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**Homework problems Classical Field Theory – SoSe 2022 – Set 11**

**due July 12 during lecture**

**Problem 37:**

Apart from the spherical coordinates one also often just uses simple cartesian coordinates for solving the Laplace equation

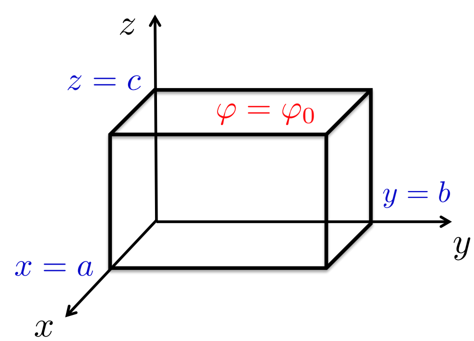
$$\Delta\varphi(\vec{x}\,) \;=\; \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\varphi(\vec{x}\,) \;=\; 0\,.$$

Consider the rectangular box shown in the figure, whose sides have the lengths $a, b, c$. We are looking for the potential in the interior of the box. There are no charges in this problem. The potential vanishes on the surface of the box, except for its lid at $z = c$, where the potential has the value $\varphi_0$.

**(a)** Try to find a solution of Laplace's equation by separation of variables in the form $\varphi(\vec{x}\,) = X(x)\,Y(y)\,Z(z)$. Find the resulting differential equations for $X, Y, Z$ and give their solutions.

**(b)** Implement the boundary conditions at $x = 0, x = a$ and $y = 0, y = b$ and $z = 0$ into the solutions.

**(c)** Finally, present the complete solution $\varphi(\vec{x}\,)$ of the problem, which also correctly satisfies the boundary condition at $z = c$. To do this, take a suitable superposition of the solutions in **(b)**.




**Problem 38:** *Electromagnetic wave packets*

In the lecture we have considered wave packets of the form

$$\psi(\vec{x},t) \;=\; \frac{1}{(2\pi)^{3/2}}\int d^3k\,\left[a_+(\vec{k}\,)\,\mathrm{e}^{-i(\omega t-\vec{k}\cdot\vec{x})}+a_-(\vec{k}\,)\,\mathrm{e}^{i(\omega t+\vec{k}\cdot\vec{x})}\right]\,,$$

with $\omega = c|\vec{k}\,|$. In the following, let us always assume for simplicity that $a_-(\vec{k}\,) = 0$.

**(a)** Let first

$$a_+(\vec{k}\,) \;=\; \delta(k_x)\,\delta(k_y)\,A(k_z)\,\Theta(k_z)\,,$$

with a function $A$ and with the usual Heaviside function $\Theta(k_z)$. Show that the wave packet moves with the speed of light in $+z$ direction, while preserving its shape.

**(b)** We now consider

$$a_+(\vec{k}\,) \;=\; \delta(k_x)\,\frac{\mathrm{e}^{-k_y^2/\sigma^2}}{\sigma\sqrt{\pi}}\,A(k_z)\,\Theta(k_z)\,,$$

with a "width" $\sigma$. Suppose the function $A(k_z)$ has a very sharp peak at a value $k_0 \neq 0$. Expand the exponent $-i(\omega t - \vec{k}\cdot\vec{x})$ to second order around the point $\vec{k} = (0, 0, k_0)$ (note that always $k_x = 0$). Use this to compute $\psi(\vec{x},t)$ as far as possible. How does the packet behave in time? Does it also maintain its shape?

**Problem 39:**

In the lecture we introduced the function

$$D(\vec{y},t) \;\equiv\; -\frac{i}{2(2\pi)^3}\int \frac{d^3k}{\omega}\,\left(\mathrm{e}^{i\omega t}-\mathrm{e}^{-i\omega t}\right)\,\mathrm{e}^{i\vec{k}\cdot\vec{y}}\,,$$

where $\omega = c\,|\vec{k}\,|$. Show that

$$D(\vec{y},t) \;=\; \frac{1}{4\pi c r}\begin{cases} \delta(r-ct) & t>0 \\ 0 & t=0 \\ -\delta(r+ct) & t<0\,, \end{cases}$$

with $r \equiv |\vec{y}\,|$.

*Help:* Use spherical coordinates for the integration.

**Problem 40:**

**(a)** Consider two vector fields of the form

$$\vec{F}(\vec{x},t) \;=\; \vec{F}_0(\vec{x}\,)\,\mathrm{e}^{-i\omega t}\,, \qquad \vec{G}(\vec{x},t) \;=\; \vec{G}_0(\vec{x}\,)\,\mathrm{e}^{-i\omega t}\,.$$

Here $\vec{F}_0, \vec{G}_0$ can be complex. Show that

$$\overline{\mathrm{Re}(\vec{F}\,)\cdot\mathrm{Re}(\vec{G}\,)} \;=\; \frac{1}{2}\mathrm{Re}\left(\vec{F}_0\cdot\vec{G}_0^*\right)\,, \qquad \overline{\mathrm{Re}(\vec{F}\,)\times\mathrm{Re}(\vec{G}\,)} \;=\; \frac{1}{2}\mathrm{Re}\left(\vec{F}_0\times\vec{G}_0^*\right)\,,$$

where we define the time average of a function that is periodic with $T = 2\pi/\omega$ by

$$\overline{f} \;\equiv\; \frac{1}{T}\int_0^T dt\,f(t)\;.$$

**(b)** Using **(a)**, show that for electromagnetic plane waves,

$$\vec{E} \;=\; \mathrm{Re}\left[\vec{E}_0\,\mathrm{e}^{i(\vec{k}\cdot\vec{x}-\omega t)}\right]\,, \qquad \vec{B} \;=\; \mathrm{Re}\left[\vec{B}_0\,\mathrm{e}^{i(\vec{k}\cdot\vec{x}-\omega t)}\right]\,,$$

the time-averaged energy density and Poynting vector are given by

$$\overline{w} \;=\; \frac{\epsilon_0}{2}\,|\vec{E}_0|^2 \;=\; \frac{1}{2\mu_0}\,|\vec{B}_0|^2\,, \qquad \overline{\vec{S}} \;=\; c\,\overline{w}\,\frac{\vec{k}}{|\vec{k}\,|}\,.$$

*Help:* Recall Sec. I.6.3 of the lecture notes.

**The following problems are meant as practice for the exam. They are optional, but you can use them to make up for earlier problems you may have missed. Note that the problems in the exam will typically be a bit simpler than the ones we consider here.**

**Problem 41:**

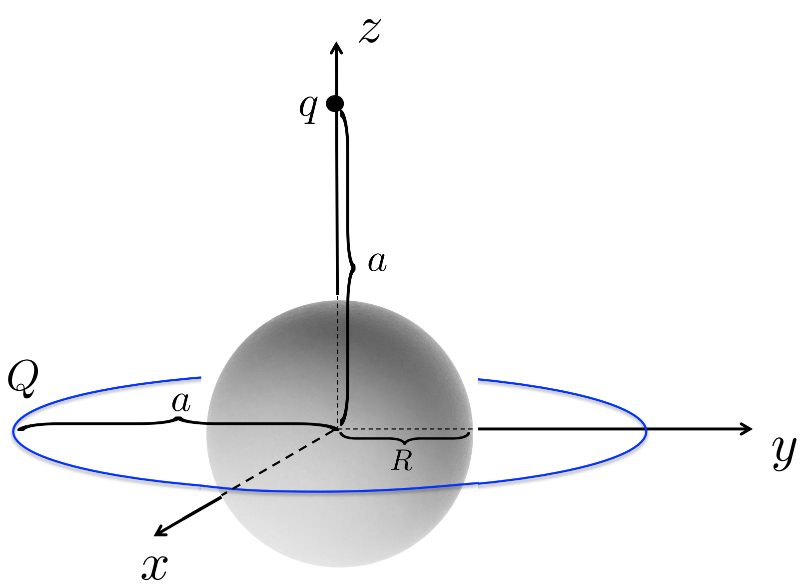
The picture below shows a conducting grounded (potential zero) sphere with radius $R$. Around the sphere, and concentric with it, is a homogeneously charged circular ring in the $(x, y)$ plane with radius $a > R$ and total charge $Q$. The ring is "infinitely" thin, so that it has no lateral extension and may be regarded as a line charge. Furthermore, on the $z$ axis we have a point charge $q$, again at distance $a$ from the origin.

**(a)** Write down the charge density $\varrho(\vec{x})$ in the exterior of the sphere.

**(b)** If one tries to solve this problem with the method of image charges, what charges are needed? What is the total charge the sphere must carry?

*Help:* It may help to picture the ring as a chain of infinitely many little charges.

**(c)** What value does $Q$ need to have so that there is no force on the point charge $q$ in $z$ direction?




**Problem 42:**

Consider the Lagrangian

$$\mathcal{L} = -\frac{1}{4\mu_0} F_{\mu\nu}F^{\mu\nu} + \Big((\partial_\mu - ieA_\mu)\phi^*\Big)\Big((\partial^\mu + ieA^\mu)\phi\Big) - m^2\phi^*\phi$$

for the fields $\phi, \phi^*, A^\mu$. Here $A^\mu$ is the electromagnetic four-potential with the usual field strength tensor $F^{\mu\nu}$. The field $\phi$ is scalar and has complex values. We treat $\phi$ and $\phi^*$ as independent fields. $\mu_0$, $e$ and $m$ are real constants.

**(a)** Derive all equations of motion of the system.

**(b)** Show that the Lagrangian is invariant under the simultaneous transformation

$$A^\mu \to A^\mu - \partial^\mu\chi(x)\,, \qquad \phi \to \mathrm{e}^{ie\chi(x)}\,\phi\,, \qquad \phi^* \to \mathrm{e}^{-ie\chi(x)}\,\phi^*\,,$$

with an arbitrary real function $\chi(x)$.

**Problem 43:**

Starting from the Lagrangian $\mathcal{L}_{\mathrm{em}}$ of electromagnetism, show that the corresponding Hamilton function is given by

$$H = \int d^3x \left[ \frac{1}{2} \left( \epsilon_0 \vec{E}^{\,2} + \frac{\vec{B}^{\,2}}{\mu_0} \right) - \vec{j} \cdot \vec{A} \right] .$$

*Help:* The Hamilton density is $\mathcal{H} = (\partial_0 A_\nu)\, \pi^\nu - \mathcal{L}_{\mathrm{em}}$, with the canonical momentum densities $\pi^\nu = \partial \mathcal{L}_{\mathrm{em}} / \partial(\partial_0 A_\nu)$. As always, all fields are assumed to vanish on the boundary of the integration volume.