Please encircle the questions you have solved and are able to present/discuss in class.

3.1 (a)	3.1 (b)	3.1(c)	3.2(d)	3.2(e)	3.2(f)

Problem 3.1 (5 points)

Consider a system described by the following canonical partition function:

$$Z = \frac{1}{N!}\left(\frac{V - Nb}{N\lambda^3}\right)^N \exp\left(aN^2/k_B VT\right). \tag{1}$$

Here N is the number of particles, V is the volume, a, b are real constants, $\lambda = \sqrt{2\pi\hbar^2\beta/m}$ is the de Broglie thermal wavelength and k_B is the Boltzmann constant.

(a) Find the equation of state $P(T, v)$ where $v = V/N$ is the volume per particle.
Hint: To obtain the equation of state, first evaluate the free energy. (**1 point**)

(b) Compute the critical point $C = (P_c, v_c, T_c)$ of the liquid-gas phase transition, with $P_c = P(T_c, v_c)$. Plot the pressure $P(T; v)$ as a function of v for different fixed values of T. These curves are the so-called isothermal curves. How do the isothermal curves for $T < T_c$ differ from the ones for $T > T_c$?

Express the reduced pressure $\bar{P} = (P - P_c)/P_c$ in terms of the order parameter $\eta = v_c/v - 1$ and the reduced temperature $\tau = (T - T_c)/T_c$. (**2 points**)
Hint: The critical point $C = (P_c, v_c, T_c)$ is given by the inflection point of the equation of state $P(T, v)$ and it is therefore given by the solution of the following system of equations

$$\left.\frac{\partial P(T, v)}{\partial v}\right|_C = 0, \qquad \text{and} \qquad \left.\frac{\partial^2 P(T, v)}{\partial v^2}\right|_C = 0. \tag{2}$$

(c) Calculate the critical exponents: $(\alpha, \beta, \gamma, \delta)$. (**2 points**)
Hint: Remember the definitions of the critical exponents. For example, given the order parameter η, the critical exponent β is defined by $\eta \sim (T_c - T)^\beta$ and the exponent α is defined as

$$C \sim (T - T_c)^{-\alpha}. \tag{3}$$

In the previous equation C is the heat capacity at constant volume of the system, which can be found from Z in Eq. (1) as

$$C = \frac{\partial \langle E \rangle}{\partial T}, \tag{4}$$

with $\langle E \rangle$ the mean energy.
Hint: You may make use of the following relations among the critical exponents (which will be derived in the lecture at a later stage)

$$\begin{aligned} \alpha + 2\beta + \gamma &= 2, \\ \alpha + \beta\delta + \beta &= 2. \end{aligned} \tag{5}$$

Problem 3.2 (5 points)

Consider the *Dietrici equation of state*, which provides a phenomenological description of a certain class of real gases:

$$P(V - b) = k_B T \exp(-a/k_B TV), \tag{6}$$

where V is the volume, P is the pressure, T the temperature, a, b are phenomenological real parameters and k_B the Boltzmann constant.

(d) Evaluate the critical constants P_c, V_c , and T_c of the given system in terms of the parameters a and b, and show that the quantity $k_B T_c/P_c V_c = e^2/2 \approx 3.695$. (**2 points**)

(e) Demonstrate that, for all values of P and for $T \geq T_c$, the Dietrici equation yields to a unique value of V.
Hint: Study the sign of the function $(\frac{\partial P}{\partial V})$ for $T \geq T_c$. (**2 points**)

(f) Demonstrate that for $T < T_c$, there are three possible values of V, for a given value of P, and the critical volume V_c is always intermediate between the largest and the smallest of the three volumes.
Hint: Study the sign of the function $P(V, T)$, for fixed $T < T_c$ as a function of V, and find the intervals where the function itself is increasing with respect to V. (**1 point**)