EBERHARD KARLS
UNIVERSITÄT
TÜBINGEN

**Mathematisch-Naturwissenschaftliche Fakultät**
**Institut für Theoretische Physik**
Prof. Dr. Igor Lesanovsky
PD Dr. Davide Campagnari

# Tutorial Advanced Quantum Mechanics

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Problem Set 6

### Problem 18 (Charged particle on a ring) (8 points)

A particle with mass $m$ and electric charge $q$ is constrained to move on a ring with radius $R$. It is convenient to work with cylindrical coordinates $\{\rho, \varphi, z\}$ and to position the ring in the $xy$-plane centred at the origin. An infinitely long solenoid whose axis coincides with the $z$-axis generates a magnetic field $\mathbf{B} = B\,\hat{\mathbf{e}}_z$ inside, while the magnetic field vanishes outside. The solenoid's radius $a$ may be smaller or larger than the ring's radius $R$, i.e. the ring itself might be immersed in the magnetic field or not.

**a. (2 points)** Verify that the magnetic field can be described by the vector potential

$$\mathbf{A} = A(\rho)\,\hat{\mathbf{e}}_\varphi, \qquad A(\rho) = \frac{B}{2}\left[\rho\,\Theta(a-\rho) + \frac{a^2}{\rho}\,\Theta(\rho-a)\right].$$

**b. (3 points)** Show that—no matter whether $a > R$ or $a < R$—the Hamilton operator of the particle reads

$$H = \frac{\hbar^2}{2mR^2}\left(\mathrm{i}\frac{\partial}{\partial\varphi} + \frac{q\Phi[B]}{2\pi\hbar}\right)^2$$

with $\Phi[B]$ being the magnetic flux through the ring's surface.

**c. (3 points)** Solve the time-independent Schrödinger equation and give the energy eigenvalues and eigenfuntions. What is the ground state energy? What is the expectation value of the momentum in the ground state?

### Problem 19 (Radiation field and the field operator) (13 points)

After quantization the vector potential $\mathbf{A}$ becomes an operator, whose mode expansion reads

$$\begin{aligned}\mathbf{A}(\mathbf{x},t) &= \frac{1}{L^{3/2}}\sum_{\mathbf{k}} \mathbf{A}(\mathbf{k},t)\,\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}},\\ \mathbf{A}(\mathbf{k},t) &= \sqrt{\frac{\hbar}{2\varepsilon_0\omega_{\mathbf{k}}}}\sum_{\lambda}\Big(a_\lambda(\mathbf{k})\,\boldsymbol{\varepsilon}_\lambda(\mathbf{k})\,\mathrm{e}^{-\mathrm{i}\omega_{\mathbf{k}}t} + a_\lambda^\dagger(-\mathbf{k})\,\boldsymbol{\varepsilon}_\lambda^*(-\mathbf{k})\,\mathrm{e}^{\mathrm{i}\omega_{\mathbf{k}}t}\Big).\end{aligned} \tag{19.1}$$

Here, the dispersion relation $\omega_{\mathbf{k}} = c|\mathbf{k}|$ holds, and the ladder operators of the modes of the radiation field $a_\lambda(\mathbf{k})$ and $a_\lambda^\dagger(\mathbf{k})$ satisfy the canonical commutation relations

$$\big[a_\lambda(\mathbf{k}), a_{\lambda'}(\mathbf{k}')\big] = 0 = \big[a_\lambda^\dagger(\mathbf{k}), a_{\lambda'}^\dagger(\mathbf{k}')\big] \qquad \text{and} \qquad \big[a_\lambda(\mathbf{k}), a_{\lambda'}^\dagger(\mathbf{k}')\big] = \delta_{\mathbf{k},\mathbf{k}'}\delta_{\lambda,\lambda'}. \tag{19.2}$$

The polarization vectors are orthonormal and transverse to the propagation direction

$$\boldsymbol{\varepsilon}^*_\lambda(\mathbf{k}) \cdot \boldsymbol{\varepsilon}_{\lambda'}(\mathbf{k}) = \delta_{\lambda,\lambda'}, \qquad \boldsymbol{\varepsilon}_\lambda(\mathbf{k}) \cdot \mathbf{k} = 0.$$

The electric and magnetic fields are given as usual by

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t}, \qquad \mathbf{B} = \nabla \times \mathbf{A}.$$

**a. (3 points)** Show the identities

$$\int_{L^3} \mathrm{d}^3x\, \mathbf{E}^2(\mathbf{x},t) = \sum_{\mathbf{k}} \dot{\mathbf{A}}(\mathbf{k},t)\, \dot{\mathbf{A}}(-\mathbf{k},t), \qquad \int_{L^3} \mathrm{d}^3x\, \mathbf{B}^2(\mathbf{x},t) = \sum_{\mathbf{k}} \mathbf{k}^2\, \mathbf{A}(\mathbf{k},t) \cdot \mathbf{A}(-\mathbf{k},t),$$

$$\int_{L^3} \mathrm{d}^3x\, \mathbf{E}(\mathbf{x},t) \times \mathbf{B}(\mathbf{x},t) = -\mathrm{i} \sum_{\mathbf{k}} \mathbf{k} \Big( \dot{\mathbf{A}}(\mathbf{k},t) \cdot \mathbf{A}(-\mathbf{k},t) \Big).$$

**b. (4 points)** Show that the Hamiltonian of the radiation field is given by

$$H_{\text{rad}} \equiv \frac{\varepsilon_0}{2} \int_{L^3} \mathrm{d}^3x \left( \mathbf{E}^2 + c^2 \mathbf{B}^2 \right) = \sum_{\mathbf{k},\lambda} \hbar\omega_{\mathbf{k}} \left( a^\dagger_\lambda(\mathbf{k}) a_\lambda(\mathbf{k}) + \frac{1}{2} \right).$$

**c. (4 points)** Show that the momentum based on the Poynting vector is given by

$$\mathbf{P}_{\text{rad}} \equiv \varepsilon_0 \int_{L^3} \mathrm{d}^3x\, \mathbf{E} \times \mathbf{B} = \sum_{\mathbf{k},\lambda} \hbar\, \mathbf{k}\, a^\dagger_\lambda(\mathbf{k}) a_\lambda(\mathbf{k}).$$

**d. (2 points)** Calculate the expectation values of $\mathbf{E}$, $\mathbf{B}$, $H_{\text{rad}}$ and $\mathbf{P}_{\text{rad}}$ for a given occupation number state.

## Problem 20 (Selection rules for dipole transitions) (7 points)

Let us consider the hydrogen atom, ignoring the spins of the electron and of the proton. In order to have a non-zero transition probability from an initial energy state $|i\rangle \equiv |n,\ell,m\rangle$ to a final one $|f\rangle \equiv |n',\ell',m'\rangle$, the matrix element of the dipole moment operator $\mathbf{D}_{fi} \equiv e\langle f|\mathbf{x}|i\rangle$ must be non-zero. The conditions to have non-zero transition probabilities, called *selection rules*, have been discussed in the lecture. Their derivation is based upon the operator identity

$$\left[\mathbf{L}^2, \left[\mathbf{L}^2, x_i\right]\right] = 2\hbar^2 \left(x_i\, \mathbf{L}^2 + \mathbf{L}^2 x_i\right),$$

which we are now going to prove.

**a. (1 point)** Calculate the commutator $\left[L_i, x_j\right]$ of the angular momentum with the position operator.

**b. (2 points)** In a first step show

$$\left[\mathbf{L}^2, x_i\right] = 2\hbar^2 x_i + 2\mathrm{i}\hbar\, \varepsilon_{ijk}\, x_j\, L_k. \tag{20.1}$$

**c. (2 points)** Now we have to take the commutator of Eq. (20.1) with $\mathbf{L}^2$,

$$\left[\mathbf{L}^2, \left[\mathbf{L}^2, x_i\right]\right] = 2\hbar^2 \left[\mathbf{L}^2, x_i\right] + 2\mathrm{i}\hbar\, \varepsilon_{ijk} \left[\mathbf{L}^2, x_j\, L_k\right]. \tag{20.2}$$

Use now Eq. (20.1) again *only* for the second term on the right-hand side of Eq. (20.2), and show

$$\left[\mathbf{L}^2, \left[\mathbf{L}^2, x_i\right]\right] = 2\hbar^2 \left(x_i\, \mathbf{L}^2 + \mathbf{L}^2 x_i\right) + 4\hbar^2 x_j (\mathrm{i}\hbar\, \varepsilon_{ijk}\, L_k - L_i\, L_j). \tag{20.3}$$

**d. (2 points)** Prove that the last term on the right-hand side of Eq. (20.3) vanishes.