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Tutorial Advanced Quantum Mechanics

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Problem Set 14

Problem 43 (Klein–Gordon equation) (7 points)

A spinless relativistic particle of mass m and electric charge $-e$ moves in the potential of a hydrogen-like nucleus $V(r) = -Z\alpha/r$, where $\alpha \approx 1/137$ is the fine structure constant. The Klein–Gordon equation with coupling to the electromagnetic field reads

$$\left[D_\mu D^\mu + m^2\right]\phi(\mathbf{x}, t) = 0, \qquad D_\mu \equiv \partial_\mu - \mathrm{i}eA_\mu .$$

a. **(4 points)** Separate the Klein–Gordon equation in spherical coordinates and find from the radial equation the energies of the bound states. Which values can the occurring quantum numbers take? What is the degeneracy of the states? What happens for $Z > 1/(2\alpha) \approx 68$?
Hint: Compare the resulting radial equation to the one from the non-relativistic Schrödinger equation.

Solution: $E = m\left\{1 + \frac{(Z\alpha)^2}{\left[n-(\ell+1/2)+\sqrt{(\ell+1/2)^2-Z^2\alpha^2}\right]^2}\right\}^{-1/2}$

b. **(3 points)** Expand the energy eigenvalues in powers of the fine structure constant up to including terms of order $\mathcal{O}(\alpha^4)$. Calculate for hydrogen ($Z = 1$) the size of the energy splitting between the $2s$ and $2p$ level to this order in α. Compare the result to the measured fine structure of the Lyman α-line: $\lambda_1 = 121.566\,824\,\mathrm{nm}$ and $\lambda_2 = 121.567\,364\,\mathrm{nm}$.

Problem 44 (Klein paradox) (9 points)

A free Dirac-particle with rest mass m, momentum $k \equiv p^3 > 0$, and positive energy $E_k \equiv p^0 = \sqrt{k^2 + m^2}$, which is moving along the z-axis, can be represented—depending on the orientation of its spin—through one of the two linearly independent solutions

$$\psi_\uparrow(z,t) = \mathrm{e}^{\mathrm{i}(kz - E_k t)}\begin{pmatrix} 1 \\ 0 \\ \frac{k}{E_k+m} \\ 0 \end{pmatrix} \qquad \text{and} \qquad \psi_\downarrow(z,t) = \mathrm{e}^{\mathrm{i}(kz - E_k t)}\begin{pmatrix} 0 \\ 1 \\ 0 \\ \frac{-k}{E_k+m} \end{pmatrix}.$$

(Note that these solutions are not normalized and that we set $\hbar = c = 1$.)

a. **(4 points)** The particle shall be prepared in the half-space $z < 0$ in the state $\psi_{\text{in}}(z,t) \equiv \psi_\uparrow(z,t)$ (spin up). At $z \geq 0$ there shall be an (electrostatic) potential barrier with constant height V off

which the particle is scattered. Calculate the scattered and reflected wave and determine the incoming, transmitted and reflected current densities $j(z, t)$ along the z-axis.
Show that the current densities are time-independent and that the total current is conserved, i.e. $j_{\text{in}} = j_{\text{trans}} + |j_{\text{ref}}|$.

b. **(2 points)** Assume, at first, that $V < 2m$. Discuss the dependence of the wave function on the energy of the incoming wave.

c. **(3 points)** Repeat the discussion of **b** but for $V > 2m$. Show, in particular, that despite the high potential barrier there can be a transmitted particle current. Moreover, show that the reflected current can be larger than the incoming one.

Problem 45 (Foldy–Wouthuysen transformation) (7 points)

Consider the free Dirac equation in non-covariant form

$$\mathrm{i}\partial_t \psi = h_{\mathrm{D}}\psi, \qquad h_{\mathrm{D}} \equiv \boldsymbol{\alpha} \cdot \mathbf{p} + \beta m.$$

Perform the transformation

$$\psi' = \mathrm{e}^{\beta \boldsymbol{\alpha} \cdot \hat{\mathbf{p}}\, \theta(\mathbf{p})} \psi,$$

where $\theta(\mathbf{p})$ is a real function. ($\hat{\mathbf{p}}$ denotes the unit vector.)

a. **(2 points)** Show that this transformation is unitary. Prove furthermore

$$\mathrm{e}^{\beta \boldsymbol{\alpha} \cdot \hat{\mathbf{p}}\, \theta} = \cos\theta(\mathbf{p}) + \beta \boldsymbol{\alpha} \cdot \hat{\mathbf{p}} \sin\theta(\mathbf{p}).$$

b. **(3 points)** Fix the function $\theta(\mathbf{p})$ from the requirement that the transformed Hamilton operator does not depend on the Dirac α matrices any more.