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Homework problems Classical Field Theory – SoSe 2022 – Set 10

due July 05 in lecture

Problem 33: *(Part (c) is voluntary)*

(a) We show here the *mean value theorem* of electrostatics. Consider a spatial region that contains no charges. Show that at every point of this region the potential is given by the average of the potential over the surface of a sphere with arbitrary radius R around the point:

$$\varphi(\vec{x}\,) \;=\; \frac{1}{4\pi R^2} \oint\limits_{|\vec{x}-\vec{x}\,'|=R} da'\,\varphi(\vec{x}\,')\,.$$

Hint: Make suitable use of Green's second theorem with φ und $\psi = 1/|\vec{x}-\vec{x}\,'|$.

(b) Consider now the potential φ on the points of a discrete three-dimensional lattice. Let a be the distance between two neighboring lattice points. By performing a Taylor expansion for small a, show that if there are no charges we have

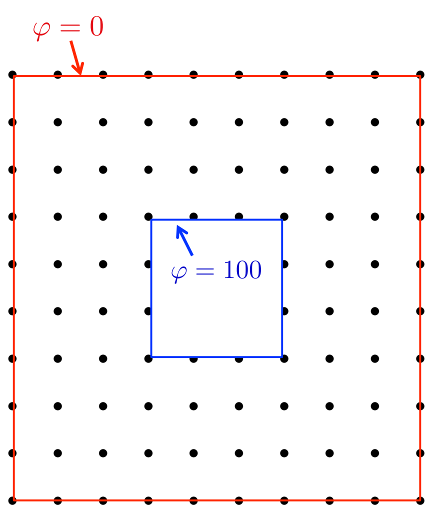
$$\varphi(x,y,z) \;=\; \frac{1}{6}\Big[\varphi(x+a,y,z)+\varphi(x-a,y,z)+\varphi(x,y+a,z)+\varphi(x,y-a,z)+\varphi(x,y,z+a)+\varphi(x,y,z-a)\Big],$$

up to corrections of order a^3. Compare to **(a)**.

(c) The result in **(b)** may be used to obtain numerical solutions of boundary value problems. In a two-dimensional case the expression for φ in **(b)** translates to

$$\varphi(x,y) \;=\; \frac{1}{4}\left[\varphi(x+a,y)+\varphi(x-a,y)+\varphi(x,y+a)+\varphi(x,y-a)\right].$$

One iterates this equation by successively replacing the potential at every point by the average value of the potentials at the neighboring points, until the result converges. (This method is known as the *relaxation method*.) Apply this procedure to the two-dimensional arrangement shown in the figure. Choose the boundary conditions such that the potential vanishes on the outer boundary of the square and has the value 100 V on the inner one. Adopt suitable starting values for the iteration and use the symmetry of the arrangement.



Problem 34:

In the lecture we defined the spherical multipole moments as

$$q_{\ell m} = \int d^3x'\, \varrho(\vec{x}\,')\, r'^{\ell}\, Y^*_{\ell m}(\theta', \phi')\,.$$

Write the quadrupole moments ($\ell = 2$) as linear combinations of the cartesian quadrupole moments ($x_1, x_2, x_3 = x, y, z$)

$$Q_{ij} = \int d^3x'\, \varrho(\vec{x}\,') \left(3x'_i x'_j - r'^2\, \delta_{ij}\right)\,.$$

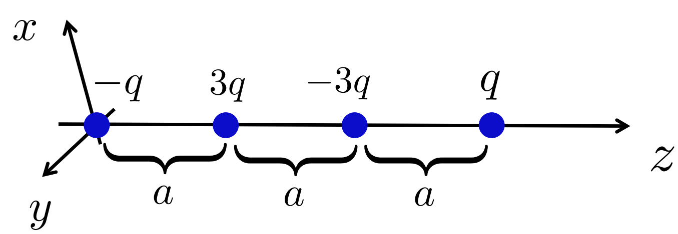
Use this to show that the quadrupole term in the potential $\varphi(\vec{x}\,)$ has the form

$$\varphi_{\text{Quadr.}}(\vec{x}\,) = \frac{1}{8\pi\epsilon_0} \sum_{i,j=1}^{3} Q_{ij}\, \frac{x_i x_j}{r^5}\,.$$

To do this, start from the expression for $\varphi(\vec{x}\,)$ in the spherical multipole expansion (see lecture). We already derived the monopole term and the dipole term in the lecture.

Problem 35:

Consider the charge distribution shown in the figure. Determine the charge density $\varrho(\vec{x}\,)$ and examine the potential $\varphi(\vec{x}\,)$ for $|\vec{x}\,| \gg a$ via a Taylor expansion. What is the leading term? Compute its contribution to the potential explicitly.



Problem 36:

The multipole moments depend in general on the choice of the coordinate origin. In this example we will see that the *leading* moment is however always independent of that choice.

(a) Consider first the simplest case: Show that the dipole moment of a charge distribution is independent of the choice of origin if the total charge Q of the distribution vanishes. As a “corollary”, show that for $Q \neq 0$ one can choose the origin in such a way that the dipole moment vanishes.

(b) An arbitrary nth moment is constructed from terms with (in total) n factors of the components of the position vector,

$$M^{(n)}_{\alpha\beta\gamma} = \int d^3x\, \varrho(\vec{x}\,)\, x^{\alpha}\, y^{\beta}\, z^{\gamma}\,,$$

where $\alpha, \beta, \gamma \in \mathbb{N}_0$ and $\alpha + \beta + \gamma = n$. Show that $M^{(n)}_{\alpha\beta\gamma}$ is precisely independent of the choice of the coordinate origin if all lower moments (that is, moments with a lower value of n) vanish.