Prof. Dr. Werner Vogelsang, Institut für Theoretische Physik, Universität Tübingen

Homework problems Classical Field Theory – SoSe 2022 – Set 4

due May 17 in lecture

Problem 12:

The Lorentz transformation for a rotation about the x^3 axis (angle θ) is given by

$$(\Lambda^\mu{}_\nu) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} .$$

Convince yourself that $\vec{E}$ and $\vec{B}$ transform like ordinary three-dimensional vectors under this rotation. To do this, compute their transformation property from that of the field strength tensor.

Problem 13:

Show that the *Jacobi identity* for the field strength tensor,

$$\partial^\lambda F^{\mu\nu} + \partial^\mu F^{\nu\lambda} + \partial^\nu F^{\lambda\mu} \;=\; 0\,,$$

is equivalent to the relation

$$\partial_\mu \tilde{F}^{\mu\nu} \;=\; 0$$

for the dual field strength tensor.

Problem 14:

From the lecture we know that $\vec{B}^{\,2} - \vec{E}^{\,2}/c^2$ and $\vec{E}\cdot\vec{B}$ are both Lorentz invariant. Show that:

(a) If, in a certain inertial system, the fields $\vec{E}$ and $\vec{B}$ are not orthogonal, then there is *no* inertial system in which one of them vanishes.

(b) If, on the other hand, $\vec{E}\cdot\vec{B} = 0$ in a particular inertial system, and if $c^2\vec{B}^2 \neq \vec{E}^2$, then there exists an inertial system in which one of the two fields vanishes.

Help: Choose the coordinate axes in such a way that for example $\vec{E} = E\,\vec{e}_x$ and $\vec{B} = B\,\vec{e}_y$. Use a suitable Lorentz boost and the results of Problem **11**.

(c) If in a certain inertial system IS $\vec{E}\cdot\vec{B} = 0$ and $c|\vec{B}\,| = |\vec{E}\,|$, then in a system IS′ that is moving relative to IS with speed $\vec{V}$ in the direction of $\vec{E}\times\vec{B}$ both fields will be seen as from IS, but reduced by the factor

$$\sqrt{\frac{c-V}{c+V}}\,.$$

Problem 15:

The product of two arbitrary components of the ε tensor may be written as

$$\varepsilon^{\mu\nu\rho\sigma}\,\varepsilon^{\alpha\beta\gamma\delta} \;=\; -\det\begin{pmatrix} g^{\mu\alpha} & g^{\mu\beta} & g^{\mu\gamma} & g^{\mu\delta} \\ g^{\nu\alpha} & g^{\nu\beta} & g^{\nu\gamma} & g^{\nu\delta} \\ g^{\rho\alpha} & g^{\rho\beta} & g^{\rho\gamma} & g^{\rho\delta} \\ g^{\sigma\alpha} & g^{\sigma\beta} & g^{\sigma\gamma} & g^{\sigma\delta} \end{pmatrix}.$$

(a) Check this relation for one example by inserting explicit values for the indices.

(b) Show that

$$\varepsilon^{\mu\nu\rho\sigma}\,\varepsilon_{\mu}{}^{\beta\gamma\delta} \;=\; -\det\begin{pmatrix} g^{\nu\beta} & g^{\nu\gamma} & g^{\nu\delta} \\ g^{\rho\beta} & g^{\rho\gamma} & g^{\rho\delta} \\ g^{\sigma\beta} & g^{\sigma\gamma} & g^{\sigma\delta} \end{pmatrix}.$$

(c) Use the result in **(b)** to find simple relations for

$$\varepsilon^{\mu\nu\rho\sigma}\,\varepsilon_{\mu\nu}{}^{\gamma\delta}\,,$$

$$\varepsilon^{\mu\nu\rho\sigma}\,\varepsilon_{\mu\nu\rho}{}^{\delta}\,,$$

$$\varepsilon^{\mu\nu\rho\sigma}\,\varepsilon_{\mu\nu\rho\sigma}\,.$$