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Dynamic Programming and the Hamilton-Jacobi-Bellman Equation

In this chapter, we turn our attention away from the derivation of necessary and sufficient conditions that can be used to find the optimal time paths of the state, costate, and control variables, and focus on the optimal value function more closely. In particular, we will derive the fundamental first-order partial differential equation obeyed by the optimal value function, known as the *Hamilton-Jacobi-Bellman* equation. This shift in our attention, moreover, will lead us to a different *form* for the optimal value of the control vector, namely, the *feedback* or *closed-loop* form of the control. This form of the optimal control typically gives the optimal value of the control vector as a function of the current date, the current state, and the parameters of the control problem. In contrast, the form of the optimal control vector derived via the necessary conditions of optimal control theory is termed *open-loop*, and in general gives the optimal value of the control vector as a function of the independent variable time, the parameters, and the initial and/or terminal values of the planning horizon and the state vector. Essentially, the feedback form of the optimal control is a decision rule, for it gives the optimal value of the control for *any* current period and *any* admissible state in the current period that may arise. In contrast, the open-loop form of the optimal control is a curve, for it gives the optimal values of the control as the independent variable time varies over the planning horizon. We will see, however, that even though the closed-loop and open-loop controls differ in form, they yield identical values for the optimal control at each date of the planning horizon.

The approach we take in this chapter is known as *dynamic programming*. The main logic used to derive the Hamilton-Jacobi-Bellman (H-J-B) equation is the *principle of optimality*. To quote Bellman (1957, page 83):

> An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision.

Recall that we've already given two different proofs of the principle of optimality, one in Theorem 4.1 and another in Theorem 9.2. As a result, there is no need to provide another one here. Now would be an appropriate time to look back at these proofs if your memory of them is a bit vague or you skipped over them, however.

The optimal control problem under consideration is given by

$$\begin{aligned} &\max_{\mathbf{u}(\cdot),\mathbf{x}_T} \int_0^T f(s,\mathbf{x}(s),\mathbf{u}(s);\boldsymbol{\alpha})\,ds + \phi(\mathbf{x}(T),T) \\ &\text{s.t.} \quad \dot{\mathbf{x}}(s) = \mathbf{g}(s,\mathbf{x}(s),\mathbf{u}(s);\boldsymbol{\alpha}),\ \mathbf{x}(0)=\mathbf{x}_0,\ \mathbf{x}(T)=\mathbf{x}_T, \end{aligned} \tag{1}$$

where $\mathbf{u}(\cdot):\Re\to\Re^M$ is the control function, $\mathbf{x}(\cdot):\Re\to\Re^N$ is the state function, $\boldsymbol{\alpha}\in\Re^A$ is a vector of exogenous and constant parameters, $\mathbf{x}_0\in\Re^N$ is the initial state, $\phi(\cdot):\Re^{N+1}\to\Re$ is the salvage value or scrap value function, $\mathbf{g}(\cdot):\Re\times\Re^N\times\Re^M\times\Re^A\to\Re^N$ is the vector-valued transition function, $f(\cdot):\Re\times\Re^N\times\Re^M\times\Re^A\to\Re$ is the integrand function, and s is the dummy variable of integration rather than t. As we shall see, this change in the dummy variable of integration serves an important pedagogical device, for it frees us up to use t as an arbitrary initial date of the optimal control problem. Let $(\mathbf{z}(s;\boldsymbol{\alpha},\mathbf{x}_0,T), \mathbf{v}(s;\boldsymbol{\alpha},\mathbf{x}_0,T))$ be the optimal open-loop pair for problem (1), and let $\boldsymbol{\lambda}(s;\boldsymbol{\alpha},\mathbf{x}_0,T)$ be the corresponding open-loop costate vector. Note that we are now emphasizing the open-loop nature of the above solution triplet, since we intend to contrast it with its closed-loop or feedback counterpart.

Define the *optimal value function* $V(\cdot)$ as the maximum value of the objective functional that can be obtained starting at *any* time $t\in[0,T]$ and in *any* admissible state $\mathbf{x}_t$, given the parameter vector $\boldsymbol{\alpha}$. The optimal value function $V(\cdot)$ is therefore defined $\forall\, t\in[0,T]$ and for any admissible state $\mathbf{x}_t$ that may occur given the assumptions we have adopted. More formally, we have the following definition of $V(\cdot)$:

$$\begin{aligned} &V(\boldsymbol{\alpha},t,\mathbf{x}_t,T) \stackrel{\text{def}}{=} \max_{\mathbf{u}(\cdot),\mathbf{x}_T} \int_t^T f(s,\mathbf{x}(s),\mathbf{u}(s);\boldsymbol{\alpha})\,ds + \phi(\mathbf{x}(T),T) \\ &\text{s.t.} \quad \dot{\mathbf{x}}(s) = \mathbf{g}(s,\mathbf{x}(s),\mathbf{u}(s);\boldsymbol{\alpha}),\ \mathbf{x}(t)=\mathbf{x}_t,\ \mathbf{x}(T)=\mathbf{x}_T. \end{aligned} \tag{2}$$

Note that we will often suppress the dependence of $V(\cdot)$ on T for notational clarity in problem (2). The present development should be familiar up to this point; however, now the optimal value function $V(\cdot)$ is defined as starting at any date $t\in[0,T]$ and for any admissible state $\mathbf{x}_t$, given the parameter vector $\boldsymbol{\alpha}$. Given the above optimal solution to problem (1), it should be clear that $(\mathbf{z}(s;\boldsymbol{\alpha},t,\mathbf{x}_t,T), \mathbf{v}(s;\boldsymbol{\alpha},t,\mathbf{x}_t,T))$ is the optimal open-loop pair for problem (2) and $\boldsymbol{\lambda}(s;\boldsymbol{\alpha},t,\mathbf{x}_t,T)$ is the corresponding open-loop costate vector, for the *only* difference between problems (1) and (2) is the initial time and initial state. That is, the solution functions must be the same for problems (1) and (2) in view of the fact that they are structurally identical, but the values of the solution functions will in general differ because of the difference in the initial time and initial state. You should recall, however, that if we set

$\mathbf{x}_t = \mathbf{z}(t; \boldsymbol{\alpha}, \mathbf{x}_0, T, \mathbf{x}_T)$ in problem (2), then the solutions to the optimal control problems (1) and (2) are identical on the closed interval $[t, T]$ by the principle of optimality. The initial (or starting) date t in problem (2) is often referred to as the *base period*. Notice that from the definition of $V(\cdot)$ in Eq. (2),

$$V(\boldsymbol{\alpha}, T, \mathbf{x}(T)) = \phi(\mathbf{x}(T), T). \tag{3}$$

This is a boundary condition for the H-J-B partial differential equation given in Theorem 19.1 below. Let us now turn to the development of the H-J-B equation.

Theorem 19.1(H-J-B equation): *If $V(\cdot) \in C^{(1)}$, then the first-order nonlinear partial differential equation obeyed by the optimal value function $V(\cdot)$ defined in Eq. (2) is given by*

$$-V_t(\boldsymbol{\alpha}, t, \mathbf{x}_t) = \max_{\mathbf{u}}[f(t, \mathbf{x}_t, \mathbf{u}; \boldsymbol{\alpha}) + V_{\mathbf{x}}(\boldsymbol{\alpha}, t, \mathbf{x}_t)\mathbf{g}(t, \mathbf{x}_t, \mathbf{u}; \boldsymbol{\alpha})] \tag{4}$$

in vector notation, or equivalently by

$$-V_t(\boldsymbol{\alpha}, t, \mathbf{x}_t) = \max_{\mathbf{u}} \left[f(t, \mathbf{x}_t, \mathbf{u}; \boldsymbol{\alpha}) + \sum_{n=1}^{N} V_{x_n}(\boldsymbol{\alpha}, t, \mathbf{x}_t) g^n(t, \mathbf{x}_t, \mathbf{u}; \boldsymbol{\alpha}) \right]$$

in index notation, where $t \in [0, T]$ is any base period, $\mathbf{x}_t$ is any admissible state, and $\boldsymbol{\alpha}$ is a vector of constant parameters.

Proof: For any $\Delta t > 0$ and small, Eq. (2) can be rewritten as

$$\begin{aligned} V(\boldsymbol{\alpha}, t, \mathbf{x}_t) = \max_{\substack{\mathbf{u}(\cdot),\, \mathbf{x}_T \\ s \in [t, T]}} &\left[\int_t^{t+\Delta t} f(s, \mathbf{x}(s), \mathbf{u}(s); \boldsymbol{\alpha})\, ds \right. \\ &\left. + \int_{t+\Delta t}^{T} f(s, \mathbf{x}(s), \mathbf{u}(s); \boldsymbol{\alpha})\, ds + \phi(\mathbf{x}(T), T) \right] \end{aligned} \tag{5}$$

$$\text{s.t.} \quad \dot{\mathbf{x}}(s) = \mathbf{g}(s, \mathbf{x}(s), \mathbf{u}(s); \boldsymbol{\alpha}),\ \mathbf{x}(t) = \mathbf{x}_t,\ \mathbf{x}(T) = \mathbf{x}_T.$$

In arriving at Eq. (5), notice that we have broken up the interval $[t, T]$ in Eq. (2) into the subintervals $[t, t + \Delta t]$ and $(t + \Delta t, T]$. This is a basic property of integrable functions [see, e.g., Apostol (1974), Theorem 7.4]. It is simply an assertion that the integral is additive with respect to the interval of integration. Now, by the principle of optimality, the control function $\mathbf{u}(\cdot)$, $s \in (t + \Delta t, T]$ must be optimal, that is, it must maximize the objective functional for the control problem beginning at time $s = t + \Delta t$ in the state $\mathbf{x}(t + \Delta t)$; otherwise $V(\boldsymbol{\alpha}, t, \mathbf{x}_t)$ could not be the maximum value of the objective functional as defined by Eq. (2). Note that the state at time $s = t + \Delta t$, namely, $\mathbf{x}(t + \Delta t)$, depends on the state $\mathbf{x}_t$ prevailing at time $s = t$ and on the control function chosen over the first subinterval $\mathbf{u}(\cdot)$, $s \in [t, t + \Delta t]$. By the principle of optimality, Eq. (5) can therefore be rewritten as

$$V(\boldsymbol{\alpha}, t, \mathbf{x}_t) = \max_{\substack{\mathbf{u}(\cdot) \\ s\in[t,\, t+\Delta t]}} \left[\int_t^{t+\Delta t} f(s, \mathbf{x}(s), \mathbf{u}(s); \boldsymbol{\alpha})\, ds \right.$$

$$\left. + \max_{\substack{\mathbf{u}(\cdot),\, \mathbf{x}_T \\ s\in(t+\Delta t,\, T]}} \left[\int_{t+\Delta t}^{T} f(s, \mathbf{x}(s), \mathbf{u}(s); \boldsymbol{\alpha})\, ds + \phi(\mathbf{x}(T), T) \right] \right] \tag{6}$$

$$\text{s.t.} \quad \dot{\mathbf{x}}(s) = \begin{cases} \mathbf{g}(s, \mathbf{x}(s), \mathbf{u}(s); \boldsymbol{\alpha}),\ \mathbf{x}(t) = \mathbf{x}_t, & s \in [t, t + \Delta t] \\ \mathbf{g}(s, \mathbf{x}(s), \mathbf{u}(s); \boldsymbol{\alpha}),\ \mathbf{x}(t + \Delta t) \text{ given},\ \mathbf{x}(T) = \mathbf{x}_T, & s \in (t + \Delta t, T]. \end{cases}$$

Using the definition of the optimal value function $V(\cdot)$ in Eq. (2) applied to the subinterval $(t + \Delta t, T]$ and the given value of the state vector $\mathbf{x}(t + \Delta t)$, Eq. (6) can be rewritten as

$$V(\boldsymbol{\alpha}, t, \mathbf{x}_t) = \max_{\substack{\mathbf{u}(\cdot) \\ s\in[t,\, t+\Delta t]}} \left[\int_t^{t+\Delta t} f(s, \mathbf{x}(s), \mathbf{u}(s); \boldsymbol{\alpha})\, ds + V(\boldsymbol{\alpha}, t + \Delta t, \mathbf{x}(t + \Delta t)) \right]$$

$$\text{s.t.} \quad \dot{\mathbf{x}}(s) = \mathbf{g}(s, \mathbf{x}(s), \mathbf{u}(s); \boldsymbol{\alpha}),\ \mathbf{x}(t) = \mathbf{x}_t. \tag{7}$$

Equation (7) asserts that the maximum value of the optimal control problem defined in Eq. (2), which is given by the value $V(\boldsymbol{\alpha}, t, \mathbf{x}_t)$, can be broken up into the sum of the optimal return over the initial subinterval $[t, t + \Delta t]$, plus the return by continuing optimally from the terminal position $(t + \Delta t, \mathbf{x}(t + \Delta t))$ of the first subinterval. Note that the immediate return over the initial subinterval $[t, t + \Delta t]$ and the future return over the second subinterval $(t + \Delta t, T]$ are affected by the choice of the optimal control over the initial subinterval $[t, t + \Delta t]$ because of the state equation.

Equation (7) can now be put into the more useful form asserted by the theorem. First, define the function $F(\cdot)$ by

$$F(t + \Delta t) \stackrel{\text{def}}{=} V(\boldsymbol{\alpha}, t + \Delta t, \mathbf{x}(t + \Delta t)). \tag{8}$$

Given that $V(\cdot) \in C^{(1)}$ by hypothesis, use Taylor's theorem and the chain rule to expand $F(t + \Delta t)$ about the point t to get

$$\begin{aligned} F(t + \Delta t) &= F(t) + F'(t)\Delta t + o(\Delta t) \\ &= V(\boldsymbol{\alpha}, t, \mathbf{x}(t)) + [V_t(\boldsymbol{\alpha}, t, \mathbf{x}(t)) + V_{\mathbf{x}}(\boldsymbol{\alpha}, t, \mathbf{x}(t))\dot{\mathbf{x}}(t)]\, \Delta t + o(\Delta t) \\ &= V(\boldsymbol{\alpha}, t, \mathbf{x}_t) + [V_t(\boldsymbol{\alpha}, t, \mathbf{x}_t) + V_{\mathbf{x}}(\boldsymbol{\alpha}, t, \mathbf{x}_t)\mathbf{g}(t, \mathbf{x}_t, \mathbf{u}(t); \boldsymbol{\alpha})]\, \Delta t + o(\Delta t), \end{aligned} \tag{9}$$

where $o(\Delta t)$ denotes the terms in the Taylor expansion of higher order than one, that is, the remainder, with the property that $\lim_{\Delta t\to 0} [o(\Delta t)/\Delta t] = 0$. Observe that we also used the state equation evaluated at $s = t$, namely, $\dot{\mathbf{x}}(t) = \mathbf{g}(t, \mathbf{x}(t), \mathbf{u}(t); \boldsymbol{\alpha})$, and the initial condition $\mathbf{x}(t) = \mathbf{x}_t$ of problem (2) in arriving at the final form of Eq. (9). Inspection of Eqs. (8) and (9) shows that we have established that

$$\begin{aligned} V(\boldsymbol{\alpha}, t + \Delta t, \mathbf{x}(t + \Delta t)) &= V(\boldsymbol{\alpha}, t, \mathbf{x}_t) \\ &\quad + [V_t(\boldsymbol{\alpha}, t, \mathbf{x}_t) + V_{\mathbf{x}}(\boldsymbol{\alpha}, t, \mathbf{x}_t)\,\mathbf{g}(t, \mathbf{x}_t, \mathbf{u}(t); \boldsymbol{\alpha})]\,\Delta t + o(\Delta t). \end{aligned} \tag{10}$$

We will return to this equation shortly.

In a similar vein, we can use Taylor's theorem to rewrite the integral on the right-hand side of Eq. (7) as follows. To this end, define the function $G(\cdot)$ by

$$G(t + \Delta t) \stackrel{\text{def}}{=} \int_t^{t+\Delta t} f(s, \mathbf{x}(s), \mathbf{u}(s); \boldsymbol{\alpha})\,ds. \tag{11}$$

Then use Taylor's theorem and Leibniz's rule to expand $G(t + \Delta t)$ about the point t to get

$$\begin{aligned} G(t + \Delta t) &= G(t) + G'(t)\Delta t + o(t) \\ &= 0 + f(t, \mathbf{x}(t), \mathbf{u}(t); \boldsymbol{\alpha})\Delta t + o(t) \\ &= 0 + f(t, \mathbf{x}_t, \mathbf{u}(t); \boldsymbol{\alpha})\Delta t + o(t), \end{aligned} \tag{12}$$

where we have again used the initial condition $\mathbf{x}(t) = \mathbf{x}_t$ of problem (2). Inspection of Eqs. (11) and (12) shows that

$$\int_t^{t+\Delta t} f(s, \mathbf{x}(s), \mathbf{u}(s); \boldsymbol{\alpha})\,ds = f(t, \mathbf{x}_t, \mathbf{u}(t); \boldsymbol{\alpha})\Delta t + o(t). \tag{13}$$

We are now in a position to simplify Eq. (7).

Substituting Eqs. (10) and (13) into Eq. (7) yields

$$\begin{aligned} V(\boldsymbol{\alpha}, t, \mathbf{x}_t) = \max_{\substack{\mathbf{u}(\cdot) \\ s\in[t,\, t+\Delta t]}} \; & [f(t, \mathbf{x}_t, \mathbf{u}(t); \boldsymbol{\alpha})\Delta t + V(\boldsymbol{\alpha}, t, \mathbf{x}_t) \\ & + [V_t(\boldsymbol{\alpha}, t, \mathbf{x}_t) + V_{\mathbf{x}}(\boldsymbol{\alpha}, t, \mathbf{x}_t)\mathbf{g}(t, \mathbf{x}_t, \mathbf{u}(t); \boldsymbol{\alpha})]\Delta t + o(\Delta t)] \\ \text{s.t.} \quad & \dot{\mathbf{x}}(s) = \mathbf{g}(s, \mathbf{x}(s), \mathbf{u}(s); \boldsymbol{\alpha}),\ \mathbf{x}(t) = \mathbf{x}_t. \end{aligned} \tag{14}$$

By the definition of $V(\cdot)$ given in Eq. (2), $V(\cdot)$ is not a function of the control vector because it has been maximized out of problem (2). As a result, the term $V(\boldsymbol{\alpha}, t, \mathbf{x}_t)$ can be canceled from both sides of Eq. (14). Furthermore, upon dividing Eq. (14) by Δt, we have that

$$\begin{aligned} 0 = \max_{\substack{\mathbf{u}(\cdot) \\ s\in[t,\, t+\Delta t]}} & \Big[f(t, \mathbf{x}_t, \mathbf{u}(t); \boldsymbol{\alpha}) + V_t(\boldsymbol{\alpha}, t, \mathbf{x}_t) \\ & + V_{\mathbf{x}}(\boldsymbol{\alpha}, t, \mathbf{x}_t)\mathbf{g}(t, \mathbf{x}_t, \mathbf{u}(t); \boldsymbol{\alpha}) + \frac{o(\Delta t)}{\Delta t} \Big] \\ \text{s.t.} \quad & \dot{\mathbf{x}}(s) = \mathbf{g}(s, \mathbf{x}(s), \mathbf{u}(s); \boldsymbol{\alpha}),\ \mathbf{x}(t) = \mathbf{x}_t. \end{aligned} \tag{15}$$

Now let $\Delta t \to 0$ in Eq. (15). Recalling that $\lim_{\Delta t\to 0}[o(\Delta t)/\Delta t] = 0$ and that $V_t(\cdot)$ is not a function of the control vector, Eq. (15) therefore reduces to

$$-V_t(\boldsymbol{\alpha}, t, \mathbf{x}_t) = \max_{\mathbf{u}(t)}[f(t, \mathbf{x}_t, \mathbf{u}(t); \boldsymbol{\alpha}) + V_{\mathbf{x}}(\boldsymbol{\alpha}, t, \mathbf{x}_t)\mathbf{g}(t, \mathbf{x}_t, \mathbf{u}(t); \boldsymbol{\alpha})], \tag{16}$$

upon bringing $V_t(\boldsymbol{\alpha}, t, \mathbf{x}_t)$ to the left-hand side.

It is important to note that because we have let $\Delta t \to 0$ in Eq. (15) in arriving at Eq. (16), the maximization with respect to the *control function* in Eq. (15) reduces to choosing the *value* of the control function at a single point in time in Eq. (16). That is, we are choosing the *control function* $\mathbf{u}(\cdot)\,\forall\, s \in [t, t + \Delta t]$ in Eq. (15), which is equivalent to choosing a curve, whereas in Eq. (16), we are choosing the *value* of the control function $\mathbf{u}(\cdot)$ at the point $s = t$, which is equivalent to choosing a point on the curve. The latter observation means that in Eq. (16), we are solving a *static* maximization problem.

Finally, we can put Eq. (16) in the form of the theorem by letting $\mathbf{u}$ represent the *value* of the control function $\mathbf{u}(\cdot)$ at the point $s = t$, thereby yielding

$$-V_t(\boldsymbol{\alpha}, t, \mathbf{x}_t) = \max_{\mathbf{u}}[f(t, \mathbf{x}_t, \mathbf{u}; \boldsymbol{\alpha}) + V_{\mathbf{x}}(\boldsymbol{\alpha}, t, \mathbf{x}_t)\mathbf{g}(t, \mathbf{x}_t, \mathbf{u}; \boldsymbol{\alpha})].$$

This completes the proof. Q.E.D.

Theorem 19.1 gives the fundamental nonlinear first-order partial differential equation obeyed by the optimal value function $V(\cdot)$. In principle, one solves the *static* maximization problem on the right-hand side of Eq. (4) and expresses $\mathbf{u}$ as a function of $(\boldsymbol{\alpha}, t, \mathbf{x}_t)$ and the unknown function $V_{\mathbf{x}}(\cdot)$, say, $\mathbf{u} = \bar{\mathbf{u}}(\boldsymbol{\alpha}, t, \mathbf{x}_t, V_{\mathbf{x}}(\cdot))$. Then this solution for $\mathbf{u}$ is substituted back into the H-J-B equation to get the nonlinear first-order partial differential equation

$$-V_t = f(t, \mathbf{x}_t, \bar{\mathbf{u}}(\boldsymbol{\alpha}, t, \mathbf{x}_t, V_{\mathbf{x}}); \boldsymbol{\alpha}) + V_{\mathbf{x}}\,\mathbf{g}(t, \mathbf{x}_t, \bar{\mathbf{u}}(\boldsymbol{\alpha}, t, \mathbf{x}_t, V_{\mathbf{x}}); \boldsymbol{\alpha}),$$

which is to be solved for $V(\boldsymbol{\alpha}, t, \mathbf{x}_t, T)$ using the boundary condition given in Eq. (3). Finally, to determine the optimizing value of $\mathbf{u}$, say, $\mathbf{u}^c(\boldsymbol{\alpha}, t, \mathbf{x}_t, T)$, differentiate $V(\boldsymbol{\alpha}, t, \mathbf{x}_t, T)$ with respect to the state to get $V_{\mathbf{x}}(\boldsymbol{\alpha}, t, \mathbf{x}_t, T)$, and then substitute $V_{\mathbf{x}}(\boldsymbol{\alpha}, t, \mathbf{x}_t, T)$ into $\mathbf{u} = \bar{\mathbf{u}}(\boldsymbol{\alpha}, t, \mathbf{x}_t, V_{\mathbf{x}}(\cdot))$, that is to say, $\mathbf{u}^c(\boldsymbol{\alpha}, t, \mathbf{x}_t, T)$ is given by the identity $\mathbf{u}^c(\boldsymbol{\alpha}, t, \mathbf{x}_t, T) \equiv \bar{\mathbf{u}}(\boldsymbol{\alpha}, t, \mathbf{x}_t, V_{\mathbf{x}}(\boldsymbol{\alpha}, t, \mathbf{x}_t, T))$. Example 19.1 demonstrates how this is done for a simple control problem.

As one might surmise, the superscript c in the expression $\mathbf{u}^c(\boldsymbol{\alpha}, t, \mathbf{x}_t, T)$ is used to signify that $\mathbf{u}^c(\boldsymbol{\alpha}, t, \mathbf{x}_t, T)$ is the value of the optimal closed-loop (or feedback) control. Consequently, it is beneficial at this juncture to have a precise and formal definition of an optimal closed-loop control function. The ensuing definition provides one.

Definition 19.1: If, given an admissible initial state $\mathbf{x}_t$ in problem (2), there exists a unique optimal control path $\mathbf{v}(s; \boldsymbol{\alpha}, t, \mathbf{x}_t, T)$, $s \in [t, T]$, then the *optimal value of the closed-loop control function* is given by $\mathbf{v}(t; \boldsymbol{\alpha}, t, \mathbf{x}_t, T) \stackrel{\text{def}}{=} \mathbf{v}(s; \boldsymbol{\alpha}, t, \mathbf{x}_t, T)|_{s=t}$.

Definition 19.1 says that if the optimal open-loop control is unique, then the optimal value of the closed-loop control is found by evaluating the optimal open-loop control at the base period. It is important to observe that in the paragraph just before Definition 19.1, we asserted, but did not establish, that $\mathbf{u}^c(\boldsymbol{\alpha}, t, \mathbf{x}_t, T)$, derived by solving the H-J-B equation, is in fact the optimal value of the closed-loop control as given in Definition 19.1. We will address this deficiency shortly. As you may recall, most of the examples we have encountered in this book have had a unique optimal control path; consequently, Definition 19.1 may be applied to such control problems to derive the optimal closed-loop control. We will see that this is the case in Example 19.1 as well.

Theorem 19.1 may be used to derive the necessary conditions of optimal control theory, as we will now show. First observe that the expression on the right-hand side of Eq. (4) that is to be maximized with respect to $\mathbf{u}$ looks strikingly similar to the Hamiltonian. In fact, the two expressions are identical. To see this, recall that by the dynamic envelope theorem and the principle of optimality, $\boldsymbol{\lambda}(t; \boldsymbol{\alpha}) = V_{\mathbf{x}}(\boldsymbol{\alpha}, t, \mathbf{x}(t))'$, since $\mathbf{x}(t) = \mathbf{x}_t$. Hence the right-hand side of Eq. (4) instructs us to maximize the Hamiltonian with respect to the control vector $\mathbf{u}$, which is precisely the Maximum Principle of Pontryagin. In passing, observe that Eq. (4) is *identical* to that part of the dynamic envelope theorem that pertains to the effect of an increase in the initial time on the optimal value function, a result you should verify by looking back at the dynamic envelope theorem.

The state equation was also used to derive Theorem 19.1 and is also part of the admissibility criteria and therefore part of the necessary conditions of optimal control theory. So that matches up exactly too.

Finally, we want to show that the ordinary differential equation for the costate variable $\boldsymbol{\lambda}(t)$ follows from Eq. (4). First recall that $\mathbf{u} = \bar{\mathbf{u}}(\boldsymbol{\alpha}, t, \mathbf{x}_t, V_{\mathbf{x}})$ solves the maximization problem in Eq. (4). Next, assume that we have substituted this solution back into the H-J-B equation, that is, Eq. (4), resulting in the partial differential equation

$$-V_t = f(t, \mathbf{x}_t, \bar{\mathbf{u}}(\boldsymbol{\alpha}, t, \mathbf{x}_t, V_{\mathbf{x}}); \boldsymbol{\alpha}) + \sum_{n=1}^{N} V_{x_n} g^n(t, \mathbf{x}_t, \bar{\mathbf{u}}(\boldsymbol{\alpha}, t, \mathbf{x}_t, V_{\mathbf{x}}); \boldsymbol{\alpha}).$$

Now assume that we have solved this partial differential equation for $V(\boldsymbol{\alpha}, t, \mathbf{x}_t)$ and that $V(\cdot) \in C^{(2)}$. Upon substituting $V(\boldsymbol{\alpha}, t, \mathbf{x}_t)$ back into the above form of the H-J-B equation, we get the identity

$$\begin{aligned} - V_t(\boldsymbol{\alpha}, t, \mathbf{x}(t)) &\equiv f(t, \mathbf{x}(t), \bar{\mathbf{u}}(\boldsymbol{\alpha}, t, \mathbf{x}(t), V_{\mathbf{x}}(\boldsymbol{\alpha}, t, \mathbf{x}(t))); \boldsymbol{\alpha}) \\ &\quad + \sum_{n=1}^{N} V_{x_n}(\boldsymbol{\alpha}, t, \mathbf{x}(t)) g^n(t, \mathbf{x}(t), \bar{\mathbf{u}}(\boldsymbol{\alpha}, t, \mathbf{x}(t), V_{\mathbf{x}}(\boldsymbol{\alpha}, t, \mathbf{x}(t))); \boldsymbol{\alpha}), \end{aligned} \tag{17}$$

since $\mathbf{x}(t) = \mathbf{x}_t$. Because Eq. (4) is a static optimization problem, we can invoke the prototype static envelope theorem and differentiate Eq. (17) with respect to x_i using the chain rule and the assumption that $V(\cdot) \in C^{(2)}$ to get

$$-V_{tx_i} = f_{x_i} + \sum_{n=1}^{N} V_{x_n} g^n_{x_i} + \sum_{n=1}^{N} g^n V_{x_i x_n}, \quad i = 1, 2, \ldots, N. \tag{18}$$

Note that we have suppressed the arguments of the functions in Eq. (18) seeing as they are easily recoverable from Eq. (17). Now because $\lambda_i(t; \boldsymbol{\alpha}) = V_{x_i}(\boldsymbol{\alpha}, t, \mathbf{x}(t))$, $i = 1, 2, \ldots, N$, it therefore follows from the chain rule that

$$\begin{aligned}\dot{\lambda}_i(t; \boldsymbol{\alpha}) &\stackrel{\text{def}}{=} \frac{d}{dt}\lambda_i(t; \boldsymbol{\alpha}) = \frac{d}{dt}V_{x_i}(\boldsymbol{\alpha}, t, \mathbf{x}(t)) = V_{x_i t} + \sum_{n=1}^{N} V_{x_i x_n}\dot{x}_n(t) \\ &= V_{x_i t} + \sum_{n=1}^{N} V_{x_i x_n} g^n, \quad i = 1, 2, \ldots, N, \end{aligned} \tag{19}$$

upon using the state equation. Solving Eq. (18) for V_{tx_i}, recalling the assumption that $V(\cdot) \in C^{(2)}$, and substituting the expression into Eq. (19) yields

$$\begin{aligned}\dot{\lambda}_i(t; \boldsymbol{\alpha}) &= -f_{x_i} - \sum_{n=1}^{N} V_{x_n} g^n_{x_i} - \sum_{n=1}^{N} g^n V_{x_i x_n} + \sum_{n=1}^{N} V_{x_i x_n} g^n \\ &= -f_{x_i} - \sum_{n=1}^{N} V_{x_n} g^n_{x_i}, \quad i = 1, 2, \ldots, N.\end{aligned}$$

Again using the fact that $\lambda_n = V_{x_n}$, $n = 1, 2, \ldots, N$, as well as the definition of the Hamiltonian $H(t, \mathbf{x}, \mathbf{u}, \boldsymbol{\lambda}; \boldsymbol{\alpha}) \stackrel{\text{def}}{=} f(t, \mathbf{x}, \mathbf{u}; \boldsymbol{\alpha}) + \sum_{n=1}^{N} \lambda_n g^n(t, \mathbf{x}, \mathbf{u}; \boldsymbol{\alpha})$, gives the costate equation

$$\dot{\lambda}_i = -f_{x_i} - \sum_{n=1}^{N} \lambda_n g^n_{x_i} = -H_{x_i}, \quad i = 1, 2, \ldots, N.$$

Thus, under the assumptions that $V(\cdot) \in C^{(2)}$, the H-J-B equation can be used to prove the necessary conditions of optimal control theory, just as we intended to demonstrate.

Before turning to an example, let's establish the aforementioned claim that $\mathbf{u}^c(\boldsymbol{\alpha}, t, \mathbf{x}_t, T)$, derived by solving the H-J-B equation, is in fact the optimal closed-loop control as defined in Definition 19.1. In order to prove this assertion, we must show that the value of the closed-loop control $\mathbf{u}^c(\boldsymbol{\alpha}, t, \mathbf{x}_t, T)$ is identical to the value of the open-loop control in the base period t for problem (2), namely, $\mathbf{v}(t; \boldsymbol{\alpha}, t, \mathbf{x}_t, T)$. First, in order to use Definition 19.1, we assume that the optimal control path $\mathbf{v}(s; \boldsymbol{\alpha}, t, \mathbf{x}_t, T)$, $s \in [t, T]$, is unique. It then follows from the unique optimality of the control path $\mathbf{v}(s; \boldsymbol{\alpha}, t, \mathbf{x}_t, T)$, $s \in [t, T]$, that $\mathbf{v}(t; \boldsymbol{\alpha}, t, \mathbf{x}_t, T)$ is the only solution to $\max_{\mathbf{u}}[f(t, \mathbf{x}_t, \mathbf{u}; \boldsymbol{\alpha}) + V_{\mathbf{x}}(\boldsymbol{\alpha}, t, \mathbf{x}_t)\mathbf{g}(t, \mathbf{x}_t, \mathbf{u}; \boldsymbol{\alpha})]$, the maximization problem on the right-hand side of the H-J-B equation in the base period t. This follows from the heretofore established fact that $f(t, \mathbf{x}_t, \mathbf{u}; \boldsymbol{\alpha}) + V_{\mathbf{x}}(\boldsymbol{\alpha}, t, \mathbf{x}_t)\mathbf{g}(t, \mathbf{x}_t, \mathbf{u}; \boldsymbol{\alpha})$ is nothing other than the Hamiltonian in a different guise. Moreover, because $\mathbf{u}^c(\boldsymbol{\alpha}, t, \mathbf{x}_t, T)$ is also the solution to $\max_{\mathbf{u}}[f(t, \mathbf{x}_t, \mathbf{u}; \boldsymbol{\alpha}) + V_{\mathbf{x}}(\boldsymbol{\alpha}, t, \mathbf{x}_t)\mathbf{g}(t, \mathbf{x}_t, \mathbf{u}; \boldsymbol{\alpha})]$, as we established above, it follows from uniqueness that $\mathbf{u}^c(\boldsymbol{\alpha}, t, \mathbf{x}_t, T) \equiv \mathbf{v}(t; \boldsymbol{\alpha}, t, \mathbf{x}_t, T)$, which is what we wished to show. In sum, therefore, we have established if the optimal control path of problem (2) is unique, then the value of the control that maximizes the right-hand side of the H-J-B equation *is* the value of the optimal closed-loop control.

Let's now turn to a simple mathematical example to see how this all works.

Example 19.1: Consider the following purely mathematical optimal control problem:

$$V(t, x_t) \stackrel{\text{def}}{=} \min_{u(\cdot), x_T} \left[\int_t^T (u(s))^2 \, ds + (x(T))^2 \right]$$

$$\text{s.t.} \quad \dot{x}(s) = x(s) + u(s),\ x(t) = x_t,\ x(T) = x_T.$$

This problem is devoid of economic content in order to demonstrate the steps involved in solving the H-J-B equation and in constructing the optimal closed-loop control. Note that we initially suppress the dependence of $V(\cdot)$ on T so as to keep the notation relatively uncluttered.

By Theorem 19.1, the H-J-B equation is given by

$$-V_t = \min_u \left\{ u^2 + V_x[x + u] \right\}.$$

Observe that we have also dropped the subscript t on the state variable because we know the base period is t from the problem statement, and for notational clarity. In view of the fact that the Hamiltonian, which as you know is the right-hand-side expression to be minimized, is a convex function of the state and control variables, a solution of the necessary conditions is a solution of the optimal control problem. Thus the necessary and sufficient condition for the above minimization problem in the H-J-B equation is given by $2u + V_x = 0$, which is easily solved to get $u = \bar{u}(V_x) \stackrel{\text{def}}{=} -\frac{1}{2}V_x$. Substituting this solution for the control variable into the H-J-B equation yields the partial differential equation to be solved for the optimal value function $V(\cdot)$, namely,

$$-V_t = xV_x - \frac{1}{4}V_x^2. \tag{20}$$

This happens to be a partial differential equation that can be solved for the unknown optimal value function $V(\cdot)$; this is why the example was chosen. To solve this partial differential equation, we propose a general functional form for $V(\cdot)$, and then seek to determine a set of parameter values for it so that the proposed function satisfies Eq. (20).

To that end, we guess that a solution to the H-J-B partial differential equation is a polynomial in x. This guess is motivated by the fact that the integrand, transition, and salvage functions of the optimal control problem under consideration are quadratic or linear functions of the state variable. For a polynomial of order k in x, the left-hand side of Eq. (20) is a polynomial of order k, whereas the right-hand side is a polynomial of order $2(k-1)$ because of the term V_x^2. In order for the left-hand and right-hand sides to be of the same order, that is, for $k = 2(k-1)$ to hold, it follows that $k = 2$. Thus we conclude that a quadratic function of x will suffice. In contrast, we leave our guess about the functional form with respect to t in general terms. Hence our guess is that the optimal value function is of the form

$$V(t, x) = A(t)x^2,$$

where $A(\cdot)$ is an unknown function of t that we wish to ascertain. Next we seek to determine if our proposed guess is in fact a solution of the H-J-B equation.

To that end, compute the first-order partial derivatives of $V(t, x) = A(t)x^2$:

$$V_t(t, x) = \dot{A}(t)x^2 \quad \text{and} \quad V_x(t, x) = 2A(t)x.$$

Substituting these derivatives into Eq. (20) gives $-\dot{A}(t)x^2 = 2A(t)x^2 - [A(t)]^2x^2$, or

$$[\dot{A}(t) + 2A(t) - [A(t)]^2]\,x^2 = 0. \tag{21}$$

Equation (21) must hold for all admissible values of x in order for $V(t, x) = A(t)x^2$ to be a solution to Eq. (20), which is true if and only if the function $A(\cdot)$ is a solution to the nonlinear ordinary differential equation $\dot{A}(t) + 2A(t) = [A(t)]^2$. This looks to be a formidable differential equation to solve explicitly, since it is nonlinear. Recall that we want an explicit solution of the differential equation $\dot{A}(t) + 2A(t) = [A(t)]^2$ because we want an explicit expression for the optimal value function and the feedback solution for the control variable.

As you may recall from an elementary differential equation course, $\dot{A}(t) + 2A(t) = [A(t)]^2$ is a Bernoulli differential equation with $n = 2$. A standard solution procedure for this class of differential equations begins by defining a new variable $y \stackrel{\text{def}}{=} A^{1-n} = A^{-1}$, which implies that $\dot{y} = -A^{-2}\dot{A}$. Multiplying the Bernoulli equation through by $-A^{-2}$ gives $-A^{-2}\dot{A} - 2A^{-1} = -1$. This latter form of the differential equation can be rewritten in terms of the variable y as

$$\dot{y} - 2y = -1, \tag{22}$$

which is a linear first-order constant coefficient ordinary differential equation whose integrating factor is $\exp[2t]$. The general solution of Eq. (22) is thus

$$y(t) = \frac{1}{2} + ce^{2t}, \tag{23}$$

where c is a constant of integration. Recall that the boundary condition for the H-J-B equation (20) is given by $V(T, x_T) = x_T^2$. Because we've assumed that

$V(t,x) = A(t)x^2$, this implies that the boundary condition takes the simple form $A(T) = 1$. But seeing as $y \stackrel{\text{def}}{=} A^{1-n} = A^{-1}$, this translates into the boundary condition $y(T) = 1$ for Eq. (23). Using this boundary condition implies that $c = \frac{1}{2}e^{-2T}$. Hence the specific solution of Eq. (22) takes the form $y(t) = \frac{1}{2}[1 + e^{2(t-T)}]$, which in turn implies the specific solution

$$A(t) = \frac{2}{1 + e^{2(t-T)}} \tag{24}$$

for the unknown function $A(\cdot)$. Note that it satisfies the boundary condition $A(T) = 1$.

With the function $A(\cdot)$ given by Eq. (24), we now have the information to construct the optimal value function $V(\cdot)$. Because we guessed that $V(t,x) = A(t)x^2$, Eq. (24) implies that the optimal value function is of the form

$$V(t,x,T) = \frac{2x^2}{1 + e^{2(t-T)}}. \tag{25}$$

Note that we have now included T as an argument of $V(\cdot)$. Given that $A(T) = 1$, it follows that $V(T,x,T) = A(T)x^2 = x^2$; hence the boundary condition is satisfied. We leave it as a mental exercise to verify that Eq. (25) satisfies the H-J-B equation (20).

Turning to the *feedback* or *closed-loop* form of the solution for the control variable, we see that because $u = \bar{u}(V_x) \stackrel{\text{def}}{=} -\frac{1}{2}V_x$, use of Eq. (25) gives

$$u = u^c(t,x,T) \stackrel{\text{def}}{=} \frac{-2x}{1 + e^{2(t-T)}}. \tag{26}$$

Recall that the adjectives *feedback* and *closed-loop* refer to the fact that the optimal control in Eq. (26) is expressed as a function of the base period t, the value of the state variable in the base period, and the parameter T. This contrasts with the open-loop form of an optimal control that we derived in all the previous chapters dealing with optimal control theory. You should also recall that the open-loop form of the optimal control is typically expressed as a function of the independent variable time (denoted by s in the present chapter), the parameters of the problem, and the initial and/or terminal values of the planning horizon and state variables, the latter two sets of variables depending on whether they are decision variables or parametrically given.

To find the corresponding open-loop solution for the control variable, one could substitute the closed-loop solution for the control from Eq. (26) into the state equation $\dot{x} = x + u$ to get the ordinary differential equation $\dot{x} = x - [2x/(1 + e^{2(t-T)})]$. Rather than attempt to solve this differential equation, we leave it as a mental exercise to show that

$$u = u^o(s;t,x_t,T) \stackrel{\text{def}}{=} \frac{-2x_t e^{(T-t)} e^{(T-s)}}{1 + e^{2(T-t)}}, \tag{27}$$

$$\lambda = \lambda^o(s; t, x_t, T) \stackrel{\text{def}}{=} \frac{4x_t e^{(T-t)} e^{(T-s)}}{1 + e^{2(T-t)}}, \tag{28}$$

$$x = x^o(s; t, x_t, T) \stackrel{\text{def}}{=} \left[\frac{x_t e^{(T-t)}}{1 + e^{2(T-t)}}\right] [e^{(T-s)} - e^{(T+s-2t)}] + x_t e^{(s-t)}, \tag{29}$$

$$x_T = x_T^o(t, x_t, T) \stackrel{\text{def}}{=} \frac{2x_t e^{(T-t)}}{1 + e^{2(T-t)}} \tag{30}$$

is the unique optimal open-loop solution to the control problem. It is obtained using the appropriate necessary conditions of optimal control theory.

Prior to this example, we proved that if the optimal control path is unique, then the value of the control that maximizes the right-hand side of the H-J-B equation is the value of the optimal closed-loop control. That is, we have the following identity linking the value of the open-loop and closed-loop solutions for the control variable:

$$u^o(t; t, x_t, T) \equiv u^c(t, x_t, T).$$

This identity is straightforward to verify using the explicit formulas for the open-loop and closed-loop forms of the control variable given in Eqs. (28) and (26), respectively. It asserts that the value of the open-loop control in the base period, that is, the value of the closed-loop control, is identically equal to the value of the control that solves the H-J-B equation. This is as it should be, because the limiting process used to establish the H-J-B equation shows that it applies to any base period of the underlying optimal control problem.

Another noteworthy observation, akin to that just mentioned, is that when the open-loop costate is evaluated at the base period $s = t$, then its value is identical to that of the closed-loop costate. In symbols, we are asserting that

$$\lambda^o(t; t, x_t, T) \equiv V_x(t, x_t, T).$$

You are asked to verify the above two identities in a mental exercise. These identities turn out to be important results in a general sense because they help solidify one's understanding of the relationship between the open-loop and closed-loop forms of the solution. As a result, we will return to them, as well as related ones, at a later point in the chapter, when we study another class of optimal control problems.

Most economic applications of optimal control theory involve discounting, an infinite planning horizon, and functions $f(\cdot)$ and $\mathbf{g}(\cdot)$ that do not depend explicitly on time, the independent variable, as we have noted in several earlier chapters. With this mathematical structure, the H-J-B equation reduces to a simpler form, namely, an ordinary differential equation. This simplification is of great value for the dynamic duality theory to be expounded upon in the next chapter. Let us therefore proceed to establish this and other, related results.

In light of the aforementioned remarks, let us now consider the following class of discounted autonomous infinite-horizon optimal control problems:

$$\max_{\mathbf{u}(\cdot)} \int_0^{+\infty} f(\mathbf{x}(s), \mathbf{u}(s); \boldsymbol{\alpha})\, e^{-rs}\, ds$$
$$\text{s.t.} \quad \dot{\mathbf{x}}(s) = \mathbf{g}(\mathbf{x}(s), \mathbf{u}(s); \boldsymbol{\alpha}),\ \mathbf{x}(0) = \mathbf{x}_0, \tag{31}$$

where $r > 0$ is the discount rate. Note that we have not placed any condition on the limiting value of the state variables, that is, no conditions are placed on $\lim_{t\to+\infty} \mathbf{x}(t)$. For any initial time or base period $t \in [0, +\infty)$, and any admissible value of the state vector in the base period $\mathbf{x}(t) = \mathbf{x}_t$, define the *present value optimal value function* $V^{pv}(\cdot)$ by

$$V^{pv}(\boldsymbol{\alpha}, r, t, \mathbf{x}_t) \stackrel{\text{def}}{=} \max_{\mathbf{u}(\cdot)} \int_t^{+\infty} f(\mathbf{x}(s), \mathbf{u}(s); \boldsymbol{\alpha})\, e^{-rs}\, ds$$
$$\text{s.t.} \quad \dot{\mathbf{x}}(s) = \mathbf{g}(\mathbf{x}(s), \mathbf{u}(s); \boldsymbol{\alpha}),\ \mathbf{x}(t) = \mathbf{x}_t. \tag{32}$$

Thus, $V^{pv}(\boldsymbol{\alpha}, r, t, \mathbf{x}_t)$ is the maximum present value of the optimal control problem (31) that begins in the admissible state $\mathbf{x}(t) = \mathbf{x}_t$ in any base period $t \in [0, +\infty)$, given the parameter vector $(\boldsymbol{\alpha}, r)$. The adjective *present value* is required because the values of the integrand function are discounted back to time zero rather than the base period t in Eq. (32), and it is the base period t that is the current period from which planning begins. To see this, simply evaluate the integrand at time $s = t$ and observe that the value of $f(\cdot)$ is multiplied by e^{-rt}, thereby implying it is discounted back to time zero rather than the base period t. Consequently, if we take the base period to be time zero, that is, we set $t = 0$ in Eq. (32), then the values of the integrand are discounted to the base period $t = 0$, which is the same as the current period in this case. This means that if we set $t = 0$ in Eq. (32), then $V^{pv}(\cdot)$ is equal to the current value optimal value function.

Now multiply Eq. (32) by the identity $e^{-rt}e^{rt} \equiv 1$, and note that this term is independent of the dummy variable of integration s. This yields an alternative but equivalent expression for the present value optimal value function, videlicet,

$$V^{pv}(\boldsymbol{\alpha}, r, t, \mathbf{x}_t) \stackrel{\text{def}}{=} e^{-rt} \max_{\mathbf{u}(\cdot)} \int_t^{+\infty} f(\mathbf{x}(s), \mathbf{u}(s); \boldsymbol{\alpha})\, e^{-r(s-t)}\, ds$$
$$\text{s.t.} \quad \dot{\mathbf{x}}(s) = \mathbf{g}(\mathbf{x}(s), \mathbf{u}(s); \boldsymbol{\alpha}),\ \mathbf{x}(t) = \mathbf{x}_t. \tag{33}$$

This is an equation we will make use of below. Given Eq. (33), the *current value optimal value function* $V^{cv}(\cdot)$ can be defined by

$$V^{cv}(\boldsymbol{\alpha}, r, t, \mathbf{x}_t) \stackrel{\text{def}}{=} \max_{\mathbf{u}(\cdot)} \int_t^{+\infty} f(\mathbf{x}(s), \mathbf{u}(s); \boldsymbol{\alpha})\, e^{-r(s-t)}\, ds$$
$$\text{s.t.} \quad \dot{\mathbf{x}}(s) = \mathbf{g}(\mathbf{x}(s), \mathbf{u}(s); \boldsymbol{\alpha}),\ \mathbf{x}(t) = \mathbf{x}_t. \tag{34}$$

Consequently, $V^{cv}(\boldsymbol{\alpha}, r, t, \mathbf{x}_t)$ is the maximum current value of the optimal control problem (31) that begins in the admissible state $\mathbf{x}(t) = \mathbf{x}_t$ in any base period $t \in [0, +\infty)$, given the parameter vector $(\boldsymbol{\alpha}, r)$.

By examining Eqs. (33) and (34) carefully, it should be apparent that the following relationship holds between the present value and current value optimal value functions for infinite-horizon current value autonomous optimal control problems:

$$V^{pv}(\boldsymbol{\alpha}, r, t, \mathbf{x}_t) \equiv e^{-rt} V^{cv}(\boldsymbol{\alpha}, r, t, \mathbf{x}_t)\, \forall\, t \in [0, +\infty). \tag{35}$$

This says that the present value optimal value function has the same value as the current value optimal value function once we discount the latter's value back to time $s = 0$ of the control problem, which is a quite intuitive and natural result. The most important feature of the current value optimal value function $V^{cv}(\cdot)$ is that it is independent of the initial date or base period t, one of several properties presented in the ensuing theorem.

Theorem 19.2: *The ensuing properties hold for the present value optimal value function $V^{pv}(\cdot)$ and the current value optimal value function $V^{cv}(\cdot)$ for the class of discounted autonomous infinite-horizon optimal control problems defined in Eq. (31):*

(a) $V^{cv}(\boldsymbol{\alpha}, r, t, \mathbf{x}_t) \equiv V^{cv}(\boldsymbol{\alpha}, r, 0, \mathbf{x}_t)\, \forall\, t \in [0, +\infty)$,
(b) $V^{pv}(\boldsymbol{\alpha}, r, 0, \mathbf{x}_t) \equiv V^{cv}(\boldsymbol{\alpha}, r, 0, \mathbf{x}_t)$, and
(c) $V^{pv}(\boldsymbol{\alpha}, r, t, \mathbf{x}_t) \equiv e^{-rt} V^{pv}(\boldsymbol{\alpha}, r, 0, \mathbf{x}_t)\, \forall\, t \in [0, +\infty)$.

Proof: We begin by proving part (a). First, use Eq. (34), the equation defining the current value optimal value function $V^{cv}(\cdot)$, to write down the definition of $V^{cv}(\boldsymbol{\alpha}, r, 0, \mathbf{x}_t)$:

$$V^{cv}(\boldsymbol{\alpha}, r, 0, \mathbf{x}_t) \stackrel{\text{def}}{=} \max_{\mathbf{u}(\cdot)} \int_0^{+\infty} f(\mathbf{x}(s), \mathbf{u}(s); \boldsymbol{\alpha})\, e^{-rs}\, ds$$
$$\text{s.t.} \quad \dot{\mathbf{x}}(s) = \mathbf{g}(\mathbf{x}(s), \mathbf{u}(s); \boldsymbol{\alpha}),\ \mathbf{x}(s)|_{s=0} = \mathbf{x}_t. \tag{36}$$

Second, employ Eq. (36) to write down the definition of $V^{cv}(\boldsymbol{\alpha}, r, t, \mathbf{x}_t)$. We want to be extra careful in writing down the definition of $V^{cv}(\boldsymbol{\alpha}, r, t, \mathbf{x}_t)$ by keeping in mind what we intend to prove, scilicet, $V^{cv}(\boldsymbol{\alpha}, r, t, \mathbf{x}_t) \equiv V^{cv}(\boldsymbol{\alpha}, r, 0, \mathbf{x}_t)\, \forall\, t \in [0, +\infty)$. Thus, in writing down the definition of $V^{cv}(\boldsymbol{\alpha}, r, t, \mathbf{x}_t)$, we want only the third argument of it to be different from that in $V^{cv}(\boldsymbol{\alpha}, r, 0, \mathbf{x}_t)$. To accomplish this, when we advance the base period from zero to t in Eq. (36), we at the same time subtract t from the dummy variable of integration s wherever it appears in the state and control functions. Doing just that gives

$$V^{cv}(\boldsymbol{\alpha}, r, t, \mathbf{x}_t) \stackrel{\text{def}}{=} \max_{\mathbf{u}(\cdot)} \int_t^{+\infty} f(\mathbf{x}(s-t), \mathbf{u}(s-t); \boldsymbol{\alpha})\, e^{-r(s-t)}\, ds$$

$$\text{s.t.} \quad \dot{\mathbf{x}}(s-t) = \mathbf{g}(\mathbf{x}(s-t), \mathbf{u}(s-t); \boldsymbol{\alpha}),\ \mathbf{x}(s-t)|_{s=t} = \mathbf{x}_t,$$

which holds for any $t \in [0, +\infty)$. Now let's change the dummy variable of integration from s to $\tau \stackrel{\text{def}}{=} s - t$, which implies that $d\tau = ds$. Moreover, this change of variables also implies that if $s = t$, then $\tau = 0$, whereas if $s \to +\infty$, then $\tau \to +\infty$. Using these results allows us to rewrite the previous definition of $V^{cv}(\boldsymbol{\alpha}, r, t, \mathbf{x}_t)$ equivalently as

$$V^{cv}(\boldsymbol{\alpha}, r, t, \mathbf{x}_t) \stackrel{\text{def}}{=} \max_{\mathbf{u}(\cdot)} \int_0^{+\infty} f(\mathbf{x}(\tau), \mathbf{u}(\tau); \boldsymbol{\alpha}) e^{-r\tau} d\tau$$

$$\text{s.t.} \quad \dot{\mathbf{x}}(\tau) = \mathbf{g}(\mathbf{x}(\tau), \mathbf{u}(\tau); \boldsymbol{\alpha}),\ \mathbf{x}(\tau)|_{\tau=0} = \mathbf{x}_t. \tag{37}$$

Now observe that the right-hand side of Eq. (37) is identical to the right-hand side of Eq. (36), since they differ only with respect to their dummy variables of integration, which is immaterial. Hence the left-hand sides of Eqs. (36) and (37) are identical too, that is,

$$V^{cv}(\boldsymbol{\alpha}, r, t, \mathbf{x}_t) \equiv V^{cv}(\boldsymbol{\alpha}, r, 0, \mathbf{x}_t)\, \forall\, t \in [0, +\infty),$$

which is what we set out to prove.

To prove part (b), that is, $V^{pv}(\boldsymbol{\alpha}, r, 0, \mathbf{x}_t) \equiv V^{cv}(\boldsymbol{\alpha}, r, 0, \mathbf{x}_t)$, simply set $t = 0$ in the definitions in Eqs. (33) and (34) and note that identical expressions result. This result also follows immediately by evaluating Eq. (35) at $t = 0$.

To prove part (c), namely, $V^{pv}(\boldsymbol{\alpha}, r, t, \mathbf{x}_t) \equiv e^{-rt} V^{pv}(\boldsymbol{\alpha}, r, 0, \mathbf{x}_t)\, \forall\, t \in [0, +\infty)$, note that by parts (a) and (b), $V^{cv}(\boldsymbol{\alpha}, r, t, \mathbf{x}_t) \equiv V^{pv}(\boldsymbol{\alpha}, r, 0, \mathbf{x}_t)\, \forall\, t \in [0, +\infty)$. Substituting this result into Eq. (35) then produces the desired result. Q.E.D.

Part (a) of Theorem 19.2, scilicet, $V^{cv}(\boldsymbol{\alpha}, r, t, \mathbf{x}_t) \equiv V^{cv}(\boldsymbol{\alpha}, r, 0, \mathbf{x}_t) \forall\, t \in [0, +\infty)$, asserts that for a given value of the state variable in the base period

$\mathbf{x}_t$, a given vector of parameters $\boldsymbol{\alpha}$, and a given discount rate r, the base period has no effect on the value of the current value optimal value function in discounted autonomous infinite-horizon optimal control problems. This is intuitive in that no matter when you start in the future, you still have an infinite amount of time left in the optimal control problem (31). Thus the only things that determine the value of $V^{cv}(\cdot)$ are the parameter vector $\boldsymbol{\alpha}$, the discount rate r, and the value of state variable $\mathbf{x}_t$ in the base period. As a result of this part of Theorem 19.2, we are permitted to write the value of the current value optimal value function $V^{cv}(\cdot)$ as $V^{cv}(\boldsymbol{\alpha}, r, \mathbf{x}_t)$, in which we completely suppress the appearance of the base period. We will adopt this simplified notation for $V^{cv}(\cdot)$ from this point forward when discussing such a class of control problems. Finally, it is worthwhile to recognize that an *equivalent* way to write part (a) is $\partial V^{cv}(\boldsymbol{\alpha}, r, \mathbf{x}_t)/\partial t \equiv 0$, since this says that $V^{cv}(\cdot)$ is independent of the base period t.

Part (b) of Theorem 19.2, which we may now write as $V^{pv}(\boldsymbol{\alpha}, r, 0, \mathbf{x}_t) \equiv V^{cv}(\boldsymbol{\alpha}, r, \mathbf{x}_t)$ in light of the notational convention adopted in the previous paragraph, asserts that the present value and current value optimal value functions have identical values if we take as the base period the first instant of the planning horizon $t = 0$. Recall that we made a remark to this effect after Eq. (32). This result is also intuitive, for if the date we are discounting to is the current date we are starting our planning from, then there is no difference between the present value and current value of our optimal plan. In passing, note that because part (c) of Theorem 19.2 is equivalent to Eq. (35), as the proof revealed, there is no need to provide an economic interpretation of it here, for that was done when Eq. (35) was introduced.

Before establishing that the H-J-B equation of Theorem 19.1 reduces to an ordinary differential equation for the class of discounted autonomous infinite-horizon optimal control problems defined by Eq. (31), let's pause and demonstrate the results of Theorem 19.2 with an nonrenewable resource–extraction model. We leave some of the more tedious calculations for a mental exercise so as to emphasize the content of Theorem 19.2.

Example 19.2: Consider the following discounted autonomous infinite-horizon model of a nonrenewable resource–extracting firm:

$$V^{pv}(\alpha, r, t, x_t) \stackrel{\text{def}}{=} \max_{q(\cdot)} \left\{ \int_t^{+\infty} [\alpha \ln q(s)]\, e^{-rs}\, ds \right\}$$

$$\text{s.t.} \quad \dot{x}(s) = -q(s),\, x(t) = x_t,\, \lim_{s \to +\infty} x(s) = 0.$$

By this point in the text, the economic interpretation of this model should be completely clear: $x(s)$ is the stock of the nonrenewable resource in the ground at time s, $q(s)$ is the extraction rate of the nonrenewable resource at time s, and the instantaneous profit from extracting at the rate $q(s)$ is $\alpha \ln q(s)$, where $\alpha > 0$ is a parameter of the profit function. Two features of some significance in this model are (i) the infinite planning horizon, and (ii) the limiting terminal condition on the resource stock. The latter requires that the resource stock be exhausted only in the limit of the planning horizon.

In order to solve this optimal control problem, we begin, as usual, by writing down the current value Hamiltonian, scilicet, $H(x, q, \lambda; \alpha) \stackrel{\text{def}}{=} \alpha \ln q - \lambda q$. At this point in the book, you shouldn't have any problem deriving the solution to the control problem, so that will be left for a mental exercise. Consequently, we simply state the solution here for future reference:

$$q^o(s-t; r, x_t) = r x_t e^{-r(s-t)},$$

$$x^o(s-t; r, x_t) = x_t e^{-r(s-t)},$$

$$\lambda(s-t; \alpha, r, x_t) = \alpha r^{-1} x_t^{-1} e^{r(s-t)}.$$

Before we delve into verifying Theorem 19.2, let's pause to make two remarks. First, notice that the independent variable s and the base period t always enter the solution in the form $s - t$, and we have indicated this in the notation. This was to be expected if you followed the proof of Theorem 19.2 carefully. Second, the above solution is the open-loop form, since it was derived using the methods of optimal control theory.

To find the present value optimal value function, substitute the optimal path of the control variable into the objective functional and integrate by substitution and by parts to get

$$V^{pv}(\alpha, r, t, x_t) = \frac{\alpha}{r}[\ln r x_t - 1] e^{-rt}.$$

You will be asked to verify this in a mental exercise. Given that $V^{pv}(\alpha, r, 0, x_t) = \frac{\alpha}{r}[\ln r x_t - 1]$, it is immediate that $V^{pv}(\alpha, r, t, x_t) \equiv e^{-rt} V^{pv}(\alpha, r, 0, x_t) \forall\, t \in [0, +\infty)$, which is part (c) of Theorem 19.2. Moreover, because the current value optimal value function is, by definition, e^{rt} times the present value optimal value function, we have that $V^{cv}(\alpha, r, x_t) = \frac{\alpha}{r}[\ln r x_t - 1] = V^{pv}(\alpha, r, 0, x_t)$, which is part (b) of Theorem 19.2. Moreover, because $V^{cv}(\alpha, r, x_t) = \frac{\alpha}{r}[\ln r x_t - 1]$ is independent of the base period t, this verifies part (a) of Theorem 19.2.

Let's return to the basic identity linking the present value and current value optimal value functions, either Eq. (35) or a combination of parts (b) and (c) of Theorem 19.2, namely,

$$V^{pv}(\boldsymbol{\alpha}, r, t, \mathbf{x}) \equiv e^{-rt} V^{cv}(\boldsymbol{\alpha}, r, \mathbf{x}) \,\forall\, t \in [0, +\infty), \tag{38}$$

for the class of problems under consideration, that is, the discounted autonomous infinite-horizon variety defined by Eq. (31). Note that we have dropped the subscript on the initial value of the state variable since by now the base period is clear. Partially differentiating identity (38) with respect to the initial date and the corresponding state gives

$$V_t^{pv}(\boldsymbol{\alpha}, r, t, \mathbf{x}) \equiv -re^{-rt}V^{cv}(\boldsymbol{\alpha}, r, \mathbf{x})\,\forall\, t \in [0, +\infty), \tag{39}$$

$$V_{x_n}^{pv}(\boldsymbol{\alpha}, r, t, \mathbf{x}) \equiv e^{-rt}V_{x_n}^{cv}(\boldsymbol{\alpha}, r, \mathbf{x})\,\forall\, t \in [0, +\infty], \quad n = 1, 2, \ldots, N. \tag{40}$$

For the current value autonomous infinite-horizon class of control problems given by Eq. (31), the H-J-B equation given by Theorem 19.1 takes the form

$$-V_t^{pv}(\boldsymbol{\alpha}, r, t, \mathbf{x}) = \max_{\mathbf{u}}\left[f(\mathbf{x}, \mathbf{u}; \boldsymbol{\alpha})e^{-rt} + \sum_{n=1}^{N} V_{x_n}^{pv}(\boldsymbol{\alpha}, r, t, \mathbf{x})g^n(\mathbf{x}, \mathbf{u}; \boldsymbol{\alpha})\right]. \tag{41}$$

Substituting Eqs. (39) and (40) into the H-J-B equation (41), and then multiplying through by e^{rt} yields

$$rV^{cv}(\boldsymbol{\alpha}, r, \mathbf{x}) = \max_{\mathbf{u}}\left[f(\mathbf{x}, \mathbf{u}; \boldsymbol{\alpha}) + \sum_{n=1}^{N} V_{x_n}^{cv}(\boldsymbol{\alpha}, r, \mathbf{x})g^n(\mathbf{x}, \mathbf{u}; \boldsymbol{\alpha})\right], \tag{42}$$

which is an ordinary differential equation obeyed by the current value optimal value function. It is important to remember that this result holds only for the current value optimal value function for current value autonomous, infinite-horizon optimal control problems defined by Eq. (31). In other words, if the independent variable s appears explicitly in the integrand function $f(\cdot)$ or vector-valued transition function $\mathbf{g}(\cdot)$, or if the time horizon is finite, then Eq. (42) does not hold. This is a very important result because of the ubiquitous nature of current value autonomous infinite horizon optimal control problems in intertemporal economic theory. We therefore record this result in the ensuing theorem.

Theorem 19.3: *If $V^{cv}(\cdot) \in C^{(1)}$, then the current value optimal value function for the class of discounted autonomous infinite-horizon optimal control problems defined by Eq. (31) obeys the ordinary differential equation*

$$rV^{cv}(\boldsymbol{\alpha}, r, \mathbf{x}_t) = \max_{\mathbf{u}}\left[f(\mathbf{x}_t, \mathbf{u}; \boldsymbol{\alpha}) + \sum_{n=1}^{N} V_{x_n}^{cv}(\boldsymbol{\alpha}, r, \mathbf{x}_t)g^n(\mathbf{x}_t, \mathbf{u}; \boldsymbol{\alpha})\right]$$

in index notation, or equivalently

$$rV^{cv}(\boldsymbol{\alpha}, r, \mathbf{x}_t) = \max_{\mathbf{u}}\left[f(\mathbf{x}_t, \mathbf{u}; \boldsymbol{\alpha}) + V_{\mathbf{x}}^{cv}(\boldsymbol{\alpha}, r, \mathbf{x}_t)\mathbf{g}(\mathbf{x}_t, \mathbf{u}; \boldsymbol{\alpha})\right]$$

in vector notation. This holds for any base period $t \in [0, +\infty)$ and any state $\mathbf{x}_t$ that is admissible.

Let's further examine Example 19.2 in light of Theorem 19.3.

Example 19.3: Recall that we derived an explicit expression for the current value optimal value function in Example 19.2, to wit, $V^{cv}(\alpha, r, x_t) = \frac{\alpha}{r}[\ln rx_t - 1]$. The current value shadow price of the stock is therefore found by partially differentiating $V^{cv}(\alpha, r, x_t) = \frac{\alpha}{r}[\ln rx_t - 1]$ with respect to the stock, thereby yielding $V^{cv}_{x_t}(\alpha, r, x_t) = \frac{\alpha}{rx_t}$. Having an explicit expression for the current value shadow price of the stock means that in this example at least, we can derive an explicit expression for the right-hand side of the H-J-B equation given in Theorem 19.3. This, in turn, allows us to derive an explicit expression for the value of the control that maximizes the right-hand side of the H-J-B equation, which we now know to be the closed-loop form of the optimal extraction rate.

To this end, Theorem 19.3 directs us to solve the static maximization problem

$$rV^{cv}(\alpha, r, x_t) = \max_q \left\{\alpha \ln q - V^{cv}_{x_t}(\alpha, r, x_t)q\right\} = \max_q \left\{\alpha \ln q - \frac{\alpha}{rx_t}q\right\}$$

in order to find the optimal control. Given that the function to be maximized is strictly concave in q, the first-order necessary condition $\frac{\alpha}{q} - \frac{\alpha}{rx_t} = 0$ is also sufficient for determining the unique global maximum of the problem. Solving $\frac{\alpha}{q} - \frac{\alpha}{rx_t} = 0$ gives the optimal control $q^c(r, x_t) = rx_t$. Substituting $q^c(r, x_t) = rx_t$ back into the H-J-B equation above gives

$$rV^{cv}(\alpha, r, x_t) = \alpha \ln q^c(r, x_t) - \frac{\alpha}{rx_t}q^c(r, x_t) = \alpha[\ln rx_t - 1],$$

from which it follows that $V^{cv}(\alpha, r, x_t) = \frac{\alpha}{r}[\ln rx_t - 1]$, as expected.

In comparing the optimal extraction rate derived by way of the H-J-B equation, to wit, $q^c(r, x_t) = rx_t$, with the open-loop form of the optimal extraction rate given in Example 19.2 and derived via the necessary conditions of optimal control theory, scilicet, $q^o(s - t; r, x_t) = rx_t e^{-r(s-t)}$, two important relationships between the solutions emerge. First, the values of the two forms of the extraction rates are identical when one substitutes the open-loop solution of the resource stock $x^o(s - t; r, x_t) = x_t e^{-r(s-t)}$ for the base period stock x_t in the optimal extraction rate derived by way of the H-J-B equation, as is easily verified:

$$q^c(r, x^o(s - t; r, x_t)) = rx^o(s - t; r, x_t) = rx_t e^{-r(s-t)} = q^o(s - t; r, x_t). \quad (43)$$

This is as it should be, since (i) the planning horizon is infinite, thereby implying that no matter what date one takes as the base period, there is always an infinite amount of time left in the planning horizon, and (ii) the integrand function $f(\cdot)$ and the transition function $g(\cdot)$ do not depend explicitly on the independent time variable s (though $f(\cdot)$ is multiplied by the discount factor e^{-rs}). If either one of these conditions does not hold, then the relationship between the two solutions given in Eq. (43) will not, in general, hold either. A mental exercise asks you to verify this by using the two forms of the solution for the control variable of the autonomous finite-horizon control problem in Example 19.1.

One way to better understand Eq. (43) is as follows. Because the optimal control $q^c(r, x_t) = rx_t$ depends only on the value of the state variable in the base period and the discount rate, the base period has no effect on it. This is a result of properties

(i) and (ii) noted in the previous paragraph. Consequently, no matter what base period one adopts, when the base period stock is set equal to the value of the stock that corresponds to the optimal open-loop extraction rate, the optimal extraction rate determined by way of the H-J-B equation must be the same as the value of the extraction rate determined by the open-loop control at the date in the planning horizon that corresponds to the value of the resource stock used in the calculation. Another way to understand Eq. (43) is to observe that because all the parameters are known with certainty in any base period, one can either solve for the entire time path of the extraction rate in the base period and thus derive the open-loop solution or, equivalently, solve for the optimal control by repeatedly solving the H-J-B equation by varying the base period continuously over the interval $[0, +\infty)$, taking the base period stock of each static optimization problem to be the value determined by the preceding optimal extraction rate. Either way, one would have determined the identical optimal extraction rate over the planning horizon.

In passing, it should not be too surprising that the analogue to Eq. (43) holds between the open-loop $\lambda(s-t;\alpha,r,x_t)$ and current value shadow price of the stock derived by way of the H-J-B equation, namely, $V_{x_t}^{cv}(\alpha,r,x_t)$, that is to say,

$$V_{x_t}^{cv}(\alpha,r,x^o(s-t;r,x_t)) = \frac{\alpha}{rx^o(s-t;r,x_t)} = \frac{\alpha}{rx_t e^{-r(s-t)}}$$
$$= \alpha r^{-1} x_t^{-1} e^{r(s-t)} = \lambda(s-t;\alpha,r,x_t),$$

and for exactly the same reason.

The second important relationship between the closed-loop and open-loop extraction rates was noted in Example 9.1, and can similarly be determined from inspection of the two forms of the solutions, namely, $q^c(r,x_t) = rx_t$ and $q^o(s-t;r,x_t) = rx_t e^{-r(s-t)}$. In particular, evaluating the open-loop extraction rate at the base period $s=t$ yields the extraction rate determined via the H-J-B equation, that is,

$$q^o(s-t;r,x_t)\big|_{s=t} = rx_t e^{-r(s-t)}\big|_{s=t} = rx_t = q^c(r,x_t), \tag{44}$$

with the same property holding for the current value shadow price of the stock:

$$\lambda(s-t;\alpha,r,x_t)|_{s=t} = \alpha r^{-1} x_t^{-1} e^{r(s-t)}\big|_{s=t} = \frac{\alpha}{rx_t} = V_{x_t}^{cv}(\alpha,r,x_t).$$

This relationship between the two forms of the solution means that we can, in principle, avoid solving the H-J-B equation to determine the closed-loop solution. This is of limited practical value, however, for as you know, it is a formidable task to solve for the open-loop solution of a control problem for all but the most simple, and often economically uninteresting, functional forms of the integrand and transition functions. In any case, Definition 19.1 shows that one can derive the closed-loop solution by employing the necessary and sufficient conditions of optimal control theory to find the open-loop solution, and then simply evaluate the open-loop solution at the base period to arrive at the closed-loop form of the solution.

We will make use of this fact in the next chapter, when we study the qualitative and intertemporal duality properties of the adjustment cost model of the firm.

The final observation in this example concerns the closed-loop form of the optimal extraction rate. In view of the fact that $q^c(r, x_t) = rx_t$, the optimal extraction rate in the base period is expressed solely in terms of the state variable in the base period and the parameters of the problem, which in this example is the discount rate. The important observation is that the base period is not an argument of the closed-loop form of the optimal extraction rate. This feature is typical of *all* discounted autonomous infinite-horizon optimal control problems, that is, control problems of the form given by Eq. (31), as we shall shortly see. Note, however, that this property does not hold in general for autonomous finite-horizon optimal control problems, as is readily seen from the closed-loop control of Example 19.1 given in Eq. (26), namely, $u = u^c(t, x, T) \stackrel{\text{def}}{=} -2x/1 + e^{2(t-T)}$. Observe that the solution depends explicitly on the base period t.

The stationary property noted in Example 19.3, scilicet, that the closed-loop form of the optimal control is independent of the base period, is a result of the fact that the optimal value function is independent of the base period for the class of discounted autonomous infinite-horizon optimal control problems. This observation is the key to proving the following general result, the details of which we leave as a mental exercise.

Theorem 19.4: *For the class of discounted autonomous infinite-horizon optimal control problems defined by Eq. (31), the optimal values of the closed-loop control vector and the current value costate vector are a function of the value of the state variable in the base period* $\mathbf{x}_t$*, the discount rate* r*, and the parameter vector* $\boldsymbol{\alpha}$*, but not the base period* t*, that is to say,* $\mathbf{u} = \mathbf{u}^c(\boldsymbol{\alpha}, r, \mathbf{x}_t)$ *and* $\boldsymbol{\lambda} = \boldsymbol{\lambda}^c(\boldsymbol{\alpha}, r, \mathbf{x}_t)$.

Theorem 19.4 asserts that the closed-loop solution of the control vector and the current value costate vector do not vary with the base period, ceteris paribus. This is not really surprising upon reflection, since (i) no matter when one begins planning in a discounted autonomous infinite-horizon control problem, that is, no matter what base period is adopted, there is still an infinite amount of time left in the planning horizon, and (ii) the integrand and transition functions do not vary with a change in the base period, that is to say, they are stationary. Hence, the control problem looks identical from the perspective of *any* base period under these circumstances, thus resulting in the same value of the control regardless of the starting date one plans from, that is, the base period, ceteris paribus. It is worthwhile to reiterate that Theorem 19.4 does not, in general, hold for nonautonomous control problems nor for autonomous control problems with a finite-planning horizon, pointing out just how special the discounted autonomous infinite-horizon variety is.

This chapter has laid the groundwork for the ensuing one on intertemporal duality theory. Because we will rely heavily on this material, especially Theorem 19.3, it is critically important that you fully understand the contents of this chapter before pressing on. Note that even though we have solved for the closed-loop controls and optimal value functions of various optimal control problems in the examples presented herewith, this is not how we intend to use the results of this chapter. In contrast, what we intend to do in the next chapter is akin to how one goes about developing the generic comparative statics and duality properties of the prototype profit maximization and cost minimization models of the firm, albeit for a dynamic economic model. In passing, also note that we could have presented the H-J-B equation as a sufficient condition for optimality, but have chosen not to do so because we do not intend to make use of it in that form.

MENTAL EXERCISES

19.1 Recall the optimal control problem of Example 19.1:

$$V(t, x_t) \stackrel{\text{def}}{=} \min_{u(\cdot), x_T} \left[\int_t^T (u(s))^2 \, ds + (x(T))^2 \right]$$

$$\text{s.t.} \quad \dot{x}(s) = x(s) + u(s), \ x(t) = x_t, \ x(T) = x_T.$$

(a) Verify that the optimal value function given in Eq. (25) satisfies the H-J-B equation (20).

(b) Show that

$$u = u^o(s; t, x_t, T) \stackrel{\text{def}}{=} \frac{-2x_t e^{(T-t)} e^{(T-s)}}{1 + e^{2(T-t)}},$$

$$\lambda = \lambda^o(s; t, x_t, T) \stackrel{\text{def}}{=} \frac{4x_t e^{(T-t)} e^{(T-s)}}{1 + e^{2(T-t)}},$$

$$x = x^o(s; t, x_t, T) \stackrel{\text{def}}{=} \left[\frac{x_t e^{(T-t)}}{1 + e^{2(T-t)}} \right] [e^{(T-s)} - e^{(T+s-2t)}] + x_t e^{(s-t)},$$

$$x_T = x_T^o(t, x_t, T) \stackrel{\text{def}}{=} \frac{2x_t e^{(T-t)}}{1 + e^{2(T-t)}}$$

is the unique optimal open-loop solution to the control problem.

(c) Verify the identity $u^o(t; t, x_t, T) \equiv u^c(t, x_t, T)$ using the explicit expressions for the open-loop and closed-loop controls.

(d) Show that $u^o(s; t, x_t, T) \neq u^c(t, x^o(s; t, x_t, T), T)$ using the explicit expressions for the open-loop and closed-loop controls. Explain.

(e) Verify the identity $\lambda^o(t; t, x_t, T) \equiv V_x(t, x_t, T)$ using the explicit expressions for the open-loop and closed-loop costates.

(f) Show that $\lambda^o(s; t, x_t, T) \neq V_x(t, x^o(s; t, x_t, T), T)$ using the explicit expressions for the open-loop and closed-loop costates. Explain.

19.2 Recall the optimal control problem of Examples 19.2 and 19.3:

$$V^{pv}(\alpha, r, t, x_t) \stackrel{\text{def}}{=} \max_{q(\cdot)} \left\{ \int_t^{+\infty} [\alpha \ln q(s)]\, e^{-rs}\, ds \right\}$$

$$\text{s.t.} \quad \dot{x}(s) = -q(s),\, x(t) = x_t,\, \lim_{s \to +\infty} x(s) = 0.$$

(a) Prove that the optimal open-loop solution to this problem is given by

$$q^o(s - t; r, x_t) = r x_t e^{-r(s-t)},$$

$$x^o(s - t; r, x_t) = x_t e^{-r(s-t)},$$

$$\lambda(s - t; \alpha, r, x_t) = \alpha r^{-1} x_t^{-1} e^{r(s-t)}.$$

(b) Verify that the present value optimal value function is given by

$$V^{pv}(\alpha, r, t, x_t) = \frac{\alpha}{r}[\ln r x_t - 1]\, e^{-rt}.$$

19.3 Consider the following discounted autonomous infinite-horizon optimal control problem:

$$\min_{u(\cdot)} \int_0^{+\infty} [a[x(s)]^2 + b[u(s)]^2]\, e^{-rs}\, ds$$

$$\text{s.t.} \quad \dot{x}(s) = u(s),\;\; x(0) = x_0 > 0,$$

where $a > 0$ and $b > 0$ are given parameters and $r > 0$ is the discount rate. You will use Theorem 19.3 to solve this optimal control problem.

(a) For a given base period $t \in [0, +\infty)$, define the current value optimal value function $V(\cdot)$. Also write down the H-J-B equation obeyed by $V(\cdot)$ for any admissible state in the base period x_t.

(b) Determine the unique and globally optimal solution to the minimization problem dictated by the H-J-B equation, say, $u = \bar{u}(b, V_x)$.

(c) Derive the ordinary differential equation satisfied by the current value optimal value function $V(\cdot)$.

Guess that the current value optimal value function $V(\cdot)$ is of the form $V(x) = Ax^2$, where A is a constant to be determined. Note that this guess is motivated by the facts that the ordinary differential equation in part (c) is quadratic in x, that V_x is squared, and that $V(\cdot)$ is independent of the base period by Theorem 19.2.

(d) Determine the value of the constant A and thus find the precise form of the optimal value function.

(e) Find the closed-loop form of the control, say, $u = u^c(a, b, r, x)$.

(f) Using $u = u^c(a, b, r, x)$ in the state equation, find the open-loop form of the solution for the control, say, $u = u^o(s - t; a, b, r, x_t)$, using the initial condition $x(t) = x_t$ in the base period $s = t$. Take note of the fact that this solution depends on the difference between the time index dummy s and the base period t.

(g) Verify that the following two identities hold for the control problem:

$$u^o(s-t; a, b, r, x_t) \equiv u^c(a, b, r, x^o(s-t; a, b, r, x_t)),$$

$$u^o(s-t; a, b, r, x_t)|_{s=t} \equiv u^c(a, b, r, x_t).$$

19.4 Consider the following optimal control problem:

$$\max_{u(\cdot)} \int_0^T \ln u(s)\, ds$$

$$\text{s.t.} \quad \dot{x}(s) = \delta x(s) - u(s),\ x(0) = x_0,\ x(T) = x_T.$$

(a) Define the optimal value function $V(\cdot)$ and write down the H-J-B equation for the control problem for a given base period $t \in [0, +\infty)$ and any admissible state in the base period x_t.

(b) Derive the value of the control that maximizes the right-hand side of the H-J-B equation, say, $u = \bar{u}(V_x)$.

(c) Show that the optimal value function

$$V(\delta, t, x_t, T, x_T) \stackrel{\text{def}}{=} [T-t] \ln\left[\frac{x_t e^{-\delta t} - x_T e^{-\delta T}}{T-t}\right] + \frac{\delta}{2}[T^2 - t^2]$$

satisfies the H-J-B equation.

(d) Derive the optimal value function in part (c) using the necessary (and sufficient) conditions of optimal control theory.

19.5 Consider the optimal control problem

$$\max_{u(\cdot)} \int_0^T \ln u(s) e^{-rs}\, ds$$

$$\text{s.t.} \quad \dot{x}(s) = \delta x(s) - u(s),\ x(0) = x_0,\ x(T) = x_T,$$

where $r > 0$.

(a) Define the optimal value function $V(\cdot)$ and write down the H-J-B equation for the control problem for a given base period $t \in [0, +\infty)$ and any admissible state in the base period x_t.

(b) Derive the value of the control that maximizes the right-hand side of the H-J-B equation, say, $u = \bar{u}(V_x)$.

(c) Show that the optimal value function

$$V(\delta, r, t, x_t, T, x_T) \stackrel{\text{def}}{=} \frac{e^{-rt} - e^{-rT}}{r} \ln\left[\frac{re^{-\delta(t+T)}[x_T e^{\delta t} - x_t e^{\delta T}]}{e^{-rT} - e^{-rt}}\right] + \frac{\delta - r}{r}[te^{-rt} - Te^{-rt}] + r^2[\delta - r][e^{-rt} - e^{-rt}]$$

satisfies the H-J-B equation.

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19.6 Prove Theorem 19.4.

19.7 Consider the following discounted infinite-horizon optimal control problem:

$$\max_{u(\cdot)} \int_0^{+\infty} e^{-s}\sqrt{u(s)}\,ds$$

$$\text{s.t.}\quad \dot{x}(s) = x(s) - u(s),\ x(0) = x_0 > 0.$$

(a) Define the current value optimal value function $V(\cdot)$ and write down the H-J-B equation for the control problem for a given base period $t \in [0, +\infty)$ and any admissible state in the base period x_t.

(b) Derive the value of the control that maximizes the right-hand side of the H-J-B equation, say, $u = \bar{u}(V_x)$.

(c) Show that the current value optimal value function

$$V(\alpha, t, x_t) \stackrel{\text{def}}{=} \alpha x_t + \frac{1}{4\alpha}$$

satisfies the H-J-B equation for $\alpha \in \Re_{++}$.

(d) Derive the optimal value of the closed-loop control.

(e) Derive the current value optimal value function $V(\cdot)$ by examining the H-J-B equation and making an educated guess about its functional form. Show that your guess leads you to the same function as in part (c).

FURTHER READING

Bellman (1957) is the seminal reference on dynamic programming, and may still be consulted with positive net benefits. The basic approach followed in this chapter in deriving the H-J-B equation of Theorem 19.1 may be found in various guises in Nemhauser (1966), Barnett (1975), Bryson and Ho (1975), Kamien and Schwartz (1991), and Léonard and Van Long (1992). Dockner et al. (2000) present the H-J-B equation as a sufficient condition of optimality. Hadley and Kemp (1971, Chapter 4) and Leitmann (1981, Chapter 16) present excellent discussions of the so-called synthesis of an optimal feedback control, that is, the problem of converting an optimal open-loop control into an optimal closed-loop (or feedback) control.

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