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# Tutorial Advanced Quantum Mechanics

Winter semester 2021/2022 — Tübingen, 2nd November 2021

## Problem Set 3

### Problem 9 (Harmonic oscillator) (6 points)

The Hamilton operator of the one-dimensional harmonic oscillator reads

$$H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2 = \hbar\omega\left(a^\dagger a + \tfrac{1}{2}\right)$$

with

$$x = \frac{x_0}{\sqrt{2}}(a^\dagger + a), \qquad p = \frac{\mathrm{i}\hbar}{x_0\sqrt{2}}(a^\dagger - a), \qquad x_0 = \sqrt{\frac{\hbar}{m\omega}}. \tag{9.1}$$

The ladder operators satisfy the commutation relation $[a, a^\dagger] = \hat{1}$. The eigenstates and the corresponding eigenenergies are given by

$$|n\rangle = \frac{1}{\sqrt{n!}}(a^\dagger)^n|0\rangle, \qquad E_n = \hbar\omega(n + \tfrac{1}{2}),$$

whilst the ground state $|0\rangle$ obeys $a|0\rangle = 0$.

**a.** **(3 points)** Evaluate the uncertainties $(\Delta x)^2$ and $(\Delta p)^2$ in an arbitrary oscillator eigenstate $|n\rangle$. For which $n$ is the product of uncertainties minimal?

**b.** **(3 points)** At $t = 0$ the oscillator is in the state

$$|\psi(0)\rangle = C\big(2|0\rangle + |1\rangle\big).$$

Determine the normalization constant $C$ such that $\langle\psi(0)|\psi(0)\rangle = 1$. Calculate the expectation values of the ladder operators $a$ and $a^\dagger$ as a function of time and use this result to show that the position and momentum operators obey the Ehrenfest theorem.

### Problem 10 (Transition probability) (14 points)

A one-dimensional harmonic oscillator is subject to a perturbation restricted to a finite time interval

$$V(x,t) = e\,\mathcal{E}\,x\left[1 - (t/\tau)^2\right]\Theta(\tau - |t|)$$

with constants $e$, $\mathcal{E}$ and $\tau > 0$. At an initial time $t_i < -\tau$ the oscillator is prepared in an energy eigenstate $|n\rangle$; at a later time $t_f > \tau$ we perform an energy measurement.

**a.** **(3 points)** Show that the transition probability $w_{n\to k}$ of finding the oscillator in a state $|k\rangle$ is given by the squared modulus of the transition amplitude

$$\mathcal{A}_{n\to k} \equiv \langle k|U_{\mathrm{D}}(\tau,-\tau)|n\rangle,$$

where $U_{\mathrm{D}}$ is the time-evolution operator in the interaction picture.

**b.** **(2 points)** Show that the transition amplitude $\mathcal{A}_{n\to k}$ has the perturbative expression

$$\begin{aligned}\mathcal{A}_{n\to k} = \delta_{kn} &- \frac{\mathrm{i}}{\hbar}\int_{-\tau}^{\tau}\mathrm{d}t\ \mathrm{e}^{\mathrm{i}\omega_{kn}t}\langle k|V(t)|n\rangle \\ &- \frac{1}{\hbar^2}\sum_{\ell}\int_{-\tau}^{\tau}\mathrm{d}t_1\int_{-\tau}^{t_1}\mathrm{d}t_2\ \mathrm{e}^{\mathrm{i}\omega_{k\ell}t_1}\langle k|V(t_1)|\ell\rangle\,\mathrm{e}^{\mathrm{i}\omega_{\ell n}t_2}\,\langle \ell|V(t_2)|n\rangle + \cdots\end{aligned}$$

where $\hbar\omega_{kn} = E_k - E_n$.

**c.** **(4 points)** Evaluate the transition probability $w_{n\to k\neq n}$ of finding the oscillator in a *different* state $|k\rangle$ at order $\mathcal{O}(e^2)$.

**d.** **(1 point)** Evaluate the persistence probability $w_{n\to n}$ at order $\mathcal{O}(e^2)$.

**e.** **(4 points)** Evaluate the transition probability $w_{0\to 2}$ at order $\mathcal{O}(e^4)$.

### Problem 11 (Periodic perturbation) (4 points)

In Problems 7 and 8 we investigated a two-level system subject to a periodic perturbation of the form

$$V(t) = \begin{pmatrix} 0 & \gamma\,\mathrm{e}^{\mathrm{i}\omega t} \\ \gamma^*\,\mathrm{e}^{-\mathrm{i}\omega t} & 0\end{pmatrix}.$$

The transition probability $1\to 2$ was found in first-order perturbation theory to be

$$w_{1\to 2}(t) = \frac{4|\gamma|^2}{\hbar^2(\omega-\omega_{21})^2}\sin^2\!\Big(\frac{\omega-\omega_{21}}{2}\,t\Big), \qquad \hbar\omega_{21} = E_2 - E_1.$$

It has been shown in the lecture that the transition *rate* in case of a generic periodic perturbation $V(t) = V_0\,\mathrm{e}^{-\mathrm{i}\omega t} + V_0^\dagger\,\mathrm{e}^{\mathrm{i}\omega t}$ is given by

$$\Gamma_{n\to k} = \frac{2\pi}{\hbar}\Big[|V_{kn}|^2\,\delta(E_k - E_n - \hbar\omega) + |V_{nk}|^2\,\delta(E_k - E_n + \hbar\omega)\Big]$$

with $V_{nk} = \langle n|V_0|k\rangle$. Check explicitly that

$$\lim_{t\to\infty} w_{1\to 2}(t) \approx t\,\Gamma_{1\to 2}.$$

*Hint:* Note that $\frac{\varepsilon}{\pi}\frac{\sin^2(x/\varepsilon)}{x^2}$ is a nascent Dirac delta for $\varepsilon\to 0$.