SEVENTEEN

The Adjustment Cost Model of the Firm

We continue with the theme of the previous two chapters by examining the local stability, steady state comparative statics, and local comparative dynamics properties of the adjustment cost model of the firm. The analysis is carried one step further in the present chapter, however, in that we augment the local comparative dynamics phase diagram with some analytical calculations. To help hone our economic intuition, the qualitative results derived herewith will be compared and contrasted with those of the static price-taking profit-maximizing model of the firm. Because we have already spent considerable time in formulating the adjustment cost model of the firm in Examples 1.2 and 9.1, we will be a bit more succinct this time around.

To begin, the mathematical statement of the adjustment cost model of the firm, extended to the case of an infinite planning horizon, is given by

$$V(\boldsymbol{\beta}) \stackrel{\text{def}}{=} \max_{I(\cdot)} \int_0^{+\infty} [pf(K(t), I(t)) - cK(t) - gI(t)]\, e^{-rt}\, dt$$

$$\text{s.t.} \quad \dot{K}(t) = I(t) - \delta K(t),\ K(0) = K_0. \tag{1}$$

The function $f(\cdot) : \Re_+^2 \to \Re_+$ is the $C^{(2)}$ *generalized* production function of the firm, for it depends not only on the capital stock of the firm $K(t)$ at time t, but also on the gross rate of change of the capital stock, or the gross investment rate $I(t)$ at time t. It is assumed that $f_K(K, I) > 0$, $\text{sign}[f_I(K, I)] = -\text{sign}[I]$, and furthermore, that $f(\cdot)$ is concave in (K, I). The assumption that $\text{sign}[f_I(K, I)] = -\text{sign}[I]$ is equivalent to the requirement that $f_I(K, I) \lesseqgtr 0$ as $-I \lesseqgtr 0$, and asserts that output decreases as investment ($I > 0$) *or* disinvestment ($I < 0$) in the capital stock takes place. The intuition behind the negative effect of investment on the output of the firm is that when capital is purchased, it must be installed for it to become a productive asset. The process of installation, however, takes resources away from production, thereby resulting in the fall in output. Similarly, when disinvestment takes place, the process of uninstalling the capital also takes resources away from

https://doi.org/10.1017/CBO9780511806827.019 Published online by Cambridge University Press

production, thereby resulting in the decrease in output as well. The assumption that $\text{sign}[f_I(K, I)] = -\text{sign}[I]$ is therefore the backbone of the adjustment cost model. The concavity of $f(\cdot)$ in (K, I) implies that the capital stock and the investment rate display diminishing marginal productivity. These assumptions are not that strong, and moreover, they are consistent with assumptions we are familiar with from the static theory of the profit-maximizing firm. What is more, they will help sharpen our qualitative results and thus allow us to focus on the economic content of the model rather than on some of its tangential mathematical details.

The single good the firm produces via its production function is sold at the constant price of $p > 0$, $c > 0$ is the constant holding cost per unit of the capital stock, and $g > 0$ is the constant price paid per unit of investment or disinvestment in the capital stock. The firm discounts its cash flow at the constant rate $r > 0$ and begins its planning with a given capital stock $K_0 > 0$, but no assumptions are placed on $\lim_{t\to+\infty} K(t)$ at this juncture. We assume that the natural nonnegativity constraint on the capital stock $K(t) \geq 0$ is not binding for all $t \in [0, +\infty)$. We do not, however, assume that $I(t) \geq 0$ for all $t \in [0, +\infty)$. In other words, we permit the firm to disinvest, that is to say, $I(t) \leq 0$ is permitted. Note that $\dot{K}(t)$ is net investment, since depreciation, assumed proportional to the existing stock of capital with depreciation rate $\delta > 0$, is subtracted from gross investment $I(t)$ to arrive at $\dot{K}(t)$. For notational clarity, define $\boldsymbol{\beta} \stackrel{\text{def}}{=} (\boldsymbol{\alpha}, K_0) \stackrel{\text{def}}{=} (c, g, p, r, \delta, K_0) \in \Re^6_{++}$ as the vector of time-independent parameters. Finally, assume that there exists a solution of the necessary conditions of Theorem 14.3, which we denote by the pair $(K^*(t;\boldsymbol{\beta}), I^*(t;\boldsymbol{\beta}))$, with the property that $(K^*(t;\boldsymbol{\beta}), I^*(t;\boldsymbol{\beta})) \to (K^s(\boldsymbol{\alpha}), I^s(\boldsymbol{\alpha}))$ as $t \to +\infty$, where $\lambda(t;\boldsymbol{\beta})$ is the corresponding time path of the current value costate variable and $(K^s(\boldsymbol{\alpha}), I^s(\boldsymbol{\alpha}))$ is the simple steady state solution of the necessary conditions.

To begin the analysis, define the current value Hamiltonian by

$$H(K, I, \lambda; \boldsymbol{\alpha}) \stackrel{\text{def}}{=} pf(K, I) - cK - gI + \lambda[I - \delta K]. \tag{2}$$

Assuming that the objective functional exists for all admissible pairs of functions, Theorems 14.3 and 14.9 imply the following necessary conditions:

$$H_I(K, I, \lambda; \boldsymbol{\alpha}) = pf_I(K, I) - g + \lambda = 0, \tag{3}$$

$$H_{II}(K, I, \lambda; \boldsymbol{\alpha}) = pf_{II}(K, I) \leq 0, \tag{4}$$

$$\dot{\lambda} = r\lambda - H_K(K, I, \lambda; \boldsymbol{\alpha}) = [r + \delta]\lambda - pf_K(K, I) + c, \tag{5}$$

$$\dot{K} = H_\lambda(K, I, \lambda; \boldsymbol{\alpha}) = I - \delta K,\ K(0) = K_0, \tag{6}$$

$$\lim_{t\to+\infty} e^{-rt} H(K(t), I(t), \lambda(t); \boldsymbol{\alpha}) = 0. \tag{7}$$

Just like the two previous chapters, we have not assumed much specific mathematical structure on the problem, and as a result, we cannot get a closed-form solution of the necessary conditions. Nonetheless, we will be able to characterize the solution qualitatively. Before doing so, however, we deal with three preliminary features of the solution $(K^*(t;\boldsymbol{\beta}), I^*(t;\boldsymbol{\beta}))$ to the necessary conditions of Theorem 14.3.

Let us first verify that the necessary transversality condition (7) from Theorem 14.9 is satisfied by the solution $(K^*(t;\boldsymbol{\beta}), I^*(t;\boldsymbol{\beta}))$. From Eq. (3), we have $\lambda = g - pf_I(K, I)$. Because we have assumed that $(K^*(t;\boldsymbol{\beta}), I^*(t;\boldsymbol{\beta})) \to (K^s(\boldsymbol{\alpha}), I^s(\boldsymbol{\alpha}))$ as $t \to +\infty$, it then follows from $\lambda = g - pf_I(K, I)$ that $\lambda(t;\boldsymbol{\beta}) \to \lambda^s(\boldsymbol{\alpha})$ as $t \to +\infty$, where $\lambda^s(\boldsymbol{\alpha})$ is the corresponding steady state solution of the current value costate variable. Consequently, $\lim_{t\to+\infty} H(K^*(t;\boldsymbol{\beta}), I^*(t;\boldsymbol{\beta}), \lambda(t;\boldsymbol{\beta}); \boldsymbol{\alpha})$ exists, and seeing as $\lim_{t\to+\infty} e^{-rt} = 0$, the necessary transversality condition (7) of Theorem 14.9 is satisfied by the solution $(K^*(t;\boldsymbol{\beta}), I^*(t;\boldsymbol{\beta}))$ and $\lambda(t;\boldsymbol{\beta})$.

Second, even though one may believe that the capital stock is a good in this model, and therefore that $\lambda(t;\boldsymbol{\beta}) > 0$ for all $t \in [0, +\infty)$, this is not necessarily the case. To see this, we again use the necessary condition $\lambda = g - pf_I(K, I)$, the assumption that $\operatorname{sign}[f_I(K, I)] = -\operatorname{sign}[I]$, and the fact that $g > 0$. Clearly, if $I^*(t;\boldsymbol{\beta}) > 0$, then $f_I(K^*(t;\boldsymbol{\beta}), I^*(t;\boldsymbol{\beta})) < 0$, and thus $\lambda(t;\boldsymbol{\beta}) > 0$. If, however, $I^*(t;\boldsymbol{\beta}) < 0$, then $f_I(K^*(t;\boldsymbol{\beta}), I^*(t;\boldsymbol{\beta})) > 0$, and it is possible that $\lambda(t;\boldsymbol{\beta}) < 0$. These conclusions are intuitive, too, for if the firm is investing in the capital stock, then it obviously considers the capital stock a good, since it wants more of it rather than less, thus implying that current value shadow price of capital is positive. On the other hand, if the firm is disinvesting in the capital stock, then it wants less of it rather than more, in which case, if the firm wants to disinvest at a sufficiently high rate, as indicated by a "large" positive value of $f_I(K^*(t;\boldsymbol{\beta}), I^*(t;\boldsymbol{\beta}))$, then it views the capital stock as a bad, thereby implying that current value shadow price of capital is negative. Note that necessary condition (4) holds globally due to our assumption that $f(\cdot)$ is concave in (K, I). In fact, in order to rule out division by zero when we derive a differential equation for the investment rate, given in Eq. (11) below, we will assume that the second-order sufficient condition for I to maximize $H(\cdot)$ holds, that is, $H_{II}(K, I, \lambda; \boldsymbol{\alpha}) = pf_{II}(K, I) < 0$ holds when evaluated along the curves $(K^*(t;\boldsymbol{\beta}), I^*(t;\boldsymbol{\beta}))$.

Lastly, let's establish that the solution $(K^*(t;\boldsymbol{\beta}), I^*(t;\boldsymbol{\beta}))$ of the necessary conditions is a solution of the adjustment cost model under suitable additional assumptions. By Theorem 14.4, if we can verify that $\lim_{t\to+\infty} e^{-rt}\lambda(t;\boldsymbol{\beta})[K^*(t;\boldsymbol{\beta}) - K(t)] \le 0$ for all admissible paths $K(t)$ of the capital stock, then we may conclude that $(K^*(t;\boldsymbol{\beta}), I^*(t;\boldsymbol{\beta}))$ is a solution of the adjustment cost model. Since $\lim_{t\to+\infty} e^{-rt} = 0$, and $K^*(t;\boldsymbol{\beta}) \to K^s(\boldsymbol{\alpha})$ and $\lambda(t;\boldsymbol{\beta}) \to \lambda^s(\boldsymbol{\alpha})$ as $t \to +\infty$, if all admissible paths $K(t)$ are bounded, or $\lim_{t\to+\infty} K(t)$ exists for all admissible paths, then it follows that $\lim_{t\to+\infty} e^{-rt}\lambda(t;\boldsymbol{\beta})[K^*(t;\boldsymbol{\beta}) - K(t)] = 0$ and the limiting transversality condition is satisfied. In this case, therefore, the solution $(K^*(t;\boldsymbol{\beta}), I^*(t;\boldsymbol{\beta}))$ of the necessary conditions is a solution of the adjustment cost model. Let's now turn to the qualitative characterization of the model.

To begin, we reduce the necessary conditions given in Eqs. (3), (5), and (6) down to two ordinary differential equations in the variables (K, I). First, differentiate

Eq. (3) with respect to t to get

$$pf_{IK}(K,I)\dot{K} + pf_{II}(K,I)\dot{I} + \dot{\lambda} = 0. \tag{8}$$

Next, substitute $\dot{\lambda} = [r+\delta]\lambda - pf_K(K,I) + c$ from Eq. (5) and $\dot{K} = I - \delta K$ from Eq. (6) into Eq. (8) to eliminate $\dot{\lambda}$ and $\dot{K}$, respectively. This process yields the ordinary differential equation

$$pf_{IK}(K,I)[I-\delta K] + pf_{II}(K,I)\dot{I} + [r+\delta]\lambda - pf_K(K,I) + c = 0. \tag{9}$$

Finally, substitute $\lambda = g - pf_I(K,I)$ from Eq. (3) into Eq. (9) and solve for $\dot{I}$, which, in conjunction with Eq. (6) yields

$$\dot{K} = I - \delta K, \tag{10}$$

$$\dot{I} = \frac{pf_K(K,I) + [r+\delta][pf_I(K,I) - g] - c - pf_{IK}(K,I)[I-\delta K]}{pf_{II}(K,I)} \tag{11}$$

as the system of ordinary differential equations of interest. Note that $(K^*(t;\boldsymbol{\beta}), I^*(t;\boldsymbol{\beta}))$ necessarily satisfy Eqs. (10) and (11). Given this solution, we may then define the dynamic supply function $y^*(\cdot)$ by $y^*(t;\boldsymbol{\beta}) \overset{\text{def}}{=} f(K^*(t;\boldsymbol{\beta}), I^*(t;\boldsymbol{\beta}))$, completely analogous to the way the supply function of a profit-maximizing firm is defined.

Now recall that $(K^s(\boldsymbol{\alpha}), I^s(\boldsymbol{\alpha}))$ is the simple steady state solution of the necessary conditions. Therefore, by definition, $(K^s(\boldsymbol{\alpha}), I^s(\boldsymbol{\alpha}))$ is the simultaneous solution of the necessary conditions (10) and (11) when $\dot{K} = 0$ and $\dot{I} = 0$. That is to say, $(K^s(\boldsymbol{\alpha}), I^s(\boldsymbol{\alpha}))$ is the solution of the following pair of algebraic equations:

$$I - \delta K = 0, \tag{12}$$

$$pf_K(K,I) + [r+\delta][pf_I(K,I) - g] - c = 0. \tag{13}$$

Note that we used the fact that $\dot{I} = 0$ if and only if the numerator of Eq. (11) is equal to zero, along with Eq. (12), to arrive at the final form of Eq. (13). Given the steady state solution $(K^s(\boldsymbol{\alpha}), I^s(\boldsymbol{\alpha}))$, we may define the steady state supply function $y^s(\cdot)$ by $y^s(\boldsymbol{\alpha}) \overset{\text{def}}{=} f(K^s(\boldsymbol{\alpha}), I^s(\boldsymbol{\alpha}))$.

The first qualitative result for the adjustment cost model is contained in the following proposition. Its proof is left for a mental exercise.

Proposition 14.1 (Homogeneity): *The functions* $(K^*(\cdot), I^*(\cdot), y^*(\cdot))$ *and* $(K^s(\cdot), I^s(\cdot), y^s(\cdot))$ *are positively homogeneous of degree zero in the parameters* (c, g, p).

This homogeneity property is the exact intertemporal analogue of the homogeneity of the static profit-maximizing demand and supply functions with respect to input and output prices. Let's now turn to the examination of the local stability of the steady state.

As usual, we begin this endeavor by calculating the Jacobian matrix of Eqs. (10) and (11) and evaluating the result at the simple steady state solution $(K^s(\boldsymbol{\alpha}), I^s(\boldsymbol{\alpha}))$:

$$\mathbf{J}_d(K^s(\boldsymbol{\alpha}), I^s(\boldsymbol{\alpha})) \stackrel{\text{def}}{=} \left.\begin{bmatrix} \dfrac{\partial \dot{K}}{\partial K} & \dfrac{\partial \dot{K}}{\partial I} \\ \dfrac{\partial \dot{I}}{\partial K} & \dfrac{\partial \dot{I}}{\partial I} \end{bmatrix}\right|_{\substack{\dot{K}=0 \\ \dot{I}=0}}$$

$$= \begin{bmatrix} -\delta & 1 \\ \dfrac{f_{KK}(K^s(\boldsymbol{\alpha}), I^s(\boldsymbol{\alpha})) + [r+2\delta] f_{IK}(K^s(\boldsymbol{\alpha}), I^s(\boldsymbol{\alpha}))}{f_{II}(K^s(\boldsymbol{\alpha}), I^s(\boldsymbol{\alpha}))} & r+\delta \end{bmatrix}. \quad (14)$$

Because $\operatorname{tr}[\mathbf{J}_d(K^s(\boldsymbol{\alpha}), I^s(\boldsymbol{\alpha}))] = r > 0$, the eigenvalues of $\mathbf{J}_d(K^s(\boldsymbol{\alpha}), I^s(\boldsymbol{\alpha}))$ sum to the discount rate and therefore cannot both have negative real parts, thereby ruling out local asymptotic stability of the steady state. But we have already established that the optimal solution of the adjustment cost model converges to the steady state in the limit of the planning horizon. Therefore, there must exist at least one trajectory in the *KI*-phase plane that asymptotically approaches the steady state. This means that it *cannot* be the case that $|\mathbf{J}_d(K^s(\boldsymbol{\alpha}), I^s(\boldsymbol{\alpha}))| > 0$, for then both eigenvalues would have positive real parts and thus *no* trajectories would approach the steady state as $t \to +\infty$. Moreover, the assumed simplicity of the steady state implies that $|\mathbf{J}_d(K^s(\boldsymbol{\alpha}), I^s(\boldsymbol{\alpha}))| \neq 0$ by definition. Consequently, $|\mathbf{J}_d(K^s(\boldsymbol{\alpha}), I^s(\boldsymbol{\alpha}))| < 0$, or equivalently, the eigenvalues are real and of the opposite sign, since their product equals $|\mathbf{J}_d(K^s(\boldsymbol{\alpha}), I^s(\boldsymbol{\alpha}))|$. That is to say, the steady state is a local saddle point, with two trajectories approaching the steady state. These trajectories represent the stable manifold of the saddle point steady state. In sum, therefore, because we were able to establish that (i) the optimal solution of the adjustment cost model converges to the simple steady state solution of the necessary conditions, and (ii) $\operatorname{tr}[\mathbf{J}_d(K^s(\boldsymbol{\alpha}), I^s(\boldsymbol{\alpha}))] = r > 0$, we were led to the conclusion that the simple steady state is a local saddle point.

With the local stability of the steady state resolved, we turn to the construction of the phase portrait. First consider the $\dot{K} = 0$ isocline, which by definition is given by $I - \delta K = 0$. In the *KI*-phase plane, the $\dot{K} = 0$ isocline is thus a straight line emanating from the origin with slope $\delta > 0$. For points above the $\dot{K} = 0$ isocline, $\dot{I} > 0$, whereas for points below the $\dot{K} = 0$ isocline, $\dot{I} < 0$. These observations are a straightforward consequence of the simple structure of the state equation.

The $\dot{I} = 0$ isocline is much more complicated, however. Using Eq. (11), the $\dot{I} = 0$ isocline is by definition given implicitly by

$$\dot{I} = 0 \Leftrightarrow pf_K(K, I) + [r+\delta][pf_I(K, I) - g] - c - pf_{IK}(K, I)[I - \delta K] = 0. \quad (15)$$

By the implicit function theorem, we may use $\mathbf{J}_d(K^s(\boldsymbol{\alpha}), I^s(\boldsymbol{\alpha}))$ given in Eq. (14) to derive the slope of the $\dot{I} = 0$ isocline in a neighborhood of the steady state:

$$\left.\frac{\partial I}{\partial K}\right|_{\substack{\dot{K}=0\\ \dot{I}=0}} = \left.\frac{-\partial \dot{I}/\partial K}{\partial \dot{I}/\partial I}\right|_{\substack{\dot{K}=0\\ \dot{I}=0}}$$

$$= \frac{-f_{KK}(K^s(\boldsymbol{\alpha}), I^s(\boldsymbol{\alpha})) - [r+2\delta] f_{IK}(K^s(\boldsymbol{\alpha}), I^s(\boldsymbol{\alpha}))}{[r+\delta] f_{II}(K^s(\boldsymbol{\alpha}), I^s(\boldsymbol{\alpha}))} \gtrless 0. \quad (16)$$

Because we have not specified a sign for $f_{IK}(K, I)$, we cannot unequivocally determine the sign of $\partial \dot{I}/\partial K\big|_{\dot{K}=\dot{I}=0}$ and consequently the slope of the $\dot{I} = 0$ isocline in a neighborhood of the steady state. If, however, we assume that $f_{IK}(K^s(\boldsymbol{\alpha}), I^s(\boldsymbol{\alpha})) \leq 0$, then $\partial \dot{I}/\partial K\big|_{\dot{K}=\dot{I}=0} > 0$ and the slope of the $\dot{I} = 0$ isocline is negative in a neighborhood of the steady state. Even without making this assumption, we may glean useful information about the slope of the $\dot{I} = 0$ isocline in a neighborhood of the steady state by examining the condition $|\mathbf{J}_d(K^s(\boldsymbol{\alpha}), I^s(\boldsymbol{\alpha}))| < 0$.

Using Eq. (14) and the fact that $|\mathbf{J}_d(K^s(\boldsymbol{\alpha}), I^s(\boldsymbol{\alpha}))| < 0$, we have

$$\left|\mathbf{J}_d(K^s(\boldsymbol{\alpha}), I^s(\boldsymbol{\alpha}))\right| = \left.\begin{vmatrix} \dfrac{\partial \dot{K}}{\partial K} & \dfrac{\partial \dot{K}}{\partial I} \\ \dfrac{\partial \dot{I}}{\partial K} & \dfrac{\partial \dot{I}}{\partial I} \end{vmatrix}\right|_{\substack{\dot{K}=0\\ \dot{I}=0}} = \left.\left[\underbrace{\frac{\partial \dot{K}}{\partial K}}_{-}\underbrace{\frac{\partial \dot{I}}{\partial I}}_{+} - \underbrace{\frac{\partial \dot{K}}{\partial I}}_{+}\underbrace{\frac{\partial \dot{I}}{\partial K}}_{?}\right]\right|_{\substack{\dot{K}=0\\ \dot{I}=0}} < 0.$$

Noting the signs below each term of the above determinant, we have that

$$\left.\left[\underbrace{\frac{\partial \dot{K}}{\partial K}}_{-}\underbrace{\frac{\partial \dot{I}}{\partial I}}_{+} - \underbrace{\frac{\partial \dot{K}}{\partial I}}_{+}\underbrace{\frac{\partial \dot{I}}{\partial K}}_{?}\right]\right|_{\substack{\dot{K}=0\\ \dot{I}=0}} < 0 \Leftrightarrow \left.\frac{-\partial \dot{K}/\partial K}{\partial \dot{K}/\partial I}\right|_{\substack{\dot{K}=0\\ \dot{I}=0}} > \left.\frac{-\partial \dot{I}/\partial K}{\partial \dot{I}/\partial I}\right|_{\substack{\dot{K}=0\\ \dot{I}=0}}. \quad (17)$$

Invoking the implicit function theorem, Eq. (17) asserts that the slope of the $\dot{K} = 0$ isocline is greater than the slope of the $\dot{I} = 0$ isocline in a neighborhood of the steady state if and only if the steady state is a local saddle point. Thus even though the $\dot{I} = 0$ isocline may slope upward or downward in a neighborhood of the steady state, its slope must be less than that of the $\dot{K} = 0$ isocline. That is, the saddle point nature of the steady state rules out the $\dot{I} = 0$ isocline having a greater slope than the $\dot{K} = 0$ isocline in a neighborhood of the steady state.

The vector field associated with the differential equation for the investment rate in Eq. (11) is determined by the elements of the second row of the Jacobian matrix $\mathbf{J}_d(K^s(\boldsymbol{\alpha}), I^s(\boldsymbol{\alpha}))$ defined in Eq. (14). Given that

$$\left.\frac{\partial \dot{I}}{\partial I}\right|_{\substack{\dot{K}=0\\ \dot{I}=0}} = r + \delta > 0, \quad (18)$$

points above the $\dot{I} = 0$ isocline in a neighborhood of the steady state have $\dot{I} > 0$, whereas points below the $\dot{I} = 0$ isocline in a neighborhood of the steady state have $\dot{I} < 0$. This is true regardless of the slope of the $\dot{I} = 0$ isocline because the above element of $\mathbf{J}_d(K^s(\boldsymbol{\alpha}), I^s(\boldsymbol{\alpha}))$ is unambiguously signed. At first glance, this conclusion seems to be at odds with the fact that

$$\left.\frac{\partial \dot{I}}{\partial K}\right|_{\substack{\dot{K}=0\\ \dot{I}=0}} = \frac{f_{KK}(K^s(\boldsymbol{\alpha}), I^s(\boldsymbol{\alpha})) + [r + 2\delta] f_{IK}(K^s(\boldsymbol{\alpha}), I^s(\boldsymbol{\alpha}))}{f_{II}(K^s(\boldsymbol{\alpha}), I^s(\boldsymbol{\alpha}))} \gtrless 0.$$

That is to say, it may appear to be a contradiction that the vector field associated with the investment rate differential equation is fully determined in a neighborhood of the steady state, but at the same time the slope of the $\dot{I} = 0$ isocline is not unequivocally known. This is not the case, however, as we now proceed to show.

Because the slope of the $\dot{I} = 0$ isocline in a neighborhood of the steady state is given by Eq. (16) and $\partial \dot{I}/\partial I\big|_{\dot{K}=\dot{I}=0} = r + \delta > 0$, it follows that knowing the sign of $\partial \dot{I}/\partial K\big|_{\dot{K}=\dot{I}=0}$ is equivalent to knowing the slope of the $\dot{I} = 0$ isocline in a neighborhood of the steady state. For example, if the slope of the $\dot{I} = 0$ isocline is negative in a neighborhood of the steady state, then this is equivalent to $\partial \dot{I}/\partial K\big|_{\dot{K}=\dot{I}=0} > 0$. In this instance, $\partial \dot{I}/\partial K\big|_{\dot{K}=\dot{I}=0} > 0$ implies that points to the right of the $\dot{I} = 0$ isocline in a neighborhood of the steady state have $\dot{I} > 0$, which is perfectly consistent with the vector field calculation based on Eq. (18). Of course, this is not too surprising in view of the fact that with a downward sloping isocline, points to the right of it lie on the same side of it, as do points above it. If, however, the slope of the $\dot{I} = 0$ isocline is positive in a neighborhood of the steady state (but less than that of the $\dot{K} = 0$ isocline), then this is equivalent to $\partial \dot{I}/\partial K\big|_{\dot{K}=\dot{I}=0} < 0$. In this case, $\partial \dot{I}/\partial K\big|_{\dot{K}=\dot{I}=0} < 0$ implies that points to the right of the $\dot{I} = 0$ isocline in a neighborhood of the steady state have $\dot{I} < 0$. This is also perfectly consistent with the vector field calculation based on Eq. (18) because with an upward sloping isocline, points to the right of it lie on the opposite side of it, as do points above it. In sum, therefore, it is not a contradiction that the vector field associated with the investment rate differential equation is fully determined in a neighborhood of the steady state, but at the same time, the slope of the $\dot{I} = 0$ isocline is not unequivocally known.

At this juncture, we have completely determined the type and local stability of the steady state solution of the differential equations given by Eqs. (10) and (11). The information we've gathered so far is displayed in Figure 17.1 under the assumption that the slope of the $\dot{I} = 0$ isocline is negative in a neighborhood of the steady state. As should be evident by inspection of Figure 17.1, the optimal solution to the adjustment cost model is represented by trajectory **A** or **B**, as these are the only trajectories that reach the simple steady state as $t \to +\infty$. None of the other trajectories in the phase plane have this property and therefore are not optimal.

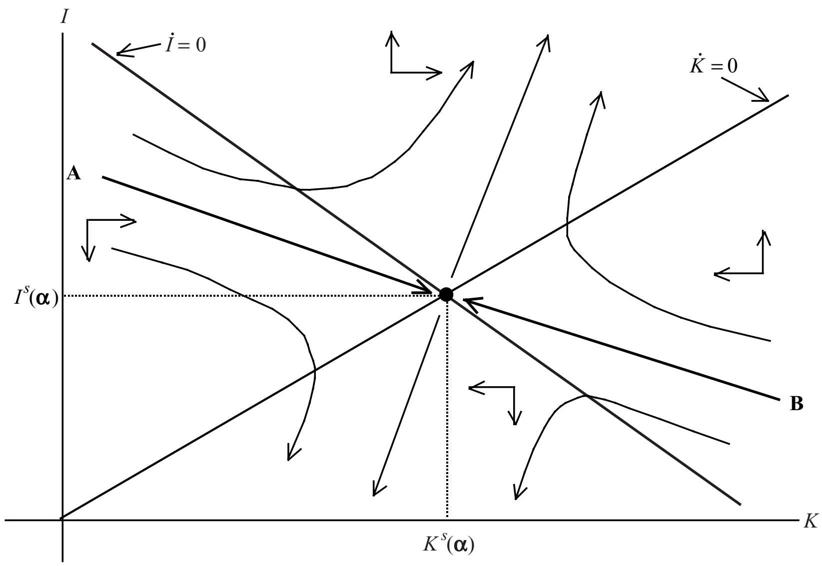

Figure 17.1

If $K_0 > K^s(\alpha)$, that is, if the initial stock of capital exceeds its steady state value, then path **B** is optimal and net disinvestment in the capital stock continually takes place until it is driven down to its steady state value $K^s(\alpha)$, that is, $\dot{K}^*(t;\beta) < 0\,\forall\, t \in [0, +\infty)$. In this case, the stock of capital declines monotonically over time until the smaller steady state stock of capital is reached. If, however, $K_0 < K^s(\alpha)$, that is, if the initial stock of capital is less than its steady state value, then path **A** is optimal and net investment in the capital stock continually takes place until it is driven up to its steady state value $K^s(\alpha)$, that is, $\dot{K}^*(t;\beta) > 0\,\forall\, t \in [0, +\infty)$. In this case, the stock of capital rises monotonically over time until the larger steady state stock of capital is reached.

As of this point, we have graphically found the solution to the adjustment cost model of the firm under the given assumptions, and have determined a few of its qualitative properties by constructing the phase diagram corresponding to the necessary conditions. It would be unfortunate, however, to stop here in the characterization of the solution of the adjustment cost model seeing as the real economic questions of interest are those that ask how the solution changes when some parameter of the model changes. In static economic theory, these questions are answered by a comparative statics analysis. In dynamic economic theory, there are two such analyses for infinite horizon problems:

(a) *steady state comparative statics*: these determine the effect of a parameter change on the steady state values of the choice variables.
(b) *comparative dynamics*: these determine the effect of a parameter change on the time path of the decision variables.

We now turn to an examination of these matters in the adjustment cost model of the firm.

To begin, by the implicit function theorem, a sufficient condition for the steady state solution $(K^s(\boldsymbol{\alpha}), I^s(\boldsymbol{\alpha}))$ to be locally well defined is that the Jacobian determinant of Eqs. (12) and (13) with respect to (K, I) be nonzero when evaluated at the steady state, that is,

$$\left|\mathbf{J}_s(K^s(\boldsymbol{\alpha}), I^s(\boldsymbol{\alpha}))\right| = \begin{vmatrix} -\delta & 1 \\ \begin{array}{l} pf_{KK}(K^s(\boldsymbol{\alpha}), I^s(\boldsymbol{\alpha})) \\ +[r+\delta]pf_{IK}(K^s(\boldsymbol{\alpha}), I^s(\boldsymbol{\alpha})) \end{array} & \begin{array}{l} pf_{KI}(K^s(\boldsymbol{\alpha}), I^s(\boldsymbol{\alpha})) \\ +[r+\delta]pf_{II}(K^s(\boldsymbol{\alpha}), I^s(\boldsymbol{\alpha})) \end{array} \end{vmatrix} \neq 0. \quad (19)$$

This nonzero Jacobian determinant condition holds by virtue of the assumption that $H_{II}(K, I, \lambda; \boldsymbol{\alpha}) = pf_{II}(K, I) < 0$ when evaluated along the optimal curves $(K^*(t; \boldsymbol{\beta}), I^*(t; \boldsymbol{\beta}))$, and the assumed simplicity of the steady state. In order to establish the veracity of the claim, you are asked in a mental exercise to prove that $|\mathbf{J}_s(K^s(\boldsymbol{\alpha}), I^s(\boldsymbol{\alpha}))| = pf_{II}(K^s(\boldsymbol{\alpha}), I^s(\boldsymbol{\alpha}))\,|\,\mathbf{J}_d(K^s(\boldsymbol{\alpha}), I^s(\boldsymbol{\alpha}))|$. Given this result, it follows from the aforementioned two assumptions that $|\mathbf{J}_s(K^s(\boldsymbol{\alpha}), I^s(\boldsymbol{\alpha}))| \neq 0$. Hence, by the implicit function theorem, the steady state necessary conditions given in Eqs. (12) and (13) can be solved, in principle, for K and I in terms of the parameters $\boldsymbol{\alpha} \overset{\text{def}}{=} (c, g, p, r, \delta)$ locally. Moreover, because $f(\cdot) \in C^{(2)}$, thereby implying that $f_K(\cdot) \in C^{(1)}$ and $f_I(\cdot) \in C^{(1)}$, the functions $K^s(\cdot)$ and $I^s(\cdot)$ are locally $C^{(1)}$ in $\boldsymbol{\alpha}$ by the implicit function theorem too. These conclusions rigorously justify the steady state comparative statics that follow. Finally, because $pf_{II}(K, I) < 0$ when evaluated along the curves $(K^*(t; \boldsymbol{\beta}), I^*(t; \boldsymbol{\beta}))$ and $|\mathbf{J}_d(K^s(\boldsymbol{\alpha}), I^s(\boldsymbol{\alpha}))| < 0$, it follows from $|\mathbf{J}_s(K^s(\boldsymbol{\alpha}), I^s(\boldsymbol{\alpha}))| = pf_{II}(K^s(\boldsymbol{\alpha}), I^s(\boldsymbol{\alpha}))\,|\,\mathbf{J}_d(K^s(\boldsymbol{\alpha}), I^s(\boldsymbol{\alpha}))|$ that $|\mathbf{J}_s(K^s(\boldsymbol{\alpha}), I^s(\boldsymbol{\alpha}))| > 0$. This is a crucial result, for $|\,\mathbf{J}_s(K^s(\boldsymbol{\alpha}), I^s(\boldsymbol{\alpha}))|$ appears in the denominator of all the steady state comparative statics expressions. In sum, therefore, the second-order sufficient condition for the investment rate to maximize the current value Hamiltonian and the local saddle point nature of the steady state play an analogous role to the second-order sufficient conditions of the static profit-maximization model in signing the comparative statics expressions.

The steady state comparative statics are found by first substituting $(K^s(\boldsymbol{\alpha}), I^s(\boldsymbol{\alpha}))$ into the steady state necessary conditions, Eqs. (12) and (13), thus creating the identities

$$I^s(\boldsymbol{\alpha}) - \delta K^s(\boldsymbol{\alpha}) \equiv 0, \quad (20)$$

$$pf_K(K^s(\boldsymbol{\alpha}), I^s(\boldsymbol{\alpha})) + [r+\delta]\left[pf_I(K^s(\boldsymbol{\alpha}), I^s(\boldsymbol{\alpha})) - g\right] - c \equiv 0, \quad (21)$$

and then differentiating these identities with respect to the parameter of interest using the multivariate chain rule. For example, differentiating identities (20) and

https://doi.org/10.1017/CBO9780511806827.019 Published online by Cambridge University Press

(21) with respect to the discount rate r gives

$$\begin{bmatrix} -\delta & 1 \\ pf_{KK}(K^s(\boldsymbol{\alpha}), I^s(\boldsymbol{\alpha})) & pf_{KI}(K^s(\boldsymbol{\alpha}), I^s(\boldsymbol{\alpha})) \\ +[r+\delta]pf_{IK}(K^s(\boldsymbol{\alpha}), I^s(\boldsymbol{\alpha})) & +[r+\delta]pf_{II}(K^s(\boldsymbol{\alpha}), I^s(\boldsymbol{\alpha})) \end{bmatrix} \begin{bmatrix} \dfrac{\partial K^s(\boldsymbol{\alpha})}{\partial r} \\ \dfrac{\partial I^s(\boldsymbol{\alpha})}{\partial r} \end{bmatrix}$$

$$\equiv \begin{bmatrix} 0 \\ g - pf_I(K^s(\boldsymbol{\alpha}), I^s(\boldsymbol{\alpha})) \end{bmatrix}. \tag{22}$$

An application of Cramer's rule thus yields the solution

$$\frac{\partial K^s(\boldsymbol{\alpha})}{\partial r} \equiv \frac{pf_I(K^s(\boldsymbol{\alpha}), I^s(\boldsymbol{\alpha})) - g}{|\mathbf{J}_s(K^s(\boldsymbol{\alpha}), I^s(\boldsymbol{\alpha}))|} < 0, \tag{23}$$

$$\frac{\partial I^s(\boldsymbol{\alpha})}{\partial r} \equiv \frac{\delta\left[pf_I(K^s(\boldsymbol{\alpha}), I^s(\boldsymbol{\alpha})) - g\right]}{|\,\mathbf{J}_s(K^s(\boldsymbol{\alpha}), I^s(\boldsymbol{\alpha}))|} = \delta\frac{\partial K^s(\boldsymbol{\alpha})}{\partial r} < 0. \tag{24}$$

The inequalities in Eqs. (23) and (24) follow from $|\mathbf{J}_s(K^s(\boldsymbol{\alpha}), I^s(\boldsymbol{\alpha}))| > 0$, $g > 0$, $p > 0$, $\delta > 0$, and the adjustment cost assumption $\text{sign}[f_I(K, I)] = -\text{sign}[I]$ in conjunction with the fact that $I^s(\boldsymbol{\alpha}) \equiv \delta K^s(\boldsymbol{\alpha}) > 0$. These comparative statics results assert that an increase in the firm's discount rate leads to a decline in its steady state stock of capital and steady state investment rate. Because the increase in the discount rate makes the firm more impatient, it prefers to do beneficial things earlier rather than later. In the adjustment cost model, this leads the firm to have less capital around in the long run.

Turning to the steady state supply function $y^s(\cdot)$, its comparative statics are found by differentiating $y^s(\boldsymbol{\alpha}) \overset{\text{def}}{=} f(K^s(\boldsymbol{\alpha}), I^s(\boldsymbol{\alpha}))$ with respect to the discount rate:

$$\frac{\partial y^s(\boldsymbol{\alpha})}{\partial r} = f_K(K^s(\boldsymbol{\alpha}), I^s(\boldsymbol{\alpha}))\frac{\partial K^s(\boldsymbol{\alpha})}{\partial r} + f_I(K^s(\boldsymbol{\alpha}), I^s(\boldsymbol{\alpha}))\frac{\partial I^s(\boldsymbol{\alpha})}{\partial r}.$$

Using Eq. (24), this may be simplified to read

$$\frac{\partial y^s(\boldsymbol{\alpha})}{\partial r} = \left[f_K(K^s(\boldsymbol{\alpha}), I^s(\boldsymbol{\alpha})) + \delta f_I(K^s(\boldsymbol{\alpha}), I^s(\boldsymbol{\alpha}))\right]\frac{\partial K^s(\boldsymbol{\alpha})}{\partial r} < 0,$$

since $f_K(K^s(\boldsymbol{\alpha}), I^s(\boldsymbol{\alpha})) + \delta f_I(K^s(\boldsymbol{\alpha}), I^s(\boldsymbol{\alpha})) \equiv p^{-1}[c + g[r+\delta]] - rf_I(K^s(\boldsymbol{\alpha}), I^s(\boldsymbol{\alpha})) > 0$ by Eq. (21). Thus the negative effect on output brought about by the lower capital stock outweighs the positive effect on output brought about by the lower investment rate and its concomitant lower adjustment costs. Consequently, output falls in the steady state when the discount rate rises.

Now consider an increase in the output price. Differentiating identities (20) and (21) with respect to p gives

$$\begin{bmatrix} -\delta & 1 \\ pf_{KK}(K^s(\boldsymbol{\alpha}), I^s(\boldsymbol{\alpha})) & pf_{KI}(K^s(\boldsymbol{\alpha}), I^s(\boldsymbol{\alpha})) \\ +[r+\delta]pf_{IK}(K^s(\boldsymbol{\alpha}), I^s(\boldsymbol{\alpha})) & +[r+\delta]pf_{II}(K^s(\boldsymbol{\alpha}), I^s(\boldsymbol{\alpha})) \end{bmatrix} \begin{bmatrix} \dfrac{\partial K^s(\boldsymbol{\alpha})}{\partial p} \\ \dfrac{\partial I^s(\boldsymbol{\alpha})}{\partial p} \end{bmatrix}$$

$$\equiv \begin{bmatrix} 0 \\ -f_K(K^s(\boldsymbol{\alpha}), I^s(\boldsymbol{\alpha})) \\ -[r+\delta]f_I(K^s(\boldsymbol{\alpha}), I^s(\boldsymbol{\alpha})) \end{bmatrix}.$$

Solving this linear system of equations via Cramer's rule yields

$$\frac{\partial K^s(\boldsymbol{\alpha})}{\partial p} \equiv \frac{f_K(K^s(\boldsymbol{\alpha}), I^s(\boldsymbol{\alpha})) + [r+\delta]f_I(K^s(\boldsymbol{\alpha}), I^s(\boldsymbol{\alpha}))}{|\mathbf{J}_s(K^s(\boldsymbol{\alpha}), I^s(\boldsymbol{\alpha}))|}$$

$$= \frac{p^{-1}[c+[r+\delta]g]}{|\mathbf{J}_s(K^s(\boldsymbol{\alpha}), I^s(\boldsymbol{\alpha}))|} > 0, \tag{25}$$

$$\frac{\partial I^s(\boldsymbol{\alpha})}{\partial p} \equiv \frac{\delta f_K(K^s(\boldsymbol{\alpha}), I^s(\boldsymbol{\alpha})) + \delta[r+\delta]f_I(K^s(\boldsymbol{\alpha}), I^s(\boldsymbol{\alpha}))}{|\mathbf{J}_s(K^s(\boldsymbol{\alpha}), I^s(\boldsymbol{\alpha}))|}$$

$$= \frac{\delta p^{-1}[c+[r+\delta]g]}{|\mathbf{J}_s(K^s(\boldsymbol{\alpha}), I^s(\boldsymbol{\alpha}))|} = \delta\frac{\partial K^s(\boldsymbol{\alpha})}{\partial p} > 0. \tag{26}$$

Note that we have employed Eq. (21) in arriving at the final form of Eqs. (25) and (26). Differentiating $y^s(\boldsymbol{\alpha}) \stackrel{\text{def}}{=} f(K^s(\boldsymbol{\alpha}), I^s(\boldsymbol{\alpha}))$ with respect to p and using Eq. (26) gives the slope of the steady state supply function

$$\frac{\partial y^s(\boldsymbol{\alpha})}{\partial p} = \left[f_K(K^s(\boldsymbol{\alpha}), I^s(\boldsymbol{\alpha})) + \delta f_I(K^s(\boldsymbol{\alpha}), I^s(\boldsymbol{\alpha}))\right]\frac{\partial K^s(\boldsymbol{\alpha})}{\partial p} > 0, \tag{27}$$

since $f_K(K^s(\boldsymbol{\alpha}), I^s(\boldsymbol{\alpha})) + \delta f_I(K^s(\boldsymbol{\alpha}), I^s(\boldsymbol{\alpha})) > 0$, as shown above. Equation (27) demonstrates that when the output price increases, the firm produces more of the now more valuable good in the steady state, a result that jibes with that from the static profit-maximizing model of the firm. In order to do so, Eq. (25) shows that the firm must increase its steady state capital stock (the only productive input the firm uses) in order to produce the larger rate of output in the steady state. Equation (26) then shows that this necessitates a higher investment rate in the steady state. The remaining steady state comparative statics and their economic interpretation are left for the mental exercises.

One limitation of a steady state comparative statics analysis is that it only explains where the position of new steady state is relative to the old steady state. In other words, it does not demonstrate how the approach path to the new steady state is affected by a change in a parameter. This is what a local comparative dynamics analysis, to which we now turn, tells us. Note that for the remainder of this chapter, our attention will be focused on the local comparative dynamics of an increase in the output price. Moreover, we will conduct the investigation under the assumption that the slope of the $\dot{I} = 0$ isocline is negative in a neighborhood of the steady state.

Let's first consider the local comparative dynamics from a graphical point of view, with the aid of the phase diagram in Figure 17.1 and a little bit of implicit function theorem work. We know from Figure 17.1 that the steady state is a local saddle point, with two paths converging to it as $t \to +\infty$ and all other paths diverging from it as $t \to +\infty$. Equations (25) and (26) show that the new steady state capital stock and investment rate are larger than their old steady state values because of the output price increase. Now we come to the important observation that the $\dot{K} = 0$ isocline is unaffected by a change in the output price. That this is true is a result of the fact that p does not appear in the $\dot{K} = 0$ isocline, as is plainly obvious because it is given by $I - \delta K = 0$. Hence, in order for the new steady state stock of capital and investment rate to be larger than their old steady values when the output price increases, the $\dot{I} = 0$ isocline must shift up.

We can verify the upward shift in the $\dot{I} = 0$ isocline by applying the implicit function theorem to the $\dot{I} = 0$ isocline defined in Eq. (15). Specifically, we wish to compute the partial derivative of I with respect to p along the $\dot{I} = 0$ isocline and evaluate the result at the steady state. Invoking the implicit function theorem, the result of this set of operations is

$$\left.\frac{\partial I}{\partial p}\right|_{\substack{\dot{K}=0\\ \dot{I}=0}} = \left.\frac{-\partial \dot{I}/\partial p}{\partial \dot{I}/\partial I}\right|_{\substack{\dot{K}=0\\ \dot{I}=0}} = \frac{-f_K(K^s(\boldsymbol{\alpha}), I^s(\boldsymbol{\alpha})) - [r+\delta] f_I(K^s(\boldsymbol{\alpha}), I^s(\boldsymbol{\alpha}))}{p[r+\delta] f_{II}(K^s(\boldsymbol{\alpha}), I^s(\boldsymbol{\alpha}))} > 0,$$

since $f_K(K^s(\boldsymbol{\alpha}), I^s(\boldsymbol{\alpha})) + [r+\delta] f_I(K^s(\boldsymbol{\alpha}), I^s(\boldsymbol{\alpha})) \equiv p^{-1}[c + g[r+\delta]] > 0$ by Eq. (21). This result demonstrates that an increase in p along the $\dot{I} = 0$ isocline, holding K and (c, g, r, δ) constant, increases I in a neighborhood of the steady state. In other words, this implicit function theorem calculation shows that the $\dot{I} = 0$ isocline shifts up when the output price increases in a neighborhood of the steady state. This means that the new $\dot{I} = 0$ isocline associated with the higher output price must lie above the old $\dot{I} = 0$ isocline, just as we argued above. The local comparative dynamics phase diagram corresponding to an increase in the output price is given in Figure 17.2.

It is worthwhile at this juncture to pause and again repeat the three remarks made in Chapters 15 and 16 concerning the construction of a local comparative dynamics phase diagram. The first remark is that the local dynamics depicted in Figure 17.1 apply to *both* of the steady states depicted in Figure 17.2. In other words, the local dynamics around the old and the new steady states are qualitatively identical, and are therefore of the saddle point variety. As a result, there is no need to fully draw in the vector field around each steady state in Figure 17.2, since the complete vector field for it can be inferred from that in Figure 17.1. Second, before the increase in the output price occurs, the firm is assumed to be at rest at the old steady state. Third,

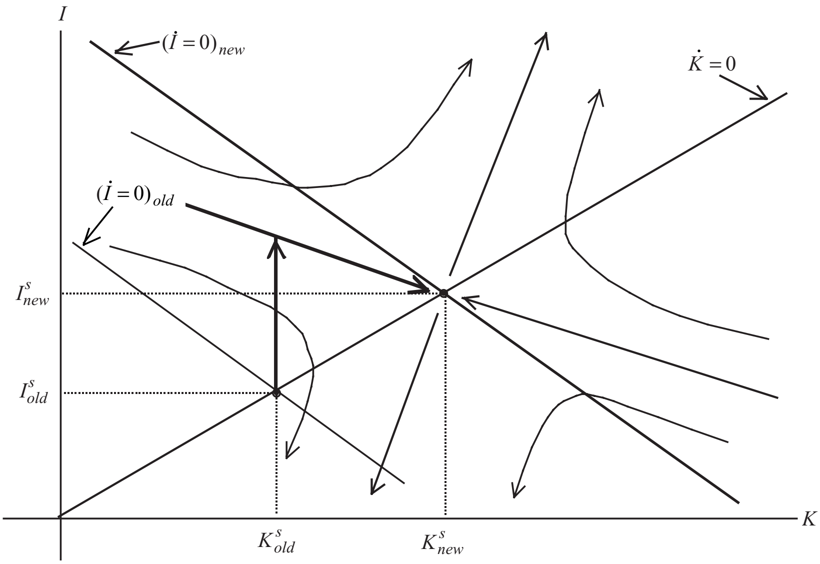

Figure 17.2

the firm is assumed to eventually come to rest at the new steady state as a result of the increase in the output price. That is, the old steady state value of the capital stock is taken as the initial condition in the local comparative dynamics exercise, whereas the new steady state value of the capital stock is taken as the terminal condition. The local comparative dynamics phase diagram therefore depicts the optimal transition path from the old to the new steady state that results from the increase in the output price.

The local comparative dynamics follow from this phase portrait. Initially, the firm is positioned at the old steady state (K^s_{old}, I^s_{old}). We know from the steady state comparative statics that when the output price increases, the steady state stock of capital and investment rate both increase. Because the capital stock is fixed at any given moment in time – in particular, the initial moment the output price increases – the capital stock cannot change the instant the output price increases. Hence, the only way the firm can get from the old steady state (K^s_{old}, I^s_{old}) to the new steady state (K^s_{new}, I^s_{new}) is for the firm to increase its initial rate of investment as soon as the output price increases. In this way, the firm can get on the trajectory in the phase plane that takes it to the new steady state. This trajectory is labeled with the thick lines in Figure 17.2. Note that this is achieved by increasing the investment rate by a *precise amount* the instant the output price increases. If when the output price increases, the initial investment rate is increased too much or too little, or if it is decreased, the firm will not be on the trajectory that allows it to reach the new steady state. Because the initial increase in the investment rate drives it above its new steady state value, the investment rate must fall over time as the new steady state is approached. All the while, the capital stock of the firm increases monotonically from its old to its new steady state value.

In order to gain some analytical insight into the aforementioned comparative dynamics results, as well as to rigorously justify the geometry, we return to the method of linearization exposited in Chapter 13 to study the local stability properties of a system of autonomous and nonlinear differential equations. Remember that the linearization approach involves using Taylor's theorem to linearly approximate a system of nonlinear ordinary differential equations in a neighborhood of the steady state. Because we already have computed the Jacobian matrix of the necessary conditions (10) and (11), we have essentially derived the corresponding linearized system of differential equations that we shall work with.

The linearized form of the necessary conditions (10) and (11) is therefore given by

$$\begin{bmatrix} \Delta\dot{K} \\ \Delta\dot{I} \end{bmatrix} = \begin{bmatrix} -\delta & 1 \\ j_{21}(\boldsymbol{\alpha}) & r+\delta \end{bmatrix} \begin{bmatrix} \Delta K \\ \Delta I \end{bmatrix}, \tag{28}$$

where

$$j_{21}(\boldsymbol{\alpha}) \stackrel{\text{def}}{=} \frac{f_{KK}(K^s(\boldsymbol{\alpha}), I^s(\boldsymbol{\alpha})) + [r+2\delta] f_{IK}(K^s(\boldsymbol{\alpha}), I^s(\boldsymbol{\alpha}))}{f_{II}(K^s(\boldsymbol{\alpha}), I^s(\boldsymbol{\alpha}))} = \left.\frac{\partial \dot{I}}{\partial K}\right|_{\substack{\dot{K}=0 \\ \dot{I}=0}},$$

and $\Delta K \stackrel{\text{def}}{=} K - K^s(\boldsymbol{\alpha})$ and $\Delta I \stackrel{\text{def}}{=} I - I^s(\boldsymbol{\alpha})$, thereby implying that $\Delta\dot{K} \stackrel{\text{def}}{=} \dot{K}$ and $\Delta\dot{I} \stackrel{\text{def}}{=} \dot{I}$. Recall that because we are working under the assumption that the slope of the $\dot{I} = 0$ isocline is negative in a neighborhood of the steady state, $j_{21}(\boldsymbol{\alpha}) > 0$. Also recall that Eq. (28) is the result of applying Taylor's theorem to Eqs. (10) and (11) and neglecting terms of order two or higher from the expansion.

Equation (28) is a linear and homogeneous system of ordinary differential equations with constant coefficients. As you may recollect from your prior coursework in elementary differential equations, because the eigenvalues (γ_1, γ_2) of the Jacobian matrix $\mathbf{J}_d(K^s(\boldsymbol{\alpha}), I^s(\boldsymbol{\alpha}))$ are real and unequal, a fact we have already established, the general solution of system (28) takes the form

$$\begin{bmatrix} \Delta K(t) \\ \Delta I(t) \end{bmatrix} = c_1\mathbf{v}^1 e^{\gamma_1 t} + c_2\mathbf{v}^2 e^{\gamma_2 t} \tag{29}$$

by Theorem 25.1 of Simon and Blume (1994), where c_1 and c_2 are constants of integration and $\mathbf{v}^i \in \Re^2$ is an eigenvector of $\mathbf{J}_d(K^s(\boldsymbol{\alpha}), I^s(\boldsymbol{\alpha}))$ corresponding to the eigenvalue $\gamma_i, i = 1, 2$. Given that $\Delta K \stackrel{\text{def}}{=} K - K^s(\boldsymbol{\alpha})$ and $\Delta I \stackrel{\text{def}}{=} I - I^s(\boldsymbol{\alpha})$, we may rewrite Eq. (29) equivalently as

$$\begin{bmatrix} K(t) \\ I(t) \end{bmatrix} = \begin{bmatrix} K^s(\boldsymbol{\alpha}) \\ I^s(\boldsymbol{\alpha}) \end{bmatrix} + c_1\mathbf{v}^1 e^{\gamma_1 t} + c_2\mathbf{v}^2 e^{\gamma_2 t}. \tag{30}$$

By definition, the eigenvector $\mathbf{v}^i \in \Re^2$, $i = 1, 2$, is the solution to the following homogeneous system of linear algebraic equations:

$$\begin{bmatrix} -\delta-\gamma_i & 1 \\ j_{21}(\boldsymbol{\alpha}) & r+\delta-\gamma_i \end{bmatrix} \begin{bmatrix} v_1^i \\ v_2^i \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{31}$$

Without loss of generality, we may let $\gamma_1 < 0$ and $\gamma_2 > 0$. Because the above coefficient matrix is singular by the definition of an eigenvalue, we may use either row of it to find the corresponding eigenvector. Setting $i = 1$ and using the second row of Eq. (31), we find that

$$\mathbf{v}^1 = \begin{bmatrix} v_1^1 \\ v_2^1 \end{bmatrix} = \begin{bmatrix} 1 \\ \dfrac{j_{21}(\boldsymbol{\alpha})}{\gamma_1 - r - \delta} \end{bmatrix}, \tag{32}$$

whereas setting $i = 2$ and using the first row of Eq. (31), we find that

$$\mathbf{v}^2 = \begin{bmatrix} v_1^2 \\ v_2^2 \end{bmatrix} = \begin{bmatrix} 1 \\ \delta + \gamma_2 \end{bmatrix}. \tag{33}$$

Because $j_{21}(\boldsymbol{\alpha}) > 0$ and $\gamma_1 < 0$, $v_2^1 < 0$, whereas $v_2^2 > 0$, since $\gamma_2 > 0$. Thus the stable manifold of the steady state, which is spanned by the eigenvector $\mathbf{v}^1$ corresponding to the eigenvalue $\gamma_1 < 0$, has a negative slope in the *KI*-phase plane, whereas the unstable manifold of the steady state, which is spanned by the eigenvector $\mathbf{v}^2$ corresponding to the eigenvalue $\gamma_2 > 0$, has a positive slope in the *KI*-phase plane. Inspection of Figure 17.1 shows that this conclusion is confirmed geometrically.

To determine the constants of integration and thus the specific solution to Eq. (28), first recall that the optimal time paths of the capital stock and investment rate satisfy the limiting properties $\lim_{t\to+\infty} K(t) = K^s(\boldsymbol{\alpha})$ and $\lim_{t\to+\infty} I(t) = I^s(\boldsymbol{\alpha})$. Using Eq. (30), this yields

$$\lim_{t\to+\infty} \begin{bmatrix} K(t) \\ I(t) \end{bmatrix} = \lim_{t\to+\infty} \left\{ \begin{bmatrix} K^s(\boldsymbol{\alpha}) \\ I^s(\boldsymbol{\alpha}) \end{bmatrix} + c_1 \mathbf{v}^1 e^{\gamma_1 t} + c_2 \mathbf{v}^2 e^{\gamma_2 t} \right\} = \begin{bmatrix} K^s(\boldsymbol{\alpha}) \\ I^s(\boldsymbol{\alpha}) \end{bmatrix}.$$

The last term in the curly bracketed expression, however, does not possess a limit if $c_2 \neq 0$ because $\gamma_2 > 0$, that is, $c_2 \mathbf{v}^2 e^{\gamma_2 t} \to \left[\begin{smallmatrix} \pm\infty \\ \pm\infty \end{smallmatrix}\right]$ as $t \to +\infty$ if $c_2 \neq 0$, thereby violating the convergence property of the optimal solution curves. Hence, in order for $\lim_{t\to+\infty} K(t) = K^s(\boldsymbol{\alpha})$ and $\lim_{t\to+\infty} I(t) = I^s(\boldsymbol{\alpha})$ to hold, $c_2 = 0$. Equation (30) therefore reduces to

$$\begin{bmatrix} K(t) \\ I(t) \end{bmatrix} = \begin{bmatrix} K^s(\boldsymbol{\alpha}) \\ I^s(\boldsymbol{\alpha}) \end{bmatrix} + c_1 \mathbf{v}^1 e^{\gamma_1 t}. \tag{34}$$

Applying the initial condition $K(0) = K_0$ to Eq. (34) and using Eq. (32) yields

$$K(0) = K^s(\boldsymbol{\alpha}) + c_1 = K_0 \Rightarrow c_1 = K_0 - K^s(\boldsymbol{\alpha}).$$

Thus the specific solution to the linearized system of differential equations given in Eq. (28) is

$$\begin{bmatrix} K^*(t;\boldsymbol{\beta}) \\ I^*(t;\boldsymbol{\beta}) \end{bmatrix} = \begin{bmatrix} K^s(\boldsymbol{\alpha}) \\ I^s(\boldsymbol{\alpha}) \end{bmatrix} + [K_0 - K^s(\boldsymbol{\alpha})] \begin{bmatrix} 1 \\ \dfrac{j_{21}(\boldsymbol{\alpha})}{\gamma_1 - r - \delta} \end{bmatrix} e^{\gamma_1 t}. \tag{35}$$

Equation (35) describes the optimal time path of the capital stock and investment rate in a neighborhood of the steady state.

The local comparative dynamics are derived from Eq. (35) by differentiating with respect to the parameter of interest and evaluating the resulting derivative at $K_0 = K^s(\boldsymbol{\alpha})$. Thus in the local comparative dynamics exercise, we take the initial capital stock to be the steady state value of the capital stock. This is exactly how we have conducted the local comparative dynamics via a phase diagram in this and the previous two chapters. As you will establish in a mental exercise, the eigenvalues (γ_1, γ_2) of the Jacobian matrix $\mathbf{J}_d(K^s(\boldsymbol{\alpha}), I^s(\boldsymbol{\alpha}))$ are functions of the parameter vector $\boldsymbol{\alpha} \stackrel{\text{def}}{=} (c, g, p, r, \delta)$. This is important to remember in the local comparative dynamics calculations.

To carry out the local comparative dynamics analysis, first differentiate Eq. (35) with respect to the output price p and recall that γ_1 is a function of $\boldsymbol{\alpha} \stackrel{\text{def}}{=} (c, g, p, r, \delta)$:

$$\frac{\partial K^*}{\partial p}(t;\boldsymbol{\beta}) = \frac{\partial K^s}{\partial p}(\boldsymbol{\alpha}) + [K_0 - K^s(\boldsymbol{\alpha})]\, e^{\gamma_1 t}\frac{\partial \gamma_1}{\partial p} t - e^{\gamma_1 t}\frac{\partial K^s}{\partial p}(\boldsymbol{\alpha}),$$

$$\frac{\partial I^*}{\partial p}(t;\boldsymbol{\beta}) = \frac{\partial I^s}{\partial p}(\boldsymbol{\alpha}) + [K_0 - K^s(\boldsymbol{\alpha})]$$
$$\times \left[\frac{j_{21}(\boldsymbol{\alpha})}{\gamma_1 - r - \delta} e^{\gamma_1 t}\frac{\partial \gamma_1}{\partial p} t + e^{\gamma_1 t}\frac{\partial}{\partial p}\left[\frac{j_{21}(\boldsymbol{\alpha})}{\gamma_1 - r - \delta}\right]\right] - e^{\gamma_1 t}\left[\frac{j_{21}(\boldsymbol{\alpha})}{\gamma_1 - r - \delta}\right]\frac{\partial K^s}{\partial p}(\boldsymbol{\alpha}).$$

Then evaluate the above derivatives at $K_0 = K^s(\boldsymbol{\alpha})$, and use Eq. (26) to derive the local comparative dynamics for an increase in the output price:

$$\left.\frac{\partial K^*}{\partial p}(t;\boldsymbol{\beta})\right|_{K_0=K^s(\boldsymbol{\alpha})} = [1 - e^{\gamma_1 t}]\frac{\partial K^s}{\partial p}(\boldsymbol{\alpha}) \geq 0\,\forall\, t \in [0, +\infty), \tag{36}$$

$$\left.\frac{\partial I^*}{\partial p}(t;\boldsymbol{\beta})\right|_{K_0=K^s(\boldsymbol{\alpha})} = \left[\delta - e^{\gamma_1 t}\left[\frac{j_{21}(\boldsymbol{\alpha})}{\gamma_1 - r - \delta}\right]\right]\frac{\partial K^s}{\partial p}(\boldsymbol{\alpha}) > 0\,\forall\, t \in [0, +\infty). \tag{37}$$

The inequalities are a result of the facts $\partial K^s(\boldsymbol{\alpha})/\partial p > 0$ from Eq. (25), $\gamma_1 < 0$, which implies that $e^{\gamma_1 t} \in (0, 1]\,\forall\, t \in [0, +\infty)$, and $j_{21}(\boldsymbol{\alpha}) > 0$. Furthermore, when $t = 0$, that is, at the moment p is increased, we have

$$\left.\frac{\partial K^*}{\partial p}(0;\boldsymbol{\beta})\right|_{K_0=K^s(\boldsymbol{\alpha})} = 0, \tag{38}$$

$$\left.\frac{\partial I^*}{\partial p}(0;\boldsymbol{\beta})\right|_{K_0=K^s(\boldsymbol{\alpha})} = \left[\delta - \left[\frac{j_{21}(\boldsymbol{\alpha})}{\gamma_1 - r - \delta}\right]\right]\frac{\partial K^s}{\partial p}(\boldsymbol{\alpha}) > 0. \tag{39}$$

Finally, letting $t \to +\infty$ in Eqs. (36) and (37) yields

$$\begin{aligned}\lim_{t\to+\infty} \frac{\partial K^*}{\partial p}(t;\boldsymbol{\beta})\bigg|_{K_0=K^s(\boldsymbol{\alpha})} &= \lim_{t\to+\infty}\left[[1-e^{\gamma_1 t}]\frac{\partial K^s}{\partial p}(\boldsymbol{\alpha})\right] \\ &= \frac{\partial K^s}{\partial p}(\boldsymbol{\alpha})\lim_{t\to+\infty}[1-e^{\gamma_1 t}] = \frac{\partial K^s}{\partial p}(\boldsymbol{\alpha}), \qquad (40)\end{aligned}$$

$$\begin{aligned}\lim_{t\to+\infty} \frac{\partial I^*}{\partial p}(t;\boldsymbol{\beta})\bigg|_{K_0=K^s(\boldsymbol{\alpha})} &= \lim_{t\to+\infty}\left[\left[\delta - e^{\gamma_1 t}\left[\frac{j_{21}(\boldsymbol{\alpha})}{\gamma_1 - r - \delta}\right]\right]\frac{\partial K^s}{\partial p}(\boldsymbol{\alpha})\right] \\ &= \frac{\partial K^s}{\partial p}(\boldsymbol{\alpha})\lim_{t\to+\infty}\left[\left[\delta - e^{\gamma_1 t}\left[\frac{j_{21}(\boldsymbol{\alpha})}{\gamma_1 - r - \delta}\right]\right]\right] \\ &= \delta\frac{\partial K^s}{\partial p}(\boldsymbol{\alpha}) = \frac{\partial I^s}{\partial p}(\boldsymbol{\alpha}). \qquad (41)\end{aligned}$$

Let us now turn to an economic interpretation of Eqs. (36) through (41). We will see that they confirm all of the qualitative implications we deduced via the local comparative dynamics phase portrait in Figure 17.2.

By Eq. (38), the moment the output price increases ($t = 0$), the capital stock remains at its old steady state level and is therefore unaffected, just as we claimed when we discussed the local comparative dynamics via the phase diagram in Figure 17.2. On the other hand, the instant the output price increases, the investment rate rises by the precise amount given in Eq. (39). In fact, because $\gamma_1 < 0$ and $j_{21}(\boldsymbol{\alpha}) > 0$, it follows from Eq. (39) that

$$\frac{\partial I^*}{\partial p}(0;\boldsymbol{\beta})\bigg|_{K_0=K^s(\boldsymbol{\alpha})} = \left[\delta - \left[\frac{j_{21}(\boldsymbol{\alpha})}{\gamma_1 - r - \delta}\right]\right]\frac{\partial K^s}{\partial p}(\boldsymbol{\alpha}) > \delta\frac{\partial K^s}{\partial p}(\boldsymbol{\alpha}) = \frac{\partial I^s}{\partial p}(\boldsymbol{\alpha}) > 0.$$

Thus the moment the output price increases, the investment rate increases so much that it exceeds its new steady state investment rate, a result that we were able to deduce with the aid of the local comparative dynamics phase diagram in Figure 17.2. After the initial increase in the output price, Eqs. (36) and (37) show that the capital stock and investment rate are both higher all along their approach to the new steady state. As $t \to +\infty$, Eqs. (40) and (41) show that the capital stock and investment rate approach their new steady state values, both of which exceed their old steady state values.

The last remark we wish to make about the local comparative dynamics of the adjustment cost model brings us back to Theorem 14.10, the dynamic envelope theorem for discounted infinite horizon optimal control models. As you may recall, in order to establish Theorem 14.10, we had to make an assumption that, in the context of the adjustment cost model, was of the form

$$\lim_{t\to+\infty}\frac{\partial K^*}{\partial p}(t;\boldsymbol{\beta}) = \frac{\partial K^s}{\partial p}(\boldsymbol{\alpha}).$$

By Eq. (40), if we take the initial stock of capital to be the steady state stock of capital, that is, $K_0 = K^s(\boldsymbol{\alpha})$, then this condition holds. The same conclusion applies to the optimal investment rate as well, as can be seen by inspection of Eq. (41).

In the next chapter, we undertake a systematic qualitative study of a general infinite horizon discounted autonomous control problem with one state variable and one control variable. The theorems developed will permit a more complete understanding of the basic mathematical structure responsible for many of the qualitative properties of numerous optimal control models used in dynamic economic theory.

MENTAL EXERCISES

17.1 Prove Proposition 17.1.

17.2 Prove that $|\mathbf{J}_s(K^s(\boldsymbol{\alpha}), I^s(\boldsymbol{\alpha}))| = pf_{II}(K^s(\boldsymbol{\alpha}), I^s(\boldsymbol{\alpha}))\,|\,\mathbf{J}_d(K^s(\boldsymbol{\alpha}),\ I^s(\boldsymbol{\alpha}))|$.

17.3 Prove that the steady state value of the current value shadow price of the capital stock is positive, that is, $\lambda^s(\boldsymbol{\alpha}) > 0$. Is the capital stock a good or a bad in the steady state? Explain.

17.4 Assume, in contrast to the chapter proper, that the slope of the $\dot{I} = 0$ isocline is positive in a neighborhood of the steady state.
   (a) Draw the phase portrait in *KI*-phase space. Compare it to that in Figure 17.1.
   (b) Draw the local comparative dynamics phase portrait corresponding to an increase in the output price. Compare it to that in Figure 17.2.

17.5 This exercise asks you to prove that eigenvectors of $\mathbf{J}_d(K^s(\boldsymbol{\alpha}), I^s(\boldsymbol{\alpha}))$ are identical regardless of which row of Eq. (31) is used to find them.
   (a) Derive $\mathbf{v}^1$ using the first row of Eq. (31).
   (b) Derive $\mathbf{v}^2$ using the second row of Eq. (31).
   (c) Show that $\mathbf{v}^1$ and $\mathbf{v}^2$ derived in parts (a) and (b) are identical to $\mathbf{v}^1$ and $\mathbf{v}^2$ derived in Eqs. (32) and (33), respectively.

17.6 Prove that the eigenvalues (γ_1, γ_2) of the Jacobian matrix $\mathbf{J}_d(K^s(\boldsymbol{\alpha}), I^s(\boldsymbol{\alpha}))$ are functions of the parameter vector $\boldsymbol{\alpha} \overset{\text{def}}{=} (c, g, p, r, \delta)$.

17.7 Derive the local comparative dynamics associated with an increase in the discount rate r for the adjustment cost model of the firm. Your derivation of the qualitative properties should be as complete and exhaustive as that given in the text. Draw the comparative dynamics phase portraits, and provide an economic explanation of the results.

17.8 Derive the steady state comparative statics and local comparative dynamics associated with an increase in the holding cost of capital c for the adjustment cost model of the firm. Your derivation of the qualitative properties should be as complete and exhaustive as that given in the text. Draw the comparative dynamics phase portraits, and provide an economic explanation of the results.

17.9 Derive the steady state comparative statics and local comparative dynamics associated with an increase in the purchase price of capital g for the adjustment cost model of the firm. Your derivation of the qualitative properties should be as complete and exhaustive as that given in the text. Draw the comparative dynamics phase portraits, and provide an economic explanation of the results.

17.10 Derive the steady state comparative statics and local comparative dynamics associated with an increase in the initial stock of capital K_o for the adjustment cost model of the firm. Your derivation of the qualitative properties should be as complete and exhaustive as that given in the text. Draw the comparative dynamics phase portraits, and provide an economic explanation of the results.

17.11 *Dynamics of the Adjustment Cost Model of the Firm in State-Costate Phase Space*. This question reexamines the adjustment cost model of the capital-accumulating firm assuming that the generalized production function is additively separable in the capital stock and investment rate. This has the effect of sharpening and simplifying some of the analysis. Let $f(\cdot):\Re_+ \to \Re_+$ be the $C^{(2)}$ production function, where $f(0)=0$, $f'(K)>0$ and $f''(K)<0\;\forall\, K\in\Re_{++}$, $\lim_{K\to 0^+} f'(K)=+\infty$, and $\lim_{K\to+\infty} f'(K)=0$, where $K(t)$ is the capital stock of the firm at time t. The output of the firm is sold at the constant price $p>0$, the capital stock has unit maintenance costs of $c>0$, and $g>0$ is the constant cost per unit of purchased capital, that is, the purchase price of investment $I(t)$. In addition, let $C(\cdot):\Re\to\Re_+$ be the $C^{(2)}$ cost of adjustment function (in dollars), with $C(0)=0$, $C'(0)=0$, $\operatorname{sign}(C'(I))=\operatorname{sign}(I)$, and $C''(I)>0\;\forall\, I\in\Re_+$. The firm is asserted to operate over the indefinite future and discounts its instantaneous profits at the constant rate $r>0$. The state equation is the prototype capital accumulation equation with a constant rate of decay $\delta>0$. Finally, the firm begins its planning at time $t=0$ with the given stock of capital $K(0)=K_0>0$. The optimal control problem the firm must solve in order to determine its optimal investment plan is therefore given by

$$V(\boldsymbol{\beta}) \stackrel{\text{def}}{=} \max_{I(\cdot)} \int_0^{+\infty} [pf(K(t)) - cK(t) - gI(t) - C(I(t))]\, e^{-rt}\, dt$$

$$\text{s.t.}\quad \dot{K}(t) = I(t) - \delta K(t),\;\; K(0) = K_0,$$

where $\boldsymbol{\beta} \stackrel{\text{def}}{=} (\boldsymbol{\alpha}, K_0) \stackrel{\text{def}}{=} (c, g, p, r, \delta, K_0) \in \Re^6_{++}$ are the time invariant parameters of the problem. Notice that we are not imposing a nonnegativity constraint on the investment rate. This means that the firm can buy capital ($I(t)>0$) or sell capital ($I(t)<0$) at the market price of $g>0$. We are therefore not considering the case of irreversible investment in this question. Assume that there exists a solution of the necessary conditions of Theorem 14.3, say, $(K^*(t;\boldsymbol{\beta}), I^*(t;\boldsymbol{\beta}))$, with the property that $(K^*(t;\boldsymbol{\beta}), I^*(t;\boldsymbol{\beta})) \to (K^s(\boldsymbol{\alpha}), I^s(\boldsymbol{\alpha}))$ as $t\to+\infty$, where $\lambda(t;\boldsymbol{\beta})$ is the corresponding time path of the current value costate variable and $(K^s(\boldsymbol{\alpha}), I^s(\boldsymbol{\alpha}))$ is the simple steady state solution of the necessary conditions. Finally, assume that the objective functional exists for all admissible pairs of functions.

(a) Write down the necessary conditions for this problem in current value form. Provide an economic interpretation of the Maximum Principle equation.
(b) Prove that the solution $(K^*(t;\boldsymbol{\beta}), I^*(t;\boldsymbol{\beta}))$ of the necessary conditions is the unique optimal solution to the adjustment cost model of the firm under a suitable additional assumption to be determined by you.
(c) Show that the necessary and sufficient conditions can be reduced to a pair of autonomous ordinary differential equations in (K, λ).
(d) Prove that the steady state is a saddle point.
(e) Prove that the steady state defines (K, λ) as locally $C^{(1)}$ functions of the parameter vector $\boldsymbol{\alpha}$, say, $K^s(\boldsymbol{\alpha})$ and $\lambda^s(\boldsymbol{\alpha})$.
(f) Derive the phase portrait for the ordinary differential equations you derived in part (c). Be sure to include your derivations of the slopes of the $\dot{K} = 0$ and $\dot{\lambda} = 0$ isoclines, as well as the vector field in the $K\lambda$-phase plane.
(g) Derive the steady state comparative statics for the output price p. Provide an economic interpretation of the comparative statics result.
(h) Draw the local comparative dynamics phase portrait for the output price p, making sure to label the optimal path from the old to the new steady state clearly. Provide an economic interpretation of the comparative dynamics result.
(i) Find the steady state comparative statics for the purchase price of the investment good g. Provide an economic interpretation of the comparative statics result.
(j) Draw the local comparative dynamics phase portrait for the purchase price of the investment good g. Provide an economic interpretation of the comparative dynamics result.
(k) Find the steady state comparative statics for the unit maintenance cost c. Provide an economic interpretation of the comparative statics result.
(l) Draw the local comparative dynamics phase portrait for the unit maintenance cost c. Provide an economic interpretation of the comparative dynamics result.

FURTHER READING

The seminal paper on the adjustment cost model of the firm is by Eisner and Strotz (1963). Treadway (1970) provides a qualitative analysis of the model akin to that developed in this chapter. Treadway (1971) and Mortensen (1974) analyze some of the qualitative properties of a multivariate extension of the adjustment cost model. Epstein (1982) and Caputo (1992) provide rather extensive comparative dynamics characterizations of the model. See the references in Chapter 20 for the fundamental papers that develop an intertemporal duality theory for the adjustment cost model and for those that attempt to empirically test it and use it for policy purposes.

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