SIXTEEN

A Dynamic Limit Pricing Model of the Firm

The limit pricing model developed in this chapter is an attempt to gain some insight into the optimal pricing strategy for a dominant firm or group of joint profit-maximizing oligopolists facing potential entry into the product market. Early research on this subject was from a static perspective, and, for the most part, concluded that the dominant firm will maximize its present value by either (i) charging the short-run profit-maximizing price and allowing its market share to decline, or (ii) setting price at the limit price thereby precluding all entry. A firm practicing short-run profit maximization would have to continually ignore the reality of entry by other firms induced by its pricing strategy. On the other hand, a firm charging the limit price has to think that its current market share is in fact its long-run optimal market share. Economic intuition suggests that the optimal pricing strategy entails a balancing of current profits and long-run market share. For example, a dominant firm currently charging a high price and earning high current profits is likely sacrificing some future profits through the erosion of its current market share. It is this dependence of the dominant firm's future market share, and thus its profits, on its current pricing strategy that makes this model inherently dynamic.

We assert that the optimal pricing strategy of the dominant firm will maximize the present discounted value of its profits, as given by the functional

$$\max_{p(\cdot)} J[x(\cdot), p(\cdot)] \stackrel{\text{def}}{=} \int_0^{+\infty} [p(t) - c]\, q(p(t), x(t))\, e^{-rt}\, dt, \tag{1}$$

where $p(t)$ is the price charged by the dominant firm at time t for its good, $c > 0$ is the constant average (and marginal) cost of production of the dominant firm, $x(t)$ is the level of competitive fringe (or rival) firms' sales at time t, $q(\cdot)$ is the residual demand function facing the dominant firm, and $r > 0$ is the constant discount rate of the dominant firm. Note that we may also interpret $x(t)$ as the number of fringe firms in the market at time t as long as we assume that each fringe firm sells just one unit of the good.

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We assume that the residual demand curve $q(p(t), x(t))$ facing the dominant firm is given by the total market demand curve for the product $f(p(t))$, less the level of rival firms' sales $x(t)$, that is to say,

$$q(p(t), x(t)) \stackrel{\text{def}}{=} f(p(t)) - x(t). \tag{2}$$

In other words, the residual demand curve at any given point in time can be found by subtracting the output of the competitive fringe from the total market demand for the product. Equation (2) therefore indicates that the net effect of rival entry into the product market is to shift the dominant firm's residual demand curve laterally.

We assume that the fringe firms are rational actors in the sense that their rate of entry, or the rate of change of fringe sales, $\dot{x}(t)$, is determined by their expected rate of return from entering the market. If potential entrants view the current product price as a proxy for the future price, then the rate of entry can be approximated by a monotonically increasing function of the current product price. We assume that such a relationship can be represented by the linear state equation and initial condition

$$\dot{x}(t) = k[p(t) - \bar{p}],\; x(0) = x_0, \tag{3}$$

where $k > 0$ is the time invariant fringe response coefficient, $\bar{p} > 0$ is the time invariant limit price, and $x_0 > 0$ is the initial level of fringe sales. The limit price $\bar{p}$ is defined as that price for which net entry is zero, that is, $\dot{x}(t) = 0 \Leftrightarrow p(t) = \bar{p}$. As Eq. (3) makes clear, any price below the limit price $\bar{p}$ causes exit from the market by fringe firms ($\dot{x}(t) < 0$), and any price above the limit price $\bar{p}$ attracts new entrants to the market ($\dot{x}(t) > 0$). Alternatively, we have that $\operatorname{sign} \dot{x}(t) = \operatorname{sign}[p(t) - \bar{p}]$.

The following assumptions are also imposed on the dynamic limit pricing model:

$$f(\cdot) \in C^{(2)}\,\forall\, p > 0,\;\; f'(p) < 0\,\forall\, p \geq 0,$$

$$p(t) > 0\,\forall\, t \in [0, +\infty) \quad \text{and} \quad x(t) > 0\,\forall\, t \in [0, +\infty),$$

$$\bar{p} \geq c.$$

The first two assumptions say that the market demand function has two continuous derivatives for all positive prices and is downward sloping for nonnegative product prices. The next two state that the dominant firm will never give the product away, even for an instant, and that there are always some fringe sales of the product. These two assumptions are made so we can more readily focus on the basic economic content of the problem. Finally, the last assumption says the dominant firm's average cost of production is never greater than the limit price. The difference between the limit price and the dominant firm's average cost of production is a measure of the cost advantage enjoyed by the dominant firm.

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Bringing all this information together, the complete statement of the dynamic limit pricing model is

$$V(c, k, \bar{p}, r, x_0) \stackrel{\text{def}}{=} \max_{p(\cdot)} \int_0^{+\infty} [p(t) - c]\,[f(p(t)) - x(t)]\,e^{-rt}\,dt \tag{4}$$

$$\text{s.t.} \quad \dot{x}(t) = k[p(t) - \bar{p}],\; x(0) = x_0.$$

Note that we have not imposed any assumptions on $\lim_{t\to+\infty} x(t)$, nor do we intend to. As you will show in a mental exercise, without imposing any other assumptions on control problem (4), we cannot invoke a sufficiency theorem to solve it. Consequently, we assume that an optimal solution exists to this limit pricing problem and denote it by $(x^*(t;\boldsymbol{\beta}), p^*(t;\boldsymbol{\beta}))$, with corresponding current value costate variable $\lambda(t;\boldsymbol{\beta})$, where $\boldsymbol{\beta} \stackrel{\text{def}}{=} (c, k, \bar{p}, r, x_0)$ is the parameter vector of problem (4).

The current value Hamiltonian for problem (4) is defined as

$$H(x, p, \lambda; c, k, \bar{p}) \stackrel{\text{def}}{=} [p - c][f(p) - x] + \lambda k[p - \bar{p}]. \tag{5}$$

Given the assumptions made so far on problem (4), and furthermore assuming that the objective functional exists for all admissible pairs of curves $(x(t), p(t))$, Theorems 14.3 and 14.9 imply the following necessary conditions:

$$H_p(x, p, \lambda; c, k, \bar{p}) = [p - c]f'(p) + f(p) - x + \lambda k = 0, \tag{6}$$

$$H_{pp}(x, p, \lambda; c, k, \bar{p}) = [p - c]f''(p) + 2f'(p) \le 0, \tag{7}$$

$$\dot{\lambda} = r\lambda - H_x(x, p, \lambda; c, k, \bar{p}) = r\lambda + p - c, \tag{8}$$

$$\dot{x} = H_\lambda(x, p, \lambda; c, k, \bar{p}) = k[p - \bar{p}],\; x(0) = x_0, \tag{9}$$

$$\lim_{t\to+\infty} e^{-rt} H(x, p, \lambda; c, k, \bar{p}) = 0. \tag{10}$$

Equation (6) says the present value profit-maximizing price charged by the dominant firm results in the marginal profit being proportional to the negative of the current value shadow price of rival sales, the constant of proportionality being the fringe response coefficient. Because fringe firms' sales are a bad (rather than a good), that is, more fringe firms' sales means less dominant firm sales, ceteris paribus, the current value shadow price of rival sales is negative in an optimal plan, that is, $\lambda(t;\boldsymbol{\beta}) < 0\,\forall\, t \in [0, +\infty)$. The transversality condition in Eq. (10) asserts that the present value of the Hamiltonian along the optimal path is zero in the limit of the planning horizon. It helps to place more structure on the optimal pair $(x^*(t;\boldsymbol{\beta}), p^*(t;\boldsymbol{\beta}))$ than we would otherwise be able to detect without it, as we now proceed to show.

To this end, assume that a simple steady state solution to the limit pricing problem exists, say, $(x^s(\boldsymbol{\alpha}), p^s(\boldsymbol{\alpha}))$, with corresponding steady state current value shadow price of fringe sales $\lambda^s(\boldsymbol{\alpha})$, where $\boldsymbol{\alpha} \stackrel{\text{def}}{=} (c, k, \bar{p}, r)$. Now if $(x^*(t;\boldsymbol{\beta}), p^*(t;\boldsymbol{\beta})) \to (x^s(\boldsymbol{\alpha}), p^s(\boldsymbol{\alpha}))$ as $t \to +\infty$, thereby implying that $\lambda(t;\boldsymbol{\beta}) \to \lambda^s(\boldsymbol{\alpha})$ as $t \to +\infty$ by

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Eq. (6), then such an optimal solution does indeed satisfy the limiting transversality condition in Eq. (10). To see this, observe that $e^{-rt} \to 0$ as $t \to +\infty$ and that the value of the current value Hamiltonian along such a converging path goes to a finite value as $t \to +\infty$, since $H(\cdot) \in C^{(2)}$. These two facts imply that Eq. (10) is satisfied for an optimal solution that converges to its steady state solution. Accordingly, we assume that the optimal solution to the limit pricing model converges to its simple steady state solution in the limit of the planning horizon, since such a solution satisfies the necessary limiting transversality condition given in Eq. (10).

We also assume that the local second-order sufficient condition $H_{pp}(x, p, \lambda; c, k, \bar{p}) < 0$ for maximizing the current value Hamiltonian holds along the optimal path. This implies that in a neighborhood of the optimal solution, marginal profit is downward sloping in the product price, since $H_{pp}(x, p, \lambda; c, k, \bar{p}) = \pi_{pp}(x, p; c)$. Given Eq. (6), the fact that $\lambda(t; \boldsymbol{\beta}) < 0 \, \forall \, t \in [0, +\infty)$, and the assumption that $H_{pp}(x, p, \lambda; c, k, \bar{p}) = \pi_{pp}(x, p; c) < 0$ in a neighborhood of the optimal solution, we can prove the following preliminary result.

Proposition 16.1: *If $H_{pp}(x, p, \lambda; c, k, \bar{p}) < 0$ along the optimal solution, then the optimal dynamic price $p^*(t; \boldsymbol{\beta})$ will always be less than the short-run profit-maximizing price p_m all along the optimal path.*

Proof: If the dominant firm has been moving along the optimal path, then its instantaneous profit at any given moment in the planning horizon is given by

$$\pi(x, p; c) \stackrel{\text{def}}{=} [p - c][f(p) - x].$$

The short-run profit-maximizing price p_m therefore necessarily satisfies

$$\pi_p(x, p; c) = [p - c]f'(p) + f(p) - x = 0, \tag{11}$$

where x is evaluated at its optimal value for the given point in time under consideration. From Eq. (6), it follows that the optimal dynamic price path $p^*(t; \boldsymbol{\beta})$ necessarily satisfies

$$\pi_p(x, p; c) = [p - c]f'(p) + f(p) - x = -\lambda k > 0, \tag{12}$$

where the strict inequality follows from the fact that $\lambda(t; \boldsymbol{\beta}) < 0 \, \forall \, t \in [0, +\infty)$ and the assumption that $k > 0$. Because $H_{pp}(x, p, \lambda; c, k, \bar{p}) = \pi_{pp}(x, p; c) < 0$ along the optimal solution by assumption, the curve $\pi_p(x, p; c)$, plotted as a function of p, is downward sloping. Hence the optimal dynamic price $p^*(t; \boldsymbol{\beta})$, which satisfies Eq. (12), must necessarily be less than the short-run profit-maximizing price p_m, which satisfies Eq. (11). Q.E.D.

Proposition 16.1 establishes that, ceteris paribus, the dynamic limit pricing firm sets its optimal price below that of a myopic monopolistic firm. The most important

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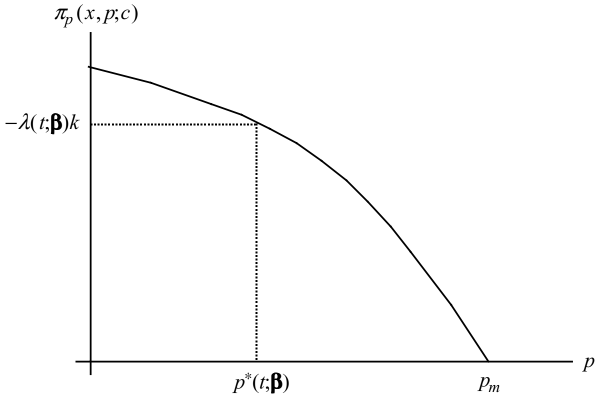

Figure 16.1

features of the model in reaching this conclusion are the presence of the current value costate variable in Eq. (6) and the fact that it is negative. This proposition therefore shows that one of the conclusions derived from a static investigation of the limit pricing model is incorrect, namely, that the dominant firm will maximize its present value by charging the short-run profit-maximizing price and allowing its market share to decline. Figure 16.1 provides a graphical representation of the proposition.

Our task now is to determine the local stability of the steady state, which the optimal solution converges to in the limit of the planning horizon. In order to do so, we must reduce the three necessary conditions (6), (8), and (9), down to two differential equations involving just two of the three variables (x, p, λ). The two variables of primary interest are the price set by the dominant firm and the level of fringe sales. As a result, we must eliminate λ from the necessary conditions. To do this, the following recipe, which is used often in the literature, is employed.

The first step is to differentiate the necessary condition $H_p(x, p, \lambda; c, k, \bar{p}) = 0$ with respect to t to get

$$\frac{d}{dt} H_p(x, p, \lambda; c, k, \bar{p}) = [[p - c] f''(p) + 2 f'(p)]\dot{p} - \dot{x} + \dot{\lambda} k = 0. \tag{13}$$

This is a valid operation because the optimal solution must necessarily satisfy $H_p(x, p, \lambda; c, k, \bar{p}) = 0$ identically for all $t \in [0, +\infty)$, and thus must satisfy $\frac{d}{dt} H_p(x, p, \lambda; c, k, \bar{p}) = 0$ as well. Heuristically, if a solution satisfies an equation for all values of t, then it must also satisfy that same equation when t is perturbed slightly.

The second step is to solve Eq. (6) for λ to get

$$\lambda = k^{-1}[x - f(p) - [p - c] f'(p)]. \tag{14}$$

This step is required seeing as we want to eliminate λ from the necessary conditions.

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The final step is to substitute Eqs. (8), (9), and (14) into Eq. (13) to eliminate $\dot{\lambda}$, $\dot{x}$, and λ from Eq. (13), thereby yielding

$$[[p-c]f''(p)+2f'(p)]\dot{p}-k[p-\bar{p}]$$
$$+k[rk^{-1}[x-f(p)-[p-c]f'(p)]+p-c]=0. \tag{15}$$

Solving Eq. (15) for $\dot{p}$ and combining the resulting differential equation with the state equation gives the pair of necessary conditions of economic interest

$$\dot{p}=\frac{r[[p-c]f'(p)+f(p)-x]+k[c-\bar{p}]}{[p-c]f''(p)+2f'(p)}, \tag{16}$$

$$\dot{x}=k[p-\bar{p}]. \tag{17}$$

Note that because we have assumed that $H_{pp}(x, p, \lambda; c, k, \bar{p})<0$ along the optimal path, the denominator of Eq. (16) is nonzero and thus the differential equation is well defined. With the reduction of the necessary conditions to a pair of simultaneous differential equations now complete, we may now turn to the determination of the local stability of the steady state.

We begin by finding the steady state solution of necessary conditions (16) and (17). Inasmuch as $\dot{p}=0$ and $\dot{x}=0$ by definition of a steady state, the steady state versions of Eqs. (16) and (17) take the form

$$\dot{p}=0 \Leftrightarrow r[[p-c]f'(p)+f(p)-x]+k[c-\bar{p}]=0, \tag{18}$$

$$\dot{x}=0 \Leftrightarrow p-\bar{p}=0. \tag{19}$$

From Eq. (19), it is clear that the steady state solution for the dominant firm's price exists and is given by

$$p=p^s(\boldsymbol{\alpha}) \stackrel{\text{def}}{=} \bar{p}>0. \tag{20}$$

Upon substituting Eq. (20) into Eq. (18) and solving for x, we find that the steady state solution for fringe sales exists and is given by

$$x=x^s(\boldsymbol{\alpha}) \stackrel{\text{def}}{=} f(\bar{p})-[\bar{p}-c]\,[r^{-1}k-f'(\bar{p})]. \tag{21}$$

Using Eq. (21), we may then define the steady state market share of the dominant firm as

$$m^s(\boldsymbol{\alpha}) \stackrel{\text{def}}{=} \frac{f(\bar{p})-x^s(\boldsymbol{\alpha})}{f(\bar{p})}=\frac{[\bar{p}-c][r^{-1}k-f'(\bar{p})]}{f(\bar{p})} \geq 0. \tag{22}$$

In passing, note that there is no need to discuss or compute the Jacobian matrix of Eqs. (18) and (19), as explicit steady state solutions for the dominant firm's price and fringe sales have been obtained. Before examining the local stability of the steady state solution, we pause and present a few observations about this solution.

The first is that if the dominant firm does not enjoy a cost advantage over the fringe firms, that is, if $\bar{p}=c$, then from Eqs. (21) and (22) it follows that the fringe

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firms supply the entire market and the market share of the dominant firm is zero in the steady state, that is to say, $x^s(\boldsymbol{\alpha})|_{\bar{p}=c} = f(c)$ and $m^s(\boldsymbol{\alpha})|_{\bar{p}=c} = 0$. Therefore, if the dominant firm does not enjoy a cost advantage over the fringe firms, then the optimal pricing strategy of the dominant firm results in it pricing itself out of the market in the long run. Because this is the optimal policy given $\bar{p} = c$, any other pricing strategy is inferior.

Second, the steady state market share of the dominant firm and fringe sales are positive in the steady state for all $\bar{p} > c$ and $\bar{p}$ sufficiently close to c. To see this, differentiate Eq. (21) with respect to $\bar{p}$ and evaluate the result at $\bar{p} = c$ to get

$$\begin{aligned}\left.\frac{\partial x^s(\boldsymbol{\alpha})}{\partial \bar{p}}\right|_{\bar{p}=c} &= [f'(\bar{p}) + [\bar{p} - c]f''(\bar{p}) - [r^{-1}k - f'(\bar{p})]]\big|_{\bar{p}=c} \\ &= 2f'(c) - r^{-1}k < 0. \end{aligned} \tag{23}$$

Given that $x^s(\boldsymbol{\alpha})|_{\bar{p}=c} = f(c)$ and $\partial x^s(\boldsymbol{\alpha})/\partial \bar{p}$ is continuous, it follows from Eq. (23) that $x^s(\boldsymbol{\alpha}) < f(\bar{p})$ for all $\bar{p} > c$ and $\bar{p}$ sufficiently close to c. This, in turn, implies that $m^s(\boldsymbol{\alpha}) > 0$ for all $\bar{p} > c$ and $\bar{p}$ sufficiently close to c using Eq. (22). Thus our assumption of an interior solution can be justified under the assumptions that $\bar{p} > c$ and that $\bar{p}$ is sufficiently close to c.

The local stability of the steady state is determined by finding the eigenvalues of the Jacobian matrix corresponding to Eqs. (16) and (17) evaluated at the steady state, namely,

$$\mathbf{J}(p^s(\boldsymbol{\alpha}), x^s(\boldsymbol{\alpha})) \stackrel{\text{def}}{=} \begin{bmatrix} \dfrac{\partial \dot{p}}{\partial p} & \dfrac{\partial \dot{p}}{\partial x} \\ \dfrac{\partial \dot{x}}{\partial p} & \dfrac{\partial \dot{x}}{\partial x} \end{bmatrix}_{\substack{\dot{p}=0 \\ \dot{x}=0}} = \begin{bmatrix} r & \dfrac{-r}{H_{pp}(\bar{p}; c)} \\ k & 0 \end{bmatrix}, \tag{24}$$

where $H_{pp}(p; c) = [p - c]f''(p) + 2f'(p) < 0$. Note that in the notation for $H_{pp}(\cdot)$, we have made explicit only those variables on which it depends. Now recall that by Mental Exercise 13.4, or equivalently, by Theorem 23.9 of Simon and Blume (1994), the product of the eigenvalues of $\mathbf{J}(p^s(\boldsymbol{\alpha}), x^s(\boldsymbol{\alpha}))$ equals its determinant. As a result, because $|\mathbf{J}(p^s(\boldsymbol{\alpha}), x^s(\boldsymbol{\alpha}))| = rk/H_{pp}(\bar{p}; c) < 0$, one eigenvalue is real and positive and the other is real and negative, thereby implying that the steady state is hyperbolic. Therefore, by Theorem 13.6 or Theorem 13.7, the steady state $(p^s(\boldsymbol{\alpha}), x^s(\boldsymbol{\alpha}))$ of the nonlinear system of differential equations comprising Eqs. (16) and (17) is an unstable saddle point, with two trajectories in the xp-phase plane converging to it as $t \to +\infty$. With the local stability of the steady state determined, we turn to the derivation of the phase diagram for the limit pricing model.

First we examine the $\dot{p} = 0$ isocline, defined as

$$\dot{p} = 0 \Leftrightarrow r[[p - c]f'(p) + f(p) - x] + k[c - \bar{p}] = 0. \tag{25}$$

Because the Jacobian of the $\dot{p} = 0$ isocline with respect to p evaluated at the steady state is $rH_{pp}(\bar{p}; c) = r[[\bar{p} - c]f''(\bar{p}) + 2f'(\bar{p})] < 0$, the implicit function theorem

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implies that we can, in principle, express p as a function of $(x; \boldsymbol{\alpha})$ along the $\dot{p} = 0$ isocline in a neighborhood of the steady state, say, $p = P(x; \boldsymbol{\alpha})$. Hence, by the implicit function theorem, the slope of the $\dot{p} = 0$ isocline in a neighborhood of the steady state is given by

$$\left.\frac{\partial p}{\partial x}\right|_{\substack{\dot{p}=0\\ \dot{x}=0}} = P_x(x^s(\boldsymbol{\alpha}); \boldsymbol{\alpha}) = \left.\frac{-\partial \dot{p}/\partial x}{\partial \dot{p}/\partial p}\right|_{\substack{\dot{p}=0\\ \dot{x}=0}} = \frac{1}{H_{pp}(\bar{p}; c)} < 0. \tag{26}$$

This means that the $\dot{p} = 0$ isocline is downward sloping in a neighborhood of the steady state in the xp-phase space.

To find the vector field associated with the $\dot{p} = 0$ isocline, differentiate Eq. (16) with respect to p or x, and then evaluate the result at the steady state to get

$$\left.\frac{\partial \dot{p}}{\partial p}\right|_{\substack{\dot{p}=0\\ \dot{x}=0}} = r > 0, \tag{27}$$

$$\left.\frac{\partial \dot{p}}{\partial x}\right|_{\substack{\dot{p}=0\\ \dot{x}=0}} = \frac{-r}{H_{pp}(\bar{p}; c)} > 0. \tag{28}$$

Thus, in a neighborhood of the steady state, a movement in the direction of p or x increases $\dot{p}$ from zero to a positive number. Because the $\dot{p} = 0$ isocline is negatively sloped in a neighborhood of the steady state, it follows that $\dot{p} > 0$ above the $\dot{p} = 0$ isocline and $\dot{p} < 0$ below the $\dot{p} = 0$ isocline, in a neighborhood of the steady state. Observe that the derivatives in Eqs. (27) and (28) are the elements of the first row of $\mathbf{J}(p^s(\boldsymbol{\alpha}), x^s(\boldsymbol{\alpha}))$, and that the negative of the ratio of the (1, 2) to (1, 1) elements of $\mathbf{J}(p^s(\boldsymbol{\alpha}), x^s(\boldsymbol{\alpha}))$ gives the slope of the $\dot{p} = 0$ isocline in a neighborhood of the steady state. Thus, as remarked in Chapter 15, once the Jacobian matrix of the dynamical system is calculated, the slopes of the nullclines and the vector field are known in a neighborhood of the steady state.

Note that in computing Eqs. (24), (27), and (28), use was made of the following insight, which you are asked to prove in a mental exercise. This is a useful trick that should be filed away in your memory for future use.

Lemma 16.1: *Consider a differential equation of the form*

$$\dot{y} = \frac{F(x, y)}{G(x, y)},$$

where $F(\cdot) \in C^{(1)}$ and $G(\cdot) \in C^{(1)}$. The derivative of $\dot{y}$ with respect to x or y evaluated where $\dot{y} = 0$, or where $\dot{x} = 0$ and $\dot{y} = 0$, is found by differentiating only the numerator function $F(\cdot)$ with respect to x or y.

The $\dot{x} = 0$ isocline is easy to determine because of its simple linear structure and its independence from the state variable:

$$\dot{x} = 0 \Leftrightarrow p - \bar{p} = 0. \tag{29}$$

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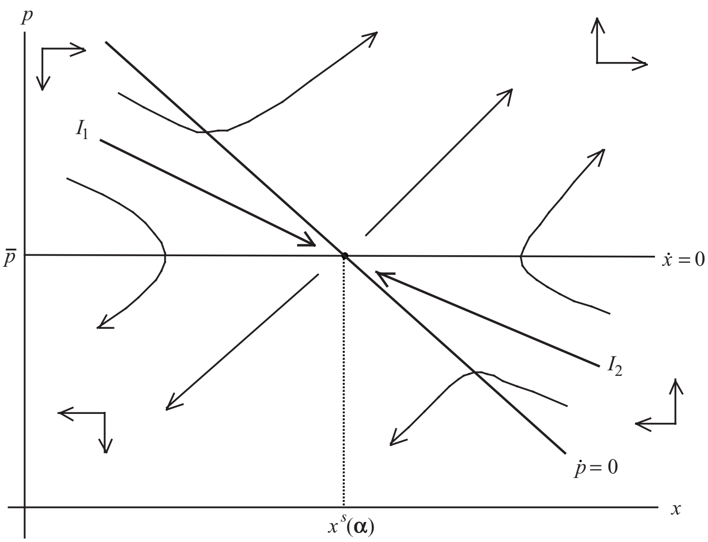

Figure 16.2

Equation (29) demonstrates that the $\dot{x} = 0$ isocline is a horizontal line at the value $p = \bar{p}$ in the xp-phase plane, and that its solution is the steady state value of the dominant firm's price. By inspection of Eq. (17), it is easy to see that if $p > \bar{p}$, then $\dot{x} > 0$, whereas if $p < \bar{p}$, then $\dot{x} < 0$. Note that these properties hold globally, not just in a neighborhood of the steady state. Figure 16.2 brings all the qualitative information gathered so far together in a phase portrait.

Inspection of Figure 16.2 reveals that the steady state, which occurs at the intersection of the $\dot{p} = 0$ and $\dot{x} = 0$ isoclines, is a local saddle point, confirming our earlier conclusion when we examined the eigenvalues of $\mathbf{J}(p^s(\boldsymbol{\alpha}), x^s(\boldsymbol{\alpha}))$. Note that only the trajectories I_1 and I_2 asymptotically approach the steady state, that is to say, as $t \to +\infty$. Because we assumed that the optimal solution to the limit pricing model converges to its steady state solution in the limit of the planning horizon, these trajectories are indeed the optimal ones. Moreover, as demonstrated earlier, they meet all the necessary conditions, including the necessary transversality condition. None of the other trajectories reach the steady state and so cannot be optimal.

Referring to Figure 16.2, we see that if $x_0 < x^s(\boldsymbol{\alpha})$, then the optimal pricing policy for the dominant firm is to set its initial price above the limit price and gradually lower it over time until the limit price is reached asymptotically. Because the dominant firm's price is above the limit price for the entire planning horizon, this optimal policy induces rival entry and thus erodes the dominant firm's market share over time. In contrast, when $x_0 > x^s(\boldsymbol{\alpha})$, the dominant firm sets its initial price below the limit price and gradually raises it over time until the limit price is reached asymptotically. Given that the dominant firm's price is below the limit price for the entire planning horizon in this instance, this policy drives rival firms from

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the market and thus increases the dominant firm's market share over time. With the local stability of the model determined and the basic qualitative nature of the optimal pricing policy laid out, we now turn toward the steady state comparative statics of the model.

Instead of considering changes in all of the parameters $(c, k, \bar{p}, r)$, we will instead focus on the parameters (c, k), and leave the remaining two for a mental exercise. Observe that since the steady state price is equal to the limit price, changes in (c, k) leave it unchanged. Differentiating Eqs. (20), (21), and (22) with respect to (c, k) yields the following steady state comparative statics:

$$\frac{\partial x^s(\boldsymbol{\alpha})}{\partial c} = -f'(\bar{p}) + \frac{k}{r} > 0, \ \frac{\partial p^s(\boldsymbol{\alpha})}{\partial c} = 0,$$
$$\frac{\partial m^s(\boldsymbol{\alpha})}{\partial c} = -\frac{\partial x^s(\boldsymbol{\alpha})/\partial c}{f(\bar{p})} = \frac{f'(\bar{p}) - r^{-1}k}{f(\bar{p})} < 0, \tag{30}$$

$$\frac{\partial x^s(\boldsymbol{\alpha})}{\partial k} = r^{-1}[c - \bar{p}] \leq 0, \ \frac{\partial p^s(\boldsymbol{\alpha})}{\partial k} = 0,$$
$$\frac{\partial m^s(\boldsymbol{\alpha})}{\partial k} = -\frac{\partial x^s(\boldsymbol{\alpha})/\partial k}{f(\bar{p})} = \frac{r^{-1}[\bar{p} - c]}{f(\bar{p})} \geq 0. \tag{31}$$

The economic interpretation of Eq. (30) is seemingly straightforward. For example, an increase in the dominant firm's average cost of production reduces its cost advantage over the fringe firms. Consequently, the dominant firm will not change its price, but sales to the fringe firms increase (or more fringe firms enter the market) and the dominant firm's market share falls. Thus the falling cost advantage of the dominant firm induces substitution by consumers toward the fringe firms' product. What is less than clear about this interpretation is why the dominant firm loses market share when its steady state price is unchanged. To fully understand the economics of this steady state comparative statics result, we must therefore determine the corresponding local comparative dynamics of an increase in the dominant firm's average cost of production.

Similarly, the economic story is not entirely clear for Eq. (31) either, the steady state comparative statics of the fringe response coefficient. An increase in k means that the fringe firms respond more quickly (or efficiently) to the price signals sent out by the dominant firm. As a result of this increased responsiveness on the part of the fringe firms, Eq. (31) shows they lose sales (assuming $\bar{p} > c$) in the steady state to the dominant firm while the steady state market share of the dominant firm rises, even though it didn't change its steady state price. As in the previous case of an increase in the dominant firm's average cost of production, the economics of these steady state comparative statics results are less than satisfactory. Therefore, let's now turn to an examination of the corresponding local comparative dynamics results so as to more fully understand the qualitative effects of increasing c and k on the dominant firm's optimal pricing policy.

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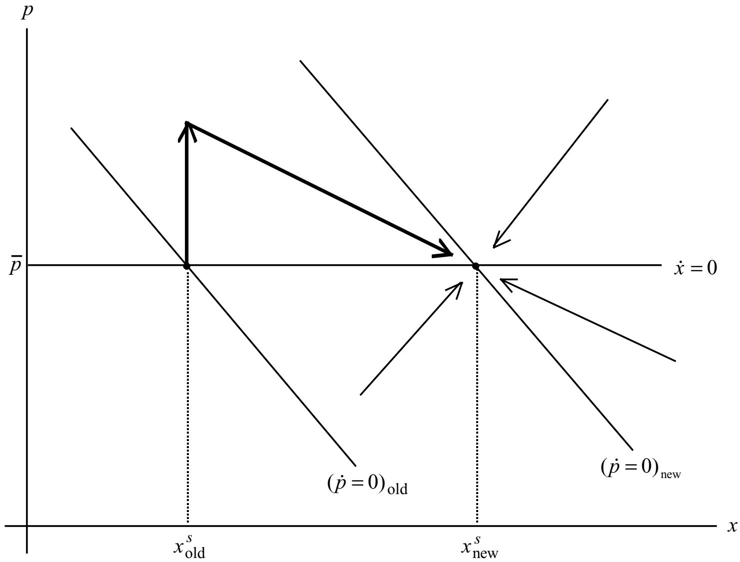

Figure 16.3

Consider an increase in the dominant firm's average cost of production c first. From Eq. (29), we know that the $\dot{x} = 0$ isocline is independent of c and therefore doesn't shift when c increases. We also know from Eq. (30) that steady state rival sales increase, whereas the steady state price of the dominant firm doesn't change, when c increases. Combining these three observations, we therefore conclude that the only way for this to occur in the phase diagram is for the $\dot{p} = 0$ isocline to shift up. This conclusion may be verified analytically by finding $\partial p/\partial c$ along the $\dot{p} = 0$ isocline and evaluating the result at the steady state. By Eq. (25) and the implicit function theorem, we find that

$$\left.\frac{\partial p}{\partial c}\right|_{\substack{\dot{p}=0\\ \dot{x}=0}} = P_c(x^s(\boldsymbol{\alpha}); \boldsymbol{\alpha}) = \left.\frac{-\partial \dot{p}/\partial c}{\partial \dot{p}/\partial p}\right|_{\substack{\dot{p}=0\\ \dot{x}=0}} = \frac{rf'(\bar{p}) - k}{rH_{pp}(\bar{p}; c)} > 0.$$

This means that along the $\dot{p} = 0$ isocline, an increase in c holding $(x, k, \bar{p}, r)$ constant causes p to increase in a neighborhood of the steady state. That is, for the same value of x, p is larger because of the larger value of c. This implies that the $\dot{p} = 0$ isocline shifts up when c increases, just as we deduced above. Figure 16.3 depicts this shift of the $\dot{p} = 0$ isocline in the phase plane.

It is worthwhile at this juncture to pause and repeat three remarks made in Chapter 15 concerning the construction of a local comparative dynamics phase diagram. The first remark is that the local dynamics depicted in Figure 16.2 apply to *both* of the steady states depicted in Figure 16.3. In other words, the local dynamics around the old and the new steady states are qualitatively identical, and are therefore of the saddle point variety. As a result, there is no need to fully draw in the vector

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field around each steady state in Figure 16.3 because the complete vector field for it can be inferred from that in Figure 16.2. Second, before the increase in the average cost of production occurs, the dominant firm is assumed to be at rest at the old steady state. Third, the dominant firm is assumed to eventually come to rest at the new steady state as a result of the increase in the average cost of production. That is, the old steady state value of rival sales is taken as the initial condition in the local comparative dynamics exercise, whereas the new steady state value of rival sales is taken as the terminal condition. The local comparative dynamics phase diagram therefore depicts the optimal transition path from the old to the new steady state that results from the increase in the average cost of production of the dominant firm.

The economic story that emanates from Figure 16.3 is not only relatively straightforward but, more importantly, is sound. The instant the dominant firm's average cost of production increases, it responds by increasing its price above the limit price. The initial increase in the dominant firm's price is such that it jumps to the stable manifold corresponding to the new steady state. Seeing as rival sales are fixed at its old steady state value initially, they do not initially respond to the higher price charged by the dominant firm, hence the initial vertical upward jump in Figure 16.3. The instant after the dominant firm raises its price to cover its higher average production costs, fringe firms enter the market, or equivalently, fringe sales increase, for the dominant firm's price is now above the limit price. Fringe firms continue to enter the market until the new steady state is reached and the dominant firm's price returns to the limit price, at which point entry of new fringe firms ceases. We therefore see that even though the dominant firm's steady state price is not affected by an increase in its average production costs, the price it charges is higher during the transition from the old to the new steady state, and this is what accounts for the larger fringe sales in the new steady state. In other words, even though there are no long-run effects on the dominant firm's price because of its higher average cost of production, there are transitory effects.

To wrap up the qualitative characterization of the effects of an increase in the dominant firm's average cost of production, let's apply Theorem 14.10, the dynamic envelope theorem for discounted autonomous infinite-horizon control problems, to the limit pricing model. In order to do so, first recall that Theorem 14.10 part (i) is stated in terms of the present value Hamiltonian, whereas we have been working with the current value Hamiltonian $H(\cdot)$, as defined in Eq. (5). Thus, we must first multiply $H(\cdot)$ by e^{-rt} to convert it into the present value Hamiltonian. Then we may differentiate $e^{-rt}H(\cdot)$ with respect to c, evaluate the resulting derivative along the optimal path, and finally, integrate the expression over the planning horizon in order to get the correct envelope expression. Doing just that yields

$$V_c(\boldsymbol{\beta}) \equiv -\int_0^{+\infty} [f(p^*(t;\boldsymbol{\beta})) - x^*(t;\boldsymbol{\beta})]\, e^{-rt}\, dt < 0.$$

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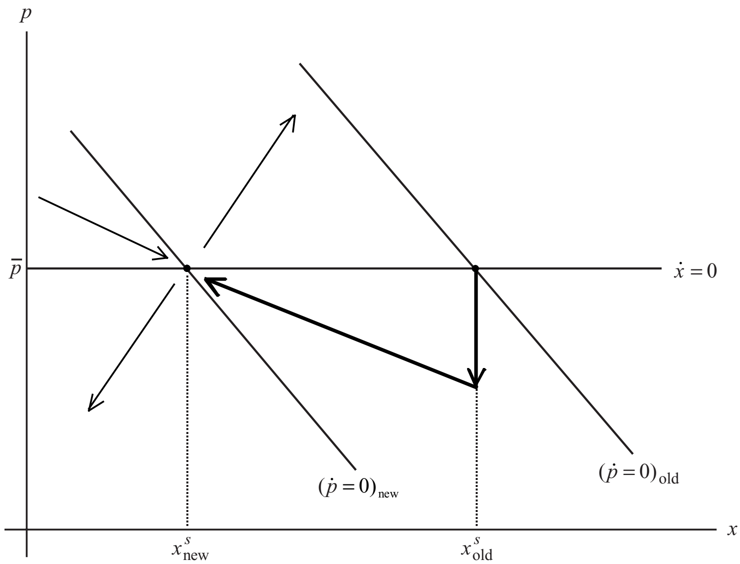

Figure 16.4

Hence, an increase in the dominant firm's average cost of production reduces its present discounted value of profits, even though the dominant firm temporarily raised its price above the limit price in an attempt to cover its higher average costs of production.

Finally, we consider the local comparative dynamics of an increase in the fringe response coefficient k. From Eq. (29), we know that the $\dot{x} = 0$ isocline is independent of k and therefore doesn't shift when k increases. In the case in which $\bar{p} > c$, we see from Eq. (31) that steady state fringe sales fall with the increase in k while the steady state price of the dominant firm is unaffected. These facts imply that the $\dot{p} = 0$ isocline shifts downward in the xp-phase plane. As before, this can be verified analytically by computing $\partial p/\partial k$ along the $\dot{p} = 0$ isocline and evaluating the result at the steady state. By Eq. (25) and the implicit function theorem, we find that

$$\left.\frac{\partial p}{\partial k}\right|_{\substack{\dot{p}=0\\ \dot{x}=0}} = P_k(x^s(\boldsymbol{\alpha}); \boldsymbol{\alpha}) = \left.\frac{-\partial \dot{p}/\partial k}{\partial \dot{p}/\partial p}\right|_{\substack{\dot{p}=0\\ \dot{x}=0}} = \frac{\bar{p} - c}{r H_{pp}(\bar{p}; c)} < 0.$$

This calculation shows that along the $\dot{p} = 0$ isocline, an increase in k holding $(x, c, \bar{p}, r)$ constant causes p to decrease in a neighborhood of the steady state. That is, for the same value of x, p is smaller because of the larger value of k. This implies that the $\dot{p} = 0$ isocline shifts down in the phase plane. The local comparative dynamics phase diagram for an increase in k is depicted in Figure 16.4.

The local comparative dynamics phase diagram provides the additional information required to fully understand the qualitative effects of an increase in k. The

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moment k increases, the dominant firm responds by lowering its price below the limit price so as to jump to the stable manifold corresponding to the new steady state. Because the dominant firm's price is now below the limit price, this action drives fringe firms out of the market, thereby increasing the dominant firm's market share. Over time, the dominant firm gradually increases its price back toward the limit price, all the while continuing to drive fringe firms from the market and increasing its market share. The local comparative dynamics thus reveal why the fringe sales are smaller in the new steady state even though the dominant firm's steady state price is unchanged: in the transition from the old steady state to the new steady state, the dominant firm lowers its price below the limit price, thereby driving fringe firms from the market. The effects of a perturbation in k thus clearly spell out the need for a local comparative dynamic analysis in addition to a steady state comparative static analysis in order to get a complete and sound qualitative understanding of a dynamic economic model.

Note that our analysis of the dynamic limit pricing model has focused on the qualitative properties of the state and control variables, not the state and costate variables. You may recall that in the previous chapter, we studied the optimal economic growth model and focused instead on the state and costate variables. Because both pairs of the ordinary differential equations used for the analysis, namely, the state/costate and state/control pairs, are derived from the same set of necessary conditions, the information contained in either pair of differential equations is identical, and thus yields identical qualitative conclusions. It is thus the economic question of interest that dictates which pair of differential equations to analyze, not some fundamental mathematical property or logical reason. Note, however, that when there is more than one control variable but still a single state variable, there is a mathematical advantage to using the state and costate differential equations for analysis, scilicet, there is only a pair of them to analyze. This means that a phase portrait may be used for the qualitative analysis, a real advantage indeed.

We undertake a systematic qualitative investigation of the adjustment cost model of the firm in the next chapter. In addition to employing a phase diagram for studying the local comparative dynamics properties of the model, we will augment our approach by introducing some analytical material.

MENTAL EXERCISES

16.1 Prove Lemma 16.1.

16.2 This exercise asks you to complete the qualitative analysis of the limit pricing model studied in this chapter by considering the remaining two parameters $(\bar{p}, r)$. Answer the questions below for the dominant firm's price, fringe sales, and market share.

(a) Derive the steady state comparative statics for the limit price $\bar{p}$.

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(b) Derive the local comparative dynamics for the limit price $\bar{p}$ using a phase diagram.
(c) Provide an economic interpretation of the above two qualitative results.
(d) Derive the steady state comparative statics for the discount rate r.
(e) Derive the local comparative dynamics phase diagram for the discount rate r.
(f) Provide an economic interpretation of the above two qualitative results.

16.3 We did not analyze any of the qualitative properties of the current value shadow price of fringe sales in this chapter. You are asked to do so in this exercise. Note that you may use any of the results or equations established in the chapter in your answer.
(a) Define the steady state current value shadow price of fringe sales.
(b) Derive the steady state comparative statics for the current value shadow price of fringe sales for the parameters $(c, k, \bar{p}, r)$.
(c) Provide an economic interpretation of the results.

16.4 This exercise asks you to reconsider the qualitative analysis of the dynamic limit pricing model presented in the chapter by deriving the phase diagram in the λx-phase plane.
(a) Reduce the three necessary conditions (6), (8), and (9) down to two differential equations involving just the two variables (x, λ).
(b) Write down the steady state version of the canonical equations from part (a). Find the steady state values of x and λ, say $x^s(\boldsymbol{\alpha})$ and $\lambda^s(\boldsymbol{\alpha})$, respectively. Is there any need to compute the Jacobian matrix of the steady state equations? Explain.
(c) Derive the steady state comparative statics $\partial x^s(\boldsymbol{\alpha})/\partial c$ and $\partial \lambda^s(\boldsymbol{\alpha})/\partial c$.
(d) Derive the steady state comparative statics $\partial x^s(\boldsymbol{\alpha})/\partial k$ and $\partial \lambda^s(\boldsymbol{\alpha})/\partial k$.
(e) Derive the local comparative dynamics phase diagram for the parameter c. Show all your work and label the diagram carefully.
(f) Derive the local comparative dynamics phase diagram for the parameter k. Show all your work and label the diagram carefully.

16.5 Apply Theorem 14.10 to derive the dynamic envelope results for the parameters k and $\bar{p}$ of the limit pricing model. Provide an economic interpretation of the results.

16.6 Consider a farmer who is concerned about the effects of production on the soil quality of her farm. It is assumed that the soil characteristic of interest can be represented by a single state variable x, say, for example, soil depth. The output of the farm is given by the production relationship $y = f(x, v)$, $f(\cdot) \in C^{(2)}$, where y is the output of the farm and v is the variable input used. The farmer is asserted to solve the static cost minimization problem given by

$$C(x, y; w) \stackrel{\text{def}}{=} \min_{v}\{w \cdot v \text{ s.t. } y = f(x, v)\},$$

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at each moment in time of the planning horizon, where w is the unit cost of the variable input. The restricted minimum cost function $C(\cdot)$ has the following properties:

$$C_x(x, y; w) < 0, C_y(x, y; w) > 0, C_{yx}(x, y; w) < 0, C_{xx}(x, y; w) > 0,$$

$$C_{yy}(x, y; w) > 0, C_{xx}(x, y; w)C_{yy}(x, y; w) - [C_{yx}(x, y; w)]^2 > 0, C \in C^{(2)}.$$

The time rate of change of the soil characteristic $\dot{x}(t)$ is assumed to be negatively affected by the output of the farm via the differential equation

$$\dot{x}(t) = B - \alpha y(t),$$

where $B > 0$ is the constant rate of soil improvement and $\alpha > 0$ measures the marginal impact output has on the rate of change of soil quality. Bringing all the information together, the control problem the farmer is asserted to solve is

$$V(\boldsymbol{\beta}) \stackrel{\text{def}}{=} \max_{y(\cdot)} \int_0^{+\infty} [py(t) - C(x(t), y(t); w)]\, e^{-rt}\, dt$$

$$\text{s.t.} \quad \dot{x}(t) = B - \alpha y(t),\ x(0) = x_0,$$

where $\boldsymbol{\beta} \stackrel{\text{def}}{=} (\boldsymbol{\gamma}, x_0) \stackrel{\text{def}}{=} (\alpha, B, p, r, w, x_0)$, $x_0 > 0$ is the initial soil quality, $r > 0$ is the farmer's discount rate, and $p > 0$ is the constant price of the output. It is assumed that the inequality constraints $x(t) \geq 0$ and $y(t) \geq 0$ are not binding $\forall\, t \in [0, +\infty)$ in an optimal plan, that there exists a solution $(x^*(t; \boldsymbol{\beta}), y^*(t; \boldsymbol{\beta}))$ of the necessary conditions that converges to the simple steady state solution of the necessary conditions $(x^s(\boldsymbol{\gamma}), y^s(\boldsymbol{\gamma}))$ as $t \to +\infty$, and that the objective functional exists for all admissible pairs of functions. Let $\lambda(t; \boldsymbol{\beta})$ be the corresponding time path of the current value costate variable, and note that no conditions are placed on $\lim_{t\to+\infty} x(t)$.

(a) Write down the necessary conditions for the problem in current value form. Provide an economic interpretation to the Maximum Principle equation.

(b) Prove that under suitable additional assumptions to be determined by you, the solution $(x^*(t; \boldsymbol{\beta}), y^*(t; \boldsymbol{\beta}))$ to the necessary conditions is the unique solution of the control problem.

(c) Reduce the necessary conditions down to two differential equations in the variables (x, y). Prove that the fixed point $(x^s(\boldsymbol{\gamma}), y^s(\boldsymbol{\gamma}))$ is a local saddle point.

(d) Find the steady state comparative statics $\partial x^s(\boldsymbol{\gamma})/\partial p$ and $\partial y^s(\boldsymbol{\gamma})/\partial p$. Provide an economic interpretation of the results.

(e) Draw a phase diagram in the xy-phase space, and label it carefully. Plot x on the horizontal axis and y on the vertical axis.

(f) Draw the local comparative dynamics phase diagram for the change in p. Label your diagram carefully, showing the optimal path from one steady

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state to another. Provide an economic interpretation. Is the firm better off facing a higher value of p? Show your work and explain.

(g) Find the steady state comparative statics $\partial x^s(\gamma)/\partial B$ and $\partial y^s(\gamma)/\partial B$. Provide an economic interpretation of the results.

(h) Draw the local comparative dynamics phase diagram for the change in B. Label your diagram carefully, showing the optimal path from one steady state to another. Provide an economic interpretation. Is the firm better off facing a higher value of B? Show your work and explain.

16.7 A society is concerned about the negative externalities associated with its consumption rate $C(t)$, namely, the accumulation of a stock of pollution $P(t)$. The instantaneous utility of the society depends on both the rate of consumption of a homogeneous good and the resulting stock of pollution generated by the consumption, say,

$$U(C, P; \alpha_1, \alpha_2) \stackrel{\text{def}}{=} U^1(C; \alpha_1) + U^2(P; \alpha_2),$$

where $U^i(\cdot) : \Re^2_{++} \to \Re,\ U^i(\cdot) \in C^{(2)},\ i = 1, 2$, and

$$U^1_C(C; \alpha_1) > 0,\ U^1_{CC}(C; \alpha_1) < 0,\ U^1_{C\alpha_1}(C; \alpha_1) > 0,$$
$$U^2_P(P; \alpha_2) < 0,\ U^2_{PP}(P; \alpha_2) < 0,\ U^2_{P\alpha_2}(P; \alpha_2) < 0.$$

In addition, it is assumed that

$$\lim_{C\to 0^+} U^1_C(C; \alpha_1) = +\infty\ \forall\, \alpha_1 > 0,\ \lim_{P\to 0^+} U^2_P(P; \alpha_2) = 0\ \forall\, \alpha_2 > 0.$$

A constant rate of output $Y > 0$ is to be divided between consumption $C(t)$ and pollution control (or pollution elimination) $E(t)$, so that $Y = C(t) + E(t)$ must hold for all t. The stock of pollution is assumed to increase with consumption at an increasing rate as given by the twice continuously differentiable function $g(\cdot)$, that is,

$$g(\cdot) \in C^{(2)}\ \forall\, C > 0,\ g(0) = 0,\ g'(C) > 0,\ \text{and } g''(C) > 0\ \forall\, C > 0.$$

Society can slow the accumulation or hasten the decline of pollution by devoting some output to pollution control. The amount of pollution cleaned up is given by the function $h(\cdot) \in C^{(2)}\ \forall\, E > 0$, where

$$h(0) = 0,\ h'(E) > 0 \text{ and } h''(E) < 0\ \forall\, E > 0,\ \lim_{h\to 0^+} h'(E) = +\infty.$$

Thus society's net contribution to the flow of pollution is given by $g(C) - h(E)$. However, given the assumption of a fixed level of output, the choice of consumption completely determines pollution control expenditure via

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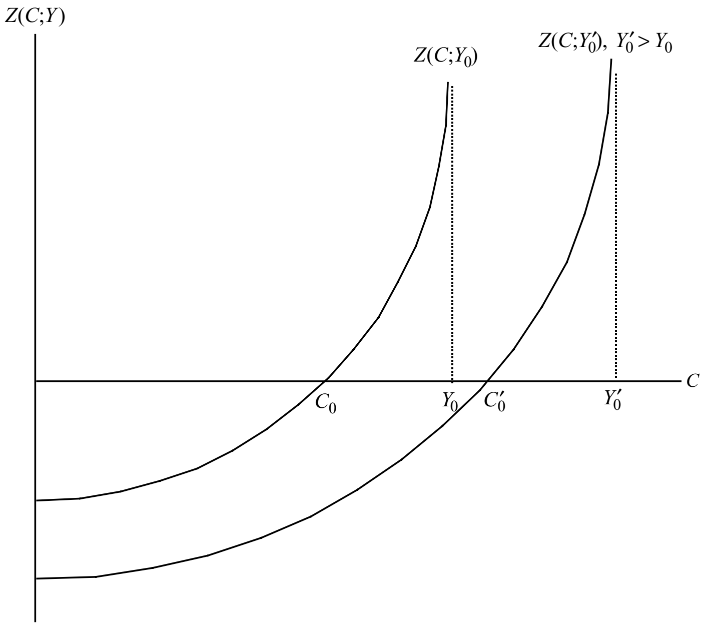

Figure 16.5

$E(t) = Y - C(t)$. Therefore the net contribution to the flow of pollution is given by $Z(C;Y) \stackrel{\text{def}}{=} g(C) - h(Y - C)$, where

$$Z_C(C;Y) = g'(C) + h'(Y - C) > 0,$$

$$Z_{CC}(C;Y) = g''(C) - h''(Y - C) > 0,$$

$$Z_Y(C;Y) = -h'(Y - C) < 0,\ \ Z_{CY}(C;Y) = h''(Y - C) < 0.$$

Hence the flow of pollution increases with consumption at an increasing rate, and an increase in the output of society reduces pollution in total and at the marginal. Finally, let $C = C_0$ be the consumption rate such that the net flow of pollution is zero, that is, $Z(C_0;Y) = 0$. Given that $Z_C(C;Y) > 0$, it follows that

$$Z(C;Y) \begin{cases} < 0\,\forall\, C \in [0, C_0) \\ = 0 \text{ at } C = C_0 \\ > \ 0\,\forall\, C \in (C_0, +\infty). \end{cases}$$

Thus for all $C \in [0, C_0)$, society is net abating, whereas for all $C \in (C_0, +\infty)$, it is net polluting. Figure 16.5 summarizes the qualitative information about the function $Z(\cdot)$.

The function $Z(\cdot)$ may be thought of as the pollution control function for a given Y. By selecting the consumption rate, society uniquely determines the

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amount of pollution it generates in net terms, since Y is constant. The pollution control function therefore has two components: (i) an *active* control given by $h(E)$, where society cleans up pollution directly by devoting part of its output to cleanup activities, and (ii) a *passive* control given by $g(C)$, where society can reduce the rate of pollution accumulation by lowering its consumption rate. It is also assumed that pollution decays at the rate $\delta > 0$. Putting all the above information together, the optimal control problem facing the planning authority is given by

$$V(\boldsymbol{\beta}) \stackrel{\text{def}}{=} \max_{C(\cdot)} \int_0^{+\infty} [U^1(C(t);\alpha_1) + U^2(P(t);\alpha_2)]e^{-rt}\,dt$$

$$\text{s.t.} \quad \dot{P}(t) = Z(C(t);Y) - \delta P(t),\ \ P(0) = P_0,$$

where $\boldsymbol{\beta} \stackrel{\text{def}}{=} (\boldsymbol{\alpha}, P_0) \stackrel{\text{def}}{=} (\alpha_1, \alpha_2, \delta, r, Y, P_0)$ is the vector of time invariant parameters of the control problem, $r > 0$ is the social rate of discount, $P_0 > 0$ is the initial stock of pollution, and no conditions are placed on $\lim_{t\to+\infty} P(t)$. Assume that there exists a solution $(P^*(t;\boldsymbol{\beta}), C^*(t;\boldsymbol{\beta}))$ of the necessary conditions that converges to the simple steady state solution of the necessary conditions $(P^s(\boldsymbol{\alpha}), C^s(\boldsymbol{\alpha}))$ as $t \to +\infty$, and that the objective functional exists for all admissible pairs of functions. Let $\lambda(t;\boldsymbol{\beta})$ be the corresponding time path of the current value costate variable.

(a) Provide an economic interpretation of the assumptions made on the instantaneous utility function.

(b) Derive the necessary conditions in current value form.

(c) What is the economic interpretation of the current value costate variable? Is it positive or negative in an optimal plan? Show your work.

(d) Prove that under suitable additional assumptions to be determined by you, the solution $(P^*(t;\boldsymbol{\beta}), C^*(t;\boldsymbol{\beta}))$ to the necessary conditions is the unique solution of the control problem.

(e) Reduce the necessary conditions down to a pair of ordinary differential equations in the variables (P, C). Prove that the fixed point $(P^s(\boldsymbol{\alpha}), C^s(\boldsymbol{\alpha}))$ is a local saddle point, and that $(P^s(\cdot), C^s(\cdot))$ are locally $C^{(1)}$ in $\boldsymbol{\alpha}$.

(f) Carefully draw the phase diagram for the system of ordinary differential equations in part (e). Label the optimal trajectories. Plot P on the horizontal axis and C on the vertical axis.

(g) Derive the steady state comparative statics $\partial P^s(\boldsymbol{\alpha})/\partial r$ and $\partial C^s(\boldsymbol{\alpha})/\partial r$. Draw the corresponding local comparative dynamics phase diagram. Provide an economic interpretation.

(h) Derive the steady state comparative statics $\partial P^s(\boldsymbol{\alpha})/\partial \alpha_1$ and $\partial C^s(\boldsymbol{\alpha})/\partial \alpha_1$. Draw the corresponding local comparative dynamics phase diagram. Is

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society better off or worse off facing a higher value of α_1? Provide an economic interpretation.

(i) Derive the steady state comparative statics $\partial P^s(\boldsymbol{\alpha})/\partial\alpha_2$ and $\partial C^s(\boldsymbol{\alpha})/\partial\alpha_2$. Draw the corresponding local comparative dynamics phase diagram. Is society better off or worse off facing a higher value of α_2? Provide an economic interpretation.

(j) Derive the steady state comparative statics $\partial P^s(\boldsymbol{\alpha})/\partial Y$ and $\partial C^s(\boldsymbol{\alpha})/\partial Y$. Draw the corresponding local comparative dynamics phase diagram. Is society better off or worse off facing a higher value of Y? Provide an economic interpretation.

16.8 A local planning council has been charged with managing a community's lake. The community has well-defined preferences over both the stock of water in the lake $w(t)$ and the consumption rate $c(t)$ of the water, say, $U(c, w; \alpha)$, where $U(\cdot) \in C^{(2)}$ and α is a taste shift parameter. The community has locally nonsatiated preferences over both the consumption rate and stock of water in the lake, thereby implying that $U_c(c, w; \alpha) > 0$ and $U_w(c, w; \alpha) > 0$. In addition, both marginal utilities are declining; hence $U_{cc}(c, w; \alpha) < 0$ and $U_{ww}(c, w; \alpha) < 0$. Assume, for simplicity, that $U_{cw}(c, w; \alpha) \equiv 0$ and that the natural nonnegativity constraints $c(t) \geq 0$ and $w(t) \geq 0$ are not binding in the optimal plan. The planning council discounts future instantaneous utility at the rate $r > 0$, and plans over the indefinite future for the community. The lake recharges at the constant rate $R > 0$, whereas consumption $c(t)$ reduces the rate of recharge, so that on net, the rate of change of the stock of water in the lake is given by $\dot{w}(t) = R - c(t)$. The initial stock of water is given as $w(0) = w_0 > 0$, and no assumptions are placed on $\lim_{t\to+\infty} w(t)$. The planning council wants to choose the consumption function $c(\cdot)$ to maximize the present discounted utility of the community over the infinite planning horizon. Assume that the objective functional exists for all admissible function pairs. For notational clarity, define $\boldsymbol{\beta} \stackrel{\text{def}}{=} (\boldsymbol{\theta}, w_0) \stackrel{\text{def}}{=} (\alpha, r, R, w_0)$.

(a) Set up the planning council's optimal control problem.

(b) Find the necessary conditions for an optimal plan using the current value Hamiltonian. Provide an economic interpretation of the current value costate variable.

(c) Assume that $(w^*(t; \boldsymbol{\beta}), c^*(t; \boldsymbol{\beta}))$ is a solution to the necessary conditions with the property that as $t \to +\infty$, $(w^*(t; \boldsymbol{\beta}), c^*(t; \boldsymbol{\beta})) \to (w^s(\boldsymbol{\theta}), c^s(\boldsymbol{\theta}))$, where $(w^s(\boldsymbol{\theta}), c^s(\boldsymbol{\theta}))$ is the simple steady state solution of the necessary conditions, and $\lambda(t; \boldsymbol{\beta})$ is the corresponding time path of the current value costate. Prove that $(w^*(t; \boldsymbol{\beta}), c^*(t; \boldsymbol{\beta}))$ is the unique optimal solution of the control problem under suitable additional assumptions to be identified by you.

(d) Reduce the necessary conditions down to a pair of ordinary differential equations in the variables (w, c). Prove that the fixed point $(w^s(\boldsymbol{\theta}), c^s(\boldsymbol{\theta}))$ is a local saddle point, and that $(w^s(\cdot), c^s(\cdot))$ are locally $C^{(1)}$ in $\boldsymbol{\theta}$.

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(e) Draw the phase diagram corresponding to the system of ordinary differential equations in part (d). Show all your work and label your phase diagram carefully. Identify the optimal trajectories in the phase diagram. Plot w on the horizontal axis and c on the vertical axis.

(f) Assume that $U_{c\alpha}(c, w; \alpha) \equiv 0$ and that $U_{w\alpha}(c, w; \alpha) > 0$. What is the economic interpretation of these assumptions?

(g) Prove that $\partial w^s(\boldsymbol{\theta})/\partial\alpha > 0$, $\partial c^s(\boldsymbol{\theta})/\partial\alpha \equiv 0$, and $\partial \mu^s(\boldsymbol{\theta})/\partial\alpha \equiv 0$ under the assumptions in part (f). Provide an economic interpretation.

(h) Draw the local comparative dynamics phase diagram for an increase in α under the assumptions in part (f). Provide an economic interpretation and carefully justify your answer. Is the community better off with a higher value of α? Explain.

16.9 This question is concerned with the adjustment cost model of the capital accumulating firm. Let $f(\cdot) : \Re_+ \to \Re_+$ be the $C^{(2)}$ production function, where $f(0) = 0$, $f'(K) > 0$ for all $K \in \Re_{++}$, $\lim_{K\to+\infty} f'(K) = 0$, and $f''(K) < 0$ for all $K \in \Re_{++}$, where K is the capital stock of the firm. The output of the firm is sold at the constant price of $p > 0$ per unit of output, the capital stock has maintenance costs of $w > 0$ per unit of capital, and $g > 0$ is the constant cost per unit of purchased capital, that is, the purchase price of investment I. In addition, let $C(\cdot) : \Re \to \Re_+$ be the $C^{(2)}$ cost of adjustment function (in dollars), where $C(0) = 0$, $C'(0) = 0$, $\operatorname{sign}(C'(I)) = \operatorname{sign}(I)$, and $C''(I) > 0$ for all $\forall I \in \Re$. The firm is asserted to operate over the indefinite future and discounts its instantaneous profits at the constant rate $r > 0$. The state equation is the prototype capital accumulation equation with a constant rate of decay of $\delta > 0$, to wit, $\dot{K} = I - \delta K$. Finally, the firm begins its planning at time $t = 0$ with the given stock of capital $K(0) = K_0 > 0$, but no restrictions are placed on $\lim_{t\to+\infty} K(t)$. The optimal control problem the firm must solve in order to determine its optimal investment plan is therefore given by

$$V(\boldsymbol{\beta}) \stackrel{\text{def}}{=} \max_{I(\cdot)} \int_0^{+\infty} [pf(K(t)) - wK(t) - gI(t) - C(I(t))]\, e^{-rt}\, dt$$

$$\text{s.t.} \quad \dot{K}(t) = I(t) - \delta K(t),\ K(0) = K_0,$$

$$I(t) \in U \stackrel{\text{def}}{=} \{I(\cdot) : I(t) \geq 0\},$$

where $\boldsymbol{\beta} \stackrel{\text{def}}{=} (K_0, \boldsymbol{\alpha}) \stackrel{\text{def}}{=} (K_0, p, w, g, r, \delta) \in \Re^6_{++}$ are the time invariant parameters of the problem. Assume that the objective functional exists for all admissible pairs of functions, and that there exists a solution $(K^*(t; \boldsymbol{\beta}), I^*(t; \boldsymbol{\beta}))$ of the necessary conditions that converges to the simple steady state solution of the necessary conditions $(K^s(\boldsymbol{\alpha}), I^s(\boldsymbol{\alpha}))$ as $t \to +\infty$, where $\lambda(t; \boldsymbol{\beta})$ is the corresponding time path of the costate variable.

(a) Write down the current value Hamiltonian with costate variable λ. What is the economic interpretation of λ?

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(b) Write down the necessary conditions for this problem. Provide an economic interpretation of the Maximum Principle equation. In particular, if $I(t) > 0$ holds at some $t \in [0, +\infty)$ along the optimal path, then interpret the necessary condition. Similarly, provide an economic interpretation of the sufficient condition for $I(t) = 0$ to hold for some $t \in [0, +\infty)$ along the optimal path.

(c) Assume that $I(t) > 0$ holds for all $t \in [0, +\infty)$ in an optimal program. Prove that under suitable additional assumptions to be determined by you, that the solution $(K^*(t; \boldsymbol{\beta}), I^*(t; \boldsymbol{\beta}))$ to the necessary conditions is the unique solution of the control problem.

(d) Assuming that $I(t) > 0$ holds for all $t \in [0, +\infty)$ in an optimal program, show that the necessary and sufficient conditions can be reduced to a pair of autonomous ordinary differential equations in the variables (K, I). Prove that the fixed point $(K^s(\boldsymbol{\alpha}), I^s(\boldsymbol{\alpha}))$ is a local saddle point, and that $(K^s(\cdot), I^s(\cdot))$ are locally $C^{(1)}$ in $\boldsymbol{\alpha}$.

(e) Derive the phase portrait corresponding to the ordinary differential equations in part (d). Show your work. Plot K on the horizontal axis and I on the vertical axis.

(f) Find the steady state comparative statics $\partial K^s(\boldsymbol{\alpha})/\partial p$ and $\partial I^s(\boldsymbol{\alpha})/\partial p$. Provide an economic interpretation.

(g) Draw the local comparative dynamics phase portrait for the output price p, and identify the optimal path from the old steady state to the new steady state. Provide an economic interpretation of the comparative dynamics result. Is the firm better off facing a higher output price? Show your work and explain.

(h) Find the steady state comparative statics $\partial K^s(\boldsymbol{\alpha})/\partial w$ and $\partial I^s(\boldsymbol{\alpha})/\partial w$. Provide an economic interpretation.

(i) Draw the comparative dynamics phase portrait for the maintenance cost w, and identify the optimal path from the old steady state to the new steady state. Provide an economic interpretation of the comparative dynamics result. Is the firm better off facing a higher maintenance cost? Show your work and explain.

FURTHER READING

The material in this chapter is based on the seminal paper by Gaskins (1971). Numerous extensions of the basic model have appeared in the literature over the past three decades. An important one is that by Kamien and Schwartz (1971), in which the entry and exit of the rival firms are stochastic from the point of view of the dominant firm. Leung (1991) presents a thorough discussion of the transversality conditions associated with the Kamien and Schwartz (1971) model. The dissertation of Ardila Vasquez (1991) is the source for the model of soil depletion in Mental Exercise 16.6. The papers by Barrett (1991) and LaFrance (1992) are closely related to this mental

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exercise too. Mental Exercise 16.7 is based on the work of Forster (1973). Neher (1990) is a good reference for optimal control theory applied to natural resource economics models, and is the reference that spurred Mental Exercise 16.8. Brown and Deacon (1972) is an early application of optimal control theory to groundwater use. For references on the adjustment cost model of the firm, please see the references in Chapter 17.

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