EIGHTEEN

Qualitative Properties of Infinite Horizon Optimal Control Problems with One State Variable and One Control Variable

Since the publication of Samuelson's *Foundations of Economic Analysis* (1947), economists have been aware of the importance of developing theories that are, at least in principle, refutable. As you may recall, refutable theories are those that lead to well-defined predictions about potentially observable variables in response to perturbations in parameters, as well as curvature properties, homogeneity restrictions, and reciprocity or symmetry results. Such empirically interesting models are the standard fare in the static theory of the consumer and firm, and via duality theory, these models have been subjected to numerous tests of their veracity.

In contrast, the economics literature using dynamic optimization models contains many analyses that are deficient in testable implications. For example, many authors usually state the problem under consideration, derive and economically interpret the necessary conditions, make some reference to sufficient conditions, discuss the existence and multiplicity of steady state equilibria, and finally, analyze the stability of the steady state equilibria. Although all these aspects are important, they do not represent a complete picture of an optimal control model, for the qualitative properties of the model are not examined.

The purpose of this chapter is to examine a ubiquitous class of optimal control models for its particular mathematical structure that leads to steady state comparative statics and local comparative dynamics refutable propositions. The class of optimal control problems under consideration is one with an infinite time horizon, time entering explicitly only through the exponential discount factor, a single state variable, a single control variable, a time-independent vector of parameters influencing the integrand and state equation, a given initial state, and an optimal solution that converges to the steady state solution as an indefinite amount of time passes. Arguably, this class of optimal control problems comprises the largest class of applied dynamic optimization models in economic theory.

We take a systematic approach to uncovering the qualitative properties of one-dimensional optimal control problems. First, the local stability of the steady state is investigated. Next, using the local stability result, a comparative statics analysis of the steady state is carried out. Finally, using the local stability property of the

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steady state and the steady state comparative statics, a local comparative dynamics analysis is performed. The theorems are applied to the adjustment cost model of the firm to demonstrate their usefulness in deriving qualitative results in optimal control problems, as well as to elucidate the underlying mathematical structure responsible for such qualitative results. We refer the reader to Caputo (1997), on which this chapter is based, for a more complete literature review and further technical details.

The optimal control problem under consideration is to find a control function $u(\cdot)$ and its associated response function $x(\cdot)$ that solve

$$\begin{aligned} &\max_{u(\cdot)} \int_0^{+\infty} f(x(t), u(t); \boldsymbol{\alpha})\, e^{-rt}\, dt \\ \text{s.t.}\quad &\dot{x}(t) = g(x(t),\ u(t); \boldsymbol{\beta}),\ x(0) = x_0, \\ &(x(t), u(t)) \in X \times U. \end{aligned} \tag{1}$$

The following assumptions are imposed on problem (1) and explained subsequently:

(A.1) $f(\cdot) : X \times U \times A \to \Re$ and $f(\cdot) \in C^{(2)}$ for all $(x, u; \boldsymbol{\alpha}) \in X \times U \times A$, where $X \subset \Re$, $U \subset \Re$, and $A \subset \Re^{K_1}$ are convex and compact sets, and $\boldsymbol{\alpha}$ is a time-independent vector of parameters.

(A.2) $g(\cdot) : X \times U \times B \to \Re$ and $g(\cdot) \in C^{(2)}$ for all $(x, u; \boldsymbol{\beta}) \in X \times U \times B$, where $B \subset \Re^{K_2}$ is a compact and convex set, $\boldsymbol{\beta}$ is a time-independent vector of parameters, and $g_u(x(t), u(t); \boldsymbol{\beta}) \neq 0$ along the optimal path.

(A.3) There exists a unique optimal solution pair to the optimal control problem (1) for all $(\boldsymbol{\alpha}, \boldsymbol{\beta}, r, x_0) \in \operatorname{int}(A \times B \times D \times X)$, denoted by $(z(t; \boldsymbol{\theta}, x_0), v(t; \boldsymbol{\theta}, x_0))$ with corresponding current value costate variable $\lambda(t; \boldsymbol{\theta}, x_0)$, where $\boldsymbol{\theta} \overset{\text{def}}{=} (\boldsymbol{\alpha}, \boldsymbol{\beta}, r)$, $D \subset \Re_+$ is a compact and convex set, and r is the time-independent discount rate.

(A.4) The optimal pair $(z(t; \boldsymbol{\theta}, x_0), v(t; \boldsymbol{\theta}, x_0)) \in \operatorname{int}(X \times U)$ for all $(\boldsymbol{\theta}, x_0) \in \operatorname{int}(A \times B \times D \times X)$ and for all $t \in [0, +\infty)$.

(A.5) $H_{uu}(x(t), u(t), \lambda(t); \boldsymbol{\alpha}, \boldsymbol{\beta}) \neq 0$ along the optimal path, where $H(\cdot)$ is the current value Hamiltonian function associated with problem (1).

(A.6) There exists a unique, interior, and simple steady state solution to problem (1) for all $\boldsymbol{\theta} \in \operatorname{int}(A \times B \times D)$, denoted by $(x^*(\boldsymbol{\theta}), u^*(\boldsymbol{\theta})) \in \operatorname{int}(X \times U)$, which is the unique solution to the steady state version of the necessary conditions.

(A.7) $\lim_{t \to +\infty}(z(t; \boldsymbol{\theta}, x_0), v(t; \boldsymbol{\theta}, x_0)) = (x^*(\boldsymbol{\theta}), u^*(\boldsymbol{\theta}))$ for all $(\boldsymbol{\theta}, x_0) \in \operatorname{int}(A \times B \times D \times X)$.

Consider assumptions (A.1) and (A.2) first. The assumed differentiability of the functions $f(\cdot)$ and $g(\cdot)$ is common to virtually all applied papers in economics that use optimal control theory. It allows the use of the differential calculus in computing the local comparative dynamics and steady state comparative statics of the model. Furthermore, by Theorem 6.29 of Protter and Morrey (1991), $f(\cdot)$ and $g(\cdot)$ and their first-and second-order partial derivatives are bounded because their domains are compact and $f(\cdot)$ and $g(\cdot)$ are $C^{(2)}$. Moreover, the bounded nature of $f(\cdot)$ along with the exponential discounting implies the objective functional of problem (1) converges for all admissible pairs. The assumption that $g_u(x(t), u(t); \boldsymbol{\beta}) \neq 0$ along the optimal path means that the control variable affects the evolution of the state variable in an optimal plan, and in conjunction with Eq. (3), it implies that $\lambda(t)$ is well defined in an optimal plan. Additionally, $g_u(x(t), u(t); \boldsymbol{\beta}) \neq 0$, Eq. (3), and the boundedness of the first partial derivatives of $f(\cdot)$ and $g(\cdot)$ imply that $\lambda(\cdot)$ is bounded in an optimal plan. Hence the present value costate variable has a value of zero in the limit of the planning horizon, that is, $\lim_{t\to+\infty} e^{-rt}\lambda(t) = 0$.

Assumption (A.3) is important in that without it, one cannot guarantee that a solution to problem (1) exists unless some other assumptions are employed. Given that the focus of this chapter is on the qualitative properties of problem (1), this assumption is innocuous from an economic (but not a mathematical) point of view. The restriction that the parameters lie in the interior of their convex and compact sets rules out the mathematical complications that arise when the optimal solution functions are differentiated with respect to a parameter that is at the boundary of its set. Furthermore, note that the parameter vector $\boldsymbol{\theta}$ is constant over time, implying that the agent that solves problem (1) has static expectations.

Assumption (A.4) allows the analysis to ignore the mathematical complications necessitated when the optimal paths are at the boundary of the feasible region. In practice, this assumption implies that, say, nonnegativity restrictions on the state and control variables are not binding for optimal paths. Furthermore, the compact nature of X and U rules out unbounded $(x(t), u(t))$ pairs as optimal.

Assumption (A.5) implies that the second-order sufficient condition for maximizing the current value Hamiltonian holds along the optimal path. Moreover, it allows the necessary conditions to be reduced to a pair of differential equations in $(x(t), u(t))$.

Assumption (A.6) asserts the existence of a unique and simple steady state solution to problem (1). Recall that a simple fixed point, that is, steady state, is one in which both of the eigenvalues associated with the Jacobian matrix of the linearization of the differential equations evaluated at the fixed point are nonzero, or equivalently, that the determinant of the said Jacobian matrix is not zero. It will be shown below that the assumed simplicity of the steady state and the mathematical structure of problem (1) imply that the steady state is hyperbolic. A hyperbolic fixed point, as you may recall, is one in which the real part of the eigenvalues associated with the Jacobian matrix of the linearization of the differential equations evaluated at the fixed point is nonzero. This means that the classical linearization theorem of ordinary differential equations, scilicet, Theorem 13.7, can be applied to study the local stability properties of the fixed point. Moreover, it implies that the phase portraits of the original nonlinear system and its linearization at the fixed point are qualitatively equivalent in a neighborhood of the fixed point.

Finally, assumption (A.7) asserts that the optimal paths of the state and control variables converge to their steady state values as $t \to +\infty$. In other words, problem (1) is a class of discounted infinite-horizon optimal control problems in which it turns out that the free terminal endpoint optimal solutions converge to their steady state values as $t \to +\infty$. It will be shown that assumptions (A.6) and (A.7) imply that the steady state is a local saddle point.

In order to derive the necessary conditions for problem (1), first define the current value Hamiltonian by

$$H(x, u, \lambda; \boldsymbol{\alpha}, \boldsymbol{\beta}) \stackrel{\text{def}}{=} f(x, u; \boldsymbol{\alpha}) + \lambda g(x, u; \boldsymbol{\beta}). \tag{2}$$

By Theorems 14.3 and 14.9, the necessary conditions that must hold along the optimal path are

$$H_u(x, u, \lambda; \boldsymbol{\alpha}, \boldsymbol{\beta}) = f_u(x, u; \boldsymbol{\alpha}) + \lambda g_u(x, u; \boldsymbol{\beta}) = 0, \tag{3}$$

$$H_{uu}(x, u, \lambda; \boldsymbol{\alpha}, \boldsymbol{\beta}) = f_{uu}(x, u; \boldsymbol{\alpha}) + \lambda g_{uu}(x, u; \boldsymbol{\beta}) \leq 0, \tag{4}$$

$$\dot{\lambda} = r\lambda - H_x(x, u, \lambda; \boldsymbol{\alpha}, \boldsymbol{\beta}) = [r - g_x(x, u; \boldsymbol{\beta})]\lambda - f_x(x, u; \boldsymbol{\alpha}), \tag{5}$$

$$\dot{x} = H_\lambda(x, u, \lambda; \boldsymbol{\alpha}, \boldsymbol{\beta}) = g(x, u; \boldsymbol{\beta}),\ x(0) = x_0, \tag{6}$$

$$\lim_{t \to +\infty} e^{-rt} H(x, u, \lambda; \boldsymbol{\alpha}, \boldsymbol{\beta}) = \lim_{t \to +\infty} e^{-rt} [f(x, u; \boldsymbol{\alpha}) + \lambda g(x, u; \boldsymbol{\beta})] = 0. \tag{7}$$

Because $f(\cdot)$ and $g(\cdot)$ are bounded and $\lambda(\cdot)$ is bounded in an optimal plan, as noted above, the limiting transversality condition (7) does indeed hold in an optimal plan. In light of the ensuing analysis, it will prove convenient to reduce the necessary conditions (3), (5), and (6) to a pair of nonlinear first-order ordinary differential equations in (x, u).

To this end, first note that $H_{uu}(x, u, \lambda; \boldsymbol{\alpha}, \boldsymbol{\beta}) < 0$ along the optimal path by Eq. (4) and assumption (A.5), which in turn permits us to differentiate Eq. (3) with respect to t to get

$$\begin{aligned} f_{uu}(x, u; \boldsymbol{\alpha})\dot{u} \;+\; & f_{ux}(x, u; \boldsymbol{\alpha})\dot{x} + \lambda[g_{uu}(x, u; \boldsymbol{\beta})\dot{u} \\ & + g_{ux}(x, u; \boldsymbol{\beta})\dot{x}] + g_u(x, u; \boldsymbol{\beta})\dot{\lambda} = 0. \end{aligned} \tag{8}$$

In view of the fact that $g_u(x(t), u(t); \boldsymbol{\beta}) \neq 0$ along the optimal path by assumption (A.2), we can solve Eq. (3) for the current value costate variable, namely, $\lambda = -f_u(x, u; \boldsymbol{\alpha})/g_u(x, u; \boldsymbol{\beta})$, and substitute it along with Eqs. (5) and (6) into Eq. (8) to get

$$\begin{aligned} \dot{u} = \; & \frac{f_u(x, u; \boldsymbol{\alpha})[r - g_x(x, u; \boldsymbol{\beta})] + g_u(x, u; \boldsymbol{\beta}) f_x(x, u; \boldsymbol{\alpha})}{f_{uu}(x, u; \boldsymbol{\alpha}) - f_u(x, u; \boldsymbol{\alpha})[g_u(x, u; \boldsymbol{\beta})]^{-1} g_{uu}(x, u; \boldsymbol{\beta})} \\ & + \frac{g(x, u; \boldsymbol{\beta}) \left[\dfrac{f_u(x, u; \boldsymbol{\alpha})}{g_u(x, u; \boldsymbol{\beta})} g_{ux}(x, u; \boldsymbol{\beta}) - f_{ux}(x, u; \boldsymbol{\alpha}) \right]}{f_{uu}(x, u; \boldsymbol{\alpha}) - f_u(x, u; \boldsymbol{\alpha})[g_u(x, u; \boldsymbol{\beta})]^{-1} g_{uu}(x, u; \boldsymbol{\beta})} \end{aligned} \tag{9}$$

$$\dot{x} = g(x, u; \boldsymbol{\beta}) \tag{10}$$

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as the pair of necessary conditions for problem (1). Because the denominator of Eq. (9) is equal to $H_{uu}(x, u, \lambda; \boldsymbol{\alpha}, \boldsymbol{\beta})$, as can be seen by solving Eq. (3) for $\lambda = -f_u(x, u; \boldsymbol{\alpha})/g_u(x, u; \boldsymbol{\beta})$ and substituting into Eq. (4), and $H_{uu}(x, u, \lambda; \boldsymbol{\alpha}, \boldsymbol{\beta}) < 0$ along the optimal path as noted above, Eq. (9) is well defined. For future reference, define $h(x, u; \boldsymbol{\theta})$ as the right-hand side of Eq. (9). This definition is of value when we discuss the local stability of the steady state solution of Eqs. (9) and (10).

Note that we could have reduced the necessary conditions (3), (5), and (6) to a pair of nonlinear first-order ordinary differential equations in (λ, x) as follows. By assumption (A.5), we can apply the implicit function theorem to Eq. (3) to express u locally as a function of $(x, \lambda; \boldsymbol{\alpha}, \boldsymbol{\beta})$, say, $u = \hat{u}(x, \lambda; \boldsymbol{\alpha}, \boldsymbol{\beta})$. Then by substituting $u = \hat{u}(x, \lambda; \boldsymbol{\alpha}, \boldsymbol{\beta})$ into Eqs. (5) and (6), we would have the desired result. Seeing as the information in the (λ, x) dynamical system is the same as that in Eqs. (9) and (10), there is no need to pursue the alternative approach as well.

The steady state of problem (1) is defined as the solution to Eqs. (9) and (10) when $\dot{x} = \dot{u} = 0$, in which case they reduce to

$$f_u(x, u; \boldsymbol{\alpha})[r - g_x(x, u; \boldsymbol{\beta})] + g_u(x, u; \boldsymbol{\beta}) f_x(x, u; \boldsymbol{\alpha}) = 0, \tag{11}$$

$$g(x, u; \boldsymbol{\beta}) = 0. \tag{12}$$

Given any value of $\boldsymbol{\theta} \in \operatorname{int}(A \times B \times D)$, say, $\boldsymbol{\theta} = \boldsymbol{\theta}^\circ$, assumption (A.6) asserts the existence of a unique point $(x^\circ, u^\circ) = (x^*(\boldsymbol{\theta}^\circ), u^*(\boldsymbol{\theta}^\circ)) \in \operatorname{int}(X \times U)$ that satisfies Eqs. (11) and (12). Thus, by assumptions (A.1), (A.2), and (A.6), and the implicit function theorem,

$$u = u^*(\boldsymbol{\theta}), \tag{13}$$

$$x = x^*(\boldsymbol{\theta}), \tag{14}$$

$$\lambda = \lambda^*(\boldsymbol{\theta}) = \frac{-f_u\left(x^*(\boldsymbol{\theta}), u^*(\boldsymbol{\theta}); \boldsymbol{\alpha}\right)}{g_u\left(x^*(\boldsymbol{\theta}), u^*(\boldsymbol{\theta}); \boldsymbol{\beta}\right)} \tag{15}$$

are the unique $C^{(1)}$ solutions to Eqs. (11) and (12) for all $\boldsymbol{\theta} \in B(\boldsymbol{\theta}^\circ; \varepsilon)$, since the steady state Jacobian determinant of Eqs. (11) and (12), scilicet,

$$\begin{aligned} |J_s| \stackrel{\text{def}}{=} {} & g_x^*[[r - g_x^*] f_{uu}^* + f_x^* g_{uu}^*] - g_{ux}^*[g_u^* f_x^* + f_u^* g_x^*] - [g_u^*]^2 f_{xx}^* \\ & + g_u^*[f_u^* g_{xx}^* - [r - 2g_x^*] f_{ux}^*], \end{aligned} \tag{16}$$

is nonzero at $(x^\circ, u^\circ, \boldsymbol{\theta}^\circ)$ by assumption (A.6) and Eq. (23), where $g_x^* \stackrel{\text{def}}{=} g_x(x^*(\boldsymbol{\theta}), u^*(\boldsymbol{\theta}); \boldsymbol{\beta})$ and so on signify the evaluation of the derivatives at the steady state.

Given the existence of the simple steady state of problem (1), attention is now turned to the local stability properties of the steady state. To this end, apply Taylor's theorem to Eqs. (9) and (10), and recall the definition of $h(x, u; \boldsymbol{\theta})$ given after Eq. (10) in order to derive the linearized system of differential equations evaluated

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at the steady state:

$$\begin{bmatrix} \Delta\dot{u} \\ \Delta\dot{x} \end{bmatrix} = \begin{bmatrix} h_u^* & h_x^* \\ g_u^* & g_x^* \end{bmatrix} \begin{bmatrix} \Delta u \\ \Delta x \end{bmatrix}, \tag{17}$$

where

$$h_u^* \stackrel{\text{def}}{=} \frac{[r - g_x^*] f_{uu}^* + f_x^* g_{uu}^*}{f_{uu}^* - f_u^* [g_u^*]^{-1} g_{uu}^*}, \tag{18}$$

$$h_x^* \stackrel{\text{def}}{=} \frac{[r - 2g_x^*] f_{ux}^* - f_u^* g_{xx}^* + g_u^* f_{xx}^* + g_{ux}^* \left[f_u^* [g_u^*]^{-1} g_x^* + f_x^* \right]}{f_{uu}^* - f_u^* [g_u^*]^{-1} g_{uu}^*}, \tag{19}$$

$$J_d \stackrel{\text{def}}{=} \begin{bmatrix} h_u^* & h_x^* \\ g_u^* & g_x^* \end{bmatrix}, \tag{20}$$

and $\Delta u \stackrel{\text{def}}{=} u - u^*(\boldsymbol{\theta})$, $\Delta x \stackrel{\text{def}}{=} x - x^*(\boldsymbol{\theta})$, $\Delta\dot{u} \stackrel{\text{def}}{=} \dot{u}$, and $\Delta\dot{x} \stackrel{\text{def}}{=} \dot{x}$. You are asked to verify these calculations in a mental exercise.

The local stability of the simple steady state is determined by finding the eigenvalues of the coefficient matrix of Eq. (17), the dynamic Jacobian J_d defined in Eq. (20). Because the steady state is assumed to be simple by assumption (A.6), it follows from the definition of simplicity that $|J_d| \neq 0$. The eigenvalues δ_ℓ, $\ell = 1, 2$, of J_d are found by solving the characteristic equation associated with J_d, which is given by

$$\begin{aligned} |J_d - \delta I| &= \begin{vmatrix} h_u^* - \delta & h_x^* \\ g_u^* & g_x^* - \delta \end{vmatrix} = [h_u^* - \delta][g_x^* - \delta] - h_x^* g_u^* \\ &= \delta^2 - [\mathrm{tr} J_d]\delta + |J_d| = 0. \end{aligned} \tag{21}$$

Equating the identity $(\delta - \delta_1)(\delta - \delta_2) \equiv \delta^2 - (\delta_1 + \delta_2)\delta + \delta_1\delta_2$ to zero and comparing it to characteristic polynomial in Eq. (21) implies that

$$\delta_1 + \delta_2 = \mathrm{tr} J_d = h_u^* + g_x^* = r > 0, \tag{22}$$

$$\delta_1\delta_2 = |J_d| = h_u^* g_x^* - h_x^* g_u^* = |J_s| \left[f_{uu}^* - f_u^* [g_u^*]^{-1} g_{uu}^* \right]^{-1} \neq 0, \tag{23}$$

where the result in Eq. (22) was obtained by solving Eq. (11) for $f_x^* = -f_u^* [g_u^*]^{-1} [r - g_x^*]$ and using this expression to substitute out f_x^* in h_u^*. Recalling that $|J_d| \neq 0$ by assumption (A.6), as noted above, Eq. (23) implies that $|J_s| [f_{uu}^* - f_u^* [g_u^*]^{-1} g_{uu}^*]^1 \neq 0$. Because $H_{uu}^* = f_{uu}^* - f_u^* [g_u^*]^{-1} g_{uu}^* < 0$ by assumption (A.5) and Eqs. (4) and (15), as also remarked above, it follows that $|J_s| \neq 0$. Note that the conclusion $|J_s| \neq 0$ was alluded to in the discussion of Eq. (16) when the implicit function theorem was applied to Eqs. (11) and (12) to arrive at the steady state solution in Eqs. (13) through (15).

From Eq. (22), the sum of the eigenvalues is positive, so that if they are complex conjugates, their real part is nonzero, which implies that the steady state is hyperbolic. Given that the sum of the eigenvalues is positive, at least one eigenvalue is positive, thus ruling out the possibility that the steady state is locally asymptotically stable. Now if both eigenvalues are positive or have positive real parts, then the steady state is an unstable node or an unstable spiral, respectively, and any trajectory that does not begin at the steady state cannot reach it, as all trajectories diverge from the steady state with the passage of time. Thus both eigenvalues cannot be positive or have positive real parts, as this violates the convergence requirement of assumption (A.7). Because the steady state is hyperbolic and must be reached as $t \to +\infty$ by assumption (A.7), the only possibility left for the eigenvalues of J_d is that one is negative, say, $\delta_1 < 0$, and the other is positive, say $\delta_2 > 0$, thereby implying that the steady state is a local saddle point and $|J_d| = \delta_1\delta_2 < 0$. From Eq. (22), it follows that $\delta_2 > r > 0$ and $\delta_2 > |\delta_1|$, and that both eigenvalues are real. Thus the steady state can be reached by following one of the two trajectories that approach it as $t \to +\infty$. The discussion of this paragraph is summarized by

Theorem 18.1 (Local Stability): *Under assumptions (A.1) through (A.7), the steady state of problem (1) is a local saddle point. Moreover, J_d has real eigenvalues $\delta_1 < 0$, $\delta_2 > r > 0$, $\delta_2 > |\delta_1|$, and $|J_d| < 0$.*

In one sense, this is quite a surprising result. The surprise is that the rather general and ubiquitous class of optimal control problems defined by problem (1) and assumptions (A.1) through (A.7) has an unstable steady state. That is to say, it is not possible for this class of control problems to have a locally asymptotically stable steady state. Inasmuch as many intertemporal economic models fit within this class of control problems, they too have unstable steady states. Thus the "best" one may expect, as far as local stability is concerned, is that the steady state is a saddle point, as this type of fixed point at least has a stable manifold that asymptotically reaches the steady state. Even so, there is a practical difficulty when a steady state is a local saddle point. To see this, observe that in the context of problem (1), the stable manifold of the saddle point steady state is a line in the two-dimensional phase space, and hence of one dimension less than the space itself. Consequently, if one were to imagine the phase portrait overlaid on a dart board, and one had to throw darts at it to establish the initial value of the control variable for a given initial value of the state variable, then the probability is zero that a dart would land on the stable manifold. This implies that for all practical purposes, it is impossible to select the initial value of the control variable to lie on the stable manifold, thereby ruling out reaching the steady state asymptotically.

On the other hand, the above conclusion is not a limitation from a theoretical point of view. To comprehend this claim, simply observe that one can always set the constant of integration associated with the positive eigenvalue of J_d equal to zero, just as we did in arriving at Eq. (35) in Chapter 17 or in Eq. (37) below. The resulting time paths of the state and control variables would therefore lie on the stable manifold of the saddle point steady state. Furthermore, the fact that the stable manifold is of dimension one for a local saddle point steady state leads to the uniqueness of an optimal solution. Moreover, the uniqueness of the optimal solution is generally considered to be a much "nicer" property than the nonuniqueness that would result if the steady state was locally asymptotically stable, that is, if both eigenvalues of J_d were negative. Because the steady state has been shown to be a local saddle point and thus can, in principle, be reached starting from a point $x_0 \neq x^*(\boldsymbol{\theta})$, attention is now turned to the steady state comparative statics of problem (1).

The saddle point nature of the steady state will now be used to provide some steady state comparative statics information about problem (1). First, recall from Eq. (23) and Theorem 18.1 that $|J_d| = |J_s|\,[f_{uu}^* - f_u^*[g_u^*]^{-1}g_{uu}^*]^{-1} < 0$. Next, recall that from assumption (A.5) and Eqs. (4) and (15), $H_{uu}^* = f_{uu}^* - f_u^*[g_u^*]^{-1}g_{uu}^* < 0$. Putting the last two conclusions together, we see that $|J_s| > 0$, that is, the saddle point nature of the steady state and satisfaction of the second-order sufficient condition for the control to maximize the current value Hamiltonian imply that the steady state Jacobian determinant is positive. This implies that a steady state comparative static analysis of problem (1) can be carried out via the implicit function theorem. The discussion of this paragraph is summarized by the following corollary.

Corollary 18.1 (Steady State Jacobian): *Under assumptions (A.1) through (A.7) on problem (1), the steady state Jacobian determinant is positive, that is,* $|J_s| > 0$.

This is an important result, for $|J_s|$ appears in *all* the steady state comparative statics expressions. Analogously, recall that it is the second-order sufficient conditions of static optimization problems that imply the Jacobian determinant of the first-order necessary conditions is nonzero, thereby allowing a comparative statics analysis of the problem to be carried out via the implicit function theorem.

The steady state comparative statics for problem (1) are found by substituting the steady state solution of the necessary conditions $(x^*(\boldsymbol{\theta}), u^*(\boldsymbol{\theta}))$ back into Eqs. (11) and (12), the equations from which they were derived by the implicit function theorem, thereby creating local identities in the parameter vector $\boldsymbol{\theta}$:

$$f_u(x^*(\boldsymbol{\theta}), u^*(\boldsymbol{\theta}); \boldsymbol{\alpha})[r - g_x(x^*(\boldsymbol{\theta}), u^*(\boldsymbol{\theta}); \boldsymbol{\beta})] + g_u(x^*(\boldsymbol{\theta}), u^*(\boldsymbol{\theta}); \boldsymbol{\beta}) f_x(x^*(\boldsymbol{\theta}), u^*(\boldsymbol{\theta}); \boldsymbol{\alpha}) \equiv 0, \tag{24}$$

$$g(x^*(\boldsymbol{\theta}), u^*(\boldsymbol{\theta}); \boldsymbol{\theta}) \equiv 0. \tag{25}$$

Differentiating Eqs. (24) and (25) with respect to the parameter of interest using the multivariate chain rule, and solving the resulting linear system with Cramer's rule yields the steady state comparative statics. For example, differentiation of Eqs. (24) and (25) with respect to the parameter α_i, $i = 1, 2, \ldots, K_1$, yields the system of linear equations

$$\begin{bmatrix} f^*_{uu}[r-g^*_x] - f^*_u g^*_{xu} & f^*_{ux}[r-g^*_x] - f^*_u g^*_{xx} \\ +g^*_u f^*_{xu} + f^*_x g^*_{uu} & +g^*_u f^*_{xx} + f^*_x g^*_{ux} \\ g^*_u & g^*_x \end{bmatrix} \begin{bmatrix} \dfrac{\partial u^*}{\partial \alpha_i} \\ \dfrac{\partial x^*}{\partial \alpha_i} \end{bmatrix}$$

$$\equiv \begin{bmatrix} -f^*_{u\alpha_i}[r-g^*_x] - g^*_u f^*_{x\alpha_i} \\ 0 \end{bmatrix}. \tag{26}$$

Applying Cramer's rule to this linear system of equations yields Eqs. (27) and (28) of Theorem 18.2. Using the same recipe for the parameters $(\boldsymbol{\beta}, r)$ yields the proof of the remaining parts of the theorem, which you are asked to provide in a mental exercise.

Theorem 18.2 (Steady State Comparative Statics): *In problem (1), with assumptions (A.1) through (A.7) holding, the steady state comparative statics are given by*

$$\frac{\partial x^*(\boldsymbol{\theta})}{\partial \alpha_i} \equiv \frac{f^*_{u\alpha_i}[r-g^*_x]g^*_u + [g^*_u]^2 f^*_{x\alpha_i}}{|J_s|}, \quad i = 1, \ldots, K_1, \tag{27}$$

$$\frac{\partial u^*(\boldsymbol{\theta})}{\partial \alpha_i} \equiv \frac{-f^*_{u\alpha_i}[r-g^*_x]g^*_x - g^*_u g^*_x f^*_{x\alpha_i}}{|J_s|}, \quad i = 1, \ldots, K_1, \tag{28}$$

$$\frac{\partial x^*(\boldsymbol{\theta})}{\partial \beta_j} \equiv \frac{g^*_{\beta_j}[f^*_u g^*_{xu} - [r-g^*_x]f^*_{uu} - g^*_u f^*_{xu} - f^*_x g^*_{uu}] + g^*_u[f^*_x g^*_{u\beta_j} - f^*_u g^*_{x\beta_j}]}{|J_s|},$$
$$j = 1, \ldots, K_2, \tag{29}$$

$$\frac{\partial u^*(\boldsymbol{\theta})}{\partial \beta_j} \equiv \frac{g^*_{\beta_j}[-f^*_u g^*_{xx} + [r-g^*_x]f^*_{ux} + g^*_u f^*_{xx} + f^*_x g^*_{ux}] + g^*_x[f^*_u g^*_{x\beta_j} - f^*_x g^*_{u\beta_j}]}{|J_s|},$$
$$j = 1, \ldots, K_2, \tag{30}$$

$$\frac{\partial x^*(\boldsymbol{\theta})}{\partial r} \equiv \frac{f^*_u g^*_u}{|J_s|}, \tag{31}$$

$$\frac{\partial u^*(\boldsymbol{\theta})}{\partial r} \equiv \frac{-f^*_u g^*_x}{|J_s|}. \tag{32}$$

At this level of generality, no signs are implied for any of the steady state comparative statics, just as one would suspect. The denominator of these expressions, however, is positive as noted in Corollary 18.1. Given this observation, the following corollary is a direct consequence of Theorem 18.2, as can be seen by inspection.

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Corollary 18.2.1 (Steady State Comparative Statics): *In problem (1) , with assumptions (A.1) through (A.7) holding,*

(A) If a perturbation in the discount rate r occurs in the steady state, then

$$\operatorname{sign}\left[\frac{\partial x^*(\boldsymbol{\theta})}{\partial r}\right] = \operatorname{sign}[f_u^* g_u^*],$$

$$\operatorname{sign}\left[\frac{\partial u^*(\boldsymbol{\theta})}{\partial r}\right] = -\operatorname{sign}[f_u^* g_x^*].$$

(B) If $f_{u\alpha_i}^* = 0, i = 1, \ldots, K_1$, *then*

$$\operatorname{sign}\left[\frac{\partial x^*(\boldsymbol{\theta})}{\partial \alpha_i}\right] = \operatorname{sign}[f_{x\alpha_i}^*], \quad i = 1, \ldots, K_1,$$

$$\operatorname{sign}\left[\frac{\partial u^*(\boldsymbol{\theta})}{\partial \alpha_i}\right] = -\operatorname{sign}[g_u^* g_x^* f_{x\alpha_i}^*], \quad i = 1, \ldots, K_1.$$

(C) If $f_{x\alpha_i}^* = 0, i = 1, \ldots, K_1$, *then*

$$\operatorname{sign}\left[\frac{\partial x^*(\boldsymbol{\theta})}{\partial \alpha_i}\right] = \operatorname{sign}[f_{u\alpha_i}^*[r - g_x^*]g_u^*], \quad i = 1, \ldots, K_1,$$

$$\operatorname{sign}\left[\frac{\partial u^*(\boldsymbol{\theta})}{\partial \alpha_i}\right] = -\operatorname{sign}[f_{u\alpha_i}^*[r - g_x^*]g_x^*], \quad i = 1, \ldots, K_1.$$

The results of Corollary 18.2.1 are important enough to be given an economic interpretation. The first part of (A) asserts that the effect of an increase in the discount rate on the steady state value of the state variable is qualitatively the same as the product of the marginal impact of the control variable on the integrand and the marginal impact of the control variable on the state variable's rate of change, evaluated at the steady state. In economic problems, if the state variable is a good, say, capital or fish, then ordinarily $\lambda^*(\boldsymbol{\theta}) = -f_u^*/g_u^* > 0$, or equivalently $f_u^* g_u^* < 0$, thereby implying that $\partial x^*/\partial r < 0$. On the other hand, if the state variable is a bad, say, toxic waste, then typically $\lambda^*(\boldsymbol{\theta}) = -f_u^*/g_u^* < 0$, or equivalently $f_u^* g_u^* > 0$, thereby implying that $\partial x^*/\partial r > 0$. In other words, if the stock is a good, then an increase in the discount rate usually decreases the stock in the steady state, whereas if the stock is a bad, then an increase in the discount rate normally increases the stock in the steady state.

The second assertion of part (A) says that if the discount rate increases, then its impact on the steady state value of the control variable is the opposite qualitatively of the product of the marginal impact of the control variable on the integrand and the marginal product of the state variable, evaluated at the steady state. If the stock is a good, then typically $f_u^* g_x^* > 0$, so that an increase in the discount rate decreases the

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steady state value of the control variable. Similarly, if the stock is a bad, then usually $f_u^* g_x^* < 0$, so that an increase in the discount rate increases the steady state value of the control variable. In light of the above steady state comparative statics for the state and control variables, it is usually the case that the state and control variables have the same qualitative steady state response to an increase in the discount rate.

Part (B) of the corollary is a conjugate pairs result. It asserts that if a parameter α_i is additively separable from the control variable at the steady state, then the effect of an increase in α_i on the steady state value of the state variable is the same qualitatively as the effect the increase in α_i has on the marginal impact the state variable has on the integrand, evaluated at the steady state. This means that if a parameter α_i enters problem (1) in such a manner, then all one has to do to compute the sign of $\partial x^*/\partial \alpha_i$ is to find the sign of $f_{x\alpha_i}^*$. Clearly, this represents a vast simplification over using Theorem 18.2 in computing the steady state comparative statics. The corresponding result for the control variable is slightly more complicated (but just as easy to use) because of the presence of the product $g_u^* g_x^*$. The conjugate pairs structure of part (B) is common to many papers in economics that use optimal control theory, and is therefore responsible for many of the refutable steady state comparative statics of such papers.

Finally, part (C) is also a conjugate pairs result, but in this case, α_i and the state variable must be additively separable at the steady state. In this instance, the sign of the steady state comparative statics depends not only on the sign of the partials g_u^*, g_x^*, and $f_{u\alpha_i}^*$, but also on the size of g_x^* relative to r. Thus although this part of the corollary is no more difficult to use than any of the other parts, it requires more information in that the magnitude of g_x^* is required. It should again be noted that the conjugate pairs structure of part (C) is common to many papers in economics that use optimal control theory, and is therefore responsible for many of the refutable steady state comparative statics in such papers.

Notice that in Theorem 18.2, g_x^* appears in every term of the numerator of $\partial u^*/\partial \alpha_i$ and $\partial u^*/\partial r$. Hence if $g_x^* = 0$, then the steady state value of the control variable is independent of any parameter that appears in the integrand of problem (1). This observation yields

Corollary 18.2.2 (Steady State Comparative Statics): *In problem (1), with assumptions (A.1) through (A.7) holding, if $g_x^* = 0$, then $\partial u^*/\partial r \equiv 0$ and $\partial u^*/\partial \alpha_i \equiv 0$, $i = 1, \ldots, K_1$.*

The economic interpretation of this result is that if the marginal product of the state variable is zero at the steady state, then the steady state value of the control variable is unaffected by changes in any parameter that enters the integrand of problem (1). An immediate application of Corollary 18.2.2 is in the dynamic limit pricing model of Gaskins (1971) examined in Chapter 16, where the state equation is independent of the state variable (rival sales), thereby implying that the dominant firm's price (the control variable) is unaffected by changes in the discount rate

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or average cost of production. This corollary is also applicable to the model of soil conservation examined in Mental Exercise 16.6, since soil quality (the state variable) does not enter the state equation, and therefore a rise in the output price, input price, or discount rate leaves steady state output (the control variable) unaffected. It is important to note, however, that just because the steady state value of the control variable is independent of $(\boldsymbol{\alpha}, r)$, this *does not* mean that the control variable's optimal trajectory is independent of these same parameters. This will be spelled out carefully in what follows.

Notice that the expressions in Theorem 18.2 for perturbations in $(\boldsymbol{\alpha}, r)$ are relatively simple compared with those that involve perturbations in $\boldsymbol{\beta}$. This is straightforward to explain. Seeing as $(\boldsymbol{\alpha}, r)$ appear *explicitly* in only one of the steady state necessary conditions, scilicet, Eq. (24), differentiating Eqs. (24) and (25) with respect to $(\boldsymbol{\alpha}, r)$ and writing in matrix notation, as in Eq. (26) for example, reveals that the right-hand side column vector has only one nonzero element. When Cramer's rule is used to solve for the steady state comparative statics, the zero element in the right-hand side vector will reduce the number of terms in the numerator of the resulting comparative statics expressions. If one follows the same recipe for perturbations in $\boldsymbol{\beta}$, however, both elements of the right-hand side vector will be nonzero, for $\boldsymbol{\beta}$ appears explicitly in both steady state necessary conditions. When Cramer's rule is applied to this system, terms in the numerator will not drop out as they did for perturbations in $(\boldsymbol{\alpha}, r)$. The important point is that when a parameter enters the state equation explicitly, it is in general more difficult to derive refutable steady state comparative statics of the form $\partial x^*/\partial \beta_j$ and $\partial u^*/\partial \beta_j$. This is important, for if the steady state comparative statics for some parameter are ambiguous, then so too will be the corresponding local comparative dynamics, as will be demonstrated later.

It is important also to keep in mind that the steady state comparative statics of Theorem 18.2 have been simplified by allowing a parameter to enter only the integrand function $f(\cdot)$ or the transition function $g(\cdot)$. If a parameter entered both of these functions, then the resulting steady state comparative statics would be more complicated, as is straightforward to show. In this case, there are no simple sufficient conditions like Corollaries 18.2.1 and 18.2.2 available to render the results more user-friendly. In passing, note that there appear to be relatively few problems in the economic literature where a parameter appears in both $f(\cdot)$ and $g(\cdot)$.

Before moving on to the local comparative dynamics, it is worthwhile to remember that the conjugacy conditions of parts (B) and (C) of Corollary 18.2.1 and the sufficient condition of Corollary 18.2.2 are local conditions, that is, they need only hold in some neighborhood of the steady state for one to use the corollaries. Now if $f_{u\alpha_i} \equiv 0$, $f_{x\alpha_i} \equiv 0$, or $g_x \equiv 0$, or in other words, if the conjugacy conditions hold over the entire domain of the integrand or the marginal product of the state variable is identically zero, then the corollaries can also be used. Naturally, the global requirements on the functions are stronger than the local requirements needed for the use of the corollaries, but in practice, the global conditions are easier to verify, since this can be done simply by inspection of the control problem under consideration.

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Finally, note that the steady state comparative statics for the current value shadow price follow from Eq. (15) and use of Theorem 18.2.

The local comparative dynamics properties of problem (1) are determined by solving the linearized pair of ordinary differential equations given in Eq. (17). Since the eigenvalues of J_d are real and unequal by Theorem 18.1, Theorem 25.1 of Simon and Blume (1994) asserts that the general solution of Eq. (17) is given by

$$\begin{bmatrix}\Delta u(t)\\ \Delta x(t)\end{bmatrix} = c_1\mathbf{Q}^1 e^{\delta_1 t} + c_2\mathbf{Q}^2 e^{\delta_2 t}, \tag{33}$$

where c_1 and c_2 are arbitrary constants of integration and $\mathbf{Q}^\ell \in \Re^2$ is an eigenvector of the coefficient matrix J_d corresponding to the eigenvalue δ_ℓ, $\ell = 1, 2$. The eigenvalues $\delta_1 < 0$ and $\delta_2 > r > 0$ were derived earlier; hence only the eigenvectors need to be determined. The eigenvectors (Q_1^ℓ, Q_2^ℓ) corresponding to the eigenvalues δ_ℓ, $\ell = 1, 2$, are by definition the solution to the following homogeneous system of linear algebraic equations:

$$\begin{bmatrix} h_u^* - \delta_\ell & h_x^* \\ g_u^* & g_x^* - \delta_\ell\end{bmatrix}\begin{bmatrix}Q_1^\ell\\ Q_2^\ell\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}, \quad \ell = 1, 2. \tag{34}$$

The rows of Eq. (34) are linearly dependent because the eigenvalues of J_d were chosen such that $|J_d - \delta_\ell I| = 0$, $\ell = 1, 2$, and this is the determinant of the coefficient matrix in Eq. (34). This means that the eigenvectors (Q_1^ℓ, Q_2^ℓ) corresponding to the eigenvalues δ_ℓ, $\ell = 1, 2$, are unique only up to a scalar multiple. Taking $Q_2^\ell = 1$, $\ell = 1, 2$, as the normalization, the eigenvectors are

$$\left(Q_1^\ell, Q_2^\ell\right) = \left(\frac{-h_x^*}{h_u^* - \delta_\ell}, 1\right) \equiv \left(\frac{\delta_\ell - g_x^*}{g_u^*}, 1\right), \quad \ell = 1, 2. \tag{35}$$

The two formulas for the Q_1^ℓ component of the eigenvectors given in Eq. (35) are identical because of the way the eigenvalues were chosen, as you are asked to demonstrate in a mental exercise.

Given that the eigenvalues and eigenvectors of J_d are now known, let us rewrite the general solution of the homogeneous first-order differential equation system (17) as

$$\begin{bmatrix}v(t;\boldsymbol{\theta}, x_0)\\ z(t;\boldsymbol{\theta}, x_0)\end{bmatrix} = \begin{bmatrix}u^*(\boldsymbol{\theta})\\ x^*(\boldsymbol{\theta})\end{bmatrix} + c_1\begin{bmatrix}Q_1^1\\ 1\end{bmatrix}e^{\delta_1 t} + c_2\begin{bmatrix}Q_1^2\\ 1\end{bmatrix}e^{\delta_2 t}, \tag{36}$$

rather than as we did in Eq. (33). The arbitrary constants of integration are determined from the initial condition $x(0) = x_0$ and the convergence requirement on the optimal path of the state variable from assumption (A.7), namely, $\lim_{t\to+\infty} z(t;\boldsymbol{\theta}, x_0) = x^*(\boldsymbol{\theta})$. Taking the latter condition first yields

$$\lim_{t\to+\infty} z(t;\boldsymbol{\theta}, x_0) = \lim_{t\to+\infty}\left[x^*(\boldsymbol{\theta}) + c_1 e^{\delta_1 t} + c_2 e^{\delta_2 t}\right] = x^*(\boldsymbol{\theta}).$$

Seeing as $\delta_1 < 0$, $c_1 e^{\delta_1 t} \to 0$ as $t \to +\infty$, but because $\delta_2 > 0$, $c_2 e^{\delta_2 t} \to \pm\infty$ as $t \to +\infty$ if $c_2 \neq 0$, in which case the limit of the entire bracketed term does not exist, thus violating the convergence condition. Hence $c_2 = 0$ for convergence to the steady state to be met. Note that the conclusion $c_2 = 0$ may also be obtained from the convergence requirement on the optimal path of the control variable from assumption (A.7). Now apply the initial condition to find c_1:

$$z(0; \boldsymbol{\theta}, x_0) = x^*(\boldsymbol{\theta}) + c_1 = x_0 \Rightarrow c_1 = x_0 - x^*(\boldsymbol{\theta}).$$

Thus the specific solution to Eq. (17) that satisfies the initial condition and convergence requirement is given by

$$\begin{bmatrix} v(t; \boldsymbol{\theta}, x_0) \\ z(t; \boldsymbol{\theta}, x_0) \end{bmatrix} = \begin{bmatrix} u^*(\boldsymbol{\theta}) \\ x^*(\boldsymbol{\theta}) \end{bmatrix} + [x_0 - x^*(\boldsymbol{\theta})] \begin{bmatrix} Q_1^1 \\ 1 \end{bmatrix} e^{\delta_1 t}, \tag{37}$$

along with its time derivative

$$\begin{bmatrix} \dot{v}(t; \boldsymbol{\theta}, x_0) \\ \dot{z}(t; \boldsymbol{\theta}, x_0) \end{bmatrix} = \delta_1 [x_0 - x^*(\boldsymbol{\theta})] \begin{bmatrix} Q_1^1 \\ 1 \end{bmatrix} e^{\delta_1 t}. \tag{38}$$

The local comparative dynamics of problem (1) follow from differentiating Eqs. (37) and (38) with respect to $(\boldsymbol{\theta}, x_0)$ and evaluating the resulting derivatives at $x_0 = x^*(\boldsymbol{\theta})$. They are summarized by the following theorem, whose proof is left for the mental exercises.

Theorem 18.3 (Local Comparative Dynamics): *In problem (1), under assumptions (A.1) through (A.7), the effects of parameter perturbations for all* $t \in [0, +\infty)$ *on the time path of the optimal solution in a neighborhood of an optimal steady state are given by*

$$\left. \frac{\partial v(t; \boldsymbol{\theta}, x_0)}{\partial x_0} \right|_{x_0 = x^*(\boldsymbol{\theta})} = Q_1^1 e^{\delta_1 t}, \tag{39}$$

$$\left. \frac{\partial z(t; \boldsymbol{\theta}, x_0)}{\partial x_0} \right|_{x_0 = x^*(\boldsymbol{\theta})} = e^{\delta_1 t} \geq 0, \tag{40}$$

$$\left. \frac{\partial \dot{v}(t; \boldsymbol{\theta}, x_0)}{\partial x_0} \right|_{x_0 = x^*(\boldsymbol{\theta})} = \delta_1 Q_1^1 e^{\delta_1 t}, \tag{41}$$

$$\left. \frac{\partial \dot{z}(t; \boldsymbol{\theta}, x_0)}{\partial x_0} \right|_{x_0 = x^*(\boldsymbol{\theta})} = \delta_1 e^{\delta_1 t} \leq 0, \tag{42}$$

$$\left. \frac{\partial v(t; \boldsymbol{\theta}, x_0)}{\partial \theta_k} \right|_{x_0 = x^*(\boldsymbol{\theta})} = \frac{\partial u^*(\boldsymbol{\theta})}{\partial \theta_k} - Q_1^1 e^{\delta_1 t} \frac{\partial x^*(\boldsymbol{\theta})}{\partial \theta_k}, \quad k = 1, \ldots, K_1 + K_2 + 1, \tag{43}$$

$$\left. \frac{\partial z(t; \boldsymbol{\theta}, x_0)}{\partial \theta_k} \right|_{x_0 = x^*(\boldsymbol{\theta})} = \frac{\partial x^*(\boldsymbol{\theta})}{\partial \theta_k} [1 - e^{\delta_1 t}], \quad k = 1, \ldots, K_1 + K_2 + 1, \tag{44}$$

$$\left.\frac{\partial \dot{v}(t;\boldsymbol{\theta},x_0)}{\partial \theta_k}\right|_{x_0=x^*(\boldsymbol{\theta})} = -\delta_1 Q_1^1 e^{\delta_1 t}\frac{\partial x^*(\boldsymbol{\theta})}{\partial \theta_k}, \quad k=1,\ldots,K_1+K_2+1, \tag{45}$$

$$\left.\frac{\partial \dot{z}(t;\boldsymbol{\theta},x_0)}{\partial \theta_k}\right|_{x_0=x^*(\boldsymbol{\theta})} = -\delta_1 e^{\delta_1 t}\frac{\partial x^*(\boldsymbol{\theta})}{\partial \theta_k}, \quad k=1,\ldots,K_1+K_2+1. \tag{46}$$

By definition, the *impact effects* of parameter perturbations follow from Theorem 18.3 by evaluating the derivatives at $t=0$. Because $\delta_1<0$, the following corollary is evident upon inspection of Theorem 18.3.

Corollary 18.3 (Local Comparative Dynamics): *In problem (1), under assumptions (A.1) through (A.7), the following local comparative dynamics results hold for all $t \in [0,+\infty)$:*

$$\operatorname{sign}\left[\left.\frac{\partial v(t;\boldsymbol{\theta},x_0)}{\partial x_0}\right|_{x_0=x^*(\boldsymbol{\theta})}\right] = \operatorname{sign}\left[Q_1^1\right], \tag{47}$$

$$\operatorname{sign}\left[\left.\frac{\partial \dot{v}(t;\boldsymbol{\theta},x_0)}{\partial x_0}\right|_{x_0=x^*(\boldsymbol{\theta})}\right] = -\operatorname{sign}\left[Q_1^1\right], \tag{48}$$

$$\operatorname{sign}\left[\left.\frac{\partial z(t;\boldsymbol{\theta},x_0)}{\partial \theta_k}\right|_{x_0=x^*(\boldsymbol{\theta})}\right] = \operatorname{sign}\left[\frac{\partial x^*(\boldsymbol{\theta})}{\partial \theta_k}\right], \quad k=1,\ldots,K_1+K_2+1, \tag{49}$$

$$\operatorname{sign}\left[\left.\frac{\partial \dot{v}(t;\boldsymbol{\theta},x_0)}{\partial \theta_k}\right|_{x_0=x^*(\boldsymbol{\theta})}\right] = \operatorname{sign}\left[Q_1^1\frac{\partial x^*(\boldsymbol{\theta})}{\partial \theta_k}\right], \quad k=1,\ldots,K_1+K_2+1, \tag{50}$$

$$\operatorname{sign}\left[\left.\frac{\partial \dot{z}(t;\boldsymbol{\theta},x_0)}{\partial \theta_k}\right|_{x_0=x^*(\boldsymbol{\theta})}\right] = \operatorname{sign}\left[\frac{\partial x^*(\boldsymbol{\theta})}{\partial \theta_k}\right], \quad k=1,\ldots,K_1+K_2+1. \tag{51}$$

Theorem 18.3 and Corollary 18.3 point out four important facts about the qualitative properties of discounted infinite-horizon optimal control models with one state and one control variable. The first point is that one must know the steady state comparative statics for the control *and* state variables before any refutable local comparative dynamics will emerge for the level of the control variable. This follows from Eq. (43), for it shows that the steady state comparative statics for the state and control variables appear in the local comparative dynamics for the level of the control variable. Simply put, one must know where one's exact destination is before the best path to take there can be determined.

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Second, the local comparative dynamics of the state variable and its time derivative depend only on the steady state comparative statics of the state variable, as shown by Eqs. (49) and (51). Thus the impact of a parameter perturbation on the optimal path of the control variable is, in general, more difficult to pin down than that on the state variable and its time derivative.

Third, it is, in general, easier to determine the impact that a parameter perturbation has on the time derivative of the control variable than on the level of the control variable. This follows from Eq. (50), and is a result of the fact that only the steady state comparative statics for the state variable and the eigenvector corresponding to the negative eigenvalue are required for computing the effects of parameter perturbations on the time derivative of the control variable. The steady state comparative statics of the control variable do not play a role here.

The fourth and final point is that even if the steady state value of the control variable is independent of some parameter, this does *not* mean that the time path of the control variable is independent of this parameter. This is readily seen from Eq. (43), for if $\partial u^*/\partial\theta_k = 0$, then as long as $\partial x^*/\partial\theta_k \neq 0$ and $Q_1^1 \neq 0$, the time path of the control variable will still be affected by a perturbation in θ_k, even though its steady state position does not change. The idea is that as long as the terminal position of the state variable is affected, the control variable must change, at least temporarily, for it to drive the state to its new destination.

Recall that Theorems 18.2 and 18.3 and their associated corollaries apply to the optimal control problem (1). Therefore, in order to find the corresponding result in a prototypical calculus of variations problem, simply set $g(x, u; \boldsymbol{\beta}) \stackrel{\text{def}}{=} u$ and note that $Q_1^1 = \delta_1 < 0$ from Eq. (35). It then follows from inspection of Theorem 18.3 that the steady state comparative statics for the state variable *completely* determine the local comparative dynamics of a prototypical calculus of variations problem.

Before presenting an example, one final remark on Theorem 18.3 is in order, as it relates to Theorem 14.10, the dynamic envelope theorem for discounted infinite-horizon optimal control models. As you may recall, in order to establish Theorem 14.10, we had to make an assumption that, in the notation of this chapter, was of the form

$$\lim_{t\to+\infty} \frac{\partial z}{\partial\theta_k}(t;\boldsymbol{\theta}, x_0) = \frac{\partial x^*}{\partial\theta_k}(\boldsymbol{\theta}), \quad k = 1, \ldots, K_1 + K_2 + 1.$$

If we take the initial value of the state variable to be the steady state value of the state variable, that is to say, $x_0 = x^*(\boldsymbol{\theta})$, then the above condition holds. This can be confirmed by letting $t \to +\infty$ in Eq. (44). Note that the same conclusion applies to the optimal control variable as well, as can be confirmed by letting $t \to +\infty$ in Eq. (43).

Let's pause briefly in order to present an example of some of the points just made. In it, we contemplate the classical model of a dynamic limit pricing dominant firm introduced into the literature by Gaskins (1971) and studied rather extensively in Chapter 16.

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Example 18.1: The dynamic limit pricing model of Gaskins (1971) can be stated in the notation of the present chapter as follows: $f(x,u;\boldsymbol{\alpha}) \stackrel{\text{def}}{=} [u-c][D(u)-x]$, $g(x,u;\boldsymbol{\beta}) \stackrel{\text{def}}{=} k[u-\bar{u}]$, $\bar{u}>0$ is the limit price, u is the dominant firm's product price, x is fringe or rival sales, $c>0$ is the dominant firm's constant average cost of production, $k>0$ is the response coefficient, and $D(u)$ is the total demand for the product. Because $g_x(\cdot) \equiv 0$, the dominant firm's steady state price is independent of its discount rate and average production cost by Corollary 18.2.2, as noted earlier. By Eq. (30), the dominant firm's steady state price is also independent of the fringe's response coefficient k, whereas by Eq. (29), steady state rival sales fall as k increases, which may seem odd without considering the local comparative dynamics of the problem. Given that $Q_1^1 = \delta_1 k^{-1} < 0$ from Eq. (35), Eq. (43) shows that as k increases, the dominant firm responds by lowering the price of its product below the limit price, which in turn drives some of its rivals from the market. As time passes, Eq. (45) shows that the dominant firm gradually raises its price above the level it fell the moment k first increased but below the limit price, all the while driving more rivals from the market. Eventually, when a long enough period of time passes, that is, as $t \to +\infty$, the dominant firm's price increases back to its original level (the limit price), and rival sales are smaller in the new steady state.

This brief example should convince you that an examination of only the steady state comparative statics of an optimal control problem may not give a sound economic understanding of its qualitative properties. Investigating the local comparative dynamics of the parameter perturbation may, in many cases, lead to a sound economic interpretation of the steady state comparative statics, which if taken in isolation, may seem counterintuitive. Furthermore, the local comparative dynamics may often yield qualitative results that show that the steady state policy implications may differ quite substantially from their impact effects. These observations will be further reinforced in the next, more extended, example.

In this final example of the chapter, we examine a classical dynamic economic problem in some detail in order to show how the theorems and corollaries of this chapter can be used to (i) simplify the derivation of its qualitative properties, and (ii) uncover the mathematical structure of the model responsible for its qualitative properties.

Example 18.2: Consider the following simplified version of the adjustment cost model of the price-taking firm we studied in Chapter 17:

$$\max_{I(\cdot)} \int_0^{+\infty} [pF(K(t)) - cK(t) - gI(t) - C(I(t);\gamma)]\, e^{-rt}\, dt$$

$$\text{s.t.} \quad \dot{K}(t) = I(t) - \delta K(t),\ K(0) = K_0,$$

where $K(t)$ is the capital stock, $I(t)$ is the investment rate, $F(\cdot)$ is the production function with $F'(K)>0$ and $F''(K)<0$, $C(\cdot)$ is the cost of adjustment function with

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$C(0;\gamma)=0$, $C_I(I;\gamma)>0$ for $I>0$, $C_I(I;\gamma)<0$ for $I<0$, $C_I(0;\gamma)=0$, and $C_{II}(I;\gamma)>0$, $p>0$ is the unit price of output, $c>0$ is the holding or maintenance cost per unit of capital, $g>0$ is the purchase price per unit of investment, $\gamma>0$ is a cost of adjustment shift parameter with $C_{I\gamma}(I;\gamma)>0$ for $I>0$, $C_{I\gamma}(I;\gamma)<0$ for $I<0$, $C_{I\gamma}(0;\gamma)=0$, $r>0$ is the discount rate on cash flows, $\delta>0$ is the depreciation rate of the capital stock, $K_0>0$ is the given initial stock of capital, $\boldsymbol{\alpha}\overset{\text{def}}{=}(c,g,p,\gamma)$, $\beta\overset{\text{def}}{=}\delta$, and $\boldsymbol{\theta}\overset{\text{def}}{=}(\boldsymbol{\alpha},\beta,r)=(c,g,p,\gamma,\delta,r)$. It is assumed that assumptions (A.1) through (A.7) hold for the adjustment cost model. Denote the optimal pair by $(K(t;\boldsymbol{\theta},K_0), I(t;\boldsymbol{\theta},K_0))$.

We begin the analysis by investigating the steady state comparative statics of an increase in the purchase price of investment $g>0$. Upon defining $f(K,I;\boldsymbol{\alpha})\overset{\text{def}}{=}pF(K)-cK-gI-C(I;\gamma)$ as the integrand of the problem and $g(K,I;\boldsymbol{\beta})\overset{\text{def}}{=}I-\delta K$ as the transition equation, Theorem 18.2, Eqs. (27) and (28), yield

$$\frac{\partial K^*(\boldsymbol{\theta})}{\partial g}\equiv\frac{-[r+\delta]}{|J_s|}<0, \tag{52}$$

$$\frac{\partial I^*(\boldsymbol{\theta})}{\partial g}\equiv\frac{-\delta[r+\delta]}{|J_s|}\equiv\delta\frac{\partial K^*(\boldsymbol{\theta})}{\partial g}<0. \tag{53}$$

These results demonstrate that an increase in the purchase price of the investment good leads to a decrease in the steady state capital stock and investment rate. Because the price of the investment good and the investment rate form a conjugate pair, that is, $f_{Kg}(\cdot)\equiv 0$, the sufficient condition of part (C) of Corollary 18.2.1 may be used to reach the same qualitative conclusion. Moreover, it follows from the definition of the steady state level of output $y^*(\boldsymbol{\theta})\overset{\text{def}}{=}F(K^*(\boldsymbol{\theta}))$ and Eq. (52) that $\partial y^*(\boldsymbol{\theta})/\partial g=F'(K^*(\boldsymbol{\theta}))\partial K^*(\boldsymbol{\theta})/\partial g<0$. In other words, the steady state level of output decreases with an increase in the purchase price of the investment good.

Before moving on to the local comparative dynamics, let's pause and construct the phase portrait in the *KI*-phase plane. This will be beneficial when we go to characterize the local comparative dynamics graphically. In order to do so efficiently, we simply employ Eqs. (9) and (10) to derive

$$\dot{I}=\frac{pF'(K)-w-[g+C_I(I;\gamma)][r+\delta]}{-C_{II}(I;\gamma)}, \tag{54}$$

$$\dot{K}=I-\delta K. \tag{55}$$

By definition, the nullclines are given by

$$\dot{I}=0\Leftrightarrow pF'(K)-w-[g+C_I(I;\gamma)][r+\delta]=0, \tag{56}$$

$$\dot{K}=0\Leftrightarrow I-\delta K=0. \tag{57}$$

The slopes of the nullclines in a neighborhood of the steady state $(K^*(\boldsymbol{\theta}), I^*(\boldsymbol{\theta}))$ are given by the implicit function theorem as

$$\left.\frac{\partial I}{\partial K}\right|_{\substack{\dot{I}=0 \\ (K^*(\boldsymbol{\theta}),\, I^*(\boldsymbol{\theta}))}} = \frac{pF''(K^*(\boldsymbol{\theta}))}{[r+\delta]C_{II}(I^*(\boldsymbol{\theta});\gamma)} < 0, \tag{58}$$

$$\left.\frac{\partial I}{\partial K}\right|_{\substack{\dot{K}=0 \\ (K^*(\boldsymbol{\theta}),\, I^*(\boldsymbol{\theta}))}} = \delta > 0. \tag{59}$$

Because the $\dot{K} = 0$ isocline is given by the linear equation $I - \delta K = 0$, it is a simple matter to determine its global properties. But seeing as we are deriving the steady state comparative statics and local comparative dynamics in a neighborhood of the steady state, all we require for the geometry to support our qualitative results is the slope of the isoclines in a neighborhood of the steady state. The same is true for the vector field associated with the differential equations (54) and (55), to which we now turn.

The vector field associated with Eqs. (54) and (55) is determined from the dynamic Jacobian matrix of these differential equations, as you should recall. We can either calculate the vector field directly from Eqs. (54) and (55) or use Eqs. (18) and (19) to aid us in this endeavor. Either way, we find the vector field to be given by

$$h_u^* = \left.\frac{\partial \dot{I}}{\partial I}\right|_{\dot{K}=\dot{I}=0} = [r+\delta] > 0,\;\; h_x^* = \left.\frac{\partial \dot{I}}{\partial K}\right|_{\dot{K}=\dot{I}=0} = \frac{pF''(K^*(\boldsymbol{\theta}))}{-C_{II}(I^*(\boldsymbol{\theta});\gamma)} > 0, \tag{60}$$

$$g_u^* = \left.\frac{\partial \dot{K}}{\partial I}\right|_{\dot{K}=\dot{I}=0} = 1 > 0,\;\; g_x^* = \left.\frac{\partial \dot{K}}{\partial K}\right|_{\dot{K}=\dot{I}=0} = -\delta < 0. \tag{61}$$

Upon comparing Eqs. (58) and (59) to Eqs. (60) and (61), the following conclusions should be evident: (i) the slope of the I nullcline in a neighborhood of the steady state is equal to the negative of the ratio of the second column entry to the first column entry of row one of the dynamic Jacobian matrix J_d associated with the differential equations (54) and (55), and (ii) the slope of the K nullcline in a neighborhood of the steady state is equal to the negative of the ratio of the second column entry to the first column entry of row two of the dynamic Jacobian matrix J_d associated with the differential equations (54) and (55). That this is true more generally for any two-dimensional system of ordinary differential equations is the subject of a mental exercise. Note that in establishing the general result, we take it that the control variable is represented on the so-called y-axis and the state variable is represented on the so-called x-axis. This convention implies that the slope of a nullcline is thus found by taking the derivative of the control variable with respect to the state variable along the said nullcline via the implicit function theorem, and then evaluating the

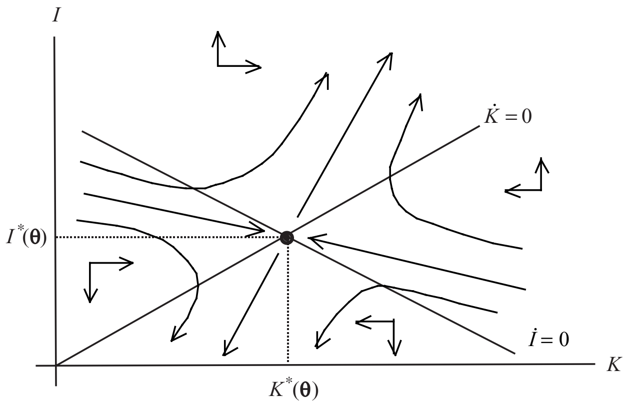

Figure 18.1

result at the steady state solution. Let's now return to the matter at hand, to wit, the derivation of the local phase portrait corresponding to the system of differential equations comprising Eqs. (54) and (55).

Using Eqs. (60) and (61), we find that $|J_d| = h_u^* g_x^* - h_x^* g_u^* < 0$. Moreover, because $\delta_1 \delta_2 = |J_d|$ by Eq. (23), one eigenvalue is negative and the other is positive, thereby implying that the steady state is a local saddle point. The local phase portrait corresponding to differential equations (54) and (55) can now be constructed using the qualitative information in Eqs. (58) through (61) and the fact that the steady state is a local saddle point. Figure 18.1 gives the local phase portrait.

Before we calculate the local comparative dynamics for g, observe that by using both forms of Eq. (35) and the aforementioned properties of the production function and adjustment cost function, that $Q_1^1 = \frac{pF''(K^*(\boldsymbol{\theta}))/C_{II}(I^*(\boldsymbol{\theta});\gamma)}{r+\delta-\delta_1} \equiv \delta_1 + \delta < 0$. Then by Eq. (43) of Theorem 18.3, it follows that

$$\left.\frac{\partial I(t;\boldsymbol{\theta}, K_0)}{\partial g}\right|_{K_0=K^*(\boldsymbol{\theta})} = \frac{\partial I^*(\boldsymbol{\theta})}{\partial g} - [\delta_1 + \delta]e^{\delta_1 t}\frac{\partial K^*(\boldsymbol{\theta})}{\partial g} < \frac{\partial I^*(\boldsymbol{\theta})}{\partial g} < 0 \tag{62}$$

for all $t \in [0, +\infty)$, the inequalities being a result of Eqs. (52) and (53). Moreover, by Eq. (45) of Theorem 18.3, it also follows that

$$\left.\frac{\partial \dot{I}(t;\boldsymbol{\theta}, K_0)}{\partial g}\right|_{K_0=K^*(\boldsymbol{\theta})} = -\delta_1[\delta_1 + \delta]e^{\delta_1 t}\frac{\partial K^*(\boldsymbol{\theta})}{\partial g} > 0 \tag{63}$$

for all $t \in [0, +\infty)$. Equations (62) and (63) demonstrate that the moment g increases, the investment rate falls by its largest amount, which is more than what is dictated by the steady state comparative statics. This impact effect is gradually reversed over time, as Eq. (63) shows, until the new (but lower) steady state investment rate is reached. All the while, the capital stock displays its typical monotonic

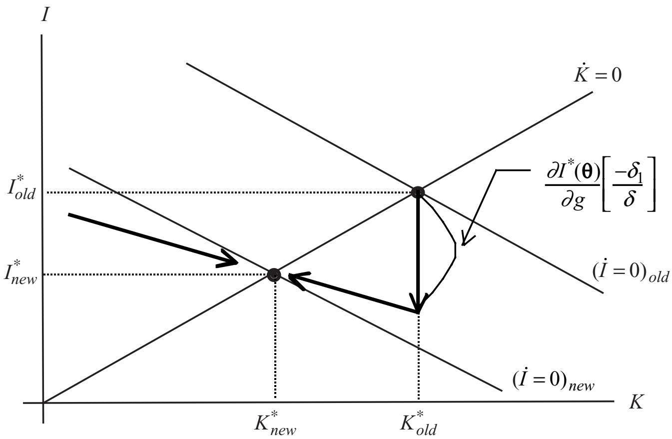

Figure 18.2

decline from its old to its new steady state level, for

$$\left.\frac{\partial K(t;\boldsymbol{\theta},K_0)}{\partial g}\right|_{K_0=K^*(\boldsymbol{\theta})}=\frac{\partial K^*(\boldsymbol{\theta})}{\partial g}[1-e^{\delta_1 t}]\leq 0\,\forall t\ \in[0,+\infty), \tag{64}$$

$$\left.\frac{\partial \dot{K}(t;\boldsymbol{\theta},K_0)}{\partial g}\right|_{K_0=K^*(\boldsymbol{\theta})}=-\delta_1 e^{\delta_1 t}\frac{\partial K^*(\boldsymbol{\theta})}{\partial g}<0\,\forall t\ \in[0,+\infty) \tag{65}$$

using Eqs. (44) and (46) of Theorem 18.3

Figure 18.2 presents the local comparative dynamics phase diagram for an increase in the purchase price of the investment good. In passing, observe that in Figure 18.2, we have indicated the precise amount by which the investment rate must fall the instant the purchase price of investment increases in order for the trajectory to asymptotically reach the new steady state. The precise amount is given by the impact effect

$$\left.\frac{\partial I(t;\boldsymbol{\theta},K_0)}{\partial g}\right|_{\substack{K_0=K^*(\boldsymbol{\theta})\\ t=0}}=\frac{\partial I^*(\boldsymbol{\theta})}{\partial g}\left[\frac{-\delta_1}{\delta}\right]<\frac{\partial I^*(\boldsymbol{\theta})}{\partial g}<0,$$

which you are asked to verify in a mental exercise.

In order to make sure that you understand the derivation of the comparative dynamics phase portrait given in Figure 18.2, several remarks are in order. First, recognize that Eqs. (52) and (53) imply that the new steady state lies to the southwest of the old steady state in the *KI*- phase plane. Second, observe that the $\dot{K}=0$ isocline given in Eq. (57) does not depend on the purchase price g of the investment good, but that the $\dot{I}=0$ isocline given in Eq. (56) does. Given these two facts, it then follows that the $\dot{I}=0$ isocline shifts down when g increases, as this is the only way for the new steady state to lie to the southwest of the old steady state. This conclusion can also be confirmed by application of the implicit function theorem to the $\dot{I} = 0$ isocline given in Eq. (56). This calculation is left as a mental exercise. Third, the local dynamics are qualitatively identical around the old and the new steady states. Fourth, the initial value of the capital stock for the local comparative dynamics exercise is taken as the old steady state value of the capital stock, and thus is fixed or given to the firm when the purchase price of the investment good first increases. This, in turn, implies that the capital stock (the state variable) does not *immediately* change when g increases, as this is the meaning of a state variable. This is easily verified by evaluating Eq. (64) at $t = 0$. As a result of this latter point, it follows that the investment rate (the control variable) must immediately respond to the increase in g if the new steady state is to be reached asymptotically. In order to find out which direction the investment rate must initially move in response to the increase in g, we simply note the two stable trajectories corresponding to the new steady state and seek to determine whether an increase or decrease in the investment rate is required to reach one of these trajectories. Using Figure 18.2 as a guide, we see that the investment rate must decrease initially when g increases, a fact we have already deduced from Eq. (62). The precise amount by which the investment rate decreases is given by the vertical distance from the old steady state to the stable arm to the right of the new steady state, and is determined by the impact effect of g on the investment rate as indicated in the Figure 18.2.

Finally, let us point out another important feature of the theorems and corollaries of this chapter as highlighted by the adjustment cost model. In particular, the steady state comparative statics for the parameters (c, γ, r) need not be discussed in detail, for Corollary 18.2.1 shows that their steady state comparative statics are qualitatively identical to that of g just discussed, and therefore by Theorem 18.3, so are their local comparative dynamics. The same conclusion does *not* hold for the depreciation rate of the capital stock, as inspection of Eq. (30) of Theorem 18.2 reveals. Furthermore, note that it is the conjugacy between g and I that is responsible for the refutable qualitative properties we uncovered in the adjustment cost model. Such a conjugacy is in fact the driving force behind many of the refutable qualitative results in the intertemporal economics literature.

For optimal control problems of the form of problem (1), which arguably comprise the largest class of models in dynamic economic theory, this chapter has provided an exhaustive qualitative characterization of their steady state comparative statics and local comparative dynamics properties. This was achieved by studying, in order, (i) the local stability of the steady state, (ii) the steady state comparative statics, and (iii) the local comparative dynamics of the problem.

The limitations of the linearization method presented in this chapter for studying the qualitative properties of optimal control problems are significant enough to be spelled out. For control problems with more than one state variable, but that otherwise meet the assumptions of the chapter, the linearization method may be applied, but as pointed out by Nagatani (1976), the qualitative results derivable by the method are few. If the control problem has a finite horizon but is otherwise consistent with the assumptions adopted in this chapter, then the linearization method cannot generally be used, for the steady state is not the focus in such problems. The variational differential equation approach of Oniki (1973) and the dynamic primal-dual formalism of Caputo (1990a, 1990b, 1992) exposited in Chapter 11, however, may be applied to such problems. Even if there are more than two states, the latter two methods may still be applied, but Oniki's (1973) approach is very difficult to extract qualitative information from, whereas Caputo's (1990a, 1990b, 1992) method is just as easy to use as when only one state is present, and in fact yields a richer set of qualitative properties because of symmetry. Finally, the method presented herein is not applicable when the control problem is nonautonomous, regardless of the dimension of the state, whereas Oniki's (1973) and Caputo's (1990a, 1990b, 1992) methods are perfectly at home in such instances.

In the next two chapters, we shift attention to the dynamic programming approach for solving optimal control problems. This will permit us to delve into intertemporal duality results that form the basis for empirical work that is based on optimal control formulations of intertemporal economic models.

MENTAL EXERCISES

18.1 Show that the necessary conditions for problem (1) can be reduced to a pair of differential equations in (λ, x).

18.2 Derive the steady state Jacobian matrix of Eqs. (11) and (12), and show that its determinant is given by Eq. (16).

18.3 Verify the expressions given in Eqs. (17) through (20).

18.4 Prove that $|J_d| = |J_s|\,[f_{uu}^* - f_u^*[g_u^*]^{-1}g_{uu}^*]^{-1}$ in Eq. (23).

18.5 Prove the remaining parts of Theorem 18.2.

18.6 Derive the steady state comparative statics of the current value shadow price.

18.7 Show that the two formulas for the Q_1^ℓ component of the eigenvectors given in Eq. (35) are identical.

18.8 Prove Theorem 18.3.

18.9 Consider the pair of autonomous ordinary differential equations

$$\dot{x}_1 = F^1(x_1, x_2),$$

$$\dot{x}_2 = F^2(x_1, x_2).$$

Imagine that x_1 is plotted horizontally and x_2 is plotted vertically in the x_1x_2-phase plane. Prove that the slope of the x_1 nullcline in a neighborhood of the steady state is equal to the negative of the ratio of the first column entry to the second column entry of row one of the Jacobian matrix associated with the differential equations evaluated at the steady state solution. Also prove the corresponding result for the x_2 nullcline.

18.10 With reference to Example 18.2:

(a) Prove that the impact effect of g on the investment rate is given by the formula in Figure 18.2.

(b) Prove that the $\dot{I} = 0$ isocline shifts down when g increases by using the implicit function theorem.

18.11 The neoclassical optimal growth model may be stated as

$$\max_{c(\cdot)} \int_0^{+\infty} U(c(t);\alpha)\, e^{-rt}\, dt$$

$$\text{s.t.} \quad \dot{k}(t) = \phi(k(t);\gamma) - c(t) - [\eta + \delta]k(t),\ k(0) = k_0,$$

where $c(t)$ is per-capita consumption, $k(t)$ is the capital-to-labor ratio, $U(\cdot)$ is the social instantaneous utility function with $U_c(c;\alpha) > 0$ and $U_{cc}(c;\alpha) < 0$, $\phi(\cdot)$ is the average product of labor function with $\phi_k(k;\gamma) > 0$ and $\phi_{kk}(k;\gamma) < 0$, $r > 0$ is the social discount rate, α is a taste change parameter such that $U_{c\alpha}(c;\alpha) > 0$, γ is a technology parameter such that $\phi_\gamma(k;\gamma) > 0$ and $\phi_{k\gamma}(k;\gamma) > 0$, $\eta > 0$ is the exogenous growth rate of the labor force, $\delta > 0$ is the depreciation rate of capital, $k_0 > 0$ is the initial capital-to-labor ratio, $\boldsymbol{\beta} \overset{\text{def}}{=} (\gamma, \delta, \eta)$, and $\boldsymbol{\theta} \overset{\text{def}}{=} (\alpha, \gamma, \delta, \eta, r)$. It is assumed that assumptions (A.1) through (A.7) hold for this model.

(a) Show that the steady state comparative statics of the social discount rate are

$$\frac{\partial k^*(\boldsymbol{\theta})}{\partial r} \equiv \frac{-U_c^*}{|J_s|} < 0, \quad \frac{\partial c^*(\boldsymbol{\theta})}{\partial r} \equiv \frac{-rU_c^*}{|J_s|} < 0.$$

(b) Show that the qualitative conclusions in part (a) could have just as easily been reached using Corollary 18.2.1.

These steady state comparative statics assert that an increase in the social discount rate reduces the steady state capital-to-labor ratio and per-capita consumption, which at first glance seems incongruous. For example, how can the capital-to-labor ratio fall if per-capita consumption has also fallen? The answer and intuition become clear only when the local comparative dynamics are investigated.

(c) In order to make complete sense of these steady state comparative statics, therefore, show that the impact effect of an increase in r is given by

$$\left.\frac{\partial c(t;\boldsymbol{\theta},k_0)}{\partial r}\right|_{\substack{k_0 = k^*(\boldsymbol{\theta}) \\ t=0}} = \delta_1 \frac{\partial k^*(\boldsymbol{\theta})}{\partial r} > 0,$$

where $\delta_1 < 0$ is the negative eigenvalue associated with the linearized system of differential equations.

(d) Now provide a complete economic interpretation of the effect of an increase in the discount rate. Be sure to draw the corresponding local comparative dynamics phase portrait.

(e) Show that $\partial k^*(\boldsymbol{\theta})/\partial\alpha = \partial c^*(\boldsymbol{\theta})/\partial\alpha \equiv 0$. Provide an economic interpretation.

(f) Show that a change in tastes has no effect on the time paths of the capital-to-labor ratio and per-capita consumption in a neighborhood of the steady state.

One's intuition may suggest that a rise in the marginal utility of per-capita consumption should affect the steady state of the system or the approach to it, but the mathematics show that such intuition is faulty in the neoclassical growth model.

(g) Show that the steady state comparative statics of the technology parameter are

$$\frac{\partial k^*(\boldsymbol{\theta})}{\partial\gamma} \equiv \frac{U_c^*\phi_{k\gamma}^*}{|J_s|} > 0,$$

$$\frac{\partial c^*(\boldsymbol{\theta})}{\partial\gamma} \equiv \frac{-\phi_\gamma^* U_c^* \phi_{kk}^* + \phi_k^* U_c^* \phi_{k\gamma}^*}{|J_s|} > 0.$$

Provide an economic interpretation of these results.

(h) Show that the impact effect of a change in γ is given by

$$\left.\frac{\partial c(t;\boldsymbol{\theta},k_0)}{\partial\gamma}\right|_{\substack{k_0=k^*(\boldsymbol{\theta})\\ t=0}} = \frac{\partial c^*(\boldsymbol{\theta})}{\partial\gamma} - [r-\delta_1]\frac{\partial k^*(\boldsymbol{\theta})}{\partial\gamma}.$$

In contrast to the case of an increase in the social discount rate, the sign of this expression cannot be determined in general. Thus the moment the increase in γ occurs, it is not possible, in general, to determine if per-capita consumption rises or falls. If the increase in steady state per-capita consumption is large relative to the increase in the steady state capital-to-labor ratio, then per-capita consumption jumps up when γ increases in order to reach its relatively larger steady state value. If, however, the increase in steady state per-capita consumption is small relative to the increase in the steady state capital-to-labor ratio, then per-capita consumption jumps down when γ increases, as there is no need to have per-capita consumption rise initially to meet the relatively smaller increase in its steady state value.

18.12 *Optimal Advertising by a Monopolist*. The basic idea of the model is that a monopolistic firm has a stock of advertising goodwill, $A(t)$, that summarizes the effects of current and past advertising expenditures by the monopolist on the demand for its products. The advertising goodwill (or capital) changes over time according to the ordinary differential equation $\dot{A}(t) = g(u(t);\alpha_2) - \delta A(t)$, where $u(t)$ is the current advertising rate in dollars, $\delta > 0$ is the constant depreciation rate of advertising goodwill, $\alpha_2 > 0$ is a shift parameter, and $A_0 > 0$ is the initial stock of goodwill. Let $g(\cdot): \Re_+ \times \Re_{++} \to \Re_+$ be the $C^{(2)}$ function that maps advertising expenditures into advertising goodwill, that is, $g(\cdot)$ is the goodwill production function. Assume that $g(0;\alpha_2) = 0$, which means that if the monopolist decides not to spend any money on advertising, then the stock of advertising goodwill does not change. In addition, assume that for all $(u, \alpha_2) \in \Re_{++} \times \Re_{++}$, $g_u(u;\alpha_2) > 0$, $g_{u\alpha_2}(u;\alpha_2) > 0$, and $g_{uu}(u;\alpha_2) < 0$, and $\lim_{u\to+\infty} g_u(u;\alpha_2) = 0$ and $\lim_{u\to0^+} g_u(u;\alpha_2) = +\infty$. The monopolist is asserted to operate over the indefinite future and discount its instantaneous profits $\pi(A;\alpha_1)$ at the constant rate $r > 0$. It is assumed that the $C^{(2)}$ profit function $\pi(\cdot):\Re_+ \times \Re_{++} \to \Re_+$ satisfies $\pi_A(A;\alpha_1) > 0$, $\pi_{A\alpha_1}(A;\alpha_1) > 0$, and $\pi_{AA}(A;\alpha_1) < 0$ for all $(A, \alpha_1) \in \Re_{++} \times \Re_{++}$. The monopolist begins its planning at time $t = 0$ with a given stock of advertising goodwill, namely, $A(0) = A_0 > 0$. The optimal control problem the firm must solve in order to determine its optimal advertising expenditure plan is thus given by

$$V(\boldsymbol{\beta}) \stackrel{\text{def}}{=} \max_{u(\cdot)} \int_0^{+\infty} [\pi(A(t);\alpha_1) - u(t)]\, e^{-rt}\, dt$$

$$\text{s.t.} \quad \dot{A}(t) = g(u(t);\alpha_2) - \delta A(t),\ A(0) = A_0,$$

$$A(t) \geq 0,\ u(t) \geq 0\ \forall\, t \in [0, +\infty),$$

where $(\boldsymbol{\theta}, A_0) \stackrel{\text{def}}{=} (\alpha_1, \alpha_2, r, \delta, A_0) \in \Re^5_{++}$ are the time invariant parameters of the problem. Assume that assumptions (A.1) through (A.7) hold for this model. Note that the problem statement explicitly includes nonnegativity restrictions on the paths of the control and state variables.

(a) Prove that if $u(t) \geq 0\ \forall\, t \in [0, +\infty)$, then the nonnegativity constraint on the state variable is not binding $\forall\, t \in [0, +\infty)$. This means that you can ignore the state variable inequality constraint.

(b) Write down the current value Hamiltonian $H(\cdot)$ with current value costate variable λ, and derive the necessary conditions for this problem. Prove that $u(t) > 0\ \forall\, t \in [0, +\infty)$ in an optimal plan.

(c) Prove that the necessary conditions are also sufficient for determining the uniquely optimal solution to the monopolist's advertising problem under suitable additional assumptions to be determined by you.

(d) Show that the necessary (and sufficient) conditions reduce to the following pair of autonomous ordinary differential equations in (u, A):

$$\dot{u} = \frac{g_u(u;\alpha_2)[g_u(u;\alpha_2)\pi_A(A;\alpha_1) - (r+\delta)]}{g_{uu}(u;\alpha_2)}$$

$$\dot{A} = g(u;\alpha_2) - \delta A.$$

(e) Derive the phase portrait for the ordinary differential equations derived in part (d). Be sure to include your derivations of the slopes of the $\dot{u} = 0$ and $\dot{A} = 0$ isoclines, as well as the vector field in the (u, A) phase plane. What type of steady state equilibrium does the model have?

(f) Let $(u^s(\boldsymbol{\alpha}), A^s(\boldsymbol{\alpha}))$ be the solution to the steady state version of the necessary and sufficient conditions in part (d). Prove that the functions $(u^s(\cdot), A^s(\cdot))$ are locally $C^{(1)}$ in $\boldsymbol{\alpha}$. Present your argument carefully using the correct theorem.

(g) Find the steady state comparative statics for the discount rate. Provide an economic interpretation.

(h) Draw the local comparative dynamics phase portrait for the discount rate, making sure to label the optimal path from the old to the new steady state clearly. Provide an economic interpretation of the comparative dynamics result.

(i) Find the steady state comparative statics for the parameter α_1. Provide an economic interpretation.

(j) Draw the local comparative dynamics phase portrait for the parameter α_1, making sure to label the optimal path from the old to the new steady state clearly. Provide an economic interpretation of the comparative dynamics result.

18.13 *The Consumer's Lifetime Allocation Process*. This question extends Mental Exercise 12.9 by considering a consumer with a nonlinear utility function and an infinite planning horizon. You should review the aforementioned mental exercise if you did not attempt it or have forgotten much of its setup. As a result, only those parts of the problem that differ from the previous version will be exposited here. In particular, we now consider the case in which the preferences of the consumer are represented by a nonlinear and additively separable function of the consumption rate and the stock of the asset, to wit, $W(a(t), c(t); \alpha_1, \alpha_2) \stackrel{\text{def}}{=} U(c(t); \alpha_1) + V(a(t); \alpha_2)$, where $U(\cdot) \in C^{(2)}$, $U_c(c; \alpha_1) > 0$, $U_{cc}(c; \alpha_1) < 0$, and $U_{c\alpha_1}(c; \alpha_1) > 0 \; \forall \, (c; \alpha_1) \in \Re^2_{++}$, $\lim_{c \to 0} U_c(c; \alpha_1) = +\infty \;\; \forall \;\; \alpha_1 \in \Re_{++}$, and $V(\cdot) \in C^{(2)}$, $V_a(a; \alpha_2) > 0$, $V_{aa}(a; \alpha_2) < 0$, and $V_{a\alpha_2}(a; \alpha_2) > 0 \;\; \forall \;\; (c; \alpha_1) \in \Re^2_{++}$. It is still assumed that $a(t) \gtrless 0$ is possible, so there is no constraint on the stock of the asset, but the consumption rate is required to be nonnegative. In sum then, the optimal control problem to be solved for the optimal consumption plan is given by

$$\max_{c(\cdot)} \int_0^{+\infty} [U(c(t); \alpha_1) + V(a(t); \alpha_2)] \, e^{-\rho t} \, dt$$

$$\text{s.t.} \quad \dot{a}(t) = y + r a(t) - c(t), \; a(0) = a_0,$$

$$c(t) \geq 0 \; \forall \, t \in [0, +\infty),$$

where $(\boldsymbol{\theta}, a_0) \stackrel{\text{def}}{=} (\alpha_1, \alpha_2, \rho, r, y, a_0) \in \Re^6_{++}$ are the time-invariant parameters of the problem. Assume that assumptions (A.1) through (A.7) hold for this model. As in the previous version of this model, also assume that $\rho > r$.

(a) Write down the current value Hamiltonian $H(\cdot)$ with current value costate variable λ, and derive the necessary conditions for this problem. Prove that $c(t) > 0\,\forall\, t \in [0, +\infty)$ in an optimal plan.
(b) Prove that the necessary conditions are also sufficient for determining the uniquely optimal solution to the consumption planning problem under suitable additional assumptions to be determined by you.
(c) Reduce the three necessary and sufficient conditions to a pair of autonomous ordinary differential equations involving only (a, c).
(d) Derive the phase portrait for the autonomous ordinary differential equations derived in part (c). Be sure to include your derivations of the slopes of the $\dot{a} = 0$ and $\dot{c} = 0$ nullclines, as well as the vector field in the ac-phase plane.

You will discover that two phase diagrams are possible based on the vector field. It is therefore your task to decide which phase portrait is the one that describes the optimal solution to the control problem. Moreover, you must justify your answer carefully and rigorously.

(e) Let $(a^s(\boldsymbol{\alpha}), c^s(\boldsymbol{\alpha}))$ be the solution to the steady state version of the necessary and sufficient conditions in part (c). Find the steady state comparative statics for an increase in income. Provide an economic interpretation.
(f) Draw the local comparative dynamics phase portrait for income, making sure to label the optimal path from the old steady state to the new steady state clearly. Provide an economic interpretation of the comparative dynamics result.
(g) Find the steady state comparative statics for an increase in the parameter α_2. Provide an economic interpretation.
(h) Draw the local comparative dynamics phase portrait for the parameter α_2, making sure to label the optimal path from the old steady state to the new steady state clearly. Provide an economic interpretation of the comparative dynamics result.

FURTHER READING

Samuelson (1947) was the first to systematically use the implicit function theorem to investigate the comparative statics properties of economic models. The method of comparative dynamics put forth by Oniki (1973) is essentially a generalization of the classical method of variational differential equations. His method and the dynamic primal-dual formalism of Caputo (1990a, 1990b, 1992) are approachs to conducting a comparative dynamics analysis of an optimal control problem that are *complementary* to the linearization approach of this chapter. Caputo (1989) provides a detailed account of the linearization approach given here in the context of the nonrenewable resource–extracting model of the firm, whereas Caputo and Ostrom (1996) provide another application in the context of drug policy. This chapter is based on the work of Caputo (1997). Otani (1982) derives comparative dynamics formulas for the multiple state class of symmetric calculus of variations problems. Mental Exercise 18.12 is motivated by the research of Nerlove and Arrow (1962). See Sethi (1977) and Feichtinger et al. (1994) for survey articles dealing with optimal control models in advertising. Dockner (1985) presents a detailed analysis of the local stability of steady states in optimal control problems with two state variables akin to that of this chapter. Tahvonen (1991) provides an extension of one of Dockner's (1985) stability theorems.

REFERENCES

Caputo, M.R. (1989), "The Qualitative Content of Renewable Resource Models," *Natural Resource Modeling*, 3, 241–259.

Caputo, M.R. (1990a), "Comparative Dynamics via Envelope Methods in Variational Calculus," *Review of Economic Studies*, 57, 689–697.

Caputo, M.R. (1990b), "How to Do Comparative Dynamics on the Back of an Envelope in Optimal Control Theory," *Journal of Economic Dynamics and Control*, 14, 655–683.

Caputo, M.R. (1992), "A Primal-Dual Approach to Comparative Dynamics with Time Dependent Parameters in Variational Calculus," *Optimal Control Applications and Methods*, 13, 73–86.

Caputo, M.R. (1997), "The Qualitative Structure of a Class of Infinite Horizon Optimal Control Problems," *Optimal Control Applications and Methods* 18, 195–215.

Caputo, M.R. and Ostrom, B.J. (1996), "Optimal Government Policy Regarding a Previously Illegal Commodity," *Southern Economic Journal*, 62, 690–709.

Dockner, E. (1985), "Local Stability Analysis in Optimal Control Problems with Two State Variables," in Feichtinger, G. (ed.), *Optimal Control Theory and Economic Analysis,* Vol 2 (Amsterdam: North-Holland).

Feichtinger, G., Hartl, R.F., and Sethi, S.P. (1994), "Dynamic Optimal Control Models in Advertising: Recent Developments," *Management Science*, 40, 195–226.

Gaskins, D.W. (1971), "Dynamic Limit Pricing: Optimal Pricing under Threat of Entry," *Journal of Economic Theory*, 3, 306–322.

Nagatani, K. (1976), "On the Comparative Statics of Temporary Equilibrium," Discussion paper, University of British Columbia, No. 76–30.

Nerlove, M. and Arrow, K.J. (1962), "Optimal Advertising Policy under Dynamic Conditions," *Economica*, 29, 129–142.

Oniki, H. (1973), "Comparative Dynamics (Sensitivity Analysis) in Optimal Control Theory," *Journal of Economic Theory*, 6, 265–283.

Otani, K. (1982), "Explicit Formulae of Comparative Dynamics," *International Economic Review*, 23, 411–419.

Protter, M.H. and Morrey, C.B. (1991, 2nd Ed.), *A First Course in Real Analysis* (New York: Springer-Verlag, Inc.).

Samuelson, P.A. (1947), *Foundations of Economic Analysis* (Cambridge, Mass.: Harvard University Press).

Sethi, S.P. (1977), “Dynamic Optimal Control Models in Advertising: A Survey,” *SIAM Review*, 19, 685–725.

Simon, C.P. and Blume, L. (1994), *Mathematics for Economists* (New York: W.W. Norton & Company, Inc.).

Tahvonen, O. (1991), “On the Dynamics of Renewable Resource Harvesting and Pollution Control,” *Environmental and Resource Economics* 1, 97–117.

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