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Comparative Dynamics via Envelope Methods

We continue our development of the dynamic envelope theorem in this chapter, but do so with a different purpose in mind as well as from a different point of view. The purpose herewith is the development of a general method of comparative dynamics, applicable to any sufficiently smooth optimal control problem. The point of view we take is that the parameters of the optimal control problem, rather than the control variables themselves, are viewed as the choice or decision variables. This *dual* point of view is fundamental to our development of a general method of comparative dynamics, in that without it, we would not be able to achieve our goal. We will see that by adopting a dual view of an optimal control problem, we can succeed in providing a one-line proof of the dynamic envelope theorem and, at the same time, more simply reveal the envelope nature of the result. More importantly, however, we will show that the comparative dynamics properties of *all* sufficiently smooth optimal control problems are contained in a symmetric and semidefinite matrix, typically subject to constraint. This matrix, in effect, is a generalized Slutsky-type matrix in integral form, and is shown to characterize the effects that parameter perturbations have on the entire time path of the optimal trajectories. We will also provide sufficient conditions for the optimal value function to be convex in the parameters. Let us now turn to the detailed development of these important results.

The primal form of the fixed endpoint and fixed-time optimal control problem under consideration is given by

$$V(\boldsymbol{\beta}) \stackrel{\text{def}}{=} \max_{\mathbf{u}(\cdot)} \int_{t_0}^{t_1} f(t, \mathbf{x}(t), \mathbf{u}(t); \boldsymbol{\alpha})\, dt \tag{P}$$

$$\text{s.t.} \quad \dot{\mathbf{x}}(t) = \mathbf{g}(t, \mathbf{x}(t), \mathbf{u}(t); \boldsymbol{\alpha}),\ \mathbf{x}(t_0) = \mathbf{x}_0,\ \mathbf{x}(t_1) = \mathbf{x}_1,$$

where $\mathbf{x}(t) \stackrel{\text{def}}{=} (x_1(t), x_2(t), \ldots, x_N(t)) \in \Re^N$ is the state vector, $\mathbf{u}(t) \stackrel{\text{def}}{=} (u_1(t), u_2(t), \ldots, u_M(t)) \in \Re^M$ is the control vector, $\dot{\mathbf{x}}(\cdot) \stackrel{\text{def}}{=} (\dot{x}_1(\cdot), \dot{x}_2(\cdot), \ldots, \dot{x}_N(\cdot))$, $\mathbf{g}(\cdot) \stackrel{\text{def}}{=} (g^1(\cdot), g^2(\cdot), \ldots, g^N(\cdot))$ is the vector of transition functions, $\boldsymbol{\alpha} \stackrel{\text{def}}{=} (\alpha_1,$

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$\alpha_2, \ldots, \alpha_A) \in \Re^A$ is a vector of time-independent parameters that affect the state equations and integrand, and $\boldsymbol{\beta} \stackrel{\text{def}}{=} (\boldsymbol{\alpha}, t_0, \mathbf{x}_0, t_1, \mathbf{x}_1) \in \Re^{2+2N+A}$. With pedagogical considerations at the forefront, we thus impose the following assumptions on problem (P).

(A.1) $f(\cdot) \in C^{(2)}$ and $g^n(\cdot) \in C^{(2)}$, $n = 1, 2, \ldots, N$, on their respective domains.
(A.2) There exists a unique optimal solution to problem (P) for each $\boldsymbol{\beta} \in B(\boldsymbol{\beta}^\circ; \delta)$, which we denote by the triplet $(\mathbf{z}(t; \boldsymbol{\beta}), \mathbf{v}(t; \boldsymbol{\beta}), \boldsymbol{\lambda}(t; \boldsymbol{\beta}))$, where $B(\boldsymbol{\beta}^\circ; \delta)$ is an open $2 + 2N + A$ – ball centered at the given value of the parameter $\boldsymbol{\beta}^\circ$ of radius $\delta > 0$.
(A.3) The vector-valued functions $(\mathbf{z}(\cdot), \mathbf{v}(\cdot), \boldsymbol{\lambda}(\cdot))$ are $C^{(1)}$ in $(t; \boldsymbol{\beta})$ for all $(t; \boldsymbol{\beta}) \in [t_0^\circ, t_1^\circ] \times B(\boldsymbol{\beta}^\circ; \delta)$.
(A.4) $V(\cdot) \in C^{(2)}$ in $\boldsymbol{\beta}$ for all $\boldsymbol{\beta} \in B(\boldsymbol{\beta}^\circ; \delta)$.

Some comments on these assumptions are required before the necessary and sufficient conditions for problem (P) are discussed.

Assumption (A.1) is required because we plan to use the differential calculus in our qualitative characterization of problem (P) and, furthermore, because we intend to use second-order necessary conditions as part of our analysis, thereby necessitating the use of second-order partial derivatives. Assumption (A.2) guarantees that a unique optimal solution exists to problem (P) when the parameter vector takes on values in some open set, a rather mild assumption given the generic nature of problem (P). Assumptions (A.3) and (A.4) are similarly required in view of the fact that we are aiming for a differential characterization of the comparative dynamics properties of the generic problem (P). Note that we are not imposing any other assumptions on problem (P) and its solution, such as concavity, separability, linearity, and the like, and consequently, the results derived henceforth are truly fundamental or intrinsic to problem (P). In passing, observe that $\dot{\mathbf{z}}(\cdot) \in C^{(1)}$ in $(t; \boldsymbol{\beta})$ for all $(t; \boldsymbol{\beta}) \in [t_0^\circ, t_1^\circ] \times B(\boldsymbol{\beta}^\circ; \delta)$, a result that will be used in this chapter, and which you are asked to prove in a mental exercise.

The Hamiltonian for problem (P) is defined as

$$H(t, \mathbf{x}, \mathbf{u}, \boldsymbol{\lambda}; \boldsymbol{\alpha}) \stackrel{\text{def}}{=} f(t, \mathbf{x}, \mathbf{u}; \boldsymbol{\alpha}) + \boldsymbol{\lambda}'\mathbf{g}(t, \mathbf{x}, \mathbf{u}; \boldsymbol{\alpha}). \tag{1}$$

By Corollary 4.2, the necessary conditions are given by

$$H_{\mathbf{u}}(t, \mathbf{x}, \mathbf{u}, \boldsymbol{\lambda}; \boldsymbol{\alpha}) = \mathbf{0}'_M, \tag{2}$$

$$\mathbf{h}' H_{\mathbf{uu}}(t, \mathbf{x}, \mathbf{u}, \boldsymbol{\lambda}; \boldsymbol{\alpha})\mathbf{h} \leq 0 \, \forall \, \mathbf{h} \in \Re^M, \tag{3}$$

$$\dot{\boldsymbol{\lambda}}' = -H_{\mathbf{x}}(t, \mathbf{x}, \mathbf{u}, \boldsymbol{\lambda}; \boldsymbol{\alpha}), \tag{4}$$

$$\dot{\mathbf{x}}' = H_{\boldsymbol{\lambda}}(t, \mathbf{x}, \mathbf{u}, \boldsymbol{\lambda}; \boldsymbol{\alpha}), \tag{5}$$

$$\mathbf{x}(t_0) = \mathbf{x}_0, \;\; \mathbf{x}(t_1) = \mathbf{x}_1. \tag{6}$$

By Theorem 4.3, we know that if $H(\cdot)$ is a concave function of $(\mathbf{x}, \mathbf{u})$ for all $t \in [t_0, t_1]$ when the costate vector is $\boldsymbol{\lambda}(t; \boldsymbol{\beta})$, then a solution of Eqs. (2) through (6) is a solution of problem (P). With these fundamentals set out, we now turn to the dual view of problem (P) and its qualitative properties.

The so-called *dynamic primal-dual problem* corresponding to the primal problem (P) is defined as

$$\max_{\boldsymbol{\beta}} D(\boldsymbol{\beta}) \stackrel{\text{def}}{=} \int_{t_0}^{t_1} f(t, \mathbf{z}(t;\bar{\boldsymbol{\beta}}), \mathbf{v}(t;\bar{\boldsymbol{\beta}}); \boldsymbol{\alpha})\, dt - V(\boldsymbol{\beta})$$

$$\text{s.t.} \quad \mathbf{g}(t, \mathbf{z}(t;\bar{\boldsymbol{\beta}}), \mathbf{v}(t;\bar{\boldsymbol{\beta}}); \boldsymbol{\alpha}) - \dot{\mathbf{z}}(t;\bar{\boldsymbol{\beta}}) = 0, \tag{P-D}$$

$$\mathbf{z}(t_0;\bar{\boldsymbol{\beta}}) = \mathbf{x}_0,\ \mathbf{z}(t_1;\bar{\boldsymbol{\beta}}) = \mathbf{x}_1,$$

where

$$V(\boldsymbol{\beta}) \stackrel{\text{def}}{=} \int_{t_0}^{t_1} f(t, \mathbf{z}(t;\boldsymbol{\beta}), \mathbf{v}(t;\boldsymbol{\beta}); \boldsymbol{\alpha})\, dt \tag{7}$$

is the constructive, and equivalent, definition of the optimal value function for problem (P), for any $\boldsymbol{\beta} \in B(\boldsymbol{\beta}^\circ;\delta)$, and $(\mathbf{z}(t;\bar{\boldsymbol{\beta}}), \mathbf{v}(t;\bar{\boldsymbol{\beta}}))$ is the optimal pair of curves given that the parameter vector $\boldsymbol{\beta}$ is fixed at the *arbitrary* value $\bar{\boldsymbol{\beta}} \in B(\boldsymbol{\beta}^\circ;\delta)$. It is important to recognize that in problem (P-D), the pair of curves $(\mathbf{z}(t;\bar{\boldsymbol{\beta}}), \mathbf{v}(t;\bar{\boldsymbol{\beta}}))$ is fixed because the parameter vector $\boldsymbol{\beta}$ is fixed at the arbitrary value $\bar{\boldsymbol{\beta}} \in B(\boldsymbol{\beta}^\circ;\delta)$, whereas the parameter vector $\boldsymbol{\beta} \stackrel{\text{def}}{=} (\boldsymbol{\alpha}, t_0, \mathbf{x}_0, t_1, \mathbf{x}_1) \in \Re^{2+2N+A}$, not $\bar{\boldsymbol{\beta}}$, can be freely chosen because it is not held fixed, as signified by the lack of an over bar. Therefore, by construction, or equivalently, by definition of problems (P) and (P-D), $D(\boldsymbol{\beta}) \leq 0\, \forall\, \boldsymbol{\beta} \in B(\boldsymbol{\beta}^\circ;\delta)$ seeing as $f(\cdot)$ is evaluated along the pair of curves $(\mathbf{z}(t;\bar{\boldsymbol{\beta}}), \mathbf{v}(t;\bar{\boldsymbol{\beta}}))$, which is optimal only when $\boldsymbol{\beta} = \bar{\boldsymbol{\beta}}$. Furthermore, when $\boldsymbol{\beta} = \bar{\boldsymbol{\beta}}$ in problem (P-D), $f(\cdot)$ is then evaluated along $(t, \mathbf{z}(t;\bar{\boldsymbol{\beta}}), \mathbf{v}(t;\bar{\boldsymbol{\beta}}), \bar{\alpha})$, thereby implying that $D(\bar{\boldsymbol{\beta}}) = 0$. That is to say,

$$D(\bar{\boldsymbol{\beta}}) = \int_{\bar{t}_0}^{\bar{t}_1} f(t, \mathbf{z}(t;\bar{\boldsymbol{\beta}}), \mathbf{v}(t;\bar{\boldsymbol{\beta}}); \bar{\boldsymbol{\alpha}})\, dt - V(\bar{\boldsymbol{\beta}}) = \max_{\boldsymbol{\beta}} D(\boldsymbol{\beta}) = 0$$

by assumption (A.2), the definition of the optimal value function $V(\cdot)$ in problem (P), and the definition of the dynamic primal-dual problem (P-D).

The above development demonstrates that the dynamic primal-dual problem (P-D) has a known solution by its very construction. Notice that we are treating the parameter vector $\boldsymbol{\beta}$ as the decision vector in the dynamic primal-dual problem (P-D), whereas the pair of curves $(\mathbf{z}(t;\bar{\boldsymbol{\beta}}), \mathbf{v}(t;\bar{\boldsymbol{\beta}}))$ are held fixed or parametric, an *exact inversion* of their roles in the primal (or usual) form of the control problem (P). It is precisely this dual view of the primal optimal control problem (P), that is, the inversion of the roles of the parameters and control variables, that leads to the powerful results derived herein. Simply put, the dynamic primal-dual problem

(P-D) treats the *explicit* appearance of the parameter vector $\boldsymbol{\beta} \stackrel{\text{def}}{=} (\boldsymbol{\alpha}, t_0, \mathbf{x}_0, t_1, \mathbf{x}_1) \in \Re^{2+2N+A}$ as the decision vector, but treats the *implicit* appearance of the parameter vector, that is, the occurrence of $\boldsymbol{\beta}$ as an argument of the state, control, and costate vectors, as fixed.

In passing, we should remark on the appellation *primal-dual* employed when defining problem (P-D). The *primal* portion of the name comes from the fact that if we adopt the perspective that the parameter vector $\boldsymbol{\beta}$ is fixed and the control vector is the decision vector in problem (P-D), just as we do in the primal problem (P), then problem (P-D) is identical to problem (P). On the other hand, if we adopt the perspective that the time paths of the state and control variables are fixed while the parameter vector $\boldsymbol{\beta}$ is the decision vector in problem (P-D), then we have a dual optimization problem (P-D). Dual, in this instance, simply means that we have two related optimization problems that can be generated from a single underlying optimization problem, and the decision vectors in each lie in different spaces. This usage of the word *dual* is in accord with that when one contemplates dual pairs of linear programming problems, for example. In sum, therefore, problem (P-D) contains the primal (or original) *and* dual optimization problems in a single problem statement. Let us now continue with the development of the dynamic primal-dual problem (P-D).

Observe that by Eqs. (1) and (7), and Eq. (5) expressed in identity form, we may obtain the revealing result that

$$\begin{aligned}
V(\boldsymbol{\beta}) &\stackrel{\text{def}}{=} \int_{t_0}^{t_1} f(t, \mathbf{z}(t;\boldsymbol{\beta}), \mathbf{v}(t;\boldsymbol{\beta}); \boldsymbol{\alpha})\, dt \\
&= \int_{t_0}^{t_1} [H(t, \mathbf{z}(t;\boldsymbol{\beta}), \mathbf{v}(t;\boldsymbol{\beta}), \boldsymbol{\lambda}(t;\boldsymbol{\beta}); \boldsymbol{\alpha}) - \boldsymbol{\lambda}(t;\boldsymbol{\beta})'\mathbf{g}(t, \mathbf{z}(t;\boldsymbol{\beta}), \mathbf{v}(t;\boldsymbol{\beta}); \boldsymbol{\alpha})]\, dt \\
&= \int_{t_0}^{t_1} [H(t, \mathbf{z}(t;\boldsymbol{\beta}), \mathbf{v}(t;\boldsymbol{\beta}), \boldsymbol{\lambda}(t;\boldsymbol{\beta}); \boldsymbol{\alpha}) - \boldsymbol{\lambda}(t;\boldsymbol{\beta})'\dot{\mathbf{z}}(t;\boldsymbol{\beta})]\, dt \\
&\neq \int_{t_0}^{t_1} H(t, \mathbf{z}(t;\boldsymbol{\beta}), \mathbf{v}(t;\boldsymbol{\beta}), \boldsymbol{\lambda}(t;\boldsymbol{\beta}); \boldsymbol{\alpha})\, dt. \qquad (8)
\end{aligned}$$

Equation (8) shows that the optimal value function $V(\cdot)$ is *not* defined as the integral of the Hamiltonian function $H(\cdot)$ evaluated along the optimal solution, but is instead defined as the integral of the Hamiltonian function $H(\cdot)$, minus the inner product of the costate vector and time derivative of the state vector, $\boldsymbol{\lambda}'\dot{\mathbf{x}}$, evaluated along the optimal solution. Therefore, even though the standard method of optimal control theory may be applied to problem (P) along with the static envelope theorem and

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static primal-dual methodology of Silberberg (1974) to derive the curvature properties of the Hamiltonian function $H(\cdot)$, Eq. (8) makes it clear that knowing the curvature properties of $H(\cdot)$ is not, in general, sufficient to demonstrate any curvature properties of $V(\cdot)$. Consequently, it is thus essential that the integral form of the solution to the optimal control problem be studied in order to ascertain the curvature properties of $V(\cdot)$. This provides the necessary motivation for studying the primal-dual problem (P-D) corresponding to the primal problem (P).

Because the parameter vector $\boldsymbol{\beta}$ is time independent, *static* Lagrangian optimization methods may be applied to the dynamic primal-dual problem (P-D) to study the comparative dynamics properties of the corresponding primal control problem (P). In order to form the proper Lagrangian function corresponding to problem (P-D), we begin with a few important observations. First, because the state equation constraint $\mathbf{g}(t, \mathbf{z}(t; \bar{\boldsymbol{\beta}}), \mathbf{v}(t; \bar{\boldsymbol{\beta}}); \boldsymbol{\alpha}) - \dot{\mathbf{z}}(t; \bar{\boldsymbol{\beta}}) = 0$ of problem (P-D) must hold for all $t \in [t_0, t_1]$, it must be integrated over the planning horizon $[t_0, t_1]$, just like the integrand function $f(\cdot)$, after it is taken with the inner product of the proper multiplier vector. Second, the correct multiplier vector for this operation is the time path of the costate vector when $\boldsymbol{\beta} = \bar{\boldsymbol{\beta}}$, scilicet, $\boldsymbol{\lambda}(t; \bar{\boldsymbol{\beta}})$, which is optimal when $\boldsymbol{\beta} = \bar{\boldsymbol{\beta}}$. To see that $\boldsymbol{\lambda}(t; \bar{\boldsymbol{\beta}})$ is the correct multiplier vector for the state equation constraint $\mathbf{g}(t, \mathbf{z}(t; \bar{\boldsymbol{\beta}}), \mathbf{v}(t; \bar{\boldsymbol{\beta}}); \boldsymbol{\alpha}) - \dot{\mathbf{z}}(t; \bar{\boldsymbol{\beta}}) = 0$ of problem (P-D), one need only recognize that $\boldsymbol{\lambda}(t; \bar{\boldsymbol{\beta}})$ is precisely the costate vector that corresponds to the pair of curves $(\mathbf{z}(t; \bar{\boldsymbol{\beta}}), \mathbf{v}(t; \bar{\boldsymbol{\beta}}))$, which is itself optimal when $\boldsymbol{\beta} = \bar{\boldsymbol{\beta}}$. Third, it is not necessary to adopt this procedure for the initial and terminal condition constraints in view of the fact that they hold only at one point in time in the planning horizon rather than over an interval. Finally, take note that the present development is wholly analogous to that employed in the proof of the necessary conditions in Chapter 2.

With the above observations in mind, the Lagrangian function $L(\cdot)$ corresponding to the dynamic primal-dual problem (P-D) is defined as

$$
\begin{aligned}
L(\boldsymbol{\beta}) &\stackrel{\text{def}}{=} \int_{t_0}^{t_1} [f(t, \mathbf{z}(t; \bar{\boldsymbol{\beta}}), \mathbf{v}(t; \bar{\boldsymbol{\beta}}); \boldsymbol{\alpha}) + \boldsymbol{\lambda}(t; \bar{\boldsymbol{\beta}})' [\mathbf{g}(t, \mathbf{z}(t; \bar{\boldsymbol{\beta}}), \mathbf{v}(t; \bar{\boldsymbol{\beta}}); \boldsymbol{\alpha}) \\
&\qquad - \dot{\mathbf{z}}(t; \bar{\boldsymbol{\beta}})]]\, dt - V(\boldsymbol{\beta}) \qquad (9) \\
&= \int_{t_0}^{t_1} [H(t, \mathbf{z}(t; \bar{\boldsymbol{\beta}}), \mathbf{v}(t; \bar{\boldsymbol{\beta}}), \boldsymbol{\lambda}(t; \bar{\boldsymbol{\beta}}); \boldsymbol{\alpha}) - \boldsymbol{\lambda}(t; \bar{\boldsymbol{\beta}})' \dot{\mathbf{z}}(t; \bar{\boldsymbol{\beta}})]\, dt - V(\boldsymbol{\beta}),
\end{aligned}
$$

which certainly smacks of, but is not identical to, Eq. (8). Integration of Eq. (9) by parts in what should now be a familiar manner, and use of the initial and terminal condition constraints on the state vector, to wit, $\mathbf{z}(t_0; \bar{\boldsymbol{\beta}}) = \mathbf{x}_0$ and $\mathbf{z}(t_1; \bar{\boldsymbol{\beta}}) = \mathbf{x}_1$, respectively, yields an alternative but equivalent form of the Lagrangian function for problem (P-D), namely,

$$L(\boldsymbol{\beta}) \overset{\text{def}}{=} \int_{t_0}^{t_1} [H(t, \mathbf{z}(t;\bar{\boldsymbol{\beta}}), \mathbf{v}(t;\bar{\boldsymbol{\beta}}), \boldsymbol{\lambda}(t;\bar{\boldsymbol{\beta}}); \boldsymbol{\alpha}) + \dot{\boldsymbol{\lambda}}(t;\bar{\boldsymbol{\beta}})'\mathbf{z}(t;\bar{\boldsymbol{\beta}})]\, dt$$
$$- \boldsymbol{\lambda}(t_1;\bar{\boldsymbol{\beta}})'\mathbf{x}_1 + \boldsymbol{\lambda}(t_0;\bar{\boldsymbol{\beta}})'\mathbf{x}_0 - V(\boldsymbol{\beta}). \tag{10}$$

This is the form of the Lagrangian we intend to work with.

Alternatively, one could arrive at Eq. (10) by integrating Eq. (9) by parts, just as we did above. But rather than substitute $\mathbf{z}(t_0;\bar{\boldsymbol{\beta}}) = \mathbf{x}_0$ and $\mathbf{z}(t_1;\bar{\boldsymbol{\beta}}) = \mathbf{x}_1$ into that result, we could add the two inner product terms $\boldsymbol{\lambda}(t_0;\bar{\boldsymbol{\beta}})'[\mathbf{x}_0 - \mathbf{z}(t_0;\bar{\boldsymbol{\beta}})]$ and $\boldsymbol{\lambda}(t_1;\bar{\boldsymbol{\beta}})'[\mathbf{z}(t_1;\bar{\boldsymbol{\beta}}) - \mathbf{x}_1]$ into the integration-by-parts Lagrangian expression to account for the endpoint constraints on the state vector, and then cancel four of the terms in the resulting expression to arrive at Eq. (10). Accordingly, there are two different, but equivalent, ways to incorporate the two endpoint constraints on the state vector into the Lagrangian function corresponding to problem (P-D).

As remarked above, problem (P-D) is a static optimization problem. Using Eq. (10), we therefore have the following first-order necessary conditions for problem (P-D):

$$L_{\boldsymbol{\alpha}}(\boldsymbol{\beta}) = \int_{t_0}^{t_1} H_{\boldsymbol{\alpha}}(t, \mathbf{z}(t;\bar{\boldsymbol{\beta}}), \mathbf{v}(t;\bar{\boldsymbol{\beta}}), \boldsymbol{\lambda}(t;\bar{\boldsymbol{\beta}}); \boldsymbol{\alpha})\, dt - V_{\boldsymbol{\alpha}}(\boldsymbol{\beta}) = \mathbf{0}'_A, \tag{11}$$

$$L_{t_0}(\boldsymbol{\beta}) = -H(t_0, \mathbf{z}(t_0;\bar{\boldsymbol{\beta}}), \mathbf{v}(t_0;\bar{\boldsymbol{\beta}}), \boldsymbol{\lambda}(t_0;\bar{\boldsymbol{\beta}}); \boldsymbol{\alpha})$$
$$- \dot{\boldsymbol{\lambda}}(t_0;\bar{\boldsymbol{\beta}})'\mathbf{z}(t_0;\bar{\boldsymbol{\beta}}) + \dot{\boldsymbol{\lambda}}(t_0;\bar{\boldsymbol{\beta}})'\mathbf{x}_0 - V_{t_0}(\boldsymbol{\beta}) = 0, \tag{12}$$

$$L_{\mathbf{x}_0}(\boldsymbol{\beta}) = \boldsymbol{\lambda}(t_0;\bar{\boldsymbol{\beta}})' - V_{\mathbf{x}_0}(\boldsymbol{\beta}) = \mathbf{0}'_N, \tag{13}$$

$$L_{t_1}(\boldsymbol{\beta}) = H(t_1, \mathbf{z}(t_1;\bar{\boldsymbol{\beta}}), \mathbf{v}(t_1;\bar{\boldsymbol{\beta}}), \boldsymbol{\lambda}(t_1;\bar{\boldsymbol{\beta}}); \boldsymbol{\alpha})$$
$$+ \dot{\boldsymbol{\lambda}}(t_1;\bar{\boldsymbol{\beta}})'\mathbf{z}(t_1;\bar{\boldsymbol{\beta}}) - \dot{\boldsymbol{\lambda}}(t_1;\bar{\boldsymbol{\beta}})'\mathbf{x}_1 - V_{t_1}(\boldsymbol{\beta}) = 0, \tag{14}$$

$$L_{\mathbf{x}_1}(\boldsymbol{\beta}) = -\boldsymbol{\lambda}(t_1;\bar{\boldsymbol{\beta}})' - V_{\mathbf{x}_1}(\boldsymbol{\beta}) = \mathbf{0}'_N, \tag{15}$$

$$\mathbf{g}(t, \mathbf{z}(t;\bar{\boldsymbol{\beta}}), \mathbf{v}(t;\bar{\boldsymbol{\beta}}); \boldsymbol{\alpha}) - \dot{\mathbf{z}}(t;\bar{\boldsymbol{\beta}}) = 0, \tag{16}$$

$$\mathbf{x}_0 - \mathbf{z}(t_0;\bar{\boldsymbol{\beta}}) = \mathbf{0}_N, \tag{17}$$

$$\mathbf{z}(t_1;\bar{\boldsymbol{\beta}}) - \mathbf{x}_1 = \mathbf{0}_N, \tag{18}$$

which all hold at $\boldsymbol{\beta} = \bar{\boldsymbol{\beta}}$ by construction. In fact, because $(\mathbf{z}(t;\boldsymbol{\beta}), \mathbf{v}(t;\boldsymbol{\beta}))$ is an optimal pair and $\boldsymbol{\lambda}(t;\boldsymbol{\beta})$ is the corresponding costate vector for all $\boldsymbol{\beta} \in B(\boldsymbol{\beta}^\circ;\delta)$, and because $\bar{\boldsymbol{\beta}} \in B(\boldsymbol{\beta}^\circ;\delta)$ is an *arbitrary* value of $\boldsymbol{\beta}$, the fact that Eqs. (11) through (18) hold for $\boldsymbol{\beta} = \bar{\boldsymbol{\beta}}$ implies that Eqs. (11) through (18) also hold for *any* $\boldsymbol{\beta} \in B(\boldsymbol{\beta}^\circ;\delta)$ as long as the domain of all the functions are evaluated at the same value of $\boldsymbol{\beta} \in$

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$B(\boldsymbol{\beta}^\circ;\delta)$. Moreover, seeing as $\mathbf{z}(t_0;\boldsymbol{\beta}) \equiv \mathbf{x}_0$ and $\mathbf{z}(t_1;\boldsymbol{\beta}) \equiv \mathbf{x}_1$ for all $\boldsymbol{\beta} \in B(\boldsymbol{\beta}^\circ;\delta)$ from Eqs. (17) and (18), respectively, the two inner product terms in Eqs. (12) and (14) cancel, thereby leaving a simplified expression for these equations and resulting in an alternative proof of Theorem 9.1, the dynamic envelope theorem for the optimal control problem (P).

Theorem 11.1 (Dynamic Envelope Theorem): *For optimal control problem (P), with assumptions (A.1) through (A.4) holding, the following envelope results exist and are $C^{(1)}$ for all $\boldsymbol{\beta} \in B(\boldsymbol{\beta}^\circ;\delta)$:*

$$V_{\boldsymbol{\alpha}}(\boldsymbol{\beta}) \equiv \int_{t_0}^{t_1} H_{\boldsymbol{\alpha}}(t, \mathbf{z}(t;\boldsymbol{\beta}), \mathbf{v}(t;\boldsymbol{\beta}), \boldsymbol{\lambda}(t;\boldsymbol{\beta}); \boldsymbol{\alpha})\, dt,$$

$$V_{t_0}(\boldsymbol{\beta}) \equiv -H(t_0, \mathbf{z}(t_0;\boldsymbol{\beta}), \mathbf{v}(t_0;\boldsymbol{\beta}), \boldsymbol{\lambda}(t_0;\boldsymbol{\beta}); \boldsymbol{\alpha}),$$

$$V_{\mathbf{x}_0}(\boldsymbol{\beta}) \equiv \boldsymbol{\lambda}(t_0;\boldsymbol{\beta})',$$

$$V_{t_1}(\boldsymbol{\beta}) \equiv H(t_1, \mathbf{z}(t_1;\boldsymbol{\beta}), \mathbf{v}(t_1;\boldsymbol{\beta}), \boldsymbol{\lambda}(t_1;\boldsymbol{\beta}); \boldsymbol{\alpha}),$$

$$V_{\mathbf{x}_1}(\boldsymbol{\beta}) \equiv -\boldsymbol{\lambda}(t_1;\boldsymbol{\beta})'.$$

Recall that in Chapter 9, we proved the dynamic envelope theorem, that is, Theorem 9.1, by differentiating the constructive definition of the optimal value function, given by Eq. (7) in the present chapter, applying the integration-by-parts formula, and making use of the necessary condition of Corollary 4.2. That proof, therefore, may be considered the dynamic equivalent of the proof of the prototype static envelope theorem by Silberberg and Suen (2001, pp. 160–161). In contrast, the above proof of the dynamic envelope theorem may be considered the dynamic equivalent of the primal-dual proof of the static envelope theorem by Silberberg (1974).

Experience with static optimization theory strongly suggests that in searching for the qualitative properties of any mathematical model, the second-order or curvature properties of the model must be investigated. That is, although first-order properties such as the above dynamic envelope theorem do provide useful qualitative information about a particular problem, it is the second-order necessary conditions that provide the vast majority of qualitative information about an optimization problem, constrained or not.

Remembering that problem (P-D) is a static optimization problem, its second-order necessary conditions, which hold at $\boldsymbol{\beta} = \bar{\boldsymbol{\beta}}$ by construction, are given by

$$\mathbf{a}' L_{\boldsymbol{\beta}\boldsymbol{\beta}}(\boldsymbol{\beta})\mathbf{a} \le 0 \,\forall\, \mathbf{a} \in \Re^{2+2N+A} \ni G_{\boldsymbol{\beta}}(\boldsymbol{\beta})\mathbf{a} = \mathbf{0}_{3N}, \tag{19}$$

where

$$\underset{1\times(2+2N+A)}{\mathbf{a}'} \overset{\text{def}}{=} \begin{pmatrix} \underset{1\times A}{\mathbf{a}^{1\prime}} & \underset{1\times 1}{a^{2\prime}} & \underset{1\times N}{\mathbf{a}^{3\prime}} & \underset{1\times 1}{a^{4\prime}} & \underset{1\times N}{\mathbf{a}^{5\prime}} \end{pmatrix}, \tag{20}$$

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$$\underset{(2+2N+A)\times(2+2N+A)}{L_{\beta\beta}(\boldsymbol{\beta})} = \begin{bmatrix} \underset{A\times A}{L_{\alpha\alpha}(\boldsymbol{\beta})} & \underset{A\times 1}{L_{\alpha t_0}(\boldsymbol{\beta})} & \underset{A\times N}{L_{\alpha \mathbf{x}_0}(\boldsymbol{\beta})} & \underset{A\times 1}{L_{\alpha t_1}(\boldsymbol{\beta})} & \underset{A\times N}{L_{\alpha \mathbf{x}_1}(\boldsymbol{\beta})} \\ \underset{1\times A}{L_{t_0\alpha}(\boldsymbol{\beta})} & \underset{1\times 1}{L_{t_0 t_0}(\boldsymbol{\beta})} & \underset{1\times N}{L_{t_0 \mathbf{x}_0}(\boldsymbol{\beta})} & \underset{1\times 1}{L_{t_0 t_1}(\boldsymbol{\beta})} & \underset{1\times N}{L_{t_0 \mathbf{x}_1}(\boldsymbol{\beta})} \\ \underset{N\times A}{L_{\mathbf{x}_0\alpha}(\boldsymbol{\beta})} & \underset{N\times 1}{L_{\mathbf{x}_0 t_0}(\boldsymbol{\beta})} & \underset{N\times N}{L_{\mathbf{x}_0 \mathbf{x}_0}(\boldsymbol{\beta})} & \underset{N\times 1}{L_{\mathbf{x}_0 t_1}(\boldsymbol{\beta})} & \underset{N\times N}{L_{\mathbf{x}_0 \mathbf{x}_1}(\boldsymbol{\beta})} \\ \underset{1\times A}{L_{t_1\alpha}(\boldsymbol{\beta})} & \underset{1\times 1}{L_{t_1 t_0}(\boldsymbol{\beta})} & \underset{1\times N}{L_{t_1 \mathbf{x}_0}(\boldsymbol{\beta})} & \underset{1\times 1}{L_{t_1 t_1}(\boldsymbol{\beta})} & \underset{1\times N}{L_{t_1 \mathbf{x}_1}(\boldsymbol{\beta})} \\ \underset{N\times A}{L_{\mathbf{x}_1\alpha}(\boldsymbol{\beta})} & \underset{N\times 1}{L_{\mathbf{x}_1 t_0}(\boldsymbol{\beta})} & \underset{N\times N}{L_{\mathbf{x}_1 \mathbf{x}_0}(\boldsymbol{\beta})} & \underset{N\times 1}{L_{\mathbf{x}_1 t_1}(\boldsymbol{\beta})} & \underset{N\times N}{L_{\mathbf{x}_1 \mathbf{x}_1}(\boldsymbol{\beta})} \end{bmatrix}, \tag{21}$$

$$\underset{1\times 3N}{G(\boldsymbol{\beta})'} \stackrel{\text{def}}{=} \Big[\underset{1\times N}{\mathbf{g}(t, \mathbf{z}(t;\bar{\boldsymbol{\beta}}), \mathbf{v}(t;\bar{\boldsymbol{\beta}}); \boldsymbol{\alpha})' - \dot{\mathbf{z}}(t;\bar{\boldsymbol{\beta}})'} \quad \underset{1\times N}{\mathbf{x}_0' - \mathbf{z}(t_0;\bar{\boldsymbol{\beta}})'} \quad \underset{1\times N}{\mathbf{z}(t_1;\bar{\boldsymbol{\beta}})' - \mathbf{x}_1'}\Big], \tag{22}$$

and therefore

$$\underset{3N\times(2+2N+A)}{G_{\beta}(\boldsymbol{\beta})} = \begin{bmatrix} \underset{N\times A}{\mathbf{g}_{\alpha}(t, \mathbf{z}(t;\bar{\boldsymbol{\beta}}), \mathbf{v}(t;\bar{\boldsymbol{\beta}}); \boldsymbol{\alpha})} & \underset{N\times 1}{\mathbf{0}} & \underset{N\times N}{\mathbf{0}} & \underset{N\times 1}{\mathbf{0}} & \underset{N\times N}{\mathbf{0}} \\ \underset{N\times A}{\mathbf{0}} & \underset{N\times 1}{-\dot{\mathbf{z}}(t_0;\bar{\boldsymbol{\beta}})} & \underset{N\times N}{\mathbf{I}} & \underset{N\times 1}{\mathbf{0}} & \underset{N\times N}{\mathbf{0}} \\ \underset{N\times A}{\mathbf{0}} & \underset{N\times 1}{\mathbf{0}} & \underset{N\times N}{\mathbf{0}} & \underset{N\times 1}{\dot{\mathbf{z}}(t_1;\bar{\boldsymbol{\beta}})} & \underset{N\times N}{-\mathbf{I}} \end{bmatrix}. \tag{23}$$

It is these conditions that reveal the qualitative properties of problem (P). The results achieved may be summed up as follows: the qualitative restrictions implied by the dynamic maximization assertion and the mathematical structure of problem (P), as revealed by the dynamic primal-dual methodology, are contained in a symmetric negative semidefinite matrix subject to constraint. Symmetry follows from the $C^{(2)}$ nature of $f(\cdot)$ and $\mathbf{g}(\cdot)$ from assumption (A.1), the local $C^{(1)}$ nature of the functions $(\mathbf{z}(\cdot), \mathbf{v}(\cdot), \boldsymbol{\lambda}(\cdot))$ from assumption (A.3), and the local $C^{(2)}$ nature of $V(\cdot)$ from assumption (A.4). This can be seen by the ensuing development and from inspection of the details of the appendix to this chapter, which constitutes the bulk of the proof of Theorem 11.2.

In order to turn the curvature condition in Eq. (19) into a meaningful comparative dynamics statement, we must express the elements of the matrix $L_{\beta\beta}(\boldsymbol{\beta})$ in terms of linear combinations of partial derivatives of the state, control, and costate functions with respect to the parameter vector $\boldsymbol{\beta}$. Because $\boldsymbol{\beta} \stackrel{\text{def}}{=} (\boldsymbol{\alpha}, t_0, \mathbf{x}_0, t_1, \mathbf{x}_1) \in \Re^{2+2N+A}$, this derivation is rather involved. Therefore, instead of presenting all the details in the chapter proper, we relegate most of the computations to the appendix of this chapter. We thus present a detailed development of the terms only in the submatrix $L_{\alpha\alpha}(\boldsymbol{\beta})$, which for economic applications, is usually the matrix of interest.

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To begin this process, first differentiate Eq. (11) with respect to $\boldsymbol{\alpha}$ and evaluate the result at $\boldsymbol{\beta} = \bar{\boldsymbol{\beta}}$ to get

$$\underset{A\times A}{L_{\boldsymbol{\alpha}\boldsymbol{\alpha}}}(\bar{\boldsymbol{\beta}}) = \int_{\bar{t}_0}^{\bar{t}_1} \underset{A\times A}{H_{\boldsymbol{\alpha}\boldsymbol{\alpha}}(t, \mathbf{z}(t;\bar{\boldsymbol{\beta}}), \mathbf{v}(t;\bar{\boldsymbol{\beta}}), \boldsymbol{\lambda}(t;\bar{\boldsymbol{\beta}}); \bar{\boldsymbol{\alpha}})}\, dt - \underset{A\times A}{V_{\boldsymbol{\alpha}\boldsymbol{\alpha}}}(\bar{\boldsymbol{\beta}}). \tag{24}$$

Next, use Theorem 11.1 to calculate the matrix $V_{\boldsymbol{\alpha}\boldsymbol{\alpha}}(\boldsymbol{\beta})$ and evaluate the result at $\boldsymbol{\beta} = \bar{\boldsymbol{\beta}}$:

$$\begin{aligned} \underset{A\times A}{V_{\boldsymbol{\alpha}\boldsymbol{\alpha}}}(\bar{\boldsymbol{\beta}}) \equiv \int_{\bar{t}_0}^{\bar{t}_1} \Bigg\{ & \underset{A\times N}{H_{\boldsymbol{\alpha}\mathbf{x}}(t, \mathbf{z}(t;\bar{\boldsymbol{\beta}}), \mathbf{v}(t;\bar{\boldsymbol{\beta}}), \boldsymbol{\lambda}(t;\bar{\boldsymbol{\beta}}); \bar{\boldsymbol{\alpha}})} \underset{N\times A}{\frac{\partial \mathbf{z}(t;\bar{\boldsymbol{\beta}})}{\partial \boldsymbol{\alpha}}} \\ & + \underset{A\times M}{H_{\boldsymbol{\alpha}\mathbf{u}}(t, \mathbf{z}(t;\bar{\boldsymbol{\beta}}), \mathbf{v}(t;\bar{\boldsymbol{\beta}}), \boldsymbol{\lambda}(t;\bar{\boldsymbol{\beta}}); \bar{\boldsymbol{\alpha}})} \underset{M\times A}{\frac{\partial \mathbf{v}(t;\bar{\boldsymbol{\beta}})}{\partial \boldsymbol{\alpha}}} \\ & + \underset{A\times N}{H_{\boldsymbol{\alpha}\boldsymbol{\lambda}}(t, \mathbf{z}(t;\bar{\boldsymbol{\beta}}), \mathbf{v}(t;\bar{\boldsymbol{\beta}}), \boldsymbol{\lambda}(t;\bar{\boldsymbol{\beta}}); \bar{\boldsymbol{\alpha}})} \underset{N\times A}{\frac{\partial \boldsymbol{\lambda}(t;\bar{\boldsymbol{\beta}})}{\partial \boldsymbol{\alpha}}} \\ & + \underset{A\times A}{H_{\boldsymbol{\alpha}\boldsymbol{\alpha}}(t, \mathbf{z}(t;\bar{\boldsymbol{\beta}}), \mathbf{v}(t;\bar{\boldsymbol{\beta}}), \boldsymbol{\lambda}(t;\bar{\boldsymbol{\beta}}); \bar{\boldsymbol{\alpha}})} \Bigg\} dt \end{aligned} \tag{25}$$

Recall, however, that the point $\boldsymbol{\beta} = \bar{\boldsymbol{\beta}} \in B(\boldsymbol{\beta}^\circ; \delta)$ at which Eqs. (24) and (25) are evaluated is arbitrary; thus they hold for all $\boldsymbol{\beta} \in B(\boldsymbol{\beta}^\circ; \delta)$. Noting this and then substituting Eq. (25) into Eq. (24) yields the result we are aiming for, videlicet, the elements of the submatrix $L_{\boldsymbol{\alpha}\boldsymbol{\alpha}}(\boldsymbol{\beta})$ are now expressed as linear combinations of the partial derivatives of the state, control, and costate functions with respect to the parameter vector $\boldsymbol{\alpha}$:

$$\begin{aligned} & \underset{A\times A}{L_{\boldsymbol{\alpha}\boldsymbol{\alpha}}}(\boldsymbol{\beta}) \\ & = -\int_{t_0}^{t_1} \left[\underset{A\times N}{H_{\boldsymbol{\alpha}\mathbf{x}}(t;\boldsymbol{\beta})} \underset{N\times A}{\frac{\partial \mathbf{z}(t;\boldsymbol{\beta})}{\partial \boldsymbol{\alpha}}} + \underset{A\times M}{H_{\boldsymbol{\alpha}\mathbf{u}}(t;\boldsymbol{\beta})} \underset{M\times A}{\frac{\partial \mathbf{v}(t;\boldsymbol{\beta})}{\partial \boldsymbol{\alpha}}} + \underset{A\times N}{H_{\boldsymbol{\alpha}\boldsymbol{\lambda}}(t;\boldsymbol{\beta})} \underset{N\times A}{\frac{\partial \boldsymbol{\lambda}(t;\boldsymbol{\beta})}{\partial \boldsymbol{\alpha}}} \right] dt, \end{aligned} \tag{26}$$

where $H_{\boldsymbol{\alpha}\mathbf{x}}(t;\boldsymbol{\beta}) \stackrel{\text{def}}{=} H_{\boldsymbol{\alpha}\mathbf{x}}(t, \mathbf{z}(t;\boldsymbol{\beta}), \mathbf{v}(t;\boldsymbol{\beta}), \boldsymbol{\lambda}(t;\boldsymbol{\beta}); \boldsymbol{\alpha})$, $H_{\boldsymbol{\alpha}\mathbf{u}}(t;\boldsymbol{\beta}) \stackrel{\text{def}}{=} H_{\boldsymbol{\alpha}\mathbf{u}}(t, \mathbf{z}(t;\boldsymbol{\beta}), \mathbf{v}(t;\boldsymbol{\beta}), \boldsymbol{\lambda}(t;\boldsymbol{\beta}); \boldsymbol{\alpha})$, and $H_{\boldsymbol{\alpha}\boldsymbol{\lambda}}(t;\boldsymbol{\beta}) \stackrel{\text{def}}{=} H_{\boldsymbol{\alpha}\boldsymbol{\lambda}}(t, \mathbf{z}(t;\boldsymbol{\beta}), \mathbf{v}(t;\boldsymbol{\beta}), \boldsymbol{\lambda}(t;\boldsymbol{\beta}); \boldsymbol{\alpha}) = \mathbf{g}_{\boldsymbol{\alpha}}(t, \mathbf{z}(t;\boldsymbol{\beta}), \mathbf{v}(t;\boldsymbol{\beta}); \boldsymbol{\alpha})'$. This proves the formula for the (1,1) block element of the matrix $L_{\boldsymbol{\beta}\boldsymbol{\beta}}(\boldsymbol{\beta})$. The appendix derives the formulas for the remaining 24 blocks of $L_{\boldsymbol{\beta}\boldsymbol{\beta}}(\boldsymbol{\beta})$ in the same exact manner. We have therefore proven the following fundamental comparative dynamics result for problem (P).

Theorem 11.2 (Comparative Dynamics): *For control problem (P), with assumptions (A.1) through (A.4) holding, the matrix $L_{\beta\beta}(\boldsymbol{\beta})$ is negative semidefinite subject to the constraint that $G_{\beta}(\boldsymbol{\beta})\mathbf{a} = \mathbf{0}_{3N}$ for all $\boldsymbol{\beta} \in B(\boldsymbol{\beta}^{\circ};\delta)$, where*

$$\underset{(2+2N+A)\times(2+2N+A)}{L_{\beta\beta}(\boldsymbol{\beta})} = \begin{bmatrix} \underset{A\times A}{L_{\boldsymbol{\alpha}\boldsymbol{\alpha}}(\boldsymbol{\beta})} & \underset{A\times 1}{L_{\boldsymbol{\alpha}t_0}(\boldsymbol{\beta})} & \underset{A\times N}{L_{\boldsymbol{\alpha}\mathbf{x}_0}(\boldsymbol{\beta})} & \underset{A\times 1}{L_{\boldsymbol{\alpha}t_1}(\boldsymbol{\beta})} & \underset{A\times N}{L_{\boldsymbol{\alpha}\mathbf{x}_1}(\boldsymbol{\beta})} \\ \underset{1\times A}{L_{t_0\boldsymbol{\alpha}}(\boldsymbol{\beta})} & \underset{1\times 1}{L_{t_0t_0}(\boldsymbol{\beta})} & \underset{1\times N}{L_{t_0\mathbf{x}_0}(\boldsymbol{\beta})} & \underset{1\times 1}{L_{t_0t_1}(\boldsymbol{\beta})} & \underset{1\times N}{L_{t_0\mathbf{x}_1}(\boldsymbol{\beta})} \\ \underset{N\times A}{L_{\mathbf{x}_0\boldsymbol{\alpha}}(\boldsymbol{\beta})} & \underset{N\times 1}{L_{\mathbf{x}_0t_0}(\boldsymbol{\beta})} & \underset{N\times N}{L_{\mathbf{x}_0\mathbf{x}_0}(\boldsymbol{\beta})} & \underset{N\times 1}{L_{\mathbf{x}_0t_1}(\boldsymbol{\beta})} & \underset{N\times N}{L_{\mathbf{x}_0\mathbf{x}_1}(\boldsymbol{\beta})} \\ \underset{1\times A}{L_{t_1\boldsymbol{\alpha}}(\boldsymbol{\beta})} & \underset{1\times 1}{L_{t_1t_0}(\boldsymbol{\beta})} & \underset{1\times N}{L_{t_1\mathbf{x}_0}(\boldsymbol{\beta})} & \underset{1\times 1}{L_{t_1t_1}(\boldsymbol{\beta})} & \underset{1\times N}{L_{t_1\mathbf{x}_1}(\boldsymbol{\beta})} \\ \underset{N\times A}{L_{\mathbf{x}_1\boldsymbol{\alpha}}(\boldsymbol{\beta})} & \underset{N\times 1}{L_{\mathbf{x}_1t_0}(\boldsymbol{\beta})} & \underset{N\times N}{L_{\mathbf{x}_1\mathbf{x}_0}(\boldsymbol{\beta})} & \underset{N\times 1}{L_{\mathbf{x}_1t_1}(\boldsymbol{\beta})} & \underset{N\times N}{L_{\mathbf{x}_1\mathbf{x}_1}(\boldsymbol{\beta})} \end{bmatrix},$$

$$\underset{1\times(2+2N+A)}{\mathbf{a}'} \stackrel{\text{def}}{=} \begin{pmatrix} \underset{1\times A}{\mathbf{a}^{1\prime}} & \underset{1\times 1}{a^{2\prime}} & \underset{1\times N}{\mathbf{a}^{3\prime}} & \underset{1\times 1}{a^{4\prime}} & \underset{1\times N}{\mathbf{a}^{5\prime}} \end{pmatrix},$$

$$\underset{1\times 3N}{G(\boldsymbol{\beta})'} \stackrel{\text{def}}{=} \Big[\underset{1\times N}{\mathbf{g}(t, \mathbf{z}(t;\bar{\boldsymbol{\beta}}), \mathbf{v}(t;\bar{\boldsymbol{\beta}}); \boldsymbol{\alpha})' - \dot{\mathbf{z}}(t;\bar{\boldsymbol{\beta}})'} \quad \underset{1\times N}{\mathbf{x}_0' - \mathbf{z}(t_0;\bar{\boldsymbol{\beta}})'} \quad \underset{1\times N}{\mathbf{z}(t_1;\bar{\boldsymbol{\beta}})' - \mathbf{x}_1'} \Big],$$

$$\underset{3N\times(2+2N+A)}{G_{\beta}(\boldsymbol{\beta})} = \begin{bmatrix} \underset{N\times A}{\mathbf{g}_{\boldsymbol{\alpha}}(t, \mathbf{z}(t;\bar{\boldsymbol{\beta}}), \mathbf{v}(t;\bar{\boldsymbol{\beta}}); \boldsymbol{\alpha})} & \underset{N\times 1}{\mathbf{0}} & \underset{N\times N}{\mathbf{0}} & \underset{N\times 1}{\mathbf{0}} & \underset{N\times N}{\mathbf{0}} \\ \underset{N\times A}{\mathbf{0}} & \underset{N\times 1}{-\dot{\mathbf{z}}(t_0;\bar{\boldsymbol{\beta}})} & \underset{N\times N}{\mathbf{I}} & \underset{N\times 1}{\mathbf{0}} & \underset{N\times N}{\mathbf{0}} \\ \underset{N\times A}{\mathbf{0}} & \underset{N\times 1}{\mathbf{0}} & \underset{N\times N}{\mathbf{0}} & \underset{N\times 1}{\dot{\mathbf{z}}(t_1;\bar{\boldsymbol{\beta}})} & \underset{N\times N}{-\mathbf{I}} \end{bmatrix},$$

$$\underset{A\times A}{L_{\boldsymbol{\alpha}\boldsymbol{\alpha}}(\boldsymbol{\beta})} = -\int_{t_0}^{t_1} \left[\underset{A\times N}{H_{\boldsymbol{\alpha}\mathbf{x}}(t;\boldsymbol{\beta})} \underset{N\times A}{\frac{\partial \mathbf{z}(t;\boldsymbol{\beta})}{\partial \boldsymbol{\alpha}}} + \underset{A\times M}{H_{\boldsymbol{\alpha}\mathbf{u}}(t;\boldsymbol{\beta})} \underset{M\times A}{\frac{\partial \mathbf{v}(t;\boldsymbol{\beta})}{\partial \boldsymbol{\alpha}}} + \underset{A\times N}{H_{\boldsymbol{\alpha}\boldsymbol{\lambda}}(t;\boldsymbol{\beta})} \underset{N\times A}{\frac{\partial \boldsymbol{\lambda}(t;\boldsymbol{\beta})}{\partial \boldsymbol{\alpha}}} \right] dt,$$

and where the remaining 24 blocks of $L_{\beta\beta}(\boldsymbol{\beta})$ are given by Eqs. (91) through (114) in the appendix.

The dynamic primal-dual methodology reveals that all dynamic optimization problems for which assumptions (A.1) through (A.4) are met possess a negative semidefinite matrix, subject to constraint, which captures the fundamental or intrinsic comparative dynamics properties of the model. It is important to understand that this theorem has relied only on the maximization assertion and assumptions (A.1) through (A.4), pointing to its truly basic nature. No assumptions relating to the concavity of the integrand or transitions functions, their functional forms, separability assumptions, and the like were imposed on problem (P) to derive Theorem 11.2.

Assumptions (A.1) through (A.4) are not that strong either, at least from an economic point of view, as nearly all applied modeling in economics adopts a set of assumptions similar to (A.1) through (A.4). Let's now consider a special case of the general result in Theorem 11.2 that arises frequently in intertemporal economic problems.

Consider the assumption $\mathbf{g}_{\alpha}(t, \mathbf{x}, \mathbf{u}; \boldsymbol{\alpha}) \equiv \mathbf{0}_{N\times A}$, which implies that the parameter vector $\boldsymbol{\alpha}$ does not appear in the transitions functions. Moreover, because $\boldsymbol{\alpha}$ does not appear explicitly in the endpoint constraints on the state vector in problem (P-D), the choice of $\boldsymbol{\alpha}$ in problem (P-D) is unconstrained in this instance. Said equivalently, the first column block of the matrix $G_{\beta}(\boldsymbol{\beta})$ is null, in which case, $G_{\beta}(\boldsymbol{\beta})\mathbf{a} = \mathbf{0}_{3N}$ implies that the subvector $\mathbf{a}^1$ is arbitrary, that is, not subject to constraint. Hence the matrix $L_{\boldsymbol{\alpha}\boldsymbol{\alpha}}(\boldsymbol{\beta})$ is negative semidefinite for all $\boldsymbol{\beta} \in B(\boldsymbol{\beta}^{\circ}; \delta)$ free of constraint. Seeing as $\mathbf{g}_{\alpha}(t, \mathbf{x}, \mathbf{u}; \boldsymbol{\alpha}) \equiv \mathbf{0}_{N\times A}$ in the present case, Eq. (24) and the negative semidefiniteness of $L_{\boldsymbol{\alpha}\boldsymbol{\alpha}}(\boldsymbol{\beta})$ imply

$$\mathbf{a}^{1\prime} \underset{A\times A}{L_{\boldsymbol{\alpha}\boldsymbol{\alpha}}(\boldsymbol{\beta})}\, \mathbf{a}^1 = \mathbf{a}^{1\prime} \left[\int_{t_0}^{t_1} \underset{A\times A}{f_{\boldsymbol{\alpha}\boldsymbol{\alpha}}(t, \mathbf{z}(t;\boldsymbol{\beta}), \mathbf{v}(t;\boldsymbol{\beta}); \boldsymbol{\alpha})}\, dt \right] \mathbf{a}^1 - \mathbf{a}^{1\prime} \underset{A\times A}{V_{\boldsymbol{\alpha}\boldsymbol{\alpha}}(\boldsymbol{\beta})}\, \mathbf{a}^1 \le 0. \tag{27}$$

Thus if $f(\cdot)$ is convex in $\boldsymbol{\alpha}$ for all $\boldsymbol{\beta} \in B(\boldsymbol{\beta}^{\circ}; \delta)$, then it follows from Eq. (27) and Theorem 21.5 of Simon and Blume (1994) that $V(\cdot)$ is convex in $\boldsymbol{\alpha}$ for all $\boldsymbol{\beta} \in B(\boldsymbol{\beta}^{\circ}; \delta)$, because then

$$\mathbf{a}^{1\prime} \underset{A\times A}{V_{\boldsymbol{\alpha}\boldsymbol{\alpha}}(\boldsymbol{\beta})}\, \mathbf{a}^1 \ge \mathbf{a}^{1\prime} \left[\int_{t_0}^{t_1} \underset{A\times A}{f_{\boldsymbol{\alpha}\boldsymbol{\alpha}}(t, \mathbf{z}(t;\boldsymbol{\beta}), \mathbf{v}(t;\boldsymbol{\beta}); \boldsymbol{\alpha})}\, dt \right] \mathbf{a}^1 \ge 0. \tag{28}$$

This inequality also demonstrates that under the postulated assumptions, $V(\cdot)$ is *more* convex in $\boldsymbol{\alpha}$ than is $f(\cdot)$. We have therefore proven the following corollary to Theorem 11.2.

Corollary 11.2 (Convexity of the Optimal Value Function): *For control problem (P), with assumptions (A.1) through (A.4) holding, if (i)* $\mathbf{g}_{\alpha}(t, \mathbf{x}, \mathbf{u}; \boldsymbol{\alpha}) \equiv \mathbf{0}_{N\times A}$ *and (ii)* $f(\cdot)$ *is convex in* $\boldsymbol{\alpha}$ *for all* $\boldsymbol{\beta} \in B(\boldsymbol{\beta}^{\circ}; \delta)$*, then* $V(\cdot)$ *is convex in* $\boldsymbol{\alpha}$ *for all* $\boldsymbol{\beta} \in B(\boldsymbol{\beta}^{\circ}; \delta)$.

An analogous result holds in the static profit maximizing model of the firm. The objective function contains all the model's parameters (output price and input prices) and is linear and thus convex in these parameters, and the constraint (the production function) is independent of the model's parameters. The static primal-dual methodology of Silberberg (1974) immediately reveals that the indirect profit function is convex in input prices and output price. This is simply a special case of a more general result derivable from the static primal-dual methodology, which states: if the objective function of a constrained static maximization problem is convex in the parameters of the problem and the constraints are independent of the parameters,

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then the indirect objective function is locally convex in the parameters. In passing, note that Corollary 11.2 holds for problem (P) for any subset of the parameters in $\boldsymbol{\alpha}$ that do not appear in the transition functions.

We finish up the chapter with an extended application of Theorem 11.1 and Corollary 11.2.

Example 11.1: The adjustment cost model of the firm with two capital stocks is given by

$$V(\boldsymbol{\alpha}) \stackrel{\text{def}}{=} \max_{u_1(\cdot),u_2(\cdot)} \int_0^T [pf(x_1(t), x_2(t), u_1(t), u_2(t))$$

$$- \ w_1x_1(t) - w_2x_2(t) - g_1u_1(t) - g_2u_2(t)]\, e^{-rt}\, dt$$

$$\text{s.t.} \quad \dot{x}_1(t) = u_1(t), \dot{x}_2(t) = u_2(t),$$

$$x_1(0) = x_{10},\ x_2(0) = x_{20},\ x_1(T) = x_{1T},\ x_2(T) = x_{2T},$$

where $\boldsymbol{\alpha} \stackrel{\text{def}}{=} (p, w_1, w_2, g_1, g_2, r)$ is the vector of time-independent parameters. For notational simplicity, let $\mathbf{z}(t;\boldsymbol{\alpha}) \stackrel{\text{def}}{=} (z_1(t;\boldsymbol{\alpha}), z_2(t;\boldsymbol{\alpha}))$ and $\mathbf{v}(t;\boldsymbol{\alpha}) \stackrel{\text{def}}{=} (v_1(t;\boldsymbol{\alpha}), v_2(t;\boldsymbol{\alpha}))$ be the optimal paths of the capital stocks and investment rates. We have chosen to set $N = 2$ to sharpen the exposition and have assumed that the capital stocks do not depreciate. Moreover, as the notation conveys, we will focus solely on the vector $\boldsymbol{\alpha} \stackrel{\text{def}}{=} (p, w_1, w_2, g_1, g_2, r)$ of parameters, which are those of more immediate economic interest.

Theorem 11.1, the dynamic envelope theorem, asserts that the partial derivative of the optimal value function with respect to a parameter may be obtained by (i) differentiating the Hamiltonian of the control problem directly with respect to the parameter of interest, that is, *prior to* substituting in the optimal paths, (ii) evaluating the derivative along the optimal paths, and (iii) integrating the result over the planning horizon. Defining $y(t;\boldsymbol{\alpha}) \stackrel{\text{def}}{=} f(\mathbf{z}(t;\boldsymbol{\alpha}), \mathbf{v}(t;\boldsymbol{\alpha}))$ as the value of the supply function of the firm, and then applying this recipe to the model, yields

$$V_p(\boldsymbol{\alpha}) \equiv \int_0^T y(t;\boldsymbol{\alpha})\, e^{-rt}\, dt > 0, \tag{29}$$

$$V_{w_n}(\boldsymbol{\alpha}) \equiv -\int_0^T z_n(t;\boldsymbol{\alpha})\, e^{-rt}\, dt < 0, \quad n = 1, 2, \tag{30}$$

$$V_{g_n}(\boldsymbol{\alpha}) \equiv -\int_0^T v_n(t;\boldsymbol{\alpha})\, e^{-rt}\, dt \gtrless 0, \quad n = 1, 2, \tag{31}$$

$$V_r(\boldsymbol{\alpha}) \equiv -\int_0^T t\pi(t;\boldsymbol{\alpha})\, e^{-rt}\, dt \gtrless 0, \tag{32}$$

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where $\pi(t;\boldsymbol{\alpha}) \stackrel{\text{def}}{=} pf(\mathbf{z}(t;\boldsymbol{\alpha}), \mathbf{v}(t;\boldsymbol{\alpha})) - \mathbf{w}'\mathbf{z}(t;\boldsymbol{\alpha}) - \mathbf{g}'\mathbf{v}(t;\boldsymbol{\alpha})$ is instantaneous profits along the optimal path. Rather than recovering the instantaneous demand and supply functions, as does the static envelope theorem, the dynamic envelope theorem recovers the *cumulative discounted demand and supply functions*, a feature of the dynamic envelope theorem we noted earlier in Chapter 9, when the theorem was first introduced. Notice that Eqs. (29) and (30) are unambiguously signed, for the firm must have some capital on hand to produce output if it is to be profitable. These two properties of the optimal value function are analogous to those static indirect profit function, namely, that it is increasing in the output price and decreasing in the input prices.

In contrast, Eqs. (31) and (32) are not unambiguously signed. Given that no nonnegativity restriction is imposed on the investment rates in the above version of the adjustment cost model, the firm may find it optimal to invest $(\mathbf{v}(t;\boldsymbol{\alpha}) > \mathbf{0}_2)$ or disinvest $(\mathbf{v}(t;\boldsymbol{\alpha}) < \mathbf{0}_2)$ in the capital stock at various points in the planning horizon, thereby yielding the ambiguous sign in Eq. (31). The ambiguity in the sign of Eq. (32) follows from the fact that although $V(\boldsymbol{\alpha}) > 0$ must hold for the firm to be in business, instantaneous profits along the optimal path may be positive or negative at any given point in the planning horizon. Naturally, if one is willing to assume that instantaneous profit is nonnegative at each point in time of the planning horizon along the optimal path, then $V_r(\boldsymbol{\alpha}) < 0$.

Because the integrand function of the adjustment cost model is linear in $\boldsymbol{\gamma} \stackrel{\text{def}}{=} (p, w_1, w_2, g_1, g_2)$, the model satisfies the conditions of Corollary 11.2, thereby implying that the optimal value function $V(\cdot)$ is locally convex in $\boldsymbol{\gamma}$. Thus, upon differentiating Eqs. (29) through (31) with respect to $\boldsymbol{\gamma}$ and using the convexity of $V(\cdot)$, we arrive at the own-price effects

$$V_{pp}(\boldsymbol{\alpha}) \equiv \frac{\partial}{\partial p}\int_0^T y(t;\boldsymbol{\alpha})\,e^{-rt}\,dt = \int_0^T \frac{\partial y}{\partial p}(t;\boldsymbol{\alpha})\,e^{-rt}\,dt \geq 0, \tag{33}$$

$$V_{w_n w_n}(\boldsymbol{\alpha}) \equiv -\frac{\partial}{\partial w_n}\int_0^T z_n(t;\boldsymbol{\alpha})\,e^{-rt}\,dt = -\int_0^T \frac{\partial z_n}{\partial w_n}(t;\boldsymbol{\alpha})\,e^{-rt}\,dt \geq 0, \quad n = 1, 2, \tag{34}$$

$$V_{g_n g_n}(\boldsymbol{\alpha}) \equiv -\frac{\partial}{\partial g_n}\int_0^T v_n(t;\boldsymbol{\alpha})\,e^{-rt}\,dt = -\int_0^T \frac{\partial v_n}{\partial g_n}(t;\boldsymbol{\alpha})\,e^{-rt}\,dt \geq 0, \quad n = 1, 2. \tag{35}$$

These equations have important economic interpretations and implications. For example, Eq. (33) demonstrates that cumulative discounted production (or supply) will not fall when the output price increases, that is, the cumulative discounted supply function is nondecreasing in the output price. Equivalently, it asserts that the discounted supply function slope, when integrated over the planning horizon,

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is nonnegative. In contrast, static profit maximization theory implies the slope of the supply function is nonnegative at each point in time. Although this implies the inequality in Eq. (33), the converse is certainly not true. In other words, one may find the firm behaving irrationally over some finite period of time according to static profit maximization theory, but when viewed over its entire planning horizon, its behavior may be quite rational from an intertemporal point of view, that is to say, Eq. (33) may nonetheless be satisfied. Similarly, Eq. (35) establishes that the cumulative discounted investment demand function is a nonincreasing function of its own price. In other words, the law of demand holds for cumulative discounted investment. Alternatively, the discounted own-price effect on investment, when integrated over the planning horizon, is nonpositive. Thus, at various points in the planning horizon, or even over some finite period of time in the planning horizon, $\partial v_n(t;\boldsymbol{\alpha})/\partial g_n > 0$ is possible, and in fact, it is perfectly consistent with Eq. (35). Analogous comments and interpretation apply to Eq. (34). Note that the results in Eqs. (33) through (35) may be just as easily derived from Theorem 11.2 directly, as you are asked to show in a mental exercise.

In the preceding paragraph, you may have noticed that we didn't derive or discuss the comparative dynamics of the discount rate r. This is a result of our use of Corollary 11.2 to derive the aforementioned comparative dynamics. Simply put, Corollary 11.2 can't be used to derive the comparative dynamics of the discount rate because the integrand function $F(\cdot)$ of the adjustment cost model, defined by

$$F(t, \mathbf{x}, \mathbf{u}; \boldsymbol{\alpha}) \stackrel{\text{def}}{=} [pf(x_1, x_2, u_1, u_2) - w_1x_1 - w_2x_2 - g_1u_1 - g_2u_2]\, e^{-rt},$$

is not, in general, convex in the discount rate r. Thus, in general, we must rely on Theorem 11.2 to determine the comparative dynamics of the discount rate. Note, in passing, that a mental exercise asks you to contemplate the case in which $F(\cdot)$ is locally convex in r.

Recalling that $\boldsymbol{\alpha} \stackrel{\text{def}}{=} (p, w_1, w_2, g_1, g_2, r)$, so that the sixth element of the parameter vector $\boldsymbol{\alpha}$ is the discount rate r, it follows from Theorem 11.2 that

$$L_{rr}(\boldsymbol{\beta}) = -\int_0^T \sum_{n=1}^{2} \Bigg[F_{rx_n}(t, \mathbf{z}(t;\boldsymbol{\alpha}), \mathbf{v}(t;\boldsymbol{\alpha});\boldsymbol{\alpha}) \frac{\partial z_n}{\partial r}(t;\boldsymbol{\alpha}) + F_{ru_n}(t, \mathbf{z}(t;\boldsymbol{\alpha}), \mathbf{v}(t;\boldsymbol{\alpha});\boldsymbol{\alpha}) \frac{\partial v_n}{\partial r}(t;\boldsymbol{\alpha}) \Bigg] dt \le 0.$$

Using the definition of $F(\cdot)$ given above, this last equation can be rewritten as

$$\int_0^T \sum_{n=1}^{2} \Bigg[[pf_{x_n}(\mathbf{z}(t;\boldsymbol{\alpha}), \mathbf{v}(t;\boldsymbol{\alpha})) - w_n] \frac{\partial z_n}{\partial r}(t;\boldsymbol{\alpha}) + [pf_{u_n}(\mathbf{z}(t;\boldsymbol{\alpha}), \mathbf{v}(t;\boldsymbol{\alpha})) - g_n] \frac{\partial v_n}{\partial r}(t;\boldsymbol{\alpha}) \Bigg]\, te^{-rt}\, dt \le 0. \tag{36}$$

This expression is less amenable to a clean and simple economic interpretation than are Eqs. (33) through (35) . Nevertheless, it represents the comparative dynamics of the discount rate in the adjustment cost model of the firm. We are therefore led to conclude that the comparative dynamics of the discount rate are more complicated than they are for the prices $\boldsymbol{\gamma} \stackrel{\text{def}}{=} (p, w_1, w_2, g_1, g_2)$.

To finish up this example, let's consider a few of the symmetry or reciprocity relations revealed by the dynamic primal-dual method. Because $V(\cdot) \in C^{(2)}$, it thus follows by differentiating Eqs. (29) through (31) that

$$V_{pw_n}(\boldsymbol{\alpha}) \equiv \frac{\partial}{\partial w_n}\int_0^T y(t;\boldsymbol{\alpha})\, e^{-rt}\, dt = -\frac{\partial}{\partial p}\int_0^T z_n(t;\boldsymbol{\alpha})\, e^{-rt}\, dt$$
$$\equiv V_{w_n p}(\boldsymbol{\alpha}), \quad n = 1, 2, \tag{37}$$
$$V_{w_n w_\ell}(\boldsymbol{\alpha}) \equiv -\frac{\partial}{\partial w_\ell}\int_0^T z_n(t;\boldsymbol{\alpha})\, e^{-rt}\, dt = -\frac{\partial}{\partial w_n}\int_0^T z_\ell(t;\boldsymbol{\alpha})\, e^{-rt}\, dt$$
$$\equiv V_{w_\ell w_n}(\boldsymbol{\alpha}), \quad n \neq \ell = 1, 2. \tag{38}$$

Equations (37) and (38) may be compared with their static profit maximizing counterparts, to wit,

$$\pi^*_{pw_n}(p, \mathbf{w}) \equiv \frac{\partial y^*}{\partial w_n}(p, \mathbf{w}) = -\frac{\partial x_n^*}{\partial p}(p, \mathbf{w}) \equiv \pi^*_{w_n p}(p, \mathbf{w}), \quad n = 1, 2,$$
$$\pi^*_{w_n w_\ell}(p, \mathbf{w}) \equiv -\frac{\partial x_n^*}{\partial w_\ell}(p, \mathbf{w}) = -\frac{\partial x_\ell^*}{\partial w_n}(p, \mathbf{w}) \equiv \pi^*_{w_\ell w_n}(p, \mathbf{w}), \quad n \neq \ell = 1, 2,$$

where $\pi^*(p, \mathbf{w}) \stackrel{\text{def}}{=} \max_{\mathbf{x}\in\Re^2_{++}} \{pf(x_1, x_2) - w_1 x_1 - w_2 x_2\}$ is the value of the indirect profit function, $x_n^*(p, \mathbf{w})$, $n = 1, 2$, are the optimal values of the factor demand functions, and $y^*(p, \mathbf{w})$ is the optimal value of the supply function. The simplicity of all of the above reciprocity relations is a result of the conjugate and linear nature in which the parameters and decision variables appear in the adjustment cost model and the profit maximization model. Equations (37) and (38) show that it is the cumulative discounted demand and supply functions that possess the symmetry properties in the adjustment cost model, not unlike the comparative dynamics results in Eqs. (33) through (35). Thus, unlike static models, the symmetry does not have to hold at each point in time, but only over the entire planning horizon. Again, what appears to be irrational behavior from a static perspective may be perfectly consistent with rational dynamic behavior. Finally, note that the results in Eqs. (37) and (38) may also be derived directly from Theorem 11.2. A mental exercise asks you to complete the derivation of the comparative dynamics of the adjustment cost model.

We have shown that all optimal control problems of the class defined by problem (P) that meet assumptions (A.1) through (A.4) possess rich qualitative properties.

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The dynamic primal-dual approach developed herein was used to determine the qualitative properties of such models, and in the process, an alternative proof of the dynamic envelope theorem was exhibited. The qualitative properties revealed by the dynamic primal-dual approach are contained in a symmetric negative semidefinite matrix subject to constraint. This matrix may be thought of as an intertemporal generalization of the Slutsky matrix. It places qualitative restrictions on the demand and supply functions over the entire planning horizon, rather than at each point in time. That is, the qualitative comparative dynamics properties are in terms of the cumulative discounted demand and supply functions, not to their instantaneous forms. Symmetry was shown to be a fundamental qualitative property of any optimal control problem that meets the stated assumptions. Finally, sufficient conditions for the optimal value function to be convex in the parameters were provided.

The power of the dual view is clearly revealed by the methods established herein, for it would be difficult, if not impossible, to derive Theorem 11.2, the central result of this chapter, from a strictly primal view of the optimal control problem (P). Moreover, it is not clear why anyone would even calculate such complicated expressions from a primal vista, nor is it clear why anyone would expect them to represent the intrinsic comparative dynamics properties of an optimal control problem. A dual view of problem (P), on the other hand, has led to refutable comparative dynamics results in a rather simple and elegant way.

APPENDIX

In the chapter proper, we derived the diagonal block matrix $L_{\alpha\alpha}(\boldsymbol{\beta})$ of the full comparative dynamics matrix $L_{\beta\beta}(\boldsymbol{\beta})$ in Theorem 11.2. The purpose of the appendix is to derive the remaining 24 blocks of the matrix $L_{\beta\beta}(\boldsymbol{\beta})$ in Theorem 11.2.

As a first step, differentiate Eqs. (11) through (15) with respect to the vector of parameters $\boldsymbol{\beta} \stackrel{\text{def}}{=} (\boldsymbol{\alpha}, t_0, \mathbf{x}_0, t_1, \mathbf{x}_1) \in \Re^{2+2N+A}$ and evaluate the results at $\boldsymbol{\beta} = \bar{\boldsymbol{\beta}}$ to get

$$L_{\alpha t_0}(\bar{\boldsymbol{\beta}}) = -H_{\boldsymbol{\alpha}}(\bar{t}_0; \bar{\boldsymbol{\beta}}) - V_{\alpha t_0}(\bar{\boldsymbol{\beta}}), \tag{39}$$

$$L_{\alpha \mathbf{x}_0}(\bar{\boldsymbol{\beta}}) = -V_{\alpha \mathbf{x}_0}(\bar{\boldsymbol{\beta}}), \tag{40}$$

$$L_{\alpha t_1}(\bar{\boldsymbol{\beta}}) = H_{\boldsymbol{\alpha}}(\bar{t}_1; \bar{\boldsymbol{\beta}}) - V_{\alpha t_1}(\bar{\boldsymbol{\beta}}), \tag{41}$$

$$L_{\alpha \mathbf{x}_1}(\bar{\boldsymbol{\beta}}) = -V_{\alpha \mathbf{x}_1}(\bar{\boldsymbol{\beta}}), \tag{42}$$

$$L_{t_0 \alpha}(\bar{\boldsymbol{\beta}}) = -H_{\boldsymbol{\alpha}}(\bar{t}_0; \bar{\boldsymbol{\beta}}) - V_{t_0 \alpha}(\bar{\boldsymbol{\beta}}), \tag{43}$$

$$\begin{aligned} L_{t_0 t_0}(\bar{\boldsymbol{\beta}}) &= -H_t(\bar{t}_0; \bar{\boldsymbol{\beta}}) - H_{\mathbf{x}}(\bar{t}_0; \bar{\boldsymbol{\beta}})\,\dot{\mathbf{z}}(\bar{t}_0; \bar{\boldsymbol{\beta}}) - H_{\mathbf{u}}(\bar{t}_0; \bar{\boldsymbol{\beta}})\,\dot{\mathbf{v}}(\bar{t}_0; \bar{\boldsymbol{\beta}}) - H_{\boldsymbol{\lambda}}(\bar{t}_0; \bar{\boldsymbol{\beta}})\,\dot{\boldsymbol{\lambda}}(\bar{t}_0; \bar{\boldsymbol{\beta}}) \\ &\quad - \dot{\boldsymbol{\lambda}}(\bar{t}_0; \bar{\boldsymbol{\beta}})'\dot{\mathbf{z}}(\bar{t}_0; \bar{\boldsymbol{\beta}}) - \ddot{\boldsymbol{\lambda}}(\bar{t}_0; \bar{\boldsymbol{\beta}})'\mathbf{z}(\bar{t}_0; \bar{\boldsymbol{\beta}}) + \ddot{\boldsymbol{\lambda}}(\bar{t}_0; \bar{\boldsymbol{\beta}})'\bar{\mathbf{x}}_0 - V_{t_0 t_0}(\bar{\boldsymbol{\beta}}) \\ &= -H_t(\bar{t}_0; \bar{\boldsymbol{\beta}}) - \dot{\boldsymbol{\lambda}}(\bar{t}_0; \bar{\boldsymbol{\beta}})'\dot{\mathbf{z}}(\bar{t}_0; \bar{\boldsymbol{\beta}}) - V_{t_0 t_0}(\bar{\boldsymbol{\beta}}), \end{aligned} \tag{44}$$

$$L_{t_0 \mathbf{x}_0}(\bar{\boldsymbol{\beta}}) = \dot{\boldsymbol{\lambda}}(\bar{t}_0; \bar{\boldsymbol{\beta}})' - V_{t_0 \mathbf{x}_0}(\bar{\boldsymbol{\beta}}), \tag{45}$$

$$L_{t_0t_1}(\bar{\boldsymbol{\beta}}) = -V_{t_0t_1}(\bar{\boldsymbol{\beta}}), \tag{46}$$

$$L_{t_0\mathbf{x}_1}(\bar{\boldsymbol{\beta}}) = -V_{t_0\mathbf{x}_1}(\bar{\boldsymbol{\beta}}), \tag{47}$$

$$L_{\mathbf{x}_0\boldsymbol{\alpha}}(\bar{\boldsymbol{\beta}}) = -V_{\mathbf{x}_0\boldsymbol{\alpha}}(\bar{\boldsymbol{\beta}}), \tag{48}$$

$$L_{\mathbf{x}_0t_0}(\bar{\boldsymbol{\beta}}) = \dot{\boldsymbol{\lambda}}(\bar{t}_0;\bar{\boldsymbol{\beta}}) - V_{\mathbf{x}_0t_0}(\bar{\boldsymbol{\beta}}), \tag{49}$$

$$L_{\mathbf{x}_0\mathbf{x}_0}(\bar{\boldsymbol{\beta}}) = -V_{\mathbf{x}_0\mathbf{x}_0}(\bar{\boldsymbol{\beta}}), \tag{50}$$

$$L_{\mathbf{x}_0t_1}(\bar{\boldsymbol{\beta}}) = -V_{\mathbf{x}_0t_1}(\bar{\boldsymbol{\beta}}), \tag{51}$$

$$L_{\mathbf{x}_0\mathbf{x}_1}(\bar{\boldsymbol{\beta}}) = -V_{\mathbf{x}_0\mathbf{x}_1}(\bar{\boldsymbol{\beta}}), \tag{52}$$

$$L_{t_1\boldsymbol{\alpha}}(\bar{\boldsymbol{\beta}}) = H_{\boldsymbol{\alpha}}(\bar{t}_1;\bar{\boldsymbol{\beta}}) - V_{t_1\boldsymbol{\alpha}}(\bar{\boldsymbol{\beta}}), \tag{53}$$

$$L_{t_1t_0}(\bar{\boldsymbol{\beta}}) = -V_{t_1t_0}(\bar{\boldsymbol{\beta}}), \tag{54}$$

$$L_{t_1\mathbf{x}_0}(\bar{\boldsymbol{\beta}}) = -V_{t_1\mathbf{x}_0}(\bar{\boldsymbol{\beta}}), \tag{55}$$

$$\begin{aligned} L_{t_1t_1}(\bar{\boldsymbol{\beta}}) &= H_t(\bar{t}_1;\bar{\boldsymbol{\beta}}) + H_{\mathbf{x}}(\bar{t}_1;\bar{\boldsymbol{\beta}})\,\dot{\mathbf{z}}(\bar{t}_1;\bar{\boldsymbol{\beta}}) + H_{\mathbf{u}}(\bar{t}_1;\bar{\boldsymbol{\beta}})\,\dot{\mathbf{v}}(\bar{t}_1;\bar{\boldsymbol{\beta}}) + H_{\boldsymbol{\lambda}}(\bar{t}_1;\bar{\boldsymbol{\beta}})\,\dot{\boldsymbol{\lambda}}(\bar{t}_1;\bar{\boldsymbol{\beta}}) \\ &\quad + \dot{\boldsymbol{\lambda}}(\bar{t}_1;\bar{\boldsymbol{\beta}})'\dot{\mathbf{z}}(\bar{t}_1;\bar{\boldsymbol{\beta}}) + \ddot{\boldsymbol{\lambda}}(\bar{t}_1;\bar{\boldsymbol{\beta}})'\mathbf{z}(\bar{t}_1;\bar{\boldsymbol{\beta}}) - \ddot{\boldsymbol{\lambda}}(\bar{t}_1;\bar{\boldsymbol{\beta}})'\bar{\mathbf{x}}_1 - V_{t_1t_1}(\bar{\boldsymbol{\beta}}) \\ &= H_t(\bar{t}_1;\bar{\boldsymbol{\beta}}) + \dot{\boldsymbol{\lambda}}(\bar{t}_1;\bar{\boldsymbol{\beta}})'\dot{\mathbf{z}}(\bar{t}_1;\bar{\boldsymbol{\beta}}) - V_{t_1t_1}(\bar{\boldsymbol{\beta}}), \end{aligned} \tag{56}$$

$$L_{t_1\mathbf{x}_1}(\bar{\boldsymbol{\beta}}) = -\dot{\boldsymbol{\lambda}}(\bar{t}_1;\bar{\boldsymbol{\beta}})' - V_{t_1\mathbf{x}_1}(\bar{\boldsymbol{\beta}}), \tag{57}$$

$$L_{\mathbf{x}_1\boldsymbol{\alpha}}(\bar{\boldsymbol{\beta}}) = -V_{\mathbf{x}_1\boldsymbol{\alpha}}(\bar{\boldsymbol{\beta}}), \tag{58}$$

$$L_{\mathbf{x}_1t_0}(\bar{\boldsymbol{\beta}}) = -V_{\mathbf{x}_1t_0}(\bar{\boldsymbol{\beta}}), \tag{59}$$

$$L_{\mathbf{x}_1\mathbf{x}_0}(\bar{\boldsymbol{\beta}}) = -V_{\mathbf{x}_1\mathbf{x}_0}(\bar{\boldsymbol{\beta}}), \tag{60}$$

$$L_{\mathbf{x}_1t_1}(\bar{\boldsymbol{\beta}}) = -\dot{\boldsymbol{\lambda}}(\bar{t}_1;\bar{\boldsymbol{\beta}}) - V_{\mathbf{x}_1t_1}(\bar{\boldsymbol{\beta}}), \tag{61}$$

$$L_{\mathbf{x}_1\mathbf{x}_1}(\bar{\boldsymbol{\beta}}) = -V_{\mathbf{x}_1\mathbf{x}_1}(\bar{\boldsymbol{\beta}}). \tag{62}$$

Note that in simplifying Eqs. (44) and (56), we used the necessary conditions of Corollary 4.2 in identity form, to wit,

$$H_{\mathbf{u}}(t,\mathbf{z}(t;\boldsymbol{\beta}),\mathbf{v}(t;\boldsymbol{\beta}),\boldsymbol{\lambda}(t;\boldsymbol{\beta});\boldsymbol{\alpha}) \equiv \mathbf{0}'_M, \tag{63}$$

$$\dot{\boldsymbol{\lambda}}(t;\boldsymbol{\beta})' \equiv -H_{\mathbf{x}}(t,\mathbf{z}(t;\boldsymbol{\beta}),\mathbf{v}(t;\boldsymbol{\beta}),\boldsymbol{\lambda}(t;\boldsymbol{\beta});\boldsymbol{\alpha}), \tag{64}$$

$$\dot{\mathbf{z}}(t;\boldsymbol{\beta})' \equiv H_{\boldsymbol{\lambda}}(t,\mathbf{z}(t;\boldsymbol{\beta}),\mathbf{v}(t;\boldsymbol{\beta}),\boldsymbol{\lambda}(t;\boldsymbol{\beta});\boldsymbol{\alpha}), \tag{65}$$

$$\mathbf{z}(t_0;\boldsymbol{\beta}) \equiv \mathbf{x}_0,\ \ \mathbf{z}(t_1;\boldsymbol{\beta}) \equiv \mathbf{x}_1, \tag{66}$$

which hold for all $\boldsymbol{\beta} \in B(\boldsymbol{\beta}^\circ;\delta)$.

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The next step is to differentiate the envelope results of Theorem 11.1 with respect to the parameter vector $\boldsymbol{\beta} \stackrel{\text{def}}{=} (\boldsymbol{\alpha}, t_0, \mathbf{x}_0, t_1, \mathbf{x}_1)$ and evaluate the results at $\boldsymbol{\beta} = \bar{\boldsymbol{\beta}}$ to get

$$V_{\alpha t_0}(\bar{\boldsymbol{\beta}}) \equiv -H_{\alpha}(\bar{t}_0; \bar{\boldsymbol{\beta}}) + \int_{\bar{t}_0}^{\bar{t}_1} \left[H_{\alpha \mathbf{x}}(t; \bar{\boldsymbol{\beta}}) \frac{\partial \mathbf{z}(t; \bar{\boldsymbol{\beta}})}{\partial t_0} + H_{\alpha \mathbf{u}}(t; \bar{\boldsymbol{\beta}}) \frac{\partial \mathbf{v}(t; \bar{\boldsymbol{\beta}})}{\partial t_0} + H_{\alpha \boldsymbol{\lambda}}(t; \bar{\boldsymbol{\beta}}) \frac{\partial \boldsymbol{\lambda}(t; \bar{\boldsymbol{\beta}})}{\partial t_0} \right] dt, \tag{67}$$

$$V_{\alpha \mathbf{x}_0}(\bar{\boldsymbol{\beta}}) \equiv \int_{\bar{t}_0}^{\bar{t}_1} \left[H_{\alpha \mathbf{x}}(t; \bar{\boldsymbol{\beta}}) \frac{\partial \mathbf{z}(t; \bar{\boldsymbol{\beta}})}{\partial \mathbf{x}_0} + H_{\alpha \mathbf{u}}(t; \bar{\boldsymbol{\beta}}) \frac{\partial \mathbf{v}(t; \bar{\boldsymbol{\beta}})}{\partial \mathbf{x}_0} + H_{\alpha \boldsymbol{\lambda}}(t; \bar{\boldsymbol{\beta}}) \frac{\partial \boldsymbol{\lambda}(t; \bar{\boldsymbol{\beta}})}{\partial \mathbf{x}_0} \right] dt, \tag{68}$$

$$V_{\alpha t_1}(\bar{\boldsymbol{\beta}}) \equiv H_{\alpha}(\bar{t}_1; \bar{\boldsymbol{\beta}}) + \int_{\bar{t}_0}^{\bar{t}_1} \left[H_{\alpha \mathbf{x}}(t; \bar{\boldsymbol{\beta}}) \frac{\partial \mathbf{z}(t; \bar{\boldsymbol{\beta}})}{\partial t_1} + H_{\alpha \mathbf{u}}(t; \bar{\boldsymbol{\beta}}) \frac{\partial \mathbf{v}(t; \bar{\boldsymbol{\beta}})}{\partial t_1} + H_{\alpha \boldsymbol{\lambda}}(t; \bar{\boldsymbol{\beta}}) \frac{\partial \boldsymbol{\lambda}(t; \bar{\boldsymbol{\beta}})}{\partial t_1} \right] dt, \tag{69}$$

$$V_{\alpha \mathbf{x}_1}(\bar{\boldsymbol{\beta}}) \equiv \int_{\bar{t}_0}^{\bar{t}_1} \left[H_{\alpha \mathbf{x}}(t; \bar{\boldsymbol{\beta}}) \frac{\partial \mathbf{z}(t; \bar{\boldsymbol{\beta}})}{\partial \mathbf{x}_1} + H_{\alpha \mathbf{u}}(t; \bar{\boldsymbol{\beta}}) \frac{\partial \mathbf{v}(t; \bar{\boldsymbol{\beta}})}{\partial \mathbf{x}_1} + H_{\alpha \boldsymbol{\lambda}}(t; \bar{\boldsymbol{\beta}}) \frac{\partial \boldsymbol{\lambda}(t; \bar{\boldsymbol{\beta}})}{\partial \mathbf{x}_1} \right] dt, \tag{70}$$

$$\begin{aligned} V_{t_0 \alpha}(\bar{\boldsymbol{\beta}}) &\equiv -H_{\mathbf{x}}(\bar{t}_0; \bar{\boldsymbol{\beta}}) \frac{\partial \mathbf{z}(\bar{t}_0; \bar{\boldsymbol{\beta}})}{\partial \boldsymbol{\alpha}} - H_{\mathbf{u}}(\bar{t}_0; \bar{\boldsymbol{\beta}}) \frac{\partial \mathbf{v}(\bar{t}_0; \bar{\boldsymbol{\beta}})}{\partial \boldsymbol{\alpha}} - H_{\boldsymbol{\lambda}}(\bar{t}_0; \bar{\boldsymbol{\beta}}) \frac{\partial \boldsymbol{\lambda}(\bar{t}_0; \bar{\boldsymbol{\beta}})}{\partial \boldsymbol{\alpha}} \\ &\quad - H_{\alpha}(\bar{t}_0; \bar{\boldsymbol{\beta}}) = -H_{\boldsymbol{\lambda}}(\bar{t}_0; \bar{\boldsymbol{\beta}}) \frac{\partial \boldsymbol{\lambda}(\bar{t}_0; \bar{\boldsymbol{\beta}})}{\partial \boldsymbol{\alpha}} - H_{\alpha}(\bar{t}_0; \bar{\boldsymbol{\beta}}), \end{aligned} \tag{71}$$

$$\begin{aligned} V_{t_0 t_0}(\bar{\boldsymbol{\beta}}) &\equiv -H_t(\bar{t}_0; \bar{\boldsymbol{\beta}}) - H_{\mathbf{x}}(\bar{t}_0; \bar{\boldsymbol{\beta}}) \left[\dot{\mathbf{z}}(\bar{t}_0; \bar{\boldsymbol{\beta}}) + \frac{\partial \mathbf{z}(\bar{t}_0; \bar{\boldsymbol{\beta}})}{\partial t_0} \right] \\ &\quad - H_{\mathbf{u}}(\bar{t}_0; \bar{\boldsymbol{\beta}}) \left[\dot{\mathbf{v}}(\bar{t}_0; \bar{\boldsymbol{\beta}}) + \frac{\partial \mathbf{v}(\bar{t}_0; \bar{\boldsymbol{\beta}})}{\partial t_0} \right] - H_{\boldsymbol{\lambda}}(\bar{t}_0; \bar{\boldsymbol{\beta}}) \left[\dot{\boldsymbol{\lambda}}(\bar{t}_0; \bar{\boldsymbol{\beta}}) + \frac{\partial \boldsymbol{\lambda}(\bar{t}_0; \bar{\boldsymbol{\beta}})}{\partial t_0} \right] \\ &= -H_t(\bar{t}_0; \bar{\boldsymbol{\beta}}) - H_{\boldsymbol{\lambda}}(\bar{t}_0; \bar{\boldsymbol{\beta}}) \left[\dot{\boldsymbol{\lambda}}(\bar{t}_0; \bar{\boldsymbol{\beta}}) + \frac{\partial \boldsymbol{\lambda}(\bar{t}_0; \bar{\boldsymbol{\beta}})}{\partial t_0} \right], \end{aligned} \tag{72}$$

$$\begin{aligned} V_{t_0 \mathbf{x}_0}(\bar{\boldsymbol{\beta}}) &\equiv -H_{\mathbf{x}}(\bar{t}_0; \bar{\boldsymbol{\beta}}) \frac{\partial \mathbf{z}(\bar{t}_0; \bar{\boldsymbol{\beta}})}{\partial \mathbf{x}_0} - H_{\mathbf{u}}(\bar{t}_0; \bar{\boldsymbol{\beta}}) \frac{\partial \mathbf{v}(\bar{t}_0; \bar{\boldsymbol{\beta}})}{\partial \mathbf{x}_0} - H_{\boldsymbol{\lambda}}(\bar{t}_0; \bar{\boldsymbol{\beta}}) \frac{\partial \boldsymbol{\lambda}(\bar{t}_0; \bar{\boldsymbol{\beta}})}{\partial \mathbf{x}_0} \\ &= -H_{\mathbf{x}}(\bar{t}_0; \bar{\boldsymbol{\beta}}) - H_{\boldsymbol{\lambda}}(\bar{t}_0; \bar{\boldsymbol{\beta}}) \frac{\partial \boldsymbol{\lambda}(\bar{t}_0; \bar{\boldsymbol{\beta}})}{\partial \mathbf{x}_0}, \end{aligned} \tag{73}$$

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$$V_{t_0t_1}(\bar{\boldsymbol{\beta}}) \equiv -H_{\mathbf{x}}(\bar{t}_0;\bar{\boldsymbol{\beta}})\frac{\partial \mathbf{z}(\bar{t}_0;\bar{\boldsymbol{\beta}})}{\partial t_1} - H_{\mathbf{u}}(\bar{t}_0;\bar{\boldsymbol{\beta}})\frac{\partial \mathbf{v}(\bar{t}_0;\bar{\boldsymbol{\beta}})}{\partial t_1} - H_{\boldsymbol{\lambda}}(\bar{t}_0;\bar{\boldsymbol{\beta}})\frac{\partial \boldsymbol{\lambda}(\bar{t}_0;\bar{\boldsymbol{\beta}})}{\partial t_1}$$

$$= -H_{\boldsymbol{\lambda}}(\bar{t}_0;\bar{\boldsymbol{\beta}})\frac{\partial \boldsymbol{\lambda}(\bar{t}_0;\bar{\boldsymbol{\beta}})}{\partial t_1}, \tag{74}$$

$$V_{t_0\mathbf{x}_1}(\bar{\boldsymbol{\beta}}) \equiv -H_{\mathbf{x}}(\bar{t}_0;\bar{\boldsymbol{\beta}})\frac{\partial \mathbf{z}(\bar{t}_0;\bar{\boldsymbol{\beta}})}{\partial \mathbf{x}_1} - H_{\mathbf{u}}(\bar{t}_0;\bar{\boldsymbol{\beta}})\frac{\partial \mathbf{v}(\bar{t}_0;\bar{\boldsymbol{\beta}})}{\partial \mathbf{x}_1} - H_{\boldsymbol{\lambda}}(\bar{t}_0;\bar{\boldsymbol{\beta}})\frac{\partial \boldsymbol{\lambda}(\bar{t}_0;\bar{\boldsymbol{\beta}})}{\partial \mathbf{x}_1}$$

$$= -H_{\boldsymbol{\lambda}}(\bar{t}_0;\bar{\boldsymbol{\beta}})\frac{\partial \boldsymbol{\lambda}(\bar{t}_0;\bar{\boldsymbol{\beta}})}{\partial \mathbf{x}_1}, \tag{75}$$

$$V_{\mathbf{x}_0\boldsymbol{\alpha}}(\bar{\boldsymbol{\beta}}) \equiv \frac{\partial \boldsymbol{\lambda}(\bar{t}_0;\bar{\boldsymbol{\beta}})}{\partial \boldsymbol{\alpha}}, \tag{76}$$

$$V_{\mathbf{x}_0t_0}(\bar{\boldsymbol{\beta}}) \equiv \dot{\boldsymbol{\lambda}}(\bar{t}_0;\bar{\boldsymbol{\beta}}) + \frac{\partial \boldsymbol{\lambda}(\bar{t}_0;\bar{\boldsymbol{\beta}})}{\partial t_0}, \tag{77}$$

$$V_{\mathbf{x}_0\mathbf{x}_0}(\bar{\boldsymbol{\beta}}) \equiv \frac{\partial \boldsymbol{\lambda}(\bar{t}_0;\bar{\boldsymbol{\beta}})}{\partial \mathbf{x}_0}, \tag{78}$$

$$V_{\mathbf{x}_0t_1}(\bar{\boldsymbol{\beta}}) \equiv \frac{\partial \boldsymbol{\lambda}(\bar{t}_0;\bar{\boldsymbol{\beta}})}{\partial t_1}, \tag{79}$$

$$V_{\mathbf{x}_0\mathbf{x}_1}(\bar{\boldsymbol{\beta}}) \equiv \frac{\partial \boldsymbol{\lambda}(\bar{t}_0;\bar{\boldsymbol{\beta}})}{\partial \mathbf{x}_1}, \tag{80}$$

$$V_{t_1\boldsymbol{\alpha}}(\bar{\boldsymbol{\beta}}) \equiv H_{\mathbf{x}}(\bar{t}_1;\bar{\boldsymbol{\beta}})\frac{\partial \mathbf{z}(\bar{t}_1;\bar{\boldsymbol{\beta}})}{\partial \boldsymbol{\alpha}} + H_{\mathbf{u}}(\bar{t}_1;\bar{\boldsymbol{\beta}})\frac{\partial \mathbf{v}(\bar{t}_1;\bar{\boldsymbol{\beta}})}{\partial \boldsymbol{\alpha}}$$

$$+ H_{\boldsymbol{\lambda}}(\bar{t}_1;\bar{\boldsymbol{\beta}})\frac{\partial \boldsymbol{\lambda}(\bar{t}_1;\bar{\boldsymbol{\beta}})}{\partial \boldsymbol{\alpha}} + H_{\boldsymbol{\alpha}}(\bar{t}_1;\bar{\boldsymbol{\beta}})$$

$$= H_{\boldsymbol{\lambda}}(\bar{t}_1;\bar{\boldsymbol{\beta}})\frac{\partial \boldsymbol{\lambda}(\bar{t}_1;\bar{\boldsymbol{\beta}})}{\partial \boldsymbol{\alpha}} + H_{\boldsymbol{\alpha}}(\bar{t}_1;\bar{\boldsymbol{\beta}}), \tag{81}$$

$$V_{t_1t_0}(\bar{\boldsymbol{\beta}}) \equiv H_{\mathbf{x}}(\bar{t}_1;\bar{\boldsymbol{\beta}})\frac{\partial \mathbf{z}(\bar{t}_1;\bar{\boldsymbol{\beta}})}{\partial t_0} + H_{\mathbf{u}}(\bar{t}_1;\bar{\boldsymbol{\beta}})\frac{\partial \mathbf{v}(\bar{t}_1;\bar{\boldsymbol{\beta}})}{\partial t_0} + H_{\boldsymbol{\lambda}}(\bar{t}_1;\bar{\boldsymbol{\beta}})\frac{\partial \boldsymbol{\lambda}(\bar{t}_1;\bar{\boldsymbol{\beta}})}{\partial t_0}$$

$$= H_{\boldsymbol{\lambda}}(\bar{t}_1;\bar{\boldsymbol{\beta}})\frac{\partial \boldsymbol{\lambda}(\bar{t}_1;\bar{\boldsymbol{\beta}})}{\partial t_0}, \tag{82}$$

$$V_{t_1\mathbf{x}_0}(\bar{\boldsymbol{\beta}}) \equiv H_{\mathbf{x}}(\bar{t}_1;\bar{\boldsymbol{\beta}})\frac{\partial \mathbf{z}(\bar{t}_1;\bar{\boldsymbol{\beta}})}{\partial \mathbf{x}_0} + H_{\mathbf{u}}(\bar{t}_1;\bar{\boldsymbol{\beta}})\frac{\partial \mathbf{v}(\bar{t}_1;\bar{\boldsymbol{\beta}})}{\partial \mathbf{x}_0} + H_{\boldsymbol{\lambda}}(\bar{t}_1;\bar{\boldsymbol{\beta}})\frac{\partial \boldsymbol{\lambda}(\bar{t}_1;\bar{\boldsymbol{\beta}})}{\partial \mathbf{x}_0}$$

$$= H_{\boldsymbol{\lambda}}(\bar{t}_1;\bar{\boldsymbol{\beta}})\frac{\partial \boldsymbol{\lambda}(\bar{t}_1;\bar{\boldsymbol{\beta}})}{\partial \mathbf{x}_0}, \tag{83}$$

$$V_{t_1t_1}(\bar{\boldsymbol{\beta}}) \equiv H_t(\bar{t}_1;\bar{\boldsymbol{\beta}}) + H_{\mathbf{x}}(\bar{t}_1;\bar{\boldsymbol{\beta}})\left[\dot{\mathbf{z}}(\bar{t}_1;\bar{\boldsymbol{\beta}}) + \frac{\partial \mathbf{z}(\bar{t}_1;\bar{\boldsymbol{\beta}})}{\partial t_1}\right]$$

$$+ H_{\mathbf{u}}(\bar{t}_1;\bar{\boldsymbol{\beta}})\left[\dot{\mathbf{v}}(\bar{t}_1;\bar{\boldsymbol{\beta}}) + \frac{\partial \mathbf{v}(\bar{t}_1;\bar{\boldsymbol{\beta}})}{\partial t_1}\right] + H_{\boldsymbol{\lambda}}(\bar{t}_1;\bar{\boldsymbol{\beta}})\left[\dot{\boldsymbol{\lambda}}(\bar{t}_1;\bar{\boldsymbol{\beta}}) + \frac{\partial \boldsymbol{\lambda}(\bar{t}_1;\bar{\boldsymbol{\beta}})}{\partial t_1}\right]$$

$$= H_t(\bar{t}_1;\bar{\boldsymbol{\beta}}) + H_{\boldsymbol{\lambda}}(\bar{t}_1;\bar{\boldsymbol{\beta}})\left[\dot{\boldsymbol{\lambda}}(\bar{t}_1;\bar{\boldsymbol{\beta}}) + \frac{\partial \boldsymbol{\lambda}(\bar{t}_1;\bar{\boldsymbol{\beta}})}{\partial t_1}\right], \tag{84}$$

$$V_{t_1\mathbf{x}_1}(\bar{\boldsymbol{\beta}}) \equiv H_{\mathbf{x}}(\bar{t}_1;\bar{\boldsymbol{\beta}})\frac{\partial \mathbf{z}(\bar{t}_1;\bar{\boldsymbol{\beta}})}{\partial \mathbf{x}_1} + H_{\mathbf{u}}(\bar{t}_1;\bar{\boldsymbol{\beta}})\frac{\partial \mathbf{v}(\bar{t}_1;\bar{\boldsymbol{\beta}})}{\partial \mathbf{x}_1} + H_{\boldsymbol{\lambda}}(\bar{t}_1;\bar{\boldsymbol{\beta}})\frac{\partial \boldsymbol{\lambda}(\bar{t}_1;\bar{\boldsymbol{\beta}})}{\partial \mathbf{x}_1}$$

$$= H_{\mathbf{x}}(\bar{t}_1;\bar{\boldsymbol{\beta}}) + H_{\boldsymbol{\lambda}}(\bar{t}_1;\bar{\boldsymbol{\beta}})\frac{\partial \boldsymbol{\lambda}(\bar{t}_1;\bar{\boldsymbol{\beta}})}{\partial \mathbf{x}_1}, \tag{85}$$

$$V_{\mathbf{x}_1\boldsymbol{\alpha}}(\bar{\boldsymbol{\beta}}) \equiv -\frac{\partial \boldsymbol{\lambda}(\bar{t}_1;\bar{\boldsymbol{\beta}})}{\partial \boldsymbol{\alpha}}, \tag{86}$$

$$V_{\mathbf{x}_1 t_0}(\bar{\boldsymbol{\beta}}) \equiv -\frac{\partial \boldsymbol{\lambda}(\bar{t}_1;\bar{\boldsymbol{\beta}})}{\partial t_0}, \tag{87}$$

$$V_{\mathbf{x}_1\mathbf{x}_0}(\bar{\boldsymbol{\beta}}) \equiv -\frac{\partial \boldsymbol{\lambda}(\bar{t}_1;\bar{\boldsymbol{\beta}})}{\partial \mathbf{x}_0}, \tag{88}$$

$$V_{\mathbf{x}_1 t_1}(\bar{\boldsymbol{\beta}}) \equiv -\dot{\boldsymbol{\lambda}}(\bar{t}_1;\bar{\boldsymbol{\beta}}) - \frac{\partial \boldsymbol{\lambda}(\bar{t}_1;\bar{\boldsymbol{\beta}})}{\partial t_1}, \tag{89}$$

$$V_{\mathbf{x}_1\mathbf{x}_1}(\bar{\boldsymbol{\beta}}) \equiv -\frac{\partial \boldsymbol{\lambda}(\bar{t}_1;\bar{\boldsymbol{\beta}})}{\partial \mathbf{x}_1}. \tag{90}$$

In addition to the necessary conditions of Corollary 4.2 given in Eqs. (63) through (66), we also made use of the following implications of the fixed endpoints on the state vector:

$$\mathbf{z}(t_0;\boldsymbol{\beta}) \equiv \mathbf{x}_0 \Rightarrow \begin{cases} \dfrac{\partial \mathbf{z}(t_0;\boldsymbol{\beta})}{\partial \boldsymbol{\alpha}} \equiv \mathbf{0}_{N\times A}, \\ \dot{\mathbf{z}}(t_0;\boldsymbol{\beta}) + \dfrac{\partial \mathbf{z}(t_0;\boldsymbol{\beta})}{\partial t_0} \equiv \mathbf{0}_N, \\ \dfrac{\partial \mathbf{z}(t_0;\boldsymbol{\beta})}{\partial \mathbf{x}_0} \equiv \mathbf{I}_N, \\ \dfrac{\partial \mathbf{z}(t_0;\boldsymbol{\beta})}{\partial t_1} \equiv \mathbf{0}_N, \\ \dfrac{\partial \mathbf{z}(t_0;\boldsymbol{\beta})}{\partial \mathbf{x}_1} \equiv \mathbf{0}_{N\times N}, \end{cases}$$

$$\mathbf{z}(t_1;\boldsymbol{\beta}) \equiv \mathbf{x}_1 \Rightarrow \begin{cases} \dfrac{\partial \mathbf{z}(t_1;\boldsymbol{\beta})}{\partial \boldsymbol{\alpha}} \equiv \mathbf{0}_{N\times A}, \\ \dfrac{\partial \mathbf{z}(t_1;\boldsymbol{\beta})}{\partial t_0} \equiv \mathbf{0}_N, \\ \dfrac{\partial \mathbf{z}(t_1;\boldsymbol{\beta})}{\partial \mathbf{x}_0} \equiv \mathbf{0}_{N\times N}, \\ \dot{\mathbf{z}}(t_1;\boldsymbol{\beta}) + \dfrac{\partial \mathbf{z}(t_1;\boldsymbol{\beta})}{\partial t_1} \equiv \mathbf{0}_N, \\ \dfrac{\partial \mathbf{z}(t_1;\boldsymbol{\beta})}{\partial \mathbf{x}_1} \equiv \mathbf{I}_N, \end{cases}$$

in simplifying Eqs. (67) through (90).

Now recall that Eqs. (39) through (90) hold for all $\boldsymbol{\beta} \in B(\boldsymbol{\beta}^\circ;\delta)$, as established in the chapter proper. As a result, we may drop the over bar on the parameters. Doing just that, substituting Eqs. (67) through (90) into Eqs. (39) through (62) respectively, and then canceling terms yields the results we seek, namely,

$$L_{\boldsymbol{\alpha}t_0}(\boldsymbol{\beta}) = -\int_{t_0}^{t_1}\left[H_{\boldsymbol{\alpha}\mathbf{x}}(t;\boldsymbol{\beta})\frac{\partial \mathbf{z}(t;\boldsymbol{\beta})}{\partial t_0} + H_{\boldsymbol{\alpha}\mathbf{u}}(t;\boldsymbol{\beta})\frac{\partial \mathbf{v}(t;\boldsymbol{\beta})}{\partial t_0} + H_{\boldsymbol{\alpha}\boldsymbol{\lambda}}(t;\boldsymbol{\beta})\frac{\partial \boldsymbol{\lambda}(t;\boldsymbol{\beta})}{\partial t_0}\right]dt, \tag{91}$$

$$L_{\boldsymbol{\alpha}\mathbf{x}_0}(\boldsymbol{\beta}) = -\int_{t_0}^{t_1}\left[H_{\boldsymbol{\alpha}\mathbf{x}}(t;\boldsymbol{\beta})\frac{\partial \mathbf{z}(t;\boldsymbol{\beta})}{\partial \mathbf{x}_0} + H_{\boldsymbol{\alpha}\mathbf{u}}(t;\boldsymbol{\beta})\frac{\partial \mathbf{v}(t;\boldsymbol{\beta})}{\partial \mathbf{x}_0} + H_{\boldsymbol{\alpha}\boldsymbol{\lambda}}(t;\boldsymbol{\beta})\frac{\partial \boldsymbol{\lambda}(t;\boldsymbol{\beta})}{\partial \mathbf{x}_0}\right]dt, \tag{92}$$

$$L_{\boldsymbol{\alpha}t_1}(\boldsymbol{\beta}) = -\int_{t_0}^{t_1}\left[H_{\boldsymbol{\alpha}\mathbf{x}}(t;\boldsymbol{\beta})\frac{\partial \mathbf{z}(t;\boldsymbol{\beta})}{\partial t_1} + H_{\boldsymbol{\alpha}\mathbf{u}}(t;\boldsymbol{\beta})\frac{\partial \mathbf{v}(t;\boldsymbol{\beta})}{\partial t_1} + H_{\boldsymbol{\alpha}\boldsymbol{\lambda}}(t;\boldsymbol{\beta})\frac{\partial \boldsymbol{\lambda}(t;\boldsymbol{\beta})}{\partial t_1}\right]dt, \tag{93}$$

$$L_{\boldsymbol{\alpha}\mathbf{x}_1}(\boldsymbol{\beta}) = -\int_{t_0}^{t_1}\left[H_{\boldsymbol{\alpha}\mathbf{x}}(t;\boldsymbol{\beta})\frac{\partial \mathbf{z}(t;\boldsymbol{\beta})}{\partial \mathbf{x}_1} + H_{\boldsymbol{\alpha}\mathbf{u}}(t;\boldsymbol{\beta})\frac{\partial \mathbf{v}(t;\boldsymbol{\beta})}{\partial \mathbf{x}_1} + H_{\boldsymbol{\alpha}\boldsymbol{\lambda}}(t;\boldsymbol{\beta})\frac{\partial \boldsymbol{\lambda}(t;\boldsymbol{\beta})}{\partial \mathbf{x}_1}\right]dt, \tag{94}$$

$$L_{t_0\boldsymbol{\alpha}}(\boldsymbol{\beta}) = H_{\boldsymbol{\lambda}}(t_0;\boldsymbol{\beta})\frac{\partial \boldsymbol{\lambda}(t_0;\boldsymbol{\beta})}{\partial \boldsymbol{\alpha}}, \tag{95}$$

$$L_{t_0t_0}(\boldsymbol{\beta}) = H_{\boldsymbol{\lambda}}(t_0;\boldsymbol{\beta})\frac{\partial \boldsymbol{\lambda}(t_0;\boldsymbol{\beta})}{\partial t_0}, \tag{96}$$

$$L_{t_0\mathbf{x}_0}(\boldsymbol{\beta}) = H_{\boldsymbol{\lambda}}(t_0;\boldsymbol{\beta})\frac{\partial \boldsymbol{\lambda}(t_0;\boldsymbol{\beta})}{\partial \mathbf{x}_0}, \tag{97}$$

$$L_{t_0t_1}(\boldsymbol{\beta}) = H_{\boldsymbol{\lambda}}(t_0;\boldsymbol{\beta})\frac{\partial \boldsymbol{\lambda}(t_0;\boldsymbol{\beta})}{\partial t_1}, \tag{98}$$

$$L_{t_0\mathbf{x}_1}(\boldsymbol{\beta}) = H_{\boldsymbol{\lambda}}(t_0;\boldsymbol{\beta})\frac{\partial \boldsymbol{\lambda}(t_0;\boldsymbol{\beta})}{\partial \mathbf{x}_1}, \tag{99}$$

$$L_{\mathbf{x}_0\boldsymbol{\alpha}}(\boldsymbol{\beta}) = -\frac{\partial \boldsymbol{\lambda}(t_0;\boldsymbol{\beta})}{\partial \boldsymbol{\alpha}}, \tag{100}$$

$$L_{\mathbf{x}_0t_0}(\boldsymbol{\beta}) = -\frac{\partial \boldsymbol{\lambda}(t_0;\boldsymbol{\beta})}{\partial t_0}, \tag{101}$$

$$L_{\mathbf{x}_0\mathbf{x}_0}(\boldsymbol{\beta}) = -\frac{\partial \boldsymbol{\lambda}(t_0;\boldsymbol{\beta})}{\partial \mathbf{x}_0}, \tag{102}$$

$$L_{\mathbf{x}_0t_1}(\boldsymbol{\beta}) = -\frac{\partial \boldsymbol{\lambda}(t_0;\boldsymbol{\beta})}{\partial t_1}, \tag{103}$$

$$L_{\mathbf{x}_0\mathbf{x}_1}(\boldsymbol{\beta}) = -\frac{\partial \boldsymbol{\lambda}(t_0;\boldsymbol{\beta})}{\partial \mathbf{x}_1}, \tag{104}$$

$$L_{t_1\boldsymbol{\alpha}}(\boldsymbol{\beta}) = -H_{\boldsymbol{\lambda}}(t_1;\boldsymbol{\beta})\frac{\partial \boldsymbol{\lambda}(t_1;\boldsymbol{\beta})}{\partial \boldsymbol{\alpha}}, \tag{105}$$

$$L_{t_1t_0}(\boldsymbol{\beta}) = -H_{\boldsymbol{\lambda}}(t_1;\boldsymbol{\beta})\frac{\partial \boldsymbol{\lambda}(t_1;\boldsymbol{\beta})}{\partial t_0}, \tag{106}$$

$$L_{t_1\mathbf{x}_0}(\boldsymbol{\beta}) = -H_{\boldsymbol{\lambda}}(t_1;\boldsymbol{\beta})\frac{\partial \boldsymbol{\lambda}(t_1;\boldsymbol{\beta})}{\partial \mathbf{x}_0}, \tag{107}$$

$$L_{t_1t_1}(\boldsymbol{\beta}) = -H_{\boldsymbol{\lambda}}(t_1;\boldsymbol{\beta})\frac{\partial \boldsymbol{\lambda}(t_1;\boldsymbol{\beta})}{\partial t_1}, \tag{108}$$

$$L_{t_1\mathbf{x}_1}(\boldsymbol{\beta}) = -H_{\boldsymbol{\lambda}}(t_1;\boldsymbol{\beta})\frac{\partial \boldsymbol{\lambda}(t_1;\boldsymbol{\beta})}{\partial \mathbf{x}_1}, \tag{109}$$

$$L_{\mathbf{x}_1\boldsymbol{\alpha}}(\boldsymbol{\beta}) = \frac{\partial \boldsymbol{\lambda}(t_1;\boldsymbol{\beta})}{\partial \boldsymbol{\alpha}}, \tag{110}$$

$$L_{\mathbf{x}_1t_0}(\boldsymbol{\beta}) = \frac{\partial \boldsymbol{\lambda}(t_1;\boldsymbol{\beta})}{\partial t_0}, \tag{111}$$

$$L_{\mathbf{x}_1\mathbf{x}_0}(\boldsymbol{\beta}) = \frac{\partial \boldsymbol{\lambda}(t_1;\boldsymbol{\beta})}{\partial \mathbf{x}_0}, \tag{112}$$

$$L_{\mathbf{x}_1t_1}(\boldsymbol{\beta}) = \frac{\partial \boldsymbol{\lambda}(t_1;\boldsymbol{\beta})}{\partial t_1}, \tag{113}$$

$$L_{\mathbf{x}_1\mathbf{x}_1}(\boldsymbol{\beta}) = \frac{\partial \boldsymbol{\lambda}(t_1;\boldsymbol{\beta})}{\partial \mathbf{x}_1}. \tag{114}$$

MENTAL EXERCISES

11.1 Prove that $\dot{\mathbf{z}}(\cdot) \in C^{(1)}$ in $(t;\boldsymbol{\beta})$ for all $(t;\boldsymbol{\beta}) \in [t_0^\circ, t_1^\circ] \times B(\boldsymbol{\beta}^\circ;\delta)$.

11.2 Explain why Eq. (8) is not identical to Eq. (9).

11.3 Prove that Eq. (10) can be derived from Eq. (9) by integration by parts.

11.4 Verify the assertion in the paragraph after Eq. (10).

11.5 Consider the unconstrained static optimization problem

$$\phi(\boldsymbol{\alpha}) \stackrel{\text{def}}{=} \max_{\mathbf{x}\in\Re^N} f(\mathbf{x};\boldsymbol{\alpha}),$$

where $\boldsymbol{\alpha} \in \Re^A$ and $\mathbf{x}^*(\boldsymbol{\alpha})$ is the optimal value of the decision vector for all $\boldsymbol{\alpha} \in B(\boldsymbol{\alpha}^\circ;\delta)$.

(a) Prove that the $A \times A$ matrix $f_{\boldsymbol{\alpha}\boldsymbol{\alpha}}(\mathbf{x}^*(\boldsymbol{\alpha});\boldsymbol{\alpha}) - \phi_{\boldsymbol{\alpha}\boldsymbol{\alpha}}(\boldsymbol{\alpha})$ is symmetric and negative semidefinite.

(b) Prove that the matrix $\mathbf{Q}(\boldsymbol{\alpha})$, the typical term of which is defined by

$$\mathbf{Q}_{ij}(\boldsymbol{\alpha}) \stackrel{\text{def}}{=} \sum_{n=1}^{N} f_{\alpha_i x_n}(\mathbf{x}^*(\boldsymbol{\alpha}); \boldsymbol{\alpha}) \frac{\partial x_n^*}{\partial \alpha_j}(\boldsymbol{\alpha}), \quad i, j = 1, 2, \ldots, A,$$

is symmetric and positive semidefinite.

(c) Prove that if $f(\cdot)$ is convex in $\boldsymbol{\alpha} \in \Re^A$, then $\phi(\cdot)$ is convex in $\boldsymbol{\alpha} \in \Re^A$.

11.6 Consider the constrained static optimization problem

$$\phi(\boldsymbol{\alpha}) \stackrel{\text{def}}{=} \max_{\mathbf{x} \in \Re^N} \{f(\mathbf{x}; \boldsymbol{\alpha}) \text{ s.t. } g^k(\mathbf{x}) = 0, \quad k = 1, 2, \ldots, K < N\},$$

where $\boldsymbol{\alpha} \in \Re^A$ and $\mathbf{x}^*(\boldsymbol{\alpha})$ is the optimal value of the decision vector for all $\boldsymbol{\alpha} \in B(\boldsymbol{\alpha}^\circ; \delta)$.

(a) Prove that the $A \times A$ matrix $f_{\boldsymbol{\alpha}\boldsymbol{\alpha}}(\mathbf{x}^*(\boldsymbol{\alpha}); \boldsymbol{\alpha}) - \phi_{\boldsymbol{\alpha}\boldsymbol{\alpha}}(\boldsymbol{\alpha})$ is symmetric and negative semidefinite.

(b) Prove that the matrix $\mathbf{Q}(\boldsymbol{\alpha})$, the typical term of which is defined by

$$\mathbf{Q}_{ij}(\boldsymbol{\alpha}) \stackrel{\text{def}}{=} \sum_{n=1}^{N} f_{\alpha_i x_n}(\mathbf{x}^*(\boldsymbol{\alpha}); \boldsymbol{\alpha}) \frac{\partial x_n^*}{\partial \alpha_j}(\boldsymbol{\alpha}), \quad i, j = 1, 2, \ldots, A,$$

is symmetric and positive semidefinite.

(c) Prove that if $f(\cdot)$ is convex in $\boldsymbol{\alpha} \in \Re^A$, then $\phi(\cdot)$ is convex in $\boldsymbol{\alpha} \in \Re^A$. This establishes that if the objective function of a constrained static maximization problem is convex in the parameters of the problem and the constraints are independent of the parameters, then the indirect objective function is locally convex in the parameters.

11.7 Derive Eqs. (33) through (35) directly from Theorem 11.2.

11.8 Give a simple necessary and sufficient condition for the integrand function $F(\cdot)$ of the adjustment cost model to be locally convex in the discount rate r, where

$$F(t, \mathbf{x}, \mathbf{u}; \boldsymbol{\alpha}) \stackrel{\text{def}}{=} [pf(x_1, x_2, u_1, u_2) - w_1 x_1 - w_2 x_2 - g_1 u_1 - g_2 u_2]\, e^{-rt}.$$

Then derive the comparative dynamics of the discount rate using Corollary 11.1. Compare your result with that in Eq. (36).

11.9 Derive the 6×6 matrix $L_{\boldsymbol{\alpha}\boldsymbol{\alpha}}(\boldsymbol{\beta})$ for the adjustment cost model of the firm. Note that this has already been started for you in Eqs. (33) through (38). You have 24 more terms to derive.

11.10 This question asks you to explore the comparative dynamics properties of the inventory accumulation problem from a dual point of view. Recall that the model under investigation is given by the optimal control problem

$$C(\boldsymbol{\beta}) \stackrel{\text{def}}{=} \min_{u(\cdot)} \int_0^T \left[c_1[u(t)]^2 + c_2 x(t)\right] dt$$

$$\text{s.t.} \quad \dot{x}(t) = u(t),\, x(0) = 0,\, x(T) = x_T,$$

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where $z(t;\boldsymbol{\beta}) = \frac{1}{4}c_2c_1^{-1}t[t-T] + x_T T^{-1}t$ and $v(t;\boldsymbol{\beta}) = \frac{1}{4}c_2c_1^{-1}[2t-T] + x_T T^{-1}$ are the optimal time paths of the inventory stock and production rate, respectively, and $\boldsymbol{\beta} \stackrel{\text{def}}{=} (c_1, c_2, T, x_T)$. Do *not* use the specific functional form given for $z(t;\boldsymbol{\beta})$ in what follows unless specifically asked to do so. Assume that $x_T \geq \frac{1}{4}c_2c_1^{-1}T^2$, so that $v(t;\boldsymbol{\beta}) \geq 0$ for all $t \in [0, T]$.

(a) Carefully and rigorously set up the dynamic primal-dual problem corresponding to the primal optimal control problem. Treat (T, x_T) as fixed parameters in the primal-dual problem, that is, never make them objects of choice.

(b) Derive the first-order and second-order *necessary* conditions for the dynamic primal-dual problem.

(c) Prove that

$$\frac{\partial C}{\partial c_1}(\boldsymbol{\beta}) \equiv \int_0^T [v(t;\boldsymbol{\beta})]^2\,dt > 0 \quad\text{and}\quad \frac{\partial C}{\partial c_2}(\boldsymbol{\beta}) \equiv \int_0^T z(t;\boldsymbol{\beta})\,dt > 0,$$

and provide an economic interpretation of each.

(d) Use the envelope results in part (c) to rewrite the second-order necessary conditions, and in the process, prove that the matrix

$$\begin{bmatrix} \displaystyle\int_0^T 2v(t;\boldsymbol{\beta})\frac{\partial v(t;\boldsymbol{\beta})}{\partial c_1}\,dt & \displaystyle\int_0^T 2v(t;\boldsymbol{\beta})\frac{\partial v(t;\boldsymbol{\beta})}{\partial c_2}\,dt \\ \displaystyle\int_0^T \frac{\partial z(t;\boldsymbol{\beta})}{\partial c_1}\,dt & \displaystyle\int_0^T \frac{\partial z(t;\boldsymbol{\beta})}{\partial c_2}\,dt \end{bmatrix}$$

is symmetric and negative semidefinite.

(e) Prove that the minimum cost function $C(\cdot)$ is positively homogeneous of degree one in (c_1, c_2).

(f) Prove that the minimum cost function $C(\cdot)$ is concave in (c_1, c_2).

(g) Prove that the vector (c_1, c_2) lies in the null space of the Hessian matrix of the minimum cost function $C(\cdot)$ with respect to (c_1, c_2). What does this tell you about one of the eigenvalues of the Hessian matrix of the minimum cost function $C(\cdot)$ with respect to (c_1, c_2)? What, in turn, does this tell you about the rank of the Hessian matrix of the minimum cost function $C(\cdot)$ with respect to (c_1, c_2), and thus about whether the said matrix is singular?

(h) Verify that

$$\int_0^T \frac{\partial z(t;\boldsymbol{\beta})}{\partial c_2}\,dt < 0$$

using the specific functional form for $z(t;\boldsymbol{\beta})$. Provide an economic interpretation of this result.

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FURTHER READING

The dynamic primal-dual methodology developed in this chapter is a continuous-time intertemporal equivalent of the static primal-dual methodology of Silberberg (1974) and is based upon the work of Caputo (1990a, 1990b), in which the dynamic primal-dual formalism was developed. The results of this chapter have been extended by Caputo (1992a) to the case in which the parameter vector α is a function of time. The dynamic primal-dual method has been fruitfully applied to study the comparative dynamics properties of the nonrenewable resource–extracting model of the firm [Caputo (1990c)], the adjustment cost model of the firm [Caputo (1992b)], the labor-managed model of the firm [Caputo (1992c)], and an intertemporal model of the consumer [Caputo (1994)].

REFERENCES

Caputo, M.R. (1990a), "Comparative Dynamics via Envelope Methods in Variational Calculus," *Review of Economics Studies*, 57, 689–697.

Caputo, M.R. (1990b), "How to Do Comparative Dynamics on the Back of an Envelope in Optimal Control Theory," *Journal of Economic Dynamics and Control*, 14, 655–683.

Caputo, M.R. (1990c), "New Qualitative Properties in the Competitive Nonrenewable Resource Extracting Model of the Firm," *International Economic Review*, 31, 829–839.

Caputo, M.R. (1992a), "A Primal-Dual Approach to Comparative Dynamics with Time-Dependent Parameters in Variational Calculus," *Optimal Control Applications and Methods*, 13, 73–86.

Caputo, M.R. (1992b), "Fundamental Symmetries and Qualitative Properties in the Adjustment Cost Model of the Firm," *Journal of Mathematical Economics*, 21, 99–112.

Caputo, M.R. (1992c), "Comparative Dynamics in the Labor-Managed Model of the Firm," *Journal of Comparative Economics*, 16, 272–286.

Caputo, M.R. (1994), "The Slutsky Matrix and Homogeneity in Intertemporal Consumer Theory," *Journal of Economics*, 60, 255–279.

Silberberg, E. (1974), "A Revision of Comparative Statics Methodology in Economics, or How to Do Comparative Statics on the Back of an Envelope," *Journal of Economic Theory*, 7, 159–172.

Silberberg, E. and Suen, W. (2001, 3rd Ed.), *The Structure of Economics: A Mathematical Analysis* (New York: Irwin/McGraw-Hill).

Simon, C.P. and Blume, L. (1994), *Mathematics for Economists* (New York: W.W. Norton & Company).