Aufgabe 8:

a) $[J_i, J_j] = i\hbar\, \varepsilon_{ijk} J_k$

$[J_i J_i, J_k] = i\hbar\, \varepsilon_{ikj} J_i J_j + i\hbar\, \varepsilon_{ikj} J_j J_i$

$= i\hbar\, \varepsilon_{ikj} J_i J_j + i\hbar\, \varepsilon_{jki} J_i J_j$

$= i\hbar\, \varepsilon_{ikj} (J_i J_j - J_i J_j) = 0$

b) $[J_z, J_+] = [J_z, J_x + i J_y] = i\hbar J_y - \hbar(-J_x)$

$= \hbar (J_x + i J_y) = \hbar J_+$

$[J_z, J_-] = [J_z, J_x - i J_y] = i\hbar J_y - \hbar J_x = -\hbar J_-$

$[J_+, J_-] = [J_x + i J_y, J_x - i J_y] = -i [J_x, J_y] + i [J_y, J_x]$

$= \hbar J_z + \hbar J_z = 2\hbar J_z$

c) $J_z J_+ |j, m_j\rangle = ([J_z, J_+] + J_+ J_z) |j, m_j\rangle$

$= \hbar J_+ |j, m_j\rangle + J_+ \hbar m_j |j, m_j\rangle$

$= \hbar (m_j + 1) J_+ |j, m_j\rangle$ □

$J_z J_- |j, m_j\rangle = ([J_z, J_-] + J_- J_z) |j, m_j\rangle = (-\hbar J_- + J_- \hbar m_j) |j, m_j\rangle$

$= \hbar (m_j - 1) J_- |j, m_j\rangle$

d) $J_+J_- = J_x^2 + J_y^2 + iJ_yJ_x - iJ_xJ_y$

$J_-J_+ = J_x^2 + J_y^2 - iJ_yJ_x + iJ_xJ_y$

$\Rightarrow J^2 = \frac{1}{2}(J_+J_- + J_-J_+) + J_z^2$

Aufgabe 9

a) potentielle Energie $V = -\frac{e^2}{4\pi\varepsilon_0 r}$

$$\langle V\rangle = \int d^3r\, \psi^*_{100} V \psi_{100} = \int_0^\infty dr\, r^2 \int_0^\pi d\vartheta \sin\vartheta \int_0^{2\pi} d\varphi \left(-\frac{e^2}{4\pi\varepsilon_0 r}\right)\frac{4}{a_0^3} e^{-2r/a_0}\frac{1}{4\pi}$$

$$= -\frac{e^2}{\pi\varepsilon_0 a_0^3}\underbrace{\int_0^\infty dr\, r\, e^{-2r/a_0}}$$

$$= \left[-\frac{a_0}{2} r\, e^{-2r/a_0}\right]_0^\infty + \int_0^\infty \frac{a_0}{2} e^{-2r/a_0} = \frac{a_0^2}{4}$$

$\Rightarrow \langle V\rangle = -\frac{e^2}{4\pi\varepsilon_0 a_0}$ mal Z wenn man will

b) $R_{20}(r) = \left(\frac{Z}{2a_0}\right)^{3/2}\left(1-\frac{Zr}{2a_0}\right)e^{-Zr/2a_0}$

daraus liest man die einzige Nullstelle $r_0 = \frac{2a_0}{Z}$ ab

$$R_{30}(r) = 2\left(\frac{Z}{3a_0}\right)^{3/2}\left(1 - \frac{2Zr}{3a_0} + \frac{2(Zr)^2}{27a_0^2}\right)e^{-Zr/3a_0}$$

$R_{30}(r) = 0$ wenn

$$1 - \frac{2}{3}Z\frac{r}{a_0} + \frac{2Z^2}{27}\left(\frac{r}{a_0}\right)^2 = 0 \Rightarrow \left(\frac{r}{a_0}\right)_{1/2} = \frac{9}{2Z} \pm \sqrt{\left(\frac{9}{2Z}\right)^2 - \frac{27}{2}\frac{1}{Z^2}}$$

$$= \frac{9}{2}\frac{1}{Z}\left(1 \pm \sqrt{1-\frac{2}{3}}\right)$$

c) Der Zustand im thermischen Gleichgewicht wird dann zu

$$|\psi\rangle = \frac{1}{\sqrt{4}}\left(\psi_{200} + \psi_{21-1} + \psi_{210} + \psi_{211}\right)$$

$$= \frac{1}{2}\sqrt{\frac{Z^3}{24 a_0^3}}\frac{1}{\sqrt{4\pi}}\left(\sqrt{3}\left(2-\frac{Zr}{a_0}\right) + \sqrt{3}\,\frac{Zr}{a_0}\cos\vartheta + \frac{1}{2}\sqrt{3}\sin\vartheta\, e^{-i\varphi}\right.$$

$$\left. - \frac{Zr}{a_0}\frac{1}{2}\sqrt{3}\sin\vartheta\, e^{i\varphi}\right) e^{-\frac{Zr}{2a_0}}$$

$$|\psi|^2 = \frac{1}{4}\,\frac{Z^3}{8a_0^3}\,\frac{1}{4\pi}\left[\left(2-\frac{Zr}{a_0}(1-\cos\vartheta)\right)^2 + \cancel{1}\sin^2\vartheta\,\frac{Z^2r^2}{a_0^2}\sin^2\varphi\right] e^{-\frac{Zr}{a_0}}$$

ist nicht kugelsymmetrisch, jedoch

$$|\psi_{200}|^2 + |\psi_{210}|^2 + |\psi_{21-1}|^2 + |\psi_{211}|^2 =$$

$$e^{-\frac{Zr}{a_0}}\,\frac{Z^3}{8a_0^3}\,\frac{1}{4\pi}\left[\left(2-\frac{Zr}{a_0}\right)^2 \cancel{} + \left(\frac{Zr}{a_0}\right)^2\cos^2\vartheta + \frac{1}{2}\left(\frac{Zr}{a_0}\right)^2\sin^2\vartheta + \frac{1}{2}\left(\frac{Zr}{a_0}\right)^2\sin^2\vartheta\right]$$

$$= \frac{Z^3}{8a_0^3}\,\frac{1}{4\pi}\left[\left(2-\frac{Zr}{a_0}\right)^2 + \left(\frac{Zr}{a_0}\right)^2\right] e^{-\frac{Zr}{a_0}}$$

Aufgabe 10