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Please encircle the questions you have solved and are able to present/discuss in class.

| 7.1(a) | 7.1(b) | 7.1(c) | 7.2(a) | 7.2(b) | 7.2(c) |
|---|---|---|---|---|---|

# 1 Problem 7.1: Continuum-field theory description of the Ising model (5 points)

Consider the Ising model in $d$ spatial dimensions, i.e. the spins are arranged on a hypercubic lattice in $d$ dimension with lattice spacing $a$ (the same in all the directions of the lattice). The volume $V$ of the system is then $V = Na^d$. The Hamiltonian is

$$H = -\frac{1}{2}\sum_{i,j=1}^{N} J'_{ij} S_i S_j - \sum_{i=1}^{N} h'_i S_i, \tag{1}$$

with ferromagnetic coupling matrix $J'_{ij} > 0$ between site $i$ and $j$, and with $J'_{ij} = J'_{ji}$. In the case of nearest neighbour interaction, we have $J'_{ij} = J' > 0$ if $i$ and $j$ are nearest neighbors and zero otherwise. In this exercise we shall, however, keep the matrix $J_{ij}$ general. In Eq. (1), $h'_i$ denotes an external inhomogeneous magnetic fields acting on the spin at lattice site $i$. In the following discussion we will use the quantities

$$J_{ij} = \beta J'_{ij} \quad \text{and} \quad h_i = \beta h'_i, \tag{2}$$

which are the couplings of the Hamiltonian multiplied by the inverse temperature $\beta = 1/(k_B T)$ appearing in the Boltzmann weight. As a final element of the notation we emphasize that in Eq. (1)

$$S_i = S(\vec{r}_i), \quad J_{ij} = J(|\vec{r}_i - \vec{r}_j|) = J(|\vec{r}_j - \vec{r}_i|) = J_{ji}, \tag{3}$$

where $\vec{r}_i$ is the $d$-dimensional position vector locating the lattice site $i$. We have assumed that the coupling $J_{ij}$ is homogeneous and isotropic, and thereby it depends only on the absolute value of the difference between $\vec{r}_i$ and $\vec{r}_j$.
In this exercise we want to derive the continuum-field theory description of the lattice Hamiltonian in Eq. (1) in the limit of vanishing lattice spacing $a \to 0$ and infinite number of spins $N \to \infty$.

(a) Prove the identity

$$\frac{1}{\sqrt{\det A}} e^{\frac{1}{2}\sum_{i,j=1}^{N} B_i (A^{-1})_{ij} B_j} = \int_{-\infty}^{+\infty} \prod_{i=1}^{N} \left(\frac{dx_i}{\sqrt{2\pi}}\right) \exp\left(-\frac{1}{2}\sum_{i,j=1}^{N} x_i A_{ij} x_j + \sum_{i=1}^{N} x_i B_i\right), \tag{4}$$

where $A_{ij}$ is a real symmetric positive-definite matrix and $B_i$, with $i = 1, 2 \dots N$, is an arbitrary $N$-dimensional vector. The identity in Eq. (4) is usually named Hubbard–Stratonovich transformation. (**1 point**)
*Hint*: Make the change of variables $y_i = x_i - \sum_{j=1}^{N} (A^{-1})_{ij} B_j$. Then you can exploit the fact that the matrix $A_{ij}$ is real and symmetric and therefore it can be diagonalized in terms of real eigenvalues $\lambda_i \in \mathbb{R}$, with $i = 1, 2 \dots N$.

(b) Apply the identity of Eq. (4) to the canonical partition function $Z$ associated with the Hamiltonian defined by Eqs. (1) and (2). Show that $Z$ can be written as

$$Z \propto \int_{-\infty}^{\infty} \prod_{i=1}^{N} d\varphi_i \exp(-S(\{(\varphi_i\})), \tag{5}$$

where

$$S(\{(\varphi_i\}) = \frac{1}{2}\sum_{i,j=1}^{N} (\varphi_i - h_i) K_{ij} (\varphi_j - h_j) - \sum_{i=1}^{N} \ln(\cosh \varphi_i), \tag{6}$$

and $K_{ij} = J_{ij}^{-1}$ is the inverse of the matrix $J_{ij}$. In Eq. (5) the symbol $\propto$ denotes proportionality in the sense that we are neglecting an unimportant multiplicative constant which does not depend on the integration variables $\varphi_i$. Note that this representation of the partition function $Z$ depends on the *continuous* variables $\varphi_i \in (-\infty, \infty)$ instead of the *discrete* spin variables $s_i = \pm 1$, with $i = 1, 2 \dots \infty$. In the following we set, for simplicity, the magnetic field to zero $h_i \equiv 0$. Assuming that the field is small $|\varphi_i| \ll 1$, expand the second term on the right hand side of Eq. (6) up to order $\varphi_i^4$. Verify that you get the following expression for $S(\{(\varphi_i\})$ (**1 point**)

$$S(\{(\varphi_i\}) = \frac{1}{2}\sum_{i,j=1}^{N} \varphi_i K_{ij} \varphi_j - \sum_{i=1}^{N} \left(\frac{\varphi_i^2}{2} - \frac{\varphi_i^4}{12}\right). \tag{7}$$

(c) Discuss why in the continuum limit, $V = Na^d \to \infty$ with $a \to 0$, the expression given by Eqs. (5) and (7) takes the form

$$Z \propto \int \mathcal{D}\varphi \exp(-S[\varphi]), \quad \text{with} \quad S[\varphi] = \frac{1}{2a^2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \mathrm{d}\vec{r}\,\mathrm{d}\vec{r}'\,\varphi(\vec{r})K(\vec{r}-\vec{r}')\varphi(\vec{r}') - \int_{\mathbb{R}^d}\mathrm{d}\vec{r}\left(\frac{\varphi^2(\vec{r})}{2a^2} - \mu\varphi^4(\vec{r})\right). \tag{8}$$

The previous equation is a *functional integral* over all the possible values of the smoothly varying field $\varphi(\vec{r})$. The action $S[\varphi]$ is accordingly a functional of the field $\varphi(\vec{r})$. Expand the field $\varphi(\vec{r}')$ in Eq. (8) around $\vec{r}$ keeping terms up to second order in the difference $\vec{r} - \vec{r}'$. Verify that you get at the end of the calculation the following form for the action $S[\varphi]$

$$S[\varphi] = \int_{\mathbb{R}^d} \mathrm{d}\vec{r}\,\mathcal{L}(\varphi(\vec{r})), \quad \text{with} \quad \mathcal{L}(\varphi) = \frac{1}{2}m^2\varphi^2 + \frac{1}{2}c\,(\nabla\varphi)^2 + \mu\varphi^4, \tag{9}$$

where $\mathcal{L}(\varphi)$ has the meaning of a Lagrangian in $d$ space dimension (with Euclidean metrix). Give the expression of the coefficients $m$, $c$ and $\mu$. Give the physical interpretation of the coefficient $m^2$. (**3 points**)
*Hint:* In taking the continuum limit, the field $\varphi(\vec{r})$ in Eqs. (8) and (9) is defined from the integration variables $\varphi_i$ in Eq. (7) as

$$\varphi(\vec{r}) = \frac{\varphi_i}{a^{(d-2)/2}}. \tag{10}$$

*Hint:* It is useful to consider the Fourier transform of $K(\vec{r})$. Be reminded that the definition of the Fourier transform $\hat{K}(\vec{q})$ of an arbitrary function $K(\vec{r})$ is given by

$$\hat{K}(\vec{q}) = \int_{\mathbb{R}^d} \mathrm{d}\vec{r}\,K(\vec{r})\exp(-i\vec{q}\cdot\vec{r}), \quad \text{with inverse} \quad K(\vec{r}) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\mathrm{d}\vec{q}\,\hat{K}(\vec{q})\exp(i\vec{q}\cdot\vec{r}). \tag{11}$$

# 2 Problem 7.2: Gaussian field theory-propagator and Wick theorem

**(5 points +3 bonus points)**

In this exercise we consider the Lagrangian of Eq. (9) for $\mu = 0$:

$$S_0[\varphi] = \int_{\mathbb{R}^d} \mathrm{d}\vec{r}\,\mathcal{L}_0(\varphi(\vec{r})), \quad \text{with} \quad \mathcal{L}_0(\varphi) = \frac{1}{2}m^2\varphi^2 + \frac{1}{2}\,(\nabla\varphi)^2, \tag{12}$$

where the subscript in $\mathcal{L}_0$ refers to the fact that $\mu = 0$, and we have set the constant $c = 1$ without loss of generality. The aim of this exercise is to compute the two-point correlation function $G_0(\vec{x}_1, \vec{x}_2)$, which is defined by

$$G_0(\vec{x}_1, \vec{x}_2) = \langle\varphi(\vec{x}_1)\varphi(\vec{x}_2)\rangle = \frac{1}{Z}\int \mathcal{D}\varphi\,\varphi(\vec{x}_1)\varphi(\vec{x}_2)\,e^{-S_0[\varphi]}. \tag{13}$$

The two-point correlator in field theory is also usually named *propagator*. To compute $G_0$ it is useful to modify the partition function $Z$ by including the so-called source field $h(\vec{r})$

$$Z_0[h] \propto \int \mathcal{D}\varphi \exp[-S_0[\varphi] + \int \mathrm{d}\vec{r}\,h(\vec{r})\varphi(\vec{r})]. \tag{14}$$

One can get the propagator $G_0(\vec{x}_1, \vec{x}_2)$ by taking functional derivatives of $Z_0[h]$ with respect to the source field $h$:

$$G_0(\vec{x}_1, \vec{x}_2) = \frac{1}{Z_0[h]}\left.\frac{\delta^2 Z_0[h]}{\delta h(\vec{x}_1)\delta h(\vec{x}_2)}\right|_{h=0}, \tag{15}$$

and setting $h = 0$ at the end of the calculation.

(a) Compute the partition function $Z_0[h]$ associated to $\mathcal{L}_0$ in Eq. (12). Further show that the propagator satisfies the following differential equation

$$(-\nabla^2 + m^2)G_0(\vec{r}) = \delta(\vec{r}). \tag{16}$$

Note that $G_0(\vec{x}_1, \vec{x}_2) = G_0(\vec{x}_1 - \vec{x}_2) = G_0(\vec{r}) = G_0(r)$ because of translation and rotation invariance with the same reasoning as in Eq. (3). (**1 point**)
*Hint*: The partition function $Z_0[h]$ can be computed using the following identity

$$\int \mathcal{D}\varphi \exp\left[\int \mathrm{d}\vec{r}\mathrm{d}\vec{r}'\,\frac{1}{2}\varphi(\vec{r})A(\vec{r},\vec{r}')\varphi(\vec{r}') + \int \mathrm{d}\vec{r}h(\vec{r})\varphi(\vec{r})\right] \propto \exp\left(\frac{1}{2}\int \mathrm{d}\vec{r}\mathrm{d}\vec{r}'\,h(\vec{r})G(\vec{r},\vec{r}')h(\vec{r}')\right), \tag{17}$$

where $G(\vec{r}, \vec{r}')$ is the inverse of the operator $A$ and thereby it obeys the equation

$$\int \mathrm{d}\vec{y}A(\vec{r},\vec{y})G(\vec{y},\vec{r}') = \delta^{(d)}(\vec{r}-\vec{r}'), \tag{18}$$

with $\delta^{(d)}(\vec{r} - \vec{r}')$ the $d$-dimensional Dirac-delta function. Note that Eqs. (17) and (18) are nothing but the continuum version of the discrete transformation in Eq. (4).

(b) Solve Eq. (18) for $G_0(r)$. Verify that you get the following expression

$$G_0(r) = \int_{\mathbb{R}^d} \frac{d\vec{q}}{(2\pi)^d} \frac{e^{i\vec{q}\cdot\vec{r}}}{q^2+m^2}. \tag{19}$$

(**2 points**)
*Hint:* It is useful to take the Fourier transform of Eq. (18). The Fourier transform has been defined in Eq. (11).

(c) Evaluate $G_0(r)$ from the integral in Eq. (19). Discuss the asymptotic behavior of $G_0(r)$ both for small $r \to 0$ and for large distances $r \to \infty$. (**2 points**)
*Hint:* The propagator $G_0(r)$ depends only on the modulus $r = |\vec{r}|$ of the position vector. It is therefore convenient to change variable in the integral in Eq. (19) from Cartesian to spherical ones. The solid angle $\Omega(d)$ in $d$ spatial dimension is

$$\Omega(d) = \frac{2\pi^{d/2}}{\Gamma(d/2)}, \tag{20}$$

with $\Gamma(d/2)$ the Euler-gamma function. The following definitions are also useful

$$\begin{aligned} \int \mathrm{d}\theta \sin^{2v}(\theta)\, e^{iqr\cos\theta} &= \frac{\Gamma(v+\frac{1}{2})\Gamma(\frac{1}{2})}{(\frac{kr}{2})^v} J_v(qr), \\ \int \mathrm{d}q\, q^{v+1} \frac{J_v(qr)}{q^2+m^2} &= m^v K_v(mr), \end{aligned} \tag{21}$$

where $J_v$ is the Bessel function of first kind of order $v$ and $K_v$ is the modified Bessel function of second kind of order $v$.

This is a "**bonus question**", i.e., you can gain 3 extra points from this beyond the 10 points given in the previous questions. You can then use these 3 extra points to fill some points that you could have missed in the previous (or in the following) sheets.
In this task we want to establish the *Wick theorem*. The latter is a central result valid for Gaussian Lagrangians as $\mathcal{L}_0$ in Eq. (12). The Wick theorem allows to compute $n$-point correlation function $G_0^{(n)}(\vec{x}_1, \vec{x}_2 \ldots \vec{x}_n)$ in terms of the propagator $G_0(\vec{x}_1, \vec{x}_2)$. We verify this statement here for the case $n = 4$.
Prove that the four-point function $G_0^{(4)}(\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4)$ satisfies the following equation

$$G_0^{(4)}(\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4) = G_0(\vec{x}_1, \vec{x}_2)G_0(\vec{x}_3, \vec{x}_4) + G_0(\vec{x}_1, \vec{x}_3)G_0(\vec{x}_2, \vec{x}_4) + G_0(\vec{x}_1, \vec{x}_4)G_0(\vec{x}_2, \vec{x}_3). \tag{22}$$

(**3 bonus points**)