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Tutorial Advanced Quantum Mechanics

Winter semester 2021/2022 Tübingen, 10th January 2022

Problem Set 11

Problem 35 (Green's function of the Schrödinger equation) (8 points)

The retarded Green function of the time-dependent Schrödinger equation is defined by the initial-value problem

$$(\mathrm{i}\hbar\partial_t - H)G^+(t,t') = \delta(t-t'), \qquad G^+(t,t') = 0 \text{ for } t < t',$$

where the Hamilton operator H might in general depend on time .

a. (2 points) Show that $G^+(t,t')$ is related to the time-evolution operator $U(t,t')$ by

$$G^+(t,t') = -\frac{\mathrm{i}}{\hbar}\,\Theta(t-t')\,U(t,t').$$

b. (2 points) Derive the Fourier representation

$$G^+(E) = \int_{-\infty}^{\infty} \mathrm{d}t\, \mathrm{e}^{\frac{\mathrm{i}}{\hbar}Et}\, G^+(t) = \frac{1}{E-H+\mathrm{i}\varepsilon}$$

for a time-*independent* Hamilton operator H.
Hint: Use the representation

$$\Theta(t) = \frac{\mathrm{i}}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\omega\, \frac{\mathrm{e}^{-\mathrm{i}\omega t}}{\omega+\mathrm{i}\varepsilon}$$

where the limit $\varepsilon \to 0$ is implied.

c. (4 points) Calculate explicitly $G_0^+(\mathbf{x},\mathbf{y};E) \equiv \langle\mathbf{x}|G_0^+(E)|\mathbf{y}\rangle$ for a free particle in $d = 3$ spatial dimensions.

Problem 36 (Scattering in one dimension) (12 points)

A quantum-mechanical particle of mass m moves in one spatial dimension under effect of the potential $V(x) = \frac{\hbar^2}{ma}\,\delta(x)$.

a. (3 points) Solve the one-dimensional scattering problem in the familiar way: assume for the wave function the behaviour

$$\psi(x<0) = \mathrm{e}^{\mathrm{i}kx} + r\,\mathrm{e}^{-\mathrm{i}kx}, \qquad \psi(x>0) = t\,\mathrm{e}^{\mathrm{i}kx}$$

and determine the coefficients r and t. Calculate the reflection and transmission coefficients $R = |r|^2$ and $T = |t|^2$. [*Solution:* $t = ka/(\mathrm{i}+ka)$, $r = t-1$.]

We look now at the same problem from the point of view of the Lippmann–Schwinger equation, which in one spatial dimension reads

$$u_k(x) = \mathrm{e}^{\mathrm{i}kx} + \int \mathrm{d}y\, G_0^+(x,y;k)\, V(y)\, u_k(y), \qquad E = \frac{\hbar^2 k^2}{2m} > 0,$$

where $G_0^+(x,y;k)$ is the energy representation of the Green function of the Schrödinger equation (cf. the preceding problem).

b. **(3 points)** Calculate the one-dimensional Green function $G_0^+(x,y;k)$.

[*Solution:* $G_0^+(x,y;k) = -\mathrm{i}m\, \mathrm{e}^{\mathrm{i}k|x-y|}/(\hbar^2 k)$.]

c. **(3 points)** Show that the reflection and transmission coefficients for a short-ranged potential can be written as

$$t = 1 - \frac{\mathrm{i}m}{\hbar^2 k} \int \mathrm{d}y\, \mathrm{e}^{-\mathrm{i}ky} V(y)\, u_k(y), \qquad r = -\frac{\mathrm{i}m}{\hbar^2 k} \int \mathrm{d}y\, \mathrm{e}^{\mathrm{i}ky} V(y)\, u_k(y).$$

What is the relation to the three-dimensional scattering amplitude $f_{\mathbf{k}}(\hat{\mathbf{x}})$?

d. **(3 points)** Solve the Lippmann–Schwinger equation for the potential $V(x) = \frac{\hbar^2}{ma}\,\delta(x)$. Calculate the transmission and reflection coefficients and compare them to the results of part **a**.