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Tutorial Advanced Quantum Mechanics

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Problem Set 12

Problem 37 (Separable potential) (8 points)

The Lippmann–Schwinger equation for a scattering state with wavenumber $\mathbf{k}$ and energy $E_\mathbf{k} = \hbar^2\mathbf{k}^2/(2m)$ reads

$$|\psi_\mathbf{k}^+\rangle = |\mathbf{k}\rangle + \frac{1}{E_\mathbf{k} - H_0 + \mathrm{i}\varepsilon} V|\psi_\mathbf{k}^+\rangle.$$

a. **(2 points)** Show that the momentum representation $\langle\mathbf{p}|V|\mathbf{q}\rangle$ of a *local* potential V depends only on the difference of the two momenta.

b. **(4 points)** Consider a non-local but *separable* potential, i.e.

$$\langle\mathbf{p}|V|\mathbf{q}\rangle = v(\mathbf{p})v(\mathbf{q}).$$

Start from the Lippmann–Schwinger equation and derive an equation for the quantity

$$A(\mathbf{k}) \equiv \int \frac{\mathrm{d}^3q}{(2\pi)^3}\, v(\mathbf{q})\, \langle\mathbf{q}|\psi_\mathbf{k}^+\rangle.$$

Find the exact solution for the Lippmann–Schwinger equation.

Solution: $\langle\mathbf{p}|\psi_\mathbf{k}^+\rangle = (2\pi)^3\delta(\mathbf{p}-\mathbf{k}) + \dfrac{v(\mathbf{p})\,v(\mathbf{k})}{E_\mathbf{k} - E_\mathbf{p} + \mathrm{i}\varepsilon}\left[1 - \displaystyle\int \frac{\mathrm{d}^3q}{(2\pi)^3} \frac{v^2(\mathbf{q})}{E_\mathbf{k} - E_\mathbf{q} + \mathrm{i}\varepsilon}\right]^{-1}.$

c. **(2 points)** Give an exact expression for the scattering amplitude

$$f_\mathbf{k}(\theta,\phi) = -\frac{m}{2\pi\hbar^2}\,\langle\mathbf{k}' = k\hat{\mathbf{x}}|V|\psi_\mathbf{k}^+\rangle$$

for a separable potential.

Problem 38 (Born approximation and Fermi's golden rule) (6 points)

A spinless particle with mass m is elastically scattered with energy E by a short-ranged potential $V(\mathbf{x})$. In order to avoid non-normalizable states we work in a finite box with volume L^3. The scalar product of states reads thus

$$\langle\psi_1|\psi_2\rangle = \int_{L^3} \mathrm{d}^3x\, \psi_1^*(\mathbf{x})\, \psi_2(\mathbf{x}).$$

As initial and final state we consider the normalized plane waves

$$\begin{aligned} t \to -\infty: \qquad & \psi_{\text{in}}(\mathbf{x}) = \langle \mathbf{x}|\mathbf{k}\rangle \ = L^{-3/2}\,\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}}, \\ t \to +\infty: \qquad & \psi_{\text{out}}(\mathbf{x}) = \langle \mathbf{x}|\mathbf{k}'\rangle = L^{-3/2}\,\mathrm{e}^{\mathrm{i}\mathbf{k}'\cdot\mathbf{x}}, \end{aligned}$$

where $|\mathbf{k}| = |\mathbf{k}'| = \sqrt{2mE}/\hbar$.

a. **(2 points)** Show that the density of states (number of states per energy and solid angle) of a free particle ($V(\mathbf{x}) = 0$) with momentum in a solid angle $\mathrm{d}\Omega$ is given by

$$\rho(E) = \left(\frac{L}{2\pi\hbar}\right)^3 \sqrt{2m^3E}.$$

b. **(2 points)** Use Fermi's golden rule

$$\Gamma_{i\to f} = \frac{2\pi}{\hbar}\,|\langle f|V|i\rangle|^2 \rho(E_f)$$

to calculate the transition rate $\mathrm{d}\Gamma$ into states with energy E scattered into a solid angle $\mathrm{d}\Omega$.

c. **(2 points)** The differential cross section is the ratio of the transition rate $\mathrm{d}\Gamma$ and the incoming particle current

$$\mathrm{d}\sigma = |\mathbf{j}_{\text{in}}|^{-1}\,\mathrm{d}\Gamma \qquad \text{with } \mathbf{j} = \frac{\hbar}{m}\,\Im(\psi^*\nabla\psi).$$

Calculate $\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}$. Is the limit $L \to \infty$ well defined?

Problem 39 (Properties of the cross section in Born approximation) (5 points)

Consider the differential cross section for elastic scattering in Born approximation

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \left(\frac{m}{2\pi\hbar^2}\right)^2 \left|\int \mathrm{d}^3x\,\mathrm{e}^{-\mathrm{i}\mathbf{q}\cdot\mathbf{x}}\,V(\mathbf{x})\right|^2,$$

where $\mathbf{q} \equiv \mathbf{k}' - \mathbf{k}$ is as usual the momentum transfer. Prove the following features of the cross section:

a. **(2 points)** If the scattering potential $V(\mathbf{x})$ never changes its sign, then the differential cross section $\mathrm{d}\sigma/\mathrm{d}\Omega$ takes the maximum value for $E \to 0$ (if finite).

b. **(3 points)** For a *central* potential $V(\mathbf{x}) = V(r)$ the product $E \cdot \sigma(E)$, (where $\sigma(E) = \int \mathrm{d}\Omega\,\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}$ is the *total* cross section) is a monotonic increasing function of the energy E.

Hint: Work out the angular integrations and show that $E \cdot \sigma(E)$ can be cast into the form $E \cdot \sigma(E) = \int_0^{aE} \mathrm{d}x\,|f(x)|^2$ with a positive constant $a > 0$ and an appropriate function $f(x)$.