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Homework problems Classical Field Theory – SoSe 2022 – Set 8

due June 21 in lecture

Problem 27: *Expansion in terms of eigenfunctions – recall your quantum mechanics lecture!*

Let us return to the differential equations we discussed for problems **24** and **25**. Here we will investigate yet another useful method for constructing Green's function.

(a) The differential operator $\left(\frac{d^2}{dx^2}+\frac{1}{4}\right)$ in our differential equation may be used to define an eigenvalue problem of the form

$$\left(\frac{d^2}{dx^2}+\frac{1}{4}\right) f_n(x) \;=\; \lambda_n\, f_n(x)\,, \qquad f_n(0)=f_n(\pi)=0\,.$$

The eigenfunctions f_n form a basis of the functions on $[0,\pi]$ that vanish at the boundary. The f_n are orthonormal, that is, $\int_0^\pi dx\, f_n(x)\, f_m(x) \;=\; \delta_{mn}$. Determine the f_n and the corresponding eigenvalues λ_n.

(b) Show that the Green function of the problem may be written as

$$G(x,x') \;=\; \sum_n \frac{1}{\lambda_n}\, f_n(x)\, f_n(x')\,.$$

(c) Use this expression to derive the solution to our original differential equation, again for $g(x)=x/2$ and $g(x)=\sin(2x)$.

Problem 28:

A ring of radius R made of a very thin wire carries the charge Q. The ring is lying in the (x,y)-plane; the coordinate origin is located in the center of the ring.

(a) Find an expression for the charge density $\varrho(\vec{x}\,)$. We neglect the thickness of the wire.

(b) Derive the potential φ at an arbitrary position on the symmetry axis of the ring. The potential is supposed to vanish at infinity. Also compute the field strength $\vec{E}$ on the symmetry axis. Where is it maximal?

Problem 29: *Second uniqueness theorem*

The figure below shows a volume V in which we have an arbitrary number of conductors. Let us assume we know the total charge Q_i on each of the conductors. The volume could be enclosed by yet another conductor, or it could also be infinite. Show that the electric field in V is unique.

Help: Assume that there are two solutions $\vec{E}_1, \vec{E}_2$ of the problem and consider $\vec{E}_3 \equiv \vec{E}_1 - \vec{E}_2$. Use now Green's first theorem by inserting for both functions the potential φ_3 corresponding to $\vec{E}_3$.