) = \lambda, \; \lambda(1) = 0,$$
$$\dot{x} = H_\lambda(t, x, u, \lambda; \alpha) = u - x, \; x(0) = x_0.$$

From $H_u = 0$, we have $u = \alpha t + \lambda$. Separate the variables in the costate equation to get $\frac{d\lambda}{\lambda} = dt$, which readily integrates to $\ln \lambda = t + c_1$, where c_1 is a constant of integration. This general solution can be rewritten as $\lambda(t) = k_1 e^t$, where $k_1 \stackrel{\text{def}}{=} e^{c_1}$. The transversality condition $\lambda(1) = 0$ implies that $k_1 = 0$. Hence the specific solution of the costate equation is $\lambda(t) = 0 \; \forall \, t \in [0, 1]$. This, in turn, implies that $v(t; \alpha) = \alpha t$ is the path of the control variable. Using $v(t; \alpha) = \alpha t$ in the state equation gives $\dot{x} = \alpha t - x$, or $\dot{x} + x = \alpha t$. The integrating factor for this ordinary differential equation is given by

$$\mu(t) \stackrel{\text{def}}{=} \exp\left[\int^{t} ds\right] = e^{t}.$$

Hence, upon multiplying $\dot{x} + x = \alpha t$ on both sides by e^t we have $\dot{x}e^t + xe^t = \frac{d}{dt}[xe^t] = \alpha te^t$. Integrating both sides gives $xe^t = \alpha \int te^t dt + k_2$, where k_2 is a constant of integration. Now perform an integration-by-parts exercise on this by letting $p = t$, so that $dp = dt$, and letting $dq = e^t\, dt$, so that $q = e^t$, thereby yielding

$$\int te^t dt = te^t - \int e^t dt = e^t[t-1].$$

Thus the general solution of the state equation is $x(t) = \alpha[t-1] + k_2 e^{-t}$. Using the initial condition $x(0) = x_0$ implies that $k_2 = \alpha + x_0$. The specific solution to the state equation is therefore given by $z(t; \alpha, x_0) = \alpha[t-1] + [\alpha + x_0]\, e^{-t}$. In sum, the solution to the necessary conditions is

$$v(t; \alpha) = \alpha t,$$

$$z(t; \alpha, x_0) = \alpha[t-1] + [\alpha + x_0]\, e^{-t},$$

$$\lambda(t) = 0\ \forall\, t \in [0, 1],$$

which depends on the parameters α and x_0, just like the solution of the necessary conditions in Example 2.1.

In closing out this example, it is worthwhile to pause and emphasize a somewhat unexpected feature of the solution of the necessary conditions. In particular, take note of the fact that the time path of the control variable that solves the necessary conditions, to wit, $v(t; \alpha) = \alpha t$, is *identical* to the value of the control variable that maximizes the integrand function at each t. In effect, this means that the above control problem is not really a dynamic optimization problem. Moreover, the state equation has no effect on the solution for the control variable. This is consistent with the discussion in Chapter 1, in which we emphasized that it was the state equation, and the appearance of the control variable in it, that made for a dynamic optimization problem. In this example, however, the control variable does appear in the state equation, so that is not the source of this atypical result. Two features are responsible for the peculiarity: (i) the integrand function is independent of the state variable, and (ii) the terminal value of the state variable is a decision variable. In Chapter 3, we will see in general why this is so.

Now that we have seen how to use Theorem 2.2 to find solution candidates, let's now use it to examine an intertemporal economic model of the firm and, more importantly, demonstrate how qualitative information about the model may be uncovered. This example and the manner in which it is solved and analyzed are therefore more in accord with how economists use optimal control theory in order to reach economic insights about a particular behavioral model.

Example 2.3: In this example, we examine a simplified version of the adjustment cost model of the capital accumulating firm that we introduced in Example 1.2. Moreover, by making some functional form assumptions, we will be able to explicitly calculate the solution to the necessary conditions. To that end, let the single output-producing firm have a linear production function, say, $f(K) \overset{\text{def}}{=} K$, where the capital stock at time t, $K(t)$, is measured in units such that one unit of capital produces one unit of output. The output is sold at the constant price $p > 0$, and holding (or maintenance) costs per unit of the capital stock are constant and given by $w > 0$, where it is assumed that $p - w > 0$. The cost of buying and installing new purchases of the capital stock are given by the cost of adjustment function $C(\cdot)$, with values $C(I) \overset{\text{def}}{=} \frac{1}{2}cI^2$, where $I(t)$ is the gross investment rate at time t and $c > 0$ is a cost of adjustment parameter. Assume that depreciation is nonexistent, so that $I(t)$ can then also be interpreted as the net investment rate at time t. For the purpose of obtaining a simpler solution of the necessary conditions, assume that the firm does *not* discount its flow of profit. The firm begins its operation at time $t = 0$ with the given capital stock $K_0 > 0$, but has the freedom to choose its terminal capital stock $K_T > 0$ at the end of its given planning horizon $T > 0$. The firm must have a nonnegative stock of capital on hand at all times, which implies the constraint $K(t) \geq 0 \ \forall\, t \in [0, T]$, but it may buy or sell the capital stock, thereby implying that $I(t)$ may be positive, zero, or negative in each period of the planning horizon. The firm is asserted to solve the following optimal control problem:

$$\max_{I(\cdot),k_T} J[K(\cdot), I(\cdot)] \overset{\text{def}}{=} \int_0^T \left[pK(t) - wK(t) - \frac{1}{2}c[I(t)]^2\right] dt$$

$$\text{s.t.} \quad \dot{K}(t) = I(t),\ K(0) = K_0,\ K(T) = K_T.$$

Notice that we have ignored that constraint $K(t) \geq 0 \ \forall\, t \in [0, T]$ in the problem statement. We will show that this is justified by proving that a solution of the necessary conditions will have $K(t) > 0 \ \forall\, t \in [0, T]$.

Defining the Hamiltonian as $H(K, I, \lambda) \overset{\text{def}}{=} [p - w]K - \frac{1}{2}cI^2 + \lambda I$, Theorem 2.2 yields the ensuing necessary conditions:

$$H_I(K, I, \lambda) = -cI + \lambda = 0, \tag{24}$$

$$\dot{\lambda} = -H_K(K, I, \lambda) = w - p,\ \ \lambda(T) = 0, \tag{25}$$

$$\dot{K} = H_\lambda(K, I, \lambda) = I,\ \ K(0) = K_0. \tag{26}$$

Before solving the necessary conditions, let's see what qualitative information can be gleaned from them. First, observe that because $[p - w] > 0$ by assumption $\dot{\lambda}(t) < 0 \ \forall\, t \in [0, T]$ from Eq. (25). Second, because $\lambda(T) = 0$, it in addition follows that $\lambda(t) > 0 \ \forall\, t \in [0, T)$. Third, in view of the fact that $I(t) = c^{-1}\lambda(t)$ from Eq. (24), $c > 0$ by assumption, and $\lambda(t) > 0 \ \forall\, t \in [0, T)$, it follows that $I(t) > 0 \ \forall\, t \in [0, T)$. Fourth, as $I(t) = c^{-1}\lambda(t)$ and $\lambda(T) = 0$, we have that $I(T) = 0$. Fifth, given that $\dot{K}(t) = I(t)$ and $K(0) = K_0 > 0$ from Eq. (26), and that $I(t) > 0 \ \forall\, t \in [0, T)$ and

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$I(T) = 0$, it follows that $\dot{K}(t) > 0\ \forall\, t \in [0, T)$, $\dot{K}(T) = 0$ and that $K(t) > 0\ \forall\, t \in [0, T]$, the latter demonstrating that the nonnegativity restriction on the capital stock is not binding in an optimal plan (assuming that one exists). In words, we have shown that (i) the shadow value of the capital stock is positive throughout the planning horizon and zero only at the terminal date, (ii) the shadow value of the capital stock is decreasing throughout the planning horizon, (iii) the investment rate is positive throughout the planning horizon and zero only at the terminal date, and (iv) the capital stock is positive throughout the planning horizon. Observe that the above results demonstrate the equivalence of the transversality condition $\lambda(T) = 0$ and the investment rate $I(T) = 0$. This equivalence is intuitive, for once the firm enters the last period of the planning horizon, there is no more future. Thus by purchasing capital goods in the last period, the firm would only incur costs of adjustment, but no benefits (increased future output) from doing so. Finally, note that all of these qualitative conclusions were derived without actually solving the necessary conditions. Let's now turn to the explicit solution of the necessary conditions.

Integrating the costate equation, we get the general solution $\lambda(t) = [w - p]t + a_1$, where a_1 is a constant of integration. The transversality condition $\lambda(T) = 0$ implies that $a_1 = [p - w]T > 0$, given that $[p - w] > 0$. The specific solution to the costate equation is therefore given by the expression $\lambda(t; p, w, T) = [p - w][T - t] \geq 0\ \forall\, t \in [0, T]$. Because $I(t) = c^{-1}\lambda(t)$ from Eq. (24), the time path of the investment rate is given by $I^*(t; c, p, w, T) = c^{-1}[p - w][T - t] \geq 0\ \forall\, t \in [0, T]$. Using this expression, the state equation becomes $\dot{K} = c^{-1}[p - w][T - t]$, which upon integration yields the general solution $K(t) = c^{-1}[p - w][T - (t/2)]t + a_2$, where a_2 is another constant of integration. Using the initial condition $K(0) = K_0$ gives $a_2 = K_0$ as the value of the constant of integration. The specific solution for the time path of the capital stock is therefore given by $K^*(t; c, p, w, K_0, T) = c^{-1}[p - w][T - (t/2)]t + K_0$. Recalling that the production function is given by $f(K) \stackrel{\text{def}}{=} K$, we may also interpret $K^*(t; c, p, w, K_0, T)$ equivalently as the time path of output of the firm. In passing, note that because the triplet $(K^*(t; c, p, w, K_0, T), I^*(t; c, p, w, T), \lambda(t; p, w, T))$ is the *only* solution of the necessary conditions, we know that *if* an optimal solution exists to the adjustment cost model, then this triplet is the unique optimal solution.

In the prototypical models of the competitive firm, such as cost minimization and profit maximization, the demand and supply functions are positively homogeneous of degree zero in the prices. The same is true of the investment demand function $I^*(\cdot)$ in the above adjustment cost model. To see this, simply observe that for any $\theta \in \Re_{++}$, we have that

$$\begin{aligned} I^*(t; \theta c, \theta p, \theta w, T) &= \frac{[\theta p - \theta w]}{\theta c}[T - t] = \frac{\theta[p - w]}{\theta c}[T - t] \\ &= \frac{[p - w]}{c}[T - t] = I^*(t; c, p, w, T), \end{aligned}$$

which is the definition of positive homogeneity of degree zero in (c, p, w). This asserts that if the output price, holding cost per unit of capital, and adjustment costs all rise proportionally, then there is no change in the investment rate of the firm. A mental exercise asks you to examine some other homogeneity properties present in the model. Let's now turn to some comparative dynamics calculations.

Because we have explicit solutions for the time paths of the capital stock (or equivalently, output), the investment rate, and the shadow value of the capital stock, one method of conducting a comparative dynamics analysis is to differentiate these explicit solutions with respect to the parameter of interest. By way of example, let's consider the effects of an increase in the output price p. Differentiating the triplet $(K^*(t; c, p, w, K_0, T), I^*(t; c, p, w, T), \lambda(t; p, w, T))$ with respect to p yields

$$\frac{\partial K^*}{\partial p}(t; c, p, w, K_0, T) = c^{-1}[T - (t/2)]t \begin{cases} = 0 \text{ at } t = 0 \\ > 0 \; \forall\, t \in (0, T], \end{cases} \tag{27}$$

$$\frac{\partial I^*}{\partial p}(t; c, p, w, T) = c^{-1}[T - t] \begin{cases} > 0 \; \forall\, t \in [0, T) \\ = 0 \text{ at } t = T, \end{cases} \tag{28}$$

$$\frac{\partial \lambda}{\partial p}(t; p, w, T) = [T - t] \begin{cases} > 0 \; \forall\, t \in [0, T) \\ = 0 \text{ at } t = T. \end{cases} \tag{29}$$

Equation (28) shows that an increase in the output price drives up the investment rate of the firm in every period but the last. This is not unexpected, for the capital stock of the firm is the only input to production, and Eq. (27) shows that the firm produces more of its now more valuable output. Hence the only way to achieve this end is for the firm to invest more in its only input. Finally, in view of the fact that product produced by the firm is now more valuable in the market, the firm values its capital input more highly too, just as Eq. (29) shows. We elect to end the discussion here and relegate the remaining comparative dynamics calculations and interpretations to a mental exercise.

The final theorem in this chapter is an important generalization of Theorem 2.2. Moreover, the generalization is a natural one from an economic point of view. To see this, you may have noticed that in the control problem (1) studied so far, there was no value placed on the state variable remaining at the terminal time; that is, the benefit associated with a positive stock of capital at the terminal date was zero. In other words, even though we let the decision maker optimally select the terminal amount of the state variable, we assumed that the salvage or scrap value of the state variable remaining at the terminal date was zero. In most economic situations, the capital remaining after the firm has shut down still has some value in the market. Hence, we now relax that rather simple-minded situation and consider the following *scrap value* or *salvage value* optimal control problem:

$$\max_{u(\cdot),x_1} J_S[x(\cdot),u(\cdot)] \stackrel{\text{def}}{=} \int_{t_0}^{t_1} f(t,x(t),u(t))\,dt + S(x_1)$$

$$\text{s.t.}\quad \dot{x}(t) = g(t,x(t),u(t)), \tag{30}$$

$$x(t_0) = x_0,\ x(t_1) = x_1.$$

In order to proceed, we must impose the following additional assumption on problem (30):

(A.4) $S(\cdot) \in C^{(1)}$ over an open convex set.

The proof of the following theorem is relegated to the mental exercises, as it is a straightforward extension of the proof of Theorem 2.1.

Theorem 2.3 (Necessary Conditions, Salvage Value): *Suppose that $v(\cdot) \in C^{(0)}$ and that $v(t) \in \operatorname{int} U\ \forall\, t \in [t_0,t_1]$, and let $z(\cdot) \in C^{(1)}\ \forall\, t \in [t_0,t_1]$ be the corresponding state function that satisfies the state equation $\dot{x}(t) = g(t,x(t),u(t))$ and initial condition $x(t_0) = x_0$, so that $(z(t),v(t))$ is an admissible pair. Then if $(z(t),v(t))$ yields the absolute (or global) maximum of $J_S[x(\cdot),u(\cdot)]$ when $x(t_1) = x_1$ is a decision variable, it is necessary that there exist a function $\lambda(\cdot) \in C^{(1)}\ \forall\, t \in [t_0,t_1]$ such that*

$$H_u(t,z(t),v(t),\lambda(t)) = 0\ \forall\, t \in [t_0,t_1], \tag{31}$$

$$\dot{\lambda}(t) = -H_x(t,z(t),v(t),\lambda(t)),\ \lambda(t_1) = S'(z(t_1)), \tag{32}$$

$$\dot{z}(t) = H_\lambda(t,z(t),v(t),\lambda(t)),\ z(t_0) = x_0. \tag{33}$$

In the last example of this chapter, we employ Theorem 2.3 to solve a variant of the inventory accumulation problem introduced in Example 1.4. Moreover, we will conduct a relatively comprehensive comparative dynamics analysis of the problem, without which the economic intuition and insight provided by the model would be less than what one would get from interpretation of the necessary conditions alone.

Example 2.4: Let's revisit the inventory accumulation problem originally stated in Example 1.4, but with a slight twist on it. Specifically, now contemplate the case in which the firm in question can decide on the terminal amount of the good held in inventory, that is, $x(T) = x_T$ is now a decision variable. Moreover, the firm can sell its accumulated inventory $x(T) = x_T$ at the terminal date T in one chunk to a retail firm at a price of $p > 0$, thereby generating revenue of px_T dollars. The firm still incurs production and inventory holding costs as before, but now seeks a production schedule to maximize its profit from the production and sale of the good produced.

That is, the firm now solves the following salvage value control problem:

$$\max_{u(\cdot),x_T} J_S[x(\cdot),u(\cdot)] \stackrel{\text{def}}{=} px_T - \int_0^T \left[c_1[u(t)]^2 + c_2x(t)\right] dt$$

$$\text{s.t.} \quad \dot{x}(t) = u(t),\ x(0) = 0,\ x(T) = x_T,$$

where all terms are as defined earlier. Notice that in this version of the problem, the revenue received by the firm comes only at the terminal date when all the inventory is sold, whereas production and storage costs are incurred continuously throughout the planning horizon as the good is produced and stored. Assume that $p > c_2T$, an assumption that we will return to in the course of solving this problem.

Define the Hamiltonian for this salvage value control problem as

$$H(x,u,\lambda;c_1,c_2) \stackrel{\text{def}}{=} -c_1u^2 - c_2x + \lambda u,$$

where λ is the costate variable. We will prove in Chapter 9 that λ at time t has the economic interpretation of the marginal or shadow value of the state variable at time t. Hence we may interpret $\lambda(t)$ as the shadow value of a unit of the good in inventory at time t, given that the state variable $x(t)$ is the stock of the good in inventory at time t. Note that the Hamiltonian is defined *excluding* the salvage value term.

According to Theorem 2.3, the necessary conditions are given by

$$H_u = -2c_1u + \lambda = 0,$$

$$\dot{\lambda} = -H_x = c_2,\ \lambda(T) = p,$$

$$\dot{x} = H_\lambda = u,\ x(0) = 0.$$

Seeing as the necessary condition $H_u = 0$ defines the production rate in terms of the shadow value of the stock, that is, $u = \lambda/(2c_1)$, and the state equation defines the rate of change of the stock in terms of the production rate, that is, $\dot{x} = u$, the key to solving the necessary conditions is to therefore solve the costate equation.

Before we begin solving the necessary conditions, observe that because $\dot{\lambda} = c_2 > 0$ and $\lambda(T) = p > 0$, the shadow value of the inventory increases over the planning horizon until it equals the market price of the good at the terminal date, at which time the stock of inventory is sold. This is intuitive, for an additional unit of inventory at the initial date must incur larger storage costs than an additional unit of inventory produced at a later date, so we would expect the shadow value of the inventory to rise over time. Furthermore, at the terminal date, an additional unit of inventory does not have to be held and so incurs no storage cost, and thus is valued at the market price.

To solve the costate equation $\dot{\lambda} = c_2$, separate the variables to get $d\lambda = c_2\,dt$, and then integrate to get the general solution $\lambda = a_1 + c_2t$, where a_1 is a constant of integration. Next, use the transversality condition $\lambda(T) = p$ to find that $a_1 = p - c_2T > 0$, the inequality resulting from the assumption that $p > c_2T$. The specific solution to the costate equation is therefore given by $\lambda(t;c_2,p,T) = p + c_2[t-T]$.

In view of the fact that $\lambda(0; c_2, p, T) = p - c_2T > 0$, we conclude, in addition to what we did in the previous paragraph, that the shadow value of the inventory is positive throughout the planning horizon. Moreover, notice that it is the presence of the inventory holding costs that causes the divergence between the shadow value of the inventory and the market price of the good. In other words, if the good was costless to store, then the shadow value of the inventory would be equal to the market price of the good throughout the planning horizon.

The solution of the necessary conditions for the production rate is found by substituting $\lambda(t; c_2, p, T) = p + c_2[t - T]$ into $u = \lambda/(2c_1)$, thereby yielding $v(t; \boldsymbol{\beta}) = [p + c_2[t - T]]/(2c_1)$, where $\boldsymbol{\beta} \stackrel{\text{def}}{=} (c_1, c_2, p, T)$ is the parameter vector for the problem. Since $v(0; \boldsymbol{\beta}) = [p - c_2T]/(2c_1) > 0$ and $\dot{v}(t; \boldsymbol{\beta}) = c_2/(2c_1) > 0$, the production rate is positive throughout the planning horizon and increasing over time. This latter feature is a result of the storage cost of the inventory, that is, $c_2 > 0$ rather than $c_2 = 0$. In other words, the positive cost associated with inventory storage biases the firm's production plan to produce the good at a higher rate in the latter part of the planning horizon because such goods will incur lower storage costs than those produced earlier in the planning horizon.

The time path of the inventory stock that solves the necessary conditions is found by substituting $v(t; \boldsymbol{\beta}) = [p + c_2[t - T]]/(2c_1)$ in the state equation, separating the variables and integrating, thus giving $x = t[2p + c_2[t - 2T]]/(4c_1) + a_2$ as the general solution, where a_2 is a constant of integration. The initial condition $x(0) = 0$ implies that $a_2 = 0$. Thus the specific solution for the inventory stock is $z(t; \boldsymbol{\beta}) = t[2p + c_2[t - 2T]]/(4c_1)$. In passing, note that the assumption $p > c_2T$ is sufficient to conclude that for all $t \in [0, T]$, the shadow value of the inventory is positive and the production rate is positive, and thus a positive amount of the good will be produced and sold by the firm. The assumption $p > c_2T$ means that the market price of the good is greater than the storage cost of a unit of the good produced in the initial period of the planning horizon.

To finish up this example, let's do a simple comparative dynamics exercise and provide an economic interpretation of the results, focusing on the time path of the production rate $v(t; \boldsymbol{\beta}) = [p + c_2[t - T]]/(2c_1)$, the shadow value of the inventory $\lambda(t; c_2, p, T) = p + c_2[t - T]$, and the amount of the good sold $z(T; \boldsymbol{\beta}) = T[2p - c_2T]/(4c_1)$ (given that this is a decision variable in the current version of the model). Differentiating these solutions with respect to, say, c_2, gives

$$\frac{\partial v(t; \boldsymbol{\beta})}{\partial c_2} = \frac{t - T}{2c_1} \begin{cases} < 0 \; \forall \, t \in [0, T), \\ = 0 \;\text{ at }\; t = T, \end{cases} \tag{34}$$

$$\frac{\partial \lambda(t; c_2, p, T)}{\partial c_2} = [t - T] \begin{cases} < 0 \; \forall \, t \in [0, T), \\ = 0 \;\text{ at }\; t = T, \end{cases} \tag{35}$$

$$\frac{\partial z(T; \boldsymbol{\beta})}{\partial c_2} = \frac{-T^2}{4c_1} < 0. \tag{36}$$

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Equation (36) shows that the higher inventory holding costs result in lower total production of the good. By Eq. (36), the firm accomplishes this by producing the good at a slower rate in all the periods in the planning horizon except the last. Equation (35) shows that the shadow value of the inventory is lower as a result of the increase in the inventory holding costs. This was to be expected, given that an additional unit of the good in inventory now costs the firm more to hold and is thus less valuable to the firm.

These comparative dynamics results turn out to contrast sharply with those of the simple version of the inventory accumulation model developed in Example 1.4, as we will see in Chapter 4. For example, in the simple model, an increase in holding costs does not affect total production of the good because it is fixed by assumption. Rather, it results in the production rate being lower in the first half of the planning horizon and higher in the second half of the planning horizon, with the net effect leaving total production unchanged. The remaining comparative dynamics of this model and their economic interpretation are left for the mental exercises.

In closing out this chapter, it is worthwhile to stress a few important aspects of the results developed herein. First, remember that Theorems 2.2 and 2.3 give a set of *necessary conditions* for a pair of curves $(z(t), v(t))$ to yield a maximum (or minimum) of the functional $J[\cdot]$ or $J_s[\cdot]$. Therefore, just because a pair of curves satisfies the necessary conditions of Theorems 2.2 and 2.3, we cannot, at this juncture, be sure that the pair actually solves the optimal control problem under consideration. In particular, we *cannot* conclude that the solutions of the necessary conditions of Examples 2.1–2.4 solve the posed optimal control problem. That is the nature of necessary conditions in optimization problems. Once we have studied sufficient conditions in the next chapter, however, we will be able to conclude that the solutions of the necessary conditions of Examples 2.1–2.4 are in fact the unique solutions to the posed optimal control problems. In attempting to find a solution to an optimal control problem, one therefore searches for solutions of the necessary conditions, for an optimal solution must satisfy them. In other words, if a function does not satisfy the necessary conditions, then it cannot be the solution to the optimal control problem under consideration.

Typically, a central issue in economics when employing optimal control theory is this: "Given an objective functional $J[\cdot]$, find an admissible pair of functions $(z(\cdot), v(\cdot))$ such that $J[z(\cdot), v(\cdot)] \geq J[x(\cdot), u(\cdot)]$ for all admissible function pairs $(x(\cdot), u(\cdot))$." As a result, we can employ Theorem 2.2 (or its numerous generalizations) to obtain candidate functions. *If a maximizing pair of functions exists* and the hypotheses of Theorem 2.2 are met, then a maximizing pair of functions is among the solutions to the necessary conditions of Theorem 2.2. To verify that a particular admissible pair is indeed maximizing, one may proceed in two ways:

(i) Find *all* pairs of functions that are solutions of the necessary conditions. Prove the existence of a maximizing pair of functions that are admissible and show

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that all the hypotheses of Theorem 2.2 are met. Compare the values of $J[\cdot]$ for all solution pairs of the necessary conditions.

(ii) Employ *sufficient* conditions for a maximizing pair of functions to verify that a particular solution pair of the necessary conditions is maximizing.

In any case, the usual first step is to utilize Theorem 2.2 to deduce solutions to the necessary conditions.

APPENDIX

In the appendix, we present, but do not prove, a theorem that shows how to differentiate a function that is defined by an integral when the integrand function and its domain are bounded. This theorem will be used repeatedly throughout our development of the necessary conditions of optimal control theory, as well as in developing the dynamic counterpart of the envelope theorem. It often, but not always, goes by the name of Leibniz's rule. See Protter and Morrey (1991, Theorem 11.3) for a proof of the theorem.

Theorem A.2.1 (Leibniz's Rule): *Let $f(\cdot)$ be a function with domain a rectangle given by $R \stackrel{\text{def}}{=} \{(t, y) : a \le y \le b, c \le t \le d\} \subset \Re^2$, and with a range in $\Re$. Suppose that $f(\cdot)$ and $f_y(\cdot)$ are continuous on R and that the functions $h_0(\cdot)$ and $h_1(\cdot)$ both have a continuous first derivative on the interval $I \stackrel{\text{def}}{=} \{y : a \le y \le b\} \subset \Re$ with range on $J \stackrel{\text{def}}{=} \{t : c \le t \le d\} \subset \Re$. If the function $\phi(\cdot) : I \to \Re$ is defined by*

$$\phi(y) \stackrel{\text{def}}{=} \int_{h_0(y)}^{h_1(y)} f(t, y)\, dt,$$

then its derivative is given by

$$\phi'(y) = f(h_1(y), y)h_1'(y) - f(h_0(y), y)h_0'(y) + \int_{h_0(y)}^{h_1(y)} f_y(t, y)\, dt.$$

We will demonstrate Theorem A.2.1 with some examples.

Example A.2.1: Consider the function $\phi(\cdot)$ defined by the integral

$$\phi(y) \stackrel{\text{def}}{=} \int_0^1 \frac{\sin yt}{1 + t}\, dt.$$

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Thus $f(t, y) \stackrel{\text{def}}{=} \frac{\sin yt}{1+t}$ in this case. Because $f(\cdot)$ and $f_y(\cdot) = \frac{t \cos yt}{1+t}$ are continuous for $t \in [0, 1]$ and $y \in \Re$, we can apply Theorem A.2.1 to get

$$\phi'(y) = \int_0^1 \frac{t \cos yt}{1+t}\, dt.$$

Notice that we do not have to compute the integral to find the value of the derivative, as this is one of the benefits of Leibniz's rule.

Example A.2.2: Given the function $F(\cdot)$ defined by

$$F(x, y) \stackrel{\text{def}}{=} \int_y^{\ln x} \frac{\sin xt}{t(1+y)}\, dt,$$

let's find $F_x(x, y)$. In order to find this partial derivative using Leibniz's rule, we simply treat the variable y as fixed and apply Theorem A.2.1. Doing just that, we get

$$F_x(x, y) = \left[\frac{\sin(x \ln x)}{\ln x(1+y)}\right]\frac{1}{x} + \int_y^{\ln x} \frac{\cos xt}{(1+y)}\, dt$$

as the result we are after.

MENTAL EXERCISES

2.1 Let $\phi(x) \stackrel{\text{def}}{=} \int_0^1 [(1+t)^{-1} \sin xt]\, dt$. Find $\phi'(x)$.

2.2 Let $\phi(x) \stackrel{\text{def}}{=} \int_1^2 e^{-t}(1+xt)^{-1}\, dt$. Find $\phi'(x)$.

2.3 Let $\phi(q) \stackrel{\text{def}}{=} \int_1^{q^2} \cos(t^2)\, dt$. Find $\phi'(q)$.

2.4 Let $\phi(y) \stackrel{\text{def}}{=} \int_{y^2}^{y} \sin(yt)\, dt$. Find $\phi'(y)$.

2.5 Let $\phi(x) \stackrel{\text{def}}{=} \int_{\cos x}^{1+x^2} e^{-t}(1+xt)^{-1}\, dt$. Find $\phi'(x)$.

2.6 Let $\phi(y) \stackrel{\text{def}}{=} \int_{y^2}^{y} t^{-1} \sin(yt)\, dt$. Find $\phi'(y)$.

2.7 Let $\phi(x) \stackrel{\text{def}}{=} \int_{x^m}^{x^n} (x+t)^{-1}\, dt$. Find $\phi'(x)$.

2.8 Let $F(x, y) \stackrel{\text{def}}{=} \int_y^{x^2} t^{-1} e^{xt}\, dt$. Find $F_x(x, y)$ and $F_y(x, y)$.

2.9 Let $F(x, y) \stackrel{\text{def}}{=} \int_{x^2+y^2}^{x^2-y^2} (t^2 + 2x^2 - y^2)\, dt$. Find $F_x(x, y)$ and $F_y(x, y)$.

2.10 Let $F(x, y) \stackrel{\text{def}}{=} \int_{h_0(x,y)}^{h_1(x,y)} f(t, x, y)\, dt$. Find $F_x(x, y)$ and $F_y(x, y)$.

2.11 Suppose that the equation $\int_{h_0(y)}^{h_1(x)} f(t, x, y)\, dt = 0$, which is a relation between x and y, actually defines y as a function of x. If we write $y = \phi(x)$ for this function, find the derivative $\phi'(x)$.

2.12 In the discussion contained in the appendix concerning the differentiation of functions defined by definite integrals, no mention was made of the derivative with respect to the variable of integration t. Is that a justifiable omission? Why or why not?

2.13 Prove that the comparison control curves $u(t;\varepsilon)$ defined in Eq. (2) can be made to lie arbitrarily close to the optimal control curve $v(t)$ for all $t \in [t_0, t_1]$. More formally, prove that there exists a value $\varepsilon_0(\delta)$ such that for all $|\varepsilon| < \varepsilon_0(\delta)$, $u(t;\varepsilon) \in B(v(t);\delta)\ \forall\, t \in [t_0, t_1]$. Make sure you also find the value $\varepsilon_0(\delta)$.

2.14 Prove that if $v(t) \in \operatorname{int} U\ \forall\, t \in [t_0, t_1]$, then $u(t;\varepsilon) \in \operatorname{int} U\ \forall\, t \in [t_0, t_1]$, where the comparison control curves $u(t;\varepsilon)$ are defined in Eq. (2).

2.15 Consider the optimal control problem

$$\max_{u(\cdot),x_1} J[x(\cdot),u(\cdot)] \stackrel{\text{def}}{=} \int_0^1 [x(t)+u(t)]\,dt$$

$$\text{s.t.}\quad \dot{x}(t) = 1-[u(t)]^2,\ x(0)=1,\ x(1)=x_1.$$

(a) Write down the necessary conditions for this problem.

(b) Find the paths for the state, control, and costate variables that satisfy the necessary conditions.

2.16 Find the solution of the necessary conditions for the control problem

$$\min_{u(\cdot),x_T} J[x(\cdot),u(\cdot)] \stackrel{\text{def}}{=} \int_0^T [[x(t)]^2 + ax(t) + bu(t) + c[u(t)]^2]\,dt$$

$$\text{s.t.}\quad \dot{x}(t) = u(t),\ x(0)=x_0,\ x(T)=x_T,$$

where $c > 0$ holds, but no restrictions are placed on a or b.

2.17 Find the solution of the necessary conditions for the control problem

$$\max_{u(\cdot),x_5} J[x(\cdot),u(\cdot)] \stackrel{\text{def}}{=} \int_1^5 [u(t)x(t) - [x(t)]^2 - [u(t)]^2]\,dt$$

$$\text{s.t.}\quad \dot{x}(t) = x(t)+u(t),\ x(1)=2,\ x(5)=x_5.$$

2.18 Find the solution of the necessary conditions for the control problem

$$\min_{u(\cdot),x_1} J[x(\cdot),u(\cdot)] \stackrel{\text{def}}{=} \int_0^1 [u(t)]^2\,dt$$

$$\text{s.t.}\quad \dot{x}(t) = x(t)+u(t),\ x(0)=1,\ x(1)=x_1.$$

2.19 Find the solution of the necessary conditions for the control problem

$$\max_{u(\cdot),x_2} J[x(\cdot),u(\cdot)] \stackrel{\text{def}}{=} \int_0^2 [x(t) - [u(t)]^2]\,dt$$

$$\text{s.t.} \quad \dot{x}(t) = u(t),\ x(0) = 0,\ x(1) = x_2.$$

2.20 Find the solution of the necessary conditions for the control problem

$$\max_{u(\cdot),x_1} J[x(\cdot),u(\cdot)] \stackrel{\text{def}}{=} \int_0^1 -\frac{1}{2}\,[[x(t)]^2 + [u(t)]^2]\,dt$$

$$\text{s.t.} \quad \dot{x}(t) = u(t) - x(t),\ x(0) = 1,\ x(1) = x_1.$$

2.21 Find the solution of the necessary conditions for the control problem

$$\min_{u(\cdot),x_1} J[x(\cdot),u(\cdot)] \stackrel{\text{def}}{=} \int_0^1 [[x(t)]^2 + [u(t)]^2]\,dt$$

$$\text{s.t.} \quad \dot{x}(t) = u(t),\ x(0) = 1, x(1) = x_1.$$

2.22 Find the solution of the necessary conditions for the control problem

$$\min_{u(\cdot),x_1} J[x(\cdot),u(\cdot)] \stackrel{\text{def}}{=} \int_0^1 [tu(t) + [u(t)]^2]\,dt$$

$$\text{s.t.} \quad \dot{x}(t) = u(t),\ x(0) = 1,\ x(1) = x_1.$$

2.23 Find the solution of the necessary conditions for the control problem

$$\min_{u(\cdot),x_2} J[x(\cdot),u(\cdot)] \stackrel{\text{def}}{=} \int_0^2 [t^2 + [u(t)]^2]\,dt$$

$$\text{s.t.} \quad \dot{x}(t) = u(t),\ x(0) = 4,\ x(1) = x_2.$$

2.24 Consider the following variant of the intertemporal utility maximization problem introduced in Example 1.3:

$$\max_{c(\cdot),k_T} J[k(\cdot),c(\cdot)] \stackrel{\text{def}}{=} \int_0^T e^{-rt}U(c(t))\,dt$$

$$\text{s.t.} \quad \dot{k}(t) = w + ik(t) - c(t),\ k(0) = k_0,\ k(T) = k_T.$$

Note that unlike Example 1.3, the terminal capital stock is freely chosen in this version of the problem. Also recall that $U(\cdot) \in C^{(1)}$, $U'(c) > 0$, and $U''(c) < 0$ are assumed to hold.

(a) Write down all the necessary conditions.

(b) Show that no solution to the problem exists.

(c) Can you provide an economic explanation for this nonexistence result?
(d) Determine two different ways to avoid the nonexistence result.

2.25 This mental exercise continues the qualitative investigation of the adjustment cost model of the firm started in Example 2.3.
(a) Prove that for all $\theta \in \Re_{++}$, $K^*(t; \theta c, \theta p, \theta w, K_0, T) = K^*(t; c, p, w, K_0, T)$. Provide an economic interpretation of this result.
(b) Prove that for all $\theta \in \Re_{++}$, $K^*(t; c, \theta p, \theta w, \theta K_0, T) = \theta K^*(t; c, p, w, K_0, T)$. Provide an economic interpretation of this result.
(c) Prove that for all $\theta \in \Re_{++}$, $\lambda(t; \theta p, \theta w, T) = \theta \lambda(t; p, w, T)$. Provide an economic interpretation of this result.
(d) Derive the comparative dynamics for the parameter c. Provide an economic interpretation of the results.
(e) Derive the comparative dynamics for the parameter w. Provide an economic interpretation of the results.
(f) Derive the comparative dynamics for the parameter K_0. Provide an economic interpretation of the results.
(g) Derive the comparative dynamics for the parameter T. Provide an economic interpretation of the results.
(h) Show and explain why

$$\frac{d}{dT}\dot{K}^*(T; \boldsymbol{\beta}) = \ddot{K}^*(T; \boldsymbol{\beta}) + \frac{\partial}{\partial T}\dot{K}^*(T; \boldsymbol{\beta}) \equiv 0,$$

where $\boldsymbol{\beta} \stackrel{\text{def}}{=} (c, p, w, K_0, T)$. Provide an economic interpretation of this comparative dynamics result.

2.26 Derive the necessary conditions obeyed by an optimal solution $(z(t), v(t))$ of the optimal control problem

$$\max_{u(\cdot), x_0, x_1} J[x(\cdot), u(\cdot)] \stackrel{\text{def}}{=} \int_{t_0}^{t_1} f(t, x(t), u(t))\, dt$$

$$\text{s.t.} \quad \dot{x}(t) = g(t, x(t), u(t)),\ x(t_0) = x_0,\ x(t_1) = x_1,$$

under assumptions (A.1), (A.2), and (A.3).

2.27 Consider the optimal control problem

$$\max_{u(\cdot), x_0, x_1} J[x(\cdot), u(\cdot)] \stackrel{\text{def}}{=} \int_{t_0}^{t_1} f(t, x(t), u(t))\, dt$$

$$\text{s.t.} \quad \dot{x}(t) = g(t, x(t), u(t)),\ x(t_0) = x_0,\ x(t_1) = x_1.$$

(a) Write down all the necessary conditions.

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(b) Prove that

$$\int_{t_0}^{t_1} H_x(t, z(t), v(t))\, dt = 0,$$

where $H(\cdot)$ is the Hamiltonian and $(z(t), v(t))$ is a solution of the necessary conditions.

(c) Provide an interpretation of this result.

2.28 Find the solution of the necessary conditions for the control problem

$$\min_{u(\cdot), x_0, x_1} J[x(\cdot), u(\cdot)] \stackrel{\text{def}}{=} \int_0^1 \left[\frac{1}{2}[u(t)]^2 + x(t)u(t) + x(t)\right] dt$$

$$\text{s.t.} \quad \dot{x}(t) = u(t),\ x(0) = x_0,\ x(1) = x_1.$$

2.29 Prove Theorem 2.3.

2.30 Find the solution of the necessary conditions for the optimal control problem

$$\min_{u(\cdot), x_1} J[x(\cdot), u(\cdot)] \stackrel{\text{def}}{=} \int_0^1 [u(t)]^2\, dt + [x_1]^2$$

$$\text{s.t.} \quad \dot{x}(t) = x(t) + u(t),\ x(0) = 1,\ x(1) = x_1.$$

2.31 Consider the optimal control problem

$$\max_{u(\cdot), x_1} J[x(\cdot), u(\cdot)] \stackrel{\text{def}}{=} \int_{t_0}^{t_1} f(t, x(t), u(t))\, dt$$

$$\text{s.t.} \quad \dot{x}(t) = g(t, x(t), u(t)),\ x(t_0) = x_0,\ x(t_1) = x_1.$$

Let $(z(t), v(t))$ be the optimal pair of curves, and let $\lambda(t)$ be the corresponding time path of the costate variable. Define the Hamiltonian as

$$H(t, x, u, \lambda) \stackrel{\text{def}}{=} f(t, x, u) + \lambda g(t, x, u).$$

(a) Prove that

$$\frac{d}{dt} H(t, z(t), v(t), \lambda(t)) = \frac{\partial}{\partial t} H(t, z(t), v(t), \lambda(t)).$$

(b) Prove that if the optimal control problem is autonomous, that is, the independent variable t doesn't enter $f(\cdot)$ or $g(\cdot)$ explicitly, i.e., $f_t(t, x, u) = g_t(t, x, u) \equiv 0$, then $H(\cdot)$ is constant along the optimal path.

(c) An autonomous calculus of variations problem is defined as

$$\max_{x(\cdot), x_1} J[x(\cdot)] \stackrel{\text{def}}{=} \int_{t_0}^{t_1} F(x(t), \dot{x}(t))\, dt$$

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$$\text{s.t.} \quad x(t_0) = x_0, \; x(t_1) = x_1.$$

It can be shown that the Euler equation for this autonomous calculus of variations problem may be written as

$$F(x, \dot{x}) - \dot{x} F_{\dot{x}}(x, \dot{x}) = \text{constant}.$$

Show that $F(x, \dot{x}) - \dot{x} F_{\dot{x}}(x, \dot{x}) = \text{constant}$ if and only if $H(t, x, u, \lambda) = \text{constant}$. Note that you are to rewrite the above autonomous calculus of variations problem as an optimal control problem in order to demonstrate this equivalence.

2.32 Derive and interpret the comparative dynamics corresponding to the parameters (c_1, p, T) for Example 2.4.

2.33 **True, False, Explain:** If you found two different solutions to the necessary conditions, then you would have two different solutions of the optimal control problem.

2.34 **True, False, Explain:** An admissible pair $(x(t), u(t))$ that does not satisfy the necessary conditions is not a solution of a given optimal control problem.

2.35 **True, False, Explain:** If $(x_1(t), u_1(t))$ and $(x_2(t), u_2(t))$ are the only two admissible pairs and $J[x_1(\cdot), u_1(\cdot)] > J[x_2(\cdot), u_2(\cdot)]$, then the pair $(x_1(t), u_1(t))$ would be the solution of the optimal control problem under consideration.

2.36 Let the pair of curves $(z(t), v(t))$ be the optimal solution to

$$\max_{u(\cdot)} J[x(\cdot), u(\cdot)] \stackrel{\text{def}}{=} \int_{t_0}^{t_1} f(t, x(t), u(t))\, dt$$

$$\text{s.t.} \quad \dot{x}(t) = g(t, x(t), u(t)), \; x(t_0) = x_0, \; x(t_1) = x_1.$$

Now select a point $\tau \in (t_0, t_1)$, and let $x_\tau = z(\tau)$. The fact that both end points are fixed in this control problem and that we have not examined the necessary conditions for such a class of control problems is irrelevant in answering this question.

(a) Prove that the pair of curves $(z(t), v(t))$ is the optimal solution to

$$\max_{u(\cdot)} J[x(\cdot), u(\cdot)] \stackrel{\text{def}}{=} \int_{t_0}^{\tau} f(t, x(t), u(t))\, dt$$

$$\text{s.t.} \quad \dot{x}(t) = g(t, x(t), u(t)), \; x(t_0) = x_0, \; x(\tau) = x_\tau.$$

(b) Explain in words what you have just proved.

2.37 Prove that if the pair of curves $(z(t), v(t))$ is the optimal solution to

$$\max_{u(\cdot), x_1} J[x(\cdot), u(\cdot)] \stackrel{\text{def}}{=} \int_{t_0}^{t_1} f(t, x(t), u(t))\, dt$$

$$\text{s.t.} \quad \dot{x}(t) = g(t, x(t), u(t)), \; x(t_0) = x_0, \; x(t_1) = x_1,$$

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then the pair of curves $(z(t), v(t))$ is also the optimal solution to

$$\max_{u(\cdot),x_1} \hat{J}[x(\cdot), u(\cdot)] \stackrel{\text{def}}{=} \int_{t_0}^{t_1} [f(t, x(t), u(t)) + F(t)]\, dt$$

$$\text{s.t.} \quad \dot{x}(t) = g(t, x(t), u(t)),\ x(t_0) = x_0,\ x(t_1) = x_1.$$

Explain your steps and reasoning clearly.

FURTHER READING

The advanced calculus text by Taylor and Mann (1983, 3rd Ed.) is both a highly readable treatment of the material and an outstanding reference. Their insight into some of the subtleties of advanced calculus is without peer. The real analysis book by Protter and Morrey (1991, 2nd Ed.) is also highly recommended, for much the same reasons. The gentlest introduction to the theorems of this chapter may be found in Chiang (1992). This book, for all intents and purposes, is pitched at the advanced undergraduate level, requiring only Chiang's (1984, 3rd Ed.) *Fundamental Methods of Mathematical Economics* as a mathematical prerequisite. The Kamien and Schwartz (1991, 2nd Ed.) text, on the other hand, is pitched at a higher level, arguably the same level as the exposition of the chapter proper, albeit with rather terse development of the material.

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