EBERHARD KARLS UNIVERSITÄT TÜBINGEN

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Tutorial Advanced Quantum Mechanics

Winter semester 2021/2022 Tübingen, 13th December 2021

Problem Set 9

In this sheet you will need the following relations:

$$e^X Y e^{-X} = Y + [X, Y] + \frac{1}{2}[X, [X, Y]] + \frac{1}{3!}[X, [X, [X, Y]]] + \dots$$

$$e^A e^B = e^{A+B} e^{\frac{1}{2}[A,B]} \quad \text{if } [A, [A, B]] = 0 = [B, [A, B]].$$

Problem 28 (Functions of the fermionic occupation number operator) (3 points)

Let b and $b^\dagger$ be fermionic ladder operators satisfying the usual anti-commutation relations

$$\{b, b\} = 0 = \{b^\dagger, b^\dagger\}, \qquad \{b, b^\dagger\} = 1.$$

Show that any analytic function $f(\hat{N})$ of the occupation number operator $\hat{N} = b^\dagger b$ can be written as

$$f(\hat{N}) = \alpha + \beta \hat{N}$$

and give the numbers α and β explicitly.

Problem 29 (Conservation of particle number) (3 points)

The Hamilton operator of a system of interacting fermions is

$$H = h_{ij} b_i^\dagger b_j + \frac{1}{4} V_{ij,km} b_i^\dagger b_j^\dagger b_k b_m,$$

where h and V are constant matrices (summation over repeated indices is implied). Show that the particle number operator

$$\hat{N} = b_\ell^\dagger b_\ell$$

commutes with the Hamiltonian.
Hint: Problem 5a.

Problem 30 (Coherent states) (9 points)

A *coherent state* $|\zeta\rangle$ is defined as eigenstate of the annihilation operator a of the harmonic oscillator with eigenvalue $\zeta \in \mathbb{C}$.

a. **(2 points)** Write the coherent state $|\zeta\rangle$ as linear combination of the oscillator eigenstates $|n\rangle$. Use the eigenvalue equation to find a recursion relation for the expansion coefficients, and show

$$|\zeta\rangle \propto \mathrm{e}^{a^\dagger \zeta}|0\rangle,$$

where $|0\rangle$ is the oscillator's ground state.

b. **(3 points)** Normalize the coherent state found in part **a.** and show that it can be written as

$$|\zeta\rangle = \mathrm{e}^{a^\dagger \zeta - \zeta^* a}|0\rangle.$$

c. **(2 points)** Calculate the standard deviations $(\Delta x)^2$ and $(\Delta p)^2$ in an arbitrary coherent state $|\zeta\rangle$.

d. **(2 points)** A harmonic oscillator is prepared in a coherent state $|\zeta\rangle$ at the time $t = 0$. Calculate the state of the system at a later time $t > 0$ and show that it is still a coherent state.

Problem 31 (Interacting system) (12 points)

Consider a physical system with one bosonic and one fermionic degree of freedom, whose dynamics is governed by the Hamilton operator

$$H = \varepsilon b^\dagger b + \omega a^\dagger a + g\, b^\dagger b(a^\dagger + a).$$

Here, b and $b^\dagger$ are the fermionic ladder operators (satisfying anti-commutation relations); a and $a^\dagger$ are the bosonic ones (satisfying commutation relations); ε, ω, and g are positive constants. The Hilbert space is spanned by the product states

$$|n_\mathrm{f}\rangle^{(\mathrm{F})} \otimes |n_\mathrm{b}\rangle^{(\mathrm{B})}, \qquad n_\mathrm{f} \in \{0, 1\}, \quad n_\mathrm{b} \in \mathbb{N}.$$

a. **(3 points)** Show that the Hamilton operator can be cast into the form

$$H = \alpha\, b^\dagger b + \beta\, A^\dagger A$$

where α and β are constants depending on ε, ω and g, and A, $A^\dagger$ are bosonic ladder operators satisfying $[A, A^\dagger] = \hat{1}$. [*Solution:* $A = a + g\, b^\dagger b/\omega$.]

b. **(4 points)** Give the spectrum of the Hamiltonian, and show that the energy eigenstates are

$$|k; n\rangle = \frac{1}{\sqrt{n!}}\left(a^\dagger + \frac{g}{\omega}k\right)^n \mathrm{e}^{-\frac{g}{\omega}k(a^\dagger - a)}|k\rangle^{(\mathrm{F})} \otimes |0\rangle^{(\mathrm{B})}.$$

Hint: Since the operators $A^\dagger A$ and $b^\dagger b$ commute, they can be simultaneously diagonalized. Let $|\Omega(k)\rangle$ be the ground state of the 'A-oscillator' for a given fermionic occupation number k, i.e. the state satisfying

$$b^\dagger b|\Omega(k)\rangle = k|\Omega(k)\rangle, \qquad A|\Omega(k)\rangle = 0.$$

Determine $|\Omega(k)\rangle$ and construct the excited states on top of that.

c. **(3 points)** Show that the energy eigenstates from part **b.** can be written as

$$|k; n\rangle = \mathrm{e}^{-\frac{g}{\omega}k(a^\dagger - a)}|k\rangle^{(\mathrm{F})} \otimes |n\rangle^{(\mathrm{B})}.$$

d. **(2 points)** What is the ground state of the system?