Please encircle the questions you have solved and are able to present/discuss in class.

1.1 (a)	1.1 (b)	1.1(c)	1.2(a)	1.2(b)	1.2(c)

Problem 1.1 (5 points)

Following the **Transfer matrix method** outlined in the lectures:

(a) Find the matrix S which diagonalizes the transfer matrix T, i.e., find its eigenvalues and eigenvectors, for a $1d$ Ising model with periodic boundary conditions (the notation is analogous to the one used in the lectures):

$$E = -\epsilon \sum_{i=1}^{N} s_i s_{i+1} - H \sum_{i=1}^{N} s_i. \tag{1}$$

At the boundary one has $s_{N+1} = s_1$. (**1 point**)

Hint: To simplify the calculations, make use of the following substitution: $\cot(2\phi) = e^{2k} \sinh(h)$, where $k = \beta\epsilon$ and $h = \beta H$.
Hint: To simplify the calculations check that the two eigenvectors $\vec{v}_1$ and $\vec{v}_2$ of the transfer matrix T can be parametrized in terms of the angle ϕ as follows

$$\begin{aligned} T\vec{v}_1 &= \lambda_1 \vec{v}_1, \quad \vec{v}_1 = \begin{pmatrix} \cos(\phi) \\ \sin(\phi) \end{pmatrix}, \\ T\vec{v}_2 &= \lambda_2 \vec{v}_2, \quad \vec{v}_2 = \begin{pmatrix} -\sin(\phi) \\ \cos(\phi), \end{pmatrix} \end{aligned} \tag{2}$$

with $\lambda_1 > \lambda_2$.

(b) Evaluate the expectation value of a single spin variable $\langle s_i \rangle$ and show that, in the thermodynamic limit, $N \to \infty$, this is equal to

$$\langle s_i \rangle = \cos(2\phi). \tag{3}$$

(**2 points**)

(c) Evaluate in the thermodynamic limit, $N \to \infty$, the connected two-point correlation function $G(r)$

$$G(r) = \langle s_1 s_{1+r} \rangle - \langle s_1 \rangle \langle s_{1+r} \rangle , \tag{4}$$

and show that it is equal to

$$G(r) = \sin^2(2\phi) e^{-r/\xi}, \tag{5}$$

where ξ is the correlation length. Write the expression of ξ in the case of zero magnetic field $H = 0$. (**2 points**)

Problem 1.2 (5 points)

In this problem we consider again the $1d$ Ising model in Eq. (1), but we do not consider periodic boundary conditions.

(a) Calculate, by making use of the transfer matrix method and of the results of **Problem 1.1**, the partition function Z_f in the case of *free boundary conditions*. Here, the chain has no periodicity at the border. Perform the calculation via the following steps:
(i) Calculate the partition function Z_{++}, which refers to the case where the first and the last spin are fixed to 1, namely $s_1 = 1$ and $s_N = 1$.
(ii) Calculate the partition function Z_{+-}, which refers to the case where the first and the last spin are fixed $s_1 = 1$ and $s_N = -1$.
(iii) Calculate the partition function Z_{-+}, which refers to the case where the first and the last spin are fixed to $s_1 = -1$ and $s_N = 1$.
(iv) Calculate the partition function Z_{--}, which refers to the case where the first and the last spin are fixed to -1,

namely $s_1 = -1$ and $s_N = -1$.
(v) What is the relation between Z_f and Z_{++}, Z_{--}, Z_{+-} and Z_{--}?
Outline briefly the differences of the partition function Z_f with respect to the case of periodic boundary conditions analyzed in **Problem 1.1**. (**3 points**)

Hint: Note that, compared to the case with periodic boundaries, the transfer matrix here, for the first and the last spin, does not contain any term proportional to $s_1 s_N$.

(b) Calculate the free energy

$$F = -\frac{1}{\beta} \log Z. \tag{6}$$

for all the cases discussed in the previous task, i.e., for $Z = Z_f, Z_{++}, Z_{-+}, Z_{+-}, Z_{--}$. (**1 point**)

(c) Show that, when $N \to \infty$, the free energies F computed in the previous task are identical to the one obtained in the case of periodic boundary conditions apart from a constant term, which is independent of the system size N and which depends on the particular choice of the boundary conditions. (**1 point**)