TWENTY

Intertemporal Duality in the Adjustment Cost Model of the Firm

This chapter builds directly on the last in developing a duality for the adjustment cost model of the firm. In particular, the current value form of the H-J-B equation given in Theorem 19.3 will be exploited to develop a method to derive the duality properties of the adjustment cost model of the firm. Moreover, we will establish envelope results that will allow the *explicit* construction of the *feedback* or *closed-loop* forms of the investment demand, variable input demand, and output supply functions, given a functional form for the current value optimal value function with known properties. The importance of such a development is monumental in dynamic economic theory for the reasons well summarized by Epstein (1981, page 82):

> In static models, duality is a convenience. Demand functions cannot generally be determined explicitly from the technology but they are defined implicitly by first order conditions which can serve as the basis for estimation, though perhaps requiring complicated simultaneous equations techniques. Explicit solutions for calculus of variations problems are even rarer and the implicit representation of solutions generally involves a second order nonlinear differential equation (system) and non-trivial boundary conditions. The differential equation system can serve as the basis for estimation only if the generally unrealistic assumption is made that the firm does not revise its plans for several periods and continues along the same optimal path. Thus duality is indispensable for empirical work based on functional forms that are too complicated to be derived directly from the technology as explicit solutions of a problem of intertemporal optimization.

The basic assertion that will be used throughout the chapter is that at any point in time $t \in [0, +\infty)$, called a *base period* (or starting date), the firm solves the following discounted autonomous infinite-horizon optimal control problem:

$$\begin{aligned} V(\mathbf{K}_t, \mathbf{c}, \mathbf{w}) \stackrel{\text{def}}{=} \max_{\mathbf{L}(\cdot),\mathbf{I}(\cdot)} \int_t^{+\infty} [F(\mathbf{L}(s), \mathbf{K}(s), \mathbf{I}(s)) - \mathbf{w}'\mathbf{L}(s) - \mathbf{c}'\mathbf{K}(s)]\, e^{-r(s-t)}\, ds \\ \text{s.t.} \quad \dot{\mathbf{K}}(s) = \mathbf{I}(s) - \delta\mathbf{K}(s), \mathbf{K}(t) = \mathbf{K}_t > 0, \\ (\mathbf{K}(s), \mathbf{c}, \mathbf{w}) \in \Theta\, \forall\, s \in [t, +\infty), \end{aligned} \tag{1}$$

where $F(\cdot)$ is a production function giving the maximum amount of the scalar output y that can be produced from the variable input $\mathbf{L}(s) \in \Re_{+}^{M}$ and the quasi-fixed factor $\mathbf{K}(s) \in \Re_{++}^{N}$, the N capital stocks, given that the gross investment rate is $\mathbf{I}(s) \in \Re_{+}^{N}$. The vector $\mathbf{w} \in \Re_{++}^{M}$ is the normalized rental price of the variable input vector $\mathbf{L}(s) \in \Re_{+}^{M}$, whereas the vector $\mathbf{c} \in \Re_{++}^{N}$ is the normalized rental price of the capital stock vector $\mathbf{K}(s) \in \Re_{++}^{N}$. The normalization is implicit in the statement of the adjustment cost problem (1), since we have set the scalar output price equal to unity for the entire planning period. The prices denote actual market prices at time $s = t$, which are expected to persist indefinitely. This is the static expectations assumption often made in the literature, whereby current prices are expected to remain constant for the foreseeable future. As the base period changes and new market prices are observed, the firm revises its expectations and its previous plans, thus only the $s = t$ part of the solution to the control problem (1) is in general carried out. This is a crucial assumption that will be maintained throughout this chapter. The discount rate is $r > 0$, and $\boldsymbol{\delta}$ is a diagonal $N \times N$ matrix of depreciation rates $\delta_n > 0$ for the nth capital stock, $n = 1, 2, \ldots, N$, whereas $\mathbf{K}_t \in \Re_{++}^{N}$ is the initial vector of capital stocks. Note that $V(\cdot)$ is the *current value* optimal value function for problem (1), and as such, by Theorem 19.3, does not depend explicitly on the base period $t \in [0, +\infty)$ in which the optimization problem begins (or starts). Hence one could set $t = 0$ in problem (1) without loss of generality, as is often done in the literature. The set $\Theta \subset \Re_{++}^{2N+M}$ is assumed to be bounded and open, and will be taken to be the domain of the current value optimal value function $V(\cdot)$. By way of a reminder, recall that all vectors are taken to be column vectors, the superscript symbol $'$ denotes transposition, and we adopt the convention that the derivative of a scalar valued function with respect to a column vector is a row vector that has the dimension of the vector variable the derivative was taken with respect to.

Two additional assumptions are maintained for the remainder of the chapter. First, the same real rate of discount and same depreciation matrix are used by the firm in all base periods to discount future profits and depreciate the capital stocks. Hence the discount rate r and the depreciation matrix $\boldsymbol{\delta}$ are constants and therefore may be suppressed as arguments of $V(\cdot)$, a simplification we have already employed. Second, the domain of definition of $F(\cdot)$ is restricted to a bounded open set $\Phi \subset \Re_{++}^{2N+M}$, and thus defines an implicit constraint in problem (1). In particular, because $\Phi \subset \Re_{++}^{2N+M}$, we are assuming that the natural nonnegativity constraints on $(\mathbf{L}(s), \mathbf{K}(s), \mathbf{I}(s))$ are not binding at any point in time in the planning horizon. Such an assumption is typically not restrictive for empirical work based on aggregate data.

Now define the following two sets:

$$\Phi(\mathbf{K}) \stackrel{\text{def}}{=} \{(\mathbf{L}, \mathbf{I}) : (\mathbf{L}, \mathbf{K}, \mathbf{I}) \in \Phi\},$$

$$\Theta(\mathbf{K}) \stackrel{\text{def}}{=} \{(\mathbf{c}, \mathbf{w}) : (\mathbf{K}, \mathbf{c}, \mathbf{w}) \in \Theta\}.$$

These sets will be the domains of the primal and dual optimization problems given in Eqs. (2) and (3), respectively. For each $\mathbf{K} \in \Re_{++}^{N}$, it is assumed that $\Phi(\mathbf{K})$ is empty if and only if $\Theta(\mathbf{K})$ is empty. Denote the empirically relevant decisions corresponding to problem (1) by $\dot{\mathbf{K}}^*(\mathbf{K}_t, \mathbf{c}, \mathbf{w})$, $\mathbf{L}^*(\mathbf{K}_t, \mathbf{c}, \mathbf{w})$, and $y^*(\mathbf{K}_t, \mathbf{c}, \mathbf{w})$. That is, $\dot{\mathbf{K}}^*(\mathbf{K}_t, \mathbf{c}, \mathbf{w})$, $\mathbf{L}^*(\mathbf{K}_t, \mathbf{c}, \mathbf{w})$, and $y^*(\mathbf{K}_t, \mathbf{c}, \mathbf{w})$ are the optimal solution to problem (1) at $s = t$. The functions $(\dot{\mathbf{K}}^*(\cdot), \mathbf{L}^*(\cdot), y^*(\cdot))$ are called the *policy functions*. We will also refer to them as the net investment demand function, the variable input demand function, and the output supply function, respectively. Given that policy functions are expressed solely as functions of the capital stock in the base period and the parameters of the problem, or because they are optimal in the base period $s = t$, you should immediately recognize them as the closed-loop or feedback form of the solution. Finally, let $\boldsymbol{\lambda}^*(\mathbf{K}_t, \mathbf{c}, \mathbf{w})$ denote the corresponding current value shadow price at time $s = t$ in problem (1).

Our main goal in this chapter is to establish a duality between the production function $F(\cdot)$ and the current value optimal value function $V(\cdot)$. The dynamic duality results of Epstein (1981) to be expounded upon below are local. This, however, normally suffices for empirical purposes, for our interest is usually focused on a neighborhood of prices and quantities defined by the data we have at hand. This is the reason that the solution of problem (1) is restricted to an open and bounded set.

The following regularity conditions, valid throughout Φ, are imposed on the production function $F(\cdot)$:

(T.1) $F(\cdot) : \Phi \to \Re_+$, $F(\cdot) \in C^{(1)}$, $F_{\mathbf{L}}(\cdot) \in C^{(1)}$, and $F_{\mathbf{I}}(\cdot) \in C^{(1)}$.
(T.2) $F_{\mathbf{L}}(\mathbf{L}, \mathbf{K}, \mathbf{I}) > \mathbf{0}'_M$, $F_{\mathbf{K}}(\mathbf{L}, \mathbf{K}, \mathbf{I}) > \mathbf{0}'_N$, and $F_{\mathbf{I}}(\mathbf{L}, \mathbf{K}, \mathbf{I}) < \mathbf{0}'_N$.
(T.3) $F(\cdot)$ is strongly concave in $(\mathbf{L}, \mathbf{I})$.
(T.4) For each $(\mathbf{K}_t, \mathbf{c}, \mathbf{w}) \in \Theta$, a unique solution exists for problem (1) in the sense of convergent integrals; the policy functions $(\dot{\mathbf{K}}^*(\cdot), \mathbf{L}^*(\cdot), y^*(\cdot))$ are $C^{(1)}$ on Θ, and the current value shadow price function $\boldsymbol{\lambda}^*(\cdot) \in C^{(2)}$ on Θ.
(T.5) $\boldsymbol{\lambda}^*_{\mathbf{c}}(\mathbf{K}, \mathbf{c}, \mathbf{w})$ is nonsingular for each $(\mathbf{K}, \mathbf{c}, \mathbf{w}) \in \Theta$.
(T.6) For each $(\mathbf{L}^\circ, \mathbf{K}_t, \mathbf{I}^\circ) \in \Phi$, there exists $(\mathbf{K}_t, \mathbf{c}^\circ, \mathbf{w}^\circ) \in \Theta$ such that $(\mathbf{L}^\circ, \mathbf{I}^\circ)$ is optimal in problem (1) at $s = t$ given the initial capital stock $\mathbf{K}_t$ and prices $(\mathbf{c}^\circ, \mathbf{w}^\circ)$.
(T.7) For each $(\mathbf{K}_t, \mathbf{c}, \mathbf{w}) \in \Theta$, problem (1) has a unique steady state capital stock $\mathbf{K}^s(\mathbf{c}, \mathbf{w}) \in \Theta$ that is globally asymptotically stable, that is, optimal paths converge to $\mathbf{K}^s(\mathbf{c}, \mathbf{w})$ regardless of the initial stock $\mathbf{K}_t$.

Generalizations of assumptions (T.1) through (T.7) are possible and consistent with a theory of duality. They were chosen by Epstein (1981) so as to simplify the exposition without doing undue violence to potential empirical applications.

Assumptions (T.1) through (T.3) are more or less standard. In particular, because we intend to use the differential calculus to characterize the duality between $F(\cdot)$ and $V(\cdot)$, some smoothness assumptions are required. Assumption (T.2) means that the marginal product of every variable input and every capital stock is positive whereas the marginal product of every investment rate is negative. In particular, the assumption $F_{\mathbf{I}}(\mathbf{L}, \mathbf{K}, \mathbf{I}) < \mathbf{0}'_N$, where $\mathbf{0}_N$ is the null column N-vector, implies that each component of the vector is negative, and thus reflects the internal adjustment costs associated with gross investment. The results given here would not be materially altered if adjustment costs were external and/or depended on net investment. Condition (T.3) holds if the Hessian matrix of $F(\cdot)$ with respect to $(\mathbf{L}, \mathbf{I})$ is negative definite throughout Φ. Note that we must be careful in interpreting the curvature properties of $F(\cdot)$ with respect to $(\mathbf{L}, \mathbf{I})$ because $\Phi(\mathbf{K})$ may not be a convex set. Thus, $F(\cdot)$ concave in $(\mathbf{L}, \mathbf{I})$ for each $\mathbf{K}$ should be taken to mean that the appropriate Hessian matrix is negative semidefinite throughout $\Phi(\mathbf{K})$ for each $\mathbf{K}$. An analogous interpretation applies to the convexity of $V(\cdot)$ in $(\mathbf{c}, \mathbf{w})$ because $\Theta(\mathbf{K})$ need not be a convex set. Condition (T.4) asserts the existence of well-defined and differentiable solutions associated with problem (1), and that $V(\mathbf{K}_t, \mathbf{c}, \mathbf{w})$ is finite for each $(\mathbf{K}_t, \mathbf{c}, \mathbf{w}) \in \Theta$. The nonsingularity of $\boldsymbol{\lambda}^*_{\mathbf{c}}(\cdot)$ asserted in (T.5) could be dispensed with, but at the cost of considerable additional complexity in the exposition. Moreover, the nonsingularity of $\boldsymbol{\lambda}^*_{\mathbf{c}}(\cdot)$ could not be refuted empirically, and it is a sufficient condition for the functional relationship $\boldsymbol{\lambda} = \boldsymbol{\lambda}^*(\mathbf{K}, \mathbf{c}, \mathbf{w})$ to be locally invertible in $\mathbf{c}$ for given $(\mathbf{K}, \mathbf{w})$ by the implicit function theorem. Points $(\mathbf{L}^\circ, \mathbf{K}_t, \mathbf{I}^\circ) \in \Phi$ that violated (T.6) would never be observed, and so there is no loss in ruling them out. Condition (T.7) could be weakened to require only that any given capital stock profile that is optimal in problem (1) lie in a compact subset of $\{\mathbf{K} : (\mathbf{K}, \mathbf{c}, \mathbf{w}) \in \Theta\}$. We are now in a position to derive the main results of Epstein (1981).

Assume that the production function $F(\cdot)$ satisfies assumptions (T.1) through (T.7), and let the current value optimal value function $V(\cdot)$ be defined by problem (1). Then by Theorem 19.3, $V(\cdot)$ satisfies the H-J-B equation

$$\begin{aligned} rV(\mathbf{K}, \mathbf{c}, \mathbf{w}) = \max_{(\mathbf{L},\mathbf{I})\in\Phi(\mathbf{K})} &\{F(\mathbf{L}, \mathbf{K}, \mathbf{I}) - \mathbf{w}'\mathbf{L} - \mathbf{c}'\mathbf{K} + V_{\mathbf{K}}(\mathbf{K}, \mathbf{c}, \mathbf{w})[\mathbf{I} - \delta\mathbf{K}]\}, \\ &(\mathbf{K}, \mathbf{c}, \mathbf{w}) \in \Theta, \end{aligned} \tag{2}$$

where $t \in [0, +\infty)$ is *any* base period and $\mathbf{K}$ is *any admissible* vector of capital in the base period. It is important to understand that the maximizing values of $\mathbf{L}$ and $\mathbf{I}$ in problem (2) when $\mathbf{K} = \mathbf{K}_t$ and $(\mathbf{K}_t, \mathbf{c}^\circ, \mathbf{w}^\circ) \in \Theta$ are precisely the demands that are optimal in problem (1) at $s = t$ by assumption (T.6). More generally, for each $(\mathbf{K}, \mathbf{c}, \mathbf{w}) \in \Theta$, the maximizing values of $\mathbf{L}$ and $\mathbf{I}$ in problem (2) are given by the values of the policy functions $(\mathbf{L}^*(\mathbf{K}, \mathbf{c}, \mathbf{w}), \mathbf{I}^*(\mathbf{K}, \mathbf{c}, \mathbf{w}))$, which are the optimal values of the control variables in the optimal control problem (1) in any base period $t \in [0, +\infty)$, given that the corresponding value of the capital stock in the base period is $\mathbf{K}$. This is precisely what we observed in the last chapter: the value of the open-loop form of the control vector in the base period is identical to the value of the feedback form of the control vector.

The most important aspect of the H-J-B equation (2) is that it is a static optimization problem relating the functions $F(\cdot)$ and $V(\cdot)$. Therefore, the duality theory of Silberberg (1974) for static optimization problems may be applied to establish a duality between the functions $F(\cdot)$ and $V(\cdot)$. This is the intertemporal duality we seek to establish below.

The *dual* (or inverse) problem of Eq. (2) defines a production function $F^*(\cdot)$, given a function $V(\cdot)$ that satisfies an appropriate set of regularity conditions, and is given by

$$F^*(\mathbf{L}, \mathbf{K}, \mathbf{I}) = \min_{(\mathbf{c},\mathbf{w})\in\Theta(\mathbf{K})} \{rV(\mathbf{K}, \mathbf{c}, \mathbf{w}) + \mathbf{w}'\mathbf{L} + \mathbf{c}'\mathbf{K} - V_{\mathbf{K}}(\mathbf{K}, \mathbf{c}, \mathbf{w})[\mathbf{I} - \delta\mathbf{K}]\},$$
$$(\mathbf{L}, \mathbf{K}, \mathbf{I}) \in \Phi. \tag{3}$$

If problem (3) seems unnatural or otherwise strange, simply rewrite problem (2) as a primal-dual optimization problem, maximizing with respect to $(\mathbf{c}, \mathbf{w}) \in \Theta(\mathbf{K})$. By doing so, problem (3) immediately follows with $F^*(\cdot)$ replaced by $F(\cdot)$. We will see such details below when we prove Theorems 20.1 and 20.2.

Before presenting the regularity conditions that will be shown to characterize the current value optimal value function $V(\cdot)$, we first present the following formulas:

$$\tilde{\mathbf{I}}(\mathbf{K}, \mathbf{c}, \mathbf{w}) \overset{\text{def}}{=} V_{\mathbf{cK}}^{-1}(\mathbf{K}, \mathbf{c}, \mathbf{w})[rV_{\mathbf{c}}'(\mathbf{K}, \mathbf{c}, \mathbf{w}) + \mathbf{K}] + \delta\mathbf{K}, \tag{4}$$

$$\tilde{\mathbf{L}}(\mathbf{K}, \mathbf{c}, \mathbf{w}) \overset{\text{def}}{=} -rV_{\mathbf{w}}'(\mathbf{K}, \mathbf{c}, \mathbf{w}) + V_{\mathbf{wK}}(\mathbf{K}, \mathbf{c}, \mathbf{w})V_{\mathbf{cK}}^{-1}(\mathbf{K}, \mathbf{c}, \mathbf{w})[rV_{\mathbf{c}}'(\mathbf{K}, \mathbf{c}, \mathbf{w}) + \mathbf{K}], \tag{5}$$

$$\begin{aligned}\tilde{y}(\mathbf{K}, \mathbf{c}, \mathbf{w}) &\overset{\text{def}}{=} rV(\mathbf{K}, \mathbf{c}, \mathbf{w}) + \mathbf{w}'\tilde{\mathbf{L}}(\mathbf{K}, \mathbf{c}, \mathbf{w}) + \mathbf{c}'\mathbf{K} \\ &\quad - V_{\mathbf{K}}(\mathbf{K}, \mathbf{c}, \mathbf{w})[\tilde{\mathbf{I}}(\mathbf{K}, \mathbf{c}, \mathbf{w}) - \delta\mathbf{K}] \\ &= r[V(\mathbf{K}, \mathbf{c}, \mathbf{w}) - V_{\mathbf{w}}(\mathbf{K}, \mathbf{c}, \mathbf{w})\mathbf{w} \\ &\quad - V_{\mathbf{c}}(\mathbf{K}, \mathbf{c}, \mathbf{w})\mathbf{c}] - [V_{\mathbf{K}}(\mathbf{K}, \mathbf{c}, \mathbf{w}) - \mathbf{w}'V_{\mathbf{wK}}(\mathbf{K}, \mathbf{c}, \mathbf{w}) \\ &\quad - \mathbf{c}'V_{\mathbf{cK}}(\mathbf{K}, \mathbf{c}, \mathbf{w})\mathbf{c}]\left[V_{\mathbf{cK}}^{-1}(\mathbf{K}, \mathbf{c}, \mathbf{w})[rV_{\mathbf{c}}'(\mathbf{K}, \mathbf{c}, \mathbf{w}) + \mathbf{K}]\right].\end{aligned} \tag{6}$$

Note that these are simply definitions of the left-hand-side functions and hence hold for all $(\mathbf{K}, \mathbf{c}, \mathbf{w}) \in \Theta$. What we intend to show is that they describe optimal behavior in problem (1), that is, the policy or closed-loop solution functions for problem (1) are equal to them. One way to get some intuition on them is to recognize that they are simply the envelope results for the primal H-J-B problem (2).

Given these definitions, the following conditions will be shown to characterize the current value optimal value function $V(\cdot)$ for problem (1):

(V.1) $V(\cdot)$ is a real-valued, bound-from-below function defined on Θ; $V(\cdot) \in C^{(2)}$ and $V_{\mathbf{K}}(\cdot) \in C^{(2)}$.

(V.2) (i) $(r\mathrm{I}_N + \delta)V_{\mathbf{K}}'(\mathbf{K}, \mathbf{c}, \mathbf{w}) + \mathbf{c} - V_{\mathbf{KK}}(\mathbf{K}, \mathbf{c}, \mathbf{w})[\tilde{\mathbf{I}}(\mathbf{K}, \mathbf{c}, \mathbf{w}) - \delta\mathbf{K}] > \mathbf{0}_N$, and (ii) $V_{\mathbf{K}}'(\mathbf{K}, \mathbf{c}, \mathbf{w}) > \mathbf{0}_N$.

(V.3) For each $(\mathbf{K}, \mathbf{c}, \mathbf{w}) \in \Theta$, $\tilde{y}(\mathbf{K}, \mathbf{c}, \mathbf{w}) \geq 0$; for each $\mathbf{K}$ such that $\Theta(\mathbf{K})$ is nonempty, $(\tilde{\mathbf{L}}(\mathbf{K}, \cdot, \cdot), \mathbf{K}, \tilde{\mathbf{I}}(\mathbf{K}, \cdot, \cdot))$ maps $\Theta(\mathbf{K})$ onto $\Phi(\mathbf{K})$.

(V.4) The dynamical system $\dot{\mathbf{K}} = \tilde{\mathbf{I}}(\mathbf{K}, \mathbf{c}, \mathbf{w}) - \delta\mathbf{K}$, $\mathbf{K}(t) = \mathbf{K}_t$, $(\mathbf{K}_t, \mathbf{c}, \mathbf{w}) \in \Theta$, defines a curve $\mathbf{K}(s)$ such that $(\mathbf{K}(s), \mathbf{c}, \mathbf{w}) \in \Theta\,\forall\, s \in [t, +\infty)$, and $\mathbf{K}(s) \to \mathbf{K}^s(\mathbf{c}, \mathbf{w}) \in \Theta$ as $s \to +\infty$, a globally asymptotically stable steady state.

(V.5) $V_{\mathbf{cK}}(\mathbf{K}, \mathbf{c}, \mathbf{w})$ is nonsingular for each $(\mathbf{K}, \mathbf{c}, \mathbf{w}) \in \Theta$.

(V.6) For $(\mathbf{K}, \mathbf{c}^{\circ}, \mathbf{w}^{\circ}) \in \Theta$, the minimum in problem (3) is attained at $(\mathbf{c}^{\circ}, \mathbf{w}^{\circ})$ if $(\mathbf{I}, \mathbf{L}) = (\tilde{\mathbf{I}}(\mathbf{K}, \mathbf{c}^{\circ}, \mathbf{w}^{\circ}), \tilde{\mathbf{L}}(\mathbf{K}, \mathbf{c}^{\circ}, \mathbf{w}^{\circ}))$.

(V.7) The $(M+N) \times (M+N)$ matrix

$$\underbrace{\begin{bmatrix} \underbrace{\dfrac{\partial \tilde{\mathbf{L}}}{\partial \mathbf{w}}(\mathbf{K}, \mathbf{c}, \mathbf{w})}_{M \times M} & \underbrace{\dfrac{\partial \tilde{\mathbf{L}}}{\partial \mathbf{c}}(\mathbf{K}, \mathbf{cw})}_{M \times N} \\ \underbrace{\dfrac{\partial \tilde{\mathbf{I}}}{\partial \mathbf{w}}(\mathbf{K}, \mathbf{c}, \mathbf{w})}_{N \times M} & \underbrace{\dfrac{\partial \tilde{\mathbf{I}}}{\partial \mathbf{c}}(\mathbf{K}, \mathbf{c}, \mathbf{w})}_{N \times N} \end{bmatrix}}_{(M+N)\times(M+N)}$$

is nonsingular for $(\mathbf{K}, \mathbf{c}, \mathbf{w}) \in \Theta$.

Let us remark that the "onto" property in (V.3) simply means that the range of the functions is the target space of the functions and that I_N is the $N \times N$ identity matrix.

The following two theorems are the main results of Epstein (1981). They establish a formal intertemporal duality between the production function $F(\cdot)$ and the current value optimal value function $V(\cdot)$, as well as give specific formulas for the empirically relevant policy functions, that is, the feedback or closed-loop form of the optimal control functions.

Theorem 20.1 (Intertemporal Duality):

(a) Let $F(\cdot)$ satisfy (T.1) through (T.7) and define $V(\cdot)$ by Eq. (1). Then $V(\cdot)$ satisfies (V.1) through (V.7). If further $V(\cdot)$ is used to define $F^*(\cdot)$ by Eq. (3), then $F^*(\cdot) \equiv F(\cdot)$.

(b) Let $V(\cdot)$ satisfy (V.1) through (V.7) and define $F(\cdot)$ by Eq. (3). Then $F(\cdot)$ satisfies (T.1) through (T.7). If further $F(\cdot)$ is used to define $V^*(\cdot)$ by Eq. (1), then $V^*(\cdot) \equiv V(\cdot)$.

Theorem 20.2 (Policy Function Formulae): *Let $F(\cdot)$ satisfy (T.1) through (T.7) and let $V(\cdot)$ be the current value optimal value function defined by Eq. (1). Then the policy functions are given by*

$$\dot{\mathbf{K}}^*(\mathbf{K}, \mathbf{c}, \mathbf{w}) = \tilde{\mathbf{I}}(\mathbf{K}, \mathbf{c}, \mathbf{w}) - \delta \mathbf{K},$$

$$\mathbf{L}^*(\mathbf{K}, \mathbf{c}, \mathbf{w}) = \tilde{\mathbf{L}}(\mathbf{K}, \mathbf{c}, \mathbf{w}),$$

$$y^*(\mathbf{K}, \mathbf{c}, \mathbf{w}) = \tilde{y}(\mathbf{K}, \mathbf{c}, \mathbf{w}),$$

for all $(\mathbf{K}, \mathbf{c}, \mathbf{w}) \in \Theta$, where the functions $\tilde{\mathbf{I}}(\cdot)$, $\tilde{\mathbf{L}}(\cdot)$, and $\tilde{y}(\cdot)$ are defined by Eqs. (4), (5), and (6), respectively.

Before we embark on the proof of these theorems, we pause and make several remarks. First, the ensuing proof is long and detailed, so don't expect to be able to breeze through it and fully understand it on the first reading. Second, in the course of proving Theorem 20.1, we will also prove Theorem 20.2. Third, there are at least two different ways of proving several parts of the theorems. We offer one proof in the text and leave the alternative methods of proof for the mental exercises.

Proof of both theorems: We begin by proving part (a) of Theorem 20.1, and in the process prove Theorem 20.2. To this end, assume that $F(\cdot)$ satisfies assumptions (T.1) through (T.7) and define $V(\cdot)$ by Eq. (1).

(V.1) The current value optimal value function $V(\cdot)$ is real-valued on Θ because for each $(\mathbf{K}_t, \mathbf{c}, \mathbf{w}) \in \Theta$, a unique solution exists for problem (1) in the sense of convergent integrals by assumption (T.4). The assumed boundedness of the sets Φ and Θ implies that $V(\cdot)$ is bounded below over Θ. By the dynamic envelope theorem and the principle of optimality, we know that $V'_{\mathbf{K}}(\mathbf{K}, \mathbf{c}, \mathbf{w}) \equiv \boldsymbol{\lambda}^*(\mathbf{K}, \mathbf{c}, \mathbf{w})$. Moreover, given that $\boldsymbol{\lambda}^*(\cdot) \in C^{(2)}$ on Θ by assumption (T.4), it follows that $V_{\mathbf{K}}(\cdot) \in C^{(2)}$ on Θ too. To finish this part, we must show that $V(\cdot) \in C^{(2)}$. Actually, all we have to show is that the second-order partial derivatives of $V(\cdot)$ with respect to the prices $(\mathbf{c}, \mathbf{w})$ are continuous, because we already know that $V_{\mathbf{K}}(\cdot) \in C^{(2)}$ on Θ. Thus, applying the static envelope theorem to problem (2) yields

$$rV_{c_i}(\mathbf{K}, \mathbf{c}, \mathbf{w}) \equiv -K_i + \sum_{n=1}^{N} V_{K_n c_i}(\mathbf{K}, \mathbf{c}, \mathbf{w})[I_n^*(\mathbf{K}, \mathbf{c}, \mathbf{w}) - \delta_n K_n], \ i = 1, 2, \ldots, N,$$

$$rV_{w_j}(\mathbf{K}, \mathbf{c}, \mathbf{w}) \equiv -L_j^*(\mathbf{K}, \mathbf{c}, \mathbf{w}) + \sum_{n=1}^{N} V_{K_n w_j}(\mathbf{K}, \mathbf{c}, \mathbf{w})[I_n^*(\mathbf{K}, \mathbf{c}, \mathbf{w}) - \delta_n K_n],$$
$$j = 1, 2, \ldots, M.$$

The right-hand sides of these two equations are a $C^{(1)}$ function of $(\mathbf{K}, \mathbf{c}, \mathbf{w}) \in \Theta$ seeing as $V_{\mathbf{K}}(\cdot) \in C^{(2)}$ on Θ by assumption (T.1) and the policy functions $(\mathbf{L}^*(\cdot), \mathbf{I}^*(\cdot)) = (\mathbf{L}^*(\cdot), \dot{\mathbf{K}}^*(\cdot) + \boldsymbol{\delta}\mathbf{K})$ are $C^{(1)}$ on Θ by assumption (T.4). It therefore follows that the second-order partial derivatives of $V(\cdot)$ with respect to the prices $(\mathbf{c}, \mathbf{w})$ are continuous functions on Θ, thereby implying that $V(\cdot) \in C^{(2)}$ on Θ. You will be asked to compute these second-order partial derivatives in a mental exercise to further enhance your understanding of this part of the proof.

(V.5) By the dynamic envelope theorem and the principle of optimality, we know that $V'_{\mathbf{K}}(\mathbf{K}, \mathbf{c}, \mathbf{w}) \equiv \boldsymbol{\lambda}^*(\mathbf{K}, \mathbf{c}, \mathbf{w})$. Differentiating this identity with respect to $\mathbf{c}$ results in the identity $V_{\mathbf{Kc}}(\mathbf{K}, \mathbf{c}, \mathbf{w}) \equiv \boldsymbol{\lambda}^*_{\mathbf{c}}(\mathbf{K}, \mathbf{c}, \mathbf{w})$. Because $\boldsymbol{\lambda}^*_{\mathbf{c}}(\mathbf{K}, \mathbf{c}, \mathbf{w})$ is nonsingular for each $(\mathbf{K}, \mathbf{c}, \mathbf{w}) \in \Theta$ by assumption (T.5), it follows from the last identity that the same is true for $V_{\mathbf{Kc}}(\mathbf{K}, \mathbf{c}, \mathbf{w})$. But because $V_{\mathbf{Kc}}(\mathbf{K}, \mathbf{c}, \mathbf{w}) \equiv V'_{\mathbf{cK}}(\mathbf{K}, \mathbf{c}, \mathbf{w})$, as you will show in a mental exercise, $V'_{\mathbf{cK}}(\mathbf{K}, \mathbf{c}, \mathbf{w})$ is nonsingular for each $(\mathbf{K}, \mathbf{c}, \mathbf{w}) \in \Theta$. Finally, you should recall a basic theorem of linear algebra that states that a matrix is invertible, that is, nonsingular, if and only if its transpose is invertible. Applying this theorem yields the desired conclusion that $V_{\mathbf{cK}}(\mathbf{K}, \mathbf{c}, \mathbf{w})$ is nonsingular for each $(\mathbf{K}, \mathbf{c}, \mathbf{w}) \in \Theta$.

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(V.6) and Theorem 20.2 Let $(\mathbf{K}, \mathbf{c}^\circ, \mathbf{w}^\circ) \in \Theta$. By assumption (T.6), $\mathbf{I}^\circ = \mathbf{I}^*(\mathbf{K}, \mathbf{c}^\circ, \mathbf{w}^\circ)$ and $\mathbf{L}^\circ = \mathbf{L}^*(\mathbf{K}, \mathbf{c}^\circ, \mathbf{w}^\circ)$ solve the primal H-J-B problem (2) when $(\mathbf{c}, \mathbf{w}) = (\mathbf{c}^\circ, \mathbf{w}^\circ)$. Given this fact, the primal-dual problem corresponding to problem (2) is

$$0 = \max_{(\mathbf{c},\mathbf{w})\in\Theta(\mathbf{K})} \{F(\mathbf{L}^\circ, \mathbf{K}, \mathbf{I}^\circ) - \mathbf{w}'\mathbf{L}^\circ - \mathbf{c}'\mathbf{K} + V_{\mathbf{K}}(\mathbf{K}, \mathbf{c}, \mathbf{w})[\mathbf{I}^\circ - \delta\mathbf{K}] - rV(\mathbf{K}, \mathbf{c}, \mathbf{w})\},\ (\mathbf{L}^\circ, \mathbf{K}, \mathbf{I}^\circ) \in \Phi, \tag{7}$$

where, by construction, the price vector $(\mathbf{c}^\circ, \mathbf{w}^\circ)$ is optimal in problem (7) given that we have set $(\mathbf{I}, \mathbf{L}) = (\mathbf{I}^\circ, \mathbf{L}^\circ) = (\mathbf{I}^*(\mathbf{K}, \mathbf{c}^\circ, \mathbf{w}^\circ), \mathbf{L}^*(\mathbf{K}, \mathbf{c}^\circ, \mathbf{w}^\circ))$. Because $F(\mathbf{L}^\circ, \mathbf{K}, \mathbf{I}^\circ)$ is independent of the decision variables $(\mathbf{c}, \mathbf{w})$ for the primal-dual optimization problem (7), we can subtract $F(\mathbf{L}^\circ, \mathbf{K}, \mathbf{I}^\circ)$ from both sides to get

$$-F(\mathbf{L}^\circ, \mathbf{K}, \mathbf{I}^\circ) = \max_{(\mathbf{c},\mathbf{w})\in\Theta(\mathbf{K})} \{-\mathbf{w}'\mathbf{L}^\circ - \mathbf{c}'\mathbf{K} + V_{\mathbf{K}}(\mathbf{K}, \mathbf{c}, \mathbf{w})[\mathbf{I}^\circ - \delta\mathbf{K}] - rV(\mathbf{K}, \mathbf{c}, \mathbf{w})\},\ (\mathbf{L}^\circ, \mathbf{K}, \mathbf{I}^\circ) \in \Phi.$$

Now we can use the fact that $-\min_{\mathbf{x}} f(\mathbf{x}) = \max_{\mathbf{x}}[-f(\mathbf{x})]$ and multiply the preceding equation on both sides by minus unity to get

$$F(\mathbf{L}^\circ, \mathbf{K}, \mathbf{I}^\circ) = \min_{(\mathbf{c},\mathbf{w})\in\Theta(\mathbf{K})} \{rV(\mathbf{K}, \mathbf{c}, \mathbf{w}) + \mathbf{w}'\mathbf{L}^\circ + \mathbf{c}'\mathbf{K} - V_{\mathbf{K}}(\mathbf{K}, \mathbf{c}, \mathbf{w})[\mathbf{I}^\circ - \delta\mathbf{K}]\},\ (\mathbf{L}^\circ, \mathbf{K}, \mathbf{I}^\circ) \in \Phi. \tag{8}$$

Finally, note that our choice of the point $(\mathbf{K}, \mathbf{c}^\circ, \mathbf{w}^\circ) \in \Theta$ was arbitrary, hence making problem (8) equivalent to problem (3), thereby proving that $F(\mathbf{L}, \mathbf{K}, \mathbf{I}) \equiv F^*(\mathbf{L}, \mathbf{K}, \mathbf{I})\,\forall\,(\mathbf{L}, \mathbf{K}, \mathbf{I}) \in \Phi$ and establishing property (V.6).

We can say more at this juncture. In particular, the first-order necessary conditions for an optimum in problem (3), or equivalently, problem (8), are given by

$$rV'_{\mathbf{c}}(\mathbf{K}, \mathbf{c}, \mathbf{w}) + \mathbf{K} - V'_{\mathbf{Kc}}(\mathbf{K}, \mathbf{c}, \mathbf{w})[\mathbf{I}^\circ - \delta\mathbf{K}] = \mathbf{0}_N, \tag{9}$$

$$rV'_{\mathbf{w}}(\mathbf{K}, \mathbf{c}, \mathbf{w}) + \mathbf{L}^\circ - V'_{\mathbf{Kw}}(\mathbf{K}, \mathbf{c}, \mathbf{w})[\mathbf{I}^\circ - \delta\mathbf{K}] = \mathbf{0}_M, \tag{10}$$

which hold at $(\mathbf{K}, \mathbf{c}^\circ, \mathbf{w}^\circ) \in \Theta$ by construction. Because $V_{\mathbf{Kc}}(\mathbf{K}, \mathbf{c}, \mathbf{w}) \equiv V'_{\mathbf{cK}}(\mathbf{K}, \mathbf{c}, \mathbf{w})$, as noted in the proof of property (V.5), the first-order necessary condition (9) can be solved for $\mathbf{I}^\circ = \mathbf{I}^*(\mathbf{K}, \mathbf{c}^\circ, \mathbf{w}^\circ)$ upon setting $(\mathbf{c}, \mathbf{w}) = (\mathbf{c}^\circ, \mathbf{w}^\circ)$ and recalling that $V_{\mathbf{cK}}^{-1}(\mathbf{K}, \mathbf{c}, \mathbf{w})$ exists for each $(\mathbf{K}, \mathbf{c}, \mathbf{w}) \in \Theta$ by property (V.5), that is,

$$\mathbf{I}^*(\mathbf{K}, \mathbf{c}^\circ, \mathbf{w}^\circ) = V_{\mathbf{cK}}^{-1}(\mathbf{K}, \mathbf{c}^\circ, \mathbf{w}^\circ)[rV'_{\mathbf{c}}(\mathbf{K}, \mathbf{c}^\circ, \mathbf{w}^\circ) + \mathbf{K}] + \delta\mathbf{K}. \tag{11}$$

Inspection of Eqs. (4) and (11) reveals that $\tilde{\mathbf{I}}(\mathbf{K}, \mathbf{c}^\circ, \mathbf{w}^\circ) = \mathbf{I}^*(\mathbf{K}, \mathbf{c}^\circ, \mathbf{w}^\circ)$. Inasmuch as our choice of the point $(\mathbf{K}, \mathbf{c}^\circ, \mathbf{w}^\circ) \in \Theta$ was arbitrary, however, we have established that

$$\mathbf{I}^*(\mathbf{K}, \mathbf{c}, \mathbf{w}) \equiv \tilde{\mathbf{I}}(\mathbf{K}, \mathbf{c}, \mathbf{w})\,\forall\,(\mathbf{K}, \mathbf{c}, \mathbf{w}) \in \Theta,$$

which proves the first formula of Theorem 20.2, upon recalling that $\mathbf{I}^*(\mathbf{K}, \mathbf{c}, \mathbf{w}) \equiv \dot{\mathbf{K}}^*(\mathbf{K}, \mathbf{c}, \mathbf{w}) + \delta\mathbf{K}$.

Similarly, in view of the fact that $V_{\mathbf{Kw}}(\mathbf{K}, \mathbf{c}, \mathbf{w}) \equiv V'_{\mathbf{wK}}(\mathbf{K}, \mathbf{c}, \mathbf{w})$, the first-order necessary condition (10) can be solved for $\mathbf{L}^\circ = \mathbf{L}^*(\mathbf{K}, \mathbf{c}^\circ, \mathbf{w}^\circ)$ upon setting $(\mathbf{c}, \mathbf{w}) = (\mathbf{c}^\circ, \mathbf{w}^\circ)$, that is,

$$\begin{aligned}\mathbf{L}^*(\mathbf{K}, \mathbf{c}^\circ, \mathbf{w}^\circ) &= -rV'_{\mathbf{w}}(\mathbf{K}, \mathbf{c}^\circ, \mathbf{w}^\circ) + V_{\mathbf{wK}}(\mathbf{K}, \mathbf{c}^\circ, \mathbf{w}^\circ)[\mathbf{I}^\circ - \delta\mathbf{K}] \\ &= -rV'_{\mathbf{w}}(\mathbf{K}, \mathbf{c}^\circ, \mathbf{w}^\circ) + V_{\mathbf{wK}}(\mathbf{K}, \mathbf{c}^\circ, \mathbf{w}^\circ)V_{\mathbf{cK}}^{-1}(\mathbf{K}, \mathbf{c}^\circ, \mathbf{w}^\circ) \\ &\quad \times [rV'_{\mathbf{c}}(\mathbf{K}, \mathbf{c}^\circ, \mathbf{w}^\circ) + \mathbf{K}],\end{aligned}$$

where we used the fact that $\mathbf{I}^\circ = \mathbf{I}^*(\mathbf{K}, \mathbf{c}^\circ, \mathbf{w}^\circ)$ and Eq. (11). Inspection of Eq. (5) reveals that $\tilde{\mathbf{L}}(\mathbf{K}, \mathbf{c}^\circ, \mathbf{w}^\circ) = \mathbf{L}^*(\mathbf{K}, \mathbf{c}^\circ, \mathbf{w}^\circ)$. However, because our choice of the point $(\mathbf{K}, \mathbf{c}^\circ, \mathbf{w}^\circ) \in \Theta$ was arbitrary, we have actually shown that

$$\mathbf{L}^*(\mathbf{K}, \mathbf{c}, \mathbf{w}) \equiv \tilde{\mathbf{L}}(\mathbf{K}, \mathbf{c}, \mathbf{w}) \,\forall\, (\mathbf{K}, \mathbf{c}, \mathbf{w}) \in \Theta,$$

which proves the second formula of Theorem 20.2. Consequently, there is no need to distinguish between the functions $(\mathbf{I}^*(\cdot), \mathbf{L}^*(\cdot))$ and $(\tilde{\mathbf{I}}(\cdot), \tilde{\mathbf{L}}(\cdot))$, for they are identical for all $(\mathbf{K}, \mathbf{c}, \mathbf{w}) \in \Theta$. As remarked earlier, these are intertemporal envelope results for the adjustment cost model.

Finally, to complete the proof of Theorem 20.2, first note that the supply function $y^*(\cdot)$ is defined as $y^*(\mathbf{K}, \mathbf{c}, \mathbf{w}) \stackrel{\text{def}}{=} F(\mathbf{L}^*(\mathbf{K}, \mathbf{c}, \mathbf{w}), \mathbf{K}, \mathbf{I}^*(\mathbf{K}, \mathbf{c}, \mathbf{w}))$, and $F(\mathbf{L}, \mathbf{K}, \mathbf{I}) \equiv F^*(\mathbf{L}, \mathbf{K}, \mathbf{I}) \,\forall\, (\mathbf{L}, \mathbf{K}, \mathbf{I}) \in \Phi$, as shown above. The latter identity means that we can find the supply function in identity form from problem (2) or problem (3), that is,

$$\begin{aligned}y^*(\mathbf{K}, \mathbf{c}, \mathbf{w}) \equiv{}& rV(\mathbf{K}, \mathbf{c}, \mathbf{w}) + \mathbf{w}'\mathbf{L}^*(\mathbf{K}, \mathbf{c}, \mathbf{w}) \\ &+ \mathbf{c}'\mathbf{K} - V_{\mathbf{K}}(\mathbf{K}, \mathbf{c}, \mathbf{w})[\mathbf{I}^*(\mathbf{K}, \mathbf{c}, \mathbf{w}) - \delta\mathbf{K}].\end{aligned} \tag{12}$$

This is the first expression for the supply function in Theorem 20.2. Recalling the preceding two proofs regarding the policy functions $(\mathbf{I}^*(\cdot), \mathbf{L}^*(\cdot))$ and then substituting in their formulas from Eqs. (4) and (5) allows us to rewrite Eq. (12) as

$$\begin{aligned}y^*(\mathbf{K}, \mathbf{c}, \mathbf{w}) \equiv{}& rV(\mathbf{K}, \mathbf{c}, \mathbf{w}) + \mathbf{w}'\big[-rV'_{\mathbf{w}}(\mathbf{K}, \mathbf{c}, \mathbf{w}) \\ &+ V_{\mathbf{wK}}(\mathbf{K}, \mathbf{c}, \mathbf{w})V_{\mathbf{cK}}^{-1}(\mathbf{K}, \mathbf{c}, \mathbf{w})[rV'_{\mathbf{c}}(\mathbf{K}, \mathbf{c}, \mathbf{w}) + \mathbf{K}]\big] \\ &+ \mathbf{c}'\mathbf{K} - V_{\mathbf{K}}(\mathbf{K}, \mathbf{c}, \mathbf{w})V_{\mathbf{cK}}^{-1}(\mathbf{K}, \mathbf{c}, \mathbf{w})[rV'_{\mathbf{c}}(\mathbf{K}, \mathbf{c}, \mathbf{w}) + \mathbf{K}].\end{aligned} \tag{13}$$

From the first formula of Theorem 20.2, to wit, $\mathbf{I}^*(\mathbf{K}, \mathbf{c}, \mathbf{w}) \equiv V_{\mathbf{cK}}^{-1}(\mathbf{K}, \mathbf{c}, \mathbf{w}) \times [rV'_{\mathbf{c}}(\mathbf{K}, \mathbf{c}, \mathbf{w}) + \mathbf{K}] + \delta\mathbf{K}$, we can derive the identity

$$\mathbf{c}'\mathbf{K} \equiv -r\mathbf{c}'V'_{\mathbf{c}}(\mathbf{K}, \mathbf{c}, \mathbf{w}) + \mathbf{c}'V_{\mathbf{cK}}(\mathbf{K}, \mathbf{c}, \mathbf{w})V_{\mathbf{cK}}^{-1}(\mathbf{K}, \mathbf{c}, \mathbf{w})[rV'_{\mathbf{c}}(\mathbf{K}, \mathbf{c}, \mathbf{w}) + \mathbf{K}], \tag{14}$$

which you are asked to show in a mental exercise. Upon substituting Eq. (14) into Eq. (13), and using the fact that, for example, $\mathbf{c}'V'_{\mathbf{c}}(\mathbf{K}, \mathbf{c}, \mathbf{w}) = V_{\mathbf{c}}(\mathbf{K}, \mathbf{c}, \mathbf{w})\mathbf{c}$ as it is a scalar, we get

$$\begin{aligned}y^*(\mathbf{K}, \mathbf{c}, \mathbf{w}) \equiv r[V(\mathbf{K}, \mathbf{c}, \mathbf{w}) - V_{\mathbf{c}}(\mathbf{K}, \mathbf{c}, \mathbf{w})\mathbf{c} - V_{\mathbf{w}}(\mathbf{K}, \mathbf{c}, \mathbf{w})\mathbf{w}]\\ - [V_{\mathbf{K}}(\mathbf{K}, \mathbf{c}, \mathbf{w}) - \mathbf{c}'V_{\mathbf{cK}}(\mathbf{K}, \mathbf{c}, \mathbf{w})\\ - \mathbf{w}'V_{\mathbf{wK}}(\mathbf{K}, \mathbf{c}, \mathbf{w})]V_{\mathbf{cK}}^{-1}(\mathbf{K}, \mathbf{c}, \mathbf{w})[rV_{\mathbf{c}}'(\mathbf{K}, \mathbf{c}, \mathbf{w}) + \mathbf{K}].\end{aligned}$$

This equation is the second formula for the supply function $y^*(\cdot)$ and therefore completes the proof of Theorem 20.2 on the formulas for the policy functions.

(V.2) Differentiate problem (2) with respect to $\mathbf{K}$ and apply the static envelope theorem to get

$$\begin{aligned}rV_{\mathbf{K}}(\mathbf{K}, \mathbf{c}, \mathbf{w}) \equiv F_{\mathbf{K}}(\mathbf{L}^*(\mathbf{K}, \mathbf{c}, \mathbf{w}), \mathbf{K}, \mathbf{I}^*(\mathbf{K}, \mathbf{c}, \mathbf{w})) - \mathbf{c}' - V_{\mathbf{K}}(\mathbf{K}, \mathbf{c}, \mathbf{w})\boldsymbol{\delta}\\ + [\mathbf{I}^*(\mathbf{K}, \mathbf{c}, \mathbf{w}) - \boldsymbol{\delta}\mathbf{K}]'V_{\mathbf{KK}}(\mathbf{K}, \mathbf{c}, \mathbf{w}).\end{aligned}$$

Take the transpose of the preceding equation to get an $N \times 1$ column vector, observe that $rV_{\mathbf{K}}'(\mathbf{K}, \mathbf{c}, \mathbf{w}) = r\mathrm{I}_N V_{\mathbf{K}}'(\mathbf{K}, \mathbf{c}, \mathbf{w})$, and then rearrange it to arrive at

$$\begin{aligned}(r\mathrm{I}_N + \boldsymbol{\delta})V_{\mathbf{K}}'(\mathbf{K}, \mathbf{c}, \mathbf{w}) + \mathbf{c} - V_{\mathbf{KK}}(\mathbf{K}, \mathbf{c}, \mathbf{w})[\mathbf{I}^*(\mathbf{K}, \mathbf{c}, \mathbf{w}) - \boldsymbol{\delta}\mathbf{K}]\\ \equiv F_{\mathbf{K}}'(\mathbf{L}^*(\mathbf{K}, \mathbf{c}, \mathbf{w}), \mathbf{K}, \mathbf{I}^*(\mathbf{K}, \mathbf{c}, \mathbf{w})) > \mathbf{0}_N,\end{aligned}$$

where we have used the facts that $\boldsymbol{\delta}$ and $V_{\mathbf{KK}}(\mathbf{K}, \mathbf{c}, \mathbf{w})$ are $N \times N$ symmetric matrices and that $F_{\mathbf{K}}'(\mathbf{L}, \mathbf{K}, \mathbf{I}) > \mathbf{0}_N$ throughout Φ by assumption (T.2). This equation thus proves part (i).

Given that we have assumed that the optimal solution is interior, the first-order necessary condition for $\mathbf{I}$ from problem (2) in identity form is given by

$$F_{\mathbf{I}}(\mathbf{L}^*(\mathbf{K}, \mathbf{c}, \mathbf{w}), \mathbf{K}, \mathbf{I}^*(\mathbf{K}, \mathbf{c}, \mathbf{w})) + V_{\mathbf{K}}(\mathbf{K}, \mathbf{c}, \mathbf{w}) \equiv \mathbf{0}_N'.$$

This identity can be rearranged to read

$$V_{\mathbf{K}}(\mathbf{K}, \mathbf{c}, \mathbf{w}) \equiv -F_{\mathbf{I}}(\mathbf{L}^*(\mathbf{K}, \mathbf{c}, \mathbf{w}), \mathbf{K}, \mathbf{I}^*(\mathbf{K}, \mathbf{c}, \mathbf{w})) > \mathbf{0}_N',$$

because $F_{\mathbf{I}}'(\mathbf{L}, \mathbf{K}, \mathbf{I}) < \mathbf{0}_N$ throughout Φ by assumption (T.2). This completes the proof of part (ii).

(V.3) Seeing as $F(\cdot): \Phi \to \Re_+$ by assumption (T.1), $F(\mathbf{L}, \mathbf{K}, \mathbf{I}) \geq 0$ throughout Φ, which implies that $y^*(\mathbf{K}, \mathbf{c}, \mathbf{w}) \geq 0$ for each $(\mathbf{K}, \mathbf{c}, \mathbf{w}) \in \Theta$ because $y^*(\mathbf{K}, \mathbf{c}, \mathbf{w}) \stackrel{\text{def}}{=} F(\mathbf{L}^*(\mathbf{K}, \mathbf{c}, \mathbf{w}), \mathbf{K}, \mathbf{I}^*(\mathbf{K}, \mathbf{c}, \mathbf{w}))$. The rest of property (V.3) is implied by assumption (T.6).

(V.4) This is easy to prove because assumptions (T.4) and (T.7) imply property (V.4). In particular, assumption (T.4) asserts the existence of a unique solution to the optimal control problem (1) for each $(\mathbf{K}_t, \mathbf{c}, \mathbf{w}) \in \Theta$, whereas assumption (T.7) asserts that the corresponding time path of the capital stock converges to the unique steady state value of the capital stock for any initial value of the capital stock, which is exactly what property (V.4) asserts.

(V.7) Because $\mathbf{L}^*(\mathbf{K}, \mathbf{c}, \mathbf{w}) > \mathbf{0}_M$ and $\mathbf{I}^*(\mathbf{K}, \mathbf{c}, \mathbf{w}) > \mathbf{0}_N$ are the optimal solution to the primal H-J-B problem (2) for all $(\mathbf{K}, \mathbf{c}, \mathbf{w}) \in \Theta$, they satisfy the following first-order necessary conditions identically:

$$F'_{\mathbf{L}}(\mathbf{L}^*(\mathbf{K},\mathbf{c},\mathbf{w}),\mathbf{K},\mathbf{I}^*(\mathbf{K},\mathbf{c},\mathbf{w})) - \mathbf{w} \equiv \mathbf{0}_M,$$
$$F'_{\mathbf{I}}(\mathbf{L}^*(\mathbf{K},\mathbf{c},\mathbf{w}),\mathbf{K},\mathbf{I}^*(\mathbf{K},\mathbf{c},\mathbf{w})) + V'_{\mathbf{K}}(\mathbf{K},\mathbf{c},\mathbf{w}) \equiv \mathbf{0}_N.$$

Differentiating these identities with respect to $\mathbf{w}$ yields

$$\begin{bmatrix} \underset{M\times M}{F_{\mathbf{LL}}(\mathbf{L}^*(\mathbf{K},\mathbf{c},\mathbf{w}),\mathbf{K},\mathbf{I}^*(\mathbf{K},\mathbf{c},\mathbf{w}))} & \underset{M\times N}{F_{\mathbf{LI}}(\mathbf{L}^*(\mathbf{K},\mathbf{c},\mathbf{w}),\mathbf{K},\mathbf{I}^*(\mathbf{K},\mathbf{c},\mathbf{w}))} \\ \underset{N\times M}{F_{\mathbf{IL}}(\mathbf{L}^*(\mathbf{K},\mathbf{c},\mathbf{w}),\mathbf{K},\mathbf{I}^*(\mathbf{K},\mathbf{c},\mathbf{w}))} & \underset{N\times N}{F_{\mathbf{II}}(\mathbf{L}^*(\mathbf{K},\mathbf{c},\mathbf{w}),\mathbf{K},\mathbf{I}^*(\mathbf{K},\mathbf{c},\mathbf{w}))} \end{bmatrix}$$
$$\times \begin{bmatrix} \underset{M\times M}{\dfrac{\partial \mathbf{L}^*}{\partial \mathbf{w}}(\mathbf{K},\mathbf{c},\mathbf{w})} \\ \underset{N\times M}{\dfrac{\partial \mathbf{I}^*}{\partial \mathbf{w}}(\mathbf{K},\mathbf{c},\mathbf{w})} \end{bmatrix} \equiv \begin{bmatrix} \mathrm{I}_M \\ -\underset{N\times M}{V_{\mathbf{Kw}}(\mathbf{K},\mathbf{c},\mathbf{w})} \end{bmatrix}.$$

Differentiating the above identity form of the first-order necessary conditions again, but now with respect to $\mathbf{c}$, yields

$$\begin{bmatrix} \underset{M\times M}{F_{\mathbf{LL}}(\mathbf{L}^*(\mathbf{K},\mathbf{c},\mathbf{w}),\mathbf{K},\mathbf{I}^*(\mathbf{K},\mathbf{c},\mathbf{w}))} & \underset{M\times N}{F_{\mathbf{LI}}(\mathbf{L}^*(\mathbf{K},\mathbf{c},\mathbf{w}),\mathbf{K},\mathbf{I}^*(\mathbf{K},\mathbf{c},\mathbf{w}))} \\ \underset{N\times M}{F_{\mathbf{IL}}(\mathbf{L}^*(\mathbf{K},\mathbf{c},\mathbf{w}),\mathbf{K},\mathbf{I}^*(\mathbf{K},\mathbf{c},\mathbf{w}))} & \underset{N\times N}{F_{\mathbf{II}}(\mathbf{L}^*(\mathbf{K},\mathbf{c},\mathbf{w}),\mathbf{K},\mathbf{I}^*(\mathbf{K},\mathbf{c},\mathbf{w}))} \end{bmatrix}$$
$$\times \begin{bmatrix} \underset{M\times N}{\dfrac{\partial \mathbf{L}^*}{\partial \mathbf{c}}(\mathbf{K},\mathbf{c},\mathbf{w})} \\ \underset{N\times N}{\dfrac{\partial \mathbf{I}^*}{\partial \mathbf{c}}(\mathbf{K},\mathbf{c},\mathbf{w})} \end{bmatrix} \equiv \begin{bmatrix} \mathbf{0}_{M\times N} \\ -\underset{N\times N}{V_{\mathbf{Kc}}(\mathbf{K},\mathbf{c},\mathbf{w})} \end{bmatrix},$$

where $\mathbf{0}_{M\times N}$ is an $M \times N$ null matrix. We can combine the previous two matrix equations into one expression to get

$$\begin{bmatrix} \underset{M\times M}{F^*_{\mathbf{LL}}} & \underset{M\times N}{F^*_{\mathbf{LI}}} \\ \underset{N\times M}{F^*_{\mathbf{IL}}} & \underset{N\times N}{F^*_{\mathbf{II}}} \end{bmatrix} \begin{bmatrix} \underset{M\times M}{\dfrac{\partial \mathbf{L}^*}{\partial \mathbf{w}}(\mathbf{K},\mathbf{c},\mathbf{w})} & \underset{M\times N}{\dfrac{\partial \mathbf{L}^*}{\partial \mathbf{c}}(\mathbf{K},\mathbf{c},\mathbf{w})} \\ \underset{N\times M}{\dfrac{\partial \mathbf{I}^*}{\partial \mathbf{w}}(\mathbf{K},\mathbf{c},\mathbf{w})} & \underset{N\times N}{\dfrac{\partial \mathbf{I}^*}{\partial \mathbf{c}}(\mathbf{K},\mathbf{c},\mathbf{w})} \end{bmatrix}$$
$$\equiv \begin{bmatrix} \mathrm{I}_M & \mathbf{0}_{M\times N} \\ -\underset{N\times M}{V_{\mathbf{Kw}}(\mathbf{K},\mathbf{c},\mathbf{w})} & -\underset{N\times N}{V_{\mathbf{Kc}}(\mathbf{K},\mathbf{c},\mathbf{w})} \end{bmatrix},$$

where $F^*_{\mathbf{LL}} \stackrel{\text{def}}{=} F_{\mathbf{LL}}(\mathbf{L}^*(\mathbf{K},\mathbf{c},\mathbf{w}),\mathbf{K},\mathbf{I}^*(\mathbf{K},\mathbf{c},\mathbf{w}))$, and similarly for the other terms. Because $F(\cdot)$ is strongly concave in $(\mathbf{L},\mathbf{I})$ by assumption (T.3), or equivalently, the

Hessian matrix of $F(\cdot)$ with respect to $(\mathbf{L}, \mathbf{I})$ is negative definite, the Hessian matrix of $F(\cdot)$ with respect to $(\mathbf{L}, \mathbf{I})$ is invertible. As a result, the above matrix equation can be rewritten as

$$\begin{bmatrix} \underset{M\times M}{\dfrac{\partial \mathbf{L}^*}{\partial \mathbf{w}}(\mathbf{K},\mathbf{c},\mathbf{w})} & \underset{M\times N}{\dfrac{\partial \mathbf{L}^*}{\partial \mathbf{c}}(\mathbf{K},\mathbf{c},\mathbf{w})} \\ \underset{N\times M}{\dfrac{\partial \mathbf{I}^*}{\partial \mathbf{w}}(\mathbf{K},\mathbf{c},\mathbf{w})} & \underset{N\times N}{\dfrac{\partial \mathbf{I}^*}{\partial \mathbf{c}}(\mathbf{K},\mathbf{c},\mathbf{w})} \end{bmatrix}$$

$$\equiv \begin{bmatrix} \underset{M\times M}{F^*_{\mathbf{LL}}} & \underset{M\times N}{F^*_{\mathbf{LI}}} \\ \underset{N\times M}{F^*_{\mathbf{IL}}} & \underset{N\times N}{F^*_{\mathbf{II}}} \end{bmatrix}^{-1} \begin{bmatrix} \mathrm{I}_M & \mathbf{0}_{M\times N} \\ -\underset{N\times M}{V_{\mathbf{Kw}}(\mathbf{K},\mathbf{c},\mathbf{w})} & -\underset{N\times N}{V_{\mathbf{Kc}}(\mathbf{K},\mathbf{c},\mathbf{w})} \end{bmatrix}. \tag{15}$$

Now recall that in proving property (V.5), we showed that $V_{\mathbf{Kc}}(\mathbf{K}, \mathbf{c}, \mathbf{w}) \equiv \boldsymbol{\lambda}^*_{\mathbf{c}}(\mathbf{K}, \mathbf{c}, \mathbf{w})$, which, because of assumption (T.5), implies that $V_{\mathbf{Kc}}(\mathbf{K}, \mathbf{c}, \mathbf{w})$ is nonsingular. The second matrix on the right-hand side in Eq. (15) is thus nonsingular seeing as its determinant equals $(-1)^N |V_{\mathbf{Kc}}(\mathbf{K}, \mathbf{c}, \mathbf{w})|$, which is nonzero because $V_{\mathbf{Kc}}(\mathbf{K}, \mathbf{c}, \mathbf{w})$ is nonsingular. Therefore, each of the matrices on the right-hand side of Eq. (15) is nonsingular. But the product of nonsingular matrices is nonsingular, and thus the matrix

$$\begin{bmatrix} \underset{M\times M}{\dfrac{\partial \mathbf{L}^*}{\partial \mathbf{w}}(\mathbf{K},\mathbf{c},\mathbf{w})} & \underset{M\times N}{\dfrac{\partial \mathbf{L}^*}{\partial \mathbf{c}}(\mathbf{K},\mathbf{c},\mathbf{w})} \\ \underset{N\times M}{\dfrac{\partial \mathbf{I}^*}{\partial \mathbf{w}}(\mathbf{K},\mathbf{c},\mathbf{w})} & \underset{N\times N}{\dfrac{\partial \mathbf{I}^*}{\partial \mathbf{c}}(\mathbf{K},\mathbf{c},\mathbf{w})} \end{bmatrix}$$

is nonsingular for $(\mathbf{K}, \mathbf{c}, \mathbf{w}) \in \Theta$. Hence the proof of part (a) of Theorem 20.1 is complete.

Let us now turn to the proof of part (b) of Theorem 20.1. To that end, assume that $V(\cdot)$ satisfies assumptions (V.1) through (V.7) and define $F(\cdot)$ by Eq. (3). We intend to show that $F(\cdot)$ satisfies properties (T.1) through (T.7). First observe that because of assumptions (V.3) and (V.6), the function $F(\cdot)$ is well defined.

(T.4) Let $(\mathbf{K}_t, \mathbf{c}^\circ, \mathbf{w}^\circ) \in \Theta$. By the definition of $F(\cdot)$, that is to say, Eq. (3), we have that

$$\begin{aligned} F(\mathbf{L}, \mathbf{K}, \mathbf{I}) - \mathbf{w}^{\circ\prime}\mathbf{L} - \mathbf{c}^{\circ\prime}\mathbf{K} &\le rV(\mathbf{K}, \mathbf{c}^\circ, \mathbf{w}^\circ) \\ &\quad - V_{\mathbf{K}}(\mathbf{K}, \mathbf{c}^\circ, \mathbf{w}^\circ)[\mathbf{I} - \delta\mathbf{K}],\ (\mathbf{L}, \mathbf{I}) \in \Phi(\mathbf{K}). \end{aligned} \tag{16}$$

Equation (16) holds with equality if and only if $(\mathbf{L}, \mathbf{I}) = (\mathbf{L}^*(\mathbf{K}, \mathbf{c}^\circ, \mathbf{w}^\circ), \mathbf{I}^*(\mathbf{K}, \mathbf{c}^\circ, \mathbf{w}^\circ))$, in which case, $(\mathbf{c}^\circ, \mathbf{w}^\circ)$ is the optimal solution to problem (3) by assumption (V.6). Therefore, for any finite $T > t$ and any admissible pair, it follows from the inequality in Eq. (16) that

$$\int_t^T [F(\mathbf{L}(s), \mathbf{K}(s), \mathbf{I}(s)) - \mathbf{w}^{\circ\prime}\mathbf{L}(s) - \mathbf{c}^{\circ\prime}\mathbf{K}(s)]\, e^{-r(s-t)}\, ds$$

$$\leq \int_t^T [rV(\mathbf{K}(s), \mathbf{c}^\circ, \mathbf{w}^\circ) - V_{\mathbf{K}}(\mathbf{K}(s), \mathbf{c}^\circ, \mathbf{w}^\circ)\dot{\mathbf{K}}(s)]\, e^{-r(s-t)}\, ds$$

$$= -\int_t^T \frac{d}{ds}\, [e^{-r(s-t)} V(\mathbf{K}(s), \mathbf{c}^\circ, \mathbf{w}^\circ)]\, ds$$

$$= V(\mathbf{K}_t, \mathbf{c}^\circ, \mathbf{w}^\circ) - e^{-r(T-t)} V(\mathbf{K}(T), \mathbf{c}^\circ, \mathbf{w}^\circ),$$

where we have used the fact that the state equation $\dot{\mathbf{K}}(s) = \mathbf{I}(s) - \delta\mathbf{K}(s)$ holds for all admissible pairs. The above inequality shows that the value of all admissible pairs is bounded above by $V(\mathbf{K}_t, \mathbf{c}^\circ, \mathbf{w}^\circ) - e^{-r(T-t)} V(\mathbf{K}(T), \mathbf{c}^\circ, \mathbf{w}^\circ)$. By assumption (V.6), this upper bound is uniquely obtained by setting $(\mathbf{L}, \mathbf{I}) = (\mathbf{L}^*(\mathbf{K}, \mathbf{c}^\circ, \mathbf{w}^\circ), \mathbf{I}^*(\mathbf{K}, \mathbf{c}^\circ, \mathbf{w}^\circ))$, in which case, the above inequality becomes the equality

$$\begin{aligned}\int_t^T &[F(\mathbf{L}^*(\mathbf{K}^*(s), \mathbf{c}^\circ, \mathbf{w}^\circ), \mathbf{K}^*(s), \mathbf{I}^*(\mathbf{K}^*(s), \mathbf{c}^\circ, \mathbf{w}^\circ)) \\ &\quad - \mathbf{w}^{\circ\prime}\mathbf{L}^*(\mathbf{K}^*(s), \mathbf{c}^\circ, \mathbf{w}^\circ) - \mathbf{c}^{\circ\prime}\mathbf{K}^*(s)]\, e^{-r(s-t)}\, ds \\ &= V(\mathbf{K}_t, \mathbf{c}^\circ, \mathbf{w}^\circ) - e^{-r(T-t)} V(\mathbf{K}^*(T), \mathbf{c}^\circ, \mathbf{w}^\circ),\end{aligned} \tag{17}$$

where $\mathbf{K}^*(s)$ is the time path of the capital stock determined by solving the state equation and initial condition using the optimal value of the investment rate, that is, it is the solution to $\dot{\mathbf{K}} = \mathbf{I}^*(\mathbf{K}, \mathbf{c}^\circ, \mathbf{w}^\circ) - \delta\mathbf{K},\ \mathbf{K}(t) = \mathbf{K}_t$. By assumption (V.4), we have $\lim_{T\to+\infty} \mathbf{K}^*(T) = \mathbf{K}^s(\mathbf{c}^\circ, \mathbf{w}^\circ)$, which implies that $\lim_{T\to+\infty} V(\mathbf{K}^*(T), \mathbf{c}^\circ, \mathbf{w}^\circ) = V(\mathbf{K}^s(\mathbf{c}^\circ, \mathbf{w}^\circ), \mathbf{c}^\circ, \mathbf{w}^\circ)$ as $V(\cdot) \in C^{(2)}$ by assumption (V.1). Moreover, $\lim_{T\to+\infty} e^{-r(T-t)} = 0$ because $r > 0$. Using these two results and the fact that the limit of a product of functions is the product of their individual limits when such limits exist, it follows that $\lim_{T\to+\infty} e^{-r(T-t)} V(\mathbf{K}^*(T), \mathbf{c}^\circ, \mathbf{w}^\circ) = 0$. Letting $T \to +\infty$ in Eq. (17) and using this latter result therefore yields

$$\begin{aligned}\int_t^{+\infty} &[F(\mathbf{L}^*(\mathbf{K}^*(s), \mathbf{c}^\circ, \mathbf{w}^\circ), \mathbf{K}^*(s), \mathbf{I}^*(\mathbf{K}^*(s), \mathbf{c}^\circ, \mathbf{w}^\circ)) \\ &\quad - \mathbf{w}^{\circ\prime}\mathbf{L}^*(\mathbf{K}^*(s), \mathbf{c}^\circ, \mathbf{w}^\circ) - \mathbf{c}^{\circ\prime}\mathbf{K}^*(s)]\, e^{-r(s-t)}\, ds = V(\mathbf{K}_t, \mathbf{c}^\circ, \mathbf{w}^\circ).\end{aligned}$$

This demonstrates that $V(\mathbf{K}_t, \mathbf{c}^\circ, \mathbf{w}^\circ)$ is the value of the optimal plan corresponding to the production function $F(\cdot)$. This proves the first part of property (T.4).

To prove the differentiability of the policy functions, recall that by Eqs. (4) through (6) and Theorem 20.2,

$$\mathbf{I}^*(\mathbf{K},\mathbf{c},\mathbf{w}) = V_{\mathbf{cK}}^{-1}(\mathbf{K},\mathbf{c},\mathbf{w})[rV_{\mathbf{c}}'(\mathbf{K},\mathbf{c},\mathbf{w})+\mathbf{K}]+\boldsymbol{\delta}\mathbf{K},$$

$$\mathbf{L}^*(\mathbf{K},\mathbf{c},\mathbf{w}) = -rV_{\mathbf{w}}'(\mathbf{K},\mathbf{c},\mathbf{w})+V_{\mathbf{wK}}(\mathbf{K},\mathbf{c},\mathbf{w})V_{\mathbf{cK}}^{-1}(\mathbf{K},\mathbf{c},\mathbf{w})[rV_{\mathbf{c}}'(\mathbf{K},\mathbf{c},\mathbf{w})+\mathbf{K}],$$

$$\begin{aligned} y^*(\mathbf{K},\mathbf{c},\mathbf{w}) &= rV(\mathbf{K},\mathbf{c},\mathbf{w})+\mathbf{w}'\tilde{\mathbf{L}}(\mathbf{K},\mathbf{c},\mathbf{w})+\mathbf{c}'\mathbf{K}-V_{\mathbf{K}}(\mathbf{K},\mathbf{c},\mathbf{w})[\tilde{\mathbf{I}}(\mathbf{K},\mathbf{c},\mathbf{w})-\boldsymbol{\delta}\mathbf{K}] \\ &= r[V(\mathbf{K},\mathbf{c},\mathbf{w})-V_{\mathbf{w}}(\mathbf{K},\mathbf{c},\mathbf{w})\mathbf{w}-V_{\mathbf{c}}(\mathbf{K},\mathbf{c},\mathbf{w})\mathbf{c}] \\ &\quad -[V_{\mathbf{K}}(\mathbf{K},\mathbf{c},\mathbf{w})-\mathbf{w}'V_{\mathbf{wK}}(\mathbf{K},\mathbf{c},\mathbf{w})-\mathbf{c}'V_{\mathbf{cK}}(\mathbf{K},\mathbf{c},\mathbf{w})] \\ &\quad \times \left[V_{\mathbf{cK}}^{-1}(\mathbf{K},\mathbf{c},\mathbf{w})[rV_{\mathbf{c}}'(\mathbf{K},\mathbf{c},\mathbf{w})+\mathbf{K}]\right]. \end{aligned}$$

Because $V(\cdot)\in C^{(2)}$ and $V_{\mathbf{K}}(\cdot)\in C^{(2)}$ on Θ by assumption (V.1), and $\dot{\mathbf{K}}^*(\mathbf{K},\mathbf{c},\mathbf{w}) = \mathbf{I}(\mathbf{K},\mathbf{c},\mathbf{w})-\boldsymbol{\delta}\mathbf{K}$, inspection of the above formulae shows that $(\dot{\mathbf{K}}^*(\cdot), \mathbf{L}^*(\cdot), y^*(\cdot)) \in C^{(1)}$ on Θ. To prove the differentiability of the current value shadow price function, first recall that $V_{\mathbf{K}}'(\mathbf{K},\mathbf{c},\mathbf{w}) \equiv \boldsymbol{\lambda}^*(\mathbf{K},\mathbf{c},\mathbf{w})$ by the principle of optimality and the dynamic envelope theorem. Then, because $V_{\mathbf{K}}(\cdot)\in C^{(2)}$ on Θ by assumption (V.1), it immediately follows that $\boldsymbol{\lambda}(\cdot)\in C^{(2)}$ on Θ.

(T.5) By assumption (V.5), $V_{\mathbf{cK}}(\mathbf{K},\mathbf{c},\mathbf{w})$ is nonsingular for each $(\mathbf{K},\mathbf{c},\mathbf{w})\in\Theta$, thus so is its transpose $V_{\mathbf{cK}}'(\mathbf{K},\mathbf{c},\mathbf{w})$. Given that $V_{\mathbf{Kc}}(\mathbf{K},\mathbf{c},\mathbf{w}) \equiv \boldsymbol{\lambda}_{\mathbf{c}}^*(\mathbf{K},\mathbf{c},\mathbf{w})$ and $V_{\mathbf{Kc}}(\mathbf{K},\mathbf{c},\mathbf{w}) \equiv V_{\mathbf{cK}}'(\mathbf{K},\mathbf{c},\mathbf{w})$, as we showed in the proof of property (V.5), it therefore follows that $\boldsymbol{\lambda}_{\mathbf{c}}^*(\mathbf{K},\mathbf{c},\mathbf{w})$ is nonsingular for each $(\mathbf{K},\mathbf{c},\mathbf{w})\in\Theta$.

(T.6) Let $(\mathbf{L}^\circ,\mathbf{K}_t,\mathbf{I}^\circ)\in\Phi$, and let $(\mathbf{c}^\circ,\mathbf{w}^\circ)\in\Theta(\mathbf{K}_t)$ be optimal in problem (3). In the proof of property (T.4), we showed that given $(\mathbf{c}^\circ,\mathbf{w}^\circ)\in\Theta(\mathbf{K}_t)$, $(\mathbf{L}^\circ,\mathbf{I}^\circ) = (\mathbf{L}^*(\mathbf{K},\mathbf{c}^\circ,\mathbf{w}^\circ), \mathbf{I}^*(\mathbf{K},\mathbf{c}^\circ,\mathbf{w}^\circ))$, where $\mathbf{K}=\mathbf{K}^*(s)$ is the solution to $\dot{\mathbf{K}} = \mathbf{I}^*(\mathbf{K},\mathbf{c}^\circ,\mathbf{w}^\circ)-\boldsymbol{\delta}\mathbf{K}$, $\mathbf{K}(t)=\mathbf{K}_t$, is the solution to optimal control problem (1). It is therefore optimal in problem (1) in the base period $s=t$, which is what we wished to demonstrate.

(T.7) Property (T.7) is simply a restatement of assumption (V.4), for they are both asserting the global asymptotic stability of the steady state capital stock.

(T.1) and (T.2) We have yet to prove the differentiability of $F(\cdot)$, though we did note that it is well defined prior to starting the proof of property (T.4). Consider, therefore, the following system of $M+N$ equations:

$$\mathbf{L} = \mathbf{L}^*(\mathbf{K},\mathbf{c},\mathbf{w}), \tag{18}$$

$$\mathbf{I} = \mathbf{I}^*(\mathbf{K},\mathbf{c},\mathbf{w}). \tag{19}$$

The Jacobian of Eqs. (18) and (19) is given by the $(M+N)\times(M+N)$ matrix

$$\begin{bmatrix} \dfrac{\partial\mathbf{L}^*}{\partial\mathbf{w}}(\mathbf{K},\mathbf{c},\mathbf{w}) & \dfrac{\partial\mathbf{L}^*}{\partial\mathbf{c}}(\mathbf{K},\mathbf{c},\mathbf{w}) \\ \dfrac{\partial\mathbf{I}^*}{\partial\mathbf{w}}(\mathbf{K},\mathbf{c},\mathbf{w}) & \dfrac{\partial\mathbf{I}^*}{\partial\mathbf{c}}(\mathbf{K},\mathbf{c},\mathbf{w}) \end{bmatrix}.$$

By assumption (V.7), this matrix is nonsingular for $(\mathbf{K}, \mathbf{c}, \mathbf{w}) \in \Theta$. Thus, by the implicit function theorem and the fact that $(\mathbf{I}^*(\cdot), \mathbf{L}^*(\cdot)) \in C^{(1)}$ on Θ, the solution $(\mathbf{c}, \mathbf{w}) = (\mathbf{c}^*(\mathbf{L}, \mathbf{K}, \mathbf{I}), \mathbf{w}^*(\mathbf{L}, \mathbf{K}, \mathbf{I}))$ to Eqs. (18) and (19) is locally well defined and $C^{(1)}$ on Φ. Moreover, by assumption (V.6), for $(\mathbf{K}, \mathbf{c}, \mathbf{w}) \in \Theta$, the minimum of problem (3) is attained at $(\mathbf{c}, \mathbf{w}) = (\mathbf{c}^*(\mathbf{L}, \mathbf{K}, \mathbf{I}), \mathbf{w}^*(\mathbf{L}, \mathbf{K}, \mathbf{I}))$ when $(\mathbf{L}, \mathbf{I}) = (\mathbf{L}^*(\mathbf{K}, \mathbf{c}, \mathbf{w}), \mathbf{I}^*(\mathbf{K}, \mathbf{c}, \mathbf{w}))$.

Given that $F(\cdot)$ is defined by problem (3), the above facts applied to problem (3) imply that $F(\cdot) \in C^{(1)}$ on Φ. To see this, apply the static envelope theorem to problem (3) to get

$$F'_{\mathbf{L}}(\mathbf{L}, \mathbf{K}, \mathbf{I}) = \mathbf{w}^*(\mathbf{L}, \mathbf{K}, \mathbf{I}) > \mathbf{0}_M, \tag{20}$$

$$\begin{aligned} F'_{\mathbf{K}}(\mathbf{L}, \mathbf{K}, \mathbf{I}) = {} & (r\mathrm{I}_N + \delta)V'_{\mathbf{K}}(\mathbf{K}, \mathbf{c}^*(\mathbf{L}, \mathbf{K}, \mathbf{I}), \mathbf{w}^*(\mathbf{L}, \mathbf{K}, \mathbf{I})) + \mathbf{c}^*(\mathbf{L}, \mathbf{K}, \mathbf{I}) \\ & - V_{\mathbf{KK}}(\mathbf{K}, \mathbf{c}^*(\mathbf{L}, \mathbf{K}, \mathbf{I}), \mathbf{w}^*(\mathbf{L}, \mathbf{K}, \mathbf{I}))[\mathbf{I} - \delta\mathbf{K}] > \mathbf{0}_N, \end{aligned} \tag{21}$$

$$F'_{\mathbf{I}}(\mathbf{L}, \mathbf{K}, \mathbf{I}) = -V'_{\mathbf{K}}(\mathbf{K}, \mathbf{c}^*(\mathbf{L}, \mathbf{K}, \mathbf{I}), \mathbf{w}^*(\mathbf{L}, \mathbf{K}, \mathbf{I})) < \mathbf{0}_N, \tag{22}$$

where we have used assumption (V.2) and the basic assumption that $\mathbf{w}^*(\mathbf{L}, \mathbf{K}, \mathbf{I}) \in \Re^M_{++}$ in order to sign the gradients. The inequalities in Eqs. (20) through (22) prove property (T.2). Because $(\mathbf{c}^*(\cdot), \mathbf{w}^*(\cdot)) \in C^{(1)}$ on Φ, as established above, and $V_{\mathbf{K}}(\cdot) \in C^{(2)}$ by assumption (V.1), inspection of Eqs. (20) through (22) implies that $F(\cdot) \in C^{(1)}$, $F_{\mathbf{L}}(\cdot) \in C^{(1)}$, and $F_{\mathbf{I}}(\cdot) \in C^{(1)}$ on Φ. This completes the proof of property (T.1).

(T.3) Application of the primal-dual method to problem (3) readily establishes that $F(\cdot)$ is concave in $(\mathbf{L}, \mathbf{I})$, the details of which you are asked to provide in a mental exercise. To prove the strong concavity of $F(\cdot)$ with respect to $(\mathbf{L}, \mathbf{I})$, we will show that its Hessian matrix with respect to $(\mathbf{L}, \mathbf{I})$ is nonsingular. To this end, differentiate Eqs. (20) and (22) with respect to $\mathbf{L}$ to get

$$F_{\mathbf{LL}}(\mathbf{L}, \mathbf{K}, \mathbf{I}) = \frac{\partial \mathbf{w}^*}{\partial \mathbf{L}},$$

$$\begin{aligned} F_{\mathbf{IL}}(\mathbf{L}, \mathbf{K}, \mathbf{I}) = {} & -V_{\mathbf{Kc}}(\mathbf{K}, \mathbf{c}^*(\mathbf{L}, \mathbf{K}, \mathbf{I}), \mathbf{w}^*(\mathbf{L}, \mathbf{K}, \mathbf{I})) \frac{\partial \mathbf{c}^*}{\partial \mathbf{L}} \\ & - V_{\mathbf{Kw}}(\mathbf{K}, \mathbf{c}^*(\mathbf{L}, \mathbf{K}, \mathbf{I}), \mathbf{w}^*(\mathbf{L}, \mathbf{K}, \mathbf{I})) \frac{\partial \mathbf{w}^*}{\partial \mathbf{L}}. \end{aligned}$$

Differentiating Eqs. (20) and (22) with respect to $\mathbf{I}$ this time yields

$$F_{\mathbf{LI}}(\mathbf{L}, \mathbf{K}, \mathbf{I}) = \frac{\partial \mathbf{w}^*}{\partial \mathbf{I}},$$

$$\begin{aligned} F_{\mathbf{II}}(\mathbf{L}, \mathbf{K}, \mathbf{I}) = {} & -V_{\mathbf{Kc}}(\mathbf{K}, \mathbf{c}^*(\mathbf{L}, \mathbf{K}, \mathbf{I}), \mathbf{w}^*(\mathbf{L}, \mathbf{K}, \mathbf{I})) \frac{\partial \mathbf{c}^*}{\partial \mathbf{I}} \\ & - V_{\mathbf{Kw}}(\mathbf{K}, \mathbf{c}^*(\mathbf{L}, \mathbf{K}, \mathbf{I}), \mathbf{w}^*(\mathbf{L}, \mathbf{K}, \mathbf{I})) \frac{\partial \mathbf{w}^*}{\partial \mathbf{I}}. \end{aligned}$$

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The previous four matrix equations may be combined to form one matrix equation, namely,

$$\begin{bmatrix} F_{\mathbf{LL}}(\mathbf{L},\mathbf{K},\mathbf{I}) & F_{\mathbf{LI}}(\mathbf{L},\mathbf{K},\mathbf{I}) \\ F_{\mathbf{IL}}(\mathbf{L},\mathbf{K},\mathbf{I}) & F_{\mathbf{II}}(\mathbf{L},\mathbf{K},\mathbf{I}) \end{bmatrix} = \begin{bmatrix} \mathrm{I}_M & \mathbf{0}_{M\times N} \\ -V^*_{\mathbf{Kw}} & -V^*_{\mathbf{Kc}} \end{bmatrix} \begin{bmatrix} \dfrac{\partial \mathbf{w}^*}{\partial \mathbf{L}} & \dfrac{\partial \mathbf{w}^*}{\partial \mathbf{I}} \\ \dfrac{\partial \mathbf{c}^*}{\partial \mathbf{L}} & \dfrac{\partial \mathbf{c}^*}{\partial \mathbf{I}} \end{bmatrix}, \tag{23}$$

where $V^*_{\mathbf{Kw}} \stackrel{\text{def}}{=} V_{\mathbf{Kw}}(\mathbf{K}, \mathbf{c}^*(\mathbf{L},\mathbf{K},\mathbf{I}), \mathbf{w}^*(\mathbf{L},\mathbf{K},\mathbf{I}))$ and similarly for the other term. Next, differentiate the identity form of Eqs. (18) and (19) with respect to $\mathbf{L}$ and $\mathbf{I}$, and then similarly combine them into a single matrix equation. This process yields the matrix equation

$$\begin{bmatrix} \mathrm{I}_M & \mathbf{0}_{M\times N} \\ \mathbf{0}_{N\times M} & \mathrm{I}_N \end{bmatrix} = \begin{bmatrix} \dfrac{\partial \mathbf{L}^*}{\partial \mathbf{w}} & \dfrac{\partial \mathbf{L}^*}{\partial \mathbf{c}} \\ \dfrac{\partial \mathbf{I}^*}{\partial \mathbf{w}} & \dfrac{\partial \mathbf{I}^*}{\partial \mathbf{c}} \end{bmatrix} \begin{bmatrix} \dfrac{\partial \mathbf{w}^*}{\partial \mathbf{L}} & \dfrac{\partial \mathbf{w}^*}{\partial \mathbf{I}} \\ \dfrac{\partial \mathbf{c}^*}{\partial \mathbf{L}} & \dfrac{\partial \mathbf{c}^*}{\partial \mathbf{I}} \end{bmatrix}, \tag{24}$$

a result you are asked to prove in a mental exercise. Seeing as the $(M+N)\times(M+N)$ identity matrix on the left-hand side of Eq. (24) is nonsingular, both of the matrices on the right-hand side of Eq. (24) are nonsingular too. As $V_{\mathbf{cK}}(\mathbf{K},\mathbf{c},\mathbf{w})$ is nonsingular for each $(\mathbf{K},\mathbf{c},\mathbf{w}) \in \Theta$ by assumption (V.5), and $V_{\mathbf{Kc}}(\mathbf{K},\mathbf{c},\mathbf{w}) \equiv V'_{\mathbf{cK}}(\mathbf{K},\mathbf{c},\mathbf{w})$ as shown in the proof of property (V.5), $V_{\mathbf{Kc}}(\mathbf{K},\mathbf{c},\mathbf{w})$ is nonsingular for each $(\mathbf{K},\mathbf{c},\mathbf{w}) \in \Theta$ as well because the transpose of a nonsingular matrix is nonsingular. Thus *both* matrices on the right-hand side of Eq. (23) are nonsingular, thereby implying that the Hessian matrix of $F(\cdot)$ with respect to $(\mathbf{L},\mathbf{I})$ on the left-hand side of Eq. (23) is nonsingular too.

To complete the proof of part (b), use $F(\cdot)$ to define $V^*(\cdot)$ by way of Eq. (1). We showed in the proof of property (T.4) above that $V^*(\cdot) \equiv V(\cdot)$. Hence the proofs of Theorems 20.1 and 20.2 are complete. Q.E.D.

Condition (V.6) is central in the above development of intertemporal duality in that it asserts the existence of a solution to the dual problem (3). To get a better feel for it, note that it could have been expressed in the following equivalent manner:

(V.6′) Given $(\mathbf{K},\mathbf{c}^\circ,\mathbf{w}^\circ) \in \Theta$, there exists $(\mathbf{I}^\circ,\mathbf{L}^\circ) \in \Phi(\mathbf{K})$ such that $(\mathbf{c}^\circ,\mathbf{w}^\circ)$ is optimal in problem (3) given $(\mathbf{L}^\circ,\mathbf{K},\mathbf{I}^\circ)$.

Stated in this form, it should be apparent that property (V.6) is dual to property (T.6), for the two are worded in essentially symmetrical ways. Another interpretation of property (V.6) is that it requires that the first-order necessary conditions be sufficient for a global minimum in problem (3) over $\Theta(\mathbf{K})$. This is clearly a curvature restriction in the sense that the attainment of a global minimum implies that second-order necessary conditions hold at the optimum, and as you know from your prior work in microeconomic theory, such second-order necessary conditions are equivalent to the local convexity of the objective function.

Note, in passing, that an alternative derivation of the policy function formulas in Theorem 20.2 is given by differentiating the identity form of problem (2) and invoking the static envelope theorem, since problem (2) is a static optimization problem. The derivation of the policy function formulae via this route is left for a mental exercise.

We close our theoretical discussion of intertemporal duality by looking more closely at the monotonicity and curvature properties of the current value optimal value function $V(\cdot)$ defined in problem (1).

Theorem 20.3: *Let* $V(\cdot)$ *be defined by problem* (1) *and let* $F(\cdot)$ *satisfy conditions (T.1) through (T.7). Then* $V(\cdot)$ *is*

(a) increasing in $\mathbf{K}_t$*,*
(b) decreasing in $\mathbf{c}$*,*
(c) decreasing in $\mathbf{w}$*,*
(d) convex in $(\mathbf{c}, \mathbf{w}) \in \Theta(\mathbf{K})$*,*
(e) concave in $\mathbf{K}_t$ *if* $F(\cdot)$ *is concave in* $(\mathbf{L}, \mathbf{K}, \mathbf{I}) \in \Phi$.
(f) For any $(\mathbf{K}, \mathbf{c}^\circ, \mathbf{w}^\circ) \in \Theta$*, the function* $Z(\cdot)$ *defined by*

$$Z(\mathbf{c}, \mathbf{w}) \stackrel{\text{def}}{=} rV(\mathbf{K}, \mathbf{c}, \mathbf{w}) - V_{\mathbf{K}}(\mathbf{K}, \mathbf{c}, \mathbf{w})[\mathbf{I}^*(\mathbf{K}, \mathbf{c}^\circ, \mathbf{w}^\circ) - \delta\mathbf{K}]$$

is convex in $(\mathbf{c}, \mathbf{w})$ *locally around* $(\mathbf{c}^\circ, \mathbf{w}^\circ)$.

Proof: There are a few ways to go about proving some of the parts of this theorem. We will adopt one strategy here and ask you to provide an alternative proof of some parts of the theorem in a mental exercise.
(a) We already proved this property of $V(\cdot)$ in proving property (V.2) part (ii) of Theorem 20.1(a). You are asked to prove it by way of the dynamic envelope theorem and the Maximum Principle applied to problem (1) in a mental exercise.
(b) Let $\mathbf{c}^1 \leq \mathbf{c}^2$, where the inequality $\leq$ for vectors $\mathbf{a}, \mathbf{b} \in \Re^n$ is defined as follows: $\mathbf{a} \leq \mathbf{b}$ if and only if $a_\ell \leq b_\ell$, $\ell = 1, 2, \ldots, n$, and $a_k < b_k$ for at least one index $k \in \{1, 2, \ldots, n\}$. Let the triplet $(\mathbf{L}^i(s), \mathbf{K}^i(s), \mathbf{I}^i(s))$ be optimal for $\mathbf{c} = \mathbf{c}^i$ and $\mathbf{K}^i(t) = \mathbf{K}_t$, $i = 1, 2$. Then we have the following string of identities and inequalities, the explanation of each being given below each step in the proof:

$$V(\mathbf{K}_t, \mathbf{c}^2, \mathbf{w}) \equiv \int_t^{+\infty} [F(\mathbf{L}^2(s), \mathbf{K}^2(s), \mathbf{I}^2(s)) - \mathbf{w}'\mathbf{L}^2(s) - \mathbf{c}^{2\prime}\mathbf{K}^2(s)]\, e^{-r(s-t)}\, ds$$

by the definition of the optimal value function $V(\cdot)$ in problem (1)

$$< \int_t^{+\infty} [F(\mathbf{L}^2(s), \mathbf{K}^2(s), \mathbf{I}^2(s)) - \mathbf{w}'\mathbf{L}^2(s) - \mathbf{c}^{1\prime}\mathbf{K}^2(s)]\, e^{-r(s-t)}\, ds$$

the integrand is a strictly decreasing function of $\mathbf{c}$, since the optimal triplet is interior and $\mathbf{c}^1 \leq \mathbf{c}^2$

$$\le \underbrace{\int_t^{+\infty} [F(\mathbf{L}^1(s), \mathbf{K}^1(s), \mathbf{I}^1(s)) - \mathbf{w}'\mathbf{L}^1(s) - \mathbf{c}^{1\prime}\mathbf{K}^1(s)]\, e^{-r(s-t)}\, ds}_{(\mathbf{L}^2(s),\mathbf{K}^2(s),\mathbf{I}^2(s)) \text{ is not necessarily optimal for } \mathbf{c}=\mathbf{c}^1, \text{ but } (\mathbf{L}^1(s),\mathbf{K}^1(s),\mathbf{I}^1(s)) \text{ is}}$$

$$\underset{\text{definition of } V(\cdot) \text{ in Eq. (1)}}{\equiv} V(\mathbf{K}_t, \mathbf{c}^1, \mathbf{w}).$$

That is, $V(\mathbf{K}_t, \mathbf{c}^1, \mathbf{w}) > V(\mathbf{K}_t, \mathbf{c}^2, \mathbf{w})$ for $\mathbf{c}^1 \le \mathbf{c}^2$, which is what we set out to prove. You are asked to provide another proof of this result via the dynamic envelope theorem in a mental exercise.

(c) The proof of this part of Theorem 20.3 is left for a mental exercise in which you are asked to prove it in two different ways, one as in the proof of part (b), and the other via the dynamic envelope theorem.

(d) Consider price vectors $(\mathbf{c}^i, \mathbf{w}^i)$, $i = 1, 2$, and $(\mathbf{c}^\omega, \mathbf{w}^\omega) \stackrel{\text{def}}{=} \omega(\mathbf{c}^1, \mathbf{w}^1) + [1 - \omega](\mathbf{c}^2, \mathbf{w}^2)$, where $\omega \in [0, 1]$. As you should recall, the vector $(\mathbf{c}^\omega, \mathbf{w}^\omega)$ is called a *convex combination* of the vectors $(\mathbf{c}^1, \mathbf{w}^1)$ and $(\mathbf{c}^2, \mathbf{w}^2)$. Let the optimal time paths corresponding to these three price vectors be given by $(\mathbf{L}^j(s), \mathbf{K}^j(s), \mathbf{I}^j(s))$ for $\mathbf{K}^j(t) = \mathbf{K}_t$ and $j = 1, 2, \omega$. Then we have the following string of identities and inequalities, the explanation of each again being given below each step in the proof:

$$V(\mathbf{K}_t, \mathbf{c}^\omega, \mathbf{w}^\omega)$$

$$\equiv \underbrace{\int_t^{+\infty} [F(\mathbf{L}^\omega(s), \mathbf{K}^\omega(s), \mathbf{I}^\omega(s)) - \mathbf{w}^{\omega\prime}\mathbf{L}^\omega(s) - \mathbf{c}^{\omega\prime}\mathbf{K}^\omega(s)]\, e^{-r(s-t)}\, ds}_{\text{by the definition of the optimal value function } V(\cdot) \text{ in problem (1)}}$$

$$\equiv \omega \int_t^{+\infty} [F(\mathbf{L}^\omega(s), \mathbf{K}^\omega(s), \mathbf{I}^\omega(s)) - \mathbf{w}^{1\prime}\mathbf{L}^\omega(s) - \mathbf{c}^{1\prime}\mathbf{K}^\omega(s)]\, e^{-r(s-t)}\, ds$$

$$+ \underbrace{[1-\omega] \int_t^{+\infty} [F(\mathbf{L}^\omega(s), \mathbf{K}^\omega(s), \mathbf{I}^\omega(s)) - \mathbf{w}^{2\prime}\mathbf{L}^\omega(s) - \mathbf{c}^{2\prime}\mathbf{K}^\omega(s)]\, e^{-r(s-t)}\, ds}_{\text{the definition } (\mathbf{c}^\omega,\mathbf{w}^\omega) \stackrel{\text{def}}{=} \omega(\mathbf{c}^1,\mathbf{w}^1)+[1-\omega](\mathbf{c}^2,\mathbf{w}^2),\, \omega\in[0,1], \text{ was used to rewrite the integrand}}$$

$$\le \omega \int_t^{+\infty} [F(\mathbf{L}^1(s), \mathbf{K}^1(s), \mathbf{I}^1(s)) - \mathbf{w}^{1\prime}\mathbf{L}^1(s) - \mathbf{c}^{1\prime}\mathbf{K}^1(s)]\, e^{-r(s-t)}\, ds$$

$$+ \underbrace{[1-\omega] \int_t^{+\infty} [F(\mathbf{L}^2(s), \mathbf{K}^2(s), \mathbf{I}^2(s)) - \mathbf{w}^{2\prime}\mathbf{L}^2(s) - \mathbf{c}^{2\prime}\mathbf{K}^2(s)]\, e^{-r(s-t)}\, ds}_{(\mathbf{L}^\omega(s),\mathbf{K}^\omega(s),\mathbf{I}^\omega(s)) \text{ is not necessarily optimal for } (\mathbf{c}^i,\mathbf{w}^i),\, i=1,2, \text{ but } (\mathbf{L}^i(s),\mathbf{K}^i(s),\mathbf{I}^i(s)),\, i=1,2, \text{ is}}$$

$$\underset{\text{by definition of the optimal value function } V(\cdot) \text{ in Eq. (1)}}{\equiv \omega V(\mathbf{K}_t, \mathbf{c}^1, \mathbf{w}^1) + [1-\omega] V(\mathbf{K}_t, \mathbf{c}^2, \mathbf{w}^2)}.$$

In other words, we have shown that

$$V(\mathbf{K}_t, \omega\mathbf{c}^1 + [1-\omega]\mathbf{c}^2, \omega\mathbf{w}^1 + [1-\omega]\mathbf{w}^2) \le \omega V(\mathbf{K}_t, \mathbf{c}^1, \mathbf{w}^1) + [1-\omega]V(\mathbf{K}_t, \mathbf{c}^2, \mathbf{w}^2),$$

which is the definition for $V(\cdot)$ to be convex in $(\mathbf{c}, \mathbf{w}) \in \Theta(\mathbf{K})$. Note that this result can also be established by applying the dynamic primal-dual formalism to problem (1), as you are asked to show in a mental exercise. Note that this proof does not rely on the fact that $V(\cdot) \in C^{(2)}$ on Θ, whereas the proof by way of the dynamic envelope theorem does.

(e) Let $(\mathbf{L}^i(s), \mathbf{K}^i(s), \mathbf{I}^i(s))$ be the optimal triplet for $\mathbf{K}^i(t) = \mathbf{K}_t^i$, $i = 1, 2$. Define the vector $\mathbf{K}_t^\omega \stackrel{\text{def}}{=} \omega\mathbf{K}_t^1 + [1-\omega]\mathbf{K}_t^2$, where $\omega \in [0, 1]$, and let $(\mathbf{L}^\omega(s), \mathbf{K}^\omega(s), \mathbf{I}^\omega(s))$ be the optimal triplet for $\mathbf{K}^\omega(t) = \mathbf{K}_t^\omega$. To begin, we must first establish that the triplet

$$\omega(\mathbf{L}^1(s), \mathbf{K}^1(s), \mathbf{I}^1(s)) + [1-\omega](\mathbf{L}^2(s), \mathbf{K}^2(s), \mathbf{I}^2(s)),\ \ \omega \in [0, 1]$$

is an admissible solution when the initial capital stock is $\mathbf{K}_t^\omega \stackrel{\text{def}}{=} \omega\mathbf{K}_t^1 + [1-\omega]\mathbf{K}_t^2$. Recall that in order to demonstrate admissibility, we must verify that the proposed control vector satisfies any constraints placed on it and that the proposed state and control curves satisfy the state equation and given initial condition. The only constraint on the values of the state and control variables, which is implicit, is that they are positive. This constraint is satisfied by the above triplet because it is a convex combination of optimal curves. Next, recall that $(\mathbf{K}^i(s), \mathbf{I}^i(s))$, $i = 1, 2$, satisfy the state equation by their virtue of being optimal for $\mathbf{K}^i(t) = \mathbf{K}_t^i$, $i = 1, 2$. Using this observation, we therefore have that

$$\begin{aligned}
&\frac{d}{ds}[\omega\mathbf{K}^1(s) + [1-\omega]\mathbf{K}^2(s)] \\
&\quad = \omega\dot{\mathbf{K}}^1(s) + [1-\omega]\dot{\mathbf{K}}^2(s) = \omega[\mathbf{I}^1(s) - \boldsymbol{\delta}\mathbf{K}^1(s)] + [1-\omega][\mathbf{I}^2(s) - \boldsymbol{\delta}\mathbf{K}^2(s)] \\
&\quad = [\omega\mathbf{I}^1(s) + [1-\omega]\mathbf{I}^2(s)] - \boldsymbol{\delta}[\omega\mathbf{K}^1(s) + [1-\omega]\mathbf{K}^2(s)],
\end{aligned}$$

which demonstrates that the above triplet satisfies the state equation. Finally, we have that

$$\omega\mathbf{K}^1(t) + [1-\omega]\mathbf{K}^2(t) = \omega\mathbf{K}_t^1 + [1-\omega]\mathbf{K}_t^2 \stackrel{\text{def}}{=} \mathbf{K}_t^\omega$$

in view of the fact that $\mathbf{K}^i(t) = \mathbf{K}_t^i$, $i = 1, 2$. This shows that the initial condition is also satisfied by the above triplet and thus completes the proof of admissibility. Note, in passing, that we did not have to perform the verification of admissibility in the proof of part (d) because the state equation and initial condition are independent of the price vectors $\mathbf{c}$ and $\mathbf{w}$.

To complete the proof, we have the following string of identities and inequalities, the explanation of each being given below each step in the proof:

$$V(\mathbf{K}_t^\omega, \mathbf{c}, \mathbf{w})$$

$$\equiv \int_t^{+\infty} [F(\mathbf{L}^\omega(s), \mathbf{K}^\omega(s), \mathbf{I}^\omega(s)) - \mathbf{w}'\mathbf{L}^\omega(s) - \mathbf{c}'\mathbf{K}^\omega(s)]\, e^{-r(s-t)}\, ds$$

by the definition of the optimal value function $V(\cdot)$ in problem (1)

$$\geq \int_t^{+\infty} [F(\omega(\mathbf{L}^1(s), \mathbf{K}^1(s), \mathbf{I}^1(s)) + [1-\omega](\mathbf{L}^2(s), \mathbf{K}^2(s), \mathbf{I}^2(s)))]\, e^{-r(s-t)}\, ds$$

$$- \int_t^{+\infty} [\mathbf{w}'(\omega\mathbf{L}^1(s) + [1-\omega]\mathbf{L}^2(s)) + \mathbf{c}'(\omega\mathbf{K}^1(s) + [1-\omega]\mathbf{K}^2(s))]\, e^{-r(s-t)}\, ds$$

$\omega(\mathbf{L}^1(s),\mathbf{K}^1(s),\mathbf{I}^1(s))+[1-\omega](\mathbf{L}^2(s),\mathbf{K}^2(s),\mathbf{I}^2(s))$ is admissible but not necessarily optimal for $\mathbf{K}_t^\omega \stackrel{\text{def}}{=} \omega\mathbf{K}_t^1+[1-\omega]\mathbf{K}_t^2$

$$\geq \omega \int_t^{+\infty} [F(\mathbf{L}^1(s), \mathbf{K}^1(s), \mathbf{I}^1(s)) - \mathbf{w}'\mathbf{L}^1(s) - \mathbf{c}'\mathbf{K}^1(s)]\, e^{-r(s-t)}\, ds$$

$$+ [1-\omega] \int_t^{+\infty} [F(\mathbf{L}^2(s), \mathbf{K}^2(s), \mathbf{I}^2(s)) - \mathbf{w}'\mathbf{L}^2(s) - \mathbf{c}'\mathbf{K}^2(s)]\, e^{-r(s-t)}\, ds$$

follows from the assumed concavity of $F(\cdot)$ in $(\mathbf{L},\mathbf{K},\mathbf{I})$

$$\equiv \omega V\left(\mathbf{K}_t^1, \mathbf{c}, \mathbf{w}\right) + [1-\omega] V\left(\mathbf{K}_t^2, \mathbf{c}, \mathbf{w}\right).$$

by definition of the optimal value function $V(\cdot)$ in Eq. (1)

In other words, we have shown that

$$V\left(\omega\mathbf{K}_t^1 + [1-\omega]\mathbf{K}_t^2, \mathbf{c}, \mathbf{w}\right) \geq \omega V\left(\mathbf{K}_t^1, \mathbf{c}, \mathbf{w}\right) + [1-\omega] V\left(\mathbf{K}_t^2, \mathbf{c}, \mathbf{w}\right),$$

which is the definition for $V(\cdot)$ to be concave in $\mathbf{K}_t$.

(f) First recall the dual optimization problem (3), that is,

$$F(\mathbf{L}, \mathbf{K}, \mathbf{I}) = \min_{(\mathbf{c},\mathbf{w})\in\Theta(\mathbf{K})} \{rV(\mathbf{K}, \mathbf{c}, \mathbf{w}) + \mathbf{w}'\mathbf{L} + \mathbf{c}'\mathbf{K} - V_{\mathbf{K}}(\mathbf{K}, \mathbf{c}, \mathbf{w})[\mathbf{I} - \delta\mathbf{K}]\},$$
$$(\mathbf{L}, \mathbf{K}, \mathbf{I}) \in \Phi.$$

The function on the right-hand side that is to be minimized, namely, the Hamiltonian, is a $C^{(2)}$ function of $(\mathbf{c}, \mathbf{w}) \in \Theta(\mathbf{K})$ by Theorem 20.1. If we set

$$(\mathbf{L}, \mathbf{I}) = (\mathbf{L}^\circ, \mathbf{I}^\circ) = (\mathbf{L}^*(\mathbf{K}, \mathbf{c}^\circ, \mathbf{w}^\circ), \mathbf{I}^*(\mathbf{K}, \mathbf{c}^\circ, \mathbf{w}^\circ)), (\mathbf{c}^\circ, \mathbf{w}^\circ) \in \Theta(\mathbf{K}),$$

in problem (3), then by Theorem 20.1, it follows that the solution of problem (3) occurs at $(\mathbf{c}^\circ, \mathbf{w}^\circ)$. Because $Z(\mathbf{c}, \mathbf{w}) \stackrel{\text{def}}{=} rV(\mathbf{K}, \mathbf{c}, \mathbf{w}) - V_{\mathbf{K}}(\mathbf{K}, \mathbf{c}, \mathbf{w})[\mathbf{I}^*(\mathbf{K}, \mathbf{c}^\circ, \mathbf{w}^\circ) - \delta\mathbf{K}]$, we can rewrite problem (3) in the form

$$F(\mathbf{L}^\circ, \mathbf{K}, \mathbf{I}^\circ) = \min_{(\mathbf{c},\mathbf{w})\in\Theta(\mathbf{K})} \{Z(\mathbf{c}, \mathbf{w}) + \mathbf{w}'\mathbf{L}^\circ + \mathbf{c}'\mathbf{K}^\circ\},\ (\mathbf{L}^\circ, \mathbf{K}, \mathbf{I}^\circ) \in \Phi.$$

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Inasmuch as this problem is an unconstrained static minimization problem that attains a global and hence a local minimum at $(\mathbf{c}, \mathbf{w}) = (\mathbf{c}^\circ, \mathbf{w}^\circ)$, the second-order necessary conditions for this problem immediately imply that the Hessian matrix of $Z(\mathbf{c}, \mathbf{w})$ with respect to $(\mathbf{c}, \mathbf{w})$ is negative semidefinite at $(\mathbf{c}^\circ, \mathbf{w}^\circ)$. Because the choice of the point $(\mathbf{c}^\circ, \mathbf{w}^\circ) \in \Theta(\mathbf{K})$ that we used to fix the values of $(\mathbf{L}, \mathbf{I})$ in problem (3) was arbitrary, $Z(\cdot)$ is convex in $(\mathbf{c}, \mathbf{w})$ for all $(\mathbf{c}, \mathbf{w}) \in \Theta(\mathbf{K})$, and the proof is complete. Q.E.D.

Part (f) is an important result, in that unlike the static profit-maximizing model of the firm, convexity of $V(\cdot)$ in the prices $(\mathbf{c}, \mathbf{w})$ is *not* sufficient to completely characterize the optimal value function in the adjustment cost model of the firm. In particular, third-order properties of $V(\cdot)$, that is, third-order partial derivatives of $V(\cdot)$, are required to obtain a complete characterization of it. This follows because convexity of

$$Z(\mathbf{c}, \mathbf{w}) \stackrel{\text{def}}{=} rV(\mathbf{K}, \mathbf{c}, \mathbf{w}) - V_{\mathbf{K}}(\mathbf{K}, \mathbf{c}, \mathbf{w})[\mathbf{I}^*(\mathbf{K}, \mathbf{c}^\circ, \mathbf{w}^\circ) - \delta\mathbf{K}]$$

in $(\mathbf{c}, \mathbf{w})$ requires computation of its Hessian matrix with respect to $(\mathbf{c}, \mathbf{w})$, and thus will involve third-order partial derivatives of the value function $V(\cdot)$. This is an unusual result from the perspective of static optimization theory, in which second-order properties always suffice in characterizing indirect objective functions.

The significance of Theorems 20.1 and 20.2 is that they provide a straightforward, though possibly tedious, way to derive systems of factor demand and output supply equations fully consistent with the intertemporal optimization problem (1). The recipe is simple: hypothesize a functional from for $V(\cdot)$ and then use Theorem 20.2 to derive the closed-loop factor demand and supply functions, that is, the policy functions. The resulting policy functions are then estimated with data on $(\mathbf{L}, \mathbf{I}, \mathbf{K}, \mathbf{c}, \mathbf{w})$. The resulting parameter estimates can then be used to determine if $V(\cdot)$ satisfies properties (V.1) through (V.7), and thus if the data are consistent with the adjustment cost model of the firm.

From the viewpoint of the practitioner, the ease with which properties (V.1) through (V.7) may be verified for a particular functional form of $V(\cdot)$ is of some importance because it can determine which functional forms for $V(\cdot)$ are practical to work with. The set Θ, which is the domain of $V(\cdot)$, is determined by the capital stock and normalized price data one has at hand. The verification of properties (V.1) through (V.5) and (V.7) is a relatively straightforward matter, as we will see in an ensuing example. Recall from the proof of Theorem 20.1 that property (V.7) was necessary only to show that $F(\cdot)$ was sufficiently smooth and strongly concave in $(\mathbf{L}, \mathbf{I})$. As a result, in the example to follow, we will ignore this property.

Before presenting the example, however, a few more remarks concerning the crucial property (V.6) are warranted. Assume that $\Theta(\mathbf{K})$ is a convex set. Then a sufficient condition for property (V.6) to hold is that $Z(\mathbf{c}, \mathbf{w}) \stackrel{\text{def}}{=} rV(\mathbf{K}, \mathbf{c}, \mathbf{w})$

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$-\,V_{\mathbf{K}}(\mathbf{K}, \mathbf{c}, \mathbf{w})[\mathbf{I}^*(\mathbf{K}, \mathbf{c}^\circ, \mathbf{w}^\circ) - \boldsymbol{\delta}\mathbf{K}]$ is convex in $(\mathbf{c}, \mathbf{w})$ over $\Theta(\mathbf{K})$. Thus if $V_K(\cdot)$ is linear in $(\mathbf{c}, \mathbf{w})$, then property (V.6) is equivalent to the convexity of $V(\cdot)$ in $(\mathbf{c}, \mathbf{w})$, an observation of great practical importance. In either of these instances, only the convexity of a function need be checked, which requires the examination of an appropriate Hessian matrix. We will rely on the linearity of $V_{\mathbf{K}}(\cdot)$ in $(\mathbf{c}, \mathbf{w})$ in the ensuing example, as well as assume that $\Theta(\mathbf{K})$ is a convex set for each $\mathbf{K}$.

In the static theory of duality, the so-called flexible functional forms take center stage in empirical applications of the theory. In this context, a flexible functional form is one that may provide a second-order approximation to an arbitrary function. For example, the quadratic or translog functional forms are flexible, since for suitable parameter values, they may assume any given theoretically consistent set of values for zero-, first-, and second-order partial derivatives at a point. In the dynamic theory of duality presented in this chapter, flexibility must be defined in a slightly different manner.

Let's now examine the issue of flexibility of the optimal value function $V(\cdot)$ in the context of the adjustment cost model of the firm defined in Eq. (1). Because the objective of most intertemporal empirical work is the estimation of demand, supply, and shadow price elasticities, we adopt the following definition of flexibility.

Definition 20.1: A functional form for a value function is said to be *flexible* if the derived policy and shadow price functions can provide a first-order approximation at a point to a corresponding set of functions generated by an arbitrary value function that satisfies properties (V.1) through (V.7).

To render this definition in specific terms, recall Theorem 20.2, the formulae for the policy functions $(\mathbf{I}^*(\cdot), \mathbf{L}^*(\cdot), y^*(\cdot))$ given in Eqs. (4) through (6), respectively, and the fact that $\boldsymbol{\lambda}^*(\cdot) = V'_{\mathbf{K}}(\cdot)$. It then follows that a functional form is flexible if and only if it can assume, at a point, any given set of theoretically consistent values for $V(\cdot)$, all first- and second-order partial derivatives for $V(\cdot)$, and all first-order partial derivatives of $V_{\mathbf{cK}}(\cdot)$ and $V_{\mathbf{wK}}(\cdot)$. This observation again highlights the importance of the third-order properties of the optimal value function in achieving a complete qualitative characterization of the adjustment cost model.

In order to achieve the required flexibility for $V(\cdot)$, therefore, one must typically estimate a large number of parameters because of the importance of third-order properties. For example, if $M = 2$ and $N = 1$, then one must estimate 24 parameters in a flexible functional form for $V(\cdot)$, whereas if $M = 1$ and $N = 2$, then one must estimate 42 parameters, numbers you are asked to verify in a mental exercise. Thus, even with rather low dimensions for the control and state spaces, numerous parameters must be estimated when employing flexible functional forms for $V(\cdot)$. The example that follows does not use a flexible functional form for $V(\cdot)$, but it is nonetheless useful in applied work.

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Example 20.1: Let $M = 1$ and $N = 1$ for clarity of exposition, and consider the following candidate for an optimal value function:

$$V(K, c, w) = a_0 + a_1 K + a_2 c + a_3 w + \frac{1}{2}A_{11}K^2 + A_{12}Kc + A_{13}Kw + \frac{1}{2}A_{22}c^2 + A_{23}cw + \frac{1}{2}A_{33}w^2, \tag{25}$$

where a_0, a_i, and A_{ij}, $i, j = 1, 2, 3$, are the parameters to be estimated. Using Theorem 20.2, Eqs. (4) through (6), and the state equation, we can derive the policy functions

$$\dot{K}^*(K, c, w) = \left(\frac{a_2 r}{A_{12}}\right) + \left(\frac{A_{22}r}{A_{12}}\right)c + \left(\frac{A_{23}r}{A_{12}}\right)w + \left(\frac{1 + rA_{12}}{A_{12}}\right)K, \tag{26}$$

$$L^*(K, c, w) = \left(\frac{(a_2A_{13} - a_3A_{12})r}{A_{12}}\right) + \left(\frac{(A_{13}A_{22} - A_{12}A_{23})r}{A_{12}}\right)c + \left(\frac{(A_{13}A_{23} - A_{12}A_{33})r}{A_{12}}\right)w + \left(\frac{A_{13}}{A_{12}}\right)K, \tag{27}$$

$$y^*(K, c, w) = a_0 r + a_1 rK - \frac{1}{2}A_{22}rc^2 - \frac{1}{2}A_{33}rw^2 - A_{23}rcw + \frac{1}{2}A_{11}rK^2 - a_1\dot{K}^*(K, c, w) - A_{11}K\dot{K}^*(K, c, w) \tag{28}$$

using the optimal value function in Eq. (25). You are asked to verify these calculations in a mental exercise. In carrying out the estimation of the parameters of the optimal value function, one must jointly estimate all three policy functions, for this is the only way to get estimates of all the parameters of the optimal value function *and* carry out the statistical tests of properties (V.1) through (V.6). Furthermore, recall that the discount rate and the rate of depreciation are assumed to be given constants in the analysis. As a result, when carrying out the estimation of the parameters in Eqs. (26) through (28), the values of r and δ are inserted in the policy functions as constants and then the estimation is carried out. Thus r and δ are not variables in the way that (K, c, w) are, nor are they parameters to be estimated.

Let's now investigate the restrictions imposed on the policy functions by properties (V.1) through (V.6). To that end, first consider property (V.1). That $V(\cdot) \in C^{(2)}$ and $V_K(\cdot) \in C^{(2)}$ follows from the fact that the hypothesized optimal value function in Eq. (25) is a quadratic function of (K, c, w). The strict inequalities in property (V.2) require that

$$[r + \delta]V_K(K, c, w) + c - V_{KK}(K, c, w)\dot{K}^*(K, c, w) = [r + \delta][a_1 + A_{11}K + A_{12}c + A_{13}w] + c - A_{11}\dot{K}^*(K, c, w) > 0,$$

$$V_K(K, c, w) = a_1 + A_{11}K + A_{12}c + A_{13}w > 0.$$

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Notice that these restrictions involve the parameters we are trying to estimate and the data. As such, statistical testing of them is a nontrivial matter, for they are not inequality restrictions involving just the parameters to be estimated. The empirically relevant property in condition (V.3) is that $y^*(K, c, w) \geq 0$, which is certainly met in all the data one would encounter. Property (V.4) requires that the steady state is globally asymptotically stable. Because the policy function $\dot{K}^*(K, c, w)$ given in Eq. (26) is a constant coefficient linear differential equation in K, the steady state is globally asymptotically stable if its Jacobian is negative, that is, if $\partial \dot{K}^*(K, c, w)/\partial K < 0$. Using the policy function for the net investment rate in Eq. (26), this condition becomes

$$\left(\frac{1 + rA_{12}}{A_{12}}\right) = r + A_{12}^{-1} < 0.$$

Thus the steady state is globally asymptotically stable if and only if the reciprocal of the coefficient A_{12} is less than the negative of the firm's discount rate. Seeing as $V_{cK}(K, c, w) = A_{12}$, property (V.5) is met if $A_{12} \neq 0$, which it will be if the steady state is globally asymptotically stable. Finally, property (V.6), the generalized curvature condition derived from the dual H-J-B equation, turns out to be equivalent to the positive semidefiniteness of the coefficient matrix

$$\begin{bmatrix} A_{22} & A_{23} \\ A_{23} & A_{33} \end{bmatrix}.$$

To see this, first observe that $V_K(K, c, w)$ is a linear function of (c, w). Now recall our earlier observation that when such a condition holds, the curvature condition (V.6) is equivalent to the convexity of the optimal value function $V(\cdot)$ in (c, w). Because the Hessian matrix of $V(\cdot)$ with respect to (c, w) is given by the preceding matrix, we now see why its positive semidefiniteness is equivalent to property (V.6) when $V_K(K, c, w)$ is a linear function of (c, w).

The steady state value of the capital stock $K^s(c, w)$ is found by setting $\dot{K}^*(K, c, w) = 0$ in Eq. (26) and solving for K, that is,

$$K^s(c, w) = -r[1 + rA_{12}]^{-1}[a_2 + A_{22}c + A_{23}w]. \tag{29}$$

Using Eq. (29), Eq. (26) can be written in the so-called accelerator form, to wit,

$$\dot{K}^*(K, c, w) = \left[r + A_{12}^{-1}\right][K - K^s(c, w)], \tag{30}$$

a result you are asked to verify in a mental exercise. Now recall that $r + A_{12}^{-1} < 0$ for global asymptotic stability of the steady state. Hence, by differentiating Eq. (30) with respect to c and w, we find that

$$\frac{\partial \dot{K}^*(K, c, w)}{\partial c} = -\left[r + A_{12}^{-1}\right] \frac{\partial K^s(c, w)}{\partial c},$$

$$\frac{\partial \dot{K}^*(K, c, w)}{\partial w} = -\left[r + A_{12}^{-1}\right] \frac{\partial K^s(c, w)}{\partial w}.$$

These two equations demonstrate that the effect of an increase in c or w on the steady state capital stock is qualitatively the same as it is on the base period net investment rate. This is an implication of the particular functional form adopted for the optimal value function, and is not, in general, implied by the adjustment cost model, as we have previously seen in Chapter 17. This is one particular inflexibility exhibited by the optimal value function given in Eq. (25).

Following Epstein (1981), we have developed a dynamic duality theory in the context of the adjustment cost model of the firm. Numerous perturbations and generalizations of the basic theory presented here are possible. One perturbation of the theory is its extension to the nonrenewable resource–extracting model of the firm. A generalization of some importance would extend the intertemporal duality theory to cover the case of nonstatic price expectations. Complementary to a dynamic duality theory is the comparative dynamics properties of the feedback form of the optimal control vector. Consequently, you are asked to consider the comparative dynamics properties of the policy functions for a simplified version of the adjustment cost model in a mental exercise.

MENTAL EXERCISES

20.1 It is common for the adjustment cost model to include the cost of gross investment in the objective functional rather than the rental cost of the capital stock, that is, the optimal control problem often takes the alternative form

$$\max_{\mathbf{I}(\cdot),\mathbf{L}(\cdot)} \hat{J}[\mathbf{K}(\cdot),\mathbf{L}(\cdot),\mathbf{I}(\cdot)] \stackrel{\text{def}}{=} \int_t^{+\infty} [F(\mathbf{L}(s),\mathbf{K}(s),\mathbf{I}(s)) - \mathbf{w}'\mathbf{L}(s) - \mathbf{q}'\mathbf{I}(s)]\, e^{-r(s-t)}\, ds$$

$$\text{s.t.} \quad \dot{K}_n(s) = I_n(s) - \delta_n K_n(s), \;\; K_n(t) = K_{nt} > 0, \quad n = 1, 2, \ldots, N,$$

where $\mathbf{q} \in \Re^N_{++}$ is the normalized purchase price of the investment goods. Define the objective functional of problem (1) by $J[\mathbf{K}(\cdot),\mathbf{L}(\cdot),\mathbf{I}(\cdot)]$, and recall that problem (1) is stated in terms of the rental cost of capital. Assume that all admissible paths of the capital stocks are bounded.

(a) Using integration by parts, prove that $\hat{J}[\mathbf{K}(\cdot),\mathbf{L}(\cdot),\mathbf{I}(\cdot)] = J[\mathbf{K}(\cdot),\mathbf{L}(\cdot),\mathbf{I}(\cdot)] + \mathbf{q}'\mathbf{K}_t$, where $\mathbf{c} \stackrel{\text{def}}{=} (r\mathbf{I}_N + \boldsymbol{\delta})\mathbf{q}$.

(b) What does the result in part (a) imply about the solutions of the two optimal control problems? Explain clearly.

20.2 Does assumption (T.1) imply that $F(\cdot) \in C^{(2)}$? Does $F(\cdot) \in C^{(2)}$ imply the smoothness assumptions in (T.1)? Explain clearly.

20.3 Compute the second-order partial derivatives of the current value optimal value function $V(\cdot)$ by using the equations

$$rV_{c_i}(\mathbf{K}, \mathbf{c}, \mathbf{w}) \equiv -K_i + \sum_{n=1}^{N} V_{K_n c_i}(\mathbf{K}, \mathbf{c}, \mathbf{w})[I_n^*(\mathbf{K}, \mathbf{c}, \mathbf{w}) - \delta_n K_n],$$

$$i = 1, 2, \ldots, N,$$

$$rV_{w_j}(\mathbf{K}, \mathbf{c}, \mathbf{w}) \equiv -L_j^*(\mathbf{K}, \mathbf{c}, \mathbf{w}) + \sum_{n=1}^{N} V_{K_n w_j}(\mathbf{K}, \mathbf{c}, \mathbf{w})[I_n^*(\mathbf{K}, \mathbf{c}, \mathbf{w}) - \delta_n K_n],$$

$$j = 1, 2, \ldots, M.$$

Now prove that $V(\cdot) \in C^{(2)}$.

20.4 Prove that $V_{\mathbf{Kc}}(\mathbf{K}, \mathbf{c}, \mathbf{w}) \equiv V'_{\mathbf{cK}}(\mathbf{K}, \mathbf{c}, \mathbf{w})$ by differentiating with respect to the components of the vectors and writing out the associated matrix in detail. Note that this is the matrix form of Young's theorem.

20.5 Derive the identity $\mathbf{c}'\mathbf{K} \equiv -r\mathbf{c}'V'_{\mathbf{c}}(\mathbf{K}, \mathbf{c}, \mathbf{w}) + \mathbf{c}'V_{\mathbf{cK}}(\mathbf{K}, \mathbf{c}, \mathbf{w})V_{\mathbf{cK}}^{-1}(\mathbf{K}, \mathbf{c}, \mathbf{w})$ $[rV'_{\mathbf{c}}(\mathbf{K}, \mathbf{c}, \mathbf{w}) + \mathbf{K}]$ in Eq. (14) by using the result that $\mathbf{I}^*(\mathbf{K}, \mathbf{c}, \mathbf{w}) \equiv V_{\mathbf{cK}}^{-1}(\mathbf{K}, \mathbf{c}, \mathbf{w})[rV'_{\mathbf{c}}(\mathbf{K}, \mathbf{c}, \mathbf{w}) + \mathbf{K}] + \boldsymbol{\delta}\mathbf{K}$.

20.6 Prove that $F(\cdot)$ is concave in $(\mathbf{L}, \mathbf{I})$ by applying the primal-dual method to problem (3).

20.7 Show that by differentiating the identity form of Eqs. (18) and (19) with respect to $\mathbf{L}$ and $\mathbf{I}$, and then combining them into a single matrix equation, you arrive at Eq. (24), namely,

$$\begin{bmatrix} \mathrm{I}_M & \mathbf{0}_{M\times N} \\ \mathbf{0}_{N\times M} & \mathrm{I}_N \end{bmatrix} = \begin{bmatrix} \dfrac{\partial \mathbf{L}^*}{\partial \mathbf{w}} & \dfrac{\partial \mathbf{L}^*}{\partial \mathbf{c}} \\ \dfrac{\partial \mathbf{I}^*}{\partial \mathbf{w}} & \dfrac{\partial \mathbf{I}^*}{\partial \mathbf{c}} \end{bmatrix} \begin{bmatrix} \dfrac{\partial \mathbf{w}^*}{\partial \mathbf{L}} & \dfrac{\partial \mathbf{w}^*}{\partial \mathbf{I}} \\ \dfrac{\partial \mathbf{c}^*}{\partial \mathbf{L}} & \dfrac{\partial \mathbf{c}^*}{\partial \mathbf{I}} \end{bmatrix}.$$

20.8 Recall the primal H-J-B equation (2)

$$rV(\mathbf{K}, \mathbf{c}, \mathbf{w}) = \max_{(\mathbf{L},\mathbf{I})\in\Phi(\mathbf{K})} \{F(\mathbf{L}, \mathbf{K}, \mathbf{I}) - \mathbf{w}'\mathbf{L} - \mathbf{c}'\mathbf{K} + V_{\mathbf{K}}(\mathbf{K}, \mathbf{c}, \mathbf{w})[\mathbf{I} - \boldsymbol{\delta}\mathbf{K}]\},$$

$$(\mathbf{K}, \mathbf{c}, \mathbf{w}) \in \Theta,$$

where $(\mathbf{L}, \mathbf{I}) = (\mathbf{L}^*(\mathbf{K}, \mathbf{c}, \mathbf{w}), \mathbf{I}^*(\mathbf{K}, \mathbf{c}, \mathbf{w}))$ are the optimal solutions to this optimization problem, known as the policy functions, feedback, or closed-loop controls. Use the static envelope theorem on the above H-J-B equation to prove that

(a) $\mathbf{I}^*(\mathbf{K}, \mathbf{c}, \mathbf{w}) \equiv V_{\mathbf{cK}}^{-1}(\mathbf{K}, \mathbf{c}, \mathbf{w})[rV'_{\mathbf{c}}(\mathbf{K}, \mathbf{c}, \mathbf{w}) + \mathbf{K}] + \boldsymbol{\delta}\mathbf{K}$,

(b) $\mathbf{L}^*(\mathbf{K}, \mathbf{c}, \mathbf{w})$
$\equiv -rV'_{\mathbf{w}}(\mathbf{K}, \mathbf{c}, \mathbf{w}) + V_{\mathbf{wK}}(\mathbf{K}, \mathbf{c}, \mathbf{w})V_{\mathbf{cK}}^{-1}(\mathbf{K}, \mathbf{c}, \mathbf{w})[rV'_{\mathbf{c}}(\mathbf{K}, \mathbf{c}, \mathbf{w}) + \mathbf{K}]$.

For the next two parts of this question, you do not have to use the static envelope theorem to establish the result.

(c) $y^*(\mathbf{K}, \mathbf{c}, \mathbf{w}) \equiv rV(\mathbf{K}, \mathbf{c}, \mathbf{w}) + \mathbf{w}'\mathbf{L}^*(\mathbf{K}, \mathbf{c}, \mathbf{w}) + \mathbf{c}'\mathbf{K}$
$- V_{\mathbf{K}}(\mathbf{K}, \mathbf{c}, \mathbf{w})[\mathbf{I}^*(\mathbf{K}, \mathbf{c}, \mathbf{w}) - \delta\mathbf{K}]$.

(d) $y^*(\mathbf{K}, \mathbf{c}, \mathbf{w}) \equiv r[V(\mathbf{K}, \mathbf{c}, \mathbf{w}) - V_{\mathbf{w}}(\mathbf{K}, \mathbf{c}, \mathbf{w})\mathbf{w} - V_{\mathbf{c}}(\mathbf{K}, \mathbf{c}, \mathbf{w})\mathbf{c}]$
$- [V_{\mathbf{K}}(\mathbf{K}, \mathbf{c}, \mathbf{w}) - \mathbf{w}'V_{\mathbf{wK}}(\mathbf{K}, \mathbf{c}, \mathbf{w}) - \mathbf{c}'V_{\mathbf{cK}}(\mathbf{K}, \mathbf{c}, \mathbf{w})]$
$\times [V_{\mathbf{cK}}^{-1}(\mathbf{K}, \mathbf{c}, \mathbf{w})[rV_{\mathbf{c}}'(\mathbf{K}, \mathbf{c}, \mathbf{w}) + \mathbf{K}]]$.

20.9 In this exercise, you are asked to provide the alternative proofs of parts of Theorem 20.3. To this end, let $V(\cdot)$ be defined by problem (1) and let $F(\cdot)$ satisfy conditions (T.1) through (T.7).

(a) Prove that $V(\cdot)$ is increasing in $\mathbf{K}_t$ by way of the dynamic envelope theorem and the Maximum Principle applied to problem (1).
(b) Prove that $V(\cdot)$ is decreasing in $\mathbf{c}$ via the dynamic envelope theorem.
(c) Prove that $V(\cdot)$ is decreasing in $\mathbf{w}$ by mimicking the proof given for Theorem 20.3 part (b).
(d) Prove that $V(\cdot)$ is decreasing in $\mathbf{w}$ via the dynamic envelope theorem.
(e) Prove that $V(\cdot)$ is convex in $(\mathbf{c}, \mathbf{w}) \in \Theta(\mathbf{K})$ by applying the dynamic primal-dual formalism to problem (1).

20.10 As remarked in the chapter, in order to achieve the required flexibility for $V(\cdot)$, one must typically estimate a large number of parameters because of the importance of third-order properties. Show that if $M = 2$ and $N = 1$, then one must estimate 24 parameters in a flexible functional form for $V(\cdot)$, whereas if $M = 1$ and $N = 2$, then one must estimate 42 parameters. How many parameters must one estimate in a flexible functional form for $V(\cdot)$ if $M = 1$ and $N = 1$?

20.11 Verify that the three policy functions given in Eqs. (26) through (28) of Example 20.1 can be derived from the optimal value function given in Eq. (25). Also verify the accelerator form of the net investment rate policy function given in Eq. (30).

20.12 In this exercise, you will explore the comparative dynamics properties of the policy functions of the adjustment cost model. To this end, assume that $M = N = 1$ for simplicity. In this case, problem (1) takes the form

$$V(K_t, c, w) \stackrel{\text{def}}{=} \max_{L(\cdot), I(\cdot)} \int_t^{+\infty} [F(L(s), K(s), I(s)) - wL(s) - cK(s)]\, e^{-r(s-t)}\, ds$$

$$\text{s.t.} \quad \dot{K}(s) = I(s) - \delta K(s),\ K(t) = K_t > 0.$$

Assume that properties (T.1) through (T.7) hold on the production function $F(\cdot)$.

(a) Write down the primal form of the H-J-B equation corresponding to the above optimal control problem.
(b) Derive the first-order necessary and second-order sufficient conditions for the maximization problem in the H-J-B equation. How do you know that the second-order sufficient conditions hold for the maximization

problem? Denote the optimal solutions of the H-J-B maximization problem by $(L^*(K, c, w), I^*(K, c, w))$. Recall that these are the values of the policy functions.

(c) Derive the comparative dynamics of the policy functions with respect to w. Can you sign either of the comparative dynamics results? Why? If adjustment costs are additively separable so that $F_{LI}(L, K, I) = F_{KI}(L, K, I) \equiv 0$, then show that

$$\frac{\partial L^*(K, c, w)}{\partial w} < 0 \text{ and } \operatorname{sign}\left[\frac{\partial I^*(K, c, w)}{\partial w}\right] = \operatorname{sign}[V_{Kw}(K, c, w)].$$

(d) Derive the comparative dynamics of the policy functions with respect to c. Can you sign either of the comparative dynamics results? Why? If adjustment costs are additively separable, then show that

$$\frac{\partial L^*(K, c, w)}{\partial c} \equiv 0 \text{ and } \operatorname{sign}\left[\frac{\partial I^*(K, c, w)}{\partial c}\right] = \operatorname{sign}[V_{Kc}(K, c, w)].$$

(e) Derive the comparative dynamics of the policy functions with respect to K. Can you sign either of the comparative dynamics results? Why? If adjustment costs are additively separable, then show that

$$\operatorname{sign}\left[\frac{\partial L^*(K, c, w)}{\partial K}\right] = \operatorname{sign}[F_{LK}(L^*(K, c, w), K, I^*(K, c, w))]$$

and that

$$\operatorname{sign}\left[\frac{\partial I^*(K, c, w)}{\partial K}\right] = \operatorname{sign}[V_{KK}(K, c, w)].$$

FURTHER READING

The subject matter of this chapter, videlicet, intertemporal or dynamic duality, is a relatively recent development, dating to the papers of Cooper and McLaren (1980), McLaren and Cooper (1980), and Epstein (1981). Cooper and McLaren (1993) survey the different approaches one may take in solving and characterizing the solution of an intertemporal model of the consumer. Lasserre and Ouellette (1999) present a rather general duality theory for an expected cost-minimizing firm facing costs of adjustment in discrete time. Though we have not discussed the details of the estimation of the policy functions in much detail in this chapter, an excellent place to begin such an inquiry is the paper by Epstein and Denny (1983), the first one to empirically implement and test the adjustment cost model via the duality laid down here. Galeotti (1996) provides a nice survey of dynamic production theory. Closely related to the duality theory presented here is the comparative dynamics properties of the policy functions or feedback controls. On this subject, one may consult the recent paper by Caputo (2003), which derives general comparative dynamics results for the feedback or closed-loop form of the optimal control vector for a ubiquitous class of optimal control problems, and then applies the theorems to the nonrenewable resource–extracting model of the firm.

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