EBERHARD KARLS UNIVERSITÄT TÜBINGEN

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Tutorial Advanced Quantum Mechanics

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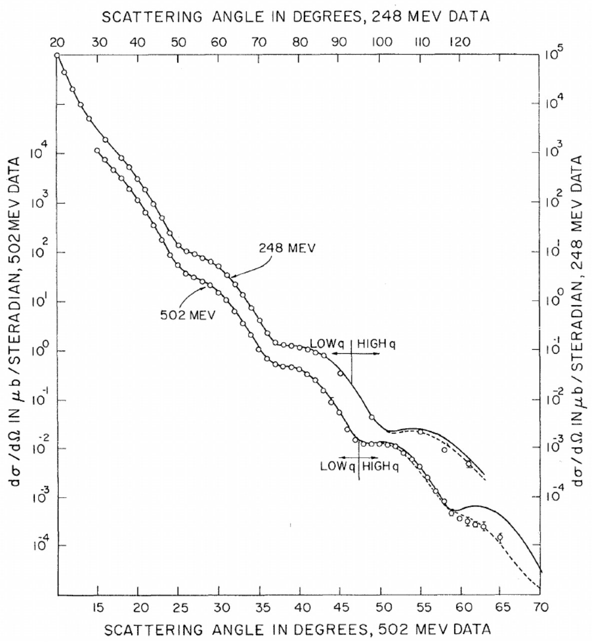
Problem Set 13

Problem 40 (Formfaktor) (7 points)

The picture [Phys. Rev. Lett. **23**, 1402 (1969)] shows the differential cross sections for elastic scattering of electrons on ^{208}Pb nuclei at energies 248 MeV and 502 MeV. Since the nucleus has spin 0 we can ignore complications arising from the spin coupling. Please pay attention to the different scales of the two data sets! In order to explain the oscillations we will investigate the effect of a finite nuclear size on the cross section.



Let the interaction of the electron with a *single* scattering centre be described by a potential $V_0(\mathbf{x})$. The target consists of *several* scattering centres, whose spatial distribution is described by a normalized density function $\rho(\mathbf{x})$.

a. (2 points) Write down an expression for the total potential $V_{\text{tot}}(\mathbf{x})$ made up of the single scattering potential $V_0(\mathbf{x})$ and the distribution $\rho(\mathbf{x})$. Show that the differential cross section in Born approximation is given by

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = F(\mathbf{q}) \left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} \right)_0$$

where $(\mathrm{d}\sigma / \mathrm{d}\Omega)_0$ is the differential cross section of the single scattering centre, $\mathbf{q} = \mathbf{k}' - \mathbf{k}$ is the familiar momentum transfer, and $F(\mathbf{q})$ is the so-called *form factor*.

[*Solution:* $F(\mathbf{q}) = \left| \int \mathrm{d}^3x \, \mathrm{e}^{-\mathrm{i}\mathbf{q}\cdot\mathbf{x}} \rho(\mathbf{x}) \right|^2$.]

b. (2 points) Calculate the form factor for the Coulomb scattering on a homogeneously charged sphere of radius R. [*Solution:* $F(\mathbf{q}) = 9\left[\sin(qR) - qR\cos(qR)\right]^2/(qR)^6$.]

c. (1 point) Show that the cross section should have minima at the angles θ_i satisfying

$$z_i = 2kR \sin\frac{\theta_i}{2},$$

where z_i are the solutions of the equation $\tan(z) = z$.

d. (2 points) Use the plotted data to estimate the radius R of a lead nucleus.

Hint: At the given energies the electrons are ultrarelativistic, i.e. $\hbar k \approx E/c$. The first few solutions of the trascendental equation $\tan(z) = z$ are $z_1 \simeq 4.49$; $z_2 \simeq 7.72$; $z_3 \simeq 10.9$; $z_4 \simeq 14.1$.

Problem 41 (Gamma matrices) (7 points)

a. (2 points) Show the relation $(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0$.

b. (2 points) Show for a four vector a^μ the identity $\not{a}^2 = a^2$.

c. (3 points) Consider the matrix $\gamma_5 := \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3$. Calculate $(\gamma_5)^2$, $\gamma_5^\dagger$, and $\{\gamma_5, \gamma^\mu\}$.

Problem 42 (Clifford algebra) (8 points)

Prove for the 16 matrices of the Clifford algebra

$$\Gamma_n \in \left\{\mathbb{1},\ \gamma^\mu,\ \gamma_5 \equiv \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3,\ \gamma_5\gamma^\mu,\ \sigma^{\mu\nu} \equiv \frac{\mathrm{i}}{2}[\gamma^\mu, \gamma^\nu]\right\}$$

the following relations:

a. (1 point) $(\Gamma_n)^2 = \pm\mathbb{1}$;

b. (1 point) for every matrix Γ_n with the exception of the identity $\mathbb{1}$ there exists a Γ_m that anti-commutes with Γ_n;

c. (2 points) every matrix Γ_n with the exception of the identity $\mathbb{1}$ is traceless;

d. (2 points) for each pair Γ_m, Γ_n with $m \neq n$ there exists a Γ_k such that $\Gamma_m\Gamma_n = \lambda\Gamma_k$, whereby $\lambda = \pm 1$ or $\pm\mathrm{i}$;

e. (2 points) the 16 matrices Γ_n are linearly independent.