NINE

The Dynamic Envelope Theorem and Economic Interpretations

One objective of this chapter is to prove the intertemporal equivalent of the prototype envelope theorem, namely, the *dynamic envelope theorem*, for a general class of fixed endpoint optimal control problems. The second objective is to use this important theorem to impart a deeper economic interpretation to the Hamiltonian and costate variables than we were heretofore able to. In the next chapter, we will use the dynamic envelope theorem to provide simple and motivated proofs of the transversality conditions corresponding to various endpoint conditions in optimal control problems. If your understanding of the prototype envelope theorem is less than ideal, then it would be best to pause at this juncture and deepen your understanding of it before tackling the ensuing material.

To begin, we proceed with the *literal* definition of the optimal value function $V(\cdot)$ for the following fixed endpoints optimal control problem:

$$V(\boldsymbol{\beta}) \stackrel{\text{def}}{=} \max_{\mathbf{u}(\cdot)} \int_{t_0}^{t_1} f(t, \mathbf{x}(t), \mathbf{u}(t); \boldsymbol{\alpha})\, dt \qquad \text{(OC)}$$

$$\text{s.t.} \quad \dot{\mathbf{x}}(t) = \mathbf{g}(t, \mathbf{x}(t), \mathbf{u}(t); \boldsymbol{\alpha}),\ \mathbf{x}(t_0) = \mathbf{x}_0,\ \mathbf{x}(t_1) = \mathbf{x}_1,$$

where $\mathbf{x}(t) \stackrel{\text{def}}{=} (x_1(t), x_2(t), \ldots, x_N(t)) \in \Re^N$ is the state vector, $\mathbf{u}(t) \stackrel{\text{def}}{=} (u_1(t), u_2(t), \ldots, u_M(t)) \in \Re^M$ is the control vector, $\boldsymbol{\alpha} \stackrel{\text{def}}{=} (\alpha_1, \alpha_2, \ldots, \alpha_A) \in \Re^A$ is a vector of time-independent parameters that affect both the state equation and the integrand, $\mathbf{g}(\cdot) \stackrel{\text{def}}{=} (g^1(\cdot), g^2(\cdot), \ldots, g^N(\cdot))$ is the transition function, and $\boldsymbol{\beta} \stackrel{\text{def}}{=} (\boldsymbol{\alpha}, t_0, \mathbf{x}_0, t_1, \mathbf{x}_1) \in \Re^{2+2N+A}$ is the vector of parameters of the problem. Assuming that an optimal pair of curves $(\mathbf{z}(t; \boldsymbol{\beta}), \mathbf{v}(t; \boldsymbol{\beta}))$ exists to problem (OC) for all $\boldsymbol{\beta} \in B(\boldsymbol{\beta}^\circ; \delta)$, where $B(\boldsymbol{\beta}^\circ; \delta)$ is an open $2 + 2N + A$ – ball centered at the given value of the parameter $\boldsymbol{\beta}^\circ \in \Re^{2+2N+A}$ of radius $\delta > 0$, we can also define the optimal value function $V(\cdot)$ of problem (OC) *constructively* as

$$V(\boldsymbol{\beta}) \equiv \int_{t_0}^{t_1} f(t, \mathbf{z}(t; \boldsymbol{\beta}), \mathbf{v}(t; \boldsymbol{\beta}); \boldsymbol{\alpha})\, dt. \qquad (1)$$

https://doi.org/10.1017/CBO9780511806827.011 Published online by Cambridge University Press

Notice that Eq. (1) is exactly how one would go about constructing (or deriving) the optimal value function in practice if one were able to perform all the mathematical operations explicitly. That is, Eq. (1) shows how one would construct $V(\cdot)$ if one could explicitly find the optimal solution $(\mathbf{z}(t;\boldsymbol{\beta}), \mathbf{v}(t;\boldsymbol{\beta}))$ to the control problem, assuming it exists, and then substitute it into the objective functional and integrate over the planning horizon.

Because of the introduction of the parameter vector $\boldsymbol{\alpha}$, and the importance of it in the dynamic envelope theorem, we must modify our basic assumptions on the integrand function and transition functions to take this feature into account. Consequently, we now impose the ensuing assumptions on the functions $f(\cdot)$ and $\mathbf{g}(\cdot)$ throughout the remainder of this chapter:

(A.1) A param-
eters $\boldsymbol{\alpha}$.

(A.2) A param-
eters $\boldsymbol{\alpha}$.

Given these preliminaries, we are now in a position to prove the following dynamic envelope theorem for the fixed endpoint class of optimal control problems defined by problem (OC).

Theorem 9.1 (Dynamic Envelope Theorem): *Let* $(\mathbf{z}(t;\boldsymbol{\beta}), \mathbf{v}(t;\boldsymbol{\beta}))$ *be the optimal pair for problem (OC), and let* $\boldsymbol{\lambda}(t;\boldsymbol{\beta})$ *be the corresponding time path of the costate vector. Define the Hamiltonian as* $H(t,\mathbf{x},\mathbf{u},\boldsymbol{\lambda};\boldsymbol{\alpha}) \stackrel{\text{def}}{=} f(t,\mathbf{x},\mathbf{u};\boldsymbol{\alpha}) + \sum_{\ell=1}^{N} \lambda_\ell g^\ell(t,\mathbf{x},\mathbf{u};\boldsymbol{\alpha})$. *If* $\mathbf{z}(\cdot) \in C^{(1)}$ *and* $\mathbf{v}(\cdot) \in C^{(1)}$ *in* $(t;\boldsymbol{\beta}) \,\forall\, (t;\boldsymbol{\beta}) \in [t_0^\circ, t_1^\circ] \times B(\boldsymbol{\beta}^\circ;\delta)$, *then* $V(\cdot) \in C^{(1)} \,\forall\, \boldsymbol{\beta} \in B(\boldsymbol{\beta}^\circ;\delta)$, *and furthermore,* $\forall\, \boldsymbol{\beta} \in B(\boldsymbol{\beta}^\circ;\delta)$:

$$V_{\alpha_i}(\boldsymbol{\beta}) \stackrel{\text{def}}{=} \frac{\partial V(\boldsymbol{\beta})}{\partial \alpha_i} \equiv \int_{t_0}^{t_1} H_{\alpha_i}(t,\mathbf{z}(t;\boldsymbol{\beta}),\mathbf{v}(t;\boldsymbol{\beta}),\boldsymbol{\lambda}(t;\boldsymbol{\beta});\boldsymbol{\alpha})\,dt, \quad i = 1,2,\ldots,A, \tag{i}$$

$$V_{t_0}(\boldsymbol{\beta}) \equiv -H(t_0,\mathbf{x}_0,\mathbf{v}(t_0;\boldsymbol{\beta}),\boldsymbol{\lambda}(t_0;\boldsymbol{\beta});\boldsymbol{\alpha}), \tag{ii}$$

$$V_{x_{0j}}(\boldsymbol{\beta}) \equiv \lambda_j(t_0;\boldsymbol{\beta}), \quad j = 1,2,\ldots,N, \tag{iii}$$

$$V_{t_1}(\boldsymbol{\beta}) \equiv H(t_1,\mathbf{x}_1,\mathbf{v}(t_1;\boldsymbol{\beta}),\boldsymbol{\lambda}(t_1;\boldsymbol{\beta});\boldsymbol{\alpha}), \tag{iv}$$

$$V_{x_{1j}}(\boldsymbol{\beta}) \equiv -\lambda_j(t_1;\boldsymbol{\beta}), \quad j = 1,2,\ldots,N. \tag{v}$$

Proof: We will prove parts (i) and (ii) and leave the rest for the mental exercises. In the proof of part (i), we employ index notation, whereas in the proof of part (ii), we employ vector notation. We employ both types of notation, since the literature uses both.

(i) Differentiate the optimal value function $V(\cdot)$ as defined constructively in Eq. (1) with respect to α_i, and use the chain rule and Leibniz's rule to get

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$$V_{\alpha_i}(\boldsymbol{\beta}) \equiv \int_{t_0}^{t_1} \left[\sum_{n=1}^{N} f_{x_n}(t, \mathbf{z}(t;\boldsymbol{\beta}), \mathbf{v}(t;\boldsymbol{\beta}); \boldsymbol{\alpha}) \frac{\partial z_n}{\partial \alpha_i}(t;\boldsymbol{\beta}) + \sum_{m=1}^{M} f_{u_m}(t, \mathbf{z}(t;\boldsymbol{\beta}), \right.$$
$$\left. \mathbf{v}(t;\boldsymbol{\beta}); \boldsymbol{\alpha}) \frac{\partial v_m}{\partial \alpha_i}(t;\boldsymbol{\beta}) + f_{\alpha_i}(t, \mathbf{z}(t;\boldsymbol{\beta}), \mathbf{v}(t;\boldsymbol{\beta}); \boldsymbol{\alpha}) \right] dt, \qquad (2)$$

for $i = 1, 2, \ldots, A$. To simplify the notation, define $f^*_{x_n}(t) \stackrel{\text{def}}{=} f_{x_n}(t, \mathbf{z}(t;\boldsymbol{\beta}), \mathbf{v}(t;\boldsymbol{\beta}); \boldsymbol{\alpha})$, $n = 1, 2, \ldots, N$, and likewise for the other partial derivatives. Because $(\mathbf{z}(t;\boldsymbol{\beta}), \mathbf{v}(t;\boldsymbol{\beta}))$ is an optimal pair, it must satisfy the state equation and boundary conditions identically $\forall\, \boldsymbol{\beta} \in B(\boldsymbol{\beta}^\circ;\delta)$ and $\forall\, t \in [t_0, t_1]$, thereby implying the identities

$$g^\ell(t, \mathbf{z}(t;\boldsymbol{\beta}), \mathbf{v}(t;\boldsymbol{\beta}); \boldsymbol{\alpha}) - \dot{z}_\ell(t;\boldsymbol{\beta}) \equiv 0, \quad \ell = 1, 2, \ldots, N,$$
$$z_\ell(t_0;\boldsymbol{\beta}) \equiv x_{0\ell},\, z_\ell(t_1;\boldsymbol{\beta}) \equiv x_{1\ell}, \quad \ell = 1, 2, \ldots, N.$$

Differentiate these three identities with respect to α_i to arrive at

$$\sum_{n=1}^{N} g^\ell_{x_n}(t, \mathbf{z}(t;\boldsymbol{\beta}), \mathbf{v}(t;\boldsymbol{\beta}); \boldsymbol{\alpha}) \frac{\partial z_n}{\partial \alpha_i}(t;\boldsymbol{\beta}) + \sum_{m=1}^{M} g^\ell_{u_m}(t, \mathbf{z}(t;\boldsymbol{\beta}), \mathbf{v}(t;\boldsymbol{\beta}); \boldsymbol{\alpha}) \frac{\partial v_m}{\partial \alpha_i}(t;\boldsymbol{\beta})$$
$$+ g^\ell_{\alpha_i}(t, \mathbf{z}(t;\boldsymbol{\beta}), \mathbf{v}(t;\boldsymbol{\beta}); \boldsymbol{\alpha}) - \frac{\partial \dot{z}_\ell}{\partial \alpha_i}(t;\boldsymbol{\beta}) \equiv 0, \quad \ell = 1, 2, \ldots, N, \qquad (3)$$
$$\frac{\partial z_\ell}{\partial \alpha_i}(t_0;\boldsymbol{\beta}) \equiv 0,\, \frac{\partial z_\ell}{\partial \alpha_i}(t_1;\boldsymbol{\beta}) \equiv 0, \quad \ell = 1, 2, \ldots, N. \qquad (4)$$

As we did above, define $g^{\ell*}_{u_m}(t) \stackrel{\text{def}}{=} g^\ell_{u_m}(t, \mathbf{z}(t;\boldsymbol{\beta}), \mathbf{v}(t;\boldsymbol{\beta}); \boldsymbol{\alpha})$, $m = 1, 2, \ldots, M$, $\ell = 1, 2, \ldots, N$, and similarly for the other partial derivatives. Now multiply the ℓth identity in Eq. (3) by its corresponding costate variable $\lambda_\ell(t;\boldsymbol{\beta})$ and sum the resulting expression over ℓ, from $\ell = 1$ to $\ell = N$. Because the N identities in Eq. (3) are identically zero $\forall\, \boldsymbol{\beta} \in B(\boldsymbol{\beta}^\circ;\delta)$ and $\forall\, t \in [t_0, t_1]$, they are clearly still zero $\forall\, \boldsymbol{\beta} \in B(\boldsymbol{\beta}^\circ;\delta)$ and $\forall\, t \in [t_0, t_1]$ when multiplied by $\lambda_\ell(t;\boldsymbol{\beta})$ and summed over ℓ. Moreover, when the resulting sum is integrated over $[t_0, t_1]$, it is still identically zero and therefore may be added to Eq. (2) without changing the latter's value, thereby yielding

$$V_{\alpha_i}(\boldsymbol{\beta}) \equiv \int_{t_0}^{t_1} \left[\sum_{n=1}^{N} \left[f^*_{x_n}(t) + \sum_{\ell=1}^{N} \lambda_\ell(t;\boldsymbol{\beta}) g^{\ell*}_{x_n}(t) \right] \frac{\partial z_n}{\partial \alpha_i}(t;\boldsymbol{\beta}) \right.$$
$$+ \sum_{m=1}^{M} \left[f^*_{u_m}(t) + \sum_{\ell=1}^{N} \lambda_\ell(t;\boldsymbol{\beta}) g^{\ell*}_{u_m}(t) \right] \frac{\partial v_m}{\partial \alpha_i}(t;\boldsymbol{\beta})$$
$$\left. + \left[f^*_{\alpha_i}(t) + \sum_{\ell=1}^{N} \lambda_\ell(t;\boldsymbol{\beta}) g^{\ell*}_{\alpha_i}(t) \right] - \sum_{\ell=1}^{N} \lambda_\ell(t;\boldsymbol{\beta}) \frac{\partial \dot{z}_\ell}{\partial \alpha_i}(t;\boldsymbol{\beta}) \right] dt. \qquad (5)$$

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Now define the partial derivatives of the Hamiltonian along the optimal path by

$$H^*_{x_n}(t) \stackrel{\text{def}}{=} f^*_{x_n}(t) + \sum_{\ell=1}^{N} \lambda_\ell(t;\boldsymbol{\beta}) g^{\ell*}_{x_n}(t), \quad n = 1, 2 \ldots, N,$$

$$H^*_{u_m}(t) \stackrel{\text{def}}{=} f^*_{u_m}(t) + \sum_{\ell=1}^{N} \lambda_\ell(t;\boldsymbol{\beta}) g^{\ell*}_{u_m}(t), \quad m = 1, 2 \ldots, M,$$

$$H^*_{\alpha_i}(t) \stackrel{\text{def}}{=} f^*_{\alpha_i}(t) + \sum_{\ell=1}^{N} \lambda_\ell(t;\boldsymbol{\beta}) g^{\ell*}_{\alpha_i}(t), \quad i = 1, 2 \ldots, A.$$

These definitions allow Eq. (5) to be rewritten more cleanly as

$$V_{\alpha_i}(\boldsymbol{\beta}) \equiv \int_{t_0}^{t_1} \left[\sum_{n=1}^{N} H^*_{x_n}(t) \frac{\partial z_n}{\partial \alpha_i}(t;\boldsymbol{\beta}) + \sum_{m=1}^{M} H^*_{u_m}(t) \frac{\partial v_m}{\partial \alpha_i}(t;\boldsymbol{\beta}) + H^*_{\alpha_i}(t) \right.$$
$$\left. - \sum_{\ell=1}^{N} \lambda_\ell(t;\boldsymbol{\beta}) \frac{\partial \dot{z}_\ell}{\partial \alpha_i}(t;\boldsymbol{\beta}) \right] dt. \tag{6}$$

Upon letting

$$p_\ell = \lambda_\ell(t;\boldsymbol{\beta}), \qquad dq_\ell = \frac{\partial \dot{z}_\ell}{\partial \alpha_i}(t;\boldsymbol{\beta})\, dt = \frac{d}{dt}\left[\frac{\partial z_\ell}{\partial \alpha_i}(t;\boldsymbol{\beta})\right] dt,$$
$$dp_\ell = \dot{\lambda}_\ell(t;\boldsymbol{\beta})\, dt, \qquad q_\ell = \frac{\partial z_\ell}{\partial \alpha_i}(t;\boldsymbol{\beta}),$$

$\ell = 1, 2, \ldots, N$, we can integrate each term in the last sum of Eq. (6) by parts, thus yielding

$$-\int_{t_0}^{t_1} \sum_{\ell=1}^{N} \lambda_\ell(t;\boldsymbol{\beta}) \frac{\partial \dot{z}_\ell}{\partial \alpha_i}(t;\boldsymbol{\beta})\, dt = -\sum_{\ell=1}^{N} \lambda_\ell(t;\boldsymbol{\beta}) \frac{\partial z_\ell}{\partial \alpha_i}(t;\boldsymbol{\beta}) \Bigg|_{t=t_0}^{t=t_1}$$
$$+ \int_{t_0}^{t_1} \sum_{\ell=1}^{N} \dot{\lambda}_\ell(t;\boldsymbol{\beta}) \frac{\partial z_\ell}{\partial \alpha_i}(t;\boldsymbol{\beta})\, dt = \int_{t_0}^{t_1} \sum_{\ell=1}^{N} \dot{\lambda}_\ell(t;\boldsymbol{\beta}) \frac{\partial z_\ell}{\partial \alpha_i}(t;\boldsymbol{\beta})\, dt, \tag{7}$$

where we have used Eq. (4) to simplify Eq. (7). Substituting Eq. (7) into Eq. (6) yields

$$V_{\alpha_i}(\boldsymbol{\beta})$$
$$\equiv \int_{t_0}^{t_1} \left[\sum_{n=1}^{N} [H^*_{x_n}(t) + \dot{\lambda}_n(t;\boldsymbol{\beta})] \frac{\partial z_n}{\partial \alpha_i}(t;\boldsymbol{\beta}) + \sum_{m=1}^{M} H^*_{u_m}(t) \frac{\partial v_m}{\partial \alpha_i}(t;\boldsymbol{\beta}) + H^*_{\alpha_i}(t) \right] dt. \tag{8}$$

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Note that in arriving at the final form of Eq. (8), the dummy index of summation ℓ from Eq. (7) was replaced by the dummy index of summation n, which is legitimate in view of the fact that they are both dummy indices and range over identical values. Because $(\mathbf{z}(t;\boldsymbol{\beta}), \mathbf{v}(t;\boldsymbol{\beta}))$ is the optimal pair for problem (OC) and $\boldsymbol{\lambda}(t;\boldsymbol{\beta})$ is the corresponding time path of the costate vector, it follows from Corollary 4.2 that $H^*_{x_n}(t) + \dot{\lambda}_n(t;\boldsymbol{\beta}) \equiv 0, n = 1, 2, \ldots, N$, and that $H^*_{u_m}(t) \equiv 0, m = 1, 2, \ldots, M$, for all $t \in [t_0, t_1]$; hence Eq. (8) simplifies to

$$V_{\alpha_i}(\boldsymbol{\beta}) \equiv \int_{t_0}^{t_1} H^*_{\alpha_i}(t)\,dt \overset{\text{def}}{=} \int_{t_0}^{t_1} H_{\alpha_i}(t, \mathbf{z}(t;\boldsymbol{\beta}), \mathbf{v}(t;\boldsymbol{\beta}), \boldsymbol{\lambda}(t;\boldsymbol{\beta}); \boldsymbol{\alpha})\,dt,\ i = 1, 2, \ldots, A,$$

which is what we set out to demonstrate. Q.E.D.

(ii) We employ vector notation and avoid integration by parts in this proof. To begin, differentiate the optimal value function $V(\cdot)$ defined constructively in Eq. (1) with respect to t_0, using the chain rule and Leibniz's rule to get

$$V_{t_0}(\boldsymbol{\beta}) \equiv -f(t_0, \mathbf{z}(t_0;\boldsymbol{\beta}), \mathbf{v}(t_0;\boldsymbol{\beta}); \boldsymbol{\alpha}) + \int_{t_0}^{t_1} \left[\underbrace{f_{\mathbf{x}}(t, \mathbf{z}(t;\boldsymbol{\beta}), \mathbf{v}(t;\boldsymbol{\beta}); \boldsymbol{\alpha})}_{1\times N} \underbrace{\frac{\partial \mathbf{z}}{\partial t_0}(t;\boldsymbol{\beta})}_{N\times 1} + \underbrace{f_{\mathbf{u}}(t, \mathbf{z}(t;\boldsymbol{\beta}), \mathbf{v}(t;\boldsymbol{\beta}); \boldsymbol{\alpha})}_{1\times M} \underbrace{\frac{\partial \mathbf{v}}{\partial t_0}(t;\boldsymbol{\beta})}_{M\times 1} \right] dt. \tag{9}$$

As we did above, simplify the notation by defining $f^*_{\mathbf{x}}(t) \overset{\text{def}}{=} f_{\mathbf{x}}(t, \mathbf{z}(t;\boldsymbol{\beta}), \mathbf{v}(t;\boldsymbol{\beta}); \boldsymbol{\alpha})$ and likewise for the other vector derivatives. Because $(\mathbf{z}(t;\boldsymbol{\beta}), \mathbf{v}(t;\boldsymbol{\beta}))$ is an optimal pair, it must satisfy the state equation and boundary conditions identically $\forall\, \boldsymbol{\beta} \in B(\boldsymbol{\beta}^\circ;\delta)$ and $\forall\, t \in [t_0, t_1]$; hence we have the following identities:

$$\mathbf{g}(t, \mathbf{z}(t;\boldsymbol{\beta}), \mathbf{v}(t;\boldsymbol{\beta}); \boldsymbol{\alpha}) - \dot{\mathbf{z}}(t;\boldsymbol{\beta}) \equiv \mathbf{0}_N,$$

$$\mathbf{z}(t_0;\boldsymbol{\beta}) \equiv \mathbf{x}_0,\ \mathbf{z}(t_1;\boldsymbol{\beta}) \equiv \mathbf{x}_1.$$

Differentiating these identities with respect to t_0 yields

$$\underbrace{\mathbf{g}_{\mathbf{x}}(t, \mathbf{z}(t;\boldsymbol{\beta}), \mathbf{v}(t;\boldsymbol{\beta}); \boldsymbol{\alpha})}_{N\times N} \underbrace{\frac{\partial \mathbf{z}}{\partial t_0}(t;\boldsymbol{\beta})}_{N\times 1} + \underbrace{\mathbf{g}_{\mathbf{u}}(t, \mathbf{z}(t;\boldsymbol{\beta}), \mathbf{v}(t;\boldsymbol{\beta}); \boldsymbol{\alpha})}_{N\times M} \underbrace{\frac{\partial \mathbf{v}}{\partial t_0}(t;\boldsymbol{\beta})}_{M\times 1} - \underbrace{\frac{\partial \dot{\mathbf{z}}}{\partial t_0}(t;\boldsymbol{\beta})}_{N\times 1} \equiv \mathbf{0}_N, \tag{10}$$

$$\underbrace{\dot{\mathbf{z}}(t_0;\boldsymbol{\beta})}_{N\times 1} + \underbrace{\frac{\partial \mathbf{z}}{\partial t_0}(t_0;\boldsymbol{\beta})}_{N\times 1} \equiv \mathbf{0}_N,\quad \underbrace{\frac{\partial \mathbf{z}}{\partial t_0}(t_1;\boldsymbol{\beta})}_{N\times 1} \equiv \mathbf{0}_N. \tag{11}$$

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Note that because t_0 appears as the value of the time argument and as a parameter in the vector $\boldsymbol{\beta}$, we get the sum of two vectors in the first expression of Eq. (11) upon differentiating the identity $\mathbf{z}(t_0;\boldsymbol{\beta}) \equiv \mathbf{x}_0$ with respect to t_0. Again, define $\mathbf{g}_{\mathbf{x}}^*(t) \stackrel{\text{def}}{=} \mathbf{g}_{\mathbf{x}}(t,\mathbf{z}(t;\boldsymbol{\beta}),\mathbf{v}(t;\boldsymbol{\beta});\boldsymbol{\alpha})$ and likewise for the other Jacobian matrix in Eq. (10). Now premultiply the identity in Eq. (10) by its corresponding costate vector $\boldsymbol{\lambda}(t;\boldsymbol{\beta})'$. Because the vector identity in Eq. (10) is identically zero $\forall\, \boldsymbol{\beta} \in B(\boldsymbol{\beta}^\circ;\delta)$ and $\forall\, t \in [t_0,t_1]$, it is still zero $\forall\, \boldsymbol{\beta} \in B(\boldsymbol{\beta}^\circ;\delta)$ and $\forall\, t \in [t_0,t_1]$ when it is premultiplied by $\boldsymbol{\lambda}(t;\boldsymbol{\beta})'$. Moreover, when the resulting scalar is integrated over $[t_0,t_1]$, it is still identically zero and therefore may be added to Eq. (9) without changing the latter's value, thereby yielding

$$V_{t_0}(\boldsymbol{\beta}) \equiv -\, f(t_0,\mathbf{z}(t_0;\boldsymbol{\beta}),\mathbf{v}(t_0;\boldsymbol{\beta});\boldsymbol{\alpha}) + \int_{t_0}^{t_1} \left[\left[\underbrace{f_{\mathbf{x}}^*(t)}_{1\times N} + \underbrace{\boldsymbol{\lambda}(t;\boldsymbol{\beta})'}_{1\times N} \underbrace{\mathbf{g}_{\mathbf{x}}^*(t)}_{N\times N} \right] \underbrace{\frac{\partial \mathbf{z}}{\partial t_0}(t;\boldsymbol{\beta})}_{N\times 1} \right.$$

$$\left. + \left[\underbrace{f_{\mathbf{u}}^*(t)}_{1\times M} + \underbrace{\boldsymbol{\lambda}(t;\boldsymbol{\beta})'}_{1\times N} \underbrace{\mathbf{g}_{\mathbf{u}}^*(t)}_{N\times M} \right] \underbrace{\frac{\partial \mathbf{v}}{\partial t_0}(t;\boldsymbol{\beta})}_{M\times 1} - \underbrace{\boldsymbol{\lambda}(t;\boldsymbol{\beta})'}_{1\times N} \underbrace{\frac{\partial \dot{\mathbf{z}}}{\partial t_0}(t;\boldsymbol{\beta})}_{N\times 1} \right] dt. \tag{12}$$

To ease the notational burden, define the gradient vectors of the Hamiltonian along the optimal path by

$$\underbrace{H_{\mathbf{x}}^*(t)}_{1\times N} \stackrel{\text{def}}{=} \underbrace{f_{\mathbf{x}}^*(t)}_{1\times N} + \underbrace{\boldsymbol{\lambda}(t;\boldsymbol{\beta})'}_{1\times N} \underbrace{\mathbf{g}_{\mathbf{x}}^*(t)}_{N\times N},$$

$$\underbrace{H_{\mathbf{u}}^*(t)}_{1\times M} \stackrel{\text{def}}{=} \underbrace{f_{\mathbf{u}}^*(t)}_{1\times M} + \underbrace{\boldsymbol{\lambda}(t;\boldsymbol{\beta})'}_{1\times N} \underbrace{\mathbf{g}_{\mathbf{u}}^*(t)}_{N\times M}.$$

These definitions permit Eq. (12) to be rewritten more cleanly as

$$V_{t_0}(\boldsymbol{\beta}) \equiv -f(t_0,\mathbf{z}(t_0;\boldsymbol{\beta}),\mathbf{v}(t_0;\boldsymbol{\beta});\boldsymbol{\alpha})$$
$$+ \int_{t_0}^{t_1} \left[\underbrace{H_{\mathbf{x}}^*(t)}_{1\times N} \underbrace{\frac{\partial \mathbf{z}}{\partial t_0}(t;\boldsymbol{\beta})}_{N\times 1} + \underbrace{H_{\mathbf{u}}^*(t)}_{1\times M} \underbrace{\frac{\partial \mathbf{v}}{\partial t_0}(t;\boldsymbol{\beta})}_{M\times 1} - \underbrace{\boldsymbol{\lambda}(t;\boldsymbol{\beta})'}_{1\times N} \underbrace{\frac{\partial \dot{\mathbf{z}}}{\partial t_0}(t;\boldsymbol{\beta})}_{N\times 1} \right] dt. \tag{13}$$

Given that $H_{\mathbf{u}}^*(t) \equiv \mathbf{0}_M'$ and $\dot{\boldsymbol{\lambda}}(t;\boldsymbol{\beta})' \equiv -H_{\mathbf{x}}^*(t) \forall\, t \in [t_0,t_1]$ by Corollary 4.2,

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Eq. (13) reduces to

$$V_{t_0}(\boldsymbol{\beta}) \equiv -\ f(t_0, \mathbf{z}(t_0;\boldsymbol{\beta}), \mathbf{v}(t_0;\boldsymbol{\beta}); \boldsymbol{\alpha}) \\ -\int_{t_0}^{t_1} \left[\underbrace{\dot{\boldsymbol{\lambda}}(t;\boldsymbol{\beta})'}_{1\times N} \underbrace{\frac{\partial \mathbf{z}}{\partial t_0}(t;\boldsymbol{\beta})}_{N\times 1} + \underbrace{\boldsymbol{\lambda}(t;\boldsymbol{\beta})'}_{1\times N} \underbrace{\frac{\partial \dot{\mathbf{z}}}{\partial t_0}(t;\boldsymbol{\beta})}_{N\times 1} \right] dt. \tag{14}$$

Now observe that

$$\frac{d}{dt}\left[\underbrace{\boldsymbol{\lambda}(t;\boldsymbol{\beta})'}_{1\times N} \underbrace{\frac{\partial \mathbf{z}}{\partial t_0}(t;\boldsymbol{\beta})}_{N\times 1} \right] = \underbrace{\dot{\boldsymbol{\lambda}}(t;\boldsymbol{\beta})'}_{1\times N} \underbrace{\frac{\partial \mathbf{z}}{\partial t_0}(t;\boldsymbol{\beta})}_{N\times 1} + \underbrace{\boldsymbol{\lambda}(t;\boldsymbol{\beta})'}_{1\times N} \underbrace{\frac{\partial \dot{\mathbf{z}}}{\partial t_0}(t;\boldsymbol{\beta})}_{N\times 1}$$

by the product rule of differentiation. Substituting this result and Eq. (11) into Eq. (14) yields

$$\begin{aligned} V_{t_0}(\boldsymbol{\beta}) &\equiv -f(t_0, \mathbf{z}(t_0;\boldsymbol{\beta}), \mathbf{v}(t_0;\boldsymbol{\beta}); \boldsymbol{\alpha}) - \int_{t_0}^{t_1} \frac{d}{dt}\left[\underbrace{\boldsymbol{\lambda}(t;\boldsymbol{\beta})'}_{1\times N} \underbrace{\frac{\partial \mathbf{z}}{\partial t_0}(t;\boldsymbol{\beta})}_{N\times 1} \right] dt \\ &= -f(t_0, \mathbf{z}(t_0;\boldsymbol{\beta}), \mathbf{v}(t_0;\boldsymbol{\beta}); \boldsymbol{\alpha}) - \left[\underbrace{\boldsymbol{\lambda}(t;\boldsymbol{\beta})'}_{1\times N} \underbrace{\frac{\partial \mathbf{z}}{\partial t_0}(t;\boldsymbol{\beta})}_{N\times 1} \right]_{t=t_0}^{t=t_1} \\ &= -f(t_0, \mathbf{z}(t_0;\boldsymbol{\beta}), \mathbf{v}(t_0;\boldsymbol{\beta}); \boldsymbol{\alpha}) - \boldsymbol{\lambda}(t_0;\boldsymbol{\beta})'\dot{\mathbf{z}}(t_0;\boldsymbol{\beta}). \end{aligned} \tag{15}$$

Because $\dot{\mathbf{z}}(t_0;\boldsymbol{\beta}) \equiv \mathbf{g}(t_0, \mathbf{z}(t_0;\boldsymbol{\beta}), \mathbf{v}(t_0;\boldsymbol{\beta}); \boldsymbol{\alpha})$ and $\mathbf{z}(t_0, \boldsymbol{\beta}) \equiv \mathbf{x}_0$ by the admissibility of the optimal pair, Eq. (15) reduces to

$$V_{t_0}(\boldsymbol{\beta}) \equiv -H(t_0, \mathbf{x}_0, \mathbf{v}(t_0;\boldsymbol{\beta}), \boldsymbol{\lambda}(t_0;\boldsymbol{\beta}); \boldsymbol{\alpha})$$

upon using the definition of the Hamiltonian. Q.E.D.

Before presenting an example to demonstrate how to use Theorem 9.1 to bring out the economic content of an optimal control problem, let's lay out the recipe for applying part (i) of it to a particular optimal control problem, as well as provide an economic interpretation of all its results. For the interpretations below, we assume that the optimal value function can be interpreted as the maximum present discounted profit of a capital accumulating firm. Naturally, if the optimal value function has a different interpretation dictated by the economic content of the control problem under consideration, then the ensuing economic interpretations will be slightly altered. Nonetheless, any economic interpretation will exhibit the same flavor, as future examples and mental exercises will demonstrate.

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Part (i): To *apply* this part of the theorem to an optimal control problem, one complies with the following four-step recipe:

(a) Define the Hamiltonian for the control problem under consideration.
(b) Differentiate the Hamiltonian *directly* with respect to the parameter of interest; that is, differentiate the Hamiltonian with respect to the parameter of interest *prior to* substituting in the optimal paths of the state, costate, and control variables.
(c) Substitute the optimal paths of the state, costate, and control variables into the derivative of the Hamiltonian from step (b).
(d) Integrate the result in step (c) over the planning horizon.

This part of the dynamic envelope theorem asserts that the effect of an increase in a time-independent parameter that enters the integrand and state equations on the optimal value function is equivalent to the impact that the parameter has explicitly (or directly) on the Hamiltonian evaluated along the optimal solution paths and integrated over the planning horizon. It should be clear that this part of the dynamic envelope theorem smacks of the classical static envelope theorem, even as far as the "recipe" is concerned, save for the fact that the effect of the parameter change is integrated over a planning horizon. Also note that this dynamic envelope result will recover the dynamic supply and demand functions in the spirit of Shepherd's lemma and Hotelling's lemma, albeit in altered forms, as we will demonstrate.

Part (ii): This part asserts that the effect of an increase in the initial date of the planning horizon on the optimal value function is equal to the negative of the value of the Hamiltonian evaluated at the optimal solution at the initial date of the planning horizon. In other words, an increase in t_0 is like starting the plan a "little later." Assuming that the value of the Hamiltonian is positive at the initial date in the optimal plan, this envelope result shows the effect of a later starting date is to decrease the present discounted value of profits. This means that the value of the Hamiltonian at the initial date is the shadow value of time for the firm, since it tells the firm the maximum amount its present discounted value of profits would increase if it starts the optimizing plan a little sooner. Note that, in general, the value of the Hamiltonian could just as well be negative at the initial date of the planning horizon, as it depends on the economic problem under consideration.

Part (iii): This envelope result asserts that the effect of an increase in the initial value of the jth state variable (or jth stock) on the optimal value function is equal to the value of the corresponding costate variable at the initial date, namely, $\lambda_j(t_0; \boldsymbol{\beta})$. Given that we are assuming the optimal value function can be interpreted as the maximum present discounted profit of a firm, $\lambda_j(t_0; \boldsymbol{\beta})$ measures the most a firm would pay for another unit of the jth state variable at the initial date, since $\lambda_j(t_0; \boldsymbol{\beta})$ gives the change in the maximum present discounted profit of a firm at the initial date due to a marginal increase in the jth initial stock. Hence $\lambda_j(t_0; \boldsymbol{\beta})$ has the economic interpretation of an *imputed* or *shadow* price of the jth state variable at the initial date in an optimal plan. Intuitively, one expects $\lambda_j(t_0; \boldsymbol{\beta})$ to be positive if the jth

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stock is a good, such as the stock of productive capital or the stock of some valuable asset in general, whereas one expects $\lambda_j(t_0;\boldsymbol{\beta})$ to be negative if the jth stock is a bad, such as the stock of waste or pollution.

Part (iv): This is the analogue of part (ii). Here we find that the effect of an increase in the terminal date of the planning horizon on the optimal value function is equal to the value of the Hamiltonian evaluated at the optimal solution at the terminal date of the planning horizon. Analogous to part (ii), an increase in t_1 is like terminating the plan a "little later." Assuming that the value of the Hamiltonian is positive at the terminal date in the optimal plan, the effect of a later ending date results in an increase in the firm's present discounted profit. That is, by finishing the plan a little later, the firm would gain the instantaneous cash flow at the terminal date. As a result, we may interpret the value of the Hamiltonian at the terminal date as the shadow value of time for the firm.

Part (v): This is the analogue of part (iii). In this case, we find that the effect of an increase in the value of the jth state variable at the terminal date on the optimal value function is equal to the negative of the corresponding costate variable at the terminal date, namely, $-\lambda_j(t_1;\boldsymbol{\beta})$. Given that we are assuming the optimal value function can be interpreted as the maximum present discounted profit of a firm, $-\lambda_j(t_1;\boldsymbol{\beta})$ measures the additional cost incurred by the firm in reaching the higher terminal stock requirement if the stock is a good. If, however, the stock is a bad, say pollution, then $-\lambda_j(t_1;\boldsymbol{\beta})$ measures the increase in profits that results from having to clean up less because of the higher stock of pollution permitted at the terminal date.

Let's now hone our understanding of the dynamic envelope theorem by applying it to the capital accumulating model of the firm facing costs of adjustment. We will then examine in some detail the technical issue of the differentiability of the optimal value function.

Example 9.1: Consider the following intertemporal model of a firm:

$$V(\boldsymbol{\beta}) \stackrel{\text{def}}{=} \max_{I(\cdot)} \int_0^T [pf(K(t), I(t)) - cK(t) - gI(t)]\, e^{-rt}\, dt$$

$$\text{s.t.} \quad \dot{K}(t) = I(t) - \delta K(t),\ K(0) = K_0,\ K(T) = K_T.$$

The economic interpretation of this model, known as the *adjustment cost model of the firm*, is similar to the static profit maximizing price-taking model of the firm, as we noted in Example 1.2 when a version of it was first introduced. The function $f(\cdot): \Re_+^2 \to \Re_+$ is the $C^{(2)}$ *generalized* production function of the firm, for it depends not only on the capital stock of the firm $K(t)$ at any time $t \in [0, T]$, but also on the gross rate of change of the capital stock, or the gross investment rate $I(t)$ at any time $t \in [0, T]$. It is assumed that $f_K(K, I) > 0$, $\text{sign}[f_I(K, I)] = -\text{sign}[I]$, and $f(\cdot)$ is concave in $(K, I)\ \forall\, t \in [0, T]$. In a mental exercise, you are asked to show that the assumption of concavity implies that a solution of the necessary conditions is

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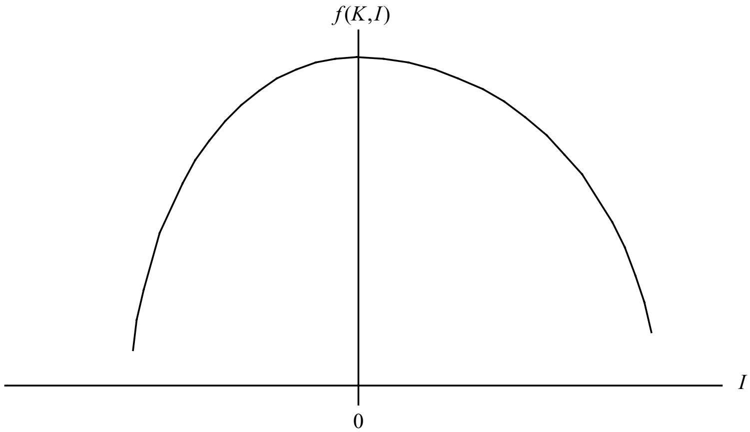

Figure 9.1

a solution of the adjustment cost problem. Note that the assumption $\operatorname{sign}[f_I(K, I)] = -\operatorname{sign}[I]$ is equivalent to the requirement that $f_I(K, I) \gtrless 0$ as $-I \gtrless 0$ and asserts that output decreases as investment $(I > 0)$ *or* disinvestment $(I < 0)$ in the capital stock takes place. The intuition behind the negative effect of investment on the output of the firm is that when capital is purchased, it must be installed for it to become a productive asset. The process of installation, however, takes resources away from production, thereby resulting in the fall in output. Similarly, when disinvestment takes place, the process of uninstalling the capital also takes resources away from production, thereby resulting in the decrease in output as well. See Figure 9.1 for the geometry of this characterization.

The single good the firm produces via its production function is sold at the constant price of $p > 0$, $c > 0$ is the constant holding cost per unit of the capital stock, and $g > 0$ is the constant price paid per unit of investment or disinvestment in the capital stock. The firm discounts its cash flow at the constant rate $r > 0$, begins its planning with a given capital stock $K_0 > 0$, and terminates its planning with the required capital stock $K_T > 0$. We assume that the natural nonnegativity constraint on the capital stock $K(t) \geq 0$ is not binding for all $t \in [0, T]$. We do not, however, assume that $I(t) \geq 0$ for all $t \in [0, T]$. In other words, we permit the firm to disinvestment; that is, $I(t) \leq 0$ is permitted. Note that $\dot{K}(t)$ is net investment, since depreciation, assumed proportional to the existing stock of capital with depreciation rate $\delta > 0$, is subtracted from gross investment $I(t)$ to arrive at $\dot{K}(t)$. Finally, define $\boldsymbol{\beta} \overset{\text{def}}{=} (c, g, p, r, \delta, K_0, T, K_T)$ as the vector of time-independent parameters and let $(K^*(t; \boldsymbol{\beta}), I^*(t; \boldsymbol{\beta}))$ denote the optimal pair, with $\lambda(t; \boldsymbol{\beta})$ being the corresponding time path of the costate variable.

First, consider the output price as the parameter of interest, and apply the four-step recipe after Theorem 9.1 to derive the pertinent dynamic envelope result:

$$H(t, K, I, \lambda; c, g, p, r, \delta) \overset{\text{def}}{=} [pf(K, I) - cK - gI]\, e^{-rt} + \lambda[I - \delta K],$$

$$\frac{\partial H}{\partial p} = f(K, I)e^{-rt},$$

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$$\left.\frac{\partial H}{\partial p}\right|_{\substack{\text{optimal}\\ \text{path}}} = f(K^*(t;\boldsymbol{\beta}), I^*(t;\boldsymbol{\beta}))\,e^{-rt} \stackrel{\text{def}}{=} y^*(t;\boldsymbol{\beta})\,e^{-rt},$$

$$\frac{\partial V(\boldsymbol{\beta})}{\partial p} = \int_0^T \left.\frac{\partial H}{\partial p}\right|_{\substack{\text{optimal}\\ \text{path}}} dt = \int_0^T y^*(t;\boldsymbol{\beta})\,e^{-rt}\,dt > 0.$$

Note that the definition $y^*(t;\boldsymbol{\beta}) \stackrel{\text{def}}{=} f(K^*(t;\boldsymbol{\beta}), I^*(t;\boldsymbol{\beta}))$ of the instantaneous supply function was used in the above dynamic envelope derivation. The same recipe may be applied to the parameters (c, g, r, δ), but these are left for a mental exercise because they are of the same ilk as p. Given the above economic interpretation of the adjustment cost model, the optimal value function $V(\cdot)$ represents the maximum present discounted value of profit for the firm. Hence the dynamic envelope result $\partial V(\boldsymbol{\beta})/\partial p > 0$ means that an increase in the output price of the firm will increase its maximum present discounted value of profit, not a surprising result. That is, if the output of the firm is more highly valued in the market, then the maximum present value of profit the firm can earn is correspondingly higher. Moreover, note that the dynamic envelope result $\partial V(\boldsymbol{\beta})/\partial p$ recovers the *cumulative discounted supply function* for the adjustment cost firm. That is, the dynamic envelope result $\partial V(\boldsymbol{\beta})/\partial p$ recovers the amount of the good produced by the firm over its planning horizon rather than at a single date in its planning horizon, appropriately discounted. To put this result in perspective, recall that in the competitive model of the static profit maximizing firm, the partial derivative of the indirect profit function with respect to the output price recovers, via the static envelope theorem, the firm's profit maximizing supply function (this result often goes under the heading of Hotelling's lemma). Finally, we note that the recovery of the cumulative discounted supply function by the dynamic envelope theorem is not a complete surprise. This is because the output price is constant, so that an increase in it represents a uniformly higher price for the entire planning horizon, and not just simply a higher price in a given time period.

The dynamic envelope results for the parameters (K_0, K_T, T) are given by

$$\frac{\partial V(\boldsymbol{\beta})}{\partial K_0} = \lambda(0;\boldsymbol{\beta}) \gtrless 0,$$

$$\frac{\partial V(\boldsymbol{\beta})}{\partial K_T} = -\lambda(T;\boldsymbol{\beta}) \gtrless 0,$$

$$\begin{aligned}\frac{\partial V(\boldsymbol{\beta})}{\partial T} &= H(T, K^*(T;\boldsymbol{\beta}), I^*(T;\boldsymbol{\beta}), \lambda(T;\boldsymbol{\beta}); c, g, p, r, \delta)\\ &= \pi^*(T;\boldsymbol{\beta})\,e^{-rT} + \lambda(T;\boldsymbol{\beta})[I^*(T;\boldsymbol{\beta}) - \delta K^*(T;\boldsymbol{\beta})] \gtrless 0,\end{aligned}$$

where $\pi^*(t;\boldsymbol{\beta}) \stackrel{\text{def}}{=} py^*(t;\boldsymbol{\beta}) - cK^*(t;\boldsymbol{\beta}) - gI^*(t;\boldsymbol{\beta})$ is the optimal instantaneous profit flow of the firm. The dynamic envelope result $\partial V(\boldsymbol{\beta})/\partial K_0 = \lambda(0;\boldsymbol{\beta})$ reinforces our earlier interpretation that $\lambda(0;\boldsymbol{\beta})$ is the shadow value of a unit of the

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capital stock at the beginning of the planning horizon. In the above adjustment cost model, however, investment *or* disinvestment can take place, so it is not necessarily true that $\lambda(0;\boldsymbol{\beta}) > 0$. To see this formally, first compute the necessary condition

$$H_I(t, K, I, \lambda; c, g, p, r, \delta) = [pf_I(K, I) - g]\, e^{-rt} + \lambda = 0.$$

At $t = 0$ along the optimal path, this equation reduces to $\lambda^*(0;\boldsymbol{\beta}) = g - pf_I(K_0, I^*(0;\boldsymbol{\beta})) \gtrless 0$ because $f_I(K, I) \gtrless 0$ as $-I \gtrless 0$, as remarked above. Thus, if $I^*(0;\boldsymbol{\beta}) \geq 0$, then $f_I(K_0, I^*(0;\boldsymbol{\beta})) \leq 0$, and as a result, we may conclude that $\lambda(0;\boldsymbol{\beta}) > 0$. In this case, therefore, $\lambda(0;\boldsymbol{\beta})$ is the maximum amount the firm would pay for an additional unit of capital at the start of the planning horizon, since $\lambda(0;\boldsymbol{\beta})$ measures the amount by which its maximum present discounted value of profit would increase if it had the additional unit of capital at the beginning of its planning horizon and employed it optimally. If $I^*(0;\boldsymbol{\beta}) < 0$, however, then $f_I(K_0, I^*(0;\boldsymbol{\beta})) > 0$ and the sign of $\lambda(0;\boldsymbol{\beta})$ cannot be determined without knowing the magnitudes of $g > 0$ and $f_I(K_0, I^*(0;\boldsymbol{\beta})) > 0$. Thus, in this instance, it is possible that $\lambda(0;\boldsymbol{\beta}) < 0$. This all makes economic sense, for if the firm prefers to disinvest at $t = 0$, then it deems the initial capital stock to be too large. Hence, an increase in the initial capital stock would lower the firm's present discounted value of profit because it would have to rid itself of more capital, a costly endeavor. Essentially, the reciprocal economic interpretation applies to the dynamic envelope result $\partial V(\boldsymbol{\beta})/\partial K_T = -\lambda(T;\boldsymbol{\beta})$, and so is left for a mental exercise.

The third dynamic envelope result $\partial V(\boldsymbol{\beta})/\partial T = H(T, K^*(T;\boldsymbol{\beta}), I^*(T;\boldsymbol{\beta}), \lambda(T;\boldsymbol{\beta}); c, g, p, r, \delta)$ asserts that the value of the Hamiltonian evaluated at the optimal solution and the terminal date is the marginal value (or marginal cost, if negative) to the firm of extending the planning horizon. In other words, if $\partial V(\boldsymbol{\beta})/\partial T > 0$, then the shadow value of extending the planning horizon is positive and the firm would pay up to the amount $H(T, K^*(T;\boldsymbol{\beta}), I^*(T;\boldsymbol{\beta}), \lambda(T;\boldsymbol{\beta}); c, g, p, r, \delta)$ to have the planning horizon extended, since its present discounted value of profits would increase by the amount $H(T, K^*(T;\boldsymbol{\beta}), I^*(T;\boldsymbol{\beta}), \lambda(T;\boldsymbol{\beta}); c, g, p, r, \delta)$ by such an extension. The Hamiltonian evaluated at the optimal solution and the terminal date therefore has the interpretation of the shadow value of time to the firm.

In proving the dynamic envelope theorem, that is, Theorem 9.1, we assumed that $\mathbf{z}(\cdot)$ and $\mathbf{v}(\cdot)$ were locally $C^{(1)}$ functions of $(t;\boldsymbol{\beta})\ \forall\,(t;\boldsymbol{\beta}) \in [t_0^\circ, t_1^\circ] \times B(\boldsymbol{\beta}^\circ;\delta)$, which in turn implied that the optimal value function $V(\cdot)$ was a locally $C^{(1)}$ function of the parameters $\boldsymbol{\beta} \overset{\text{def}}{=} (\boldsymbol{\alpha}, t_0, \mathbf{x}_0, t_1, \mathbf{x}_1)$. Though these assumptions on $\mathbf{z}(\cdot)$ and $\mathbf{v}(\cdot)$ may appear to be innocuous from an economic point of view, they are certainly not from a purely mathematical one. For example, even seemingly "simple" optimal control problems may violate these assumptions and thus result in an optimal value function that is not differentiable everywhere, though it usually is at least locally. Let's turn to an example of just such a case.

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Example 9.2: Consider the following optimal control problem:

$$V(x_0) \stackrel{\text{def}}{=} \max_{u(\cdot),x_1} \int_0^1 x(t)u(t)\,dt$$

$$\text{s.t.} \quad \dot{x}(t) = 0,\, x(0) = x_0,\, x(1) = x_1,$$

$$0 \le u(t) \le 1.$$

The Hamiltonian is given by $H(x, u, \lambda) \stackrel{\text{def}}{=} xu$, since $g(t, x, u) \equiv 0$. Because $\dot{x}(t) = 0$ and x_1 is a decision variable, all admissible paths of the state variable are given by $z(t; x_0) = x_0$. Therefore, if $x_0 > 0$, then the maximum of $H(x, u, \lambda) \stackrel{\text{def}}{=} xu$ with respect to u occurs at $v(t; x_0) = 1$, whereas if $x_0 < 0$, then the maximum of $H(x, u, \lambda) \stackrel{\text{def}}{=} xu$ with respect to u occurs at $v(t; x_0) = 0$. Finally, observe that if $x_0 = 0$, then the objective functional is identically zero and any admissible value of the control variable is optimal. Putting these three cases together, we may conclude that

$$V(x_0) = \begin{cases} x_0 & \forall x_0 \ge 0, \\ 0 & \forall x_0 < 0, \end{cases} \quad \text{and} \quad v(t; x_0) = \begin{cases} 1 & \forall x_0 \ge 0, \\ 0 & \forall x_0 < 0. \end{cases}$$

It therefore follows that $\lim_{x_0 \to 0^+} V'(x_0) = 1$ while $\lim_{x_0 \to 0^-} V'(x_0) = 0$. Because these limits are not equal, it follows from the definition of differentiability that $V(\cdot)$ is not differentiable at the point $x_0 = 0$. Thus $V(\cdot)$ is not differentiable for all $x_0 \in \Re$, but it *is* differentiable in any neighborhood of $x_0 \ne 0$ not containing the origin. Similarly, because $\lim_{x_0 \to 0^+} v(t; x_0) = 1$ while $\lim_{x_0 \to 0^-} v(t; x_0) = 0$, $v(\cdot)$ is not continuous with respect to x_0 for all $x_0 \in \Re$, but it *is* continuous with respect to x_0 in any neighborhood of $x_0 \ne 0$ not containing the origin. Note that local differentiability is all that is generally required in many applications of optimal control theory to economics. This is because one can often argue that local differentiability of $v(\cdot)$ or $V(\cdot)$ is all that is required for a qualitative characterization of an economic model, just as it typically is when one studies the comparative statics properties of static economic models.

Let's now return to our optimal control problem (OC). Recall that by part (iii) of Theorem 9.1, the dynamic envelope theorem, the effect of an increase in the jth initial value of the state variable on the optimal value function is given by

$$V_{x_{0j}}(\boldsymbol{\beta}) \equiv \lambda_j(t_0; \boldsymbol{\beta}), \qquad j = 1, 2, \ldots, N.$$

From this envelope result, we can legitimately interpret $\lambda_j(t_0; \boldsymbol{\beta})$ as the shadow value of the jth state variable in an optimal program at time t_0, the initial time. By part (iii) of Theorem 9.1, we also know that such an interpretation is valid at the terminal time t_1. Given this valid economic interpretation of $\lambda_j(t_0; \boldsymbol{\beta})$ and $\lambda_j(t_1; \boldsymbol{\beta})$, we would like to extend it so that $\lambda_j(t; \boldsymbol{\beta})$ can be *legitimately* interpreted as the shadow value of the jth state variable in an optimal program at any time $t \in (t_0, t_1)$

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for problem (OC). It is important that you understand that Theorem 9.1, parts (i) and (iii), do not permit that interpretation of $\lambda_j(t;\boldsymbol{\beta})$ for all $t \in (t_0, t_1)$, for problem (OC).

In view of this intention, let $s \in (t_0, t_1)$ be a fixed but arbitrary initial or starting date of the following *truncation* of the original optimal control problem (OC):

$$V(\boldsymbol{\gamma}) \stackrel{\text{def}}{=} \max_{\mathbf{u}(\cdot)} \int_{s}^{t_1} f(t, \mathbf{x}(t), \mathbf{u}(t); \boldsymbol{\alpha})\, dt$$

$$\text{s.t.} \quad \dot{\mathbf{x}}(t) = \mathbf{g}(t, \mathbf{x}(t), \mathbf{u}(t); \boldsymbol{\alpha}), \ \mathbf{x}(s) = \mathbf{x}_s, \ \mathbf{x}(t_1) = \mathbf{x}_1, \qquad (\text{OC}')$$

where $\boldsymbol{\gamma} \stackrel{\text{def}}{=} (\boldsymbol{\alpha}, s, \mathbf{x}_s, t_1, \mathbf{x}_1)$ is the time-independent parameter vector of problem (OC′). Problem (OC′) is a family of optimal control problems embedded within the control problem (OC), parameterized by the starting date $s \in (t_0, t_1)$ and initial value of the state vector $\mathbf{x}_s$. Assume that an optimal pair exists to problem (OC′) and denote it by $(\mathbf{z}(t;\boldsymbol{\gamma}), \mathbf{v}(t;\boldsymbol{\gamma}))$, and let $\boldsymbol{\lambda}(t;\boldsymbol{\gamma})$ be the corresponding time path of the costate vector. Notice that the optimal value functions in problems (OC) and (OC′), as well as the solution functions and corresponding costate vector function, are denoted with the same symbols. That this is the case follows immediately from the observation that the integrand and transition functions are identical in problems (OC) and (OC′) as is the terminal value of the state vector, whereas all that differ are the initial date and initial value of the state vector. That is, problems (OC) and (OC′) are structurally identical. What this means is that the values of the optimal solution functions and corresponding costate vector function may differ, but not their functional forms. This will be seen in Example 9.3 below. With this in mind, we are now in a position to prove the following fundamental theorem.

Theorem 9.2 (The Principle of Optimality): *Let* $(\mathbf{z}(t;\boldsymbol{\beta}), \mathbf{v}(t;\boldsymbol{\beta}))$ *be the optimal pair for problem* (OC) *with corresponding costate vector time path* $\boldsymbol{\lambda}(t;\boldsymbol{\beta})$*, and let* $(\mathbf{z}(t;\boldsymbol{\gamma}), \mathbf{v}(t;\boldsymbol{\gamma}))$ *be the optimal pair for problem* (OC′) *with corresponding costate vector time path* $\boldsymbol{\lambda}(t;\boldsymbol{\gamma})$*. Suppose that* $s \in (t_0, t_1)$ *is a fixed but arbitrary starting date for problem* (OC′)*. If* $\mathbf{x}_s = \mathbf{z}(s;\boldsymbol{\beta})$*, then the optimal pair for problem* (OC′) *is* $(\mathbf{z}(t;\boldsymbol{\beta}), \mathbf{v}(t;\boldsymbol{\beta}))$ *with corresponding costate vector time path* $\boldsymbol{\lambda}(t;\boldsymbol{\beta})$ *for all* $t \in [s, t_1]$*, namely, the same optimal pair and corresponding costate vector for problem* (OC) *for the interval* $[s, t_1]$*. That is, if* $\mathbf{x}_s = \mathbf{z}(s;\boldsymbol{\beta})$*, then for all* $t \in [s, t_1]$*, we have the identities*

$$\mathbf{z}(t;\boldsymbol{\beta}) \equiv \mathbf{z}(t;\boldsymbol{\alpha}, s, \mathbf{z}(s;\boldsymbol{\beta}), t_1, \mathbf{x}_1),$$

$$\mathbf{v}(t;\boldsymbol{\beta}) \equiv \mathbf{v}(t;\boldsymbol{\alpha}, s, \mathbf{z}(s;\boldsymbol{\beta}), t_1, \mathbf{x}_1),$$

$$\boldsymbol{\lambda}(t;\boldsymbol{\beta}) \equiv \boldsymbol{\lambda}(t;\boldsymbol{\alpha}, s, \mathbf{z}(s;\boldsymbol{\beta}), t_1, \mathbf{x}_1).$$

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Proof: We use a proof by contraposition to establish the first two identities. Suppose that $(\mathbf{z}(t;\boldsymbol{\beta}), \mathbf{v}(t;\boldsymbol{\beta}))$ is *not* the optimal pair for problem (OC′); in other words, the optimal pair $(\mathbf{z}(t;\boldsymbol{\alpha}, s, \mathbf{z}(s;\boldsymbol{\beta}), t_1, \mathbf{x}_1), \mathbf{v}(t;\boldsymbol{\alpha}, s, \mathbf{z}(s;\boldsymbol{\beta}), t_1, \mathbf{x}_1))$ for problem (OC′) is not identically equal to the optimal pair $(\mathbf{z}(t;\boldsymbol{\beta}), \mathbf{v}(t;\boldsymbol{\beta}))$ for problem (OC) for all $t \in [s, t_1]$. This implies that the objective functional of problem (OC) could be improved upon by following the pair $(\mathbf{z}(t;\boldsymbol{\beta}), \mathbf{v}(t;\boldsymbol{\beta}))$ from $t = t_0$ until $t = s$, and then switching to the pair $(\mathbf{z}(t;\boldsymbol{\alpha}, s, \mathbf{z}(s;\boldsymbol{\beta}), t_1, \mathbf{x}_1), \mathbf{v}(t;\boldsymbol{\alpha}, s, \mathbf{z}(s;\boldsymbol{\beta}), t_1, \mathbf{x}_1))$ from $t = s$ until $t = t_1$, rather than following the pair $(\mathbf{z}(t;\boldsymbol{\beta}), \mathbf{v}(t;\boldsymbol{\beta}))$ for all $t \in [t_0, t_1]$. But this contradicts the assumed optimality of the pair $(\mathbf{z}(t;\boldsymbol{\beta}), \mathbf{v}(t;\boldsymbol{\beta}))$ for problem (OC), thereby establishing the first two identities.

To prove the third identity, we proceed as follows. By the two identities just established, the values of the optimal value functions for problems (OC) and (OC′) are identical over the truncated horizon $[s, t_1]$, and thus so too are their derivatives with respect to the initial value of the state vector, assuming that such derivatives exist. By part (iii) of the dynamic envelope theorem and the fact that the initial time s of the problem (OC′) is a fixed but arbitrary element of the interval (t_0, t_1), the third identity then follows. Q.E.D.

The interpretation of Theorem 9.2 is straightforward and important, for what follows hinges on a sound understanding of it. The principle of optimality, as given above, asserts that if you break up an optimal control problem into two parts, a "beginning" interval over which to plan and an "ending" interval over which to plan, the break point being any value of time in the original planning horizon, then the plan that is optimal over the entire planning horizon is also optimal over the truncated ending planning horizon, provided you begin the ending planning interval with the value of the state vector that occurred at the end of the beginning planning interval. More simply, the principle of optimality asserts that any portion of an optimal path is optimal, which is pretty obvious when stated in this simple form. Recall that we proved the principle of optimality in Chapter 4 when we provided a rigorous proof of the Maximum Principle. This would be an excellent time to return to that statement and proof.

Let's now show that the dynamic envelope theorem and principle of optimality, that is, Theorems 9.1 and 9.2, do not permit us to interpret the costate vector in the manner we wish. To begin, apply part (iii) of the dynamic envelope theorem to the truncated control problem (OC′) to obtain the result

$$V_{x_{sj}}(\boldsymbol{\gamma}) \equiv \lambda_j(s;\boldsymbol{\gamma}), \quad j = 1, 2, \ldots, N. \tag{16}$$

This envelope result means that we are justified in interpreting $\lambda_j(s;\boldsymbol{\gamma})$ as the shadow value of the jth state variable at time s, the initial date of the truncated control problem (OC′). However, because the starting date s for the truncated control problem (OC′) is arbitrary, save for the fact that $s \in (t_0, t_1)$, we are therefore permitted on the basis of this observation and Eq. (16) to interpret $\lambda_j(s;\boldsymbol{\gamma})$ as the shadow value of the jth

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state variable at any starting time $s \in (t_0, t_1)$ of the corresponding truncated control problem. In other words, the interpretation based on the principle of optimality and the dynamic envelope theorem shows that $\lambda_j(s; \gamma)$ is the marginal contribution of $x_{sj} = z_j(s; \boldsymbol{\beta})$ to the value $V(\gamma)$, which is the maximum value of the objective functional over the truncated horizon $[s, t_1]$, where $s \in (t_0, t_1)$. It is important to realize, however, that these two theorems *do not* establish the claim that $\lambda_j(s; \boldsymbol{\beta})$ is the marginal contribution of $x_{sj} = z_j(s; \boldsymbol{\beta})$ to the value of $V(\boldsymbol{\beta})$, which is the maximum value of the objective functional over the entire horizon $[t_0, t_1]$. The difficulty of establishing this intended interpretation of the costate vector is that we cannot prove that the derivative of $V(\boldsymbol{\beta})$ with respect to $x_{sj} = z_j(s; \boldsymbol{\beta})$ equals $\lambda_j(s; \boldsymbol{\beta})$ because $x_{sj} = z_j(s; \boldsymbol{\beta})$ is not an exogenous parameter, but rather it is indirectly and optimally chosen via the state equation in solving the optimal control problem (OC). Hence, we must alter our approach to establishing the above economic interpretation of the costate vector in such a way as to formalize the notion that at some time $s \in (t_0, t_1)$, a small amount of the jth state variable is suddenly added to the existing value of the jth state variable. In passing, note that the dynamic envelope result in Eq. (16) is exactly how we *defined* the costate variable in our rigorous proof of the Maximum Principle in Chapter 4.

Before moving on to a rigorously justified economic interpretation of the costate vector, let's pause and consider the following simple mathematical example to demonstrate the principle of optimality explicitly in an optimal control setting.

Example 9.3: Consider the simple control problem

$$V(t_0, x_0, t_1, x_1) \stackrel{\text{def}}{=} \min_{u(\cdot)} \int_{t_0}^{t_1} \frac{1}{2}[u(t)]^2 \, dt$$

$$\text{s.t.} \quad \dot{x}(t) = u(t), x(t_0) = x_0, x(t_1) = x_1.$$

The Hamiltonian is given by $H(x, u, \lambda) \stackrel{\text{def}}{=} \frac{1}{2}u^2 + \lambda u$. Because $H(\cdot)$ is convex in (x, u), Theorem 4.3 asserts that the following equations are both necessary and sufficient to determine the global minimum to the optimal control problem:

$$\begin{aligned} H_u(x, u, \lambda) &= u + \lambda = 0, \\ \dot{x} &= H_\lambda(x, u, \lambda) = u, x(t_0) = x_0, x(t_1) = x_1, \\ \dot{\lambda} &= -H_x(x, u, \lambda) = 0. \end{aligned}$$

Because $\dot{\lambda} = 0$, the costate variable is a constant, say, $\lambda(t) = c_1$. The necessary condition $H_u = 0$ gives $u(t) = -\lambda(t)$, and when combined with the prior observation, it implies that $u(t) = -c_1$. The state equation therefore becomes $\dot{x} = -c_1$, which, when integrated, yields the general solution $x(t) = c_2 - c_1 t$, where c_2 is another constant of integration. Using the boundary conditions in the general solution of the state equation gives the ensuing system of linear equations to be solved for the

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constants of integration:

$$x(t_0) = c_2 - c_1 t_0 = x_0,$$
$$x(t_1) = c_2 - c_1 t_1 = x_1.$$

Applying Cramer's rule to this system of linear equations yields

$$c_1 = \frac{x_0 - x_1}{t_1 - t_0},$$
$$c_1 = \frac{t_1 x_0 - t_0 x_1}{t_1 - t_0},$$

as the constants of integration. Plugging the values of the constants of integration into the general solutions for the state, control, and costate variables yields the specific solution to the control problem, namely,

$$\begin{aligned} z(t; t_0, x_0, t_1, x_1) &= \left[\frac{t_1 x_0 - t_0 x_1}{t_1 - t_0}\right] - \left[\frac{x_0 - x_1}{t_1 - t_0}\right] t, \\ v(t; t_0, x_0, t_1, x_1) &= \frac{x_1 - x_0}{t_1 - t_0}, \\ \lambda(t; t_0, x_0, t_1, x_1) &= \frac{x_0 - x_1}{t_1 - t_0}. \end{aligned} \tag{17}$$

This is the unique solution of the control problem because it is the only solution of the necessary and sufficient conditions.

Now consider the structurally identical but truncated control problem

$$V(s, x_s, t_1, x_1) \stackrel{\text{def}}{=} \min_{u(\cdot)} \int_s^{t_1} \frac{1}{2}[u(t)]^2\, dt$$
$$\text{s.t.} \quad \dot{x}(t) = u(t), x(s) = x_s, x(t_1) = x_1,$$

where $s \in (t_0, t_1)$ and x_s is an admissible value of the state variable. Because the truncated optimal control problem is identical in every way to the original optimal control problem, except for the fact that the truncated problem begins at time $t = s$ in state $x(s) = x_s$, the solution *functions* (but not necessarily their values) are identical too, a point we made earlier, as you may recall. The solution of the truncated control problem is therefore found by replacing t_0 with s and x_0 with x_s to get

$$\begin{aligned} z(t; s, x_s, t_1, x_1) &= \left[\frac{t_1 x_s - s x_1}{t_1 - s}\right] - \left[\frac{x_s - x_1}{t_1 - s}\right] t, \\ v(t; s, x_s, t_1, x_1) &= \frac{x_1 - x_s}{t_1 - s}, \\ \lambda(t; s, x_s, t_1, x_1) &= \frac{x_s - x_1}{t_1 - s}. \end{aligned} \tag{18}$$

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As above, this is the unique solution of the truncated control problem because it is the only solution of the necessary and sufficient conditions.

The principle of optimality asserts that the solution triplet in Eq. (17) is identically equal to the solution triplet in Eq. (18) for all $t \in [s, t_1]$ when, of course, the latter triplet is evaluated at $x_s = z(s; t_0, x_0, t_1, x_1)$. Let's verify that this is true for the optimal path of the state variable:

$$
\begin{aligned}
&z(t; s, x_s, t_1, x_1)|_{x_s = z(s;t_0,x_0,t_1,x_1)} \\
&\quad = \left[\frac{t_1\left[\frac{t_1x_0 - t_0x_1}{t_1 - t_0}\right] - t_1\left[\frac{x_0 - x_1}{t_1 - t_0}\right]s - sx_1}{t_1 - s}\right] - \left[\frac{\left[\frac{t_1x_0 - t_0x_1}{t_1 - t_0}\right] - \left[\frac{x_0 - x_1}{t_1 - t_0}\right]s - x_1}{t_1 - s}\right]t \\
&\quad = \left[\frac{t_1[t_1x_0 - t_0x_1] - st_1x_0 + st_0x_1}{[t_1 - s][t_1 - t_0]}\right] - \left[\frac{t_1x_0 - s[x_0 - x_1] - t_1x_1}{[t_1 - s][t_1 - t_0]}\right]t \\
&\quad = \left[\frac{[t_1 - s][t_1x_0 - t_0x_1]}{[t_1 - s][t_1 - t_0]}\right] - \left[\frac{[t_1 - s][x_0 - x_1]}{[t_1 - s][t_1 - t_0]}\right]t \\
&\quad = \left[\frac{t_1x_0 - t_0x_1}{t_1 - t_0}\right] - \left[\frac{x_0 - x_1}{t_1 - t_0}\right]t = z(t; t_0, x_0, t_1, x_1).
\end{aligned}
$$

The proof that the optimal control and costate paths satisfy the principle of optimality is left for a mental exercise.

Let's now turn to the problem of formally establishing the claim that $\lambda_j(s; \boldsymbol{\beta})$ is the marginal contribution of $x_{sj} = z_j(s; \boldsymbol{\beta})$ to the value of $V(\boldsymbol{\beta})$, which is the maximum value of the objective functional over the entire horizon $[t_0, t_1]$. That is, we wish to rigorously prove that the jth costate variable at any time t in the planning horizon of the original optimal control problem (OC) is the shadow value of the jth state variable at time t. To this end, redefine the state equation as

$$\dot{\mathbf{x}}(t) = \mathbf{g}(t, \mathbf{x}(t), \mathbf{u}(t); \boldsymbol{\alpha}) + \mathbf{h}(a), \tag{19}$$

where $\mathbf{h}(a) \in \Re^N$ is a vector of zeros save for the jth component, the latter of which is defined by

$$h^j(a) \stackrel{\text{def}}{=} \begin{cases} 0 & t \in [t_0, s), \\ a\varepsilon^{-1} & t \in [s, s + \varepsilon), \\ 0 & t \in [s + \varepsilon, t_1). \end{cases} \tag{20}$$

The number ε is an arbitrarily small and positive number that we will eventually allow to shrink to zero in the limit, whereas $a \in \Re$ is a parameter. The injection to the jth state variable takes place during the "short" interval $[s, s + \varepsilon)$, where the smaller is ε, the more abrupt the injection. When $\varepsilon \to 0^+$, this mimics the addition

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of a units to the jth state variable at time $t = s$, since

$$\lim_{\varepsilon\to 0^+} \int_s^{s+\varepsilon} h^j(a)\,dt = \lim_{\varepsilon\to 0^+} \int_s^{s+\varepsilon} a\varepsilon^{-1}\,dt = \lim_{\varepsilon\to 0^+} a\varepsilon^{-1}[s+\varepsilon-s] = \lim_{\varepsilon\to 0^+} a\varepsilon^{-1}\varepsilon = a.$$

Alternatively, we can see that when $\varepsilon \to 0^+$, this mimics the addition of a units to the jth state variable at time $t = s$ by the following sequence of mathematical operations. First, note that the jth state equation can be written as

$$\int_s^{s+\varepsilon} \dot{x}_j(t)\,dt = \int_s^{s+\varepsilon} [g^j(t,\mathbf{x}(t),\mathbf{u}(t);\boldsymbol{\alpha}) + a\varepsilon^{-1}]\,dt = \int_s^{s+\varepsilon} g^j(t,\mathbf{x}(t),\mathbf{u}(t);\boldsymbol{\alpha})\,dt + a \tag{21}$$

upon using the previous integration result. Because $dx_j = \dot{x}_j(t)dt$, it then follows that

$$\int_s^{s+\varepsilon} \dot{x}_j(t)\,dt = \int_{x_j(s)}^{x_j(s+\varepsilon)} dx_j = x_j(s+\varepsilon) - x_j(s).$$

Thus, Eq. (21) reduces to

$$x_j(s+\varepsilon) = x_j(s) + \int_s^{s+\varepsilon} g^j(t,\mathbf{x}(t),\mathbf{u}(t);\boldsymbol{\alpha})\,dt + a.$$

Now let $\varepsilon \to 0^+$ in the last equation to get

$$\begin{aligned}\lim_{\varepsilon\to 0^+} x_j(s+\varepsilon) &= \lim_{\varepsilon\to 0^+} \left[x_j(s) + \int_s^{s+\varepsilon} g^j(t,\mathbf{x}(t),\mathbf{u}(t);\boldsymbol{\alpha})\,dt + a \right] \\ &= \lim_{\varepsilon\to 0^+} x_j(s) + \lim_{\varepsilon\to 0^+} \int_s^{s+\varepsilon} g^j(t,\mathbf{x}(t),\mathbf{u}(t);\boldsymbol{\alpha})\,dt + \lim_{\varepsilon\to 0^+} a \\ &= x_j(s) + 0 + a \neq x_j(s),\end{aligned}$$

since the limit of a sum is the sum of the limits, provided each of the individual limits exists, which they clearly do here. The last equation asserts that the value of the jth state variable at the instant of time "just after" $t = s$ differs from the value of the jth state variable at time $t = s$ by the amount a. Therefore, a represents the injection of a units to the jth state variable at time $t = s$. Figure 9.2 gives the geometry of this construction.

The Hamiltonian for the optimal control problem formed with the state equation given by Eq. (19), which we define as the *perturbed* control problem, is defined as

$$\hat{H}(t,\mathbf{x},\mathbf{u},\boldsymbol{\mu};\boldsymbol{\alpha},a) \stackrel{\text{def}}{=} f(t,\mathbf{x},\mathbf{u};\boldsymbol{\alpha}) + \boldsymbol{\mu}'[\mathbf{g}(t,\mathbf{x},\mathbf{u};\boldsymbol{\alpha}) + \mathbf{h}(a)].$$

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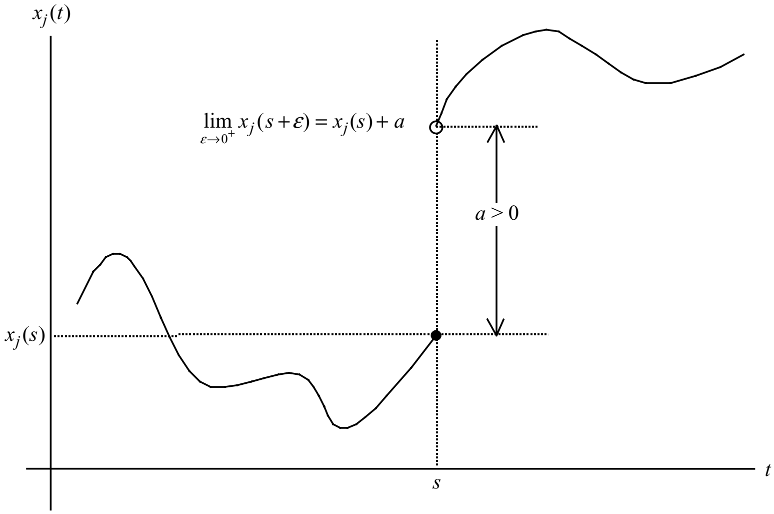

Figure 9.2

The optimal pair for the perturbed control problem, which is assumed to exist, is denoted by $(\hat{\mathbf{x}}(t; \boldsymbol{\beta}, a), \hat{\mathbf{u}}(t; \boldsymbol{\beta}, a))$, with corresponding costate vector $\boldsymbol{\mu}(t; \boldsymbol{\beta}, a)$. Also define the optimal value function for the perturbed control problem constructively as

$$\hat{V}(\boldsymbol{\beta}, a) \equiv \int_{t_0}^{t_1} f(t, \hat{\mathbf{x}}(t; \boldsymbol{\beta}, a), \hat{\mathbf{u}}(t; \boldsymbol{\beta}, a); \boldsymbol{\alpha})\, dt.$$

Note that when $a = 0$, $(\hat{\mathbf{x}}(t; \boldsymbol{\beta}, 0), \hat{\mathbf{u}}(t; \boldsymbol{\beta}, 0), \boldsymbol{\mu}(t; \boldsymbol{\beta}, 0)) \equiv (\mathbf{z}(t; \boldsymbol{\beta}), \mathbf{v}(t; \boldsymbol{\beta}), \boldsymbol{\lambda}(t; \boldsymbol{\beta}))$, that is, the perturbed control problem's solution is identically equal to the original control problem's solution. This, in turn, implies that the optimal value functions of the perturbed and original optimal control problems are also identically equal in value when $a = 0$, that is, $\hat{V}(\boldsymbol{\beta}, 0) \equiv V(\boldsymbol{\beta})$.

Our goal is to compute the derivative of $\hat{V}(\cdot)$ with respect to a evaluated at $a = 0$, that is, to compute $\hat{V}_a(\boldsymbol{\beta}, a)\big|_{a=0}$. For sufficiently small ε, this will indicate the correct marginal value of the jth state variable at any time $s \in (t_0, t_1)$ along the *original* optimal path. We ultimately will show that $\hat{V}_a(\boldsymbol{\beta}, a)\big|_{a=0} = \lambda_j(s; \boldsymbol{\beta})$ for any $s \in (t_0, t_1)$. This result will therefore permit us to rigorously interpret $\lambda_j(s; \boldsymbol{\beta})$ as the shadow value of the jth state variable at any time $s \in (t_0, t_1)$ along the optimal path of the *original* optimal control problem (OC), which is how we'd like to interpret it.

To derive $\hat{V}_a(\boldsymbol{\beta}, a)$, apply part (i) of the dynamic envelope theorem to $\hat{V}(\cdot)$ to get

$$\hat{V}_a(\boldsymbol{\beta}, a) = \int_{t_0}^{t_1} \frac{\partial \hat{H}}{\partial a}\bigg|_{\substack{\text{optimal}\\ \text{path}}} dt = \int_{t_0}^{t_1} \hat{\mu}_j(t; \boldsymbol{\beta}, a) \frac{d}{da} h^j(a)\, dt = \int_{s}^{s+\varepsilon} \hat{\mu}_j(t; \boldsymbol{\beta}, a)\varepsilon^{-1}\, dt, \tag{22}$$

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because $\frac{d}{da}h^j(a) = 0\,\forall\, t \notin [s, s+\varepsilon)$ and $\frac{d}{da}h^j(a) = \varepsilon^{-1}\,\forall\, t \in [s, s+\varepsilon)$ from Eq. (20). Now evaluate Eq. (22) at $a = 0$ and recall that $\boldsymbol{\lambda}(t;\boldsymbol{\beta}) \equiv \boldsymbol{\mu}(t;\boldsymbol{\beta}, 0)$ to get

$$\hat{V}_a(\boldsymbol{\beta}, a)\big|_{a=0} = \int_s^{s+\varepsilon} \lambda_j(t;\boldsymbol{\beta})\varepsilon^{-1}\, dt. \tag{23}$$

As previously demonstrated, we can approximate an instantaneous injection to the jth state variable at time s by letting ε get arbitrarily small in Eq. (23). More formally,

$$\lim_{\varepsilon\to 0^+} \hat{V}_a(\boldsymbol{\beta}, a)\big|_{a=0} = \lim_{\varepsilon\to 0^+} \int_s^{s+\varepsilon} \lambda_j(t;\boldsymbol{\beta})\varepsilon^{-1}\, dt. \tag{24}$$

Define $\Lambda_j(t) \overset{\text{def}}{=} \int_{t_0}^{t} \lambda_j(\tau;\boldsymbol{\beta})\, d\tau$, so that by Leibniz's rule, $\dot{\Lambda}_j(t) = \lambda_j(t;\boldsymbol{\beta})$. Using this definition, Eq. (24) becomes

$$\begin{aligned}\lim_{\varepsilon\to 0^+} \hat{V}_a(\boldsymbol{\beta}, a)\big|_{a=0} &= \lim_{\varepsilon\to 0^+} \int_s^{s+\varepsilon} \dot{\Lambda}_j(t)\varepsilon^{-1}\, dt \\ &= \lim_{\varepsilon\to 0^+} \frac{\Lambda_j(s+\varepsilon) - \Lambda_j(s)}{\varepsilon} = \dot{\Lambda}_j(s) = \lambda_j(s;\boldsymbol{\beta}) \end{aligned} \tag{25}$$

by the definition of the right-hand derivative. Seeing as $\hat{V}(\boldsymbol{\beta}, 0) \equiv V(\boldsymbol{\beta})$ is the optimal value function over the entire horizon, we have therefore shown that the rate of change of the optimal value function with respect to an injection in the jth state variable at any time $s \in (t_0, t_1)$ is the value of the jth costate variable at time s for the *original* optimal control problem (OC). Our proof is now complete.

To see the significance of Eq. (25), apply part (iii) of the dynamic envelope theorem to the truncated optimal control problem (OC$'$) and evaluate the result at $\mathbf{x}_s = \mathbf{z}(s;\boldsymbol{\beta})$ to get

$$\begin{aligned} V_{x_{sj}}(\boldsymbol{\alpha}, s, \mathbf{z}(s;\boldsymbol{\beta}), t_1, \mathbf{x}_1) &= \lambda_j(s;\boldsymbol{\alpha}, s, \mathbf{z}(s;\boldsymbol{\beta}), t_1, \mathbf{x}_1) \\ &\equiv \lambda_j(s;\boldsymbol{\beta}), \quad j = 1, 2, \ldots, N, \end{aligned} \tag{26}$$

where we have used the identity $\boldsymbol{\lambda}(t;\boldsymbol{\beta}) \equiv \boldsymbol{\lambda}(t;\boldsymbol{\alpha}, s, \mathbf{z}(s;\boldsymbol{\beta}), t_1, \mathbf{x}_1)$ from the principle of optimality (Theorem 9.2). The dynamic envelope results in Eqs. (25) and (26) are the same, but that in Eq. (25) applies to the original control problem (OC) and thus the entire planning horizon $[t_0, t_1]$, whereas that in Eq. (26) applies to the truncated control problem (OC$'$) and thus the truncated planning horizon $[s, t_1]$. This therefore means that the marginal gain from an increase in the jth stock at any time $s \in (t_0, t_1)$ is the same over the truncated planning horizon $[s, t_1]$ as it is over the entire planning horizon $[t_0, t_1]$.

So far, we have established the dynamic envelope theorem for a class of optimal control problems with many state and control variables, but without any constraints.

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For the sake of completeness, we now present and discuss the dynamic envelope theorem for the general class of constrained control problems studied in Chapter 6, namely,

$$V(\boldsymbol{\beta}) \stackrel{\text{def}}{=} \max_{\mathbf{u}(\cdot)} \int_{t_0}^{t_1} f(t, \mathbf{x}(t), \mathbf{u}(t); \boldsymbol{\alpha})\, dt$$

$$\begin{aligned} \text{s.t.} \quad & \dot{\mathbf{x}}(t) = \mathbf{g}(t, \mathbf{x}(t), \mathbf{u}(t); \boldsymbol{\alpha}),\ \mathbf{x}(t_0) = \mathbf{x}_0,\ \mathbf{x}(t_1) = \mathbf{x}_1, \\ & h^k(t, \mathbf{x}(t), \mathbf{u}(t); \boldsymbol{\alpha}) \geq 0, \quad k = 1, 2, \ldots, K', \\ & h^k(t, \mathbf{x}(t), \mathbf{u}(t); \boldsymbol{\alpha}) = 0, \quad k = K'+1, K'+2, \ldots, K, \end{aligned} \tag{27}$$

where $\mathbf{x}(t) \stackrel{\text{def}}{=} (x_1(t), x_2(t), \ldots, x_N(t)) \in \Re^N$ is the state vector; $\mathbf{u}(t) \stackrel{\text{def}}{=} (u_1(t), u_2(t), \ldots, u_M(t)) \in \Re^M$ is the control vector; $\boldsymbol{\alpha} \stackrel{\text{def}}{=} (\alpha_1, \alpha_2, \ldots, \alpha_A) \in \Re^A$ is a vector of time-independent parameters that affect the state equation, integrand, and constraint functions; $\mathbf{g}(\cdot) \stackrel{\text{def}}{=} (g^1(\cdot), g^2(\cdot), \ldots, g^N(\cdot))$ is the transition function; $\dot{\mathbf{x}}(\cdot) \stackrel{\text{def}}{=} (\dot{x}_1(\cdot), \dot{x}_2(\cdot), \ldots, \dot{x}_N(\cdot))$, $\mathbf{h}(\cdot) \stackrel{\text{def}}{=} (h^1(\cdot), h^2(\cdot), \ldots, h^K(\cdot))$ is the vector of constraint functions; and $\boldsymbol{\beta} \stackrel{\text{def}}{=} (\boldsymbol{\alpha}, t_0, \mathbf{x}_0, t_1, \mathbf{x}_1) \in \Re^{2+2N+A}$ is the vector of parameters of the problem.

With the introduction of the constraints, we must now impose a few more assumptions on problem (27), in addition to assumptions (A.1) and (A.2) noted earlier. First of all, we assume that the rank constraint qualification of Chapter 6 holds for problem (27). Second, we impose the following assumption on the constraint functions:

(A.3) $\mathbf{h}(\cdot) \in C^{(1)}$ with respect to the $1 + N + M$ variables $(t, \mathbf{x}, \mathbf{u})$ and the A parameters $\boldsymbol{\alpha}$.

For the general class of control problems defined in Eq. (27), the dynamic envelope theorem takes on the following form.

Theorem 9.3 (Dynamic Envelope Theorem, Constraints): *Let* $(\mathbf{z}(t; \boldsymbol{\beta}), \mathbf{v}(t; \boldsymbol{\beta}))$ *be the optimal pair for problem (27),* $\boldsymbol{\lambda}(t; \boldsymbol{\beta})$ *be the corresponding time path of the costate vector, and* $\boldsymbol{\mu}(t; \boldsymbol{\beta})$ *be the corresponding time path of the Lagrange multiplier vector. Define the Hamiltonian as*

$$H(t, \mathbf{x}, \mathbf{u}, \boldsymbol{\lambda}; \boldsymbol{\alpha}) \stackrel{\text{def}}{=} f(t, \mathbf{x}, \mathbf{u}; \boldsymbol{\alpha}) + \sum_{\ell=1}^{N} \lambda_\ell g^\ell(t, \mathbf{x}, \mathbf{u}; \boldsymbol{\alpha}),$$

and the Lagrangian as

$$L(t, \mathbf{x}, \mathbf{u}, \boldsymbol{\lambda}, \boldsymbol{\mu}; \boldsymbol{\alpha}) \stackrel{\text{def}}{=} f(t, \mathbf{x}, \mathbf{u}; \boldsymbol{\alpha}) + \sum_{\ell=1}^{N} \lambda_\ell g^\ell(t, \mathbf{x}, \mathbf{u}; \boldsymbol{\alpha}) + \sum_{k=1}^{K} \mu_k h^k(t, \mathbf{x}, \mathbf{u}; \boldsymbol{\alpha}).$$

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If $\mathbf{z}(\cdot) \in C^{(1)}$, $\mathbf{v}(\cdot) \in C^{(1)}$, *and* $\mu_k(\cdot) \in C^{(1)}$, $k = K'+1, K'+2, \ldots, K$, *in* $(t;\boldsymbol{\beta})\,\forall\,(t;\boldsymbol{\beta}) \in [t_0^\circ, t_1^\circ]B(\boldsymbol{\beta}^\circ;\delta)$, *then* $V(\cdot) \in C^{(1)}\,\forall\,\boldsymbol{\beta} \in B(\boldsymbol{\beta}^\circ;\delta)$, *and furthermore*, $\forall\,\boldsymbol{\beta} \in B(\boldsymbol{\beta}^\circ;\delta)$:

$$V_{\alpha_i}(\boldsymbol{\beta}) \overset{\text{def}}{=} \frac{\partial V(\boldsymbol{\beta})}{\partial \alpha_i}$$

$$\equiv \int_{t_0}^{t_1} L_{\alpha_i}(t, \mathbf{z}(t;\boldsymbol{\beta}), \mathbf{v}(t;\boldsymbol{\beta}), \boldsymbol{\lambda}(t;\boldsymbol{\beta}), \boldsymbol{\mu}(t;\boldsymbol{\beta}); \boldsymbol{\alpha})\, dt, \quad i = 1, 2, \ldots, A, \tag{i}$$

$$V_{t_0}(\boldsymbol{\beta}) \equiv -H(t_0, \mathbf{x}_0, \mathbf{v}(t_0;\boldsymbol{\beta}), \boldsymbol{\lambda}(t_0;\boldsymbol{\beta}); \boldsymbol{\alpha}), \tag{ii}$$

$$V_{x_{0j}}(\boldsymbol{\beta}) \equiv \lambda_j(t_0;\boldsymbol{\beta}), \quad j = 1, 2, \ldots, N, \tag{iii}$$

$$V_{t_1}(\boldsymbol{\beta}) \equiv H(t_1, \mathbf{x}_1, \mathbf{v}(t_1;\boldsymbol{\beta}), \boldsymbol{\lambda}(t_1;\boldsymbol{\beta}); \boldsymbol{\alpha}), \tag{iv}$$

$$V_{x_{1j}}(\boldsymbol{\beta}) \equiv -\lambda_j(t_1;\boldsymbol{\beta}), \quad j = 1, 2, \ldots, N. \tag{v}$$

The proof of this theorem is rather technical for the following reasons. Recall from Chapter 6 that the presence of inequality constraints means that we can expect there to be intervals in the planning horizon when some of them are binding and some are not. As a result, there exists the possibility that the set of binding inequality constraints changes over the course of the planning horizon. That is, there exist times, referred to as *switching times*, when the set of binding inequality constraints change. We also know from Chapter 6 that at such switching times, the optimal control may not be differentiable with respect to a parameter. For these reasons, the proof of Theorem 9.3 is a bit technical and involved. We do, however, present a mental exercise in which you are asked to demonstrate some of the above assertions. A mental exercise also asks you to provide a proof of Theorem 9.3 when there are only equality constraints present, as the above features that cause the technical difficulties are not present in this instance.

In comparing Theorem 9.3 with Theorem 9.1, notice that there is actually very little difference in their conclusions. In particular, conclusions (ii) through (v) of both theorems are identical, except for the important fact that the solutions of the two different control problems are not, in general, equal. This is not surprising, for results (ii) through (v) are just the shadow values of time and the stocks and, as such, should not have fundamentally different expressions for them. It is important to recognize, though, that their values generally differ in the two types of control problems. The real difference between Theorem 9.3 and Theorem 9.1, therefore, pertains to result (i). This is not surprising either, for Theorem 9.3 deals with a constrained control problem whereas Theorem 9.1 does not, and the parameter under consideration is a general one. Hence, by analogy with the static envelope theorem, one would not expect the resulting dynamic envelope expressions to be identical, because they are not in the analogous static circumstance.

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Before closing out this chapter, let's examine an intertemporal consumption problem in order to see the dynamic envelope theorem in action once again, and to help improve our understanding of it and its relation to the archetype static envelope theorem.

Example 9.4: Consider an individual who expects to live $T > 0$ years and who is contemplating a lifetime utility maximizing consumption plan. Preferences over the rates of consumption of the M goods $\mathbf{c}(t) \in \Re_{+}^{M}$ are given by the $C^{(2)}$ instantaneous utility function $U(\cdot)$, where $U_{c_m}(\mathbf{c}(t)) > 0$ and $U_{c_m c_m}(\mathbf{c}(t)) < 0$ for $m = 1, 2, \ldots, M$. The time-invariant price vector of the goods is given by $\mathbf{p} \in \Re_{++}^{M}$, whereas the individual's present value of lifetime wealth is $w_0 > 0$. Assuming the person has an intertemporal rate of discount given by $\rho > 0$, the isoperimetric problem faced by the individual is given by

$$V(\boldsymbol{\beta}) \stackrel{\text{def}}{=} \max_{\mathbf{c}(\cdot)} \int_0^T U(\mathbf{c}(t))\, e^{-\rho t}\, dt$$

$$\text{s.t.} \quad \int_0^T \mathbf{p}'\mathbf{c}(t)\, e^{-rt}\, dt = w_0,$$

where $r > 0$ is the discount rate on expenditures, say, the rate of interest earned on a money market account, and $\boldsymbol{\beta} \stackrel{\text{def}}{=} (\mathbf{p}, r, \rho, w_0, T)$. The intertemporal budget constraint states that the present discounted value of expenditures on the goods is equal to the individual's present discounted value of wealth, the intertemporal equivalent of the budget constraint from static consumer theory when the capital markets are perfect. The dynamic analogue of the indirect utility function from static consumer theory is the optimal value function $V(\cdot)$. It may be interpreted as the individual's discounted lifetime indirect utility function in the present setting.

As stated, the above intertemporal utility maximization problem is not in standard form. In order to transform it into standard form, define a state variable as follows:

$$E(t) \stackrel{\text{def}}{=} \int_0^t \mathbf{p}'\mathbf{c}(s)\, e^{-rs}\, ds.$$

This new state variable has the interpretation of the present discounted value of expenditures up to time t, or the present value of cumulative expenditures at time t. By Leibniz's rule, we have that $\dot{E}(t) = \mathbf{p}'\mathbf{c}(t)e^{-rt}$, with boundary conditions $E(0) = 0$ and $E(T) = w_0$. We may therefore restate the above isoperimetric problem in the

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ensuing equivalent standard form:

$$V(\boldsymbol{\beta}) \stackrel{\text{def}}{=} \max_{\mathbf{c}(\cdot)} \int_0^T U(\mathbf{c}(t))\, e^{-\rho t}\, dt$$

$$\text{s.t.} \quad \dot{E}(t) = \mathbf{p}'\mathbf{c}(t)e^{-rt},\, E(0) = 0,\, E(T) = w_0.$$

This is the form of the problem we shall work on with the dynamic envelope theorem. We assume that the consumption rate of each of the M goods is positive in every period of the individual's lifetime.

To begin the analysis, let's first establish that the costate variable is a negative constant in an optimal consumption plan (assuming that one actually exists). To see this, first define the Hamiltonian as $H(t, E, \mathbf{c}, \lambda; \mathbf{p}, r, \rho) \stackrel{\text{def}}{=} U(\mathbf{c})e^{-\rho t} + \lambda e^{-rt} \sum_{m=1}^{M} p_m c_m$, and note that the state variable E does not appear explicitly in it. As a result of this observation, the costate equation becomes $\dot{\lambda} = -H_E(t, E, \mathbf{c}, \lambda; \mathbf{p}, r, \rho) = 0$, thereby implying that λ is a constant and thus only a function of the parameter vector $\boldsymbol{\beta} \stackrel{\text{def}}{=} (\mathbf{p}, r, \rho, w_0, T)$ and not t, say, $\lambda(\boldsymbol{\beta})$. Moreover, the necessary condition $H_{c_j}(t, E, \mathbf{c}, \lambda; \mathbf{p}, r, \rho) = 0$ implies that $U_{c_j}(\mathbf{c})e^{-\rho t} = -\lambda p_j e^{-rt}$, $j = 1, 2, \ldots, M$. Because $U_{c_j}(\mathbf{c}) > 0$ and $p_j > 0$, $j = 1, 2, \ldots, M$, and $e^{-\rho t} > 0$ and $e^{-rt} > 0$, this necessary condition implies that $\lambda(\boldsymbol{\beta}) < 0$, as was to be established. This makes economic sense too, for the state variable $E(t)$ is the present value of cumulative expenditures at time t and thus is a "bad," in that the higher is $E(t)$ at any t, the lower is the remaining wealth of the individual for spending on future consumption. With this result established, we now turn to the implications of the dynamic envelope theorem for the intertemporal consumption problem. Let us denote the optimal pair by $(E^*(t; \boldsymbol{\beta}), \mathbf{c}^*(t; \boldsymbol{\beta}))$ in what follows.

By Theorem 9.1, we have the following dynamic envelope results:

$$\frac{\partial V}{\partial p_j}(\boldsymbol{\beta}) = \int_0^T \frac{\partial H}{\partial p_j}(t, E, \mathbf{c}, \lambda; \mathbf{p}, r, \rho)\Big|_{\substack{\text{optimal}\\ \text{path}}} dt$$

$$= \lambda(\boldsymbol{\beta}) \int_0^T c_j^*(t; \boldsymbol{\beta})e^{-rt}\, dt < 0, \quad j = 1, 2, \ldots, M, \tag{28}$$

$$\frac{\partial V}{\partial r}(\boldsymbol{\beta}) = -\lambda(\boldsymbol{\beta}) \int_0^T t\mathbf{p}'\mathbf{c}^*(t; \boldsymbol{\beta})e^{-rt}\, dt > 0, \tag{29}$$

$$\frac{\partial V}{\partial \rho}(\boldsymbol{\beta}) = -\int_0^T tU(\mathbf{c}^*(t; \boldsymbol{\beta}))e^{-\rho t}\, dt \gtrless 0, \tag{30}$$

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$$\frac{\partial V}{\partial w_0}(\boldsymbol{\beta}) = -\lambda(\boldsymbol{\beta}) > 0, \tag{31}$$

$$\begin{aligned}\frac{\partial V}{\partial T}(\boldsymbol{\beta}) &= H(T, E^*(T;\boldsymbol{\beta}), \mathbf{c}^*(T;\boldsymbol{\beta}), \lambda(\boldsymbol{\beta}); \mathbf{p}, r, \rho) \\ &= U(\mathbf{c}^*(T;\boldsymbol{\beta}))e^{-\rho T} + \lambda(\boldsymbol{\beta})\mathbf{p}'\mathbf{c}^*(T;\boldsymbol{\beta})e^{-rT} \gtrless 0.\end{aligned} \tag{32}$$

Equation (28) asserts that an increase in the price of any good makes the individual worse off, a completely analogous result to that in static consumer theory. This qualitative result is just as one would expect, for a price increase in effect lowers the purchasing power of the individual's wealth, which, in turn, lowers discounted lifetime utility. Similarly, Eq. (29) shows that an increase in the interest rate r lowers the present value of consumption expenditures, and thus effectively increases the purchasing power of the individual's wealth, thereby increasing lifetime discounted utility. These economic interpretations follow rigorously from Eq. (31), which demonstrates that an increase in the individual's present discounted value of wealth makes the individual better off, just as one would expect. In other words, $\lambda(\boldsymbol{\beta}) < 0$ is behind these qualitative results, and as shown above, this fact is a result of our assumption that $U_{c_m}(\mathbf{c}(t)) > 0$, $m = 1, 2, \ldots, M$.

To finish up this example, we derive the intertemporal equivalent of the Antonelli-Roy lemma. In order to do so, simply divide the negative of Eq. (28) by Eq. (31) to get

$$\begin{aligned}\frac{-\partial V(\boldsymbol{\beta})/\partial p_j}{\partial V(\boldsymbol{\beta})/\partial w_0} &= \frac{-\lambda(\boldsymbol{\beta})\int_0^T c_j^*(t;\boldsymbol{\beta})\,e^{-rt}\,dt}{-\lambda(\boldsymbol{\beta})} \\ &= \int_0^T c_j^*(t;\boldsymbol{\beta})\,e^{-rt}\,dt > 0, \quad j = 1, 2, \ldots, M.\end{aligned}$$

In the intertemporal theory of the consumer, therefore, the equivalent of the Antonelli-Roy lemma recovers the cumulative discounted demand function for a good. Referring back to Example 9.1, this is consistent with the results of the dynamic envelope theorem applied to the adjustment cost model of the firm, in which case, the dynamic envelope theorem also recovered the cumulative discounted demand functions.

In the next chapter, we use the dynamic envelope theorem to derive the necessary transversality conditions for a class of optimal control problems that are more or less ubiquitous in intertemporal economics. The dynamic envelope theorem permits this to be done in a simple and palatable manner. In the process, we will further hone our economic intuition about optimal control problems and their necessary conditions, as well as our understanding of the dynamic envelope theorem.

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MENTAL EXERCISES

9.1 Starting at the bottom of page 169 through the top of page 172, Kamien and Schwartz (1991, 2nd Edition, *first* printing only) prove the dynamic envelope theorem for a parameter r entering the integrand function only. Find the error in their proof and explain it clearly. This error was corrected in the second printing of the 2nd edition.

9.2 Prove part (iii) of Theorem 9.1.

9.3 Prove part (iv) of Theorem 9.1.

9.4 Prove part (v) of Theorem 9.1.

9.5 Prove that $V(\cdot) \in C^{(1)}\ \forall\, \boldsymbol{\beta} \in B(\boldsymbol{\beta}^{\circ}; \delta)$ in Theorem 9.1.

9.6 In our economic interpretation of parts (ii) and (iv) of Theorem 9.1, we assumed that the value of the Hamiltonian was positive at the initial and terminal dates in the optimal plan.
   (a) Given this assumption, if you had the choice, would you prefer to start your planning sooner or later? Why?
   (b) Given this assumption, if you had the choice, would you prefer to end your planning sooner or later? Why?

9.7 For Example 9.1, derive the dynamic envelope results for the parameters (c, g, r, δ) by invoking Theorem 9.1, and then provide an economic interpretation of each. Also provide a thorough economic interpretation of the dynamic envelope result for the parameter K_T.

9.8 Consider the optimal control problem

$$V(\alpha, t_0, x_0, t_1) \stackrel{\text{def}}{=} \max_{u(\cdot), x_1} \int_{t_0}^{t_1} \alpha x(t) u(t)\, dt$$

$$\text{s.t.} \quad \dot{x}(t) = 0,\ x(t_0) = x_0,\ x(t_1) = x_1,$$

$$0 \le u(t) \le 1,$$

where $\alpha > 0$ is a time-independent parameter. Note that this is a generalization of the control problem in Example 9.2.
   (a) Determine the pair $(z(t; \alpha, t_0 x_0, t_1), v(t; \alpha, t_0 x_0, t_1))$ that solves the necessary conditions. Consider the cases $x_0 > 0$ and $x_0 < 0$ separately.
   (b) Prove that $(z(t; \alpha, t_0, x_0, t_1), v(t; \alpha, t_0, x_0, t_1))$ is a solution of the control problem for each case ($x_0 > 0$ and $x_0 < 0$) separately.
   (c) Prove that $V(\cdot)$ is not differentiable with respect to x_0 for all $x_0 \in \Re$, but that it is differentiable with respect to x_0 in any neighborhood of $x_0 \neq 0$ not containing the value $x_0 = 0$.

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(d) Prove, however, that $V(\cdot)$ is differentiable with respect to (α, t_0, t_1) for all $x_0 \in \Re$.

9.9 For Example 9.3, show that

(a) $v(t; s, x_s, t_1, x_1)|_{x_s = z(s;t_0,x_0,t_1,x_1)} = v(t; t_0, x_0, t_1, x_1)\,\forall\, t \in [s, t_1]$,

(b) $\lambda(t; s, x_s, t_1, x_1)|_{x_s = z(s;t_0,x_0,t_1,x_1)} = \lambda(t; t_0, x_0, t_1, x_1)\,\forall\, t \in [s, t_1]$.

(c) Why does one need the restriction $t \in [s, t_1]$?

9.10 Recall our prototype inventory accumulation problem:

$$V(\boldsymbol{\beta}) \stackrel{\text{def}}{=} \min_{u(\cdot)} \int_0^T \left[c_1[u(t)]^2 + c_2 x(t)\right] dt$$

$$\text{s.t.} \quad \dot{x}(t) = u(t),\, x(0) = 0,\, x(T) = x_T,$$

where $\boldsymbol{\beta} \stackrel{\text{def}}{=} (c_1, c_2, T, x_T)$. Recall that in Example 4.5, we found the optimal solution to be given by the curves $z(t; \boldsymbol{\beta}) = \frac{1}{4} c_2 c_1^{-1} t[t - T] + x_T T^{-1} t$, $v(t; \boldsymbol{\beta}) = \frac{1}{4} c_2 c_1^{-1}[2t - T] + x_T T^{-1}$, and $\lambda(t; \boldsymbol{\beta}) = \frac{1}{2} c_2 [T - 2t] - 2c_1 x_T T^{-1}$. Do not use these explicit formulas in answering this question unless asked. You may *invoke* Theorem 9.1 in what follows, unless specifically asked not to.

(a) Prove that $z(t; \boldsymbol{\beta})$ and $v(t; \boldsymbol{\beta})$ are positively homogeneous of degree zero in (c_1, c_2), Provide an economic interpretation of these results.

(b) Give an alternative definition of $V(\boldsymbol{\beta})$, and prove that $V(\boldsymbol{\beta})$ is positively homogeneous of degree one in (c_1, c_2). Do not derive an explicit expression for $V(\boldsymbol{\beta})$.

(c) Find the partial derivative $V_{c_1}(\boldsymbol{\beta})$, determine its sign, and provide an economic interpretation of it.

(d) Find the partial derivative $V_{c_2}(\boldsymbol{\beta})$, determine its sign, and provide an economic interpretation of it.

(e) Find the partial derivative $V_T(\boldsymbol{\beta})$, determine its sign, and provide an economic interpretation of it.

(f) Find the partial derivative $V_{x_T}(\boldsymbol{\beta})$, determine its sign, and provide an economic interpretation of it.

(g) Now derive an explicit formula for $V(\boldsymbol{\beta})$.

(h) Verify that parts (c) and (d) hold by using the explicit formulas for $V(\boldsymbol{\beta})$, $z(t; \boldsymbol{\beta})$, and $v(t; \boldsymbol{\beta})$.

9.11 Let $(z(t), v(t))$ be the optimal pair for the following control problem:

$$V(t_0, x_0, t_1, x_1) \stackrel{\text{def}}{=} \max_{u(\cdot)} \int_{t_0}^{t_1} f(t, x(t), u(t))\, dt$$

$$\text{s.t.}\ \dot{x}(t) = g(t, x(t), u(t)),\, x(t_0) = x_0,\, x(t_1) = x_1.$$

Explain clearly why the dynamic envelope theorem and principle of optimality *cannot* be used to interpret the costate variable $\lambda(t)$ corresponding to

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$(z(t), v(t))$ as the shadow value of the state variable at any time $t \in [t_0, t_1]$ for the above optimal control problem.

9.12 Prove Theorem 9.3 when there are only *equality* constraints present.

9.13 In order to demonstrate some of the assertions made after Theorem 9.3, consider the following inequality constrained optimal control problem:

$$V(\alpha) \stackrel{\text{def}}{=} \max_{u(\cdot),x_2} \int_0^2 \left[\alpha x(t) + tu(t) - \frac{1}{2}(u(t))^2\right] dt$$

$$\text{s.t.} \quad \dot{x}(t) = u(t),\, x(0) = 0,\, x(2) = x_2,$$

$$u(t) \leq 1,$$

where $\alpha \in (-\infty, \frac{1}{2})$.

(a) Prove that a solution of the necessary conditions is a solution of the optimal control problem.

(b) Derive the necessary and sufficient conditions.

(c) Find $\lambda(t; \alpha)$, the solution of the costate equation and transversality condition.

(d) Notice that the integrand is an increasing function of the control variable for $t \in [0, 1]$ if $u(t) < t$, and for all $t \in (1, 2]$, since $u(t) \leq 1$ must hold. Hence it is natural to conjecture the existence of a unique switching time that is a function of $\alpha \in (-\infty, \frac{1}{2})$, say, $s(\alpha)$. Conjecture, therefore, that for $t \in [0, s(\alpha))$, the constraint on the control is not binding, whereas for $t \in (s(\alpha), 2]$ it is, so that $u(t) = 1$. Find the solution to the control problem using this conjecture, including the switching time.

(e) Show that the optimal control, Lagrange multiplier, and time derivative of the state variable are *not* differentiable in α at the switching time $s(\alpha)$, but that all three are $C^{(1)}$ in α for $t \in [0, 2] \sim s(\alpha)$.

(f) Show, however, that the optimal state and costate variables are $C^{(1)}$ in α for $t \in [0, 2]$.

(g) Find the value of $V(\cdot)$.

(h) Confirm the veracity of Theorem 9.1 part (i).

FURTHER READING

A superb (and, arguably, the best) reference on the static envelope theorem and comparative statics is Silberberg and Suen (2001). Besides the seminal paper by Eisner and Strotz (1963) on the adjustment cost model of the firm, Treadway (1970) is another excellent reference. Example 9.2 is culled from Seierstad and Sydsæter (1987). The rigorous proof that the *j*th costate variable at any time *t* in the planning horizon of the original optimal control problem (OC) is the shadow value of the *j*th state variable at time *t* is a result of Léonard (1987). A proof of Theorem 9.3 (under assumptions that are stronger than those employed here) can be found in

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LaFrance and Barney (1991), which is also the source for Example 9.4 and Mental Exercise 9.13. Seierstad (1982) provides a thorough analysis of the differentiability of the optimal value function. Dual proofs of the static and dynamic envelope theorems can be found in Silberberg (1974) and Caputo (1990a, 1990b), respectively. In Chapter 11, we will explore the modern envelope approach to the dynamic envelope theorem and the concomitant implications it has for the comparative dynamics properties of optimal control problems.

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