EIGHT

Economic Characterization of Reciprocal Isoperimetric Problems

Microeconomic theorists have learned to take advantage of the symmetry afforded by reciprocal pairs of static optimization problems. Recall that the adjective *reciprocal* signifies that the second (or reciprocal) optimization problem reverses the roles of the original (or primal) problem's objective function and constraint function, and substitutes the minimization hypothesis for the maximization hypothesis. The classical economic example of this occurs in the archetype pair of reciprocal (but not dual) consumer problems: utility maximization and expenditure minimization.

A powerful advantage in working with reciprocal pairs of optimization problems is that one has a choice of which problem to analyze in order to extract the economic information, for the information in one problem can always be used to extract the information in the other. For example, in the modern proof of the negative semidefiniteness of the Slutsky matrix one first establishes the negative semidefiniteness of the substitution matrix, which comprises the first partial derivatives of the Hicksian demand functions with respect to the prices, by invoking the concavity of the expenditure function and the envelope theorem. Then one uses this result along with the Slutsky equation to establish the negative semidefiniteness of the Slutsky matrix. Thus the modern proof of the negative semidefiniteness of the Slutsky matrix works off the reciprocal expenditure minimization problem rather than the primal utility maximization problem, even though the theorem to be proven pertains to the utility maximization problem's solution. This avenue of proof is easier and more economically intuitive, which accounts for its prevalence in textbook expositions of the theory of the consumer. Therefore, when working with reciprocal pairs of optimization problems, the choice of which problem to analyze often comes down to determining which problem yields the results of interest with the greatest clarity and most appealing economic intuition.

We showed in Theorem 7.2 that under certain conditions, the solution of the primal isoperimetric problem is an extremal of the reciprocal isoperimetric problem, and that the associated multipliers are reciprocals of one another. We extend the results of Theorem 7.2 in this chapter by developing a fundamental set of identities linking the *optimal* solution functions and optimal value functions of a reciprocal

https://doi.org/10.1017/CBO9780511806827.010 Published online by Cambridge University Press

pair of isoperimetric problems. In addition, we elucidate the qualitative relationships between the optimal solution functions and optimal value functions for such a class of problems, and apply the results so obtained to the nonrenewable resource–extracting model of the firm. Those interested in pursuing the technical matters more deeply are referred to Caputo (1998, 1999), where the proofs of the ensuing theorems can be found.

Under consideration in this chapter is a general pair of reciprocal isoperimetric problems, the functions and parameters of which will be discussed below when the assumptions are laid out. In particular, for $\mathbf{x}(t) \stackrel{\text{def}}{=} (x_1(t), x_2(t), \ldots, x_N(t))$, consider the *primal* maximization problem

$$F^M(\varepsilon, \beta) \stackrel{\text{def}}{=} \max_{\mathbf{x}(\cdot)} \left\{ \int_0^T F(t, \mathbf{x}(t), \dot{\mathbf{x}}(t); \boldsymbol{\alpha})\, dt \text{ s.t. } \int_0^T G(t, \mathbf{x}(t), \dot{\mathbf{x}}(t); \boldsymbol{\alpha})\, dt = \beta, \right.$$
$$\left. \mathbf{x}(0) = \mathbf{x}_0, \mathbf{x}(T) = \mathbf{x}_T \right\}, \tag{P}$$

where $F^M(\cdot)$ is the *optimal value function* and $\mathbf{x}^M(\cdot)$ is its associated *optimal solution function*, defined as

$$\mathbf{x}^M(t; \varepsilon, \beta) \stackrel{\text{def}}{=} \arg\max_{\mathbf{x}(\cdot)} \left\{ \int_0^T F(t, \mathbf{x}(t), \dot{\mathbf{x}}(t); \boldsymbol{\alpha})\, dt \text{ s.t. } \int_0^T G(t, \mathbf{x}(t), \dot{\mathbf{x}}(t); \boldsymbol{\alpha})\, dt = \beta, \right.$$
$$\left. \mathbf{x}(0) = \mathbf{x}_0, \mathbf{x}(T) = \mathbf{x}_T \right\}. \tag{1}$$

Furthermore, also consider the *reciprocal* minimization problem

$$G^m(\varepsilon, \gamma) \stackrel{\text{def}}{=} \min_{\mathbf{x}(\cdot)} \left\{ \int_0^T G(t, \mathbf{x}(t), \dot{\mathbf{x}}(t); \boldsymbol{\alpha})\, dt \text{ s.t. } \int_0^T F(t, \mathbf{x}(t), \dot{\mathbf{x}}(t); \boldsymbol{\alpha})\, dt = \gamma, \right.$$
$$\left. \mathbf{x}(0) = \mathbf{x}_0, \mathbf{x}(T) = \mathbf{x}_T \right\}, \tag{R}$$

where $G^m(\cdot)$ is the optimal value function and $\mathbf{x}^m(\cdot)$ is its associated optimal solution function, defined as

$$\mathbf{x}^m(t; \varepsilon, \gamma) \stackrel{\text{def}}{=} \arg\min_{\mathbf{x}(\cdot)} \left\{ \int_0^T G(t, \mathbf{x}(t), \dot{\mathbf{x}}(t); \boldsymbol{\alpha})\, dt \text{ s.t. } \int_0^T F(t, \mathbf{x}(t), \dot{\mathbf{x}}(t); \boldsymbol{\alpha})\, dt = \gamma, \right.$$
$$\left. \mathbf{x}(0) = \mathbf{x}_0, \mathbf{x}(T) = \mathbf{x}_T \right\}. \tag{2}$$

Note that the reciprocal of the reciprocal problem (R) is the primal problem (P).

https://doi.org/10.1017/CBO9780511806827.010 Published online by Cambridge University Press

The ensuing assumptions are imposed on the isoperimetric problems (P) and (R), and are discussed subsequently. They are sufficient for the following analysis to hold, but are not necessarily the weakest possible.

(A.1) $F(\cdot) : \Re \times X \times \dot{X} \times A \to \Re$, $F(\cdot) \in C^{(2)}$ on its domain, $G(\cdot) : \Re \times X \times \dot{X} \times A \to \Re$, and $G(\cdot) \in C^{(2)}$ on its domain, where $X \subset \Re^N$ and $\dot{X} \subset \Re^N$ are convex and open sets, and $A \subset \Re^L$ is an open set.

(A.2) For each $t \in [0, T]$ and each $\boldsymbol{\alpha} \in A$, $F(\cdot)$ is a concave function of $(\mathbf{x}, \dot{\mathbf{x}})$, $\forall (\mathbf{x}, \dot{\mathbf{x}}) \in X \times \dot{X}$, and $G(\cdot)$ is a convex function of $(\mathbf{x}, \dot{\mathbf{x}})$, $\forall (\mathbf{x}, \dot{\mathbf{x}}) \in X \times \dot{X}$.

(A.3) There exists a unique admissible solution to the augmented Euler equation of problem (P) denoted by $\mathbf{x}^M(t; \varepsilon, \beta)$, along with the multiplier or conjugate variable $\psi^M(\varepsilon, \beta) > 0$, when the time-independent parameters

$$\begin{aligned}(\varepsilon, \beta) &\stackrel{\text{def}}{=} (\boldsymbol{\alpha}, \mathbf{x}_0, T, \mathbf{x}_T, \beta) = (\varepsilon^\circ, \beta^\circ) \\ &\stackrel{\text{def}}{=} (\boldsymbol{\alpha}^\circ, \mathbf{x}_0^\circ, T^\circ, \mathbf{x}_T^\circ, \beta^\circ) \in A \times X \times \Re \times X \times \Re,\end{aligned}$$

where $(\varepsilon^\circ, \beta^\circ)$ is a given value of the parameter vector (ε, β).

(A.4) There exists a unique admissible solution to the augmented Euler equation of problem (R) denoted by $\mathbf{x}^m(t; \varepsilon, \gamma)$, along with the multiplier or conjugate variable $\psi^m(\varepsilon, \gamma) > 0$, when the time-independent parameters

$$\begin{aligned}(\varepsilon, \gamma) &\stackrel{\text{def}}{=} (\boldsymbol{\alpha}, \mathbf{x}_0, T, \mathbf{x}_T, \gamma) = (\varepsilon^\circ, \gamma^\circ) \\ &\stackrel{\text{def}}{=} (\boldsymbol{\alpha}^\circ, \mathbf{x}_0^\circ, T^\circ, \mathbf{x}_T^\circ, \gamma^\circ) \in A \times X \times \Re \times X \times \Re,\end{aligned}$$

where $(\varepsilon^\circ, \gamma^\circ)$ is a given value of the parameter vector (ε, γ).

(A.5) The augmented integrand function for problem (P) has a nonzero Hessian determinant with respect to $\dot{\mathbf{x}}(t)$ when evaluated at $(\mathbf{x}^M(t; \varepsilon^\circ, \beta^\circ), \psi^M(\varepsilon^\circ, \beta^\circ))$.

(A.6) The augmented integrand function for problem (R) has a nonzero Hessian determinant with respect to $\dot{\mathbf{x}}(t)$ when evaluated at $(\mathbf{x}^m(t; \varepsilon^\circ, \gamma^\circ), \psi^m(\varepsilon^\circ, \gamma^\circ))$.

Assumption (A.1) imposes our standard $C^{(2)}$ assumption on the functions $F(\cdot)$ and $G(\cdot)$. In addition, the domains X and $\dot{X}$ are required to be convex because of the postulated concavity of $F(\cdot)$ and convexity of $G(\cdot)$ in $(\mathbf{x}, \dot{\mathbf{x}})$ from assumption (A.2). Assumptions (A.3) and (A.4) assert the existence of a unique admissible solution to the augmented Euler equation of isoperimetric problems (P) and (R) for a given value of the time-independent parameter vector of each problem, respectively. By Theorem 7.4, the concavity of $F(\cdot)$ and the convexity of $G(\cdot)$ in $(\mathbf{x}, \dot{\mathbf{x}})$ from assumption (A.2), along with the assumption that the multipliers are positive from assumptions (A.3) and (A.4), imply that the curve $\mathbf{x}^M(t; \varepsilon^\circ, \beta^\circ)$ is the unique solution to problem (P) and that the curve $\mathbf{x}^m(t; \varepsilon^\circ, \gamma^\circ)$ is the unique solution to problem (R). Assumptions (A.5) and (A.6) are technical assumptions that permit us to further conclude that the curves $\mathbf{x}^M(t; \varepsilon, \beta)$ and $\mathbf{x}^m(t; \varepsilon, \gamma)$ are the optimal solutions to problems (P) and (R) for all values of (ε, β) and (ε, γ) in an open neighborhood of $(\varepsilon^\circ, \beta^\circ)$ and $(\varepsilon^\circ, \gamma^\circ)$, respectively. The parameter vector $\boldsymbol{\alpha}$ typically represents a vector of market prices and the discount rate that the economic agent in question faces, whereas the parameter β, the isoperimetric constraint parameter for (P), may represent a given amount of some resource by which the economic agent is constrained. The parameter vector ε is defined so as to keep the notation as palatable as possible, but it excludes the isoperimetric constraint parameters (β, γ) because they play a different role in the qualitative analysis than the parameters defined in ε. Finally, we remark that the solution functions $(\mathbf{x}^M(\cdot), \dot{\mathbf{x}}^M(\cdot), \psi^M(\cdot), \mathbf{x}^m(\cdot), \dot{\mathbf{x}}^m(\cdot), \psi^m(\cdot))$ are locally $C^{(1)}$ and that the optimal value functions $F^M(\cdot)$ and $G^m(\cdot)$ are locally $C^{(2)}$.

As noted above, assumptions (A.1) through (A.6) are not the most general sufficient conditions under which the following results will hold. As a result, remarks will be offered after the proof of Theorem 8.1 as to which assumptions may be relaxed and which are crucial for its conclusions. That being said, it is still true that when the focus is on the qualitative properties of a model, as it is here, these assumptions are often (but not always) maintained either implicitly or explicitly in dynamic optimization problems in economics. Given these assumptions, we now state the main result of this chapter, linking the optimal solution functions and optimal value functions of the reciprocal pair of isoperimetric problems (P) and (R).

Theorem 8.1 (Reciprocal Identities): *Under assumptions (A.1) through (A.6), the following identities link the values of the optimal solution functions and optimal value functions of the reciprocal pair of isoperimetric problems (P) and (R):*

$$(\mathbf{x}^M(t;\varepsilon,\beta), \dot{\mathbf{x}}^M(t;\varepsilon,\beta)) \equiv (\mathbf{x}^m(t;\varepsilon, F^M(\varepsilon,\beta)), \dot{\mathbf{x}}^m(t;\varepsilon, F^M(\varepsilon,\beta)))$$
$$\forall (t,\varepsilon,\beta) \in [0, T^\circ] \times B((\varepsilon^\circ, \beta^\circ); \delta_P), \tag{a}$$

$$G^m(\varepsilon, F^M(\varepsilon,\beta)) \equiv \beta \, \forall (\varepsilon,\beta) \in B((\varepsilon^\circ, \beta^\circ); \delta_P), \tag{b}$$

$$(\mathbf{x}^m(t;\varepsilon,\gamma), \dot{\mathbf{x}}^m(t;\varepsilon,\gamma)) \equiv (\mathbf{x}^M(t;\varepsilon, G^m(\varepsilon,\gamma)), \dot{\mathbf{x}}^M(t;\varepsilon, G^m(\varepsilon,\gamma)))$$
$$\forall (t,\varepsilon,\gamma) \in [0, T^\circ] \times B((\varepsilon^\circ, \gamma^\circ); \delta_R), \tag{c}$$

$$F^M(\varepsilon, G^m(\varepsilon,\gamma)) \equiv \gamma \ \forall \ (\varepsilon,\gamma) \in B((\varepsilon^\circ, \gamma^\circ); \delta_R). \tag{d}$$

The interpretation of Theorem 8.1 is important and therefore deserves comment. Part (a) asserts that the value of the function and its time derivative that solves the primal problem (P) is identically equal to the value of the function and its time derivative that solves the reciprocal problem (R) when the value of the isoperimetric constraint in problem (R) is set equal to the maximum value of the functional, that is, the optimal value function, in problem (P). Symmetrically, part (c) asserts that the value of the function and its time derivative that solves the reciprocal problem (R) is identically equal to the value of the function and its time derivative that solves the primal problem (P), when the value of the isoperimetric constraint in problem (P) is set equal to the optimal value function in problem (R). Parts (b) and (d) show that for ε fixed, the optimal value functions are inverses of one another with respect to the isoperimetric constraint parameters β and γ, respectively. Theorem 8.1, therefore, is the generalized intertemporal equivalent of the identities linking the Marshallian and Hicksian demand functions, and the identities linking the indirect utility and expenditure functions, of the archetype utility maximization and expenditure minimization problems of consumer theory.

As noted earlier, Theorem 8.1 holds under more general conditions than those stated in assumptions (A.1) through (A.6). For example, the differentiability assumption in (A.1) can be weakened, without necessarily invalidating Theorem 8.1. Uniqueness of the solutions, however, is needed in the proof of parts (a) and (c) of Theorem 8.1. Parts (b) and (d) of Theorem 8.1, on the other hand, will hold under more general conditions than parts (a) and (c). For instance, the presence of multiple solutions to problems (P) and (R) would not affect parts (b) and (d), but it would invalidate parts (a) and (c) as stated.

By making explicit use of the differentiability of the optimal value functions and optimal solution functions and invoking the chain rule, the following corollary, whose proof is relegated to the mental exercises, links the derivatives of the aforementioned functions. Please observe that in the statement of the corollary, we have selectively noted the dimensionality of the vectors and matrices for clarity.

Corollary 8.1 (Derivative Decomposition): *Under assumptions (A.1) through (A.6), the following derivative decompositions hold for the reciprocal pair of isoperimetric problems (P) and (R):*

$$\underbrace{\frac{\partial \mathbf{x}^M}{\partial \varepsilon}(t;\varepsilon,\beta)}_{N\times(L+2N+1)} \equiv \underbrace{\frac{\partial \mathbf{x}^m}{\partial \varepsilon}(t;\varepsilon,F^M(\varepsilon,\beta))}_{N\times(L+2N+1)} + \underbrace{\frac{\partial \mathbf{x}^m}{\partial \gamma}(t;\varepsilon,F^M(\varepsilon,\beta))}_{N\times 1}\underbrace{\frac{\partial F^M}{\partial \varepsilon}(\varepsilon,\beta)}_{1\times(L+2N+1)}, \tag{a}$$

$$\frac{\partial \dot{\mathbf{x}}^M}{\partial \varepsilon}(t;\varepsilon,\beta) \equiv \frac{\partial \dot{\mathbf{x}}^m}{\partial \varepsilon}(t;\varepsilon,F^M(\varepsilon,\beta)) + \frac{\partial \dot{\mathbf{x}}^m}{\partial \gamma}(t;\varepsilon,F^M(\varepsilon,\beta))\frac{\partial F^M}{\partial \varepsilon}(\varepsilon,\beta),$$

$$\underbrace{\frac{\partial \mathbf{x}^M}{\partial \beta}(t;\varepsilon,\beta)}_{N\times 1} \equiv \underbrace{\frac{\partial \mathbf{x}^m}{\partial \gamma}(t;\varepsilon,F^M(\varepsilon,\beta))}_{N\times 1}\underbrace{\frac{\partial F^M}{\partial \beta}(\varepsilon,\beta)}_{1\times 1},$$

$$\frac{\partial \dot{\mathbf{x}}^M}{\partial \beta}(t;\varepsilon,\beta) \equiv \frac{\partial \dot{\mathbf{x}}^m}{\partial \gamma}(t;\varepsilon,F^M(\varepsilon,\beta))\frac{\partial F^M}{\partial \beta}(\varepsilon,\beta)$$

hold $\forall\,(t,\varepsilon,\beta) \in [0,T^\circ] \times B((\varepsilon^\circ,\beta^\circ);\delta_P)$, and

$$\underbrace{\frac{\partial G^m}{\partial \varepsilon}(\varepsilon,F^M(\varepsilon,\beta))}_{1\times(L+2N+1)} + \underbrace{\frac{\partial G^m}{\partial \gamma}(\varepsilon,F^M(\varepsilon,\beta))}_{1\times 1}\underbrace{\frac{\partial F^M}{\partial \varepsilon}(\varepsilon,\beta)}_{1\times(L+2N+1)} \equiv \mathbf{0}'_{L+2N+1}, \tag{b}$$

$$\underbrace{\frac{\partial G^m}{\partial \gamma}(\varepsilon,F^M(\varepsilon,\beta))}_{1\times 1}\underbrace{\frac{\partial F^M}{\partial \beta}(\varepsilon,\beta)}_{1\times 1} \equiv 1$$

https://doi.org/10.1017/CBO9780511806827.010 Published online by Cambridge University Press

hold $\forall\,(\varepsilon, \beta) \in B((\varepsilon^\circ, \beta^\circ); \delta_P)$, and

$$\frac{\partial \mathbf{x}^m}{\partial \varepsilon}(t; \varepsilon, \gamma) \equiv \frac{\partial \mathbf{x}^M}{\partial \varepsilon}(t; \varepsilon, G^m(\varepsilon, \gamma)) + \frac{\partial \mathbf{x}^M}{\partial \beta}(t; \varepsilon, G^m(\varepsilon, \gamma))\frac{\partial G^m}{\partial \varepsilon}(\varepsilon, \gamma), \quad \text{(c)}$$

$$\frac{\partial \dot{\mathbf{x}}^m}{\partial \varepsilon}(t; \varepsilon, \gamma) \equiv \frac{\partial \dot{\mathbf{x}}^M}{\partial \varepsilon}(t; \varepsilon, G^m(\varepsilon, \gamma)) + \frac{\partial \dot{\mathbf{x}}^M}{\partial \beta}(t; \varepsilon, G^m(\varepsilon, \gamma))\frac{\partial G^m}{\partial \varepsilon}(\varepsilon, \gamma),$$

$$\frac{\partial \mathbf{x}^m}{\partial \gamma}(t; \varepsilon, \gamma) \equiv \frac{\partial \mathbf{x}^M}{\partial \beta}(t; \varepsilon, G^m(\varepsilon, \gamma))\frac{\partial G^m}{\partial \gamma}(\varepsilon, \gamma),$$

$$\frac{\partial \dot{\mathbf{x}}^m}{\partial \gamma}(t; \varepsilon, \gamma) \equiv \frac{\partial \dot{\mathbf{x}}^M}{\partial \beta}(t; \varepsilon, G^m(\varepsilon, \gamma))\frac{\partial G^m}{\partial \gamma}(\varepsilon, \gamma),$$

hold $\forall\,(t, \varepsilon, \gamma) \in [0, T^\circ] \times B((\varepsilon^\circ, \gamma^\circ); \delta_R)$, and

$$\frac{\partial F^M}{\partial \varepsilon}(\varepsilon, G^m(\varepsilon, \gamma)) + \frac{\partial F^M}{\partial \beta}(\varepsilon, G^m(\varepsilon, \gamma))\frac{\partial G^m}{\partial \varepsilon}(\varepsilon, \gamma) \equiv \mathbf{0}, \quad \text{(d)}$$

$$\frac{\partial F^M}{\partial \beta}(\varepsilon, G^m(\varepsilon, \gamma))\frac{\partial G^m}{\partial \gamma}(\varepsilon, \gamma) \equiv 1,$$

hold $\forall\,(\varepsilon, \gamma) \in B((\varepsilon^\circ, \gamma^\circ); \delta_R)$.

The first two identities in parts (a) and (c) of Corollary 8.1 are the reciprocal pair of generalized Slutsky-like decompositions for the "prices" of the reciprocal pair of isoperimetric problems (P) and (R), and the last two identities of (a) and (c) are the generalized Slutsky-like decompositions for "income" and "utility level." Similarly, the first identities of parts (b) and (d) are the generalized Roy-like identities, whereas the second identities of parts (b) and (d) demonstrate the reciprocal nature of the conjugate variables for the isoperimetric constraints. To see the latter assertion, simply note that by assumptions (A.3) and (A.4) and Theorem 7.3, $\partial F^M(\varepsilon, \beta)/\partial \beta \equiv \psi^M(\varepsilon, \beta) > 0$ and $\partial G^m(\varepsilon, \gamma)/\partial \gamma \equiv \psi^m(\varepsilon, \gamma) > 0$ are the constant conjugate variables for problems (P) and (R), respectively. This reciprocal relationship between the conjugate variables of problems (P) and (R) should not be too surprising at this juncture of the chapter, since it was noted above that parts (b) and (d) of Theorem 8.1 show that the optimal value functions $F^M(\cdot)$ and $G^m(\cdot)$ are inverses of one another with respect to the isoperimetric constraint parameters (β, γ). Rather than dwell any further on the interpretation of the generic results of Theorem 8.1 and Corollary 8.1, a more complete and intuitive economic interpretation of them is given in the context of the competitive nonrenewable resource–extracting model of the firm, which we now proceed to analyze. In passing, note that not all of the identities in Corollary 8.1 are independent of one another, as you are asked to verify in a mental exercise.

To begin, let $q(t)$ be the extraction rate of the nonrenewable resource at time t and let $p > 0$ be the constant market price of the extracted product, that is, the output price. Define the minimum cost of extracting the resource at the rate $q(t)$ at the constant input price of $w > 0$ by

$$C(q; w) \overset{\text{def}}{=} \min_{v}\{w \cdot v \text{ s.t. } q = f(v)\},$$

where $f(\cdot) \in C^{(2)}$ is the production function with the standard properties $f'(v) > 0$ and $f''(v) < 0$, and $v > 0$ is the variable input used to extract the nonrenewable resource. In a mental exercise, you are asked to prove that these assumptions imply that $C_q(q; w) > 0$, $C_{qq}(q; w) > 0$, $C_w(q; w) > 0$, and $C_{qw}(q; w) > 0$. The firm has a fixed planning horizon of $T > 0$ years and a discount rate of $r > 0$. The initial stock of the resource is $s > 0$, and the firm is assumed to extract all of it by the close of the planning horizon.

The primal isoperimetric problem asserts that the firm chooses an extraction function $q(\cdot)$ to maximize the present discounted value of profit over the planning horizon, such that cumulative extraction equals the initial stock:

$$\Pi(\varepsilon, s) \overset{\text{def}}{=} \max_{q(\cdot)} \left\{ \int_0^T [pq(t) - C(q(t); w)]\, e^{-rt}\, dt \text{ s.t. } \int_0^T q(t)\, dt = s \right\}, \tag{3}$$

where $\Pi(\varepsilon, s)$ is the maximum present discounted value of profit that can be earned from complete extraction of the initial resource stock s at prices (p, w), discount rate r, and planning horizon T; $q^M(t; \varepsilon, s)$ is the optimal extraction path; $\psi^M(\varepsilon, s)$ is the corresponding shadow value of the initial resource stock, that is, the optimal value of the primal conjugate variable; and $\varepsilon \overset{\text{def}}{=} (p, r, w, T)$. The reciprocal problem asserts that the firm chooses an extraction function $q(\cdot)$ to minimize the cumulative amount of the resource extracted over the planning horizon, subject to producing a present discounted value of profit equal to a predetermined level $\pi > 0$:

$$Q(\varepsilon, \pi) \overset{\text{def}}{=} \min_{q(\cdot)} \left\{ \int_0^T q(t)\, dt \text{ s.t. } \int_0^T [pq(t) - C(q(t); w)]\, e^{-rt}\, dt = \pi \right\}, \tag{4}$$

where $Q(\varepsilon, \pi)$ is the minimum cumulative extraction that will yield the present discounted value of profit π at prices (p, w), discount rate r, and planning horizon T; $q^m(t; \varepsilon, \pi)$ is the optimal extraction path; and $\psi^m(\varepsilon, s)$ is the corresponding marginal change in cumulative extraction due to an increase in the required present value of profit, that is, the optimal value of the reciprocal conjugate variable.

Applying Theorem 8.1 to the reciprocal isoperimetric problems (3) and (4) yields their fundamental identities:

$$q^M(t; \varepsilon, s) \equiv q^m(t; \varepsilon, \Pi(\varepsilon, s)), \tag{5}$$

$$Q(\varepsilon, \Pi(\varepsilon, s)) \equiv s, \tag{6}$$

$$q^m(t; \varepsilon, \pi) \equiv q^M(t; \varepsilon, Q(\varepsilon, \pi)), \tag{7}$$

$$\Pi(\varepsilon, Q(\varepsilon, \pi)) \equiv \pi. \tag{8}$$

Equation (5) asserts that at the given prices (p, w), discount rate r, planning horizon T, and initial stock size s, the extraction rate that maximizes the present discounted value of profit subject to complete extraction of the initial resource stock is exactly the same extraction rate that minimizes the cumulative amount of the resource extracted subject to earning a present discounted value of profit that is equal to the maximum attainable at prices (p, w), discount rate r, horizon length T, and initial stock size s. Reciprocally, Eq. (7) asserts that at the given prices (p, w), discount rate r, planning horizon T, and required present discounted value of profit π, the extraction rate that minimizes the cumulative amount of the resource extracted subject to earning a given present discounted value of profit is exactly the same extraction rate that maximizes the present discounted value of profit subject to complete extraction of an initial resource stock that is equal to the minimum cumulative amount of the resource one would extract at prices (p, w), discount rate r, planning horizon T, and required present discounted value of profit π. Equation (6) asserts that given the prices (p, w), discount rate r, and planning horizon T, the minimum cumulative extraction required to reach the maximum present discounted value of profit obtainable at prices (p, w), discount rate r, planning horizon T, and initial stock s is identically the initial stock s. Symmetrically, identity (8) asserts that at the given prices (p, w), discount rate r, and planning horizon T, the maximum present discounted value of profit obtainable when the initial stock size is the minimum cumulative extraction that can be achieved at prices (p, w), discount rate r, planning horizon T, and present discounted value of profit π is exactly the present discounted value of profit π. That is, for fixed ε, identities (6) and (8) show that the optimal value functions $\Pi(\cdot)$ and $Q(\cdot)$ are inverses of one another with respect to s and π, respectively.

Though the economic interpretation of the identities (5) through (8) is important, the relationship between the derivatives of the functions in Eqs. (5) through (8) and their economic interpretation is even more so. Using Corollary 8.1, or simply differentiating the identities in Eqs. (5) through (8) with respect to, say, (p, s, π), the chain rule gives

$$\frac{\partial q^M}{\partial p}(t;\varepsilon,s) \equiv \frac{\partial q^m}{\partial p}(t;\varepsilon,\Pi(\varepsilon,s)) + \frac{\partial q^m}{\partial \pi}(t;\varepsilon,\Pi(\varepsilon,s))\frac{\partial \Pi}{\partial p}(\varepsilon,s), \tag{9}$$

$$\frac{\partial q^M}{\partial s}(t;\varepsilon,s) \equiv \frac{\partial q^m}{\partial \pi}(t;\varepsilon,\Pi(\varepsilon,s))\frac{\partial \Pi}{\partial s}(\varepsilon,s), \tag{10}$$

$$\frac{\partial Q}{\partial p}(\varepsilon,\Pi(\varepsilon,s)) + \frac{\partial Q}{\partial \pi}(\varepsilon,\Pi(\varepsilon,s))\frac{\partial \Pi}{\partial p}(\varepsilon,s) \equiv 0, \tag{11}$$

$$\frac{\partial Q}{\partial \pi}(\varepsilon,\Pi(\varepsilon,s))\frac{\partial \Pi}{\partial s}(\varepsilon,s) \equiv 1, \tag{12}$$

$$\frac{\partial q^m}{\partial p}(t;\varepsilon,\pi) \equiv \frac{\partial q^M}{\partial p}(t;\varepsilon,Q(\varepsilon,\pi)) + \frac{\partial q^M}{\partial s}(t;\varepsilon,Q(\varepsilon,\pi))\frac{\partial Q}{\partial p}(\varepsilon,\pi), \tag{13}$$

https://doi.org/10.1017/CBO9780511806827.010 Published online by Cambridge University Press

$$\frac{\partial q^m}{\partial \pi}(t; \varepsilon, \pi) \equiv \frac{\partial q^M}{\partial s}(t; \varepsilon, Q(\varepsilon, \pi))\frac{\partial Q}{\partial \pi}(\varepsilon, \pi), \tag{14}$$

$$\frac{\partial \Pi}{\partial p}(\varepsilon, Q(\varepsilon, \pi)) + \frac{\partial \Pi}{\partial s}(\varepsilon, Q(\varepsilon, \pi))\frac{\partial Q}{\partial p}(\varepsilon, \pi) \equiv 0, \tag{15}$$

$$\frac{\partial \Pi}{\partial s}(\varepsilon, Q(\varepsilon, \pi))\frac{\partial Q}{\partial \pi}(\varepsilon, \pi) \equiv 1. \tag{16}$$

The remaining derivative decompositions for the parameters (r, w, T) are similar and are thus left for a mental exercise. The identities (5) through (8) and (9) through (16) are heretofore undiscovered for this intertemporal model.

Equations (9) and (13) are the reciprocal pair of Slutsky-like decompositions for the output price in the nonrenewable resource–extracting model of the firm. In particular, Eq. (9) asserts that the slope of the present value profit maximizing supply function is composed of two parts: (i) $\partial q^m/\partial p$, the effect of a price change on supply holding constant the present value of profit as well as all other parameters except the market price of the extracted good, that is, the slope of the supply function from the reciprocal problem (4); and (ii) $(\partial q^m/\partial \pi)(\partial \Pi/\partial p)$, the effect the price change has on the maximum present value of profit when it adjusts optimally, and the concomitant effect the change in the present value of profit has on supply. The first effect could be called the *iso-wealth* effect, because, by the definition of the partial derivative, it explicitly holds the present value of profit (i.e., wealth) constant when the output price changes. The second term could be called the *wealth* effect, seeing as it measures the effect the price change has on supply by optimally varying the present discounted value of profit.

Similarly, Eq. (13) asserts that the slope of the cumulative extraction–minimizing supply function consists of two parts: (i) an *iso-stock* effect $\partial q^M/\partial p$, which measures the effect of a price change on supply holding constant the initial stock of the resource as well as all other parameters except the market price of the extracted good, that is, the slope of the supply function from the primal problem (3), and (ii) a *stock* effect $(\partial q^M/\partial s)(\partial Q/\partial p)$, which measures the effect of a price change on minimum cumulative extraction when it adjusts optimally, and the concomitant effect the change in cumulative extraction has on supply. The economic interpretation of Eqs. (10) and (14) is similar to that just given and is therefore left for a mental exercise.

The identities (11) and (15) are the Roy-like identities for the nonrenewable resource–extracting model of the firm. To see this, simply note that by Theorem 7.3, it follows that $\partial \Pi(\varepsilon, s)/\partial p \equiv \int_0^T q^M(t; \varepsilon, s)e^{-rt}\,dt$, so that a rearrangement of Eq. (11) yields

$$\int_0^T q^M(t; \varepsilon, s)e^{-rt}\,dt \equiv \frac{-\frac{\partial Q}{\partial p}(\varepsilon, \Pi(\varepsilon, s))}{\frac{\partial Q}{\partial \pi}(\varepsilon, \Pi(\varepsilon, s))}.$$

One can also arrive at the identical formula by first evaluating Eq. (15) at $\pi = \Pi(\varepsilon, s)$ and using Eq. (6), then using $\partial \Pi(\varepsilon, s)/\partial p \equiv \int_0^T q^M(t; \varepsilon, s)e^{-rt}\,dt$, and finally substituting out $\partial \Pi(\varepsilon, s)/\partial s$ using Eq. (12), as you are asked to verify in a mental exercise. Notice that in this model, the Roy-like identity forces one's attention on the cumulative discounted supply function.

By Theorem 7.3, $\partial \Pi(\varepsilon, s)/\partial s \equiv \psi^M(\varepsilon, s)$ and $\partial Q(\varepsilon, \pi)/\partial \pi \equiv \psi^m(\varepsilon, \pi)$, so Eqs. (12) and (16) demonstrate the reciprocal nature of the time-independent conjugate variables for problems (3) and (4), a result that should now appear intuitive given the earlier observation that $\Pi(\cdot)$ and $Q(\cdot)$ are inverse functions of one another with respect to the isoperimetric constraint parameters (π, s). In other words, at the optimum, the shadow value of the initial stock, $\psi^M(\varepsilon, s)$, is the reciprocal of the marginal change in cumulative extraction due to an increase in the required present value of profit, $\psi^m(\varepsilon, \pi)$, when $s = Q(\varepsilon, \pi)$ or $\pi = \Pi(\varepsilon, s)$.

The above derivative decompositions can yield further insight into the reciprocal pair of isoperimetric problems (3) and (4) upon exploring the comparative dynamics properties of *one* of the problems. Only one of the problems needs to be examined for comparative dynamics results, for the identities in Eqs. (9) through (16) yield the comparative dynamics of the reciprocal problem. Problem (3) is investigated here because it is the primal problem and thus the more common formulation. It is important to point out that the comparative dynamics method used below is quite general, the only limitation being that $\dot{q}$ can't explicitly appear in the problem. Moreover, the method permits the derivation of the comparative dynamics properties of the problem *without* the aid of a phase portrait or specific functional forms.

By Theorem 7.1, the necessary conditions for problem (3) are found by defining the augmented integrand, $\tilde{F}(t, q, \psi; p, r, w) \overset{\text{def}}{=} [pq - C(q; w)]\, e^{-rt} - \psi q$, and applying the Euler equation to $\tilde{F}(\cdot)$. Since $\partial \tilde{F}/\partial \dot{q} \equiv 0$, this yields the *algebraic* equation

$$\frac{\partial \tilde{F}}{\partial q} = [p - C_q(q; w)]\, e^{-rt} - \psi = 0. \tag{17}$$

It should be clear that $\psi > 0$, because if $\psi < 0$, then $p < C_q(q; w)$ from Eq. (17), in which case the firm would lose money on every unit of the resource extracted and hence go out of business. Also note that when $\psi = 0$, the firm acts as a static profit maximizer setting $p = C_q(q; w)$. This case is uninteresting, however, because the isoperimetric problem (3) is no longer dynamic when $\psi = 0$ holds. Finally, note that because $\psi > 0$, $\tilde{F}(\cdot)$ is a concave function of $(q, \dot{q})$ for all $t \in [0, T]$ by assumption (A.2). Therefore, by Theorem 7.4, an admissible solution of the augmented Euler equation (17) is a globally optimal solution to the isoperimetric problem (3).

Given that $C(\cdot) \in C^{(2)}$ and $C_{qq}(q; w) > 0$, the Jacobian of Eq. (17) with respect to q is negative everywhere, that is, $\tilde{F}_{qq}(t, q, \psi; p, r, w) = -C_{qq}(q; w)e^{-rt} < 0$. Hence, by the implicit function theorem, Eq. (17) can be solved in principle for the extraction rate as a $C^{(1)}$ function of $(t; \psi, p, r, w)$:

$$q = \hat{q}(t; \psi, p, r, w). \tag{18}$$

The function $\hat{q}(\cdot)$ gives the extraction rate that satisfies the augmented Euler equation *conditional* on the conjugate variable ψ, the shadow value of the initial stock, which is a constant by Theorem 7.1. Thus the function $\hat{q}(\cdot)$ is *not* necessarily admissible as we have yet to verify that it satisfies the integral constraint of problem (3). Therefore, in order to find the *optimal* extraction rate, namely, $q^M(t; \varepsilon, s)$, the value of the constant multiplier ψ must be found so as to make $\hat{q}(\cdot)$ an *admissible* solution of the augmented Euler equation. Before doing so, however, it is instructive to examine the qualitative properties of the function $\hat{q}(\cdot)$ first.

Using the implicit function theorem on Eq. (17) yields the qualitative properties of the function $\hat{q}(\cdot)$:

$$\frac{\partial \hat{q}}{\partial t} \equiv \frac{-r[p - C_q(q; w)]}{C_{qq}(q; w)} < 0, \tag{19}$$

$$\frac{\partial \hat{q}}{\partial \psi} \equiv \frac{-1}{C_{qq}(q; w)e^{-rt}} < 0, \tag{20}$$

$$\frac{\partial \hat{q}}{\partial p} \equiv \frac{1}{C_{qq}(q; w)} > 0, \tag{21}$$

$$\frac{\partial \hat{q}}{\partial r} \equiv \frac{-t\,[p - C_q(q; w)]}{C_{qq}(q; w)} \leq 0, \tag{22}$$

$$\frac{\partial \hat{q}}{\partial w} \equiv \frac{-C_{qw}(q; w)}{C_{qq}(q; w)} < 0 \quad (\because C_{qw}(q; w) > 0), \tag{23}$$

where all the functions in Eqs. (19) through (23) are evaluated at $q = \hat{q}(t; \psi, p, r, w)$. Equation (19) says that the extraction rate declines over the planning horizon, holding constant the shadow value of the initial stock. The qualitative result in Eq. (26) asserts that an increase in the shadow value of the initial stock, ceteris paribus, lowers the extraction rate, which is intuitive, because one would like to leave more of the resource "in the ground" if its shadow value in the unextracted state is higher. Equations (21) and (23) assert, respectively, that holding the shadow value of the initial stock constant, the extraction rate rises with an increase in the output price and a decrease in the input price. These qualitative results are also intuitive, because if the extracted resource is worth more in the market or it is cheaper to extract, then it is sensible for the firm to extract the resource at a faster rate in order to maximize its present discounted value of profit. The comparative dynamics result in Eq. (22) appears counterintuitive, in that normally an increase in the discount rate is thought to increase the extraction rate early in the planning horizon and decrease it later.

Remember, however, that ψ is held fixed in Eqs. (21), (22), and (23), so none of these comparative dynamics results measure the change in the *optimal* extraction rate due to a change in the parameter, for the optimal value of ψ must be determined, and it will be a function of all the parameters except for t.

To find the optimal value of ψ, substitute the extraction rate $q = \hat{q}(t; \psi, p, r, w)$, the solution to the augmented Euler equation (17), into the isoperimetric constraint of problem (3):

$$\int_0^T \hat{q}(t; \psi, p, r, w)\, dt = s. \tag{24}$$

By Eq. (20), the Jacobian of Eq. (24) with respect to ψ is negative, that is,

$$\frac{\partial}{\partial \psi} \int_0^T \hat{q}(t; \psi, p, r, w)\, dt = \int_0^T \frac{\partial \hat{q}}{\partial \psi}(t; \psi, p, r, w)\, dt < 0.$$

Thus, by the implicit function theorem, we can in principle solve Eq. (24) for the shadow value of the initial stock as a $C^{(1)}$ function of the parameters (ε, s):

$$\psi = \psi^M(\varepsilon, s). \tag{25}$$

Recall that Theorem 7.1 asserts that the optimal value of ψ given in Eq. (25) is constant. This is evident by Eq. (24), since t is the dummy variable of integration and is thus integrated out of Eq. (24) when solving for ψ. This fact is also reflected in the notation of Eq. (25) by observing that $\psi^M(\cdot)$ is independent of t.

Applying the implicit function theorem to Eq. (24) yields the comparative dynamics of the shadow value of the initial stock:

$$\frac{\partial \psi^M}{\partial p} \equiv \frac{-\int_0^T \frac{\partial \hat{q}}{\partial p}(t; \psi, p, r, w)\, dt}{\int_0^T \frac{\partial \hat{q}}{\partial \lambda}(t; \psi, p, r, w)\, dt} \in (0, 1), \tag{26}$$

$$\frac{\partial \psi^M}{\partial r} \equiv \frac{-\int_0^T \frac{\partial \hat{q}}{\partial r}(t; \psi, p, r, w)\, dt}{\int_0^T \frac{\partial \hat{q}}{\partial \lambda}(t; \psi, p, r, w)\, dt} < 0, \tag{27}$$

$$\frac{\partial \psi^M}{\partial w} \equiv \frac{-\int_0^T \frac{\partial \hat{q}}{\partial w}(t; \psi, p, r, w)\, dt}{\int_0^T \frac{\partial \hat{q}}{\partial \lambda}(t; \psi, p, r, w)\, dt} < 0, \tag{28}$$

$$\frac{\partial \psi^M}{\partial s} \equiv \frac{1}{\int_0^T \frac{\partial \hat{q}}{\partial \lambda}(t; \psi, p, r, w)\, dt} < 0, \tag{29}$$

$$\frac{\partial \psi^M}{\partial T} \equiv \frac{-\hat{q}(T; \psi, p, r, w)}{\int_0^T \frac{\partial \hat{q}}{\partial \lambda}(t; \psi, p, r, w)\, dt} \geq 0, \tag{30}$$

https://doi.org/10.1017/CBO9780511806827.010 Published online by Cambridge University Press

where the functions in Eqs. (26) through (30) are evaluated at $\psi = \psi^M(\varepsilon, s)$ and the *quantitative* result in Eq. (26) follows from Eqs. (20) and (21). The comparative dynamics given in Eqs. (26) through (30) are the expected economic intuitions. For example, Eq. (26) asserts that an increase in the output price drives up the shadow value of the initial stock, but not by as much as the output price rose, because if the market now values the extracted product more, the rational firm recognizes this *and* the fact that it is costly to extract the resource, and thus places only a fractionally higher shadow value on the unextracted product. In other words, if the extracted good is worth more, then so is its unextracted form, but only fractionally, since the former is derived at a cost from the latter. If the firm becomes more impatient, implying an increase in the discount rate, then Eq. (27) shows that the shadow value of the initial stock falls because the present discounted value of the asset itself is lower too. If the input price rises, then the firm's marginal cost of extraction rises. The firm thus places a lower shadow value on the initial stock in view of the fact that the net profit on each unit extracted is lower, which is the intuition behind Eq. (28). Equation (29) asserts that the shadow value of the initial stock is lower the larger the initial stock of the resource, that is, with a larger initial stock available, the firm values each additional unit less. Alternatively, $\psi^M(\cdot)$ is the firm's inverse demand function for the initial stock of the resource; hence Eq. (29) is simply an assertion that the law of demand holds for the initial resource stock in inverse demand form. Recalling that $\partial\Pi(\varepsilon, s)/\partial s \equiv \psi^M(\varepsilon, s) > 0$, another interpretation of Eq. (29) is that the optimal value function $\Pi(\cdot)$ is strictly concave in the initial resource stock. Finally, if the extraction rate is zero (positive) in the terminal period of the planning horizon, then Eq. (30) shows that the firm places the same (a higher) shadow value on the initial resource stock when the length of the planning horizon increases. This is intuitive because if the firm is not extracting in the last period of the planning horizon, then it has already extracted all of the stock available before the terminal date, and hence there is no additional value to the firm in having more time available to extract the same amount of the stock.

With the shadow value of the initial stock now determined, the optimal extraction rate can be found by substituting Eq. (25) into Eq. (18), thus *defining* the optimal extraction rate:

$$q^M(t;\varepsilon, s) \stackrel{\text{def}}{=} \hat{q}(t;\psi^M(\varepsilon, s), p, r, w). \tag{31}$$

The comparative dynamics of the optimal extraction rate follow from differentiating Eq. (31) with respect to the parameter of interest using the chain rule:

$$\frac{\partial q^M}{\partial t}(t;\varepsilon, s) = \frac{\partial \hat{q}}{\partial t}(t;\psi^M(\varepsilon, s), p, r, w) < 0\,\forall\, t \in [0, T], \tag{32}$$

$$\frac{\partial q^M}{\partial p}(t;\varepsilon, s) = \frac{\partial \hat{q}}{\partial \psi}(t;\psi^M(\varepsilon, s), p, r, w)\frac{\partial \psi^M}{\partial p}(\varepsilon, s) + \frac{\partial \hat{q}}{\partial p}(t;\psi^M(\varepsilon, s), p, r, w), \tag{33}$$

https://doi.org/10.1017/CBO9780511806827.010 Published online by Cambridge University Press

$$\frac{\partial q^M}{\partial r}(t;\varepsilon,s)=\frac{\partial\hat{q}}{\partial\psi}(t;\psi^M(\varepsilon,s),p,r,w)\frac{\partial\psi^M}{\partial r}(\varepsilon,s)+\frac{\partial\hat{q}}{\partial r}(t;\psi^M(\varepsilon,s),p,r,w), \tag{34}$$

$$\frac{\partial q^M}{\partial w}(t;\varepsilon,s)=\frac{\partial\hat{q}}{\partial\psi}(t;\psi^M(\varepsilon,s),p,r,w)\frac{\partial\psi^M}{\partial w}(\varepsilon,s)+\frac{\partial\hat{q}}{\partial w}(t;\psi^M(\varepsilon,s),p,r,w), \tag{35}$$

$$\frac{\partial q^M}{\partial s}(t;\varepsilon,s)=\frac{\partial\hat{q}}{\partial\psi}(t;\psi^M(\varepsilon,s),p,r,w)\frac{\partial\psi^M}{\partial s}(\varepsilon,s)>0\,\forall\, t\in[0,T], \tag{36}$$

$$\frac{\partial q^M}{\partial T}(t;\varepsilon,s)=\frac{\partial\hat{q}}{\partial\psi}(t;\psi^M(\varepsilon,s),p,r,w)\frac{\partial\psi^M}{\partial T}(\varepsilon,s)\leq 0\,\forall\, t\in[0,T]. \tag{37}$$

The sign of Eq. (32) follows from Eq. (19), whereas the signs in Eqs. (36) and (37) follow from Eqs. (20), (29), and (30). Equation (32) is the well-known result that the optimal extraction rate falls over the planning period, and the equality exhibited between $\partial q^M/\partial t$ and $\partial\hat{q}/\partial t$ is a result of the fact that $\psi^M(\cdot)$ is independent of t.

In general, Eqs. (33), (34), and (35) cannot be signed $\forall\, t\in[0,T]$, but this is not surprising. To see why, recall that $q^M(t;\varepsilon,s)$ satisfies the isoperimetric constraint identically; hence

$$\int_0^T q^M(t;\varepsilon,s)\,dt\equiv s.$$

Differentiating this identity with respect to, say, p, yields

$$\frac{\partial}{\partial p}\int_0^T q^M(t;\varepsilon,s)\,dt=\int_0^T\frac{\partial q^M}{\partial p}(t;\varepsilon,s)\,dt\equiv 0 \tag{38}$$

and says that the effect of an output price increase on cumulative extraction is zero, since cumulative extraction must equal the given initial stock, the latter of which is unaffected by the output price increase. Therefore, if $\partial q^M/\partial p$ is positive for some period of time, it must be negative over some other period of time for Eq. (38) to hold. Hence a uniform sign for $\partial q^M/\partial p$ for all $t\in[0,T]$ is precluded by Eq. (38), that is, by the fixed initial stock and the integral constraint that all of the resource be extracted. Note, however, that upon evaluating Eq. (33) at $t=0$, using Eqs. (20) and (21) to deduce that $\partial\hat{q}(0;\psi,p,r,w)/\partial\psi=-\partial\hat{q}(0;\psi,p,r,w)/\partial p<0$, and then recalling Eq. (26), it follows that Eq. (33) can be written as

$$\frac{\partial q^M}{\partial p}(0;\varepsilon,s)=\frac{\partial\hat{q}}{\partial p}(0;\psi^M(\varepsilon,s),p,r,w)\left[1-\frac{\partial\psi^M}{\partial p}(\varepsilon,s)\right]>0. \tag{39}$$

Thus, at the initial date of the planning horizon, an increase in the output price results in an increase in the extraction rate. By continuity of $\partial q^M(\cdot)/\partial p$ in t, $\partial q^M(t;\varepsilon,s)/\partial p > 0$ holds for some finite interval of time near the initial date. It therefore follows from this observation and Eqs. (32) and (38) that $\partial q^M(t;\varepsilon,s)/\partial p < 0$ holds for some finite interval of time near the end of the planning horizon. You are asked to prove this latter result in a mental exercise following the steps and logic used to deduce Eq. (39). In sum, the firm would like to extract more of the resource because it is now worth more in the market. But seeing as the initial stock of the resource is fixed and the extraction rate declines over the planning period, the only way the firm can take advantage of the price increase is by rearranging its extraction profile so that more of the resource is extracted early in the planning horizon and less is extracted later. A similar analysis may be done for Eq. (34), as you are asked to verify in a mental exercise.

Equation (36) asserts that an increase in the initial resource stock increases the optimal extraction rate at each date of the planning horizon. This is judicious, for a larger initial stock necessitates a higher extraction rate given that the planning horizon is unchanged. Equation (37) asserts that if the firm is extracting the resource in the last period of the planning horizon and the length of the planning horizon increases so that it now has more time to extract the same amount of the resource, then it will do so at a slower rate. Moreover, if the firm is not extracting in the last period of the planning horizon, implying that it has already extracted all of the stock available before the terminal date, then an increase in the length of the planning horizon would have no effect on its optimal extraction plan.

Given that the comparative dynamics properties of the primal problem (3) have been deduced, all one has to do now is exploit the identities in Eqs. (9) through (16) to uncover the comparative dynamics properties of the reciprocal problem (4). For example, by Eqs. (13) and (14), it follows that

$$\frac{\partial q^m}{\partial p}(t;\varepsilon,\pi) \equiv \frac{\partial q^M}{\partial p}(t;\varepsilon,Q(\varepsilon,\pi)) + \frac{\partial q^M}{\partial s}(t;\varepsilon,Q(\varepsilon,\pi))\frac{\partial Q}{\partial p}(\varepsilon,\pi), \tag{40}$$

$$\frac{\partial q^m}{\partial \pi}(t;\varepsilon,\pi) \equiv \frac{\partial q^M}{\partial s}(t;\varepsilon,Q(\varepsilon,\pi))\frac{\partial Q}{\partial \pi}(\varepsilon,\pi) > 0\,\forall\, t \in [0,T]. \tag{41}$$

Recalling that $\partial Q(\varepsilon,\pi)/\partial\pi \equiv \psi^m(\varepsilon,\pi) > 0$ by Theorem 7.3 and that $\partial q^M(t;\varepsilon,s)/\partial s > 0$ for all $t \in [0,T]$ by Eq. (36), Eq. (41) asserts that an increase in the required wealth target for the cumulative extraction–minimizing firm will lead to an increase in its extraction rate at every date of the planning horizon, thereby resulting in higher cumulative extraction. That is, given that the firm must now produce a higher level of wealth from the same initial stock of the resource, it must extract a larger amount of the stock over its planning horizon, and in order to do so, it must extract the stock at a faster rate in every period because its planning horizon is unchanged.

Equation (40), on the other hand, has an ambiguous sign even at $t = 0$, but can be manipulated so that an economically intuitive necessary and sufficient condition for signing it emerges. First, evaluate Eqs. (36) and (40) at $t = 0$, and then evaluate Eqs. (36) and (39) at $s = Q(\varepsilon, \pi)$. Next, substitute Eqs. (36) and (39) into Eq. (40) to get

$$\frac{\partial q^m}{\partial p}(0; \varepsilon, \pi) = \frac{\partial \hat{q}}{\partial p}(0; \psi^M(\varepsilon, Q(\varepsilon, \pi)), p, r, w)\left[1 - \frac{\partial \psi^M}{\partial p}(\varepsilon, Q(\varepsilon, \pi))\right] + \frac{\partial \hat{q}}{\partial \psi}(0; \psi^M(\varepsilon, Q(\varepsilon, \pi)), p, r, w)\frac{\partial \psi^M}{\partial s}(\varepsilon, Q(\varepsilon, \pi))\frac{\partial Q}{\partial p}(\varepsilon, \pi). \tag{42}$$

Finally, recall that Eqs. (20) and (21) imply that $\partial \hat{q}(0; \psi, p, r, w)/\partial \psi = -\partial \hat{q}(0; \psi, p, r, w)/\partial p$, thereby implying that Eq. (42) reduces to

$$\frac{\partial q^m}{\partial p}(0; \varepsilon, \pi) = \frac{\partial \hat{q}}{\partial p}(0; \psi^M(\varepsilon, Q(\varepsilon, \pi)), p, r, w) \times \left[1 - \frac{\partial \psi^M}{\partial p}(\varepsilon, Q(\varepsilon, \pi)) - \frac{\partial \psi^M}{\partial s}(\varepsilon, Q(\varepsilon, \pi))\frac{\partial Q}{\partial p}(\varepsilon, \pi)\right]. \tag{43}$$

This is the expression sought. Because $\partial \hat{q}/\partial p > 0$ from Eq. (21), $\partial q^m(0; \varepsilon, \pi)/\partial p > 0$ if and only if the bracketed expression in Eq. (43) is positive.

To impart some economic intuition to the bracketed expression in Eq. (43), define the *compensated* shadow value of the initial stock by

$$\psi^c(\varepsilon, \pi) \stackrel{\text{def}}{=} \psi^M(\varepsilon, Q(\varepsilon, \pi)). \tag{44}$$

Thus $\psi^c(\varepsilon, \pi)$ is the shadow value of the initial stock along a level curve of the present value of profit, that is, wealth, constraint. Differentiating Eq. (44) with respect to the output price yields a Slutsky-like equation for the shadow value of the initial stock, namely,

$$\frac{\partial \psi^c}{\partial p}(\varepsilon, \pi) = \frac{\partial \psi^M}{\partial p}(\varepsilon, Q(\varepsilon, \pi)) + \frac{\partial \psi^M}{\partial s}(\varepsilon, Q(\varepsilon, \pi))\frac{\partial Q}{\partial p}(\varepsilon, \pi). \tag{45}$$

It follows from Theorem 7.3 that $\partial Q(\varepsilon, \pi)/\partial p \equiv -\int_0^T \psi^m(\varepsilon, \pi) q^m(t; \varepsilon, \pi) e^{-rt} dt < 0$, implying that minimum cumulative extraction falls with an increase in the output price. This is an intuitive conclusion, because a higher output price allows the firm to reach its target level of wealth by extracting less of the resource over its given planning horizon. Using the above dynamic envelope result and Eqs. (26) and (29), it follows that $\partial \psi^c(\varepsilon, \pi)/\partial p > 0$, but it may be greater than or less than unity. Finally, substituting Eqs. (44) and (45) into Eq. (43) yields the economically more revealing version of Eq. (43), to wit,

$$\frac{\partial q^m}{\partial p}(0; \varepsilon, \pi) = \frac{\partial \hat{q}}{\partial p}(0; \psi^c(\varepsilon, \pi), p, r, w)\left[1 - \frac{\partial \psi^c}{\partial p}(\varepsilon, \pi)\right]. \tag{46}$$

Equation (46) shows that the cumulative extraction–minimizing supply response at the initial date of the planning horizon is positive if and only if the effect of an output price increase on the compensated shadow value of the initial stock lies inside the unit interval, that is, $\partial q^m(0; \varepsilon, \pi)/\partial p > 0 \Leftrightarrow \partial \psi^c(\varepsilon, \pi)/\partial p \in (0, 1)$. In passing, note the wonderful symmetry in the comparative dynamics of the output price at the initial date exhibited by Eqs. (39) and (46).

It is important to note that we could have derived the comparative dynamics properties of the reciprocal problem (4) first, and then used the identities in Eqs. (9) and (10) to uncover the comparative dynamics properties of the primal problem (3). More generally, whether one wants to derive the comparative dynamics of the primal problem (P) first, as was done here, and then use Theorem 8.1 and Corollary 8.1 to derive the qualitative properties of the reciprocal problem (R), or derive the comparative dynamics properties of the reciprocal problem (R) first, and then use Theorem 8.1 and Corollary 8.1 to derive the qualitative properties of the primal problem (P), is simply a matter of choice. The qualitative information extracted is identical regardless of the route taken. That is the beauty, in fact, of dealing with reciprocal pairs of isoperimetric problems. Though the qualitative comparative dynamics properties of the reciprocal problem can be obtained from the primal problem through the use of Corollary 8.1, and vice versa, it may well turn out that deriving the comparative dynamics properties of the primal problem may be easier by first deriving the comparative dynamics properties of the reciprocal problem, and then invoking Corollary 8.1 to recover those of the primal problem, as it is with a reciprocal pair of intertemporal consumer problems. In fact, the application of Theorem 8.1 and Corollary 8.1 may yield new qualitative insights into economic problems that have been extensively analyzed by other methods, as has been demonstrated here with the nonrenewable resource–extracting model of the firm.

In the ensuing chapter, we return to our study of optimal control theory. In particular, we develop the continuous-time intertemporal generalization of the prototype envelope theorem, to wit, the dynamic envelope theorem. This important theorem permits us to achieve a deeper economic understanding of an optimal control problem. Furthermore, in Chapter 10, we use the dynamic envelope theorem to achieve simple and intuitive proofs of a general set of transversality conditions that are ubiquitous in dynamic economic problems.

MENTAL EXERCISES

8.1 In this question, we return to the R&D problem you analyzed in Mental Exercise 7.8. Recall that the R&D project is subject to diminishing marginal productivity of research expenditures. In other words, defining $e(t)$ as the effort rate expended on R&D at time t, the relationship between the effort rate expended on the project at time t and the spending rate $s(t)$ at time t

is given by

$$e(t) = f(s(t); \alpha),$$

where $f(\cdot) \in C^{(2)}$, $f_s(s; \alpha) > 0$, $f_{ss}(s; \alpha) < 0$, $f_\alpha(s; \alpha) > 0$, and $f_{s\alpha}(s; \alpha) > 0$. An increase in the parameter α therefore represents an increase in the total and marginal product of R&D expenditures. Because $f_s(s; \alpha) > 0$ and $f_{ss}(s; \alpha) < 0$, higher R&D expenditures lead to higher effort, but at a decreasing rate, as asserted above. The cumulative or total effort required to complete the R&D project by the predetermined time $T > 0$ is given by $A > 0$. It is assumed that cumulative effort expended on the R&D project equals the total effort required to complete the R&D project; hence

$$\int_0^T f(s(t); \alpha)\, dt = A.$$

The objective for the firm is to minimize the present discounted cost of completing the R&D project by time T, where $r > 0$ is the discount rate. More formally, the firm is asserted to solve the isoperimetric problem

$$C(\alpha, A, r, T) \stackrel{\text{def}}{=} \min_{s(\cdot)} \left\{ \int_0^T e^{-rt} s(t) dt \text{ s.t. } \int_0^T f(s(t); \alpha)\, dt = A \right\}.$$

(a) Solve, in principle, the augmented Euler equation for its general solution, say, $s = \hat{s}(t; \alpha, r, \psi)$, using a theorem you deem appropriate.
(b) Find $\partial \hat{s}(t; \alpha, r, \psi)/\partial \alpha$ and provide an economic interpretation.
(c) Find $\partial \hat{s}(t; \alpha, r, \psi)/\partial r$ and provide an economic interpretation.
(d) Find $\partial \hat{s}(t; \alpha, r, \psi)/\partial \psi$ and provide an economic interpretation.
(e) Using an appropriate theorem, solve, in principle, for the optimal value of the conjugate variable, say, $\psi = \psi^*(\alpha, A, r, T)$. What is the economic interpretation of $\psi = \psi^*(\alpha, A, r, T)$?
(f) Find $\partial \psi^*(\alpha, A, r, T)/\partial \alpha$ and provide an economic interpretation.
(g) Find $\partial \psi^*(\alpha, A, r, T)/\partial r$ and provide an economic interpretation.
(h) Now write down an identity linking $\hat{s}(t; \alpha, r, \psi)$ to the optimal spending rate, say, $s^*(t; \alpha, A, r, T)$, using the information at hand.
(i) Prove that $\partial s^*(t; \alpha, A, r, T)/\partial \alpha$ can be broken up into the sum of two Slutsky-like terms. Provide an economic interpretation of the result. Can you sign it? Why or why not?
(j) Show that at this level of generality, $\partial s^*(t; \alpha, A, r, T)/\partial r$ cannot be signed. Show, however, that $\partial s^*(0; \alpha, A, r, T)/\partial r < 0$. Provide an economic interpretation of this result.

8.2 Prove Corollary 8.1.

https://doi.org/10.1017/CBO9780511806827.010 Published online by Cambridge University Press

8.3 Prove that not all of the identities in Corollary 8.1 are independent of one another.

8.4 Prove that

$$C(q;w) \stackrel{\text{def}}{=} \min_{v} \{w \cdot v \text{ s.t. } q = f(v)\}$$

satisfies $C_q(q;w) > 0$, $C_{qq}(q;w) > 0$, $C_w(q;w) > 0$, and $C_{qw}(q;w) > 0$.

8.5 Derive the derivative decompositions for the nonrenewable resource–extracting model of the firm for the parameters (r, w, T) analogous to those in Eqs. (9) through (16).

8.6 Provide an economic interpretation of Eqs. (10) and (14).

8.7 Show that one can also arrive at the formula

$$\int_0^T q^M(t;\varepsilon,s)\, e^{-rt}\, dt \equiv \frac{-\frac{\partial Q}{\partial p}(\varepsilon, \Pi(\varepsilon,s))}{\frac{\partial Q}{\partial \pi}(\varepsilon, \Pi(\varepsilon,s))}$$

via Eq. (15), as noted in the text.

8.8 Prove that $\partial q^M(t;\varepsilon,s)/\partial p < 0$ holds for some finite interval of time near the end of the planning horizon following the steps and logic used to deduce Eq. (39).

8.9 Using Eq. (34), prove that

$$\frac{\partial q^M}{\partial r}(0;\varepsilon,s) = \frac{\partial \hat{q}}{\partial \psi}(0;\psi^M(\varepsilon,s), p, r, w)\frac{\partial \psi^M}{\partial r}(\varepsilon,s) > 0,$$

and provide an economic interpretation of the result comparable to Eq. (39).

FURTHER READING

As remarked in the third paragraph of the chapter, the references for the results contained herewith are Caputo (1998, 1999). The paper by Caputo (1994) examines the intertemporal Slutsky matrix for an isoperimetric consumer problem and demonstrates an assertion made in the next to last paragraph in the chapter, videlicet, that it may be easier to derive the comparative dynamics properties of the primal problem by first deriving the comparative dynamics properties of the reciprocal problem. Further pertinent references for this chapter are those noted in Chapter 7. Of related interest are the papers by Newman (1982), Weber (1998), and Caputo (2000, 2001). These papers deal with reciprocal constrained optimization problems of the static variety and their associated comparative statics properties. Hotelling (1931) is the source for the nonrenewable resource–extracting model of the firm.

REFERENCES

Caputo, M.R. (1994), "The Slutsky Matrix and Homogeneity in Intertemporal Consumer Theory," *Journal of Economics*, 60, 255–279.

Caputo, M.R. (1998), "Economic Characterization of Reciprocal Isoperimetric Control Problems," *Journal of Optimization Theory and Applications*, 98, 325–350.

Caputo, M.R. (1999), "Economic Characterization of Reciprocal Isoperimetric Control Problems Revisited," *Journal of Optimization Theory and Applications*, 101, 723–730.

Caputo, M.R. (2000), "Lagrangian Transposition Identities and Reciprocal Pairs of Constrained Optimization Problems," *Economics Letters*, 66, 265–273.

Caputo, M.R. (2001), "Further Results on Lagrange Multipliers with Several Binding Constraints," *Economics Letters*, 70, 335–340.

Hotelling, H. (1931), "The Economics of Exhaustible Resources," *Journal of Political Economy*, 39, 137–175.

Newman, P. (1982), "Mirrored Pairs of Optimization Problems," *Economica*, 49, 109–119.

Weber, C.E. (1998), "A Note on Lagrange Multipliers with Several Binding Constraints," *Economics Letters*, 59, 71–75.