Please encircle the questions you have solved and are able to present/discuss in class.

10.1(a)	10.1(b)	10.1(c)	10.1(d)	10.1(e)	10.2(a)	10.2(b)	10.2(c)	10.2(d)

Problem 10.1: Real-space renormalization group of the Ising model on a triangular lattice (5 points)

Consider a $d = 2$ dimensional Ising model on a triangular lattice, with periodic boundary conditions. The Hamiltonian is:

$$-H = K \sum_{\langle i,j \rangle} S_i S_j, \tag{1}$$

with $K = J/(k_B T)$, $J > 0$ the ferromagnetic interaction between nearest neighbors pairs $\langle i, j\rangle$. In the following we shall denote with S_i^I, with $i = 1, 2, 3$, the spin on the lattice site i of the block $I = 1, 2 \dots M$, as shown in Fig. 1.

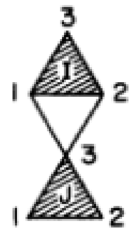

Figure 1: Interaction between the spins on the triangular lattice. In the Figure we denote with I and J the triangular blocks of spins (in black dashed). We denote with M the number of triangular blocks. In the Figure $M = 2$. We further denote with S_i^I the spin belonging to the block I at the lattice site i, with $i = 1, 2, 3$.

In real space, on the lattice, the coarse graining step of the renormalization group procedure is obtained upon defining "block spins" upon grouping together the spins S_i^I at the vertices of each triangular block I of the lattice. The block spin S_I in block I is computed from the *majority rule*:

$$S_I = \text{sign}(S_1^I + S_2^I + S_3^I). \tag{2}$$

We shall denote with

$$\sigma_I = \{S_1^I, S_2^I, S_3^I\}, \tag{3}$$

the microscopic configurations of the spins within a block I.
The aim of this exercise is to write the effective/coarse-grained Hamiltonian $H'(\{S_I\})$ of the block spins S_I degrees of freedom in Eq. (2). The effective Hamiltonian is obtained upon summing over the microscopic configurations $\{\sigma_I\}$ $(I = 1, 2 \dots M)$:

$$Z = \sum_{\{S_I\}} \sum_{\{\sigma_I\}} e^{-H} = \sum_{\{S_I\}} e^{-H'(\{S_I\})}. \tag{4}$$

(a) Show that the Hamiltonian $-H$ in Eq. (1) can be separated

$$-H = H_0 + V, \tag{5}$$

into a term H_0 containing interactions between spins within the same block and another V that accounts for interactions between different blocks. We will treat the term V as a perturbation. Write the expression of H_0 and V. Further show that

$$e^{-H'(\{S_I\})} = \langle e^V \rangle_0 \sum_{\{\sigma_I\}} e^{H_0(\{S_I\},\{\sigma_I\})}, \tag{6}$$

where we have denoted with $\langle \dots \rangle_0$ the average over the Boltzmann weight of H_0 as

$$\langle A(\{S_I\}) \rangle_0 = \frac{\sum_{\{\sigma_I\}} e^{H_0(\{S_I\},\{\sigma_I\})} A(\{S_I\},\{\sigma_I\})}{\sum_{\{\sigma_I\}} e^{H_0(\{S_I\},\{\sigma_I\})}}. \tag{7}$$

(**1 point**)

(b) Compute the sum

$$Z_0^M(K) = \sum_{\{\sigma_I\}} e^{H_0(\{S_I\},\{\sigma_I\})}, \tag{8}$$

in Eq. (6).
Hint: $Z_0(M)$ has the physical meaning of the partition function of one block of spins with a fixed value of the block spin S_I in Eq. (2) (**1 point**).

(c) To compute the term $\langle e^V \rangle_0$ in Eq. (6) we perform the *cumulant expansion*. Show that

$$\langle e^V \rangle_0 = \exp\left\{ \langle V \rangle_0 + \frac{1}{2}[\langle V^2 \rangle_0 - \langle V \rangle_0^2] + \mathcal{O}(V^3) \right\}. \tag{9}$$

Use the result in Eq. (9) to write the effective Hamiltonian $H'(\{S_I\})$ at first order in V. You should get the following

$$-H'(\{S_I\}) = M \log Z_0(K) + \langle V \rangle_0 + \mathcal{O}(V^2). \tag{10}$$

(**1 point**)

(d) Compute the term $\langle V \rangle_0$ in Eq. (10). You should get the following

$$-H'(\{S_I\}) = M \log Z_0(K) + K' \sum_{\langle i,j \rangle} S^I S^J + \mathcal{O}(V^2), \tag{11}$$

with the RG iterative relation for the coupling constant K' in terms of K

$$K' = 2K\phi^2(K), \qquad \phi(K) = \frac{e^{3K} + e^{-K}}{e^{3K} + 3e^{-K}}. \tag{12}$$

(**1 point**)

(e) Compute the fixed points K^* of the RG iterative relation (12). Give the physical interpretation of your result. Compute the critical temperature T_c of the model from the knowledge of the fixed points of the RG transformation (12). (**1 point**)

Problem 10.2: Momentum-space renormalization group of the Ising field theory (5 points)

In this exercise we consider the φ^4 field theory description of the Ising universality class. The action has the usual form

$$S[\varphi] = \int_V \mathrm{d}^d\vec{r}\, \mathcal{L}(\varphi(\vec{r})), \quad \text{with} \quad \mathcal{L}(\varphi) = \frac{1}{2}(\nabla\varphi)^2 + \frac{1}{2}r_0\varphi^2 + \frac{\mu_0}{4}\varphi^4, \tag{13}$$

with V the total volume of the system. We shall also consider the simpler case of the Gaussian theory $S_G[\varphi]$ obtained upon setting $\mu = 0$ in Eq. (13). The Gaussian action is

$$S_G[\varphi] = \int_V \mathrm{d}^d\vec{r} \left(\frac{1}{2}(\nabla\varphi)^2 + \frac{1}{2}r_0\varphi^2 \right) = \frac{1}{V} \sum_{0<|\vec{k}|<\Lambda} \frac{1}{2} |\hat{\phi}_k|^2 (r_0 + k^2), \tag{14}$$

where in the second step we have performed the Fourier transform to write the action in Fourier-k space. In Eq. (14), Λ is called "ultraviolet cutoff" and it is introduced in order to regularize possible divergencies at high values of k (in the ultraviolet region at high frequencies). This cutoff is of order $\Lambda \sim 1/a$, where a is the lattice spacing. We remember also for future convenience the definition of the Fourier transform for a system in a finite volume V

$$\varphi(\vec{r}) = \frac{1}{V} \sum_{\vec{k}} \hat{\varphi}_k\, e^{i\vec{k}\cdot\vec{r}}, \quad \text{with inverse} \quad \hat{\varphi}_k = \int_V \mathrm{d}^d\vec{r}\, \varphi(\vec{r})\, e^{-i\vec{k}\cdot\vec{r}}. \tag{15}$$

The aim of the exercise is to perform the momentum-space renormalization group on the action $S[\varphi]$ in Eq. (13). The momentum-space renormalization group is based on a coarse graining procedure in momentum space, thereby looking at the behavior of the action $S[\varphi]$ for different values of the cutoff Λ. We define an RG transformation in momentum space by integrating out the degrees of freedom with high frequencies:

$$\Lambda/l < |\vec{k}| < \Lambda, \tag{16}$$

with $l > 1$ parametrizing the flow in momentum space of the action. The field $\varphi(\vec{r})$ is therefore decomposed into high $\varphi'_l(\vec{r})$ and low frequency $\sigma_l(\vec{r})$ components

$$\varphi_l(\vec{r}) = \frac{1}{V} \sum_{0<|\vec{k}|<\Lambda/l} \hat{\varphi}_k e^{i\vec{k}\cdot\vec{r}} + \frac{1}{V} \sum_{\Lambda/l<|\vec{k}|<\Lambda} \hat{\varphi}_k e^{i\vec{k}\cdot\vec{r}} = \varphi'_l(\vec{r}) + \sigma_l(\vec{r}). \tag{17}$$

Our aim is to write an effective action $S'[\varphi']$ only for the low frequency components:

$$Z \propto \int D\varphi'_l \, D\sigma_l \, e^{-S[\varphi]} \propto \int D\varphi'_l \, e^{-S'[\varphi']}, \tag{18}$$

thereby integrating out the degrees of freedom $\sigma_l(\vec{r})$. Note that this procedure is conceptually analogous to what we did in the previous exercise where we integrated out the degrees of freedom σ_I inside a block.
We start by considering the Gaussian action $S_G[\varphi]$, which allows for simpler calculations than the ones for $S[\varphi]$. We will get back to the action in Eq. (13) in the last part of the exercise.

(a) Show that the Gaussian action satisfies the following property

$$S_G[\varphi] = S_G[\varphi' + \sigma] = S_G[\varphi'] + S_G[\sigma]. \tag{19}$$

(**1 point**)

(b) Using the propery in Eq. (19) show that you can exactly integrate out the high frequency components σ_l. You should get the following after integrating out the $\{\sigma_l\}$ degrees of freedom

$$Z \propto \int \prod_{0<|\vec{k}|<\Lambda/l} \mathrm{d}\hat{\varphi}'_k \exp\left[-\frac{1}{V} \sum_{0<|\vec{k}|<\Lambda/l} \frac{1}{2}(r_0 + k^2)|\hat{\varphi}'_k|^2 \right] = \int D\varphi' \exp(-S'[\varphi']). \tag{20}$$

(**1 point**)

(c) Take the infinite volume limit $V \to \infty$ in Eq. (20). Perform the *rescaling* transformations

$$\vec{k}_l = l\,\vec{k}, \tag{21}$$

$$\hat{\varphi}_l(\vec{k}_l) = z^{-1} \hat{\varphi}'_k. \tag{22}$$

Determine z as a function of l such that the action $S'[\hat{\varphi}_l(\vec{k}_l)]$ has the *same* form as S_G we started from, but with a different coupling r_l. You should get the following

$$S'[\hat{\varphi}_l(\vec{k}_l)] = \frac{1}{2} \int_0^\Lambda \frac{\mathrm{d}^d \vec{k}_l}{(2\pi)^d} (r_l + k_l^2) |\hat{\varphi}_l(\vec{k})|^2. \tag{23}$$

What is the recursive relation giving r_l as a function of r_0? Make the change of variable

$$l = e^s, \tag{24}$$

and write the recursive equation for r_l in differential form. You should get the following

$$\frac{\mathrm{d}r(s)}{\mathrm{d}s} = 2r(s). \tag{25}$$

Find the fixed point r^* of the previous equation. Interpret and discuss the physical meaning of the rescaling step in Eq. (22) and of the fixed point. Try also to highlight the analogies and the differences of the present approach with the one followed in the previous exercise. Try to be as precise as you can. (**2 points**)

(d) In this last point we consider the effect of the term φ^4 to the RG flow equation in Eq. (25). For the complete, non-Gaussian, action $S[\varphi]$ the coarse-graining and rescaling procedure are conceptually analogous to the steps we did now for the Gaussian model. The calculations get, however, much more difficult and therefore we skip them and we start directly from the flow equations of the φ^4 theory. The equations are found to be

$$\frac{\mathrm{d}r(s)}{\mathrm{d}s} = 2r(s) + A\frac{\mu(s)}{1 + r(s)}, \tag{26}$$

$$\frac{\mathrm{d}\mu(s)}{\mathrm{d}s} = \mu(s)\left[\epsilon - B\frac{\mu(s)}{(1 + r(s))^2}\right], \tag{27}$$

with the definitions (d is the space dimensionality)

$$\epsilon = 4 - d, \quad A = 12K_4, \quad B = 36K_4, \quad K_4 = \frac{1}{8\pi^2}. \tag{28}$$

Eqs. (26) and (27) are the leading order of a perturbative expansion in ϵ. Find the fixed points (r^*, μ^*) of the system of equations in Eqs. (26) and (27) at the leading order in ϵ. Give the physical interpretation of your result. (**1 point**)
Hint: there can be more than one critical fixed point. Try to understand which fixed point controls the large-scale physics of the model as a function of the space dimensionality d.

This is a "**bonus question**", i.e., you can gain extra points from this beyond the 10 points given in the previous questions. You can then use these extra points to fill some points that you could have missed in the previous (or in the following) sheets. **The number of extra points assigned is left undertemined**, i.e., it depends on how precise you answer and how well you interpret physically the results. The more you show you have understood the above concepts, the more you get in terms of points.
Linearize the RG transformation in Eqs. (26) and (27) around the critical fixed points found in the previous point at leading order in ϵ. Sketch the RG flow in the plane (r, u) of the coupling constants for different values of the space dimensionality d. Give the physical meaning of your result. Try to explain, on the basis of the plot you have done, the concepts of universality, upper critical dimension and critical exponents. Try to be as complete and precise as you can. (**undetermined number of bonus points**)