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Tutorial Advanced Quantum Mechanics

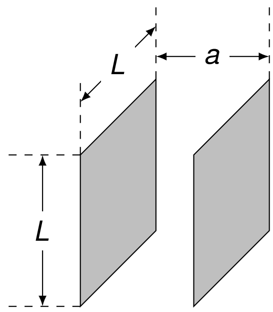
Winter semester 2021/2022 Tübingen, 29th November 2021

Problem Set 7

Problem 21 (Casimir effect) **(10 points)**

The Hamiltonian of the quantized radiation field confined to a box with volume V and with periodic boundary conditions involves a divergent zero-point energy

$$E_{0,\text{free}} = \sum_{\mathbf{k},\lambda} \frac{\hbar\omega_{\mathbf{k}}}{2} \xrightarrow{V\to\infty} \hbar c V \int \frac{\mathrm{d}^3 k}{(2\pi)^3}\, |\mathbf{k}|.$$



Whilst this divergent zero-point energy is not directly measurable, its dependence on the boundaries does lead to observable phenomena. To investigate this, we consider in the following two ideal conducting plates with surface areas $A = L^2$ separated by a distance $a \ll L$. In the plane of the plates we will still be using periodic boundary conditions and consider the limit $L^2 \to \infty$. However, the tangential component of the electric field and the normal component of the magnetic field must vanish on the plates: quantization of the radiation field with these modified boundary conditions leads to a (also diverging) zero-point energy

$$E_{0,\text{plates}} = \hbar c L^2 \int \frac{\mathrm{d}^2\kappa}{(2\pi)^2} \left[\frac{|\boldsymbol{\kappa}|}{2} + \sum_{n=1}^{\infty} \sqrt{\boldsymbol{\kappa}^2 + (n\pi/a)^2} \right],$$

where $\boldsymbol{\kappa}$ is the component of the wavenumber parallel to the plates. We are interested in the *difference* between the energy in the presence and absence of the plates; in order to make the computations feasible we introduce an exponential cut-off and work with

$$\sigma(a) \equiv \frac{E_{0,\text{plates}} - E_{0,\text{free}}}{L^2} = \hbar c \int \frac{\mathrm{d}^2\kappa}{(2\pi)^2} \left\{ \frac{|\boldsymbol{\kappa}|}{2}\, \mathrm{e}^{-\varepsilon|\boldsymbol{\kappa}|} + \sum_{n=1}^{\infty} \sqrt{\boldsymbol{\kappa}^2 + (n\pi/a)^2}\, \mathrm{e}^{-\varepsilon\sqrt{\boldsymbol{\kappa}^2+(n\pi/a)^2}} \right\}$$
$$- \hbar c a \int \frac{\mathrm{d}^3 k}{(2\pi)^3}\, |\mathbf{k}|\, \mathrm{e}^{-\varepsilon|\mathbf{k}|},$$

where the limit $\varepsilon \to 0$ should be taken at the end of the calculation.

a. (3 points) Show that $\sigma(a)$ can be written as

$$\sigma(a) = \frac{\hbar c}{2\pi\varepsilon^3} - \frac{3\hbar c a}{\pi^2\varepsilon^4} - \hbar c \frac{\partial}{\partial\varepsilon} \sum_{n=1}^{\infty} \int \frac{\mathrm{d}^2\kappa}{(2\pi)^2}\, \mathrm{e}^{-\varepsilon\sqrt{\boldsymbol{\kappa}^2+(n\pi/a)^2}}.$$

b. **(3 points)** Work out the integration over κ and show

$$\sigma(a) = \frac{\hbar c}{2\pi\varepsilon^3} - \frac{3\hbar c a}{\pi^2\varepsilon^4} + \frac{\hbar c}{2\pi}\frac{\partial^2}{\partial\varepsilon^2}\left[\frac{1}{\varepsilon}\sum_{n=1}^{\infty} \mathrm{e}^{-n\pi\varepsilon/a}\right].$$

c. **(3 points)** Evaluate the remaining sum, expand the result for small ε, work out the derivatives and and take finally the limit $\varepsilon \to 0$. Evaluate the *pressure* $p(a) = -\partial\sigma(a)/\partial a$ between the plates, and give its order of magnitude for a plate distance in the range of 1µm.
Hint: Use $1/(\mathrm{e}^x - 1) = 1/x - 1/2 + x/12 - x^3/720 + \mathcal{O}(x^5)$.

d. **(1 point)** We have been working under the assumption of ideal conducting plates; in real metals this holds for electromagnetic waves with a frequency below the *plasma frequency* $\omega_{\mathrm{pl}}^2 \simeq n_{\mathrm{e}}e^2/(\varepsilon_0 m_{\mathrm{e}})$, where m_{e} is the electron mass, and n_{e} is the density of free electrons. The dominating frequencies in the Casimir effect are the ones of order c/a, so the approximation should be good as long as $2\pi c/a \ll \omega_{\mathrm{pl}}$: rewrite this inequality as a condition for the fine-structure constant $\alpha = e^2/(4\pi\varepsilon_0\hbar c)$. Is the condition satisfied for copper plates ($n_{\mathrm{e}} = 8.5 \cdot 10^{28}\,\mathrm{m}^{-3}$) at a distance of 1 µm?

Problem 22 (Polarization vectors) (2 points)

The polarization vectors $\boldsymbol{\varepsilon}_\lambda(\mathbf{k})$ for electromagnetic waves are orthogonal to each other and to the wavenumber $\mathbf{k}$

$$\mathbf{k}\cdot\boldsymbol{\varepsilon}_\lambda(\mathbf{k}) = 0, \qquad \boldsymbol{\varepsilon}_\lambda(\mathbf{k})\cdot\boldsymbol{\varepsilon}^*_{\lambda'}(\mathbf{k}) = \delta_{\lambda,\lambda'}.$$

Show the relation

$$\sum_\lambda \left[\boldsymbol{\varepsilon}_\lambda(\mathbf{k})\right]_i \left[\boldsymbol{\varepsilon}^*_\lambda(\mathbf{k})\right]_j = t_{ij}(\mathbf{k})$$

where $t_{ij}(\mathbf{k}) = \delta_{ij} - k_i k_j/\mathbf{k}^2$ is the *transverse projector*.
Hint: Motivate that the most general form for t_{ij} is $A\delta_{ij} + Bk_ik_j$ and determine the coefficients A and B by suitable tensor contractions.

Problem 23 ($2p \to 1s$ transition of the hydrogen atom) (14 points)

In the dipole approximation, the leading-order expression for atomic transitions in the hydrogen atom is given by

$$\Gamma_{i\to f} = \frac{4\pi^2\alpha\hbar c}{L^3}\sum_{\mathbf{q},\lambda'}\omega_{\mathbf{q}}\Big\{\left|\langle f|a_{\lambda'}(\mathbf{q})\,\boldsymbol{\varepsilon}_{\lambda'}(\mathbf{q})\cdot\mathbf{x}|i\rangle\right|^2\delta(E_f - E_i - \hbar\omega_{\mathbf{q}}) + \left|\langle f|a^\dagger_{\lambda'}(\mathbf{q})\,\boldsymbol{\varepsilon}^*_{\lambda'}(\mathbf{q})\cdot\mathbf{x}|i\rangle\right|^2\delta(E_f - E_i + \hbar\omega_{\mathbf{q}})\Big\},$$

where $\alpha = e^2/(4\pi\varepsilon_0\hbar c) \simeq 1/137$ is the fine structure constant. In this exercise we investigate the spontaneous decay of the first excited state $|n=2,\ell=1,m=0\rangle$ of the hydrogen atom into the ground state $|n=1,\ell=0,m=0\rangle$. The initial and final states are hence

$$|i\rangle = |210\rangle\otimes|0\rangle \qquad \text{and} \qquad |f\rangle = |100\rangle\otimes|\mathbf{k},\lambda\rangle,$$

where $|0\rangle$ is the vacuum of the photon field, and

$$|\mathbf{k},\lambda\rangle = a^\dagger_\lambda(\mathbf{k})|0\rangle$$

is the state of one photon with momentum $\hbar\mathbf{k}$, frequency $\omega_{\mathbf{k}} = |\mathbf{k}|c$, and polarization λ.

a. (3 points) Show that the probability rate $\Gamma(\mathbf{k}, \lambda)$ of the atom emitting a photon with momentum $\hbar\mathbf{k}$ and polarization λ is given by

$$\Gamma(\mathbf{k}, \lambda) = \frac{4\pi^2 \alpha \hbar c}{L^3}\, \omega_{\mathbf{k}} \left| \langle 100 | \boldsymbol{\varepsilon}^*_\lambda(\mathbf{k}) \cdot \mathbf{x} | 210 \rangle \right|^2 \delta(E_1 - E_2 + \hbar\omega_{\mathbf{k}}).$$

b. (3 points) Since we are only interested in the decay rate of the atom, the photon's exact momentum and polarization are irrelevant: the decay rate is thus found by summing the result of part **a.** over all possible polarizations and momenta. At this point we can now send the quantization volume to infinity: using $\frac{1}{L^3}\sum_{\mathbf{k}} \to \int \frac{\mathrm{d}^3k}{(2\pi)^3} = \int \frac{\mathrm{d}k\, k^2\, \mathrm{d}\Omega}{(2\pi)^3}$ show that the decay rate is given by the expression

$$\Gamma = \frac{\alpha}{2\pi} \frac{(E_2 - E_1)^3}{\hbar^3 c^2} \int \mathrm{d}\Omega \sum_\lambda \left| \langle 100 | \boldsymbol{\varepsilon}^*_\lambda(\mathbf{k}) \cdot \mathbf{x} | 210 \rangle \right|^2.$$

c. (2 points) Use the result of Problem 22 to work out the sum over polarizations and show

$$\Gamma = \frac{\alpha}{2\pi} \frac{(E_2 - E_1)^3}{\hbar^3 c^2} \sum_{i,j} \langle 100 | x_i | 210 \rangle \langle 210 | x_j | 100 \rangle \int \mathrm{d}\Omega\, t_{ij}(\mathbf{k}).$$

d. (1 point) Show that the remaining integral over the photon's possible directions is

$$\int \mathrm{d}\Omega\, t_{ij}(\mathbf{k}) = \frac{8\pi}{3}\, \delta_{ij}.$$

Hint: Proceed analogously to Problem 22.

e. (3 points) Calculate the matrix elements of the dipole operator. Show that the only non-vanishing contribution comes from the z coordinate and reads

$$d \equiv \langle 100 | z | 210 \rangle = \sqrt{2}\, \frac{2^7}{3^5} \frac{\hbar}{m_e c \alpha},$$

where m_e is the electron's mass.
Hint: $Y_{00}(\theta, \phi) = 1/\sqrt{4\pi}$, $Y_{10}(\theta, \phi) = \sqrt{3/(4\pi)}\, \cos\theta$, $R_{10}(r) = \frac{2}{a^{3/2}}\, \mathrm{e}^{-r/a}$, $R_{21}(r) = \frac{1}{2\sqrt{6}\, a^{3/2}} (r/a)\, \mathrm{e}^{-r/(2a)}$, $a = \hbar/(m_e c \alpha)$.

f. (2 points) Insert the results of parts **d.** and **e.** as well as the eigenenergies $E_n = -m_e c^2 \alpha^2/(2n^2)$ of the hydrogen atom into the expression for Γ from part **c.** and find the decay rate. Compare the resulting average lifetime $\tau = 1/\Gamma$ to the the experimental value $\tau_{exp} = 1.60(1)$ ns.

[*Solution:* $\Gamma = m_e c^2 \alpha^5 (2/3)^8/\hbar$]