The Dynamic Envelope Theorem and Transversality Conditions

We now exploit some benefits of the dynamic envelope theorem established in Chapter 9 by deriving the necessary transversality conditions corresponding to various endpoint conditions that are relatively common in economic applications of optimal control theory. The use of the dynamic envelope theorem renders this a relatively simple and economically revealing process. That is to say, the conclusions established will help build additional economic intuition about optimal control problems and their solutions, as well as the dynamic envelope theorem itself. Furthermore, we provide a rather general sufficiency theorem that is of value in solving optimal control problems.

To begin, consider the *variable endpoint* and *variable time* optimal control problem

$$V^*(\boldsymbol{\alpha}) \stackrel{\text{def}}{=} \max_{\mathbf{u}(\cdot),t_0,\mathbf{x}_0,t_1,\mathbf{x}_1} \int_{t_0}^{t_1} f(t,\mathbf{x}(t),\mathbf{u}(t);\boldsymbol{\alpha})\,dt \tag{1}$$

$$\begin{aligned} \text{s.t.}\quad & \dot{\mathbf{x}}(t) = \mathbf{g}(t,\mathbf{x}(t),\mathbf{u}(t);\boldsymbol{\alpha}),\ \ \mathbf{x}(t_0)=\mathbf{x}_0,\ \ \mathbf{x}(t_1)=\mathbf{x}_1, \\ & h^k(t,\mathbf{x}(t),\mathbf{u}(t);\boldsymbol{\alpha}) \ge 0, \quad k=1,2,\ldots,K', \\ & h^k(t,\mathbf{x}(t),\mathbf{u}(t);\boldsymbol{\alpha}) = 0, \quad k=K'+1,K'+2,\ldots,K, \end{aligned}$$

where $\mathbf{x}(t) \stackrel{\text{def}}{=} (x_1(t),x_2(t),\ldots,x_N(t)) \in \Re^N$ is the state vector; $\mathbf{u}(t) \stackrel{\text{def}}{=} (u_1(t), u_2(t),\ldots,u_M(t)) \in \Re^M$ is the control vector; $\dot{\mathbf{x}}(\cdot) \stackrel{\text{def}}{=} (\dot{x}_1(\cdot),\dot{x}_2(\cdot),\ldots,\dot{x}_N(\cdot))$, $\mathbf{g}(\cdot) \stackrel{\text{def}}{=} (g^1(\cdot),g^2(\cdot),\ldots,g^N(\cdot))$ is the vector of transition functions; $\mathbf{h}(\cdot) \stackrel{\text{def}}{=} (h^1(\cdot), h^2(\cdot),\ldots,h^K(\cdot))$ is the vector of constraint functions, both inequality and equality; and $\boldsymbol{\alpha} \stackrel{\text{def}}{=} (\alpha_1,\alpha_2,\ldots,\alpha_A) \in \Re^A$ is a vector of time-independent parameters that affect the state equations, integrand, and constraint functions. Assume that an optimal solution $(\mathbf{z}^*(t;\boldsymbol{\alpha}),\mathbf{v}^*(t;\boldsymbol{\alpha}))$ exists to problem (1) for all $\boldsymbol{\alpha} \in B(\boldsymbol{\alpha}^\circ;\delta_1)$, with corresponding costate vector $\boldsymbol{\lambda}^*(t;\boldsymbol{\alpha})$ and Lagrange multiplier vector $\boldsymbol{\mu}^*(t;\boldsymbol{\alpha})$, and let $\boldsymbol{\gamma}^*(\boldsymbol{\alpha}) \stackrel{\text{def}}{=} (t_0^*(\boldsymbol{\alpha}),\mathbf{x}_0^*(\boldsymbol{\alpha}),t_1^*(\boldsymbol{\alpha}),\mathbf{x}_1^*(\boldsymbol{\alpha}))$ be the optimal solution for the initial and terminal values of the horizon and state vector, where $B(\boldsymbol{\alpha}^{\circ};\delta_1)$ is an open A – ball centered at the given value of the parameter vector $\boldsymbol{\alpha}^{\circ}$ of radius $\delta_1 > 0$. Given that the initial and terminal dates and states are decision variables, the optimal value function and solution functions for problem (1) depend only on the parameter vector $\boldsymbol{\alpha}$. Given the above definitions, it should be apparent that the identities $\mathbf{x}_0^*(\boldsymbol{\alpha}) \equiv \mathbf{z}^*(t_0^*(\boldsymbol{\alpha});\boldsymbol{\alpha})$ and $\mathbf{x}_1^*(\boldsymbol{\alpha}) \equiv \mathbf{z}^*(t_1^*(\boldsymbol{\alpha});\boldsymbol{\alpha})$ hold for the optimal initial and terminal values of the state vector.

Now recall our canonical *fixed endpoint* and *fixed-time* optimal control problem

$$\hat{V}(\boldsymbol{\alpha},\boldsymbol{\gamma}) \stackrel{\text{def}}{=} \max_{\mathbf{u}(\cdot)} \int_{t_0}^{t_1} f(t,\mathbf{x}(t),\mathbf{u}(t);\boldsymbol{\alpha})\,dt \tag{2}$$

$$\text{s.t.}\quad \dot{\mathbf{x}}(t) = \mathbf{g}(t,\mathbf{x}(t),\mathbf{u}(t);\boldsymbol{\alpha}),\ \ \mathbf{x}(t_0)=\mathbf{x}_0,\ \ \mathbf{x}(t_1)=\mathbf{x}_1,$$

$$h^k(t,\mathbf{x}(t),\mathbf{u}(t);\boldsymbol{\alpha}) \ge 0, \quad k=1,2,\ldots,K',$$

$$h^k(t,\mathbf{x}(t),\mathbf{u}(t);\boldsymbol{\alpha}) = 0, \quad k=K'+1,K'+2,\ldots,K,$$

where $\boldsymbol{\gamma} \stackrel{\text{def}}{=} (t_0,\mathbf{x}_0,t_1,\mathbf{x}_1) \in \Re^{2+2N}$ and $\boldsymbol{\beta} \stackrel{\text{def}}{=} (\boldsymbol{\alpha},t_0,\mathbf{x}_0,t_1,\mathbf{x}_1) \in \Re^{2+2N+A}$. Assume that an optimal solution $(\hat{\mathbf{z}}(t;\boldsymbol{\beta}),\hat{\mathbf{v}}(t;\boldsymbol{\beta}))$ exists to problem (2) for all $\boldsymbol{\beta} \in B(\boldsymbol{\beta}^{\circ};\delta_2)$, with corresponding costate vector $\hat{\boldsymbol{\lambda}}(t;\boldsymbol{\beta})$ and Lagrange multiplier vector $\hat{\boldsymbol{\mu}}(t;\boldsymbol{\beta})$, where $B(\boldsymbol{\beta}^{\circ};\delta_2)$ is an open $2+2N+A$ – ball centered at the given value of the parameter $\boldsymbol{\beta}^{\circ}$ of radius $\delta_2 > 0$. In view of the fact that $\boldsymbol{\gamma} \stackrel{\text{def}}{=} (t_0,\mathbf{x}_0,t_1,\mathbf{x}_1)$ is parametrically given in problem (2), the optimal value function and solution functions depend on these parameters in addition to $\boldsymbol{\alpha}$.

Because problems (1) and (2) depend on exogenous parameters, and because we intend to make use of the dynamic envelope theorem, we take it that assumptions (A.1) through (A.3) in Chapter 9 are in force throughout this chapter as well. Moreover, because of the presence of constraints in problems (1) and (2), we assume that the rank constraint qualification given in Chapter 6 holds throughout this chapter too. Given these preliminaries, we have the following set of necessary conditions for selecting the initial and terminal values of the planning horizon and state vector in problem (1).

Theorem 10.1 (Free Transversality Conditions): *If $\hat{V}(\cdot) \in C^{(1)}\ \forall\, \boldsymbol{\beta} \in B(\boldsymbol{\beta}^{\circ};\delta_2)$, then in addition to the necessary conditions of Theorem 6.1, the following transversality conditions are necessary for the variable endpoint and variable time optimal control problem (1):*

$$H(t_0^*(\boldsymbol{\alpha}),\mathbf{z}^*(t_0^*(\boldsymbol{\alpha});\boldsymbol{\alpha}),\mathbf{v}^*(t_0^*(\boldsymbol{\alpha});\boldsymbol{\alpha}),\boldsymbol{\lambda}^*(t_0^*(\boldsymbol{\alpha});\boldsymbol{\alpha});\boldsymbol{\alpha}) = 0\,[t_0\,\text{free}], \tag{3}$$

$$\lambda_j^*(t_0^*(\boldsymbol{\alpha});\boldsymbol{\alpha}) = 0, \quad j=1,2,\ldots,N\ [\mathbf{x}_0\ \text{free}], \tag{4}$$

$$H(t_1^*(\boldsymbol{\alpha}),\mathbf{z}^*(t_1^*(\boldsymbol{\alpha});\boldsymbol{\alpha}),\mathbf{v}^*(t_1^*(\boldsymbol{\alpha});\boldsymbol{\alpha}),\boldsymbol{\lambda}^*(t_1^*(\boldsymbol{\alpha});\boldsymbol{\alpha});\boldsymbol{\alpha}) = 0\ [t_1\text{free}], \tag{5}$$

$$\lambda_j^*(t_1^*(\boldsymbol{\alpha});\boldsymbol{\alpha}) = 0, \quad j=1,2,\ldots,N\ [\mathbf{x}_1\text{free}]. \tag{6}$$

https://doi.org/10.1017/CBO9780511806827.012 Published online by Cambridge University Press

Proof: Given that $V^*(\boldsymbol{\alpha})$ is the value of the optimal value function for problem (1), in which no constraints are placed on the horizon or endpoints, it must be at least as large as the value of the optimal value function $\hat{V}(\boldsymbol{\alpha}, \boldsymbol{\gamma})$ for problem (2), in which the endpoints and horizon are constrained to be fixed. This observation follows directly from the very definition of an optimization problem. More formally, we have that $V^*(\boldsymbol{\alpha}) \geq \hat{V}(\boldsymbol{\alpha}, \boldsymbol{\gamma}) \forall \boldsymbol{\gamma} \stackrel{\text{def}}{=} (t_0, \mathbf{x}_0, t_1, \mathbf{x}_1)$ in a neighborhood of $\boldsymbol{\gamma}^*(\boldsymbol{\alpha}) \stackrel{\text{def}}{=} (t_0^*(\boldsymbol{\alpha}), \mathbf{x}_0^*(\boldsymbol{\alpha}), t_1^*(\boldsymbol{\alpha}), \mathbf{x}_1^*(\boldsymbol{\alpha}))$ for each $\boldsymbol{\alpha} \in B(\boldsymbol{\alpha}^\circ; \delta_1)$. The identity $V^*(\boldsymbol{\alpha}) \equiv \hat{V}(\boldsymbol{\alpha}, \boldsymbol{\gamma})$ therefore holds by definition when $\boldsymbol{\gamma} = \boldsymbol{\gamma}^*(\boldsymbol{\alpha})$, that is, $V^*(\boldsymbol{\alpha}) \equiv \hat{V}(\boldsymbol{\alpha}, \boldsymbol{\gamma}^*(\boldsymbol{\alpha})) \forall \boldsymbol{\alpha} \in B(\boldsymbol{\alpha}^\circ; \delta_1)$. In other words, $V^*(\boldsymbol{\alpha}) \stackrel{\text{def}}{=} \max_{\boldsymbol{\gamma}} \hat{V}(\boldsymbol{\alpha}, \boldsymbol{\gamma})$ and $\boldsymbol{\gamma}^*(\boldsymbol{\alpha}) \stackrel{\text{def}}{=} \arg\max_{\boldsymbol{\gamma}} \hat{V}(\boldsymbol{\alpha}, \boldsymbol{\gamma})$. Because $\hat{V}(\cdot) \in C^{(1)} \forall \boldsymbol{\beta} \in B(\boldsymbol{\beta}^\circ; \delta_2)$ by assumption and no constraints are placed on the choice of $\boldsymbol{\gamma}$ in the unconstrained optimization problem $V^*(\boldsymbol{\alpha}) \stackrel{\text{def}}{=} \max_{\boldsymbol{\gamma}} \hat{V}(\boldsymbol{\alpha}, \boldsymbol{\gamma})$, it follows from static optimization theory that the ensuing conditions are necessary:

$$\hat{V}_{t_0}(\boldsymbol{\alpha}, \boldsymbol{\gamma})\big|_{\boldsymbol{\gamma}=\boldsymbol{\gamma}^*(\boldsymbol{\alpha})} = 0, \;\; \hat{V}_{\mathbf{x}_0}(\boldsymbol{\alpha}, \boldsymbol{\gamma})\big|_{\boldsymbol{\gamma}=\boldsymbol{\gamma}^*(\boldsymbol{\alpha})} = \mathbf{0}'_N,$$

$$\hat{V}_{t_1}(\boldsymbol{\alpha}, \boldsymbol{\gamma})\big|_{\boldsymbol{\gamma}=\boldsymbol{\gamma}^*(\boldsymbol{\alpha})} = 0, \;\; \hat{V}_{\mathbf{x}_1}(\boldsymbol{\alpha}, \boldsymbol{\gamma})\big|_{\boldsymbol{\gamma}=\boldsymbol{\gamma}^*(\boldsymbol{\alpha})} = \mathbf{0}'_N.$$

By Theorem 9.3, the dynamic envelope theorem, these derivatives are given by

$$\hat{V}_{t_0}(\boldsymbol{\alpha}, \boldsymbol{\gamma})\big|_{\boldsymbol{\gamma}=\boldsymbol{\gamma}^*(\boldsymbol{\alpha})} = -H(t_0^*(\boldsymbol{\alpha}), \hat{\mathbf{z}}(t_0^*(\boldsymbol{\alpha}); \boldsymbol{\alpha}, \boldsymbol{\gamma}^*(\boldsymbol{\alpha})), \hat{\mathbf{v}}(t_0^*(\boldsymbol{\alpha}); \boldsymbol{\alpha}, \boldsymbol{\gamma}^*(\boldsymbol{\alpha})),$$
$$\hat{\boldsymbol{\lambda}}(t_0^*(\boldsymbol{\alpha}); \boldsymbol{\alpha}, \boldsymbol{\gamma}^*(\boldsymbol{\alpha})); \boldsymbol{\alpha}) = 0,$$
$$\hat{V}_{x_{0j}}(\boldsymbol{\alpha}, \boldsymbol{\gamma})\big|_{\boldsymbol{\gamma}=\boldsymbol{\gamma}^*(\boldsymbol{\alpha})} = \hat{\lambda}_j(t_0^*(\boldsymbol{\alpha}); \boldsymbol{\alpha}, \boldsymbol{\gamma}^*(\boldsymbol{\alpha})) = 0, \quad j = 1, 2, \ldots, N,$$
$$\hat{V}_{t_1}(\boldsymbol{\alpha}, \boldsymbol{\gamma})\big|_{\boldsymbol{\gamma}=\boldsymbol{\gamma}^*(\boldsymbol{\alpha})} = H(t_1^*(\boldsymbol{\alpha}), \hat{\mathbf{z}}(t_1^*(\boldsymbol{\alpha}); \boldsymbol{\alpha}, \boldsymbol{\gamma}^*(\boldsymbol{\alpha})), \hat{\mathbf{v}}(t_1^*(\boldsymbol{\alpha}); \boldsymbol{\alpha}, \boldsymbol{\gamma}^*(\boldsymbol{\alpha})),$$
$$\hat{\boldsymbol{\lambda}}(t_1^*(\boldsymbol{\alpha}); \boldsymbol{\alpha}, \boldsymbol{\gamma}^*(\boldsymbol{\alpha})); \boldsymbol{\alpha}) = 0,$$
$$\hat{V}_{x_{1j}}(\boldsymbol{\alpha}, \boldsymbol{\gamma})\big|_{\boldsymbol{\gamma}=\boldsymbol{\gamma}^*(\boldsymbol{\alpha})} = -\hat{\lambda}_j(t_1^*(\boldsymbol{\alpha}); \boldsymbol{\alpha}, \boldsymbol{\gamma}^*(\boldsymbol{\alpha})) = 0, \quad j = 1, 2, \ldots, N.$$

Notice that the above derivatives are evaluated along the triplet $(\hat{\mathbf{z}}(t; \boldsymbol{\alpha}, \boldsymbol{\gamma}), \hat{\mathbf{v}}(t; \boldsymbol{\alpha}, \boldsymbol{\gamma}), \hat{\boldsymbol{\lambda}}(t; \boldsymbol{\alpha}, \boldsymbol{\gamma}))$, since it corresponds to the optimal value function $\hat{V}(\cdot)$ being differentiated. This means that we have not yet established the transversality conditions of Theorem 10.1, which apply to problem (1) and its solution. In order to do so, we must establish that the solutions to problems (1) and (2) are identical when $\boldsymbol{\gamma} = \boldsymbol{\gamma}^*(\boldsymbol{\alpha})$. If we therefore set $\boldsymbol{\gamma} = \boldsymbol{\gamma}^*(\boldsymbol{\alpha})$ in the fixed endpoint and fixed-time optimal control problem (2), then the horizon and endpoints in problem (2) are held fixed at the values that are optimal in the variable endpoint and variable time optimal control problem (1). In other words, if $\boldsymbol{\gamma} = \boldsymbol{\gamma}^*(\boldsymbol{\alpha})$ in problem (2), then the optimal pair that solves problem (2) passes through the same endpoints over the same time interval as the optimal pair that solves problem (1), and vice versa. Hence, seeing as problems (1) and (2) are otherwise identical, their solutions must be identical too when $\boldsymbol{\gamma} = \boldsymbol{\gamma}^*(\boldsymbol{\alpha})$, which in turn implies that the costate and Lagrange multiplier vectors are identical as well. This reasoning thus establishes the following identities for all $t \in [t_0^*(\boldsymbol{\alpha}), t_1^*(\boldsymbol{\alpha})]$:

$$(\mathbf{z}^*(t;\boldsymbol{\alpha}), \mathbf{v}^*(t;\boldsymbol{\alpha}), \boldsymbol{\lambda}^*(t;\boldsymbol{\alpha}), \boldsymbol{\mu}^*(t;\boldsymbol{\alpha}))$$
$$\equiv (\hat{\mathbf{z}}(t;\boldsymbol{\alpha}, \boldsymbol{\gamma}^*(\boldsymbol{\alpha})), \hat{\mathbf{v}}(t;\boldsymbol{\alpha}, \boldsymbol{\gamma}^*(\boldsymbol{\alpha})), \hat{\boldsymbol{\lambda}}(t;\boldsymbol{\alpha}, \boldsymbol{\gamma}^*(\boldsymbol{\alpha})), \hat{\boldsymbol{\mu}}(t;\boldsymbol{\alpha}, \boldsymbol{\gamma}^*(\boldsymbol{\alpha}))).$$

Substituting these identities into the four dynamic envelope derivatives above completes the proof of the necessity of the transversality conditions.

To complete the proof of Theorem 10.1, we must demonstrate that the quadruplet $(\mathbf{z}^*(t;\boldsymbol{\alpha}), \mathbf{v}^*(t;\boldsymbol{\alpha}), \boldsymbol{\lambda}^*(t;\boldsymbol{\alpha}), \boldsymbol{\mu}^*(t;\boldsymbol{\alpha}))$ satisfies the necessary conditions of Theorem 6.1, which pertain to the fixed endpoint and fixed-time optimal control problem (2). This conclusion follows immediately from the above identities, as we now intend to show. Because the quadruplet $(\hat{\mathbf{z}}(t;\boldsymbol{\alpha}, \boldsymbol{\gamma}^*(\boldsymbol{\alpha})), \hat{\mathbf{v}}(t;\boldsymbol{\alpha}, \boldsymbol{\gamma}^*(\boldsymbol{\alpha})), \hat{\boldsymbol{\lambda}}(t;\boldsymbol{\alpha}, \boldsymbol{\gamma}^*(\boldsymbol{\alpha})), \hat{\boldsymbol{\mu}}(t;\boldsymbol{\alpha},\boldsymbol{\gamma}^*(\boldsymbol{\alpha})))$ is the solution of the fixed endpoint and fixed-time optimal control problem (2) when $\boldsymbol{\gamma} = \boldsymbol{\gamma}^*(\boldsymbol{\alpha})$, it satisfies the necessary conditions of Theorem 6.1. By the above identities, the quadruplet $(\mathbf{z}^*(t;\boldsymbol{\alpha}), \mathbf{v}^*(t;\boldsymbol{\alpha}), \boldsymbol{\lambda}^*(t;\boldsymbol{\alpha}), \boldsymbol{\mu}^*(t;\boldsymbol{\alpha}))$ is identically equal to the quadruplet $(\hat{\mathbf{z}}(t;\boldsymbol{\alpha}, \boldsymbol{\gamma}^*(\boldsymbol{\alpha})), \hat{\mathbf{v}}(t;\boldsymbol{\alpha},\boldsymbol{\gamma}^*(\boldsymbol{\alpha})), \hat{\boldsymbol{\lambda}}(t;\boldsymbol{\alpha}, \boldsymbol{\gamma}^*(\boldsymbol{\alpha})), \hat{\boldsymbol{\mu}}(t;\boldsymbol{\alpha}, \boldsymbol{\gamma}^*(\boldsymbol{\alpha})))$. As a result, $(\mathbf{z}^*(t;\boldsymbol{\alpha}), \mathbf{v}^*(t;\boldsymbol{\alpha}), \boldsymbol{\lambda}^*(t;\boldsymbol{\alpha}), \boldsymbol{\mu}^*(t;\boldsymbol{\alpha}))$ also satisfies the necessary conditions of Theorem 6.1. Q.E.D.

The economic interpretation of these transversality conditions is straightforward and reinforces the economic intuition developed in the last chapter about the dynamic envelope theorem. This is not really surprising as the transversality conditions rely heavily on the dynamic envelope theorem. For example, Eq. (3) of Theorem 10.1 asserts that the optimal starting date for an intertemporal plan should be chosen such that all instantaneous profit opportunities are exhausted at the margin, both *current* through the integrand $f(t, \mathbf{x}(t), \mathbf{u}(t); \boldsymbol{\alpha})$, and *indirect* through the total shadow value of investment $\sum_{\ell=1}^{N} \lambda_\ell g^\ell(t, \mathbf{x}(t), \mathbf{u}(t); \boldsymbol{\alpha})$. Otherwise, there would be a gain to either delaying the start of the program or beginning it earlier. Equation (4) says that the optimal initial state vector should be chosen such that, at the margin, a unit of capital has no value in an optimal program. In other words, the firm should start with a capital stock such that it would not be willing to pay anything for an additional unit. In general, therefore, the transversality conditions reflect nothing other than the marginal principle that the optimal amount of an activity occurs when the net marginal value of it is zero. The economic interpretation of Eqs. (5) and (6) is left for a mental exercise.

Consider now the following *inequality constrained variable endpoint* and *variable time* optimal control problem:

$$\bar{V}(\boldsymbol{\theta}) \overset{\text{def}}{=} \max_{\mathbf{u}(\cdot), t_1, \mathbf{x}_1} \int_{t_0}^{t_1} f(t, \mathbf{x}(t), \mathbf{u}(t); \boldsymbol{\alpha})\, dt \tag{7}$$

$$\text{s.t.}\quad \dot{\mathbf{x}}(t) = \mathbf{g}(t, \mathbf{x}(t), \mathbf{u}(t); \boldsymbol{\alpha}),\ \ \mathbf{x}(t_0) = \mathbf{x}_0,\ \ \mathbf{x}(t_1) = \mathbf{x}_1 \geq \mathbf{x}_T,\ \ t_1 \leq T,$$
$$h^k(t, \mathbf{x}(t), \mathbf{u}(t); \boldsymbol{\alpha}) \geq 0, \quad k = 1, 2, \ldots, K',$$
$$h^k(t, \mathbf{x}(t), \mathbf{u}(t); \boldsymbol{\alpha}) = 0, \quad k = K'+1, K'+2, \ldots, K,$$

where $(\boldsymbol{\alpha}, t_0, \mathbf{x}_0)$ are fixed (or given) parameters, but $(t_1, \mathbf{x}_1)$ are chosen subject to the stated inequality constraints, and $\boldsymbol{\theta} \stackrel{\text{def}}{=} (\boldsymbol{\alpha}, t_0, \mathbf{x}_0, T, \mathbf{x}_T)$. As we did above, assume that an optimal solution $(\bar{\mathbf{z}}(t;\boldsymbol{\theta}), \bar{\mathbf{v}}(t;\boldsymbol{\theta}))$ exists to problem (7) for all $\boldsymbol{\theta} \in B(\boldsymbol{\theta}^\circ; \delta_3)$, with corresponding costate vector $\bar{\boldsymbol{\lambda}}(t;\boldsymbol{\theta})$ and Lagrange multiplier vector $\bar{\boldsymbol{\mu}}(t;\boldsymbol{\theta})$, and let $\boldsymbol{\omega}(\boldsymbol{\theta}) \stackrel{\text{def}}{=} (\bar{t}_1(\boldsymbol{\theta}), \bar{\mathbf{x}}_1(\boldsymbol{\theta}))$ be the optimal solution for the terminal values of the time horizon and the state vector, where $B(\boldsymbol{\theta}^\circ; \delta_3)$ is an open $2 + 2N + A$ – ball centered at the given value of the parameter vector $\boldsymbol{\theta}^\circ$ of radius $\delta_3 > 0$.

Problem (7) is a common form for optimal control problems to take in economics. It turns out to be relatively rare for the initial date or the initial state vector to be decision variables. We have therefore stuck with one of the principles enunciated in the preface, namely, that the theorems presented in the book will typically be those that are the most useful in economic applications of optimal control theory.

As demonstrated in the proof of Theorem 10.1, it is not only correct, but it is also very useful, to think of problem (7) as being solved in three distinct stages:

Stage 1: Treat $(t_1, \mathbf{x}_1)$ as fixed parameters in problem (7) and solve it as if it were a fixed endpoint and fixed-time control problem. Then construct its corresponding optimal value function. That is, literally solve problem (2) to find the vector $(\hat{\mathbf{z}}(t;\boldsymbol{\alpha},\boldsymbol{\gamma}), \hat{\mathbf{v}}(t;\boldsymbol{\alpha},\boldsymbol{\gamma}), \hat{\boldsymbol{\lambda}}(t;\boldsymbol{\alpha},\boldsymbol{\gamma}), \hat{\boldsymbol{\mu}}(t;\boldsymbol{\alpha},\boldsymbol{\gamma}))$, and then derive the associated value of the optimal value function $\hat{V}(\boldsymbol{\alpha},\boldsymbol{\gamma})$.

Stage 2: Find $(t_1, \mathbf{x}_1)$ by solving the ensuing *static* constrained optimization problem:

$$\bar{V}(\boldsymbol{\theta}) \stackrel{\text{def}}{=} \max_{t_1,\mathbf{x}_1}\{\hat{V}(\boldsymbol{\alpha},\boldsymbol{\gamma}) \text{ s.t. } t_1 \le T, \mathbf{x}_1 \ge \mathbf{x}_T\}. \tag{8}$$

This yields the solution $\bar{\boldsymbol{\omega}}(\boldsymbol{\theta}) \stackrel{\text{def}}{=} (\bar{t}_1(\boldsymbol{\theta}), \bar{\mathbf{x}}_1(\boldsymbol{\theta}))$ and value of the optimal value function $\bar{V}(\boldsymbol{\theta})$. In principle, $\bar{V}(\boldsymbol{\theta})$ is found by substituting the solution $\bar{\boldsymbol{\omega}}(\boldsymbol{\theta}) \stackrel{\text{def}}{=} (\bar{t}_1(\boldsymbol{\theta}), \bar{\mathbf{x}}_1(\boldsymbol{\theta}))$ into $\hat{V}(\boldsymbol{\alpha},\boldsymbol{\gamma})$, that is, by employing the identity $\bar{V}(\boldsymbol{\theta}) \equiv \hat{V}(\boldsymbol{\alpha}, t_0, \mathbf{x}_0, \bar{t}_1(\boldsymbol{\theta}), \bar{\mathbf{x}}_1(\boldsymbol{\theta}))$.

Stage 3: Derive the solution to problem (7) by substituting $\bar{\boldsymbol{\omega}}(\boldsymbol{\theta}) \stackrel{\text{def}}{=} (\bar{t}_1(\boldsymbol{\theta}), \bar{\mathbf{x}}_1(\boldsymbol{\theta}))$ into the quadruplet $(\hat{\mathbf{z}}(t;\boldsymbol{\alpha},\boldsymbol{\gamma}), \hat{\mathbf{v}}(t;\boldsymbol{\alpha},\boldsymbol{\gamma}), \hat{\boldsymbol{\lambda}}(t;\boldsymbol{\alpha},\boldsymbol{\gamma}), \hat{\boldsymbol{\mu}}(t;\boldsymbol{\alpha},\boldsymbol{\gamma}))$ from Stage 1, that is, use the identities

$$(\bar{\mathbf{z}}(t;\boldsymbol{\theta}), \bar{\mathbf{v}}(t;\boldsymbol{\theta})) \equiv (\hat{\mathbf{z}}(t;\boldsymbol{\alpha}, t_0, \mathbf{x}_0, \bar{\boldsymbol{\omega}}(\boldsymbol{\theta})), \hat{\mathbf{v}}(t;\boldsymbol{\alpha}, t_0, \mathbf{x}_0, \bar{\boldsymbol{\omega}}(\boldsymbol{\theta}))),$$

$$(\bar{\boldsymbol{\lambda}}(t;\boldsymbol{\theta}), \bar{\boldsymbol{\mu}}(t;\boldsymbol{\theta})) \equiv (\hat{\boldsymbol{\lambda}}(t;\boldsymbol{\alpha}, t_0, \mathbf{x}_0, \bar{\boldsymbol{\omega}}(\boldsymbol{\theta})), \hat{\boldsymbol{\mu}}(t;\boldsymbol{\alpha}, t_0, \mathbf{x}_0, \bar{\boldsymbol{\omega}}(\boldsymbol{\theta}))),$$

which hold for all $t \in [t_0, \bar{t}_1(\boldsymbol{\theta})]$. To see why the above identities must hold, we reason as follows. If the constraints in problem (8) [and thus problem (7)] are not binding, so that $\bar{t}_1(\boldsymbol{\theta}) < T$ and $\bar{\mathbf{x}}_1(\boldsymbol{\theta}) > \mathbf{x}_T$, then the solution to problem (2) when t_1 is held fixed at the value $\bar{t}_1(\boldsymbol{\theta})$ and $\mathbf{x}_1$ is held fixed at the value $\bar{\mathbf{x}}_1(\boldsymbol{\theta})$ is identical to the solution to problem (7), for in this instance, problems (2) and (7) are identical. On the other hand, if the constraints in problem (8) [and thus problem (7)] are binding, so that $\bar{t}_1(\boldsymbol{\theta}) = T$ and $\bar{\mathbf{x}}_1(\boldsymbol{\theta}) = \mathbf{x}_T$, then the solution to problem (2) when t_1 is held fixed at the value T and $\mathbf{x}_1$ is held fixed at the value $\mathbf{x}_T$ is identical to the solution to problem (7), for in this instance, problems (2) and (7) are again identical. Hence, the above identities hold whether the constraints in problem (7) bind or not.

Note that the identities in Stage 3 prove that the solution to problem (7) must satisfy the necessary conditions of Theorem 6.1, which as you may recall, apply to a fixed endpoint and fixed-time optimal control problem. To see this, recall that the quadruplet

$$(\hat{\mathbf{z}}(t;\boldsymbol{\alpha},t_0,\mathbf{x}_0,\bar{\boldsymbol{\omega}}(\boldsymbol{\theta})),\hat{\mathbf{v}}(t;\boldsymbol{\alpha},t_0,\mathbf{x}_0,\bar{\boldsymbol{\omega}}(\boldsymbol{\theta})),\hat{\boldsymbol{\lambda}}(t;\boldsymbol{\alpha},t_0,\mathbf{x}_0,\bar{\boldsymbol{\omega}}(\boldsymbol{\theta})),\hat{\boldsymbol{\mu}}(t;\boldsymbol{\alpha},t_0,\mathbf{x}_0,\bar{\boldsymbol{\omega}}(\boldsymbol{\theta})))$$

satisfies the necessary conditions of Theorem 6.1 because it solves the fixed endpoint and fixed-time problem (2) when $(t_1, \mathbf{x}_1) = (\bar{t}_1(\boldsymbol{\theta}), \bar{\mathbf{x}}_1(\boldsymbol{\theta}))$. Since the quadruplet $(\bar{\mathbf{z}}(t;\boldsymbol{\theta}), \bar{\mathbf{v}}(t;\boldsymbol{\theta}), \bar{\boldsymbol{\lambda}}(t;\boldsymbol{\theta}), \bar{\boldsymbol{\mu}}(t;\boldsymbol{\theta}))$ is identically equal to $(\hat{\mathbf{z}}(t;\boldsymbol{\alpha},t_0,\mathbf{x}_0,\bar{\boldsymbol{\omega}}(\boldsymbol{\theta})),$ $\hat{\mathbf{v}}(t;\boldsymbol{\alpha},t_0,\mathbf{x}_0,\bar{\boldsymbol{\omega}}(\boldsymbol{\theta})),$ $\hat{\boldsymbol{\lambda}}(t;\boldsymbol{\alpha},t_0,\mathbf{x}_0,\bar{\boldsymbol{\omega}}(\boldsymbol{\theta})),$ $\hat{\boldsymbol{\mu}}(t;\boldsymbol{\alpha},t_0,\mathbf{x}_0,\bar{\boldsymbol{\omega}}(\boldsymbol{\theta})))$, the quadruplet $(\bar{\mathbf{z}}(t;\boldsymbol{\theta}), \bar{\mathbf{v}}(t;\boldsymbol{\theta}), \bar{\boldsymbol{\lambda}}(t;\boldsymbol{\theta}), \bar{\boldsymbol{\mu}}(t;\boldsymbol{\theta}))$ must also satisfy the necessary conditions of Theorem 6.1. We are now in a position to state and prove the following theorem.

Theorem 10.2 (Inequality Constrained Transversality Conditions): *If $\hat{V}(\cdot) \in C^{(1)}\ \forall\, \boldsymbol{\beta} \in B(\boldsymbol{\beta}^{\circ};\delta_2)$, then in addition to the necessary conditions of Theorem 6.1, the following transversality conditions are necessary for the inequality-constrained variable endpoint and variable time optimal control problem (7):*

$$\left.\begin{array}{l} H(\bar{t}_1(\boldsymbol{\theta}), \bar{\mathbf{z}}(\bar{t}_1(\boldsymbol{\theta});\boldsymbol{\theta}), \bar{\mathbf{v}}(\bar{t}_1(\boldsymbol{\theta});\boldsymbol{\theta}), \bar{\boldsymbol{\lambda}}(\bar{t}_1(\boldsymbol{\theta});\boldsymbol{\theta});\boldsymbol{\alpha}) \geq 0 \\ T - \bar{t}_1(\boldsymbol{\theta}) \geq 0 \\ H(\bar{t}_1(\boldsymbol{\theta}), \bar{\mathbf{z}}(\bar{t}_1(\boldsymbol{\theta});\boldsymbol{\theta}), \bar{\mathbf{v}}(\bar{t}_1(\boldsymbol{\theta});\boldsymbol{\theta}), \bar{\boldsymbol{\lambda}}(\bar{t}_1(\boldsymbol{\theta});\boldsymbol{\theta});\boldsymbol{\alpha})\,[T - \bar{t}_1(\boldsymbol{\theta})] = 0 \end{array}\right\}[t_1 \leq T], \tag{9}$$

$$\left.\begin{array}{l} \bar{\lambda}_j(\bar{t}_1(\boldsymbol{\theta});\boldsymbol{\theta}) \geq 0, \quad j = 1, 2, \ldots, N \\ \bar{z}_j(\bar{t}_1(\boldsymbol{\theta});\boldsymbol{\theta}) \geq x_{Tj}, \quad j = 1, 2, \ldots, N \\ \bar{\lambda}_j(\bar{t}_1(\boldsymbol{\theta});\boldsymbol{\theta})\,[\bar{z}_j(\bar{t}_1(\boldsymbol{\theta});\boldsymbol{\theta}) - x_{Tj}] = 0, \quad j = 1, 2, \ldots, N \end{array}\right\}[\mathbf{x}_1 \geq \mathbf{x}_T]. \tag{10}$$

Proof: Recall the static constrained optimization problem defined in Stage 2 above, namely,

$$\bar{V}(\boldsymbol{\theta}) \stackrel{\text{def}}{=} \max_{t_1,\mathbf{x}_1}\{\hat{V}(\boldsymbol{\alpha},\boldsymbol{\gamma}) \text{ s.t. } t_1 \leq T, \mathbf{x}_1 \geq \mathbf{x}_T\},$$

and form the Lagrangian function associated with it:

$$L(t_1,\mathbf{x}_1,\chi,\boldsymbol{\varphi}) \stackrel{\text{def}}{=} \hat{V}(\boldsymbol{\alpha},t_0,\mathbf{x}_0,t_1,\mathbf{x}_1) + \chi[T - t_1] + \sum_{n=1}^{N} \varphi_n[x_{1n} - x_{Tn}],$$

where $(\chi, \boldsymbol{\varphi})$ are the Lagrange multipliers for the $N+1$ inequality constraints of the problem. The necessary conditions for the optimal choice of $(t_1, \mathbf{x}_1)$ are given

https://doi.org/10.1017/CBO9780511806827.012 Published online by Cambridge University Press

by Theorem 18.4 of Simon and Blume (1994):

$$L_{t_1}(t_1, \mathbf{x}_1, \chi, \boldsymbol{\varphi}) = \hat{V}_{t_1}(\boldsymbol{\alpha}, t_0, \mathbf{x}_0, t_1, \mathbf{x}_1) - \chi = 0, \tag{11}$$

$$L_{x_{1j}}(t_1, \mathbf{x}_1, \chi, \boldsymbol{\varphi}) = \hat{V}_{x_{1j}}(\boldsymbol{\alpha}, t_0, \mathbf{x}_0, t_1, \mathbf{x}_1) + \varphi_j = 0, \quad j = 1, 2, \ldots, N, \tag{12}$$

$$L_{\chi}(t_1, \mathbf{x}_1, \chi, \boldsymbol{\varphi}) = T - t_1 \geq 0, \chi \geq 0, [T - t_1]\chi = 0, \tag{13}$$

$$L_{\varphi_j}(t_1, \mathbf{x}_1, \chi, \boldsymbol{\varphi}) = x_{1j} - x_{Tj} \geq 0, \varphi_j \geq 0, [x_{1j} - x_{Tj}]\varphi_j = 0, \quad j = 1, 2, \ldots, N. \tag{14}$$

These necessary conditions hold at $(t_1, \mathbf{x}_1) = (\bar{t}_1(\boldsymbol{\theta}), \bar{\mathbf{x}}_1(\boldsymbol{\theta}))$ by assumption. Now use part (iv) of Theorem 9.3 (the dynamic envelope theorem) and combine Eqs. (11) and (13) to derive

$$\begin{gathered} H(\bar{t}_1(\boldsymbol{\theta}), \hat{\mathbf{z}}(\bar{t}_1(\boldsymbol{\theta}); \boldsymbol{\alpha}, t_0, \mathbf{x}_0, \bar{\boldsymbol{\omega}}(\boldsymbol{\theta})), \hat{\mathbf{v}}(\bar{t}_1(\boldsymbol{\theta}); \boldsymbol{\alpha}, t_0, \mathbf{x}_0, \bar{\boldsymbol{\omega}}(\boldsymbol{\theta})), \\ \hat{\boldsymbol{\lambda}}(\bar{t}_1(\boldsymbol{\theta}); \boldsymbol{\alpha}, t_0, \mathbf{x}_0, \bar{\boldsymbol{\omega}}(\boldsymbol{\theta})); \boldsymbol{\alpha}) \geq 0, T - \bar{t}_1(\boldsymbol{\theta}) \geq 0, \\ H(\bar{t}_1(\boldsymbol{\theta}), \hat{\mathbf{z}}(\bar{t}_1(\boldsymbol{\theta}); \boldsymbol{\alpha}, t_0, \mathbf{x}_0, \bar{\boldsymbol{\omega}}(\boldsymbol{\theta})), \hat{\mathbf{v}}(\bar{t}_1(\boldsymbol{\theta}); \boldsymbol{\alpha}, t_0, \mathbf{x}_0, \bar{\boldsymbol{\omega}}(\boldsymbol{\theta})), \\ \hat{\boldsymbol{\lambda}}(\bar{t}_1(\boldsymbol{\theta}); \boldsymbol{\alpha}, t_0, \mathbf{x}_0, \bar{\boldsymbol{\omega}}(\boldsymbol{\theta})); \boldsymbol{\alpha})\,[T - \bar{t}_1(\boldsymbol{\theta})] = 0. \end{gathered}$$

Note that the above derivatives are evaluated along the triplet $(\hat{\mathbf{z}}(t; \boldsymbol{\alpha}, \boldsymbol{\gamma}), \hat{\mathbf{v}}(t; \boldsymbol{\alpha}, \boldsymbol{\gamma}), \hat{\boldsymbol{\lambda}}(t; \boldsymbol{\alpha}, \boldsymbol{\gamma}))$, seeing as it corresponds to the optimal value function $\hat{V}(\cdot)$ being differentiated. Next, use part (v) of Theorem 9.3 and combine Eqs. (12) and (14) to derive

$$\begin{gathered} \hat{\lambda}_j(\bar{t}_1(\boldsymbol{\theta}); \boldsymbol{\alpha}, t_0, \mathbf{x}_0, \bar{\boldsymbol{\omega}}(\boldsymbol{\theta})) \geq 0, \bar{x}_{1j}(\boldsymbol{\theta}) - x_{Tj} \geq 0, \\ \hat{\lambda}_j(\bar{t}_1(\boldsymbol{\theta}); \boldsymbol{\alpha}, t_0, \mathbf{x}_0, \bar{\boldsymbol{\omega}}(\boldsymbol{\theta}))\,[\bar{x}_{1j}(\boldsymbol{\theta}) - x_{Tj}] = 0, \end{gathered}$$

for $j = 1, 2, \ldots, N$. Using the identities of Stage 3 completes the proof. Q.E.D.

It should be clear that Theorem 10.2 is a generalization of Theorem 10.1. For example, if $\bar{t}_1(\boldsymbol{\theta}) < T$, implying that the terminal time constraint is not binding in problem (7), then Eq. (9) of Theorem 10.2 implies that value of the Hamiltonian vanishes at the terminal time, which is Eq. (5) of Theorem 10.1. Likewise, if $\bar{\mathbf{x}}_1(\boldsymbol{\theta}) > \mathbf{x}_T$, so that the terminal state constraint is not binding in problem (7), then Eq. (10) of Theorem 10.2 implies that the value of the costate vector is zero at the terminal time, which is Eq. (6) of Theorem 10.1. In other words, if the constraint is not binding, we are in the variable endpoint and variable time world, so Theorem 10.1 applies.

Conversely, if the value of the Hamiltonian is positive at the terminal date, then Eq. (9) of Theorem 10.2 implies that the terminal time constraint is binding in an optimal plan. Similarly, if the value of the costate vector is positive at the terminal date, then Eq. (10) of Theorem 10.2 implies that the terminal state constraint is binding in an optimal plan. Hence, Theorem 10.2 is just a *complementary slackness* theorem for inequality-constrained variable endpoints and variable time control problems, not unlike that encountered in linear and nonlinear programming.

In passing, observe that if a subset of the elements of the vector $\boldsymbol{\gamma} \stackrel{\text{def}}{=} (t_0, \mathbf{x}_0, t_1, \mathbf{x}_1)$ are decision variables in problem (1), say, $(t_1, \mathbf{x}_1)$, so that we are dealing with a variable terminal endpoint and terminal time optimal control problem, then only a subset of the necessary conditions of Theorem 10.1 apply, videlicet, Eqs. (5) and (6) in this instance. The veracity of this claim follows from the fact that the first-order necessary conditions are identical for the choice of $(t_1, \mathbf{x}_1)$ regardless of whether $(t_0, \mathbf{x}_0)$ are held parametrically fixed or are decision variables. The values of $(t_1, \mathbf{x}_1)$, however, differ, in general, depending on whether $(t_0, \mathbf{x}_0)$ are held parametrically fixed or are decision variables. This observation obviously carries over for any subset of the elements of the vector $\gamma \stackrel{\text{def}}{=} (t_0, \mathbf{x}_0, t_1, \mathbf{x}_1)$ that are choice or decision variables.

We now pause to illustrate how Theorem 10.2 is often used in economics. Note that we will provide an economic interpretation of the resulting necessary conditions, but will not actually solve them for an explicit solution.

Example 10.1: Let's reconsider the Hotelling model of a resource extracting firm, but now allow the firm to choose how much of the natural resource to leave in the ground when it decides to terminate its mining operation. Assume that the mining firm is a price taker in both the input and output markets. The firm is asserted to solve the static cost minimization problem

$$C(x, q; w) \stackrel{\text{def}}{=} \min_{L} \{w L \text{ s.t. } q = f(x, L)\}$$

at each point in the planning horizon, where $L > 0$ is the variable input used in extracting the resource from the ground, $w > 0$ is the unit price of the variable input, q is the extraction rate of the resource from the ground (or the output from the mine), x is the stock of the natural resource in the ground, $C(\cdot)$ is the minimum cost function, and $f(\cdot) \in C^{(2)}$ is the production function, assumed to have the typical properties, scilicet,

$$f_x(x, L) > 0,\ f_L(x, L) > 0,\ f_{xL}(x, L) \geq 0,$$

$$f_{xx}(x, L) < 0,\ f_{LL}(x, L) < 0,\ f_{xx}(x, L) f_{LL}(x, L) - [f_{xL}(x, L)]^2 > 0.$$

Given these properties for $f(\cdot)$, you will show in a mental exercise that $C(\cdot)$ satisfies

$$C \in C^{(2)},\ C_x(x, q; w) < 0,\ C_q(x, q; w) > 0,\ C_{qx}(x, q; w) < 0,$$

$$C_{xx}(x, q; w) > 0,\ C_{qq}(x, q; w) > 0,\ C_{xx}(x, q; w) C_{qq}(x, q; w) - [C_{qx}(x, q; w)]^2 > 0.$$

The optimal control problem can therefore be stated as

$$V(p, r, w, x_0) \stackrel{\text{def}}{=} \max_{q(\cdot), T, x_T} \int_0^T [p q(t) - C(x(t), q(t); w)] \, e^{-rt} \, dt$$

$$\text{s.t.} \quad \dot{x}(t) = -q(t),\ x(0) = x_0,\ x(T) = x_T \geq 0,$$

$$q(t) \in U \stackrel{\text{def}}{=} \{q(\cdot) : q(t) \geq 0\},$$

where $p > 0$ is the constant output price and $r > 0$ is the discount rate. Note that we are implicitly assuming that the firm wants to be in the extraction business, in that we are presuming that the optimal T satisfies $T > 0$. This allows us to ignore the natural inequality constraint $T \geq 0$. You are asked to investigate the consequences of this constraint in a mental exercise. Notice that the optimal value function $V(\cdot)$ does not depend on (T, x_T), because they are decision variables in the present formulation of the mining problem.

To solve this problem, first form the Hamiltonian $H(t, x, q, \lambda) \stackrel{\text{def}}{=} [pq - C(x, q; w)]\, e^{-rt} - \lambda q$. Applying Theorem 10.2 yields the necessary conditions

$$H_q(t, x, q, \lambda) = [p - C_q(x, q; w)]\, e^{-rt} - \lambda \leq 0,\ \ q \geq 0,$$

$$\big[[p - C_q(x, q; w)]\, e^{-rt} - \lambda\big]\, q = 0,$$

$$\dot{x} = H_\lambda(t, x, q, \lambda) = -q,\ \ x(0) = x_0,$$

$$\dot{\lambda} = -H_x(t, x, q, \lambda) = C_x(x, q; w)\, e^{-rt},$$

$$H(T, x(T), q(T), \lambda(T)) \stackrel{\text{def}}{=} [pq(T) - C(x(T), q(T); w)]\, e^{-rt} - \lambda(T)q(T) = 0,$$

$$\lambda(T) \geq 0,\ \ x(T) \geq 0,\ \ \lambda(T)x(T) = 0.$$

Recall that the necessary conditions of Theorem 6.1 are part of those from Theorem 10.2. Let's now turn to an economic interpretation of these necessary conditions.

To begin, first note that $\lambda(t)$ is the *present value* shadow price of the unextracted resource at time t, since the optimal value function of the mining firm represents the maximum value of profits from its optimal mining plan discounted to time zero, the initial period of the planning horizon. If $q(t) > 0$ at any moment $t \in [0, T]$, then the necessary condition $H_q q = 0$ implies that $H_q = 0$, or equivalently, that $p - C_q(x, q; w) = \lambda e^{rt} \stackrel{\text{def}}{=} \mu$, where μ is the *current value* shadow price of the unextracted natural resource. This equation asserts that if at any moment in time it is optimal to extract the resource at a positive rate, then marginal profit must equal the current value shadow price of a unit of the resource in situ, that is, the resource in its natural environment. Thus the current value shadow price μ of the unextracted natural resource represents the amount by which the price of the extracted nonrenewable resource exceeds its marginal cost of extraction. That μ is positive follows intuitively from the observation that the stock of the unextracted natural resource is a good to the mining firm. In a mental exercise, you are asked to show that this intuition can be confirmed mathematically.

Turning to the costate equation $\dot{\lambda} = C_x(x, q; w)\, e^{-rt}$ and recalling that $C_x(x, q; w) < 0$, we see that the present value shadow price of the unextracted resource falls over the planning horizon. This, however, does not imply that the current value shadow price μ falls over the planning horizon. To see this, recall that $\mu \stackrel{\text{def}}{=} \lambda e^{rt}$, so that upon using the costate equation, we find that $\dot{\mu} = r\lambda e^{rt} + \dot{\lambda} e^{rt} = r\mu +$

$C_x(x, q; w)$. Because $C_x(x, q; w) < 0$ and $\mu > 0$, it is clear that the current value shadow price μ may be rising or falling over the planning horizon. The state equation implies that the rate of change of the stock can't be positive, since $\dot{x} = -q$ and $q \geq 0$. This is intuitive too, in view of the fact that without discovery of new resource deposits (which we have implicitly assumed to be zero), the stock of the resource in the ground cannot increase over time.

Now consider the transversality condition for the terminal stock of the resource. If the firm finds it optimal to leave some of the resource stock in the ground as the horizon comes to a close, that is, if $x(T) = x_T > 0$, then $\lambda(T)x(T) = 0$ implies that the present value shadow price of the stock in situ is zero at the terminal date of the planning horizon, that is, $\lambda(T) = 0$. Intuitively, this result says that if the firm decides to leave some of the resource in the ground as the planning horizon comes to a close, then they must not place any value on having another unit in situ, because they didn't find it optimal to extract all of what was there to begin with. Given that $C_x(x, q; w) < 0$ and $C_{qx}(x, q; w) < 0$, that is, because extraction costs and marginal extraction costs rise as the stock is depleted, this is likely to be the outcome in the model. Conversely, if the firm places a positive present value shadow price on the stock in situ at the terminal date, that is, if $\lambda(T) > 0$, then $\lambda(T)x(T) = 0$ implies that the entire stock of the resource will be extracted from the ground as the horizon comes to a close, that is, $x(T) = 0$. This makes economic sense too, for if the firm finds it optimal to extract all of the resource from its environment by the end of the planning period, then it would surely place a positive shadow value on another unit of the resource in situ.

Finally, turn to the transversality condition for the choice of the planning horizon, namely, $[pq(T) - C(x(T), q(T); w)]\, e^{-rt} - \lambda(T)q(T) = 0$. If we continue to assume that $x(T) = x_T > 0$ in an optimal plan, then $\lambda(T) = 0$, as noted above. In this instance, the transversality condition reduces to the simpler form

$$pq(T) - C(x(T), q(T); w) = 0.$$

This necessary condition asserts that if the firm finds it optimal to leave some of the resource stock in the ground at the terminal date of the planning horizon, then the optimal terminal date should be chosen such that total revenue equals total cost, or equivalently, such that profit is zero. Furthermore, if in addition, the firm finds it optimal to extract in the final period, that is, $q(T) > 0$, then $H_q(T, x(T), q(T), \lambda(T)) = 0$, as noted above. Upon combining the previous simplified transversality condition with this latter one, we have

$$p = C_q(x(T), q(T); w),$$

$$p = \frac{C(x(T), q(T); w)}{q(T)},$$

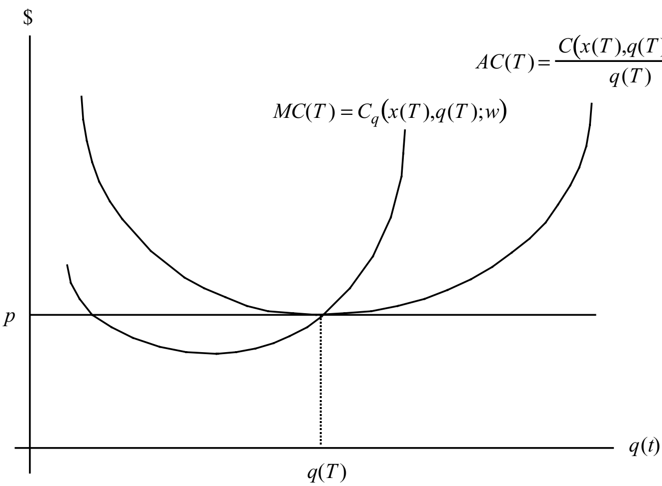

Figure 10.1

as the form of the two transversality conditions under the stated assumptions. Equating the previous two transversality conditions yields

$$MC(T) \overset{\text{def}}{=} C_q(x(T), q(T); w) = \frac{C(x(T), q(T); w)}{q(T)} \overset{\text{def}}{=} AC(T).$$

To sum up, we have shown that if $x(T) = x_T > 0$ and $q(T) > 0$ in an optimal plan, then at the optimal terminal date of the firm's planning horizon, its profit is zero and its marginal cost of production equals its average cost of production. Therefore, in the final period of its planning horizon under the given assumptions, the mining firm acts just like a prototypical static profit maximizing price-taking firm in long-run competitive industry equilibrium. Figure 10.1 presents a graphical representation of these conclusions.

Notice that because $MC(T) = AC(T)$, the mining firm must be at the minimum of $AC(T)$. Moreover, for the minimum of $AC(T)$ to occur at some $q(T) > 0$, the $AC(T)$ curve must be U-shaped, which can only occur if the mining firm faces some fixed cost of extraction. Finally, note that there are other configurations for the values of the resource stock, the extraction rate, and the current value shadow price of the stock that may occur in the terminal period of the planning horizon. You are asked to explore some of these in a mental exercise.

Before considering a more general set of transversality conditions, let's examine another economic problem and actually use the transversality conditions to find the explicit solution of the necessary conditions. The ubiquitous inventory accumulation problem turns out to be an excellent problem to which to apply

Theorem 10.1 for the reason that we can solve the necessary conditions for an explicit solution.

Example 10.2: A simple but rich variant of the workhorse inventory accumulation problem has the delivery date as a decision variable:

$$C(c_1, c_2, x_T) \stackrel{\text{def}}{=} \min_{u(\cdot),T} \int_0^T \left[c_1[u(t)]^2 + c_2 x(t)\right] dt$$
$$\text{s.t.} \quad \dot{x}(t) = u(t),\ x(0) = 0,\ x(T) = x_T.$$

Let's ignore the nonnegativity constraint on the production rate to begin with. If it turns out that the production rate is negative for some interval of time, we will then go back and explicitly include it in the control problem. Note that because T is a choice variable in this version of the problem, it does not appear as an argument of the minimum cost function $C(\cdot)$.

The Hamiltonian is given by $H(x, u, \lambda) \stackrel{\text{def}}{=} c_1 u^2 + c_2 x + \lambda u$, so that by Theorem 10.1, the necessary conditions are

$$H_u(x, u, \lambda) = 2c_1 u + \lambda = 0,$$
$$\dot{\lambda} = -H_x(x, u, \lambda) = -c_2,$$
$$\dot{x} = H_\lambda(x, u, \lambda) = u,\ x(0) = 0,\ x(T) = x_T,$$
$$H(x(T), u(T), \lambda(T)) = c_1[u(T)]^2 + c_2 x(T) + \lambda(T)u(T) = 0.$$

Solving $H_u(x, u, \lambda) = 2c_1 u + \lambda = 0$ for the control gives $u = -\lambda/2c_1$. The solution to the costate equation is easily found to be $\lambda(t) = -c_2 t + k_1$, where k_1 is a constant of integration. The general solution for the production rate is then given by $v(t) = c_2 t/2c_1 - k_1/2c_1$. Substituting this into the state equation yields $\dot{x} = c_2 t/2c_1 - k_1/2c_1$, which, upon separating the variables and integrating, gives the general solution $z(t) = c_2 t^2/4c_1 - k_1 t/2c_1 + k_2$, where k_2 is another constant of integration. Notice our strategy: we solve the necessary conditions, treating T as if it was a known constant, after which we solve for the constants of integration and the delivery date T.

The three equations we will use to find the three unknowns (k_1, k_2, T) are the initial condition $x(0) = 0$, the terminal endpoint condition $x(T) = x_T$, and the free terminal-time transversality condition $H(z(T), v(T), \lambda(T)) = 0$. First of all, it is easy to see that the initial condition $x(0) = 0$ implies that $k_2 = 0$. Next, substitute the general solution of the necessary conditions into the free terminal-time transversality condition $H(z(T), v(T), \lambda(T)) = 0$, and then simplify the result to get the equation

$$-\frac{1}{4c_1}k_1^2 + k_2 = 0.$$

Given that $k_2 = 0$, it follows from the above equation that $k_1 = 0$ too. Finally, apply the (fixed) terminal endpoint condition $x(T) = x_T$ to the general solution $z(t) = c_2 t^2/4c_1 - k_1 t/2c_1 + k_2$ of the state equation to arrive at

$$\frac{c_2 T^2}{4c_1} - \frac{k_1 T}{2c_1} + k_2 = x_T.$$

Because $k_1 = 0$ and $k_2 = 0$, this equation simplifies to $T^2 = 4c_1 x_T/c_2$. Clearly, there are two values of T that satisfy this equation, namely, $T = \pm 2\sqrt{c_1 x_T/c_2}$. For the control problem to make economic sense, we take the positive square root; thus $T^*(c_1, c_2, x_T) = 2\sqrt{c_1 x_T/c_2} > 0$. Recalling that $k_1 = 0$ and $k_2 = 0$, the specific solution of the necessary conditions is given by

$$\begin{aligned}
T^*(c_1, c_2, x_T) &= 2\sqrt{c_1 x_T/c_2} > 0, \\
z^*(t; c_1, c_2) &= \frac{1}{4}c_1^{-1}c_2 t^2 \geq 0\,\forall\, t \in [0, T^*(c_1, c_2, x_T)], \\
v^*(t; c_1, c_2) &= \frac{1}{2}c_1^{-1}c_2 t \geq 0\,\forall\, t \in [0, T^*(c_1, c_2, x_T)], \\
\lambda^*(t; c_2) &= -c_2 t \leq 0\,\forall\, t \in [0, T^*(c_1, c_2, x_T)].
\end{aligned}$$

Take note of the fact that the shadow cost of the inventory stock $\lambda^*(t; c_2) = -c_2 t$ is less than or equal to zero for the entire planning horizon. You are asked in a mental exercise to provide the economic intuition for this result. Also observe that the production rate is nonnegative throughout the planning horizon, so our strategy of ignoring the nonnegativity constraint on the production rate was without problems.

Let's now investigate the comparative dynamic properties of the solution to the necessary conditions of the above inventory accumulation problem, and compare them with those derived from the fixed time horizon version of the model given in Example 4.5 and Mental Exercise 4.15. To this end, differentiate the above solution of the necessary conditions with respect to the size of the order x_T to get

$$\begin{aligned}
\frac{\partial T^*}{\partial x_T}(c_1, c_2, x_T) &= \left[c_1 c_2^{-1} x_T\right]^{-\frac{1}{2}} c_1 c_2^{-1} > 0, \\
\frac{\partial z^*}{\partial x_T}(t; c_1, c_2) = \frac{\partial v^*}{\partial x_T}(t; c_1, c_2) &= \frac{\partial \lambda^*}{\partial x_T}(t; c_2) \equiv 0.
\end{aligned}$$

The second set of equations implies that an increase in the size of the order doesn't affect the inventory level, the production rate, or the shadow value of the inventory. Hence the way the firm elects to meet the larger order is to produce over a longer period. This result contrasts sharply with that when the delivery date is fixed (see Mental Exercise 4.15), in which case, all the adjustments in the plan come from increasing the production rate. Moreover, by the dynamic envelope theorem,

$$\frac{\partial C}{\partial x_T}(c_1, c_2, x_T) = -\lambda^*(T^*(c_1, c_2, x_T); c_2) > 0.$$

Thus, not surprisingly, an increase in the size of the order drives up minimum production costs. This is true whether the horizon is fixed or a choice variable.

Now differentiate the solution of the necessary conditions with respect to the unit cost of holding inventory c_2:

$$\begin{aligned}
\frac{\partial T^*}{\partial c_2}(c_1, c_2, x_T) &= -\left[c_1 c_2^{-1} x_T\right]^{-\frac{1}{2}} c_1 c_2^{-2} x_T < 0, \\
\frac{\partial z^*}{\partial c_2}(t; c_1, c_2) &= \frac{1}{4} c_1^{-1} t^2 \geq 0 \,\forall\, t \in [0, T^*(c_1, c_2, x_T)], \\
\frac{\partial v^*}{\partial c_2}(t; c_1, c_2) &= \frac{1}{2} c_1^{-1} t \geq 0 \,\forall\, t \in [0, T^*(c_1, c_2, x_T)], \\
\frac{\partial \lambda^*}{\partial c_2}(t; c_2) &= -t \leq 0 \,\forall\, t \in [0, T^*(c_1, c_2, x_T)].
\end{aligned}$$

The economic interpretation here is that an increase in the unit holding cost of inventory results in the firm shortening the delivery date, as holding the goods in inventory now costs more. Given the shorter planning period and the fact that the same amount of the good is still required to be produced, the higher unit holding cost results in a higher production rate and thus a higher inventory level at each moment in the planning horizon, except for the initial date, when neither is affected. In addition, the shadow value of the inventory falls. In contrast, when the planning horizon is fixed, Example 4.5 showed that the increase in the unit holding cost of inventory caused the production rate to decrease and the shadow value of the inventory to increase in the first half of the planning horizon, and the production rate to increase and the shadow value of the inventory to decrease in the second half of the planning horizon, such that total production was unchanged. Moreover, by the dynamic envelope theorem

$$\frac{\partial C}{\partial c_2}(c_1, c_2, x_T) = \int_0^T \frac{\partial H}{\partial c_2}(x, u, \lambda)\bigg|_{\substack{\text{optimal}\\\text{solution}}} dt = \int_0^{T^*(c_1, c_2, x_T)} z^*(t; c_1, c_2)\, dt > 0.$$

This calculation confirms the intuitive conclusion that the higher holding cost of inventory also drives up the firm's minimum cost of production.

https://doi.org/10.1017/CBO9780511806827.012 Published online by Cambridge University Press

Finally, differentiate the solution of the necessary conditions with respect to the production cost coefficient c_1:

$$\frac{\partial T^*}{\partial c_1}(c_1, c_2, x_T) = \left[c_1 c_2^{-1} x_T\right]^{-\frac{1}{2}} c_2^{-1} x_T > 0,$$

$$\frac{\partial z^*}{\partial c_1}(t; c_1, c_2) = -\frac{1}{4} c_2 c_1^{-2} t^2 \leq 0 \, \forall \, t \in [0, T^*(c_1, c_2, x_T)],$$

$$\frac{\partial v^*}{\partial c_1}(t; c_1, c_2) = -\frac{1}{2} c_2 c_1^{-2} t \leq 0 \, \forall \, t \in [0, T^*(c_1, c_2, x_T)],$$

$$\frac{\partial \lambda^*}{\partial c_1}(t; c_2) \equiv 0.$$

These comparative dynamic results also have a straightforward economic interpretation. With higher production costs, the firm spreads them out over a longer period. Given the longer production period and the fact that the same number of units are required to be produced, the firm will produce them at a slower rate and carry less inventory at each moment of the planning horizon, thereby resulting in no effect on the shadow value of the inventory. In contrast, with the planning horizon fixed, Mental Exercise 4.15 showed that an increase in the production cost coefficient increased the production rate in the first half of the planning horizon and decreased it in the second half of the planning horizon, with the shadow value of the inventory being lower throughout. Regardless of whether T is fixed or a decision variable, however, the firm's minimum cost of production will increase, since

$$\frac{\partial C}{\partial c_1}(c_1, c_2, x_T) = \int_0^T \frac{\partial H}{\partial c_1}(x, u, \lambda)\bigg|_{\substack{\text{optimal} \\ \text{solution}}} dt = \int_0^{T^*(c_1, c_2, x_T)} [v^*(t; c_1, c_2)]^2 \, dt > 0$$

by the dynamic envelope theorem.

In Example 10.2, we applied the dynamic envelope theorem, that is, Theorem 9.1, to it to discern the effects of various parameters on the firm's minimum cost of production. Recall that the dynamic envelope theorems proven in Chapter 9 were established for a fixed-time and fixed endpoints optimal control problem. In the production planning problem of Example 10.2, however, the terminal time was a decision variable. Is it correct, therefore, to apply Theorem 9.1 to an optimal control problem that is not, strictly speaking, covered by it? The answer is yes, as we now proceed to demonstrate. More generally, we will show that the pertinent formulas of Theorems 9.1 and 9.3 remain valid when any subset of the $2n + 2$ element vector $(t_0, \mathbf{x}_0, t_1, \mathbf{x}_1)$ are decision variables. This claim follows directly from the Stage 2 argument underlying the proof of the transversality condition and the prototype static envelope theorem.

To see this, consider, for example, a control problem that has $(t_1, \mathbf{x}_1)$ as decision variables. Then the Stage 2 problem to find the relevant transversality conditions is given by

$$\breve{V}(\boldsymbol{\alpha}, t_0, \mathbf{x}_0) \stackrel{\text{def}}{=} \max_{t_1, \mathbf{x}_1} \hat{V}(\boldsymbol{\alpha}, t_0, \mathbf{x}_0, t_1, \mathbf{x}_1),$$

where $\breve{V}(\cdot)$ is the optimal value function for the control problem with $(t_1, \mathbf{x}_1)$ as decision variables and $\hat{V}(\cdot)$ is the optimal value function for control problem (2), where $(t_0, \mathbf{x}_0, t_1, \mathbf{x}_1)$ are fixed. An application of the prototype envelope theorem to the above unconstrained static optimization problem relating $\breve{V}(\cdot)$ to $\hat{V}(\cdot)$ yields

$$\breve{V}_{\boldsymbol{\alpha}}(\boldsymbol{\alpha}, t_0, \mathbf{x}_0) = \hat{V}_{\boldsymbol{\alpha}}(\boldsymbol{\alpha}, t_0, \mathbf{x}_0, t_1, \mathbf{x}_1)\Big|_{\substack{\text{optimal}\\ \text{solution}}},$$

$$\breve{V}_{t_0}(\boldsymbol{\alpha}, t_0, \mathbf{x}_0) = \hat{V}_{t_0}(\boldsymbol{\alpha}, t_0, \mathbf{x}_0, t_1, \mathbf{x}_1)\Big|_{\substack{\text{optimal}\\ \text{solution}}},$$

$$\breve{V}_{\mathbf{x}_0}(\boldsymbol{\alpha}, t_0, \mathbf{x}_0) = \hat{V}_{\mathbf{x}_0}(\boldsymbol{\alpha}, t_0, \mathbf{x}_0, t_1, \mathbf{x}_1)\Big|_{\substack{\text{optimal}\\ \text{solution}}}.$$

This proves that the envelope results for the parameters $(\boldsymbol{\alpha}, t_0, \mathbf{x}_0)$ of the optimal control problem with $(t_1, \mathbf{x}_1)$ as decision variables are *identical* to the envelope results for the fixed-time and fixed endpoints optimal control problems of Theorems 9.1 and 9.3, as claimed.

Before turning to sufficient conditions, let's examine the transversality conditions for one more economically important class of optimal control problems. Specifically, the last generalization of importance for economic problems is the variable endpoint and variable time *salvage value* or *scrap value* class of control problems that we encountered earlier. This class of control problems is given by

$$\tilde{V}(\boldsymbol{\alpha}) \stackrel{\text{def}}{=} \max_{\mathbf{u}(\cdot), t_0, \mathbf{x}_0, t_1, \mathbf{x}_1} \left\{ \int_{t_0}^{t_1} f(t, \mathbf{x}(t), \mathbf{u}(t); \boldsymbol{\alpha})\, dt + S(t_0, \mathbf{x}_0, t_1, \mathbf{x}_1; \boldsymbol{\alpha}) \right\} \tag{15}$$

$$\text{s.t.} \quad \dot{\mathbf{x}}(t) = \mathbf{g}(t, \mathbf{x}(t), \mathbf{u}(t); \boldsymbol{\alpha}),\ \ \mathbf{x}(t_0) = \mathbf{x}_0,\ \ \mathbf{x}(t_1) = \mathbf{x}_1,$$

$$h^k(t, \mathbf{x}(t), \mathbf{u}(t); \boldsymbol{\alpha}) \geq 0, \quad k = 1, 2, \ldots, K',$$

$$h^k(t, \mathbf{x}(t), \mathbf{u}(t); \boldsymbol{\alpha}) = 0, \quad k = K'+1, K'+2, \ldots, K,$$

where $S(\cdot) \in C^{(1)}$ is the salvage value function. Assume that an optimal solution $(\tilde{\mathbf{z}}(t; \boldsymbol{\alpha}), \tilde{\mathbf{v}}(t; \boldsymbol{\alpha}))$ exists to problem (15) for all $\boldsymbol{\alpha} \in B(\boldsymbol{\alpha}^{\circ}; \delta_4)$, with corresponding costate vector $\tilde{\boldsymbol{\lambda}}(t; \boldsymbol{\alpha})$ and Lagrange multiplier vector $\tilde{\boldsymbol{\mu}}(t; \boldsymbol{\alpha})$, and let $\tilde{\boldsymbol{\gamma}}(\boldsymbol{\alpha}) \stackrel{\text{def}}{=} (\tilde{t}_0(\boldsymbol{\alpha}), \tilde{\mathbf{x}}_0(\boldsymbol{\alpha}), \tilde{t}_1(\boldsymbol{\alpha}), \tilde{\mathbf{x}}_1(\boldsymbol{\alpha}))$ be the optimal solution for the initial and terminal values of the horizon and state vector, where $B(\boldsymbol{\alpha}^{\circ}; \delta_4)$ is an open A – ball centered at the given value of the parameter vector $\boldsymbol{\alpha}^{\circ}$ of radius $\delta_4 > 0$. Because the initial and terminal dates and states are optimally chosen, the optimal value function and solution functions for problem (15) depend only on the parameter vector $\boldsymbol{\alpha}$. Given the above definitions, it should be clear that we have the identities $\tilde{\mathbf{x}}_0(\boldsymbol{\alpha}) \equiv \tilde{\mathbf{z}}(\tilde{t}_0(\boldsymbol{\alpha}); \boldsymbol{\alpha})$ and $\tilde{\mathbf{x}}_1(\boldsymbol{\alpha}) \equiv \tilde{\mathbf{z}}(\tilde{t}_1(\boldsymbol{\alpha}); \boldsymbol{\alpha})$.

The ensuing theorem is not unexpected at this point if you have been following and understanding the prior development of the transversality conditions. Consequently, its proof is left for a mental exercise.

Theorem 10.3 (Scrap Value Transversality Conditions): *If $\hat{V}(\cdot) \in C^{(1)}\ \forall\, \boldsymbol{\beta} \in B(\boldsymbol{\beta}^{\circ}; \delta_2)$, then in addition to the necessary conditions of Theorem 6.1, the following transversality conditions are necessary for the variable endpoint and variable time scrap value optimal control problem (15):*

$$H(\tilde{t}_0(\boldsymbol{\alpha}), \tilde{\mathbf{z}}(\tilde{t}_0(\boldsymbol{\alpha}); \boldsymbol{\alpha}), \tilde{\mathbf{v}}(\tilde{t}_0(\boldsymbol{\alpha}); \boldsymbol{\alpha}), \tilde{\boldsymbol{\lambda}}(\tilde{t}_0(\boldsymbol{\alpha}); \boldsymbol{\alpha}); \boldsymbol{\alpha}) = \frac{\partial S}{\partial t_0}(\tilde{t}_0(\boldsymbol{\alpha}), \tilde{\mathbf{x}}_0(\boldsymbol{\alpha}), \tilde{t}_1(\boldsymbol{\alpha}), \tilde{\mathbf{x}}_1(\boldsymbol{\alpha}); \boldsymbol{\alpha})\ [t_0 \text{ free}], \tag{16}$$

$$\tilde{\lambda}_j(\tilde{t}_0(\boldsymbol{\alpha}); \boldsymbol{\alpha}) = -\frac{\partial S}{\partial x_{0j}}(\tilde{t}_0(\boldsymbol{\alpha}), \tilde{\mathbf{x}}_0(\boldsymbol{\alpha}), \tilde{t}_1(\boldsymbol{\alpha}), \tilde{\mathbf{x}}_1(\boldsymbol{\alpha})), \quad j = 1, 2, \ldots, N\ [\mathbf{x}_0 \text{ free}], \tag{17}$$

$$H(\tilde{t}_1(\boldsymbol{\alpha}), \tilde{\mathbf{z}}(\tilde{t}_1(\boldsymbol{\alpha}); \boldsymbol{\alpha}), \tilde{\mathbf{v}}(\tilde{t}_1(\boldsymbol{\alpha}); \boldsymbol{\alpha}), \tilde{\boldsymbol{\lambda}}(\tilde{t}_1(\boldsymbol{\alpha}); \boldsymbol{\alpha}); \boldsymbol{\alpha}) = -\frac{\partial S}{\partial t_1}(\tilde{t}_0(\boldsymbol{\alpha}), \tilde{\mathbf{x}}_0(\boldsymbol{\alpha}), \tilde{t}_1(\boldsymbol{\alpha}), \tilde{\mathbf{x}}_1(\boldsymbol{\alpha}); \boldsymbol{\alpha})\ [t_1 \text{ free}], \tag{18}$$

$$\tilde{\lambda}_j(\tilde{t}_1(\boldsymbol{\alpha}); \boldsymbol{\alpha}) = \frac{\partial S}{\partial x_{1j}}(\tilde{t}_0(\boldsymbol{\alpha}), \tilde{\mathbf{x}}_0(\boldsymbol{\alpha}), \tilde{t}_1(\boldsymbol{\alpha}), \tilde{\mathbf{x}}_1(\boldsymbol{\alpha})), \quad j = 1, 2, \ldots, N\ [\mathbf{x}_1 \text{free}]. \tag{19}$$

The final issue in this chapter concerns sufficiency conditions for the free endpoint version of the salvage value control problem (15). It turns out that it is difficult to find sufficient conditions of any practical value for control problems in which the starting and/or ending dates of the planning horizon are choice variables because of an inherent lack of convexity properties in such problems. We will therefore not present such a theorem, but instead refer you to Seierstad and Sydsæter (1987, Chapter 2, Section 9) for a promising sufficiency theorem.

The final theorem of the chapter contains what should not be an unexpected result for the following *variable endpoint* version of the scrap value optimal control problem (15):

$$\widehat{V}(\boldsymbol{\varphi}) \stackrel{\text{def}}{=} \max_{\mathbf{u}(\cdot), \mathbf{x}_0, \mathbf{x}_1} \left\{ \int_{t_0}^{t_1} f(t, \mathbf{x}(t), \mathbf{u}(t); \boldsymbol{\alpha})\, dt + S(t_0, \mathbf{x}_0, t_1, \mathbf{x}_1; \boldsymbol{\alpha}) \right\}$$
$$\text{s.t.} \quad \dot{\mathbf{x}}(t) = \mathbf{g}(t, \mathbf{x}(t), \mathbf{u}(t); \boldsymbol{\alpha}), \mathbf{x}(t_0) = \mathbf{x}_0, \mathbf{x}(t_1) = \mathbf{x}_1, \tag{20}$$

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$$h^k(t, \mathbf{x}(t), \mathbf{u}(t); \boldsymbol{\alpha}) \geq 0,\ k = 1, 2, \ldots, K',$$

$$h^k(t, \mathbf{x}(t), \mathbf{u}(t); \boldsymbol{\alpha}) = 0,\ k = K' + 1, K' + 2, \ldots, K,$$

where $\boldsymbol{\varphi} \overset{\text{def}}{=} (\boldsymbol{\alpha}, t_0, t_1)$ are fixed (or given) parameters but $(\mathbf{x}_0, \mathbf{x}_1)$ are now decision variables. Assume that an optimal solution $(\widehat{\mathbf{z}}(t; \boldsymbol{\varphi}), \widehat{\mathbf{v}}(t; \boldsymbol{\varphi}))$ exists to problem (20) for all $\boldsymbol{\varphi} \in B(\boldsymbol{\varphi}^\circ; \delta_5)$, with corresponding costate vector $\widehat{\boldsymbol{\lambda}}(t; \boldsymbol{\varphi})$ and Lagrange multiplier vector $\widehat{\boldsymbol{\mu}}(t; \boldsymbol{\varphi})$, and furthermore let $\widehat{\boldsymbol{\Psi}}(\boldsymbol{\varphi}) \overset{\text{def}}{=} (\widehat{\mathbf{x}}_0(\boldsymbol{\varphi}), \widehat{\mathbf{x}}_1(\boldsymbol{\varphi}))$ be the optimal solution for the initial and terminal values of the state vector, where $B(\boldsymbol{\varphi}^\circ; \delta_5)$ is an open $2 + A$ – ball centered at the given value of the parameter vector $\boldsymbol{\varphi}^\circ$ of radius $\delta_5 > 0$. Given the above definitions, it should again be clear that we have the identities $\widehat{\mathbf{x}}_0(\boldsymbol{\varphi}) \equiv \widehat{\mathbf{z}}(t_0; \boldsymbol{\varphi})$ and $\widehat{\mathbf{x}}_1(\boldsymbol{\varphi}) \equiv \widehat{\mathbf{z}}(t_1; \boldsymbol{\varphi})$.

The following sufficiency theorem can be proven following the approach given in Chapter 6, and so is left for a mental exercise.

Theorem 10.4 (Mangasarian Sufficient Conditions, Scrap Value): *Let* $(\widehat{\mathbf{z}}(t; \boldsymbol{\varphi}), \widehat{\mathbf{v}}(t; \boldsymbol{\varphi}))$ *be an admissible pair for problem* (20). *Suppose that* $(\widehat{\mathbf{z}}(t; \boldsymbol{\varphi}), \widehat{\mathbf{v}}(t; \boldsymbol{\varphi}))$ *satisfies the appropriate subset of the necessary conditions of Theorem 10.3 with corresponding costate vector* $\widehat{\boldsymbol{\lambda}}(t; \boldsymbol{\varphi})$ *and Lagrange multiplier vector* $\widehat{\boldsymbol{\mu}}(t; \boldsymbol{\varphi})$*, and let* $L(t, \mathbf{x}, \mathbf{u}, \boldsymbol{\lambda}, \boldsymbol{\mu}) \overset{\text{def}}{=} f(t, \mathbf{x}, \mathbf{u}) + \boldsymbol{\lambda}'\mathbf{g}(t, \mathbf{x}, \mathbf{u}) + \boldsymbol{\mu}'\mathbf{h}(t, \mathbf{x}, \mathbf{u})$ *be the value of the Lagrangian function. If* $L(\cdot)$ *is a concave function of* $(\mathbf{x}, \mathbf{u})\, \forall\, t \in [t_0, t_1]$ *over an open convex set containing all the admissible values of* $(\mathbf{x}(\cdot), \mathbf{u}(\cdot))$ *when the costate vector is* $\widehat{\boldsymbol{\lambda}}(t; \boldsymbol{\varphi})$ *and Lagrange multiplier vector is* $\widehat{\boldsymbol{\mu}}(t; \boldsymbol{\varphi})$*, and* $S(\cdot)$ *is a concave function of* $(\mathbf{x}_0, \mathbf{x}_1)$ *over an open convex set containing all the admissible values of* $\mathbf{x}(\cdot)$*, then* $\widehat{\mathbf{v}}(t; \boldsymbol{\varphi})$ *is an optimal control and* $(\widehat{\mathbf{z}}(t; \boldsymbol{\varphi}), \widehat{\mathbf{v}}(t; \boldsymbol{\varphi}))$ *yields the absolute maximum of problem* (20). *If* $L(\cdot)$ *and* $S(\cdot)$ *are strictly concave functions under the same conditions, then* $(\widehat{\mathbf{z}}(t; \boldsymbol{\varphi}), \widehat{\mathbf{v}}(t; \boldsymbol{\varphi}))$ *yields the unique absolute maximum of problem* (20).

It is worthwhile to remember that the theorems developed in this chapter do not exhaust the possibilities for transversality conditions, yet they do cover those classes of optimal control problems that rear their head most often in economics. Note, however, that by following the strategy of proof established herein, one can establish the necessary transversality conditions for virtually any type of finite horizon optimal control problem of interest in economics.

In closing out this chapter, it is important to observe that the transversality conditions presented here are really just first-order necessary conditions corresponding to a Stage 2 static optimization problem. Naturally, this observation leads one to conjecture that there must exist a set of companion second-order necessary and second-order sufficient conditions of the Stage 2 static optimization problem. Moreover, by further analogy with static optimization theory, one would also conjecture that there exist comparative statics results for the endpoints that are decision variables. These conjectures are correct and constitute the subject matter of the article by Caputo and Wilen (1995), which you are now prepared to study.

MENTAL EXERCISES

10.1 Provide an economic interpretation of parts (iii) and (iv) of Theorem 10.1.

10.2 In Example 10.1, show that

$$C(x, q; w) \stackrel{\text{def}}{=} \min_{L}\{wL \text{ s.t. } q = f(x, L)\}$$

has the properties asserted, given those assumed about $f(\cdot)$.

10.3 In Example 10.1, show that if $x(T) = x_T > 0$ in an optimal plan, then $\mu(t) > 0\, \forall\, t \in [0, T)$. Provide an economic interpretation of this result.

10.4 Provide an economic interpretation of the necessary conditions of Example 10.1 under the assumptions that $\lambda(T) > 0$ and $q(T) > 0$ in an optimal plan.

10.5 Impose the inequality constraint $T \geq 0$ on the mining problem of Example 10.1. Under what condition will the firm choose to set $T = 0$? Provide an economic interpretation of this result.

10.6 What does it mean to say that the Maximum Principle, canonical differential equations, and appropriate transversality conditions are necessary conditions? Be precise.

10.7 What does it mean to say that concavity of the Lagrangian with respect to the state and control variables for all t, plus the appropriate necessary conditions, are sufficient conditions? Be precise.

10.8 What is the role played by transversality conditions in optimal control problems? How do they arise?

10.9 Consider the autonomous optimal control problem

$$\max_{\mathbf{u}(\cdot), T} \int_0^T f(\mathbf{x}(t), \mathbf{u}(t))\, dt$$

$$\text{s.t.} \quad \dot{\mathbf{x}}(t) = g(\mathbf{x}(t), \mathbf{u}(t)),\ \mathbf{x}(0) = \mathbf{x}_0,\ \mathbf{x}(T) = \mathbf{x}_T.$$

(a) Prove that the Hamiltonian for this control problem is constant at zero $\forall\, t \in [0, T]$ along an optimal solution path.

(b) Would the result in part (a) change if either or both of the endpoints for the state vector were freely chosen? Why or why not?

(c) Verify part (a) using Example 10.2.

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10.10 Find the solution of the necessary conditions for the following free terminal time optimal control problem:

$$\max_{u(\cdot),T} \int_0^T -[t^2 + [u(t)]^2]\, dt$$

$$\text{s.t.} \quad \dot{x}(t) = u(t),\ x(0) = 4,\ x(T) = 5.$$

Only consider solutions where $T > 0$.

10.11 Provide the economic intuition for why the shadow cost of the inventory is less than or equal to zero for the entire planning horizon in Example 10.2. In addition, provide an economic interpretation of the comparative dynamics for the shadow cost of the inventory.

10.12 Consider the *time-optimal* control problem

$$\max_{u(\cdot),T} J[x(\cdot), u(\cdot)] \overset{\text{def}}{=} \int_0^T -1\, dt$$

$$\text{s.t.} \quad \dot{x}(t) = x(t) + u(t),\ x(0) = 5,\ x(T) = 11,$$

$$u(t) \in U \overset{\text{def}}{=} \{u(t) : 0 \leq u(t) \leq 1\}.$$

The objective of such optimal control problems is to reach the given terminal value of the state in the least amount of time. This can be seen more clearly by integrating the objective functional to get

$$J[x(\cdot), u(\cdot)] \overset{\text{def}}{=} \int_0^T -1 dt = -t|_{t=0}^{t=T} = -T.$$

Thus in order to *maximize* $J[x(\cdot), u(\cdot)]$, we want to *minimize* the time T required to reach the given terminal value of the state variable.

(a) Derive the necessary conditions for the time-optimal control problem.

(b) Find the general solution of the costate variable, letting c_1 be the constant of integration. Can you determine the sign of the costate variable at this juncture in the problem? Explain what it depends on.

(c) Assuming that $c_1 \neq 0$, describe qualitatively the nature of the optimal solution for the control variable.

(d) Use the transversality condition and terminal endpoint condition to determine the *sign* of c_1. What, therefore, is the optimal path of the control variable?

(e) Find the specific solution of the state equation.

(f) Return to the transversality condition to find the value of c_1, and thus the specific solution of the costate equation.

(g) Use the terminal endpoint condition to find the value of the planning horizon.

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(h) Prove that you have found the unique optimal solution to the time-optimal control problem.

10.13 Prove Theorem 10.3.

10.14 Prove Theorem 10.4.

10.15 In Example 10.1, we examined the nonrenewable resource–extracting model of the firm when $x(t)$ was defined as the stock of the asset remaining in the ground at time t and the amount of the stock left in the ground at time T was a choice variable, as was the terminal date of the planning horizon T. In this exercise, we define $x(t)$ differently and only let the terminal date of the planning horizon T be a choice. To this end, let $x_0 > 0$ be the total quantity of a nonrenewable resource stock controlled by a monopolist who discounts continuously at the rate $r > 0$. The monopolist is asserted to maximize the present discounted value of revenue, defined as $R(q(t)) \stackrel{\text{def}}{=} p(q(t))q(t)$, where $q(t)$ is the extraction rate of the resource at time t, and $p(\cdot) \in C^{(2)}$ is the inverse demand function with $p'(q) < 0$ (the law of demand) and $p(0)$ finite. Cumulative extraction at time t, namely, $x(t)$, is defined as the "sum" of all the extraction rates up to and including time t, that is to say,

$$x(t) \stackrel{\text{def}}{=} \int_0^t q(s)\,ds.$$

By Leibniz's rule, $\dot{x}(t) = q(t)$ is the differential equation governing the rate of change of cumulative extraction, and hence the state equation. We assume that the stock is completely exhausted at the endogenously chosen terminal time T, thereby implying that

$$x(T) \stackrel{\text{def}}{=} \int_0^T q(s)\,ds = x_0$$

and $x(0) = 0$ are the boundary conditions. Thus, the optimal control problem to be solved by the monopolist is given by

$$\max_{q(\cdot),T} J[x(\cdot), q(\cdot)] \stackrel{\text{def}}{=} \int_0^T e^{-rt} p(q(t))q(t)\,dt$$

$$\text{s.t.} \quad \dot{x}(t) = q(t), \; x(0) = 0, x(T) = x_0.$$

Explicitly show, in the order given, that:

(a) $e^{-rt}R'(q(t)) = c$, a constant.

(b) $R''(q(t)) \leq 0$ necessarily, along an optimal extraction path.

(c) $q(T) = 0$.

(d) At $t = T$, marginal revenue equals average revenue (per unit of extraction).

(e) $c = e^{-rt}p(0)$.

(f) $R'(q(t)) = e^{-r(T-t)}p(0)$ along an optimal extraction path.
(g) Assuming that $R''(q(t)) < 0$, the optimal rate of extraction declines over the planning horizon.
(h) The price of the extracted good rises over time.
(i) Assume that $p(q) \stackrel{\text{def}}{=} [1 - e^{-kq}]/q$ is the inverse demand curve, where $k > 0$ is a parameter. Find an explicit solution to the necessary conditions with the given inverse demand curve. Let $T^*(x_0, r)$ be the value of the terminal time.
(j) Show that $\partial T^*(x_0, r)/\partial x_0 > 0$ and that $\partial T^*(x_0, r)/\partial r < 0$. Provide an economic interpretation.
(k) Now suppose price depends on the cumulative amount of the resource extracted as well as on the current extraction rate, say, $p(x, q) \stackrel{\text{def}}{=} a - bx - cq$, where $a > 0, b > 0$, and $c > 0$ are parameters. Find an explicit solution to the necessary conditions with the new inverse demand curve. State the constants of integration and T implicitly as the solution of a simultaneous system of equations.

10.16 This mental exercise asks you to compare and contrast the comparative dynamics properties of the inventory accumulation problem under the assumptions of a fixed delivery date and an optimally chosen delivery date in some more detail.
(a) Set up identities linking the solution functions $(z^*(\cdot), v^*(\cdot), \lambda^*(\cdot))$ of Example 10.2 to the solution functions $(z(\cdot), v(\cdot), \lambda(\cdot))$ of its fixed horizon counterpart in Example 4.5. Do this symbolically first, and then with the explicit solutions to verify that you have the correct identities.
(b) Provide an economic interpretation of both identities.
(c) Differentiate both symbolic identities with respect to x_T to derive a Slutsky-like equation. Provide an economic interpretation of the results.
(d) Plug in the specific functions to verify the comparative dynamic results of Example 10.2.
(e) Repeat parts (c) and (d) for the production cost parameter c_1.
(f) Repeat parts (c) and (d) for the inventory-holding cost parameter c_2.

10.17 This question asks you to further explore the comparative dynamics properties of the inventory accumulation problem, but this time, *without* the aid of the transversality condition for a salvage function, that is, Theorem 10.3. To begin, define $\boldsymbol{\beta} \stackrel{\text{def}}{=} (c_1, c_2, T, x_T)$ and recall that the fixed endpoints version of the model is given by the optimal control problem

$$C(\boldsymbol{\beta}) \stackrel{\text{def}}{=} \min_{u(\cdot)} \int_0^T [c_1[u(t)]^2 + c_2 x(t)]\,dt$$

$$\text{s.t.} \quad \dot{x}(t) = u(t),\ x(0) = 0,\ x(T) = x_T,$$

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where $z(t;\boldsymbol{\beta}) = \frac{1}{4}c_2c_1^{-1}t[t-T] + x_T T^{-1}t$ and $v(t;\boldsymbol{\beta}) = \frac{1}{4}c_2c_1^{-1}[2t-T] + x_T T^{-1}$. A more realistic version of this fixed endpoints problem, as you may recall from Example 2.4, is one in which the firm has control over the terminal stock of the good in inventory x_T that it is able to sell in a competitive market at price $p > 0$. This version of the inventory accumulation problem can thus be stated as the following scrap (or salvage) value optimal control problem:

$$\Pi(\boldsymbol{\gamma}) \stackrel{\text{def}}{=} \max_{u(\cdot),x_T} \left\{ px_T - \int_0^T \left[c_1[u(t)]^2 + c_2x(t)\right] dt \right\}$$

$$\text{s.t.} \quad \dot{x}(t) = u(t),\ x(0) = 0,\ x(T) = x_T,$$

where $\boldsymbol{\gamma} \stackrel{\text{def}}{=} (c_1, c_2, p, T)$. Let $(\hat{z}(t;\boldsymbol{\gamma}), \hat{v}(t;\boldsymbol{\gamma}))$ be the optimal pair of curves to the salvage value problem. Do *not* use the specific functional form given for the pair $(z(t;\boldsymbol{\beta}), v(t;\boldsymbol{\beta}))$ in what follows unless the question specifically asks you to. Note that even though we have a theorem that allows us to handle salvage value control problems, we will solve the above salvage value problem via a different and arguably more economically insightful route.

(a) One way to solve the salvage value problem is to first solve the fixed endpoints problem for its optimal pair, scilicet, $(z(t;\boldsymbol{\beta}), v(t;\boldsymbol{\beta}))$, which we have done already, and derive the optimal value function $C(\boldsymbol{\beta})$, and then solve a *static* maximization problem to find the optimal value of x_T. Write down the second-stage static maximization problem used to find x_T.

(b) Derive the first-order necessary and second-order sufficient conditions for this static optimization problem, and provide an economic interpretation of each.

(c) Let the solution to the first-order necessary condition be $\hat{x}_T(\boldsymbol{\gamma})$, and find an explicit formula for it. Provide an economic interpretation of the necessary and sufficient condition for $\hat{x}_T(\boldsymbol{\gamma}) > 0$ to hold. **Hint:** You will find the dynamic envelope theorem useful here, as well as the explicit functional form of $z(t;\boldsymbol{\beta})$.

(d) Find $\partial\hat{x}_T(\boldsymbol{\gamma})/\partial c_2$ and provide an economic interpretation.

(e) Write down an identity that defines $\hat{v}(t;\boldsymbol{\gamma})$ in terms of $v(t;\boldsymbol{\beta})$ and $\hat{x}_T(\boldsymbol{\gamma})$ without using the explicit functions. In other words, write down the identity in general terms.

(f) Prove that

$$\frac{\partial\hat{v}}{\partial c_2}(t;\boldsymbol{\gamma}) \equiv \frac{\partial v}{\partial c_2}(t;c_1,c_2,T,x_T^*(\boldsymbol{\gamma})) + \frac{\partial v}{\partial x_T}(t;c_1,c_2,T,x_T^*(\boldsymbol{\gamma}))\frac{\partial\hat{x}_T}{\partial c_2}(\boldsymbol{\gamma})\,\forall\, t \in [0,T],$$

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and provide an economic interpretation. Do *not* use the explicit functional forms for $v(t;\boldsymbol{\beta})$ and $\hat{x}_T(\gamma)$.

(g) Prove that

$$\frac{\partial \hat{v}}{\partial c_2}(t;\gamma) \leq 0 \, \forall \, t \in [0, T].$$

Explain why this result differs from that obtained for $\partial v(t;\boldsymbol{\beta})/\partial c_2$. You should make use of the explicit functional forms for $v(t;\boldsymbol{\beta})$ and $\hat{x}_T(\gamma)$.

(h) Prove that

$$\frac{\partial \hat{v}}{\partial p}(t;\gamma) \equiv \frac{\partial v}{\partial x_T}(t; c_1, c_2, T, x_T^*(\gamma))\frac{\partial \hat{x}_T}{\partial p}(\gamma) > 0 \, \forall \, t \in [0, T],$$

and provide an economic interpretation. You should again make use of the explicit functional forms for $v(t;\boldsymbol{\beta})$ and $\hat{x}_T(\gamma)$.

10.18 This question asks you to explore the comparative dynamics properties of the adjustment cost model of the firm when it faces a known cyclical fluctuation in the market price of the good it produces. To begin, let the market-determined time varying output price be given by $P(t) \stackrel{\text{def}}{=} p + \alpha_1 \sin(\alpha_2 t)$, where $p > 0$ is the time-invariant portion of the output price, $\alpha_1 > 0$ is a parameter that determines the amplitude of the periodic motion of the output price, and $\alpha_2 > 0$ is a parameter that determines the period of the output price time path. The production function is assumed to be a linear function of the capital stock $k(t)$, say, $f(k(t)) \stackrel{\text{def}}{=} k(t)$, and the capital stock is assumed *not* to depreciate. The purchase price per unit of investment $I(t)$ is given by the constant $q > 0$, which is set by the market. Adjustment costs are taken to be a quadratic function of the investment rate, say, $C(I(t)) \stackrel{\text{def}}{=} \frac{1}{2}c[I(t)]^2$, where $c > 0$ is the adjustment cost parameter. The initial stock of capital $k_0 > 0$ is given to the firm, but the terminal stock of capital k_T is to be optimally chosen by the firm. The firm lives over the fixed and finite interval $[0, T]$ and does not discount its cash flow. The optimal control problem facing the firm can therefore be stated as

$$\Pi(\gamma) \stackrel{\text{def}}{=} \max_{I(\cdot), k_T} \int_0^T \left[[p + \alpha_1 \sin(\alpha_2 t)]k(t) - qI(t) - \frac{1}{2}c[I(t)]^2 \right] dt$$

$$\text{s.t.} \quad \dot{k}(t) = I(t), \; k(0) = k_0, \; k(T) = k_T,$$

where $\gamma \stackrel{\text{def}}{=} (\alpha_1, \alpha_2, c, p, q, k_0, T)$ is the parameter vector. Assume that the nonnegativity restriction on the capital stock, namely, $k(t) \geq 0 \, \forall \, t \in [0, T]$, is not binding.

(a) Determine the pair of curves $(k^*(t;\gamma), I^*(t;\gamma))$ and corresponding time path of the shadow value of the capital stock $\lambda(t;\gamma)$ that satisfy the necessary conditions.

(b) Prove that the pair of curves $(k^*(t;\gamma), I^*(t;\gamma))$ is the unique optimal solution of the control problem.

(c) Clearly explain how the optimal investment path of the firm is forward looking with respect to the output price path, or equivalently, how it anticipates the changes in direction of the output price path.

(d) Prove that

$$\Pi(\theta\alpha_1, \alpha_2, \theta c, \theta p, \theta q, k_0, T) = \theta\Pi(\alpha_1, \alpha_2, c, p, q, k_0, T)\,\forall\theta > 0$$

and provide an economic interpretation.

(e) Prove that

$$I^*(t; \theta\alpha_1, \alpha_2, \theta c, \theta p, \theta q, k_0, T) = I^*(t; \alpha_1, \alpha_2, c, p, q, k_0, T)\,\forall\theta > 0$$

and provide an economic interpretation.

(f) Prove that

$$\frac{\partial I^*(t;\gamma)}{\partial p} \geq 0\,\forall\, t \in [0, T],$$

and provide an economic interpretation. Is the firm better off facing a higher value of p? Show your work and explain.

(g) Prove that

$$\frac{\partial I^*(t;\gamma)}{\partial q} < 0\,\forall\, t \in [0, T],$$

and provide an economic interpretation. Is the firm better off facing a higher value of q? Show your work and explain.

(h) Prove that

$$\left.\frac{\partial I^*(t;\gamma)}{\partial \alpha_1}\right|_{t=0} \geq 0,$$

and provide an economic interpretation. Is the firm better off facing a higher value of α_1? Show your work and explain.

FURTHER READING

Léonard and Van Long (1992, Chapter 7) and Seierstad and Sydsæter (1987) present additional theorems giving necessary transversality conditions for other classes of optimal control problems. Caputo and Wilen (1995) establish second-order necessary and sufficient conditions for choosing the horizon and state endpoints for a general class of control problems, as well as prove some general comparative statics theorems about such choices using both primal and dual methods of comparative statics.

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Seierstad, A. and Sydsaeter, K. (1987), *Optimal Control Theory with Economic Applications* (New York: Elsevier Science Publishing Co., Inc.).

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https://doi.org/10.1017/CBO9780511806827.012 Published online by Cambridge University Press