SEVEN

Necessary and Sufficient Conditions for Isoperimetric Problems

Mathematically, isoperimetric problems are a class of optimal control problems that involve finding an extremum of one integral, subject to another integral having a prescribed value. One such economic problem of this class was presented in Example 3.5 and Mental Exercise 4.21, videlicet, the nonrenewable resource–extracting model of the firm. In each instance, we transformed the given isoperimetric problem into an optimal control problem in standard form so that the control problem could be solved with the theorems developed to that point. The goal of this chapter is to develop theorems, both necessary and sufficient, that will help us solve isoperimetric problems directly.

An important reason for studying this class of control problems is that they provide a unified view of principal-agent problems as well as a general method for their solution. We demonstrate this in Example 7.3 for the optimal contracting problem when the effort (or action) of the agent is observable by the principal. A byproduct of solving the principal-agent problem via this method is that the independent variable t, which we have heretofore always referred to as time, is now the realized value of the random variable profit. As remarked at the end of Chapter 6, one may safely skip this chapter and the next on a first read without loss of continuity. If, however, one wishes to read the chapter, it is recommended that Mental Exercise 4.22 be worked at this juncture, as it introduces basic ideas and concepts for what follows.

A general form of isoperimetric problems is given by

$$V(\boldsymbol{\beta}) \stackrel{\text{def}}{=} \max_{\mathbf{x}(\cdot)} \left\{ J[\mathbf{x}(\cdot)] \stackrel{\text{def}}{=} \int_{t_0}^{t_1} F(t, \mathbf{x}(t), \dot{\mathbf{x}}(t))\, dt \right\}$$

$$\text{s.t.} \quad K[\mathbf{x}(\cdot)] \stackrel{\text{def}}{=} \int_{t_0}^{t_1} G(t, \mathbf{x}(t), \dot{\mathbf{x}}(t))\, dt = c, \tag{1}$$

$$\mathbf{x}(t_0) = \mathbf{x}_0, \ \mathbf{x}(t_1) = \mathbf{x}_1,$$

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where $c \in \Re$ is a given parameter that we treat as fixed, just like the endpoints $(t_0, \mathbf{x}_0, t_1, \mathbf{x}_1)$; $\boldsymbol{\beta} \stackrel{\text{def}}{=} (c, t_0, \mathbf{x}_0, t_1, \mathbf{x}_1)$ is the parameter vector; and $\mathbf{x}(\cdot) \stackrel{\text{def}}{=} (x_1(\cdot), x_2(\cdot), \ldots, x_N(\cdot))$. The defining feature of isoperimetric problems is the presence of an integral constraint on the curve $\mathbf{x}(t)$, its slope $\dot{\mathbf{x}}(t)$, or either one singly. As such, the value of the function $\mathbf{x}(\cdot)$ and/or the value of the slope function $\dot{\mathbf{x}}(\cdot)$ are *not* restricted at each point in the planning horizon; only the integral of some function of these functions is constrained. Such integral constraints are therefore less constraining than constraints that restrict the value of the function $\mathbf{x}(\cdot)$ and/or the value of the slope function $\dot{\mathbf{x}}(\cdot)$ at each $t \in [t_0, t_1]$. Hence it is perfectly sound for an isoperimetric problem with only one decision function to have any finite number of integral constraints.

As you will recall from Chapter 4, the function $V(\cdot)$ is called the *optimal value function*. It represents the maximum value of the objective functional $J[\cdot]$ conditional on the parameters $\boldsymbol{\beta} \stackrel{\text{def}}{=} (c, t_0, \mathbf{x}_0, t_1, \mathbf{x}_1)$ of the isoperimetric problem (1). The optimal value function is the dynamic analogue of the indirect objective function in static optimization theory. In static microeconomic theory, for example, analogous functions would be the firm's profit function, the firm's cost function, the consumer's expenditure function, or the consumer's indirect utility function. We refer to problem (1) as the *primal* isoperimetric problem, as it is the original form of the problem.

Because the isoperimetric problem (1) differs from the prototype optimal control problem we have been dealing with up until this juncture, we must modify our definition of admissibility. To this end, we have the following definition.

Definition 7.1: Any $C^{(2)}$ function $\mathbf{x}(\cdot)$ on $[t_0, t_1]$, such that $\mathbf{x}(t_0) = \mathbf{x}_0$, $\mathbf{x}(t_1) = \mathbf{x}_1$, and $\int_{t_0}^{t_1} G(t, \mathbf{x}(t), \dot{\mathbf{x}}(t))\, dt = c$, will be called an *admissible function* for the isoperimetric problem (1).

As before, admissibility requires that the candidate functions be sufficiently smooth and satisfy the fixed endpoints of the problem. In addition, admissibility now requires that the candidate functions satisfy the integral constraint of the isoperimetric problem (1). This is a natural extension of our earlier definition of admissibility to isoperimetric problems.

The following assumptions are imposed on the primal isoperimetric problem (1) so as to guarantee that the ensuing analysis is free of tangential mathematical details:

(I.1) $F(\cdot) \in C^{(2)}$ and $G(\cdot) \in C^{(2)}$ on their domains.
(I.2) $\exists$ an optimal solution function $\mathbf{z}(\cdot)$, where $\mathbf{z}(\cdot) \in C^{(2)}\ \forall\, t \in [t_0, t_1]$.
(I.3) The set of admissible curves $\mathbf{x}(t)$ is *not* an empty set.

If assumption (I.3) is not satisfied, then we have no problem, for then there are no curves from which to choose the optimum. In general, we shall assume that there are infinitely many admissible curves $\mathbf{x}(t)$.

Given the basic mathematical structure of the isoperimetric problem (1) and the assumptions imposed on it, we have the following theorem delineating its necessary conditions. The theorem is stated for $\mathbf{x}(t) \in \Re^N$ but proven for the case $N = 1$ for ease of exposition.

Theorem 7.1 (Necessary Conditions): *Let $\mathbf{z}(\cdot)$ be an admissible function that yields an interior maximum of the isoperimetric problem (1) but is not an extremal function for the constraining functional $K[\cdot]$. Then there exists a constant ψ, such that if*

$$\tilde{F}(t, \mathbf{x}, \dot{\mathbf{x}}, \psi) \stackrel{\text{def}}{=} F(t, \mathbf{x}, \dot{\mathbf{x}}) - \psi G(t, \mathbf{x}, \dot{\mathbf{x}}),$$

then it is necessary that the curve $\mathbf{z}(t)$ satisfy the system of augmented Euler equations,

$$\tilde{F}_{x_n}(t, \mathbf{x}, \dot{\mathbf{x}}, \psi) = \frac{d}{dt}\tilde{F}_{\dot{x}_n}(t, \mathbf{x}, \dot{\mathbf{x}}, \psi), \quad n = 1, 2, \ldots, N.$$

In other words, the curve $\mathbf{z}(t)$ must satisfy the necessary conditions for an unconstrained maximum when the integrand function is $\tilde{F}(\cdot)$ not $F(\cdot)$.

Proof: Suppose $z(\cdot)$ is an admissible function that solves problem (1). Let $\eta_1(\cdot)$ and $\eta_2(\cdot)$ be two independent functions of class $C^{(2)}\, \forall\, t \in [t_0, t_1]$ that are arbitrary save for satisfying

$$\eta_1(t_0) = \eta_1(t_1) = 0 \quad \text{and} \quad \eta_2(t_0) = \eta_2(t_1) = 0. \tag{2}$$

Now consider the varied curve given by

$$x(t; \varepsilon_1, \varepsilon_2) \stackrel{\text{def}}{=} z(t) + \varepsilon_1\eta_1(t) + \varepsilon_2\eta_2(t), \tag{3}$$

which is defined for sufficiently small ε_1 and ε_2. Assuming that the varied curve $x(t; \varepsilon_1, \varepsilon_2)$ defined in Eq. (3) is admissible, we have that

$$\varphi(\varepsilon_1, \varepsilon_2) \stackrel{\text{def}}{=} \int_{t_0}^{t_1} G(t, x(t; \varepsilon_1, \varepsilon_2),\ \dot{x}(t; \varepsilon_1, \varepsilon_2))\, dt - c = 0, \tag{4}$$

where $\dot{x}(t; \varepsilon_1, \varepsilon_2) \stackrel{\text{def}}{=} \dot{z}(t) + \varepsilon_1\dot{\eta}_1(t) + \varepsilon_2\dot{\eta}_2(t)$. Observe that Eq. (4) is equivalent to some functional relationship between ε_1 and ε_2, as we will establish below. Because the curve $z(t)$ is the optimal solution to problem (1) by assumption (I.2), it satisfies the integral constraint identically. Moreover, because $x(t; 0, 0) = z(t)$ from Eq. (3), it follows from Eq. (4) and the assumed optimality of the curve $x(t; 0, 0) = z(t)$ that $\varphi(0, 0) = 0$. Differentiating $\varphi(\varepsilon_1, \varepsilon_2)$ defined in Eq. (4) with respect to ε_2 using Leibniz's rule, evaluating the resulting derivative at $(\varepsilon_1, \varepsilon_2) = (0, 0)$, and then using the definition of the varied curve given in Eq. (3) yields

$$\frac{\partial \varphi}{\partial \varepsilon_2}(0,0) = \int_{t_0}^{t_1} [G_x(t, z(t), \dot{z}(t))\eta_2(t) + G_{\dot{x}}(t, z(t), \dot{z}(t))\dot{\eta}_2(t)]\, dt.$$

Given that the curve $z(t)$ is not an extremal for the constraining functional $K[\cdot]$ by assumption, it follows that the function $\eta_2(\cdot)$ can be chosen such that

$$\frac{\partial \varphi}{\partial \varepsilon_2}(0,0) = \int_{t_0}^{t_1} [G_x(t, z(t),\ \dot{z}(t))\eta_2(t) + G_{\dot{x}}(t, z(t),\ \dot{z}(t))\dot{\eta}_2(t)]\ dt \neq 0. \quad (5)$$

This is precisely what it means for the curve $z(t)$ to not be an extremal for the constraining functional $K[\cdot]$, for if $z(t)$ was an extremal, the derivative in Eq. (5) would vanish. Now note that $\varphi(\cdot) \in C^{(2)}$ in some neighborhood of $(\varepsilon_1, \varepsilon_2) = (0, 0)$ by assumption (I.1) and the fact that $\eta_i(\cdot) \in C^{(2)}$, $i = 1, 2$. Because Eq. (5) holds, the implicit function theorem implies that Eq. (4) defines ε_2 as a function of ε_1 in some neighborhood of $(\varepsilon_1, \varepsilon_2) = (0, 0)$, as asserted above. This is because Eq. (5) is the nonvanishing Jacobian of Eq. (4) evaluated at the solution to Eq. (4), which is required for use of the implicit function theorem. For sufficiently small ε_1 and ε_2, therefore, we may regard Eq. (3) as defining a one-parameter family of curves, just as we did for varied curves in proving the necessary conditions of free endpoint optimal control problems in Chapter 2. Thus Eq. (3) defines admissible curves for the isoperimetric problem (1).

Given that the curve $z(t)$ solves problem (1) by assumption (I.2), the function $\Phi(\cdot)$, with values given by

$$\Phi(\varepsilon_1, \varepsilon_2) \stackrel{\text{def}}{=} \int_{t_0}^{t_1} F(t, x(t; \varepsilon_1, \varepsilon_2), \dot{x}(t; \varepsilon_1, \varepsilon_2))\, dt, \quad (6)$$

has a constrained maximum at $(\varepsilon_1, \varepsilon_2) = (0, 0)$ subject to $\varphi(\varepsilon_1, \varepsilon_2) = 0$, since $x(t; 0, 0) = z(t)$ by Eq. (3). Thus the optimal values of ε_1 and ε_2, namely, 0 and 0, are the solution to the following static constrained optimization problem:

$$\max_{\varepsilon_1, \varepsilon_2} \{\Phi(\varepsilon_1, \varepsilon_2) \text{ s.t. } \varphi(\varepsilon_1, \varepsilon_2) = 0\}. \quad (7)$$

Because Eq. (5) is the nondegenerate constraint qualification for problem (7), it can be solved by the method of Lagrange. Forming the Lagrangian function

$$L(\varepsilon_1, \varepsilon_2, \psi) \stackrel{\text{def}}{=} \Phi(\varepsilon_1, \varepsilon_2) - \psi\varphi(\varepsilon_1, \varepsilon_2)$$

and differentiating with respect to ε_1, ε_2, and ψ gives the first-order necessary conditions, which hold at $(\varepsilon_1, \varepsilon_2) = (0, 0)$ by construction. The first two of these necessary conditions are given by

$$\frac{\partial L}{\partial \varepsilon_i}(\varepsilon_1, \varepsilon_2, \psi) = \frac{\partial \Phi}{\partial \varepsilon_i}(\varepsilon_1, \varepsilon_2) - \psi \frac{\partial \varphi}{\partial \varepsilon_i}(\varepsilon_1, \varepsilon_2) = 0, \quad i = 1, 2. \tag{8}$$

A straightforward rearrangement of Eq. (8) yields the familiar tangency conditions of constrained optimization theory, to wit,

$$\frac{\partial \Phi(0,0)/\partial \varepsilon_1}{\partial \varphi(0,0)/\partial \varepsilon_1} = \frac{\partial \Phi(0,0)/\partial \varepsilon_2}{\partial \varphi(0,0)/\partial \varepsilon_2} = \psi. \tag{9}$$

Equation (9) can be rewritten as

$$\frac{\int_{t_0}^{t_1} [F_x(t)\eta_1(t) + F_{\dot{x}}(t)\dot{\eta}_1(t)]\,dt}{\int_{t_0}^{t_1} [G_x(t)\eta_1(t) + G_{\dot{x}}(t)\dot{\eta}_1(t)]\,dt} = \frac{\int_{t_0}^{t_1} [F_x(t)\eta_2(t) + F_{\dot{x}}(t)\dot{\eta}_2(t)]\,dt}{\int_{t_0}^{t_1} [G_x(t)\eta_2(t) + G_{\dot{x}}(t)\dot{\eta}_2(t)]\,dt} = \psi, \tag{10}$$

upon using Eqs. (3), (4), and (6), as you are asked to verify in a mental exercise. Note that the derivatives of the functions $F(\cdot)$ and $G(\cdot)$ are evaluated along the optimal path $z(t)$ because Eq. (9) is evaluated at $(\varepsilon_1, \varepsilon_2) = (0, 0)$, the optimal solution to problem (7). Given that Eq. (10) holds for arbitrary choices of $\eta_1(\cdot)$ and $\eta_2(\cdot)$, ψ is constant. More explicitly, because the variable t is integrated out of Eq. (10), ψ is not a function of t and is therefore constant, as was to be proven.

A simple rearrangement of Eq. (10) yields

$$\begin{aligned}\int_{t_0}^{t_1} & [[F_x(t, z(t), \dot{z}(t)) - \psi G_x(t, z(t), \dot{z}(t))]\eta_i(t) \\ & + [F_{\dot{x}}(t, z(t), \dot{z}(t)) - \psi G_{\dot{x}}(t, z(t), \dot{z}(t))]\dot{\eta}_i(t)]\,dt = 0, \end{aligned} \tag{11}$$

for $i = 1, 2$. Defining $\tilde{F}(t, x, \dot{x}, \psi) \stackrel{\text{def}}{=} F(t, x, \dot{x}) - \psi G(t, x, \dot{x})$ and integrating Eq. (11) by parts in the usual manner using Eq. (2) gives

$$\int_{t_0}^{t_1} \left[\tilde{F}_x(t, z(t), \dot{z}(t), \psi) - \frac{d}{dt}[\tilde{F}_{\dot{x}}(t, z(t), \dot{z}(t), \psi)]\right]\eta_i(t)\,dt = 0, \quad i = 1, 2, \tag{12}$$

which you are also asked to verify in a mental exercise. To complete the proof, we must show that the term $\tilde{F}_x(t, z(t), \dot{z}(t), \psi) - \frac{d}{dt}[\tilde{F}_{\dot{x}}(t, z(t), \dot{z}(t), \psi)] \equiv 0$ for all $t \in [t_0, t_1]$ in Eq. (12).

We will use a contrapositive proof to establish this result. This is achieved by assuming that $\tilde{F}_x(t, z(t), \dot{z}(t), \psi) - \frac{d}{dt}[\tilde{F}_{\dot{x}}(t, z(t), \dot{z}(t), \psi)] \not\equiv 0$ and then showing that this assumption violates Eq. (12). Suppose, therefore, that $\tilde{F}_x(t, z(t), \dot{z}(t), \psi) - \frac{d}{dt}[\tilde{F}_{\dot{x}}(t, z(t), \dot{z}(t), \psi)] \neq 0$ at some point $\bar{t} \in [t_0, t_1]$, say, $\tilde{F}_x(t, z(t), \dot{z}(t), \psi) - \frac{d}{dt}[\tilde{F}_{\dot{x}}(t, z(t), \dot{z}(t), \psi)] > 0$ without loss of generality. Then because $\tilde{F}_x(t, z(t), \dot{z}(t), \psi) - \frac{d}{dt}[\tilde{F}_{\dot{x}}(t, z(t), \dot{z}(t), \psi)]$ is a continuous function of t by assumption (I.1), $\tilde{F}_x(t, z(t), \dot{z}(t), \psi) - \frac{d}{dt}[\tilde{F}_{\dot{x}}(t, z(t), \dot{z}(t), \psi)] > 0$ in some interval $[a, b]$, with $a < b$, $\bar{t} \in [a, b]$, and $[a, b] \subset [t_0, t_1]$. Now select $\eta_i(\cdot) \in C^{(2)}$ such that $\eta_i(t) > 0 \; \forall\, t \in (a, b)$ and $\eta_i(t) = 0$ for $t \notin (a, b)$, $i = 1, 2$. See Figure 7.1 for

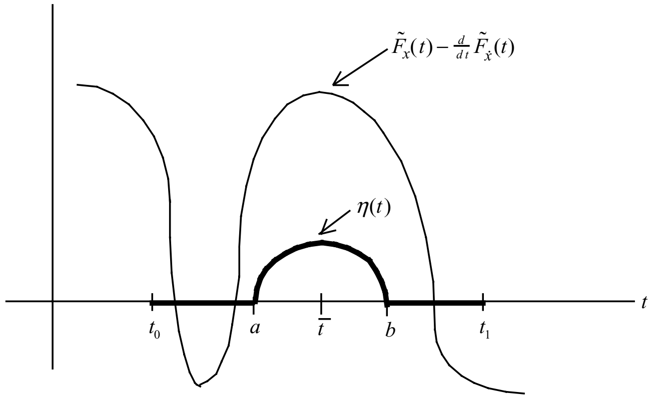

Figure 7.1

the geometry of these restrictions. Under these conditions, it follows that

$$\int_{t_0}^{t_1} \left[\tilde{F}_x(t, z(t), \dot{z}(t), \psi) - \frac{d}{dt} [\tilde{F}_{\dot{x}}(t, z(t), \dot{z}(t), \psi)] \right] \eta_i(t)\, dt$$

$$= \int_a^b \left[\tilde{F}_x(t, z(t), \dot{z}(t), \psi) - \frac{d}{dt} [\tilde{F}_{\dot{x}}(t, z(t), \dot{z}(t), \psi)] \right] \eta_i(t)\, dt > 0,$$

$i = 1, 2,$ which contradicts Eq. (12), thereby completing the proof. Q.E.D.

Theorem 7.1 essentially asserts that the curve $\mathbf{z}(t)$ that solves the isoperimetric problem (1) is an extremal for the functional

$$\int_{t_0}^{t_1} \tilde{F}(t, \mathbf{x}(t), \dot{\mathbf{x}}(t), \psi)\, dt \stackrel{\text{def}}{=} \int_{t_0}^{t_1} [F(t, \mathbf{x}(t), \dot{\mathbf{x}}(t)) - \psi G(t, \mathbf{x}(t), \dot{\mathbf{x}}(t))]\, dt.$$

The two constants of integration that result from integrating the augmented Euler equation, and the constant multiplier ψ, are found using the two boundary conditions and the integral constraint.

Before turning to an example to demonstrate the application of Theorem 7.1, an important property of isoperimetric problems will be established. To this end, consider the following *reciprocal*, or *transposed*, or *mirrored* isoperimetric problem:

$$W(\boldsymbol{\theta}) \stackrel{\text{def}}{=} \min_{\mathbf{x}(\cdot)} \left\{ K[\mathbf{x}(\cdot)] \stackrel{\text{def}}{=} \int_{t_0}^{t_1} G(t, \mathbf{x}(t), \dot{\mathbf{x}}(t))\, dt \right\}$$

$$\text{s.t.} \quad J[\mathbf{x}(\cdot)] \stackrel{\text{def}}{=} \int_{t_0}^{t_1} F(t, \mathbf{x}(t), \dot{\mathbf{x}}(t))\, dt = b, \tag{13}$$

$$\mathbf{x}(t_0) = \mathbf{x}_0, \mathbf{x}(t_1) = \mathbf{x}_1,$$

where $\boldsymbol{\theta} \stackrel{\text{def}}{=} (b, t_0, \mathbf{x}_0, t_1, \mathbf{x}_1)$ is the parameter vector and $W(\cdot)$ is the optimal value function for the reciprocal isoperimetric problem (13). The reciprocal problem is simply a rearrangement of the primal isoperimetric problem (1) whereby the objective functional and the integral constraint are interchanged and the objective of maximization is replaced with the objective of minimization. We note that it is *incorrect* to refer to isoperimetric problem (13) as the dual of isoperimetric problem (1) in view of the fact that the decision functions are identical in problems (1) and (13), and thus are members of the same function space. Strictly speaking, the prototype definition of a dual problem is one whose solution lies in a vector space that is different from, or dual to, that of the primal problem, as in, for example, dual pairs of linear programming problems. This definition of *dual* clearly does not hold for isoperimetric problems (1) and (13).

The ensuing theorem and a significant extension of it play an important role in performing a relatively complete comparative dynamics characterization of the nonrenewable resource–extracting model of the firm, as will be seen in Chapter 8.

Theorem 7.2: *Let $b = J[\mathbf{z}(\cdot)]$, or equivalently, let $b = V(\boldsymbol{\beta})$, in the reciprocal isoperimetric problem (13). If the curve $\mathbf{z}(t)$ is a solution to problem (1), then the curve $\mathbf{z}(t)$ is an extremal for problem (13). Moreover, if the corresponding multiplier in problem (1) is ψ, then ψ^{-1} is the corresponding multiplier in problem (13).*

Proof: First observe that $F(t, \mathbf{x}, \dot{\mathbf{x}}) - \psi G(t, \mathbf{x}, \dot{\mathbf{x}}) \equiv -\psi[G(t, \mathbf{x}, \dot{\mathbf{x}}) - \psi^{-1} F(t, \mathbf{x}, \dot{\mathbf{x}})]$, which immediately yields the conclusion regarding the multipliers because $F(t, \mathbf{x}, \dot{\mathbf{x}}) - \psi G(t, \mathbf{x}, \dot{\mathbf{x}})$ is the augmented integrand for problem (1) and $G(t, \mathbf{x}, \dot{\mathbf{x}}) - \psi^{-1} F(t, \mathbf{x}, \dot{\mathbf{x}})$ is the augmented integrand for problem (13). Thus the augmented integrands of isoperimetric problems (1) and (13) differ from each other by a constant, namely, $-\psi$. Next, observe that if the curve $\mathbf{z}(t)$ satisfies the system of augmented Euler equations

$$\begin{aligned} &F_{x_n}(t, \mathbf{x}, \dot{\mathbf{x}}) - \psi G_{x_n}(t, \mathbf{x}, \dot{\mathbf{x}}) \\ &\quad = \frac{d}{dt}[F_{\dot{x}_n}(t, \mathbf{x}, \dot{\mathbf{x}}) - \psi G_{\dot{x}_n}(t, \mathbf{x}, \dot{\mathbf{x}})], \quad n = 1, 2, \ldots, N, \end{aligned}$$

then it surely satisfies the system of augmented Euler equations

$$\begin{aligned} &G_{x_n}(t, \mathbf{x}, \dot{\mathbf{x}}) - \psi^{-1} F_{x_n}(t, \mathbf{x}, \dot{\mathbf{x}}) \\ &\quad = \frac{d}{dt}\left[G_{\dot{x}_n}(t, \mathbf{x}, \dot{\mathbf{x}}) - \psi^{-1} F_{\dot{x}_n}(t, \mathbf{x}, \dot{\mathbf{x}})\right], \quad n = 1, 2, \ldots, N, \end{aligned}$$

for the latter is simply the former multiplied by the constant $-\psi^{-1}$. To complete the proof, we must show that the curve $\mathbf{z}(t)$ is admissible in problem (13). Because the endpoints in the pair of problems (1) and (13) are identical, all that is left to demonstrate is that the curve $\mathbf{z}(t)$ satisfies the integral constraint in problem (13).

When $b = V(\boldsymbol{\beta})$, the integral constraint in problem (13) is

$$J[\mathbf{x}(\cdot)] \stackrel{\text{def}}{=} \int_{t_0}^{t_1} F(t, \mathbf{x}(t), \dot{\mathbf{x}}(t))\, dt = V(\boldsymbol{\beta}).$$

By the definition of the optimal value function $V(\cdot)$ in problem (1), the curve $\mathbf{z}(t)$ obviously satisfies this integral condition seeing as it is a solution to problem (1). Q.E.D.

Notice that Theorem 7.1 smacks of the reciprocal nature of the pair of static consumer problems, utility maximization and expenditure minimization. Because we have not imposed much structure on the isoperimetric problem in this chapter, we cannot claim that the solution of the primal maximization isoperimetric problem (1) is also a solution of the reciprocal minimization isoperimetric problem (13), and vice versa, as we can for the aforementioned static consumer problems. This conclusion will have to wait for the next chapter. Similarly, we cannot claim at this juncture that the solution of the augmented Euler equation is the solution of the isoperimetric problem, since the augmented Euler equation is but a necessary condition for optimality. This conclusion will have to wait until the end of this chapter, when a sufficiency theorem is introduced.

Let's now demonstrate how to use Theorems 7.1 and 7.2 in the context of a relatively simple example devoid of economic content.

Example 7.1: Consider the following simple *primal* isoperimetric problem:

$$\min_{x(\cdot)} J[x(\cdot)] \stackrel{\text{def}}{=} \int_0^1 [\dot{x}(t)]^2\, dt$$

$$\text{s.t.} \quad K[x(\cdot)] \stackrel{\text{def}}{=} \int_0^1 x(t)\, dt = c,$$

$$x(0) = 0,\ x(1) = 0,$$

where $c > 0$ is the given parameter of the problem. We begin by defining the augmented integrand by $\tilde{F}(t, x, \dot{x}, \psi) \stackrel{\text{def}}{=} F(t, x, \dot{x}) - \psi G(t, x, \dot{x}) = \dot{x}^2 - \psi x$, where ψ is the constant multiplier from the primal problem, and then computing

$$\tilde{F}_x(t, x, \dot{x}, \psi) = -\psi,\ \tilde{F}_{\dot{x}}(t, x, \dot{x}, \psi) = 2\dot{x},\ \frac{d}{dt}\tilde{F}_{\dot{x}}(t, x, \dot{x}, \psi) = 2\ddot{x}.$$

The augmented Euler equation $\tilde{F}_x(t, x, \dot{x}, \psi) - \frac{d}{dt}\tilde{F}_{\dot{x}}(t, x, \dot{x}, \psi) = 0$ is therefore given by $\ddot{x} = -\frac{1}{2}\psi$. Separating the variables and integrating twice gives the general solution to the augmented Euler equation, to wit, $x(t) = -\frac{1}{4}\psi t^2 + c_1 t + c_2$, where

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(c_1, c_2) are the constants of integration. Using the boundary conditions yields

$$x(0) = 0 \Rightarrow c_2 = 0,$$
$$x(1) = 0 \Rightarrow c_1 = \frac{1}{4}\psi.$$

Plugging these two results into the general solution of the augmented Euler equation gives $x(t) = -\frac{1}{4}\psi t^2 + \frac{1}{4}\psi t = \frac{1}{4}\psi[t - t^2]$, and then using this in the integral constraint yields

$$\int_0^1 x(t)\,dt = \frac{1}{4}\psi \int_0^1 [t - t^2]\,dt = \left[\frac{1}{8}\psi t^2 - \frac{1}{12}\psi t^3\right]\Bigg|_{t=0}^{t=1}$$
$$= \psi\left[\frac{1}{8} - \frac{1}{12}\right] = \frac{1}{24}\psi = c.$$

It thus follows that $\psi = 24c$ and $c_1 = 6c$. Plugging the solutions for (c_1, c_2, ψ) into the general solution of the augmented Euler equation yields the definite solution, its rate of change, and the multiplier:

$$z_1(t; c) = -6ct^2 + 6ct,\ \ \dot{z}_1(t; c) = -12ct + 6c,\ \ \psi = 24c. \tag{14}$$

We will present a sufficiency theorem at the end of this chapter that will allow us to conclude that the curve $z_1(t; c)$ is the unique solution of the primal isoperimetric problem.

To finish up this part of the example, plug $\dot{z}_1(t; c) = -12ct + 6c$ in the objective functional $J[\cdot]$ to find its optimal value:

$$V(c) \stackrel{\text{def}}{=} J[z_1(\cdot)] = \int_0^1 [\dot{z}_1(t; c)]^2\,dt = \int_0^1 [-12ct + 6c]^2\,dt$$
$$= \int_0^1 [144c^2t^2 - 144c^2t + 36c^2]\,dt$$
$$= \left[\frac{144}{3}c^2t^3 - 72c^2t^2 + 36c^2t\right]\Bigg|_{t=0}^{t=1} = 12c^2.$$

Recall that the function $V(\cdot)$ is the *optimal value function* and gives the optimal value of the objective functional conditional on the parameters of the variational problem.

Now let's solve the *reciprocal* isoperimetric problem

$$\max_{x(\cdot)} K[x(\cdot)] \stackrel{\text{def}}{=} \int_0^1 x(t)\,dt$$

$$\text{s.t.} \quad J[x(\cdot)] \stackrel{\text{def}}{=} \int_0^1 [\dot{x}(t)]^2\,dt = V(c) = 12c^2,$$
$$x(0) = 0,\ x(1) = 0.$$

Notice that the integral constraint has been forced to have a value equal to that of the optimal value function from the primal isoperimetric problem, exactly as is required by Theorem 7.2. The augmented integrand function $\hat{F}(\cdot)$ for the reciprocal problem has values defined by $\hat{F}(t, x, \dot{x}, \theta) \stackrel{\text{def}}{=} G(t, x, \dot{x}) - \theta F(t, x, \dot{x}) = x - \theta\dot{x}^2$, where θ is the constant multiplier for the reciprocal problem. Differentiating $\hat{F}(t, x, \dot{x}, \theta) \stackrel{\text{def}}{=} x - \theta\dot{x}^2$ gives

$$\hat{F}_x(t, x, \dot{x}, \theta) = 1,\ \hat{F}_{\dot{x}}(t, x, \dot{x}, \theta) = -2\theta\dot{x},\ \frac{d}{dt}\hat{F}_{\dot{x}}(t, x, \dot{x}, \theta) = -2\theta\ddot{x},$$

and hence the augmented Euler equation is given by $\ddot{x} = -\frac{1}{2}\theta^{-1}$. Separating the variables and integrating twice yields the general solution $x(t) = -\frac{1}{4}\theta^{-1}t^2 + k_1 t + k_2$, where (k_1, k_2) are the constants of integration. The three unknowns (k_1, k_2, θ) are found with the help of the two boundary conditions and the integral constraint. Using the boundary conditions gives

$$x(0) = 0 \Rightarrow k_2 = 0,$$
$$x(1) = 0 \Rightarrow k_1 = \frac{1}{4}\theta^{-1}.$$

Substituting $k_2 = 0$ and $k_1 = \frac{1}{4}\theta^{-1}$ into the general solution of the augmented Euler equation gives $x(t) = -\frac{1}{4}\theta^{-1}t^2 + \frac{1}{4}\mu^{-1}t = \frac{1}{4}\theta^{-1}[t - t^2]$, and then using this in the integral constraint yields

$$\int_0^1 [\dot{x}(t)]^2\,dt = \frac{1}{16}\theta^{-2}\int_0^1 [1 - 2t]^2\,dt = \frac{1}{16}\theta^{-2}\int_0^1 [1 - 4t + 4t^2]dt$$
$$= \frac{1}{16}\theta^{-2}\left[t - 2t^2 + \frac{4}{3}t^3\right]\Bigg|_{t=0}^{t=1} = \frac{1}{16}\theta^{-2}\left[1 - 2 + \frac{4}{3}\right]$$
$$= \frac{1}{48}\theta^{-2} = V(c) = 12c^2.$$

Clearly, there are two solutions for θ to the above equation. Because $\theta^{-2} = 12(48)c^2 = 12^2 2^2 c^2$, this implies that $\theta^{-1} = \pm 24c$ and thus $k_1 = \pm 6c$. As a result, the definite solution to the augmented Euler equation of the reciprocal isoperimetric problem is given by *either* of the following two curves and associated multipliers:

$$z_2(t; c) = -6ct^2 + 6ct,\ \dot{z}_2(t; c) = -12ct + 6c,\ \theta = \frac{1}{24c}, \tag{15}$$

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$$z_3(t;c) = 6ct^2 - 6ct,\ \dot{z}_3(t;c) = 12ct - 6c,\ \theta = -\frac{1}{24c}. \tag{16}$$

Note that the solution to the augmented Euler equation is not unique, because we have found two solutions. Such nonuniqueness is *not* inconsistent with Theorem 7.2, however, for all Theorem 7.2 asserts is that the solution of the primal isoperimetric problem is an extremal for the reciprocal isoperimetric problem when $b = V(c)$. That this is true in this example can be seen by inspecting Eqs. (14) and (15), which reveals that $z_1(t;c) \equiv z_2(t;c)\, \forall\, t \in [0, 1]$. Theorem 7.2 also asserts that the corresponding multiplier for the primal isoperimetric problem is the reciprocal of a corresponding multiplier in the reciprocal isoperimetric problem when $b = V(c)$. That this is also true in this example can be seen by inspecting Eqs. (14) and (15) again. It is also important to note that Theorem 7.2 does *not* assert that the extremal for the primal isoperimetric problem is the only extremal for the reciprocal isoperimetric problem.

Now substitute the curve $z_2(t;c) = -6ct^2 + 6ct$ in the objective functional $K[\cdot]$ of the reciprocal isoperimetric problem to find its optimal value:

$$\begin{aligned} K[z_2(\cdot)] &= \int_0^1 z_2(t;c)\,dt = \int_0^1 [-6ct^2 + 6ct]\,dt \\ &= [-2ct^3 + 3ct^2]\Big|_{t=0}^{t=1} = -2c + 3c = c > 0. \end{aligned} \tag{17}$$

A similar substitution using the curve $z_3(t;c) = 6ct^2 - 6ct$ yields

$$\begin{aligned} K[z_3(\cdot)] &= \int_0^1 z_3(t)\,dt = \int_0^1 [6ct^2 - 6ct]\,dt \\ &= [2ct^3 - 3ct^2]\Big|_{t=0}^{t=1} = 2c - 3c = -c < 0. \end{aligned} \tag{18}$$

Inspection of Eqs. (17) and (18) shows that the curve $z_2(t;c) = -6ct^2 + 6ct$ yields the *maximum* of the reciprocal isoperimetric problem (which is how it is stated), whereas the curve $z_3(t;c) = 6ct^2 - 6ct$ yields the *minimum* of the reciprocal isoperimetric problem. We will also verify these conclusions in Example 7.3, after we develop a sufficiency theorem.

Another interesting feature of this example can be uncovered by recalling the definition of the optimal value function for the reciprocal isoperimetric problem, namely, $W(b) \stackrel{\text{def}}{=} K[z_2(\cdot)]$, where b is the value of the integral constraint in the reciprocal isoperimetric problem. Equation (17) shows that optimal value function for the maximizing reciprocal isoperimetric problem is the inverse of the optimal value function for the primal isoperimetric problem when $b = V(c)$. To see this, simply observe that $W(V(c)) \stackrel{\text{def}}{=} K[z_2(\cdot)]|_{b=V(c)}$ and that $K[z_2(\cdot)]|_{b=V(c)} = c$ from Eq. (17). These two results imply that $W(V(c)) = c$, or that $W(\cdot) = V^{-1}(\cdot)$. Hence the optimal value functions for the *stated* reciprocal pair of isoperimetric problems are inverses of one another. This inverse relationship between the optimal value functions is in fact true more generally, and not at all an artifact of the simple mathematical structure of this example, as we will see in the next chapter.

The next chapter will explore in more detail, and at a higher level of generality, the qualitative relationships between reciprocal pairs of isoperimetric problems that Theorem 7.2 and Example 7.1 have exposed. In the process of doing so, we will also rather exhaustively study the comparative dynamics properties of the nonrenewable resource–extracting model of the firm.

It is appropriate at this point to turn to an economic interpretation of the multiplier ψ in Theorems 7.1 and 7.2. The analysis that follows smacks of the classical envelope theorem of static microeconomic theory. It is therefore wise to take mental notes on the similarities and differences between the ensuing derivation and the more familiar static envelope theorem, for the envelope theorem plays just as important a role in uncovering the qualitative properties of dynamic optimization problems as it does in static optimization problems.

To begin the analysis, define the *optimal value function* as

$$V(\boldsymbol{\beta}) \stackrel{\text{def}}{=} \max_{\mathbf{x}(\cdot)} \int_{t_0}^{t_1} F(t, \mathbf{x}(t), \dot{\mathbf{x}}(t); \boldsymbol{\alpha})\, dt$$

$$\text{s.t.} \int_{t_0}^{t_1} G(t, \mathbf{x}(t), \dot{\mathbf{x}}(t); \boldsymbol{\alpha})\, dt = c,\ \mathbf{x}(t_0) = \mathbf{x}_0,\ \mathbf{x}(t_1) = \mathbf{x}_1, \tag{19}$$

The solution curve to this problem, which is assumed to exist, is denoted by $\mathbf{z}(t; \boldsymbol{\beta})$, where $\psi^*(\boldsymbol{\beta})$ is the value of the corresponding multiplier and $\boldsymbol{\beta} \stackrel{\text{def}}{=} (\boldsymbol{\alpha}, c, t_0, x_0, t_1, x_1)$ is the vector of parameters of the isoperimetric problem (19). Note that problem (19) is almost identical to problem (1), the difference being that now the functions $F(\cdot)$ and $G(\cdot)$ depend on the parameter vector $\boldsymbol{\alpha} \in \Re^A$.

Observe that the optimal value function $V(\cdot)$ is defined in a manner that is perfectly analogous to the way the indirect objective function of static optimization theory is defined. In other words, the optimal value function is the intertemporal analogue of the indirect objective function. Moreover, the optimal value function $V(\cdot)$ can also be defined *constructively* as

$$V(\boldsymbol{\beta}) \equiv \int_{t_0}^{t_1} F(t, \mathbf{z}(t; \boldsymbol{\beta}),\ \dot{\mathbf{z}}(t; \boldsymbol{\beta}); \boldsymbol{\alpha})\, dt. \tag{20}$$

Note that this constructive way of defining the optimal value function $V(\cdot)$ is completely analogous to the constructive definition of the indirect objective function. By a constructive definition, we mean literally how one would go about finding or constructing the optimal value function $V(\cdot)$ in practice if one had an explicit formula for the optimal path $\mathbf{z}(t; \boldsymbol{\beta})$, its time derivative $\dot{\mathbf{z}}(t; \boldsymbol{\beta})$, and the integrand function $F(\cdot)$.

To get an economic interpretation of the multiplier $\psi^*(\boldsymbol{\beta})$, first recall that by assumption (I.1), $F(\cdot) \in C^{(2)}$ and $G(\cdot) \in C^{(2)}$ on their domains, and that $z(\cdot) \in C^{(2)}$ by assumption (I.2). This means that the partial derivative $\partial V(\boldsymbol{\beta})/\partial c$ exists and can be computed from Eq. (20) by differentiating under the integral sign by way of Leibniz's rule. Doing just that using the heretofore established vector notation yields

$$\frac{\partial V}{\partial c}(\boldsymbol{\beta}) \equiv \int_{t_0}^{t_1} \left[\underbrace{F_{\mathbf{x}}(t, \mathbf{z}(t;\boldsymbol{\beta}), \dot{\mathbf{z}}(t;\boldsymbol{\beta}); \boldsymbol{\alpha})}_{1\times N} \underbrace{\frac{\partial \mathbf{z}}{\partial c}(t;\boldsymbol{\beta})}_{N\times 1} + \underbrace{F_{\dot{\mathbf{x}}}(t, \mathbf{z}(t;\boldsymbol{\beta}), \dot{\mathbf{z}}(t;\boldsymbol{\beta}); \boldsymbol{\alpha})}_{N\times 1} \underbrace{\frac{\partial \dot{\mathbf{z}}}{\partial c}(t;\boldsymbol{\beta})}_{N\times 1} \right] dt. \tag{21}$$

Because the curve $\mathbf{z}(t;\boldsymbol{\beta})$ is the optimal solution to isoperimetric problem (19) it must therefore satisfy the integral constraint, the initial condition, and the terminal condition *identically*. Upon differentiating these three identities with respect to c, we have

$$c - \int_{t_0}^{t_1} G(t, z(t;\boldsymbol{\beta}),\ \dot{z}(t;\boldsymbol{\beta}); \boldsymbol{\alpha})\, dt \equiv 0 \Rightarrow$$

$$1 - \int_{t_0}^{t_1} \left[\underbrace{G_{\mathbf{x}}(t, \mathbf{z}(t;\boldsymbol{\beta}), \dot{\mathbf{z}}(t;\boldsymbol{\beta}); \boldsymbol{\alpha})}_{1\times N} \underbrace{\frac{\partial \mathbf{z}}{\partial c}(t;\boldsymbol{\beta})}_{N\times 1} + \underbrace{G_{\dot{\mathbf{x}}}(t, \mathbf{z}(t;\boldsymbol{\beta}), \dot{\mathbf{z}}(t;\boldsymbol{\beta}); \boldsymbol{\alpha})}_{1\times N} \underbrace{\frac{\partial \dot{\mathbf{z}}}{\partial c}(t;\boldsymbol{\beta})}_{N\times 1} \right] dt \equiv 0, \tag{22}$$

$$\mathbf{z}(t_0;\boldsymbol{\beta}) \equiv \mathbf{x}_0 \Rightarrow \frac{\partial \mathbf{z}}{\partial c}(t_0;\boldsymbol{\beta}) \equiv \mathbf{0}_N, \tag{23}$$

$$\mathbf{z}(t_1;\boldsymbol{\beta}) \equiv \mathbf{x}_1 \Rightarrow \frac{\partial \mathbf{z}}{\partial c}(t_1;\boldsymbol{\beta}) \equiv \mathbf{0}_N. \tag{24}$$

Now multiply Eq. (22) by $\psi^*(\boldsymbol{\beta})$, the multiplier's solution to problem (19), and add the result to Eq. (21) to get

$$\frac{\partial V}{\partial c}(\boldsymbol{\beta}) \equiv \int_{t_0}^{t_1} \left[\tilde{F}_{\mathbf{x}}(t, \mathbf{z}(t;\boldsymbol{\beta}), \dot{\mathbf{z}}(t;\boldsymbol{\beta}), \psi^*(\boldsymbol{\beta}); \boldsymbol{\alpha}) \frac{\partial \mathbf{z}}{\partial c}(t;\boldsymbol{\beta}) + \tilde{F}_{\dot{\mathbf{x}}}(t, \mathbf{z}(t;\boldsymbol{\beta}), \dot{\mathbf{z}}(t;\boldsymbol{\beta}), \psi^*(\boldsymbol{\beta}); \boldsymbol{\alpha}) \frac{\partial \dot{\mathbf{z}}}{\partial c}(t;\boldsymbol{\beta}) \right] dt + \psi^*(\boldsymbol{\beta}), \tag{25}$$

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where $\tilde{F}(t, \mathbf{x}, \dot{\mathbf{x}}, \psi; \boldsymbol{\alpha}) \stackrel{\text{def}}{=} F(t, \mathbf{x}, \dot{\mathbf{x}}; \boldsymbol{\alpha}) - \psi G(t, \mathbf{x}, \dot{\mathbf{x}}; \boldsymbol{\alpha})$ is the augmented integrand function defined previously. Note that the step taken in going from Eq. (21) to Eq. (25) is perfectly valid, for all we have done is simply add a term that is *identically zero* to Eq. (21) to arrive at Eq. (25).

Next, integrate the second term of the integrand of Eq. (25) by parts to get

$$\left.\begin{aligned} \mathbf{p}' &= \tilde{F}_{\dot{\mathbf{x}}}(t, \mathbf{z}(t;\boldsymbol{\beta}), \dot{\mathbf{z}}(t;\boldsymbol{\beta}), \psi^*(\boldsymbol{\beta}); \boldsymbol{\alpha}), & \mathbf{dq} &= \frac{\partial \dot{\mathbf{z}}}{\partial c}(t;\boldsymbol{\beta})\,dt = \frac{d}{dt}\left[\frac{\partial \mathbf{z}}{\partial c}(t;\boldsymbol{\beta})\right]dt \\ \mathbf{dp}' &= \frac{d}{dt}\tilde{F}_{\dot{\mathbf{x}}}(t, \mathbf{z}(t;\boldsymbol{\beta}), \dot{\mathbf{z}}(t;\boldsymbol{\beta}), \psi^*(\boldsymbol{\beta}); \boldsymbol{\alpha})\,dt, & \mathbf{q} &= \frac{\partial \mathbf{z}}{\partial c}(t;\boldsymbol{\beta}) \end{aligned}\right\} \Rightarrow$$

$$\begin{aligned} &\int_{t_0}^{t_1}\left[\tilde{F}_{\dot{\mathbf{x}}}(t, \mathbf{z}(t;\boldsymbol{\beta}), \dot{\mathbf{z}}(t;\boldsymbol{\beta}), \psi^*(\boldsymbol{\beta}); \boldsymbol{\alpha})\frac{\partial \dot{\mathbf{z}}}{\partial c}(t;\boldsymbol{\beta})\right]dt \\ &\quad = \tilde{F}_{\dot{\mathbf{x}}}(t, \mathbf{z}(t;\boldsymbol{\beta}), \dot{\mathbf{z}}(t;\boldsymbol{\beta}), \psi^*(\boldsymbol{\beta}); \boldsymbol{\alpha})\frac{\partial \mathbf{z}}{\partial c}(t;\boldsymbol{\beta})\Big|_{t=t_0}^{t=t_1} \\ &\qquad - \int_{t_0}^{t_1}\left[\frac{d}{dt}\tilde{F}_{\dot{\mathbf{x}}}(t, \mathbf{z}(t;\boldsymbol{\beta}), \dot{\mathbf{z}}(t;\boldsymbol{\beta}), \psi^*(\boldsymbol{\beta}); \boldsymbol{\alpha})\right]\frac{\partial \mathbf{z}}{\partial c}(t;\boldsymbol{\beta})\,dt. \end{aligned} \tag{26}$$

Because $\partial \mathbf{z}(t_0;\boldsymbol{\beta})/\partial c \equiv \mathbf{0}_N$ from Eq. (23) and $\partial \mathbf{z}(t_1;\boldsymbol{\beta})/\partial c \equiv \mathbf{0}_N$ from Eq. (24), Eq. (26) reduces to

$$\begin{aligned} &\int_{t_0}^{t_1}\left[\tilde{F}_{\dot{\mathbf{x}}}(t, \mathbf{z}(t;\boldsymbol{\beta}), \dot{\mathbf{z}}(t;\boldsymbol{\beta}), \psi^*(\boldsymbol{\beta}); \boldsymbol{\alpha})\frac{\partial \dot{\mathbf{z}}}{\partial c}(t;\boldsymbol{\beta})\right]dt \\ &\quad = -\int_{t_0}^{t_1}\left[\frac{d}{dt}\tilde{F}_{\dot{\mathbf{x}}}(t, \mathbf{z}(t;\boldsymbol{\beta}), \dot{\mathbf{z}}(t;\boldsymbol{\beta}), \psi^*(\boldsymbol{\beta}); \boldsymbol{\alpha})\right]\frac{\partial \mathbf{z}}{\partial c}(t;\boldsymbol{\beta})\,dt. \end{aligned} \tag{27}$$

Substituting Eq. (27) into Eq. (25) gives the penultimate expression we are after, to wit,

$$\begin{aligned} \frac{\partial V}{\partial c}(\boldsymbol{\beta}) \equiv \int_{t_0}^{t_1}&\Big[\tilde{F}_{\mathbf{x}}(t, \mathbf{z}(t;\boldsymbol{\beta}), \dot{\mathbf{z}}(t;\boldsymbol{\beta}), \psi^*(\boldsymbol{\beta}); \boldsymbol{\alpha}) \\ &- \frac{d}{dt}\tilde{F}_{\dot{\mathbf{x}}}(t, \mathbf{z}(t;\boldsymbol{\beta}), \dot{\mathbf{z}}(t;\boldsymbol{\beta}), \psi^*(\boldsymbol{\beta}); \boldsymbol{\alpha})\Big]\frac{\partial \mathbf{z}}{\partial c}(t;\boldsymbol{\beta})\,dt + \psi^*(\boldsymbol{\beta}). \end{aligned} \tag{28}$$

Because the curve $\mathbf{z}(t;\boldsymbol{\beta})$ is the solution to isoperimetric problem (19) and $\psi^*(\boldsymbol{\beta})$ is the corresponding value of the multiplier, $\tilde{F}_{\mathbf{x}}(t, \mathbf{z}(t;\boldsymbol{\beta}), \dot{\mathbf{z}}(t;\boldsymbol{\beta}), \psi^*(\boldsymbol{\beta}); \boldsymbol{\alpha}) -$

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$\frac{d}{dt}\tilde{F}_{\dot{\mathbf{x}}}(t, \mathbf{z}(t;\boldsymbol{\beta}), \dot{\mathbf{z}}(t;\boldsymbol{\beta}), \psi^*(\boldsymbol{\beta}); \boldsymbol{\alpha}) \equiv \mathbf{0}'_N$ by Theorem 7.1. Using this result, Eq. (28) simplifies to

$$\frac{\partial V}{\partial c}(\boldsymbol{\beta}) \equiv \psi^*(\boldsymbol{\beta}), \tag{29}$$

which is what we wished to show.

The economic interpretation of $\psi^*(\boldsymbol{\beta})$ comes from looking at the left-hand side of Eq. (29), since it is identically equal to $\psi^*(\boldsymbol{\beta})$. Equation (29) thus asserts that $\psi^*(\boldsymbol{\beta})$ measures the rate of change in the optimal value function when the integral constraint parameter c changes. Thus $\lambda^*(\boldsymbol{\beta})$ has the interpretation of the *marginal value* or *shadow price* of the parameter c.

In sum, we have proven part (ii) of the following important theorem, a dynamic envelope theorem, the remaining parts of which you are asked to prove in the mental exercises.

Theorem 7.3 (Dynamic Envelope Theorem): *For the isoperimetric problem (19), with assumptions (I.1), (I.2), and (I.3) holding, the following dynamic envelope results hold for the optimal value function $V(\cdot)$:*

(i) $V_{\alpha_i}(\boldsymbol{\beta}) \equiv \int_{t_0}^{t_1} \tilde{F}_{\alpha_i}(t, \mathbf{z}(t;\boldsymbol{\beta}), \dot{\mathbf{z}}(t;\boldsymbol{\beta}), \psi^*(\boldsymbol{\beta}); \boldsymbol{\alpha})\,dt, \quad i = 1, 2, \ldots, A,$

(ii) $V_c(\boldsymbol{\beta}) \equiv \psi^*(\boldsymbol{\beta}),$

(iii) $V_{t_0}(\boldsymbol{\beta}) \equiv -\tilde{F}(t_0, \mathbf{x}_0, \dot{\mathbf{z}}(t_0;\boldsymbol{\beta}), \psi^*(\boldsymbol{\beta}); \boldsymbol{\alpha}) + \tilde{F}_{\dot{\mathbf{x}}}(t_0, \mathbf{x}_0, \dot{\mathbf{z}}(t_0;\boldsymbol{\beta}), \psi^*(\boldsymbol{\beta}); \boldsymbol{\alpha})\,\dot{\mathbf{z}}(t_0;\boldsymbol{\beta}),$

(iv) $V_{\mathbf{x}_0}(\boldsymbol{\beta}) \equiv -\tilde{F}_{\dot{\mathbf{x}}}(t_0, \mathbf{x}_0, \dot{\mathbf{z}}(t_0;\boldsymbol{\beta}), \psi^*(\boldsymbol{\beta}); \boldsymbol{\alpha}),$

(v) $V_{t_1}(\boldsymbol{\beta}) \equiv \tilde{F}(t_1, \mathbf{x}_1, \dot{\mathbf{z}}(t_1;\boldsymbol{\beta}), \psi^*(\boldsymbol{\beta}); \boldsymbol{\alpha}) - \tilde{F}_{\dot{\mathbf{x}}}(t_1, \mathbf{x}_1, \dot{\mathbf{z}}(t_1;\boldsymbol{\beta}), \psi^*(\boldsymbol{\beta}); \boldsymbol{\alpha})\,\dot{\mathbf{z}}(t_1;\boldsymbol{\beta}),$

(vi) $V_{\mathbf{x}_1}(\boldsymbol{\beta}) \equiv \tilde{F}_{\dot{\mathbf{x}}}(t_1, \mathbf{x}_1, \dot{\mathbf{z}}(t_1;\boldsymbol{\beta}), \psi^*(\boldsymbol{\beta}); \boldsymbol{\alpha}),$

where $\tilde{F}(t, \mathbf{x}, \dot{\mathbf{x}}, \psi; \boldsymbol{\alpha}) \overset{\text{def}}{=} F(t, \mathbf{x}, \dot{\mathbf{x}}; \boldsymbol{\alpha}) - \psi G(t, \mathbf{x}, \dot{\mathbf{x}}; \boldsymbol{\alpha})$ *is the augmented integrand function.*

Let's now consider an economic example in order to drive home some of the points just made, namely, the Hotelling model of the nonrenewable resource–extracting firm.

Example 7.2: An owner of a piece of land knows that it has $x_0 > 0$ units of some nonrenewable asset, such as oil, in the ground. The owner wants to determine the extraction path $q(t)$ over a fixed planning period $[0, T]$ that will maximize the present discounted value of profit associated with extracting and selling the asset. Let $\pi(q(t))$ be the instantaneous profit from extracting and selling the nonrenewable resource at rate $q(t)$, which is discounted at rate $r > 0$. Assume that $\pi(\cdot) \in C^{(2)}$ on its domain and that $\pi'(q) > 0$ and $\pi''(q) < 0$. The owner has decided that all of the resource will be extracted by the end of the planning period, implying that cumulative extraction must equal the initial stock. Putting all this information together, we can formally state the isoperimetric problem corresponding to this scenario as

$$\Pi(r, x_0, T) \stackrel{\text{def}}{=} \left\{ \max_{q(\cdot)} J[q(\cdot)] \stackrel{\text{def}}{=} \int\limits_0^T \pi(q(t))\, e^{-rt}\, dt \right\}$$

$$\text{s.t.} \quad K[q(\cdot)] \stackrel{\text{def}}{=} \int\limits_0^T q(t)\, dt = x_0.$$

Note that this formulation is identical to that in Mental Exercise 4.21.

The augmented integrand for this problem is defined as $\tilde{F}(t, q, \psi) \stackrel{\text{def}}{=} \pi(q)e^{-rt} - \psi q$, where by Theorem 7.3, ψ can be interpreted as the present value shadow price of the unextracted resource, the adjective *present value* arising from the presence of the discount factor in the integrand. The partial derivatives of the augmented integrand function are therefore given by $\tilde{F}_q(t, q, \psi) = \pi'(q)e^{-rt} - \psi$ and $\tilde{F}_{\dot{q}}(t, q, \psi) \equiv 0$, thereby implying that the augmented Euler equation takes the form

$$\tilde{F}_q(t, q, \psi) = \pi'(q)\, e^{-rt} - \psi = 0, \tag{30}$$

where ψ is a constant by Theorem 7.1.

Equation (30) says that the optimal extraction rate is such that the present value of marginal profit, to wit, $\pi'(q)e^{-rt}$, is constant over the planning period and equal to the multiplier ψ, the present value shadow price of the unextracted resource. Alternatively, Eq. (30) can be rearranged as $\pi'(q) = \lambda e^{rt}$, implying that marginal profit grows at the discount rate $r > 0$. Because Eq. (30) holds identically in t for all $t \in [0, T]$ along the optimal path, it is valid to differentiate it with respect to t. Doing so and recalling that ψ is a constant gives $\pi''(q)\dot{q}e^{-rt} - r\pi'(q)\, e^{-rt} = 0$. Multiplying this differential equation through by e^{rt}, recognizing that $\frac{d}{dt}\pi'(q) = \pi''(q)\dot{q}$, and then rearranging it gives

$$\frac{\frac{d}{dt}[\pi'(q)]}{\pi'(q)} = \frac{\pi''(q)\dot{q}}{\pi'(q)} = r > 0. \tag{31}$$

Equation (31) is known as *Hotelling's rule*. It asserts that the optimal extraction rate equates the relative rate of change of the marginal profit from extracting and selling a unit of the nonrenewable resource, with the rate at which an alternative asset would grow if it was placed in an interest-bearing account earning interest at the rate $r > 0$.

Solving Hotelling's rule (31) for $\dot{q}$ and recalling that $\pi'(q) > 0$ and $\pi''(q) < 0$ by assumption gives

$$\dot{q} = \frac{\pi'(q)}{\pi''(q)} r < 0. \tag{32}$$

Equation (32) shows that the optimal extraction rate of the nonrenewable resource is declining over the planning horizon $[0, T]$. This implies that the extraction rate

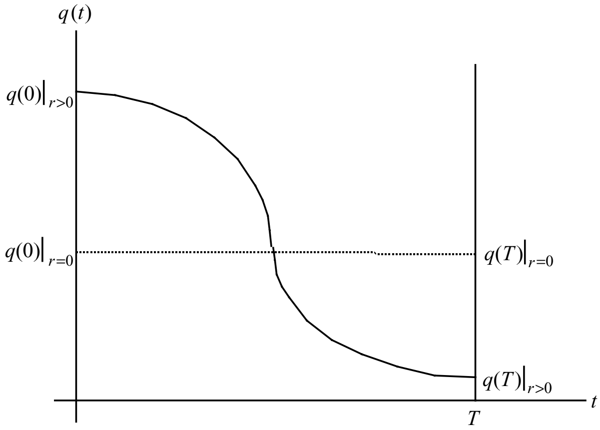

Figure 7.2

is the largest at the initial date of the planning horizon and smallest at the terminal date of the planning horizon. Equation (32) also shows that if $r = 0$, then $\dot{q} = 0$, implying that the extraction rate is constant over the planning horizon if the firm does not discount its future flow of profits. We can therefore conclude from these two observations that the discount rate has the effect of shifting the extraction profile toward the present, a typical impatience result. In fact, since the planning horizon is fixed and the owner is required to extract all the resource from the ground, it follows that the initial extraction rate when $r > 0$ must exceed the extraction rate when $r = 0$, because otherwise, the owner would never extract all of the asset when $r > 0$. Moreover, as the terminal date of the planning horizon approaches, the extraction rate with $r > 0$ must be less than the extraction rate with $r = 0$. Finally, the area under the extraction path when $r > 0$ is identical to the area under the extraction path when $r = 0$ because of the integral constraint that requires the owner to extract all of the asset from the ground over the fixed planning horizon, that is, because of the fact that $\int_0^T q(t)\,dt = x_0$ and T and x_0 are fixed parameters. See Figure 7.2 for the geometry behind these qualitative conclusions.

Let's finish up this example by providing a more motivated and careful economic interpretation of the optimal value of the constant multiplier $\psi^*(r, x_0, T)$ using Theorem 7.3. To begin, recall that the optimal value function $\Pi(\cdot)$ in this model is the maximum present discounted value of profit from extracting and selling the resource stock, and that the equivalent of the parameter c in Theorem 7.3 is the initial size of the nonrenewable resource deposit x_0. By Theorem 7.3, we have that $\partial\Pi(r, x_0, T)/\partial x_0 = \psi^*(r, x_0, T)$. Thus $\psi^*(r, x_0, T)$ is the *present value shadow price* of the initial stock of the resource, since it is the amount by which the present discounted profits of the firm increase when the initial size of the resource deposit increases. In other words, $\psi^*(r, x_0, T)$ is the maximum amount the owner of the firm would pay for a small increase in the initial size of the nonrenewable resource deposit, since $\psi^*(r, x_0, T)$ is precisely the amount by which the present discounted value of profits would rise because of the increase in the initial resource deposit. We can therefore think of $\psi^*(r, x_0, T)$ as the present value of one more unit of the resource stock to the owner of the firm, and in deciding upon whether to purchase one more unit of the stock *in situ*, the owner would compare the value of the stock to her, as given by $\psi^*(r, x_0, T)$, to the market price of the resource stock.

In the third example, we investigate the principal-agent problem with hidden actions. We will make use of the necessary conditions in Theorem 7.1 as well as the dynamic envelope results of Theorem 7.3 to solve the problem and deduce its comparative statics properties.

Example 7.3: Suppose that the owner of a firm, the *principal*, desires to hire a manager, the *agent*, for a one-time project. (It is this sort of conceptualization that has resulted in the appellation *principal-agent* for the resulting optimal control problem.) The profitability of the project is given by the random variable $\tilde{\pi}$ and is determined, in part, by the actions of, or choices made by, the manager. Let π be the project's observed (or realized) profit and $e \in E$ be the manager's effort on the project, where $E \subset \Re$ is the set of possible effort levels of the manager. It is assumed that the profit of the project is influenced by the effort of the manager, though not fully determined by it. Consequently, it is assumed that profit may take on any value in the closed interval $[\underline{\pi}, \bar{\pi}]$, and that $\tilde{\pi}$ is stochastically related to $e \in E$ by way of the conditional probability density function $(\pi, e) \mapsto f(\pi \mid e)$, where $f(\pi \mid e) > 0$ and $f(\cdot) \in C^{(2)}$ for all $e \in E$ and $\pi \in [\underline{\pi}, \bar{\pi}]$. Therefore, any potential realization of $\tilde{\pi}$ may arise following any given effort choice $e \in E$ by the manager.

The manager is an expected utility maximizer with a $C^{(2)}$ von Neumann–Morgenstern utility function $u(\cdot)$, with values $u(w, e)$, where w is the wage received by the manager. It is assumed that $u_w(w, e) > 0$, $u_{ww}(w, e) < 0$, and $u_e(w, e) < 0$ for all (w, e). In other words, the manager prefers more income to less, is strictly risk averse over income lotteries, and prefers lower effort. The owner receives the project's profit less the wage payment to the manager, and is assumed to be risk neutral, thereby implying that his objective is the maximization of expected net profit.

Let's examine the so-called optimal contracting problem when the manager's effort is observable to the owner. A contract in this context specifies not only the manager's effort $e \in E$, but also the manager's wage payment as a function of observed (or realized) profit, say, $w(\pi)$. Assuming that a competitive market for managers exists, the owner must therefore provide the manager with an expected utility of at least $\bar{u}$, the manager's reservation utility, if the manager is to accept the owner's contract offer. Note that if the manager rejects the owner's contract offer, then the owner receives a payoff of zero. We henceforth assume that the owner finds it worthwhile to make the manager an offer that he will accept.

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Putting all this information together, we see that the optimal contract for the owner solves the ensuing isoperimetric problem:

$$V^*(\underline{\pi}, \bar{\pi}, \bar{u}) \stackrel{\text{def}}{=} \max_{e \in E, w(\cdot)} \left\{ \int_{\underline{\pi}}^{\bar{\pi}} [\pi - w(\pi)] f(\pi \mid e)\, d\pi \ \text{ s.t. } \int_{\underline{\pi}}^{\bar{\pi}} u(w(\pi), e) f(\pi \mid e)\, d\pi \geq \bar{u} \right\}. \tag{33}$$

At this juncture, it is best to pause and make three rather important remarks about problem (33). First, notice that the independent variable is the realized (or observed) profit of the project. This stands in sharp contrast to every other optimal control problem encountered in the book, where the independent variable has been (or will be) time. Nonetheless, this in no way discredits the optimal contracting problem (33) from being solved by the theorems developed herewith. What is important, therefore, is that problem (33) seeks to determine a *function* of the independent variable; the nature of the independent variable itself, however, is unimportant. Second, problem (33) is a special case of the general isoperimetric problem given in Eq. (1). To see this, simply observe that the choice function in problem (33) is scalar valued, whereas in problem (1), it is vector valued, and that in problem (33), the derivative function $\dot{w}(\cdot)$ is absent. Moreover, it is precisely because problem (33) does not involve the derivative function $\dot{w}(\cdot)$ that no initial and/or terminal conditions on $w(\cdot)$ need to be specified. Third, the isoperimetric constraint always binds at a solution of problem (33). To verify this, note that if it does not bind at a solution $w(\pi)$ for the wage profile, then there exists a lower wage schedule, say, $w(\pi) - \varepsilon$ for sufficiently small $\varepsilon > 0$, such that the manager would still accept the wage schedule $w(\pi) - \varepsilon$, that is, for which, the isoperimetric constraint still does not bind, and that also results in higher expected profit for the owner. This contradicts the optimality of the wage schedule $w(\pi)$ and thus establishes the claim. Henceforth we will write the isoperimetric constraint as an equality constraint.

Rather than tackle problem (33) directly, it is advantageous to solve it in two stages. In the first stage, we fix e at an arbitrary value in the set E, and then seek to determine the best wage function $w(\cdot)$ that the owner should offer the manager. Given this optimal wage function, the second-stage problem determines the optimal choice of the manager's effort e.

The first-stage isoperimetric problem is therefore given by

$$V(e, \underline{\pi}, \bar{\pi}, \bar{u}) \stackrel{\text{def}}{=} \max_{w(\cdot)} \left\{ \int_{\underline{\pi}}^{\bar{\pi}} [\pi - w(\pi)] f(\pi \mid e)\, d\pi \ \text{ s.t. } \int_{\underline{\pi}}^{\bar{\pi}} u(w(\pi), e) f(\pi \mid e)\, d\pi = \bar{u} \right\}. \tag{34}$$

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Notice that the optimal value functions in problems (33) and (34) differ, as they should, in view of the fact that problem (34) holds effort fixed whereas problem (33) does not. The augmented integrand of problem (34) is given by $\tilde{F}(\pi, w, \psi; e) \stackrel{\text{def}}{=} [\pi - w] f(\pi \mid e) - \psi u(w, e) f(\pi \mid e)$. By Theorem 7.1, it is necessary that a solution of problem (34) satisfy the Euler equation based on the augmented integrand function, videlicet, $\tilde{F}_w(\pi, w, \psi; e) = \frac{d}{dt}\tilde{F}_{\dot{w}}(\pi, w, \psi; e)$. Because $\tilde{F}_{\dot{w}}(\pi, w, \psi; e) \equiv 0$, the Euler equation reduces to $\tilde{F}_w(\pi, w, \psi; e) = -f(\pi \mid e) - \psi u_w(w, e) f(\pi \mid e) = 0$. Moreover, because $f(\pi | e) > 0$ for all $e \in E$ and $\pi \in [\underline{\pi}, \bar{\pi}]$, the augmented Euler equation simplifies to

$$-1 - \psi u_w(w, e) = 0. \tag{35}$$

Furthermore, the multiplier ψ is a constant, that is to say, it is not a function of the independent variable π, by Theorem 7.1. Seeing as e is a given value of effort, these two conclusions therefore imply that the optimal wage function, which necessarily satisfies Eq. (35), is not a function of π either. In other words, the optimal wage contract that the owner offers the manager is independent of the realized profit on the project. This conclusion is a risk-sharing result in that the risk-neutral owner fully insures the risk-averse manager against any income risk by making the manager's wage independent of the profit of the project on which the manager is contracted to work.

Seeing as $u(\cdot) \in C^{(2)}$ and $u_{ww}(w, e) < 0$ for all (w, e), the implicit function theorem implies that Eq. (35) may be solved, in principle, for the wage rate as a locally $C^{(1)}$ function of the effort of the manager and the multiplier, that is,

$$w = \hat{w}(e, \psi). \tag{36}$$

It is important to understand that the solution in Eq. (36) is not the optimal wage rate of the manager, for the corresponding value of the multiplier has yet to be determined. Nonetheless, it is still worthwhile to investigate the comparative statics properties of the function $\hat{w}(\cdot)$. Before doing so, it is prudent to first provide an economic interpretation of the multiplier.

To this end, we begin by recalling Theorem 7.3 part (ii), a dynamic envelope theorem, which when applied to problem (34) gives the effect of a change in the reservation utility of the manager on the owner's maximum expected profit (conditional on effort). Specifically, applying Theorem 7.3 part (ii) to problem (34) gives

$$\frac{\partial V}{\partial \bar{u}}(e, \underline{\pi}, \bar{\pi}, \bar{u}) \equiv \psi^*(e, \underline{\pi}, \bar{\pi}, \bar{u}) < 0, \tag{37}$$

where $\psi^*(e, \underline{\pi}, \bar{\pi}, \bar{u})$ is the yet-to-be-determined optimal value of the multiplier. The sign of $\psi^*(e, \underline{\pi}, \bar{\pi}, \bar{u})$ follows from Eq. (35) and the assumption that $u_w(w, e) > 0$ for all (w, e). Given that $\psi^*(e, \underline{\pi}, \bar{\pi}, \bar{u}) < 0$, Eq. (37) therefore permits us to interpret $\psi^*(e, \underline{\pi}, \bar{\pi}, \bar{u})$ as the marginal cost of the manager's reservation utility. The fact that $\partial V(e, \underline{\pi}, \bar{\pi}, \bar{u})/\partial \bar{u} < 0$ is intuitive, because for a given effort, an improvement in the manager's outside opportunities means that the owner must now compensate the

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manager with a higher wage if the owner wishes to get the manager to accept the contract. This increase in the wage offered to the manager raises the costs of the project and thus reduces the owner's profit on the project.

Returning to the qualitative properties of the solution $w = \hat{w}(e, \psi)$ of Eq. (35), we find, by way of the implicit function theorem, that

$$\frac{\partial \hat{w}}{\partial e}(e, \psi) \equiv -\frac{u_{we}(\hat{w}(e, \psi), e)}{u_{ww}(\hat{w}(e, \psi), e)} \gtrless 0, \tag{38}$$

$$\frac{\partial \hat{w}}{\partial \psi}(e, \psi) \equiv -\frac{u_{w}(\hat{w}(e, \psi), e)}{\psi u_{ww}(\hat{w}(e, \psi), e)} < 0, \tag{39}$$

because $u_w(w, e) > 0$ and $u_{ww}(w, e) < 0$ for all (w, e) and $\psi < 0$, whereas no assumption was made regarding the sign of $u_{we}(w, e)$. Equation (38) asserts that the wage offered the manager may rise or fall with an increase in the manager's effort, holding constant the marginal cost of the manager's reservation utility. This may seem like an odd result, but it is wise to remember that $w = \hat{w}(e, \psi)$ is *not* the optimal wage, so it is best to reserve judgment about the ambiguity in the sign of Eq. (38). Equation (39), on the other hand, has a definite and intuitive sign. In particular, because $\psi < 0$, it shows that a fall in the marginal cost of the manager's reservation utility reduces the wage offered to the manager. This is intuitive in that as the outside opportunities of the manager decrease, the owner can offer the manager a lower wage and still get the manager to accept the contract.

To determine the optimal value of the marginal cost of the manager's reservation utility, we make use of the remaining necessary condition, to wit, the isoperimetric constraint. Substituting $w = \hat{w}(e, \psi)$ in the isoperimetric constraint thus yields

$$\int_{\underline{\pi}}^{\bar{\pi}} u(\hat{w}(e, \psi), e) f(\pi \mid e)\, d\pi = \bar{u}.$$

But in view of the fact that $w = \hat{w}(e, \psi)$ is independent of π, so is $u(\hat{w}(e, \psi), e)$. Consequently, $u(\hat{w}(e, \psi), e)$ may be factored out in front of the integral, which gives

$$u(\hat{w}(e, \psi), e) \int_{\underline{\pi}}^{\bar{\pi}} f(\pi \mid e)\, d\pi = \bar{u}.$$

Recalling that $f(\pi \mid e)$ is the value of the conditional probability density function, it must, by definition, integrate to unity over the interval $[\underline{\pi}, \bar{\pi}]$. As a result, the above equation reduces to

$$u(\hat{w}(e, \psi), e) = \bar{u}. \tag{40}$$

The Jacobian of Eq. (40) with respect to ψ is given by

$$u_w(\hat{w}(e, \psi), e) \frac{\partial \hat{w}}{\partial \psi}(e, \psi) < 0,$$

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the sign of which follows from the assumption that $u_w(w, e) > 0$ for all (w, e) and Eq. (39). Consequently, the implicit function theorem implies that Eq. (40) implicitly determines ψ as a locally $C^{(1)}$ function $(e, \bar{u})$, say,

$$\psi = \psi^*(e, \bar{u}). \tag{41}$$

Equation (41) thus expresses the optimal value of the marginal cost of the manager's reservation utility as a function of the manager's effort and his reservation utility, but not the endpoints $\underline{\pi}$ and $\bar{\pi}$, as we had initially indicated in Eq. (37).

Differentiating the identity $u(\hat{w}(e, \psi^*(e, \bar{u})), e) \equiv \bar{u}$ with respect to e and $\bar{u}$ gives the qualitative properties of the function $\psi^*(\cdot)$, videlicet,

$$\frac{\partial \psi^*}{\partial e}(e, \bar{u}) \equiv -\frac{u_w(\hat{w}(e, \psi^*(e, \bar{u})), e)\frac{\partial \hat{w}}{\partial e}(e, \psi^*(e, \bar{u})) + u_e(\hat{w}(e, \psi^*(e, \bar{u})), e)}{u_w(\hat{w}(e, \psi^*(e, \bar{u})), e)\frac{\partial \hat{w}}{\partial \psi}(e, \psi^*(e, \bar{u}))} \gtrless 0, \tag{42}$$

$$\frac{\partial \psi^*}{\partial \bar{u}}(e, \bar{u}) \equiv \frac{1}{u_w(\hat{w}(e, \psi^*(e, \bar{u})), e)\frac{\partial \hat{w}}{\partial \psi}(e, \psi^*(e, \bar{u}))} < 0. \tag{43}$$

Equation (43) asserts that the marginal cost of the manager's reservation utility is a strictly decreasing function of the manager's reservation utility. Moreover, by Eqs. (37) and (43), it follows that

$$\frac{\partial^2 V}{\partial \bar{u}^2}(e, \underline{\pi}, \bar{\pi}, \bar{u}) \equiv \frac{\partial \psi^*}{\partial \bar{u}}(e, \bar{u}) < 0,$$

thereby implying that the owner's expected profit is a strictly decreasing and strictly concave function of the manager's reservation utility.

Now turn to the determination of the optimal wage rate of the manager, say $w^*(e, \bar{u})$, conditional on his effort. In particular, substituting $\psi = \psi^*(e, \bar{u})$ into $w = \hat{w}(e, \psi)$ gives $w^*(e, \bar{u})$, that is, the optimal conditional wage rate of the manager is given by the identity

$$w^*(e, \bar{u}) \equiv \hat{w}(e, \psi^*(e, \bar{u})). \tag{44}$$

Differentiating Eq. (44) with respect to e and $\bar{u}$ using the chain rule, and making use of Eqs. (39) and (42) through (44) yields

$$\begin{aligned}\frac{\partial w^*}{\partial e}(e, \bar{u}) &\equiv \frac{\partial \hat{w}}{\partial e}(e, \psi^*(e, \bar{u})) + \frac{\partial \hat{w}}{\partial \psi}(e, \psi^*(e, \bar{u}))\frac{\partial \psi^*}{\partial e}(e, \bar{u}) \\ &= \frac{\partial \hat{w}}{\partial e}(e, \psi^*(e, \bar{u})) - \frac{\partial \hat{w}}{\partial \psi}(e, \psi^*(e, \bar{u})) \\ &\quad \times \left[\frac{u_w(w^*(e, \bar{u}), e)\frac{\partial \hat{w}}{\partial e}(e, \psi^*(e, \bar{u})) + u_e(w^*(e, \bar{u}), e)}{u_w(w^*(e, \bar{u}), e)\frac{\partial \hat{w}}{\partial \psi}(e, \psi^*(e, \bar{u}))}\right]\end{aligned} \tag{45}$$

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$$= -\frac{u_e(w^*(e,\bar{u}),e)}{u_w(w^*(e,\bar{u}),e)} > 0,$$

$$\frac{\partial w^*}{\partial \bar{u}}(e,\bar{u}) \equiv \frac{\partial \hat{w}}{\partial \psi}(e,\psi^*(e,\bar{u}))\frac{\partial \psi^*}{\partial \bar{u}}(e,\bar{u}) > 0. \tag{46}$$

Equation (45) shows that an increase in the effort exerted by the manager will be met with a higher conditional wage by the owner, the intuitive result that was lacking in Eq. (38). Equation (46) also yields the intuitive conclusion that the conditional wage rate of the manager increases with an increase in the manager's reservation utility, that is, with his outside opportunities. This completes the examination of the first stage of the optimal contracting problem.

The second stage of the optimal contracting problem seeks to determine the choice of the managers effort, which, as you should recall, was held fixed at an arbitrary value $e \in E$ throughout stage one. In view of the fact that we have broken the optimal contracting problem given in Eq. (33) into two stages, the optimal value function $V^*(\cdot)$ of problem (33) is defined as the maximum value of the optimal value function $V(\cdot)$ found by solving the first-stage problem (34) with respect to the manager's effort, to wit,

$$V^*(\underline{\pi},\bar{\pi},\bar{u}) \overset{\text{def}}{=} \max_{e\in E} V(e,\underline{\pi},\bar{\pi},\bar{u}). \tag{47}$$

Assuming that $e = e^*(\underline{\pi},\bar{\pi},\bar{u})$ is a solution to optimization problem (47), the first-order necessary condition this solution must satisfy is therefore $\partial V(e,\underline{\pi},\bar{\pi},\bar{u})/\partial e = 0$. Remembering that $\tilde{F}(\pi,w,\psi;e) \overset{\text{def}}{=} [\pi - w]f(\pi \mid e) - \psi u(w,e)f(\pi \mid e)$, Theorem 7.3 part (i) implies that the first-order necessary condition $\partial V(e,\underline{\pi},\bar{\pi},\bar{u})/\partial e = 0$ takes the form

$$\frac{\partial V}{\partial e}(e,\underline{\pi},\bar{\pi},\bar{u}) = \int_{\underline{\pi}}^{\bar{\pi}} \frac{\partial \tilde{F}}{\partial e}(\pi,w,\psi;e)\Big|_{\substack{w=w^*(e,\bar{u})\\ \psi=\psi^*(e,\bar{u})}} d\pi$$

$$= \int_{\underline{\pi}}^{\bar{\pi}} [[\pi - w^*(e,\bar{u})]f_e(\pi \mid e) - \psi^*(e,\bar{u})[u(w^*(e,\bar{u}),e)f_e(\pi \mid e)$$

$$+ u_e(w^*(e,\bar{u}),e)f(\pi \mid e)]]\,d\pi = 0.$$

Given that $f(\cdot)$ is a probability density function, it satisfies

$$\int_{\underline{\pi}}^{\bar{\pi}} f(\pi \mid e)\,d\pi = 1 \tag{48}$$

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for all $e \in E$, and thus is an identity in e. In view of this conclusion, we may differentiate Eq. (48) with respect to e to get

$$\int_{\underline{\pi}}^{\bar{\pi}} f_e(\pi \mid e)\, d\pi = 0. \tag{49}$$

Substituting Eqs. (48) and (49) into the above first-order necessary condition, and using the fact that $w^*(e, \bar{u})$ and $\psi^*(e, \bar{u})$ are independent π, we arrive at

$$\frac{\partial V}{\partial e}(e, \underline{\pi}, \bar{\pi}, \bar{u}) = \int_{\underline{\pi}}^{\bar{\pi}} \pi f_e(\pi \mid e)\, d\pi - \psi^*(e, \bar{u}) u_e(w^*(e, \bar{u}), e) = 0. \tag{50}$$

Rearranging Eq. (50) yields a simpler form to interpret, scilicet,

$$\int_{\underline{\pi}}^{\bar{\pi}} \pi f_e(\pi \mid e)\, d\pi = \psi^*(e, \bar{u}) u_e(w^*(e, \bar{u}), e). \tag{51}$$

The left-hand side of Eq. (51) is the expected marginal gross profit of the manager's effort, whereas the right-hand side is the product of the marginal cost of the manager's reservation utility and his marginal disutility of effort. Thus the optimal choice of the manager's effort by the owner obeys an intuitive marginal condition, namely, that the expected marginal gross profit of managerial effort equals the marginal cost of managerial effort.

The next step in the analysis is the determination of the qualitative properties of optimal managerial effort, $e = e^*(\underline{\pi}, \bar{\pi}, \bar{u})$. To this end, assume that the second-order sufficient condition holds at the solution to problem (47), that is, $V_{ee}(e^*(\underline{\pi}, \bar{\pi}, \bar{u}), \underline{\pi}, \bar{\pi}, \bar{u}) < 0$. Then substituting $e = e^*(\underline{\pi}, \bar{\pi}, \bar{u})$ in Eq. (50) and differentiating with respect to $(\underline{\pi}, \bar{\pi}, \bar{u})$, using both the chain rule and Leibniz's rule gives

$$\frac{\partial e^*}{\partial \underline{\pi}}(\underline{\pi}, \bar{\pi}, \bar{u}) \equiv \frac{\underline{\pi} f_e(\underline{\pi} \mid e^*(\underline{\pi}, \bar{\pi}, \bar{u}))}{V_{ee}(e^*(\underline{\pi}, \bar{\pi}, \bar{u}), \underline{\pi}, \bar{\pi}, \bar{u})} \gtrless 0, \tag{52}$$

$$\frac{\partial e^*}{\partial \bar{\pi}}(\underline{\pi}, \bar{\pi}, \bar{u}) \equiv \frac{-\bar{\pi} f_e(\bar{\pi} \mid e^*(\underline{\pi}, \bar{\pi}, \bar{u}))}{V_{ee}(e^*(\underline{\pi}, \bar{\pi}, \bar{u}), \underline{\pi}, \bar{\pi}, \bar{u})} \gtrless 0, \tag{53}$$

$$\frac{\partial e^*}{\partial \bar{u}}(\underline{\pi}, \bar{\pi}, \bar{u}) \equiv \frac{\psi^*(e, \bar{u}) u_{ew}(w^*(e, \bar{u}), e) \frac{\partial w^*}{\partial \bar{u}}(e, \bar{u}) + u_e(w^*(e, \bar{u}), e) \frac{\partial \psi^*}{\partial \bar{u}}(e, \bar{u})}{V_{ee}(e^*(\underline{\pi}, \bar{\pi}, \bar{u}), \underline{\pi}, \bar{\pi}, \bar{u})} \gtrless 0. \tag{54}$$

Without further assumptions on the principal-agent problem (33), none of these expressions can be signed, in general. It is certainly plausible that $\underline{\pi} < 0$ and $\bar{\pi} > 0$,

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since one would expect that negative and positive realizations of gross profit are possible on the project. Under these two conditions, the sign of the expressions in Eqs. (52) and (53) is the same as the sign of the terms $f_e(\underline{\pi} \mid e^*(\underline{\pi}, \bar{\pi}, \bar{u}))$ and $f_e(\bar{\pi} \mid e^*(\underline{\pi}, \bar{\pi}, \bar{u}))$, respectively. Equation (54), on the other hand, can be signed with the aid of the assumption $u_{we}(w, e) \equiv 0$ for all $e \in E$ and w, an assumption that is maintained in much of the literature related to the principal-agent problem. In this case, it follows from Eq. (43) and the aforementioned assumption that $u_e(w, e) < 0$ for all (w, e), that $\partial e^*(\underline{\pi}, \bar{\pi}, \bar{u})/\partial \bar{u} < 0$. Thus, under the standard assumptions made in the literature, an increase in the manager's reservation utility results in a decrease in his effort. In other words, with better outside opportunities for the manager, the owner must offer a contract that requires less effort on the part of the manager if the owner is to get the manager to accept the contract.

The final phase of the analysis is the determination of the *unconditional* optimal wage of the manager, unconditional in the sense that the optimal effort, rather than an arbitrarily given effort, is used to compute it. The unconditional optimal wage rate, say, $w^{**}(\underline{\pi}, \bar{\pi}, \bar{u})$, is thus given by an identity akin to Eq. (44), namely,

$$w^{**}(\underline{\pi}, \bar{\pi}, \bar{u}) \equiv w^*(e^*(\underline{\pi}, \bar{\pi}, \bar{u}), \bar{u}). \tag{55}$$

We terminate our analysis of the principal-agent problem at this juncture and thus relegate the qualitative analysis of identity (55) to a mental exercise.

To close out this chapter, we present a sufficiency theorem for the primal isoperimetric problem (1). It is an obvious extension of earlier sufficiency theorems for optimal control problems, and as such, its proof is left for a mental exercise.

Theorem 7.4 (Sufficient Conditions): *Let $\mathbf{z}(\cdot)$ be an admissible function and ψ be the corresponding value of the multiplier that satisfies the system of augmented Euler equations*

$$\tilde{F}_{x_n}(t, \mathbf{z}(t), \dot{\mathbf{z}}(t), \psi) - \frac{d}{dt}\tilde{F}_{\dot{x}_n}(t, \mathbf{z}(t), \dot{\mathbf{z}}(t), \psi) \equiv 0, \quad n = 1, 2, \ldots, N,$$

where $\tilde{F}(t, \mathbf{x}, \dot{\mathbf{x}}, \psi) \stackrel{\text{def}}{=} F(t, \mathbf{x}, \dot{\mathbf{x}}) - \psi G(t, \mathbf{x}, \dot{\mathbf{x}})$. Suppose that $\tilde{F}(\cdot) \in C^{(2)}$ is a concave function of $(\mathbf{x}, \dot{\mathbf{x}})\,\forall\, t \in [t_0, t_1]$ over an open convex set containing all the admissible values of $(\mathbf{x}(\cdot), \dot{\mathbf{x}}(\cdot))$ for the above value of ψ, then $J[\mathbf{z}(\cdot)] \geq J[\mathbf{x}(\cdot)]$ for all admissible functions $\mathbf{x}(\cdot)$. That is, the function $\mathbf{z}(\cdot)$ provides the global maximum to $J[\cdot]$ over the space of admissible functions. Furthermore, if $\tilde{F}(\cdot) \in C^{(2)}$ is a strictly concave function under the same conditions, then $J[\mathbf{z}(\cdot)] > J[\mathbf{x}(\cdot)]$ for all admissible functions $\mathbf{x}(\cdot)$, and the function $\mathbf{z}(\cdot)$ is unique.

For an application of this theorem, we return to Example 7.1.

Example 7.4: Recall the primal isoperimetric problem from Example 7.1:

$$\min_{x(\cdot)} J[x(\cdot)] \stackrel{\text{def}}{=} \int_0^1 [\dot{x}(t)]^2\, dt$$

$$\text{s.t.} \quad K[x(\cdot)] \stackrel{\text{def}}{=} \int_0^1 x(t)dt = c,$$

$$x(0) = 0, \; x(1) = 0,$$

where $c > 0$ is a parameter. Also recall that the augmented integrand function $\tilde{F}(\cdot)$ has values given by $\tilde{F}(t, x, \dot{x}, \psi) \stackrel{\text{def}}{=} F(t, x, \dot{x}) - \psi G(t, x, \dot{x}) = \dot{x}^2 - \psi x$. Now set $\psi = 24c > 0$ and then compute

$$\tilde{F}_{xx}(t, x, \dot{x}, 24c) = 0, \tilde{F}_{\dot{x}\dot{x}}(t, x, \dot{x}, 24c) = 2 > 0, \tilde{F}_{x\dot{x}}(t, x, \dot{x}, 24c) = 0,$$

$$\tilde{F}_{xx}(t, x, \dot{x}, 24c) \cdot \tilde{F}_{\dot{x}\dot{x}}(t, x, \dot{x}, 24c) - [\tilde{F}_{x\dot{x}}(t, x, \dot{x}, 24c)]^2 = 0.$$

Thus the Hessian matrix of $\tilde{F}(\cdot)$ with respect to $(x, \dot{x})$ is positive semidefinite for all $t \in [0, 1]$ given $\psi = 24c > 0$, which is equivalent to the convexity of the function $\tilde{F}(\cdot)$ by Theorem 21.5 of Simon and Blume (1994). Hence, by Theorem 7.4, the solution curve $z_1(t; c) = -6ct^2 + 6ct$ to the augmented Euler equation of this isoperimetric problem does in fact solve the problem. Moreover, the solution is unique because it is the *only* solution to the augmented Euler equation.

Turning to the reciprocal isoperimetric problem, we have

$$\max_{x(\cdot)} K[x(\cdot)] \stackrel{\text{def}}{=} \int_0^1 x(t)\, dt$$

$$\text{s.t.} \quad J[x(\cdot)] \stackrel{\text{def}}{=} \int_0^1 [\dot{x}(t)]^2\, dt = V(c) = 12c^2,$$

$$x(0) = 0, \; x(1) = 0.$$

Recall that the augmented integrand for the reciprocal problem is $\hat{F}(t, x, \dot{x}, \theta) \stackrel{\text{def}}{=} x - \theta \dot{x}^2$, and so

$$\hat{F}_{xx}(t, x, \dot{x}, \theta) = 0, \; \hat{F}_{\dot{x}\dot{x}}(t, x, \dot{x}, \theta) = -2\theta, \; \hat{F}_{x\dot{x}}(t, x, \dot{x}, \theta) = 0,$$

$$\hat{F}_{xx}(t, x, \dot{x}, \theta) \cdot \hat{F}_{\dot{x}\dot{x}}(t, x, \dot{x}, \theta) - [\hat{F}_{x\dot{x}}(t, x, \dot{x}, \theta)]^2 = 0.$$

Thus for $\theta = \frac{1}{24c} > 0$, the Hessian matrix of $\hat{F}(\cdot)$ with respect to $(x, \dot{x})$ is negative semidefinite for all $t \in [0, 1]$, which is equivalent to the concavity of the function $\hat{F}(\cdot)$ in $(x, \dot{x})$. Hence, by Theorem 7.4, the solution curve $z_2(t; c) = -6ct^2 + 6ct$ to

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the augmented Euler equation of the reciprocal isoperimetric problem, which corresponds to the multiplier $\theta = \frac{1}{24c} > 0$, yields the global maximum of the reciprocal isoperimetric problem. In contrast, for $\theta = -\frac{1}{24c} < 0$, the Hessian matrix of $\hat{F}(\cdot)$ with respect to $(x, \dot{x})$ is positive semidefinite for all $t \in [0, 1]$, which is equivalent to the convexity of the function $\hat{F}(\cdot)$ in $(x, \dot{x})$. Hence, by Theorem 7.4, the solution curve $z_3(t; c) = 6ct^2 - 6ct$ to the augmented Euler equation of the reciprocal isoperimetric problem, which corresponds to the multiplier $\theta = -\frac{1}{24c} < 0$, yields the global minimum of the reciprocal isoperimetric problem. These are the same conclusions we reached in Example 7.1 by direct computation of the optimal value function for the curves $z_2(t; c) = -6ct^2 + 6ct$ and $z_3(t; c) = 6ct^2 - 6ct$. Furthermore, note that $z_2(t; c) = -6ct^2 + 6ct$ is the unique curve that yields the global maximum in the reciprocal problem, whereas $z_3(t; c) = 6ct^2 - 6ct$ is the unique curve that yields the global minimum in the reciprocal problem.

The next chapter develops further results for reciprocal pairs of isoperimetric problems that are of economic importance. We demonstrate the power and reach of the general theorems by conducting a rather exhaustive comparative dynamics analysis of a general form of the nonrenewable resource–extracting model of the firm.

MENTAL EXERCISES

7.1 Consider the isoperimetric problem

$$\min_{x(\cdot)} J[x(\cdot)] \stackrel{\text{def}}{=} \int_0^1 [\dot{x}(t)]^2\, dt$$

$$\text{s.t.} \quad K[x(\cdot)] \stackrel{\text{def}}{=} \int_0^1 [x(t)]^2 dt = 2,\ x(0) = 0,\ x(1) = 0.$$

(a) Show that if the multiplier for the integral constraint is zero, that is, $\psi = 0$, then the curve $z(t) \equiv 0\, \forall\, t \in [0, 1]$ is the only solution of the augmented Euler equation satisfying the boundary conditions.

(b) Show, however, that the curve $z(t) \equiv 0\, \forall\, t \in [0, 1]$ is not admissible and thus not optimal.

(c) Show that if $\psi < 0$, then the curve $z(t) \equiv 0\, \forall\, t \in [0, 1]$ is the only solution of the Euler equation satisfying the boundary conditions. By part (b), this solution is not admissible or optimal.

(d) We now know that the curve $z(t) \equiv 0\, \forall\, t \in [0, 1]$ is not admissible or optimal. Show that if $\psi > 0$, then the optimal value of ψ is $\psi^* = n^2\pi^2$,

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$n = 1, 2, \ldots,$ and the optimal path is given by

$$z(t) = \pm \left[\frac{2}{\int_0^1 [\sin n\pi t]^2 \, dt} \right]^{\frac{1}{2}} \sin n\pi t.$$

7.2 Consider the isoperimetric problem

$$\min_{x(\cdot)} J[x(\cdot)] \stackrel{\text{def}}{=} \int_0^T e^{-rt} x(t) \, dt$$

$$\text{s.t.} \quad K[x(\cdot)] \stackrel{\text{def}}{=} \int_0^T [x(t)]^{\frac{1}{2}} \, dt = c.$$

(a) Find the extremal using Theorem 7.1.
(b) Find the extremal by eliminating the isoperimetric constraint. You must come up with a transformation that allows you to do this.
(c) Why aren't there any boundary conditions for this isoperimetric problem?
(d) Prove that the extremal solves the isoperimetric problem and is unique.

7.3 Consider the isoperimetric problem

$$\min_{x(\cdot)} J[x(\cdot)] \stackrel{\text{def}}{=} \int_0^b [1 + [x(t)]^2]^{\frac{1}{2}} \, dt$$

$$\text{s.t.} \quad K[x(\cdot)] \stackrel{\text{def}}{=} \int_0^b x(t) \, dt = c,$$

where $b > 0$ and $c > 0$ are given parameters.
(a) Find an extremal for the isoperimetric problem.
(b) Does the extremal solve the problem? Show your work.
(c) Is the extremal unique? Explain clearly.

7.4 Consider the isoperimetric problem

$$\max_{x(\cdot)} J[x(\cdot)] \stackrel{\text{def}}{=} \int_0^1 [2x(t) - [x(t)]^2] \, dt$$

$$\text{s.t.} \quad K[x(\cdot)] \stackrel{\text{def}}{=} \int_0^1 tx(t) \, dt = 1.$$

(a) Find an extremal for the isoperimetric problem.
(b) Does the extremal solve the problem? Show your work.
(c) Is the extremal unique? Explain.

7.5 Find the extremals for the functional

$$\min_{x(\cdot)} J[x(\cdot)] \stackrel{\text{def}}{=} \int_1^2 [t^2 + [\dot{x}(t)]^2]dt$$

$$\text{s.t.} \quad K[x(\cdot)] \stackrel{\text{def}}{=} \int_1^2 [x(t)]^2\,dt = 5,\ x(1) = 0,\ x(2) = 0.$$

7.6 Find the extremals for the functional

$$\min_{x(\cdot)} J[x(\cdot)] \stackrel{\text{def}}{=} \int_0^1 tx(t)\,dt$$

$$\text{s.t.} \quad K[x(\cdot)] \stackrel{\text{def}}{=} \int_0^1 [\dot{x}(t)]^2\,dt = 1,\ x(0) = 0,\ x(1) = 0.$$

7.7 Recall the archetype nonrenewable resource–extracting model of the firm in isoperimetric form, given in Example 7.2:

$$\Pi(r, x_0, T) \stackrel{\text{def}}{=} \max_{q(\cdot)} \int_0^T \pi(q(t))\,e^{-rt}\,dt$$

$$\text{s.t.} \quad K[q(\cdot)] \stackrel{\text{def}}{=} \int_0^T q(t)\,dt = x_0,$$

where $q(t)$ is the extraction rate of the nonrenewable resource, $r > 0$ is the discount rate, $\pi(q(t))$ is the profit flow from extracting at the rate $q(t)$, and $x_0 > 0$ is the initial stock of the nonrenewable resource. Assume that $\pi(\cdot) \in C^{(2)}, \pi'(q) > 0$, and $\pi''(q) < 0$. This question asks you to derive results you've already shown in Mental Exercise 4.21, but this time by solving the problem using Theorems 7.1 and 7.4. You will also extend the previous qualitative properties of the model to account for the results of this chapter, particularly Theorem 7.3. Let $\pi(q) \stackrel{\text{def}}{=} \ln q$ be the profit flow for this problem.

(a) Find explicit formulas for the optimal extraction rate, say, $q^*(t; r, x_0, T)$, and the value of the multiplier, say, $\psi^*(r, x_0, T)$. Prove that $q^*(t; r, x_0, T)$ is the unique optimal solution.

(b) Find an explicit formula for the optimal value function $\Pi(r, x_0, T)$. You may *not* leave $\Pi(r, x_0, T)$ expressed as an integral.

(c) Compute the partial derivative $\partial\Pi(r, x_0, T)/\partial x_0$ and show that Theorem 7.3 holds in this model. Provide an economic interpretation of this dynamic envelope result.

(d) Compute the comparative dynamics $\partial q^*(t; r, x_0, T)/\partial x_0$ and $\partial \psi^*(r, x_0, T)/\partial x_0$. Can you sign these? Provide an economic explanation of the results.

(e) Compute the comparative dynamics $\partial q^*(t; r, x_0, T)/\partial T$ and $\partial \psi^*(r, x_0, T)/\partial T$. Can you sign these? Provide an economic explanation of the results.

(f) Compute the comparative dynamics $\partial q^*(t; r, x_0, T)/\partial r$ and $\partial \psi^*(r, x_0, T)/\partial r$. Can you sign these? Provide an economic explanation of the results that can be signed.

(g) Consider $\partial q^*(0; r, x_0, T)/\partial r$, the so-called impact effect of the parameter change. Can you sign this? Provide an economic explanation of the result.

7.8 Imagine a research and development (R&D) project in which there are decreasing returns to spending money faster. That is, the more rapidly the money is spent, the less it contributes to total effective effort. For example, more rapid spending may be used for overtime payments, for less productive factors, or for greater use of parallel rather than sequential effort. Let $s(t)$ denote the rate of spending on R&D in dollars at time t, and let $e(t)$ denote the effort rate on R&D at time t. The link between spending and effort alluded to above is given by the production function $e(t) = [s(t)]^{\frac{1}{2}}$, which exhibits a positive but declining marginal product of spending (as the story above asserted). The cumulative effort required to complete the project by the given time $T > 0$ is $A > 0$. It is therefore required that the cumulative effort expended on the R&D project equal the total effort required to complete the project; hence

$$\int_0^T e(t)\,dt = A.$$

The firm is asserted to minimize the present discounted development cost of completing the R&D project. Formally, the isoperimetric problem is given by

$$C(A, r, T) \stackrel{\text{def}}{=} \min_{s(\cdot)} \int_0^T e^{-rt} s(t)\,dt$$

$$\text{s.t.} \quad \int_0^T [s(t)]^{\frac{1}{2}}\,dt = A,$$

where $r > 0$ is the discount rate, and the production function $e(t) = [s(t)]^{\frac{1}{2}}$ has been used to substitute into the integral constraint.

(a) Find the general solution of the augmented Euler equation.

(b) What do you notice unusual about the augmented Euler equation? Is this related to the fact that there are no boundary conditions for the isoperimetric problem?

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(c) Find the specific solution to the augmented Euler equation and denote it by $s^*(t; A, r, T)$. Find the corresponding value of the constant multiplier and designate it by $\psi^*(A, r, T)$. Also determine the optimal path of effort and denote it by $e^*(t; A, r, T)$.

(d) Prove that the solution you found in part (c) does indeed solve the isoperimetric problem.

(e) What is the economic interpretation of $\psi^*(A, r, T)$? What theorem did you invoke to get this interpretation?

(f) Find the comparative dynamics $\partial e^*(t; A, r, T)/\partial A$ and $\partial \psi^*(A, r, T)/\partial A$. Provide an economic interpretation of these results.

(g) Find the comparative dynamics $\partial e^*(t; A, r, T)/\partial T$ and $\partial \psi^*(A, r, T)/\partial T$. Provide an economic interpretation of these results.

(h) Find the comparative dynamics $\partial e^*(t; A, r, T)/\partial r$ and $\partial \lambda^*(A, r, T)/\partial r$. Show that $\partial \psi^*(A, r, T)/\partial r < 0$ and provide an economic interpretation.

(i) Show that $\partial e^*(0; A, r, T)/\partial r < 0$. What does this tell you about the effect of the interest rate increase?

7.9 Verify that Eq. (10) follows from Eq. (9) by using Eqs. (3), (4), and (6).

7.10 Verify that Eq. (12) follows from Eq. (11) by integrating Eq. (11) by parts and using Eq. (2).

7.11 Prove part (i) of Theorem 7.3. Provide an economic interpretation of it.

7.12 Prove part (iii) of Theorem 7.3. Provide an economic interpretation of it. You may want to wait until after you have read Chapter 9 to do this proof.

7.13 Prove part (iv) of Theorem 7.3. Provide an economic interpretation of it.

7.14 Prove part (v) of Theorem 7.3. Provide an economic interpretation of it. You may want to wait until after you have read Chapter 9 to do this proof.

7.15 Prove part (vi) of Theorem 7.3. Provide an economic interpretation of it.

7.16 Prove that a solution to the necessary conditions of Example 7.2 is a solution of the isoperimetric problem.

7.17 Prove Theorem 7.4.

7.18 Consider an individual whose consumption rate at time t is given by $c(t)$, and whose *instantaneous* utility function is given by $u(c(t)) \stackrel{\text{def}}{=} \ln c(t)$. Let $\rho > 0$ be the individual's subjective rate of time preference, and let $r > 0$ be the market interest rate, the latter being the rate at which cash flows are discounted. The constant price of the single composite consumption good is $p > 0$, whereas $w > 0$ is the individual's present discounted value of wealth. The consumer lives over the fixed and finite interval $[0, T]$, $T > 0$, and is asserted to maximize the present discounted value of utility by choosing a consumption function $c(\cdot)$ subject to an intertemporal budget constraint that requires the equality of the present discounted value of expenditures with the

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present discounted value of wealth. Formally, the problem is given by the isoperimetric problem

$$V(\boldsymbol{\beta}) \stackrel{\text{def}}{=} \max_{c(\cdot)} \left\{ U[c(\cdot)] \stackrel{\text{def}}{=} \int_0^T \ln c(t)\, e^{-\rho t}\, dt \text{ s.t.} E[c(\cdot)] \stackrel{\text{def}}{=} \int_0^T pc(t)\, e^{-rt}\, dt = w \right\},$$

where $\boldsymbol{\beta} \stackrel{\text{def}}{=} (p, \rho, r, w, T)$ is the constant parameter vector. Note that this version of the intertemporal utility maximization problem has a lifetime budget constraint, whereas that in Example 1.3 has a budget constraint that must hold for each date in the planning horizon.

(a) Derive the necessary conditions for this variational problem and solve them for the extremal, say, $c^*(t; \boldsymbol{\beta})$, as well as the associated conjugate variable (or multiplier), say, $\psi^*(\boldsymbol{\beta})$.

(b) Prove that the extremal you found in part (a) is the solution to the isoperimetric problem. Is it the unique solution? Explain why or why not.

(c) Prove that $\text{sign}[\dot{c}^*(t; \boldsymbol{\beta})] = \text{sign}[r - \rho]$. Provide an economic interpretation.

(d) Derive the optimal value function $V(\cdot)$. Do not leave $V(\cdot)$ expressed as an integral. Provide an economic interpretation of $V(\cdot)$.

(e) Confirm that Theorem 7.3 holds for this problem and that $V(\cdot)$ is a strictly concave function of w. Provide an economic interpretation of the strict concavity of $V(\cdot)$.

(f) Prove that $\partial c^*(t; \boldsymbol{\beta})/\partial w > 0\, \forall\, t \in [0, T]$ and provide an economic interpretation.

(g) Prove that $\partial c^*(t; \boldsymbol{\beta})/\partial r \geq 0\, \forall\, t \in [0, T]$ and provide an economic interpretation.

(h) Prove that $\partial c^*(t; \boldsymbol{\beta})/\partial p < 0\, \forall\, t \in [0, T]$ and provide an economic interpretation.

7.19 This question asks you to consider some comparative *statics* of the nonrenewable resource–extracting model of the firm in a two-period discrete time framework. Consider, therefore, a firm that “lives” for two periods: period 1, the present, and period 2, the future. Let $s > 0$ be the stock of a nonrenewable resource (say, gold) that is buried under the ground and that the firm has ownership rights to. The extraction of the nonrenewable resource is costly, and such costs may be summarized by the minimum cost function $C(\cdot) \in C^{(2)}$, which is a function of the rate of resource extraction in each period, namely, $q_i i = 1, 2$. Hence $C(q_i)$, $i = 1, 2$, are the costs incurred in each period to extract the resource. The firm is a price taker in the output market, facing prices $p_i > 0$, $i = 1, 2$, for the asset extracted in each period. The firm is asserted to

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maximize the present discounted value of the profit received from extracting and selling the nonrenewable resource, subject to the requirement that it extract the entire stock of the resource in the two periods in which it lives. The discount rate $r > 0$ is used by the firm in discounting future profit.

(a) Set up the profit maximization problem faced by the firm, including the definition of the indirect (or maximized) objective function. Denote the indirect objective function by $\pi(\cdot)$.

(b) Derive the first-order necessary conditions for this problem and provide an economic interpretation. How would you *in principle* find the optimal extraction rate in each period, say, $q_i = q_i^*(p_1, p_2, r, s)$, $i = 1, 2$, and the optimal value of the Lagrange multiplier, say, $\lambda = \lambda^*(p_1, p_2, r, s)$?

(c) *Prove* that

$$\frac{\partial \pi}{\partial s}(p_1, p_2, r, s) \equiv \lambda^*(p_1, p_2, r, s),$$

and provide an economic interpretation of the result.

(d) Derive the second-order sufficient condition in the form of a determinant condition rather than as a quadratic form. Is increasing marginal cost of extraction implied by the second-order sufficient condition?

(e) Find the comparative statics

$$\frac{\partial q_1^*}{\partial p_2}, \frac{\partial q_2^*}{\partial p_2}, \quad \text{and} \quad \frac{\partial \lambda^*}{\partial p_2},$$

and provide an economic interpretation.

(f) Find the comparative statics

$$\frac{\partial q_1^*}{\partial r}, \frac{\partial q_2^*}{\partial r}, \quad \text{and} \quad \frac{\partial \lambda^*}{\partial r},$$

and provide an economic interpretation.

7.20 *Rational Procrastination*. Here's a typical situation faced by students at universities all across the world. A research paper is assigned by the professor of a class the first day of the term, say, $t = 0$, and is due at the end of the term, say $T > 0$ hours later. At University of California, Davis, this would mean that the research paper is due in $24 \times 7 \times 10 = 1{,}680$ hours, or 10 weeks, from the present. The total effort required by the typical student to complete the paper is known to be $\varepsilon \in (0, T)$ hours. Define $e(t) \in [0, 1]$ as the proportion of each hour that the student devotes to working on the paper (i.e., research effort), and define $\ell(t) \in [0, 1]$ as the proportion of each hour that the student devotes to leisure activities. It is assumed that the student will complete the term paper by the required date, thereby implying the isoperimetric constraint

$$\int_0^T e(t)\, dt = \varepsilon.$$

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Given that each hour is made up entirely of leisure time and research effort, we also have the equality constraint that $e(t) + \ell(t) = 1$ for all $t \in [0, T]$. The instantaneous preferences of the typical student are defined over leisure time, a good, and denoted by $U(\ell)$, where $U(\cdot) \in C^{(2)}$, $U'(\ell) > 0$, and $U''(\ell) < 0$ for all $\ell(t) \in (0, 1)$. We assume that $e(t) \in (0, 1)$ and $\ell(t) \in (0, 1)$ for all $t \in [0, T]$ in an optimal plan, thereby ruling out these constraints from binding. These two assumptions will simplify the analysis considerably. The student is asserted to maximize the present discounted value of utility over the term, subject to completing the research paper. Hence the constrained isoperimetric problem faced by the typical student can be stated as

$$V(\boldsymbol{\beta}) \stackrel{\text{def}}{=} \max_{e(\cdot),\ell(\cdot)} \int_0^T U(\ell(t))\, e^{-rt}\, dt$$

$$\text{s.t.} \quad \int_0^T e(t)\, dt = \varepsilon,$$

$$e(t) + \ell(t) = 1,$$

where $r > 0$ is the student's intertemporal rate of time preference and $\boldsymbol{\beta} \stackrel{\text{def}}{=} (\varepsilon, r, T) \in \Re^3_{++}$. Assume that the pair of curves $(e^*(t; \boldsymbol{\beta}), \ell^*(t; \boldsymbol{\beta}))$ is a solution to the necessary conditions of the constrained isoperimetric problem, with corresponding multiplier for the integral constraint $\psi(\boldsymbol{\beta})$.

(a) Convert the above constrained isoperimetric problem into an unconstrained isoperimetric problem by using the equality constraint $e(t) + \ell(t) = 1$ to eliminate $e(t)$ from the problem.

(b) Derive the necessary conditions for the unconstrained isoperimetric problem.

(c) Prove that $(\ell^*(t; \boldsymbol{\beta}), \psi(\boldsymbol{\beta}))$ is a solution of the unconstrained isoperimetric problem.

(d) Prove that $\psi(\boldsymbol{\beta}) < 0$. Provide an economic interpretation of this result. Does it make sense? Explain.

(e) Prove that $\dot{\ell}^*(t; \boldsymbol{\beta}) < 0\, \forall\, t \in [0, T]$ and $\dot{e}^*(t; \boldsymbol{\beta}) > 0\, \forall\, t \in [0, T]$. Provide an economic interpretation. Is this the rational procrastination result alluded to in the problem? Why or why not?

7.21 *Maximum Entropy and Isoperimetric Problems*. Using a set of four axioms about information, it can be shown if $p \in (0, 1)$ is the probability of an event E occurring, then $h(p) \stackrel{\text{def}}{=} \ln \frac{1}{p} = -\ln p$ is the information contained in the observation that the event E actually occurred. For example, if I attach a low probability to the event that it will rain today, say, $p = 0.01$, then the amount of information transmitted by the message that it has in fact rained today is relatively large, namely, $h(0.01) \stackrel{\text{def}}{=} \ln \frac{1}{0.01} = -\ln 0.01 \approx 4.60517$, since I essentially didn't think it was going to rain today. On the other hand, if I

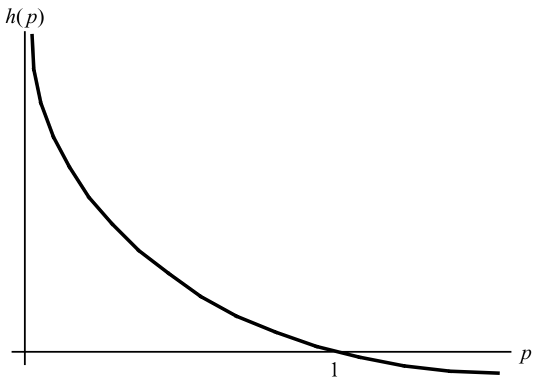

Figure 7.3

attach a high probability that it will rain today, say, $p = 0.99$, then the amount of information transmitted by the message that it has in fact rained today is relatively small, namely, $h(0.99) \overset{\text{def}}{=} \ln \frac{1}{0.99} = -\ln 0.99 \approx 0.0100503$, since I was almost certain that it was going to rain anyway. A graph of the function $h(\cdot)$ appears in Figure 7.3. The classical maximum entropy problem is to find a probability density function $p(\cdot)$ that maximizes the expected value of the information contained in a continuum of messages received from the events. It can be formulated as the following isoperimetric problem:

$$\max_{p(\cdot)} J[p(\cdot)] \overset{\text{def}}{=} -\int_{t_0}^{t_1} p(t) \ln p(t)\, dt$$

$$\text{s.t.} \quad \int_{t_0}^{t_1} p(t)\, dt = 1.$$

(a) Prove that the uniform probability density function yields the unique global optimum of the maximum entropy problem.
(b) Can you provide an intuitive explanation for this result?

7.22 *Minimum Cross-Entropy and Isoperimetric Problems*. The classical maximum entropy problem is to find a probability density function $p(\cdot)$ that maximizes the expected value of the information contained in a continuum of messages received from the events. This was the problem studied in the previous mental exercise. In many situations, however, the researcher may have nonsample or presample information about the probability density function $p(\cdot)$ in the form of a prior probability density function, say, $q(\cdot)$. In other words, the researcher may have an initial hypothesis that $q(\cdot)$ is a plausible probability density function. When such prior knowledge exists, the researcher will often wish to incorporate it into the maximum entropy formalism. This situation is handled via the principle of minimum cross-entropy. This principle implies that one should choose an estimate of the probability density function $p(\cdot)$ that can be discriminated from $q(\cdot)$ with a minimum difference. The principle of minimum cross-entropy thus leads to the following isoperimetric problem:

$$\min_{p(\cdot)} J[p(\cdot)]$$

$$\stackrel{\text{def}}{=} \left[\int_{t_0}^{t_1} p(t) \ln\left(\frac{p(t)}{q(t)}\right) dt = \int_{t_0}^{t_1} p(t) \ln p(t)\, dt - \int_{t_0}^{t_1} p(t) \ln q(t)\, dt \right]$$

$$\text{s.t.} \quad \int_{t_0}^{t_1} p(t)\, dt = 1.$$

Recall that because $q(\cdot)$ is a probability density function, it must satisfy $\int_{t_0}^{t_1} q(t)\, dt = 1$.

(a) Find the unique global optimum of the minimum cross-entropy problem.
(b) Interpret your result.

7.23 Differentiate the identity in Eq. (55) with respect to $\bar{u}$. Can you sign the resulting comparative statics expression for the unconditional optimal wage? Explain. Provide an economic interpretation of the Slutsky-like equation you just derived.

FURTHER READING

Clegg (1967) and Kamien and Schwartz (1991) contain complementary discussions of isooperimetric problems. Many intertemporal problems in economics naturally result in the formulation of an isoperimetric problem. Besides the seminal paper on the nonrenewable resource–extracting model of the firm by Hotelling (1931), other examples include the project planning model of Cullingford and Prideaux (1973), numerous R&D models by Kamien and Schwartz (1971, 1974a, 1974b, 1978), continuous–time formulations of maximum entropy problems as in Golan, Judge, and Miller (1996), and models of procrastination by Fischer (2001). An excellent place to commence further study of isoperimetric formulations of principal-agent problems is the Mas-Colell, Whinston, and Green (1995) textbook. More advanced work along this line includes Schättler and Sung (1993), Müller (1998), and Theilen (2003).

REFERENCES

Clegg, J.C. (1967), *Calculus of Variations* (New York: Interscience).

Cullingford, G. and Prideaux, J.D.C.A. (1973), “A Variational Study of Optimal Resource Profiles,” *Management Science*, 19, 1067–1081.

Fischer, C. (2001), "Read This Paper Later: Procrastination with Time-Consistent Preferences," *Journal of Economic Behavior and Organization*, 46, 249–269.

Golan, A., Judge, G., and Miller, D. (1996), *Maximum Entropy Econometrics* (New York: John Wiley & Sons).

Hotelling, H. (1931), "The Economics of Exhaustible Resources," *Journal of Political Economy*, 39, 137–175.

Kamien, M.I. and Schwartz, N.L. (1971), "Expenditure Patterns for Risky R&D Projects," *Journal of Applied Probability*, 8, 60–73.

Kamien, M.I. and Schwartz, N.L. (1974a), "Risky R&D with Rivalry," *Annals of Economic and Social Measurement*, 3, 267–277.

Kamien, M.I. and Schwartz, N.L. (1974b), "Patent Life and R&D Rivalry," *American Economic Review*, 64, 183–187.

Kamien, M.I. and Schwartz, N.L. (1978), "Self-Financing of an R&D Project," *American Economic Review*, 68, 252–261.

Kamien, M.I. and Schwartz, N.L. (1991, 2nd Ed.), *Dynamic Optimization: The Calculus of Variations and Optimal Control in Economics and Management* (New York: Elsevier Science Publishing Co., Inc.).

Mas-Colell, A., Whinston, M.D., and Green, J.R. (1995), *Microeconomic Theory* (Oxford: Oxford University Press).

Müller, H.M. (1998), "The First-Best Sharing Rule in the Continuous-Time Principal Agent Problem with Exponential Utility," *Journal of Economic Theory*, 79, 276–280.

Schättler, H. and Sung, J. (1993), "The First-Order Approach to the Continuous-Time Principal-Agent Problem with Exponential Utility," *Journal of Economic Theory*, 61, 331–371.

Simon, C.P. and Blume, L. (1994), *Mathematics for Economists* (New York: W.W. Norton & Company, Inc.).

Theilen, B. (2003), "Simultaneous Moral Hazard and Adverse Selection with Risk Averse Agents," *Economics Letters*, 79, 283–289.