Problem 1:

(a)
$$\partial_i (f g) = g \partial_i f + f \partial_i g \qquad i \in \{x, y, z\}$$

$$\Rightarrow \vec{\nabla}(f g) = g \vec{\nabla} f + f \vec{\nabla} g$$

(b)
$$\cancel{\vec{\nabla} \cdot (\vec{F} \vec{G}) = \sum_j \sum_i \partial_j F_i G_i \hat{e}_j}$$

$$\cancel{= \sum_j \sum_i (F_i \partial_j G_i + G_i \partial_j F_i) \hat{e}_j}$$

$$(\vec{F} \cdot \vec{\nabla}) \vec{G} + (\vec{G} \cdot \vec{\nabla}) \vec{F} + \vec{F} \times (\vec{\nabla} \times \vec{G}) + \vec{G} \times (\vec{\nabla} \times \vec{F})$$

$$= \sum_j \hat{e}_j \sum_i F_i \partial_i G_j + \sum_j \hat{e}_j \sum_i G_i \partial_i F_j$$

$$\sum_{ijkml} \varepsilon_{jki} \hat{e}_j F_k \varepsilon_{ilm} \partial_l G_m + \sum_{ijklm} \varepsilon_{jki} \hat{e}_j G_k \varepsilon_{ilm} \partial_l F_m$$

Einsteinsche Summenkonvention

$$= \hat{e}_j F_i \partial_i G_j + \hat{e}_j G_i \partial_i F_j + \varepsilon_{jki} \varepsilon_{ilm} \hat{e}_j F_k \partial_l G_m$$

$$+ \varepsilon_{jki} \varepsilon_{ilm} \hat{e}_j G_k \partial_l F_m$$

$$= \hat{e}_j F_i \partial_i G_j + \hat{e}_j G_i \partial_i F_j + \hat{e}_j G_i \partial_j F_i - \hat{e}_j G_i \partial_i F_j$$

$$+ \hat{e}_j F_i \partial_j G_i - \hat{e}_j F_i \partial_i G_j$$

$$= \hat{e}_j G_i \partial_j F_i + \hat{e}_j F_i \partial_j G_i = \hat{e}_j \partial_j (F_i G_i) = \vec{\nabla}(\vec{F} \cdot \vec{G})$$

c) $\vec{\nabla} \cdot (f \vec{F}) = \partial_i (f F_i) = F_i \partial_i f + f \partial_i F_i = \vec{F} \cdot (\vec{\nabla} f) + f (\vec{\nabla} \cdot \vec{F})$

d)

d) $\vec{G}\cdot(\vec{\nabla}\times\vec{F}) - \vec{F}\cdot(\vec{\nabla}\times\vec{G}) = \sum_{ijk}\varepsilon G_i\,\partial_j F_k - \sum_{ijk}\varepsilon F_i\,\partial_j G_k$

$= \sum_{jki}\varepsilon\,\partial_j(G_i F_k) - \sum_{jki}\varepsilon F_k\,\partial_j G_i - \sum_{ijk}\varepsilon F_i\,\partial_j G_k$

$= \sum_{jki}\varepsilon\,\partial_j(F_k G_i) - \sum_{ijk}\varepsilon F_i\,\partial_j G_k + \sum_{ijk}\varepsilon F_i\,\partial_j G_k$

$= \sum_{jki}\varepsilon\,\partial_j(F_k G_i) = \vec{\nabla}\cdot(\vec{F}\times\vec{G})$

e) $\vec{\nabla}\times(f\vec{F}) = \sum_{ijk}\varepsilon\,\hat{e}_i\,\partial_j(fF_k) = f\sum_{ijk}\varepsilon\,\hat{e}_i\,\partial_j F_k$

$+ \sum_{ijk}\varepsilon\,\hat{e}_i F_k\,\partial_j f = f(\vec{\nabla}\times\vec{F}) - \vec{F}\times(\vec{\nabla}f) = (\vec{\nabla}f)\times\vec{F} + f(\vec{\nabla}\times\vec{F})$

f) $\vec{\nabla}\times(\vec{F}\times\vec{G}) = \sum_{ijk}\varepsilon\,\hat{e}_i\,\partial_j\sum_{klm}\varepsilon F_l G_m = \sum_{ijk}\varepsilon\sum_{klm}\varepsilon\,\hat{e}_i\,\partial_j(F_l G_m)$

$= \hat{e}_i\,\partial_j(F_i G_j) - \hat{e}_i\,\partial_j(F_j G_i)$

$= \hat{e}_i G_j\,\partial_j F_i + \hat{e}_i F_i\,\partial_j G_j - \hat{e}_i F_j\,\partial_j G_i - \hat{e}_i G_i\,\partial_j F_j$

$= (G\cdot\vec{\nabla})\vec{F} + \vec{F}(\vec{\nabla}G) - (\vec{F}\cdot\vec{\nabla})\vec{G} - \vec{G}(\vec{\nabla}\cdot\vec{F})$

g) $\vec{\nabla}\times(\vec{\nabla}f) = \sum_{ijk}\varepsilon\,\hat{e}_i\,\partial_j\partial_k f = \frac{1}{2}\hat{e}_i\left(\sum_{ijk}\varepsilon\,\partial_j\partial_k f + \sum_{ikj}\varepsilon\,\partial_k\partial_j f\right)$

$= \vec{0}$

h) $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{F}) = \sum_{ijk}\varepsilon\,\partial_i\partial_j F_k = \frac{1}{2}\left(\sum_{ijk}\varepsilon\,\partial_i\partial_j F_k + \sum_{jik}\varepsilon\,\partial_j\partial_i F_k\right)$

$= \frac{1}{2}\left(\sum_{ijk}\varepsilon\,\partial_i\partial_j F_k - \sum_{ijk}\varepsilon\,\partial_i\partial_j F_k\right) = 0$

i) $\vec{\nabla}\times(\vec{\nabla}\times\vec{F})\ \sum_{ijk}\varepsilon\,\hat{e}_i\,\partial_j\sum_{klm}\varepsilon\,\partial_l F_m = \sum_{ijk}\varepsilon\sum_{klm}\varepsilon\,\hat{e}_i\,\partial_j\partial_l F_m$

$$= \hat{e}_i \partial_j \partial_i F_j - \hat{e}_i \partial_j \partial_j F_i = \vec{\nabla} (\vec{\nabla} \cdot \vec{F}) - \Delta \vec{F}$$

Problem 2

(a) $$\mathcal{L}(\phi, \dot\phi, \vec\nabla\phi) = \frac{1}{2}\left[\frac{1}{c^2}\left(\frac{\partial\phi}{\partial t}\right)^2 - \left(\vec\nabla\phi\right)^2 - \mu^2\phi^2\right]$$

The Euler-Lagrange equations of motions have the form

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot\phi} + \frac{d}{dx}\frac{\partial\mathcal{L}}{\partial\frac{\partial\phi}{\partial x}} + \frac{d}{dy}\frac{\partial\mathcal{L}}{\partial\left(\frac{\partial\phi}{\partial y}\right)} + \frac{d}{dz}\frac{\partial\mathcal{L}}{\partial\left(\frac{\partial\phi}{\partial z}\right)} - \frac{\partial\mathcal{L}}{\partial\phi} = 0$$

for this Lagrangian it follows:

$$\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} - \frac{\partial^2\phi}{\partial x^2} - \frac{\partial^2\phi}{\partial y^2} - \frac{\partial^2\phi}{\partial z^2} + \mu^2\phi = 0$$

The generalized impuls is derived from

$$\pi = \frac{\partial\mathcal{L}}{\partial\dot\phi} = \frac{1}{c^2}\dot\phi \quad\Rightarrow\quad \dot\phi = c^2\pi$$

$$\Rightarrow \mathcal{H} = \pi\cdot\dot\phi - \mathcal{L} = c^2\pi^2 - \frac{1}{2}\left[c^2\pi^2 - \left(\vec\nabla\phi\right)^2 - \mu^2\phi^2\right]$$

$$= \frac{1}{2}\left[c^2\pi^2 + \left(\vec\nabla\phi\right)^2 + \mu^2\phi^2\right]$$

b.) with the same derivation as for one field it holds

$$\delta S = \int_{t_1}^{t_2} dt \int_0^D dV\, \mathcal{L}\left(\psi+\delta\psi, \dot\psi+(\delta\dot\psi), \vec\nabla\psi+\vec\nabla\delta\psi, \phi+\delta\phi, \dot\phi+(\delta\dot\phi), \vec\nabla\phi+\vec\nabla\delta\phi\right)$$

$$= \int_{t_1}^{t_2} dt \int_0^D d^3r \left[\left(\frac{\partial\mathcal{L}}{\partial\psi} - \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot\psi} - \frac{d}{dx}\frac{\partial\mathcal{L}}{\partial\left(\frac{\partial\psi}{\partial x}\right)} - \frac{d}{dy}\frac{\partial\mathcal{L}}{\partial\left(\frac{\partial\psi}{\partial y}\right)} - \frac{d}{dz}\left(\frac{\partial\mathcal{L}}{\partial\left(\frac{\partial\psi}{\partial z}\right)}\right)\right)\delta\psi\right.$$

$$\left. + \left(\frac{\partial\mathcal{L}}{\partial\phi} - \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot\phi} - \frac{d}{dx}\frac{\partial\mathcal{L}}{\partial\left(\frac{\partial\phi}{\partial x}\right)} - \frac{d}{dy}\frac{\partial\mathcal{L}}{\partial\left(\frac{\partial\phi}{\partial y}\right)} - \frac{d}{dz}\frac{\partial\mathcal{L}}{\partial\left(\frac{\partial\phi}{\partial z}\right)}\right)\delta\phi\right]$$

$\Rightarrow$ From the arbitrariness of $\delta\phi$, $\delta\psi$ follows that

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot\psi} + \frac{d}{dx}\frac{\partial \mathcal{L}}{\partial(\frac{\partial\psi}{\partial x})} + \frac{d}{dy}\frac{\partial \mathcal{L}}{\partial(\frac{\partial\psi}{\partial y})} + \frac{d}{dz}\frac{\partial \mathcal{L}}{\partial(\frac{\partial\psi}{\partial z})} - \frac{\partial \mathcal{L}}{\partial \psi} = 0$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot\phi} + \frac{d}{dx}\frac{\partial \mathcal{L}}{\partial(\partial_x\phi)} + \frac{d}{dy}\frac{\partial \mathcal{L}}{\partial(\partial_y\phi)} + \frac{d}{dz}\frac{\partial \mathcal{L}}{\partial(\partial_z\phi)} - \frac{\partial \mathcal{L}}{\partial \phi} = 0$$

$$\Rightarrow \frac{iK}{2}\frac{\partial\phi}{\partial t} - \frac{K^2}{2m}\Delta\phi + V(\vec{x},t)\phi = 0 \qquad (1)$$

$$-\frac{iK}{2}\frac{\partial\psi}{\partial t} - \frac{K^2}{2m}\Delta\psi + V(\vec{x},t)\psi = 0 \qquad (2)$$

looks familiar to the Schrödinger Equation. For K being $\hbar$

(1) is the SEQ and (2) is the complex ajoined SEQ.

Problem 3

For a Lagrangian $\mathcal{L}(\phi, \dot\phi, \vec\nabla\phi)$

$$\frac{\partial \mathcal{L}}{\partial\dot\phi} = \pi \quad \text{and}$$

$$H = \pi\cdot\dot\phi - \mathcal{L}$$

the physical path will extremize the action S

$$S = \int_{t_1}^{t_2} dt \int d^3x\, \mathcal{L} = \int_{t_1}^{t_2} dt \int d^3x \left(\pi\dot\phi - H(\pi,\phi,\vec\nabla\phi)\right)$$

$$\delta S = \int_{t_1}^{t_2} dt \int d^3x \left[(\pi + \delta\pi)(\dot\phi + (\delta\dot\phi)) - H(\pi+\delta\pi, \phi+\delta\phi, \vec\nabla\phi + \vec\nabla\delta\phi) - (\pi\dot\phi - H(\pi,\phi,\vec\nabla\phi))\right]$$

$$= \int_{t_1}^{t_2} dt \int d^3x \left[\pi(\delta\dot\phi) + \dot\phi(\delta\pi) - \frac{\partial H}{\partial\pi}\delta\pi - \frac{\partial H}{\partial\phi}\delta\phi - \frac{\partial H}{\partial\vec\nabla\phi}\vec\nabla\delta\phi\right]$$

$$= \int_{t_1}^{t_2}\int d^3x \left(\dot\phi - \frac{\partial H}{\partial\pi}\right)\delta\pi - \left(\dot\pi + \frac{\partial H}{\partial\phi} - \frac{d}{dx}\frac{\partial H}{\partial(\partial_x\phi)} - \frac{d}{dy}\frac{\partial H}{\partial(\partial_y\phi)} - \frac{d}{dz}\frac{\partial H}{\partial(\partial_z\phi)}\right)\delta\phi$$

$$+\left[\pi\,\delta\phi - \frac{\partial H}{\partial(\partial_x\phi)}\delta\phi - \frac{\partial H}{\partial(\partial_y\phi)}\delta\phi - \frac{\partial H}{\partial(\partial_z\phi)}\delta\phi\right]_{\text{Randbedingungen}}$$

$(=0)$

Again ~~from the arbitra~~ because $\delta\pi, \delta\phi$ are arbitrary it has to hold, in ~~of~~ order to get $\delta S = 0$ for all $\delta\pi, \delta\phi$

$$\dot\phi - \frac{\partial H}{\partial\pi} = 0 \quad ; \quad \dot\pi + \frac{\partial H}{\partial\phi} - \frac{d}{dx}\frac{\partial H}{\partial(\partial_x\phi)} - \frac{d}{dy}\frac{\partial H}{\partial(\partial_y\phi)} - \frac{d}{dz}\frac{\partial H}{\partial(\partial_z\phi)} = 0$$

for $\mathcal{L} = \frac{\rho}{2}\left(\frac{\partial\phi}{\partial t}\right)^2 - \frac{\sigma}{2}\left(\vec\nabla\phi\right)^2$

$$\pi = \frac{\partial\mathcal{L}}{\partial\dot\phi} = \rho\,\dot\phi \Rightarrow \dot\phi = \frac{1}{\rho}\pi$$

$$H = \pi\dot\phi - \mathcal{L} = \frac{1}{\rho}\pi^2 - \frac{1}{2\rho}\pi^2 + \frac{\sigma}{2}\left(\vec\nabla\phi\right)^2 = \frac{1}{2}\frac{1}{\rho}\pi^2 + \frac{1}{2}\sigma\left(\vec\nabla\phi\right)^2$$

$$\Rightarrow \dot\phi = \frac{\pi}{\rho} \quad ; \quad \dot\pi = \sigma\,\Delta\phi$$

$$\Rightarrow \frac{\partial^2\phi}{\partial t^2} = \frac{\dot\pi}{\rho} = \frac{\sigma}{\rho}\Delta\phi \qquad \text{(Wave equation)}$$

$$= \frac{1}{c^2}\Delta\phi \qquad \left(c := \sqrt{\frac{\rho}{\sigma}}\right)$$

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