10 The adiabatic elimination principle

The adiabatic elimination principle [91, 92] is universal in the area of nonlinear dissipative dynamical systems, and its importance arises from the fact that it allows one to reduce the number of the equations which govern the dynamics of the system. We introduce it at this stage in the book in order to be able to describe the optical pumping mechanisms which can be exerted to attain population inversion. However, this principle will be applied repeatedly in the following chapters for other purposes.

In the first subsection we describe the principle in general, while the following subsections concern the issue of optical pumping. We show first that it is not possible to obtain population inversion between the two levels of the lasing transition if the pump involves only these two levels. Next we illustrate the three-level pumping scheme and the four-level pumping scheme. We demonstrate that, by adiabatically eliminating the probabilities of the additional energy levels, one arrives at an equation identical to Eq. (4.12), which, for appropriate ranges of values of the parameters involved in the pumping, allows one to obtain population inversion.

10.1 General formulation of the principle

In a dissipative dynamical system the dynamical variables undergo relaxation processes associated with suitable rate constants such as, for example, the rate $\gamma_\perp$ for the normalized atomic polarization P and the rate $\gamma_\parallel$ for the normalized population difference D. The inverses of these rates provide the time scales which characterize the evolution of each variable. Therefore we can distinguish a set of *fast variables*, which evolve over short time scales and are therefore characterized by large relaxation rates, and a set of *slow variables*, which display small relaxation rates. A remarkable simplification in the analysis of dynamical systems is obtained by using the technique of *adiabatic elimination* of fast variables, which Haken considered as a basis for the discipline which he formulated and called Synergetics [91, 92]. The method is based on the identification of two distinct stages in the time evolution of the system. The first develops over the fast time scales which characterize the fast variables; the second develops over the slow time scales. During the first stage only the fast variables evolve significantly, while the slow variables remain practically unchanged. Independently of their initial values, the fast variables tend to assume, at the end of the first stage, values that are uniquely determined by the values of the slow variables. As a result of the first stage one has a functional dependence of the fast variables on the slow variables, which can be obtained mathematically by setting the derivatives of the slow

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variables equal to zero, a sort of "stationary state" of the fast variables conditioned by the values of the slow variables. This procedure leads to an algebraic system of equations that can be solved with respect to the fast variables and provides their functional dependence on the slow variables. Haken calls this process "slaving" of the fast variables by the slow variables. Next, one introduces into the dynamical equations for the slow variables the functional expressions of the fast variables, and in this way one obtains a closed set of dynamical equations for the slow variables. This set, which has a smaller dimension than the original set of equations, governs the evolution of the system during the second stage, which occurs over the long time scales. In this way one has brought about the elimination of the fast variables, which follow adiabatically, i.e. without retardation, the evolution of the slow variables.

In order to illustrate how the adiabatic elimination principle works in practice, let us consider the simplest case of a dynamical system described by two dynamical variables x_1 and x_2 that obey the equations

$$\dot{x}_1 = \gamma_1 f_1(x_1, x_2), \tag{10.1}$$

$$\dot{x}_2 = \gamma_2 f_2(x_1, x_2), \tag{10.2}$$

where f_1 and f_2 are in general nonlinear functions. Let us assume that x_2 is the fast variable, i.e. that $\gamma_2 \gg \gamma_1$. By neglecting what happens in the first stage of the evolution let us go directly to consider the second, which is characterized by the temporal scale $1/\gamma_1$. By introducing the dimensionless time $t_1 = \gamma_1 t$ we recast the equations in the forms

$$\frac{dx_1}{dt_1} = f_1(x_1, x_2), \tag{10.3}$$

$$\varepsilon \frac{dx_2}{dt_1} = f_2(x_1, x_2), \tag{10.4}$$

with $\varepsilon = \gamma_1/\gamma_2$. Since $\varepsilon \ll 1$ while all other quantities are of order unity, we can neglect the l.h.s. term in Eq. (10.4). By solving the algebraic equation $f_2(x_1, x_2) = 0$ with respect to the fast variable x_2 we get $x_2 = x_2(x_1)$ and then a closed dynamical equation for the slow variable x_1,

$$\frac{dx_1}{dt_1} = f_1[x_1, x_2(x_1)]. \tag{10.5}$$

An alternative way of proceeding, which illustrates also what happens in the first stage of the evolution, including the loss of memory of the initial conditions for the fast variables, is the following. In this context, it is useful to rewrite the dynamical equations in such a way as to make evident the relaxation terms of the two variables,

$$\dot{x}_1 = -\gamma_1 x_1 + g_1(x_1, x_2), \tag{10.6}$$

$$\dot{x}_2 = -\gamma_2 x_2 + g_2(x_1, x_2), \tag{10.7}$$

where we have set $g_i(x_1, x_2) = \gamma_i f_i(x_1, x_2) + \gamma_i x_i$, $i = 1, 2$. We can cast the solution of Eq. (10.7) in the form of an integral equation as follows:

$$x_2(t) = x_2(0)\mathrm{e}^{-\gamma_2 t} + \int_0^t \mathrm{e}^{-\gamma_2(t-t')} g_2[x_1(t'), x_2(t')]dt'. \tag{10.8}$$

We see that for times on the order of $1/\gamma_2$ there is memory of the initial condition $x_2(0)$, but for longer times this is lost because the first term on the r.h.s. vanishes. In this limit, by passing to the integration variable $\tau = t - t'$ we can write

$$x_2(t) = \int_0^t e^{-\gamma_2 \tau} g_2[x_1(t-\tau), x_2(t-\tau)] d\tau. \tag{10.9}$$

The integrand still contains memory of the past evolution of the system because it depends on the values of the dynamical variables for times $t - \tau$ with $0 \leq \tau \leq t$. However, in the integral the function g_2 is multiplied by an exponential term that tends to zero for values of the delay τ much larger than $1/\gamma_2$. Therefore the meaningful contribution to the integral arises for $0 < \tau \lesssim 1/\gamma_2$. On the other hand, over this time scale the slow variable x_1 is practically constant and the same can be said about the fast variable x_2 because it is slaved by x_1. Hence we can conclude that g_2 varies slowly in the time interval $t - \tau$ in which the exponential is significantly different from 0, and therefore we can take g_2 out of the integral, evaluating it at time t,

$$x_2(t) = g_2[x_1(t), x_2(t)] \int_0^t e^{-\gamma_2 \tau} \, d\tau. \tag{10.10}$$

For $t \gg 1/\gamma_2$ we can safely extend the integral to infinity, obtaining the equation

$$x_2(t) = \frac{g_2[x_1(t), x_2(t)]}{\gamma_2}, \tag{10.11}$$

which coincides with the equation $f_2(x_1, x_2) = 0$ because of the definition $g_2(x_1, x_2) = \gamma_2 f_2(x_1, x_2) + \gamma_2 x_2$.

We observe finally that this technique does not work in all cases, because it may happen that the equation $f_2(x_1, x_2) = 0$ does not provide a functional relation between x_2 and x_1. In these cases one must utilize more sophisticated techniques to realize the adiabatic elimination, such as, for example, the central-manifold technique [93].

10.2 Adiabatic elimination of the atomic polarization in the Bloch equations. Limits of the optical pumping between two levels

A common procedure to realize population inversion consists in optically pumping the medium with an external electric field. Let us consider first an external field resonant with the laser transition.

Let us return to Eqs. (4.36) and (4.37) in the laser case $\sigma > 0$ assuming perfect resonance between the frequency of the field and the atomic transition frequency, so that $\Delta = 0$, and assuming that the normalized field envelope is constant in time and real. Under these conditions, if P is initially real it remains real throughout the time evolution, and the dynamical equations reduce to

$$\dot{P} = \gamma_\perp (FD - P), \tag{10.12}$$

$$\dot{D} = -\gamma_\parallel (FP + D - 1), \tag{10.13}$$

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Next, let us assume that $\gamma_\perp \gg \gamma_\parallel$, so that P becomes the fast variable and D the slow variable. Hence by setting $\dot{P} = 0$ we obtain the functional expression for P,

$$P = FD. \tag{10.14}$$

By inserting Eq. (10.14) into Eq. (10.13) we adiabatically eliminate the variable P, obtaining the following closed equation for the slow variable D:

$$\dot{D} = -\gamma_\parallel[D(1+F^2) - 1] = -\gamma_\parallel'\left(D - \frac{1}{1+F^2}\right), \tag{10.15}$$

with

$$\gamma_\parallel' = \gamma_\parallel(1+F^2). \tag{10.16}$$

Let us focus on the strong saturation limit in which $F^2 \gg 1$ so that Eq. (10.15) becomes

$$\dot{D} = -\gamma_\parallel' D, \qquad \gamma_\parallel' = \gamma_\parallel F^2. \tag{10.17}$$

Note that the damped Rabi oscillations of Section 4.3 exist only when $\gamma_\perp$ and $\gamma_\parallel$ are of the same order of magnitude. In the adiabatic elimination limit, instead, Rabi oscillations, which arise from the coherence of atom–field interaction, disappear, the time evolution of D is monotonic and, in the strong-saturation limit, D decays exponentially to zero. By taking Eq. (4.33) into account we see that the population inversion per atom r_3 also tends to zero and therefore, since $r_3 = p_1 - p_2$, the two levels are equally populated. This demonstrates that it is not possible to achieve population inversion if the pumping involves only the two levels 1 and 2 of the laser transition.

For the following section it is useful to recast Eq. (10.17) in the form of coupled rate equations for the probabilities p_1 and p_2. Since r_3 is proportional to D, it obeys the same equation, (10.17), as D. By taking into account that $r_3 = p_1 - p_2 = 2p_1 - 1 = 1 - 2p_2$ we obtain the rate equations

$$\dot{p}_1 = wp_2 - wp_1, \qquad \dot{p}_2 = -wp_2 + wp_1, \tag{10.18}$$

with

$$w = \frac{\gamma_\parallel'}{2} = \frac{\gamma_\parallel}{2}F^2. \tag{10.19}$$

By comparing this with the rate equations (4.9) and (4.10), we see that the adiabatic elimination of the atomic polarization, combined with the strong-saturation limit $F^2 \gg 1$, leads to rate equations with identical upward and downward transition rates between the two levels.

10.3 The three-level optical-pumping scheme

In order to achieve population inversion it is necessary that the pump field is resonant with two levels, at least one of which is different from the two levels of the laser transition [6].

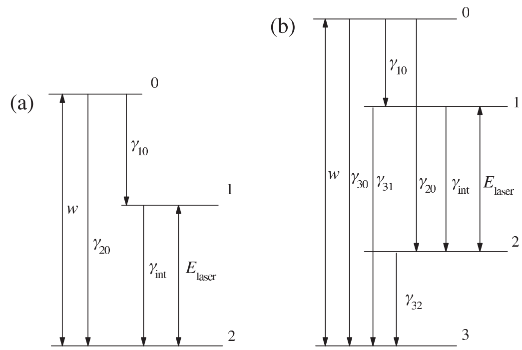

Figure 10.1 Pumping schemes involving (a) three energy levels and (b) four energy levels.

The energy-level scheme for a three-level laser is illustrated in Fig. 10.1(a). Let us indicate the three levels by 0, 1 and 2, and let us assume that $E_0 > E_1 > E_2$. The laser transition is between levels 1 and 2, while the pumping involves levels 0 and 2. Level 0 decays spontaneously to level 1 and to level 2 with rates γ_{10} and γ_{20}, respectively. Level 1 decays spontaneously to level 2, and we call the rate of this decay $\gamma_{\rm int}$, to indicate that it is a decay that occurs internally between the two levels of the laser transition. This scheme describes well some lasers such as the ruby laser (the first laser that was realized, in 1960 by Maiman [94]) and the erbium-doped fiber laser. The optical pumping produces a transition probability w from level 2 to level 0 and an equal transition probability from level 0 to level 2, similarly to what is described by Eqs. (10.18).

In order to attain population inversion between levels 1 and 2 of the laser transition it is necessary that the condition

$$\gamma_{10} \gg \gamma_{20},\ \gamma_{\rm int},\ w \tag{10.20}$$

is satisfied, because in this case the atoms which are transferred by optical pumping from level 2 to level 0 decay rapidly to level 1, where they stay for a much longer time because $\gamma_{\rm int}$ is much smaller than γ_{10}. In this way level 1 can attain a population larger than level 2, which is depopulated by the optical pumping.

Let us now describe mathematically the dynamics of the populations in terms of the following three rate equations for the probabilities p_0, p_1 and p_2 of the three levels:

$$\dot{p}_0 = -(\gamma_{10} + \gamma_{20} + w)p_0 + wp_2, \tag{10.21}$$

$$\dot{p}_1 = -\gamma_{\rm int} p_1 + \gamma_{10} p_0, \tag{10.22}$$

$$\dot{p}_2 = -wp_2 + \gamma_{\rm int} p_1 + (w + \gamma_{20})p_0. \tag{10.23}$$

Obviously $\dot{p}_0 + \dot{p}_1 + \dot{p}_2 = 0$ because the sum of the three probabilities is constant and equal to 1. The decay constants of the three variables are $\gamma_{10} + \gamma_{20} + w$, $\gamma_{\rm int}$ and w, respectively. From condition (10.20) we see that the variable p_0 is much faster than the others. Hence we eliminate it adiabatically by setting $\dot{p}_0 = 0$, which gives

$$p_0 = \frac{w}{\gamma_{10} + \gamma_{20} + w} p_2 \simeq \frac{w}{\gamma_{10}} p_2. \tag{10.24}$$

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Therefore as a consequence of condition (10.20) the occupation probability of level 0 is very small; in practice the pumping transfers atoms from level 2 to level 1. Next we insert Eq. (10.24) into Eqs. (10.22) and (10.23), obtaining

$$\dot{p}_1 = w p_2 - \gamma_{\text{int}} p_1, \tag{10.25}$$

$$\dot{p}_2 = -w \left(1 - \frac{w + \gamma_{20}}{\gamma_{10}}\right) p_2 + \gamma_{\text{int}} p_1. \tag{10.26}$$

By taking Eq. (10.20) into account again, we see that Eq. (10.26) reduces to

$$\dot{p}_2 = -w p_2 + \gamma_{\text{int}} p_1, \tag{10.27}$$

and it is immediately evident that Eqs. (10.27) and (10.25) are identical to Eqs. (4.9) and (4.10) if we take $\gamma_\uparrow = w$ and $\gamma_\downarrow = \gamma_{\text{int}}$.

Thus, we recover Eq. (4.12) for the inversion $r_3 = p_1 - p_2$, with $\gamma_\parallel = w + \gamma_{\text{int}}$ and

$$\sigma = \frac{w - \gamma_{\text{int}}}{w + \gamma_{\text{int}}}. \tag{10.28}$$

Therefore one has population inversion, provided that $w > \gamma_{\text{int}}$.

10.4 The four-level optical-pumping scheme

Let us now turn to a four-level pumping scheme [6], in which the laser transition occurs between levels 1 and 2 as before, but the pumping involves also levels 0 and 3, with $E_0 > E_1 > E_2 > E_3$. A scheme of this type, described in Fig. 10.1(b), applies for example to the case of a neodymium laser. In this case, in addition to the pumping rate w and the decay rates γ_{int} from level 1 to level 2, and γ_{10} and γ_{20} from level 0 to levels 1 and 2, respectively, we must consider also the rates γ_{32}, γ_{31} and γ_{30}, which describe the decay to the lowest level, level 3, from, respectively, levels 0, 1 and 2, respectively.

The rate equations for the four populations are

$$\dot{p}_0 = -(w + \gamma_{30} + \gamma_{20} + \gamma_{10}) p_0 + w p_3, \tag{10.29}$$

$$\dot{p}_1 = -(\gamma_{31} + \gamma_{\text{int}}) p_1 + \gamma_{10} p_0, \tag{10.30}$$

$$\dot{p}_2 = -\gamma_{32} p_2 + \gamma_{\text{int}} p_1 + \gamma_{20} p_0, \tag{10.31}$$

$$\dot{p}_3 = -w p_3 + \gamma_{32} p_2 + \gamma_{31} p_1 + (w + \gamma_{30}) p_0. \tag{10.32}$$

Let us assume that the rates γ_{10} and γ_{32} are much larger than all others. Physically this means that level 0 (the upper level of the pump transition) quickly becomes empty, decaying to level 1 (the upper level of the laser transition), and that level 2 (the lower level of the laser transition) quickly becomes empty, decaying to level 3 (the lower level of the pump transition). These conditions ensure the population inversion between the two levels of the laser transition, because they grant an accumulation of population in the upper level, while the lower level is continuously emptied.

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In this limit the probabilities p_0 and p_2 can be adiabatically eliminated, giving

$$p_0 = \frac{w}{w + \gamma_{30} + \gamma_{20} + \gamma_{10}} p_3 \approx \frac{w p_3}{\gamma_{10}}, \tag{10.33}$$

$$p_2 = \frac{\gamma_{\text{int}} p_1 + \gamma_{20} p_0}{\gamma_{32}}. \tag{10.34}$$

These equations show that p_0 and p_2 are very small. Hence we can think that the population is concentrated in levels 1 and 3, and write

$$1 = p_1 + p_3. \tag{10.35}$$

Furthermore, we can identify $r_3 = p_1 - p_2$ with p_1. By taking into account Eqs. (10.30), (10.33) and (10.35) we can therefore write

$$\begin{aligned} \dot{r}_3 = \dot{p}_1 &= -(\gamma_{31} + \gamma_{\text{int}}) r_3 + w p_3 = -(\gamma_{31} + \gamma_{\text{int}}) r_3 + w(1 - r_3) \\ &= w - (w + \gamma_{31} + \gamma_{\text{int}}) r_3 = -\gamma_{\parallel}(r_3 - \sigma), \end{aligned} \tag{10.36}$$

with

$$\gamma_{\parallel} = w + \gamma_{31} + \gamma_{\text{int}}, \tag{10.37}$$

$$\sigma = \frac{w}{w + \gamma_{31} + \gamma_{\text{int}}}, \tag{10.38}$$

in agreement with Eq. (4.12). In this case σ is always positive, i.e. one always has population inversion irrespective of the value of the pump rate w. We can note that both in the case of the three-level laser and in the case of the four-level laser the stationary value of the population difference σ tends to 1 for large values of w.