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Advanced Statistical Physics
Problem Class 5
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Please encircle the questions you have solved and are able to present/discuss in class.

5.1(a)	5.1(b)	5.1(c)	5.1(d)	5.1(e)	5.2(a)	5.2(b)	5.2(c)

Problem 5.1 (6 points)

Consider an isolated spin-$1/2$ quantum degree of freedom. The spin operators S^z and ($S^{\pm} = S^x \pm iS^y$ and $S^{\alpha} = \sigma^{\alpha}/2$ with $\alpha = x, y, z$ and σ^{α} the usual spin $1/2$ Pauli matrices) associated to this obey to the commutation relations, $[S^z, S^{\pm}] = \pm S^{\pm}, [S^+, S^-] = 2S^z$. For the case of spin-1/2, the representation is two-dimensional with the two basis states $|\pm\rangle$ (using the same notation as in the Problem Class 4) such that $S^z|\pm\rangle = \pm\frac{1}{2}|\pm\rangle$, $S^+|+\rangle = 0$, $S^-|+\rangle = |-\rangle$, $S^+|-\rangle = |+\rangle$, $S^-|-\rangle = 0$. Another possible realization of a two-state quantum system is to consider a fermionic degree of freedom, whose annihilation and creation operators we write as c and $c^\dagger$, respectively. These obey the anti-commutation relation $\{c, c^\dagger\} = 1$. The space of states is spanned by two basis states, $|0\rangle$ and $|1\rangle = c^\dagger|0\rangle$, where $|0\rangle$ is the vacuum of $c : c|0\rangle = 0$. Since the space dimensionalities coincide, can we go further and explicitly map spins to/from fermions?

(a) Let us choose to associate 'spin up' with 'no fermion', and 'spin down' with 'one fermion'. Show that rewriting the fermion operators in terms of spins according to

$$S^z = \frac{1}{2} - c^\dagger c, \quad S^+ = c, \qquad S^- = c^\dagger, \tag{1}$$

allows to reproduce the aforementioned spin commutators, from the fermionic anticommutation relations. Note, that the notation is not the same as in the lecture, but both are actually used in the literature. (**1 point**)

(b) Let us now consider the slightly more complicated case of two spin-$1/2$ degrees of freedom, with operators S_j, $j = 1, 2$ obeying:

$$[S_j^z, S_{j'}^z] = 0, \quad [S_j^z, S_{j'}^{\pm}] = \pm\delta_{j,j'}S_j^{\pm}, \quad [S_j^+, S_{j'}^-] = 2\delta_{j,j'}S_j^z. \tag{2}$$

Show that the following slightly more complicated mapping, in which we dress fermionic operators on site 2 with a string involving operators on its left (i.e., on site 1), allows to reproduce the desired spin algebra from the fermionic algebra:

$$S_j^z = \frac{1}{2} - n_j, \quad S_1^+ = c_1, \quad S_1^- = c_1^\dagger, \quad S_2^+ = (1 - 2n_1)c_2, \quad S_2^- = (1 - 2n_1)c_2^\dagger, \tag{3}$$

in which we used the shorthand notation, $n_j = c_j^\dagger c_j$ (**1 point**).
Hint: Note that the factor $1 - 2n_1$ can only take values ± 1. It can be therefore equivalently written as $e^{\pm i\pi n_1}$.

(c) This idea immediately generalizes to an arbitrary number of spin-$1/2$ operators S_j^a with a definite ordering $j = 1, 2, \ldots L$. This correspondence is known as *Jordan-Wigner transformation*:

$$S_j^z = \frac{1}{2} - n_j, \quad S_j^+ = \left(\prod_{l=1}^{j-1}(1 - 2n_l)\right)c_j, \quad S_j^- = \left(\prod_{l=1}^{j-1}(1 - 2n_l)\right)c_j^\dagger. \tag{4}$$

Show that indeed, under this mapping, the fermionic anticommutation relations on multiple sites implies the spin commutation relations on multiple sites in Eq. (2). (**1 point**)

(d) A particularly important model in magnetism is the so-called **XX model**. It is defined on a one-dimensional chain of L sites, with spin-$1/2$ operators on each site. The Hamiltonian is

$$H_{XX} = J\sum_{j=1}^{L}(S_j^x S_{j+1}^x + S_j^y S_{j+1}^y) - h\sum_{j=1}^{L} S_j^z, \tag{5}$$

where we consider the antiferromagnetic case $J > 0$ and we have also included an external field h in the z direction. We consider periodic boundary conditions on the spin operators, namely $S_{j+L}^a = S_j^a$.

Show that under the Jordan-Wigner transformation, the XX model in Eq. (5) can be written as a quadratic fermionic Hamiltonian,

$$H_{XX} = \frac{J}{2}\sum_{j=1}^{L}\left(c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j\right) + h\sum_{j=1}^{L} c_j^\dagger c_j - h\frac{L}{2}. \tag{6}$$

It is also useful to define the total number N of lattice fermions

$$N = \sum_{j=1}^{L} c_j^\dagger c_j, \tag{7}$$

in order to derive the form in Eq. (6) of the XX Hamiltonian. Which boundary conditions do the lattice fermion operators c_j in Eq. (6) obey? Note that the original magnetic field h now takes the role of the chemical potential for the fermions. (**1 point**)
Hint: The boundary conditions for the lattice fermion operators c_j in Eq. (6) depend on the parity of the fermion number N in Eq. (7).

(e) In this point you can assume, for the sake of simplicity, that the number N of fermions in Eq. (7) is odd. We shall comment at the end of this point about the case where N is even.
Using a Fourier transformation to momentum space, i.e., introducing the new set of fermionic operators,

$$a_k = \frac{1}{\sqrt{L}}\sum_{j=1}^{L} c_j e^{-i2\pi kj/L}, \quad \text{with inverse} \quad c_j = \frac{1}{\sqrt{L}}\sum_{k=1}^{L} a_k\, e^{i2\pi kj/L}, \quad \text{with } k = 1, 2 \dots L. \tag{8}$$

diagonalize the fermionic version of the XX model in Eq. (6), obtaining the form,

$$H_{XX} = \sum_k \varepsilon_k a_k^\dagger a_k - h\frac{L}{2}. \tag{9}$$

Write the explicit form of ε_k. Discuss how the definition of the Fourier transform in Eq. (8) changes if one works in the sector of the Hilber space where the number N of fermions in Eq. (7) is even. What is the ground state for large fields, i.e, for $h \gg J$? (**2 points**)

Problem 5.2 (4 points +2 bonus points)

In this exercise we want to study — using the transfer matrix — the relationship between the one-dimensional quantum Ising chain and the two-dimensional square lattice Ising model. In particular we want to find the relation between the ground-state energy of the quantum chain and the free energy of the classical model. To do this we start with a general discussion concerning a generic two-dimensional classical lattice model. In Fig. 1 we draw a two-dimensional lattice. The vertical direction is named ”time“, while the horizontal one “space”. The lattice spacing in the time direction will be denoted henceforth as τ, while the lattice spacing in the horizontal direction is referred to as a. Notice that in general τ can be different from a and the lattice can be therefore ”squeezed” in the horizontal ($a > \tau$) or in the vertical direction ($\tau > a$). In the following we will set the lattice spacing in the space direction equal to one, $a = 1$, since we will be interested in the scaling limit $\tau \to 0$ with respect to the lattice spacing in the time direction.
Consider a generic model whose partition function Z can be written as

$$Z = \text{Tr } T^N = \text{Tr } e^{-\tau N H}, \quad \text{with} \quad T = e^{-\tau H}. \tag{10}$$

Based on Eq. (10) we see that the transfer matrix T propagates the state of the system on the first row $t = 1$ row by row along the time direction up to the final row $t = N$. The operator H is a one-dimensional quantum spin Hamiltonian, which describes a quantum system defined on a row ($n = 1, 2 \dots M$) of the lattice. The partition function is thus interpreted as a sum over the histories of the spin configurations on a row and, hence, it can be regarded as a path integral with a discretized imaginary (Euclidean) time-step τ. We have seen in the previous problem class this construction for the one-dimensional classical Ising model.

(a) Prove that in the thermodynamic limit $N, M \to \infty$ the free energy density f of the classical two-dimensional model

$$f = \lim_{N,M\to\infty} \frac{F}{MN\tau}, \tag{11}$$

can be written as

$$\beta f = e_0, \tag{12}$$

with e_0 the ground-state energy density of the one-dimensional quantum Hamiltonian H. The free energy density of the classical two-dimensional model is, therefore, equal to the ground-state energy density of the corresponding one-dimensional quantum Hamiltonian H. (**2 points**)

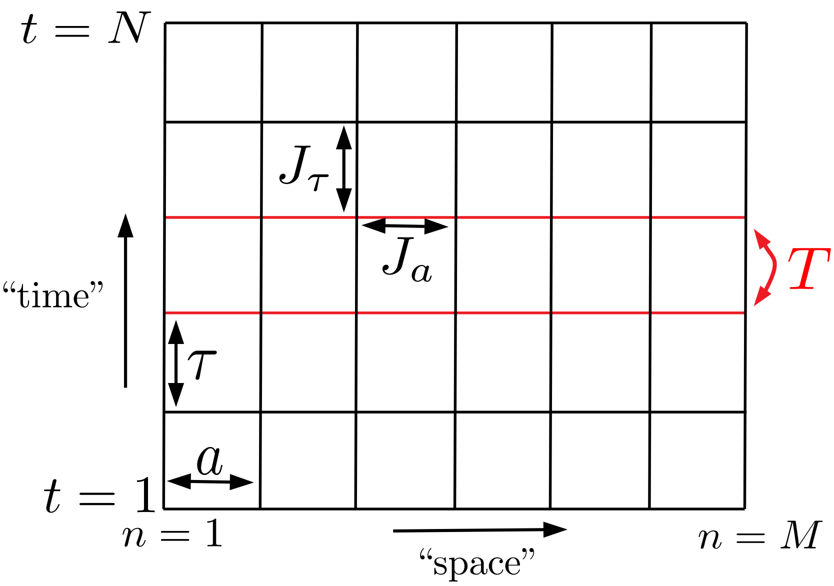

Figure 1: Two dimensional lattice with lattice spacing in the time direction denoted as τ and in the horizontal direction as a. The lattice sites in the time direction are labeled as $t = 1, 2 \ldots N$, while in the horizontal direction we use the label $n = 1, 2 \ldots M$. For the specific case of the Ising model, the nearest-neighbor interaction constant in the time direction is denoted as J_τ, while that one in the space direction is J_a.

We now verify Eq. (12) for the specific case of the Ising model. We shall denote with J_τ the nearest-neighbor interaction coupling (including the factor β of the Boltzmann weight) in the time direction and with J_a (again including the factor β of the Boltzmann weight) the neareset-neighbor interaction coupling in the space direction (see Fig. 1). For the two-dimensional classical Ising model, the partition function has been computed in the lecture script using the transfer matrix formalism. We take the following result as a starting point for our calculations:

$$
\begin{aligned}
Z &= \left(2\sinh(2J_\tau)\right)^{NM/2} \operatorname{Tr}\left(T^N\right), \\
T &= \exp\left(J_a \sum_n \sigma_n^z \sigma_{n+1}^z\right) \exp\left(\tilde{J}_\tau \sum_n \sigma_n^x\right), \\
e^{-2\tilde{J}_\tau} &= \tanh(J_\tau).
\end{aligned}
\tag{13}
$$

Just note that in the script it is assumed $N = M$, while here we consider the more general case where the number of lattice sites N in time direction can differ from the one in the space direction M.

(b) Show that parametrizing the couplings J_a and J_τ in Eq. (13) in terms of the temporal lattice spacing τ as

$$
J_a = \lambda\tau, \quad e^{-2J_\tau} = \tau, \tag{14}
$$

and considering the limit $\tau \to 0$, one can write the transfer matrix T in Eq. (13) as

$$
T = \exp(-\tau H), \quad \text{with} \quad H = -\lambda \sum_{n=1}^{M} \sigma_n^z \sigma_{n+1}^z - \sum_{n=1}^{M} \sigma_n^x. \tag{15}
$$

Interpret and discuss the physical meaning of Eq. (15). What is the physical meaning of λ? What is the physical meaning of the scaling of the couplings J_a and J_τ in Eq. (14)? (**1 point**).

(c) Show that in the limit $\tau \to 0$, with the couplings as in Eq. (14), the free energy density f, from Eqs. (13) and (15), can be written in the thermodynamic limit $N, M \to \infty$ as

$$
\beta f = -\frac{J_\tau}{\tau} - \frac{1}{2}\int_{-\pi}^{\pi} \frac{\mathrm{d}k}{2\pi} 2\sqrt{\lambda^2 - 2\lambda\cos k + 1}. \tag{16}
$$

Comment on the relation of this equation with Eq. (12) proved in point (a). Are there differences ? (**1 point**).

Hint: Use that the ground state energy density e_0 of H in Eq. (15) is given by

$$
e_0 = \lim_{M\to\infty} \frac{E_0}{M} = -\frac{1}{2}\int_{-\pi}^{\pi} \frac{\mathrm{d}k}{2\pi}\, 2\sqrt{\lambda^2 - 2\lambda\cos k + 1}, \tag{17}
$$

with E_0 the ground-state energy. The result in Eq. (17) can be derived from the exact solution for the spectrum of H in Eq. (15). Note, however, that you do not need to rederive Eq. (17). Just use Eq. (17) in order to prove Eq. (16).

This is a "**bonus question**", i.e., you can gain 2 extra points from this beyond the 10 points given in the previous questions. You can then use these 2 extra points to fill some points that you could have missed in the previous (or in the following) sheets.
Here we rederive Eq. (16) directly from the exact solution of Onsager (valid for arbitrary values of J_a and J_τ). The latter reads as follows:

$$\beta f = -\frac{1}{2\tau}\int_0^{2\pi}\frac{\mathrm{d}k}{2\pi}\log\left\{2\left[\cosh(2J_a)\cosh(2J_\tau)+(\lambda^2-2\lambda\cos k+1)^{1/2}\right]\right\}, \tag{18}$$

with

$$\frac{1}{\lambda}=\frac{1}{\sinh(2J_a)\sinh(2J_\tau)}. \tag{19}$$

Verify that rescaling the couplings J_a and J_τ as in Eq. (14), Eqs. (18) and (19) reduce to Eq. (16). (**2 bonus points**).