TWELVE

Discounting, Current Values, and Time Consistency

The majority of optimal control problems of interest to economists have the future values of the integrand, whether it is profit, cost, or utility, discounted at some positive constant rate, say, $r > 0$, which is often referred to as the *discount rate*. Because we are dealing with continuous-time optimization problems, the corresponding *discount factor* takes the exponential form e^{-rt}. The purpose of this chapter is to examine the implications this modification of the archetypal optimal control problem has on the economic interpretation of the optimal value function and costate vector, as well as the form that the necessary and sufficient conditions take. In addition, we will provide the logical and rigorous justification for the ubiquitous nature of the exponential form of the discount factor.

In light of the opening remarks, we must consider the following general class of optimal control problems in this chapter:

$$V(\boldsymbol{\alpha}, r, t_0, \mathbf{x}_0, t_1) \stackrel{\text{def}}{=} \max_{\mathbf{u}(\cdot),\mathbf{x}_1} \int_{t_0}^{t_1} f(t, \mathbf{x}(t), \mathbf{u}(t); \boldsymbol{\alpha})\, e^{-rt}\, dt$$

$$\text{s.t.} \quad \dot{\mathbf{x}}(t) = \mathbf{g}(t, \mathbf{x}(t), \mathbf{u}(t); \boldsymbol{\alpha}),\ \mathbf{x}(t_0) = \mathbf{x}_0,\ \mathbf{x}(t_1) = \mathbf{x}_1, \tag{1}$$

$$h^k(t, \mathbf{x}(t), \mathbf{u}(t); \boldsymbol{\alpha}) \geq 0, \quad k = 1, 2, \ldots, K',$$

$$h^k(t, \mathbf{x}(t), \mathbf{u}(t); \boldsymbol{\alpha}) = 0, \quad k = K' + 1, K' + 2, \ldots, K,$$

where $\mathbf{x}(t) \stackrel{\text{def}}{=} (x_1(t), x_2(t), \ldots, x_N(t)) \in \Re^N$ is the state vector, $\mathbf{u}(t) \stackrel{\text{def}}{=} (u_1(t), u_2(t), \ldots, u_M(t)) \in \Re^M$ is the control vector, $\dot{\mathbf{x}}(\cdot) \stackrel{\text{def}}{=} (\dot{x}_1(\cdot), \dot{x}_2(\cdot), \ldots, \dot{x}_N(\cdot))$, $\mathbf{g}(\cdot) \stackrel{\text{def}}{=} (g^1(\cdot), g^2(\cdot), \ldots, g^N(\cdot))$ is the vector of transition functions, $\mathbf{h}(\cdot) \stackrel{\text{def}}{=} (h^1(\cdot), h^2(\cdot), \ldots, h^K(\cdot))$ is the vector of constraint functions, both inequality and equality, and $\boldsymbol{\alpha} \stackrel{\text{def}}{=} (\alpha_1, \alpha_2, \ldots, \alpha_A) \in \Re^A$ is a vector of time-independent parameters that affect the state equation, integrand, and constraint functions. Let $(\mathbf{z}(t; \boldsymbol{\alpha}, r, t_0, \mathbf{x}_0, t_1), \mathbf{v}(t; \boldsymbol{\alpha}, r, t_0, \mathbf{x}_0, t_1))$ be the optimal pair with corresponding costate vector $\boldsymbol{\lambda}(t; \boldsymbol{\alpha}, r, t_0, \mathbf{x}_0, t_1) \in \Re^N$ and Lagrange multiplier vector $\boldsymbol{\mu}(t; \boldsymbol{\alpha}, r, t_0, \mathbf{x}_0, t_1) \in \Re^K$.

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Let us first tackle the issue of the proper economic interpretation of the optimal value function and costate vector for problem (1).

Because of the discount factor e^{-rt} in the integrand, we must be careful to interpret the optimal value function $V(\cdot)$ suitably. In the case of problem (1), $V(\cdot)$ is referred to as the *present value optimal value function*. This is because $V(\cdot)$ has its value discounted back to time $t = 0$ rather than the initial date of the planning horizon, videlicet, $t = t_0$. To see this, first observe that at the initial date of the planning horizon, namely, $t = t_0$, the discount factor e^{-rt_0} is applied to the value $f(t_0, \mathbf{x}(t_0), \mathbf{u}(t_0); \boldsymbol{\alpha})$, thereby discounting it to some time *before* the initial date $t = t_0$. Second, note that at time $t = 0$, we have that $e^{-r0} = 1$, thereby implying that at time $t = 0$, no discounting takes place. Consequently, from these two facts, we may infer that the period to which the value $f(t_0, \mathbf{x}(t_0), \mathbf{u}(t_0); \boldsymbol{\alpha})\, e^{-rt_0}$ is discounted to is $t = 0$. In view of the fact that the same is true in any period of the planning horizon, $f(t, \mathbf{x}(t), \mathbf{u}(t); \boldsymbol{\alpha})\, e^{-rt}$ is discounted to period $t = 0$ for all $t \in [t_0, t_1]$ as well. Accordingly, the value of $V(\cdot)$ is also discounted to period $t = 0$, just as we asserted above.

Now recall that by the dynamic envelope theorem applied to problem (1), we have that $\partial V(\boldsymbol{\alpha}, r, t_0, \mathbf{x}_0, t_1)/\partial x_{0n} = \lambda_n(t_0; \boldsymbol{\alpha}, r, t_0, \mathbf{x}_0, t_1)$, $n = 1, 2, \ldots, N$, thereby implying that $\lambda_n(t_0; \boldsymbol{\alpha}, r, t_0, \mathbf{x}_0, t_1)$ is the *present value* shadow price of a unit of the nth stock at time $t = t_0$ in an optimal plan. In other words, $\lambda_n(t_0; \boldsymbol{\alpha}, r, t_0, \mathbf{x}_0, t_1)$ is the shadow value of a unit of the nth stock at time $t = t_0$ discounted back to time $t = 0$, since $V(\boldsymbol{\alpha}, r, t_0, \mathbf{x}_0, t_1)$ is the total value of the stock at time $t = t_0$ discounted back to period $t = 0$. Alternatively, $\lambda_n(t_0; \boldsymbol{\alpha}, r, t_0, \mathbf{x}_0, t_1)$ is what the owner of a firm who solves problem (1) would pay for a marginal increase in his capital stock at the initial time $t = t_0$ but discounted to period $t = 0$, since the period $t = 0$ is that which the present value optimal value function $V(\cdot)$ is discounted to, as noted above. Moreover, by the development in Chapter 9 in which we rigorously established the economic interpretation of the costate vector and the reasoning just applied to interpret $\lambda_n(t_0; \boldsymbol{\alpha}, r, t_0, \mathbf{x}_0, t_1)$, we may therefore conclude that for any time $t \in [t_0, t_1]$ in problem (1), $\lambda_n(t; \boldsymbol{\alpha}, r, t_0, \mathbf{x}_0, t_1)$ is the shadow value of the nth state variable at time $t \in [t_0, t_1]$ discounted to period $t = 0$. That is to say, $\lambda_n(t; \boldsymbol{\alpha}, r, t_0, \mathbf{x}_0, t_1)$ is the present value shadow price of the nth state variable at any time $t \in [t_0, t_1]$ in an optimal plan.

Let's now take this economic interpretation one step further by considering the following perturbation of problem (1):

$$\tilde{V}(\boldsymbol{\alpha}, r, t_0, \mathbf{x}_0, t_1) \overset{\text{def}}{=} \max_{\mathbf{u}(\cdot), \mathbf{x}_1} \int_{t_0}^{t_1} f(t, \mathbf{x}(t), \mathbf{u}(t); \boldsymbol{\alpha})\, e^{-r(t-t_0)}\, dt$$

$$\text{s.t.} \quad \dot{\mathbf{x}}(t) = \mathbf{g}(t, \mathbf{x}(t), \mathbf{u}(t); \boldsymbol{\alpha}),\ \ \mathbf{x}(t_0) = \mathbf{x}_0,\ \ \mathbf{x}(t_1) = \mathbf{x}_1, \tag{2}$$

$$h^k(t, \mathbf{x}(t), \mathbf{u}(t); \boldsymbol{\alpha}) \geq 0, \quad k = 1, 2, \ldots, K',$$

$$h^k(t, \mathbf{x}(t), \mathbf{u}(t); \boldsymbol{\alpha}) = 0, \quad k = K'+1, K'+2, \ldots, K.$$

The only difference between problems (1) and (2) is that problem (2) has its objective functional multiplied by e^{rt_0}. This observation yields the identity relating the optimal value functions in problems (1) and (2), to wit, $\tilde{V}(\boldsymbol{\alpha}, r, t_0, \mathbf{x}_0, t_1) \equiv e^{rt_0}V(\boldsymbol{\alpha}, r, t_0, \mathbf{x}_0, t_1)$. Given the interpretation of $V(\cdot)$, we may therefore interpret $\tilde{V}(\cdot)$ as the *current value optimal value function*, for its value is discounted to the initial date $t = t_0$ of the planning horizon. Consequently, by the dynamic envelope theorem, we have that $\partial\tilde{V}(\boldsymbol{\alpha}, r, t_0, \mathbf{x}_0, t_1)/\partial x_{0n} = \tilde{\lambda}_n(t_0; \boldsymbol{\alpha}, r, t_0, \mathbf{x}_0, t_1)$, $n = 1, 2, \ldots, N$, thereby implying that $\tilde{\lambda}_n(t_0; \boldsymbol{\alpha}, r, t_0, \mathbf{x}_0, t_1)$ is the *current value shadow price* of a unit of the nth stock at time $t = t_0$ in an optimal plan. That is, $\tilde{\lambda}_n(t_0; \boldsymbol{\alpha}, r, t_0, \mathbf{x}_0, t_1)$ is what the owner of a firm who solves problem (2) would pay for a marginal increase in her capital stock at time $t = t_0$ in period $t = t_0$, because the period $t = t_0$ is the one to which the current value optimal value function $\tilde{V}(\cdot)$ is discounted. Moreover, by the reasoning just applied to interpret $\tilde{\lambda}_n(t_0; \boldsymbol{\alpha}, r, t_0, \mathbf{x}_0, t_1)$ and the development in Chapter 9 in which we rigorously established the economic interpretation of the costate vector, we may conclude that for any time $t \in [t_0, t_1]$ of problem (2), $\tilde{\lambda}_n(t; \boldsymbol{\alpha}, r, t_0, \mathbf{x}_0, t_1)$ is the shadow value of the nth state variable at time $t \in [t_0, t_1]$ discounted to period $t = t_0$. That is to say, $\tilde{\lambda}_n(t; \boldsymbol{\alpha}, r, t_0, \mathbf{x}_0, t_1)$ is the current value shadow price of the nth state variable at any time $t \in [t_0, t_1]$ in an optimal plan. Thus the identity linking the current value and present value shadow values is given by $\tilde{\boldsymbol{\lambda}}(t; \boldsymbol{\alpha}, r, t_0, \mathbf{x}_0, t_1) \equiv e^{rt}\boldsymbol{\lambda}(t; \boldsymbol{\alpha}, r, t_0, \mathbf{x}_0, t_1)$ for all $t \in [t_0, t_1]$. In passing, observe that because $e^{rt_0} > 0$ and is independent of the state and control variables, the optimal values of the state and control vectors to problems (1) and (2) are identical.

In wrapping up the distinction between current values and present values, an important feature in our interpretation should be emphasized. In particular, we defined $\tilde{V}(\cdot)$ as the current value optimal value function *because* its value is discounted to the initial date $t = t_0$ of the planning horizon in problem (2). We are thus treating the initial date of the planning horizon as the current period. This should not be too surprising, inasmuch as when solving an optimal control problem, we take the point of view that we are placed at the initial date of the planning horizon when we select our optimal plan. That is, at the initial date $t = t_0$ of the planning horizon, we select the optimal time path of the control vector for the entire planning horizon, and therefore view the initial date $t = t_0$ of the planning horizon as the current period from which we plan.

With the interpretation of the optimal value function and costate vector of problem (1) finished, we now seek to express the necessary conditions of problem (1) in their current value form, that is, without explicit representation of the discount factor but still accounting for its presence in the optimal control problem. Recall that in order to find the necessary conditions of problem (1), we first form the present value Hamiltonian

$$H(t, \mathbf{x}, \mathbf{u}, \boldsymbol{\lambda}; \boldsymbol{\alpha}, r) \stackrel{\text{def}}{=} f(t, \mathbf{x}, \mathbf{u}; \boldsymbol{\alpha})e^{-rt} + \sum_{n=1}^{N} \lambda_n g^n(t, \mathbf{x}, \mathbf{u}; \boldsymbol{\alpha}),$$

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and then the present value Lagrangian

$$L(t, \mathbf{x}, \mathbf{u}, \boldsymbol{\lambda}, \boldsymbol{\mu}; \boldsymbol{\alpha}, r) \stackrel{\text{def}}{=} f(t, \mathbf{x}, \mathbf{u}; \boldsymbol{\alpha})\, e^{-rt} + \sum_{n=1}^{N} \lambda_n g^n(t, \mathbf{x}, \mathbf{u}; \boldsymbol{\alpha}) + \sum_{k=1}^{K} \mu_k h^k(t, \mathbf{x}, \mathbf{u}; \boldsymbol{\alpha}). \tag{3}$$

The necessary conditions are then given by Theorem 10.1:

$$L_{u_m}(t, \mathbf{x}, \mathbf{u}, \boldsymbol{\lambda}, \boldsymbol{\mu}; \boldsymbol{\alpha}, r) = 0, \quad m = 1, 2, \ldots, M, \tag{4}$$

$$L_{\mu_\ell}(t, \mathbf{x}, \mathbf{u}, \boldsymbol{\lambda}, \boldsymbol{\mu}; \boldsymbol{\alpha}, r) \geq 0,\ \mu_\ell \geq 0,\ \mu_\ell L_{\mu_\ell}(t, \mathbf{x}, \mathbf{u}, \boldsymbol{\lambda}, \boldsymbol{\mu}; \boldsymbol{\alpha}, r) = 0, \quad \ell = 1, 2, \ldots, K', \tag{5}$$

$$L_{\mu_\ell}(t, \mathbf{x}, \mathbf{u}, \boldsymbol{\lambda}, \boldsymbol{\mu}; \boldsymbol{\alpha}, r) = 0, \quad \ell = K' + 1, K' + 2, \ldots, K, \tag{6}$$

$$\dot{\lambda}_i = -L_{x_i}(t, \mathbf{x}, \mathbf{u}, \boldsymbol{\lambda}, \boldsymbol{\mu}; \boldsymbol{\alpha}, r),\ \lambda_i(t_1) = 0, \quad i = 1, 2, \ldots, N, \tag{7}$$

$$\dot{x}_i = L_{\lambda_i}(t, \mathbf{x}, \mathbf{u}, \boldsymbol{\lambda}, \boldsymbol{\mu}; \boldsymbol{\alpha}, r),\ x_i(t_0) = x_{0i}, \quad i = 1, 2, \ldots, N. \tag{8}$$

In many economic problems, it is often more natural and convenient to conduct the analysis, as well as the discussion of the problem and its economic interpretation, in terms of current values, that is, in terms of the values of $V(\cdot)$ and $\boldsymbol{\lambda}(\cdot)$ discounted to the initial (or current) period $t = t_0$ rather than in terms of their values discounted back to time $t = 0$. Another advantage from a mathematical point of view is that if the integrand function $f(\cdot)$, transition function $\mathbf{g}(\cdot)$, and constraint function $\mathbf{h}(\cdot)$ do not depend explicitly on t, then the canonical differential equations will be autonomous if the costate vector $\boldsymbol{\lambda}$ is transformed to its current value form. Let's first tackle the form of the necessary conditions in the current value format and then establish the latter result.

In order to express the necessary conditions for problem (1) in their current value form, we begin by rewriting the Lagrangian defined in Eq. (3) as

$$L(t, \mathbf{x}, \mathbf{u}, \boldsymbol{\lambda}, \boldsymbol{\mu}; \boldsymbol{\alpha}, r) \stackrel{\text{def}}{=} e^{-rt}\left[f(t, \mathbf{x}, \mathbf{u}; \boldsymbol{\alpha}) + \sum_{n=1}^{N} e^{rt}\lambda_n g^n(t, \mathbf{x}, \mathbf{u}; \boldsymbol{\alpha}) + \sum_{k=1}^{K} e^{rt}\mu_k h^k(t, \mathbf{x}, \mathbf{u}; \boldsymbol{\alpha})\right], \tag{9}$$

and then define

$$\tilde{\boldsymbol{\lambda}}(t) \stackrel{\text{def}}{=} e^{rt}\boldsymbol{\lambda}(t) \tag{10}$$

as the current value costate vector and

$$\tilde{\boldsymbol{\mu}}(t) \stackrel{\text{def}}{=} e^{rt}\boldsymbol{\mu}(t) \tag{11}$$

as the current value Lagrange multiplier vector. Recall that $\boldsymbol{\lambda}(t)$ gives the shadow value of the state vector at time t discounted to time zero, that is, it is the present value shadow price vector. Therefore, $\tilde{\boldsymbol{\lambda}}(t)$ gives the shadow value of the state vector at time t discounted to the initial date of the planning horizon, that is, it is the current value shadow price vector.

Using Eqs. (9), (10), and (11), define the *current value Lagrangian* by

$$\begin{aligned}\tilde{L}(t, \mathbf{x}, \mathbf{u}, \tilde{\boldsymbol{\lambda}}, \tilde{\boldsymbol{\mu}}; \boldsymbol{\alpha}) &\stackrel{\text{def}}{=} e^{rt} L(t, \mathbf{x}, \mathbf{u}, \boldsymbol{\lambda}, \boldsymbol{\mu}; \boldsymbol{\alpha}, r) \\ &= f(t, \mathbf{x}, \mathbf{u}; \boldsymbol{\alpha}) + \sum_{n=1}^{N} \tilde{\lambda}_n g^n(t, \mathbf{x}, \mathbf{u}; \boldsymbol{\alpha}) + \sum_{k=1}^{K} \tilde{\mu}_k h^k(t, \mathbf{x}, \mathbf{u}; \boldsymbol{\alpha}).\end{aligned} \tag{12}$$

Differentiating Eq. (10) with respect to t and using necessary condition (7) and Eq. (10) gives

$$\frac{d}{dt}\tilde{\lambda}_i(t) = e^{rt}\dot{\lambda}_i(t) + re^{rt}\lambda_i(t) = r\tilde{\lambda}_i - e^{rt} L_{x_i}(t, \mathbf{x}, \mathbf{u}, \boldsymbol{\lambda}, \boldsymbol{\mu}; \boldsymbol{\alpha}, r), \tag{13}$$

for $i = 1, 2, \ldots, N$. From Eq. (12), it follows that $\tilde{L}_{x_i}(t, \mathbf{x}, \mathbf{u}, \tilde{\boldsymbol{\lambda}}, \tilde{\boldsymbol{\mu}}; \boldsymbol{\alpha}) = e^{rt} L_{x_i}$ $(t, \mathbf{x}, \mathbf{u}, \boldsymbol{\lambda}, \boldsymbol{\mu}; \boldsymbol{\alpha}, r)$, thus implying that Eq. (13) can be written as

$$\dot{\tilde{\lambda}}_i = r\tilde{\lambda}_i - \tilde{L}_{x_i}(t, \mathbf{x}, \mathbf{u}, \tilde{\boldsymbol{\lambda}}, \tilde{\boldsymbol{\mu}}; \boldsymbol{\alpha}), \quad i = 1, 2, \ldots, N.$$

This is the costate equation in current value form. Using Eq. (10), the free boundary transversality condition $\boldsymbol{\lambda}(t_1) = \mathbf{0}_N$ implies that $\boldsymbol{\lambda}(t_1) = e^{-rt_1}\tilde{\boldsymbol{\lambda}}(t_1) = \mathbf{0}_N$. Moreover, seeing as $e^{-rt_1} \neq 0$, $\boldsymbol{\lambda}(t_1) = \mathbf{0}_N$ is equivalent to $\tilde{\boldsymbol{\lambda}}(T) = \mathbf{0}_N$. We are therefore finished with the conversion of necessary condition (7) to its current value form.

Turning to the necessary condition (8), that is, the state equation and initial condition, it follows easily from Eq. (12) that

$$\dot{x}_i = \tilde{L}_{\tilde{\lambda}_i}(t, \mathbf{x}, \mathbf{u}, \tilde{\boldsymbol{\lambda}}, \tilde{\boldsymbol{\mu}}; \boldsymbol{\alpha}) = g^i(t, \mathbf{x}, \mathbf{u}; \boldsymbol{\alpha}), \quad i = 1, 2, \ldots, N.$$

As far as the initial condition $\mathbf{x}(t_0) = \mathbf{x}_0$ is concerned, it clearly still holds in current value form, for the transformation to current values leaves the initial value of the state vector unaffected.

By employing Eq. (12), necessary condition (4) can be written as

$$L_{u_m}(t, \mathbf{x}, \mathbf{u}, \boldsymbol{\lambda}, \boldsymbol{\mu}; \boldsymbol{\alpha}, r) = \frac{\partial}{\partial u_m}[e^{-rt}\tilde{L}(t, \mathbf{x}, \mathbf{u}, \tilde{\boldsymbol{\lambda}}, \tilde{\boldsymbol{\mu}}; \boldsymbol{\alpha})] = 0, \quad m = 1, 2, \ldots, M.$$

Because $e^{-rt} \neq 0$ and is independent of the control vector, necessary condition (4) can be written equivalently in terms of the current value Lagrangian as

$$\tilde{L}_{u_m}(t, \mathbf{x}, \mathbf{u}, \tilde{\boldsymbol{\lambda}}, \tilde{\boldsymbol{\mu}}; \boldsymbol{\alpha}) = 0, \quad m = 1, 2, \ldots, M.$$

All that remains is to transform the necessary conditions (5) and (6) to current value form.

By employing Eq. (12) again, necessary condition (6) can be written as

$$L_{\mu_\ell}(t, \mathbf{x}, \mathbf{u}, \boldsymbol{\lambda}, \boldsymbol{\mu}; \boldsymbol{\alpha}, r) = \frac{\partial}{\partial \mu_\ell}[e^{-rt}\tilde{L}(t, \mathbf{x}, \mathbf{u}, \tilde{\boldsymbol{\lambda}}, \tilde{\boldsymbol{\mu}}; \boldsymbol{\alpha})] = 0,$$
$$\ell = K' + 1, K' + 2, \ldots, K.$$

Because $e^{-rt} \neq 0$ and is independent of the Lagrange multiplier vector, necessary condition (6) can be written equivalently in terms of the current value Lagrangian as

$$\tilde{L}_{\mu_\ell}(t, \mathbf{x}, \mathbf{u}, \tilde{\boldsymbol{\lambda}}, \tilde{\boldsymbol{\mu}}; \boldsymbol{\alpha}) = 0, \quad \ell = K' + 1, K' + 2, \ldots, K.$$

Let's finish up with necessary condition (5).

With respect to necessary condition (5), notice that we have to develop three equivalent current value expressions. By employing Eq. (12) again, the first necessary condition of Eq. (5) can be written as

$$L_{\mu_\ell}(t, \mathbf{x}, \mathbf{u}, \boldsymbol{\lambda}, \boldsymbol{\mu}; \boldsymbol{\alpha}, r) = \frac{\partial}{\partial \mu_\ell}[e^{-rt}\tilde{L}(t, \mathbf{x}, \mathbf{u}, \tilde{\boldsymbol{\lambda}}, \tilde{\boldsymbol{\mu}}; \boldsymbol{\alpha})] \geq 0, \quad \ell = 1, 2, \ldots, K'.$$

Because $e^{-rt} > 0$ and is independent of the Lagrange multiplier vector, the first necessary condition of Eq. (5) can be written equivalently in terms of the current value Lagrangian as

$$\tilde{L}_{\mu_\ell}(t, \mathbf{x}, \mathbf{u}, \tilde{\boldsymbol{\lambda}}, \tilde{\boldsymbol{\mu}}; \boldsymbol{\alpha}) \geq 0, \quad \ell = 1, 2, \ldots, K'.$$

Using Eq. (11) and the second necessary condition of Eq. (5), it follows that $\boldsymbol{\mu}(t) = e^{-rt}\tilde{\boldsymbol{\mu}}(t) \geq 0$. In view of the fact that $e^{-rt} > 0$, $\boldsymbol{\mu}(t) \geq 0$ is equivalent to $\tilde{\boldsymbol{\mu}}(t) \geq 0$. Finally, employing Eqs. (11) and (12) another time, the third necessary condition of Eq. (5) can be written as

$$\mu_\ell L_{\mu_\ell}(t, \mathbf{x}, \mathbf{u}, \boldsymbol{\lambda}, \boldsymbol{\mu}; \boldsymbol{\alpha}, r) = e^{-rt}\tilde{\mu}_\ell \frac{\partial}{\partial \mu_\ell}[e^{-rt}\tilde{L}(t, \mathbf{x}, \mathbf{u}, \tilde{\boldsymbol{\lambda}}, \tilde{\boldsymbol{\mu}}; \boldsymbol{\alpha})] = 0,$$
$$\ell = 1, 2, \ldots, K'.$$

Because $e^{-rt} \neq 0$ and is independent of the Lagrange multiplier vector, the third necessary condition of Eq. (5) can be written equivalently in terms of the current value Lagrangian and the current value Lagrange multiplier as

$$\tilde{\mu}_\ell \tilde{L}_{\mu_\ell}(t, \mathbf{x}, \mathbf{u}, \tilde{\boldsymbol{\lambda}}, \tilde{\boldsymbol{\mu}}; \boldsymbol{\alpha}) = 0, \quad \ell = 1, 2, \ldots, K'.$$

The above results have established the following theorem, which we will use quite often in our application of optimal control theory to economic problems.

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Theorem 12.1: *In optimal control problem (1), the necessary conditions in Eqs. (4) through (8) from Theorem 10.1 can be written equivalently in current value form as*

$$\tilde{L}_{u_m}(t, \mathbf{x}, \mathbf{u}, \tilde{\boldsymbol{\lambda}}, \tilde{\boldsymbol{\mu}}; \boldsymbol{\alpha}) = 0, \quad m = 1, 2, \ldots, M, \tag{14}$$

$$\tilde{L}_{\mu_\ell}(t, \mathbf{x}, \mathbf{u}, \tilde{\boldsymbol{\lambda}}, \tilde{\boldsymbol{\mu}}; \boldsymbol{\alpha}) \geq 0,\ \tilde{\mu}_\ell \geq 0,\ \tilde{\mu}_\ell \tilde{L}_{\mu_\ell}(t, \mathbf{x}, \mathbf{u}, \tilde{\boldsymbol{\lambda}}, \tilde{\boldsymbol{\mu}}; \boldsymbol{\alpha}) = 0,$$

$$\ell = 1, 2, \ldots, K', \tag{15}$$

$$\tilde{L}_{\mu_\ell}(t, \mathbf{x}, \mathbf{u}, \tilde{\boldsymbol{\lambda}}, \tilde{\boldsymbol{\mu}}; \boldsymbol{\alpha}) = 0, \quad \ell = K'+1, K'+2, \ldots, K, \tag{16}$$

$$\dot{\tilde{\lambda}}_i = r\tilde{\lambda}_i - \tilde{L}_{x_i}(t, \mathbf{x}, \mathbf{u}, \tilde{\boldsymbol{\lambda}}, \tilde{\boldsymbol{\mu}}; \boldsymbol{\alpha}),\ \tilde{\lambda}_i(t_1) = 0, \quad i = 1, 2, \ldots, N, \tag{17}$$

$$\dot{x}_i = \tilde{L}_{\tilde{\lambda}_i}(t, \mathbf{x}, \mathbf{u}, \tilde{\boldsymbol{\lambda}}, \tilde{\boldsymbol{\mu}}; \boldsymbol{\alpha}),\ x_i(t_0) = x_{0i}, \quad i = 1, 2, \ldots, N. \tag{18}$$

Take note of the remarkable similarity between the necessary conditions in their present value and current value forms. The essential difference lies in the costate equation. The effect of the transformation to current values on the sufficiency theorems of Chapter 6 is in effect trivial. This is because the concavity of the present value Lagrangian $L(\cdot)$ in the state and control vectors is equivalent to the concavity of the current value Lagrangian $\tilde{L}(\cdot)$ in the state and control vectors, since $\tilde{L}(\cdot) \stackrel{\text{def}}{=} e^{rt}L(\cdot)$, $e^{rt} > 0$, and e^{rt} is independent of the state and control vectors. This is all that really needs to be said about sufficiency results when the current value formulation of an optimal control problem is analyzed.

Let's now turn to an analytical advantage of the current value formulation alluded to earlier. We formulate the advantage in the following theorem.

Theorem 12.2: *In optimal control problem (1), if the integrand function $f(\cdot)$, transition function $\mathbf{g}(\cdot)$, and constraint function $\mathbf{h}(\cdot)$ are not explicit functions of the independent variable t, that is, if $f_t(\cdot) \equiv 0$, $\mathbf{g}_t(\cdot) \equiv \mathbf{0}_N$, and $\mathbf{h}_t(\cdot) \equiv \mathbf{0}_K$, and furthermore, if the necessary conditions (14) through (16) of Theorem 12.1 determine the control vector and current value Lagrange multiplier vector as functions of the state vector, current value costate vector, and parameters, then the canonical differential equations in current value form are autonomous, that is, they do not depend explicitly on the independent variable t.*

Proof: First, observe that the current value Lagrangian for problem (1) under the assumptions $f_t(\cdot) \equiv 0$, $\mathbf{g}_t(\cdot) \equiv \mathbf{0}_N$, and $\mathbf{h}_t(\cdot) \equiv \mathbf{0}_K$ does not depend explicitly on the independent variable t. The necessary conditions of Theorem 12.1 thus reduce to

$$\tilde{L}_{u_m}(\mathbf{x}, \mathbf{u}, \tilde{\boldsymbol{\lambda}}, \tilde{\boldsymbol{\mu}}; \boldsymbol{\alpha}) = 0, \quad m = 1, 2, \ldots, M, \tag{19}$$

$$\tilde{L}_{\mu_\ell}(\mathbf{x}, \mathbf{u}, \tilde{\boldsymbol{\lambda}}, \tilde{\boldsymbol{\mu}}; \boldsymbol{\alpha}) \geq 0,\ \tilde{\mu}_\ell \geq 0,\ \tilde{\mu}_\ell \tilde{L}_{\mu_\ell}(\mathbf{x}, \mathbf{u}, \tilde{\boldsymbol{\lambda}}, \tilde{\boldsymbol{\mu}}; \boldsymbol{\alpha}) = 0,$$

$$\ell = 1, 2, \ldots, K', \tag{20}$$

$$\tilde{L}_{\mu_\ell}(\mathbf{x}, \mathbf{u}, \tilde{\boldsymbol{\lambda}}, \tilde{\boldsymbol{\mu}}; \boldsymbol{\alpha}) = 0, \quad \ell = K'+1, K'+2, \ldots, K, \tag{21}$$

$$\dot{\tilde{\lambda}}_i = r\tilde{\lambda}_i - \tilde{L}_{x_i}(\mathbf{x}, \mathbf{u}, \tilde{\boldsymbol{\lambda}}, \tilde{\boldsymbol{\mu}}; \boldsymbol{\alpha}),\ \tilde{\lambda}_i(t_1) = 0, \quad i = 1, 2, \ldots, N, \tag{22}$$

$$\dot{x}_i = \tilde{L}_{\tilde{\lambda}_i}(\mathbf{x}, \mathbf{u}, \tilde{\boldsymbol{\lambda}}, \tilde{\boldsymbol{\mu}}; \boldsymbol{\alpha}),\ x_i(t_0) = x_{0i}, \quad i = 1, 2, \ldots, N. \tag{23}$$

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By assumption, necessary conditions (19) through (21) can in principle be solved for the control vector and Lagrange multiplier vector as functions of the state vector, current value costate vector, and parameters, say $\mathbf{u} = \mathbf{u}^*(\mathbf{x}, \tilde{\boldsymbol{\lambda}}; \boldsymbol{\alpha})$ and $\tilde{\boldsymbol{\mu}} = \tilde{\boldsymbol{\mu}}^*(\mathbf{x}, \tilde{\boldsymbol{\lambda}}; \boldsymbol{\alpha})$. Therefore, by substituting $\mathbf{u} = \mathbf{u}^*(\mathbf{x}, \tilde{\boldsymbol{\lambda}}; \boldsymbol{\alpha})$ and $\tilde{\boldsymbol{\mu}} = \tilde{\boldsymbol{\mu}}^*(\mathbf{x}, \tilde{\boldsymbol{\lambda}}; \boldsymbol{\alpha})$ into the canonical differential equations (22) and (23), they can be expressed as

$$\dot{\tilde{\lambda}}_i = r\tilde{\lambda}_i - \tilde{L}_{x_i}(\mathbf{x}, \mathbf{u}^*(\mathbf{x}, \tilde{\boldsymbol{\lambda}}; \boldsymbol{\alpha}), \tilde{\boldsymbol{\lambda}}, \tilde{\boldsymbol{\mu}}^*(\mathbf{x}, \tilde{\boldsymbol{\lambda}}; \boldsymbol{\alpha}); \boldsymbol{\alpha}),\ \tilde{\lambda}_i(t_1) = 0, \quad i = 1, 2, \ldots, N,$$

$$\dot{x}_i = \tilde{L}_{\tilde{\lambda}_i}(\mathbf{x}, \mathbf{u}^*(\mathbf{x}, \tilde{\boldsymbol{\lambda}}; \boldsymbol{\alpha}), \tilde{\boldsymbol{\lambda}}, \tilde{\boldsymbol{\mu}}^*(\mathbf{x}, \tilde{\boldsymbol{\lambda}}; \boldsymbol{\alpha}); \boldsymbol{\alpha}),\ x_i(t_0) = x_{0i}, \quad i = 1, 2, \ldots, N.$$

Because t does not appear explicitly in this pair of ordinary differential equations, they are, by definition, autonomous. Q.E.D.

In general, autonomous differential equations are easier to solve than nonautonomous differential equations. Even if an explicit solution is not possible, one can use a phase diagram to qualitatively analyze the solution of an autonomous differential equation system. We will demonstrate this feature of autonomous systems of differential equations in several of the ensuing chapters, in which we will also delve more deeply into this issue.

The next issue we wish to explore in this chapter is the form of the discount factor exhibited here and in virtually every optimal control problem in economics, to wit, e^{-rt}, where the discount rate $r > 0$ is constant, that is, it is not a function of the independent variable t. One generalization is a time varying discount rate, say, $\rho(t) > 0$, in which case the discount factor is of the form $\exp[-\int_0^t \rho(\tau)d\tau]$. Note that if $\rho(t) = r > 0$ is constant, then the time varying discount factor reduces to the familiar form, because then $\exp[-\int_0^t \rho(\tau)d\tau] = \exp[-\int_0^t r\, d\tau] = e^{-rt}$. In a mental exercise, you are asked to prove that optimal control problem (1) with the discount factor $\exp[-\int_0^t \rho(\tau)d\tau]$ can be reduced to current value form, but that the resulting canonical differential equations are not autonomous even when the integrand function $f(\cdot)$, transition function $\mathbf{g}(\cdot)$, and constraint function $\mathbf{h}(\cdot)$ do not depend explicitly on the independent variable t. This means that the current value approach is relatively less useful for an optimal control problem with a time varying discount rate than it is for one in which the discount rate is a positive constant. This, however, is not a good reason to avoid the use of a time varying discount rate. A sound and logically consistent reason is provided by the principle of *time consistency* or *dynamic consistency*, to which we now turn.

Consider, therefore, the ensuing optimal control problem without the presence of inequality or equality constraints:

$$\max_{\mathbf{u}(\cdot), \mathbf{x}_T} J_0[\mathbf{x}(\cdot), \mathbf{u}(\cdot)] \stackrel{\text{def}}{=} \int_0^T f(t, \mathbf{x}(t), \mathbf{u}(t); \boldsymbol{\alpha})\, \delta(t)\, dt \tag{24}$$

$$\text{s.t.} \quad \dot{\mathbf{x}}(t) = \mathbf{g}(t, \mathbf{x}(t), \mathbf{u}(t); \boldsymbol{\alpha}),\ \mathbf{x}(0) = \mathbf{x}_0, \mathbf{x}(T) = \mathbf{x}_T,$$

where $\delta(t)$ is a *time varying discount factor* that we normalize (without loss of generality) by setting $\delta(0) = 1$. Let $(\mathbf{z}(t;\boldsymbol{\alpha}, \mathbf{x}_0, T), \mathbf{v}(t;\boldsymbol{\alpha}, \mathbf{x}_0, T))$ be the optimal pair, which is assumed to exist. The subscript on the functional $J[\cdot]$ is there to indicate planning begins in period $t = 0$. Though the change from the discount factor e^{-rt} to the time varying one $\delta(t)$ may appear innocuous, it may lead to a problem of internal inconsistency of sorts. Let's now proceed to precisely define the internal consistency, demonstrate why the internal inconsistency exists and the crucial role played by the discount factor, and determine a necessary and sufficient condition to render the individual's plan internally consistent.

Suppose that the decision maker who solves problem (24) is allowed to recalculate her optimal plan starting at a later date, say, $s \in (0, T)$. Let's also suppose that the decision maker uses the discount factor $\delta(t - s)$ in the truncated problem and starts the planning given the state that was optimal at time $t = s$ in problem (24). All other functions and parameters, however, remained unchanged. She would use the discount factor $\delta(t - s)$ if it reflected the weight attached to the value of the integrand (which could be instantaneous utility) at time t because of its distance from the starting date of the planning horizon, *not* by virtue of its calendar date. The truncated control problem to be solved by the decision maker is therefore given by

$$\begin{aligned} &\max_{\mathbf{u}(\cdot),\mathbf{x}_T} J_s[\mathbf{x}(\cdot), \mathbf{u}(\cdot)] \stackrel{\text{def}}{=} \int_s^T f(t, \mathbf{x}(t), \mathbf{u}(t); \boldsymbol{\alpha})\, \delta(t - s)\, dt \\ &\text{s.t.} \quad \dot{\mathbf{x}}(t) = \mathbf{g}(t, \mathbf{x}(t), \mathbf{u}(t); \boldsymbol{\alpha}),\ \mathbf{x}(s) = \mathbf{x}_s,\ \mathbf{x}(T) = \mathbf{x}_T, \end{aligned} \tag{25}$$

where $\mathbf{x}_s = \mathbf{z}(s;\boldsymbol{\alpha}, \mathbf{x}_0, T)$ is the optimal value of the state vector at time $t = s$ from problem (24) and $s \in (0, T)$ is any given initial or starting date at which the planning begins. Because the discount factor has been linearly shifted, the optimal pair for problem (25) is in general different from the optimal pair for problem (24). In concrete terms, this observation implies that if I plan optimally on Monday for consumption on Monday, Tuesday, and Wednesday [problem (24)], then when Tuesday rolls around and I reconsider my optimal consumption plan for Tuesday and Wednesday [problem (25)], I will generally find that my initial (i.e., from Monday's perspective) consumption plans for Tuesday and Wednesday are no longer optimal now that Tuesday has arrived! More generally, this observation means that it is not rational to obey a plan that was optimal when viewed from an earlier date [problem (24)] if it is not the optimal one at the present date [problem (25)]. Said differently, the optimal plan will in general change with a change in the initial date $s \in (0, T)$ from which one plans. Stated in this last manner, it is not at all surprising that the optimal pair for problem (24) is not, in general, the same as the optimal pair for problem (25), as comparative dynamics has taught us that, in general, a change in any parameter in an optimal control problem, including the starting date, affects the optimal solution pair.

At this point, it is important to make note of the fact that the relative weight a person may assign to the value of a future action or decision, that is, the manner of discounting, may depend on either or both of the following two things: (i) the *time distance* of the future date from the present date, or (ii) the *calendar date* of the future decision. For example, the weight I assign to the pleasure of drinking my favorite bottle of Petite Sirah on June 12, 2004, may depend on the fact that this date is a certain length of time away from the present date, or the fact that it is my birthday. To the extent that time-distance is important, I will likely assign a higher weight to June 12, 2004, as it draws near. If only the calendar date matters, then the weight will not change as June 12, 2004, approaches. Both bases for discounting a future date are included in the objective functional $J_s[\mathbf{x}(\cdot), \mathbf{u}(\cdot)]$. The importance of the calendar date enters through the explicit appearance of t in the integrand function $f(\cdot)$, whereas the importance of time-distance is given by the discount factor $\delta(t - s)$.

Let's provide a geometric view of the aforementioned divergence concerning the optimal solution pairs for problems (24) and (25) before moving on to the more formal aspects of the issue. Consider, therefore, an individual who reevaluates his optimal plan periodically, say, at times $s = s_1$, $s = s_2$, and $s = s_3$, where $0 < s_1 < s_2 < s_3 < T$, and set $M = 1$ for the purpose of graphing. Let $v_0(\cdot)$ be the optimal control function for this individual given $s = 0$, that is, it is the optimal control function for problem (24). Similarly, let $v_i(\cdot)$, $i = 1, 2, 3$, be the optimal control functions for $s = s_1$, $s = s_2$, and $s = s_3$, respectively, that is, they are the respective optimal control functions for problem (25) for initial dates $s = s_1$, $s = s_2$, and $s = s_3$.

Let's now consider Figure 12.1. If the individual does not reconsider his original optimal plan during the period $0 \le t < s_1$, then he abides by it and follows the thicker curve $v_0(t)$ for this period of time. At s_1, however, he reconsiders his plan and chooses $v_1(\cdot)$ as the optimal control function and thus follows the curve $v_1(t)$ over the period $s_1 \le t < s_2$. Therefore, at time s_1, the value of the individual's optimal control jumps from the thicker curve $v_0(t)$ to the thinner curve $v_1(t)$. The same argument applies when the next time comes for reevaluation of the optimal plan, and so on. This process of reevaluation of the original optimal plan at discrete intervals of the planning horizon thus leads to the optimal control path taking on the sawtooth pattern displayed in Figure 12.1. If the original plan is continuously reevaluated, then any single plan is optimal only at an instant $t = s$. In this case, actual behavior is given by the locus of the optimal control for $t = s$ as determined by the necessary conditions for the truncated control problem as s proceeds from 0 to T. We would say in this case that the individual's original optimal plan is *time inconsistent* or *dynamically inconsistent*, since the original plan would not be followed through if the individual reevaluated it at a later date in the planning horizon. A *time consistent* or *dynamically consistent* plan, therefore, is one in which the individual would stick to the original plan even when it was reevaluated at a later date in the planning horizon.

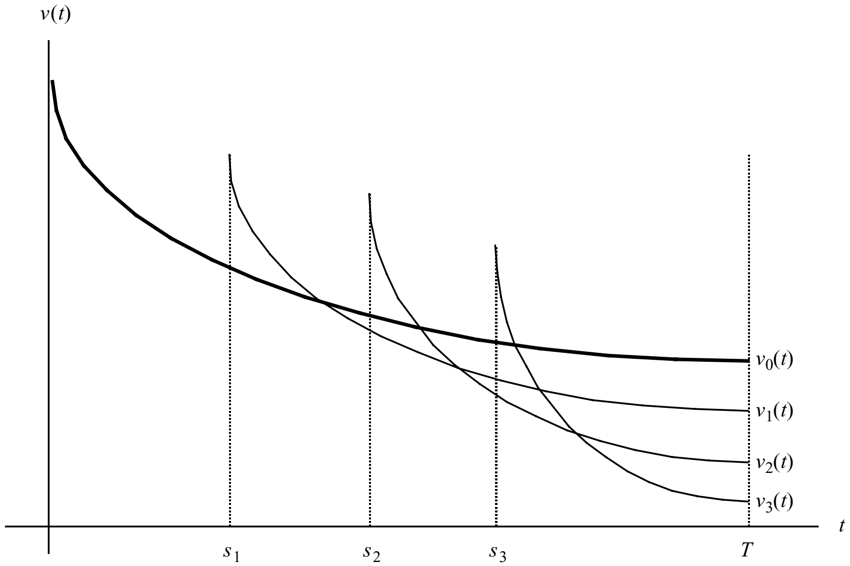

Figure 12.1

It is important to note that what we have been discussing here is the actual dynamics of intertemporal maximization, as opposed to the plan for the future that is made at a given moment. It is also equally important to remember that when the individual reconsiders his optimal plan at a later date, the only thing in the optimal control problem that has changed is the discount factor that weights the values of the integrand function. The parameters and functions are identical to those when the initial plan was set down. Moreover, there is no uncertainty in the problem, so that is not the source of time inconsistency. The time inconsistency is a result of the form of the discount factor. This is an extremely important observation to keep in mind when trying to gain intuition about time consistency.

To summarize, the question raised in the above discussion may be stated as follows: Is the optimal path for the control variables in problem (24) the same as that in problem (25)? Our answer has been *no* in general.

The following important theorem gives a simple but powerful necessary and sufficient condition for a plan to be time consistent or dynamically consistent. In other words, the theorem gives a precise condition under which an individual who continuously reevaluates his planned course of action will in fact confirm his earlier choices, and therefore carry out the plan of action originally selected. You will be asked to prove the sufficiency part of it in a mental exercise. The necessity part of the proof is given below.

Theorem 12.3 (Time Consistency): *The optimal pair* $(\mathbf{z}(t;\boldsymbol{\alpha},\mathbf{x}_0,T),\mathbf{v}(t;\boldsymbol{\alpha},\mathbf{x}_0,T))$ *for problem (24) is also the optimal pair for problem (25) over the planning horizon* $[s,T]$, $s\in[0,T)$, *if and only if* $\delta(t)=e^{-rt}$ *for some constant* r.

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Proof: For expository purposes, we prove necessity under the assumptions $M = N = 1$, so that we are dealing with one control variable and one state variable. We seek to prove that if the solution to control problems (24) and (25) is the same over the planning horizon $[s, T]$, then the discount factor function $\delta(\cdot)$ takes the form $\delta(t) = e^{-rt}$ for some constant r. This implies that the necessary conditions for control problems (24) and (25) must be the same over the planning horizon $[s, T]$ for the answers to be identical. This equivalence will imply a differential equation that the discount factor must satisfy. Solution of this differential equation will yield the desired conclusion.

To begin, form the present value Hamiltonian function for problem (24), namely,

$$H(t, x, u, \lambda; \boldsymbol{\alpha}) \stackrel{\text{def}}{=} \delta(t) f(t, x, u; \boldsymbol{\alpha}) + \lambda\, g(t, x, u; \boldsymbol{\alpha}),$$

and compute the necessary conditions

$$H_u(t, x, u, \lambda; \boldsymbol{\alpha}) = \delta(t) f_u(t, x, u; \boldsymbol{\alpha}) + \lambda g_u(t, x, u; \boldsymbol{\alpha}) = 0, \tag{26}$$

$$\dot{\lambda} = -H_x(t, x, u, \lambda; \boldsymbol{\alpha}) = -\delta(t) f_x(t, x, u; \boldsymbol{\alpha}) - \lambda g_x(t, x, u; \boldsymbol{\alpha}),\ \ \lambda(T) = 0, \tag{27}$$

$$\dot{x} = H_\lambda(t, x, u, \lambda; \boldsymbol{\alpha}) = g(t, x, u; \boldsymbol{\alpha}),\ \ x(0) = x_0, \tag{28}$$

which hold over the planning horizon $[0, T]$. Given that the optimal pair must satisfy Eq. (26) identically for each value of t in the planning horizon $[0, T]$, it is valid to differentiate Eq. (26) with respect to t and the resulting differential equation will still be equal to zero. Doing just that yields

$$\frac{d}{dt} H_u(t, x, u, \lambda; \boldsymbol{\alpha}) = \delta(t) \dot{f}_u(t) + \dot{\delta}(t) f_u(t) + \lambda \dot{g}_u(t) + \dot{\lambda} g_u(t) = 0, \forall\, t \in [0, T], \tag{29}$$

where, for example, $\dot{f}_u(t) \stackrel{\text{def}}{=} \frac{d}{dt} f_u(t, x, u; \boldsymbol{\alpha}) = f_{ut}(t, x, u; \boldsymbol{\alpha}) + f_{ux}(t, x, u; \boldsymbol{\alpha})\dot{x} + f_{uu}(t, x, u; \boldsymbol{\alpha})\dot{u}$. Upon substituting for λ from Eq. (26) and $\dot{\lambda}$ from Eq. (27), we arrive at the following differential equation:

$$f_u(t)\left[\frac{\dot{\delta}(t)}{\delta(t)}\right] = -\dot{f}_u(t) + \frac{f_u(t)\dot{g}_u(t)}{g_u(t)} + g_u(t) f_x(t) - f_u(t) g_x(t), \forall\, t \in [0, T], \tag{30}$$

which you will be asked to verify in a mental exercise.

Because control problem (25) is identical to (24) except for the fact that $\delta(t - s)$ replaces $\delta(t)$ in the integrand and the planning horizon is $[s, T]$, it immediately follows that the recipe used in deriving Eq. (30), when applied to problem (25), gives

$$f_u(t)\left[\frac{\dot{\delta}(t-s)}{\delta(t-s)}\right] = -\dot{f}_u(t) + \frac{f_u(t)\dot{g}_u(t)}{g_u(t)} + g_u(t) f_x(t) - f_u(t) g_x(t), \forall\, t \in [s, T]. \tag{31}$$

Because the right-hand sides of Eqs. (30) and (31) are equal, so must be the left-hand sides. Equating the left-hand sides of Eqs. (30) and (31) and canceling the term $f_u(t)$ gives the differential equation that must be obeyed by the discount function, assuming that the solutions to control problems (24) and (25) are identical over the planning horizon $[s, T]$, videlicet,

$$\frac{\dot{\delta}(t)}{\delta(t)} = \frac{\dot{\delta}(t-s)}{\delta(t-s)}, \; \forall\, t \in [s, T]. \tag{32}$$

This must hold for *every* $s \in [0, T)$ seeing as s was arbitrary but fixed in the foregoing derivation. But the requirement that Eq. (32) hold for every $s \in [0, T)$ means that Eq. (32) must be constant, because the left-hand side of Eq. (32) is independent of s. Thus for $s = 0$, the differential equation obeyed by the discount factor in the time consistent case is

$$\frac{\dot{\delta}(t)}{\delta(t)} = c_1, \; \forall\, t \in [0, T],$$

where c_1 is a constant. This says the logarithmic rate of change of the discount factor must be a constant in a time consistent plan. Integrating this ordinary differential equation, either by separating the variables or by use of the integrating factor $e^{-c_1 t}$, gives the general solution of the differential equation, scilicet, $\delta(t) = c_2 e^{c_1 t}$, where c_2 is a constant of integration. Using the normalization condition $\delta(0) = 1$ implies that $c_2 = 1$. Thus the specific form of the discount factor is $\delta(t) = e^{c_1 t}$, where c_1 is a constant, which was what we wished to demonstrate. Q.E.D.

This theorem asserts that a decision maker who solves problem (24), if given an opportunity to reconsider her original optimal plan at a later date, as reflected by the truncated control problem (25), will find the continuation of the original optimal plan to be optimal in the truncated control problem if and only if the discount factor is of the form $\delta(t) = e^{-rt}$, the constant exponential variety. More concretely, a discount function of the form given by Theorem 12.3 implies that the relative importance of 2004 and 2005 is the same in 2004 as it is in 2003. Consequently, when one decides in 2003 how to apportion one's wealth between 2004 and 2005, this is the same decision one would make in 2004. Thus, in 2004, the plan laid down in 2003 is confirmed.

Note that the functions and parameters are identical in problems (24) and (25), and that the initial value of the state vector in the truncated control problem (25) is the optimal value of the state vector from the control problem (24). At first glance, therefore, one might *mistakenly* think that the principle of optimality can be applied to problem (25) to conclude that the optimal pairs for problems (24) and (25) are identical over $[s, T]$. The problem here is that the discount factor in problems (24) and (25) differ in how they weigh the integrand at common dates in the planning horizon; hence one cannot appeal to the principle of optimality to claim that the optimal pair for problems (24) and (25) are identical over $[s, T]$. For example, at time $t = s$, the value of the integrand in problem (24) is given by $f(\mathbf{x}(s), \mathbf{u}(s); \boldsymbol{\alpha})\, \delta(s)$, whereas that in problem (25) is given by $f(\mathbf{x}(s), \mathbf{u}(s); \boldsymbol{\alpha})\,\delta(0)$, which are not, in general, equal. Note that even if $\delta(t) = e^{-rt}$, the values of the integrands are not equal, as they differ by the constant $\delta(s) = e^{rs}$, since t is the dummy variable of integration. Note that this observation is the key to establishing the sufficiency part of Theorem 12.3.

To sum up, an original plan is time consistent if and only if the discount factor function $\delta(\cdot)$ is of the form $\delta(t) = e^{-rt}$ for some constant r. Any other form of the discount factor would render the original optimal plan time inconsistent. Thus one formal and economically valid justification for constant exponential discounting in dynamic economic theory is the principle of time consistency. This principle, therefore, is in large part responsible for the prevalence of $\delta(t) = e^{-rt}$ as a discount factor in continuous-time optimal control problems in economics. It also has the added advantage that if the integrand function $f(\cdot)$ and the transition function $\mathbf{g}(\cdot)$ do not depend explicitly on the independent variable t, then we may conclude from Theorem 12.2 that the canonical equations are autonomous. This simplification yields a nontrivial advantage for determining the qualitative properties of the solution of a control problem via a phase diagram.

In closing out this chapter, let's place a little more structure on problem (1) by assuming that the integrand function $f(\cdot)$ and the transition function $\mathbf{g}(\cdot)$ do not depend explicitly on the independent variable t, there are no inequality or equality constraints, the planner uses the time-consistent discount factor $\delta(t) = e^{-rt}$, $t_0 = 0$, and $t_1 = T$. That is, we are now interested in the following class of control problems:

$$\tilde{V}(\boldsymbol{\alpha}, r, \mathbf{x}_0, T) \stackrel{\text{def}}{=} \max_{\mathbf{u}(\cdot), \mathbf{x}_T} \int_0^T f(\mathbf{x}(t), \mathbf{u}(t); \boldsymbol{\alpha})\, e^{-rt}\, dt \tag{33}$$

$$\text{s.t.} \quad \dot{\mathbf{x}}(t) = \mathbf{g}(\mathbf{x}(t), \mathbf{u}(t); \boldsymbol{\alpha}),\ \mathbf{x}(0) = \mathbf{x}_0,\ \mathbf{x}(T) = \mathbf{x}_T.$$

Let $(\mathbf{z}(t; \boldsymbol{\alpha}, r, \mathbf{x}_0, T), \mathbf{v}(t; \boldsymbol{\alpha}, r, \mathbf{x}_0, T))$ be the optimal pair with corresponding current value costate vector $\tilde{\boldsymbol{\lambda}}(t; \boldsymbol{\alpha}, r, \mathbf{x}_0, T)$. Define the current value Hamiltonian as

$$\tilde{H}(\mathbf{x}, \mathbf{u}, \tilde{\boldsymbol{\lambda}}; \boldsymbol{\alpha}) \stackrel{\text{def}}{=} f(\mathbf{x}, \mathbf{u}; \boldsymbol{\alpha}) + \sum_{n=1}^{N} \tilde{\lambda}_n g^n(\mathbf{x}, \mathbf{u}; \boldsymbol{\alpha}),$$

which, by Theorem 12.1, produces the necessary conditions

$$\tilde{H}_{u_m}(\mathbf{x}, \mathbf{u}, \tilde{\boldsymbol{\lambda}}; \boldsymbol{\alpha}) = 0, \quad m = 1, 2, \ldots, M,$$

$$\dot{\tilde{\lambda}}_i = r\tilde{\lambda}_i - \tilde{H}_{x_i}(\mathbf{x}, \mathbf{u}, \tilde{\boldsymbol{\lambda}}; \boldsymbol{\alpha}),\ \tilde{\lambda}(T) = 0, \quad i = 1, 2, \ldots, N,$$

$$\dot{x}_i = \tilde{H}_{\tilde{\lambda}_i}(\mathbf{x}, \mathbf{u}, \tilde{\boldsymbol{\lambda}}; \boldsymbol{\alpha}),\ x_i(0) = x_{0i}, \quad i = 1, 2, \ldots, N.$$

Letting $\hat{\mathbf{u}}(\mathbf{x}, \tilde{\boldsymbol{\lambda}}; \boldsymbol{\alpha})$ be the solution to the necessary conditions $\tilde{H}_{u_m}(\mathbf{x}, \mathbf{u}, \tilde{\boldsymbol{\lambda}}; \boldsymbol{\alpha}) = 0$, $m = 1, 2, \ldots, M$, the canonical equations can be written as

$$\dot{\tilde{\lambda}}_i = r\tilde{\lambda}_i - \tilde{H}_{x_i}(\mathbf{x}, \hat{\mathbf{u}}(\mathbf{x}, \tilde{\boldsymbol{\lambda}}; \boldsymbol{\alpha}), \tilde{\boldsymbol{\lambda}}; \boldsymbol{\alpha}),\ \tilde{\lambda}(T) = 0, \quad i = 1, 2, \ldots, N, \tag{34}$$

$$\dot{x}_i = \tilde{H}_{\tilde{\lambda}_i}(\mathbf{x}, \hat{\mathbf{u}}(\mathbf{x}, \tilde{\boldsymbol{\lambda}}; \boldsymbol{\alpha}), \tilde{\boldsymbol{\lambda}}; \boldsymbol{\alpha}),\ x_i(0) = x_{0i}, \quad i = 1, 2, \ldots, N. \tag{35}$$

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Now define the maximized current value Hamiltonian function $\tilde{M}(\cdot)$ by

$$\tilde{M}(\mathbf{x}, \tilde{\boldsymbol{\lambda}}; \boldsymbol{\alpha}) \stackrel{\text{def}}{=} \max_{\mathbf{u}} \tilde{H}(\mathbf{x}, \mathbf{u}, \tilde{\boldsymbol{\lambda}}; \boldsymbol{\alpha}).$$

By the prototype envelope theorem applied to this static optimization problem defining $\tilde{M}(\cdot)$, it follows that Eqs. (34) and (35) may be rewritten equivalently as

$$\dot{\tilde{\lambda}}_i = r\tilde{\lambda}_i - \tilde{M}_{x_i}(\mathbf{x}, \tilde{\boldsymbol{\lambda}}; \boldsymbol{\alpha}),\ \tilde{\lambda}(T) = 0, \quad i = 1, 2, \ldots, N, \tag{36}$$

$$\dot{x}_i = \tilde{M}_{\tilde{\lambda}_i}(\mathbf{x}, \tilde{\boldsymbol{\lambda}}; \boldsymbol{\alpha}),\ x_i(0) = x_{0i}, \quad i = 1, 2, \ldots, N, \tag{37}$$

a result you are asked to confirm in a mental exercise. We are now in a position to establish an important result that relates the value of the current value optimal value function $\tilde{V}(\boldsymbol{\alpha}, r, \mathbf{x}_0, T)$ in problem (33) to the value of the maximized current value Hamiltonian $\tilde{M}(\mathbf{x}, \tilde{\boldsymbol{\lambda}}; \boldsymbol{\alpha})$ evaluated at the optimal solution $(\mathbf{z}(t; \boldsymbol{\alpha}, r, \mathbf{x}_0, T), \tilde{\boldsymbol{\lambda}}(t; \boldsymbol{\alpha}, r, \mathbf{x}_0, T))$.

Theorem 12.4: *In problem (33), the value of the current value optimal value function* $\tilde{V}(\boldsymbol{\alpha}, r, \mathbf{x}_0, T)$ *is related to the value of the maximized current value Hamiltonian* $\tilde{M}(\mathbf{x}, \tilde{\boldsymbol{\lambda}}; \boldsymbol{\alpha})$ *evaluated at the optimal solution* $(\mathbf{z}(t; \boldsymbol{\alpha}, r, \mathbf{x}_0, T), \tilde{\boldsymbol{\lambda}}(t; \boldsymbol{\alpha}, r, \mathbf{x}_0, T))$ *by the formula*

$$\begin{aligned}\tilde{V}(\boldsymbol{\alpha}, r, \mathbf{x}_0, T) = \frac{1}{r}\,&[\tilde{M}(\mathbf{x}_0, \tilde{\boldsymbol{\lambda}}(0; \boldsymbol{\alpha}, r, \mathbf{x}_0, T); \boldsymbol{\alpha})\\ &- e^{-rT}\tilde{M}(\mathbf{z}(T; \boldsymbol{\alpha}, r, \mathbf{x}_0, T), \tilde{\boldsymbol{\lambda}}(T; \boldsymbol{\alpha}, r, \mathbf{x}_0, T); \boldsymbol{\alpha})].\end{aligned}$$

Proof: For notational ease, define $\boldsymbol{\beta} \stackrel{\text{def}}{=} (\boldsymbol{\alpha}, r, \mathbf{x}_0, T)$ as the vector of parameters. Begin the proof by observing that

$$\begin{aligned}&-\frac{1}{r}\int_0^T \frac{d}{dt}[e^{-rt}\tilde{M}(\mathbf{z}(t; \boldsymbol{\beta}), \tilde{\boldsymbol{\lambda}}(t; \boldsymbol{\beta}); \boldsymbol{\alpha})]\,dt\\ &\quad= \frac{1}{r}[\tilde{M}(\mathbf{z}(0; \boldsymbol{\beta}), \tilde{\boldsymbol{\lambda}}(0; \boldsymbol{\beta}); \boldsymbol{\alpha}) - e^{-rT}\tilde{M}(\mathbf{z}(T; \boldsymbol{\beta}), \tilde{\boldsymbol{\lambda}}(T; \boldsymbol{\beta}); \boldsymbol{\alpha})]\end{aligned}$$

is a result of the inverse operations of integration and differentiation, whereas

$$\begin{aligned}-\frac{1}{r}\int_0^T \frac{d}{dt}[e^{-rt}\tilde{M}(\mathbf{z}(t; \boldsymbol{\beta}), \tilde{\boldsymbol{\lambda}}(t; \boldsymbol{\beta}); \boldsymbol{\alpha})]\,dt &= \int_0^T e^{-rt}\tilde{M}(\mathbf{z}(t; \boldsymbol{\beta}), \tilde{\boldsymbol{\lambda}}(t; \boldsymbol{\beta}); \boldsymbol{\alpha})\,dt\\ -\frac{1}{r}\int_0^T e^{-rt}\sum_{i=1}^N &[\tilde{M}_{x_i}(\mathbf{z}(t; \boldsymbol{\beta}), \tilde{\boldsymbol{\lambda}}(t; \boldsymbol{\beta}); \boldsymbol{\alpha})\dot{z}_i(t; \boldsymbol{\beta})\\ &+ \tilde{M}_{\tilde{\lambda}_i}(\mathbf{z}(t; \boldsymbol{\beta}), \tilde{\boldsymbol{\lambda}}(t; \boldsymbol{\beta}); \boldsymbol{\alpha})\dot{\tilde{\lambda}}_i(t; \boldsymbol{\beta})]\,dt,\end{aligned}$$

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is a result of carrying out the differentiation. In view of the fact that the left-hand sides of the last two equations are identical, so are the right-hand sides. Upon equating the right-hand sides and using Eqs. (36) and (37), we arrive at

$$\int_0^T e^{-rt}\tilde{M}(\mathbf{z}(t;\boldsymbol{\beta}), \tilde{\boldsymbol{\lambda}}(t;\boldsymbol{\beta}); \boldsymbol{\alpha})\,dt$$

$$-\frac{1}{r}\int_0^T e^{-rt}\sum_{i=1}^{N}[r\tilde{M}_{\tilde{\lambda}_i}(\mathbf{z}(t;\boldsymbol{\beta}), \tilde{\boldsymbol{\lambda}}(t;\boldsymbol{\beta}); \boldsymbol{\alpha})\tilde{\lambda}_i(t;\boldsymbol{\beta})]\,dt$$

$$= \frac{1}{r}[\tilde{M}(\mathbf{z}(0;\boldsymbol{\beta}), \tilde{\boldsymbol{\lambda}}(0;\boldsymbol{\beta}); \boldsymbol{\alpha}) - e^{-rT}\tilde{M}(\mathbf{z}(T;\boldsymbol{\beta}), \tilde{\boldsymbol{\lambda}}(T;\boldsymbol{\beta}); \boldsymbol{\alpha})]. \quad (38)$$

Recall that the optimal path of the control vector is given by $\mathbf{v}(t;\boldsymbol{\beta}) \stackrel{\text{def}}{=} \hat{\mathbf{u}}(\mathbf{z}(t;\boldsymbol{\beta}), \tilde{\boldsymbol{\lambda}}(t;\boldsymbol{\beta}); \boldsymbol{\alpha})$. Using this observation and the definition of the maximized current value Hamiltonian, we have

$$\begin{aligned}\tilde{M}(\mathbf{z}(t;\boldsymbol{\beta}), \tilde{\boldsymbol{\lambda}}(t;\boldsymbol{\beta}); \boldsymbol{\alpha}) &= \tilde{H}(\mathbf{z}(t;\boldsymbol{\beta}), \mathbf{v}(t;\boldsymbol{\beta}), \tilde{\boldsymbol{\lambda}}(t;\boldsymbol{\beta}); \boldsymbol{\alpha})\\ &= f(\mathbf{z}(t;\boldsymbol{\beta}), \mathbf{v}(t;\boldsymbol{\beta}); \boldsymbol{\alpha})\\ &\quad + \sum_{n=1}^{N}\tilde{\lambda}_n(t;\boldsymbol{\beta})g^n(\mathbf{z}(t;\boldsymbol{\beta}), \mathbf{v}(t;\boldsymbol{\beta}); \boldsymbol{\alpha}). \quad (39)\end{aligned}$$

The archetype static envelope theorem applied to the definition of the maximized current value Hamiltonian $\tilde{M}(\cdot)$, along with the definition of the current value Hamiltonian $\tilde{H}(\cdot)$, yields

$$\begin{aligned}\tilde{M}_{\tilde{\lambda}_i}(\mathbf{z}(t;\boldsymbol{\beta}), \tilde{\boldsymbol{\lambda}}(t;\boldsymbol{\beta}); \boldsymbol{\alpha}) &= \tilde{H}_{\tilde{\lambda}_i}(\mathbf{z}(t;\boldsymbol{\beta}), \mathbf{v}(t;\boldsymbol{\beta}), \tilde{\boldsymbol{\lambda}}(t;\boldsymbol{\beta}); \boldsymbol{\alpha})\\ &= g^i(\mathbf{z}(t;\boldsymbol{\beta}), \mathbf{v}(t;\boldsymbol{\beta}); \boldsymbol{\alpha}), \quad i = 1, 2, \ldots, N. \quad (40)\end{aligned}$$

Substituting Eqs. (39) and (40) into Eq. (38) therefore yields

$$\int_0^T e^{-rt}f(\mathbf{z}(t;\boldsymbol{\beta}), \mathbf{v}(t;\boldsymbol{\beta}); \boldsymbol{\alpha})\,dt$$

$$= \frac{1}{r}[\tilde{M}(\mathbf{z}(0;\boldsymbol{\beta}), \tilde{\boldsymbol{\lambda}}(0;\boldsymbol{\beta}); \boldsymbol{\alpha}) - e^{-rT}\tilde{M}(\mathbf{z}(T;\boldsymbol{\beta}), \tilde{\boldsymbol{\lambda}}(T;\boldsymbol{\beta}); \boldsymbol{\alpha})].$$

Noting that the current value optimal value function $\tilde{V}(\cdot)$ can be defined constructively as

$$\tilde{V}(\boldsymbol{\beta}) \equiv \int_0^T e^{-rt}f(\mathbf{z}(t;\boldsymbol{\beta}), \mathbf{v}(t;\boldsymbol{\beta}); \boldsymbol{\alpha})\,dt,$$

and that $\mathbf{z}(0;\boldsymbol{\beta}) \equiv \mathbf{x}_0$, completes the proof. Q.E.D.

This is an important result for two reasons. First, the result of Theorem 12.4 turns out to be useful in determining which of several admissible paths are optimal for the adjustment cost model of the firm in the presence of certain types of nonconvexities [see, e.g., Davidson and Harris (1981)]. Second, it relates, in a very simple manner, the optimal value of the objective functional, which in general is difficult to explicitly compute, to the value of the maximized Hamiltonian at the initial and terminal dates of the planning horizon. Consequently, if T is a decision variable in problem (33), then we have the even simpler result given in the ensuing theorem, which you are asked to prove in a mental exercise. Observe that in this instance, T does not appear as an argument of the optimal pair or the current value optimal value function seeing as it is a choice variable and not a parameter.

Theorem 12.5: *In problem (33) with T a decision variable, the value of the current value optimal value function* $\tilde{V}(\boldsymbol{\alpha}, r, \mathbf{x}_0)$ *is related to the value of the maximized current value Hamiltonian* $\tilde{M}(\mathbf{x}, \tilde{\boldsymbol{\lambda}}; \boldsymbol{\alpha})$ *evaluated at the optimal solution* $(\mathbf{z}(t; \boldsymbol{\alpha}, r, \mathbf{x}_0), \tilde{\boldsymbol{\lambda}}(t; \boldsymbol{\alpha}, r, \mathbf{x}_0))$ *by the formula*

$$V(\boldsymbol{\alpha}, r, \mathbf{x}_0) = \frac{1}{r}[\tilde{M}(\mathbf{x}_0, \tilde{\boldsymbol{\lambda}}(0; \boldsymbol{\alpha}, r, \mathbf{x}_0); \boldsymbol{\alpha})].$$

We will return to an important variant of Theorem 12.5 two chapters hence, when infinite horizon problems, arguably the largest class of optimal control problems of interest in economic theory, are presented in some detail. In fact, we will encounter essentially the same theorem again in several other chapters in which we study intertemporal duality theory, pointing to its fundamental nature. Though it may not be readily apparent at present, this theorem permits empirical implementation of intertemporal economic models, an important feature that will be expanded upon in some detail in a later chapter as well.

MENTAL EXERCISES

12.1 Consider the standard model of the competitive nonrenewable resource–extracting firm without stock effects:

$$\max_{q(\cdot), T, x_T} \int_0^T [p(t)q(t) - C(q(t))]\, e^{-rt}\, dt$$

$$\text{s.t.} \quad \dot{x}(t) = -q(t),\ x(0) = x_0, x(T) = x_T \geq 0,\ q(t) \geq 0,$$

where $q(t)$ is the extraction rate, $x(t)$ is the stock of the resource in the ground, and $p(t)$ is the time-varying output price. Note the nonnegativity restriction on the terminal stock and extraction rate.

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(a) Write down the necessary conditions for this problem in current value form.

(b) Show that if $x(T) > 0$, then $\lambda(t) \equiv 0 \, \forall \, t \in [0, T]$, where $\lambda(t)$ is the *current value* costate variable. Assuming that $q(t) > 0 \, \forall \, t \in [0, T]$, provide an economic interpretation of the result $\lambda(t) \equiv 0 \, \forall \, t \in [0, T]$.

(c) Show that if $C(0) = 0$, then $q(T) > 0$ is optimal regardless of whether $x(T) = 0$ or $x(T) > 0$. Provide an economic interpretation of this result and draw a graph of the situation at $t = T$.

(d) Show that if $q(T) = 0$, then $C(0) = 0$ regardless of whether $x(T) > 0$ or $x(T) = 0$. Provide an economic interpretation of this result.

(e) Show that if $\lambda(T) > 0$, then $x(T) = 0$. Provide an economic interpretation of this result.

12.2 Let's reconsider the prototype control problem studied in this chapter, but with the added twist that the discount rate is time dependent, say $\rho(t) > 0$, rather than a constant $r > 0$. That is, consider the optimal control problem

$$V(\boldsymbol{\alpha}, \mathbf{x}_0, T) \stackrel{\text{def}}{=} \max_{\mathbf{u}(\cdot), \mathbf{x}_T} \int_0^T f(t, \mathbf{x}(t), \mathbf{u}(t); \boldsymbol{\alpha})\, \delta(t)\, dt$$

$$\text{s.t.} \quad \dot{\mathbf{x}}(t) = \mathbf{g}(t, \mathbf{x}(t), \mathbf{u}(t); \boldsymbol{\alpha}), \;\; \mathbf{x}(t_0) = \mathbf{x}_0, \;\; \mathbf{x}(T) = \mathbf{x}_T,$$

where $\delta(t) \stackrel{\text{def}}{=} \exp[-\int_0^t \rho(\tau)\, d\tau]$ is the discount factor and $\rho(t) > 0$ is the time-varying discount rate.

(a) Write down the necessary conditions for the problem in present value form.

(b) Define the current value Hamiltonian and current value costate vector for the control problem.

(c) Write down the necessary conditions for the problem in current value form. Show all of your work.

(d) Show that even if the integrand function $f(\cdot)$ and state function $\mathbf{g}(\cdot)$ do not depend explicitly on the independent variable t, that is, even if $f_t(\cdot) \equiv 0$ and $\mathbf{g}_t(\cdot) \equiv \mathbf{0}_N$, the canonical differential equations in current value form are *not* autonomous.

12.3 Prove the sufficiency part of Theorem 12.3.

12.4 Consider the control problem

$$\max_{\mathbf{u}(\cdot), \mathbf{x}_T} \int_0^T f(\mathbf{x}(t), \mathbf{u}(t))\, e^{-rt}\, dt$$

$$\text{s.t.} \quad \dot{\mathbf{x}}(t) = \mathbf{g}(\mathbf{x}(t), \mathbf{u}(t)), \;\; \mathbf{x}(t_0) = \mathbf{x}_0, \;\; \mathbf{x}(T) = \mathbf{x}_T.$$

Let $(\mathbf{z}(t), \mathbf{v}(t))$ be the optimal pair with corresponding present value costate vector $\boldsymbol{\lambda}(t)$. Define the present value Hamiltonian by

$$H(t, \mathbf{x}, \mathbf{u}, \boldsymbol{\lambda}) \stackrel{\text{def}}{=} f(\mathbf{x}, \mathbf{u})e^{-rt} + \sum_{n=1}^{N} \lambda_n g^n(\mathbf{x}, \mathbf{u}).$$

Note that even though the functions $f(\cdot)$ and $\mathbf{g}(\cdot)$ do not depend explicitly on the independent variable t, the present value Hamiltonian does because of the appearance of the discount factor.

(a) Write down the necessary conditions for the control problem.

(b) Prove that

$$\frac{d}{dt}H(t, \mathbf{z}(t), \mathbf{v}(t), \boldsymbol{\lambda}(t)) = -rf(\mathbf{z}(t), \mathbf{v}(t))\, e^{-rt}.$$

This is very similar to the envelope result established in Mental Exercises 2.31, 4.13, and 6.11. In fact, the above result is a special case of Mental Exercises 4.13 and 6.11. Why?

(c) Define the current value Hamiltonian $\tilde{H}(\cdot)$ and current value costate vector $\tilde{\boldsymbol{\lambda}}(\cdot)$.

(d) Prove that

$$\frac{d}{dt}\tilde{H}(\mathbf{z}(t), \mathbf{v}(t), \tilde{\boldsymbol{\lambda}}(t)) = r\sum_{n=1}^{N} \tilde{\lambda}_n(t) g^n(\mathbf{z}(t), \mathbf{v}(t)).$$

This proves that even if the functions $f(\cdot)$ and $\mathbf{g}(\cdot)$ do not depend explicitly on the independent variable t, but a discount factor of the form e^{-rt} multiplies $f(\cdot)$, the current value Hamiltonian is not constant along the optimal path, in contrast to assertions made in the literature [see, e.g., Chiang (1992, p. 212)]. Note also that this result was used in the proof of Theorem 12.4.

(e) To ease the notation, define $\Omega(t) \stackrel{\text{def}}{=} \tilde{H}(\mathbf{z}(t), \mathbf{v}(t), \tilde{\boldsymbol{\lambda}}(t))$, which is simply the value of the current value Hamiltonian along the optimal path. Show that the result in part (d) can be written equivalently as $\dot{\Omega}(t) = r\Omega(t) - rf(\mathbf{z}(t), \mathbf{v}(t))$.

(f) Find the general solution of the differential equation $\dot{\Omega}(t) = r\Omega(t) - rf(\mathbf{z}(t), \mathbf{v}(t))$. Assuming that T is a decision variable, find the specific solution. Use $t = 0$ as the lower limit of integration.

(g) Prove that Theorem 12.5 is a special case of the specific solution you derived in part (f).

12.5 Derive Eq. (30).

12.6 Prove that Eqs. (34) and (35) can be written equivalently as Eqs. (36) and (37), respectively.

12.7 Prove Theorem 12.5.

12.8 *Optimal Consumption of a Stock of Wine by an Impatient Drunk*. You are told by your physician that you have $T > 0$ years to live at time $t = 0$, the present, and you know this to be true. Over the course of your life, you have been an avid collector of wine for the purpose of consuming it rather than using it as an investment. Given the news from your physician, you would like to develop a consumption plan that maximizes the utility of consumption of your stock of wine over your remaining lifetime. At present, you have $w(0) = w_0 > 0$ bottles of wine in your cellar, and because you cannot take the wine with you when you die, you've decided that you will consume it all by the time you die; hence $w(T) = 0$. The instantaneous utility that you derive from the consumption of wine is given by $U(c; \alpha_1) \stackrel{\text{def}}{=} \alpha_1 c$, where $\alpha_1 > 0$ is the marginal utility of wine consumption. This set of preferences implies that you only get utility from the consumption of wine, not from the mere presence of the stock of wine in your cellar, that is, in contrast to Mental Exercise 5.6, you do not like to brag about the quantity and quality of wine in your cellar because you get no enjoyment from that. The stock of wine in your cellar at any moment $t \in [0, T]$ is given by the archetype depletion equation in integral form, namely,

$$w(t) = w_0 - \int_0^t c(s)\, ds,$$

so that by Leibniz's rule,

$$\dot{w}(t) = -c(t),\ \ w(0) = w_0,\ w(T) = 0$$

are the state equation and boundary conditions for the optimal control problem that you will solve in order to determine your optimal time path of wine consumption. Given that your preferences are linear in the consumption rate, one must place lower and upper bounds on your consumption rate, say, $0 \leq c(t) \leq \bar{c}$, where $\bar{c} > w_0/T > 0$. Also in contrast to Mental Exercise 5.6, you *do* discount your instantaneous utility at the rate $r > 0$. Defining $\boldsymbol{\beta} \stackrel{\text{def}}{=} (\alpha_1, \bar{c}, r, w_0, T) \in \Re^5_{++}$ as the vector of time-independent parameters, the optimal control problem that defines the optimal consumption rate of your stock of wine is therefore given by

$$V(\boldsymbol{\beta}) \stackrel{\text{def}}{=} \max_{c(\cdot)} \int_0^T \alpha_1 c(t) e^{-rt}\, dt$$

$$\text{s.t.} \quad \dot{w}(t) = -c(t),\ \ w(0) = w_0,\ \ w(T) = 0,$$

$$c(t) \in U \stackrel{\text{def}}{=} \left\{ c(\cdot) : 0 \leq c(t) \leq \bar{c}\, \forall\, t \in [0, T],\ \bar{c} > w_0 T^{-1} > 0 \right\}.$$

(a) Provide an *economic* interpretation of the inequality $\bar{c} > w_0/T$. What would result if the inequality ran in the other direction?

(b) Write down the current value Hamiltonian for the optimal control problem, say $H(\cdot)$, letting λ be the current value costate variable. What is

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the economic interpretation of λ? Is it positive or negative? How do you know?

(c) Assuming that an optimal solution to the control problem exists, find the decision rule governing the selection of the optimal rate of wine consumption in terms of (α_1, λ). Provide an economic interpretation of the decision rule.

(d) Prove that $c(t) = 0 \,\forall\, t \in [0, T]$ is not an admissible solution. Explain why it therefore cannot be an optimal solution.

(e) Prove that $c(t) = \bar{c} \,\forall\, t \in [0, T]$ is not an admissible solution.

(f) Find the general solution for current value costate $\lambda(t)$, letting a_1 be the constant of integration. Prove that $\lambda(t) < \alpha_1 \,\forall\, t \in [0, T]$ and $\lambda(t) > \alpha_1 \,\forall\, t \in [0, T]$ cannot hold for $\lambda(t)$ in an optimal program.

(g) Based on your answer to the earlier parts of the question, what, therefore, is the nature of the solution for the optimal consumption rate? Justify your answer fully.

(h) Show that

$$c(t) = \begin{cases} \bar{c} \,\forall\, t \in [0, \tau] \\ 0 \,\forall\, t \in (\tau, T], \end{cases} \quad w(t) = \begin{cases} w_0 - \bar{c}t & \forall\, t \in [0, \tau] \\ 0 & \forall\, t \in (\tau, T], \end{cases}$$

$$\lambda(t) = \alpha_1 e^{-r(\tau - t)}, \ \tau = \frac{w_0}{\bar{c}} < T,$$

where τ is the switch time, is a solution to the necessary conditions. Note that you must *derive* this solution, not just simply verify it satisfies the necessary conditions. Explain all the steps and reasoning clearly in your derivation.

(i) Provide an economic interpretation of the solution to the necessary conditions.

12.9 *The Baby Boomer Problem*. You are told by your physician that you have $T > 0$ years to live at time $t = 0$, the present, and you know this to be true. At present, you have an initial stock of assets $a(0) = a_0 > 0$, which may be thought of as a stock of money in some interest-bearing account, where $r > 0$ is the interest rate for the account. You also have a constant income of $y > 0$ for each $t \in [0, T]$. Letting $c(t)$ be your consumption expenditures (i.e., the rate at which you spend your money on consumption) at time $t \in [0, T]$, the dynamics of your asset are given by the first-order ordinary differential equation

$$\dot{a}(t) = y + ra(t) - c(t),$$

where $a(t)$ is your stock of the asset in the interest-bearing account at time $t \in [0, T]$. Note that we are allowing $a(t)$ to be negative, zero, or positive. The instantaneous utility that you derive from consumption is given

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by $U(c;\alpha_1) \stackrel{\text{def}}{=} \alpha_1 c$, where $\alpha_1 > 0$ is the marginal utility of consumption. This set of preferences implies that you only get utility from consumption, not from the mere presence of the stock of the asset you own, that is, you do not get enjoyment from bragging about how wealthy you are. Given that your preferences are linear in the consumption rate, one must place lower and upper bounds on your consumption rate, say, $0 \le c(t) \le \bar{c}$, where $\bar{c} > 0$. You discount your instantaneous utility at the rate $\rho > 0$, that is, your intertemporal rate of time preference is $\rho > 0$. In addition, you can choose how much of the asset to leave your spouse and/or children when you die, that is, you are free to choose the value $a(T) = a_T$, which, because it may be negative, zero, or positive, is *not* subject to any constraint. The utility that you receive from such a bequest is given by the instantaneous utility function $B(a_T;\alpha_2) \stackrel{\text{def}}{=} \alpha_2 a_T$, where $\alpha_2 > 0$ is a preference shift parameter. Defining $\boldsymbol{\beta} \stackrel{\text{def}}{=} (a_0, T, \alpha_1, \alpha_2, \bar{c}, \rho, r) \in \Re^7_{++}$ as the vector of time-independent parameters, the optimal control problem that defines the optimal consumption rate is therefore given by

$$V(\boldsymbol{\beta}) \stackrel{\text{def}}{=} \max_{c(\cdot),a_T} \left\{ \int_0^T \alpha_1 c(t) e^{-\rho t}\, dt + \alpha_2 a_T e^{-\rho T} \right\}$$

$$\text{s.t.} \quad \dot{a}(t) = y + ra(t) - c(t),\ a(0) = a_0,\ a(T) = a_T,$$

$$c(t) \in U \stackrel{\text{def}}{=} \{c(\cdot) : 0 \le c(t) \le \bar{c}\, \forall\, t \in [0, T],\ \bar{c} > 0\}.$$

Assume that $\rho > r$, and note that $\rho > 0$ is the discount factor in the integrand, *not* $r > 0$.

(a) Provide an economic interpretation of the inequality $\rho > r$.

(b) What is the economic implication of the inequality $a(t) < 0$? Moreover, what is the economic implication of having $r > 0$ be the same regardless of whether $a(t) > 0$ or $a(t) < 0$?

(c) Write down the *current value* Hamiltonian for the optimal control problem, say, $H(\cdot)$, letting λ be the *current value* costate variable. What is the economic interpretation of λ? Is it positive or negative? How do you know?

(d) Assuming that an optimal solution to the control problem exists, say, $c^*(t)$, find the decision rule governing the selection of the optimal consumption rate in terms of (α_1, λ). Provide an economic interpretation of the decision rule.

(e) Find the general *and* specific solution of the costate differential equation.

(f) Prove that if $\alpha_2 < \alpha_1$, then $c^*(t) = \bar{c}\, \forall\, t \in [0, T]$ is the solution to the necessary conditions for the consumption rate. Provide an economic interpretation of this solution.

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(g) Find the general *and* specific solution of the state equation given that $c^*(t) = \bar{c}\ \forall\, t \in [0, T]$.

(h) Prove that if $\frac{1}{r}[\bar{c} - y][e^{rT} - 1] > a_0 e^{rT}$, then $a(T) = a_T < 0$ given that $c^*(t) = \bar{c}\ \forall\, t \in [0, T]$. Provide an economic interpretation of this circumstance. This is the "baby boomer" result alluded to in the problem.

(i) Derive sufficient conditions for the following bang-bang consumption plan to be a solution to the necessary conditions:

$$c^*(t) = \begin{cases} \bar{c}\ \forall\, t \in [0, \tau] \\ 0\ \forall\, t \in (\tau, T], \end{cases}$$

where τ is the switching time. You are *not* required to solve for the corresponding time paths of the state and costate variables.

12.10 *Optimal Waste Cleanup with Residual Consequences*. A spill of a toxic substance has occurred at time $t = 0$ (the present) in the known amount of $x(0) = x_0 > 0$, where $x(t)$ is the stock of the toxic substance left in the environment at time t. The federal government has a contract with your company for a fixed interval of time, say, $[0, T]$, to reduce the size of the toxic waste stock from its initial size of x_0. Compared with the previous version of this problem examined in Example 5.3, there are two significant changes. First, the firm is given the choice of how much to clean up, that is, $x(T) = x_T$ is a choice variable for the firm. Second, after the cleanup period comes to an end ($t > T$), the firm must pay a penalty for any of the stock they failed to clean up by time T, say, to the tune of $P(x_T; \beta, r, T) \stackrel{\text{def}}{=} \beta x_T e^{-rt}$ dollars, where $\beta > 0$ is a given parameter. As head economist of the company, you are charged with determining the cleanup rate path $u(t)$ that minimizes the present discounted cost of cleaning up a variable amount of the toxic waste over the fixed horizon, including any penalty associated with the stock you failed to clean up by time T. The cleanup technology is linear in the cleanup rate, say, $cu(t)$, where $c > 0$, and the discount rate is $r > 0$. The cleanup rate is bounded below by zero and bounded above by some fixed finite rate $\bar{u} > 0$, where $\bar{u} \le x_0/T$. Because $x(t)$ is defined as the toxic stock left in the environment at time t, it follows that

$$x(t) \stackrel{\text{def}}{=} x_0 - \int_0^t u(s)\, ds.$$

Hence, by Leibniz's rule,

$$\dot{x}(t) = -u(t),\ x(0) = x_0,\ x(T) = x_T$$

are the state equation and boundary conditions for the problem. In full, the optimal control problem is

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$$C(\beta, c, r, \bar{u}, x_0, T) \stackrel{\text{def}}{=} \min_{u(\cdot), x_T} \int_0^T cu(t)e^{-rt}\,dt + \beta x_T e^{-rT}$$

$$\text{s.t.} \quad \dot{x}(t) = -u(t), x(0) = x_0, \; x(T) = x_T,$$

$$u(t) \in U \stackrel{\text{def}}{=} \{u(\cdot) : 0 \le u(t) \le \bar{u}, \bar{u} \le x_0/T\}.$$

Assume that $r^{-1} \ln c\beta^{-1} + T > 0$. The relevance of this assumption will become clear in the course of solving the problem.

(a) Write down the necessary conditions for this problem in current value form, letting λ be the current value costate variable. What is the economic interpretation of λ?

(b) Assuming that an optimal solution to the control problem exists, find the decision rule governing selection of the optimal control in terms of (c, λ).

(c) Find the specific solution for $\lambda(t)$. Prove that $\lambda(t) > 0 \,\forall\, t \in [0, T]$ and that $\dot{\lambda}(t) > 0 \,\forall\, t \in [0, T]$.

(d) Prove that if $\beta < c$, then $u(t) = 0 \,\forall\, t \in [0, T]$ is the solution to the necessary conditions. Provide an economic interpretation of this scenario.

(e) Prove that if $\beta e^{-rt} > c$, then $u(t) = \bar{u} \,\forall\, t \in [0, T]$ is the solution to the necessary conditions. Provide an economic interpretation of this scenario.

(f) Under what conditions is

$$u(t) = \begin{cases} 0 \;\forall\, t \in [0, s] \\ \bar{u} \;\forall\, t \in (s, \mathrm{T}], \end{cases}$$

where s is the switching time, a solution to the necessary conditions? Find the switching time s in this case. Provide an economic interpretation of this scenario.

(g) Prove that a solution to the necessary conditions is a solution to the control problem.

(h) What is the effect of an increase in β on the optimal cleanup policy, assuming that the bang-bang policy is optimal? Your economic interpretation must be backed up with formal mathematical calculations.

FURTHER READING

The seminal article by Strotz (1956) is the source for the material on time consistency. A brief discussion of time consistency appears in Léonard and Von Long (1992). Theorem 12.4 originates with Davidson and Harris (1981), who also provide an application of the infinite horizon version of Theorem 12.5 in the context of the adjustment cost model of the firm with nonconvexities.

REFERENCES

Chiang, A.C. (1992), *Elements of Dynamic Optimization* (New York: McGraw-Hill, Inc.).

Davidson, R. and Harris, R. (1981), "Non-Convexities in Continuous-Time Investment Theory," *Review of Economic Studies*, 48, 235–253.

Léonard, D. and Van Long, N. (1992), *Optimal Control Theory and Static Optimization in Economics* (New York: Cambridge University Press).

Strotz, R. (1956), "Myopia and Inconsistency in Dynamic Utility Maximization," *Review of Economic Studies*, 23, 165–180.