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Homework problems Classical Field Theory – SoSe 2022 – Set 1

due April 26 in lecture

Problem 1: *Differentiation rules for fields*

Study first Appendix A in the lecture notes. Consider then scalar fields $f(x,y,z)$, $g(x,y,z)$ and vector fields $\vec{F}(x,y,z)$, $\vec{G}(x,y,z)$ that are sufficiently differentiable. Show that the identities given below hold. In this problem, we will also practice working with three-dimensional indices 1,2,3. Write for example F^i for a component of $\vec{F}$. Use Einstein's summation convention and use $(\vec{a}\times\vec{b})^i = \varepsilon^{ijk} a^j b^k$. In the end, write equations **(a)**–**(i)** also in terms of "grad", "div", "rot" (or "curl") instead of $\vec{\nabla}$.

(a)

$$\vec{\nabla}(f\,g) \,=\, f\,\vec{\nabla} g + g\,\vec{\nabla} f\,,$$

(b)

$$\vec{\nabla}(\vec{F}\cdot\vec{G}\,) \,=\, (\vec{F}\cdot\vec{\nabla}\,)\vec{G} + (\vec{G}\cdot\vec{\nabla}\,)\vec{F} + \vec{F}\times(\vec{\nabla}\times\vec{G}\,) + \vec{G}\times(\vec{\nabla}\times\vec{F}\,)\,,$$

(c)

$$\vec{\nabla}\cdot(f\,\vec{F}\,) \,=\, f\,\vec{\nabla}\cdot\vec{F} + \vec{F}\cdot\vec{\nabla} f\,,$$

(d)

$$\vec{\nabla}\cdot(\vec{F}\times\vec{G}\,) \,=\, \vec{G}\cdot(\vec{\nabla}\times\vec{F}\,) - \vec{F}\cdot(\vec{\nabla}\times\vec{G}\,)\,,$$

(e)

$$\vec{\nabla}\times(f\,\vec{F}\,) \,=\, (\vec{\nabla} f)\times\vec{F} + f\,\vec{\nabla}\times\vec{F}\,,$$

(f)

$$\vec{\nabla}\times(\vec{F}\times\vec{G}\,) \,=\, (\vec{\nabla}\cdot\vec{G}\,)\vec{F} + (\vec{G}\cdot\vec{\nabla})\vec{F} - (\vec{\nabla}\cdot\vec{F}\,)\vec{G} - (\vec{F}\cdot\vec{\nabla}\,)\vec{G}\,,$$

(g)

$$\vec{\nabla}\times(\vec{\nabla} f) \,=\, \vec{0}\,,$$

(h)

$$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{F}\,) \,=\, 0\,,$$

(i)

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{F}\,) \,=\, \vec{\nabla}(\vec{\nabla}\cdot\vec{F}\,) - \Delta\vec{F}\,,$$

Help: $\varepsilon^{ijk}\varepsilon^{i'j'k} = \delta^{ii'}\delta^{jj'} - \delta^{ij'}\delta^{ji'}$. It is convenient to use the short-hand notation $\frac{\partial}{\partial x^i} \equiv \partial_i$.

Problem 2:

(a) We consider the Lagrangian

$$\mathcal{L}(\phi, \dot{\phi}, \vec{\nabla}\phi) \;=\; \frac{1}{2}\left[\frac{1}{c^2}\left(\frac{\partial \phi}{\partial t}\right)^2 - \left(\vec{\nabla}\phi\right)^2 - \mu^2\phi^2\right]$$

for a field $\phi(\vec{x}, t)$. c and μ are constants. Derive the equations of motion for the field. Also compute the corresponding Hamilton density $\mathcal{H}$.

(b) Consider now two fields $\psi(\vec{x}, t)$, $\phi(\vec{x}, t)$ with the Lagrangian

$$\mathcal{L}(\psi, \dot{\psi}, \vec{\nabla}\psi, \phi, \dot{\phi}, \vec{\nabla}\phi) \;=\; \frac{i\kappa}{2}\left(\phi\frac{\partial \psi}{\partial t} - \frac{\partial \phi}{\partial t}\psi\right) - \frac{\kappa^2}{2m}(\vec{\nabla}\phi)\cdot(\vec{\nabla}\psi) - V(\vec{x}, t)\,\phi\psi\;.$$

Here, κ and m are constants, and V is a potential. Derive the equations of motion for ψ and ϕ. Do they look familiar?

Problem 3: *(a little more difficult)*

Consider the Lagrangian density

$$\mathcal{L} \;=\; \mathcal{L}(\phi, \partial\phi/\partial t, \vec{\nabla}\phi)$$

for a field $\phi = \phi(\vec{x}, t)$. Show that the Hamilton equations of motion read

$$\begin{aligned} \frac{\partial \phi}{\partial t} &= \frac{\partial \mathcal{H}}{\partial \pi}\,, \\ \frac{\partial \pi}{\partial t} &= -\frac{\partial \mathcal{H}}{\partial \phi} + \vec{\nabla}\cdot\frac{\partial \mathcal{H}}{\partial(\vec{\nabla}\phi)}\,, \end{aligned}$$

where $\mathcal{H} \equiv \pi\frac{\partial\phi}{\partial t} - \mathcal{L}$ is the Hamilton density and $\pi(\vec{x}, t) \equiv \frac{\partial \mathcal{L}}{\partial\left(\frac{\partial\phi}{\partial t}\right)}$ is the canonical momentum density. Verify that for $\mathcal{L} = \frac{\varrho}{2}(\partial_t\phi)^2 - \frac{\sigma}{2}(\vec{\nabla}\phi)^2$ the Hamilton equations of motion lead to a wave equation.
Help: $\mathcal{H}$ is a function of ϕ, π and $\partial\phi/\partial x$, $\partial\phi/\partial y$, $\partial\phi/\partial z$. Consider the derivatives with respect to these quantities. Use the chain rule.