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Homework problems Classical Field Theory – SoSe 2022 – Set 5

due May 24 in lecture

Problem 16:

Let us suppose that there is an additional term in the Lagrangian for electrodynamics so that

$$\mathcal{L}_{\rm em} \;=\; -\frac{1}{4\mu_0}\,F_{\mu\nu}\,F^{\mu\nu}\;-\;j^\mu A_\mu\;+\;\frac{1}{4\mu_0}\,\vartheta(x)\,F_{\mu\nu}\tilde{F}^{\mu\nu}\,,$$

where $\vartheta(x)$ is a scalar function.

(a) Compute the influence of such a term on the equations of motion for the four-vector potential A and for the fields $\vec{E}$ und $\vec{B}$. What do you find when ϑ is just a constant number?

(b) Show that $F_{\mu\nu}\tilde{F}^{\mu\nu}$ may be written as a four-divergence:

$$F_{\mu\nu}\tilde{F}^{\mu\nu}\;=\;\partial_\mu K^\mu\;.$$

Problem 17:

Consider a point charge q sitting at rest in the origin as seen from some inertial system IS. As we know from the introductory Physics 2 course, its electric field is given by

$$\vec{E}\;=\;\frac{q}{4\pi\epsilon_0}\,\frac{\vec{e}_r}{r^2}\;=\;\frac{q}{4\pi\epsilon_0}\,\frac{1}{\left(x^2+y^2+z^2\right)^{3/2}}\begin{pmatrix} x\\ y\\ z\end{pmatrix}\,,$$

where for the present problem we label the coordinates as x, y, z. Let us now consider a second inertial system IS$'$ that is moving with velocity V in x-direction (this is the system we have always been using in the lecture). Compute explicitly the fields $\vec{E}\,'(\vec{x}\,',t')$ and $\vec{B}\,'(\vec{x}\,',t')$ as seen from IS$'$.

Problem 18:

Suppose that the four-potential has the form

$$\Big(A^\mu(x)\Big)\;=\;\begin{pmatrix}\frac{1}{c}\varphi(x^0)\\ A^1(x^1)\\ A^2(x^2)\\ A^3(x^3)\end{pmatrix}\,.$$

(a) Compute the fields $\vec{E}$ and $\vec{B}$ for this potential.

(b) Find a gauge transformation such that the new four-potential vanishes.

Problem 19: *(optional – maybe you'll find this interesting)*

Suppose that in nature also a magnetic charge exists, with a corresponding magnetic charge density ϱ_{m}. Moving magnetic charges then produce a magnetic current density $\vec{j}_{\mathrm{m}}$. We generalize Maxwell's equations to

$$\vec{\nabla}\cdot\vec{E} = \frac{\varrho_{\mathrm{e}}}{\epsilon_0}\,, \qquad \vec{\nabla}\cdot\vec{B} = \frac{\varrho_{\mathrm{m}}}{c\epsilon_0}\,,$$

$$\vec{\nabla}\times\vec{B} - \frac{1}{c^2}\frac{\partial\vec{E}}{\partial t} = \mu_0\,\vec{j}_{\mathrm{e}}\,, \qquad \vec{\nabla}\times\vec{E} + \frac{\partial\vec{B}}{\partial t} = -c\mu_0\,\vec{j}_{\mathrm{m}}\,.$$

(For a better distinction, we have given the electric charge and current densities the label "e".)

(a) Use the generalized Maxwell equations to show that the corresponding new four-current $j_{\mathrm{m}} = (c\varrho_{\mathrm{m}}, \vec{j}_{\mathrm{m}})$ must satisfy a continuity equation.

(b) We now define for a pair (X, Y) of two quantities the *duality transformation*

$$\begin{pmatrix} X \\ Y \end{pmatrix} \longrightarrow \begin{pmatrix} X' \\ Y' \end{pmatrix} \equiv \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix},$$

where α is a fixed angle. Show that under a simultaneous transformation of the pairs $(\vec{E}, c\vec{B})$, $(\varrho_{\mathrm{e}}, \varrho_{\mathrm{m}})$, $(\vec{j}_{\mathrm{e}}, \vec{j}_{\mathrm{m}})$ the transformed quantities again satisfy the exact same Maxwell equations.

(c) We now further assume that for *all* objects in nature the ratio of magnetic and electric charge has the same value R. Show that in this case one can bring the Maxwell equations into the familiar form, in which only one type of charge appears.

(d) In addition to the usual four-potential (now denoted as $A_{\mathrm{e}} \equiv (\frac{1}{c}\varphi_{\mathrm{e}}, \vec{A}_{\mathrm{e}})$) we introduce a new potential $A_{\mathrm{m}} = (\frac{1}{c}\varphi_{\mathrm{m}}, \vec{A}_{\mathrm{m}})$ and define

$$\vec{E} = -\left(\vec{\nabla}\varphi_{\mathrm{e}} + \frac{\partial\vec{A}_{\mathrm{e}}}{\partial t}\right) - c\,\vec{\nabla}\times\vec{A}_{\mathrm{m}}\,,$$

$$\vec{B} = \vec{\nabla}\times\vec{A}_{\mathrm{e}} - \frac{1}{c}\left(\vec{\nabla}\varphi_{\mathrm{m}} + \frac{\partial\vec{A}_{\mathrm{m}}}{\partial t}\right).$$

Show that with these relations the generalized Maxwell equations decouple into separate equations for A_{e} and A_{m}.

(e) We finally define the generalized field strength tensor

$$G^{\mu\nu} \equiv \partial^\mu A_{\mathrm{e}}^\nu - \partial^\nu A_{\mathrm{e}}^\mu - \frac{1}{2}\,\varepsilon^{\mu\nu}{}_{\rho\sigma}\left(\partial^\rho A_{\mathrm{m}}^\sigma - \partial^\sigma A_{\mathrm{m}}^\rho\right),$$

and the Lagrangian

$$\mathcal{L} \equiv -\frac{1}{4\mu_0}\,G_{\mu\nu}G^{\mu\nu} \;-\; j_{\mathrm{e}}^\mu A_{\mathrm{e},\mu} \;+\; j_{\mathrm{m}}^\mu A_{\mathrm{m},\mu}\,.$$

Show that the Euler-Lagrange equations for A_{e}^μ and A_{m}^μ lead to the equations of motion in **(d)**.

Help: You will need a result from problem **15(c)**.