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Homework problems Classical Field Theory – SoSe 2022 – Set 7

due June 14 in lecture

Problem 23:

In the lecture we have seen that the electric field may be written as

$$\vec{E}(\vec{x},t) \;=\; -\vec{\nabla}\varphi(\vec{x},t) - \frac{\partial}{\partial t}\vec{A}(\vec{x},t)\,,$$

with the scalar potential φ and the vector potential $\vec{A}$. We also learned about the Helmholtz decomposition, according to which

$$\vec{E}(\vec{x},t) \;=\; -\vec{\nabla}\phi(\vec{x},t) + \vec{\nabla}\times\vec{\mathcal{A}}(\vec{x},t)\,.$$

(In order to avoid confusion, we have introduced the symbols ϕ and $\vec{\mathcal{A}}$ for the fields appearing in this decomposition.) We consider the fields in $\mathbb{R}^3$; all fields are supposed to vanish sufficiently fast at infinity.

Choose the Coulomb gauge for φ and $\vec{A}$ and show that the two decompositions of $\vec{E}$ are identical.

Help: Recall first (see Sec. I.5.5) how φ is obtained in the Coulomb gauge. Determine $\vec{\mathcal{A}}$ and show that $\vec{\nabla}\times\vec{\mathcal{A}}$ corresponds to $-\partial\vec{A}/\partial t$ where $\vec{\nabla}\times\vec{A}=\vec{B}$.

For the following two problems we consider the ordinary differential equation

$$f''(x) + \frac{1}{4}f(x) \;=\; g(x)\,, \qquad f(0)=f(\pi)=0\,. \tag{1}$$

Here $g(x)$ is a given function; we are looking for the solution $f(x)$. We will consider the two cases

$$\begin{aligned} \textbf{(i)} \quad g(x) &= \frac{x}{2}\,, \\ \textbf{(ii)} \quad g(x) &= \sin(2x)\,. \end{aligned}$$

Convince yourself that in the two cases we have the solutions

$$\begin{aligned} \textbf{(i)} \quad f(x) &= 2x - 2\pi\sin\left(\frac{x}{2}\right)\,, \\ \textbf{(ii)} \quad f(x) &= -\frac{4}{15}\sin(2x)\,, \end{aligned}$$

of Eq. (1). (It may be a useful exercise to derive these solutions using the methods you learned about in your math lectures.)

We will now use the new concept of a *Green's function* to solve (1). We will consider three different approaches that are all of practical relevance. Our Green's function G solves the differential equation

$$\frac{d^2G(x,x')}{dx^2} + \frac{1}{4}G(x,x') \;=\; \delta(x-x')\,, \tag{2}$$

where $x' \in [0, \pi]$. Using G, we obtain a solution of the original differential equation as

$$f(x) = \int_0^\pi dx' \, G(x, x') \, g(x') \, .$$

In order to satisfy the boundary conditions, we require $G(0, x') = G(\pi, x') = 0$.

Problem 24: *Direct computation of the Green's function*

(a) Make an ansatz for the Green's function in Eq. (2), separately for $x < x'$ and $x > x'$. Employ the boundary conditions at $x = 0$ and $x = \pi$ to simplify the solution.

(b) We require the Green's function to be continuous at $x = x'$. Furthermore, there is a condition that the first derivative of G needs to satisfy. To obtain this condition, integrate Eq. (2) over a small interval $[x' - \epsilon, x' + \epsilon]$ and take the limit $\epsilon \to 0$. Use the two conditions to completely fix the Green's function. Sketch the function $G(x, x')$ for $x' = 3\pi/4$.

(c) Using G, derive now the solution of the original differential equation (1), for the two cases $g(x) = x/2$ and $g(x) = \sin(2x)$.

Problem 25: *Derivation of G via the Fourier transform*

(a) Ignore the boundary conditions for the moment and make an ansatz for the Green's function of the form $G_0(x - x')$, that is, as a function of just $y \equiv x - x'$. Next, write G_0 as a Fourier transform,

$$G_0(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk \, \tilde{G}_0(k) \, \mathrm{e}^{iky} \, , \tag{3}$$

and find $\tilde{G}_0(k)$.

(b) When you insert the result into Eq. (3) to compute $G_0(y)$ you will find that that the integrand has poles at $k = \pm\frac{1}{2}$ and that the integral is therefore not well-defined. Move the poles to $k = \frac{1}{2} + i\epsilon$ and $k = -\frac{1}{2} - i\epsilon$ and use Cauchy's theorem to compute the integral in the limit $\epsilon \to 0$. (Watch out: you will need to distinguish the cases $y < 0$, $y > 0$.)

(c) The result for G_0 does not yet satisfy the boundary conditions. Find a function $F(x, x')$ that solves the homogeneous differential equation, so that

$$G(x, x') = G_0(x - x') + F(x, x')$$

satisfies the correct boundary conditions $G(0, x') = G(\pi, x') = 0$.

Problem 26: *(This problem is optional. Some parts are a little harder ...)*
One may define the action for a free relativistic particle (rest mass m) as

$$S = \int_{t_1}^{t_2} dt\, L = -mc^2 \int_{t_1}^{t_2} dt \sqrt{1 - \dot{\vec{X}}^{\,2}/c^2}\,,$$

with the coordinates $ct, \vec{X}(t)$ of the particle in a given inertial system.

(a) Convince yourself first that in the non-relativistic limit L becomes equivalent to the familiar non-relativistic Lagrange function for a free particle.

(b) Show next that from L we find the correct equation of motion for the trajectory $\vec{X}(t)$ of a relativistic free particle.

(c) In order to write the formalism in a covariant way, we eliminate the time t in favor of the particle's proper time τ. Show that the action becomes

$$S = -mc \int_{\tau_0}^{\tau_1} d\tau \sqrt{u_\mu u^\mu} \equiv \int_{\tau_0}^{\tau_1} d\tau\, \tilde{L}\,,$$

where $u^\mu = dX^\mu/d\tau$. If we regard all four components of $(X^\mu) = (ct(\tau), \vec{X}(\tau))$ as independent variables, the Euler-Lagrange equations become

$$\frac{d}{d\tau}\frac{\partial \tilde{L}}{\partial u_\mu} - \frac{\partial \tilde{L}}{\partial X_\mu} = 0\,.$$

Apply them and find the resulting equation of motion of the particle. (Set $u_\mu u^\mu = c^2$ only *after* you have used the Euler-Lagrange equations!)

(d) The particle has the charge q. Its four-vector current at $x = (ct, \vec{x}\,)$ is

$$j^\mu(x) = q\,\delta^3(\vec{x} - \vec{X}(t))\,\frac{dX^\mu}{dt}\,.$$

Verify that j satisfies the continuity equation.

(e) We now consider the particle in an electromagnetic field $A^\mu(x)$. Show that the action of the full system is given by

$$S = \int_{\tau_0}^{\tau_1} d\tau \left(-mc\sqrt{u_\mu u^\mu} - q\,u_\mu A^\mu\right) + \frac{1}{c}\int d^4x \left(-\frac{1}{4\mu_0}\,F_{\mu\nu}\,F^{\mu\nu}\right).$$

(f) Use this action and the Euler-Lagrange equation of **(c)** to show that the equation of motion for the particle becomes

$$m\,\frac{du^\mu}{d\tau} = q\,F^\mu{}_\nu\,u^\nu\,.$$

(g) We finally define the energy-momentum tensor of the particle:

$$T^{\mu\nu}_{\text{part}}(x) \equiv m\,\delta^3(\vec{x} - \vec{X}(t))\,\frac{dX^\mu}{dt}\,\frac{dX^\nu}{d\tau}\,.$$

Convince yourself that T^{00}_{part} and $\frac{1}{c}T^{0i}_{\text{part}}$ may be interpreted as energy density and momentum density, respectively.

(h) In the lecture we encountered the energy-momentum tensor of the field, $T^{\mu\nu}_{\text{field}}$. Show that

$$\partial_\mu\left(T^{\mu\nu}_{\text{part}} + T^{\mu\nu}_{\text{field}}\right) = 0\,.$$