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Tutorial Advanced Quantum Mechanics

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Problem Set 2

Problem 5 (Useful relations) (8 points)

Let A, B, and C be arbitrary operators on a Hilbert space. As usual, we define

$$\text{commutator} \quad [A,B] \equiv AB - BA,$$
$$\text{anti-commutator} \quad \{A,B\} \equiv AB + BA.$$

a. (1 point) Show the identities

$$[AB,C] = A[B,C] + [A,C]B \qquad \text{and} \qquad [AB,C] = A\{B,C\} - \{A,C\}B.$$

b. (2 points) Let A, B be operators which do not commute but which satisfy $[[A,B],B] = 0$. Show for an arbitrary $n \in \mathbb{N}$

$$[A,B^n] = nB^{n-1}[A,B].$$

Hint: Mathematical induction.

c. (2 points) Let $[A,B] = z\,\hat{1}$ with $z \in \mathbb{C}$, and let f be a function with a well-defined Taylor expansion. Show that

$$[A,f(B)] = z\,f'(B).$$

Let U be a unitary operator ($UU^\dagger = \hat{1}$).

d. (1 point) Show that

$$U[A,B]U^\dagger = [UAU^\dagger, UBU^\dagger].$$

e. (2 points) Let $f(A)$ be an operator-valued function, defined by a Taylor expansion. Show that

$$U\,f(A)U^\dagger = f(UAU^\dagger).$$

Problem 6 (Interaction picture) (2 points)

A quantum-mechanical system is described by a Hamilton operator $H = H_0 + V(t)$. An observable A in the Schrödinger picture becomes in the interaction picture

$$A_\mathrm{D}(t) \equiv U_0^\dagger(t)\,A(t)\,U_0(t),$$

with U_0 being the time-evolution operator of the "free" Hamiltonian H_0. Derive the quantum equation of motion for A_D.

Problem 7 (Time-dependent two-level system I) **(8 points)**

Consider a two-level system described by a Hamilton operator H_0

$$H_0 = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix}$$

with eigenvalues $E_1 < E_2$. The system is initially prepared in the eigenstate with energy E_1; from the time $t = 0$ onwards a periodic perturbation $V(t)$ is applied onto the system. In the basis of eigenstates of H_0 the perturbation has the matrix representation

$$V(t) = \begin{pmatrix} 0 & \gamma\, \mathrm{e}^{\mathrm{i}\omega t} \\ \gamma^*\, \mathrm{e}^{-\mathrm{i}\omega t} & 0 \end{pmatrix}.$$

a. **(2 points)** Give explicitly the perturbation V_D in the interaction picture.

b. **(3 points)** The state for $t > 0$ in the interaction picture can be written in matrix form as $\left(\begin{smallmatrix} c_1 \\ c_2 \end{smallmatrix}\right)$. Write down the time-dependent Schrödinger equation in the interaction picture and show that the coefficient c_2 satisfies the differential equation

$$\ddot{c}_2 = -\mathrm{i}\Omega\, \dot{c}_2 - \frac{|\gamma|^2}{\hbar^2}\, c_2 \qquad \text{with} \quad \Omega \equiv \omega - (E_2 - E_1)/\hbar.$$

c. **(3 points)** Show that the probability $w(t)$ of finding the system in the eigenstate with energy E_2 at a time $t > 0$ is given by

$$w(t) = \frac{|\gamma|^2}{(\hbar\omega_R)^2}\, \sin^2(\omega_R t) \qquad \text{where} \quad \omega_R = \sqrt{\frac{\Omega^2}{4} + \frac{|\gamma|^2}{\hbar^2}}.$$

Problem 8 (Time-dependent two-level system II) **(3 points)**

Consider again the system investigated in Problem 7. Calculate again the probability $w(t)$ of measuring the energy E_2 in first-order perturbation theory, and compare the result to the exact result of Problem 7.

Hint: Use the time-evolution operator in the interaction picture.