EBERHARD KARLS UNIVERSITÄT TÜBINGEN

Mathematisch-Naturwissenschaftliche Fakultät
Institut für Theoretische Physik
Prof. Dr. Igor Lesanovsky
PD Dr. Davide Campagnari

Tutorial Advanced Quantum Mechanics

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Problem Set 1

Please encircle the questions you have solved and are able to present/discuss in class.

1a	1b	1c	2a	2b	2c	2d	2e	3a	3b	3c	4a	4b

Problem 1 (Time-evolution operator I) (6 points)

a. (2 points) Show explicitly that

$$\int_{t_a}^{t_b} \mathrm{d}t_1 \int_{t_a}^{t_1} \mathrm{d}t_2\, H(t_1)\, H(t_2) = \frac{1}{2} \int_{t_a}^{t_b} \mathrm{d}t_1\, \mathrm{d}t_2\, \mathrm{T}\{H(t_1)H(t_2)\},$$

with T being the time-ordering symbol.

b. (2 points) Derive from the differential equation for the time evolution operator $U(t_b, t_a)$ an integral equation and show that the Dyson series

$$U(t_a, t_b) = \hat{1} + \sum_{n=1}^{\infty} \left(-\frac{\mathrm{i}}{\hbar}\right)^n \int_{t_a}^{t_b} \mathrm{d}t_1 \int_{t_a}^{t_1} \mathrm{d}t_2 \cdots \int_{t_a}^{t_{n-1}} \mathrm{d}t_n\, H(t_1) \ldots H(t_n)$$

is its formal solution.

c. (2 points) Show that by using the time ordering operator the Dyson series can be written as

$$U(t_b, t_a) = \mathrm{T} \exp\left\{ -\frac{\mathrm{i}}{\hbar} \int_{t_a}^{t_b} \mathrm{d}t\, H(t) \right\}.$$

Problem 2 (Pauli matrices) (10 points)

The Pauli matrices are defined as

$$\sigma^1 = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^2 = \sigma_y = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \qquad \sigma^3 = \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

a. (2 points) Verify by explicit matrix multiplication the following identities:

$$(\sigma^i)^2 = \mathbb{1}, \qquad \sigma^1 \sigma^2 = \mathrm{i}\sigma^3,$$

where $\mathbb{1}$ is the two-dimensional unit matrix.

b. **(3 points)** Use the results of part **a.** to show

$$\sigma^j \sigma^k = \delta_{jk} + \mathrm{i} \sum_{m=1}^{3} \varepsilon_{jkm} \sigma^m,$$

where δ_{jk} is the Kronecker delta and ε_{jkm} is the (completely anti-symmetric) Levi-Civita symbol. Show that the above expression is equivalent to the following (anti-)commutator relations

$$[\sigma^j, \sigma^k] = 2\mathrm{i} \sum_m \varepsilon_{jkm} \sigma^m, \qquad \{\sigma^j, \sigma^k\} = 2\delta_{jk}.$$

c. **(1 point)** Show that

$$(\boldsymbol{\sigma} \cdot \mathbf{a})(\boldsymbol{\sigma} \cdot \mathbf{b}) = \mathbf{a} \cdot \mathbf{b} + \mathrm{i}\boldsymbol{\sigma} \cdot (\mathbf{a} \times \mathbf{b}),$$

where $\boldsymbol{\sigma} = (\sigma^1, \sigma^2, \sigma^3)$ and $\mathbf{a}$, $\mathbf{b}$ are constant vectors with three components.

d. **(2 points)** Show that the three Pauli matrices, together with the 2×2 identity matrix, form a complete set, i.e. that any 2×2 matrix can be expressed as a linear combination of these four matrices.

e. **(2 points)** Show for an arbitrary vector $\mathbf{a} \in \mathbb{R}^3$ the equation

$$\mathrm{e}^{\mathrm{i}\boldsymbol{\sigma}\cdot\mathbf{a}} = \cos|\mathbf{a}| + \mathrm{i}\boldsymbol{\sigma} \cdot \hat{\mathbf{a}} \sin|\mathbf{a}|,$$

where $|\mathbf{a}| \equiv \sqrt{\mathbf{a}^2}$ and $\hat{\mathbf{a}} \equiv \mathbf{a}/|\mathbf{a}|$.

Hint: The exponential of an operator is defined by its Taylor series.

Problem 3 (Time-evolution operator II) (6 points)

A two-level system is described by the Hamiltonian

$$H_1 = \hbar \boldsymbol{\omega} \cdot \boldsymbol{\sigma},$$

where $\boldsymbol{\omega}$ is a constant vector and $\boldsymbol{\sigma}$ denotes the vector of Pauli matrices.

a. **(3 points)** Provide an expression for the time evolution operator. Assuming that the system is initially (time $t = 0$) prepared in the eigenstate $|\uparrow\rangle$ with eigenvalue $+1$ of σ^3, calculate the probability to find the system in the *same* state at a later time $t > 0$.

Hint: Use the result of Problem **2e**.

b. **(2 points)** Repeat the calculations of the previous question for the Hamiltonian

$$H_2 = f(t)\, \hbar \boldsymbol{\omega} \cdot \boldsymbol{\sigma},$$

where $f(t)$ is a time-dependent real function.

c. **(1 point)** Assume that the Hamiltonian has the form

$$H_3 = \hbar \boldsymbol{\omega}(t) \cdot \boldsymbol{\sigma},$$

with an arbitrary time-dependent vector $\boldsymbol{\omega}(t)$. Is it possible to give a general closed form of the time evolution operator in this case?

Problem 4 (Heisenberg picture) **(4 points)**

The Hamiltonian of the one-dimensional harmonic oscillator is given by

$$H = \frac{p^2}{2m} + \frac{1}{2}\, m\omega^2 x^2,$$

where m is the mass and ω the oscillator's frequency.

a. **(2 points)** Calculate the time evolution of the position operator $x_{\mathrm{H}}(t)$ and the momentum operator $p_{\mathrm{H}}(t)$ in the Heisenberg picture.

b. **(2 points)** Calculate the commutators $[x_{\mathrm{H}}(t), p_{\mathrm{H}}(t')]$, $[x_{\mathrm{H}}(t), x_{\mathrm{H}}(t')]$ and $[p_{\mathrm{H}}(t), p_{\mathrm{H}}(t')]$.