Problem 4.1

I worked every exercise of this sheet.

Christian [illegible]

a) $T = \begin{pmatrix} e^{h+k} & e^{-k} \\ e^{-k} & e^{-h+k} \end{pmatrix}$

~~In order to~~ The Eigenvalues were calculated in the lecture as

$$\lambda_1 = e^{k}\left(\cosh(h) + \sqrt{\sinh^2(h) + e^{-4k}}\right)$$
$$\lambda_2 = e^{k}\left(\cosh(h) - \sqrt{\sinh^2(h) + e^{-4k}}\right)$$

In order to calculate the ~~Eigenstates~~ Eigenvectors I use the algorithm learned in linear algebra. Bringing T in "Zeilen-Stufen-Form"

$$T - \lambda_1 \mathbb{1}_2 = \begin{pmatrix} e^{k}\left(e^{h} - \cosh(h) - \sqrt{\sinh^2(h) + e^{-4k}}\right) & e^{-k} \\ e^{-k} & e^{k}\left(e^{-h} - \cosh(h) - \sqrt{\sinh^2(h) + e^{-4k}}\right) \end{pmatrix}$$

as we know from the fact, that T has two different Eigenvalues, we can eliminate the second row of T by adding two it the first row, multiplied with a factor.

this factor is

$e^{-2k}\left(e^{h} - \cosh(h) - \sqrt{\sinh^2(h) + e^{-4k}}\right)^{-1}$

I show the calculation for the element T_{22} for it's own, because it's a bit larger

$$-e^{-3k}\left(e^{h} - \cosh(h) - \sqrt{\sinh^2(h) + e^{-4k}}\right)^{-1} + e^{k}\left(e^{-h} - \cosh(h) - \sqrt{\sinh^2(h) + e^{-4k}}\right)$$

$$= -\left(e^{h} - \cosh(h) - \sqrt{\sinh^2(h) + e^{-4k}}\right)^{-1}\left[e^{-3k} - e^{k}\left(e^{-h} - \cosh(h) - \sqrt{\sinh^2(h) + e^{-4k}}\right)\cdot\left(e^{h} - \cosh(h) - \sqrt{\sinh^2(h)}\right)\right]$$

using $e^{h} + e^{-h} = 2\cosh(h)$

$$= -(\dots)\left[e^{-3k} - \left(1 + e^{k}\left(\cosh + \sqrt{\sinh^2(h) + e^{-4k}}\right)^2 - 2\cosh(h)\left(\cosh(h) + \sqrt{\sinh^2(h) + e^{-4k}}\right)\right)\right]$$

$$= -(\dots)\left[e^{-3k} - e^{k}\left(1 - \cosh^2(h) + \sinh^2(h) + e^{-4k}\right)\right]$$

$$= -(\dots)\left[e^{-3k} - e^{-3k}\right] = 0$$

Thus we ~~can~~ ~~see~~ get

$$\begin{pmatrix} e^{R}\left(e^{h}-\cosh(h)-\sqrt{\sinh^2(h)+e^{-4R}}\right) & e^{-R} \\ 0 & 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & e^{-2R}\left(e^{h}-\cosh(h)-\sqrt{\sinh^2(h)+e^{-4R}}\right)^{-1} \\ 0 & 0 \end{pmatrix}$$

$\Rightarrow$ ~~ein Eigenvektor~~ one Eigenvector is

$$\tilde{e}_1 = \begin{pmatrix} e^{-2R}\left(e^{h}-\cosh(h)-\sqrt{\sinh^2(h)+e^{-4R}}\right)^{-1} \\ -1 \end{pmatrix}$$

we can instantly ~~wri~~ write down the second

$$\tilde{e}_2 = \begin{pmatrix} e^{-2R}\left(e^{h}-\cosh(h)+\sqrt{\sinh^2(h)+e^{-4R}}\right)^{-1} \\ -1 \end{pmatrix}$$

$$e^{h}-\cosh(h)=\tfrac{1}{2}\left(e^{h}-e^{-h}\right)=\sinh(h)$$

Furthermore I substitute $\sinh(h)\,e^{2R}=\cot(2\phi)$

$$\Rightarrow e^{+2R}\left(e^{h}-\cosh(h)-\sqrt{\sinh^2(h)+e^{-4R}}\right)$$

$$=e^{+2R}\left(\sinh(h)-e^{-2R}\sqrt{\cot^2(2\phi)+1}\right)$$

$$=\left(\cot(2\phi)-\frac{1}{\sin(2\phi)}\right)=\left(\frac{\cos(2\phi)-1}{\sin(2\phi)}\right)$$

$$=-\left(\frac{2\sin^2\phi}{2\sin\phi\cdot\cos\phi}\right)$$

$$=-\tan\phi$$

$\Rightarrow \quad \tilde{e}_1 = \begin{pmatrix} -\cot\phi \\ -1 \end{pmatrix} \qquad \tilde{e}_2 = \begin{pmatrix} \tan(\phi) \\ -1 \end{pmatrix}$

$\left(\frac{\cos(2\phi)+1}{\sin(2\phi)} = \frac{2\cos^2\phi}{2\sin\phi\cos\phi} \right)$

$\Rightarrow \| \sin\phi \, \tilde{e}_1 \|_2 = 1 \;;\; \| \cos\phi \, \tilde{e}_2 \|_2 = 1$

~~die normierten Eigen~~ The normalized Eigenvector are

$\hat{e}_1 = \begin{pmatrix} -\cos\phi \\ -\sin\phi \end{pmatrix}$ bzw $\begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix}$ if you wanted

$\hat{e}_2 = \begin{pmatrix} -\sin\phi \\ \cos\phi \end{pmatrix} \quad \Rightarrow \quad S = \begin{pmatrix} e_1^t \\ e_2^t \end{pmatrix} \leftarrow$ (that's not defined the same way as in the lecture and as it's convention, but I stuck to it for the whole sheet and left it that way. Excuse me please ;)

b) $\langle s_i \rangle = \frac{1}{\lambda_1^N + \lambda_2^N} \sum_{s_1=\pm1} \cdots \sum_{s_N=\pm1} T_{s_1 s_2} \cdots T_{s_{i-1} s_i} \, s_i \, T_{s_i s_{i+1}} \cdots T_{s_N s_1}$

$= \frac{1}{\lambda_1^N + \lambda_2^N} \operatorname{Tr} \left(T^i \sigma T^{N-i} \right) \qquad \sigma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$= \frac{1}{\lambda_1^N + \lambda_2^N} \operatorname{Tr} \left(S^{-1} \begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix}^i S \sigma S^{-1} \begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix}^{N-i} S \right)$

$= \frac{1}{\lambda_1^N + \lambda_2^N} \operatorname{Tr} \left(S \sigma S^{-1} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}^N \right)$

$S \sigma S^{-1} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \cancel{\begin{pmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{pmatrix}} \begin{pmatrix} \cos\phi & -\sin\phi \\ -\sin\phi & -\cos\phi \end{pmatrix}$

$= \begin{pmatrix} \cos^2\phi - \sin^2\phi & -2\cos\phi\sin\phi \\ -2\cos\phi\sin\phi & \sin^2\phi - \cos^2\phi \end{pmatrix}$

$\cos^2\phi - \sin^2\phi = 2\cos^2\phi - 1 = \cos(2\phi)$

$\Rightarrow \langle s_i \rangle = \frac{1}{\lambda_1^N + \lambda_2^N} \left(\lambda_1^N - \lambda_2^N \right) \cos(2\phi) \xrightarrow[N\to\infty]{} \cos(2\phi)$

4 c) as known from the lecture

$$G(r) = \langle S_1 S_{1+r} \rangle - \langle S_1 \rangle \langle S_{1+r} \rangle$$

$$= \alpha_{12} \alpha_{22} \left(\frac{\lambda_2}{\lambda_1} \right)^{r} \quad \text{with} \quad \alpha = S \sigma S^{-1}$$

$$\alpha_{12} \alpha_{22} = (2 \sin\phi \cos\phi)^2 \left(\frac{\lambda_2}{\lambda_1} \right)^{r}$$

$$= \sin^2(2\phi) \left(\frac{\cosh(h) - \sqrt{\sinh^2(h) + e^{-4K}}}{\cosh(h) + \sqrt{\sinh^2(h) + e^{-4K}}} \right)^{r}$$

$$=: \sin^2(2\phi) \, e^{-r/\xi}$$

$$\text{mit} \quad \frac{1}{\xi} = \ln \left(\frac{\cosh(h) + \sqrt{\sinh^2(h) + e^{-4K}}}{\cosh(h) - \sqrt{\sinh^2(h) + e^{-4K}}} \right)$$

for $H = 0$ $h = 0$

$$\frac{1}{\xi} = \left(\ln \left(\frac{1 + e^{-2K}}{1 - e^{-2K}} \right) \right) = -K \ln \left(\frac{\cosh(K)}{\sinh(K)} \right)$$

Problem 1,2

a) $Z_f = \sum_{s_1=\pm 1} \dots \sum_{s_N=\pm 1} e^{\beta E \sum_{i=1}^{N-1} s_i s_{i+1} + \beta H \sum_{i=1}^{N} s_i}$

$\beta E = k \qquad \beta H = h$ (ohne periodische Randbedingungen)

$$Z_f = \sum_{s_1=\pm 1} \dots \sum_{s_N=\pm 1} e^{\frac{h}{2}(s_1+s_2) + k s_1 s_2} \quad e^{\frac{h}{2}(s_{N-1}+s_N) + k s_{N-1} s_N} \quad e^{\frac{h}{2}(s_1+s_N)}$$

$$= \sum_{s_1=\pm 1} \dots \sum_{s_N=\pm 1} e^{\frac{h}{2}s_1} \, T_{s_1 s_2} T_{s_2 s_3} \dots T_{s_{N-1} s_N} \, e^{\frac{h}{2}s_N}$$

$$T^{N-1} =: \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix}$$

$$\Rightarrow Z_f = e^{h}\alpha_1 + \alpha_2 + \alpha_3 + e^{-h}\alpha_4 \quad (= Z_{++} + Z_{+-} + Z_{-+} + Z_{--})$$

$$T^{N-1} = S^{-1} \begin{pmatrix} \lambda_1^{N-1} & 0 \\ 0 & \lambda_2^{N-1} \end{pmatrix} S$$

$$= \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} \lambda_1^{N-1} & 0 \\ 0 & \lambda_2^{N-1} \end{pmatrix} \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}$$

$$= \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} \lambda_1^{N-1}\cos\phi & \lambda_1^{N-1}\sin\phi \\ -\lambda_2^{N-1}\sin\phi & \lambda_2^{N-1}\cos\phi \end{pmatrix}$$

$$\Rightarrow Z_f = e^{h}\left(\lambda_1^{N-1}\cos^2\phi + \lambda_2^{N-1}\sin^2\phi\right) + 2\left(\lambda_1^{N-1}\cos\phi\sin\phi - \lambda_2^{N-1}\sin\phi\cos\phi\right)$$
$$+ e^{-h}\left(\lambda_1^{N-1}\sin^2\phi + \lambda_2^{N-1}\cos^2\phi\right)$$

$$\lambda_1 = e^{k}\left(\cosh(h) + e^{-2k}\frac{1}{\sin\phi}\right)$$

$$\lambda_2 = e^{k}\left(\cosh(h) - e^{-2k}\frac{1}{\sin\phi}\right)$$

$$Z_f = e^h \left(\lambda_1^{N-1} \left(\cos\phi + e^{-h}\sin\phi\right)^2 + \lambda_2^{N-1} \left(\cos\phi - e^{-h}\sin\phi\right)^2 \right)$$

b) + c) für $N \to \infty$ λ_1^{N-1} dominates against λ_2^{N-1}

Further $N \approx N-1$

$$\Rightarrow F_{Z_f} \approx -\frac{N}{\beta} \ln \lambda_1 - \frac{1}{\beta} \ln\left(e^h \left(\cos\phi + e^{-h}\sin\phi\right)\right)$$

$$\approx -\frac{N}{\beta} \ln(\lambda_1) = F_Z \quad \text{for very large } N$$

$$\left(\begin{aligned} Z_{++} &= e^h\left(\lambda_1^{N-1}\cos^2\phi + \lambda_2^{N-1}\sin^2\phi\right) \\ Z_{+-} &= Z_{-+} = \dots \cos\phi \sin\phi \left(\lambda_1^{N-1} - \lambda_2^{N-1}\right) \\ Z_{--} &= e^{-h}\left(\lambda_1^{N-1}\sin^2\phi + \lambda_2^{N-1}\cos^2\phi\right) \end{aligned} \right)$$

$\rightsquigarrow F_{Z_i} = -\frac{1}{\beta}\ln(Z_i)$ for very large N the same approximation holds and therefore

$F_{Z_i} \approx -\frac{N}{\beta}\ln(\lambda_1) \quad \forall i \in \{f, ++, +-, -+, --\}$