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Homework problems Classical Field Theory – SoSe 2022 – Set 13

due July 26 during lecture

Problem 51:

We consider a charge density of the form $\varrho(\vec{x},t) = \varrho_\omega(\vec{x})\,\mathrm{e}^{-i\omega t}$.

(a) Using the results from the lecture, show that the "radiation" vector potential for electric quadrupole radiation is given by

$$\vec{A}_{\mathrm{E2}}(\vec{x},t) \;=\; -\frac{\mu_0}{8\pi c}\,\omega^2\,\frac{\mathrm{e}^{i(kr-\omega t)}}{r}\int d^3x'\,\vec{x}\,'\,(\vec{n}\cdot\vec{x}\,')\,\varrho_\omega(\vec{x}\,')\,,$$

where $\vec{n} = \vec{e}_r$, $r = |\vec{x}\,|$ and $k = \omega/c$.

(b) Show that $\vec{A}_{\mathrm{E2}}$ may be written as

$$\vec{A}_{\mathrm{E2}}(\vec{x},t) \;=\; -\frac{\mu_0}{24\pi}\,ck^2\,\frac{\mathrm{e}^{i(kr-\omega t)}}{r}\left(\vec{Q}(\vec{n}\,) + \vec{n}\int d^3x'\,r'^{\,2}\,\varrho_\omega(\vec{x}\,')\right),$$

where $r' = |\vec{x}\,'|$ and

$$\vec{Q}(\vec{n}\,) \;=\; \big(Q_1(\vec{n}\,), Q_2(\vec{n}\,), Q_3(\vec{n}\,)\big)\,, \qquad Q_i(\vec{n}\,) \;\equiv\; \sum_{j=1}^{3} Q_{ij}\,n_j\,,$$

with the elements of the quadrupole tensor

$$Q_{ij} \;\equiv\; \int d^3x'\,\left(3x'_i\,x'_j - r'^{\,2}\,\delta_{ij}\right)\varrho_\omega(\vec{x}\,')\,.$$

(c) Use this expression to compute the radiation fields $\vec{E}_{\mathrm{E2}}$ and $\vec{B}_{\mathrm{E2}}$. (Discard all terms that are suppressed by powers of $1/r$ relative to the leading term.)

(d) Show that the radiant power, averaged over time, is given by

$$\frac{dP_{\mathrm{E2}}}{d\Omega} \;=\; \frac{1}{4\pi\epsilon_0}\,\frac{ck^6}{288\pi}\,|\vec{n}\times\vec{Q}(\vec{n}\,)|^2\,.$$

(e) What angular dependence $dP_{\mathrm{E2}}/d\Omega$ does one obtain for an oscillating quadrupole with

$$\varrho_\omega(\vec{x}\,) \;=\; q\,\delta(x)\,\delta(y)\,\big(\delta(z) - 2\delta(z-a) + \delta(z-2a)\big)\ ?$$

Problem 52: *(Für die Ferien ;))*

(a) Derive the components of the metric tensor $g_{\mu\nu}$ in Minkowski space in spherical coordinates and in cylindrical coordinates.

(b) Use the results to compute the corresponding Christoffel symbols.

(c) In the lecture we derived the metric tensor for a coordinate system that rotates uniformly around the z axis:

$$(g_{\mu\nu}) = \begin{pmatrix} 1 - \frac{\omega^2}{c^2}\left((x^1)^2 + (x^2)^2\right) & \frac{\omega}{c}\,x^2 & -\frac{\omega}{c}\,x^1 & 0 \\ \frac{\omega}{c}\,x^2 & -1 & 0 & 0 \\ -\frac{\omega}{c}\,x^1 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Compute the Christoffel symbols also for this case. Show that in this reference frame the equations of motion for a particle that is not subject to an external force will contain the known centrifugal and Coriolis terms.

(d) In case of **(c)**, compute the elements of the so-called *curvature tensor* which is defined as

$$R^{\lambda}{}_{\mu\nu\kappa} = \frac{\partial\Gamma^{\lambda}_{\mu\nu}}{\partial x^{\kappa}} - \frac{\partial\Gamma^{\lambda}_{\mu\kappa}}{\partial x^{\nu}} + \Gamma^{\eta}_{\mu\nu}\,\Gamma^{\lambda}_{\kappa\eta} - \Gamma^{\eta}_{\mu\kappa}\,\Gamma^{\lambda}_{\nu\eta}\,.$$