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Tutorial Advanced Quantum Mechanics

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Problem Set 10

Problem 32 (Dynamics of field operators) **(7 points)**

Consider the second-quantized Hamiltonian

$$H = \int \mathrm{d}^3x\, \psi^\dagger(\mathbf{x},t)\Big[-\frac{\hbar^2}{2m}\nabla^2 + U(\mathbf{x})\Big]\psi(\mathbf{x},t) + \frac{1}{2}\int \mathrm{d}^3x\,\mathrm{d}^3y\, \psi^\dagger(\mathbf{x},t)\,\psi^\dagger(\mathbf{y},t)\,V(\mathbf{x}-\mathbf{y})\,\psi(\mathbf{y},t)\,\psi(\mathbf{x},t)$$

of a field operator which might be bosonic or fermionic

$$\big[\psi(\mathbf{x},t),\psi^\dagger(\mathbf{y},t)\big]_\mp = \delta(\mathbf{x}-\mathbf{y}),$$

where the subscript $\mp$ stands for the commutator $(-)$ in the case of bosons, and for the anti-commutator $(+)$ in the case of fermions. The time evolution of a field operator is governed as usual by the Heisenberg equation of motion

$$\mathrm{i}\hbar\,\partial_t\psi(\mathbf{x},t) = \big[\psi(\mathbf{x},t),H\big]_-.$$

a. **(1 point)** Show for arbitrary operators A, B, C the identity

$$\big[A,BC\big]_- = \pm B\big[A,C\big]_\mp + \big[A,B\big]_\mp C.$$

b. **(3 points)** Evaluate the commutator and show for both bosons and fermions the evolution equation

$$\mathrm{i}\hbar\,\partial_t\psi(\mathbf{x},t) = \Big[-\frac{\hbar^2}{2m}\nabla^2 + U(\mathbf{x}) + \int \mathrm{d}^3y\,\psi^\dagger(\mathbf{y},t)\,\psi(\mathbf{y},t)\,V(\mathbf{x}-\mathbf{y})\Big]\psi(\mathbf{x},t).$$

c. **(3 points)** Show that the particle density operator $\rho(\mathbf{x},t) \equiv \psi^\dagger(\mathbf{x},t)\,\psi(\mathbf{x},t)$ satisfies a continuity equation

$$\partial_t\rho(\mathbf{x},t) + \nabla\cdot\mathbf{j}(\mathbf{x},t) = 0$$

and give the current density operator $\mathbf{j}(\mathbf{x},t)$ explicitly.

Problem 33 (Fermionic pair correlation function) **(8 points)**

Consider a fermion field operator with the usual mode expansion

$$\psi_s(\mathbf{x}) = \frac{1}{\sqrt{V}} \sum_{\mathbf{p}} \mathrm{e}^{\mathrm{i}\mathbf{p}\cdot\mathbf{x}/\hbar} a_{\mathbf{p},s},$$

where the ladder operators satisfy the familiar anti-commutation relations. In this exercise we will calculate the single-particle and pair correlation functions

$$G_s(\mathbf{x},\mathbf{y}) \equiv \langle \phi_0 | \psi_s^\dagger(\mathbf{x}) \psi_s(\mathbf{y}) | \phi_0 \rangle, \qquad G_{ss'}(\mathbf{x},\mathbf{y}) \equiv \langle \phi_0 | \psi_{s'}^\dagger(\mathbf{y})\, \psi_s^\dagger(\mathbf{x})\, \psi_s(\mathbf{x})\, \psi_{s'}(\mathbf{y}) | \phi_0 \rangle$$

for a free Fermi gas. Here $|\phi_0\rangle$ is the ground state of the Fermi gas, where all single-particle states are filled up to the Fermi momentum p_F,

$$|\phi_0\rangle = \prod_{|\mathbf{p}|<p_\mathrm{F}} \prod_s a_{\mathbf{p},s}^\dagger |0\rangle, \qquad p_\mathrm{F}^3 = 3\pi^2\hbar^3\nu, \qquad \nu = N/V.$$

a. (2 points) Show for the single-particle correlation function the expression

$$G_s(\mathbf{x},\mathbf{y}) = \frac{1}{V} \sum_{\mathbf{p}} \Theta(p_\mathrm{F} - |\mathbf{p}|)\, \mathrm{e}^{-\mathrm{i}\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})/\hbar}.$$

b. (3 points) Show for the pair correlation function the relation

$$G_{ss'}(\mathbf{x},\mathbf{y}) = \left(\frac{1}{V} \sum_{\mathbf{p}} \Theta(p_\mathrm{F} - |\mathbf{p}|) \right)^2 - \delta_{s,s'} \left[G_s(\mathbf{x},\mathbf{y}) \right]^2.$$

c. (3 points) Take the thermodynamic limit $\frac{1}{V}\sum_{\mathbf{p}} \to \int \frac{\mathrm{d}^3 p}{(2\pi\hbar)^3}$ and evaluate the arising integrals.

Problem 34 (Fermi gas) **(6 points)**

We can regard the valence electrons of a copper lattice as a free Fermi gas, where we assume that each atom has exactly one valence electron. The density of copper is $8.96\,\mathrm{g/cm^3}$ and its atomic weight is $63.5\,\mathrm{g/mol}$.

a. (3 points) Calculate the Fermi energy E_F and the Fermi temperature $T_\mathrm{F} = E_\mathrm{F}/k_\mathrm{B}$ of the system. Can we treat the electrons as non-relativistic? Can we assume that the system is cold?
Hint: $m_e c^2 = 511\,\mathrm{keV}$; $k_\mathrm{B} = 8.617 \cdot 10^{-5}\,\mathrm{eV/K}$; copper melts at $1083\,°\mathrm{C}$.

b. (3 points) The *bulk modulus* K of a body is defined by $K := -V\,\partial p/\partial V$, where $p = -\partial E/\partial V$ is the pressure. Calculate the electrons' contribution to the bulk modulus of copper and compare your result to the measured value $K_\mathrm{exp} = 13.4 \cdot 10^{10}\,\mathrm{N/m^2}$. How do you explain the difference?

Weihnachten 2019: Zimtstern
Weihnachten 2020: Hanfstern
Weihnachten 2021: Boostern

Wir wünschen Ihnen besinnliche Weihnachten und einen guten Rutsch ins neue Jahr!

Frohes Rechnen!