Problem 7.1

We consider the following expression for a symmetric positive definite $n \times n$ Matrix A and a N-dimensional vector B

$$F := \int_{-\infty}^{\infty} \prod_{i=1}^{N} \left(\frac{dx_i}{\sqrt{2\pi}} \right) \exp\left(-\frac{1}{2} \sum_{i,j}^{N} x_i A_{ij} x_j + \sum_{i=1}^{N} x_i B_i \right)$$

as A is symmetric we can write $A = S^T \underbrace{\mathrm{diag}(\lambda_1, \dots, \lambda_N)}_{=:D} S$ (S must not be misunderstood as a spin)

where S are orthogonal matrizies and $\lambda_1, \dots, \lambda_N$ are the Eigenvalues of A

we have terms

$$x_i S^T_{ik} \quad \text{and} \quad S_{lj} x_j = S^T_{jl} x_j$$

so we can

substitute $y_k = x_i S^T_{ik}$ as a variable transformation

$$\Rightarrow F = \underbrace{\frac{1}{|\det(S^T)|}}_{=1} \int_{-\infty}^{\infty} \prod_{i=1}^{N} \left(\frac{dy_i}{\sqrt{2\pi}} \right) \exp\left(-\frac{1}{2} \sum_{i,j} y_i D y_j + \sum_{i,j} y_i S_{ij} B_j \right)$$

$$= \int_{-\infty}^{\infty} \prod_{i=1}^{N} \left(\frac{dy_i}{\sqrt{2\pi}} \right) \exp\left(-\frac{1}{2} \sum_i \lambda_i y_i^2 + \sum_{i,j} y_i S_{ij} B_j \right)$$

$$= \int_{-\infty}^{\infty} \prod_{i=1}^{N} \left(\frac{dy_i}{\sqrt{2\pi}} \right) \exp\left(-\frac{1}{2} \sum_i \lambda_i \left(y_i - \sum_j \frac{1}{\lambda_i} S_{ij} B_j \right)^2 + \frac{1}{2} \sum_i \frac{1}{\lambda_i} \sum_{k,l} S_{ik} B_k S_{il} B_l \right)$$

as A is positive definite all λ_i are positive, and we can execute the gauss integral

$$\Rightarrow Z = \underbrace{\prod_{i=1}^{N} \frac{1}{\sqrt{\lambda_i}}}_{=\frac{1}{\sqrt{\det(A)}}} \exp\Big(\frac{1}{2} \sum_{i,k,l} B_k \underbrace{S_{ik}}_{=S^T_{ki}} \frac{1}{\lambda_i} S_{il} B_l \Big)$$

the term in the exponent be written differently again, when memorizing that

$$A^{-1} = S^T \operatorname{diag}\Big(\frac{1}{\lambda_1}, \dots, \frac{1}{\lambda_N}\Big) S$$

$$\hookrightarrow \sum_{i,k,l=1}^{N} B_k S^T_{ki} \frac{1}{\lambda_i} S_{il} B_l = B^T S^T D^{-1} B = B^T A^{-1} B$$

$$= \sum_{i,j} B_i (A^{-1})_{ij} B_j$$

$$\Rightarrow Z = \frac{1}{\sqrt{\det(A)}} \exp\Big(\frac{1}{2} \sum_{i,j=1}^{N} B_i (A^{-1})_{ij} B_j \Big)$$

b) for a Hamiltonian

$$H = -\frac{1}{2} \sum_{i,j=1}^{N} J'_{ij} S_i S_j - \sum_{i=1}^{N} h'_i S_i \quad \text{with } J'_{ij} = J'_{ji}$$

$$J_{ij} = \beta J'_{ij} \; ; \; h_i = \beta h'_i$$

we want to calculate the partition function

$$Z = \sum_{S_1=\pm 1} \cdots \sum_{S_N=\pm 1} \exp\Big(\frac{1}{2} \sum_{i,j} S_i J_{ij} S_j + \sum_i h_i S_i \Big)$$

$$= \sum_{S_1=\pm 1} \cdots \sum_{S_N=\pm 1} \frac{1}{\sqrt{\det(J)}} \int_{-\infty}^{\infty} \prod_{i=1}^{N} \Big(\frac{dx_i}{\sqrt{2\pi}}\Big) \exp\Big(-\frac{1}{2} \sum_{k,l} x_k (J^{-1})_{kl} x_l$$

$$+\sum_{k=1}^{N}(x_k+h_k)S_k\Big)$$

$$=\frac{1}{\sqrt{\det(J)}}\int_{-\infty}^{\infty}\prod_{i=1}^{N}\left(\frac{dx_i}{\sqrt{2\pi}}\right)\exp\Big(-\frac{1}{2}\sum_{k,l}x_k(J^{-1})_{kl}x_l\Big)\underbrace{\sum_{S_1=\pm1}\dots\sum_{S_N=\pm1}e^{\sum_{k=1}^{N}(x_k+h_k)S_k}}_{=\prod_{k=1}^{N}2\cosh(x_k+h_k)}$$

$$\frac{1}{\sqrt{\det(J)}}\int_{-\infty}^{\infty}\prod_{i=1}^{N}\left(\frac{2\,dx_i}{\sqrt{2\pi}}\right)\exp\Big(-\frac{1}{2}\sum_{k,l}x_k(J^{-1})_{kl}\,x_l+\sum_{k=1}^{N}\ln(\cosh(x_k+h_k))\Big)$$

$$=\frac{2^N}{\sqrt{(2\pi)^N\det(J)}}\int_{-\infty}^{\infty}\prod_{i=1}^{N}d\varphi_i\exp\Big(-\frac{1}{2}\sum_{k,l=1}^{N}(\varphi_k-h_k)K_{kl}(\varphi_l+h_l)$$

$$+\sum_{k=1}^{N}\ln(\cosh(\varphi_k))\Big)$$

$K_{kl}:=(J^{-1})_{kl}$

we now expand $S(\{\varphi_i\}):=\frac{1}{2}\sum_{i,j=1}^{N}(\varphi_i-h_i)K_{ij}(\varphi_j-h_j)-\sum_{i=1}^{N}\ln[\cosh\varphi_i]$

for small φ_i up to the order φ_i^4 for zero magnetic field $h_i=0$

Thus I have to expand $\ln(\cosh\varphi)=:y(\varphi)$

$y'(\varphi)=\tanh(\varphi)\quad y''(\varphi)=1-\tanh^2(\varphi)\quad y'''(\varphi)=-2\tanh(\varphi)+2\tanh^3(\varphi)$

$y^{IV}(\varphi)=-2\tanh^2(\varphi)-2+6\tanh^2(\varphi)-6\tanh^4(\varphi)$

$\rightsquigarrow\ \ln(\cosh\varphi)\approx\frac{\varphi^2}{2}-\frac{1}{12}\varphi^4$

$$\Rightarrow S(\{\varphi_i\})\underset{h_i=0,\,\varphi_i\text{ small}}{\approx}\frac{1}{2}\sum_{i,j=1}^{N}\varphi_iK_{ij}\varphi_j-\sum_{i=1}^{N}\left(\frac{\varphi_i^2}{2}-\frac{\varphi_i^4}{12}\right)$$

c) in the limit $V = Na^d \to \infty$, $a \to 0$

$$Z \propto \int_{-\infty}^{\infty} \lim_{N\to\infty} \prod_{i=1}^{N} dx_i \; e^{-\frac{1}{2} S(\{\varphi_i\})} = \int_{-\infty}^{\infty} D\varphi \, \exp(-S[\varphi])$$

as $\lim_{N\to\infty} S(\{\varphi_i\}) = \lim_{N\to\infty} \frac{1}{2} \frac{1}{a^2} \sum_{i=1}^{N} a^d \sum_{j=1}^{N} a^d \frac{\varphi_i}{a^{(d-2)/2}} \frac{K_{ij}}{a^{d-1}} \frac{\varphi_j}{a^{(d-2)/2}}$

$$- \lim_{N\to\infty} \sum_{i=1}^{N} a^d \left(\frac{1}{2a^2} \frac{\varphi_i^2}{a^{d-2}} - \frac{a^d}{12a^8} \frac{\varphi_i^4}{a^{3d-4}} \right)$$

$$= \frac{1}{2a^2} \int_{\mathbb{R}^d} d^d r \int_{\mathbb{R}^d} d^d r' \; \varphi(\vec{r}) \, K(\vec{r}, \vec{r}\,') \, \varphi(\vec{r}\,')$$

$$- \int_{\mathbb{R}^d} d^d r \left(\frac{\varphi^2(\vec{r})}{2a^2} - \underbrace{\frac{a^{d-4}}{12}}_{=:\mu} \varphi^4(\vec{r}) \right)$$

$K(\vec{r}, \vec{r}\,') = K(|\vec{r} - \vec{r}\,'|)$

I consider the term $\iint d^d r \, d^d r' \; \varphi(\vec{r}) \, K(\vec{r} - \vec{r}\,') \, \varphi(\vec{r}\,')$ and

expand this $\varphi(\vec{r}\,')$ up to the second order in $(\vec{r}\,' - \vec{r})$

$$\varphi(\vec{r}\,') = \varphi(\vec{r} + \vec{r}\,' - \vec{r}) = \varphi(\vec{r}) + \underbrace{(\vec{r}\,' - \vec{r}) \vec{\nabla} \varphi(\vec{r})}_{1} + \underbrace{\frac{1}{2} \sum_{i=1}^{d} (\vec{r}\,' - \vec{r})_i (\vec{r}\,' - \vec{r})_j \partial_i \partial_j \varphi(\vec{r})}_{2}$$

1 gives us

$$\iint_{\mathbb{R}^d} d^d r \, d^d r' \; \varphi(\vec{r}) \, \underset{K(\vec{r} - \vec{r}\,')}{\overset{\shortparallel}{K(\vec{r}\,' - \vec{r})}} \, (\vec{r}\,' - \vec{r}) \cdot \vec{\nabla} \varphi(\vec{r}) \underset{\text{substitution}}{=} \int_{\mathbb{R}^d} d^d r' \; \vec{r}\,' K(\vec{r}\,') \underbrace{\int_{\mathbb{R}^d} d^d r \; \underbrace{\varphi(\vec{r}) \vec{\nabla} \varphi(\vec{r})}_{= \frac{1}{2} \vec{\nabla} (\varphi(\vec{r}))^2}}_{= 0}$$

I assume here that the fields vanish if $|\vec{r}|$ goes to infinity.

(2):

$$\sum_{i,j} \int_{\mathbb{R}^d} d^d r \int_{\mathbb{R}^d} d^d r' \, (\vec{r}\,'-\vec{r})_i \, (\vec{r}\,'-\vec{r})_j \, K(\vec{r}\,'-\vec{r}) \, \phi(\vec{r}) \, \partial_i \partial_j \phi(\vec{r})$$

substitution $\vec{r}\,' \to \vec{r}\,' + \vec{r}$

$$= \sum_{i,j} \int_{\mathbb{R}^d} d^d r' \, r'_i r'_j \, K(|\vec{r}\,'|) \underbrace{\int_{\mathbb{R}^d} d^d r \, \phi(\vec{r}) \, \partial_i \partial_j \phi(\vec{r})}_{\hat{=}\, -\int_{\mathbb{R}^d} d^d r \, (\partial_i \phi(\vec{r}))\,(\partial_j (\phi(\vec{r}))}$$

assuming again that the fields ϕ vanish quickly enough, that also the derivatives of ϕ vanish

$$= -\sum_{i,j} \underbrace{\int_{\mathbb{R}^d} d^d r' \, r'_i r'_j \, K(|\vec{r}\,'|)}_{=:\gamma'} \int_{\mathbb{R}^d} d^d r \, \partial_i \phi(\vec{r}) \, \partial_j (\phi(\vec{r}))$$

examine γ':

$$\gamma' = \int_{\mathbb{R}^{d-2}} d^{d-2} r \int_{\mathbb{R}} dr_i \, r_i \int_{\mathbb{R}} dr_j \, r_j \, K(|\vec{r}|) = 0 \text{ for}$$

as r_j is an odd function and $K(|\vec{r}|)$ is an even function in r_j

γ is 0 for $i \neq j$

$$\Rightarrow (2) = \gamma \int_{\mathbb{R}^d} d^d r \sum_i (\partial_i \phi_i)^2 = \gamma \int_{\mathbb{R}^d} d^d r \, (\vec{\nabla} \phi)^2$$

with $\gamma := \frac{1}{d}\int_{\mathbb{R}^d} d^d r \; r^2 K(r)$ $\quad (r^2 = \sum_i r_i^2)$

(as $\int_{\mathbb{R}^d} d^d r \; r_i^2 K(r)$ is the same for all i)

and therefore

$$\int_{\mathbb{R}^d} d^d r \int_{\mathbb{R}^d} d^d r' \; \varphi(\vec{r})\, K(\vec{r}-\vec{r}')\, \varphi(\vec{r}') = \underbrace{\int_{\mathbb{R}^d} d^d r' \; K(r')}_{=:\alpha} \int_{\mathbb{R}^d} d^d r \; \varphi^2(\vec{r})$$

(Substitution)

$$+ \gamma \int_{\mathbb{R}^d} d^d r \, (\vec{\nabla}\varphi(\vec{r}))^2$$

$$\Rightarrow S[\varphi] = \int_{\mathbb{R}^d} d^d r \; \frac{1}{2} m^2 \varphi(\vec{r})^2 + \frac{1}{2} c \left(\vec{\nabla}\varphi(\vec{r})\right)^2 + \mu \, \varphi^4(\vec{r})$$

with $m = \frac{\alpha - 1}{a^2} = \frac{1}{a^2}\left[\int_{\mathbb{R}^d} d^d r \; K(r)\right] - \frac{1}{a^2}$

$\frac{1}{2} c = \frac{\gamma}{2a^2} = \frac{1}{2a^2 d}\int_{\mathbb{R}^d} d^d r \; r^2 K(r)$

$\mu = \frac{a^{d-4}}{12}$

if we compare this result to the Landau Theory of phase transitions and want to apply the method of steepest decent, we have to minimize $S[\varphi]$ which gives us the Euler-Lagrange equation in φ

$$c\,\Delta\varphi - m^2\varphi - 4\mu\varphi^3 = 0$$

Problem 2.2

We consider the same Lagrangian, but with $\mu = 0$

$$\mathcal{L}_0(\varphi) = \frac{1}{2} m^2 \varphi^2 + \frac{1}{2} (\vec{\nabla} \varphi)^2$$

a) in order to calculate the two point correlation function

$$G_0(\vec{x}_1, \vec{x}_2) = \langle \varphi(\vec{x}_1) \varphi(\vec{x}_2) \rangle$$

we introduce a source field $h(\vec{r})$, so that

$$Z \propto \int D\varphi \exp\big(\underbrace{-S_0[\varphi] + \int d^d r \, h(\vec{r}) \varphi(\vec{r})}_{=: S'} \big)$$

we write S' in the form

$$S'(\varphi) = -\frac{1}{2} \int d^d r \int d^d r' \, \varphi(\vec{r}) A(\vec{r}, \vec{r}') \varphi(\vec{r}') + \int d^d r \, h(\vec{r}) \varphi(\vec{r})$$

we choose $A(\vec{r}, \vec{r}') = \delta(\vec{r} - \vec{r}') \underbrace{[m^2 - \Delta']}_{D^{-1}}$

because this way it holds

$$\int d^d r \int d^d r' \, \varphi(\vec{r}) \delta(\vec{r} - \vec{r}') [m^2 - \Delta'] \varphi(\vec{r}') = \int d^d r \, m^2 \varphi(\vec{r})^2$$

$$- \int d^d r \, \varphi(\vec{r}) \Delta \varphi(\vec{r}) \underset{\text{Gauß}}{=} \int d^d r \, m^2 \varphi(\vec{r})^2 + (\vec{\nabla} \varphi(\vec{r}))^2$$

in order to use the identity given on the sheet we have to determine $G(\vec{r}, \vec{r}')$ so that

$$\int d^d r'' \, A(\vec{r}, \vec{r}'') \, G(\vec{r}'', \vec{r}') = \delta(\vec{r} - \vec{r}')$$

Therefore the inverse of D^{-1} is needed which is

$$D = \frac{1}{m^2} \sum_{k=0}^{\infty} \left(\frac{1}{m^2}\right)^k \Delta^k \qquad \left(= \frac{1}{m^2 - \Delta}\right)$$

check it first

$$D^{-1} D = (m^2 - \Delta) \frac{1}{m^2} \sum_{k=0}^{\infty} \left(\frac{1}{m^2}\right)^k \Delta^k = \frac{1}{m^2} \left[\sum_{k=0}^{\infty} \left(\frac{1}{m^2}\right)^{k-1} \Delta^k - \left(\frac{1}{m^2}\right)^k \Delta^{k+1} \right]$$

$$= \frac{1}{m^2} \left[\sum_{k=0}^{\infty} \left(\frac{1}{m^2}\right)^{k-1} \Delta^k - \sum_{k=1}^{\infty} \left(\frac{1}{m^2}\right)^{k-1} \Delta^k \right] = 1$$

so $G(\vec{x}, \vec{x}') = D(\delta(\vec{x} - \vec{x}'))$

$$\Rightarrow \int d^d y \, A(\vec{x}, \vec{y}) \, G(\vec{y}, \vec{x}') = \int d^d y \, \delta(\vec{x} - \vec{y}) \, \delta(\vec{y} - \vec{x}') = \delta(\vec{x} - \vec{x}')$$

Using this we achieve

$$Z \propto \exp\left(\frac{1}{2} \int d^d x \int d^d x' \, h(\vec{x}) \, g(\vec{x}, \vec{x}') \, h(\vec{x}')\right)$$

$$= \exp\left(\frac{1}{2} \int d^d x \int d^d x' \, h(\vec{x}) \, D(\delta(\vec{x} - \vec{x}')) \, h(\vec{x}')\right)$$

we now use that according to Gauß' theorem

$$\int d^d x \, f(\vec{x}) \, \Delta^k g(\vec{x}) = \int d^d x \, g(\vec{x}) \, \Delta f(\vec{x}) \qquad (7)$$

$$\Rightarrow Z \propto \exp\left(\frac{1}{2} \int d^d x \int d^d x' \, h(\vec{x}) \, \delta(\vec{x} - \vec{x}') \, D(h(\vec{x}'))\right)$$

$$= \exp\left(\frac{1}{2} \int d^d x \, h(\vec{x}) \, D(h(\vec{x}))\right)$$

we now have to generalize the functional derivative to functions which depend also on $F = \int d^d x \, f(\eta, \Delta\eta, \dots, \Delta^n \eta)$

with the same calculation as in the lecture and using again (1) we get

$$\frac{\delta F}{\delta \eta} = \sum_{q=0}^{n} \Delta^q \frac{\partial f}{\partial(\Delta^q \eta)}$$

writing $D = \frac{1}{m^2} + D_1 = \frac{1}{m^2} + \sum_{q=1}^{\infty} \left(\frac{1}{m^2}\right)^q \Delta^q$

we can compute

$$\frac{1}{Z}\frac{\delta Z}{\partial h(\vec{x}_1)} \qquad \frac{\delta}{\delta h(\vec{x}_1)} \int h \cdot D_1(h)\, d^dx = \frac{\delta}{\delta h(\vec{x}_1)} \int \sum_{q=1}^{\infty} h \left(\frac{1}{m^2}\right)^q \Delta^q h \, d^dx$$

$$= 2\, D_1(h(\vec{x}_1))$$

$$\frac{\delta}{\delta h(\vec{x}_1)} \int \frac{1}{m^2} h^2 \, d^dx = 2 \frac{1}{m^2} h(\vec{x}_1)$$

$$\Rightarrow \frac{\delta}{\delta h(\vec{x}_1)} \int h\, D(h)\, d^dx = 2\, D(h(\vec{x}_1))$$

$$\Rightarrow \frac{1}{Z}\frac{\delta Z}{\partial h(\vec{x}_1)} = D(h(\vec{x}_1)) = \int d^dx'\; h(\vec{x}')\, D(\delta(\vec{x}' - \vec{x}_1))$$

$$\Rightarrow G_0(\vec{x}_1, \vec{x}_2) = \frac{\delta}{\delta h(\vec{x}_2)} \left[\frac{1}{Z}\frac{\delta Z}{\delta h(\vec{x}_1)}\right] = D(\delta(\vec{x}_2 - \vec{x}_1))$$

(we would actually still get an additional term $+ \frac{1}{Z^2} \frac{\delta Z}{\partial h(\vec{x}_2)} \frac{\delta Z}{\delta h(\vec{x}_1)}$, which is 0 for $h=0$)

$$\Rightarrow D^{-1}\, G_0(\vec{x}_1, \vec{x}_2) = (m^2 - \Delta)\, G_0(\vec{x}_1, \vec{x}_2) = \delta(\vec{x}_1 - \vec{x}_2)$$

$$\rightsquigarrow (m^2 - \Delta)\, G_0(\vec{r}) = \delta(\vec{r})$$

b) Now this equation has to be solved. For this we execute a fourier transform

$$G_0(\vec{r}) = \frac{1}{(2\pi)^d} \int d^dq\; \hat{G}_0(\vec{q})\, e^{i\vec{q}\vec{r}} \quad \text{and} \quad \hat{\delta} = \int d\vec{r}\; \delta(\vec{r})\, e^{-i\vec{q}\vec{r}} = 1$$

$$\Rightarrow (m^2 - \Delta)\, G_0(\vec{r}) = (m^2 - \Delta_r) \frac{1}{(2\pi)^d} \int d^dq\; \hat{G}_0(\vec{q})\, e^{i\vec{q}\vec{r}} = \frac{1}{(2\pi)^d} \int d^dq\; \hat{G}_0(\vec{q})\, (m^2 + \vec{q}^2)\, e^{i\vec{q}\vec{r}}$$

$= \frac{1}{(2\pi)^d} \int d^d_q \; e^{-i\vec{q}\vec{r}}$

$\Rightarrow \hat{G}_0(\vec{q}) = \frac{1}{m^2+q^2} \Rightarrow G_0(\vec{r}) = \frac{1}{(2\pi)^d} \int d^d_q \; \frac{e^{i\vec{q}\vec{r}}}{m^2+q^2}$

c) n dimensional spheric coordinates have the form

$x_n = r\cos(\phi_{n-1})$

$x_{n-1} = r\sin(\phi_{n-1}) \cdots\cdots$

$x_{n-2} = r\sin(\phi_{n-1}) \cdots$

$\vdots$

$x_1 = r\sin(\phi_{n-1}) \cdots$

and $d^n x = r^{n-1} \sin(\phi_{n-1})^{n-2} \sin(\phi_{n-2})^{n-3} \cdots \sin\phi_2 \, dr \, d\phi_{n-1} \cdots d\phi_1$

and therefore if we aline $\vec{r}$ in the integral with the axis belonging to ϕ_{n-1} we get

$$G_0(\vec{r}) = \frac{1}{(2\pi)^d} \int_0^\infty dq \; q^{d-1} \int_0^\pi d\vartheta \sin^{d-2}\vartheta \; \Omega(d-1) \frac{e^{i qr\cos\vartheta}}{q^2+m^2}$$

$$= \frac{1}{(2\pi)^{\frac{d}{2}+\frac{1}{2}}} \frac{1}{\Gamma(\frac{d}{2}-\frac{1}{2})} \int_0^\infty d_q \frac{q^{d-1}}{m^2+q^2} \int_0^\pi d\vartheta \sin^{d-2}\vartheta \; e^{iqr\cos\vartheta}$$

$$= \frac{1}{(2\pi)^{\frac{d}{2}+\frac{1}{2}}} \frac{1}{\Gamma(\frac{d}{2}-\frac{1}{2})} \frac{\Gamma(\frac{d}{2}-1)\Gamma(\frac{1}{2})}{(\frac{r}{2})^{\frac{d}{2}-1}} \int_0^\infty d_q \underbrace{\frac{q^{d-1}}{q^{\frac{d}{2}-1}}}_{q^{d/2}} \frac{J_{\frac{d}{2}-1}(qr)}{m^2+q^2}$$

$$= \frac{1}{2^{d/2}\,\pi^{\frac{d}{2}+\frac{1}{2}}} \frac{1}{r^{\frac{d}{2}-1}} \Gamma\left(\frac{1}{2}\right) m^{\frac{d}{2}-1} K_{\frac{d}{2}-1}(mr) =: \gamma \frac{K_{\frac{d}{2}-1}(mr)}{r^{\frac{d}{2}-1}}$$

for the asymptotic behavior I got from Wikipedia

for large r

$$G_0(r) \propto \gamma \frac{1}{r^{\frac{d}{2}-1}} \sqrt{\frac{\pi}{2mr}}\, e^{-mr} \left(1 + \frac{4(\frac{d}{2}-1)^2-1}{8mr} + \frac{(4(\frac{d}{2}-1)^2-1)(4(\frac{d}{2}-1)^2-9)}{2(8mr)^2} + \dots\right)$$

$\to 0$ for $r \to \infty$

for small r for $d>2$

$$G_0(r) \propto \gamma \frac{1}{r^{\frac{d}{2}-1}} \frac{\Gamma(\frac{d}{2}-1)}{2} \left(\frac{2}{r}\right)^{\frac{d}{2}-1} \propto \frac{1}{r^{d-2}} \to \infty$$ for $r \to 0$

d) bonus question:

$$G_0(\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4) = \langle \varphi(\vec{x}_1)\, \varphi(\vec{x}_2)\, \varphi(\vec{x}_3)\, \varphi(\vec{x}_4) \rangle = \frac{1}{Z} \frac{\delta^4 Z}{\delta h(\vec{x}_1)\, \delta h(\vec{x}_2)\, \delta h(\vec{x}_3)\, \delta h(\vec{x}_4)}$$

$$\frac{\delta Z}{\delta h(\vec{x}_1)} = D(h(\vec{x}_1))\, \exp\left(\frac{1}{2}\int d^d r\, h(\vec{r})\, D(h(\vec{r}))\right)$$

$$\frac{\delta^2 Z}{\delta h(\vec{x}_2)\, \delta h(\vec{x}_1)} = D(\delta(\vec{x}_1 - \vec{x}_2))\, \exp[\dots] + D(h(\vec{x}_1))\, D(h(\vec{x}_2))\, \exp[\dots]$$

$$\frac{\delta^3 Z}{\delta h(\vec{x}_3)\, \delta h(\vec{x}_2)\, \delta h(\vec{x}_1)} = \Big\{ D(\delta(\vec{x}_1 - \vec{x}_2))\, D(h(\vec{x}_3)) + D(\delta(\vec{x}_1 - \vec{x}_3))\, D(h(\vec{x}_2))$$
$$+ D(h(\vec{x}_1))\, D(\delta(\vec{x}_2 - \vec{x}_3)) + D(h(\vec{x}_1))\, D(h(\vec{x}_2))$$
$$D(h(\vec{x}_3)) \Big\}\, \exp[\dots]$$

$$\frac{\delta^4 Z}{\delta h(\vec{x}_4)\,\delta h(\vec{x}_3)\,\delta h(\vec{x}_2)\,\delta h(\vec{x}_1)} = \Big\{ D(\delta(\vec{x}_1-\vec{x}_2))\, D(\delta(\vec{x}_3-\vec{x}_4))$$

$$+ D(\delta(\vec{x}_1-\vec{x}_3))\, D(\delta(\vec{x}_2-\vec{x}_4)) + D(\delta(\vec{x}_1-\vec{x}_4))\, D(\delta(\vec{x}_2-\vec{x}_3))\Big\}$$

$$\exp[\cdots] + \frac{\delta}{\delta h(\vec{x}_4)}\Big[D(h(\vec{x}_1))\, D(h(\vec{x}_2))\, D(h(\vec{x}_3))\, \exp[\cdots]\Big]$$

$$G_0(\vec{x}_1,\vec{x}_2,\vec{x}_3,\vec{x}_4) = \left[\frac{1}{Z}\,\frac{\delta^4 Z}{\delta h(\vec{x}_4)\cdots\delta h(\vec{x}_1)}\right]_{h=0} = D(\delta(\vec{x}_1-\vec{x}_2))\, D(\delta(\vec{x}_3-\vec{x}_4))$$

$$+ D(\delta(\vec{x}_1-\vec{x}_3))\, D(\delta(\vec{x}_2-\vec{x}_4)) + D(\delta(\vec{x}_1-\vec{x}_4))\, D(\delta(\vec{x}_2-\vec{x}_3))$$

$$= G_0(\vec{x}_1,\vec{x}_2)\, G_0(\vec{x}_3,\vec{x}_4) + G_0(\vec{x}_1,\vec{x}_3)\, G_0(\vec{x}_2,\vec{x}_4) + G_0(\vec{x}_1,\vec{x}_4)\, G_0(\vec{x}_2,\vec{x}_3)$$