FOURTEEN

Necessary and Sufficient Conditions for Infinite Horizon Control Problems

In many optimal control problems in economics, the planning horizon is assumed to be of infinite length. This means that the person who solves such an optimal control problem is choosing the time path of the control variables for eternity at the initial date of the planning horizon. At first glance, the infinite horizon assumption may seem to be an arbitrary and extreme one, but in fact it is often less extreme than it first appears. For example, it is often just as arbitrary and extreme to assume that a firm would stop planning at some finite date in the future. This issue is especially pertinent if one takes the view of a planner making decisions for an entire economy. Thus the infinite horizon assumption is no more or less extreme, in general, than the assumption of a finite planning horizon. In the end, the choice of the horizon length should be made based on the appropriateness of the assumption for the economic question under consideration as well as the qualitative implications it implies, and their consistency with observed behavior. It is also important to point out that infinite horizon control problems have certain properties that help to considerably simplify the analysis of them that can render an otherwise intractable problem tractable, as this chapter and several others will demonstrate. On the other hand, infinite horizon control problems present two bodacious difficulties of their own.

One difficulty that rears its head for infinite planning horizon problems is whether the objective functional even exists, since it is now an improper integral rather than a proper one. That is, we now have to be more concerned with whether the improper integral converges for all admissible pairs, since our basic assumptions on the functions $f(\cdot)$, $\mathbf{g}(\cdot)$, and $\mathbf{h}(\cdot)$ are no longer sufficient to guarantee that it does. For example, what should our optimality criterion be when the objective functional diverges for some of the admissible pairs? Said differently, how do we select the optimal pair when one or several of the admissible pairs lead to the improper integral diverging to $+\infty$? Probing the convergence issue of the objective functional and thus the choice of optimality criteria will take us into somewhat deep mathematical territory, which we wish to avoid. Our adversity to probing this issue is a result of our focus on the most common and important classes of optimal control problems for economists. That said, we will therefore restrict the domain

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of the objective functional to those admissible pairs, if any, for which the objective functional converges, thereby completely sidestepping the convergence issue. As a practical matter, it is highly recommended to proceed by solving infinite horizon control problems under the assumption that the objective functional converges for all admissible pairs. Moreover, in seeing this material for the first time, it is of considerable pedagogical value to make this assumption. If, however, convergence fails to occur, then one would look for weaker optimality criteria for which an optimum may exist.

The other difficulty that arises when studying infinite horizon control problems is the appropriate set of *necessary* transversality conditions. A famous example given by Halkin (1974) shows that, in general, the natural transversality conditions for the finite horizon case do not carry over in the expected fashion to the infinite horizon case. A bit of time will be spent on this issue seeing as it is central to many problems in economics. It is important to note, however, that the sufficient transversality conditions for the finite horizon case do in fact carry over to the infinite horizon case in the expected manner.

The general class of optimal control problems of interest in this chapter is of the form

$$
\begin{gathered}
J[\mathbf{x}(\cdot), \mathbf{u}(\cdot)] \stackrel{\text{def}}{=} \int_0^{+\infty} f(t, \mathbf{x}(t), \mathbf{u}(t); \boldsymbol{\alpha})\, dt \\
\text{s.t.} \quad \dot{\mathbf{x}}(t) = \mathbf{g}(t, \mathbf{x}(t), \mathbf{u}(t); \boldsymbol{\alpha}),\ \ \mathbf{x}(0) = \mathbf{x}_0, \qquad (1) \\
\lim_{t \to +\infty} x_n(t) = x_n^s, \quad n = 1, 2, \ldots, n_1, \\
\lim_{t \to +\infty} x_n(t) \geq x_n^s, \quad n = n_1 + 1, n_1 + 2, \ldots, n_2, \\
\text{no conditions on } x_n(t) \text{ as } t \to +\infty,\ n = n_2 + 1, n_2 + 2, \ldots, N, \\
h^k(t, \mathbf{x}(t), \mathbf{u}(t); \boldsymbol{\alpha}) \geq 0, \quad k = 1, 2, \ldots, K', \\
h^k(t, \mathbf{x}(t), \mathbf{u}(t); \boldsymbol{\alpha}) = 0, \quad k = K' + 1, K' + 2, \ldots, K,
\end{gathered}
$$

in which we seek to optimize the objective functional $J[\mathbf{x}(\cdot), \mathbf{u}(\cdot)]$ in an as yet unspecified manner, where, just as in previous chapters, $\mathbf{x}(t) \stackrel{\text{def}}{=} (x_1(t), x_2(t), \ldots, x_N(t)) \in \Re^N$ is the state vector, $\mathbf{u}(t) \stackrel{\text{def}}{=} (u_1(t), u_2(t), \ldots, u_M(t)) \in \Re^M$ is the control vector, $\mathbf{g}(\cdot) \stackrel{\text{def}}{=} (g^1(\cdot), g^2(\cdot), \ldots, g^N(\cdot))$ is the vector of transition functions, $\dot{\mathbf{x}}(\cdot) \stackrel{\text{def}}{=} (\dot{x}_1(\cdot), \dot{x}_2(\cdot), \ldots, \dot{x}_N(\cdot))$, $\mathbf{h}(\cdot) \stackrel{\text{def}}{=} (h^1(\cdot), h^2(\cdot), \ldots, h^K(\cdot))$ is the vector of constraint functions, x_n^s, $n = 1, 2, \ldots, n_2$, are fixed values of the terminal value of the state vector, and $\boldsymbol{\alpha} \stackrel{\text{def}}{=} (\alpha_1, \alpha_2, \ldots, \alpha_A) \in \Re^A$ is a vector of time-independent parameters that affect the state equation, integrand, and constraint functions. Let $(\mathbf{z}(t; \boldsymbol{\alpha}, \mathbf{x}_0), \mathbf{v}(t; \boldsymbol{\alpha}, \mathbf{x}_0))$ be the optimal pair with corresponding costate vector $\boldsymbol{\lambda}(t; \boldsymbol{\alpha}, \mathbf{x}_0)$ and Lagrange multiplier vector $\boldsymbol{\mu}(t; \boldsymbol{\alpha}, \mathbf{x}_0)$. Note that we have yet to specify the precise optimality criterion and have simply written down the objective functional $J[\mathbf{x}(\cdot), \mathbf{u}(\cdot)]$. As in the finite horizon case, we have imposed various terminal conditions on the state vector as $t \to +\infty$. Those listed above are the most common forms encountered in dynamic economic theory when an infinite planning horizon is postulated, and are chosen for that reason. Before moving on to the necessary and sufficient conditions for this class of optimal control problems, let's first be precise about what we mean by admissibility for infinite horizon control problems.

Definition 14.1: We call $(\mathbf{x}(t), \mathbf{u}(t))$ an *admissible pair* if $\mathbf{u}(\cdot)$ is any piecewise continuous control vector function such that $\mathbf{u}(t) \in U(t, \mathbf{x}(t)) \forall\, t \in [0, +\infty)$ and $\mathbf{x}(\cdot)$ is a piecewise smooth state vector function such that $\dot{\mathbf{x}}(t) = \mathbf{g}(t, \mathbf{x}(t), \mathbf{u}(t))$, $\mathbf{x}(t_0) = \mathbf{x}_0$, and in addition, if the terminal boundary conditions in control problem (1) are satisfied, where the control set $U(t, \mathbf{x}(t))$ is defined as

$$\begin{aligned} U(t, \mathbf{x}(t)) \stackrel{\text{def}}{=} \{\ & \mathbf{u}(\cdot) : h^k(t, \mathbf{x}(t), \mathbf{u}(t)) \geq 0,\ k = 1, \ldots, K', \\ & h^k(t, \mathbf{x}(t), \mathbf{u}(t)) = 0, \quad k = K' + 1, \ldots, K\}. \end{aligned}$$

For the terminal endpoint conditions $\lim_{t\to+\infty} x_n(t) = x_n^s$, $n = 1, 2, \ldots, n_1$, the requirement for admissibility is that the $\lim_{t\to+\infty} x_n(t)$ exists and equals x_n^s, $n = 1, 2, \ldots, n_1$. The requirement for admissibility for the terminal conditions $\lim_{t\to+\infty} x_n(t) \geq x_n^s$, $n = n_1 + 1, n_1 + 2, \ldots, n_2$, is that the $\lim_{t\to+\infty} x_n(t)$ exists and that it must be at least as large as x_n^s, $n = n_1 + 1, n_1 + 2, \ldots, n_2$. This terminal condition implies that paths of the state vector with some components tending to $+\infty$ or with some components exhibiting undamped cyclical movements are not admissible, since the appropriate limits do not exist in such circumstances. The last terminal condition, namely, that no conditions are placed on $x_n(t)$ as $t \to +\infty$ for $n = n_2 + 1, n_2 + 2, \ldots, N$, is analogous to the case in which $\mathbf{x}_T$ is a choice variable in finite horizon optimal control problems. In general, this terminal condition allows for the *nonexistence* of $\lim_{t\to+\infty} x_n(t)$ for $n = n_2 + 1, n_2 + 2, \ldots, N$. This may occur if $x(t) = \sin t$ or $x(t) = e^t$, for example.

As we noted above, one concern in connection with infinite horizon problems is the choice of the optimality criterion, especially when the objective functional does not converge for all admissible pairs. We will consider only one possibility here, but refer the reader to Seierstad and Sydsæter (1987, Chapter 3, sections 7, 8, 9) for four other optimality criteria that apply when the objective functional does not converge for all admissible pairs in the infinite horizon case.

The case we are interested in is a direct generalization of the finite horizon case, to wit, we look for an admissible pair of functions $(\mathbf{x}(\cdot), \mathbf{u}(\cdot))$ that *maximizes* the improper integral $J[\mathbf{x}(\cdot), \mathbf{u}(\cdot)]$. In other words, our objective is to

$$\max_{\mathbf{u}(\cdot)} J[\mathbf{x}(\cdot), \mathbf{u}(\cdot)] \stackrel{\text{def}}{=} \int_0^{+\infty} f(t, \mathbf{x}(t), \mathbf{u}(t); \boldsymbol{\alpha})\, dt, \tag{2}$$

subject to the constraints listed in problem (1). Because our objective functional is an improper integral, for the maximization to make sense, the integral in Eq. (2) must converge for all admissible pairs. We *assume* this to be the case. Although this may seem like a reasonable and possibly innocuous assumption, do not be misled, for this is a reasonable assumption in most economic models, but it is certainly not innocuous, as Seierstad and Sydsæter (1987, Chapter 3) show.

Given that we are assuming that the objective functional $J[\mathbf{x}(\cdot), \mathbf{u}(\cdot)]$ exists for all admissible pairs, that is, the improper integral converges, we begin the technical discussion by examining sufficient conditions for convergence of $J[\mathbf{x}(\cdot), \mathbf{u}(\cdot)]$. We do this by presenting two theorems that are simple to understand and use. The second theorem is the most relevant for the kinds of infinite horizon optimal control models that economists study. The proof of the first theorem is straightforward and is therefore left as a mental exercise, whereas the proof of the second is provided.

Theorem 14.1: *Given the improper integral*

$$J[\mathbf{x}(\cdot), \mathbf{u}(\cdot)] \stackrel{\text{def}}{=} \int_0^{+\infty} f(t, \mathbf{x}(t), \mathbf{u}(t); \boldsymbol{\alpha})\, dt,$$

if the integrand function $f(\cdot)$ *is finite* $\forall\ t \in [0, +\infty)$*, and if* $f(t, \mathbf{x}(t), \mathbf{u}(t); \boldsymbol{\alpha}) = 0\, \forall\, t \geq \tau$*, where* $\tau \in [0, +\infty)$*, then the improper integral will converge.*

In words, this theorem says that if the integrand function of an improper integral is finite and takes on a value of zero at some finite point in time and remains at zero thereafter, then the improper integral will converge. In effect, under the conditions of Theorem 14.1, the improper integral has a finite upper limit of integration of τ, and is therefore really a proper integral whose value is finite. This is a sufficient condition for convergence of the improper integral.

The next sufficient condition for convergence of an improper integral is an extremely useful one in dynamic economic theory. We will make use of it in later chapters.

Theorem 14.2: *Given the improper integral*

$$J[\mathbf{x}(\cdot), \mathbf{u}(\cdot)] \stackrel{\text{def}}{=} \int_0^{+\infty} f(t, \mathbf{x}(t), \mathbf{u}(t); \boldsymbol{\alpha})\, dt,$$

if $f(t, \mathbf{x}(t), \mathbf{u}(t); \boldsymbol{\alpha}) \stackrel{\text{def}}{=} \phi(t, \mathbf{x}(t), \mathbf{u}(t); \boldsymbol{\alpha})\, e^{-rt}$*, where* $r > 0$*, and if the function* $\phi(\cdot)$ *is bounded, say,* $|\phi(t, \mathbf{x}(t), \mathbf{u}(t); \boldsymbol{\alpha})| \leq B \in \Re_{++}$*, then the improper integral will converge and not exceed* $\frac{B}{r}$*.*

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Proof: The proof is a straightforward application of some elementary properties of the absolute value operator, which yield a string of equalities and inequalities, the explanations of which are below the ensuing equation:

$$\begin{aligned}
&\left| \int_0^{+\infty} f(t, \mathbf{x}(t), \mathbf{u}(t); \boldsymbol{\alpha})\, dt \right| \\
&\quad = \left| \int_0^{+\infty} \phi(t, \mathbf{x}(t), \mathbf{u}(t); \boldsymbol{\alpha}) e^{-rt}\, dt \right| \leq \int_0^{+\infty} \left| \phi(t, \mathbf{x}(t), \mathbf{u}(t); \boldsymbol{\alpha}) e^{-rt} \right| dt \\
&\quad = \int_0^{+\infty} |\phi(t, \mathbf{x}(t), \mathbf{u}(t); \boldsymbol{\alpha})| \left| e^{-rt} \right| dt \leq B \int_0^{+\infty} e^{-rt}\, dt = \frac{B}{r}.
\end{aligned}$$

The first equality follows from the definition of the function $f(\cdot)$; the first weak inequality follows from the theorem that asserts that the absolute value of the integral of a function is less than or equal to the integral of the absolute value of that function; the next equality follows from the fact that the absolute value of a product of functions is equal to the product of their absolute values; the last inequality follows from the assumed boundedness of the function $\phi(\cdot)$. Q.E.D.

The bodacious feature of Theorem 14.2 is the form of the integrand function, which we have denoted by $f(t, \mathbf{x}(t), \mathbf{u}(t); \boldsymbol{\alpha}) \stackrel{\text{def}}{=} \phi(t, \mathbf{x}(t), \mathbf{u}(t); \boldsymbol{\alpha})\, e^{-rt}$, $r > 0$ and constant. In particular, it is the presence of the exponential discount factor e^{-rt} that is the most noteworthy. This is because the function $\phi(\cdot)$ is bounded and the exponential discount factor e^{-rt} goes to zero sufficiently fast as $t \to +\infty$, thereby providing the driving force behind the convergence of the improper integral. Moreover, the appearance of the exponential discount factor e^{-rt} is common to nearly every dynamic optimization problem in economics, whether it is a firm discounting its profits in computing its present value, or an individual discounting her instantaneous utility to arrive at her wealth maximizing consumption plan. Furthermore, after Chapter 12, we now understand precisely why this is so, videlicet, the principle of time consistency or dynamic consistency. Thus, Theorem 14.2 is very useful in that it gives a simple sufficient condition for convergence of an improper integral for a class of optimal control problems that are central to, and ubiquitous in, dynamic economic theory.

We are now in a position to state and prove the necessary conditions for optimal control problem (1). First, we note that if $(\mathbf{z}(t; \boldsymbol{\alpha}, \mathbf{x}_0), \mathbf{v}(t; \boldsymbol{\alpha}, \mathbf{x}_0))$ is an optimal pair for problem (1) under our assumption of convergence of the objective functional for all admissible pairs, then for any $T < +\infty$, it follows from the principle of optimality that it must also be an optimal pair for the truncated control problem

$$\max_{\mathbf{u}(\cdot)} J[\mathbf{x}(\cdot), \mathbf{u}(\cdot)] \stackrel{\text{def}}{=} \int_0^T f(t, \mathbf{x}(t), \mathbf{u}(t); \boldsymbol{\alpha})\, dt$$

$$\text{s.t.} \quad \dot{\mathbf{x}}(t) = \mathbf{g}(t, \mathbf{x}(t), \mathbf{u}(t)),\ \mathbf{x}(0) = \mathbf{x}_0,\ \mathbf{x}(T) = \mathbf{z}(T; \boldsymbol{\alpha}, \mathbf{x}_0), \tag{3}$$

$$h^k(t, \mathbf{x}(t), \mathbf{u}(t); \boldsymbol{\alpha}) \geq 0, \quad k = 1, 2, \ldots, K',$$

$$h^k(t, \mathbf{x}(t), \mathbf{u}(t); \boldsymbol{\alpha}) = 0, \quad k = K'+1, K'+2, \ldots, K.$$

But this is the prototype finite-horizon fixed–endpoint optimal control problem, whose necessary conditions are given by Theorem 6.1. Thus the pair $(\mathbf{z}(t; \boldsymbol{\alpha}, \mathbf{x}_0), \mathbf{v}(t; \boldsymbol{\alpha}, \mathbf{x}_0))$ must satisfy all the necessary conditions of Theorem 6.1, *except for the transversality conditions that go with it,* since problem (3) does not have the same endpoint conditions as problem (1). That is, all the necessary conditions for finite horizon problems must also hold for infinite horizon problems with the exception of the transversality conditions. For notational clarity, we suppress the dependence of the optimal pair and associated costate and Lagrange multiplier vectors on the parameters $(\boldsymbol{\alpha}, \mathbf{x}_0)$, as well as suppress the dependence of the functions $f(\cdot)$, $\mathbf{g}(\cdot)$, $\mathbf{h}(\cdot)$, $H(\cdot)$, and $L(\cdot)$ on $\boldsymbol{\alpha}$. With this in mind, we have established the following theorem.

Theorem 14.3 (Necessary Conditions, Infinite Horizon): *Let $(\mathbf{z}(t), \mathbf{v}(t))$ be an admissible pair for problem (1), and assume that the rank constraint qualification is satisfied. Then if $(\mathbf{z}(t), \mathbf{v}(t))$ yields the absolute maximum of $J[\cdot]$, it is necessary that there exist a piecewise smooth vector-valued function $\boldsymbol{\lambda}(\cdot) \stackrel{\text{def}}{=} (\lambda_1(\cdot), \lambda_2(\cdot), \ldots, \lambda_N(\cdot))$ and a piecewise continuous vector-valued Lagrange multiplier function $\boldsymbol{\mu}(\cdot) \stackrel{\text{def}}{=} (\mu_1(\cdot), \mu_2(\cdot), \ldots, \mu_K(\cdot))$, such that for all $t \in [0, +\infty)$,*

$$\mathbf{v}(t) = \arg\max_{\mathbf{u}} \{H(t, \mathbf{z}(t), \mathbf{u}, \boldsymbol{\lambda}(t)) \text{ s.t. } \mathbf{u} \in U(t, \mathbf{z}(t))\},$$

that is, if

$$M(t, \mathbf{z}(t), \boldsymbol{\lambda}(t)) \stackrel{\text{def}}{=} \max_{\mathbf{u}} \{H(t, \mathbf{z}(t), \mathbf{u}, \boldsymbol{\lambda}(t)) \text{ s.t. } \mathbf{u} \in U(t, \mathbf{z}(t))\},$$

then

$$M(t, \mathbf{z}(t), \boldsymbol{\lambda}(t)) \equiv H(t, \mathbf{z}(t), \mathbf{v}(t), \boldsymbol{\lambda}(t)),$$

or equivalently

$$H(t, \mathbf{z}(t), \mathbf{v}(t), \boldsymbol{\lambda}(t)) \geq H(t, \mathbf{z}(t), \mathbf{u}, \boldsymbol{\lambda}(t))\, \forall\, \mathbf{u} \in U(t, \mathbf{z}(t)),$$

where $U(t, \mathbf{x}(t)) \stackrel{\text{def}}{=} \{\mathbf{u}(\cdot) : h^k(t, \mathbf{x}(t)\mathbf{u}(t)) \geq 0,\ k = 1, \ldots, K',\ h^k(t, \mathbf{x}(t), \mathbf{u}(t)) = 0,$ $k = K'+1, \ldots, K\}$. Because the rank constraint qualification is assumed to hold,

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the above necessary condition implies that

$$\frac{\partial L}{\partial u_m}(t, \mathbf{z}(t), \mathbf{v}(t), \boldsymbol{\lambda}(t), \boldsymbol{\mu}(t)) = 0, \quad m = 1, 2, \ldots, M,$$

$$\frac{\partial L}{\partial \mu_\ell}(t, \mathbf{z}(t), \mathbf{v}(t), \boldsymbol{\lambda}(t), \boldsymbol{\mu}(t)) \geq 0, \quad \mu_\ell(t) \geq 0,$$

$$\mu_\ell(t)\frac{\partial L}{\partial \mu_\ell}(t, \mathbf{z}(t), \mathbf{v}(t), \boldsymbol{\lambda}(t), \boldsymbol{\mu}(t)) = 0, \quad \ell = 1, 2, \ldots, K',$$

$$\frac{\partial L}{\partial \mu_\ell}(t, \mathbf{z}(t), \mathbf{v}(t), \boldsymbol{\lambda}(t), \boldsymbol{\mu}(t)) = 0, \quad \ell = K' + 1, K' + 2, \ldots, K,$$

where

$$\begin{aligned} L(t, \mathbf{x}, \mathbf{u}, \boldsymbol{\lambda}, \boldsymbol{\mu}) &\stackrel{\text{def}}{=} H(t, \mathbf{x}, \mathbf{u}, \boldsymbol{\lambda}) + \boldsymbol{\mu}'\mathbf{h}(t, \mathbf{x}, \mathbf{u}) \\ &= f(t, \mathbf{x}, \mathbf{u}) + \sum_{n=1}^{N} \lambda_n g^n(t, \mathbf{x}, \mathbf{u}) + \sum_{k=1}^{K} \mu_k h^k(t, \mathbf{x}, \mathbf{u}) \end{aligned}$$

is the Lagrangian. Furthermore, except for the points of discontinuities of $\mathbf{v}(t)$*,*

$$\dot{z}_i(t) = \frac{\partial L}{\partial \lambda_i}(t, \mathbf{z}(t), \mathbf{v}(t), \boldsymbol{\lambda}(t), \boldsymbol{\mu}(t)) = g^i(t, \mathbf{z}(t), \mathbf{v}(t)), \quad i = 1, 2, \ldots, N,$$

$$\dot{\lambda}_i(t) = -\frac{\partial L}{\partial x_i}(t, \mathbf{z}(t), \mathbf{v}(t), \boldsymbol{\lambda}(t), \boldsymbol{\mu}(t)), \quad i = 1, 2, \ldots, N,$$

where the above notation means that the functions are first differentiated with respect to the particular variable and then evaluated at $(t, \mathbf{z}(t), \mathbf{v}(t), \boldsymbol{\lambda}(t), \boldsymbol{\mu}(t))$.

It is important to note that this theorem is valid for *any* type of terminal boundary condition. In fact, if the three terminal boundary conditions in problem (1) were replaced with $\lim_{t\to+\infty} x_n(t) = x_n^s$, $n = 1, 2, \ldots, N$, so that problem (1) becomes a fixed–endpoint infinite-horizon control problem, then Theorem 14.3 contains enough information to single out one or a few candidates for optimality. To see why, simply observe that in this case, the canonical equations generate $2N$ constants of integration when solved, which can in principle be found by using the $2N$ boundary conditions $\mathbf{x}(t_0) = \mathbf{x}_0$ and $\lim_{t\to+\infty} x_n(t) = x_n^s$, $n = 1, 2, \ldots, N$.

If the terminal boundary conditions are as given in problem (1), however, then Theorem 14.3 does not contain enough information to single out one or a few candidates for optimality. This is because the $2N$ constants of integration generated when solving the canonical equations cannot be completely determined because there are only $N + n_1$ endpoint conditions given in problem (1), namely, $\mathbf{x}(t_0) = \mathbf{x}_0$ and $\lim_{t\to+\infty} x_n(t) = x_n^s$, $n = 1, 2, \ldots, n_1$, and no transversality conditions are provided by Theorem 14.3 to determine the remaining $N - n_1$ constants of integration. For this reason, some infinite horizon optimal control problems in economics simply assume that the terminal endpoint condition is $\lim_{t\to+\infty} x_n(t) = x_n^s$

for all $n = 1, 2, \ldots, N$, thereby permitting Theorem 14.3 to single out one or a few candidates for optimality, as noted in the prior paragraph. In this case, the terminal endpoints of the state variables are typically assumed to be their steady state values.

As far as the transversality conditions for problem (1) are concerned, let us state again that there are *none in general*. Given the three types of terminal endpoint conditions in problem (1), videlicet,

$$\lim_{t \to +\infty} x_n(t) = x_n^s, \quad n = 1, 2, \ldots, n_1,$$

$$\lim_{t \to +\infty} x_n(t) \geq x_n^s, \quad n = n_1 + 1, n_1 + 2, \ldots, n_2,$$

$$\text{no conditions on } x_n(t) \text{ as } t \to +\infty, \quad n = n_2 + 1, n_2 + 2, \ldots, N,$$

one might be tempted to infer by analogy with the finite horizon case that the corresponding transversality conditions are

$$\text{no conditions on } \lambda_n(t) \text{ as } t \to +\infty, \quad n = 1, 2, \ldots, n_1,$$

$$\lim_{t \to +\infty} \lambda_n(t) \geq 0, \quad n = n_1 + 1,\ n_1 + 2, \ldots, n_2,$$

$$\lim_{t \to +\infty} \lambda_n(t) = 0, \quad n = n_2 + 1,\ n_2 + 2, \ldots, N,$$

respectively. Unfortunately, this is not true in general, as we have repeatedly emphasized. Only by imposing additional and somewhat strong restrictions on the functions $f(\cdot)$, $\mathbf{g}(\cdot)$, and $\mathbf{h}(\cdot)$, does one obtain transversality conditions similar to these. The interested reader is referred to Benveniste and Scheinkman (1982), Michel (1982), Araujo and Scheinkman (1983), and of course, Seierstad and Sydsæter (1987, Chapter 3) for such matters. We will, however, present and prove the necessary transversality condition in Michel (1982), for it is useful in analyzing a large class of infinite horizon optimal control models in economics under mild assumptions that are typically encountered in dynamic economic theory. But first, we consider two sufficiency theorems and a necessary one.

The first sufficiency theorem extends the Mangasarian sufficiency theorem to infinite horizon problems, whereas the second extends the Arrow sufficiency theorem to infinite horizon problems. We prove the first and leave the proof of the second for a mental exercise. We again elect to suppress the dependence of the functions on the parameters for notational clarity.

Theorem 14.4 (Mangasarian Sufficient Conditions, Infinite Horizon): *Let* $(\mathbf{z}(t), \mathbf{v}(t))$ *be an admissible pair for problem (1). Suppose that* $(\mathbf{z}(t), \mathbf{v}(t))$ *satisfies the necessary conditions of Theorem 14.3 for problem* (1) *with costate vector* $\boldsymbol{\lambda}(t)$ *and Lagrange multiplier vector* $\boldsymbol{\lambda}(t)$*, and let* $L(t, \mathbf{x}, \mathbf{u}, \boldsymbol{\lambda}, \boldsymbol{\mu}) \stackrel{\text{def}}{=} f(t, \mathbf{x}, \mathbf{u}) + \boldsymbol{\lambda}'\mathbf{g}(t, \mathbf{x}, \mathbf{u}) + \boldsymbol{\mu}'\mathbf{h}(t, \mathbf{x}, \mathbf{u})$ *be the value of the Lagrangian function. If* $L(\cdot)$ *is a concave function of* $(\mathbf{x}, \mathbf{u})\, \forall\, t \in [0, +\infty)$ *over an open convex set containing all the*

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admissible values of $(\mathbf{x}(\cdot), \mathbf{u}(\cdot))$ *when the costate vector is* $\boldsymbol{\lambda}(t)$ *and Lagrange multiplier vector is* $\boldsymbol{\mu}(t)$*, and if for every admissible control path* $\mathbf{u}(t)$, $\lim_{t\to+\infty} \boldsymbol{\lambda}(t)'[\mathbf{z}(t) - \mathbf{x}(t)] \leq 0$*, where* $\mathbf{x}(t)$ *is the time path of the state variable corresponding to* $\mathbf{u}(t)$*, then* $\mathbf{v}(t)$ *is an optimal control and* $(\mathbf{z}(t), \mathbf{v}(t))$ *yields the global maximum of* $J[\cdot]$*. If* $L(\cdot)$ *is a strictly concave function under the same conditions, then* $(\mathbf{z}(t), \mathbf{v}(t))$ *yields the unique global maximum of* $J[\cdot]$.

Proof: Let $(\mathbf{x}(t), \mathbf{u}(t))$ be any admissible pair. By hypothesis, $L(\cdot)$ is a $C^{(1)}$ concave function of $(\mathbf{x}, \mathbf{u})\, \forall\, t \in [0, +\infty)$. It therefore follows from Theorem 21.3 of Simon and Blume (1994) that

$$\begin{aligned} L(t, \mathbf{x}(t), \mathbf{u}(t), \boldsymbol{\lambda}(t), \boldsymbol{\mu}(t)) &\leq L(t, \mathbf{z}(t), \mathbf{v}(t), \boldsymbol{\lambda}(t), \boldsymbol{\mu}(t)) \\ &\quad + L_{\mathbf{x}}(t, \mathbf{z}(t), \mathbf{v}(t), \boldsymbol{\lambda}(t), \boldsymbol{\mu}(t))\,[\mathbf{x}(t) - \mathbf{z}(t)] \\ &\quad + L_{\mathbf{u}}(t, \mathbf{z}(t), \mathbf{v}(t), \boldsymbol{\lambda}(t), \boldsymbol{\mu}(t))\,[\mathbf{u}(t) - \mathbf{v}(t)], \end{aligned} \tag{4}$$

for every $t \in [0, +\infty)$. Using the fact that $L_{\mathbf{u}}(t, \mathbf{z}(t), \mathbf{v}(t), \boldsymbol{\mu}(t), \boldsymbol{\mu}(t)) \equiv \mathbf{0}'_M$ by Theorem 14.3 in Eq. (4), and then integrating both sides of the resulting reduced inequality over the interval $[0, +\infty)$ using the definitions of $L(\cdot)$ and $J[\cdot]$, yields

$$\begin{aligned} J[\mathbf{x}(\cdot), \mathbf{u}(\cdot)] \leq J[\mathbf{z}(\cdot), \mathbf{v}(\cdot)] &+ \int_0^{+\infty} \boldsymbol{\lambda}(t)'\,[\mathbf{g}(t, \mathbf{z}(t), \mathbf{v}(t)) - \mathbf{g}(t, \mathbf{x}(t), \mathbf{u}(t))]\,dt \\ &+ \int_0^{+\infty} \boldsymbol{\mu}(t)'\,[\mathbf{h}(t, \mathbf{z}(t), \mathbf{v}(t)) - \mathbf{h}(t, \mathbf{x}(t), \mathbf{u}(t))]\,dt \\ &+ \int_0^{+\infty} L_{\mathbf{x}}(t, \mathbf{z}(t), \mathbf{v}(t), \boldsymbol{\lambda}(t), \boldsymbol{\mu}(t))\,[\mathbf{x}(t) - \mathbf{z}(t)]\,dt. \end{aligned} \tag{5}$$

By admissibility, $\dot{\mathbf{z}}(t) \equiv \mathbf{g}(t, \mathbf{z}(t), \mathbf{v}(t))$ and $\dot{\mathbf{x}}(t) \equiv \mathbf{g}(t, \mathbf{x}(t), \mathbf{u}(t))$ for every $t \in [0, +\infty)$, whereas Theorem 14.3 implies that $\dot{\boldsymbol{\lambda}}(t)' \equiv -L_{\mathbf{x}}(t, \mathbf{z}(t), \mathbf{v}(t), \boldsymbol{\lambda}(t), \boldsymbol{\mu}(t))$ for every $t \in [0, +\infty)$. Substituting these three results in Eq. (5) gives

$$\begin{aligned} J[\mathbf{x}(\cdot), \mathbf{u}(\cdot)] \leq J[\mathbf{z}(\cdot), \mathbf{v}(\cdot)] &+ \int_0^{+\infty} [\boldsymbol{\lambda}(t)'[\dot{\mathbf{z}}(t) - \dot{\mathbf{x}}(t)] + \dot{\boldsymbol{\lambda}}(t)'[\mathbf{z}(t) - \mathbf{x}(t)]]\,dt \\ &+ \int_0^{+\infty} \boldsymbol{\mu}(t)'\,[\mathbf{h}(t, \mathbf{z}(t), \mathbf{v}(t)) - \mathbf{h}(t, \mathbf{x}(t), \mathbf{u}(t))]\,dt. \end{aligned} \tag{6}$$

Moreover, Theorem 14.3 also implies that (i) $\mu_k(t)h^k(t, \mathbf{z}(t), \mathbf{v}(t)) \equiv 0$ for $k = 1, 2, \ldots, K$ because $\mu_k(t)h^k(t, \mathbf{z}(t), \mathbf{v}(t)) \equiv 0$ for $k = 1, 2, \ldots, K'$ and

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$h^k(t, \mathbf{z}(t), \mathbf{v}(t)) \equiv 0$ for $k = K' + 1, K' + 2, \ldots, K$, (ii) $\mu_k(t)h^k(t, \mathbf{x}(t), \mathbf{u}(t)) \geq 0$ for $k = 1, 2, \ldots, K'$ by virtue of $\mu_k(t) \geq 0$ and $h^k(t, \mathbf{x}(t), \mathbf{u}(t)) \geq 0$ for $k = 1, 2, \ldots, K'$, and (iii) $\mu_k(t)h^k(t, \mathbf{x}(t), \mathbf{u}(t)) \equiv 0$ for $k = K' + 1, K' + 2, \ldots, K$ on account of $h^k(t, \mathbf{x}(t), \mathbf{u}(t)) \equiv 0$ for $k = K' + 1, K' + 2, \ldots, K$. These three implications of Theorem 14.3 therefore imply that

$$\int_0^{+\infty} \boldsymbol{\mu}(t)' \left[\mathbf{h}(t, \mathbf{z}(t), \mathbf{v}(t)) - \mathbf{h}(t, \mathbf{x}(t), \mathbf{u}(t))\right] dt \leq 0. \tag{7}$$

Using the inequality in Eq. (7) permits Eq. (6) to be rewritten in the reduced form

$$J[\mathbf{x}(\cdot), \mathbf{u}(\cdot)] \leq J[\mathbf{z}(\cdot), \mathbf{v}(\cdot)] + \int_0^{+\infty} [\boldsymbol{\lambda}(t)'[\dot{\mathbf{z}}(t) - \dot{\mathbf{x}}(t)] + \dot{\boldsymbol{\lambda}}(t)'[\mathbf{z}(t) - \mathbf{x}(t)]]\, dt. \tag{8}$$

To wrap up the proof, simply note that

$$\frac{d}{dt}\left[\boldsymbol{\lambda}(t)'[\mathbf{z}(t) - \mathbf{x}(t)]\right] = \boldsymbol{\lambda}(t)'[\dot{\mathbf{z}}(t) - \dot{\mathbf{x}}(t)] + \dot{\boldsymbol{\lambda}}(t)'[\mathbf{z}(t) - \mathbf{x}(t)],$$

and substitute this result into Eq. (8) to get

$$\begin{aligned}
J[\mathbf{x}(\cdot), \mathbf{u}(\cdot)] &\leq J[\mathbf{z}(\cdot), \mathbf{v}(\cdot)] + \int_0^{+\infty} \frac{d}{dt}\left[\boldsymbol{\lambda}(t)'[\mathbf{z}(t) - \mathbf{x}(t)]\right] dt \\
&= J[\mathbf{z}(\cdot), \mathbf{v}(\cdot)] + \lim_{t\to+\infty} \boldsymbol{\lambda}(t)'\left[\mathbf{z}(t) - \mathbf{x}(t)\right] - \boldsymbol{\lambda}(0)'\left[\mathbf{z}(0) - \mathbf{x}(0)\right] \\
&= J[\mathbf{z}(\cdot), \mathbf{v}(\cdot)] + \lim_{t\to+\infty} \boldsymbol{\lambda}(t)'\left[\mathbf{z}(t) - \mathbf{x}(t)\right],
\end{aligned}$$

since by admissibility we have $\mathbf{x}(0) = \mathbf{x}_0$ and $\mathbf{z}(0) = \mathbf{x}_0$. Now if for every admissible control path $\mathbf{u}(t)$, $\lim_{t\to+\infty} \boldsymbol{\lambda}(t)'[\mathbf{z}(t) - \mathbf{x}(t)] \leq 0$, where $\mathbf{x}(t)$ is the time path of the state variable corresponding to $\mathbf{u}(t)$, then it follows that $J[\mathbf{x}(\cdot), \mathbf{u}(\cdot)] \leq J[\mathbf{z}(\cdot), \mathbf{v}(\cdot)]$ for all admissible functions $(\mathbf{x}(\cdot), \mathbf{u}(\cdot))$, just as we wished to show. If $L(\cdot)$ is a strictly concave function of $(\mathbf{x}, \mathbf{u})\,\forall\, t \in [0, +\infty)$, then the inequality in Eq. (4) becomes strict if either $\mathbf{x}(t) \neq \mathbf{z}(t)$ or $\mathbf{u}(t) \neq \mathbf{v}(t)$ for some $t \in [0, +\infty)$. In this instance, $J[\mathbf{x}(\cdot), \mathbf{u}(\cdot)] < J[\mathbf{z}(\cdot), \mathbf{v}(\cdot)]$ follows. This shows that any admissible pair of functions $(\mathbf{x}(\cdot), \mathbf{u}(\cdot))$ that is not identically equal to $(\mathbf{z}(\cdot), \mathbf{v}(\cdot))$ is suboptimal. Q.E.D.

It is important to emphasize that the transversality condition

$$\lim_{t\to+\infty} \boldsymbol{\lambda}(t)'[\mathbf{z}(t) - \mathbf{x}(t)] \leq 0$$

is an inner product expression. As a result, it can be written equivalently as

$$\lim_{t\to+\infty} \boldsymbol{\lambda}(t)'[\mathbf{z}(t) - \mathbf{x}(t)] = \lim_{t\to+\infty} \sum_{n=1}^{N} \lambda_n(t)[z_n(t) - x_n(t)] \leq 0,$$

using index notation. This form may often be useful in checking whether it is satisfied in particular control problems.

Let's determine sufficient conditions under which $\lim_{t\to+\infty} \boldsymbol{\lambda}(t)'[\mathbf{z}(t) - \mathbf{x}(t)] \leq 0$ is satisfied for the fixed endpoints version of problem (1), that is, the case in which $\lim_{t\to+\infty} x_n(t) = x_n^s$, $n = 1, 2, \ldots, N$, is the terminal endpoint condition. In this case, for the pairs of curves $(\mathbf{x}(t), \mathbf{u}(t))$ and $(\mathbf{z}(t), \mathbf{v}(t))$ to be admissible, $\lim_{t\to+\infty} x_n(t) = x_n^s$ and $\lim_{t\to+\infty} z_n(t) = x_n^s$, $n = 1, 2, \ldots, N$, must hold, respectively. This implies that $\lim_{t\to+\infty}[z_n(t) - x_n(t)] = x_n^s - x_n^s = 0$, $n = 1, 2, \ldots, N$. Therefore, if $\lambda_n(t)$, $n = 1, 2, \ldots, N$ is bounded, or if $\lim_{t\to+\infty} \lambda_n(t)$ exists for $n = 1, 2, \ldots, N$, then it follows that $\lim_{t\to+\infty} \boldsymbol{\lambda}(t)'[\mathbf{z}(t) - \mathbf{x}(t)] = 0$. Hence the transversality conditions of the Mangasarian sufficiency theorem are satisfied for the fixed endpoints version of problem (1) if either $\lambda_n(t)$ is bounded, $n = 1, 2, \ldots, N$, or if $\lim_{t\to+\infty} \lambda_n(t)$ exists for $n = 1, 2, \ldots, N$.

Obviously, there are many sufficient conditions that ensure that the transversality condition $\lim_{t\to+\infty} \boldsymbol{\lambda}(t)'[\mathbf{z}(t) - \mathbf{x}(t)] \leq 0$ holds in problem (1). These sufficient conditions are often of great value in checking the sufficient transversality condition in infinite horizon problems of interest to economists. The following is one such set of sufficient conditions.

Lemma 14.1: *In problem (1), if the following conditions hold for all admissible* $\mathbf{x}(t)$

(i) for $n = 1, 2, \ldots, n_1$, *either* $|\lambda_n(t)| < P$ *for some number P, or* $\lim_{t\to+\infty} \lambda_n(t)$ *exists, or* $\lim_{t\to+\infty} \lambda_n(t)[z_n(t) - x_n(t)] \leq 0$, *and*

(ii) $\lim_{t\to+\infty} \lambda_n(t) \geq 0$ *for* $n = n_1 + 1, n_1 + 2, \ldots, N$, *and*

(iii) $\lim_{t\to+\infty} \lambda_n(t) z_n(t) = 0$ *for* $n = n_1 + 1, n_1 + 2, \ldots, N$, *and*

(iv) $0 \leq x_n(t) < Q$ *for some number* $Q \in \Re_{++}$ *for* $n = n_1 + 1, n_1 + 2, \ldots, N$,

then $\lim_{t\to+\infty} \boldsymbol{\lambda}(t)'[\mathbf{z}(t) - \mathbf{x}(t)] \leq 0$ *for all admissible* $\mathbf{x}(t)$.

Theorem 14.4 is the Mangasarian sufficiency theorem for unbounded time horizons. Following our discussion in Chapter 6, we now present the necessary conditions for problem (1) using the maximized Hamiltonian, and then the Arrow sufficiency theorem. The proofs of both theorems are by now straightforward and thus will be left as mental exercises.

Theorem 14.5 (Necessary Conditions, Infinite Horizon): *Let* $(\mathbf{z}(t), \mathbf{v}(t))$ *be an admissible pair for problem (1), and assume that the rank constraint qualification is satisfied. Then if* $(\mathbf{z}(t), \mathbf{v}(t))$ *yields the absolute maximum of* $J[\cdot]$, *it is necessary that there exist a piecewise smooth vector-valued function* $\boldsymbol{\lambda}(\cdot) \overset{\text{def}}{=} (\lambda_1(\cdot), \lambda_2(\cdot), \ldots, \lambda_N(\cdot))$ *and a piecewise continuous vector-valued Lagrange multiplier function* $\boldsymbol{\mu}(\cdot) \overset{\text{def}}{=} (\mu_1(\cdot), \mu_2(\cdot), \ldots, \mu_K(\cdot))$, *such that for all* $t \in [0, +\infty)$,

$$\mathbf{v}(t) = \arg\max_{\mathbf{u}} \{H(t, \mathbf{z}(t), \mathbf{u}, \boldsymbol{\lambda}(t)) \text{ s.t. } \mathbf{u} \in U(t, \mathbf{z}(t))\},$$

that is, if

$$M(t, \mathbf{z}(t), \boldsymbol{\lambda}(t)) \stackrel{\text{def}}{=} \max_{\mathbf{u}}\{H(t, \mathbf{z}(t), \mathbf{u}, \boldsymbol{\lambda}(t)) \text{ s.t. } \mathbf{u} \in U(t, \mathbf{z}(t))\},$$

then

$$M(t, \mathbf{z}(t), \boldsymbol{\lambda}(t)) \equiv H(t, \mathbf{z}(t), \mathbf{v}(t), \boldsymbol{\lambda}(t)),$$

or equivalently

$$H(t, \mathbf{z}(t), \mathbf{v}(t), \boldsymbol{\lambda}(t)) \geq H(t, \mathbf{z}(t), \mathbf{u}, \boldsymbol{\lambda}(t))\ \forall\ \mathbf{u} \in U(t, \mathbf{z}(t)),$$

where $U(t, \mathbf{x}(t)) \stackrel{\text{def}}{=} \{\mathbf{u}(\cdot) : h^k(t, \mathbf{x}(t)\mathbf{u}(t)) \geq 0,\ k = 1, \ldots, K', h^k(t, \mathbf{x}(t), \mathbf{u}(t)) = 0,$ $k = K' + 1, \ldots, K\}$ *and* $\mathbf{v}(t) \stackrel{\text{def}}{=} \hat{\mathbf{u}}(t, \mathbf{z}(t), \boldsymbol{\lambda}(t))$. *Because the rank constraint qualification is assumed to hold, the above necessary condition implies that*

$$\frac{\partial L}{\partial u_m}(t, \mathbf{z}(t), \mathbf{v}(t), \boldsymbol{\lambda}(t), \boldsymbol{\mu}(t)) = 0, \quad m = 1, 2, \ldots, M,$$

$$\frac{\partial L}{\partial \mu_\ell}(t, \mathbf{z}(t), \mathbf{v}(t), \boldsymbol{\lambda}(t), \boldsymbol{\mu}(t)) \geq 0, \quad \mu_\ell(t) \geq 0,$$

$$\mu_\ell(t)\frac{\partial L}{\partial \mu_\ell}(t, \mathbf{z}(t), \mathbf{v}(t), \boldsymbol{\lambda}(t), \boldsymbol{\mu}(t)) = 0, \quad \ell = 1, 2, \ldots, K',$$

$$\frac{\partial L}{\partial \mu_\ell}(t, \mathbf{z}(t), \mathbf{v}(t), \boldsymbol{\lambda}(t), \boldsymbol{\mu}(t)) = 0, \quad \ell = K' + 1, K' + 2, \ldots, K,$$

where

$$\begin{aligned} L(t, \mathbf{x}, \mathbf{u}, \boldsymbol{\lambda}, \boldsymbol{\mu}) &\stackrel{\text{def}}{=} H(t, \mathbf{x}, \mathbf{u}, \boldsymbol{\lambda}) + \boldsymbol{\mu}'\mathbf{h}(t, \mathbf{x}, \mathbf{u}) = f(t, \mathbf{x}, \mathbf{u}) \\ &\quad + \sum_{n=1}^{N} \lambda_n g^n(t, \mathbf{x}, \mathbf{u}) + \sum_{k=1}^{K} \mu_k h^k(t, \mathbf{x}, \mathbf{u}) \end{aligned}$$

is the Lagrangian. Furthermore, except for the points of discontinuities of $\mathbf{v}(t)$,

$$\dot{z}_i(t) = \frac{\partial M}{\partial \lambda_i}(t, \mathbf{z}(t), \boldsymbol{\lambda}(t)) = g^i(t, \mathbf{z}(t), \mathbf{v}(t)), \quad i = 1, 2, \ldots, N,$$

$$\dot{\lambda}_i(t) = -\frac{\partial M}{\partial x_i}(t, \mathbf{z}(t), \boldsymbol{\lambda}(t)), \quad i = 1, 2, \ldots, N,$$

where the above notation means that the functions are first differentiated with respect to the particular variable and then evaluated at $(t, \mathbf{z}(t), \boldsymbol{\lambda}(t))$.

Theorem 14.6 (Arrow Sufficiency Theorem, Infinite Horizon): *Let* $(\mathbf{z}(t), \mathbf{v}(t))$ *be an admissible pair for problem (1). Suppose that* $(\mathbf{z}(t), \mathbf{v}(t))$ *satisfies the necessary conditions of Theorem 14.5 for problem (1) with costate vector* $\boldsymbol{\lambda}(t)$ *and Lagrange multiplier vector* $\boldsymbol{\mu}(t)$, *and let* $M(t, \mathbf{x}, \boldsymbol{\lambda}) \stackrel{\text{def}}{=} \max_{\mathbf{u}}\{H(t, \mathbf{x}, \mathbf{u}, \boldsymbol{\lambda}) \text{ s.t. } \mathbf{u} \in U(t, \mathbf{x})\}$ *be the value of the maximized Hamiltonian function. If* $M(\cdot)$ *is a concave function* *of* $\mathbf{x}\,\forall\, t \in [0, +\infty)$ *over an open convex set containing all the admissible values of* $\mathbf{x}(\cdot)$ *when the costate vector is* $\boldsymbol{\lambda}(t)$*, and if for every admissible control path* $\mathbf{u}(t)$, $\lim_{t\to+\infty} \boldsymbol{\lambda}(t)'[\mathbf{z}(t) - \mathbf{x}(t)] \leq 0$*, where* $\mathbf{x}(t)$ *is the time path of the state vector corresponding to* $\mathbf{u}(t)$*, then* $\mathbf{v}(t)$ *is an optimal control and* $(\mathbf{z}(t), \mathbf{v}(t))$ *yields the global maximum of* $J[\cdot]$*. If* $M(\cdot)$ *is a strictly concave function of* $\mathbf{x}\,\forall\, t \in [0, +\infty)$ *under the same conditions, then* $(\mathbf{z}(t), \mathbf{v}(t))$ *yields the unique global maximum of* $J[\cdot]$ *and* $\mathbf{z}(t)$ *is unique, but* $\mathbf{v}(t)$ *is not necessarily unique.*

Let's now turn to a detailed examination of the famous Halkin (1974) counterexample, which demonstrates that *in general*, there are no necessary transversality conditions for infinite horizon optimal control problems. It is important to note that initially in this example we will *not* assume that the objective functional converges for all admissible pairs, thereby breaking with a basic maintained assumption of this chapter. We then reexamine the example under the assumption that the objective functional converges for all admissible pairs. Even under this stronger assumption, we will still conclude that the Halkin (1974) counterexample is a valid counterexample for demonstrating that the intuitive transversality condition $\lim_{t+\infty} \lambda(t) = 0$ is not necessary, contrary to the claim of Chiang (1992, Chapter 9).

Example 14.1: The Halkin (1974) counterexample is given by the optimal control problem

$$\max_{u(\cdot)} J[x(\cdot), u(\cdot)] \stackrel{\text{def}}{=} \int_0^{+\infty} u(t)[1 - x(t)]\, dt$$

$$\text{s.t.} \quad \dot{x}(t) = u(t)[1 - x(t)], \; x(0) = 0,$$

$$u(t) \in U \stackrel{\text{def}}{=} \{u(\cdot) : 0 \leq u(t) \leq 1\}.$$

It is worthwhile to emphasize that we are not imposing any conditions on $\lim_{t\to+\infty} x(t)$, so this limit may not even exist. Also recall that at least initially, we are not assuming that $J[x(\cdot), u(\cdot)]$ exists for all admissible function pairs $(x(\cdot), u(\cdot))$.

Given that the constraints on the control variable are independent of the state variable and do not vary with t, we do not have to introduce a Lagrangian function in order to compute the necessary conditions. Consequently, we may simply define the Hamiltonian for this problem as $H(x, u, \lambda) \stackrel{\text{def}}{=} u[1 - x] + \lambda\, u[1 - x] = [1 - x][1 + \lambda]\, u$, and then appeal to Theorem 14.3 to compute the necessary conditions:

$$\max_{u\in[0,1]} [1 - x][1 + \lambda]\, u,$$

$$\dot{\lambda} = -H_x(x, u, \lambda) = [1 + \lambda]\, u,$$

$$\dot{x} = H_\lambda(x, u, \lambda) = u\,[1 - x], \; x(0) = 0.$$

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Inspection of the objective functional, the state equation, and the initial condition reveals that the objective functional can be expressed as

$$J[x(\cdot), u(\cdot)] \stackrel{\text{def}}{=} \int_0^{+\infty} u(t)[1 - x(t)]\, dt = \int_0^{+\infty} \dot{x}(t)\, dt = \lim_{t \to +\infty} x(t).$$

To get a handle on the value of the objective functional $J[x(\cdot), u(\cdot)]$, therefore, we will first solve the first-order differential equation $\dot{x} = [1 - x]\, u(t)$.

As a first step in this process, separate the variables and rewrite the state equation in differential form as

$$\frac{dx}{1 - x} = u(t)\, dt,$$

which readily integrates to yield

$$-\ln[1 - x] = \int_0^t u(s)\, ds + k,$$

where k is a constant of integration. Note that we have chosen the lower limit of integration to be zero because that is where our initial condition applies. A little bit of straightforward algebra then gives the general solution to the state equation

$$x(t) = 1 - e^{-k} e^{-\int_0^t u(s)\, ds}.$$

The specific solution is found by applying the initial condition $x(0) = 0$ to the above general solution, which implies that $k = 0$. Hence the specific solution of the state equation that satisfies the initial condition is

$$x(t) = 1 - e^{-\omega(t)},$$

where $\omega(t) \stackrel{\text{def}}{=} \int_0^t u(s)\, ds$. This specific solution is admissible for all values of $u(t)$ that satisfy the control constraint, that is, for all $u(t) \in [0, 1]$. Note that $J[x(\cdot), u(\cdot)] = \lim_{t \to +\infty} x(t)$ does not necessarily exist for all admissible pairs, since one could select an admissible control that smoothly oscillates between zero and unity but does not have a limit as $t \to +\infty$. This is consistent with our initial assumption for this example, to wit, that $J[x(\cdot), u(\cdot)]$ does not necessarily exist for all admissible pairs. Also note that because $u(t) \in [0, 1]$ for all $t \in [0, +\infty)$, $x(t) = 1 - e^{-\omega(t)} \in [0, 1)$ for all $t \in [0, +\infty)$.

Given that $J[x(\cdot), u(\cdot)] = \lim_{t \to +\infty} x(t)$ and $u(t) \in [0, 1]\, \forall\, t \in [0, +\infty)$, it follows from the specific solution of the state equation, to wit, $x(t) = 1 - e^{-\omega(t)}$, that *any* admissible control path such that $\omega(t) \stackrel{\text{def}}{=} \int_0^t u(s)\, ds \to +\infty$ as $t \to +\infty$ is optimal, since this implies that $x(t) \to 1$ as $t \to +\infty$, which is its least upper bound, that is, supremum. This observation implies that there are infinitely many optimal

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controls for this problem. We intend to pick a particularly simple one that will aid in the solution to this problem, namely,

$$v(t) = \frac{1}{2} \, \forall \, t \in [0, +\infty].$$

This optimal control has two nice features: (i) it is constant, and (ii) it is interior to the control region [0, 1]. Given this optimal control, the corresponding state trajectory is therefore given by $z(t) = 1 - e^{-\frac{1}{2}t}$. You are asked to verify the optimality of the pair $(z(t), v(t))$ in a mental exercise. Note that we have yet to find the corresponding time path of the costate variable.

To find $\lambda(t)$, observe that because $v(t) = \frac{1}{2}$ is an optimal control and is in the interior of the control region, it is necessarily a solution to $H_u(z(t), u, \lambda) = [1 - z(t)][1 + \lambda] = 0$. But seeing as $z(t) = 1 - e^{-\frac{1}{2}t} \in [0, 1)$ for all $t \in [0, +\infty]$, it follows that $H_u(z(t), u, \lambda) = [1 - z(t)][1 + \lambda] = 0$ if and only if $1 + \lambda = 0$, thereby implying that $\lambda(t) = -1 \, \forall \, t \in [0, +\infty]$ is the corresponding time path of the costate variable. We conclude, therefore, that $\lim_{t\to+\infty} \lambda(t) = -1$. Recalling that no conditions are placed on $x(t)$ as $t \to +\infty$, we see that the costate function does not satisfy the transversality condition $\lim_{t\to+\infty} \lambda(t) = 0$ that one might expect to hold based on analogy with the finite-horizon case. This example has thus shown that *in general*, there are no necessary transversality conditions. It is important to remember that we have *not* assumed that the objective functional converges for all admissible pairs in reaching this conclusion, and this is what has allowed us to claim the generality of the conclusion concerning the lack of a necessary transversality condition in infinite horizon optimal control problems.

To get some additional qualitative insight into this problem, let's construct the phase portrait corresponding to the canonical differential equations in the $x\lambda$-phase space. Recalling that $v(t) = \frac{1}{2}$ is an optimal control, the canonical differential equations are given by $\dot{\lambda} = \frac{1}{2}[1 + \lambda]$ and $\dot{x} = \frac{1}{2}[1 - x]$. Hence the nullclines are given by

$$\dot{x} = 0 \quad \Leftrightarrow \quad x = 1,$$
$$\dot{\lambda} = 0 \quad \Leftrightarrow \quad \lambda = -1.$$

Because $\dot{\lambda} = \frac{1}{2}[1 + \lambda]$, it follows that $\dot{\lambda} > 0$ if and only if $\lambda > -1$. Similarly, because $\dot{x} = \frac{1}{2}[1 - x]$, it follows that $\dot{x} > 0$ if and only if $x < 1$. These observations yield the vector field for the canonical equations, and Figure 14.1 depicts the completed phase diagram.

The steady state or fixed point of the canonical equations is $(x^s, \lambda^s) = (1, -1)$, as is easily verified. Notice that two paths converge to the fixed point and that both occur along the $\dot{\lambda} = 0$ isocline, whereas all other paths diverge from the steady state. The phase diagram therefore suggests that the fixed point is a saddle point, with the stable manifold given by the λ nullcline and the unstable manifold given

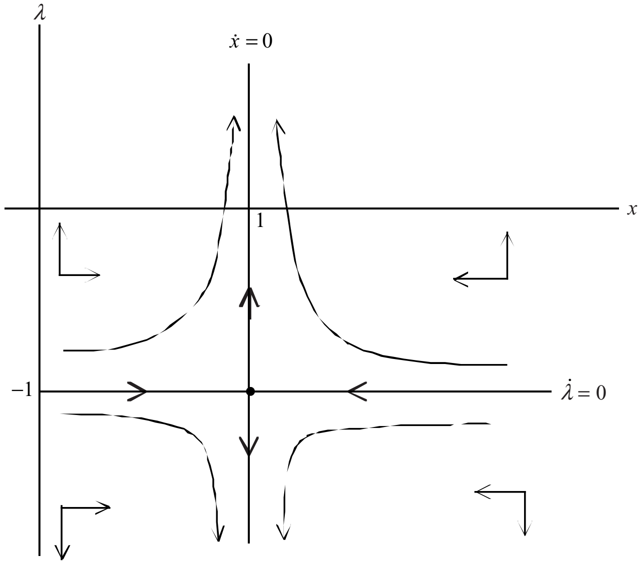

Figure 14.1

by the x nullcline. You are asked to prove that this is in fact the case in a mental exercise. Given that $x(0) = 0$, the optimal path is the one that at $t = 0$ has the value $(x, \lambda) = (0, -1)$, and goes to $(x^s, \lambda^s) = (1, -1)$ as $t \to +\infty$. Any path that does not begin at $(x, \lambda) = (0, -1)$ is not optimal because such paths have $\lambda(t) \to \pm\infty$ as $t \to +\infty$, and we know that $\lambda(t) = -1$ corresponds to the optimal pair.

Let us reconsider our conclusions under the assumption that $J[x(\cdot), u(\cdot)]$ exists for all admissible pairs of functions $(x(\cdot), u(\cdot))$, a basic assumption we have maintained throughout this chapter. As derived earlier, $x(t) = 1 - e^{-\omega(t)}$, where $\omega(t) \stackrel{\text{def}}{=} \int_0^t u(s)\, ds$, is the specific solution of the state equation. It therefore represents all the admissible time paths of the state variable when $u(t) \in [0, 1]$ for all $t \in [0, +\infty]$. Because $J[x(\cdot), u(\cdot)] = \lim_{t\to+\infty} x(t)$, the additional assumption that $J[x(\cdot), u(\cdot)]$ exists for all admissible pairs of functions $(x(\cdot), u(\cdot))$ furthermore requires that $\lim_{t\to+\infty} x(t)$ exist for all admissible pairs, thereby ruling out, for example, control paths that smoothly oscillate between zero and unity but do not have a limit as $t \to +\infty$. Thus the piecewise continuous infinite family of control functions $u(\cdot)$ defined by

$$u(t) \stackrel{\text{def}}{=} \begin{cases} k \in [0, 1] \ \forall t \in [0, \tau],\ \tau < +\infty \\ 0 \qquad\qquad \forall t \in (\tau, +\infty) \end{cases}$$

is admissible and generates a corresponding infinite family of admissible state variable time paths given by $x(t) = 1 - e^{-\omega(t)}$, where $\omega(t) \stackrel{\text{def}}{=} \int_0^t u(s)\, ds$. Moreover, for each member of this family of control functions, $\lim_{t\to+\infty} x(t)$ exists and equals a *different* value depending on the value of the constant k. In other words, the admissible

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time paths of the state variable do not converge to the same limiting value, thereby implying that the Halkin counterexample does *not* have a fixed terminal endpoint under the assumption that the objective functional exists for all admissible pairs, contrary to the claims of Chiang (1992, p. 246). Thus the Halkin counterexample remains a true counterexample to the necessity of the transversality condition $\lim_{t+\infty} \lambda(t) = 0$. This observation finishes up our examination of the Halkin counterexample.

Let's now contemplate the slightly less general class of infinite horizon optimal control problems defined by

$$V(\boldsymbol{\alpha}, r, 0, \mathbf{x}_0) \stackrel{\text{def}}{=} \max_{\mathbf{u}(\cdot)} \int_0^{+\infty} f(\mathbf{x}(t), \mathbf{u}(t); \boldsymbol{\alpha})\, e^{-rt}\, dt$$

$$\text{s.t.} \quad \dot{\mathbf{x}}(t) = \mathbf{g}(\mathbf{x}(t), \mathbf{u}(t); \boldsymbol{\alpha}),\ \ \mathbf{x}(0) = \mathbf{x}_0, \tag{9}$$

$$\lim_{t\to+\infty} x_n(t) = x_n^s, \quad n = 1, 2, \ldots, n_1,$$

$$\lim_{t\to+\infty} x_n(t) \geq x_n^s, \quad n = n_1 + 1, n_1 + 2, \ldots, n_2,$$

$$\text{no conditions on } x_n(t) \text{ as } t \to +\infty, \quad n = n_2 + 1, n_2 + 2, \ldots, N,$$

$$h^k(\mathbf{x}(t), \mathbf{u}(t); \boldsymbol{\alpha}) \geq 0, \quad k = 1, 2, \ldots, K',$$

$$h^k(\mathbf{x}(t), \mathbf{u}(t); \boldsymbol{\alpha}) = 0, \quad k = K' + 1, K' + 2, \ldots, K.$$

It is important to observe that for this class of optimal control problems, the functions $f(\cdot)$, $\mathbf{g}(\cdot)$, and $\mathbf{h}(\cdot)$ do not depend explicitly on the independent variable t. The zero appearing as the third argument of the *current value* optimal value function $V(\cdot)$ is placed there explicitly to reflect the fact that the initial time or starting date is $t = 0$ in problem (9). It is imperative that you understand why $V(\cdot)$ is the current value (as opposed to present value) optimal value function. As you may recall from Chapter 12, one simple way to understand why is to recognize that at the initial date of the planning horizon ($t = 0$), the value of the discount factor is unity, thereby implying that no discounting takes place in the initial period, the time period in which the decisions are made. This means that all future values of the integrand are discounted back to the initial date of the planning horizon ($t = 0$). This class of optimal control problems is known as the *infinite-horizon current-value autonomous* variety, because when put in current value form, the canonical equations do not depend explicitly on the independent variable t, a fact we established in Theorem 12.2. Without a doubt, this is the most prevalent class of optimal control problems in dynamic economic theory. Note that we continue to assume that the objective functional converges for all admissible pairs.

In order to establish several important results about this class of control problems, we introduce the following *family of control problems*, parameterized by the starting date $t_0 \in [0, +\infty)$:

$$
\begin{aligned}
V(\boldsymbol{\alpha}, r, t_0, \mathbf{x}_0) &\overset{\text{def}}{=} \max_{\mathbf{u}(\cdot)} \int_{t_0}^{+\infty} f(\mathbf{x}(t-t_0), \mathbf{u}(t-t_0); \boldsymbol{\alpha})\, e^{-r(t-t_0)}\, dt \\
\text{s.t.} \quad & \dot{\mathbf{x}}(t-t_0) = \mathbf{g}(\mathbf{x}(t-t_0), \mathbf{u}(t-t_0); \boldsymbol{\alpha}), \;\; \mathbf{x}(t-t_0)|_{t=t_0} = \mathbf{x}_0, \\
& \lim_{t\to+\infty} x_n(t-t_0) = x_n^s, \quad n = 1, 2, \ldots, n_1, \\
& \lim_{t\to+\infty} x_n(t-t_0) \geq x_n^s, \quad n = n_1+1, n_1+2, \ldots, n_2, \\
& \text{no conditions on } x_n(t-t_0) \text{ as } t \to +\infty, \quad n = n_2+1, n_2+2, \ldots, N, \\
& h^k(\mathbf{x}(t-t_0), \mathbf{u}(t-t_0); \boldsymbol{\alpha}) \geq 0, \quad k = 1, 2, \ldots, K', \\
& h^k(\mathbf{x}(t-t_0), \mathbf{u}(t-t_0); \boldsymbol{\alpha}) = 0, \quad k = K'+1, K'+2, \ldots, K.
\end{aligned} \tag{10}
$$

Notice that in advancing the starting date from 0 in problem (9) to t_0 in problem (10), we have correspondingly subtracted t_0 from the independent variable t wherever the latter occurs in the problem, whether that be explicitly in the exponential discount factor or implicitly as the argument of the state and control variables. Such an operation implies that the value of the current value optimal value functions in problems (9) and (10) are identically equal. This follows from the facts that both problems (i) begin in state $\mathbf{x}_0$, (ii) last indefinitely, (iii) have identical integrand and transition functions, and (iv) the delay prompted by starting problem (10) at time t_0 is exactly compensated for by a forward translation of t_0 units in the time dimension of every state and control variable and the discount function. In passing, note that problem (9) can be generated from problem (10) by setting $t_0 = 0$ in the latter.

To prove that $V(\boldsymbol{\alpha}, r, t_0, \mathbf{x}_0) \equiv V(\boldsymbol{\alpha}, r, 0, \mathbf{x}_0)\,\forall\, t_0 \in [0, +\infty)$ in a formal manner, first define a new variable $s \overset{\text{def}}{=} t - t_0$, which is precisely the forward translation of t_0 units in the time dimension noted above. It then follows that $t = t_0 \Leftrightarrow s = 0$, $t \to +\infty \Leftrightarrow s \to +\infty$, and that $ds = dt$, since t_0 is a given parameter. Substituting these results in problem (10) gives an *equivalent* representation of it, scilicet,

$$
\begin{aligned}
V(\boldsymbol{\alpha}, r, t_0, \mathbf{x}_0) &\overset{\text{def}}{=} \max_{\mathbf{u}(\cdot)} \int_{0}^{+\infty} f(\mathbf{x}(s), \mathbf{u}(s); \boldsymbol{\alpha})\, e^{-rs}\, ds \\
\text{s.t.} \quad & \dot{\mathbf{x}}(s) = \mathbf{g}(\mathbf{x}(s), \mathbf{u}(s); \boldsymbol{\alpha}), \;\; \mathbf{x}(s)|_{s=0} = \mathbf{x}_0, \\
& \lim_{s\to+\infty} x_n(s) = x_n^s, \quad n = 1, 2, \ldots, n_1, \\
& \lim_{s\to+\infty} x_n(s) \geq x_n^s, \quad n = n_1+1, n_1+2, \ldots, n_2, \\
& \text{no conditions on } x_n(s) \text{ as } s \to +\infty, \quad n = n_2+1, n_2+2, \ldots, N, \\
& h^k(\mathbf{x}(s), \mathbf{u}(s); \boldsymbol{\alpha}) \geq 0, \quad k = 1, 2, \ldots, K', \\
& h^k(\mathbf{x}(s), \mathbf{u}(s); \boldsymbol{\alpha}) = 0, \quad k = K'+1, K'+2, \ldots, K.
\end{aligned} \tag{11}
$$

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Observing that the independent variables t in problem (9) and s in problem (11) are dummy variables of integration and hence arbitrary, it follows that optimal control problems (9) and (11) are identical. This, in turn, implies that the value of their respective current value optimal value functions are identical too, thereby yielding the identity $V(\boldsymbol{\alpha}, r, t_0, \mathbf{x}_0) \equiv V(\boldsymbol{\alpha}, r, 0, \mathbf{x}_0) \forall\, t_0 \in [0, +\infty)$, just as we wished to show. Given that the identity holds *for all* $t_0 \in [0, +\infty)$, this implies that the current value optimal value function $V(\cdot)$ *does not depend explicitly* on the initial date or starting time t_0. To see this in perhaps a more transparent way, recall that $\mathbf{x}(t_0) = \mathbf{x}_0$ and substitute it into the identity $V(\boldsymbol{\alpha}, r, t_0, \mathbf{x}_0) \equiv V(\boldsymbol{\alpha}, r, 0, \mathbf{x}_0)\, \forall\, t_0 \in [0, +\infty)$ to arrive at the alternative, but equivalent, identity $V(\boldsymbol{\alpha}, r, t_0, \mathbf{x}(t_0)) \equiv V(\boldsymbol{\alpha}, r, 0, \mathbf{x}(t_0))\, \forall\, t_0 \in [0, +\infty)$. This latter form of the identity should make it clear that $V(\cdot)$ does not vary with direct or explicit changes in t_0, since the third argument of the right-hand-side of the identity is zero. The change in the value of $V(\cdot)$ that comes about by changing the initial time t_0 is thus solely a result of $\mathbf{x}(t_0) = \mathbf{x}_0$ changing with the initial time t_0. As a result, we are justified in dropping the explicit argument t_0 of $V(\cdot)$, and therefore may write its value as $V(\boldsymbol{\alpha}, r, \mathbf{x}_0)$, a practice we shall adhere to from now on. This is a crucial property of the *current value* optimal value function because it paves the way for dynamic duality theory in an ensuing chapter. In sum, therefore, the value of the current value optimal value function depends on the initial value of the state vector, the discount rate, and the parameter vector, but *not* explicitly on the initial date or starting time.

Using problem (10) as a benchmark, we may now define the *present value* optimal value function $\hat{V}(\cdot)$ by the following problem:

$$
\begin{aligned}
\hat{V}(\boldsymbol{\alpha}, r, t_0, \mathbf{x}_0) &\stackrel{\text{def}}{=} \max_{\mathbf{u}(\cdot)} e^{-rt_0} \int_{t_0}^{+\infty} f(\mathbf{x}(t-t_0), \mathbf{u}(t-t_0); \boldsymbol{\alpha})\, e^{-r(t-t_0)}\, dt \\
\text{s.t.} \quad & \dot{\mathbf{x}}(t-t_0) = \mathbf{g}(\mathbf{x}(t-t_0), \mathbf{u}(t-t_0); \boldsymbol{\alpha}), \ \ \mathbf{x}(t-t_0)|_{t=t_0} = \mathbf{x}_0, \\
& \lim_{t\to+\infty} x_n(t-t_0) = x_n^s, \quad n = 1, 2, \ldots, n_1, \\
& \lim_{t\to+\infty} x_n(t-t_0) \geq x_n^s, \quad n = n_1+1, n_1+2, \ldots, n_2, \\
& \text{no conditions on } x_n(t-t_0) \text{ as } t \to +\infty, \quad n = n_2+1, n_2+2, \ldots, N, \\
& h^k(\mathbf{x}(t-t_0), \mathbf{u}(t-t_0); \boldsymbol{\alpha}) \geq 0, \quad k = 1, 2, \ldots, K', \\
& h^k(\mathbf{x}(t-t_0), \mathbf{u}(t-t_0); \boldsymbol{\alpha}) = 0, \quad k = K'+1, K'+2, \ldots, K.
\end{aligned}
\tag{12}
$$

It should be evident from inspection of problem (12) that the value of the integral is discounted to time t_0, whereas the presence of the discount factor e^{-rt_0} in front of the integral further discounts these values back to time $t = 0$. It is this observation that justifies the use of the adjective present value in describing the function $\hat{V}(\cdot)$. That is, although the initial period in problem (12) is t_0, the values of the objective functional are discounted back to time $t = 0$, hence making $\hat{V}(\cdot)$ the present value optimal value function. Defining $s \stackrel{\text{def}}{=} t - t_0$, just as we did above, we may rewrite problem (12) in an *equivalent* manner, namely,

$$\hat{V}(\boldsymbol{\alpha}, r, t_0, \mathbf{x}_0) \stackrel{\text{def}}{=} e^{-rt_0} \max_{\mathbf{u}(\cdot)} \int_0^{+\infty} f(\mathbf{x}(s), \mathbf{u}(s); \boldsymbol{\alpha})\, e^{-rs}\, ds$$

$$\text{s.t.} \quad \dot{\mathbf{x}}(s) = \mathbf{g}(\mathbf{x}(s), \mathbf{u}(s); \boldsymbol{\alpha}), \;\; \mathbf{x}(s)|_{s=0} = \mathbf{x}_0, \tag{13}$$

$$\lim_{s\to+\infty} x_n(s) = x_n^s, \quad n = 1, 2, \ldots, n_1,$$

$$\lim_{s\to+\infty} x_n(s) \geq x_n^s, \quad n = n_1 + 1, n_1 + 2, \ldots, n_2,$$

$$\text{no conditions on } x_n(s) \text{ as } s \to +\infty, \quad n = n_2 + 1, n_2 + 2, \ldots, N,$$

$$h^k(\mathbf{x}(s), \mathbf{u}(s); \boldsymbol{\alpha}) \geq 0, \quad k = 1, 2, \ldots, K',$$

$$h^k(\mathbf{x}(s), \mathbf{u}(s); \boldsymbol{\alpha}) = 0, \quad k = K' + 1, K' + 2, \ldots, K.$$

Because of the presence of the discount factor e^{-rt_0} in problem (13), it follows that in general, the present value optimal value function $\hat{V}(\cdot)$ depends explicitly on the initial date or starting time t_0, in sharp contrast with the current value optimal value function $V(\cdot)$. Upon inspecting problems (11) and (13), and recalling the fact that $V(\cdot)$ does not depend explicitly on t_0, it should be clear that the identity $\hat{V}(\boldsymbol{\alpha}, r, t_0, \mathbf{x}_0) \equiv e^{-rt_0} V(\boldsymbol{\alpha}, r, \mathbf{x}_0)\, \forall\, t_0 \in [0, +\infty)$ also holds. This is an intuitive result, for it asserts that the value of the present value optimal value function is identically equal to the value of the current value optimal value function discounted back to time $t = 0$. Hence when $t_0 = 0$, the values of the present value and current value optimal value functions are one and the same. We pause momentarily and summarize the two results thus far established for this class of control problems.

Theorem 14.7: *Let $V(\cdot)$ be defined as in problem (10) and let $\hat{V}(\cdot)$ be defined as in problem (12). Then (i) the current value optimal value function $V(\cdot)$ does not depend explicitly on the initial time t_0, $\forall\, t_0 \in [0, +\infty)$, and (ii) $\hat{V}(\boldsymbol{\alpha}, r, t_0, \mathbf{x}_0) \equiv e^{-rt_0} V(\boldsymbol{\alpha}, r, \mathbf{x}_0)\, \forall\, t_0 \in [0, +\infty)$.*

The ensuing result is an implication of part (i) of Theorem 14.7. Its proof, which you are asked to provide in a mental exercise, will test if you have fully internalized the results of Theorem 14.7 and the necessary conditions for problems (9) and (10).

Corollary 14.1: *In optimal control problems (9) and (10), the optimal values of the current value costate function, the optimal control function, and the Lagrange multiplier function at any time t can be expressed solely as functions of the corresponding value of the state vector at time t and the parameters $(\boldsymbol{\alpha}, r)$.*

It is worthwhile to emphasize that Theorem 14.7(i), and consequently Corollary 14.1, do not hold for the finite-horizon version of optimal control problems (9) and

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(10), nor do they hold if any of the functions $f(\cdot)$, $\mathbf{g}(\cdot)$, or $\mathbf{h}(\cdot)$ are explicit functions of the independent variable t and time consistency of the optimal plan is assumed. A mental exercise probes these aspects more deeply.

In wrapping up this chapter, let us consider a further simplification of the general optimal control problem (1), namely,

$$V(\boldsymbol{\alpha}, r, \mathbf{x}_0) \stackrel{\text{def}}{=} \max_{\mathbf{u}(\cdot)} \int_0^{+\infty} f(\mathbf{x}(t), \mathbf{u}(t); \boldsymbol{\alpha})\, e^{-rt}\, dt \tag{14}$$

$$\text{s.t.} \quad \dot{\mathbf{x}}(t) = \mathbf{g}(\mathbf{x}(t), \mathbf{u}(t); \boldsymbol{\alpha}), \;\; \mathbf{x}(0) = \mathbf{x}_0,$$

where *no conditions* are imposed on $x_n(t)$ as $t \to +\infty$, $n = 1, 2, \ldots, N$. We continue to assume that the objective functional converges for all admissible pairs. Our goals in the remainder of this chapter are to (i) derive a fundamental equation for problem (14) linking the value of the current value optimal value function to the value of the current value maximized Hamiltonian, (ii) derive a general transversality condition that is necessary for the class of optimal control problems defined by problem (14), and (iii) revisit the dynamic envelope theorem.

To begin, let $(\mathbf{z}(t; \boldsymbol{\beta}), \mathbf{v}(t; \boldsymbol{\beta}))$ be the optimal pair for problem (14), and let $\boldsymbol{\lambda}(t; \boldsymbol{\beta})$ be the corresponding current value costate vector, where $\boldsymbol{\beta} \stackrel{\text{def}}{=} (\boldsymbol{\alpha}, r, \mathbf{x}_0)$. Then for *any* $\tau \in [0, +\infty)$, it follows from the definition of $V(\cdot)$ in Eq. (14), the fact that an integral is additive with respect to the limits of integration, and the principle of optimality that

$$V(\boldsymbol{\alpha}, r, \mathbf{x}_0) = \int_0^{\tau} f(\mathbf{z}(t; \boldsymbol{\beta}), \mathbf{v}(t; \boldsymbol{\beta}); \boldsymbol{\alpha})\, e^{-rt}\, dt + e^{-r\tau} V(\boldsymbol{\alpha}, r, \mathbf{z}(\tau; \boldsymbol{\beta})). \tag{15}$$

Because Eq. (15) holds for all $\tau \in [0, +\infty)$, that is, it is an *identity* in τ, we may differentiate it with respect to τ and it still holds identically. Doing just that, we obtain

$$\begin{aligned} 0 = {} & f(\mathbf{z}(\tau; \boldsymbol{\beta}), \mathbf{v}(\tau; \boldsymbol{\beta}); \boldsymbol{\alpha})\, e^{-r\tau} - r e^{-r\tau} V(\boldsymbol{\alpha}, r, \mathbf{z}(\tau; \boldsymbol{\beta})) \\ & + e^{-r\tau} V_{\mathbf{x}}(\boldsymbol{\alpha}, r, \mathbf{z}(\tau; \boldsymbol{\beta}))\, \dot{\mathbf{z}}(\tau; \boldsymbol{\beta}). \end{aligned}$$

Upon multiplying through by $e^{r\tau}$ and using the necessary condition $\dot{\mathbf{z}}(\tau; \boldsymbol{\beta}) \equiv \mathbf{g}(\mathbf{z}(\tau; \boldsymbol{\beta}), \mathbf{v}(\tau; \boldsymbol{\beta}); \boldsymbol{\alpha})$, the above equation simplifies to

$$\begin{aligned} rV(\boldsymbol{\alpha}, r, \mathbf{z}(\tau; \boldsymbol{\beta})) = {} & f(\mathbf{z}(\tau; \boldsymbol{\beta}), \mathbf{v}(\tau; \boldsymbol{\beta}); \boldsymbol{\alpha}) \\ & + V_{\mathbf{x}}(\boldsymbol{\alpha}, r, \mathbf{z}(\tau; \boldsymbol{\beta}))\, \mathbf{g}(\mathbf{z}(\tau; \boldsymbol{\beta}), \mathbf{v}(\tau; \boldsymbol{\beta}); \boldsymbol{\alpha}). \end{aligned} \tag{16}$$

Now recall that $V_{\mathbf{x}}(\boldsymbol{\alpha}, r, \mathbf{z}(\tau; \boldsymbol{\beta}))$ is the current value costate vector $\boldsymbol{\lambda}(\tau; \boldsymbol{\beta})'$ by the dynamic envelope theorem and the principle of optimality, and that the maximized current value Hamiltonian for problem (14) is defined as

$$M(\mathbf{x}, \boldsymbol{\lambda}; \boldsymbol{\alpha}) \stackrel{\text{def}}{=} \max_{\mathbf{u}} H(\mathbf{x}, \mathbf{u}, \boldsymbol{\lambda}; \boldsymbol{\alpha}),$$

where $H(\mathbf{x}, \mathbf{u}, \boldsymbol{\lambda}; \boldsymbol{\alpha}) \stackrel{\text{def}}{=} f(\mathbf{x}, \mathbf{u}; \boldsymbol{\alpha}) + \boldsymbol{\lambda}'\mathbf{g}(\mathbf{x}, \mathbf{u}; \boldsymbol{\alpha})$ is the current value Hamiltonian for problem (14). These two observations therefore permit Eq. (16) to be rewritten as

$$rV(\boldsymbol{\alpha}, r, \mathbf{z}(\tau; \boldsymbol{\beta})) = M(\mathbf{z}(\tau; \boldsymbol{\beta}), \boldsymbol{\lambda}(\tau; \boldsymbol{\beta}); \boldsymbol{\alpha}), \tag{17}$$

or equivalently as

$$rV(\boldsymbol{\alpha}, r, \mathbf{z}(\tau; \boldsymbol{\beta})) = H(\mathbf{z}(\tau; \boldsymbol{\beta}), \mathbf{v}(\tau; \boldsymbol{\beta}), \boldsymbol{\lambda}(\tau; \boldsymbol{\beta}); \boldsymbol{\alpha}), \tag{18}$$

both of which hold for all $\tau \in [0, +\infty)$. We have therefore established an important relationship between the value of the current value optimal value function and the value of the maximized current value Hamiltonian evaluated at the optimal solution. Equations (17) and (18) are known as the *Hamilton-Jacobi-Bellman* equation. You may recall that we derived another, more general form of the Hamilton-Jacobi-Bellman equation in our rigorous proof of the Maximum Principle in Chapter 4. As alluded to earlier, this equation plays a fundamental role in dynamic duality theory, as we shall see in two ensuing chapters. We summarize this fundamental result in the following theorem.

Theorem 14.8 (Hamilton-Jacobi-Bellman Equation): *Let* $(\mathbf{z}(t; \boldsymbol{\beta}), \mathbf{v}(t; \boldsymbol{\beta}))$ *be the optimal pair for problem (14), and let* $\boldsymbol{\lambda}(t; \boldsymbol{\beta})$ *be the corresponding current value costate vector, where* $\boldsymbol{\beta} \stackrel{\text{def}}{=} (\boldsymbol{\alpha}, r, \mathbf{x}_0)$. *Define* $V(\cdot)$ *by problem (14), assume that* $V_{\mathbf{x}}(\cdot) \in C^{(0)}$, *and let* $M(\cdot)$ *be the corresponding maximized current value Hamiltonian function. Then for all* $\tau \in [0, +\infty)$,

$$rV(\boldsymbol{\alpha}, r, \mathbf{z}(\tau; \boldsymbol{\beta})) = M(\mathbf{z}(\tau; \boldsymbol{\beta}), \boldsymbol{\lambda}(\tau; \boldsymbol{\beta}); \boldsymbol{\alpha}),$$

or equivalently,

$$rV(\boldsymbol{\alpha}, r, \mathbf{z}(\tau; \boldsymbol{\beta})) = H(\mathbf{z}(\tau; \boldsymbol{\beta}), \mathbf{v}(\tau; \boldsymbol{\beta}), \boldsymbol{\lambda}(\tau; \boldsymbol{\beta}); \boldsymbol{\alpha}).$$

Let's now turn to the derivation of a general transversality condition that is necessary for the class of problems defined by Eq. (14). To that end, first recall the result of Theorem 14.7(ii), namely, the identity $\hat{V}(\boldsymbol{\alpha}, r, t_0, \mathbf{x}_0) \equiv e^{-rt_0} V(\boldsymbol{\alpha}, r, \mathbf{x}_0) \, \forall \, t_0 \in [0, +\infty)$, which clearly applies to problem (14) as it is a special case of problem (10). Substituting $\mathbf{x}(t_0) = \mathbf{x}_0$ in the identity and then letting $t_0 \to +\infty$ gives

$$\lim_{t_0 \to +\infty} \hat{V}(\boldsymbol{\alpha}, r, t_0, \mathbf{x}(t_0)) \equiv \lim_{t_0 \to +\infty} e^{-rt_0} V(\boldsymbol{\alpha}, r, \mathbf{x}(t_0)) = 0,$$

since $V(\cdot)$ exists for all admissible pairs. Replacing t_0 with τ and applying this result to Theorem 14.8 gives

$$\lim_{\tau \to +\infty} e^{-r\tau} M(\mathbf{z}(\tau; \boldsymbol{\beta}), \boldsymbol{\lambda}(\tau; \boldsymbol{\beta}); \boldsymbol{\alpha}) = \lim_{\tau \to +\infty} re^{-r\tau} V(\boldsymbol{\alpha}, r, \mathbf{z}(\tau; \boldsymbol{\beta})) = 0, \tag{19}$$

which is the necessary transversality condition we were after for the class of optimal control problems defined by problem (14). Equation (19) asserts that along the optimal path of problem (14), the *present value* of the maximized Hamiltonian goes to zero as time goes to infinity. Note that because we used Theorem 14.8 in the proof of this transversality condition, we have therefore assumed that $V_{\mathbf{x}}(\cdot) \in C^{(0)}$.

An alternative, and possibly more intuitive, proof follows from Eq. (15). Given that the value $V(\boldsymbol{\alpha}, r, \mathbf{x}_0)$ is independent of τ, take the limit of Eq. (15) as $\tau \to +\infty$ to get

$$\begin{aligned} V(\boldsymbol{\alpha}, r, \mathbf{x}_0) &= \lim_{\tau \to +\infty} \left[\int_0^{\tau} f(\mathbf{z}(t;\boldsymbol{\beta}), \mathbf{v}(t;\boldsymbol{\beta}); \boldsymbol{\alpha})\, e^{-rt}\, dt + e^{-r\tau} V(\boldsymbol{\alpha}, r, \mathbf{z}(\tau;\boldsymbol{\beta})) \right] \\ &= \lim_{\tau \to +\infty} \int_0^{\tau} f(\mathbf{z}(t;\boldsymbol{\beta}), \mathbf{v}(t;\boldsymbol{\beta}); \boldsymbol{\alpha})\, e^{-rt}\, dt + \lim_{\tau \to +\infty} e^{-r\tau} V(\boldsymbol{\alpha}, r, \mathbf{z}(\tau;\boldsymbol{\beta})) \\ &= \int_0^{+\infty} f(\mathbf{z}(t;\boldsymbol{\beta}), \mathbf{v}(t;\boldsymbol{\beta}); \boldsymbol{\alpha})\, e^{-rt}\, dt + \lim_{\tau \to +\infty} e^{-r\tau} V(\boldsymbol{\alpha}, r, \mathbf{z}(\tau;\boldsymbol{\beta})) \\ &= V(\boldsymbol{\alpha}, r, \mathbf{x}_0) + \lim_{\tau \to +\infty} e^{-r\tau} V(\boldsymbol{\alpha}, r, \mathbf{z}(\tau;\boldsymbol{\beta})). \end{aligned}$$

Note that we have used the definition of $V(\cdot)$ from Eq. (14) and the fact that $V(\cdot)$ is assumed to exist for all admissible pairs. Upon canceling $V(\boldsymbol{\alpha}, r, \mathbf{x}_0)$ from both sides of the above equation, we arrive at the result $\lim_{\tau \to +\infty} e^{-r\tau} V(\boldsymbol{\alpha}, r, \mathbf{z}(\tau;\boldsymbol{\beta})) = 0$. Applying this result to Theorem 14.8 yields the transversality condition $\lim_{\tau \to +\infty} e^{-r\tau} M(\mathbf{z}(\tau;\boldsymbol{\beta}), \boldsymbol{\lambda}(\tau;\boldsymbol{\beta}); \boldsymbol{\alpha}) = 0$, just as in the previous proof. We summarize this important result in the following theorem.

Theorem 14.9 (Transversality Condition): *Let* $(\mathbf{z}(t;\boldsymbol{\beta}), \mathbf{v}(t;\boldsymbol{\beta}))$ *be the optimal pair for problem (14), and let* $\boldsymbol{\lambda}(t;\boldsymbol{\beta})$ *be the corresponding current value costate vector, where* $\boldsymbol{\beta} \stackrel{\text{def}}{=} (\boldsymbol{\alpha}, r, \mathbf{x}_0)$. *Let* $M(\cdot)$ *be the maximized current value Hamiltonian function for problem (14). Then*

$$\lim_{\tau \to +\infty} e^{-r\tau} M(\mathbf{z}(\tau;\boldsymbol{\beta}), \boldsymbol{\lambda}(\tau;\boldsymbol{\beta}); \boldsymbol{\alpha}) = 0,$$

or equivalently,

$$\lim_{\tau \to +\infty} e^{-r\tau} H(\mathbf{z}(\tau;\boldsymbol{\beta}), \mathbf{v}(\tau;\boldsymbol{\beta}), \boldsymbol{\lambda}(\tau;\boldsymbol{\beta}); \boldsymbol{\alpha}) = 0,$$

is a necessary condition for problem (14).

Rather than present an example that uses Theorem 14.9 here, we prefer to use it in the following three chapters when we study several fundamental dynamic economics models.

For the final theorem of this chapter, we return to the dynamic envelope theorem. We reexamine this central theorem for the class of optimal control problems defined by problem (14) and its associated assumptions, a ubiquitous class of control problems in dynamic economic theory. As a result, we now impose the ensuing assumptions on the functions $f(\cdot)$ and $\mathbf{g}(\cdot)$:

(A.1) $f(\cdot) \in C^{(1)}$ with respect to the $N+M$ variables $(\mathbf{x}, \mathbf{u})$ and the A parameters $\boldsymbol{\alpha}$.

(A.2) $\mathbf{g}(\cdot) \in C^{(1)}$ with respect to the $N+M$ variables $(\mathbf{x}, \mathbf{u})$ and the A parameters $\boldsymbol{\alpha}$.

Given these additional assumptions, we are now in a position to state the dynamic envelope theorem for the discounted infinite horizon class of optimal control problems defined by problem (14). The proof of the theorem is left for a mental exercise.

Theorem 14.10 (Dynamic Envelope Theorem): *Let* $(\mathbf{z}(t;\boldsymbol{\beta}), \mathbf{v}(t;\boldsymbol{\beta}))$, $\boldsymbol{\beta} \stackrel{\text{def}}{=} (\boldsymbol{\alpha}, r, \mathbf{x}_0)$*, be the optimal pair for problem (14), with the property that as* $t \to +\infty$, $(\mathbf{z}(t;\boldsymbol{\beta}), \mathbf{v}(t;\boldsymbol{\beta})) \to (\mathbf{x}^s(\boldsymbol{\alpha}, r), \mathbf{u}^s(\boldsymbol{\alpha}, r))$*, where* $(\mathbf{x}^s(\boldsymbol{\alpha}, r), \mathbf{u}^s(\boldsymbol{\alpha}, r))$ *is the locally* $C^{(1)}$ *steady state solution of the necessary conditions, and let* $\boldsymbol{\lambda}^{pv}(t;\boldsymbol{\beta}) \stackrel{\text{def}}{=} \boldsymbol{\lambda}(t;\boldsymbol{\beta})e^{-rt}$ *be the corresponding time path of the present value costate vector. Define the present value Hamiltonian as* $H^{pv}(t, \mathbf{x}, \mathbf{u}, \boldsymbol{\lambda}^{pv}; \boldsymbol{\alpha}) \stackrel{\text{def}}{=} f(\mathbf{x}, \mathbf{u}; \boldsymbol{\alpha})e^{-rt} + \sum_{\ell=1}^{N} \lambda_\ell^{pv} g^\ell(\mathbf{x}, \mathbf{u}; \boldsymbol{\alpha})$*. If* $\mathbf{z}(\cdot) \in C^{(1)}$ *and* $\mathbf{v}(\cdot) \in C^{(1)}$ *in* $(t;\boldsymbol{\beta})\,\forall\,(t;\boldsymbol{\beta}) \in [0, +\infty) \times B(\boldsymbol{\beta}^\circ;\delta)$*, then* $V(\cdot) \in C^{(1)}\,\forall\,\boldsymbol{\beta} \in B(\boldsymbol{\beta}^\circ;\delta)$*. Furthermore, if* $\partial\mathbf{z}(t;\boldsymbol{\beta})/\partial\boldsymbol{\beta} \to \partial\mathbf{x}^s(\boldsymbol{\alpha}, r)/\partial\boldsymbol{\beta}$ *as* $t \to +\infty$*, then for all* $\boldsymbol{\beta} \in B(\boldsymbol{\beta}^\circ;\delta)$*:*

$$V_{\alpha_i}(\boldsymbol{\beta}) \stackrel{\text{def}}{=} \frac{\partial V(\boldsymbol{\beta})}{\partial \alpha_i} \equiv \int_0^{+\infty} H_{\alpha_i}^{pv}(\mathbf{z}(t;\boldsymbol{\beta}), \mathbf{v}(t;\boldsymbol{\beta}), \boldsymbol{\lambda}^{pv}(t;\boldsymbol{\beta}); \boldsymbol{\alpha})\,dt, \quad i = 1, 2, \ldots, A, \tag{i}$$

$$V_r(\boldsymbol{\beta}) \equiv \int_0^{+\infty} -tf(\mathbf{z}(t;\boldsymbol{\beta}), \mathbf{v}(t;\boldsymbol{\beta}); \boldsymbol{\alpha})\,e^{-rt}\,dt, \tag{ii}$$

$$V_{x_{0j}}(\boldsymbol{\beta}) \equiv \lambda_j^{pv}(0;\boldsymbol{\beta}) = \lambda_j(0;\boldsymbol{\beta}), \quad j = 1, 2, \ldots, N. \tag{iii}$$

Several remarks concerning the proof of Theorem 14.10 are in order. First, inasmuch as $V(\cdot)$ is defined by an improper integral, which we have assumed to exist for all admissible pairs of functions, an alternative version of Leibniz's rule must be used in the proof. This version of Leibniz's rule, appropriate for improper integrals, is given in the appendix to this chapter as Theorem A.14.1. Second, as part of the proof, one must establish that the current value costate vector $\boldsymbol{\lambda}(t;\boldsymbol{\beta})$ converges to its steady state solution $\boldsymbol{\lambda}^s(\boldsymbol{\alpha}, r)$ in the limit of the planning horizon. This result, in conjunction with the assumption that as $t \to +\infty$, $\partial\mathbf{z}(t;\boldsymbol{\beta})/\partial\boldsymbol{\beta} \to \partial\mathbf{x}^s(\boldsymbol{\alpha}, r)/\partial\boldsymbol{\beta}$, is crucial in eliminating an inner product expression resulting from the integration-by-parts operation. Finally, note that one may easily rewrite Theorem 14.10 using the current value Hamiltonian.

One purpose of this chapter has been to introduce necessary and sufficient conditions for a general optimal control problem with an infinite planning horizon. For the most part, these theorems are very similar to their finite horizon counterparts, the necessary transversality conditions being the exception. We demonstrated via the Halkin (1974) counterexample that *in general*, there are no necessary transversality conditions in infinite-horizon optimal control problems. The sufficiency theorems, however, make use of a transversality condition, and there is no controversy surrounding its veracity. We also studied a slightly less general class of infinite-horizon optimal control problems, namely, those in which the integrand, transition, and constraint functions do not depend explicitly on time, and the integrand is exponentially discounted. For this class of problems, we derived some fundamental properties of the current value optimal value function that we will return to in a later chapter. Finally, for another less general, but quite common, class of infinite-horizon control problems, we derived a general necessary transversality condition and established the dynamic envelope theorem.

In the next three chapters, we employ the theorems developed herein to study several infinite-horizon current-value autonomous optimal control problems of fundamental importance in intertemporal economic theory. In particular, we examine in great detail the local stability, steady state comparative statics, and local comparative dynamics properties of these models.

APPENDIX

In order to be in a position to extend Leibniz's rule to integrals with infinite intervals of integration, we first require a definition. To prepare for the definition, let $F(\cdot) : S \to \Re$ be continuous on the infinite strip $S \stackrel{\text{def}}{=} \{(t, y) : c \le t < +\infty, a \le y \le b\}$, and suppose that

$$\lim_{t \to +\infty} F(t, y)$$

exists for each $y \in I \stackrel{\text{def}}{=} \{y : a \le y \le b\}$. We denote the above limit by $\phi(y)$.

Definition A.14.1: The function $F(\cdot)$ *tends to (or converges to)* $\phi(\cdot)$ *uniformly* on $I \stackrel{\text{def}}{=} \{y : a \le y \le b\}$ as $t \to +\infty$ if and only if for every $\varepsilon > 0$, there is a number T depending on ε such that

$$|F(t, y) - \phi(y)| < \varepsilon$$

holds for all $t > T$ and all $y \in I$. The number T depends on ε but not on y.

Theorem A.14.1 (Leibniz's Rule for Infinite Intervals of Integration): *Suppose that* $f(\cdot) : S \to \Re$ *is continuous on* $S \stackrel{\text{def}}{=} \{(t, y) : c \le t < +\infty,\ a \le y \le b\}$*. Define*

$$F(t, y) \stackrel{\text{def}}{=} \int_c^t f(\tau, y)\, d\tau.$$

Also suppose that the improper integral

$$\phi(y) \stackrel{\text{def}}{=} \int_c^{+\infty} f(\tau, y)\, d\tau$$

exists for all $y \in I \stackrel{\text{def}}{=} \{y : a \leq y \leq b\}$*, that* $\lim_{t\to+\infty} F(t, y) = \phi(y)$ *exists uniformly for* $y \in I$*, and that* $f_y(\cdot)$ *is continuous on S. If* $F_y(\cdot)$ *converges to* $\psi(\cdot)$ *as* $t \to +\infty$ *uniformly in y, then*

$$\psi(y) = \phi'(y) = \int_c^{+\infty} f_y(\tau, y)\, d\tau.$$

Example A.14.1: Define the function $\phi(\cdot)$ by the integral

$$\phi(r) \stackrel{\text{def}}{=} \int_0^{+\infty} e^{-rt}\, dt,$$

where $r > 0$, and the function $F(\cdot)$ by the integral

$$F(t, r) \stackrel{\text{def}}{=} \int_0^{t} e^{-rs}\, ds = \frac{1 - e^{-rt}}{r}.$$

It therefore follows that

$$F_r(t, r) = \frac{r(te^{-rt}) - (1 - e^{-rt})}{r^2} = -\frac{1 - e^{-rt} - rte^{-rt}}{r^2}.$$

We thus have the following limits:

$$\lim_{t\to+\infty} F(t, r) = \lim_{t\to+\infty} \frac{1 - e^{-rt}}{r} = \frac{1}{r},$$
$$\lim_{t\to+\infty} F_r(t, r) = -\lim_{t\to+\infty} \frac{1 - e^{-rt} - rte^{-rt}}{r^2} = -\frac{1}{r^2}.$$

To show that the convergence is uniform, observe that for $h > 0$,

$$\left| F(t, r) - \frac{1}{r} \right| = \frac{e^{-rt}}{r} \leq \frac{e^{-ht}}{h} \text{ for all } r \geq h,$$
$$\left| F_r(t, r) - \left(-\frac{1}{r^2}\right) \right| = \frac{e^{-rt}(1 + rt)}{r^2} \leq \frac{e^{-ht}(1 + ht)}{h^2} \text{ for all } r \geq h.$$

Therefore the convergence is uniform on any interval $r \geq h$ for $h > 0$. Now define $f(s, r) \stackrel{\text{def}}{=} e^{-rs}$ so that $f_r(s, r) = -se^{-rs}$, both of which are continuous on an infinite strip. Thus, all the hypotheses of Theorem A.14.1 are met. We may therefore apply

Theorem A.14.1 to the function $\phi(\cdot)$ to compute its derivative with respect to r, namely,

$$\phi'(r) = \int_0^{+\infty} -te^{-rt}\,dt,$$

which is what we were after to begin with.

MENTAL EXERCISES

14.1 Prove Theorem 14.1.

14.2 Prove Lemma 14.1.

14.3 Prove Theorem 14.5.

14.4 Prove the infinite horizon version of the Arrow sufficiency theorem, Theorem 14.6.

14.5 In the Halkin counterexample, is $u(t) = 0\,\forall\, t \in [0, +\infty)$ an optimal control? Explain clearly why or why not. Is it possible for the optimal control to be equal to zero for some *finite* period of time? Explain clearly why or why not.

14.6 Recall Example 14.1, the Halkin counterexample.
   (a) Prove that the Hamiltonian $H(\cdot)$ for the Halkin counterexample is *not* concave in (x, u), and thus that the Mangasarian sufficiency theorem cannot be applied to this problem to deduce optimality of the pair $(z(t), v(t)) = (1 - e^{-\frac{1}{2}t}, \frac{1}{2})$.
   (b) Prove that $(z(t), v(t)) = (1 - e^{-\frac{1}{2}t}, \frac{1}{2})$ is an optimal pair by using the infinite-horizon Arrow sufficiency theorem.

14.7 Prove that the fixed point of the canonical equations of the Halkin counterexample is a saddle point.

14.8 Let $\lambda(t) = -e^{-t}$, $x(t) = e^{t}$, and $z(t) = 1$. Prove that the following implication is incorrect:

$$\lim_{t\to+\infty} \lambda(t) \geq 0,\ \lim_{t\to+\infty} \lambda(t)z(t) = 0,\ \text{and}$$
$$x(t) \geq 0\,\forall\, t \in [t_0, +\infty) \Rightarrow \lim_{t\to+\infty} \lambda(t)[x(t) - z(t)] \geq 0.$$

This is thought to be true by some authors; see, for example, Arrow and Kurz (1970, p. 46).

14.9 Prove Corollary 14.1.

14.10 This exercise asks you to show that Theorem 14.7(i), and consequently Corollary 14.1, do not hold for the finite horizon version of problem (10), nor do they hold if any of the functions $f(\cdot)$, $\mathbf{g}(\cdot)$, or $\mathbf{h}(\cdot)$ are explicit functions of the independent variable t and time consistency of the optimal plan is assumed.

(a) Determine which step of the proof of Theorem 14.7(i) breaks down if problem (10) has a finite horizon. That is, prove that $V(\cdot)$ depends on the initial date or starting time if problem (10) has a finite planning horizon. Show your work and explain your reasoning.

Now assume that $f(\cdot)$, $\mathbf{g}(\cdot)$, and $\mathbf{h}(\cdot)$ are explicit functions of the independent variable t in problem (10), and that the planning horizon is infinite.

(b) Argue that if the *explicit* appearance of t in the functions $f(\cdot)$, $\mathbf{g}(\cdot)$, and $\mathbf{h}(\cdot)$ undergoes the linear shift to $t - t_0$ in problem (10) (in addition to the variables and discount function), then the resulting optimal plan is *not* time consistent.

(c) Assume that the optimal solution to problem (10) is time consistent. Prove that $V(\cdot)$ depends on the initial date or starting time in this case.

14.11 *Professor Halkin and his Counterexample Redux.* The famous Halkin counterexample in Example 14.1 shows that the "natural" or intuitive transversality condition, videlicet, $\lim_{t\to+\infty} \lambda(t) = 0$, is *not*, in general, a necessary condition for infinite-horizon optimal control problems when no conditions are placed on $\lim_{t\to+\infty} x(t)$ and when convergence of the objective functional for all admissible pairs is not assumed. This question asks you to reexamine this famous problem, incorporating a small but significant change in its mathematical structure.

Consider, therefore, the following perturbation of the Halkin counterexample:

$$\max_{u(\cdot)} J[x(\cdot), u(\cdot)] \stackrel{\text{def}}{=} \int_0^{+\infty} u(t)[1 - x(t)]\, e^{-rt}\, dt$$

$$\text{s.t.} \quad \dot{x}(t) = u(t)[1 - x(t)],\ x(0) = 0,$$

$$u(t) \in U \stackrel{\text{def}}{=} \{u(\cdot) : 0 \le u(t) \le 1\ \forall\, t \in [0, +\infty)\},$$

where no conditions are placed on $\lim_{t\to+\infty} x(t)$ and $r > 0$ is the discount rate. Compared with the Halkin counterexample, this problem differs from it only by the inclusion of the discount factor e^{-rt} in the integrand.

(a) Prove that $J[\cdot]$ exists for all admissible function pairs $(x(\cdot), u(\cdot))$.

(b) Show that the admissible path of the state variable satisfies

$$x(t) = 1 - e^{-\int_0^t u(s)\, ds},$$

just as in the Halkin counterexample.

(c) Verify that the triplet

$$v(t) = 1,\ z(t) = 1 - e^{-t},\ \lambda(t) = \frac{-1}{1+r} e^{-rt}$$

is a solution to the necessary conditions of the perturbation of the Halkin counterexample. Do *not* use the current value formulation of the problem; this means that $\lambda(t)$ represents the present value shadow price of the state.

(d) Can you use the Mangasarian sufficiency theorem to prove the optimality of the above triplet? Show your work and explain carefully.

(e) Can you use the Arrow sufficiency theorem to prove the optimality of the above triplet? Show your work and explain carefully.

(f) Does the solution to the perturbation of the Halkin counterexample represent a counterexample to the necessity of the "natural" or intuitive transversality condition $\lim_{t\to+\infty} \lambda(t) = 0$? Explain.

14.12 *A Seemingly Standard Optimal Control Problem*. This question shows that a seemingly standard optimal control problem, that is, one with a quadratic integrand, linear dynamics, and a positive discount rate, has only one finite steady state and it is an *unstable proper node*. Without further ado, the optimal control problem under consideration is given by

$$\max_{u(\cdot)} J[x(\cdot), u(\cdot)] \stackrel{\text{def}}{=} \int_0^{+\infty} \left[-\frac{1}{2}(u(t))^2 + x(t)u(t) - \frac{1}{2}(x(t))^2 \right] e^{-rt}\, dt$$

$$\text{s.t.} \quad \dot{x}(t) = u(t),\ x(0) = x_0 > 0.$$

Assume that $1 < r < 2$.

(a) Write down the necessary conditions for this problem in *current value* form.

(b) Prove that the current value Hamiltonian is a concave function of the state and control variables. Is it a strictly concave function of the state and control? Why or why not? Note that this condition alone is not sufficient to claim that the solution of the necessary conditions is a solution of the optimal control problem, since the planning horizon is infinite.

(c) Reduce the necessary conditions down to a pair of linear differential equations involving only (x, λ).

(d) Prove that the origin is the *only* fixed point of the ordinary differential equations (ODEs) in part (c).

(e) Prove that the origin is an unstable proper node.

(f) Find the general solution of the linear system of ODEs in part (c).

(g) Draw the phase portrait for the linear system of ODEs in part (c).

(h) Prove that

$$x^*(t; x_0) = x_0 e^t,\ u^*(t; x_0) = x_0 e^t,\ \lambda(t; x_0) = 0,\ J[x^*(\cdot), u^*(\cdot)] = 0$$

is the optimal solution of the control problem.

14.13 This question probes your understanding of infinite-horizon optimal control problems by asking you to compare them to their finite-horizon counterparts. Even though you are not asked to prove anything in this question, it may behoove you to write down a few equations in answering the question.

(a) What two features or aspects of finite horizon problems do not, in general, carry over to infinite-horizon problems? Explain precisely.

(b) Explain carefully how economists typically deal with these two complications posed by infinite-horizon problems.

14.14 Consider the following class of infinite-horizon optimal control problems:

$$\max_{u(\cdot)} J[x(\cdot), u(\cdot)] \stackrel{\text{def}}{=} \int_0^{+\infty} f(x(t), u(t))\, e^{-rt}\, dt$$

$$\text{s.t.} \quad \dot{x}(t) = u(t),\ x(0) = x_0,$$

where $f(\cdot) \in C^{(2)}$, $r > 0$ is the discount rate, and no conditions are placed on $\lim_{t\to+\infty} x(t)$. Assume that $(z(t; r, x_0), v(t; r, x_0))$ are the optimal pair of curves to this problem, with corresponding current value costate curve $\lambda(t; r, x_0)$. Furthermore, let $(x^*(r), u^*(r))$ be the simple steady state of the control problem, with $\lambda^*(r)$ being the corresponding steady state value of the costate. Assume that the objective functional exists for all admissible pairs of curves and that $f_{uu}(z(t; r, x_0), v(t; r, x_0)) < 0\, \forall\, t \in [0, +\infty)$.

(a) Derive the necessary conditions for this control problem in current value form.

(b) Transform the necessary conditions into a pair of autonomous ordinary differential equations in (x, u). **Hint:** Differentiate $H_u(x(t), u(t), \lambda(t)) = 0$ with respect to t, and use the necessary conditions to derive a differential equation for $u(t)$.

(c) Prove that $\text{tr}(\mathbf{J}(x^*(r), u^*(r))) = r$ and $|\mathbf{J}(x^*(r), u^*(r))| \neq 0$, where $\mathbf{J}(x^*(r), u^*(r))$ is the Jacobian of the system of ordinary differential equations in part (b) evaluated at the fixed point. Is the steady state locally stable? Explain.

(d) In principle, how do you find the steady state values $(x^*(r), u^*(r))$? What condition must hold for $(x^*(r), u^*(r))$ to be well defined by the implicit function theorem? Does this condition hold? Explain.

(e) Prove that if $\lim_{t\to+\infty} z(t; r, x_0) = x^*(r)$, then $|\mathbf{J}(x^*(r), u^*(r))| < 0$. Interpret this result in words.

(f) Now assume that $f_{xu}(x^*(r), u^*(r)) = 0$ and that $f_{xx}(x^*(r), u^*(r)) < 0$. Prove that $|\mathbf{J}(x^*(r), u^*(r))| < 0$. Interpret this result in words.

(g) Assume that the steady state is a local saddle point. Prove that

$$\text{sign}\left[\frac{\partial x^*(r)}{\partial r}\right] = -\text{sign}[\lambda^*(r)].$$

Provide an interpretation of this steady state comparative statics result.

14.15 Prove Theorem 14.10.

FURTHER READING

Lemma 14.1 is nearly identical to that given in Léonard and Van Long (1992, Chapter 9, Corollary 9.3.2). Other sufficient conditions for $\lim_{t\to+\infty} \boldsymbol{\lambda}(t)'[\mathbf{z}(t) - \mathbf{x}(t)] = 0$ to hold in problem (1) can be found in Seierstad and Sydsæter (1987, Chapter 3, note 16). Michel (1982) has shown that Theorem 14.9 holds without assuming that $V_{\mathbf{x}}(\cdot) \in$

$C^{(0)}$. This relaxation is unimportant for understanding the transversality condition, and, more often than not, its application to dynamic economic problems. Benveniste and Scheinkman (1982) and Araujo and Scheinkman (1983) study concave infinite-horizon control problems using advanced mathematical tools from convex analysis, and establish necessary transversality conditions. Kamihigashi (2001) generalizes and unifies much of the existing work on the necessary transversality conditions. Makris (2001) derives necessary conditions for discounted infinite-horizon control problems in which the time to switch between alternative and consecutive regimes is a decision variable, the so-called two-stage optimal control problem. Romer (1986) establishes an existence theorem for a class of infinite-horizon control problems. The material in the appendix is drawn from Protter and Morrey (1991).

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