EBERHARD KARLS UNIVERSITÄT TÜBINGEN

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Tutorial Advanced Quantum Mechanics

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Problem Set 4

Problem 12 (Gauge invariance of the Schrödinger equation) (8 points)

The Schrödinger equation for a quantum-mechanical particle of mass m and electric charge q in an external electromagnetic field described by the potential $A^\mu \equiv (\frac{1}{c}\phi, \mathbf{A})$ reads

$$\mathrm{i}\hbar \frac{\partial}{\partial t} |A; t\rangle = H(\mathbf{A}, \phi)|A; t\rangle = \left[\frac{1}{2m}\big(\mathbf{p} - q\,\mathbf{A}(\mathbf{x}, t)\big)^2 + q\phi(\mathbf{x}, t) + V(\mathbf{x})\right]|A; t\rangle,$$

where V is a possible additional, non-electromagnetic external potential. Under a gauge transformation the potentials change according to

$$\phi \to \phi' = \phi + \frac{\partial \chi}{\partial t}, \qquad \mathbf{A} \to \mathbf{A}' = \mathbf{A} - \nabla\chi,$$

with an arbitrary function $\chi(t, \mathbf{x})$. Let $\mathcal{G}$ be the unitary operator in Hilbert space describing the gauge transformation, i.e.

$$|A'; t\rangle = \mathcal{G}|A; t\rangle, \qquad \mathcal{G}^\dagger\mathcal{G} = \hat{1}.$$

We are going to fix the operator $\mathcal{G}$ by the requirement that the state $|A'; t\rangle$ satisfies the Schrödinger equation with the gauge-transformed potentials:

$$\mathrm{i}\hbar \frac{\partial}{\partial t} |A'; t\rangle = H(\mathbf{A}', \phi')|A'; t\rangle.$$

a. (2 points) Show that the above requirement leads to the condition

$$\mathrm{i}\hbar\,\mathcal{G}^\dagger \frac{\partial \mathcal{G}}{\partial t} + H(\mathbf{A}, \phi) = \mathcal{G}^\dagger H(\mathbf{A}', \phi')\mathcal{G}.$$

b. (3 points) Motivate that $\mathcal{G}$ can depend on time t and on the position operator $\mathbf{x}$ but *not* on the momentum operator $\mathbf{p}$. Make the ansatz $\mathcal{G} = \exp\{\mathrm{i}\Lambda(\mathbf{x}, t)\}$ with an Hermitian operator Λ and show the condition

$$\frac{1}{2m}\Big[\mathbf{p} - q\,\mathbf{A} + \nabla\big(\hbar\Lambda + q\,\chi\big)\Big]^2 + \frac{\partial}{\partial t}\big(\hbar\Lambda + q\,\chi\big) = \frac{1}{2m}\big(\mathbf{p} - q\,\mathbf{A}\big)^2.$$

c. (2 points) Determine Λ, and give $\mathcal{G}$ explicitly.

d. (1 point) How does the wave function $\psi_A(\mathbf{x}, t) \equiv \langle \mathbf{x}|A; t\rangle$ change under a gauge transformation?

Problem 13 (Two-dimensional harmonic oscillator) **(5 points)**

Consider an isotropic harmonic oscillator in two spatial dimensions. The *polar ladder operators* are defined as

$$b_\sigma := \frac{1}{\sqrt{2}}\big(a_x - \mathrm{i}\sigma a_y\big), \qquad \sigma = \pm,$$

with a_x and a_y being the usual annihilation operators for the coordinates x and y.

a. (2 points) Evaluate the commutators $[b_\sigma, b_\tau]$, $[b_\sigma^\dagger, b_\tau^\dagger]$, and $[b_\sigma, b_\tau^\dagger]$.

b. (3 points) Express the Hamilton operator as well as the orbital angular momentum L_z as function of the polar occupation number operators $\hat{n}_\pm \equiv b_\pm^\dagger b_\pm$.

Problem 14 (Charged harmonic oscillator in an external magnetic field) **(11 points)**

A particle with mass m and electric charge q moves in a region of space with a uniform magnetic field of strength $\mathbf{B}$. The particle is further subject to a harmonic central potential $V(r) = m\omega_0^2\mathbf{x}^2/2$.

a. (3 points) Show that a uniform magnetic field $\mathbf{B}$ can be described by a vector potential of the form $\mathbf{A} = \frac{1}{2}\,\mathbf{B}\times\mathbf{x}$. Check that such a potential satisfies the Coulomb gauge condition $\nabla\cdot\mathbf{A} = 0$.

b. (4 points) Show that the Hamilton operator for the particle can be cast into the form

$$H = \frac{\mathbf{p}^2}{2M} - \boldsymbol{\omega}_\mathrm{L}\cdot\mathbf{L} + \frac{m}{2}\,\omega_\mathrm{L}^2\,\mathbf{x}_\perp^2 + \frac{m}{2}\,\omega_0^2\mathbf{x}^2,$$

where $\mathbf{L}$ is the orbital angular momentum and $\mathbf{x}_\perp$ is the component of the position orthogonal to the magnetic field. Give an explicit expression for the *Larmor frequency* $\boldsymbol{\omega}_\mathrm{L}$.

c. (4 points) Choose the z axis parallel to the magnetic field. Express the Hamilton operator in terms of the polar ladder operators introduced in Problem 13 and evaluate the energy spectrum of the system (only the eigenenergies, not the degeneracy).