FIFTEEN

The Neoclassical Optimal Economic Growth Model

This chapter is the first in a series of three to use the theorems developed in Chapter 14 to study the qualitative properties of a classical intertemporal economic model. The focus of this chapter is on the neoclassical model of optimal economic growth. As is typical in most economic applications of optimal control theory, we provide a thorough economic interpretation of the necessary and sufficient conditions of the optimal economic growth model. More important, however, is the fact that we rather exhaustively study the local stability, steady state comparative statics, and local comparative dynamics properties of the model as well. This later feature is oftentimes neglected in economic research, which is a shame, for such a qualitative analysis really lies at the core of economic policy discussions.

The model we now proceed to develop is neoclassical because its analytical framework revolves around a neoclassical production function, say, $K, L \mapsto F(K, L)$, where $K > 0$ is the capital stock and $L > 0$ is the labor force, the latter of which we assume to be equal to the population of the economy under consideration. The single output of the economy Y is produced using this production function, thereby implying that $Y = F(K, L)$. It is assumed that $F(\cdot) \in C^{(2)}$ and has the following neoclassical properties:

$$F_K(K, L) > 0,\ \ F_L(K, L) > 0,\ \ F_{KK}(K, L) < 0, \tag{1}$$

$$F(\mu K, \mu L) \equiv \mu F(K, L)\,\forall\,\mu > 0. \tag{2}$$

Because the production function $F(\cdot)$ is assumed to be positively homogeneous of degree unity in capital and labor, and $L > 0$, we can let $\mu = L^{-1}$. This and Eq. (2) allow us to rewrite the production function $F(\cdot)$ as $F(L^{-1}K, 1) \equiv L^{-1}F(K, L)$. Note that the expression on the right-hand side of this identity is the average product of labor. Defining $k \overset{\text{def}}{=} K/L$ as the capital-labor ratio and letting $f(k) \overset{\text{def}}{=} F(L^{-1}K, 1)$, we can rewrite $F(L^{-1}K, 1) \equiv L^{-1}F(K, L)$ so as to express the average product of labor as

$$f(k) \equiv L^{-1}F(K, L). \tag{3}$$

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Using the identity $F(K, L) \equiv Lf(k)$, you are asked to establish the ensuing technical result in a mental exercise. It plays an important role in the development of the optimal growth model, as we will see.

Lemma 15.1: *Under the assumptions in Eqs. (1) and (2), it follows that*

(a) $f'(k) > 0$ *if and only if* $F_K(K, L) > 0$.
(b) $f(k) - kf'(k) > 0$ *if and only if* $F_L(K, L) > 0$.
(c) $f''(k) < 0$ *if and only if* $F_{KK}(K, L) < 0$.
(d) $f''(k) < 0$ *if and only if* $F_{LL}(K, L) < 0$.
(e) $F_{KK}(K, L) < 0$ *if and only if* $F_{KL}(K, L) > 0$ *if and only if* $F_{LL}(K, L) < 0$.

An interesting feature of this lemma is that we need only assume that the marginal product of capital is declining, *or* the marginal product of labor is declining, *or* that labor and capital are complements in production, in order to have the average product of labor increase at a decreasing rate with respect to the capital-to-labor ratio. The positive homogeneity of degree one of $F(\cdot)$ is the crucial assumption behind all of these results.

In addition to the assumptions in Eqs. (1) and (2), we also assume that

$$f(0) = 0, \ \lim_{k \to 0} f'(k) = +\infty, \ \lim_{k \to +\infty} f'(k) = 0. \tag{4}$$

The first assumption means that capital is essential for production because without it, no output can be produced. The second implies that the marginal product of capital gets infinitely large as the capital stock shrinks to zero, whereas the third assumption asserts that the marginal product of capital goes to zero as the capital stock becomes arbitrarily large.

Total output Y is allocated between gross investment I and consumption C. Assuming the goods market is in equilibrium therefore implies the market clearing condition $Y = C + I$. Gross investment consists of net investment $\dot{K}$ and replacement investment δK, which is a result of depreciation at the constant exponential rate $\delta > 0$. Hence the goods market equilibrium condition can be rewritten as $F(K, L) = C + \dot{K} + \delta K$, since $Y = F(K, L)$ and $I = \dot{K} + \delta K$. Dividing this differential equation by $L > 0$, defining $c \stackrel{\text{def}}{=} C/L$ as per-capita consumption, using Eq. (3), and rearranging yields

$$\frac{\dot{K}}{L} = f(k) - \delta k - c. \tag{5}$$

Observe that the right-hand side of this differential equation is in per-capita terms, but that the left-hand side is not. To reconcile this, first note that $\dot{K} \stackrel{\text{def}}{=} \frac{d}{dt}[kL] = k\dot{L} + L\dot{k}$. Now assume that the labor force (or equivalently, the population) is growing at the constant exponential $\eta > 0$, that is, $\dot{L}/L = \eta$. This assumption permits us to rewrite the differential equation $\dot{K} = k\dot{L} + L\dot{k}$ as $\dot{K} = k\eta L + L\dot{k}$, or

$$\frac{\dot{K}}{L} = \eta k + \dot{k}. \tag{6}$$

Equating Eq. (5) and Eq. (6) reduces the differential equation to per-capita terms, that is,

$$\dot{k} = f(k) - [\delta + \eta]k - c. \tag{7}$$

Equation (7) is the fundamental differential equation of neoclassical growth theory, and describes how the capital-labor ratio k varies over time. It is the state equation in the neoclassical optimal growth model. The value of the capital-labor ratio is given to the economy at the initial date of the planning horizon $t = 0$, say, from the past actions of the planner, and is denoted by $k(0) = k_0$. It represents the initial condition for the state equation.

The welfare of the society at each instant of the planning horizon is assumed to depend exclusively on society's per-capita consumption c of the single good via the instantaneous utility function $U(\cdot)$, which is assumed to be $C^{(2)}$ on $(0, +\infty)$ and to have the following properties:

$$U'(c) > 0 \,\forall\, c > 0,\; U''(c) < 0 \,\forall\, c > 0,\; \lim_{c\to 0} U'(c) = +\infty,\; \lim_{c\to+\infty} U'(c) = 0. \tag{8}$$

In words, we assume that the social instantaneous utility function has positive and declining marginal utility of per-capita consumption, that the marginal utility of the first unit of consumption is arbitrarily large, and that the marginal utility of consumption goes to zero as per-capita consumption gets arbitrarily large. These are known as the Inada conditions. In a mental exercise, you are asked to show that the function $c \mapsto \ln c$ exhibits such properties. The social utility function is an *instantaneous utility function*, that is, it measures the social value at a point in time of a given per-capita rate of consumption. Consequently, it must be integrated over the entire future in order to obtain the correct social welfare index, that is, the social utility functional. At the end of this chapter, we return to a more technical discussion of the *utility functional* we use, and examine some of its properties.

Recall that we have assumed that the population (or equivalently, the labor force) is growing at the constant exponential $\eta > 0$, that is to say, $\dot{L}/L = \eta$, thereby implying that $L(t) = L_0 e^{\eta t}$, where $L(0) = L_0$ is the size of the population at time $t = 0$. As a result of this assumption, we weight the instantaneous utility function by the population size before integrating it over the planning horizon. Assuming a positive social discount rate $\rho > 0$ and an infinite planning horizon, the utility functional is given by the integral

$$\int_0^{+\infty} U(c(t))L(t)\, e^{-\rho t}\, dt = \int_0^{+\infty} U(c(t))L_0 e^{\eta t} e^{-\rho t}\, dt = L_0 \int_0^{+\infty} U(c(t))\, e^{-[\rho-\eta]t}\, dt.$$

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Because of the assumed infinite planning horizon, we furthermore assume that $r \stackrel{\text{def}}{=} \rho - \eta > 0$. If we also assume that $U(\cdot)$ is bounded, then by Theorem 14.2, the utility functional converges for all admissible pairs and our control problem is in principle solvable by the theorems we've developed in Chapter 14. What's more, if we let $L_0 = 1$ by an appropriate choice of units, then our utility functional can be written as

$$\Psi[c(\cdot)] \stackrel{\text{def}}{=} \int_0^{+\infty} U(c(t))\, e^{-rt}\, dt.$$

Notice that upon comparing the previous two utility functionals, we see that weighting instantaneous utility by an exponentially growing population size and simultaneously requiring that the social discount rate exceed the growth rate of the population is mathematically equivalent to *not* weighting social utility by the exponentially growing population size but adopting a new positive social discount rate $r > 0$.

It is natural to include some constraints on the state and control variables. There are the rather obvious economic ones that constrain the values of per-capita consumption $c(t)$ and the capital-labor ratio $k(t)$ to be nonnegative at every instant in the planning horizon. We will explicitly take into account the control restriction $c(t) \geq 0 \,\forall\, t \in [0, +\infty)$, and in the course of the analysis show that an optimal path of per-capita consumption, assuming one exists, always involves positive consumption at each instant of the planning horizon. In contrast, we will initially ignore the state variable restriction $k(t) \geq 0 \,\forall\, t \in [0, +\infty)$, yet by the end of the analysis, we will similarly conclude that the optimal path of the capital-labor ratio is positive at every instant of the planning horizon. In addition, we could also place an upper bound on per-capita consumption at each instant of the planning horizon, namely, per-capita output, resulting in the inequality constraint $c(t) \leq f(k(t)) \,\forall\, t \in [0, +\infty)$. This constraint means that all consumption $c(t)$ must come out of current production $f(k(t))$, thereby implying that the capital stock $k(t)$ cannot be eaten up. For the sake of exposition and economic insight into this model, however, we ignore this constraint.

The complete statement of the optimal growth problem is therefore to find a per-capita consumption function $c(\cdot)$ that solves the infinite-horizon optimal control problem

$$\max_{c(\cdot)} \Psi[c(\cdot)] \stackrel{\text{def}}{=} \int_0^{+\infty} U(c(t))\, e^{-rt}\, dt$$

$$\text{s.t.} \quad \dot{k}(t) = f(k(t)) - [\delta + \eta]k(t) - c(t),\ k(0) = k_0, \tag{9}$$

$$c(t) \geq 0 \,\forall\, t \in [0, +\infty).$$

Notice that we have not imposed any conditions on $\lim_{t \to +\infty} k(t)$. Given the rather general nature of problem (9), we assume that a solution exists to the necessary conditions and denote it by the pair $(k^*(t; \delta, \eta, r), c^*(t; \delta, \eta, r))$, with corresponding current value costate variable $\lambda(t; \delta, \eta, r)$. Because we are interested in a qualitative characterization of the optimal solution, which is all we can hope for given the lack of specific functional forms for the instantaneous utility function and the production function, we assume that the solution $(k^*(t;\delta,\eta,r), c^*(t;\delta,\eta,r))$ of the necessary conditions converges to a simple, finite, and positive steady state solution of the necessary conditions, that is, $k^*(t;\delta,\eta,r) \to k^s(\delta,\eta,r) > 0$ and $c^*(t;\delta,\eta,r) \to c^s(\delta,\eta,r) > 0$ as $t \to +\infty$, where the superscript s indicates that it is the steady state solution. This implies the same is true for the current value shadow price of the per-capita capital stock, that is, $\lambda(t;\delta,\eta,r) \to \lambda^s(\delta,\eta,r) > 0$ as $t \to +\infty$. Notice that we have *not* assumed that the pair $(k^*(t;\delta,\eta,r), c^*(t;\delta,\eta,r))$ is the solution of the optimal control problem, however. We now proceed to the analysis of the model.

The current value Hamiltonian for this problem is defined as

$$H(k,c,\lambda) \stackrel{\text{def}}{=} U(c) + \lambda[f(k) - c - [\delta+\eta]k],$$

whereas the current value Lagrangian is defined as

$$L(k,c,\lambda,\mu) \stackrel{\text{def}}{=} U(c) + \lambda[f(k) - c - [\delta+\eta]k] + \mu c,$$

where μ is the Lagrange multiplier associated with the nonnegativity constraint on per-capita consumption. Note that the rank constraint qualification is satisfied because the constraint is linear in the control variable. Given that we have used a current value formulation, the necessary conditions of Theorems 14.3 take the form

$$L_c(k,c,\lambda,\mu) = U'(c) - \lambda + \mu = 0, \tag{10}$$

$$L_\mu(k,c,\lambda,\mu) = c \geq 0,\ \mu \geq 0,\ L_\mu(k,c,\lambda,\mu)\cdot\mu = c\mu = 0, \tag{11}$$

$$\dot\lambda = r\lambda - L_k(k,c,\lambda,\mu) = \lambda[\delta+\eta+r-f'(k)], \tag{12}$$

$$\dot k = L_\lambda(k,c,\lambda,\mu) = f(k) - [\delta+\eta]k - c,\ k(0) = k_0. \tag{13}$$

We now proceed to analyze these necessary conditions.

To begin, it should be obvious that $c = 0$, whether for an instant, a finite interval of time, or the entire planning period, satisfies the necessary conditions (11). But as we now show, $c = 0$ does not satisfy the necessary condition (10) under our assumptions, and thus cannot be optimal even for an instant. To see this, first recall that we assumed the instantaneous utility function satisfied the Inada condition $\lim_{c\to 0} U'(c) = +\infty$. Next observe that if $c = 0$ at any finite point of time in the planning horizon or over any finite period of time in the planning horizon, then the necessary condition (10) would imply that $\lambda(t) = \mu + \infty$. This, however, cannot hold because $\lambda(\cdot)$ is a continuous function of time by Theorem 14.3 and thus can only take on finite values at finite points of time in the planning horizon, or over any finite period of time in the planning horizon. Moreover, as $t \to +\infty$, it cannot be the case that $c \to 0$ as we have assumed that the per-capita capital stock, and thus per-capita consumption, converge to finite and positive steady state values. Hence $c(t) > 0\,\forall\, t \in [0,+\infty)$ in an optimal plan. By Eq. (10), this implies that $\mu(t) = 0\,\forall\, t \in [0,+\infty)$ in an optimal plan, and therefore that $U'(c(t)) = \lambda(t)\,\forall\, t \in [0,+\infty)$ in an optimal plan as well. That is, the marginal utility of per-capita consumption

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should be equated with the current value shadow price of the per-capita capital stock in an optimal plan. What's more, because $U'(c) > 0 \, \forall \, c > 0$ from Eq. (8), it follows from the necessary condition $U'(c(t)) = \lambda(t) \, \forall \, t \in [0, +\infty)$ that $\lambda(t) > 0 \, \forall \, t \in [0, +\infty)$ in an optimal plan. In other words, the capital stock is viewed as a good by society in an optimal plan.

The sufficiency conditions of Theorem 14.4 can be checked in a straightforward fashion. This is done by examining the principle minors of the Hessian matrix of the Lagrangian with respect to the state and control variables, and recalling that $U''(c) < 0$, $f''(k) < 0$, and that $\lambda(t) > 0 \, \forall \, t \in [0, +\infty)$:

$$L_{cc}(k, c, \lambda, \mu) = U''(c) < 0,$$

$$L_{KK}(k, c, \lambda, \mu) = \lambda f''(k) < 0,$$

$$L_{cc}(k, c, \lambda, \mu) L_{KK}(k, c, \lambda, \mu) - [L_{ck}(k, c, \lambda, \mu)]^2 = \lambda f''(k) U''(c) > 0.$$

These calculations demonstrate that the current value Lagrangian is a strictly concave function of the state and control variables. Note that although we are dealing with the current value Lagrangian, Theorem 14.4 is instead stated in terms of the present value Lagrangian. This is of no consequence for the above conclusion, however, as the two Lagrangians differ only by the positive constant e^{-rt}.

The last condition to check in Theorem 14.4 is the limiting transversality condition that is part of the sufficient conditions, scilicet, $\lim_{t \to +\infty} e^{-rt} \lambda(t; \delta, \eta, r)[k(t) - k^*(t; \delta, \eta, r)] \geq 0$ for all admissible curves $k(t)$. If this transversality condition holds, then we may conclude from Theorem 14.4 that $(k^*(t; \delta, \eta, r), c^*(t; \delta, \eta, r))$ is the unique solution to the optimal growth problem. Observe that we have multiplied $\lambda(t; \delta, \eta, r)$ by e^{-rt} in the statement of the limiting transversality condition in order to convert $\lambda(t; \delta, \eta, r)$ to a present value costate variable, as is required by Theorem 14.4. Now recall that the assumptions $k^*(t; \delta, \eta, r) \to k^s(\delta, \eta, r)$ and $c^*(t; \delta, \eta, r) \to c^s(\delta, \eta, r)$ as $t \to +\infty$ imply that $\lambda(t; \delta, \eta, r) \to \lambda^s(\delta, \eta, r)$ as $t \to +\infty$. Therefore, given that $\lim_{t \to +\infty} e^{-rt} = 0$, it follows that $\lim_{t \to +\infty} e^{-rt} \lambda(t; \delta, \eta, r) = \lim_{t \to +\infty} e^{-rt} \lim_{t \to +\infty} \lambda(t; \delta, \eta, r) = 0$, where we have used the fact that the limit of a product is equal to the product of the limits when the individual limits exist, as they do here. Consequently, if all the admissible paths of the per-capita capital stock $k(t)$ are bounded, or if $\lim_{t \to +\infty} k(t)$ exists for all admissible paths, then the limiting transversality condition holds with an equality. Note, however, that under the assumptions adopted herein, all the admissible paths of the per-capita capital stock are indeed bounded, as you will be asked to demonstrate in a mental exercise. Accordingly, we may conclude that the solution of the necessary conditions that converges to the fixed point of the necessary conditions is the unique solution to the optimal growth problem. We now turn to a discussion of the stability of the steady state solution of the necessary and sufficient conditions.

To begin, first recall that because $\mu(t) = 0 \, \forall \, t \in [0, +\infty)$ in an optimal plan, the current value Lagrangian and current value Hamiltonian functions are identical in an optimal plan. Because $H_{cc}(k, c, \lambda) = U''(c) < 0$ and $U(\cdot) \in C^{(2)} \, \forall \, c > 0$, we may use the implicit function theorem to solve the equation $H_c(k, c, \lambda) = U'(c) - \lambda = 0$, in principle, for c as a locally $C^{(1)}$ function of λ, say $c = \hat{c}(\lambda)$. We may also use the implicit function theorem to find the derivative of $\hat{c}(\lambda)$ with respect to λ, that is to say,

$$\hat{c}'(\lambda) = \left.\frac{-H_{c\lambda}}{H_{cc}}\right|_{c=\hat{c}(\lambda)} = \frac{1}{U''(\hat{c}(\lambda))} < 0. \tag{14}$$

This comparative statics result says that an increase in the current value shadow price of the per-capita capital stock reduces per-capita consumption, ceteris paribus. Note that this inverse relationship between the per-capita consumption rate and the current value shadow price of the per-capita capital stock is entirely a result of the assumption of declining marginal utility.

The next step is to eliminate c from the canonical differential equations (12) and (13) using the expression $c = \hat{c}(\lambda)$:

$$\dot{\lambda} = \lambda[\delta + \eta + r - f'(k)], \tag{15}$$
$$\dot{k} = f(k) - [\delta + \eta]k - \hat{c}(\lambda). \tag{16}$$

Observe that this dynamical system is a function of only (λ, k) and their time derivatives, as c has been substituted out of the system via the Maximum Principle and the implicit function theorem. By definition, the steady state solution $(k^s(\delta, \eta, r), \lambda^s(\delta, \eta, r))$ of the necessary and sufficient conditions (15) and (16) is found by setting $\dot{\lambda} = 0$ and $\dot{k} = 0$ in Eqs. (15) and (16) to get

$$\dot{\lambda} = 0 \Leftrightarrow \lambda[\delta + \eta + r - f'(k)] = 0, \tag{17}$$
$$\dot{k} = 0 \Leftrightarrow f(k) - [\delta + \eta]k - \hat{c}(\lambda) = 0, \tag{18}$$

and then, in principle, simultaneously solving these two equations for k and λ in terms of the parameters (δ, η, r). Because $\lambda(t) > 0\,\forall\, t \in [0, +\infty)$ in an optimal plan, as discussed above, one could simplify the expression in Eq. (17) to read $\delta + \eta + r - f'(k) = 0$, but this results in a loss of symmetry in the exposition, something we wish to avoid.

To determine the local stability of the fixed point $(k^s(\delta, \eta, r), \lambda^s(\delta, \eta, r))$, we compute the Jacobian matrix of the dynamical system (15) and (16) with respect to (λ, k) and evaluate it at the fixed point $(k^s(\delta, \eta, r), \lambda^s(\delta, \eta, r))$, resulting in

$$\begin{aligned}
&\mathbf{J}(k^s(\delta, \eta, r), \lambda^s(\delta, \eta, r)) \\
&\quad\stackrel{\text{def}}{=} \left.\begin{bmatrix} \dfrac{\partial \dot{\lambda}}{\partial \lambda} & \dfrac{\partial \dot{\lambda}}{\partial k} \\[2ex] \dfrac{\partial \dot{k}}{\partial \lambda} & \dfrac{\partial \dot{k}}{\partial k} \end{bmatrix}\right|_{\substack{\dot{\lambda}=0 \\ \dot{k}=0}} = \left.\begin{bmatrix} \delta + \eta + r - f'(k) & -\lambda f''(k) \\ -\hat{c}'(\lambda) & f'(k) - [\delta + \eta] \end{bmatrix}\right|_{\substack{\dot{\lambda}=0 \\ \dot{k}=0}} \\
&\quad= \begin{bmatrix} 0 & -\lambda^s(\delta, \eta, r) f''(k^s(\delta, \eta, r)) \\ -\hat{c}'(\lambda^s(\delta, \eta, r)) & r \end{bmatrix},
\end{aligned} \tag{19}$$

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where we have used Eq. (17) to simplify the (2, 2) element. Because $\hat{c}'(\lambda^s(\delta, \eta, r)) < 0$ from Eq. (14), $\lambda^s(\delta, \eta, r) > 0$, and $f''(k^s(\delta, \eta, r)) < 0$, it follows from Eq. (19) that

$$\mathrm{tr}\mathbf{J}(k^s(\delta, \eta, r),\ \lambda^s(\delta, \eta, r)) = r > 0, \tag{20}$$

$$\left|\mathbf{J}(k^s(\delta, \eta, r), \lambda^s(\delta, \eta, r))\right| = -\lambda^s(\delta, \eta, r)\hat{c}'(\lambda^s(\delta, \eta, r))f''(k^s(\delta, \eta, r)) < 0. \tag{21}$$

Now recall that by Mental Exercise 13.4, or equivalently, by Theorem 23.9 of Simon and Blume (1994), the product of the eigenvalues of $\mathbf{J}(k^s(\delta, \eta, r), \lambda^s(\delta, \eta, r))$ equals its determinant. As a result, because $|\mathbf{J}(k^s(\delta, \eta, r), \lambda^s(\delta, \eta, r))| < 0$, one eigenvalue is real and positive and the other is real and negative, thereby implying that the steady state is hyperbolic. Thus by Theorem 13.6 or Theorem 13.7, the steady state $(k^s(\delta, \eta, r), \lambda^s(\delta, \eta, r))$ of the nonlinear system of differential equation composed of Eqs. (15) and (16) is an unstable saddle point, with two trajectories in the $k\lambda$ phase plane converging to it as $t \to +\infty$.

With the local stability of the steady state settled, let's now proceed to show that the steady state functions $(k^s(\cdot), \lambda^s(\cdot))$ are well defined and locally $C^{(1)}$ in the parameters (δ, η, r). This conclusion is a straightforward application of the implicit function theorem and the fact that the steady state is a saddle point. To see this, first observe that the Jacobian determinant of the steady state equations (17) and (18) with respect to (k, λ) evaluated at the steady state solution $(k^s(\delta, \eta, r), \lambda^s(\delta, \eta, r))$ is identical to $\mathbf{J}(k^s(\delta, \eta, r), \lambda^s(\delta, \eta, r))$ given in Eq. (19), as is easily verified. This result implies that the Jacobian determinant of the steady state equations (17) and (18) with respect to (k, λ) evaluated at the steady state is not zero, that is, $|\mathbf{J}(k^s(\delta, \eta, r), \lambda^s(\delta, \eta, r))| \neq 0$. Hence, by the implicit function theorem, we can in principle solve Eqs. (17) and (18) simultaneously for the per-capita capital stock and the current value shadow price of the per-capita capital stock as functions of the parameters (δ, η, r), say, $(k^s(\delta, \eta, r), \lambda^s(\delta, \eta, r))$, just as we assumed. Furthermore, the implicit function theorem guarantees that the functions $(k^s(\cdot), \lambda^s(\cdot))$ are locally $C^{(1)}$ in the parameters seeing as the implicit equations (17) and (18) defining the steady state are at least locally $C^{(1)}$ functions of (k, λ) and (δ, η, r), thus ensuring that we can conduct a comparative statics analysis of the steady state. It is worthwhile to emphasize that $(k^s(\delta, \eta, r), \lambda^s(\delta, \eta, r))$ are the values that the optimal paths of the per-capita capital stock and the current value shadow price of the per-capita capital stock converge to as $t \to +\infty$. With these matters behind us, we turn to the construction of the phase portrait corresponding to the canonical equations (15) and (16).

To construct the phase diagram, first consider the $\dot{\lambda} = 0$ isocline, which is defined by

$$\dot{\lambda} = 0 \Leftrightarrow \lambda[\delta + \eta + r - f'(k)] = 0. \tag{22}$$

Because $\lambda > 0$, the $\dot{\lambda} = 0$ isocline can be equivalently written as $\delta + \eta + r - f'(k) = 0$. This form of the $\dot{\lambda} = 0$ isocline shows that it is independent of λ, thereby

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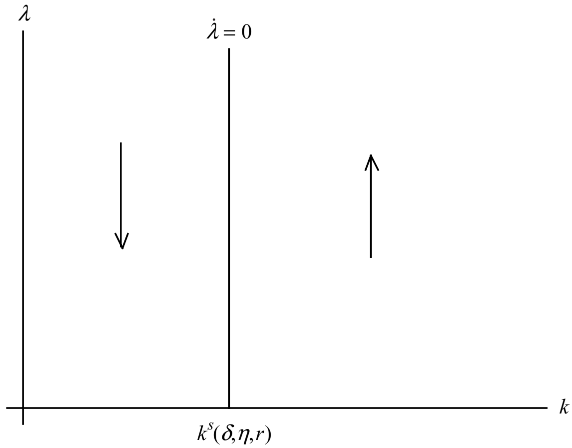

Figure 15.1

implying that the solution to Eq. (22) is the steady state solution for the per-capita capital stock, that is, $k = k^s(\delta, \eta, r) > 0$. This same fact also implies that the $\dot{\lambda} = 0$ isocline is a vertical line positioned at the steady state per-capita capital stock level in the $k\lambda$-phase space, just as Figure 15.1 depicts. Because the Jacobian determinant of Eq. (22) with respect to k is nonzero at $k = k^s(\delta, \eta, r)$, the implicit function theorem implies that we can in principle solve it for k in terms of the parameters (δ, η, r), and furthermore, that $k^s(\cdot) \in C^{(1)}$ locally, since $f'(\cdot) \in C^{(1)}$. This is a redundant observation in light of the above remark that the steady state functions $(k^s(\cdot), \lambda^s(\cdot))$ are at least locally $C^{(1)}$ functions of the parameters. A similar remark will not prove to be redundant when we study the $\dot{k} = 0$ isocline.

To find the vector field acting on λ just off the $\dot{\lambda} = 0$ isocline in a neighborhood of the steady state, differentiate Eq. (15) with respect to λ or k and evaluate the result at the steady state solution $(k^s(\delta, \eta, r), \lambda^s(\delta, \eta, r))$ to get

$$\left.\frac{\partial \dot{\lambda}}{\partial \lambda}\right|_{\substack{\dot{\lambda}=0\\ \dot{k}=0}} = \delta + \eta + r - f'(k)\big|_{\dot{\lambda}=\dot{k}=0} = \delta + \eta + r - f'(k^s(\delta, \eta, r)) \equiv 0, \tag{23}$$

$$\left.\frac{\partial \dot{\lambda}}{\partial k}\right|_{\substack{\dot{\lambda}=0\\ \dot{k}=0}} = -\lambda f''(k)\big|_{\dot{\lambda}=\dot{k}=0} = -\lambda^s(\delta, \eta, r) f''(k^s(\delta, \eta, r)) > 0. \tag{24}$$

These derivatives are the elements of the first row of the Jacobian matrix $\mathbf{J}(k^s(\delta, \eta, r), \lambda^s(\delta, \eta, r))$. Hence, once we compute the Jacobian matrix of the canonical differential equations (15) and (16) and evaluate it at the steady state solution $(k^s(\delta, \eta, r), \lambda^s(\delta, \eta, r))$, the vector field of the canonical differential equations is known in a neighborhood of the steady state.

Equation (23) asserts that in a neighborhood of the steady state, an increase in λ does not change $\dot{\lambda}$ from zero, exactly what the vertical $\dot{\lambda} = 0$ isocline shows geometrically. In contrast, Eq. (24) shows that in a neighborhood of the steady state,

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an increase in k increases $\dot{\lambda}$ from zero to a positive number, thereby implying that points to the right of the $\dot{\lambda} = 0$ isocline in a neighborhood of the steady state have $\dot{\lambda} > 0$. Symmetric reasoning applies to points to the left of the $\dot{\lambda} = 0$ isocline in a neighborhood of the steady state, resulting in $\dot{\lambda} < 0$ for such points. This is how the vector field in Figure 15.1 is derived.

Now turn to the $\dot{k} = 0$ isocline, which is defined by

$$\dot{k} = 0 \Leftrightarrow f(k) - [\delta + \eta]k - \hat{c}(\lambda) = 0. \tag{25}$$

Let $\lambda = \Lambda(k; \delta, \eta)$ be the solution to Eq. (25). Now observe that the Jacobian determinant of Eq. (25) evaluated at the solution $\lambda = \Lambda(k; \delta, \eta)$ is $-\hat{c}'(\Lambda(k; \delta, \eta)) > 0$. Thus the implicit function theorem guarantees that if a solution to the $\dot{k} = 0$ isocline exists, then it can in principle be expressed in the form $\lambda = \Lambda(k; \delta, \eta)$ as claimed, and furthermore, that $\Lambda(\cdot) \in C^{(1)}$ locally, since the implicit equation (25) defining $\lambda = \Lambda(k; \delta, \eta)$ is at least a locally $C^{(1)}$ function. Note, however, that $\lambda = \Lambda(k; \delta, \eta)$ is not the steady state value of λ, in contrast to the solution of Eq. (22) (the $\dot{\lambda} = 0$ isocline), for both k and λ appear in Eq. (25). Moreover, $\lambda = \Lambda(k; \delta, \eta)$ is not the simultaneous solution of both isoclines, as is required for it to be a steady state solution. Hence $\lambda = \Lambda(k; \delta, \eta)$ is the solution to the $\dot{k} = 0$ isocline and not the steady state value of λ.

By the implicit function theorem, the slope of the $\dot{k} = 0$ isocline in a neighborhood of the steady state solution is given by

$$\begin{aligned} \left.\frac{\partial \lambda}{\partial k}\right|_{\substack{\dot{\lambda}=0 \\ \dot{k}=0}} &= \Lambda_k(k; \delta, \eta)\big|_{\dot{\lambda}=0, \dot{k}=0} = \left.\frac{-\partial \dot{k}/\partial k}{\partial \dot{k}/\partial \lambda}\right|_{\substack{\dot{\lambda}=0 \\ \dot{k}=0}} \\ &= \frac{f'(k^s(\delta, \eta, r)) - [\delta + \eta]}{\hat{c}'(\lambda^s(\delta, \eta, r))} = \frac{r}{\hat{c}'(\lambda^s(\delta, \eta, r))} < 0, \end{aligned}$$

where use has been made of Eq. (17), or equivalently, Eq. (22), in arriving at the last equality. This calculation shows that in a neighborhood of the steady state, the $\dot{k} = 0$ isocline is negatively sloped in the $k\lambda$-phase space. Although it is possible to get a relatively global picture of the $\dot{k} = 0$ isocline for this model, we do not do so here, for the theorems we have invoked concerning stability are local in nature, as is the implicit function theorem, the tool we shall employ to conduct the steady state comparative statics. Therefore, in order to be consistent, we characterize the dynamics only locally via the phase diagram. The qualitative information about the $\dot{k} = 0$ isocline is displayed in Figure 15.2.

We note, in passing, that there is a unique value of k for which the slope of the $\dot{k} = 0$ isocline is zero, namely, that which is the solution to

$$\begin{aligned} \left.\frac{\partial \lambda}{\partial k}\right|_{\dot{k}=0} &= \Lambda_k(k; \delta, \eta)\big|_{\dot{k}=0} = \left.\frac{f'(k) - [\delta + \eta]}{\hat{c}'(\lambda)}\right|_{\dot{k}=0} = \frac{f'(k) - [\delta + \eta]}{\hat{c}'(\Lambda(k; \delta, \eta))} \\ &= 0 \Leftrightarrow f'(k) - [\delta + \eta] = 0. \end{aligned} \tag{26}$$

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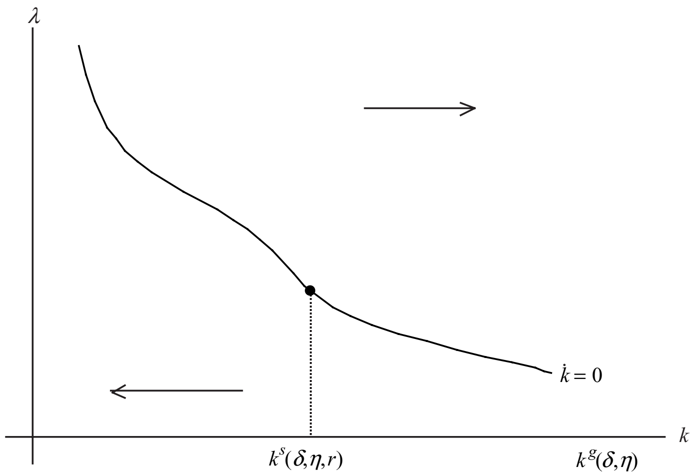

Figure 15.2

We denote the unique solution to this equation by $k = k^g(\delta, \eta)$, the superscript g standing for the "golden-rule" level of the per-capita capital stock. In a mental exercise, you are asked to establish an important relationship between the steady state level of the per-capita capital stock $k = k^s(\delta, \eta, r)$ and the golden-rule level of the per-capita capital stock $k = k^g(\delta, \eta)$, videlicet, that $k^s(\delta, \eta, r) < k^g(\delta, \eta)$, along with a few other properties concerning the shape of the $\dot{k} = 0$ isocline.

To determine the vector field for the $\dot{k} = 0$ isocline in a neighborhood of the steady state, we compute the following derivatives from Eq. (16):

$$\left.\frac{\partial \dot{k}}{\partial \lambda}\right|_{\substack{\dot{\lambda}=0\\ \dot{k}=0}} = -\hat{c}'(\lambda)\big|_{\dot{\lambda}=\dot{k}=0} = -\hat{c}'(\lambda^s(\delta, \eta, r)) > 0, \tag{27}$$

$$\left.\frac{\partial \dot{k}}{\partial k}\right|_{\substack{\dot{\lambda}=0\\ \dot{k}=0}} = f'(k) - [\delta + \eta]\big|_{\dot{\lambda}=\dot{k}=0} = f'(k^s(\delta, \eta, r)) - [\delta + \eta] = r > 0. \tag{28}$$

As expected, these are just the elements in the second row of the Jacobian $\mathbf{J}(k^s(\delta, \eta, r), \lambda^s(\delta, \eta, r))$. Equation (27) asserts that in a neighborhood of the steady state, an increase in λ increases $\dot{k}$ from zero to a positive value, thereby implying that all points above the $\dot{k} = 0$ isocline in a neighborhood of the steady state have k increasing over time. Symmetric reasoning applies for points below the $\dot{k} = 0$ isocline. Similarly, Eq. (28) demonstrates that in a neighborhood of the steady state, an increase in k increases $\dot{k}$ from zero to a positive value. Hence all points to the right of the $\dot{k} = 0$ isocline in a neighborhood of the steady state have k increasing over time, which is identical to the conclusion drawn from Eq. (27). This is just as it should be, since the $\dot{k} = 0$ isocline has a negative slope in a neighborhood of the steady state.

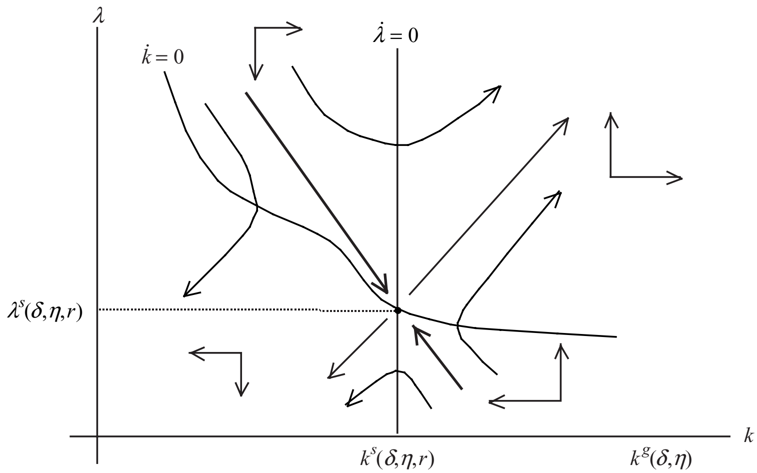

Figure 15.3

Bringing Figures 15.1 and 15.2 together, we arrive at Figure 15.3, the phase diagram for the neoclassical optimal growth problem. The phase paths consistent with the vector field in Figure 15.3 indicate that the steady state $(k^s(\delta, \eta, r), \lambda^s(\delta, \eta, r))$ is a saddle point with two paths approaching it as $t \to +\infty$, just as we found when we studied the Jacobian of the canonical equations (15) and (16). In passing, note that the steady state values of (k, λ) are found at the intersection of the $\dot{k} = 0$ and $\dot{\lambda} = 0$ isoclines.

The optimal phase paths to the neoclassical optimal growth problem are the pair of trajectories that converge to the steady state. Which of these is the unique optimal path depends on the initial condition $k(0) = k_0 > 0$ relative to the steady state value $k^s(\delta, \eta, r)$. For example, if $k_0 < k^s(\delta, \eta, r)$, then the optimal trajectory is the one in which λ falls over time and k increases over time until their steady state values are reached asymptotically. Because the optimal time path of per-capita consumption is defined by $c^*(t; \delta, \eta, r) \stackrel{\text{def}}{=} \hat{c}(\lambda(t; \delta, \eta, r))$, it follows from the chain rule that $\dot{c}^*(t; \delta, \eta, r) = \hat{c}'(\lambda(t; \delta, \eta, r))\dot{\lambda}(t; \delta, \eta, r) > 0$ when $k_0 < k^s(\delta, \eta, r)$. Thus, in this case, per-capita consumption rises over time until its steady state value, defined by $c^s(\delta, \eta, r) \stackrel{\text{def}}{=} \hat{c}(\lambda^s(\delta, \eta, r))$, is attained asymptotically. A symmetric conclusion is obtained if $k_0 > k^s(\delta, \eta, r)$, as you are asked to show in a mental exercise. Note that in the steady state, (k, c, λ) are constant at their steady state values, as is per-capita output, since it is defined as $y^s(\delta, \eta, r) \stackrel{\text{def}}{=} f(k^s(\delta, \eta, r))$. Furthermore, because (k, c, λ) are per-capita variables and the labor force (or population) is growing at the constant rate $\eta > 0$, the consumption rate, capital stock, and output rate must all be growing at the rate η in the steady state too if their per-capita values are constant.

Let's return to a remark made earlier concerning the nonnegativity constraint on the per-capita capital stock, that is, $k(t) \geq 0\, \forall\, t \in [0, +\infty)$. It should be clear

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from Figure 15.3 that this constraint never binds in an optimal solution as long as $k(0) = k_0 > 0$. To see this, simply recall that the two trajectories that converge to the steady state are the only optimal ones, and that if $k(0) = k_0 > 0$, then all along these two paths, we have that $k(t) > 0 \, \forall \, t \in [0, +\infty)$. Hence the nonnegativity constraint on the per-capita capital stock is not binding in the optimal plan as long as $k(0) = k_0 > 0$.

At this point, one might be tempted to come to the conclusion that this is all the information that can be gleaned from the neoclassical optimal growth model without specifying the functional forms of the instantaneous utility function $U(\cdot)$ and the per-capita production function $f(\cdot)$. But you should know by now that this is false, for the steady state comparative statics and the local comparative dynamics have yet to be investigated. It is in this direction that we now turn. First we will investigate the steady state comparative statics of an increase in the discount rate r, and then turn to the corresponding local comparative dynamics, leaving the remaining parameters for you to contemplate in a mental exercise.

In identity form, the steady state necessary and sufficient conditions (17) and (18) are given by

$$\lambda^s(\delta, \eta, r)[\delta + \eta + r - f'(k^s(\delta, \eta, r))] \equiv 0, \tag{29}$$

$$f(k^s(\delta, \eta, r)) - [\delta + \eta]k^s(\delta, \eta, r) - \hat{c}(\lambda^s(\delta, \eta, r)) \equiv 0. \tag{30}$$

Differentiating these identities with respect to r yields

$$\begin{bmatrix} 0 & -\lambda^s(\delta, \eta, r) f''(k^s(\delta, \eta, r)) \\ -\hat{c}'(\lambda^s(\delta, \eta, r)) & r \end{bmatrix} \begin{bmatrix} \partial \lambda^s(\delta, \eta, r)/\partial r \\ \partial k^s(\delta, \eta, r)/\partial r \end{bmatrix} \equiv \begin{bmatrix} -\lambda^s(\delta, \eta, r) \\ 0 \end{bmatrix},$$

where we have used Eq. (29) to simplify the (2, 2) element of the Jacobian. Observe that the Jacobian matrix of the above system of comparative statics equations is identical to the Jacobian matrix $\mathbf{J}(k^s(\delta, \eta, r), \lambda^s(\delta, \eta, r))$ of the canonical equations (15) and (16) given in Eq. (19), just as it should be. Solving the above linear equations via Cramer's rule yields the steady state comparative statics

$$\frac{\partial \lambda^s}{\partial r} \equiv \frac{-r\lambda^s(\delta, \eta, r)}{|\mathbf{J}(k^s(\delta, \eta, r), \lambda^s(\delta, \eta, r))|} > 0,$$

$$\frac{\partial k^s}{\partial r} \equiv \frac{-\lambda^s(\delta, \eta, r)\hat{c}'(\lambda^s(\delta, \eta, r))}{|\mathbf{J}(k^s(\delta, \eta, r), \lambda^s(\delta, \eta, r))|} < 0,$$

which you should verify before reading on. Using the definitions $c^s(\delta, \eta, r) \stackrel{\text{def}}{=} \hat{c}(\lambda^s(\delta, \eta, r))$ and $y^s(\delta, \eta, r) \stackrel{\text{def}}{=} f(k^s(\delta, \eta, r))$, we can compute the steady state comparative statics of an increase in the social discount rate on the per-capita consumption rate and the per-capita output rate:

$$\frac{\partial c^s}{\partial r} = \hat{c}'(\lambda^s(\delta, \eta, r))\frac{\partial \lambda^s}{\partial r} = \frac{-r\hat{c}'(\lambda^s(\delta, \eta, r))\lambda^s(\delta, \eta, r)}{|\mathbf{J}(k^s(\delta, \eta, r), \lambda^s(\delta, \eta, r))|} < 0,$$

$$\frac{\partial y^s}{\partial r} = f'(k^s(\delta, \eta, r))\frac{\partial k^s}{\partial r} = \frac{-f'(k^s(\delta, \eta, r))\lambda^s(\delta, \eta, r)\hat{c}'(\lambda^s(\delta, \eta, r))}{|\mathbf{J}(k^s(\delta, \eta, r), \lambda^s(\delta, \eta, r))|} < 0.$$

Observe that in each of the above four steady state comparative statics expressions, the Jacobian determinant of the canonical equations (15) and (16) appears in the denominator. This is an elemental feature of the steady state comparative statics of an optimal control problem. That is to say, it is not specific to the optimal economic growth model, as we shall see in the next three chapters. This conclusion therefore implies two noteworthy facts: (i) one must determine the local stability of a steady state before the steady state comparative statics are derived, and (ii) the local stability of a steady state plays an analogous role in the signing of the steady state comparative statics as does the second-order sufficient condition of static optimization problems. Let's now turn to the economic interpretation of these steady state comparative statics results.

An increase in the social discount rate $r > 0$ lowers the steady state stock of per-capita capital, and consequently, steady state per-capita output falls as well. With a smaller per-capita capital stock, society values it more at the margin, and hence the steady state current value shadow price of the per-capita capital stock rises. This all seems intuitively plausible, since an increase in the social discount rate implies that society prefers to receive benefits sooner rather than later, since future benefits are valued lower in present value terms. What may seem a bit odd is that both the per-capita capital stock and the per-capita consumption rate are lowered by an increase in the social discount rate. That is, how can the per-capita capital stock be lower when per-capita consumption is also lower? This apparent puzzle is resolved once we investigate the local comparative dynamics of the model. More generally, a complete economic interpretation of an optimal control model can only be achieved by a thorough examination of its steady state comparative statics and comparative dynamics properties.

To find the local comparative dynamics of an increase in the social discount rate, we make use of Figure 15.3 and the above steady state comparative statics. The first thing to notice is that the solution to the $\dot{k} = 0$ isocline, namely, $\lambda = \Lambda(k; \delta, \eta)$, is independent of the social discount rate r in view of the fact that the $\dot{k} = 0$ isocline defined by Eq. (25) is independent of the social discount rate. Hence, any change in the social discount rate does not affect the $\dot{k} = 0$ isocline. This means that the $\dot{\lambda} = 0$ isocline must be the one that shifts when the social discount rate r increases, since we know from the steady state comparative statics that all of the model's variables do indeed change when the social discount rate r increases. This is consistent with the observation that the $\dot{\lambda} = 0$ isocline, defined by Eq. (22), depends explicitly

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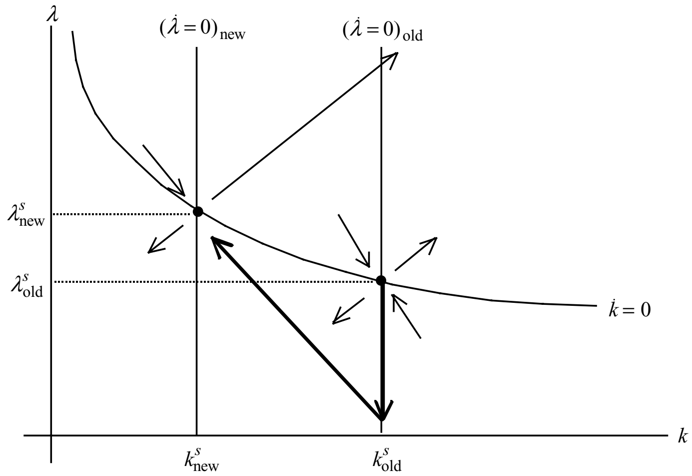

Figure 15.4

on the social discount rate. Specifically, because $\partial k^s/\partial r < 0$ and $\partial \lambda^s/\partial r > 0$, an increase in the social discount rate must shift the $\dot{\lambda} = 0$ isocline to the left in order for the steady state comparative statics to be verified in the phase diagram. The local comparative dynamics phase diagram must therefore appear as in Figure 15.4. Note that only the most relevant portion of the phase diagram is drawn, namely, the part of it in a neighborhood of the old and new steady states.

In drawing the comparative dynamics phase diagram, three important factors must be kept in mind. The first is that the local dynamics depicted in Figure 15.3 apply to *both* of the steady states depicted in Figure 15.4. In other words, the local dynamics around the old and the new steady states are qualitatively identical and are therefore of the saddle point variety. As a result, there is no need to fully draw in the vector field around each steady state in Figure 15.4 (which would clutter up the diagram severely), since the complete vector field for Figure 15.4 can be inferred from that in Figure 15.3. Second, before the increase in the social discount rate occurs, the economy is assumed to be at rest at the old steady state. Third, the economy is assumed to come to rest at the new steady state as a result of the increase in the social discount rate. That is, the old steady state value of per-capita capital is taken as the initial condition in the local comparative dynamics exercise, whereas the new steady state value of per-capita capital is taken as the terminal condition. The comparative dynamics phase diagram therefore depicts the optimal transition path from the old steady state to the new steady state that results from the increase in the social discount rate.

To begin a more detailed look at the construction of the local comparative dynamics phase portrait, first recall that in general, at any given moment in the planning horizon, the state variable is given and thus does not change when a parameter of the optimal control model initially changes. After the initial change in the parameter, however, the state variable does indeed generally change. This observation means that when the social discount rate initially increases, the per-capita capital stock is unaffected. Hence the only variable in the phase diagram that can move initially is the current value shadow price of per-capita capital. This implies that the initial movement in Figure 15.4 has to be vertical. Because the local dynamics of the new steady state are operative as soon as the social discount rate increases, a vertical upward movement in Figure 15.4 cannot lead to the new steady state, for it results in a subsequent trajectory pointing northeast and thus away from the new steady state. Consequently, a downward vertical movement must occur initially, as Figure 15.4 shows. In this case, the economy asymptotically reaches the new steady state using the stable manifold. The heavy lines in Figure 15.4 therefore depict the local comparative dynamics of the increase in the social discount rate in the neoclassical optimal growth model.

With the local comparative dynamics settled, a comprehensive economic interpretation of an increase in the social discount rate can now be given. Figure 15.4 shows that the initial effect of the social discount rate increase is to decrease the current value shadow price of per-capita capital. By differentiating the definition $c^*(t;\delta,\eta,r) \stackrel{\text{def}}{=} \hat{c}(\lambda(t;\delta,\eta,r))$ of the optimal path of per-capita consumption with respect to r and evaluating the result at $t=0$, we find that

$$\left.\frac{\partial c^*(t;\delta,\eta,r)}{\partial r}\right|_{t=0} = \underbrace{\hat{c}'(\lambda(t;\delta,\eta,r))\big|_{t=0}}_{(-)}\ \underbrace{\left.\frac{\partial \lambda(t;\delta,\eta,r)}{\partial r}\right|_{t=0}}_{(-)} > 0.$$

This shows that the initial effect of the increase in the social discount rate is to increase the per-capita consumption rate. This makes economic sense too, for the higher social discount rate means that current consumption is favored over future consumption. Inasmuch as the per-capita consumption rate is inversely related to the current value shadow price of per-capita capital by Eq. (14), this is exactly what we see in Figure 15.4. After this initial effect, the per-capita capital stock begins its monotonic decline toward its new steady state level. The current value shadow price of per-capita capital, however, begins to rise but remains below its old steady state value for some finite period of time. This implies that the per-capita consumption rate begins to fall but still remains above its old steady state rate for some finite period of time. After enough time has passed since the initial increase in the social discount rate, the current value shadow price of per-capita capital increases above its old steady state value and asymptotically converges to its new higher steady state value. This implies that the per-capita consumption rate eventually falls below its old steady state rate and asymptotically converges to its new lower steady state rate. Because the per-capita capital stock declines monotonically from its old to its new steady state level, so too does the per-capita output rate of the economy. This can be seen by differentiating the definition $y^*(t;\delta,\eta,r) \stackrel{\text{def}}{=} f(k^*(t;\delta,\eta,r))$ of the optimal time path of per-capita output with respect to r, thereby yielding $\partial y^*(t;\delta,\eta,r)/\partial r = f'(k^*(t;\delta,\eta,r))\partial k^*(t;\delta,\eta,r)/\partial r < 0$. The local comparative dynamics computations clearly show that the reason for the lower per-capita output rate and per-capita capital stock in the steady state is the higher initial rate of per-capita consumption. Without examining the local comparative dynamics, therefore, it would not have been clear as to why the higher social discount rate leads to a lower per-capita stock of capital and a lower per-capita consumption rate in the steady state. A mental exercise asks you to derive and interpret the steady state comparative statics and local comparative dynamics of the depreciation rate and the growth rate of the labor force.

We close out this chapter by returning to the *utility functional* we used for the analysis of the neoclassical optimal growth model, videlicet, $\Psi[\cdot]$, which is a real-valued functional defined on some function space of consumption functions $c(\cdot)$ (we no longer need to think in per-capita terms). The utility functional provides a numerical measure of merit or worth for a given time path of consumption $c(t)$ over a given time interval τ, $\tau \stackrel{\text{def}}{=} \{t : t_0 \leq t \leq t_1 \text{ (or } +\infty)\}$, or over the union of several such intervals. In particular, the utility functional we used was a special case of

$$\Psi[c(\cdot)] \stackrel{\text{def}}{=} \int_{t_0}^{t_1} U(c(t))\, e^{-rt}\, dt, \tag{31}$$

where $U(\cdot)$ is a real-valued function, referred to as an instantaneous utility function, and $r > 0$ is the social discount rate. Our intention here is to examine some properties of the utility functional and relate them to what we know about the archetype utility function from neoclassical consumer theory. To begin, we provide a general definition of time additivity for utility functionals.

Let $c_1(\cdot)$ be a consumption function, the domain of which is the time interval τ_1, and let $c_2(\cdot)$ be a consumption function, the domain of which is the time interval τ_2. Suppose the time intervals do not overlap, so that $\tau_1 \cap \tau_2 = \emptyset$. Furthermore, define a consumption function $c_3(\cdot)$ on the set of time values $\tau_1 \cup \tau_2$ by

$$c_3(t) \stackrel{\text{def}}{=} \begin{cases} c_1(t), & t \in \tau_1 \\ c_2(t), & t \in \tau_2. \end{cases} \tag{32}$$

We therefore have the following definition of time additivity.

Definition 15.1: The utility functional $\Psi[\cdot]$ is *time additive* if

$$\Psi[c_3(\cdot)] = \Psi[c_1(\cdot)] + \Psi[c_2(\cdot)] \tag{33}$$

for all τ_1 and τ_2 and all functions $c_1(\cdot)$ and $c_2(\cdot)$.

Essentially, this definition asserts that the contributions to the measure of utility from consumption in different time intervals are additive. For example, the particular type of utility functional adopted in Eq. (31) has the property

$$\begin{aligned}\Psi[c_3(\cdot)] &\stackrel{\text{def}}{=} \int_{\tau_1 \cup \tau_2} U(c_3(t))\, e^{-rt}\, dt = \int_{\tau_1} U(c_1(t))\, e^{-rt}\, dt + \int_{\tau_2} U(c_2(t))\, e^{-rt}\, dt \\ &= \Psi[c_1(\cdot)] + \Psi[c_2(\cdot)],\end{aligned} \tag{34}$$

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and is therefore time additive, since the integral is additive with respect to its interval of integration.

We now establish the following fundamental theorem.

Theorem 15.1: *Let $\Psi[\cdot]$ be a time additive utility functional.*

(i) If $\Psi^*[c(\cdot)] \stackrel{\text{def}}{=} h(\Psi[c(\cdot)])$ is a time additive utility functional, where $h(\cdot):\Re \to \Re$ and $h(\cdot) \in C^{(2)}$, then $\Psi^*[\cdot] = a\Psi[\cdot] + b,\ a > 0$.
(ii) Any functional $\Psi^*[\cdot]$ defined by $\Psi^*[\cdot] = a\Psi[\cdot] + b,\ a > 0$, is a time additive utility functional.
(iii) Define the utility functionals $\Psi[\cdot]$ and $\Psi^*[\cdot]$ by

$$\Psi[c(\cdot)] \stackrel{\text{def}}{=} \int_{t_0}^{t_1} U(c(t))\, e^{-rt}\, dt \quad \text{and} \quad \Psi^*[c(\cdot)] \stackrel{\text{def}}{=} \int_{t_0}^{t_1} U^*(c(t))\, e^{-rt}\, dt.$$

Then $U^*(\cdot) = aU(\cdot),\ a > 0$.

Proof: Start with part (i). Let $c_1(\cdot)$ be any consumption function not identically zero and defined over the time interval τ_1, and let $c_2(\cdot)$ be any consumption function not identically zero and defined over the time interval τ_2, $\tau_1 \cap \tau_2 = \emptyset$. Define the modified consumption functions $c_i^*(\cdot) \stackrel{\text{def}}{=} \xi_i c_i(\cdot)$, where $\xi_i > 0$, $i = 1, 2$, and also define $c_3^*(\cdot)$ in a manner analogous to Eq. (32):

$$c_3^*(t) \stackrel{\text{def}}{=} \begin{cases} c_1^*(t),\ t \in \tau_1 \\ c_2^*(t),\ t \in \tau_2. \end{cases}$$

This construction implies that $\Psi[c_i^*(\cdot)]$ is a *function* of the parameter $\xi_i > 0$, say, $\Lambda_i(\xi_i)$, $i = 1, 2$, an observation that is analogous to our framework for deriving the necessary conditions for a simplified class of optimal control problems in Chapter 2. Given that $\Psi[\cdot]$ is time additive, $\Psi[c_3^*(\cdot)]$ is a function of both parameters, say, $\Omega(\xi_1, \xi_2) \stackrel{\text{def}}{=} \Lambda_1(\xi_1) + \Lambda_2(\xi_2)$. Assume that the functions $\Lambda_i(\cdot) \in C^{(1)}$, $i = 1, 2$. Similarly, if $\Psi^*[\cdot]$ is time additive, then $\Psi^*[c_3^*(\cdot)]$ is also a function of both parameters, say, $\Xi(\xi_1, \xi_2) \stackrel{\text{def}}{=} \Delta_1(\xi_1) + \Delta_2(\xi_2)$. Thus, since $\Psi^*[c(\cdot)] \stackrel{\text{def}}{=} h(\Psi[c(\cdot)])$, it follows that

$$\Xi(\xi_1, \xi_2) = h(\Omega(\xi_1, \xi_2)) = h(\Lambda_1(\xi_1) + \Lambda_2(\xi_2)) = \Delta_1(\xi_1) + \Delta_2(\xi_2).$$

Taking the cross-partial derivative of this equation yields

$$\Xi_{\xi_1\xi_2}(\xi_1, \xi_2) = h''(\Omega(\xi_1, \xi_2))\Lambda_1'(\xi_1)\Lambda_1'(\xi_2) \equiv 0, \tag{35}$$

since $\Omega_{\xi_1\xi_2}(\xi_1, \xi_2) \equiv 0$ and $\Xi_{\xi_1\xi_2}(\xi_1, \xi_2) \equiv 0$ by construction. Furthermore, because $c_1(\cdot)$ and $c_2(\cdot)$ are not identically zero, $\Lambda_1'(\xi_1) \neq 0$ and $\Lambda_2'(\xi_2) \neq 0$. Because Eq. (35) holds for every $c_1(\cdot)$ and $c_2(\cdot)$, and on account of $\Lambda_1'(\xi_1) \neq 0$ and $\Lambda_2'(\xi_2) \neq 0$, Eq. (35) implies that $h''(\cdot) \equiv 0$ over the range of the functional $\Psi[\cdot]$, which is the differential

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equation of interest. The solution to the differential equation $h''(\cdot) \equiv 0$ is the linear function $h(x) = ax + b$, where a and b are constants of integration. This observation leads to the conclusion that $\Psi^*[\cdot] = a\Psi[\cdot] + b$ if $\Psi[\cdot]$ and $\Psi^*[\cdot]$ are time additive. To complete the proof of part (i), we must show that $a > 0$. But this is straightforward, since if $\Psi^*[\cdot]$ is a utility functional, then when $\Psi[c_\ell(\cdot)] > \Psi[c_m(\cdot)]$, it must also be true that $\Psi^*[c_\ell(\cdot)] > \Psi^*[c_m(\cdot)]$, thereby yielding the conclusion that $a > 0$.

To prove part (ii), let $\Psi^*[\cdot] = a\Psi[\cdot] + b$, $a > 0$. Then $\Psi^*[\cdot]$ is time additive if $\Psi[\cdot]$ is time additive, and the same ordering of consumption paths is provided by $\Psi[\cdot]$ and $\Psi^*[\cdot]$, since $a > 0$. Thus $\Psi[\cdot]$ and $\Psi^*[\cdot]$ are equivalent as utility functionals.

Finally, to prove part (iii), simply note that

$$\Psi^*[c(\cdot)] \stackrel{\text{def}}{=} \int_{t_0}^{t_1} U^*(c(t))\, e^{-rt}\, dt = a\Psi[c(\cdot)] + b$$

$$= a \int_{t_0}^{t_1} U(c(t))\, e^{-rt}\, dt + b,\ a > 0.$$

This means that $U^*(\cdot) = aU(\cdot)$, $a > 0$. Q.E.D.

Parts (i) and (ii) of Theorem 15.1 assert that every time additive utility functional is a positive affine transformation of a time additive utility functional. In other words, a time additive utility functional is uniquely determined to the same extent that a thermometer scale is determined, namely, only the zero point and the unit of measurement can be arbitrarily chosen. This conclusion, as you may recall, is exactly analogous to that obtained in expected utility theory, scilicet, the expected utility function is unique up to a positive affine transformation. This is not all that surprising when you consider that the operation of expectation is performed by integration for continuous random variables.

Part (iii) of the theorem asserts that the instantaneous utility function corresponding to a time additive utility functional is a positive linear transformation of an instantaneous utility function corresponding to any other time additive utility functional. That is, the instantaneous utility function corresponding to a time additive utility functional is unique up to a positive scalar multiple, or equivalently, is unique up to the unit of measure adopted.

In the next chapter, we examine a seminal dynamic model from industrial organization theory, that of the limit pricing firm. The analysis again focuses on the local stability of the steady state, the steady state comparative statics, and the local comparative dynamics of the model. Rather than conduct the analysis in the state-costate phase plane, we elect to conduct the analysis of this model in the state-control phase plane. The choice of the phase space in which to conduct the investigation is a matter of taste when only one state variable and one control variable are present in an optimal control model.

MENTAL EXERCISES

15.1 Prove Lemma 15.1.

15.2 Show that the function $c \mapsto \ln c$ satisfies

$$U'(c) > 0 \,\forall\, c > 0,\ U''(c) < 0 \,\forall\, c > 0,\ \lim_{c \to 0} U'(c) = +\infty,\ \lim_{c \to +\infty} U'(c) = 0.$$

15.3 This question asks you to prove some global properties concerning the shape of the $\dot{k} = 0$ isocline.

(a) Prove that $k^s(\delta, \eta, r) < k^g(\delta, \eta)$. Draw a simple graph that provides the geometric support for your proof.

(b) Prove that

$$\left.\frac{\partial \lambda}{\partial k}\right|_{\dot{k}=0} = \Lambda_k(k; \delta, \eta)|_{\dot{k}=0} \begin{cases} < 0 \,\forall\, k < k^g(\delta, \eta) \\ > 0 \,\forall\, k > k^g(\delta, \eta) \end{cases}.$$

Note that this result is not evaluated at the steady state, but only along the $\dot{k} = 0$ isocline.

(c) Prove that the $\dot{k} = 0$ isocline is convex in a neighborhood of $k = k^g(\delta, \eta)$.

15.4 Define the optimal path of the per-capita output rate in the neoclassical optimal growth model. For $k_0 < k^s(\delta, \eta, r)$, is the optimal path of the per-capita output rate rising or falling on its approach to the steady state? Explain.

15.5 For the case $k_0 > k^s(\delta, \eta, r)$, determine the rates of change of the per-capita output rate, per-capita consumption rate, and the per-capita capital stock on their approach to the steady state in the neoclassical optimal growth model. Show your work.

15.6 For the neoclassical optimal growth model:

(a) Derive the steady state comparative statics of an increase in the growth rate of the population (or labor force) η on $(k^s(\delta, \eta, r), c^s(\delta, \eta, r), y^s(\delta, \eta, r), \lambda^s(\delta, \eta, r))$. Provide an economic interpretation of your results.

(b) Determine the local comparative dynamics of an increase in the growth rate of the population (or labor force) η on $(k^*(t; \delta, \eta, r), c^*(t; \delta, \eta, r), y^*(t; \delta, \eta, r), \lambda(t; \delta, \eta, r))$ using a phase diagram. Provide an economic interpretation of your results.

(c) Explain clearly why you do not have to recompute parts (a) and (b) for an increase in the depreciation rate of the capital stock δ.

15.7 Recall the neoclassical optimal growth model discussed in this chapter.

(a) Define the optimal value function, say, $V(\cdot)$, in two ways, being careful to denote the parameters on which it depends.

(b) Is the economy better off or worse off with a higher growth rate of the labor force (or population)? Show your work and provide an economic explanation.

(c) Is the economy better off or worse off with a higher depreciation rate of the capital stock? Show your work and provide an economic explanation.

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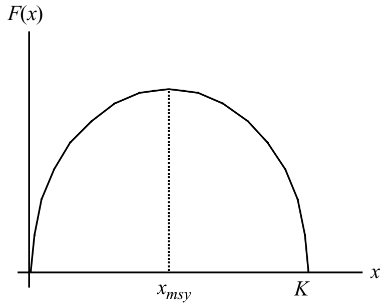

Figure 15.5

15.8 Consider a lake in which edible fish live, and denote the stock of fish at time t by $x(t)$. If undisturbed by humans, the fish grow according to the ordinary differential equation

$$\dot{x}(t) = F(x(t)),$$

where $F(\cdot)$, the growth function of the fish, has the following properties:

$$F(\cdot) \in C^{(2)},\ F(0) = 0,\ F(K) = 0,\ F'(x_{msy}) = 0,\ F''(x) < 0.$$

A prototypical graph of $F(x)$ is given by Figure 15.5, where x_{msy} is known as the maximum sustainable yield level of the stock. A local community situated by the lake can catch and consume the fish at rate c, yielding instantaneous utility $u(c;\alpha)$, where α is a parameter representing the community's taste for the fish. It is assumed that

$$u(\cdot) \in C^{(2)},\ u_c(c;\alpha) > 0,\ u_{cc}(c;\alpha) < 0,$$
$$u_{c\alpha}(c;\alpha) < 0,\ |u(c;\alpha)| \leq u_{\max} < +\infty.$$

Thus an increase in α lowers the community's enjoyment from catching and consuming fish at the margin; otherwise, the instantaneous utility function has the expected positive but declining marginal utility of consumption. Given the consumption of the fish, the rate of change of the stock is now given by the differential equation

$$\dot{x}(t) = F(x(t)) - c(t).$$

The community wishes to choose its consumption rate function $c(\cdot)$ so as to maximize their present discounted utility of consumption, that is, the community solves the problem

$$\max_{c(\cdot)} \int_0^{+\infty} u(c(t);\alpha)\, e^{-rt}\, dt$$

$$\text{s.t.}\quad \dot{x}(t) = F(x(t)) - c(t),\ \ x(0) = x_0,$$
$$c(t) \geq 0\ \forall\, t \in [0, +\infty),$$

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where $r > 0$ is the community's discount rate, $x_0 > 0$ is the initial stock of fish in the lake, and no conditions are imposed on $\lim_{t\to+\infty} x(t)$. Assume that the nonnegativity constraint $x(t) \geq 0\,\forall\, t \in [0, +\infty)$ is not binding, and that there exists a simple, finite, and positive steady state solution of the necessary conditions, say, $(x^s(\alpha, r), c^s(\alpha, r))$, with corresponding current value costate variable $\lambda^s(\alpha, r)$.

(a) Write down the current value Hamiltonian and Lagrangian. Does the objective functional converge for all admissible pairs? Show your work. Derive the necessary conditions for this problem. Provide an economic interpretation of the costate variable.

(b) Assume that an admissible solution exists to the community's planning problem that satisfies the necessary conditions and for which $\lim_{t\to+\infty} x(t) = x^s(\alpha, r) > 0$. Prove, under suitable additional assumptions to be identified by you, that this solution is the unique optimal solution of the planning problem. Show your work and explain clearly.

(c) Assume that in an optimal plan $c(t) > 0\,\forall\, t \in [0, +\infty)$. Show that the necessary conditions can be reduced to the following pair of ordinary differential equations:

$$\dot{c} = \frac{u_c(c;\alpha)[r - F'(x)]}{u_{cc}(c;\alpha)},$$

$$\dot{x} = F(x) - c.$$

Hint: Differentiate the necessary condition $H_c = 0$ with respect to t, and use the other necessary conditions to get a differential equation for c.

(d) Prove that the steady state solution of the necessary conditions in part (c), namely, $(x^s(\alpha, r), c^s(\alpha, r))$, is a local saddle point.

(e) Derive the phase diagram corresponding to the system of ordinary differential equations in part (c). Show your work and label the diagram carefully, indicating the optimal trajectory.

(f) Write down the steady state version of the necessary conditions in part (c). Explain how you would derive, in principle of course, the steady state solution $(x^s(\alpha, r), c^s(\alpha, r))$ of these necessary conditions. Explain carefully and rigorously how you know that such functions are locally $C^{(1)}$.

(g) Interpret an increase in α as an increase in the community's awareness about the negative effects of eating fish from this now polluted lake. The sign $u_{c\alpha}(c;\alpha) < 0$ reflects this kind of interpretation. *Without any mathematics*, provide an economic explanation for the signs of the steady state comparative statics, to wit,

$$\frac{\partial x^s}{\partial \alpha}, \frac{\partial c^s}{\partial \alpha}, \quad \text{and} \quad \frac{\partial \lambda^s}{\partial \alpha},$$

that you expect *a priori*. Now show that your so-called economic intuition was wrong for $\partial x^s(\alpha, r)/\partial\alpha$ and $\partial c^s(\alpha, r)/\partial\alpha$! Also compute $\partial\lambda^s(\alpha, r)/\partial\alpha$.

(h) Derive the steady state comparative statics for an increase in r.

(i) Derive the local comparative dynamics phase portrait for an increase in the discount rate. Label the diagram carefully and provide an economic interpretation.

15.9 Environmental resources provide services that range from essential life support, such as air and water, to unadulterated amenities, such as a field of wild mustard in the Napa Valley. Using a simple optimal control model, this question examines the inherent trade-off between consumption, which depletes the environmental resource stock and thus reduces its amenity value, and preservation of the resource stock, which maintains its amenity value. We assume, therefore, that society has preferences over both its rate of consumption $c(t)$ and the stock of the amenity value $a(t)$, which are represented by the $C^{(2)}$ instantaneous utility function $U(\cdot)$ with values $U(a, c)$. Assume that $U(\cdot)$ has the following properties:

$$U_a(a, c) > 0,\ U_c(a, c) > 0,\ U_{aa}(a, c) < 0,\ U_{cc}(a, c) < 0,\ U_{ac}(a, c) \equiv 0.$$

The differential equation governing the dynamics of the amenity value of the resource stock is given by $\dot{a}(t) = \gamma[\bar{a} - a(t)] - c(t)$, where $\gamma > 0$ is a constant parameter that indicates the natural vigor of the environment, and $\bar{a} > 0$ is the globally asymptotically stable fixed point of the differential equation in the absence of consumption, as is easily verified via a phase portrait. Define $a(0) = a_0 > 0$ as the initial stock of the amenity. Then the optimal control problem under consideration can be stated as

$$V(\boldsymbol{\beta}) \stackrel{\text{def}}{=} \max_{c(\cdot)} \int_0^{+\infty} U(a(t), c(t))\, e^{-rt}\, dt$$

$$\text{s.t.} \quad \dot{a}(t) = \gamma[\bar{a} - a(t)] - c(t),\ a(0) = a_0 > 0,$$

where $\boldsymbol{\beta} \stackrel{\text{def}}{=} (\bar{a}, \gamma, r, a_0) \in \Re^4_{++}$. Assume that the pair $(a^*(t; \boldsymbol{\beta}), c^*(t; \boldsymbol{\beta}))$ is an admissible solution of the necessary conditions of the optimal control problem, with corresponding current value costate variable $\lambda^*(t; \boldsymbol{\beta})$, such that $(a^*(t; \boldsymbol{\beta}), c^*(t; \boldsymbol{\beta})) \to (a^s(\boldsymbol{\alpha}), c^s(\boldsymbol{\alpha}))$ as $t \to +\infty$, where $(a^s(\boldsymbol{\alpha}), c^s(\boldsymbol{\alpha}))$ are the simple steady state values of the amenity value and consumption rate, and $\boldsymbol{\alpha} \stackrel{\text{def}}{=} (\bar{a}, \gamma, r) \in \Re^3_{++}$. Finally, assume that $a^*(t; \boldsymbol{\beta}) > 0$ and that $c^*(t; \boldsymbol{\beta}) > 0$ for all $t \in [0, +\infty)$.

(a) Write down the necessary conditions for this problem in *current value* form.

(b) Make an assumption so as to guarantee that the solution $(a^*(t; \boldsymbol{\beta}), c^*(t; \boldsymbol{\beta}))$ to the necessary conditions is the unique solution of the control problem. Make sure you verify that your assumption does indeed yield the desired conclusion and show your work.

(c) Reduce the necessary conditions down to a pair of differential equations involving only (a, c).

(d) Prove that the steady state $(a^s(\boldsymbol{\alpha}), c^s(\boldsymbol{\alpha}))$ is a local saddle point.

(e) Draw the phase portrait for the system of ordinary differential equations in part (c). Show your work and label the diagram carefully.

(f) Derive the steady state comparative statics for the discount rate r, that is, find

$$\frac{\partial a^s(\boldsymbol{\alpha})}{\partial r}, \quad \frac{\partial c^s(\boldsymbol{\alpha})}{\partial r}, \quad \frac{\partial \lambda^s(\boldsymbol{\alpha})}{\partial r}.$$

(g) Derive the local comparative dynamics phase diagram for the discount rate. Provide an economic interpretation of your results, taking into account the steady state comparative statics from part (f).

(h) Derive the steady state comparative statics for the natural vigor of the environment γ, that is to say, find

$$\frac{\partial a^s(\boldsymbol{\alpha})}{\partial \gamma}, \quad \frac{\partial c^s(\boldsymbol{\alpha})}{\partial \gamma}, \quad \frac{\partial \lambda^s(\boldsymbol{\alpha})}{\partial \gamma}.$$

(i) Derive the local comparative dynamics phase diagrams for the natural vigor of the environment. Provide an economic interpretation of your results, taking into account the steady state comparative statics from part (h).

15.10 Prove that all the admissible paths of the per-capita capital stock $k(t)$ are bounded in the neoclassical optimal economic growth model. **Hints:** In order to do so, you should employ Eqs. (4) and (7), the fact that $\delta + \eta > 0$, and the nonnegativity constraints on $c(t)$ and $k(t)$. Set $c(t) \equiv 0 \, \forall \, t \in [0, +\infty)$ so Eq. (7) reduces to $\dot{k} + [\delta + \eta]k = f(k)$. Use an appropriate integrating factor for the left-hand side of $\dot{k} + [\delta + \eta]k = f(k)$ and then integrate it using the initial condition to arrive at an implicit solution. Use this implicit solution to prove the result.

FURTHER READING

The neoclassical model of optimal economic growth has a rather long and important history in intertemporal economic theory. The seminal paper on the subject is the classic by Ramsey (1928), published when he was only 25 years old. Takayama (1985, Chapter 5, Section D) and Takayama (1993, Chapter 9) contain numerous references to much of the ensuing literature that sought to generalize the basic Ramsey (1928) model, one of the most important of which is Cass (1965). The more recent works of Romer (1986, 1990) examining increasing returns and growth, and endogenous technical change, are arguably the most important of the recent extensions of the basic model. Mental Exercise 15.8 is drawn from Plourde (1970), whereas Mental Exercise 15.9 is based on an exercise in Neher (1990). More recent work on the renewable resource–extracting model of the firm includes Caputo (1989), who presents a rather complete qualitative analysis of the model akin to that of this chapter, and Levhari and Withagen (1992), who study the model when the growth rate of the resource can be influenced directly by the actions of the exploiting firm. The material on the properties of the utility functional is drawn from Hadley and Kemp (1971).

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