ONE

Essential Elements of Continuous Time Dynamic Optimization

In order to motivate the following introductory material on dynamic optimization problems, it will be advantageous to draw heavily on your knowledge of static optimization theory. To that end, we begin by recalling the definition of the prototype unconstrained static optimization problem, namely,

$$\phi(\boldsymbol{\alpha}) \stackrel{\text{def}}{=} \max_{\mathbf{x}\in\Re^N} f(\mathbf{x};\boldsymbol{\alpha}), \tag{1}$$

where $\mathbf{x} \in \Re^N$ is a vector of *decision* or *choice variables*, $\boldsymbol{\alpha} \in \Re^A$ is a vector of *parameters*, $f(\cdot)$ is the twice continuously differentiable *objective function*, that is, $f(\cdot) \in C^{(2)}$, and $\phi(\cdot)$ is the *indirect* or *maximized objective function*. This is terminology you should be more or less familiar with from prior courses.

Because we will deal repeatedly with vectors and matrices as well as the derivatives of scalar- and vector-valued functions in this book, we pause momentarily to establish three notational conventions that we shall adhere to throughout. First, all vectors are treated as column vectors. To denote a row vector, we therefore employ the transpose operator, denoted by the symbol $'$. Thus $\mathbf{x} \in \Re^N$ is taken to be an N-element column vector, whereas $\mathbf{x}'$ is an N-element row vector. Note also that vectors appear in **boldface** type.

Second, if $\mathbf{g}(\cdot) : \Re^N \to \Re^M$ is a $C^{(1)}$ vector-valued function, thereby implying that $\mathbf{g}(\cdot) \stackrel{\text{def}}{=} (g^1(\cdot), g^2(\cdot), \ldots, g^M(\cdot))'$, then at any $\mathbf{x} \in \Re^N$, we define the $M \times N$ Jacobian matrix of $\mathbf{g}(\cdot)$ by

$$\underbrace{\mathbf{g}_{\mathbf{x}}(\mathbf{x})}_{M\times N} \stackrel{\text{def}}{=} \begin{bmatrix} g^1_{x_1}(\mathbf{x}) & g^1_{x_2}(\mathbf{x}) & \cdots & g^1_{x_N}(\mathbf{x}) \\ g^2_{x_1}(\mathbf{x}) & g^2_{x_2}(\mathbf{x}) & \cdots & g^2_{x_N}(\mathbf{x}) \\ \vdots & \vdots & \ddots & \vdots \\ g^M_{x_1}(\mathbf{x}) & g^M_{x_2}(\mathbf{x}) & \cdots & g^M_{x_N}(\mathbf{x}) \end{bmatrix}, \tag{2}$$

where $g^m_{x_n}(\mathbf{x})$ is the partial derivative of $g^m(\cdot)$ with respect to x_n evaluated at the point $(\mathbf{x})$. It is also the element in the mth row and nth column of $\mathbf{g}_{\mathbf{x}}(\mathbf{x})$. This definition implies that if $M = 1$, so that $g(\cdot) : \Re^N \to \Re$ is now a scalar-valued function, then

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$g_{\mathbf{x}}(\mathbf{x}) = (g_{x_1}(\mathbf{x}), g_{x_2}(\mathbf{x}), \ldots, g_{x_N}(\mathbf{x}))$ is a row vector, or equivalently, a $1 \times N$ matrix. This means that the derivative of a scalar-valued function with respect to a column vector is a row vector. As an extension of this notation, if we now assume that $\mathbf{g}(\cdot) : \Re^{N+A} \to \Re^M$ is a $C^{(1)}$ function whose arguments are the vectors $\mathbf{x} \in \Re^N$ and $\boldsymbol{\alpha} \in \Re^A$, then $\mathbf{g}_{\mathbf{x}}(\mathbf{x}; \boldsymbol{\alpha})$ is the $M \times N$ Jacobian matrix given in Eq. (2), whereas $\mathbf{g}_{\boldsymbol{\alpha}}(\mathbf{x}; \boldsymbol{\alpha})$ is an $M \times A$ Jacobian matrix defined similarly.

Third, if $g(\cdot) : \Re^{N+A} \to \Re$ is a $C^{(2)}$ scalar-valued function whose arguments are the vectors $\mathbf{x} \in \Re^N$ and $\boldsymbol{\alpha} \in \Re^A$, then there are four Hessian matrices that can be defined based on $g(\cdot)$ because of the two different sets of variables that it depends on, scilicet,

$$\underbrace{g_{\mathbf{xx}}(\mathbf{x}; \boldsymbol{\alpha})}_{N \times N} \stackrel{\text{def}}{=} \begin{bmatrix} g_{x_1x_1}(\mathbf{x}; \boldsymbol{\alpha}) & g_{x_1x_2}(\mathbf{x}; \boldsymbol{\alpha}) & \cdots & g_{x_1x_N}(\mathbf{x}; \boldsymbol{\alpha}) \\ g_{x_2x_1}(\mathbf{x}; \boldsymbol{\alpha}) & g_{x_2x_2}(\mathbf{x}; \boldsymbol{\alpha}) & \cdots & g_{x_2x_N}(\mathbf{x}; \boldsymbol{\alpha}) \\ \vdots & \vdots & \ddots & \vdots \\ g_{x_Nx_1}(\mathbf{x}; \boldsymbol{\alpha}) & g_{x_Nx_2}(\mathbf{x}; \boldsymbol{\alpha}) & \cdots & g_{x_Nx_N}(\mathbf{x}; \boldsymbol{\alpha}) \end{bmatrix},$$

$$\underbrace{g_{\mathbf{x}\boldsymbol{\alpha}}(\mathbf{x}; \boldsymbol{\alpha})}_{N \times A} \stackrel{\text{def}}{=} \begin{bmatrix} g_{x_1\alpha_1}(\mathbf{x}; \boldsymbol{\alpha}) & g_{x_1\alpha_2}(\mathbf{x}; \boldsymbol{\alpha}) & \cdots & g_{x_1\alpha_A}(\mathbf{x}; \boldsymbol{\alpha}) \\ g_{x_2\alpha_1}(\mathbf{x}; \boldsymbol{\alpha}) & g_{x_2\alpha_2}(\mathbf{x}; \boldsymbol{\alpha}) & \cdots & g_{x_2\alpha_A}(\mathbf{x}; \boldsymbol{\alpha}) \\ \vdots & \vdots & \ddots & \vdots \\ g_{x_N\alpha_1}(\mathbf{x}; \boldsymbol{\alpha}) & g_{x_N\alpha_2}(\mathbf{x}; \boldsymbol{\alpha}) & \cdots & g_{x_N\alpha_A}(\mathbf{x}; \boldsymbol{\alpha}) \end{bmatrix},$$

$$\underbrace{g_{\boldsymbol{\alpha\alpha}}(\mathbf{x}; \boldsymbol{\alpha})}_{A \times A} \stackrel{\text{def}}{=} \begin{bmatrix} g_{\alpha_1\alpha_1}(\mathbf{x}; \boldsymbol{\alpha}) & g_{\alpha_1\alpha_2}(\mathbf{x}; \boldsymbol{\alpha}) & \cdots & g_{\alpha_1\alpha_A}(\mathbf{x}; \boldsymbol{\alpha}) \\ g_{\alpha_2\alpha_1}(\mathbf{x}; \boldsymbol{\alpha}) & g_{\alpha_2\alpha_2}(\mathbf{x}; \boldsymbol{\alpha}) & \cdots & g_{\alpha_2\alpha_A}(\mathbf{x}; \boldsymbol{\alpha}) \\ \vdots & \vdots & \ddots & \vdots \\ g_{\alpha_A\alpha_1}(\mathbf{x}; \boldsymbol{\alpha}) & g_{\alpha_A\alpha_2}(\mathbf{x}; \boldsymbol{\alpha}) & \cdots & g_{\alpha_A\alpha_A}(\mathbf{x}; \boldsymbol{\alpha}) \end{bmatrix},$$

and the $A \times N$ matrix $g_{\boldsymbol{\alpha}\mathbf{x}}(\mathbf{x}; \boldsymbol{\alpha})$, the computation of which we leave for a mental exercise. We remark in passing that there is a matrix version of the invariance of the second-order partial derivatives to the order of differentiation, and this too is left for a mental exercise.

With the notational matters settled, let's now return to the unconstrained static optimization problem defined in Eq. (1). Assume that an optimal solution exists to problem (1), say $\mathbf{x} = \mathbf{x}^*(\boldsymbol{\alpha})$. Typically, we would find the solution by simultaneously solving the first-order necessary conditions (FONCs) of problem (1), which are given by

$$f_{\mathbf{x}}(\mathbf{x}; \boldsymbol{\alpha}) = \mathbf{0}'_N$$

in vector notation, where $\mathbf{0}_N$ is the null (column) vector in $\Re^N$, or by

$$f_{x_n}(\mathbf{x}; \boldsymbol{\alpha}) = 0, \quad n = 1, 2, \ldots, N$$

in index notation. Unless the objective function $f(\cdot)$ happened to be of a particularly simple functional form, an explicit solution for $\mathbf{x} = \mathbf{x}^*(\boldsymbol{\alpha})$ is more often than not rare. If, however, we assume that the second-order sufficient condition (SOSC) holds at $\mathbf{x} = \mathbf{x}^*(\boldsymbol{\alpha})$, that is,

$$\mathbf{h}' f_{\mathbf{xx}}(\mathbf{x}^*(\boldsymbol{\alpha}); \boldsymbol{\alpha})\mathbf{h} < 0 \,\forall\, \mathbf{h} \in \Re^N, \quad \mathbf{h} \neq \mathbf{0}_N,$$

or

$$\sum_{i=1}^{N} \sum_{j=1}^{N} f_{x_i x_j}(\mathbf{x}^*(\boldsymbol{\alpha}); \boldsymbol{\alpha}) h_i h_j < 0, \quad \text{not all } h_i = 0,$$

then we can apply the implicit function theorem to the FONCs to solve for the optimal choice vector $\mathbf{x} = \mathbf{x}^*(\boldsymbol{\alpha})$ *in principle*. To see why this is so, recall that the Jacobian matrix of the FONCs is given by the $N \times N$ matrix $f_{\mathbf{xx}}(\mathbf{x}; \boldsymbol{\alpha})$, which is identical to the Hessian matrix of the objective function. Moreover, the SOSC implies that the Hessian determinant of the FONCs is nonvanishing at the optimal solution, that is, that $|f_{\mathbf{xx}}(\mathbf{x}; \boldsymbol{\alpha})| \neq 0$ when evaluated at $\mathbf{x} = \mathbf{x}^*(\boldsymbol{\alpha})$. Because this is equivalent to the nonvanishing of the Jacobian determinant of the FONCs, the implicit function theorem may be applied to the FONCs to solve, in principle, for the optimal choice vector $\mathbf{x} = \mathbf{x}^*(\boldsymbol{\alpha})$. Again, this line of reasoning should be familiar to you from prior courses in microeconomic theory.

In light of the dynamic problems that will occupy us in this book, the most important aspect of the above discussion concerning problem (1) is that for a given value of the parameters, say, $\boldsymbol{\alpha} = \boldsymbol{\alpha}^\circ$, we typically solve for a particular value for each of the decision or choice variables, say, $x_n^\circ = x_n^*(\boldsymbol{\alpha}^\circ)$, $n = 1, 2, \ldots, N$. We do this by solving a set of algebraic equations, that is to say, the FONCs. If the parameter vector is different, say, $\boldsymbol{\alpha} = \boldsymbol{\alpha}^1$, then usually a different value of the choice variables is implied, say, $x_n^1 = x_n^*(\boldsymbol{\alpha}^1)$, $n = 1, 2, \ldots, N$. By their very nature, therefore, static optimization problems ask the decision maker to pick out a particular value of the decision variables given the parameters of the problem. For $A = N = 1$, Figure 1.1 depicts this situation graphically.

To add an economic spin to all of this, recall the prototype profit-maximizing model of the price-taking firm:

$$\pi^*(p, w_1, w_2) \stackrel{\text{def}}{=} \max_{x_1, x_2} \{pF(x_1, x_2) - w_1 x_1 - w_2 x_2\},$$

where $F(\cdot)$ is the twice continuously differentiable production function, x_1 and x_2 are the decision variables representing the inputs of the firm, w_1 and w_2 are the market prices of the inputs, p is the output price, and $\pi^*(\cdot)$ is the indirect profit function. Given a particular set of prices, say, $(p, w_1, w_2) = (p^\circ, w_1^\circ, w_2^\circ)$, the firm seeks to determine the values of the inputs that maximize its profit, say, $x_n^\circ = x_n^*(p^\circ, w_1^\circ, w_2^\circ)$, $n = 1, 2$. If such optimal values exist, then they are found, in principle, by simultaneously solving the FONCs, given by

$$p^\circ F_{x_1}(x_1, x_2) - w_1^\circ = 0,$$

$$p^\circ F_{x_2}(x_1, x_2) - w_2^\circ = 0.$$

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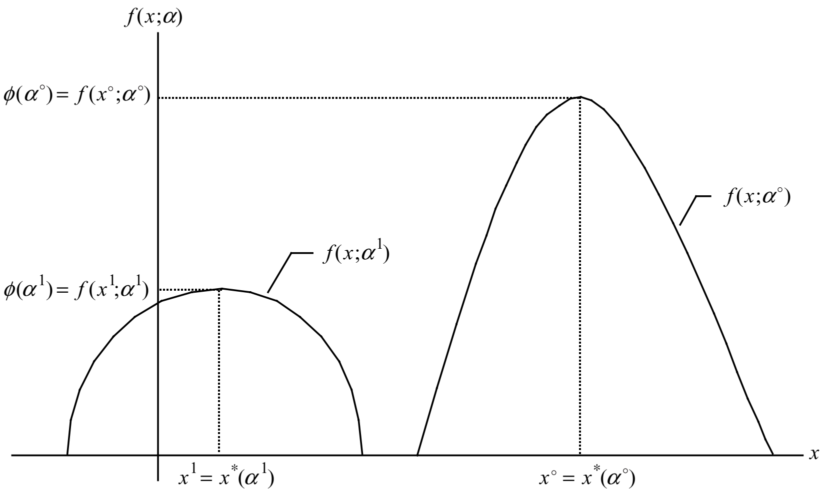

Figure 1.1

Note that these are *algebraic equations* that in general must be solved simultaneously for the optimal values of the inputs. To repeat, the most important point to take away from this discussion is that the choice of the optimal input combination is made just one time: there is no planning for the future, nor are there future decisions to be made in this problem. This is exactly as the static framework of the problem dictates.

In contrast, given the parameters of a dynamic optimization problem, its solution is a *sequence* of optimal decisions in discrete time, or a *time path* or *curve* of optimal decisions in continuous time, over the relevant *planning period* or *planning horizon*, not just one particular value for each of the decision variables like the solution to a static optimization problem. The optimal time path or curve is, by definition, the one that optimizes some type of objective function. The type of objective function in dynamic problems, however, is quite different from that in static problems. Because the solution to a continuous time dynamic optimization problem is a time path or curve, it appears to be reasonable and even natural for the objective function to place a value on the decision variables at each point in time of the planning horizon and to add up the resulting values over the relevant planning period, akin to what is done when one computes the present value of some stream of net benefits that are received over time.

To better motivate the form of the objective function in dynamic problems, consider Figure 1.2. Here three typical time paths of a function $x(\cdot)$, or curves $x(t)$ associated with the function $x(\cdot)$, are displayed along with the resulting value of the objective function associated with each time path $J[x(\cdot)]$, the latter of which we refer to as a *path value*. We have denoted the independent variable by the letter t and refer to it as time, as this is the natural interpretation of the independent variable in intertemporal problems in economics. Notice that all time paths or curves begin

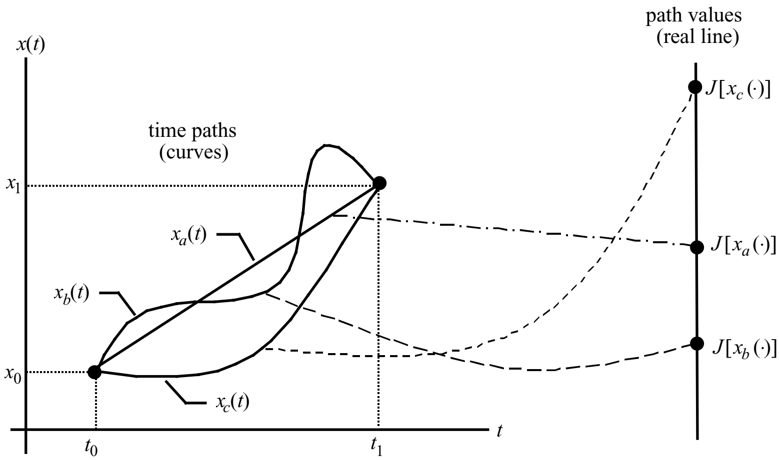

Figure 1.2

at time $t = t_0$ at the point $x = x_0$ and end at time $t = t_1$ at the point $x = x_1$, all four of which are given or fixed, thereby requiring that the paths being compared begin and end at the same position and time. The typical problem in dynamic optimization seeks to find a time path or curve $x(t)$, or equivalently a function $x(\cdot)$, that, say, maximizes the objective function $J[\cdot]$. Thus to each time path or curve $x_i(t)$, $i = a, b, c$, or function $x_i(\cdot)$, $i = a, b, c$, there is a corresponding value of the objective function $J[x_i(\cdot)]$, $i = a, b, c$.

The relationship between paths $x(t)$ or functions $x(\cdot)$ and the resulting value $J[x(\cdot)]$ is quite different from that encountered in typical lower division mathematics courses. It represents a mapping from paths or curves to real numbers, or equivalently, from functions to real numbers, and therefore is not a mapping from real numbers to real numbers as in the case of functions. Such a mapping from paths or curves to path values, or from functions to real numbers, is what Figure 1.2 depicts and is called a *functional*. The general notation we shall employ for such a mapping from functions to real numbers is $J[x(\cdot)]$, which has been employed above. This notation emphasizes that the functional $J[\cdot]$ depends on the *function* $x(\cdot)$, or equivalently, on the *entire curve* $x(t)$. Moreover, it highlights the fact that it is a change in the position of the entire path or curve $x(t)$, that is, the *variation* in the path or curve $x(t)$, rather than the change in t, that results in a change in the path value or functional $J[\cdot]$. Thus, a dynamic optimization problem in continuous time seeks to find a path or curve $x(t)$, or equivalently, a function $x(\cdot)$, that optimizes an objective functional $J[\cdot]$.

Next we consider in more detail the form of an archetype objective functional $J[\cdot]$. Because the optimal solution to a continuous time dynamic optimization problem is a path or curve $x(t)$, as noted above, associated with the path is its slope $\dot{x}(t) \stackrel{\text{def}}{=} dx(t)/dt$ at each point in time t in the planning horizon, assuming, of course, that the path $x(t)$ is smooth enough so that $\dot{x}(t)$ is defined. Suppose, moreover, that there exists a function, say, $F(\cdot)$, that assigns or imputes a value to the path and its associated derivative at each point in time in the planning horizon, the latter represented by the closed interval $[t_0, t_1]$, $0 < t_0 < t_1$. The imputed value of the path at each point in the planning horizon, therefore, depends on the moment of time t the decision is made and the value of the decision or choice variable at that time $x(t)$, as well as on the slope at that time $\dot{x}(t)$. Hence we have $F(t, x(t), \dot{x}(t))$ as the value of the function that imputes a value to the path $x(t)$ with slope $\dot{x}(t)$ at time t. Because the path $x(t)$ must necessarily travel through an interval of time, namely, the planning horizon, its total value as represented by the functional $J[x(\cdot)]$ is given by the "sum" of all the imputed values $F(t, x(t), \dot{x}(t))$ for each t in the planning horizon $[t_0, t_1]$. Moreover, because we are operating in continuous time, the appropriate notion of summation is represented by a definite integral over the closed interval $[t_0, t_1]$. Thus, the value of the functional we wish to optimize is given by

$$J[x(\cdot)] \stackrel{\text{def}}{=} \int_{t_0}^{t_1} F(t, x(t), \dot{x}(t))\, dt. \tag{3}$$

In economics, $J[x(\cdot)]$ often represents the present value of net benefits from pursuing the policy $x(t)$, with instantaneous net benefits given by $F(t, x(t), \dot{x}(t))$. In general, one calls $x(t)$ the state of the system, or value of the state variable, at time t, and $\dot{x}(t)$ the rate of change of the system, or velocity, at time t. Typically, the *explicit* appearance of t as an argument of $F(\cdot)$ is a result of discounting in economic problems. Equation (3) represents the prototype form of the objective functional for *calculus of variations* problems, which are but one class of continuous time dynamic optimization problems. More general objective functionals will be introduced shortly, when we commence with the study of optimal control theory. But for now, the present discussion and motivation are sufficient, for the idea of a mapping from paths or curves to the real line and the form of the objective functional is what one must come away with.

Before moving on to some additional motivational material, a few remarks about the form of the objective functional in Eq. (3) are warranted. First, the functional $J[\cdot]$ is *not*, in general, the area under the curve $x(t)$ between the points $t = t_0$ and $t = t_1$, the latter of which is represented by the integral $\int_{t_0}^{t_1} x(t)dt$. Thus, in optimizing $J[\cdot]$, we are *not* making decisions to optimize the area under the curve $x(t)$. Rather, we are picking paths $x(t)$ such that the "sum" of the values imputed to the path $x(t)$ and its derivative $\dot{x}(t)$ at each point in time in the planning horizon is optimized. Second, the form of $J[\cdot]$ given in Eq. (3) is not the most general form of the objective functional for calculus of variations problems, but it is the most common or canonical in economic theory, as examples to be introduced latter will confirm. In motivating the form of $J[\cdot]$, for example, there is no particular reason why $F(\cdot)$ would not, in general, depend on the second or higher derivatives of $x(t)$. What dictates which derivatives of the path $x(t)$ that are relevant to the form of $J[\cdot]$

is the particular economic phenomena under study, and as we will shortly see, most of the problems of relevance to economists dictate the presence of $\dot{x}(t)$ in $F(\cdot)$, but rarely higher derivatives.

The question you may now be wrestling with is: How does one know when to construct a dynamic model, as opposed to a static model, to study the economic events under investigation? The answer is most easily explained and motivated within the context of a simple example. Imagine an individual on an isolated island on which fresh water is available in an unlimited supply, and for which essentially no effort is required to collect the water. You may picture a brook or stream of fresh water passing next to the individual's hut. The only source of food is fish, which does require the expenditure of effort on the part of this individual. Clearly this individual can survive on this island, and the problem that this person faces is how much fish to catch each day for consumption.

Initially, let's assume that the harvested fish are impossible to store for any length of time, either because of the extreme heat of the environment or because of the lack of materials necessary to build a suitable storage facility. As this individual gets up on any day, the decision to be made is how much fish to catch *for this day only*, as storage has been ruled out. Any fish caught beyond the amount to be consumed that day simply rots and is wasted. Because fishing is costly to the individual, the catch on any day will not exceed the individual's consumption per day. Notice that the decision of how much fish to catch this day is independent of fish caught on previous days or fish expected to be caught on future days, as the lack of storage prevents any carryover of the fish. This lack of storage breaks any link between past decisions and the present decision, and any link between the present decision and future decisions. In other words, the absence of any durable asset or the inability of this person to store any of the asset (fish) renders current choices or actions independent of those made in the past, or those to be made in the future. For example, even if this person caught twice as many fish as could be consumed in a day, this would not relieve the individual of fishing the following day because the fish simply spoil, leaving zero edible fish for tomorrow. Thus the decision of how much fish to catch on any day is dependent only on the circumstances or environmental conditions of that day. As the reader may have guessed, this is exactly what dictates the decision problem faced by this individual as simply a sequence of static choice problems, each day's decision being independent of past and future decisions, and identical to that to be made on any other day, save for differing environmental conditions.

The reader may wonder why this situation is not a dynamic optimization problem since a sequence of optimal decisions must be made through time. It is not the sequence of decisions or the introduction of time per se that defines a dynamic choice problem but the link between past, present, and future decisions that makes a problem dynamic. In the scenario above, the lack of storage breaks this link, reducing the problem to a sequence of independent static optimization problems. So to have a dynamic optimization problem, *there must be some systematic link between past, present, and future decisions*.

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Now let's assume that fish caught on any day in excess of that day's consumption can be stored. Because we are concerned here about the structure of a problem that makes it dynamic, the actual period of time in which the fish can be stored or preserved is not important; the fact that fish caught on one day can be stored for future consumption is the important idea. Just as in the previous scenario, when this individual wakes up on any given day, a decision about how much fish to catch that day must be made. What is different, however, is that a stock of fish may exist in storage from the previous day's catch, and this stock must be taken into account in today's harvesting decision. Thus, the assumption of storage (or a durable good) provides a direct link between past decisions and the current decision, a link that was absent when storage was ruled out. Likewise, the decision to catch fish (or not) today impacts the amount of fish in storage for future consumption, and therefore impacts future decisions about catching fish. Storage provides a link between current decisions and future decisions as well. It is exactly this intertemporal linking of decisions that makes this second scenario a dynamic choice problem: decisions made in the past affect the current choice, which, in turn, affects future choices.

Although it may appear that in this simple example, there is only one variable, scilicet, fish, there are actually two: catching fish and fish in storage. The storage of fish is not a variable that is controlled directly by the individual; it responds to the amount caught, amount eaten, and time elapsed between catches. In macroeconomic terminology, the amount of fish in storage is a stock variable, or a state variable in the language of optimal control theory; that is, it is defined at a point in time, not over a period or length of time. The act of fishing, on the other hand, is defined over a period of time (a flow variable) and is directly under the control of the individual. In the language of optimal control theory, the catch rate of the fish is the control variable.

With the essence of a continuous time dynamic optimization problem now conveyed, let's turn to the motivation and basic mathematical setup of an optimal control problem. We therefore elect to proceed directly to optimal control theory rather than first formally introducing the calculus of variations and *then* optimal control theory.

Optimal control theory is based on a new way of viewing and formulating calculus of variations problems, and thus enables one to see them in a different light. In particular, optimal control theory often brings the economic intuition and content of a continuous time dynamic optimization problem to the surface more readily than does the calculus of variations, thereby enhancing one's economic understanding of the problem. It is this change of vista that makes optimal control theory a powerful tool for solving dynamic economic problems, for the calculus of variations can solve all problems that can be solved with optimal control theory, though not necessarily as easily, in which case both theories yield equivalent results. In fact, some textbooks, such as Hadley and Kemp (1971), develop the calculus of variations in its full generality and then use the results to prove those in optimal control theory.

The focus in optimal control theory is on some system. In economic problems, this may be the economy, an individual, or a firm. As is usual in intertemporal problems, we are interested in optimizing, in some specified sense, the behavior of the system through time. It is assumed that the manner in which the system changes through time can be described by specifying the time behavior of certain variables, say $\mathbf{x}(t) \in \Re^N$, called *state variables*, where t is the independent variable that we will almost always refer to as time. In general, the vector-valued function $\mathbf{x}(\cdot) : \Re \to \Re^N$ is assumed to be a *piecewise smooth* function of time with not more than a finite number of corners. This means that the component functions $x_n(\cdot)$, $n = 1, 2, \ldots, N$, are continuous but that the derivative functions $\dot{x}_n(\cdot)$, $n = 1, 2, \ldots, N$, are *piecewise continuous* in the sense that $\dot{x}_n(\cdot)$ has at most a finite number of discontinuities on each finite interval with finite jumps (i.e., one-sided limits) at each point of discontinuity. In economic problems, a capital stock, a stock of money or any asset for that matter (e.g., wealth), a stock of fish, the amount of some mineral in the ground, the stock of water in an aquifer, the number of chairs in a classroom, or even the distribution function of a random variable may represent a state variable. Generally, the state variables are defined at a given point in time, as the aforementioned examples indicate. This is why state variables are often referred to as stocks by economists.

Before pressing on, it is prudent at this juncture to pause momentarily and give a precise definition of a piecewise continuous function and a piecewise smooth function, for such functions will be encountered with some regularity in optimal control theory. To that end, we have the following definition.

Definition 1.1: A function $\phi(\cdot)$ is said to be *piecewise continuous* on an interval $\alpha \le t \le \beta$ if the interval can be partitioned by a finite number of points $\alpha = t_0 < t_1 < \cdots < t_K = \beta$ so that

1. $\phi(\cdot)$ is continuous on each open subinterval $t_{k-1} < t < t_k$, $k = 1, 2, \ldots, K$, and
2. $\phi(\cdot)$ approaches a finite limit as the end points of each subinterval are approached from within the subinterval.

In other words, a function $\phi(\cdot)$ is piecewise continuous on an interval $\alpha \le t \le \beta$ if it is continuous there except for a finite number of jump discontinuities. An example of a piecewise continuous function is shown in Figure 1.3. Given this definition, it is now a relatively simple matter to define a piecewise smooth function.

Definition 1.2: A function $\Phi(\cdot)$ is said to be *piecewise smooth* on an interval $\alpha \le t \le \beta$ if its derivative function $\dot{\Phi}(\cdot)$ is piecewise continuous on the interval $\alpha \le t \le \beta$.

This definition therefore implies that the derivative of a piecewise smooth function is piecewise continuous, and that the integral of a piecewise continuous function is piecewise smooth.

It is also assumed that there exists another class of variables known as *control variables*, say $\mathbf{u}(t) \in \Re^M$, where $M \ne N$ in general. The control variables may undergo jump changes and are therefore only restricted to be *piecewise continuous* in general, that is, the function $\mathbf{u}(\cdot) : \Re \to \Re^M$ is assumed to be a *piecewise continuous*

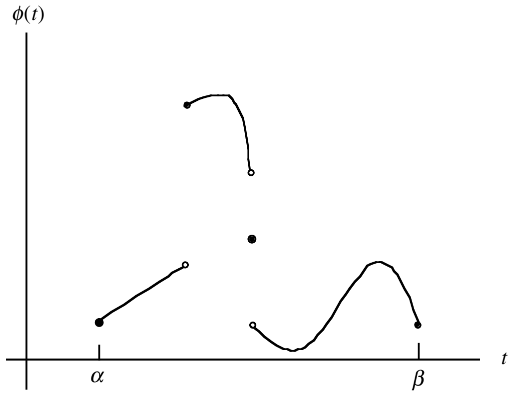

Figure 1.3

function of time. In economic problems, control variables are usually represented by flow variables that are typically defined over an interval of time. Archetypal control variables in economic problems include the investment rate in an asset, the consumption rate of a good, and the harvest rate of a resource. Control variables are those variables that the decision maker has explicit control of in the optimal control problem, as the name implies. Control variables are thus the analogues of the choice or decision variables in static optimization theory. The control variables will not, in general, be allowed to take on arbitrary values. Generally, it is assumed that for each t in the planning horizon, the control variables are restricted to vary in a fixed and prespecified set $U \subseteq \Re^M$, called the *control set* or *control region*. It is typically required that $\mathbf{u}(t) \in U$ for the entire planning horizon. The control set U may be any fixed set in $\Re^M$, say, an open or closed set, but is not restricted in any special way. Of particular importance is the case in which U is a closed set in $\Re^M$. In this case, the control variables are allowed to take values at the boundary of the control set, a situation that the classical calculus of variations cannot handle so easily. In typical economic problems, the control set may be represented by nonnegativity restrictions on the control variables or by fixed inequality constraints that bound the control variables. Finally, it is assumed that every variable of interest can be classified as a state variable or a control variable.

In view of the restrictions placed on the two classes of variables, we may think of the control variables as governing not the values of the state variables, but their rate of change. More specifically, it will be assumed that the dependence of the state variables on the control variables can be described by a first-order differential equation system, namely,

$$\dot{\mathbf{x}}(t) = \mathbf{g}(t, \mathbf{x}(t), \mathbf{u}(t))$$

in vector notation, or

$$\dot{x}_n(t) = g^n(t, \mathbf{x}(t), \mathbf{u}(t)), \quad n = 1, 2, \ldots, N$$

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in index notation, known as the *state equation*. The *transition functions* $g^n(\cdot)$, $n = 1, 2, \ldots, N$, are given functions that describe the dynamics of the system. In general, the rate of change of each state variable depends on all of the state variables, all of the control variables, various economic and technical parameters, and explicitly on time t, though the specific problem under consideration will dictate the exact form of the state equation and the variables appearing in it. The explicit dependence of the transition functions on t allows for the evolution of the state variables to depend on important exogenous factors, such as technological progress. Furthermore, suppose that the state of the system is known at time t_0, so that $\mathbf{x}(t_0) = \mathbf{x}_0$, where $\mathbf{x}_0 \in \Re^N$ is a given vector. If the time path of the control variables is specified by a certain control function, say, $\mathbf{u}(\cdot)$, defined for $t \geq t_0$, and we substitute it into the state equation, we obtain a system of N first-order ordinary differential equations for the N unknown functions $x_n(\cdot)$, $n = 1, 2, \ldots, N$. Because the initial value $\mathbf{x}_0 \in \Re^N$ of the state variable is given, the state equation will have a unique solution $\mathbf{x}(t)$ under rather mild assumptions, given by the fundamental existence and uniqueness theorem for ordinary differential equations. This solution is represented geometrically by a curve in $\Re^N$. Because this solution is essentially a response to the control function $\mathbf{u}(\cdot)$, it would be appropriate to denote it by $\mathbf{x}_{\mathbf{u}}(t)$, but we will drop the subscript $\mathbf{u}$ as is customarily done. Clearly, we could have selected another control function, and a corresponding time path of the state variable would be generated. Thus, in general, for each control function selected, there corresponds a path for the state variables that represents the solution to the state equation and initial condition. As a result of this observation, it follows that the control and state variables are essentially paired, in that once a control function is specified, the corresponding time path of the state variables is completely determined via the state equation.

To complete our statement of an optimal control problem, we must define a measure of the effectiveness of a control. Such a measure is provided by the functional

$$J[\mathbf{x}(\cdot), \mathbf{u}(\cdot)] \overset{\text{def}}{=} \int_{t_0}^{t_1} f(t, \mathbf{x}(t), \mathbf{u}(t))\, dt,$$

where the closed interval $[t_0, t_1]$ is called the *planning horizon*, t_0 is the *initial date* of the planning horizon, and t_1 is the *terminal date* of the planning horizon. The integrand function $f(\cdot)$ depends, in general, on all of the state variables, all of the control variables, various economic and technical parameters, and explicitly on time t, though as in the case of the state equation, the specific problem under consideration will dictate the exact form of the integrand function and the variables appearing in it. In economic problems, the integrand function typically represents the net benefits at each instant of time t in the planning horizon resulting from the control $\mathbf{u}(t)$ and its corresponding state $\mathbf{x}(t)$. In most economic problems, the explicit appearance of t in the integrand function is a result of the process of discounting the future net benefits using the function $t \mapsto e^{-rt}$, where $r > 0$ is the discount rate. Note, in passing, that t is a *dummy variable of integration* in the objective functional, and as such, we are free to replace it with another symbol that suits our purpose, say, s or τ, without affecting the optimal control problem. We will in fact do so in later chapters.

The archetypal optimal control problem can now be stated as follows: select a control function $\mathbf{u}(\cdot)$ such that $\mathbf{u}(t) \in U \; \forall \, t \in [t_0, t_1]$, so that when the state vector $\mathbf{x}(t)$ is determined from the state equation $\dot{\mathbf{x}}(t) = \mathbf{g}(t, \mathbf{x}(t), \mathbf{u}(t))$ and an initial condition $\mathbf{x}(t_0) = \mathbf{x}_0$ is given, the functional $J[\mathbf{x}(\cdot), \mathbf{u}(\cdot)]$ is maximized. Such a control function is called *an optimal control*, and the associated path $\mathbf{x}(t)$ is called *an optimal path*. That is, the problem is to find a control *function* $\mathbf{u}(\cdot)$ that solves the following problem:

$$
\begin{aligned}
\max_{\mathbf{u}(\cdot), \mathbf{x}_1} J[\mathbf{x}(\cdot), \mathbf{u}(\cdot)] &\overset{\text{def}}{=} \int_{t_0}^{t_1} f(t, \mathbf{x}(t), \mathbf{u}(t))\, dt \\
\text{s.t.} \quad \dot{\mathbf{x}}(t) &= \mathbf{g}(t, \mathbf{x}(t), \mathbf{u}(t)), \\
\mathbf{x}(t_0) = \mathbf{x}_0, \; \mathbf{x}(t_1) &= \mathbf{x}_1, \; \mathbf{u}(t) \in U,
\end{aligned}
\tag{4}
$$

where $\mathbf{x}_0$ is the *initial value* of the state vector and is taken as given in Eq. (4), whereas $\mathbf{x}_1$ is the *terminal value* of the state vector, which, as the notation of Eq. (4) should convey, is a choice variable rather than being fixed (or given or predetermined) like $\mathbf{x}_0$. The initial value of the state variable is normally thought to be "inherited" from the past actions of the decision maker and is thus not subject to choice. Notice that the control variables influence the value of the objective functional $J[\mathbf{x}(\cdot), \mathbf{u}(\cdot)]$ both directly through their own values and indirectly through their impact on the evolution of the state variables through the state equation. Though in general all of the state and control variables enter the integrand function $f(\cdot)$ and the transition function $\mathbf{g}(\cdot)$, as noted above, it is perfectly acceptable for $f(\cdot)$ and $\mathbf{g}(\cdot)$ to be a function of only some of the state variables or some of the control variables. As remarked above, the economic problem under investigation will dictate the form of $f(\cdot)$ and $\mathbf{g}(\cdot)$. It is important to note, however, that at least one control variable must appear in at least one state equation; otherwise the state equation could not be controlled by the choices of the decision maker, who has explicit command of only the control variables. It is not necessary for at least one control variable to appear in every state equation, however. To see this, consider an example where $M = 1$ and $N = 2$, and where the state equations take the form $\dot{x}_1(t) = u(t)$ and $\dot{x}_2(t) = x_1(t)$. In this instance, the decision maker directly influences $x_1(t)$ and thus indirectly influences $x_2(t)$, even though the control variable does not appear in the second state equation. In other words, the definition of an optimal control problem requires that the decision maker can influence the evolution of the state of the system, which dictates that at least one control variable appear in at least one state equation. Also notice that the highest derivative appearing in the problem formulation is the first derivative of the state variable, and it appears only as the left-hand side of the state equation. No derivatives of the control variables appear in the formulation of the optimal control problem. This does not result in any loss of generality, for one can always redefine the variables so that an optimal control problem takes on the prototype form given in Eq. (4). This is analogous to converting, say, a second-order differential equation into a system of two first-order differential equations. It is important to emphasize that because an optimal control problem may have several state variables and several control variables in general, each state variable would evolve according to a differential equation. Moreover, the number of control variables may be greater than, less than, or equal to the number of state variables.

Thus far, we have omitted a statement of the fundamental continuity restrictions to be placed on the integrand function $f(\cdot)$ and the transition function $\mathbf{g}(\cdot)$. The minimum assumptions that we shall impose on the aforementioned functions are as follows:

(A.1) $f(\cdot) \in C^{(0)}$ with respect to the $1 + N + M$ variables $(t, \mathbf{x}, \mathbf{u})$.
(A.2) $g^{\ell}(\cdot) \in C^{(0)}$ with respect to the $1 + N + M$ variables $(t, \mathbf{x}, \mathbf{u})$ for $\ell = 1, 2, \ldots, N$.
(A.3) $\partial f(\cdot)/\partial x_n \in C^{(0)}$ with respect to the $1 + N + M$ variables $(t, \mathbf{x}, \mathbf{u})$ for $n = 1, 2, \ldots, N$.
(A.4) $\partial g^{\ell}(\cdot)/\partial x_n \in C^{(0)}$ with respect to the $1 + N + M$ variables $(t, \mathbf{x}, \mathbf{u})$ for $\ell, n = 1, 2, \ldots, N$.

We will maintain at least these four assumptions throughout the remainder of the book. Notice that differentiability of the integrand function $f(\cdot)$ and the transition function $\mathbf{g}(\cdot)$ with respect to the control variables is not required. Although this is true in general, we will often impose stronger continuity assumptions on the functions, especially in the beginning of our development of the theory, and often for pedagogical reasons. Such instances will be noted in the ensuing chapters.

A prototypical calculus of variations problem is of the form

$$\begin{aligned} &\max_{\mathbf{x}(\cdot),\mathbf{x}_1} J[\mathbf{x}(\cdot)] \stackrel{\text{def}}{=} \int_{t_0}^{t_1} F(t, \mathbf{x}(t), \dot{\mathbf{x}}(t))\, dt \\ &\text{s.t.} \quad \mathbf{x}(t_0) = \mathbf{x}_0, \ \mathbf{x}(t_1) = \mathbf{x}_1. \end{aligned} \tag{5}$$

Note that the terminal value of the state vector is a decision variable in this formulation, as is signified by the appearance of $\mathbf{x}_1$ under the max operator. This calculus of variations problem can be transformed into the prototypical optimal control problem

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by defining the control variable as $\mathbf{u}(t) \stackrel{\text{def}}{=} \dot{\mathbf{x}}(t)$, so that Eq. (5) becomes

$$\max_{\mathbf{u}(\cdot),\mathbf{x}_1} J[\mathbf{x}(\cdot),\mathbf{u}(\cdot)] \stackrel{\text{def}}{=} \int_{t_0}^{t_1} F(t,\mathbf{x}(t),\mathbf{u}(t))\,dt$$
$$\text{s.t.} \quad \dot{\mathbf{x}}(t) = \mathbf{u}(t), \tag{6}$$
$$\mathbf{x}(t_0) = \mathbf{x}_0,\ \mathbf{x}(t_1) = \mathbf{x}_1.$$

Hence the state vector is $\mathbf{x}(t)$, $\dot{\mathbf{x}}(t) = \mathbf{u}(t)$ is the state equation, and $\mathbf{u}(t)$ is the control vector. Because there are no constraints on the control variables in Eq. (6), the control set U is the entire real N-dimensional space, that is, $U = \Re^N$.

With some of the essentials of optimal control problems now in place, we can turn to the presentation of some canonical continuous time dynamic optimization problems in economics. In particular, we intend to provide a rather detailed formulation and discussion of four dynamic optimization problems, all of which will be solved and generalized in later chapters.

Example 1.1 (Fish Farming): Let's imagine that the sole owner of a fish farm with well-defined property rights (an aquaculture environment) wants to determine the optimal rate at which to harvest and sell the fish through time. Remember that what makes a particular problem a dynamic one is that past, present, and future choices are linked in some manner, not that we have some dimension of time in the problem per se. The assumption of sole ownership with well-defined property rights allows us to avoid the problems associated with common ownership, such as in the case of ocean fishing. The problems associated with such circumstances are not properly handled through this first model.

First, let's consider the production function associated with harvesting. Clearly, the amount of fish that can be harvested, say, h, depends on the stock of fish in the body of water (e.g., a lake), which we denote by x. As the stock of fish grows, ceteris paribus, more fish can be harvested, but as is typical with production functions, this process is subject to a declining marginal product. In addition, the harvest rate is dependent on the quantity of variable inputs used. In the fisheries literature, it is commonplace to assume that all such variable inputs can be combined into one homogeneous index called effort, which we denote by e. An increase in effort, ceteris paribus, leads to a bigger catch, but this input is subject to a declining marginal product as well. Overall, diminishing returns to scale are prevalent for the $C^{(2)}$ production function $f(\cdot)$ (which is not to be confused with the integrand function in the above exposition). These assumptions can be stated mathematically as

$$h = f(x,e),\ f_x(x,e) > 0,\ f_e(x,e) > 0,\ f_{ex}(x,e) > 0,$$
$$f_{xx}(x,e) < 0,\ f_{ee}(x,e) < 0,\ f_{xx}(x,e)f_{ee}(x,e) - [f_{ex}(x,e)]^2 > 0.$$

Notice that $f_{ex}(x, e) > 0$ is also assumed, which says that the marginal product of effort is larger the more fish there are in the lake. This effect and $f_x(x, e) > 0$ are known as the prototypical stock effect in resource harvesting problems.

The cost function $C(\cdot)$ dual to the production function $f(\cdot)$ is defined in the archetypal manner by

$$C(x, h; w) \overset{\text{def}}{=} \min_e \{w \cdot e \text{ s.t. } h = f(x, e)\}, \tag{7}$$

where $w > 0$ is defined as the time-independent (i.e., constant over time) per-unit cost of effort. The cost function $C(\cdot)$ is called a *minimum restricted cost function*, the adjective *restricted* being employed because the stock of fish, x, is treated as fixed in the static optimization problem (7). It is left for the reader, as a mental exercise, to show that the following properties (and more) hold for the minimum restricted cost function $C(\cdot)$:

$$C_x(x, h; w) < 0,\ C_h(x, h; w) > 0,\ C_{hx}(x, h; w) < 0,$$

$$C_{xx}(x, h; w) > 0,\ C_{hh}(x, h; w) > 0,$$

$$C_{xx}(x, h; w)C_{hh}(x, h; w) - [C_{hx}(x, h; w)]^2 > 0,$$

$$C_w(x, h; w) > 0,\ C_{ww}(x, h; w) \equiv 0.$$

The proof of these results is an application of the implicit function theorem because problem (7) has but one decision variable and one constraint.

The owner of the fish farm is assumed to be a price taker in both the output and input markets. The market price of the harvested fish is $p > 0$ and is time independent. The owner is interested in maximizing the present discounted value of profits over the planning horizon of T years. Here we are taking time to be a continuous variable. Profit per period is the integrand function and is given by the expression $\pi(x(t), h(t); p, w) \overset{\text{def}}{=} ph(t) - C(x(t), h(t); w)$. Moreover, profit flow is discounted at the constant rate $r > 0$. Because we are in continuous time, the usual summation and discounting of per-period profits associated with discrete time is replaced with integration and continuous discounting over the planning horizon of T years. Thus, the objective functional is given by

$$J[x(\cdot), h(\cdot)] \overset{\text{def}}{=} \int_0^T [ph(t) - C(x(t), h(t); w)]\, e^{-rt}\, dt.$$

The problem statement is not complete, however, as the growth of the fish population and the effects of harvesting on it must be taken into account. Because $x(t)$ is the stock (or number) of fish at a point in time, $\dot{x}(t) \overset{\text{def}}{=} \frac{d}{dt}x(t)$ is the stock's rate of growth or decline. The growth rate of the fish stock obviously depends on the stock of fish in the lake, say, $\dot{x}(t) = F(x(t))$, where $F(x(t))$ is the stock's natural growth rate in the absence of harvesting. One common functional form for $F(\cdot)$ is the logistic, given by $F(x; \gamma, K) \overset{\text{def}}{=} \gamma x[1 - xK^{-1}]$. So far, this formulation of the rate of change of the fish stock ignores the effects of harvesting. Harvesting simply reduces the stock's growth rate by the number of fish harvested over a period of time. Thus, the stock's net growth becomes $\dot{x}(t) = F(x(t)) - h(t)$. This is a simplifying assumption because the stock's natural growth rate $F(x(t))$ may be influenced by the age of the species harvested. The natural growth function $F(\cdot)$ is concave, is twice continuously differentiable, and has a unique maximum at $x = x_{MSY}$, the maximum sustainable yield level of the fish stock. The stock x_{MSY} is therefore the solution to $F'(x) = 0$. Moreover, the natural growth function is increasing in the fish stock for $x < x_{MSY}$ and is decreasing in the fish stock for $x > x_{MSY}$. In addition, the individual owner begins the operations at time $t = 0$ with a fixed number of fry, say, $x(0) = x_0 > 0$, but may choose the stock of fish to terminate with, say x_T, when the given terminal period T of the planning horizon arrives.

Armed with this information, the optimal control problem the manager must solve to maximize the present discounted value of profits is

$$\max_{h(\cdot),x_T} J[x(\cdot), h(\cdot)] \stackrel{\text{def}}{=} \int_0^T [ph(t) - C(x(t), h(t); w)]\, e^{-rt}\, dt$$

$$\text{s.t.} \quad \dot{x}(t) = F(x(t)) - h(t), \tag{8}$$

$$x(0) = x_0, \;\; x(T) = x_T.$$

The solution (if it exists) to problem (8) yields the optimal time paths of the harvest rate and fish stock over the owner's planning horizon. Denote these solutions by $h^*(t; \boldsymbol{\beta})$ and $x^*(t; \boldsymbol{\beta})$, respectively, where $\boldsymbol{\beta} \stackrel{\text{def}}{=} (p, w, r, x_0, T) \in \Re^5_{++}$. Notice that the solutions depend not only on the parameters of the problem, as do the solutions to static optimization problems, but they also depend on the independent variable time. It is important to note that it is the differential equation that makes problem (8) a dynamic optimization problem because the decision to harvest today leaves fewer fish in the future to reproduce, thus impacting future stock levels and future harvesting decisions. The same is true regarding how past harvesting decisions affect the current harvesting decision.

In the next example, we formulate an intertemporal model of a profit-maximizing firm that uses capital and labor to produce its output and faces costs of adjusting its capital stock.

Example 1.2 (Adjustment Cost Model): This example presents a typical price-taking (in both input and output markets), capital-accumulating model of the firm, known as the *adjustment cost model of the firm*. For simplicity, assume there are two inputs to the production process, one variable input that we will call labor (L), purchased at the constant unit price of $w > 0$, and the other, a quasi-fixed input that we will call the capital stock (K), which incurs a constant per-unit maintenance cost of $c > 0$. The $C^{(2)}$ production function of this firm, say, $f(\cdot)$, depends on L and K

in the prototypical manner. In addition, the firm invests in the capital stock at the rate I and pays the constant price of $g > 0$ per unit purchased. The price of the good produced by the firm is also constant and given by $p > 0$. The firm discounts its cash flow at the rate $r > 0$. The installation of capital is costly to the firm in that it uses up resources to install the equipment and get it ready for productive uses, resources that would have otherwise been used to produce the final good. Such adjustment costs, denoted by $C(I)$, are assumed to have the following archetypal properties, to wit, $\text{sign}(C'(I)) = \text{sign}(I)$, $C(0) = C'(0) = 0$, and $C''(I) > 0$. The cash flow per period for this firm is therefore given by

$$\begin{aligned}&\pi(K(t), I(t), L(t); c, p, g, w)\\ &\quad\stackrel{\text{def}}{=} pf(K(t), L(t)) - wL(t) - cK(t) - gI(t) - C(I(t)).\end{aligned}$$

Capital depreciates at a rate $\delta > 0$ proportional to its existing stock. The rate of change of the capital stock, scilicet, $\dot{K}(t) \stackrel{\text{def}}{=} \frac{d}{dt}K(t)$, depends on the rate at which new investment goods are purchased, $I(t)$, and the rate at which the existing capital depreciates, $\delta K(t)$. Thus the net rate of change in the capital stock is given by the ordinary differential equation $\dot{K}(t) = I(t) - \delta K(t)$. Finally, this firm expects to operate its business for a period of T years, which it takes as given, and begins its operations at time $t = 0$ with an initial stock of capital $K(0) = K_0 > 0$, and ends its operations with a terminal stock of capital $K(T) = K_T > 0$. Notice that in this example, in contrast to Example 1.1 and the prototypical form of the optimal control problem given in Eq. (4), the terminal value of the state variable is fixed rather than chosen by the decision maker.

The optimal control problem the manager must solve in order to determine the optimal time paths of the capital stock, labor input, and investment rate throughout the firm's operating life is therefore given by

$$\begin{aligned}&\max_{I(\cdot),L(\cdot)} J[K(\cdot), I(\cdot), L(\cdot)]\\ &\quad\stackrel{\text{def}}{=} \int_0^T [pf(K(t), L(t)) - wL(t) - cK(t) - gI(t) - C(I(t))]e^{-rt}\,dt\\ &\qquad\qquad\text{s.t.}\quad \dot{K}(t) = I(t) - \delta K(t),\\ &\qquad\qquad K(0) = K_0,\ K(T) = K_T.\end{aligned} \tag{9}$$

The solution to the optimal control problem (9) is denoted by $K^*(t; \boldsymbol{\alpha})$, $I^*(t; \boldsymbol{\alpha})$, and $L^*(t; \boldsymbol{\alpha})$, where $\boldsymbol{\alpha} \stackrel{\text{def}}{=} (\delta, c, g, p, r, w, K_0, T, K_T) \in \Re^9_{++}$ are the parameters of the problem, and gives the optimal time paths for the capital stock, investment rate, and labor input, respectively. As in Example 1.1, it is the state equation of problem (9) that makes it dynamic. To see this, simply note that the state equation implies that the optimal investment rate depends on the existing stock of capital, which itself is a result of investment decisions in prior periods. Additionally, the rate at which capital depreciates depends on the existing capital stock. In sum then, the current decision of how much investment to undertake depends on the current capital stock, which itself is dependent on previous investment decisions, thus providing the link between current and past decisions that typifies dynamic problems.

Notice that in Examples 1.1 and 1.2, we made explicit the dependence of the optimal solutions on the parameters of the control problem. This a practice we shall generally adhere to and one that is in accord with your experience with static optimization theory. Moreover, it is a necessary prerequisite if one is to do any qualitative analysis of an optimal control model. Such qualitative analysis, known as *comparative dynamics* in dynamic optimization problems, often, but not always, seeks to answer the following question: What is the effect of, say, an increase in the price of the finished good on the time path of the investment rate of the firm? In other words, what is the sign of $\partial I^*(t;\boldsymbol{\alpha})/\partial p$ for all t in the planning horizon? A comparative dynamics investigation thus may seek to determine how a parameter change influences the decision variables at every point in the planning horizon. Recall that comparative statics is performed in a timeless world, so once the sign of a comparative statics expression is determined (or not), the result continues to hold for all time periods. In contrast, a comparative dynamics analysis may show, for example, that $\partial I^*(t;\boldsymbol{\alpha})/\partial p$ is initially positive, then turns negative for some time, only to go to zero as the terminal period approaches. For this reason, it can be more difficult to derive unequivocal or refutable results from a comparative dynamics analysis than from a comparative statics analysis. This does not mean, however, that refutable results do not exist in comparative dynamics. Quite the contrary, in fact, as we shall see in many of the later chapters.

For our next example, we present an intertemporal version of the prototype utility maximization problem.

Example 1.3 (Intertemporal Utility Maximization): Consider an individual who wants to choose her consumption rate $c(t)$ at each moment in time t so as to maximize her discounted stream of utility over her known lifetime of length $T > 0$ years. The instantaneous utility function of consumption $U(\cdot)$ is assumed to have positive but declining marginal utility of consumption, that is, $U'(c) > 0$ and $U''(c) < 0$. She discounts her future utility of consumption at the rate $r > 0$ and derives income at any given moment from an exogenously determined and time-invariant wage $w > 0$ and interest earnings of $ik(t)$ on her capital asset $k(t)$, where $i > 0$ is the constant rate of interest earned on the capital asset. One may think of the capital asset as a savings account or a money market fund. The individual may borrow capital, implying that $k(t) < 0$, as well as lend capital, implying that $k(t) > 0$, at the interest rate $i > 0$. Because $k(t)$ is the stock of her capital asset at time t, $\dot{k}(t)$ is, by definition of the derivative, the rate of change of the stock of her asset at time t. Therefore, when $\dot{k}(t) > 0$, she is purchasing or buying the capital asset, whereas $\dot{k}(t) < 0$ means she is selling the capital asset. The price at which capital is traded is normalized at unity. Her income in each period thus consists of interest earnings from her capital asset $ik(t)$ and wage income $w > 0$, whereas expenditures in each period consist of consumption expenditures $c(t)$ and investment expenditures $\dot{k}(t)$. Her budget constraint requires that in each period $t \in [0, T]$, income equals expenditures, thereby implying that her instantaneous budget constraint is given by $w + ik(t) = c(t) + \dot{k}(t)$. She begins her consumption planning with a given amount of the capital asset, say, $k(0) = k_0 > 0$, and ends up with a given amount at the time of her death, say, $k(T) = k_T > 0$. The assumptions that she knows the date of her death and that she must end up with a given amount of the asset at the time of her death are artificial and will be relaxed in the course of our development of the theory of optimal control. The planning problem she must solve is therefore given by

$$\max_{c(\cdot)} J[k(\cdot), c(\cdot)] \stackrel{\text{def}}{=} \int_0^T e^{-rt} U(c(t))\, dt$$

$$\text{s.t.} \quad \dot{k}(t) = w + ik(t) - c(t),$$

$$k(0) = k_0, \;\; k(T) = k_T.$$

That is, she must choose a consumption *function* $c(\cdot)$ so as to maximize her discounted lifetime utility over her known lifetime, given an initial and terminal stock of the capital asset. An important feature of this intertemporal utility maximization problem is that the independent variable time enters the integrand in an *explicit* fashion via the discount factor only. In other words, t enters the integrand explicitly only through the discount term e^{-rt}. This is typical of most problems in economics and is a property shared by the previous two examples as well.

We finish this introductory chapter by developing one more dynamic economic model to further reinforce some of the points made earlier. This economic model will be a workhorse for us, as you will see. In contrast to the three previous examples, we will develop an explicit functional form for the integrand, thereby allowing us to solve it for an explicit solution.

Example 1.4 (Optimal Inventory Accumulation): A firm has received an order for $x_T > 0$ units of product to be delivered at time $T > 0$. It seeks a production schedule or plan for filling this order by the specified delivery date at minimum cost. Let $x(t)$ denote the inventory accumulated by time $t \in [0, T]$. Because the firm does not have any of the good in its inventory when the order is first placed at $t = 0$, we have $x(0) = 0$ as the initial condition. By the delivery date $t = T$, however, the contract calls for delivery of $x_T > 0$ units, so the firm is required by the contract to have $x_T > 0$ units in its inventory at time $t = T$ in order to meet its obligation; hence $x(T) = x_T$ is the terminal condition. The inventory level $x(t)$ at any moment $t \in [0, T]$ is the cumulated past production; hence the rate of change of the inventory, namely, $\dot{x}(t)$, is the production rate. More formally, the relationship between the inventory level $x(t)$ and production rate $\dot{x}(t)$ at any time $t \in [0, T]$ is given by $x(t) \stackrel{\text{def}}{=} \int_0^t \dot{x}(s)ds$. The cost of producing the good consists of two components, a unit production cost that rises linearly with the production rate, and a unit holding cost of inventory per unit time that is assumed to be constant. Define $c_2 > 0$ as the unit cost of holding inventory and $c_1\dot{x}(t)$ as the unit cost of production, where $c_1 > 0$. Hence total cost at any date $t \in [0, T]$ is defined as

$$TC(t) \stackrel{\text{def}}{=} c_1\dot{x}(t) \cdot \dot{x}(t) + c_2x(t) = c_1[\dot{x}(t)]^2 + c_2x(t).$$

The firm's objective is to determine a production rate path $\dot{x}(t)$ and therefore an inventory accumulation path $x(t)$, so as to minimize its total costs over the planning period $[0, T]$. The firm, therefore, seeks to solve the following calculus of variations problem:

$$\min_{x(\cdot)} J[x(\cdot)] \stackrel{\text{def}}{=} \int_0^T \left[c_1[\dot{x}(t)]^2 + c_2x(t)\right] dt$$
$$\text{s.t.} \quad x(0) = 0, \; x(T) = x_T.$$

Observe that the production rate is not constrained to be nonnegative in this version of the problem. We will address this complication later, when we have the tools to handle it. Also notice that only the first-order derivative of $x(\cdot)$ appears in the integrand, consistent with an earlier observation that the first-order derivative is the highest that typically appears in economic applications of the calculus of variations. Finally, take note of the fact that the integrand does not depend explicitly on the independent variable time, that is, t does not enter the integrand $F(t, x(t), \dot{x}(t)) \stackrel{\text{def}}{=} c_1[\dot{x}(t)]^2 + c_2x(t)$ independently of the functions $x(\cdot)$ and $\dot{x}(\cdot)$. Since t does not appear explicitly in the integrand, this calculus of variations problem is called *autonomous*.

To convert this calculus of variations problem into an optimal control problem, all we are required to do is to select a control variable. This is straightforward, for we have emphasized all along that $\dot{x}(t)$ is the production rate, a flow variable, so it is the natural choice for the control variable. More formally, define $u(t) \stackrel{\text{def}}{=} \dot{x}(t)$ as the control variable. This differential equation thus becomes our state equation in the optimal control formulation of the problem. But remember that no derivatives of the state variable (or control variable for that matter) appear in the integrand of the objective functional of an optimal control problem; hence we must replace $\dot{x}(t)$ in the objective functional with the control variable $u(t)$. Doing just that allows us to rewrite the calculus of variations form of the inventory accumulation problem as an equivalent optimal control problem, namely,

$$\min_{u(\cdot)} J[x(\cdot), u(\cdot)] \overset{\text{def}}{=} \int_0^T \left[c_1[u(t)]^2 + c_2 x(t)\right] dt$$

$$\text{s.t.} \quad \dot{x}(t) = u(t),$$

$$x(0) = 0,\ x(T) = x_T.$$

This is a problem that will aid us greatly in our understanding of optimal control theory.

Now that we have seen the essential distinction between static and dynamic optimization problems, the generic form of the objective functional for continuous time dynamic optimization problems, and several economic models that fall naturally in the realm of continuous time dynamic optimization, the next chapter will develop the necessary conditions that will be the centerpiece of much of the ensuing analysis. In particular, the ensuing chapter presents the necessary conditions for the simplest optimal control problem, which consists of a single state variable, a single control variable, and a free terminal value of the state variable. We will offer a simple and motivated proof of the necessary conditions under stronger assumptions on the integrand function $f(\cdot)$, the transition function $\mathbf{g}(\cdot)$, and the state and control variables than those given in this chapter.

MENTAL EXERCISES

1.1 Let $g(\cdot) : \Re^{N+A} \to \Re$ be a $C^{(2)}$ scalar-valued function whose arguments are the vectors $\mathbf{x} \in \Re^N$ and $\boldsymbol{\alpha} \in \Re^A$.
   (a) Compute the $A \times N$ matrix $g_{\boldsymbol{\alpha}\mathbf{x}}(\mathbf{x}; \boldsymbol{\alpha})$.
   (b) Prove that $g_{\mathbf{x}\boldsymbol{\alpha}}(\mathbf{x}; \boldsymbol{\alpha}) = g_{\boldsymbol{\alpha}\mathbf{x}}(\mathbf{x}; \boldsymbol{\alpha})'$. This is the matrix version of the invariance of the second-order partial derivatives to the order of differentiation.

1.2 Assume that the production function $f(\cdot) \in C^{(2)}$ has the properties

$$f_x(x, e) > 0,\ f_e(x, e) > 0,\ f_{ex}(x, e) > 0,$$

$$f_{xx}(x, e) < 0,\ f_{ee}(x, e) < 0,\ f_{xx}(x, e) f_{ee}(x, e) - [f_{ex}(x, e)]^2 > 0.$$

Prove that the minimum restricted cost function $C(\cdot)$ defined by

$$C(x, h; w) \overset{\text{def}}{=} \min_e \{w \cdot e \text{ s.t. } h = f(x, e)\}$$

has the following properties

$$C_x(x, h; w) < 0,\ C_h(x, h; w) > 0,\ C_{hx}(x, h; w) < 0,\ C_{xx}(x, h; w) > 0,$$

$$C_{hh}(x, h; w) > 0,$$

$$C_{xx}(x,h;w)C_{hh}(x,h;w) - [C_{hx}(x,h;w)]^2 > 0,$$

$$C_w(x,h;w) > 0,\; C_{ww}(x,h;w) \equiv 0.$$

1.3 Draw a graph of a piecewise smooth function that is consistent with Figure 1.1 being a graph of its piecewise continuous derivative function.

FURTHER READING

There are several textbook treatments of optimal control theory that are suitable for students of economics, depending on your mathematical preparation and that intangible quality that often goes by the moniker *mathematical maturity*. The Chiang (1992) textbook covers both the calculus of variations and optimal control theory and is written at the level of his mathematical economics textbook, which is essentially the prerequisite. That is to say, it is an introductory exposition of the material emphasizing intuition and conceptual understanding over mathematical rigor. The next step up in mathematical rigor are the books by Kamien and Schwartz (1981, 1991 2nd Ed.) and Léonard and Van Long (1992). The former is a classic in economics but is showing its age somewhat, even considering the revision more than a decade ago. It is a self-contained, tightly written book aimed at graduate students and practitioners of the subjects of the calculus of variations and optimal control. The Léonard and Van Long (1992) book is well written and more modern in its approach, as it eschews the calculus of variations and proceeds directly to optimal control theory (after a brief chapter on ordinary differential equations). The Hadley and Kemp (1971) book is a step up in mathematical rigor and prerequisites from the last two. It is very well written but a bit dated, however, in that it develops the calculus of variations in its full generality and then uses those theorems to prove their counterparts in optimal control. This makes for a rather long stint before one gets to the modern theory of optimal control. The Seierstad and Sydsæter (1987) book is a nearly exhaustive and well-written account of the theorems in optimal control theory. It is definitely not written as an introductory exposition of optimal control, unless you have a solid mathematical background and a high level of mathematical maturity. It is an outstanding reference book. The Leitmann (1981) and Troutman (1996, 2nd Ed.) books are mathematics texts, with the former using engineering applications and the latter the same plus applications from physics, to demonstrate the methods. Both are accessible by any student of economics who has had courses in linear algebra, advanced calculus, and ordinary differential equations.

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