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Homework problems Classical Field Theory – SoSe 2022 – Set 3

due May 10 in lecture

Problem 8:

We consider a rank-2 Lorentz tensor whose components (in a certain inertial system) are given by

$$(A^{\mu\nu}) = \begin{pmatrix} 2 & 0 & 1 & -1 \\ -1 & 0 & 3 & 2 \\ -1 & 1 & 0 & 0 \\ -2 & 1 & 1 & -2 \end{pmatrix} .$$

(a) Derive the components $A^{\mu}{}_{\nu}$, $A_{\mu}{}^{\nu}$, $A_{\mu\nu}$.

(b) Compute:

(i) $A^{\mu}{}_{\mu}$,

(ii) $A_{\mu\nu}A^{\mu\nu}$,

(iii) $A_{\mu}{}^{\nu}\, x_{\nu}$, where x is the usual position four-vector.

Problem 9:

Let us return to problem **2(a)** and consider the Lagrangian

$$\mathcal{L} = \frac{1}{2}\left[\,(\partial_\mu\phi)\,(\partial^\mu\phi) - \mu^2\phi^2\,\right]$$

for a scalar field $\phi(x)$. Use the Euler-Lagrange equations, *but now in covariant form*, to derive the equation of motion for the field. When computing the derivatives, note that $(\partial_\mu\phi)\,(\partial^\mu\phi) = (\partial_\mu\phi)\,(\partial_\nu\phi)\,g^{\mu\nu}$.

Problem 10:

The *four-vector force* that acts on a particle with mass m is given by

$$K^\mu \equiv \frac{dp^\mu}{d\tau} = m\,\frac{du^\mu}{d\tau} = m\,\frac{d^2x^\mu}{d\tau^2} ,$$

where u is the four-velocity of the particle, $p = mu$ its four-momentum, and τ the proper time (see lecture).

(a) Show that

$$u_\mu K^\mu = 0 .$$

(Recall that $u_\mu u^\mu = c^2$).

(b) Use the explicit form

$$(u^\mu) = \gamma(t) \begin{pmatrix} c \\ \vec{v}(t) \end{pmatrix}, \qquad \gamma(t) = \frac{1}{\sqrt{1-(\vec{v}(t))^2/c^2}},$$

to show that

$$(K^\mu) = \gamma \begin{pmatrix} \frac{\vec{v}\cdot\vec{F}}{c} \\ \vec{F} \end{pmatrix},$$

where $\vec{F} = d\vec{p}/dt$ with $\vec{p} = m\gamma\vec{v}$.

(c) The relativistic energy is $E = m\gamma c^2$. Show that

$$\frac{dE}{dt} = \vec{F}\cdot\vec{v}\,.$$

(d) Use this to show that K may be written as

$$(K^\mu) = \frac{d}{d\tau}\begin{pmatrix} E/c \\ \vec{p} \end{pmatrix}$$

(which demonstrates that E/c is the zero component of the four-momentum).

Problem 11:

As mentioned in the lecture, under a Lorentz boost with an arbitrary boost velocity $\vec{v}$ the fields $\vec{E}$, $\vec{B}$ transform as

$$\begin{aligned} \vec{E}\,' &= \gamma\left(\vec{E} + \vec{v}\times\vec{B}\right) + (1-\gamma)\,\frac{\vec{v}}{v}\,\frac{\vec{v}\cdot\vec{E}}{v}\,, \\ \vec{B}\,' &= \gamma\left(\vec{B} - \frac{1}{c^2}\,\vec{v}\times\vec{E}\right) + (1-\gamma)\,\frac{\vec{v}}{v}\,\frac{\vec{v}\cdot\vec{B}}{v}\,, \end{aligned}$$

where $\gamma = 1/\sqrt{1-\vec{v}^{\,2}/c^2}$. We decompose the fields in the form $\vec{E} = \vec{E}^{\perp} + \vec{E}^{\parallel}$, $\vec{B} = \vec{B}^{\perp} + \vec{B}^{\parallel}$, where $\vec{E}^{\parallel}$ and $\vec{B}^{\parallel}$ are the components parallel to $\vec{v}$, and where $\vec{E}^{\perp}$ and $\vec{B}^{\perp}$ are the corresponding perpendicular components. Show that the parallel components are invariant under the boost and that the perpendicular ones transform as

$$\begin{aligned} \vec{E}\,'^{\perp} &= \gamma\left(\vec{E}^{\perp} + \vec{v}\times\vec{B}^{\perp}\right), \\ \vec{B}\,'^{\perp} &= \gamma\left(\vec{B}^{\perp} - \frac{1}{c^2}\,\vec{v}\times\vec{E}^{\perp}\right). \end{aligned}$$