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Tutorial Advanced Quantum Mechanics

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Problem Set 5

Problem 15 (Heisenberg picture of ladder operators) **(2 points)**

Consider a harmonic oscillator with Hamilton operator $H = \hbar\omega(a^\dagger a + 1/2)$. Show that the ladder operators in the Heisenberg picture have the form

$$a_{\mathrm{H}}(t) = a\,\mathrm{e}^{-\mathrm{i}\omega t}, \qquad a_{\mathrm{H}}^\dagger(t) = a^\dagger\,\mathrm{e}^{\mathrm{i}\omega t}.$$

Problem 16 (Ladder operators for a two-level system) **(5 points)**

Let the dynamics of a two-level system be described by a Hamilton operator of the form

$$H = \frac{\hbar\omega}{2}\sigma_3 = \frac{\hbar\omega}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

With this matrix representation the ground state is $|-\rangle \mapsto \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$, and the excited state is $|+\rangle \mapsto \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$.
Consider the ladder operators

$$\sigma_+ = |+\rangle\langle -|, \qquad \sigma_- = |-\rangle\langle +|.$$

a. **(1 point)** Give the matrix representation of $\sigma_\pm$.

b. **(2 points)** Show that $\sigma_\pm$ satisfy the relations

$$\sigma_\pm|\pm\rangle = 0, \qquad \sigma_\pm|\mp\rangle = |\pm\rangle, \qquad \sigma_3\sigma_\pm = \pm\sigma_\pm, \qquad \sigma_\pm\sigma_3 = \mp\sigma_\pm.$$

c. **(2 points)** Give the operators $\sigma_{\pm,\mathrm{H}}$ in the Heisenberg picture.

Problem 17 (Jaynes–Cummings model) **(13 points)**

The Hamilton operator describing the interaction of an atom with the quantized radiation field can be written as

$$H = H_{\mathrm{atom}} + H_{\mathrm{rad}} + H_{\mathrm{int}}$$

where

$$H_{\mathrm{atom}} = \sum_{i=1}^{N} \frac{\mathbf{p}_i^2}{2m_i} + U(\mathbf{x}_1, \dots, \mathbf{x}_N) \tag{17.1}$$

is the Hamilton operator of the unperturbed atom with N electrons, and

$$H_{\text{rad}} = \sum_{\mathbf{k},\lambda} \hbar\omega_{\mathbf{k}} a^{\dagger}_{\mathbf{k},\lambda} a_{\mathbf{k},\lambda} \tag{17.2}$$

is the Hamilton operator of the free radiation field (the divergent zero-point energy has been omitted). The coupling of the atom to the radiation field is encoded by the interaction term H_{int}; treating for simplicity a hydrogen atom in the dipole approximation, i.e. taking a dipole moment $-e\mathbf{x}$ and considering linearly polarized radiation, the interaction takes the form

$$H_{\text{int}} = \frac{e}{\sqrt{L^3}} \sum_{\mathbf{k},\lambda} \sqrt{\frac{2\pi\hbar}{\omega_{\mathbf{k}}}} (a^{\dagger}_{\mathbf{k},\lambda} + a_{\mathbf{k},\lambda}) \mathbf{e}_{\mathbf{k},\lambda} \cdot \frac{\mathrm{i}}{\hbar} [H_{\text{atom}}, \mathbf{x}], \tag{17.3}$$

where L^3 is the quantization volume and $\mathbf{e}_{\mathbf{k},\lambda}$ are the polarization vectors. The spontaneous decay of an excited atom is irreversible in accordance with the fact that the emitted photon escapes into infinity in space. Another physics opens up if the volume corresponds to an optical cavity, whose dimensions are comparable to the emitted photon's wavelength.

We are interested in a transition between two atomic states $|-\rangle$ and $|+\rangle$ with energy difference $\hbar\omega_0 \equiv E_+ - E_- > 0$. The cavity has a single mode $(\mathbf{k}, \lambda)$ with frequency ω close to ω_0. Retaining only these two atomic states and this one radiation mode, the atomic and radiation Hamiltonians [Eqs. (17.1) and (17.2)] reduce respectively to a two-level system and a single quantized radiation mode

$$H_{\text{atom}} \simeq \frac{\hbar\omega_0}{2} \sigma_3 \qquad \text{and} \qquad H_{\text{rad}} \simeq \hbar\omega a^{\dagger} a, \tag{17.4}$$

where a and $a^{\dagger}$ are the ladder operators of the mode with frequency ω and act onto the Hilbert space $\mathcal{H}_{\text{osc}}$ of a harmonic oscillator, spanned by the states $|n\rangle$ of a certain number of photons $n = 0, 1, \ldots$. In the basis $|+\rangle \mapsto \binom{1}{0}$, $|-\rangle \mapsto \binom{0}{1}$ the operator σ_3 is the familiar Pauli matrix

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

a. **(3 points)** Show that in the considered approximation the interaction Hamiltonian takes the form

$$H_{\text{int}} \simeq \hbar(a^{\dagger} + a)(g\,\sigma_+ + g^*\sigma_-)$$

where $\sigma_\pm$ are the ladder operators introduced in Problem 16. Give an explicit expression for the coupling $g \in \mathbb{C}$ in terms of the quantities occurring in Eq. (17.3). Show that g can be made real and positive by choosing an appropriate phase between the states $|+\rangle$ and $|-\rangle$.

Hint: The expression for g involves a matrix element which cannot be worked out with the given information.

b. **(2 points)** Consider the simplified atomic and radiation Hamiltonians [Eq. (17.4)] as the unperturbed Hamiltonian of the system. Calculate the Dirac picture representation $H_{\text{int,D}}$ of the interaction found in part **a**. Show that the contributions proportional to $a^{\dagger}\sigma_+$ and $a\sigma_-$ involve rapidly oscillating phases compared to those that come from $a\sigma_+$ and $a^{\dagger}\sigma_-$.

Hint: Remember that ω is supposed to be quite close to ω_0.

Neglecting the terms which in the interaction picture are strongly oscillating leads to the Hamilton operator of the Jaynes–Cummings model:

$$H_{\text{JC}} = \frac{\hbar\omega_0}{2} \sigma_3 + \hbar\omega a^{\dagger} a + \hbar g (a\sigma_+ + a^{\dagger}\sigma_-).$$

c. **(2 points)** Consider the operator $M = \sigma_3/2 + a^\dagger a$. Show that its lowest eigenvalue is non-degenerate, while the other ones are doubly degenerate. Give explicitly the eigenstates.

d. **(2 points)** Show that the operator M represents a symmetry, i.e. it is a conserved quantity.

e. **(4 points)** Since M and H_{JC} commute they have a common set of eigenstates. Diagonalize the Hamiltonian H_{JC} in each eigenspace of the eigenvalues of M and give so the spectrum of H_{JC}.