Prof. Dr. Werner Vogelsang, Institut für Theoretische Physik, Universität Tübingen

Homework problems Classical Field Theory – SoSe 2022 – Set 12

due July 19 during lecture

Problem 44: *(Part (d) is optional)*

A point charge q is moving along a trajectory $\vec{X}(t) = \vec{V}\,t$, with a constant vector $\vec{V}$.

(a) Show that the potentials generated by the particle are given by

$$\varphi(\vec{x},t) = \frac{qc}{4\pi\epsilon_0}\frac{1}{\sqrt{(c^2t-\vec{x}\cdot\vec{V})^2+(c^2-\vec{V}^2)(\vec{x}^{\,2}-c^2t^2)}}\,, \qquad \vec{A}(\vec{x},t) = \frac{\varphi(\vec{x},t)}{c}\,\frac{\vec{V}}{c}\,.$$

(b) Show that $\varphi(\vec{x},t)$ may also be written as

$$\varphi(\vec{x},t) = \frac{q}{4\pi\epsilon_0}\frac{1}{R_0\sqrt{1-\frac{\vec{V}^2}{c^2}\sin^2\theta}}\,,$$

with $R_0 = |\vec{R}_0|$, $\vec{R}_0 \equiv \vec{x} - \vec{V}\,t$, and where θ is the angle between $\vec{R}_0$ and $\vec{V}$.

(c) Using the corresponding expressions for the fields given in the lecture, show that

$$\vec{E}(\vec{x},t) = \frac{q}{4\pi\epsilon_0}\frac{1-\frac{\vec{V}^2}{c^2}}{\left(1-\frac{\vec{V}^2}{c^2}\sin^2\theta\right)^{3/2}}\frac{\vec{R}_0}{R_0^3}\,, \qquad \vec{B}(\vec{x},t) = \frac{1}{c^2}\vec{V}\times\vec{E}(\vec{x},t)\,.$$

(d) Consider now $\vec{V} = V\,\vec{e}_x$. Derive the fields $\vec{E}$ and $\vec{B}$ by performing a Lorentz boost from the particle's rest frame and verify that the results of **(c)** are recovered. Discuss the fields qualitatively.

Problem 45:

A thin rod antenna of length L points into z direction. The coordinate origin is at the center of the antenna. There is an alternating current through the antenna with current density given by

$$\vec{j}(\vec{x},t) = \begin{pmatrix} 0 \\ 0 \\ I(z,t)\,\delta(x)\,\delta(y) \end{pmatrix}, \qquad \text{where} \quad I(z,t) = I_0\left(1-\frac{2|z|}{L}\right)^2 \mathrm{e}^{-i\omega t}\,.$$

Here I_0 is a constant.

(a) Compute the corresponding charge density $\varrho(\vec{x},t)$.

(b) Use the result to compute the power $dP/d\Omega$ for dipole radiation (see lecture), averaged over one oscillation period. Also compute the total power P.

Problem 46:

Follow the steps in Sec. III.2 of the lecture to compute the *advanced* Green's function $G_{\mathrm{A}}(\vec{x}-\vec{x}\,',t-t')$ which satisfies the equation

$$\Box\, G_{\mathrm{A}}(\vec{x}-\vec{x}\,',t-t') \;=\; \frac{1}{c}\,\delta^3(\vec{x}-\vec{x}\,')\,\delta(t-t')$$

and vanishes for $t>t'$. Express your result by the function D of problem **39**.

The following problems are meant as practice for the exam. They are optional, but you can use them to make up for earlier problems you may have missed.

Problem 47:

Suppose the magnetic flux density $\vec{B}(\vec{x},t)$ has the form

$$\vec{B} \;=\; (\vec{\nabla}\alpha)\times(\vec{\nabla}\beta)\,,$$

with two scalar functions $\alpha(\vec{x},t),\beta(\vec{x},t)$.

(a) Show that this expression for $\vec{B}(\vec{x},t)$ satisfies the Maxwell equation $\vec{\nabla}\cdot\vec{B}=0$.

(b) Show that $\vec{A} \;=\; \alpha\vec{\nabla}\beta$ and $\vec{A}\,' \;=\; -\beta\vec{\nabla}\alpha$ are two possible vector potentials for $\vec{B}$.

(c) The two vector potentials are related by a gauge transformation $\vec{A}\,' \;=\; \vec{A}+\vec{\nabla}\chi$. Find an explicit expression for $\chi(\vec{x},t)$.

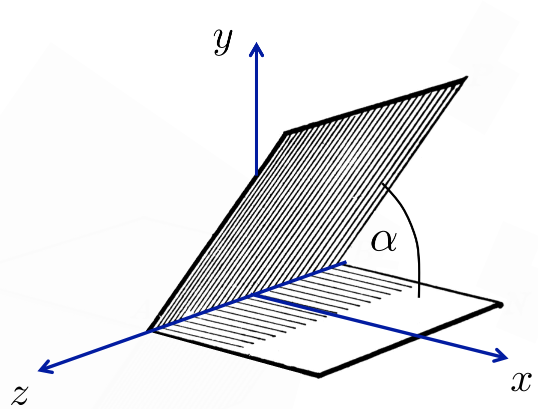
Problem 48:

The two half planes shown in the figure are on potential $\varphi=0$. The potential is also vanishing at infinity. We are interested in the region "between" the two half planes.

(a) We first assume that $\alpha=90^\circ$. Determine the Green's function $G_{\mathrm{D}}(\vec{x},\vec{x}\,')$ for this (Dirichlet-type) boundary value problem. Using G_{D}, write down an expression for the potential $\varphi(\vec{x}\,)$ for an arbitrary given charge density $\varrho(\vec{x}\,)$.

(b) Consider now the case $\alpha=60^\circ$. Present the Green's function also for this case.

(c) What do you expect for arbitrary angles α? When is the method of image charges applicable, and how many charges are needed then? (You don't need to give a proof or a detailed calculation.)



Problem 49:

In the lecture (Sec. I.5.5) we have seen that in Coulomb gauge the potentials satisfy the equations

$$\begin{aligned}\Delta\varphi(\vec{x},t) &= -\frac{\varrho(\vec{x},t)}{\epsilon_0}\,,\\ \Box\vec{A}(\vec{x},t) &= \mu_0\,\vec{j}(\vec{x},t) - \frac{1}{c^2}\vec{\nabla}\,\frac{\partial\varphi(\vec{x},t)}{\partial t} \;\equiv\; \mu_0\,\vec{J}(\vec{x},t)\,,\end{aligned}$$

where we have introduced an "effective" current $\vec{J}$ in the second equation. In the following our only boundary condition will be that all fields vanish sufficiently fast at infinity, so that boundary terms never contribute.

(a) Give the solutions for $\varphi(\vec{x},t)$ and $\vec{A}(\vec{x},t)$ in terms of ϱ and $\vec{J}$, respectively.

(b) Show that

$$\vec{\nabla}\cdot\vec{J} = 0\,.$$

(c) Use the solution for φ to show that

$$\vec{J} = \vec{j} + \vec{\nabla}\int\frac{d^3x'}{4\pi}\,\frac{\vec{\nabla}'\cdot\vec{j}(\vec{x}\,',t)}{|\vec{x}-\vec{x}\,'|}\,.$$

(d) Now use the Helmholtz decomposition of $\vec{j}$ to show that

$$\vec{J} = \vec{\nabla}\times\int\frac{d^3x'}{4\pi}\,\frac{\vec{\nabla}'\times\vec{j}(\vec{x}\,',t)}{|\vec{x}-\vec{x}\,'|}\,.$$

(e) Verify that the solution for $\vec{A}$ indeed satisfies the Coulomb gauge condition.

Help: It is helpful to write the solution for $\vec{A}$ with an integral over time that contains a suitable δ function.

Problem 50:

The figure below shows a conducting spherical shell with radius R that is on vanishing potential. The center of the sphere is the coordinate origin. In the interior of the sphere there is a dipole made of two charges $+q$ and $-q$, separated by the distance $a < R/2$. We put the dipole into the (x, y) plane; its center is at $(R/2, 0, 0)$. As shown in the figure, the dipole intersects the x axis at an angle α.

(a) Compute the corresponding dipole moment $\vec{d}$ of the two charges.

(b) Give the potential φ everywhere in the *exterior* of the sphere. We demand that it vanishes at $|\vec{x}\,| \to \infty$.

(c) Determine which image charges (give their charges and their distance from the origin!) one needs to compute the potential everywhere in the *interior* of the sphere. For which angles α do the image charges themselves form a dipole?

(d) For $\alpha = \pi/2$, write down the potential $\varphi(\vec{x}\,)$ in the interior of the sphere.