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Homework problems Classical Field Theory – SoSe 2022 – Set 2

due May 03 in lecture

Problem 4:

Write the following expressions in such a way that they do not contain the metric tensor any longer:

(a)

$$x^\mu \, y^\nu \, g_{\mu\lambda} \, g^\lambda{}_\nu \, ,$$

(b)

$$g^{\rho\sigma} \, g_\rho{}^\alpha \, x^\mu \, g_{\mu\nu} \, y_\beta \, g_{\alpha\sigma} \, .$$

Problem 5:

The Lorentz transformation for a general boost (boost velocity $\vec{V} = c\vec{\beta}$ in an arbitrary direction) is given by

$$\begin{aligned} ct' &= \gamma \left(ct - \vec{x} \cdot \vec{\beta} \right) , \\ \vec{x}\,' &= \vec{x} + (\gamma - 1) \, \frac{(\vec{x} \cdot \vec{\beta}\,) \, \vec{\beta}}{\beta^2} - \gamma \vec{\beta} \, ct \, . \end{aligned}$$

Here, $\beta^2 = \vec{\beta}^{\,2}$, and as usual $\gamma = 1/\sqrt{1-\beta^2}$.

(a) Write the transformation in the form

$$x' = \Lambda \, x \, ,$$

where x', x are the corresponding four-vectors in the two frames, and present the 4×4 matrix Λ explicitly.

(b) Convince yourself that the transformation leaves the scalar product of two four-vectors invariant.

(c) Show that in the special case $\vec{\beta} = (0, \beta^y, \beta^z)$, the transformation leaves the four-dimensional integration measure $d^4x = dx^0 dx^1 dx^2 dx^3$ invariant.

Problem 6:

(a) $x = (x^0, x^1, x^2, x^3)$ is the four-vector of a space-time point. Let a be another four-vector. As usual, $x \cdot a \equiv x_\mu a^\mu$. Which of the quantities

(i) $\phi(x) = \mathrm{e}^{-(x\cdot x)/(a\cdot a)}$,

(ii) $\phi(x) = \frac{a\cdot a}{x\cdot x + a\cdot a - x^0 a^0}$,

(iii) $\phi(x) = \frac{a\cdot x}{(x+a)\cdot(x+a)}$,

is a Lorentz scalar?

(b) For those quantities in **(a)** that are scalars, compute the gradient $\partial^\mu \phi(x)$. How does the result transform under Lorentz transformations?

Problem 7: *(a little more difficult)*

In an inertial system IS we observe the four-velocity of an object as

$$(u^\mu) = \frac{1}{\sqrt{1 - v(t)^2/c^2}} \begin{pmatrix} c \\ v(t) \\ 0 \\ 0 \end{pmatrix} .$$

Here, $v(t) = dx^1/dt$, with time and spatial components as measured in IS. We now transform to another inertial system IS$'$ that moves with speed V in positive x^1-direction relative to IS. Verify explicitly that u transforms as a four-vector under this transformation.

Help: Argue that the four-velocity in IS$'$ must be of the form

$$(u'^\mu) = \frac{1}{\sqrt{1 - (v'(t'))^2/c^2}} \begin{pmatrix} c \\ v'(t') \\ 0 \\ 0 \end{pmatrix} ,$$

and use the Lorentz transformation between IS and IS$'$ to obtain the relation between v and v'.