FIVE

Linear Optimal Control Problems

We now turn to the examination of optimal control problems that are linear in the control variables. A prominent feature of this class of problems is that the optimal control often turns out to be a piecewise continuous function of time. Recall that in Chapter 4, we defined an admissible pair of curves $(\mathbf{x}(t), \mathbf{u}(t))$ by allowing the control vector to be a piecewise continuous function of time and the state vector to be a piecewise smooth function of time. Though we allowed for this possibility in the theorems of Chapter 4, we did not solve or confront an optimal control problem whose solution exhibited these properties. You may recall, however, that in Example 4.6, where we solved the ubiquitous inventory accumulation problem subject to a nonnegativity constraint on the production rate, the optimal production rate was a continuous but not a differentiable function of time, that is, it was a piecewise smooth function of time. There are two reasons why the optimal production rate turned out to be a piecewise smooth function of time, namely, (i) the nonnegativity constraint on the production rate and (ii) the assumption of a "long" production period. One important lesson from this example, therefore, is that once an inequality constraint is imposed on the control variable, the differentiability of an optimal control function with respect to time may not hold.

Several of the optimal controls examined in this chapter are even less smooth with respect to time than the optimal production rate of Example 4.6. Generally speaking, the structural feature of an optimal control problem that is primarily responsible for this feature is its linearity in the control variables. In such linear control problems, the optimal control often, but not always, turns out to be at one boundary of the control region for some finite period of time and then it "bangs" into the other end of the control region for the remainder of the time. Optimal controls with such a characteristic are known as *bang-bang controls*. Such jump discontinuities are not just mathematical curiosities, for we will show that bang-bang controls arise quite naturally in linear optimal control problems with nontrivial economic content. It is important to note, however, that linearity in the control variable is not sufficient for a bang-bang control to be optimal. This observation follows from Example 4.3, for that problem was linear in the control variable but the optimal control turned out to be a continuous function of time (a constant in this example).

So as to avoid giving the (false) impression that all linear optimal control problems yield a bang-bang optimal control, we present an example in which the coefficient on the control variable in the Hamiltonian vanishes for a finite interval of time. In such instances, the optimal control is found in an atypical way from the necessary conditions and is referred to as a *singular control*. Accordingly, the corresponding solution of the linear control problem is known as the *singular solution*. As we shall demonstrate, the basic behavior of a singular control is essentially the opposite of a bang-bang control.

In view of the fact that there is little in the way of general theorems for dealing with the special nature of linear control problems, we take the tact of demonstrating the way to approach them via four examples. We pause after the third example, however, and make some general remarks that set the stage for the final example in which a singular control is optimal. The first example is purely mathematical and so prepares the way for the next three economic examples. This way, one can get the technicalities of the approach down first, and then hone one's skills and build economic intuition on the latter examples.

Example 5.1: Consider the following linear optimal control problem:

$$\max_{u(\cdot),x_2} \int_0^2 [2x(t) - 3u(t)]\,dt$$

$$\text{s.t.} \quad \dot{x}(t) = x(t) + u(t),\ x(0) = 4,\ x(2) = x_2,$$

$$u(t) \in U \overset{\text{def}}{=} \{u(\cdot) : 0 \le u(t) \le 2\}.$$

Because we have a closed and bounded, that is, compact, control set U, we use the general set of necessary conditions to solve the control problem, that is to say, Theorem 4.2. First, define the Hamiltonian as $H(x,u,\lambda) \overset{\text{def}}{=} 2x - 3u + \lambda[x+u] = [2+\lambda]x + [\lambda - 3]u$. Noting that the terminal value of the state variable is to be optimally chosen, the necessary conditions are

$$\max_{u\in[0,2]} H(x,u,\lambda) \overset{\text{def}}{=} [2+\lambda]x + [\lambda-3]u,$$

$$\dot{\lambda} = -H_x(x,u,\lambda) = -2 - \lambda,\ \lambda(2) = 0,$$

$$\dot{x} = H_\lambda(x,u,\lambda) = x + u,\ x(0) = 4.$$

Rewriting the costate equation as $\dot{\lambda} + \lambda = -2$, the integrating factor is seen to be e^t. Upon multiplying the costate equation through by e^t, we have that $\frac{d}{dt}[\lambda e^t] = -2e^t$. Integrating this equation yields $\lambda(t) = ce^{-t} - 2$ as the general solution, where c is a constant of integration. Applying the transversality condition $\lambda(2) = 0$ to the general solution gives $c = 2e^2$, so the specific path for the costate variable is given by

$$\lambda(t) = 2[e^{2-t} - 1].$$

Since $\dot{\lambda}(t) = -2e^{2-t} < 0\,\forall\, t \in [0, 2]$ and $\lambda(2) = 0$, it follows that $\lambda(t) > 0\,\forall\, t \in [0, 2)$.

Inspection of the Hamiltonian reveals that it is linear in the control variable u. Moreover, because the control set U is closed, setting $\partial H/\partial u = 0$ is not in general the correct necessary condition for this problem. Given that $H(\cdot)$ is linear in u, if the coefficient on u is positive, that is, $[\lambda - 3] > 0$, then in order to maximize $H(\cdot)$ with respect to u, we must choose u to be as large as possible, subject to, of course, u being in the control set U, which in our case amounts to setting $u = 2$. Likewise, if the coefficient on u is negative, that is, $[\lambda - 3] < 0$, then we choose u to be as small as possible subject to being in the control set, which amounts to setting $u = 0$. If $\lambda - 3 = 0$ at an instant or over an interval, then the choice of the control variable will not affect the value of the Hamiltonian. As a result, we are permitted to choose any admissible value of the control variable. In summary, we have shown that the optimal control must satisfy

$$v(t) = \begin{cases} 2 & \text{if } \lambda(t) > 3 \\ \in [0, 2] & \text{if } \lambda(t) = 3 \\ 0 & \text{if } \lambda(t) < 3. \end{cases} \tag{1}$$

This is, in effect, a *decision rule* for the choice of the optimal control variable, for it dictates the optimal value of the control variable at each date of the planning horizon conditional on the value of the costate variable.

The decision rule in Eq. (1) shows that in order to pin down the exact nature of the time path of the control variable, we must first determine some specific properties of the costate variable. To that end, recall that $\dot{\lambda}(t) = -2e^{2-t} < 0\,\forall\, t \in [0, 2]$, so that if $\lambda(t) = 3$, it occurs only for an instant. Also note that $\lambda(0) = 2[e^2 - 1] \approx 12.778 > 3$ and $\lambda(2) = 0$. Thus $\lambda(t)$ is greater than three for some initial period of time and then it falls to values less than three but greater than or equal to zero for the remaining period of time. Because $\dot{\lambda}(t) < 0\,\forall\, t \in [0, 2]$, there exists a unique value of t, say, $t = \tau$, such that $\lambda(\tau) = 3$. In other words, the equation $\lambda(\tau) = 3$ implicitly defines the *switching time* τ for when the control jumps from its maximum value of two to its minimum value of zero. Hence $\lambda(\tau) = 2[e^{2-\tau} - 1] = 3$ can be solved for τ to yield $\tau = 2 - \ln \frac{5}{2} \approx 1.096$. Using this information, we see that the time path of the control variable, that is, its *open-loop* form, that satisfies the necessary conditions can be precisely specified as

$$v(t) = \begin{cases} 2\,\forall\, t \in \left[0, 2 - \ln \frac{5}{2}\right] \\ 0\,\forall\, t \in \left(2 - \ln \frac{5}{2}, 2\right]. \end{cases}$$

Because $\lambda(t) = 3$ only for the instant $t = \tau$, the control variable fails to enter the Hamiltonian only at this instant too. Hence any admissible choice of the control variable is optimal at $t = \tau$. Notice that we chose $v(\tau) = 2$ so that the control variable is continuous over the closed interval $[0, 2 - \ln \frac{5}{2}]$. We thus have our first instance in which the control variable is a piecewise continuous function of time, that is to say, we have a bang-bang control. Note that this has occurred in an optimal control problem that is linear in the control variable.

To finish up determining the solution of the necessary conditions, we solve for the time path of the state variable. Given that the control is a piecewise continuous function of time, we solve the state equation separately for each interval in which the control variable is a continuous function of time. Thus the differential equation we intend to solve is given by

$$\dot{x} = \begin{cases} x + 2 \; \forall\, t \in \left[0, 2 - \ln \frac{5}{2}\right] \\ x \qquad \forall\, t \in \left(2 - \ln \frac{5}{2}, 2\right]. \end{cases}$$

The integrating factor for either state equation is e^{-t}, as you should readily discern by now. The general solution, which you should also verify, is given by

$$z(t) = \begin{cases} z_1(t) = c_1 e^t - 2, \; t \in \left[0, 2 - \ln \frac{5}{2}\right] \\ z_2(t) = c_2 e^t, \qquad t \in \left(2 - \ln \frac{5}{2}, 2\right], \end{cases}$$

where c_1 and c_2 are constants of integration. Because $x(0) = 4$ is the initial condition, it can be applied to $z_1(t)$, since this solution encompasses the initial time. Applying the initial condition $x(0) = 4$ to $z_1(t)$ yields $c_1 = 6$. The definite solution of the state equation for the first time interval is therefore

$$z_1(t) = 6e^t - 2, \; t \in \left[0, 2 - \ln \frac{5}{2}\right].$$

Notice that we do not have a terminal requirement for the state variable, as $x(2) = x_2$ is a choice variable in the problem. It therefore appears we can't definitize the constant of integration c_2 from the second time interval. Recall, however, that the state variable path is required to be a continuous function of t by the very definition of admissibility. To ensure such continuity at the switching time $t = \tau$, we equate the solution of the state variable from the first time interval at $t = \tau$, namely, $z_1(\tau)$, to the solution for the state variable from the second time interval at $t = \tau$, namely, $z_2(\tau)$. That is, we set $z_1(\tau) = z_2(\tau)$, thereby yielding $6e^{\tau} - 2 = c_2 e^{\tau}$, and solve it for c_2 to get $c_2 = 6 - 2e^{-\tau} = 6 - 2e^{-[2 - \ln \frac{5}{2}]} \approx 5.324$. Thus the specific solution to the state equation is

$$z(t) = \begin{cases} z_1(t) = 6e^t - 2, \qquad\qquad t \in \left[0, 2 - \ln \frac{5}{2}\right] \\ z_2(t) = \left[6 - 2e^{-\left[2 - \ln \frac{5}{2}\right]}\right] e^t, \; t \in \left(2 - \ln \frac{5}{2}, 2\right]. \end{cases}$$

Note that although the state variable is a continuous function of t, its time derivative is only a piecewise continuous function of t because of the piecewise continuity of the control variable with respect to t.

In order to wrap up this example, we now demonstrate that the solution to the necessary conditions is in fact an optimal solution to the optimal control problem. First, observe that the control set U is convex. Next, take note of the fact that the partial derivative of the integrand function $f(t, x, u) \stackrel{\text{def}}{=} 2x - 3u$ and transition function $g(t, x, u) \stackrel{\text{def}}{=} x + u$ with respect to u are continuous, as is required by Theorem 4.3. Hence, in order to prove that the above solution of the necessary conditions is a solution of the control problem, we are required to show that the Hamiltonian is a concave function of (x, u). This is simple, however, for $H(\cdot)$ is linear in (x, u) and hence concave in (x, u).

Let us now turn to the first economic problem that exhibits a piecewise continuous control function as its optimal solution. It is a simple model of optimal savings for an economy.

Example 5.2: Imagine a central planner who wants to choose a savings rate $s(t) \in [0, 1]$ for an economy so as to maximize its aggregate utility of consumption over a finite period of time, say, $[0, T]$, where $T > 1$. Let $k(t)$ be the economy's capital stock at time t, and let the production function be linear in the capital stock, say, $y(t) = f(k(t)) \stackrel{\text{def}}{=} k(t)$. Consumption is the fraction of output not saved, that is, $c(t) = [1 - s(t)]k(t)$. Assuming no depreciation, the investment rate $\dot{k}(t)$ is equal to the fraction of output saved, namely, $\dot{k}(t) = s(t)k(t)$. If the instantaneous utility function $U(\cdot)$ is linear in consumption, say, $U(c(t)) \stackrel{\text{def}}{=} c(t)$, then the optimal control problem to be solved by the central planner is given by

$$\max_{s(\cdot), k_T} \int_0^T [1 - s(t)]k(t)\, dt$$

$$\text{s.t.} \quad \dot{k}(t) = s(t)k(t),\ k(0) = k_0,\ k(T) = k_T,$$

$$s(t) \in U \stackrel{\text{def}}{=} \{s(\cdot) : 0 \le s(t) \le 1\},\ k(t) \ge 0,$$

where $k_0 > 0$ is the economy's initial stock of capital.

One new feature of the problem is the *explicit* appearance of the nonnegativity constraint on the stock of capital, that is, a state constraint. Inasmuch as we do not have theorems to handle state constraints, we will deal with it directly by showing that it is automatically satisfied for the control problem and can thus be ignored. To begin, note that the integrating factor for the state equation $\dot{k} - s(t)k = 0$ is $\mu(t) \stackrel{\text{def}}{=} \exp[-\int^t s(\tau) d\tau]$. After multiplying the state equation through by $\mu(t)$, the differential equation to be solved is given by $\frac{d}{dt}[\mu(t)k] = 0$. Integrating and rearranging yields $k(t) = c_1 \exp[\int_0^t s(\tau)\, d\tau]$, where c_1 is a constant of integration and where we have taken the lower value of the limit of integration to be zero, without loss of generality. Using the initial condition $k(0) = k_0$ yields $c_1 = k_0$. Hence the specific solution of the state equation for any admissible time path of the savings rate is given by $k(t; k_0) = k_0 \exp[\int_0^t s(\tau)\, d\tau]$. Because $k_0 > 0$ and the function $\exp(\cdot)$ takes on only positive values, it follows that all admissible values of the capital stock are positive. Thus the nonnegativity constraint on the capital stock never binds, as was to be shown.

The Hamiltonian for the problem is defined as $H(k, s, \lambda) \stackrel{\text{def}}{=} k + [\lambda - 1]sk$, where λ is the costate variable and has the economic interpretation of the shadow value or marginal value of the capital stock. By Theorem 4.2, the necessary condition for the choice of the savings rate is

$$\max_{s \in [0,1]} H(k, s, \lambda) \stackrel{\text{def}}{=} k + [\lambda - 1]\, sk.$$

Insofar as $k(t) > 0 \,\forall\, t \in [0, T]$, as demonstrated above, this immediately yields the decision rule that the optimal savings rate must obey:

$$s(t) = \begin{cases} 1 & \text{if } \lambda(t) > 1 \\ \in [0, 1] & \text{if } \lambda(t) = 1 \\ 0 & \text{if } \lambda(t) < 1. \end{cases} \tag{2}$$

Because $U(c) \stackrel{\text{def}}{=} c$, $U'(c) = 1$, so the decision rule asserts that if the marginal value of the capital stock is greater than the marginal utility of consuming it, then the planner should save all the output for future consumption, which is intuitive. Similarly, if the marginal value of the capital stock is less than the marginal utility of consuming it, then the planner should not save any of the output, thereby implying that the economy should consume it all.

The next step is to determine some basic properties of the shadow value of the capital stock because Eq. (2) shows that once it is determined, so too is the optimal savings rate. To that end, the costate equation and transversality condition are given by

$$\dot{\lambda} = -H_k(k, s, \lambda) = -1 - [\lambda - 1]s, \;\; \lambda(T) = 0. \tag{3}$$

Using the decision rule in Eq. (2), we see that if $\lambda(t) > 1$, then $s(t) = 1$, thereby implying that $\dot{\lambda}(t) < 0$ from Eq. (3). Similarly, if $\lambda(t) = 1$, then $s(t) = [0, 1]$, and Eq. (3) again implies that $\dot{\lambda}(t) < 0$. Finally, if $\lambda(t) < 1$, then $s(t) = 0$, again implying from Eq. (3) that $\dot{\lambda}(t) < 0$. Thus, we have shown that $\dot{\lambda}(t) < 0 \,\forall\, t \in [0, T]$. Bringing together this fact with the transversality condition $\lambda(T) = 0$ yields the intuitive conclusion that $\lambda(t) > 0 \,\forall\, t \in [0, T)$. That is, the shadow value of the capital stock is declining over the entire planning horizon and its value is positive in all periods except the last.

Looking at the decision rule in Eq. (2), we see that the key to determining the exact behavior of the optimal savings rate is determining how large the shadow value of capital is relative to the marginal utility of consuming the capital, the latter being unity, as you may recall. We will show in this paragraph and the next that the qualitative behavior of the shadow value of the capital stock is given by

$$\lambda(t)\begin{cases} > 1 \ \forall\, t \in [0, t^*) \\ = 1 \text{ for } t = t^* \\ < 1 \ \forall\, t \in (t^*, T], \end{cases} \tag{4}$$

where $t^* \in [0, T]$ is the switching time. In order to do so, we employ a proof by contradiction. To that end, we assume that $\lambda(t) \leq 1\,\forall\, t \in [0, T]$. Using the decision rule in Eq. (2) and setting $s(t) = 0$ when $\lambda(t) = 1$ (which can occur only at $t = 0$ because $\dot{\lambda}(t) < 0\,\forall\, t \in [0, T]$) implies that the costate equation reduces to $\dot{\lambda} = -1$, which readily integrates to $\lambda(t) = c_2 - t$, where c_2 is a constant of integration. Using the transversality condition $\lambda(T) = 0$ yields $c_2 = T$, thereby implying that the specific solution to the costate equation is $\lambda(t; T) = T - t$. Recalling that $T > 1$, we see that $\lambda(0; T) = T > 1$, thereby contradicting our initial assumption that $\lambda(t) \leq 1\,\forall\, t \in [0, T]$. We may therefore conclude that $\lambda(t) \leq 1\,\forall\, t \in [0, T]$ is *not* the solution to the costate equation and transversality condition. Combining this fact with the previous result that $\dot{\lambda}(t) < 0\,\forall\, t \in [0, T]$ and the transversality condition $\lambda(T) = 0$ implies that (i) there exists a unique switching time t^* at which $\lambda(t^*) = 1$, (ii) $\lambda(t) > 1\,\forall\, t \in [0, t^*)$, and (iii) $\lambda(t) \in [0, 1)\,\forall\, t \in (t^*, T]$. This establishes the veracity of Eq. (4).

To determine the exact value of the switching time t^*, we have to solve the equation $\lambda(t^*) = 1$ for t^*. In order to do so, we must first solve the costate equation. Using the decision rule (2) and Eq. (4), the costate equation (3) takes the form

$$\dot{\lambda} = \begin{cases} -\lambda \ \ \forall\, t \in [0, t^*) \\ -1 \ \ \forall\, t \in [t^*, T], \end{cases}$$

where we have chosen to set $s(t^*) = 1$, as we are free to do. Integrating this differential equation yields the general solution

$$\lambda(t) = \begin{cases} \lambda_1(t) = a_1 e^{-t} \ \ \forall\, t \in [0, t^*) \\ \lambda_2(t) = a_2 - t \ \forall\, t \in [t^*, T], \end{cases}$$

where a_1 and a_2 are constants of integration. The transversality condition $\lambda(T) = 0$ can be applied to the solution for the interval $[t^*, T]$, to wit, $\lambda_2(t)$, and implies that $a_2 = T$. Using this result and the fact that $\lambda(t^*) = \lambda_2(t^*) = 1$ gives $T - t^* = 1$, or $t^* = T - 1 > 0$. To find the constant of integration a_1, recall the requirement from Theorem 4.2 that the costate function is a piecewise smooth and hence continuous function of t. This means that $\lambda_1(t^*) = 1 = \lambda_2(t^*)$ must hold, which gives $\lambda_1(t^*) = a_1 e^{-t^*} = 1$ or $a_1 = e^{t^*} = e^{T-1}$. Hence the specific solution of the costate equation is

$$\lambda(t; T) = \begin{cases} \lambda_1(t; T) = e^{T-t-1} \ \forall\, t \in [0, T - 1) \\ \lambda_2(t; T) = T - t \ \ \forall\, t \in [T - 1, T]. \end{cases} \tag{5}$$

Notice that at $t = T - 1$, the derivatives of each portion of the solution with respect to t are equal to minus unity. That is, $\dot{\lambda}_1(T-1;T) = -1 = \dot{\lambda}_2(T-1;T)$, thereby demonstrating that $\lambda(\cdot)$ is $C^{(1)}$ with respect to t, which is more smoothness, in general, than Theorem 4.2 imposes on the costate variables.

Using the information in Eq. (4) or Eq. (5), the decision rule (2) governing the savings rate can be restated as

$$s^*(t) = \begin{cases} 1\,\forall\, t \in [0, T-1) \\ 0\,\forall\, t \in [T-1, \mathrm{T}]. \end{cases}$$

This control asserts that the central planner prescribes the maximum savings rate for the economy up until just before the switching time $t^* = T - 1$, and then immediately changes to the maximum consumption rate from the switching date until the end of the planning horizon. That is, all the output produced by the economy is initially saved. Once the switching date is reached, however, none of the output is saved. Instead, output should be consumed at the maximum possible rate from the switching date until the horizon comes to a close. We leave the solution of the state equation for a mental exercise.

The next example concerns the optimal rate at which to clean up a stock of hazardous waste. An extension of it appears in the mental exercises of a later chapter.

Example 5.3: A spill of a toxic substance has occurred at time $t = 0$ (the present) in the known amount of $x(0) = x_0 > 0$, where $x(t)$ is the stock of the toxic substance left in the environment at time t. The federal government has contracted with your company to reduce the size of the toxic waste from its initial size of x_0 to x_T by the time $t = T > 0$, where, of course, $x_0 > x_T$, so that the waste stock is smaller at $t = T$ than at $t = 0$. As head economist of the company, you are charged with determining the cleanup rate $u(t)$ that minimizes the present discounted cost of cleaning up the fixed amount $x_0 - x_T$ of the toxic substance over the fixed horizon $[0, T]$. The cleanup technology is linear in the cleanup rate, say $cu(t)$, where $c > 0$. The cleanup rate is bounded below by zero and above by a fixed finite rate $\bar{u} > 0$, where $\bar{u} > [x_0 - x_T]/T$. Given that $x(t)$ is defined as the toxic stock left in the environment at time t, it follows that

$$x(t) \stackrel{\text{def}}{=} x_0 - \int_0^t u(s)\,ds.$$

Hence, by Leibniz's rule,

$$\dot{x}(t) = -u(t),$$

$$x(0) = x_0,\ x(T) = x_T$$

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are the state equation and boundary conditions for the problem. The complete statement of the optimal control problem is given by

$$C(\boldsymbol{\beta}) \stackrel{\text{def}}{=} \min_{u(\cdot)} \int_0^T cu(t)e^{-rt}\,dt$$

$$\text{s.t.}\quad \dot{x}(t) = -u(t),\ x(0) = x_0,\ x(T) = x_T,$$

$$u(t) \in U \stackrel{\text{def}}{=} \{u(\cdot) : 0 \le u(t) \le \bar{u}, \bar{u} > [x_0 - x_T]/T\},$$

where $r > 0$ is the discount rate and $\boldsymbol{\beta} \stackrel{\text{def}}{=} (c, r, \bar{u}, x_0, T, x_T)$. Note that in contrast to the first two examples in this chapter, this is a fixed endpoints optimal control problem.

Let's begin the analysis of this model by providing an economic interpretation of the inequality $\bar{u} > [x_0 - x_T]/T$. The first thing to notice about it is that $\bar{u} > [x_0 - x_T]/T$ if and only if $\bar{u}T > [x_0 - x_T]$. The latter expression is easily interpreted. It asserts that if the firm cleans up at the maximum rate for the entire time planning horizon, then it would clean up more waste than is required by the federal government contract. Recalling that the firm is required to clean up the amount $x_0 - x_T$ (because of fixed endpoints) by time T, we reach an important conclusion, namely, that cleaning up at the maximum rate for the entire period is not admissible, and therefore not optimal; that is, $u(t) = \bar{u}\ \forall\, t \in [0, T]$ is not an admissible control and therefore not an optimal control.

The Hamiltonian for this problem is defined as $H(t, x, u, \lambda) \stackrel{\text{def}}{=} cue^{-rt} - \lambda u = [ce^{-rt} - \lambda]u$. According to Theorem 4.2, the necessary conditions are

$$\min_{u \in [0,\bar{u}]} H(t, x, u, \lambda) \stackrel{\text{def}}{=} [ce^{-rt} - \lambda]u, \tag{6}$$

$$\dot{\lambda} = -H_x(t, x, u, \lambda) = 0, \tag{7}$$

$$\dot{x} = H_\lambda(t, x, u, \lambda) = -u,\ x(0) = x_0,\ x(T) = x_T, \tag{8}$$

where λ is the *present value* shadow cost of a unit of the toxic waste at any time $t \in [0, T]$. The adjective *present value* is necessary because of the presence of the discount factor e^{-rt} in the integrand. In addition, λ is a shadow cost (rather than a shadow value) in view of the fact that the objective functional of the firm is the present discounted value of cleanup costs. Because of the fixed endpoint $x(T) = x_T$, a larger initial stock of toxic waste requires more cleanup. Moreover, cleaning up is a costly activity. Hence the present value shadow cost of a unit of toxic waste at any time $t \in [0, T]$ must be positive in an optimal plan.

The decision rule governing the optimal choice of the cleanup rate follows from inspection of Eq. (6), and is given by

$$u(t) = \begin{cases} 0 & \text{if } ce^{-rt} > \lambda \\ \in [0, \bar{u}] & \text{if } ce^{-rt} = \lambda \\ \bar{u} & \text{if } ce^{-rt} < \lambda. \end{cases} \tag{9}$$

The economic interpretation of the decision rule is straightforward. It asserts that if the present value marginal cost of cleaning up is greater than the present value shadow cost of the toxic stock, then no cleanup should take place. Likewise, if the present value marginal cost of cleaning up is less than the present value shadow cost of the toxic stock, then cleanup should take place at the maximum rate. As is typical in linear control problems, we must determine the behavior of the costate variable before the solution of the control variable is fully determined.

Before finding the solution of the costate equation, let's rule out another solution for the cleanup rate as being optimal. To that end, note that if $u(t) = 0\,\forall\, t \in [0, T]$, then $\dot{x} = 0\,\forall\, t \in [0, T]$ from the state equation (8). Using the initial condition $x(0) = x_0$ yields $x(t; x_0) = x_0\,\forall\, t \in [0, T]$ as the specific solution of the state equation. This solution, however, violates the terminal boundary requirement, scilicet, $x(T) = x_T$, since $x_0 > x_T$. Hence $u(t) = 0\,\forall\, t \in [0, T]$ is not an admissible control and therefore not an optimal control.

From Eq. (7), we see that $\dot{\lambda} = 0$, so that the general solution is $\lambda(t) = \bar{\lambda}$, where $\bar{\lambda}$ is a constant. Because the control problem has both endpoints fixed, it appears that we have no way of determining the value of the constant $\bar{\lambda}$ and hence the present value shadow cost of the toxic waste. This is not true, as we shall see shortly. What is more important at this juncture is that we can determine the optimal cleanup rate fully from the decision rule (9), and as we will now show, this does not require exact knowledge of $\bar{\lambda}$. To demonstrate this, observe that if $\bar{\lambda} < ce^{-rt}\,\forall\, t \in [0, T]$, then $u(t) = 0\,\forall\, t \in [0, T]$ from Eq. (9). But as we demonstrated above, $u(t) = 0\,\forall\, t \in [0, T]$ is not admissible; hence $\bar{\lambda} < ce^{-rt}\,\forall\, t \in [0, T]$ *cannot* hold. Similarly, if $\bar{\lambda} > ce^{-rt}\,\forall\, t \in [0, T]$, then $u(t) = \bar{u}\,\forall\, t \in [0, T]$ from Eq. (9). But $u(t) = \bar{u}\,\forall\, t \in [0, T]$ is not admissible, as shown above; therefore $\bar{\lambda} > ce^{-rt}\,\forall\, t \in [0, T]$ *cannot* hold either. Because $\lambda(t) = \bar{\lambda}$ is constant and $\frac{d}{dt}[ce^{-rt}] = -rce^{-rt} < 0$, when combined with the above two results, these latter two imply that (i) $\bar{\lambda} < ce^{-rt}$ for some finite period of time at the beginning of the planning horizon, (ii) $\bar{\lambda} = ce^{-rt}$ only for an instant, say, at $t = \tau$, the switching time, and (iii) $\bar{\lambda} > ce^{-rt}$ for some finite period of time at the end of the planning horizon. This means that our decision rule (9) now takes the form

$$v(t; \boldsymbol{\beta}) = \begin{cases} 0\,\forall\, t \in [0, \tau] \\ \bar{u}\,\forall\, t \in (\tau, \mathrm{T}], \end{cases} \tag{10}$$

thereby implying that the optimal cleanup policy is bang-bang, with no cleanup in the first time interval and the maximum rate of cleanup in the second.

Given the decision rule in Eq. (10), the state equation (8) takes the form

$$\dot{x} = \begin{cases} 0 & \forall\, t \in [0, \tau] \\ -\bar{u} & \forall\, t \in (\tau, \mathrm{T}]. \end{cases}$$

Integrating each portion of the differential equation, and using the initial condition $x(0) = x_0$ for the first and the terminal condition $x(T) = x_T$ for the second, yields

$$z(t; \boldsymbol{\beta}) = \begin{cases} x_0 & \forall\, t \in [0, \tau] \\ x_T + \bar{u}[T - t] & \forall\, t \in (\tau, \mathrm{T}] \end{cases} \tag{11}$$

as the time path of the toxic waste stock.

We still have yet to determine the value of $\bar{\lambda}$ and the switching time τ. The value of τ can be determined from the continuity of the state function $z(\cdot)$ with respect to t. Just as we did in the two previous examples, we equate the values of the two portions of the state variable solution at the switching time to get the equation $x_0 = x_T + \bar{u}[T - \tau]$, and then solve it for the switching time to find that

$$\tau = \tau^*(\boldsymbol{\beta}) \stackrel{\text{def}}{=} T - [x_0 - x_T]/\bar{u}. \tag{12}$$

As a check on the plausibility of this value, we want to ensure that $\tau^*(\boldsymbol{\beta}) \in (0, T)$, so that the switch between no cleanup and maximum cleanup actually takes place within the planning period. That $\tau^*(\boldsymbol{\beta}) \in (0, T)$ follows from our assumption $\bar{u} > [x_0 - x_T]/T$. To see this, simply rearrange it to read $T - [x_0 - x_T]/\bar{u} > 0$ and compare it to the value of the switching time given in Eq. (12).

Next, we solve for the present value shadow cost of the toxic waste by recalling that $\lambda(t) = \bar{\lambda}$, with $\bar{\lambda}$ a constant, and that $\bar{\lambda} = ce^{-rt}$ at $t = \tau^*(\boldsymbol{\beta})$, the switching time. These observations yield the constant value of the present value shadow cost of the toxic waste, to wit,

$$\lambda(t; \boldsymbol{\beta}) = ce^{-r[T - [x_0 - x_T]/\bar{u}]}, \tag{13}$$

which is positive, as argued after Eq. (8).

It is an easy matter to show that the solution of the necessary conditions given in Eqs. (10) through (13) solves the optimal control problem. As in Example 5.1, we observe that the control set U is convex, and that the partial derivatives with respect to u of the integrand function $f(\cdot)$ with values $f(t, x, u) \stackrel{\text{def}}{=} cue^{-rt}$, and transition function $g(\cdot)$ with values $g(t, x, u) \stackrel{\text{def}}{=} -u$, are continuous, as is required by Theorem 4.3. Because the Hamiltonian $H(t, x, u, \lambda) \stackrel{\text{def}}{=} cue^{-rt} - \lambda u$ is independent of x and linear in u, it is concave in (x, u); hence by Theorem 4.3, the solution of the necessary conditions given in Eqs. (10) through (13) is a solution of the optimal control problem. Moreover, seeing as the solution given in Eqs. (10) through (13) is the only solution of the necessary conditions, save for the fact that at the switching time the control variable can take on any admissible value, this solution is the unique solution of the control problem.

With the unique optimal solution now in hand, we turn to the construction of the optimal value function $C(\cdot)$, the firm's minimum present discounted value cleanup cost function. By definition, the value of $C(\cdot)$ is given by

$$C(\boldsymbol{\beta}) \stackrel{\text{def}}{=} \int_0^T cv(t;\bar{u})\,e^{-rt}\,dt = \int_{\tau^*(\boldsymbol{\beta})}^T c\bar{u}e^{-rt}\,dt = \frac{c\bar{u}}{r}\,[e^{-r[T-[x_0-x_T]/\bar{u}]} - e^{-rt}] > 0, \tag{14}$$

where we have used Eqs. (10) and (12). Differentiating Eq. (14) with respect to x_0 yields

$$\frac{\partial C(\boldsymbol{\beta})}{\partial x_0} = ce^{-r[T-[x_0-x_T]/\bar{u}]} = \lambda(t;\boldsymbol{\beta}), \tag{15}$$

thereby confirming how the costate variable was defined in the proof of the maximum principle by way of dynamic programming in Chapter 4.

In closing out this example, let's consider a few comparative dynamics properties of the solution, and leave the remainder for a mental exercise. One interesting comparative dynamics result is that the solution of the problem is virtually unaffected by the instantaneous marginal cost of cleanup c. As inspection of Eqs. (10) through (14) will confirm, an increase in c raises the present value shadow cost of the toxic waste and the minimum present discounted value of cleanup costs, but leaves all other variables unaffected. Thus, even though the instantaneous marginal cost of cleanup is higher, neither the switching time nor the cleanup rate are affected, a curious result at first glance. How can it be that an increase in the instantaneous marginal cost of cleanup does not affect the optimal cleanup rate? Inspection of the statement of the optimal control problem reveals why. Because c is a positive time invariant constant, it can be factored out of the integrand, thereby demonstrating that it has no effect on the optimal solution for the cleanup rate, switching time, or the stock of toxic waste. Seeing as c affects the optimal value function, however, it must also affect the present value shadow cost of the toxic waste stock.

For the next comparative dynamics exercise, let's investigate the effect of an increase in the maximum cleanup rate $\bar{u}$ on the optimal control problem. One may think of an increase in $\bar{u}$ as resulting from an improvement in the cleanup technology. Differentiating Eqs. (10) through (14) with respect to $\bar{u}$ yields

$$\frac{\partial v(t;\boldsymbol{\beta})}{\partial \bar{u}} = \begin{cases} 0 \; \forall\, t \in [0, \tau^*(\boldsymbol{\beta})] \\ 1 \; \forall\, t \in (\tau^*(\boldsymbol{\beta}), T], \end{cases}$$

$$\frac{\partial z(t;\boldsymbol{\beta})}{\partial \bar{u}} = \begin{cases} 0 & \forall\, t \in [0, \tau^*(\boldsymbol{\beta})] \\ [T-t] & \forall\, t \in (\tau^*(\boldsymbol{\beta}), T], \end{cases}$$

$$\frac{\partial \tau^*(\boldsymbol{\beta})}{\partial \bar{u}} = \frac{[x_0 - x_T]}{\bar{u}^2} > 0,$$

$$\frac{\partial\lambda(t;\boldsymbol{\beta})}{\partial\bar{u}} = -rce^{-r[T-[x_0-x_T]/\bar{u}]}\frac{[x_0-x_T]}{\bar{u}^2} < 0,$$

$$\frac{\partial C(\boldsymbol{\beta})}{\partial\bar{u}} = -\frac{c}{\bar{u}}e^{-r[T-[x_0-x_T]/\bar{u}]}[x_0-x_T] + \frac{c}{r}\left[e^{-r[T-[x_0-x_T]/\bar{u}]} - e^{-rt}\right] \gtrless 0.$$

This set of comparative dynamics results has an interesting economic interpretation. For example, an increase in the maximum cleanup rate permits the switching time to be delayed, that is, the switching time is pushed closer to the end of the planning horizon. This is not an unexpected occurrence in view of the fact that more waste can be cleaned up in a given period of time. In other words, the length of the period in which the firm cleans up is reduced when the maximum rate it can clean up is increased. The shorter period in which the cleanup takes place translates into a lower present value shadow cost of waste and an increase in the amount of waste in each period but the last in the cleanup period. The effect of the increase in the maximum cleanup rate on the firm's present discounted cleanup costs, however, is ambiguous. Two opposing factors are at work in this case, namely, (i) costs increase, ceteris paribus, because the maximum rate of cleanup is higher and cleaning up is a costly activity, and (ii) costs decrease, ceteris paribus, because the period in which cleanup takes place is shorter.

To finish up this example, let's consider the effects of an increase in the initial stock of the toxic waste. We thus differentiate Eqs. (10) through (14) with respect to x_0 to get

$$\frac{\partial v(t;\boldsymbol{\beta})}{\partial x_0} = 0\,\forall\, t \in [0,T],$$

$$\frac{\partial z(t;\boldsymbol{\beta})}{\partial x_0} = \begin{cases} 1\,\forall\, t \in [0,\tau^*(\boldsymbol{\beta})] \\ 0\,\forall\, t \in (\tau^*(\boldsymbol{\beta}),T], \end{cases}$$

$$\frac{\partial\tau^*(\boldsymbol{\beta})}{\partial x_0} = -\frac{1}{\bar{u}} < 0,$$

$$\frac{\partial^2 C(\boldsymbol{\beta})}{\partial x_0^2} = \frac{\partial\lambda(t;\boldsymbol{\beta})}{\partial x_0} = ce^{-r[T-[x_0-x_T]/\bar{u}]}\frac{r}{\bar{u}} > 0,$$

$$\frac{\partial C(\boldsymbol{\beta})}{\partial x_0} = ce^{-r[T-[x_0-x_T]/\bar{u}]} > 0,$$

where we used Eq. (15) in the next to last equation. These comparative dynamics results demonstrate that an increase in the initial stock of the toxic waste leaves the cleanup rate unaffected but increases the period in which the cleanup take place, that is, the switching time occurs earlier in the planning horizon. It is this latter effect that enables the firm to bring the larger initial stock down to its required level by the end of the planning horizon without increasing its rate of cleanup. This larger initial waste stock drives up both the firm's present value cost of cleanup and its present value shadow cost of the stock. Finally, observe that the last two equations show that the firm's value function $C(\cdot)$ is a strictly increasing and strictly convex function of the initial stock of toxic waste. We leave the remaining comparative dynamics results for a mental exercise.

Having fully worked three examples, we now pause to make a few general observations about linear optimal control problems. In doing so, we also pave the way for the final example in which a singular control is optimal. To begin, let us first write down a sufficiently general class of linear optimal control problems, to wit,

$$\max_{u(\cdot)} \int_{t_0}^{t_1} [f^1(t,x(t)) + f^2(t,x(t))\,u(t)]\,dt$$

$$\text{s.t.} \quad \dot{x}(t) = g^1(t,x(t)) + g^2(t,x(t))\,u(t),\ \ x(t_0) = x_0,\ \ x(t_1) = x_1, \tag{16}$$

$$u(t) \in U \overset{\text{def}}{=} \{u(\cdot) : \underline{u} \leq u(t) \leq \bar{u},\, \underline{u} < \bar{u}\}.$$

We assume that the integrand and transition functions are $C^{(1)}$ in what follows, and, as usual, let $(z(\cdot), v(\cdot))$ be the optimal pair of solution functions for problem (16) with corresponding costate function $\lambda(\cdot)$. Note that we could have alternatively taken the terminal value of the state variable to be a decision variable, seeing as this change has no essential bearing on the ensuing discussion.

The Hamiltonian for problem (16) is defined as

$$\begin{aligned} H(t,x,u,\lambda) &\overset{\text{def}}{=} f^1(t,x) + f^2(t,x)u + \lambda[g^1(t,x) + g^2(t,x)u] \\ &= f^1(t,x) + \lambda g^1(t,x) + [f^2(t,x) + \lambda g^2(t,x)]u, \end{aligned}$$

where λ is the costate variable. If we define $\sigma(\cdot)$ by the formula $\sigma(t,x,\lambda) \overset{\text{def}}{=} f^2(t,x) + \lambda g^2(t,x)$, then the Hamiltonian may be rewritten as

$$H(t,x,u,\lambda) \overset{\text{def}}{=} f^1(t,x) + \lambda g^1(t,x) + \sigma(t,x,\lambda)\,u. \tag{17}$$

The function $\sigma(\cdot)$ is known as the *switching function*. By Theorem 4.2, the necessary condition for the selection of the optimal control is $\max_{u \in U} H(t,x,u,\lambda)$, which, upon using Eq. (17), is seen to be equivalent to the decision rule

$$u(t) = \begin{cases} \underline{u} & \text{if } \sigma(t,x,\lambda) < 0 \\ \in [\underline{u}, \bar{u}] & \text{if } \sigma(t,x,\lambda) = 0 \\ \bar{u} & \text{if } \sigma(t,x,\lambda) > 0. \end{cases} \tag{18}$$

In view of the fact that the sign of $\sigma(\cdot)$ completely determines the value of the optimal control as well as its switch, if any, from its upper and lower bounds, the designation "switching function" is an apt label for $\sigma(\cdot)$.

Now recall that in Examples 5.1 through 5.3, the Hamiltonian was a linear function of the control variable, but that the coefficient on the control variable, that is, the value of the switching function, was equal to zero only for an instant. This latter conclusion was a result of the fact that $\sigma(\cdot)$ was a monotonically decreasing function of time in each example, as you will be asked to verify in a mental exercise. Consequently, in each of these examples, we were led to the conclusion that the optimal control was of the bang-bang variety. Not one of these examples exhibited a switching function that vanished for a subinterval of the planning horizon of positive length along the optimal solution. That is to say, Examples 5.1 through 5.3 did not possess the property that $\sigma(t, x, \lambda) \equiv 0 \,\forall\, t \in (t_a, t_b) \subset [t_0, t_1]$ along the optimal path. If in fact $\sigma(t, x, \lambda) \equiv 0 \,\forall\, t \in (t_a, t_b)$ along the optimal solution, then the Hamiltonian becomes independent of the control variable, and accordingly, the maximum principle does not determine the value of the optimal control by way of the typical route. Nonetheless, it is possible to manipulate the other necessary conditions to determine the value of the optimal control in such instances.

Definition 5.1: When $\sigma(t, x, \lambda) = 0$ for an interval of time, the resulting optimal solution triplet of functions, say, $(x^s(\cdot), v^s(\cdot), \lambda^s(\cdot))$, is said to be the *singular solution*.

Let's now proceed to demonstrate how, in principle, the singular solution can be determined. In view of the fact that $\sigma(t, z^s(t), \lambda^s(t)) \equiv 0 \,\forall\, t \in (t_a, t_b)$ by Definition 5.1, it follows that $\frac{d}{dt}\sigma(t, z^s(t), \lambda^s(t)) \equiv 0 \,\forall\, t \in (t_a, t_b)$ as well. Consequently, by the chain rule

$$\begin{aligned}\frac{d}{dt}\sigma(t, z^s(t), \lambda^s(t)) = {} & f_t^2(t, z^s(t)) + \lambda^s(t) g_t^2(t, z^s(t)) \\ & + \left[f_x^2(t, z^s(t)) + \lambda^s(t) g_x^2(t, z^s(t))\right] \dot{z}^s(t) \\ & + g^2(t, z^s(t))\dot{\lambda}^s(t) \equiv 0\end{aligned}$$

for all $t \in (t_a, t_b)$. Upon using $\sigma(t, z^s(t), \lambda^s(t)) \equiv 0 \,\forall\, t \in (t_a, t_b)$ and the remaining necessary conditions from Theorem 4.2, and furthermore assuming that $g^2(t, z^s(t)) \neq 0 \,\forall\, t \in (t_a, t_b)$, it can be demonstrated that the above differential equation is equivalent to

$$\begin{aligned}&\left[f_t^2(t, z^s(t)) - \frac{f^2(t, z^s(t))}{g^2(t, z^s(t))} g_t^2(t, z^s(t))\right] \\ &\quad + \left[f_x^2(t, z^s(t)) - \frac{f^2(t, z^s(t))}{g^2(t, z^s(t))} g_x^2(t, z^s(t))\right] g^1(t, z^s(t)) \\ &\quad - \left[f_x^1(t, z^s(t)) - \frac{f^2(t, z^s(t))}{g^2(t, z^s(t))} g_x^1(t, z^s(t))\right] g^2(t, z^s(t)) \equiv 0. \qquad (19)\end{aligned}$$

You are asked to confirm the veracity of Eq. (19) in a mental exercise. Said differently, the singular time path of the state variable $z^s(t)$ is the solution to

$$\left[f_t^2(t,x) - \frac{f^2(t,x)}{g^2(t,x)} g_t^2(t,x)\right] + \left[f_x^2(t,x) - \frac{f^2(t,x)}{g^2(t,x)} g_x^2(t,x)\right] g^1(t,x) - \left[f_x^1(t,x) - \frac{f^2(t,x)}{g^2(t,x)} g_x^1(t,x)\right] g^2(t,x) = 0. \tag{20}$$

Using $\sigma(t, z^s(t), \lambda^s(t)) \equiv 0 \,\forall\, t \in (t_a, t_b)$, the corresponding singular time path of the costate $\lambda^s(t)$ is therefore given by

$$\lambda^s(t) = -\frac{f^2(t, z^s(t))}{g^2(t, z^s(t))}. \tag{21}$$

Finally, using the state equation, the singular time path of the optimal control $v^s(t)$ is found as

$$v^s(t) = \frac{\dot{z}^s(t) - g^1(t, z^s(t))}{g^2(t, z^s(t))}. \tag{22}$$

Observe that Eqs. (20) through (22) reveal the importance of the assumption $g^2(t, z^s(t)) \neq 0 \,\forall\, t \in (t_a, t_b)$, along the singular solution. Moreover, note that over the interval (t_a, t_b), the optimal triplet of time paths $(z(t), v(t), \lambda(t))$ *is* the singular triplet of time paths $(z^s(t), v^s(t), \lambda^s(t))$.

The necessary conditions of Theorem 4.2 imply that, in general, an optimal control in linear control problems is a combination of bang-bang and singular controls. In Examples 5.1 through 5.3, however, the optimal controls were simply bang-bang. This was not only because of the simple mathematical structure of the problems studied but also because of the introduction of ad hoc assumptions on the parameters. For instance, in Example 5.2, the parameter restriction $T > 1$ was decisive in yielding the conclusion that the optimal control was of the bang-bang type, whereas in Example 5.3, the inequality restriction $\bar{u} > [x_0 - x_T]/T$ was responsible for the optimality of the bang-bang cleanup policy. You are asked to verify these assertions in two mental exercises. In the next (and final) example of the chapter, we study a control problem with economic content in which the optimal control is a combination of bang-bang and singular controls.

Example 5.4: As described and set up in Example 1.1, the optimal control problem faced by a fish farmer can be stated as

$$\begin{aligned}
&\max_{h(\cdot), x_T} J[x(\cdot), h(\cdot)] \stackrel{\text{def}}{=} \int_0^T [p - c(x(t); w)]\, h(t)\, e^{-rt}\, dt \\
&\text{s.t.} \quad \dot{x}(t) = F(x(t)) - h(t),\ x(0) = x_0,\ x(T) = x_T, \\
&\qquad h(t) \in U \stackrel{\text{def}}{=} \{h(\cdot) : 0 \le h(t) \le \bar{h}\}.
\end{aligned} \tag{23}$$

Observe that we have simplified the problem statement compared with that in Example 1.1. In particular, the cost function $C(\cdot)$ now takes the form $C(x, h; w) \stackrel{\text{def}}{=} c(x; w)h$. Consequently, Eq. (23) is a linear optimal control problem. Also notice that since the harvest rate appears linearly in problem (23), lower and upper bounds have been placed on it. The upper bound $\bar{h}$ is the maximum possible rate of harvest by the fish farmer, typically dictated by the technology used, whereas the lower bound rules out a negative rate of harvest. As $C_x(x, h; w) < 0$ and $C_{xx}(x, h; w) > 0$ for all $h > 0$, and because $C_x(x, h; w) = c_x(x; w)h$, it follows that $c_x(x; w) < 0$ and $c_{xx}(x; w) > 0$.

To begin the analysis, first define the Hamiltonian as

$$
\begin{aligned}
H(t, x, h, \lambda) &\stackrel{\text{def}}{=} [p - c(x; w)]he^{-rt} + \lambda[F(x) - h] \\
&= [[p - c(x; w)]\, e^{-rt} - \lambda]\, h + \lambda F(x),
\end{aligned}
$$

where λ is the present value shadow price of the fish stock, the adjective *present value* necessitated by the discount factor in the integrand. Defining

$$
\sigma(t, x, \lambda) \stackrel{\text{def}}{=} [p - c(x; w)]\, e^{-rt} - \lambda \tag{24}
$$

as the value of the switching function $\sigma(\cdot)$, the necessary condition $\max_{h \in U} H(t, x, h, \lambda)$ can be written in an equivalent form, videlicet,

$$
h(t) = \begin{cases} 0 & \text{if } \sigma(t, x, \lambda) < 0 \\ \in [0, \bar{h}] & \text{if } \sigma(t, x, \lambda) = 0 \\ \bar{h} & \text{if } \sigma(t, x, \lambda) > 0, \end{cases} \tag{25}
$$

which is a decision rule for selecting the optimal harvest rate. The remaining necessary conditions are given by

$$
\dot{\lambda} = -H_x(t, x, h, \lambda) = c_x(x; w)\, he^{-rt} - \lambda F'(x),\ \lambda(T) = 0, \tag{26}
$$

$$
\dot{x} = H_\lambda(t, x, h, \lambda) = F(x) - h,\ x(0) = x_0. \tag{27}
$$

Let $(x^*(t; \boldsymbol{\beta}), h^*(t; \boldsymbol{\beta}))$ be the optimal pair of time paths of the fish stock and harvest rate and $\lambda(t; \boldsymbol{\beta})$ be the corresponding time path of the present value shadow price of the fish stock, where $\boldsymbol{\beta} \stackrel{\text{def}}{=} (x_0, T, \bar{h}, p, r, w)$.

To center the discussion on the singular solution, we assume that $x^*(t; \boldsymbol{\beta}) > 0\, \forall\, t \in [t_0, t_1]$ and that $h^*(t; \boldsymbol{\beta}) > 0$ for at least some interval of time in the planning horizon. The first of these assumptions eliminates from consideration the issue of complete eradication of the farmer's fish stock, whereas the second is recognition of the fact that if it is indeed profitable for the farmer to be in business, then some harvest is optimal. By Lemma 3.1, or more precisely, Eq. (13) given in the proof of Lemma 3.1, and the assumption that $h^*(t; \boldsymbol{\beta}) > 0$ for at least some interval of time in the planning horizon, it follows that $\lambda(t; \boldsymbol{\beta}) > 0\, \forall\, t \in [0, T)$. We will make use of this result a little later in the example.

In line with our current interest in singular solutions, we assume that there exists an open interval of time in the planning horizon such that the value of the switching function identically vanishes along the optimal path, that is, a singular solution is optimal for some open interval in the planning horizon. More formally, we assume that $\sigma(t, x, \lambda) \equiv 0\, \forall\, t \in (t_a, t_b) \subset [t_0, t_1]$, or equivalently by Eq. (24), that $\lambda \equiv [p - c(x; w)]e^{-rt}\, \forall\, t \in (t_a, t_b)$ along the optimal path. The latter equation has a straightforward economic interpretation. If the present value shadow price of the stock of fish is equal to the present value marginal (or average) profit of harvest, then the optimal fishing rate is the singular fishing rate. Said differently, if the maximum the fish farmer would pay at time $t = 0$ for another fish in her pond at time t is exactly equal to the time $t = 0$ value of profit from harvesting the incremental fish at time t, then the singular harvest rate is optimal. We now turn to the determination of the singular harvest rate.

Because $\lambda \equiv [p - c(x; w)]e^{-rt}\, \forall\, t \in (t_a, t_b)$ along the optimal path, we may differentiate it with respect to t and use the chain rule to arrive at

$$\dot{\lambda} = -r[p - c(x; w)]e^{-rt} - c_x(x; w)\dot{x}e^{-rt}. \tag{28}$$

Equating Eq. (26) to Eq. (28) and making use of Eq. (27) and $\lambda \equiv [p - c(x; w)]e^{-rt}\, \forall\, t \in (t_a, t_b)$ gives

$$[p - c(x; w)][r - F'(x)] = -c_x(x; w)F(x), \tag{29}$$

which is the equation defining the singular solution of the stock of fish, say, $x^s(p, r, w) > 0$, which we assume to be unique. Notice that because Eq. (29) is independent of t, so too is the singular fish stock $x^s(p, r, w)$, as is clear from the notation employed. In other words, although the singular solution for the stock of fish does not vary over time, it does change as the output price, input price, or discount rate changes.

In view of the fact that the stock of fish does not change with t along the singular solution, the singular harvest rate must equal the growth rate of the fish stock. More formally, as $\dot{x}^s(p, r, w) \equiv 0$, the state equation implies that the singular harvest rate is given by

$$h^s(p, r, w) = F(x^s(p, r, w)) > 0. \tag{30}$$

Equation (30) shows that the singular harvest rate is time invariant but does indeed vary with the output price, input price, or discount rate, just like the singular value of the fish stock.

Because $\lambda \equiv [p - c(x; w)]e^{-rt}\, \forall\, t \in (t_a, t_b)$ along the singular solution, the singular time path of the present value shadow price of the fish stock is thus

$$\lambda^s(t; p, r, w) = [p - c(x^s(p, r, w); w)]e^{-rt} > 0. \tag{31}$$

In contrast to the fish stock and harvest rate, the present value shadow price of the fish stock varies with t. This is plainly obvious because t appears explicitly on the right-hand side of Eq. (31). In fact, as $\dot{\lambda}^s(t; p, r, w)/\lambda^s(t; p, r, w) = -r < 0$, the rate of change of the present value shadow price of the fish stock is the negative of the interest rate along the singular solution, and therefore declines along it.

Seeing as the transversality condition $\lambda(T) = 0$ necessarily holds along the optimal path, the singular path cannot be the optimal path at $t = T$. To see this, note that Eq. (31) shows that $\lambda^s(t; p, r, w) \to 0$ as $t \to +\infty$. Hence, for any finite value of t, $\lambda^s(t; p, r, w) > 0$ along the singular path and thus cannot meet the necessary transversality condition $\lambda(T) = 0$. Put differently, this logic demonstrates that $t_b < T$ and therefore that the fish farmer must leave the singular path before $t = T$ in order to meet the transversality condition $\lambda(T) = 0$. Continuing along this line of reasoning, as long as $x_0 \neq x^s(p, r, w)$, x_0 cannot be a solution of Eq. (29) because $x^s(p, r, w) > 0$ is assumed to be unique. This implies that the singular solution is not the optimal solution at $t = 0$ either, that is, $t_a > 0$. Hence, we have shown that there exist two intervals, namely, $[0, t_a]$ and $[t_b, T]$, for which the singular solution is not optimal. With that settled, let's now investigate the nature of the optimal solution when it doesn't coincide with the singular solution.

To that end, consider the interval $[0, t_a]$ first and assume that $x_0 > x^s(p, r, w)$. Given that $\sigma(t, x, \lambda) = 0$ along the singular solution, $x_0 > x^s(p, r, w)$, and $c_x(x; w) < 0$, it follows from Eq. (24) that $\sigma(t, x, \lambda) \stackrel{\text{def}}{=} [p - c(x; w)]e^{-rt} - \lambda > 0$ when evaluated at $x = x_0$. As a result, by Eq. (25), we may conclude that $h^*(t; \boldsymbol{\beta}) = \bar{h}\, \forall\, t \in [0, t_a]$. A mental exercise asks you to conduct the analysis under the assumption $x_0 < x^s(p, r, w)$.

Next, turn to the interval $[t_b, T]$. In this case, we know that $h^*(t; \boldsymbol{\beta}) = 0$ is admissible but results in zero profit, as examination of the integrand of the objective functional confirms. Consequently, if $\sigma(t, x, \lambda) < 0$, then $h^*(t; \boldsymbol{\beta}) = 0$ is implied by the decision rule given in Eq. (25), resulting in zero profit for the fish farmer. On the other hand, if $\sigma(t, x, \lambda) > 0$, then $h^*(t; \boldsymbol{\beta}) = \bar{h}$ is implied by Eq. (25). Moreover, by Eq. (24) $\sigma(t, x, \lambda) > 0 \Leftrightarrow [p - c(x; w)]e^{-rt} > \lambda$. This, in turn, implies that the fish farmer's profit is positive when $h^*(t; \boldsymbol{\beta}) = \bar{h}$, for we have previously established that $\lambda(t; \boldsymbol{\beta}) > 0\, \forall\, t \in [0, T)$. Our conclusion, therefore, is that $h^*(t; \boldsymbol{\beta}) = \bar{h}\, \forall\, t \in [t_b, T]$.

Summing up, we have established that the optimal harvest rate is given by a combination of bang-bang and singular controls, videlicet,

$$h^*(t; \boldsymbol{\beta}) = \begin{cases} \bar{h} & \forall\, t \in [0, t_a] \\ h^s(p, r, w) & \forall\, t \in (t_a, t_b) \\ \bar{h} & \forall\, t \in [t_b, T]. \end{cases} \tag{32}$$

It is important to point out that just as with the optimal harvest rate, one cannot determine an explicit solution for the fish stock, its present value shadow price, or the switching times, for the functional forms of the unit cost function $c(\cdot)$ and the growth function of the fish $F(\cdot)$ have not been specified. Nonetheless, at this point, the determination of the full solution for the fish stock, its present value shadow price, and the switching times t_a and t_b, is analogous to that given in the prior three examples, and so is left for a mental exercise. We wrap up the analysis of the fish farming control problem with a brief examination of the comparative dynamic properties of the singular solution.

Because $x^s(p, r, w) > 0$ is the solution of Eq. (29), we may substitute $x^s(p, r, w)$ back into Eq. (29) and get the identity

$$\begin{aligned}[p - c(x^s(p, r, w); w)]\,[r - F'(x^s(p, r, w))]& \\ + [c_x(x^s(p, r, w); w)F(x^s(p, r, w))] &\equiv 0. \end{aligned} \tag{33}$$

The comparative dynamics of the singular solution of the fish stock follow from differentiating identity (33) with respect to the parameters (p, r, w) using the chain rule. For example, differentiating identity (33) with respect to the output price p gives

$$\begin{aligned}&- [p - c(x^s(p, r, w); w)]\, F''(x^s(p, r, w))\frac{\partial x^s}{\partial p} \\ &\quad + [r - F'(x^s(p, r, w))]\left[1 - c_x(x^s(p, r, w); w)\frac{\partial x^s}{\partial p}\right] \\ &\quad + c_x(x^s(p, r, w); w)F'(x^s(p, r, w))\frac{\partial x^s}{\partial p} \\ &\quad + F(x^s(p, r, w))c_{xx}(x^s(p, r, w); w)\frac{\partial x^s}{\partial p} \equiv 0.\end{aligned}$$

A bit of algebra then yields

$$\begin{aligned}&\frac{\partial x^s}{\partial p}(p, r, w) \\ &\equiv \frac{F'(x) - r}{F'(x)c_x(x; w) + F(x)c_{xx}(x; w) + [c(x; w) - p]\, F''(x) + [F'(x) - r]\, c_x(x; w)},\end{aligned} \tag{34}$$

which, of course, is evaluated at $x = x^s(p, r, w)$. Note that because we have used the implicit function theorem to derive Eq. (34), we had to assume that the denominator of Eq. (34) is nonzero when evaluated at $x = x^s(p, r, w)$.

In an attempt to sign Eq. (34), we know from Example 1.1 that $F''(x) \leq 0$. Moreover, $F(x^s(p, r, w)) > 0$ by Eq. (30), whereas $p - c(x^s(p, r, w), w) > 0$ by Eq. (31). Finally, by identity (33) and the assumptions made in this example, it follows that $r - F'(x^s(p, r, w)) > 0$, a result you should verify before pressing on. We therefore know that the sign of the numerator of Eq. (34) is negative and all terms in the denominator except the first are nonnegative. Thus, under the assumptions adopted herein, we cannot unambiguously sign the effect of an increase in the output price on the singular stock of fish. Notice that this ambiguity arises because we do not know the sign of $F'(x^s(p, r, w))$ or its precise magnitude, for all we know is that $F'(x^s(p, r, w)) < r$. If, however, we assume that $F'(x^s(p, r, w)) \leq [r/2]$, then the denominator of Eq. (34) is positive and so $\partial x^s(p, r, w)/\partial p < 0$ follows. In this case, we reach the more or less intuitive conclusion that an increase in the price of the harvested fish results in a decline in the singular stock of fish.

In contrast, even under the additional assumption $F'(x^s(p, r, w)) \le [r/2]$, we are not able to sign the effect of an increase in the output price on the singular harvest rate. To verify this claim, differentiate Eq. (30) with respect to p to get

$$\frac{\partial h^s}{\partial p}(p, r, w) = F'(x^s(p, r, w))\frac{\partial x^s}{\partial p}(p, r, w). \tag{35}$$

Because we do not know the sign of $F'(x^s(p, r, w))$, the harvest rate of fish may increase or decrease as the market price of fish increases, thus leading to the possibility of a backward-bending supply curve for the fish along the singular path. Likewise, differentiating Eq. (31) with respect to p and using Eq. (34), we find that

$$\begin{aligned}\frac{\partial \lambda^s}{\partial p}(t; p, r, w) &= \left[1 - c_x(x^s(p, r, w); w)\frac{\partial x^s}{\partial p}(t; p, r, w)\right]e^{-rt} \\ &= \left[\frac{F'(x)c_x(x; w) + F(x)c_{xx}(x; w) + [c(x; w) - p]\,F''(x)}{F'(x)c_x(x; w) + F(x)c_{xx}(x; w) + [c(x; w) - p]\,F''(x) + [F'(x) - r]\,c_x(x; w)}\right]e^{-rt},\end{aligned} \tag{36}$$

evaluated at $x = x^s(p, r, w)$. Consequently, we cannot unambiguously determine the effect of an increase in the market price of fish on the present value shadow price of the fish stock either. Yet, if we adopt the even stronger assumption $F'(x^s(p, r, w)) \le 0$, then Eqs. (34) through (36) imply that $\partial x^s(p, r, w)/\partial p < 0$, $\partial h^s(p, r, w)/\partial p > 0$, and $\partial \lambda^s(t; p, r, w)/\partial p > 0$, respectively. In other words, if the singular stock of fish is greater than or equal to the maximum sustainable fish stock, then an increase in the market price of the fish results in an increase in the singular harvest rate, and consequently, a lower singular stock of fish. In turn, the fish farmer then places a higher present value shadow price on the smaller stock. We leave the remaining comparative dynamics calculations and interpretations for a mental exercise and thus end the example at this juncture.

The first three optimal control problems examined in this chapter were linear in the control variable and had piecewise continuous optimal control functions with respect to t. Such evidence suggests that linearity of the control problem in the control variable is sufficient for the optimal control to be a piecewise continuous function of t. This conclusion, however, is incorrect. As noted in the opening remarks of the chapter, linearity of an optimal control problem in the control variable is not sufficient for a bang-bang control to be optimal. Example 4.3 and its solution refute that claim, as does Example 5.4, in which a combination of bang-bang and singular controls was optimal.

In each of the examples in this chapter, there was also inequality constraints or bounds on the control variable. Hence an important lesson to be taken from these examples is this: whenever an optimal control problem is linear in the control variables and thus has bounds placed on them, one should be aware that the resulting optimal control function may be a piecewise continuous function of t, that is, a bang-bang control may be optimal. In addition, Example 4.6 alerts one to the fact that even if the control problem is nonlinear in the control variable but is subject to an inequality constraint (e.g., nonnegativity), the optimal control may turn out to be only a continuous function of t.

We close with a few remarks on the method we used to solve the control problems of this chapter. Recall that in each example, we solved for the optimal control by using the Maximum Principle directly, that is to say, we examined the necessary condition $\max_{\mathbf{u}\in U} H(t, \mathbf{x}, \mathbf{u}, \boldsymbol{\lambda})$ and determined the equivalent decision rule that the optimal control had to satisfy. In view of the fact that $\max_{\mathbf{u}\in U} H(t, \mathbf{x}, \mathbf{u}, \boldsymbol{\lambda})$ is a constrained static optimization problem, Theorem 4.4 asserts that rather than use the Maximum Principle directly, we may instead form the Lagrangian function associated with the constrained optimization problem $\max_{\mathbf{u}\in U} H(t, \mathbf{x}, \mathbf{u}, \boldsymbol{\lambda})$, and then use the Lagrange or Kuhn-Tucker necessary conditions to determine the optimal control. This method, of course, introduces into the analysis Lagrange multipliers that must be determined. Nonetheless, either approach, when properly executed, yields identical answers. Several mental exercises will ask you to reanalyze three of the examples using this alternative solution procedure and confirm the veracity of Theorem 4.4.

MENTAL EXERCISES

5.1 In Example 5.1, $\lambda(t)$ is a decreasing function of time, that is, $\dot{\lambda}(t) < 0\, \forall\, t \in [0, 2]$, and thus attains the value of three at only the switching time $t = \tau \approx 1.096$. What would happen *if* it turned out that $\lambda(t) = 3\, \forall\, t \in [0, 2]$?

5.2 For Example 5.2, determine the specific solution of the state equation.

5.3 Return to Example 5.3, and for the solution given in Eqs. (10) through (14):
   (a) Derive the comparative dynamics for the discount rate r and provide an economic interpretation of them.
   (b) Derive the comparative dynamics for the final time T and provide an economic interpretation of them.
   (c) Derive the comparative dynamics for the terminal stock x_T and provide an economic interpretation of them.

5.4 Consider the linear optimal control problem

$$\min_{u(\cdot),x_2} \int_0^2 [2x(t) - 3u(t)]\, dt$$

$$\text{s.t.} \quad \dot{x}(t) = x(t) + u(t),\ x(0) = 4,\ x(2) = x_2,$$

$$u(t) \in U \stackrel{\text{def}}{=} \{u(\cdot) : 0 \le u(t) \le 2\}.$$

(a) Find the solution to the necessary conditions.
(b) Show that the solution in part (a) solves the problem.

5.5 Consider the optimal control problem

$$\min_{u(\cdot)} \int_0^2 [x(t) - 1]^2 \, dt$$

$$\text{s.t.} \quad \dot{x}(t) = u(t),\ x(0) = 0,\ x(2) = 1,$$

$$u(t) \in U \stackrel{\text{def}}{=} \{u(\cdot) : 0 \leq u(t) \leq 1\}.$$

(a) Find the solution to the necessary conditions.
(b) Show that the solution in part (a) solves the problem.

5.6 *Optimal Consumption of a Stock of Wine by a Wine Snob*. You are told by your physician that you have $T > 0$ years to live at time $t = 0$, the present, and you know this to be true. Over the course of your life, you have been an avid collector of wine for the purpose of consuming it rather than using it as an investment. Given the news from your physician, you would like to develop a consumption plan that maximizes the utility of consumption of your stock of wine over your remaining lifetime. At present, you have $w(0) = w_0 > 0$ bottles of wine in your cellar, and because you cannot take the wine with you when you die, you've decided that you will consume it all by the time you die hence $w(T) = 0$. The instantaneous utility that you derive from the consumption of wine and stock of wine is given by

$$U(c, w; \alpha_1, \alpha_2) \stackrel{\text{def}}{=} \alpha_1 c + \alpha_2 w,$$

where $\alpha_1 > 0$ and $\alpha_2 > 0$ are the marginal utilities of consumption and wine capital, respectively. This set of preferences implies that you not only get utility from the consumption of wine, but also from the mere presence of the stock of wine in your cellar, that is, you like to *brag* about the quantity and quality of wine in your cellar, and get enjoyment from that. The stock of wine in your cellar at any moment $t \in [0, T]$ is given by the archetype depletion equation in integral form

$$w(t) = w_0 - \int_0^t c(s) \, ds,$$

so that by Leibniz's rule, $\dot{w}(t) = -c(t)$, $w(0) = w_0$, and $w(T) = 0$ are the state equation and boundary conditions for the optimal control problem that you will solve in order to determine your optimal time path of wine consumption. Because your preferences are linear in the consumption rate, one must place a lower and upper bound on your consumption rate, say, $0 \leq c(t) \leq \bar{c}$, where $\bar{c} > w_0/T$. Note that you do *not* discount your instantaneous utility, that is, your discount rate is zero. Defining the vector of time independent parameters as $\boldsymbol{\beta} \stackrel{\text{def}}{=} (\alpha_1, \alpha_2, \bar{c}, w_0, T) \in \Re^5_{++}$, the optimal control problem that defines the optimal consumption rate of your stock of wine is therefore given by

$$V(\boldsymbol{\beta}) \stackrel{\text{def}}{=} \max_{c(\cdot)} \int_0^T [\alpha_1 c(t) + \alpha_2 w(t)]\, dt$$

$$\text{s.t.} \quad \dot{w}(t) = -c(t),\ w(0) = w_0,\ w(T) = 0,$$

$$c(t) \in U \stackrel{\text{def}}{=} \{c(\cdot) : 0 \le c(t) \le \bar{c}, \bar{c} > w_0 T^{-1}\}.$$

(a) Provide an economic interpretation of the inequality $\bar{c} > w_0/T$.

(b) Write down the Hamiltonian for this optimal control problem, letting λ be the costate variable. What is the economic interpretation of λ? Is it positive or negative? How do you know?

(c) Assuming that an optimal solution to the control problem exists, find the decision rule governing the selection of the optimal rate of wine consumption.

(d) Prove that $c(t) = 0\, \forall\, t \in [0, T]$ is *not* an admissible solution. Explain why it therefore cannot be an optimal solution.

(e) Prove that $c(t) = \bar{c}\, \forall\, t \in [0, T]$ is *not* an admissible solution. Explain why it therefore cannot be an optimal solution.

(f) Find the general solution for $\lambda(t)$, letting a_1 be the constant of integration. Prove that $\lambda(t) < \alpha_1\, \forall\, t \in [0, T]$ and $\lambda(t) > \alpha_1\, \forall\, t \in [0, T]$ cannot hold for $\lambda(t)$ in an optimal program.

(g) What, therefore, is the nature of the solution for the optimal consumption rate?

(h) Show that

$$c^*(t; \bar{c}) = \begin{cases} 0\, \forall\, t \in [0, \tau], \\ \bar{c}\, \forall\, t \in (\tau, T], \end{cases}$$

$$w^*(t; \bar{c}, w_0, T) = \begin{cases} w_0 & \forall\, t \in [0, \tau], \\ \bar{c}[T - t] & \forall\, t \in (\tau, T], \end{cases}$$

$$\lambda(t; \alpha_1, \alpha_2, \bar{c}, w_0, T) = \alpha_1 + \alpha_2 \left[T - \frac{w_0}{\bar{c}} - t\right],$$

where

$$\tau = \tau^*(\bar{c}, w_0, T) = T - \frac{w_0}{\bar{c}} > 0$$

is the switching time, is a solution to the necessary conditions.

(i) Provide an economic interpretation of the solution to the necessary conditions. It the solution consistent with the person being a wine snob? How so?

(j) Prove that the solution of the necessary conditions solves the control problem.

(k) Derive the comparative dynamics for the optimal time paths of the consumption rate and shadow value of the wine stock, as well as the switching time, for the parameters $\boldsymbol{\beta} \stackrel{\text{def}}{=} (\alpha_1, \alpha_2, \bar{c}, w_0, T) \in \Re^5_{++}$, and provide an economic interpretation of them.

5.7 *Optimal Inventory Accumulation in a Linear World.* We return to our inventory accumulation problem for this question. Now, however, we add two twists to the prototype problem, scilicet, (i) the firm is asserted to maximize the profits from the sale of the good, and (ii) the costs of production are assumed to be linear in the production rate $u(t)$. The optimal control problem corresponding to this modified inventory accumulation problem is therefore given by

$$\max_{u(\cdot), x_T} \left\{ p x_T - \int_0^T [c_1 u(t) + c_2 x(t)]\, dt \right\}$$

$$\text{s.t.} \quad \dot{x}(t) = u(t),\ x(0) = 0,\ x(T) = x_T,$$

$$u(t) \in U \stackrel{\text{def}}{=} \{u(\cdot) : 0 \le u(t) \le \bar{u}\ \forall\, t \in [0, T]\},$$

where $p > 0$ is the market price of the good produced by the firm, $\bar{u} > 0$ is the maximum production rate the firm can achieve given its capital stock, and all the other variables and parameters have their standard (and by now well-known) interpretation. Recall that all the parameters are positive. Furthermore, assume that $p < c_2 T + c_1$ and $p > c_1$.

(a) Provide an economic interpretation of the inequalities $p < c_2 T + c_1$ and $p > c_1$.

(b) Define the Hamiltonian H with costate variable λ, and derive the decision rule for determining the optimal production rate. In defining the Hamiltonian, include the minus sign as part of the integrand so that you are solving a maximization problem.

(c) Solve the costate equation for its specific solution and show that $\dot{\lambda}(t) > 0$.

(d) Show that the production plan

$$v(t; \bar{u}) = \begin{cases} 0\ \forall\, t \in [0, s) \\ \bar{u}\ \forall\, t \in [s, T], \end{cases}$$

is a solution to the necessary conditions, where $s = T - (p - c_1)/c_2 > 0$ is the switching time.

(e) Find the time path of the state variable that solves the necessary conditions.

(f) Is the solution to the necessary conditions a solution to the control problem? Please answer the following five questions for the optimal time paths of the control and costate variables, as well as the optimal switching time.

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(g) Derive the comparative dynamics for an increase in the marginal storage cost and provide an economic interpretation.
(h) Derive the comparative dynamics for an increase in the marginal cost of production and provide an economic interpretation.
(i) Derive the comparative dynamics for an increase in the output price and provide an economic interpretation.
(j) Derive the comparative dynamics for an increase in the terminal time and provide an economic interpretation.
(k) Derive the comparative dynamics for an increase in the maximum production rate and provide an economic interpretation.

5.8 Resolve Example 5.1 by using a Lagrangian function and the Kuhn-Tucker necessary conditions of Theorem 4.4.

5.9 Resolve Example 5.2 by using a Lagrangian function and the Kuhn-Tucker necessary conditions of Theorem 4.4.

5.10 Resolve Example 5.3 by using a Lagrangian function and the Kuhn-Tucker necessary conditions of Theorem 4.4.

5.11 Establish the veracity of Eq. (19).

5.12 Solve Example 5.2 under the assumption that $T \leq 1$. Show that the resulting optimal savings rate lies on the boundary of the control set for the entire planning horizon. Provide an economic interpretation of this solution and clearly explain why it occurs.

5.13 Solve Example 5.3 under the assumption that $\bar{u} \leq [x_0 - x_T]/T$. To highlight the importance of this assumption, you are asked to proceed in two stages.
(a) Show that if $\bar{u} = [x_0 - x_T]/T$, then the resulting optimal cleanup rate lies at the boundary of the control set for the entire planning horizon. Provide an economic interpretation of this solution and clearly explain why it occurs.
(b) Now assume that $\bar{u} < [x_0 - x_T]/T$. Explain carefully and fully the technical problem you run into.

5.14 Prove that the switching function $\sigma(\cdot)$ is a monotonically decreasing function of time in Examples 5.1 through 5.3.

5.15 Show that equating Eq. (26) to Eq. (28) and using $\lambda \equiv [p - c(x; w)]e^{-rt}\, \forall\, t \in (t_a, t_b)$ and Eq. (27) yields Eq. (29). Also verify Eq. (29) by applying Eq. (20) directly to the fish farming control problem given in Example 5.4.

5.16 Determine the optimal harvest rate for Example 5.4 over the interval $[0, t_a]$ under the assumption $x_0 < x^s(p, r, w)$. Show your work.

5.17 Write down and, of course, show your work for the complete solution to Example 5.4, the fish farming problem, under the assumptions adopted in the example. Recall that just as with the optimal harvest rate given by Eq. (32), one cannot determine an explicit solution for the fish stock, its present value

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shadow price, or the switching times, as the functional forms of the unit cost function and the growth function of the fish have not been specified. The answers for the fish stock and its present value shadow price should take a form akin to Eq. (32), whereas the solution for each switching time, t_a and t_b, should be stated as the solution to an implicit equation.

5.18 Complete the comparative dynamics analysis of the singular solution of Example 5.4 under the assumption $F'(x^s(p, r, w)) \leq 0$.
(a) Derive the comparative dynamics of the singular solution for an increase in the discount rate. Provide an economic interpretation.
(b) Derive the comparative dynamics of the singular solution for an increase in the wage rate. Provide an economic interpretation.

FURTHER READING

All the referenced textbooks contain at least some discussion of piecewise continuous and bang-bang controls. In addition, they all present the material in much the same way that it has been presented here, that is, by way of examples. Kamien and Schwartz (1991) devote a section to the topic, whereas Léonard and Van Long (1992) allocate an entire chapter to control problems with discontinuous controls. There do not appear to be many published papers in the economics literature that study models that are linear in the control variable and lead to a bang-bang solution for the control. This is not that surprising in view of the fact that the assumption of linearity is quite strong. For example, if the integrand is linear in the control variable, then one is assuming that the marginal product or marginal cost of the control is constant with respect to the control (but it could still vary with time). Discussions of singular controls may be found in Kamien and Schwartz (1991) and Clark (1976).

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