Please encircle the questions you have solved and are able to present/discuss in class.

8.1(a)	8.1(b)	8.1(c)	8.1(d)	8.1(e)	8.2(a)	8.2(b)

Problem 8.1 (6 points)

Consider a system described by the the Landau free energy $f(m)$:

$$f(m) = \frac{1}{2}am^2 + \frac{1}{4}bm^4 + \frac{1}{6}cm^6 - hm, \tag{1}$$

where $c > 0$ and a and b are both linearly proportional to the pressure p and temperature T near the point (T_c, p_c). Accordingly, a and b are defined such that, $a = a_1 t + a_2 p$ and $b = b_1 t + b_2 p$, where $t = (T - T_c)/T_c$ and $p = (p - p_c)/p_c$. As p and T are varied both a and b can be made to vanish and change sign. The positivity of the coefficient c, therefore, ensures that $f(m)$ is bounded from below even in the case $b < 0$. Such system exhibits a *tricritical point*. An example is a mixture of ^{3}He and ^{4}He. The aim of this exercise is to compute the phase diagram of the model in Eq. (1) in the (a, b) plane for $h = 0$. Henceforth, in the exercise you can thus set the magnetic field h to zero: $h = 0$. In the remainder of the exercise we denote by $\bar{m}$ the absolute minimum of the Landau free energy given by Eq. (1). Furthermore, we denote by m_i the extrema of the Landau free energy, with i being an integer counting the number of extrema of Eq. (1).

(a) Consider the case $a < 0$. Find the extrema m_i of the Landau free energy, and show that two of these extrema are given by the following equation:

$$m_i^2 = \frac{-b + \sqrt{b^2 - 4ac}}{2c}, \quad \text{with} \quad i = 1, 2. \tag{2}$$

Find the absolute minimum $\bar{m}$ of $f(m)$, plot $f(m)$ for $a < 0$ and discuss the stability of the extrema that you have found. (**1 point**)

(b) Now consider first the case $a > 0$, $b > 0$, and then the case where $a > 0$, $b < 0$. Calculate for both the cases the number of extrema m_i of $f(m)$. How does $\bar{m}$ behave for $a > 0$, $b > 0$ and $a > 0$, $b < 0$? (**1 point**)

(c) Draw the Landau free energy $f(m)$ in each of the different regions of the phase diagram. (**1 point**)

(d) Draw the phase diagram of the model in the $a - b$ plane, indicating the order of any phase transitions that you have found from the previous considerations. Calculate also the position of the boundary separating the phases of the model in Eq. (1). The point $a = b = 0$ is called tricritical point. Can you suggest why?
Hint: Think about the case $h \neq 0$. (**2 points**)

(e) Calculate the critical exponents $\alpha, \beta, \delta, \gamma$, by approaching the tricritical point $a = b = 0$ along the line $b = 0$. One then has $a = (a_1 - a_2 b_1/b_2)t$. We shall consider $a_1 - a_2 b_1/b_2 > 0$. (**1 point**)
Hint: The definition of the critical exponents is

$$\bar{m} \propto (-t)^\beta, \tag{3}$$

$$\bar{m} \propto h^{1/\delta}, \quad \text{for} \quad t = 0, \tag{4}$$

$$\chi = \frac{\mathrm{d}\bar{m}}{\mathrm{d}h} \propto |t|^{-\gamma}, \tag{5}$$

$$C \propto |t|^{-\alpha}, \tag{6}$$

with $t = \frac{T - T_c}{T_c}$ being the so-called reduced temperature.

Problem 8.2 (4 points)

The aim of this exercise is to calculate the energy of a magnetic domain wall in the framework of the Landau φ^4-theory of the Ising model in one spatial dimension. Assuming translational symmetry in the (y, z)-plane, we write the Landau free energy in one spatial dimension x, with zero field, as

$$S[\varphi] = \int \mathrm{d}x\, \mathcal{L}(\varphi, \nabla\varphi), \quad \mathcal{L}(\varphi, \nabla\varphi) = \frac{r}{2}\varphi^2(x) + \frac{c}{2}(\nabla\varphi(x))^2 + \frac{\mu}{4}\varphi^4(x). \tag{7}$$

Here $\nabla\varphi(x) = \frac{\mathrm{d}\varphi}{\mathrm{d}x}$, since we are considering a one-dimensional system. In the whole exercise we assume $r < 0$, which is equivalent to considering $T < T_c$, since $r \propto T - T_c$.

(a) Solve the Euler-Lagrange equation corresponding to the action (Landau free energy) $S[\varphi]$ in Eq. (7)

$$\frac{\delta S}{\delta\varphi} = \frac{\partial\mathcal{L}}{\partial\varphi} - \frac{d}{dx}\frac{\partial\mathcal{L}}{\partial\nabla\varphi} = 0, \tag{8}$$

with boundary conditions:

$$\varphi(x \to \pm\infty) = \pm\varphi_0, \quad \nabla\varphi(x \to \pm\infty) = 0, \tag{9}$$

where φ_0 is the magnetization of the uniform solution.
Hint: We remind you the expression of the mean-field correlation length ξ in terms of the parameter r and c in Eq. (7) reads

$$\xi = \sqrt{-\frac{c}{r}}. \tag{10}$$

Hint: The solution that you should find is

$$\varphi(x) = \varphi_0 \tanh\left(\frac{x - x_0}{\sqrt{2}\xi}\right). \tag{11}$$

Plot the solution in Eq. (11) and explain its physical meaning. (**2 points**)

(b) First, find the free energy of the uniformly polarized solution φ_0 (no domain walls). Next, compute the free energy of the solution of Eq. (11) and compare it that of the uniform solution. Give the physical interpretation of the result that you have found. (**2 points**)
Hint: The free energy of a field configuration $\varphi(x)$ is computed by evaluating the action $S[\varphi]$ at that field configuration. For simplicity you can set $x_0 = 0$ in the solution in Eq. (11).