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Advanced Statistical Physics
Problem Class 9
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Please encircle the questions you have solved and are able to present/discuss in class.

9.1(a)	9.1(b)	9.1(c)	9.1(d)	9.2(a)	9.2(b)	9.2(c)	9.2(d)

Problem 9.1: Fokker-Planck equation and equilibrium distribution (5 points)

In this exercise we want to study *time-dependent fluctuations* over an equilibrium configuration. The equilibrium configuration of the system is described in terms of the usual φ^4 action $S[\varphi]$

$$S[\varphi] = \int_{\mathbb{R}^d} \mathrm{d}\vec{r}\,\mathcal{L}(\varphi(\vec{r})), \quad \text{with} \quad \mathcal{L}(\varphi) = r(t_r)\varphi^2 + \frac{1}{2}\gamma\,(\nabla\varphi)^2 + \frac{1}{2}b\varphi^4. \tag{1}$$

In the previous equation, we have denoted with $r(t_r) = r_0 t_r$, where $t_r = (T - T_c)/T_c$ is the reduced temperature. We denote it with t_r in this exercise to distinguish it from the time variable, which we shall denote as t in the following. The equilibrium configuration $\bar{\varphi}(\vec{r})$ of the field $\varphi(\vec{r})$ is given by the equation

$$\left.\frac{\delta S[\varphi]}{\delta\varphi(\vec{r})}\right|_{\bar{\varphi}(\vec{r})} = 0. \tag{2}$$

If the system is not far from equilibrium one can reasonably expect that the rate at which the system relaxes back to the equilibrium configuration $\bar{\varphi}(\vec{r})$ is proportional to the deviation from equilibrium. From this phenomenological assumption one can write the following equation for the rate of change in time t of the time-dependent field $\varphi(\vec{r}, t)$:

$$\frac{\partial\varphi(\vec{r}, t)}{\partial t} = -\Gamma\frac{\delta S[\varphi]}{\delta\varphi(\vec{r}, t)} + \zeta(\vec{r}, t). \tag{3}$$

In the previous equation, $\zeta(\vec{r}, t)$ is a noise term which accounts for the fact that thermal fluctuations will sometime cause the system to move further away from equilibrium during its time evolution. We assume that the noise $\zeta(\vec{r}, t)$ is a Gaussian random function chosen from the functional Gaussian distribution $P_\zeta[\zeta(\vec{r}, t)]$:

$$P_\zeta[\zeta(\vec{r}, t)] \propto \exp\left[-\frac{1}{2D}\int_{\mathbb{R}}\mathrm{d}t\int_{\mathbb{R}^d}\mathrm{d}\vec{r}\zeta^2(\vec{r}, t)\right], \tag{4}$$

with the variance of the distribution given by D

$$\langle\zeta(\vec{r}, t)\rangle_\zeta = 0; \quad \langle\zeta(\vec{r}, t)\zeta(\vec{r}', t')\rangle_\zeta = D\delta(\vec{r} - \vec{r}')\delta(t - t'). \tag{5}$$

In the previous equation we have denoted with $\langle\ldots\rangle_\zeta$ the average over the probability distribution P_ζ in Eq. (4). Equation (3) is usally named Langevin equation. The probability $P[\varphi(\vec{r}), t]$ of finding a field configuration $\varphi(\vec{r})$ at time t is given by

$$P[\varphi(\vec{r}), t] = \langle\delta[\varphi(\vec{r}) - \bar{\varphi}(\vec{r}, t, \{\zeta\})]\rangle_\zeta, \tag{6}$$

where $\bar{\varphi}(\vec{r}, t, \{\zeta\})$ is a solution of the Langevin equation (3) for a fixed realization $\zeta(\vec{r}, t)$ of the noise.
In the first exercise we want to derive the Fokker-Planck equation starting from Eq. (6) using the Langevin equation (3).

(a) Using the Langevin equation (3) show that the time derivative of $P[\varphi(\vec{r}), t]$ in Eq. (6) can be written as

$$\frac{\partial P[\varphi(\vec{r}), t]}{\partial t} = \int_{\mathbb{R}^d}\mathrm{d}\vec{r}'\frac{\delta}{\delta\varphi(\vec{r}', t)}\left[\Gamma P[\varphi(\vec{r}), t]\frac{\delta S}{\delta\varphi(\vec{r}', t)} - \langle\zeta(\vec{r}', t)\delta[\varphi(\vec{r}) - \bar{\varphi}(\vec{r}, t, \{\zeta\})]\rangle_\zeta\right]. \tag{7}$$

(**1 point**)

(b) Prove for the Gaussian distribution in Eqs. (4) and (5) that the following relation holds

$$\langle F[\zeta]\zeta\rangle_\zeta = D\left\langle\frac{\delta F}{\delta\zeta}\right\rangle_\zeta, \tag{8}$$

for an arbitrary functional $F[\zeta]$ of the noise ζ. Use the identity in Eq. (8) to simplify the term $\langle\zeta(\vec{r}', t)\delta[\varphi(\vec{r}) - \bar{\varphi}(\vec{r}, t, \{\zeta\})]\rangle_\zeta$ in Eq. (7). You should get the following result

$$\langle\zeta(\vec{r}', t)\delta[\varphi(\vec{r}) - \bar{\varphi}(\vec{r}, t, \{\zeta\})]\rangle_\zeta = -D\int_{\mathbb{R}^d}\mathrm{d}\vec{r}''\frac{\delta}{\delta\varphi(\vec{r}'', t)}\left\langle\frac{\delta\bar{\varphi}(\vec{r}'', t)}{\delta\zeta(\vec{r}', t)}\delta[\varphi(\vec{r}) - \bar{\varphi}(\vec{r}, t, \{\zeta\})]\right\rangle_\zeta. \tag{9}$$

(**1 point**)

(c) We want here to simplify the quantity averaged over the noise ζ inside the integral in Eq. (9). To do this, write the Langevin equation in the integral form and verify that

$$\frac{\delta\bar{\varphi}(\vec{r}'', t)}{\delta\zeta(\vec{r}', t)} = \frac{1}{2}\delta(\vec{r}' - \vec{r}''). \tag{10}$$

(**1 point**)
Hint: You can write the Langevin equation in its integral form by formally integrating the left and the right hand side of Eq. (3). Then, because of causality, one has that $\bar{\varphi}(\vec{r}, t)$ only depends on $\zeta(\vec{r}, t')$ for $t > t'$. This causes the appearance of the Heaviside theta function $\Theta(t'' - t')$. In this exercise we regularize $\Theta(0) = 1/2$, which leads to the factor $1/2$ in Eq. (10).

(d) Write the Fokker-Planck equation upon inserting the results obtained in Eqs. (9) and (10) into Eq. (7). You should obtain

$$\frac{\partial P[\varphi(\vec{r}), t]}{\partial t} = \int_{\mathbb{R}^d} \mathrm{d}\vec{r}' \, \frac{\delta}{\delta\varphi(\vec{r}', t)} \left[\Gamma P[\varphi(\vec{r}), t] \frac{\delta S}{\delta\varphi(\vec{r}', t)} + \frac{D}{2} \frac{\delta P[\varphi(\vec{r}), t]}{\delta\varphi(\vec{r}', t)}\right]. \tag{11}$$

Discuss what is the solution of the Fokker-Planck equation at long times $t \to \infty$, where the system relaxes to thermal equilibrium. What is the relation between D and Γ? Discuss and interpret the result physically. (**2 points**)

Problem 9.2: Dynamic scaling hypothesis and relaxation to the equilibrium

(5 points)

In this exercise we consider the case where $T > T_c$ and therefore the equilibrium configuration from Eq. (2), in the absence of noise, is

$$\bar{\varphi}(\vec{r}) \equiv 0. \tag{12}$$

Time dependent fluctuations on top of the solution (12) are generated by the noise term of the Langevin equation.

(a) Linearize the Langeving equation (3) by expanding up to linear order in $\delta\varphi(\vec{r}, t) = \varphi(\vec{r}, t) - \bar{\varphi}(\vec{r})$. You should get the following equation for $\delta\varphi(\vec{r}, t)$:

$$\frac{\partial \delta\varphi(\vec{r}, t)}{\partial t} = -\left[\frac{\delta\varphi(\vec{r}, t)}{\tau_0} - \gamma\Gamma\nabla^2\delta\varphi(\vec{r}, t)\right] + \zeta(\vec{r}, t). \tag{13}$$

Identify the *relaxation time* τ_0. Take the Fourier transform of Eq. (13). You should obtain

$$\frac{\partial \delta\varphi(\vec{k}, t)}{\partial t} = -\left[\frac{\delta\varphi(\vec{k}, t)}{\tau(k)}\right] + \zeta(\vec{k}, t). \tag{14}$$

Identify the relaxation time $\tau(k)$ in momentum space. (**1 point**)
Hint: Be reminded that the definition $\hat{f}(\vec{k})$ of the Fourier transform of an arbitrary function of space $f(\vec{r})$ is

$$\hat{f}(\vec{k}) = \int_{\mathbb{R}^d} \mathrm{d}\vec{r} \, f(\vec{r}) \exp(-i\vec{k}\cdot\vec{r}), \quad \text{with inverse} \quad f(\vec{r}) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \mathrm{d}\vec{k} \, \hat{f}(\vec{k}) \exp(i\vec{k}\cdot\vec{r}). \tag{15}$$

(b) We study in this point the response function $\hat{\chi}(\vec{k}, \omega)$ in Fourier space both with respect to the space and to the time coordinate:

$$\hat{\chi}(\vec{k}, \omega) = \left.\frac{\langle\delta\varphi(\vec{k}, \omega)\rangle_\zeta}{\delta\hat{h}(\vec{k}, \omega)}\right|_{h=0}. \tag{16}$$

In the previous equation $\delta\hat{h}(\vec{k}, \omega)$ is the Fourier transform of a small space and time dependent magnetic field $\delta h(\vec{r}, t)$ added to the right hand side of Eq. (3). The response function is evaluated in the linear response regime where δh is small. Compute $\hat{\chi}(\vec{k}, \omega)$ starting from the linearized Langevin equation in Eq. (13). You should obtain

$$\hat{\chi}(\vec{k}, \omega) = \frac{1}{\tau(k)^{-1} - i\omega}. \tag{17}$$

(**1 point**)
Hint: Be reminded of the definition of the Fourier transform $\hat{f}(\vec{k}, \omega)$ of an arbitrary function $f(\vec{r}, t)$ of space : and time

$$f(\vec{r}, t) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \mathrm{d}\vec{k} \, \frac{1}{2\pi} \int_{\mathbb{R}} \mathrm{d}\omega \, \hat{f}(\vec{k}, \omega) \exp(i(\vec{k}\cdot\vec{r} + \omega t)). \tag{18}$$

(c) Here we discuss the generalization of the scaling hypothesis discussed in the lectures to the non-equilibrium case analyzed here. We write the following scaling form for the relaxation time $\tau(k)$ defined after Eq. (14):

$$\tau(k) = t_r^{-y} F_\tau(\vec{k}\,\xi(t_r)), \tag{19}$$

where $F_\tau(x)$ is a scaling function and t_r is the reduced temperature defined after Eq. (1). Determine the relation between the exponents y, z and ν that the scaling equation (19) enforces. Compute the exponents y and z in the case of the linearized Langevin equation in Eq. (13).
Hint: Be reminded that ν is the critical exponent describing how the correlation length ξ diverges at the critical temperature T_c:

$$\xi(t_r) \propto t_r^{-\nu}. \tag{20}$$

Hint: The exponents y and z are defined by the following relations

$$\tau(k) \propto \begin{cases} t_r^{-y}, & \text{for} \quad \vec{k} = 0, \\ (k^z)^{-1}, & \text{at} \quad T = T_c \quad \text{and} \quad \vec{k} \neq 0. \end{cases} \tag{21}$$

(**2 points**)

(d) Here we discuss the dynamical scaling hypothesis for the response function $\hat{\chi}(\vec{k}, \omega)$. We write the following scaling form for $\hat{\chi}(\vec{k}, \omega)$:

$$\hat{\chi}(\vec{k}, \omega) = t_r^{-\gamma} F_\chi(\vec{k}\xi(t_r), \omega\tau_0), \tag{22}$$

with γ the critical exponent of the magnetic susceptibility at thermal equilibrium. Compute $\hat{\chi}(0, \omega)$ when $\vec{k} = 0$ and $T \to T_c$. Furthermore, prove that the ratio

$$\delta = \arctan\left(\frac{\mathrm{Im}\hat{\chi}(0, \omega)}{\mathrm{Re}\hat{\chi}(0, \omega)}\right), \tag{23}$$

takes a universal expression determined solely by the critical exponents z, ν and γ. (**1 point**).

Problem 9.3: Scaling hypothesis at thermal equilibrium (3 bonus points)

This is a "**bonus exercise**", i.e., you can gain 2 extra points from this beyond the 10 points given in the previous exercises. You can then use these 2 extra points to fill some points that you could have missed in the previous (or in the following) sheets.
Consider a magnetic system described by the equation of state

$$H = M^\delta F\left(\frac{t}{M^{1/\beta}}\right), \tag{24}$$

where H is a magnetic field, M is the magnetization density and F is a scaling function.

(a) Prove that δ and β in Eq. (24) are the critical exponents as defined in the lecture script. (**1 bonus point**)

(b) Prove that the scaling relation

$$\gamma = \beta(\delta - 1), \tag{25}$$

starting from Eq. (24). (**2 bonus points**)