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Homework problems Classical Field Theory – SoSe 2022 – Set 6

due May 31 in lecture

Problem 20:

(a) Compute the energy momentum tensor $T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\, \partial^\nu \phi - g^{\mu\nu}\mathcal{L}$ for the Lagrangian

$$\mathcal{L} = \frac{1}{2}\Big[(\partial_\mu \phi)\,(\partial^\mu \phi) - m^2 \phi^2 \Big] ,$$

where ϕ is a scalar field.

(b) Use the result to verify that $\partial_\mu T^{\mu\nu} = 0$.

(c) The Noether charges that we obtain for a scalar field for the invariance under homogeneous Lorentz transformations are given by

$$Q^{\nu\rho} = \frac{1}{c} \int d^3x \, \left(x^\nu\, T^{0\rho} - x^\rho\, T^{0\nu} \right) .$$

Consider the charges Q^{0i} and show that their conservation leads to

$$\frac{d}{dt} \int d^3x \, x^i \, T^{00} = \text{const.}$$

Interpret this result.

Problem 21: *Physical units in electrodynamics*

In the lecture we have chosen SI units. One also often uses the *Gauss units*. The relevant physical quantities for the two systems of units are related as follows:

$$\vec{E}_{\text{SI}} = \frac{\vec{E}_{\text{G}}}{\sqrt{4\pi\epsilon_0}} , \qquad \vec{B}_{\text{SI}} = \sqrt{\frac{\mu_0}{4\pi}}\, \vec{B}_{\text{G}} ,$$

$$\{q_{\text{SI}},\, \varrho_{\text{SI}},\, \vec{j}_{\text{SI}}\} = \{\sqrt{4\pi\epsilon_0}\, q_{\text{G}},\, \sqrt{4\pi\epsilon_0}\, \varrho_{\text{G}},\, \sqrt{4\pi\epsilon_0}\, \vec{j}_{\text{G}}\} ,$$

where q is the charge. In both systems we require that $\vec{B} = \vec{\nabla} \times \vec{A}$, as well as $\vec{E} = -\vec{\nabla}\varphi$ in the electrostatic case. Furthermore, the field strength tensor is always defined as $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$.

(a) Write the Maxwell equations and the Lorentz force in Gauss units.

(b) Derive the relation between the potentials A^μ_{SI} and A^μ_{G}. Show how in Gauss units the electric field strength is obtained from the potential in the general case.

(c) Write the Lagrangian in Gauss units and also the equations of motion (in covariant form). How does the field strength tensor look when expressed in terms of the components of $\vec{E}_{\text{G}}$ and $\vec{B}_{\text{G}}$? Finally, also write the energy density w of the field and the Poynting vector $\vec{S}$ in Gauss units.

Problem 22: *Integration in more than one dimension, and integral theorems*

(a) Study first Appendix B in the lecture notes.

(b) The surface of a torus is defined by the points

$$\vec{x}(u,v) = \begin{pmatrix} (R + r\cos u)\cos v \\ (R + r\cos u)\sin v \\ r\sin u \end{pmatrix}, \qquad u \in [0, 2\pi], \quad v \in [0, 2\pi], \quad r, R \text{ fixed}, \quad r < R.$$

(i) Compute the volume of the torus by integration.

(ii) Compute the surface of the torus using the surface integral of the first kind.

(iii) Now consider the field $\vec{F}(\vec{x}) = \vec{x}$. For the example of the field $\vec{F}$, verify the two integral theorems by Gauss, (I.299) and (I.300) in the lecture notes, when the volume is that of the full torus.

(c) Consider now the vector field

$$\vec{F}(\vec{x}) = \log(x^2 + y^2) \begin{pmatrix} -y \\ x \\ 0 \end{pmatrix}.$$

For this field, verify the two integral theorems by Stokes, (I.302) and (I.303) in the lecture notes, when the surface is a circular disk with radius R in the (x, y) plane, whose center is in the origin.