6

State-based quantum feedback control

6.1 Introduction

In the preceding chapter we introduced quantum feedback control, devoting most space to the continuous feedback control of a localized quantum system. That is, we considered feeding back the current resulting from the monitoring of that system to control a parameter in the system Hamiltonian. We described feedback both in terms of Heisenberg-picture operator equations and in terms of the stochastic evolution of the conditional state. The former formulation was analytically solvable for linear systems. However, the latter could also be solved analytically for simple linear systems, and had the advantage of giving an explanation for how well the feedback could perform.

In this chapter we develop further the theory of quantum feedback control using the conditional state. The state can be used not only as a basis for understanding feedback, but also as the basis for the feedback itself. This is a simple but elegant idea. The conditional state is, by definition, the observer's knowledge about the system. In order to control the system optimally, the observer should use this knowledge. Of course a very similar idea was discussed in Section 2.5 in the context of adaptive measurements. There, one's joint knowledge of a quantum system and a classical parameter was used to choose future measurements so as to increase one's knowledge of the classical parameter. The distinction is that in this chapter we consider state-based feedback to control the quantum system itself.

This chapter is structured as follows. Section 6.2 introduces the idea of state-based feedback by discussing the first experimental implementation of a state-based feedback protocol to control a quantum state. This experiment, in a cavity QED system, was in the 'deep' quantum regime, for which there is no classical analogue. By contrast, the remainder of the chapter is oriented towards state-based control in linear quantum systems, for which there is a classical analogue. Hence we begin this part with an analysis of state-based feedback in general classical systems, in Section 6.3, and in linear classical systems, in Section 6.4. These sections introduce ideas that will apply in the quantum case also, such as optimal control, stability, detectability and stabilizability. We contrast Markovian feedback control with optimal feedback control and also analyse a classical Markovian feedback experiment. In Sections 6.5 and 6.6 we discuss state-based control in general quantum systems and in linear quantum systems, respectively. As discussed in the preface, these

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sections contain unpublished results obtained by one of us (H. M. W.) in collaboration with Andrew Doherty and (latterly) Andy Chia. Finally, we conclude as usual with a discussion of further reading.

6.2 Freezing a conditional state

This section is devoted to the first quantum feedback experiment [SRO+02] that implemented a state-based feedback protocol for controlling a quantum system. In calling the feedback protocol state-based we mean that it would have been infeasible for it to have been invented without explicitly modelling the quantum trajectories for the conditioned system state. The quantum feedback protocol discussed here has no classical analogue. This is unlike the experiments on linear systems discussed in the preceding chapter and unlike the adaptive phase measurements discussed in Chapter 2 (where the light source was a coherent state and hence could be treated semiclassically). This jusifies its claim to be the first continuous (in time) feedback experiment in the the 'deep' quantum regime.

The experiment was performed using a cavity QED system, as shown in Fig. 6.1. A small number of atoms (in this case, in a beam) is made to interact strongly with a cavity mode, as in the experiment discussed in Section 1.5. In this case, the cavity mode is very weakly driven (by a laser) and the atoms are initially in their ground state, so that in steady state the system as a whole (atoms plus field) is always close to the ground state. Moreover, to the extent that it is excited, it is a superposition (rather than a mixture) of ground and excited states. That is, the steady state of the system can be approximated by a pure state, $|\psi_{\rm ss}\rangle$. We will show why this is the case later.

Quantum trajectories had previously been applied in this system to calculate and understand the correlation functions of the light emitted from the cavity [CCBFO00], which have also been measured experimentally [FMO00, FSRO02]. The link between correlation functions and conditioned states was explained in Section 4.3.2. Consider the $g^{(2)}(\tau)$ correlation function for the direct detection photocurrent. This is a normalized version of Glauber's second-order coherence function, $G^{(2)}(t, t+\tau)$:

$$g^{(2)}(\tau) = \frac{G^{(2)}_{\rm ss}(t, t+\tau)}{G^{(2)}_{\rm ss}(t, t+\infty)} = \frac{\langle I(t+\tau)I(t)\rangle_{\rm ss}}{\langle I(t)\rangle^2_{\rm ss}}. \tag{6.1}$$

For $\tau > 0$, we can use the approach of Section 4.3.2 to rewrite this in terms of conditional measurements as

$$g^{(2)}(\tau) = \langle I(\tau)\rangle_{\rm c}/\langle I(0)\rangle_{\rm ss}, \tag{6.2}$$

where here the subscript c means 'conditioned on a detection at time 0, when the system has reached its steady state'.

That is, $g^{(2)}(\tau)$ is the probability of getting a second detection a time τ after the first detection (which occurs when the system has reached steady state), divided by the *unconditioned* probability for the second detection. From Eq. (6.1), the function for $\tau < 0$ can be found by symmetry.

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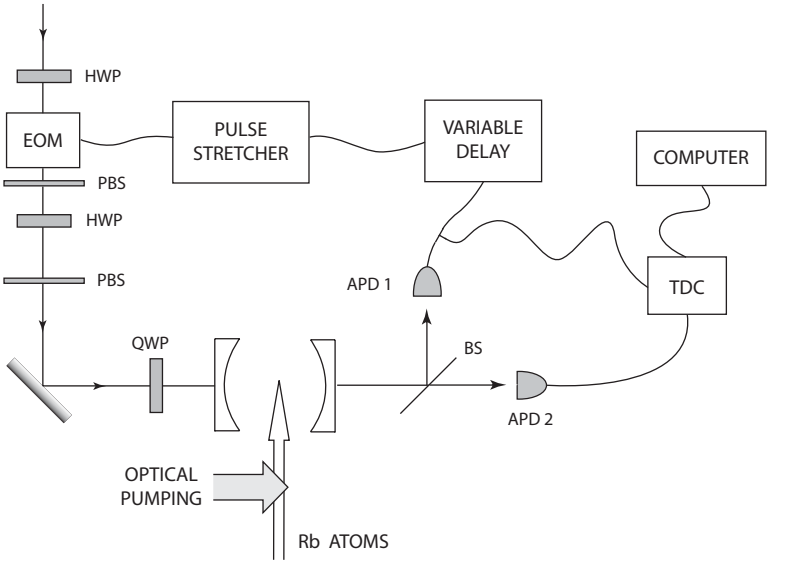

Fig. 6.1 A simplified diagram of the experimental set-up of Smith *et al.*, as depicted by Fig. 6 in Ref. [RSO+04]. Rubidium atoms in a beam are optically pumped into a ground state that couples to a cavity mode before entering the cavity. Two avalanche photo-diodes (APDs) measure the intensity of the light emitted by the cavity. The correlation between the detectors is processed using gating electronics, a time-to-digital converter (TDC) and a histogramming memory and computer. Photodetections at APD 1 trigger a change in the intensity injected into the cavity via an electro-optic modulator (EOM). The optics shown are relevant for control of the size of the intensity step and the polarization of the light injected into the cavity. HWP and QWP denote half- and quarter-wave plate, respectively, and PBS, polarization beam-splitter. Figure 6 adapted with permission from J. E. Reiner *et al.*, *Phys. Rev.* A **70**, 023819, (2004). Copyrighted by the American Physical Society.

Since the stationary system state is almost a pure state, we know from quantum trajectory theory that, immediately following the first detection, the conditional state is $|\psi_c(0)\rangle \propto \hat{a}|\psi_{ss}\rangle$, where $\hat{a}$ is the annihilation operator for the cavity mode. The correlation function (6.2) can thus be reformulated as

$$g^{(2)}(\tau) = \frac{\langle\psi_c(\tau)|\hat{a}^\dagger\hat{a}|\psi_c(\tau)\rangle}{\langle\psi_{ss}|\hat{a}^\dagger\hat{a}|\psi_{ss}\rangle}. \tag{6.3}$$

Here $|\psi_c(\tau)\rangle$ is the conditional state for $\tau > 0$, which relaxes back to $|\psi_{ss}\rangle$ as $\tau \to \infty$. In other words, measuring $g^{(2)}(\tau)$ for $\tau > 0$ is directly probing a property (the mean photon number) of the *conditional* state.

The next step taken in Ref. [SRO+02] was to *control* the conditional state (prepared by a photodetection), rather than simply *observing* it. That is, by altering the system dynamics subsequent to the first photodetection the conditional state could be altered, and hence $g^{(2)}(\tau)$ changed for $\tau > 0$. Specifically, it was shown that the dynamics of the conditional state could be *frozen* for an indefinite time, making $g^{(2)}(\tau)$ constant. The state could then

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be released to resume its (oscillatory) relaxation to $|\psi_{ss}\rangle$. This was done by changing the coherent driving of the cavity at a suitable time $\tau = T$ after the first detection.

6.2.1 The system

A simple model that captures the essential behaviour of the experimental system is the N-atom cavity QED system described by the Tavis–Cummings model [TC68]. This consists of N two-level atoms symmetrically coupled to a single mode of a cavity with annihilation operator $\hat{a}$. Assuming a resonant interaction between the cavity mode and the atoms, the Hamiltonian in the interaction frame is

$$\hat{H} = \mathrm{i}g\left(\hat{a}^\dagger \hat{J} - \hat{a}\hat{J}^\dagger\right) + \mathrm{i}E(\hat{a}^\dagger - \hat{a}) \tag{6.4}$$

(compare with Eq. (1.180)). Here we have assumed that all atoms are coupled with equal strength g, so that

$$\hat{J} = \sum_{k=1}^{N} \hat{\sigma}_k, \tag{6.5}$$

where $\hat{\sigma}_k$ is the lowering operator for atom k. We have also included coherent driving (E) of the cavity mode by a resonant laser.

Including damping of the atoms (primarily due to spontaneous emission through the sides of the cavity) and cavity (primarily due to transmission through the end mirrors), we can describe the system by the master equation (in the interaction frame)

$$\dot{\rho} = \left[E(\hat{a}^\dagger - \hat{a}) + g(\hat{a}^\dagger \hat{J} - \hat{a}\hat{J}^\dagger), \rho\right] + \kappa\mathcal{D}[\hat{a}]\rho(t) + \gamma \sum_k \mathcal{D}[\hat{\sigma}_k]\rho(t). \tag{6.6}$$

This describes a damped harmonic oscillator (the cavity) coupled to a damped anharmonic oscillator (the atoms). The anharmonicity is a result of the fact that $\hat{J}$ and $\hat{J}^\dagger$ obey different commutation relations from $\hat{a}$ and $\hat{a}^\dagger$. This is necessary since the maximum number of atomic excitations is N, which we are assuming is finite.

The evolution generated by Eq. (6.6) is very rich [Ber94]. Much simpler, but still interesting, dynamics results in the limit $E \ll \kappa \sim \gamma \sim g$ [CBR91]. In particular, in this weak-driving limit the steady state of the system is approximately a pure state. To understand why this is the case, consider a more general system consisting of damped and coupled oscillators, which could be harmonic or anharmonic. Let us denote the ground state by $|\psi_0\rangle$, and take the coupling rates and damping rates to be of order unity. For the system above, define a parameter

$$\lambda = \frac{E}{\kappa + 4\Omega^2/\gamma}, \tag{6.7}$$

where $\Omega = g\sqrt{N}$ is the N-atom single-photon Rabi frequency. For the case of weak driving, $\lambda \ll 1$. In this limit, λ is equal to the stationary value for $\langle\hat{a}\rangle$, as we will see. We now show that the steady state of the system ρ_{ss} is pure to order λ^2. That is, one can use

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the approximation

$$\rho_{\rm ss} = |\psi_{\rm ss}\rangle\langle\psi_{\rm ss}| + O(\lambda^3), \tag{6.8}$$

where

$$|\psi_{\rm ss}\rangle = |\psi_0\rangle + \lambda|\psi_1\rangle + \lambda^2|\psi_2\rangle + O(\lambda^3), \tag{6.9}$$

where $|\psi_1\rangle$ and $|\psi_2\rangle$ are states with norm of order unity having, respectively, one and two excitations (in the joint system of atom and cavity mode). Here and in the remainder of this section we are only bothering to normalize the states to lowest order in λ.

Consider unravelling the master equation of the system by unit-efficiency quantum jumps (corresponding to the emission of photons from the system). It is simple to verify that the no-jump evolution will take the system into a pure state of the form of Eq. (6.9).

Exercise 6.1 *Verify this for the master equation (6.6), by showing that the state*

$$\begin{aligned}|\psi_{\rm ss}\rangle = {} & |0,0\rangle + \lambda\left(|1,0\rangle - r|0,1\rangle\right) \\ & + \lambda^2\left(\frac{\xi_0}{\sqrt{2}}|2,0\rangle - \theta_0|1,1\rangle + \frac{\eta_0}{\sqrt{2}}|0,2\rangle\right) + O(\lambda^3)\end{aligned} \tag{6.10}$$

is an eigenstate of $-{\rm i}\hat{H} - (\kappa/2)\hat{a}^\dagger\hat{a} - (\gamma/2)\sum_k \hat{\sigma}_k^\dagger\hat{\sigma}_k$*, which generates the non-unitary evolution. Here* $|n,m\rangle$ *is a state with n photons and m excited atoms, while* $r = 2\Omega/\gamma$ *and*

$$\xi_0 = \zeta\left(1 - \frac{C}{N}\frac{2\kappa}{\kappa+\gamma}\right), \tag{6.11}$$

$$\theta_0 = -r\zeta, \tag{6.12}$$

$$\eta_0 = r^2\zeta\sqrt{1-1/N}, \tag{6.13}$$

$$\zeta = \frac{1+2C}{1+2C[1-(1/N)\kappa/(\kappa+\gamma)]}, \tag{6.14}$$

$$C = 2\Omega^2/(\kappa\gamma). \tag{6.15}$$

C is known as the co-operativity parameter. Note for later that, if $N \to \infty$ with Ω fixed, then $\zeta \to 1$ and so $\xi_0 \to 1$, $\theta_0 \to -r$ and $\eta_0 \to r^2$.

Having established Eq. (6.9) as the stationary solution of the no-jump evolution, we will have obtained the desired result if we can show that the effect of the jumps is to add to $\rho_{\rm ss}$ terms of order λ^3 and higher. That the extra terms from the jumps *are* of order λ^3 can be seen as follows.

First, the rate of jumps for the system in state (6.9) is of order λ^2. This comes from the probability of excitation of the system, which is $O(\lambda^2)$, times the damping rates, which are $O(1)$. That is to say, jumps are rare events.

Second, the effect of a jump will be once more to create a state of the form $|\psi_0\rangle + O(\lambda)$. This is because any lowering operator destroys $|\psi_0\rangle$, acts on $\lambda|\psi_1\rangle$ to turn it into $|\psi_0\rangle$ times a constant $O(\lambda)$, and acts on $\lambda^2|\psi_2\rangle$ to turn it into a state with one excitation $O(\lambda^2)$.

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Renormalizing gives the desired result: the state after the jump is different from $|\psi_{ss}\rangle$ only by an amount of order λ at most.

Third, after a jump, the system will relax back to $|\psi_{ss}\rangle$ at a rate of order unity. This is because the real part of the eigenvalues of the no-jump evolution operator will be of order the damping rates, which are of order unity. That is to say, the non-equilibrium state will persist only for a time of order unity.

On putting these together, we see that excursions from $|\psi_{ss}\rangle$ are only of order λ, and that the proportion of time the system spends making excursions is only of order λ^2. Thus Eq. (6.8) will hold, and, for the master equation Eq. (6.6), the stationary state is given by Eq. (6.10).

Exercise 6.2 *Convince yourself, if necessary, of the three points above by studying the particular example.*

6.2.2 Conditional evolution

In the actual experiment (see Fig. 6.1) only the photons emitted through one mirror are detected, and this with less than unit efficiency. The measurement operator for a photon detection is thus $\hat{M}_1 = \sqrt{\eta\kappa\,\mathrm{d}t}\,\hat{a}$ for $\eta < 1$. From the above, we know that prior to a detection we can take the system to be in state $|\psi_{ss}\rangle$. After the detection (which we take to be at time $\tau = 0$) the state is, to $O(\lambda)$,

$$|\psi_c(\tau)\rangle = |0,0\rangle + \lambda[\xi(\tau)|1,0\rangle + \theta(\tau)|0,1\rangle]\,. \tag{6.16}$$

Here the conditioned cavity field evolution, $\xi(\tau)$, and the conditioned atomic polarization evolution, $\theta(\tau)$, have the initial values ξ_0 and θ_0 as defined above.

Exercise 6.3 *Verify Eq. (6.16) for* $\tau = 0$.

The subsequent no-jump evolution of $\xi(\tau)$ and $\theta(\tau)$ is governed by the coupled differential equations

$$\dot{\xi}(\tau) = -(\kappa/2)\xi(\tau) + \Omega\theta(\tau) + E/(2\lambda), \tag{6.17}$$

$$\dot{\theta}(\tau) = -(\gamma/2)\theta(\tau) - \Omega\xi(\tau), \tag{6.18}$$

where $\xi(0) = \xi_0$ and $\theta(0) = \theta_0$. As the system relaxes to equilibrium, we have from Eq. (6.10) $\xi(\infty) = 1$ and $\theta(\infty) = -r$. These equations can be found using the no-jump evolution via the pseudo-Hamiltonian $\hat{H} - \mathrm{i}(\kappa/2)\hat{a}^\dagger\hat{a} - \mathrm{i}(\gamma/2)\sum_k \hat{\sigma}_k^\dagger\hat{\sigma}_k$.

We thus see that, to lowest order in the excitation, the post-jump evolution is equivalent to two coupled harmonic oscillators with damping and driving (remember that we are in the interaction frame where the oscillation of each oscillator at frequency $\omega \gg \Omega, \kappa, \gamma$ has been removed). This evolution can be understood classically. What is quantum in this system is all in the quantum jump that results from the detection.

The quantum nature of this jump can be seen in the atomic polarization. Upon the detection of a photon from the cavity, this changes from $-r$ to θ_0. Since the system is

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in a pure state, the only way a measurement upon one subsystem (the cavity mode) can lead to a change in the state of the second subsystem (the atoms) is if they are entangled. We have already noted above that, if $N \to \infty$, $\theta_0 \to -r$, so there would be no change in the atomic polarization. That is because in this limit there is no difference between the atomic system coupled to a harmonic oscillator and two coupled harmonic oscillators. Two coupled harmonic oscillators, driven and damped, end up in coherent states, and so cannot be entangled.

Exercise 6.4 *Show this by substituting* $\rho = |\alpha\rangle\langle\alpha| \otimes |\beta\rangle\langle\beta|$ *into the master equation*

$$\dot{\rho} = \left[(E/2)(\hat{a}^\dagger - \hat{a}) + \Omega(\hat{a}^\dagger\hat{b} - \hat{a}\hat{b}^\dagger), \rho\right] + \kappa\mathcal{D}[\hat{a}]\rho + \gamma\mathcal{D}[\hat{b}]\rho, \tag{6.19}$$

and show that $\alpha = \lambda$ *and* $\beta = -r\lambda$ *make* $\dot{\rho} = 0$. *Here* $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$ *and* $\hat{b}|\beta\rangle = \beta|\beta\rangle$.

In fact, as discussed in Section 5.4.2, there is absolutely no state change in a coherent state under photodetection, so there would be nothing at all to see in an experiment involving harmonic oscillators. Everything interesting in this experiment comes from the finite number of atoms N, which leads to an anharmonicity in the atomic oscillator (at second order in the excitation), which leads to atom–field entanglement.

The solutions to the differential equations (6.17) and (6.18), for the field and atomic excitation amplitude, respectively, are much simplified if we take $\kappa = \gamma$. This is a good approximation if $|\kappa - \gamma| \ll \Omega$, which is the case experimentally since typical values are $(\Omega, \kappa, \gamma)/(2\pi) = (48.5, 9.8, 9.1)$ MHz [RSO+04]. Under this assumption, both solutions $\xi(t)$ and $\theta(t)$ are of the form

$$f(\tau) = f_{\rm ss} + \mathrm{e}^{-(\kappa+\gamma)\tau/4}[A_f \cos(\Omega\tau) + B_f \sin(\Omega\tau)], \tag{6.20}$$

The steady-state values are, as stated above, $\xi_{\rm ss} = 1$ and $\theta_{\rm ss} = -r$. The four constants A_ξ, A_θ, B_ξ and B_θ are given by

$$A_\xi = \xi_0 - \xi_{\rm ss} = -B_\theta, \tag{6.21}$$

$$A_\theta = \theta_0 - \theta_{\rm ss} = B_\xi, \tag{6.22}$$

so that the two functions oscillate exactly out of phase.

Exercise 6.5 *Verify the above solutions.*

Using Eqs. (6.3) and (6.16) it is easy to show that $g^{(2)}(\tau)$ is given by

$$g^{(2)}(\tau) = \xi(\tau)^2. \tag{6.23}$$

That is, the correlation function measures the square of the conditioned field amplitude. It has an initial value of ξ_0^2, which, from Eq. (6.12), is always less than unity. This in itself is a nonclassical effect – it is the antibunching discussed in Section 4.6.1. For coherent light $g^{(2)}(0) = 1$, while for any classical light source (which can described as a statistical mixture of coherent states), $g^{(2)}(0)$ can only increase from unity, giving bunching.

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Exercise 6.6 *Verify Eq. (6.23), and convince yourself that antibunching is a nonclassical effect.*

6.2.3 Quantum feedback control

In this section we outline the feedback protocol used to capture and stabilize the conditional state. From Eq. (6.16) we note that the evolution of the conditional state (to first order in λ) depends on only the two functions $\xi(\tau)$ and $\theta(\tau)$. This state can be stabilized if we can, via feedback, make $\dot{\xi}(\tau) = 0$ and $\dot{\theta}(\tau) = 0$, putting it into a new steady state. From Eqs. (6.17) and (6.18) there are two parameters that are easily controlled in an experiment: the feedback time $T = \tau$ and the driving strength E. The feedback protocol simply consists of applying a different driving strength, E', from certain feedback times T_n.

To calculate the feedback times we set $\dot{\theta}(\tau)$ in Eq. (6.18) to zero. That is, we choose a time such that the magnitude of the atomic polarization is at a maximum or minimum. This is necessary because changing E directly affects only $\xi(\tau)$ (Eq. (6.17)), not $\theta(\tau)$ (Eq. (6.18)). Doing this gives the feedback time constraint $\Omega T_n = \left(n + \frac{1}{2}\right)\pi$ where $n = 0, 1, 2, \ldots$.

Exercise 6.7 *Show this.*

The change in the driving strength is then determined by substituting $\theta(T_n)$ and $\dot{\xi}(\tau) = 0$ into Eq. (6.17) and solving for E. Doing this gives

$$E'/E = \xi(T_n) = 1 + \mathrm{e}^{-(\kappa+\gamma)(n+1/2)\pi/(4\Omega)}(\theta_0 - \theta_{\mathrm{ss}}). \tag{6.24}$$

Exercise 6.8 *Show this.*

Thus, the size of the feedback $(E' - E)$ is directly proportional to the jump in the atomic polarization upon a photodetection, which is due to the entanglement. Without entanglement in the initial state, there could be no feedback stabilization.

It might seem that the last claim has not been properly justified. We have shown that for a coherent system (driven, damped harmonic oscillators) there is no change at all in the system upon a detection, and so nothing to stabilize by feedback. However, if one introduced classical noise into such a system, then the stationary state could be mixed, with classical correlations between the two harmonic oscillators. Then the detection of a photon from the first oscillator could cause a jump in the second, and there would be oscillations as the system relaxes, which one would think could be stabilized by feedback. It turns out that this is not the case, because there are not sufficient control parameters to stabilize a mixed state. This is discussed in detail in Ref. [RSO+04].

6.2.4 Experimental results

The typical experimental values, $(\Omega, \kappa, \gamma)/(2\pi) = (48.5, 9.8, 9.1)$ MHz, give a cooperativity $C \approx 53$. It is not possible to measure N, the number of atoms in the cavity

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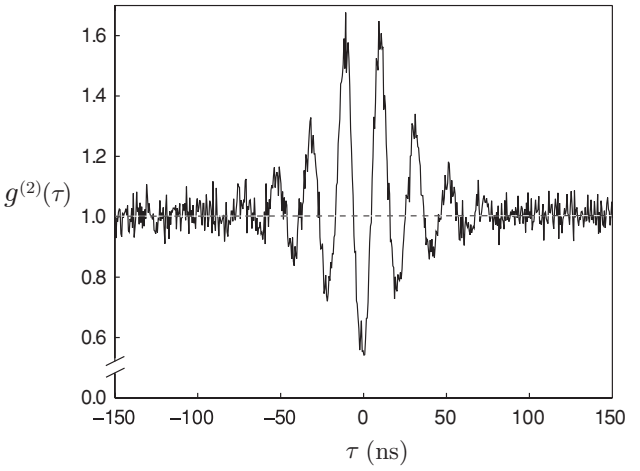

Fig. 6.2 Measured $g^{(2)}(\tau)$. $\tau = 0$ is defined by a photodetection in APD1. Data are binned into 1.0-ns bins. Figure 8 adapted with permission from J. E. Reiner *et al.*, *Phys. Rev.* A **70**, 023819, (2004). Copyrighted by the American Physical Society.

at any given time, directly. Indeed, this concept is not even well defined. First, it will fluctuate because of the random arrival times and velocities of the atoms in the beam. Second, the cavity mode is Gaussian, and so has no sharp cut-off in the transverse direction. The coupling constant g also varies longitudinally in the cavity, because it is a standing-wave mode. However, an average g can be calculated from the cavity geometry, and was found to be $g/(2\pi) = 3.7$ MHz. This implies an effective N of about 170, which is quite large.

Recall that in the limit $N \to \infty$ (with Ω fixed) there are no jumps in the system. However, from Eq. (6.11), the jump in the field amplitude scales as C/N, and C is large enough for this to be significant, with $C/N \approx 0.3$. (This parameter is known as the single-atom co-operativity.) Thus a photon detection sets up a significant oscillation in the quantum state, which is detectable by $g^{(2)}(\tau)$. A typical experimental trace of this is shown in Fig. 6.2. Referring back to Fig. 6.1, two APDs are necessary to measure $g^{(2)}(\tau)$ because the dead-time of the first detector after the detection at $\tau = 0$ (i.e. the time during which it cannot respond) is long compared with the system time-scales. Since it is very unlikely that more than two photons will be detected in the window of interest, the dead-time of the second APD does not matter.

The large value of N has a greater impact on the feedback. From Eq. (6.14), and using the approximation $\kappa \approx \gamma$, the size of the jump in the atomic polarization scales as $\sqrt{2C}/(2N) \approx 0.03$. Thus, the size of the change in the driving field in order to stabilize a conditional state, given by Eq. (6.24), is only a few per cent. Nevertheless, this small change in the driving amplitude is able to freeze the state, as shown in Fig. 6.3. When the driving is returned to its original amplitude, the relaxation of the state to $|\psi_{\rm ss}\rangle$ resumes, so

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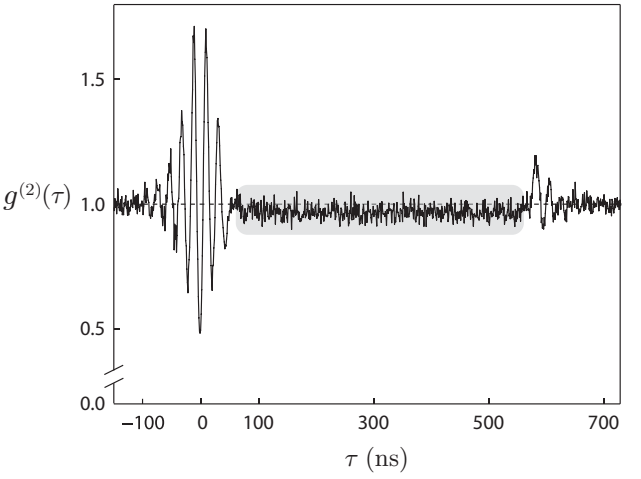

Fig. 6.3 The measured intensity correlation function with the feedback in operation. The grey box indicates the application time of the square feedback pulse, which reduced the driving amplitude by about 0.013. (This is somewhat larger than the value predicted by theory.) The pulse was turned on at $\tau = 45$ ns, in agreement with theory, and turned off 500 ns later. Data are binned into 1.0-ns bins. Figure 9 adapted with permission from J. E. Reiner *et al.*, *Phys. Rev.* A **70**, 023819, (2004). Copyrighted by the American Physical Society.

the effect of the feedback is to insert a flat line of arbitrary length, at the value $|\xi(T_n)|^2$, into the photocurrent autocorrelation function. From Eq. (6.24), this straight line can be at most of order $\sqrt{2C}/N \approx 0.06$ below the steady-state value of unity.

The smallness of the feedback signal (and the smallness of the deviation of the frozen state from the steady state) is exacerbated by the fact that the feedback cannot catch the system at the earliest time $\tau = T_1$. This is because of delays in all parts of the feedback loop, including the APDs themselves, totalling to 43 ns of delay. Since $\pi/\Omega \approx 10$ ns, this means that the earliest time at which the feedback can be applied is $T_4 \approx 45$ ns, by which time the envelope of the oscillations has decayed to about 0.25 of its original size (according to Eq. (6.24)). Even achieving feedback this fast required using electronics designed for data collection in high-energy physics experiments. This highlights the difficulties in making the delay time negligible in quantum feedback experiments.

6.3 General classical systems

Before tackling state-based feedback in general quantum systems, it is useful to review how similar concepts apply in the analogous classical systems.

6.3.1 Notation and terminology

In the remainder of this chapter we are concerned chiefly with linear systems. Hence we will be making frequent use of matrices, which we will denote by capital letters, and vectors,

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which we will denote by bold-font small letters. This greatly simplifies the equations we present, but necessitates a change in some of the conventions introduced in Chapter 1. In particular, there we used a capital letter to denote a random variable, and the corresponding small letter to act as a dummy argument in a probability distribution (or a ket). We maintain different notation for these distinct concepts, but use a new convention, explained below.

A precisely known classical system can be described by a list of real numbers that can be expressed as a vector $\mathbf{x}^\top = (x_1, x_2, \ldots, x_n)$. Here $\mathbf{v}^\top$ indicates the transpose of vector $\mathbf{v}$. We require that these variables form a *complete* set, by which we mean that any property o of the system is a function of (i.e. can be determined from) $\mathbf{x}$, and that none of the elements x_k can be determined from the remainder of them. (If some of the elements x_k could be determined from the remainder of them, then $\mathbf{x}$ would represent an *overcomplete* set of variables.) For example, for a Newtonian system with several physical degrees of freedom one could have $\mathbf{x}^\top = (\mathbf{q}^\top, \mathbf{p}^\top)$, where $\mathbf{q}$ is the vector of coordinates and $\mathbf{p}$ the vector of conjugate momenta. Taking $\mathbf{x}$ to be complete, we will refer to it as the *configuration* of the system, so that $\mathbb{R}^n$ is the system's configuration space. This coincides with the terminology introduced in Chapter 1.

We are interested in situations in which $\mathbf{x}$ is not known precisely, and is therefore a vector of random variables. An observer's *state of knowledge* of the system is then described by a probability density function $\wp(\check{\mathbf{x}})$. Here we use $\check{\mathbf{x}}$ to denote the argument of this function (a dummy variable) as opposed to $\mathbf{x}$, the random variable itself. The probability density is a non-negative function normalized according to

$$\int \mathrm{d}^n\check{\mathbf{x}}\ \wp(\check{\mathbf{x}}) = 1. \tag{6.25}$$

Here $\mathrm{d}^n\check{\mathbf{x}} \equiv \prod_{m=1}^n \mathrm{d}\check{x}_m$, and an indefinite integral indicates integration over all of configuration space. The state defines an expectation value for any property $o(\mathbf{x})$, by

$$\mathrm{E}[o] = \int \mathrm{d}^n\check{\mathbf{x}}\ \wp(\check{\mathbf{x}})o(\check{\mathbf{x}}). \tag{6.26}$$

If the notion of expectation value is taken as basic, we can instead use it to define the probability distribution:

$$\wp(\check{\mathbf{x}}) = \mathrm{E}[\delta^{(n)}(\mathbf{x} - \check{\mathbf{x}})]. \tag{6.27}$$

As in Chapter 1, we refer to $\wp(\check{\mathbf{x}})$ as the *state* of the system. Note that this differs from usual engineering practice, where $\mathbf{x}$ is sometimes called the state or (even more confusingly for quantum physicists) the state vector. Since we will soon be concerned with feedback control, there is another potential confusion worth mentioning: engineers use the term 'plant' to refer to the configuration $\mathbf{x}$ and its dynamics, reserving 'system' for the operation of the combined plant and controller.

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6.3.2 The Kushner–Stratonovich equation

We now consider conditioning the classical state upon continuous monitoring, just as we have previously done for quantum systems in Chapter 4. In fact, we have already described such monitoring for a particular case in Section 4.8.4, in the context of a classical circuit coupled to a detector for a quantum system. There we considered conditioning upon measurement of a current that contained white noise (i.e. a Wiener process). In this subsection we generalize that theory to an arbitrary measurement of this type, performed on a classical system, including back-action. (Recall that the idea of classical back-action was introduced in Section 1.1.5). Just as in the quantum case, it is also possible to consider conditioning a classical system upon a point process. However, for linear systems, as we will specialize to in Section 6.4, conditioning upon a measurement with Wiener noise can be treated semi-analytically, whereas this is not true for conditioning upon a point process. Thus we restrict our analysis to the former.

For a system with configuration $\mathbf{x}_t$ we wish to consider a measurement result that in an infinitesimal interval $[t, t + \mathrm{d}t)$ is a real number, defined by

$$y(t)\mathrm{d}t = \bar{y}(\mathbf{x}_t)\mathrm{d}t + \mathrm{d}v(t), \tag{6.28}$$

where $\bar{y}(\mathbf{x}_t)$ is an arbitrary real function of $\mathbf{x}_t$ and $\mathrm{d}v(t)$ is a Wiener process, defined as usual by

$$[\mathrm{d}v(t)]^2 = \mathrm{d}t, \tag{6.29}$$

$$\mathrm{E}[\mathrm{d}v(t)] = 0, \tag{6.30}$$

$$\mathrm{d}v(t)\mathrm{d}v(t') = 0 \text{ for } t \neq t'. \tag{6.31}$$

Note that Eqs. (6.29)–(6.30) mean that the result $y(t)$ has an infinite amount of noise in it, and so does not strictly exist. However, Eq. (6.31) means that the noise is independent from one moment to the next so that if $y(t)$ is averaged over any finite time the noise in it will be finite. Nevertheless, in this chapter we are being a little more rigorous than previously, and will always write the product $y\,\mathrm{d}t$ (which does exist) rather than y.

Using the methods of Section 4.8.4, it is not difficult to show that the equation for the conditioned classical state (commonly known as the Kushner–Stratonovich equation) is

$$\mathrm{d}\wp(\check{\mathbf{x}}|y) = \mathrm{d}w(t)\{\bar{y}(\check{\mathbf{x}}) - \mathrm{E}[\bar{y}(\mathbf{x})]\}\wp(\check{\mathbf{x}}). \tag{6.32}$$

This is a simple example of *filtering* the current to obtain information about the system, a term that will be explained in Section 6.4.3. Remember that $\mathrm{E}[\bar{y}(\mathbf{x}_t)]$ means $\int \mathrm{d}^n\check{\mathbf{x}}\,\wp(\check{\mathbf{x}};t)\bar{y}(\check{\mathbf{x}})$. Here $\mathrm{d}w(t)$ is another Wiener process defined by

$$\mathrm{d}w(t) \equiv y(t)\mathrm{d}t - \mathrm{E}[y(t)\mathrm{d}t] \tag{6.33}$$

$$= y(t)\mathrm{d}t - \mathrm{E}[\bar{y}(\mathbf{x}_t)]\mathrm{d}t \tag{6.34}$$

$$= \mathrm{d}v(t) + \bar{y}(\mathbf{x}_t)\mathrm{d}t - \mathrm{E}[\bar{y}(\mathbf{x}_t)]\mathrm{d}t. \tag{6.35}$$

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This $\mathrm{d}w(t)$ is known as the *innovation* or *residual*. It is the *unexpected* part of the result $y(t)\mathrm{d}t$, which by definition is the only part that can yield information about the system.

It may appear odd to claim that $\mathrm{d}w$ is a Wiener process (and so has zero mean) when it is equal to another Wiener process $\mathrm{d}v$ plus something non-zero, namely $\bar{y}(\mathbf{x}_t)\mathrm{d}t - \mathrm{E}[\bar{y}(\mathbf{x}_t)]\mathrm{d}t$. The point is that the observer (say Alice) whose state of knowledge is $\wp(\check{\mathbf{x}})$ does not know $\mathbf{x}$. There is no way therefore for her to discover the 'true' noise $\mathrm{d}v$. Insofar as she is concerned $\bar{y}(\mathbf{x}_t) - \mathrm{E}[\bar{y}(\mathbf{x}_t)]$ is a finite random variable of mean zero, so it makes no difference if this is added to $\mathrm{d}v/\mathrm{d}t$, which has an unbounded variance as stated above. Technically, $\mathrm{d}\mathbf{w}$ is related to $\mathrm{d}\mathbf{v}$ by a Girsanov transformation [IW89].

In general the system configuration will change in conjunction with yielding the measurement result $y(t)\mathrm{d}t$. Allowing for deterministic change as well as a purely stochastic change, the system configuration will obey the Langevin equation

$$\mathrm{d}\mathbf{x} = \mathbf{a}(\mathbf{x})\mathrm{d}t + \mathbf{b}(\mathbf{x})\mathrm{d}v \tag{6.36}$$

$$= [\mathbf{a}(\mathbf{x}) - \mathbf{b}(\mathbf{x})\bar{y}(\mathbf{x})]\mathrm{d}t + \mathbf{b}(\mathbf{x})y(t)\mathrm{d}t. \tag{6.37}$$

Note that the noise that appears in this SDE is *not* the innovation $\mathrm{d}w$, since that is an observer-dependent quantity that has no role in the dynamics of the system configuration (unless introduced by a particular observer through feedback as will be considered later). It can be shown that these dynamics alter the SDE for the system state from the 'purely Bayesian' Kushner–Stratonovich equation (6.32) to the following:

$$\begin{aligned}\mathrm{d}\wp(\check{\mathbf{x}}|y) = {} & \mathrm{d}w(t)\left\{\bar{y}(\check{\mathbf{x}}) - \sum_k \nabla_k b_k(\check{\mathbf{x}}) - \mathrm{E}[\bar{y}(\mathbf{x})]\right\}\wp(\check{\mathbf{x}}) \\ & - \mathrm{d}t \sum_k \nabla_k a_k(\check{\mathbf{x}})\wp(\check{\mathbf{x}}) \\ & + \frac{\mathrm{d}t}{2}\sum_{k,k'} \nabla_k \nabla_{k'} b_k(\check{\mathbf{x}}) b_{k'}(\check{\mathbf{x}})\wp(\check{\mathbf{x}}).\end{aligned} \tag{6.38}$$

Here $\nabla_m \equiv \partial/\partial\check{x}_m$ and the derivatives act on all functions of $\check{\mathbf{x}}$ to their right.

Exercise 6.9 *Derive Eq. (6.38) using the methods of Section 4.8.4.*

Note that this equation has a solution corresponding to complete knowledge: $\wp_{\mathrm{c}}(\check{\mathbf{x}};t) = \delta^{(n)}\big(\check{\mathbf{x}} - \mathbf{x}_t\big)$, where $\mathbf{x}_t$ obeys Eq. (6.36). This can be seen from the analysis in Section 4.8.4. For an observer who starts with complete knowledge, $\mathrm{d}v$ and $\mathrm{d}w$ *are* identical in this case.

If one were to ignore the measurement results, the resulting evolution is found from Eq. (6.38) simply by setting $\mathrm{d}w(t)$ equal to its expectation value of zero. Allowing for more than one source of noise so that $\mathrm{d}\mathbf{x} = \sum_l \mathbf{b}^{(l)}\mathrm{d}v^{(l)}$, we obtain

$$\mathrm{d}\wp(\check{\mathbf{x}}) = -\mathrm{d}t \sum_k \nabla_k a_k(\check{\mathbf{x}})\wp(\check{\mathbf{x}}) + \frac{\mathrm{d}t}{2}\sum_{k,k'} \nabla_k \nabla_{k'} \bar{D}_{k,k'}(\check{\mathbf{x}})\wp(\check{\mathbf{x}}) \tag{6.39}$$

$$\equiv \mathcal{L}\wp(\check{\mathbf{x}}). \tag{6.40}$$

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Here $\forall\, \check{\mathbf{x}}$, $\bar{D}(\check{\mathbf{x}})$ is an arbitrary positive semi-definite (PSD) matrix

$$\bar{D}(\check{\mathbf{x}}) = \sum_l \mathbf{b}^{(l)}(\check{\mathbf{x}})[\mathbf{b}^{(l)}(\check{\mathbf{x}})]^\top. \tag{6.41}$$

Equation (6.39) is known as a Fokker–Planck equation. (See Section B.5.)

If one is told that the unconditional evolution of the classical state is given by Eq. (6.39), how does that constrain the possible conditional evolution? The answer is: not much. Any 'purely Bayesian' measurement (i.e. with no back-action) can be added to the unconditional evolution without invalidating it. The unconditional evolution constrains only terms with back-action. By contrast, in the quantum case (as we will see), the unconditional evolution puts strong constraints on the conditional evolution.

6.3.3 Optimal feedback control

Before specializing to linear systems it is appropriate to make some brief comments about optimal feedback control. Control of a system means that the observer (Alice) can influence the dynamics of the system in a time-dependent fashion. Optimal control means that she implements the control so as to minimize some cost function. Feedback control means that Alice is monitoring the system and taking into account the results of that monitoring in her control.

Let the dynamical parameters that Alice can control be represented by the vector $\mathbf{u}(t)$. The dimension of $\mathbf{u}$ is independent of the dimension of the configuration $\mathbf{x}$. A completely general cost function would be the expectation value of an arbitrary functional of the time-functions $\mathbf{x}(t)$ and $\mathbf{u}(t)$. Setting a start time t_0 and a stop time t_1 for the control problem, we notate such a control cost as

$$j = \mathrm{E}\Big\{\mathcal{I}\Big[\{\mathbf{u}(t)\}_{t=t_0}^{t=t_1},\ \{\mathbf{x}(t)\}_{t=t_0}^{t=t_1}\Big]\Big\}. \tag{6.42}$$

Of course, in the minimization of j it is necessary to restrict $\mathbf{u}(t)$ to being a functional of the system output for times less than t, as well as the initial state

$$\mathbf{u}(t) = \mathcal{U}\Big[\big\{\mathbf{y}(t')\big\}_{t'=t_0}^{t'=t^-},\ \wp(\check{\mathbf{x}}; t_0)\Big]. \tag{6.43}$$

Consider the case in which the cost j that Alice is to minimize is of the form

$$j = \mathrm{E}\left[\int_{t_0}^{t_1} h(\mathbf{x}(t), \mathbf{u}(t), t)\mathrm{d}t\right], \tag{6.44}$$

which can also be written as

$$j = \int_{t_0}^{t_1} \mathrm{E}[h(\mathbf{x}(t), \mathbf{u}(t), t)]\mathrm{d}t. \tag{6.45}$$

Physically this is very reasonable since it simply says that the total cost is additive over time. In this case it is possible to show that the *separation principle* holds[1]. That is,

[1] In control theory texts (e.g. [Jac93]), the separation principle is often discussed only in the context of LQG systems, in which case it is almost identical to the concept of *certainty equivalence* – see Sec. 6.4.4. The concept introduced has therefore been called a *generalized* separation principle [Seg77].

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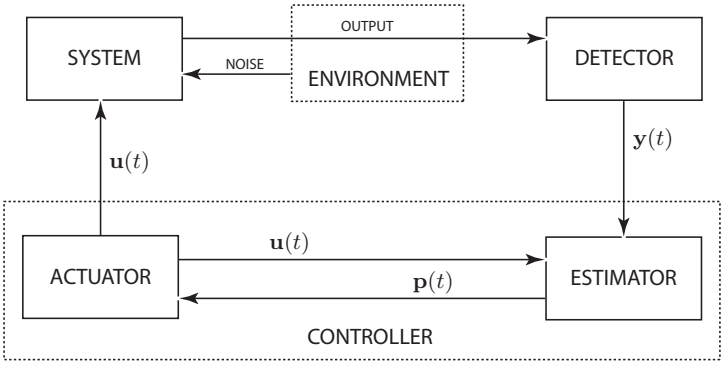

Fig. 6.4 A schematic diagram of the state-based feedback control scheme. The environment is the source of noise in the system and also mediates the system output into the detector. The controller is split into two parts: an estimator, which determines the state conditioned on the record $\mathbf{y}(t)$ from the detector, and an actuator that uses this state to control the system input $\mathbf{u}(t)$. The state, here written as $\mathbf{p}(t)$, would be the probability distribution $\wp(\check{\mathbf{x}};t)$ in the classical case and the state matrix $\rho(t)$ in the quantum case.

for a given function h, the optimal control strategy is

$$\mathbf{u}_{\text{opt}}(t) = \mathcal{U}_h\big(\wp_{\text{c}}(\check{\mathbf{x}};t),t\big). \tag{6.46}$$

In words, Alice should control the system on the basis of the control objective and her current knowledge of the system *and nothing else*. The control at time t is simply a function of the state at time t, though it is of course still a functional of the measurement record, as in Eq. (6.43). But all of the information in $\{\mathbf{y}(t')\}_{t'=t_0}^{t'=t}$ is irrelevant except insofar as it determines the present state $\wp_{\text{c}}(\check{\mathbf{x}};t)$. This is illustrated in Fig. 6.4

This is a very powerful result that gives an independent definition of $\wp_{\text{c}}(\check{\mathbf{x}};t)$. For obvious reasons, this type of feedback control is sometimes called state-based feedback, or Bayesian feedback. Determining the function $\mathcal{U}_h$ of $\wp_{\text{c}}(\check{\mathbf{x}};t)$ is nontrivial, but can be done using the technique of dynamic programming. This involves a backwards-in-time equation called the Hamilton–Jacobi–Bellman equation (or just Bellman equation for the discrete-time case) [Jac93].

6.4 Linear classical systems

In this section we specialize to the case of linear classical systems. This entails two restrictions, defined in Sections 6.4.1 and 6.4.2. In Section 6.4.3 we introduce the concept of a stabilizing solution that is important for optimal feedback control, and Markovian feedback control, which we discuss in Sections 6.4.4 and 6.4.5, respectively. Many of the results in this section we give without proof. The interested reader is referred to Ref. [ZDG96] for more details.

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6.4.1 Unconditional dynamics

The first restriction to make the system linear is explicable in terms of the dynamics of the system configuration, as follows.

(i) The system configuration obeys a linear dynamical equation

$$\mathrm{d}\mathbf{x} = A\mathbf{x}\,\mathrm{d}t + B\mathbf{u}(t)\mathrm{d}t + E\,\mathrm{d}\mathbf{v}_{\mathrm{p}}(t). \tag{6.47}$$

Here A, B and E are constant matrices, while $\mathbf{u}(t)$ is a vector of arbitrary time-dependent functions. It is known as the *input* to the system. Finally, the *process noise* $\mathrm{d}\mathbf{v}_{\mathrm{p}}$ is a vector of independent Wiener processes. That is,

$$\mathrm{E}[\mathrm{d}\mathbf{v}_{\mathrm{p}}] = \mathbf{0}, \qquad \mathrm{d}\mathbf{v}_{\mathrm{p}}(\mathrm{d}\mathbf{v}_{\mathrm{p}})^{\top} = I\,\mathrm{d}t, \tag{6.48}$$

where I is the $n \times n$ identity matrix. Thus A is square ($n \times n$), and is known as the drift matrix. The matrices B and E are not necessarily square, but can be taken to be of full column rank, so $\omega[B]$ and $\omega[E]$ can be taken to be no greater than n. (See Box 6.1.)

Strictly, the Wiener process is an example of a time-dependent function, so $\mathbf{u}(t)\mathrm{d}t$ could be extended to include $\mathrm{d}\mathbf{v}_{\mathrm{p}}(t)$ and the matrix E eliminated. This is a common convention, but we will keep the distinction because $\mathbf{u}(t)$ will later be taken to be the feedback term, which is known by the observer, whereas $\mathrm{d}\mathbf{v}_{\mathrm{p}}(t)$ is unknown.

As explained generally in Section B.5, we can turn the Langevin equation (6.47) into an equation for the state $\wp(\check{\mathbf{x}})$. With $\mathbf{u}(t)$ known but $\mathrm{d}\mathbf{v}_{\mathrm{p}}(t)$ unknown, $\wp(\check{\mathbf{x}})$ obeys a multi-dimensional OUE (Ornstein–Uhlenbeck equation – see Section 5.6) with time-dependent driving:

$$\dot{\wp}(\check{\mathbf{x}}) = \left\{-\nabla^{\top}[A\check{\mathbf{x}} + B\mathbf{u}(t)] + \tfrac{1}{2}\nabla^{\top} D\,\nabla\right\}\wp(\check{\mathbf{x}}), \tag{6.49}$$

where $D = EE^{\top}$ is called the diffusion matrix. Denote the mean system configuration by $\langle\mathbf{x}\rangle$, as usual, and the system covariance matrix $\langle\mathbf{x}\mathbf{x}^{\top}\rangle - \langle\mathbf{x}\rangle\langle\mathbf{x}\rangle^{\top}$ by V. These moments evolve as

$$\langle\dot{\mathbf{x}}\,\rangle = A\langle\mathbf{x}\rangle + B\mathbf{u}(t), \tag{6.50}$$

$$\dot{V} = AV + VA^{\top} + D. \tag{6.51}$$

Exercise 6.10 *Show this from Eq. (6.49).*
Hint: *Use integration by parts.*

Say the system state is initially a Gaussian state, which can be written in terms of the moments $\langle\mathbf{x}\rangle$ and V as

$$\wp(\check{\mathbf{x}}) = g(\check{\mathbf{x}};\langle\mathbf{x}\rangle, V) \tag{6.52}$$

$$\equiv (2\pi\,\det V)^{-1/2}\exp[-(\check{\mathbf{x}} - \langle\mathbf{x}\rangle)^{\top}(2V)^{-1}(\check{\mathbf{x}} - \langle\mathbf{x}\rangle)]. \tag{6.53}$$

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Box 6.1 Some properties of matrices

Consider a general *real* matrix A. We denote the number of columns of A by $\omega[A]$. (The vertical 'prongs' of the ω are meant to suggest columns.) Similarly, we denote the number of rows of A to be $\varepsilon[A]$. The *column space* of A is the space spanned by its column vectors, and similarly for the *row space*. The dimensionalities of these spaces are equal, and this dimensionality is called the the *rank* of A. That is, the number of *linearly independent* column vectors in A equals the number of linearly independent row vectors. Clearly

$$\text{rank}[A] \leq \min\{\omega[A], \varepsilon[A]\}.$$

The matrix A is said to be full column rank iff $\text{rank}[A] = \omega[A]$, and similarly full row rank iff $\text{rank}[A] = \varepsilon[A]$.

For a *square* real matrix, we denote the set of eigenvalues by $\{\lambda(A)\}$. Note that $\{\lambda(A)\} = \{\lambda(A^\top)\}$. We define

$$\lambda_{\max}[A] \equiv \max\{\text{Re}[\lambda(A)]\}.$$

If $\lambda_{\max}[A] < 0$ then A is said to be *Hurwitz*, or, as is appropriate in the context of dynamical systems, *strictly stable*. We say A is *stable* if $\lambda_{\max}[A] \leq 0$, *marginally stable* if $\lambda_{\max}[A] = 0$ and *unstable* if $\lambda_{\max}[A] > 0$. In the context of dynamical systems, we call the eigenvectors $\{\mathbf{x}_\lambda\colon A\mathbf{x}_\lambda = \lambda\mathbf{x}_\lambda\}$ the *dynamical modes* of the system.

For a *symmetric* real matrix, all $\lambda(A)$ are real. If they are all positive A is called positive definite, denoted $A > 0$. If they are non-negative A is called positive semi-definite (PSD), denoted $A \geq 0$. These remarks also apply to an *Hermitian* complex matrix. Any matrix P satisfying

$$P = BQB^\dagger,$$

where Q is PSD and B is an arbitrary matrix, is PSD. Also, any matrix P satisfying

$$AP + PA^\dagger + Q = 0,$$

where Q is PSD and A is *stable*, is PSD. Conversely, if P and Q are PSD then A is necessarily stable.

For an arbitrary matrix A, there exists a unique *pseudoinverse*, or *Moore–Penrose inverse*, A^+. It is defined by the four properties

1. $AA^+A = A$
2. $A^+AA^+ = A^+$
3. $(AA^+)^\dagger = AA^+$
4. $(A^+A)^\dagger = A^+A$.

If A is square and invertible then $A^+ = A^{-1}$. If A is non-square, then A^+ is also non-square, with $\varepsilon[A^+] = \omega[A]$ and $\omega[A^+] = \varepsilon[A]$. The pseudoinverse finds the 'best' solution $\mathbf{x}$ to the linear equation set $A\mathbf{x} = \mathbf{b}$, in the sense that $\mathbf{x} = A^+\mathbf{b}$ minimizes the Euclidean norm $\|A\mathbf{x} - \mathbf{b}\|^2$.

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Then it can be shown that the system state will forever remain a Gaussian state, with the moments evolving as given above. This can be shown by substitution, as discussed in Exercise 5.26

The moment evolution equations can also be obtained directly from the Langevin equation for the configuration (6.47). For example, Eq. (6.50) can be derived directly from Eq. (6.47) by taking the expectation value, while

$$\mathrm{d}(\mathbf{x}\mathbf{x}^\top) = (\mathrm{d}\mathbf{x})\mathbf{x}^\top + \mathbf{x}(\mathrm{d}\mathbf{x}^\top) + (\mathrm{d}\mathbf{x})(\mathrm{d}\mathbf{x}^\top) \tag{6.54}$$

$$= \{[A\mathbf{x}\,\mathrm{d}t + B\mathbf{u}(t)\mathrm{d}t + E\,\mathrm{d}\mathbf{v}(t)]\mathbf{x}^\top + \text{m.t.}\} + \mathrm{d}t\; EE^\top, \tag{6.55}$$

where m.t. stands for matrix transpose and we have used Eq. (6.48). Taking the expectation value and subtracting

$$\mathrm{d}\big(\langle\mathbf{x}\rangle\langle\mathbf{x}\rangle^\top\big) = \langle\mathrm{d}\mathbf{x}\rangle\langle\mathbf{x}\rangle^\top + \langle\mathbf{x}\rangle\langle\mathrm{d}\mathbf{x}^\top\rangle \tag{6.56}$$

$$= \big\{[A\langle\mathbf{x}\rangle + B\mathbf{u}(t)]\mathrm{d}t\,\langle\mathbf{x}^\top\rangle + \text{m.t.}\big\} \tag{6.57}$$

yields Eq. (6.51).

Stability. Consider the case $\mathbf{u} = \mathbf{0}$ – that is, no driving of the system. Then the system state will relax to a time-independent (stationary) state iff A is *strictly stable*. By this we mean that $\lambda_{\max}[A] < 0$ – see Box. 6.1. For linear systems we use the terminology for the dynamics corresponding to that of the matrix A: *stable* if $\lambda_{\max}(A) \leq 0$, *marginally stable* if $\lambda_{\max}(A) = 0$ and *unstable* if $\lambda_{\max}(A) > 0$. Note, however, that the commonly used terminology *asymptotically stable* describes the dynamics iff A is strictly stable. Returning to that case, the stationary state (ss) is then given by

$$\langle\mathbf{x}\rangle_{\text{ss}} = \mathbf{0}, \tag{6.58}$$

$$AV_{\text{ss}} + V_{\text{ss}}A^\top + D = 0. \tag{6.59}$$

The *linear matrix equation* (LME) for V_{ss} can be solved analytically for $n = 1$ (trivially) or $n = 2$, for which the solution is [Gar85]

$$V_{\text{ss}} = \frac{(\det A)D + (A - I\,\mathrm{tr}\,A)D(A - I\,\mathrm{tr}\,A)^\top}{2(\mathrm{tr}\,A)(\det A)}, \tag{6.60}$$

where I is the 2×2 identity matrix. Note that here we are using tr for the trace of ordinary matrices, as opposed to Tr for the trace of operators that act on the Hilbert space for a quantum system. For $n > 2$, Eq. (6.59) can be readily solved numerically. If the dynamics is asymptotically stable then all observers will end up agreeing on the state of the system: $\wp_{\text{ss}}(\check{\mathbf{x}}) = g(\check{\mathbf{x}}; \mathbf{0}, V_{\text{ss}})$.

Stabilizability and controllability. As explained above, the above concept of stability has ready applicability to a system with noise ($E \neq 0$) but with no driving ($\mathbf{u}(t) = \mathbf{0}$). However, there is another concept of stability that has ready applicability in the opposite case – that is, when there is no noise ($E = 0$) but the driving $\mathbf{u}(t)$ may be chosen arbitrarily. In that

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case, if we ignore uncertain initial conditions, the system state is essentially identical to its configuration which obeys

$$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u}(t). \tag{6.61}$$

Since $\mathbf{u}(t)$ is arbitrary and $\mathbf{x}$ is knowable to the observer, we can consider the case $\mathbf{u} = F\mathbf{x}$ so that

$$\dot{\mathbf{x}} = (A + BF)\mathbf{x}. \tag{6.62}$$

This motivates the following definition.

The pair (A, B) is said to be *stabilizable* iff there exists an F such that $A + BF$ is strictly stable.

This ensures that the observer can control the system to ensure that $\mathbf{x} \to \mathbf{0}$ in the long-time limit. As we will see later, the concept of stabilizability is useful even in the presence of noise.

Consider, for example, the free particle of mass m, with $\mathbf{x}^\top = (q, p)$. Say the observer Alice can directly affect only the momentum of the particle, using a time-dependent linear potential.

Exercise 6.11 *Show that this corresponds to the choices*

$$A = \begin{pmatrix} 0 & 1/m \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{6.63}$$

Then, with arbitrary $F = (f_q, f_p)$, we have

$$A + BF = \begin{pmatrix} 0 & 1/m \\ f_q & f_p \end{pmatrix}. \tag{6.64}$$

This matrix has eigenvalues given by $\lambda^2 - \lambda f_p - f_q/m = 0$. Clearly, for suitable f_q and f_p, $A + BF < 0$, so the system is stabilizable. This is because by affecting the momentum the observer can also indirectly affect the position, via the free evolution. By contrast, if Alice can directly affect only the position then $B = (1, 0)^\top$ and

$$A + BF = \begin{pmatrix} f_q & f_p + 1/m \\ 0 & 0 \end{pmatrix}. \tag{6.65}$$

This always has a zero eigenvalue, so the system is not stabilizable. The zero eigenvalue arises because the momentum never changes under this assumption.

A stronger property than being stabilizable is for (A, B) to be *controllable*. This allows the observer to do more: by suitable choice of $\mathbf{u}(t)$, Alice can move the configuration from

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$\mathbf{x}_0$ at t_0 to any $\mathbf{x}_1$ at any $t_1 > t_0$. It can be shown that the following holds.

> The pair (A, B) is controllable iff for any $n \times n$ real matrix O there exists a matrix F such that $\{\lambda(A + BF)\} = \{\lambda(O)\}$.

An equivalent characterization of controllability is that the *controllability matrix*

$$[B \; AB \; A^2B \; \cdots \; A^{n-1}B] \tag{6.66}$$

has full row rank (see Box. 6.1). It can be shown that this is also equivalent to the condition that $[(sI - A)B]$ has full row rank for all $s \in \mathbb{C}$. For proofs see Ref. [ZDG96].

In the above example of a free particle, the stabilizable system is also controllable because in fact the eigenvalues of Eq. (6.64) are those of an arbitrary real matrix. Note that this does *not* mean that $A + BF$ is an arbitrary real matrix – two of its elements are fixed!

6.4.2 Conditional dynamics

The second restriction necessary to obtain a linear system, in addition to Eq. (6.47), relates to the conditional dynamics.

> (ii) The system state is conditioned on the measurement result
>
> $$\mathbf{y}\, \mathrm{d}t = C\mathbf{x}\, \mathrm{d}t + \mathrm{d}\mathbf{v}_{\mathrm{m}}(t). \tag{6.67}$$

This is usually known as the *output* of the system, but we will also refer to it as the measured current. Here C is not necessarily square, but can be taken to be of full row rank. The *measurement noise* $\mathrm{d}\mathbf{v}_{\mathrm{m}}$ is a vector of independent Wiener processes. That is,

$$\mathrm{E}[\mathrm{d}\mathbf{v}_{\mathrm{m}}] = \mathbf{0}, \qquad \mathrm{d}\mathbf{v}_{\mathrm{m}}(\mathrm{d}\mathbf{v}_{\mathrm{m}})^{\top} = I\, \mathrm{d}t. \tag{6.68}$$

As explained in Section 6.3.2, the measurement noise need not be independent of the process noise (although in many control-theory texts this assumption is made). We can describe the correlations between the measurement and process noises by introducing another matrix Γ:

$$E\, \mathrm{d}\mathbf{v}_{\mathrm{p}}\, \mathrm{d}\mathbf{v}_{\mathrm{m}}^{\top} = \Gamma^{\top}\, \mathrm{d}t. \tag{6.69}$$

A cross-correlation matrix Γ is compatible with a given process noise matrix E iff we can define a matrix $\tilde{E}$ such that

$$\tilde{E}\tilde{E}^{\top} = EE^{\top} - \Gamma^{\top}\Gamma. \tag{6.70}$$

That is, iff $D - \Gamma^{\top}\Gamma$ is PSD.

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Using the theory presented in Section 6.3.2, the Kushner–Stratonovich equation appropriate to this conditioning is

$$\begin{aligned} \mathrm{d}\wp_{\mathrm{c}}(\check{\mathbf{x}}) = &\left\{-\nabla^{\top}[A\check{\mathbf{x}} + B\mathbf{u}(t)] + \tfrac{1}{2}\,\nabla^{\top} D\,\nabla\right\}\wp_{\mathrm{c}}(\check{\mathbf{x}})\mathrm{d}t \\ &+ \mathrm{d}\mathbf{w}^{\top}\{C(\check{\mathbf{x}} - \langle\mathbf{x}\rangle) - \Gamma\,\nabla\}\wp_{\mathrm{c}}(\check{\mathbf{x}}). \end{aligned} \tag{6.71}$$

Here the vector of innovations is given by

$$\mathrm{d}\mathbf{w} = \mathbf{y}\,\mathrm{d}t - C\langle\mathbf{x}\rangle_{\mathrm{c}}\,\mathrm{d}t = \mathrm{d}\mathbf{v}_{\mathrm{m}} + C(\mathbf{x} - \langle\mathbf{x}\rangle_{\mathrm{c}})\mathrm{d}t. \tag{6.72}$$

It can be shown that, like the unconditional equation (6.49), the conditional equation (6.71) admits a Gaussian state as its solution. This can be shown using the Itô calculus as in Exercise 5.34. However, it is easier to derive this solution directly from the Langevin equation (6.47) and the current equation (6.67), as we now show.

The crucial fact underlying the derivation is that, if one has two estimates $\bar{\mathbf{x}}_1$ and $\bar{\mathbf{x}}_2$ for a random variable $\mathbf{x}$, and these estimates have Gaussian uncertainties described by the covariance matrices V_1 and V_2, respectively, then the *optimal* way to combine these estimates yields a new estimate $\bar{\mathbf{x}}_3$ also with a Gaussian uncertainty V_3, given by

$$V_3 = (V_1^{-1} + V_2^{-1})^{-1}, \tag{6.73}$$

$$\bar{\mathbf{x}}_3 = V_3(V_1^{-1}\bar{\mathbf{x}}_1 + V_2^{-1}\bar{\mathbf{x}}_2). \tag{6.74}$$

Here *optimality* is defined in terms of minimizing $\mathrm{tr}[\mathrm{M}\Delta]$, where Δ is the covariance matrix $\mathrm{E}[(\bar{\mathbf{x}}_3 - \mathbf{x})(\bar{\mathbf{x}}_3 - \mathbf{x})^{\top}]$ of the error in the final estimate and M is any PD matrix. This result from standard error analysis can also be derived from Bayes' rule, with $g(\mathbf{x}; \bar{\mathbf{x}}_1, V_1)$ being the prior state and $g(\mathbf{x}; \bar{\mathbf{x}}_2, V_2)$ the forward probability (or vice versa), and $g(\mathbf{x}; \bar{\mathbf{x}}_3, V_3)$ the posterior probability. The derivation of this result in the one-dimensional case was the subject of Exercise 1.5.

Before starting the derivation it is also useful to write the problem in terms of independent noise processes. It is straightforward to check from Eq. (6.69) that this is achieved by defining

$$E\,\mathrm{d}\mathbf{v}_{\mathrm{p}} = \Gamma^{\top}\,\mathrm{d}\mathbf{v}_{\mathrm{m}} + \tilde{E}\,\mathrm{d}\mathbf{v}_{\mathrm{p:m}}, \tag{6.75}$$

where $\mathrm{d}\mathbf{v}_{\mathrm{p:m}}$ is pure process noise, *uncorrelated* with $\mathrm{d}\mathbf{v}_{\mathrm{m}}$. This allows the system Langevin equation to be rewritten as

$$\mathrm{d}\mathbf{x} = A\mathbf{x}\,\mathrm{d}t + B\mathbf{u}(t)\mathrm{d}t + \Gamma^{\top}(\mathbf{y} - C\mathbf{x})\mathrm{d}t + \tilde{E}\,\mathrm{d}\mathbf{v}_{\mathrm{p:m}}(t). \tag{6.76}$$

Now, consider a Gaussian state $\wp(\mathbf{x}, t) = g(\mathbf{x};\langle\mathbf{x}\rangle, V)$, and consider the effect of the observation of $\mathbf{y}(t)\mathrm{d}t$. Let $\bar{\mathbf{x}}_1$ be an estimate for $\mathbf{x} + \mathrm{d}\mathbf{x}$, taking into account the dynamical (back-action) effect of $\mathbf{y}$ on $\mathbf{x}$. From Eq. (6.76), this is

$$\bar{\mathbf{x}}_1 = \langle\mathbf{x}\rangle + (A - \Gamma^{\top}C)\langle\mathbf{x}\rangle\mathrm{d}t + B\mathbf{u}(t)\mathrm{d}t + \Gamma^{\top}\mathbf{y}\,\mathrm{d}t. \tag{6.77}$$

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The uncertainty in this estimate is quantified by the covariance matrix

$$V_1 = \big\langle (\mathbf{x} + \mathrm{d}\mathbf{x} - \bar{\mathbf{x}}_1)(\mathbf{x} + \mathrm{d}\mathbf{x} - \bar{\mathbf{x}}_1)^\top \big\rangle \tag{6.78}$$

$$= V + \mathrm{d}t\big\{[(A - \Gamma^\top C)V + \text{m.t.}] + \tilde{E}\tilde{E}^\top\big\}, \tag{6.79}$$

where the final term comes from the independent (and unknown) final noise term in Eq. (6.76). The estimate $\bar{\mathbf{x}}_1$ for $\mathbf{x} + \mathrm{d}\mathbf{x}$ does not take into account the fact that $\mathbf{y}$ depends upon $\mathbf{x}$ and so yields information about it. Thus from Eq. (6.67) we can form another estimate. Taking C to be invertible for simplicity (the final result holds regardless),

$$\bar{\mathbf{x}}_2 = C^{-1}\mathbf{y} \tag{6.80}$$

is an extimate with a convariance matrix

$$V_2 = \big\langle (\mathbf{x} + \mathrm{d}\mathbf{x} - \bar{\mathbf{x}}_2)(\mathbf{x} + \mathrm{d}\mathbf{x} - \bar{\mathbf{x}}_2)^\top \big\rangle \tag{6.81}$$

$$= (C^\top C\,\mathrm{d}t)^{-1} \tag{6.82}$$

to leading order. Strictly, $\bar{\mathbf{x}}_2$ as defined is an estimate for $\mathbf{x}$, not $\mathbf{x} + \mathrm{d}\mathbf{x}$. However, the infinite noise in this estimate (6.82) means that the distinction is irrelevant.

Because $\mathrm{d}\mathbf{v}_\mathrm{m}$ is independent of $\mathrm{d}\mathbf{v}_{\mathrm{p:m}}$, the estimates $\bar{\mathbf{x}}_1$ and $\bar{\mathbf{x}}_2$ are independent. Thus we can optimally combine these two estimates to obtain a new estimate $\bar{\mathbf{x}}_3$ and its variance V_3:

$$V_3 = V + \mathrm{d}t[(A - \Gamma^\top C)V + V(A - \Gamma^\top C)^\top + \tilde{E}\tilde{E}^\top - VC^\top CV], \tag{6.83}$$

$$\bar{\mathbf{x}}_3 = \langle \mathbf{x}\rangle + \mathrm{d}t[(A - \Gamma^\top C)\langle \mathbf{x}\rangle + B\mathbf{u}(t) - VC^\top C\langle \mathbf{x}\rangle] + (VC^\top + \Gamma^\top)\mathbf{y}(t)\mathrm{d}t. \tag{6.84}$$

Exercise 6.12 *Verify these by expanding Eqs. (6.73) and (6.74) to* $O(\mathrm{d}t)$.

Since $\bar{\mathbf{x}}_3$ is the optimal estimate for the system configuration, it can be identified with $\langle \mathbf{x}\rangle_\mathrm{c}(t + \mathrm{d}t)$ and V_3 with $V_\mathrm{c}(t + \mathrm{d}t)$. Thus we arrive at the SDEs for the moments which define the Gaussian state $\wp_\mathrm{c}(\mathbf{x})$,

$$\mathrm{d}\langle \mathbf{x}\rangle_\mathrm{c} = [A\langle \mathbf{x}\rangle_\mathrm{c} + B\mathbf{u}(t)]\mathrm{d}t + \big(V_\mathrm{c}C^\top + \Gamma^\top\big)\mathrm{d}\mathbf{w}, \tag{6.85}$$

$$\dot{V}_\mathrm{c} = AV_\mathrm{c} + V_\mathrm{c}A^\top + D - (V_\mathrm{c}C^\top + \Gamma^\top)(CV_\mathrm{c} + \Gamma). \tag{6.86}$$

Note that the equation for V_c is actually not stochastic, and is of the form known as a *Riccati differential equation*. Equations (6.85) and (6.86) together are known as the (generalized) Kalman filter.

Detectability and observability. The concepts of stabilizability and controllability from control engineering introduced in Section 6.4.1 are defined in terms of one's ability to control a system. There is a complementary pair of concepts, detectability and observability, that quantify one's ability to acquire information about a system.

A system is said to be *detectable* if every dynamical mode that is not strictly stable is monitored. (See Box. 6.1 for the definition of a dynamical mode.) That is, given a system described by Eqs. (6.47) and (6.67), detectability means that, if the drift matrix A leads to

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unstable or marginally stable motion, then $\mathbf{y} \propto C\mathbf{x}$ should contain information about that motion. Mathematically, it means the following.

The pair (C, A) is detectable iff

$$C\mathbf{x}_\lambda \neq \mathbf{0}\ \forall\, \mathbf{x}_\lambda\colon\ A\mathbf{x}_\lambda = \lambda\mathbf{x}_\lambda \text{ with } \mathrm{Re}(\lambda) \geq 0. \tag{6.87}$$

Clearly, if a system is not detectable then any noise in the unstable or marginally stable modes will lead to an increasing uncertainty in those modes. That is, there cannot be a stationary conditional state for the system.

A simple example is a free particle for which only the momentum is observed. That is,

$$A = \begin{pmatrix} 0 & 1/m \\ 0 & 0 \end{pmatrix}, \qquad C = (0, c), \tag{6.88}$$

for which (C, A) is not detectable since $C(1, 0)^\top = 0$ while $A(1, 0)^\top = 0(1, 0)^\top$. No information about the position will ever be obtained, so its uncertainty can only increase with time. By contrast, a free particle for which only the position is observed, that is,

$$A = \begin{pmatrix} 0 & 1/m \\ 0 & 0 \end{pmatrix}, \qquad C = (c, 0), \tag{6.89}$$

is detectable, since $(1, 0)^\top$ is the only eigenvector of A, and $C(1, 0)^\top = c$.

A very important result is the duality between detectability and stabilizability:

$$(C, A) \text{ detectable } \Longleftrightarrow (A^\top, C^\top) \text{ stabilizable.} \tag{6.90}$$

This means that the above definition of detectability gives another definition for stabilizability, while the definition of stabilizability in Section 6.4.1 gives another definition for detectability.

A stronger concept related to information gathering is *observability*. Like controllability, it has a simple definition for the case in which there is no process noise and, in this case, no measurement noise either (although there must be uncertainty in the initial conditions otherwise there is no information to gather). Thus the system is defined by $\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u}(t)$ and $\mathbf{y} = C\mathbf{x}$, and this is observable iff the initial condition $\mathbf{x}_0$ can be determined with certainty from the measurement record $\{\mathbf{y}(t)\}_{t=t_0}^{t=t_1}$ in any finite interval. This can be shown to imply the following.

The pair (C, A) is observable iff

$$C\mathbf{x}_\lambda \neq 0\ \forall\, \mathbf{x}_\lambda\colon\ A\mathbf{x}_\lambda = \lambda\mathbf{x}_\lambda. \tag{6.91}$$

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That is, even strictly stable modes are monitored. For the example of the free particle above, observability and detectability coincide because there are no stable modes.

Like the detectable–stabilizable duality, there exists an observable–controllable duality:

$$(C, A) \text{ observable} \iff (A^\top, C^\top) \text{ controllable.} \tag{6.92}$$

Thus the above definition of observability gives another definition for controllability, while the two definitions of controllability in Section 6.4.1 give another two definitions for observability.

6.4.3 Stabilizing solutions

Even if the unconditioned system evolution is unstable (see Section 6.4.1), there may be a unique stable solution to the Riccati equation (6.86) for the conditioned variance. If such a solution $V_{\rm c}^{\rm ss}$ exists, it satisfies the *algebraic* Riccati equation

$$\tilde{A}V_{\rm c}^{\rm ss} + V_{\rm c}^{\rm ss}\tilde{A}^\top + \tilde{E}\tilde{E}^\top - V_{\rm c}^{\rm ss}C^\top C V_{\rm c}^{\rm ss} = 0, \tag{6.93}$$

where

$$\tilde{A} \equiv A - \Gamma^\top C. \tag{6.94}$$

If $V_{\rm c}^{\rm ss}$ does exist, this means that, if two observers were to start with different initial Gaussian states to describe their information about the system, they would end up with the same uncertainty, described by $V_{\rm c}^{\rm ss}$.

It might be thought that this is all that could be asked for in a solution to Eq. (6.93). However, it should not be forgotten that there is more to the dynamics than the conditioned covariance matrix; there is also the conditioned mean $\langle \mathbf{x} \rangle_{\rm c}$. Consider two observers (Alice and Bob) with different initial knowledge so that they describe the system by different initial states, $g\big(\check{\mathbf{x}}; \langle \mathbf{x} \rangle^A, V^A\big)$ and $g\big(\check{\mathbf{x}}; \langle \mathbf{x} \rangle^B, V^B\big)$, respectively. Consider the equation of motion for the discrepancy between their means $\mathbf{d}_{\rm c} = \langle \mathbf{x} \rangle_{\rm c}^A - \langle \mathbf{x} \rangle_{\rm c}^B$. Assuming both observers know $\mathbf{y}(t)$ and $\mathbf{u}(t)$, we find from Eq. (6.85)

$$\mathrm{d}\mathbf{d}_{\rm c} = \mathrm{d}t\big[\tilde{A}\mathbf{d}_{\rm c} - V_{\rm c}^A C^\top C \langle \mathbf{x} \rangle_{\rm c}^A + V_{\rm c}^B C^\top C \langle \mathbf{x} \rangle_{\rm c}^B\big] + \big(V_{\rm c}^A - V_{\rm c}^B\big)C^\top \mathbf{y}(t)\mathrm{d}t. \tag{6.95}$$

Now say in the long-time limit $V_{\rm c}^A$ and $V_{\rm c}^B$ asymptotically approach $V_{\rm c}^{\rm ss}$. In this limit the equation for $\mathbf{d}_{\rm c}$ becomes deterministic:

$$\dot{\mathbf{d}}_{\rm c} = M\mathbf{d}_{\rm c}, \tag{6.96}$$

where

$$M \equiv \tilde{A} - V_{\rm c}^{\rm ss}C^\top C = A - (V_{\rm c}^{\rm ss}C^\top + \Gamma^\top)C. \tag{6.97}$$

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Thus for Alice and Bob to agree on the long-time system state it is necessary to have M strictly stable.

A solution V_c^{ss} to Eq. (6.93) that makes M strictly stable is known as a *stabilizing solution*. Because of their nice properties, we are interested in the conditions under which stabilizing solutions (rather than merely stationary solutions) to Eq. (6.93) arise. We will also introduce a new notation W to denote a stabilizing V_c^{ss}. Note that, from Eq. (6.93), if W exists then

$$-MW - WM^\top = \tilde{E}\tilde{E}^\top + WC^\top CW. \tag{6.98}$$

Now the matrix on the right-hand side is PSD, and so is W. From this it can be shown that M is necessarily stable. But to obtain a stabilizing solution we require M to be *strictly stable*.

It can be shown that a stabilizing solution exists iff $(C, \tilde{A})$ is detectable, and

$$\tilde{E}^\top \mathbf{x}_\lambda \neq \mathbf{0} \; \forall \, \mathbf{x}_\lambda \colon \; \tilde{A}^\top \mathbf{x}_\lambda = \lambda \mathbf{x}_\lambda \text{ with } \mathrm{Re}(\lambda) = 0. \tag{6.99}$$

Note that this second condition is satisfied if $(\tilde{A}, \tilde{E})$ is stabilizable (or, indeed, if $(-\tilde{A}, \tilde{E})$ is stabilizable), which also guarantees uniqueness.

Exercise 6.13 *Show that $(C, \tilde{A})$ is detectable iff (C, A) is detectable.*
Hint: *First show that $(C, \tilde{A})$ is detectable iff $(C, A - \Gamma^\top C)$ is detectable, and that the latter holds iff $\exists L$: $A - \Gamma^\top C + LC$ is strictly stable. Define $L' = L - \Gamma^\top$, to show that this holds iff $\exists L'$: $A + L'C$ is strictly stable.*

The second condition (6.99) above deserves some discussion. Recall that $\tilde{E}$ is related to the process noise in the system – if there is no diffusion ($D = 0$) then $\tilde{E} = 0$. The condition means that there is process noise in all modes of $\tilde{A}$ that are marginally stable. It might seem odd that the existence of noise helps make the system more stable, in the sense of having all observers agree on the best estimate for the system configuration $\mathbf{x}$ in the long-time limit. The reason why noise can help can be understood as follows. Consider a system with a marginally stable mode x with the dynamics $\dot{x} = 0$ (i.e. no process noise). Now say our two observers begin with *inconsistent* states of knowledge, say $\wp^\alpha(\check{x}) = \delta(\check{x} - x^\alpha)$ with $\alpha = A$ or B and $x^A \neq x^B$. Then, with no process noise, they will never come to agreement, because the noise in $\mathbf{y}(t)$ enables each of them to maintain that their initial conditions are consistent with the measurement record. By contrast, if there is process noise in x then Alice's and Bob's states will broaden, and then conditioning on the measurement record will enable them to come into agreement.

For a system with a stabilizing solution W, the terminology 'filter' for the equations describing the conditional state is easy to explain by considering the stochastic equation for the mean. For simplicity let $\mathbf{u} = \mathbf{0}$. Then, in the long-time limit, Eq. (6.85) turns into

$$\mathrm{d}\langle \mathbf{x} \rangle_c = M \langle \mathbf{x} \rangle_c \, \mathrm{d}t + \mathrm{F}^\top \mathbf{y}(t) \mathrm{d}t. \tag{6.100}$$

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Here F (a capital F) is defined as

$$\mathrm{F} = C W + \Gamma. \tag{6.101}$$

Equation (6.100) can be formally integrated to give, in the long-time limit,

$$\langle \mathbf{x} \rangle_{\mathrm{c}}(t) \to \int_{-\infty}^{t} \mathrm{e}^{M(t-s)} \mathrm{F}^{\top} \mathbf{y}(s) \mathrm{d}s. \tag{6.102}$$

Since $M < 0$, the Kalman filter for the mean is exactly a low-pass filter of the current $\mathbf{y}$.

Possible conditional steady states. For a linear system with a stabilizing solution of the algebraic Riccati equation, we have from the above analysis a simple description for the steady-state conditioned dynamics. The conditioned state is a Gaussian that jitters around in configuration space without changing 'shape'. That is, V_{c} is constant, while $\langle \mathbf{x} \rangle_{\mathrm{c}}$ evolves stochastically. For $\mathbf{u}(t) \equiv \mathbf{0}$, the evolution of $\langle \mathbf{x} \rangle_{\mathrm{c}}$ is

$$\mathrm{d}\langle \mathbf{x} \rangle_{\mathrm{c}} = A \langle \mathbf{x} \rangle_{\mathrm{c}} \, \mathrm{d}t + \mathrm{F}^{\top} \, \mathrm{d}\mathbf{w}. \tag{6.103}$$

Now the stationary variance in $\mathbf{x}$ is given by

$$V_{\mathrm{ss}} = \mathrm{E}_{\mathrm{ss}}[\mathbf{x}\mathbf{x}^{\top}] \tag{6.104}$$

$$= \mathrm{E}_{\mathrm{ss}}[(\mathbf{x} - \langle \mathbf{x} \rangle_{\mathrm{c}})(\mathbf{x} - \langle \mathbf{x} \rangle_{\mathrm{c}})^{\top} + \langle \mathbf{x} \rangle_{\mathrm{c}} \langle \mathbf{x} \rangle_{\mathrm{c}}^{\top}] \tag{6.105}$$

$$= W + \mathrm{E}_{\mathrm{ss}}[\langle \mathbf{x} \rangle_{\mathrm{c}} \langle \mathbf{x} \rangle_{\mathrm{c}}^{\top}]. \tag{6.106}$$

For a system with strictly stable A, one can find $\mathrm{E}_{\mathrm{ss}}[\langle \mathbf{x} \rangle_{\mathrm{c}} \langle \mathbf{x} \rangle_{\mathrm{c}}^{\top}]$ from the Ornstein–Uhlenbeck equation (6.103). By doing so, and remembering that W satsfies

$$AW + WA^{\top} + D = \mathrm{F}^{\top}\mathrm{F}, \tag{6.107}$$

it is easy to verify that V_{ss} as given in Eq. (6.106) does indeed satisfy the LME (6.59), which we repeat here:

$$AV_{\mathrm{ss}} + V_{\mathrm{ss}}A^{\top} + D = 0. \tag{6.108}$$

Since $\mathrm{E}_{\mathrm{ss}}[\langle \mathbf{x} \rangle_{\mathrm{c}} \langle \mathbf{x} \rangle_{\mathrm{c}}^{\top}] \geq 0$ it is clear that

$$V_{\mathrm{ss}} - W \geq 0. \tag{6.109}$$

That is, the conditioned state is more certain than the unconditioned state. It might be thought that for any given (strictly stable) unconditioned dynamics there would always be a way to monitor the system so that the conditional state is any Gaussian described by a covariance matrix W as long as W satisfies Eq. (6.109). That is, any conditional state that 'fits inside' the unconditional state would be a possible stationary conditional state. However, this is not the case. Since $\mathrm{F}^{\top}\mathrm{F} \geq 0$, it follows from Eq. (6.107) that W must satisfy the linear matrix inequality (LMI)

$$AW + WA^{\top} + D \geq 0, \tag{6.110}$$

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which is strictly stronger than Eq. (6.109). That is, it is the unconditioned dynamics (A and D), not just the unconditioned steady state $V_{\rm ss}$, that determines the possible asymptotic conditioned states.

The LMI (6.110) is easy to interpret. Say the system has reached an asymptotic conditioned state, but from time t to $t + \mathrm{d}t$ we ignore the result of the monitoring. Then the covariance matrix for the state an infinitesimal time later is, from Eq. (6.51),

$$V(t + \mathrm{d}t) = W + \mathrm{d}t(AW + WA^\top + D). \tag{6.111}$$

Now, if we had not ignored the results of the monitoring then by definition the conditioned covariance matrix at time $t + \mathrm{d}t$ would have been W. For this to be consistent with the state unconditioned upon the result $\mathbf{y}(t)$, the unconditioned state must be a convex (Gaussian) combination of the conditioned states. In simpler language, the conditioned states must 'fit inside' the unconditioned state. This will be the case iff

$$V(t + \mathrm{d}t) - W \geq 0, \tag{6.112}$$

which is identical to Eq. (6.110).

As well as being a necessary condition on W, Eq. (6.110) is a sufficient condition. That is, given a W such that $AW + WA^\top + D \geq 0$, it is always possible to find a C and Γ such that $(CW + \Gamma)^\top (CW + \Gamma) = AW + WA^\top + D$. This is easy to see for the case $W > 0$, since then Γ can be dispensed with.

Exercise 6.14 *Prove the result for the case in which some of the eigenvalues of W are zero, remembering that Γ must satisfy $D - \Gamma^\top \Gamma \geq 0$. For simplicity, assume $D > 0$.*

6.4.4 LQG optimal feedback control

LQG problems. Recall from Section 6.3.3 that for a control problem with an additive cost function (6.44) the separation principle can be used to find the optimal feedback control. To make best use of the linear systems theory we have presented above, it is desirable to put some additional restrictions on the control cost, namely that the function h in Eq. (6.44) be given by

$$h(\mathbf{x}_t, \mathbf{u}_t, t) = 2\delta(t - t_1)\mathbf{x}_t^\top P_1 \mathbf{x}_t + \mathbf{x}_t^\top P \mathbf{x}_t + \mathbf{u}_t^\top Q \mathbf{u}_t. \tag{6.113}$$

Here P_1 and P are PSD symmetric matrices, while Q is a PD symmetric matrix. In general P and Q could be time-dependent, but we will not consider that option. They represent on-going costs associated with deviation of the system configuration $\mathbf{x}(t)$ and control parameters $\mathbf{u}(t)$ from zero. The cost associated with P_1 we call the *terminal cost* (recall that t_1 is the final time for the control problem). That is, it is the cost associated with not achieving $\mathbf{x}(t_1) = \mathbf{0}$. (The factor of two before the δ-function is so that this term integrates to $\mathbf{x}_{t_1}^\top P_1 \mathbf{x}_{t_1}$.)

It is also convenient to place one final restriction on our control problem: that all noise be Gaussian. We have assumed from the start of Section 6.4 that the measurement and process

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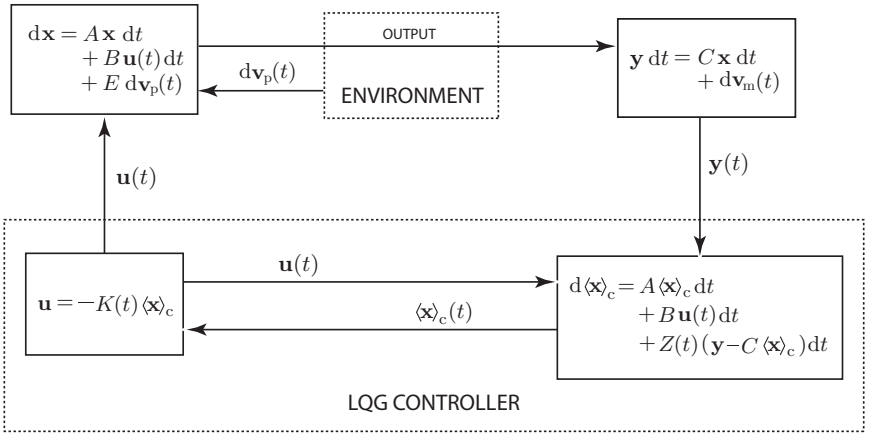

Fig. 6.5 A schematic diagram of the LQG feedback control scheme. Compare this with Fig. 6.4. Here we have introduced $Z(t) = V_\mathrm{c}(t)C^\top - \Gamma^\top$, where $V_\mathrm{c}(t)$ is the conditioned covariance matrix of $\mathbf{x}$ and $E\,\mathrm{d}\mathbf{v}_\mathrm{p}\,\mathrm{d}\mathbf{v}_\mathrm{m}^\top = \Gamma^\top\,\mathrm{d}t$. The gain K depends upon the control costs for the system and actuator. Note how the Kalman filter (the equation for $\mathrm{d}\langle\mathbf{x}\rangle$) depends upon $\mathbf{u}(t)$, the output of the actuator.

noises are Gaussian, and we now assume that the initial conditions also have Gaussian noise so that the Riccati equation (6.86) applies. With these restrictions we have defined a LQG control problem: linear dynamics for $\mathbf{x}$ and linear mapping from $\mathbf{x}$ and $\mathbf{u}$ to output $\mathbf{y}$; quadratic cost in $\mathbf{x}$ and $\mathbf{u}$, and Gaussian noise, including initial conditions.

For LQG problems the principle of *certainty equivalence* holds. This is stronger than the separation principle, and means that the optimal input $\mathbf{u}(t)$ depends upon $\wp_\mathrm{c}(\check{\mathbf{x}};t)$ only through the best estimate of the system configuration $\langle\mathbf{x}(t)\rangle_\mathrm{c}$, as if there were no noise and we were certain that the system configuration was $\langle\mathbf{x}(t)\rangle_\mathrm{c}$. Moreover, the optimal $\mathbf{u}(t)$ depends linearly upon the mean:

$$\mathbf{u}(t) = -K(t)\langle\mathbf{x}(t)\rangle_\mathrm{c}. \tag{6.114}$$

The matrix $K(t)$ is given by

$$K(t) = Q^{-1}B^\top X(t). \tag{6.115}$$

(Recall that $Q > 0$ so that Q^{-1} always exists.) Here $X(t)$ is a symmetric PSD matrix with the final condition $X(t_1) = P_1$, which is determined for $t_0 \le t < t_1$ by solving the time-reversed equation

$$\frac{\mathrm{d}X}{\mathrm{d}(-t)} = P + A^\top X + XA - XBQ^{-1}B^\top X. \tag{6.116}$$

Note that K is independent of D and C, and so is independent of the size of the process and measurement noise – this part of the feedback control problem is completely equivalent to the no-noise control problem (hence 'certainty equivalence'). The overall feedback control scheme is shown in Fig. 6.5.

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Table 6.1. *Relations between stabilizing solutions of the algebraic Riccati equation (ARE) for the cases of observing and controlling a linear system. Here 's.s.' means 'stabilizing solution', 'det.' means 'detectable' and 'stab.' means 'stabilizable'. The stabilizee is the quantity which is stabilized, and 'ss' means 'steady state'.*

	Observing	Controlling
The ARE	$\tilde{A}W + W\tilde{A}^\top + \tilde{E}\tilde{E}^\top = WC^\top CW$	$A^\top Y + YA + P = YBQ^{-1}B^\top Y$
has a unique s.s.	W: $\lambda_{\max}[M] < 0$,	Y: $\lambda_{\max}[N] < 0$,
where	$M = \tilde{A} - WC^\top C$,	$N^\top = A^\top - YBQ^{-1}B^\top$,
iff	$(C, \tilde{A})$ det. and $(\tilde{A}, \tilde{E})$ stab.	(A, B) stab. and (P, A) det.
The stabilizee is	$\mathbf{d}_c = \langle \mathbf{x} \rangle_c^A - \langle \mathbf{x} \rangle_c^B$	$\mathrm{E}[\langle \mathbf{x} \rangle_c]$
since in ss	$\dot{\mathbf{d}}_c = M\mathbf{d}_c$.	$\mathrm{d}\langle \mathbf{x} \rangle_c = N\langle \mathbf{x} \rangle_c \,\mathrm{d}t + \mathrm{F}^\top \,\mathrm{d}\mathbf{w}$.

Asymptotic LQG problems. Note that Eq. (6.116) has the form of a Riccati equation, like that for the conditioned covariance matrix V_c. As in that case, we are often interested in asymptotic problems in which $t_1 - t_0$ is much greater than any relevant relaxation time. Then, if the Riccati equation (6.116) has a unique stationary solution X_{ss} that is PSD, X will equal X_{ss} for much the greater part of the control interval, having relaxed there from P_1 (which is thus irrelevant). In such cases, the optimal control input (6.114) will be time-dependent through $\langle \mathbf{x} \rangle_c$ but K will be basically time-independent. It is often convenient to assume that P is positive definite, in which case X_{ss} will also be positive definite.

For such asymptotic problems it is natural to consider the stability and uniqueness of solutions. There is a close relation between this analysis of stability and uniqueness and that for the conditioned state in Section 6.4.3. In particular, the concept of a stabilizing solution applies here as well. We show these relations in Table 6.1, but first we motivate a few definitions. Just as we denote a stabilizing solution V_c^{ss} of the algebraic Riccati equation Eq. (6.93) as W, so we will denote a stabilizing solution X_{ss} of the Riccati equation (6.116) in steady state by Y. That is, Y is a symmetric PSD matrix satisfying

$$A^\top Y + YA + P - YBQ^{-1}B^\top Y = 0 \tag{6.117}$$

such that

$$N^\top = A^\top - YBQ^{-1}B^\top \tag{6.118}$$

is strictly stable. The relevance of this is that, for this optimal control, the conditioned system mean obeys, in the long-time limit, the linear equation

$$\mathrm{d}\langle \mathbf{x} \rangle_c = N\langle \mathbf{x} \rangle_c \,\mathrm{d}t + \mathrm{F}^\top \,\mathrm{d}\mathbf{w}, \tag{6.119}$$

where $\mathrm{F} = CW + \Gamma$ as before.

Exercise 6.15 *Show this from the control law in Eq. (6.114).*

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Since $\{\lambda(N)\} = \{\lambda(N^\top)\}$, a stabilizing solution Y ensures that the dynamics of the feedback-controlled system mean will be asymptotically stable.

The conditions under which Y is a stabilizing solution, given in Table 6.1, follow from those for W, using the duality relations of Section 6.4.2. Just as in the case of Section 6.4.3 with the noise E, it might be questioned why putting a lower bound on the cost, by requiring that (P, A) be detectable, should help make the feedback loop stable. The explanation is as follows. If (P, A) were not detectable, that would mean that there were some unstable or marginally stable modes of A to which no cost was assigned. Hence the optimal control loop would expend no resources to control such modes, and they would drift or diffuse to infinity. In theory this would not matter, since the associated cost is zero, but in practice any instability in the system is bad, not least because the linearization of the system will probably break down. Note that if $P > 0$ (as is often assumed) then (P, A) is always detectable.

In summary, for the optimal LQG controller to be strictly stable it is sufficient that (A, B) and $(\tilde{A}, \tilde{E})$ be stabilizable and that $(C, \tilde{A})$ and (P, A) be detectable. If we do not require the controller to be optimal, then it can be shown (see Lemma 12.1 of Ref. [ZDG96]) that stability can be achieved iff (A, B) is stabilizable and (C, A) is detectable.

The 'if' part (sufficiency) can be easily shown since without the requirement of optimality there is a very large family of stable controllers. By assumption we can choose F such that $A + BF$ is strictly stable and L such that $A + LC$ is strictly stable. Then, if the observer uses a (non-optimal) estimate $\bar{\mathbf{x}}$ for the system mean defined by

$$\mathrm{d}\bar{\mathbf{x}} = A\bar{\mathbf{x}}\,\mathrm{d}t + B\mathbf{u}\,\mathrm{d}t - L(\mathbf{y} - C\bar{\mathbf{x}})\mathrm{d}t \tag{6.120}$$

(compare this with Eq. (6.85)) and uses the control law

$$\mathbf{u} = F\bar{\mathbf{x}}, \tag{6.121}$$

the resulting controller is stable. This can be seen by considering the equations for the configuration $\mathbf{x}$ and the estimation error $\mathbf{e} = \mathbf{x} - \bar{\mathbf{x}}$ which obey the coupled equations

$$\mathrm{d}\mathbf{x} = (A + BF)\mathbf{x}\,\mathrm{d}t - BF\mathbf{e}\,\mathrm{d}t + E\,\mathrm{d}\mathbf{v}_\mathrm{p}, \tag{6.122}$$

$$\mathrm{d}\mathbf{e} = (A + LC)\mathbf{e}\,\mathrm{d}t + L\,\mathrm{d}\mathbf{v}_\mathrm{m} + E\,\mathrm{d}\mathbf{v}_\mathrm{p}. \tag{6.123}$$

Exercise 6.16 *Derive these.*

Strict stability of $A + BF$ and $A + LC$ guarantees that $\langle\mathbf{x}\rangle \to \mathbf{0}$ and that $\mathbf{e}$ has a bounded variance, so that $V = \langle\mathbf{x}\mathbf{x}^\top\rangle$ is bounded also.

Control costs and pacifiability. For a stable asymptotic problem the stochastic dynamics in the long-time limit are governed by the Ornstein–Uhlenbeck process (6.119). This has a closed-form solution, and thus so does the controller $\mathbf{u}$ from Eq. (6.114). Hence any statistical properties of the system and controller can be determined. For example, the stationary variance of $\mathbf{x}$ is given by Eq. (6.106). From Eq. (6.119), it thus follows that

$$N(V_\mathrm{ss} - W) + (V_\mathrm{ss} - W)N^\top + \mathrm{F}^\top\mathrm{F} = 0, \tag{6.124}$$

and hence V_ss can be determined since N is strictly stable.

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One quantity we are particularly interested in is the integrand in the cost function, which has the stationary expectation value

$$\mathrm{E}_{\mathrm{ss}}[h] = \mathrm{tr}[P V_{\mathrm{ss}}] + \mathrm{tr}[Q K (V_{\mathrm{ss}} - W) K^{\top}]. \tag{6.125}$$

Here the stationary expectation value means at a time long after t_0, but long before t_1, so that both the initial conditions on $\mathbf{x}$ and the final condition P_1 on the control are irrelevant. For ease of notation, we simply use K to denote the stationary value for $K(t)$. From the above results it is not too difficult to show that Eq. (6.125) evaluates to

$$\mathrm{E}_{\mathrm{ss}}[h] = \mathrm{tr}[Y B Q^{-1} B^{\top} Y W] + \mathrm{tr}[Y D]. \tag{6.126}$$

Note that this result depends implicitly upon A, C, Γ and P through W and Y.

Exercise 6.17 *Derive Eq. (6.125) and verify that it is equivalent to Eq. (6.126).*

It might be thought that if control is cheap ($Q \to 0$) then the gain K will be arbitrarily large, and hence $N = A - BK$ will be such that the fluctuations in $\langle \mathbf{x} \rangle_{\mathrm{c}}$ will be completely suppressed. That is, from Eq. (6.124), it might be thought that the distinction between the conditioned W and unconditioned V_{ss} covariance matrix will vanish. However, this will be the case only if B allows a sufficient degree of control over the system. Specifically, it can be seen from Eq. (6.124) (or perhaps more clearly from Eq. (6.119)) that what is required is for the columns of $\mathrm{F}^{\top}$ to be in the column space of B (see Box 6.1). This is equivalent to the condition that

$$\mathrm{rank}[B] = \mathrm{rank}[B \ \mathrm{F}^{\top}]. \tag{6.127}$$

We will call a system that satisfies this condition, for $\mathrm{F} = CW + \Gamma$ with W a stabilizing solution, *pacifiable*.

Note that, unlike the concepts of stabilizability and controllability, the notion of pacifiability relies not only upon the unconditioned evolution (matrices B and A), but also upon the measurement via C and Γ (both explicitly in F and implicitly through W). Thus it cannot be said that pacifiability is stronger or weaker than stabilizability or controllability. However, if B is full row rank then all three notions will be satisfied. In this case, for cheap control the solution Y of Eq. (6.117) will scale as $Q^{1/2}$, and we can approximate Y by the solution to the equation

$$Y B Q^{-1} B^{\top} Y = P, \tag{6.128}$$

which is independent of the system dynamics.

Exercise 6.18 *Convince yourself of this.*

In this case, the second term in Eq. (6.126) scales as $Q^{1/2}$ so that

$$\mathrm{E}_{\mathrm{ss}}[h] \to \mathrm{tr}[P W]. \tag{6.129}$$

Realistic control constraints. While it seems perfectly reasonable to consider minimizing a quadratic function of system variables (such as the energy of a harmonic oscillator), it

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might be questioned whether the quadratic cost associated with the inputs $\mathbf{u}$ is an accurate reflection of the control constraints in a given instance. For instance, in an experiment on a microscopic physical system the power consumption of the controller is typically not a concern. Rather, one tries to optimize one's control of the system within the constraints of the apparatus one has built. For example one might wish to put bounds on $\mathrm{E}[\mathbf{u}\mathbf{u}^\top]$ in order that the apparatus does produce the desired change in the system configuration, $B\mathbf{u}\,\mathrm{d}t$, for a given input $\mathbf{u}$.[2] That is, we could require

$$J - K(V - V_\mathrm{c})K^\top \geq 0 \tag{6.130}$$

for some PSD matrix J. The genuinely optimal control for this physical problem would saturate the LMI (6.130). To discover the control law that achieves this optimum it would be necessary to follow an iterative procedure to find the Q that minimizes j while respecting Eq. (6.130).

Another sort of constraint that arises naturally in manipulating microscopic systems is time delays and bandwidth problems in general. This can be dealt with in a systematic manner by introducing extra variables that are included within the system configuration $\mathbf{x}$, as discussed in Ref. [BM04]. To take a simple illustration, for feedback with a delay time τ, the Langevin equation would be

$$\mathrm{d}\mathbf{x} = A\mathbf{x}\,\mathrm{d}t + B\mathbf{u}(t-\tau)\mathrm{d}t + E\,\mathrm{d}\mathbf{v}_\mathrm{p}(t). \tag{6.131}$$

To describe this exactly would require infinite order derivatives, and hence an infinite number of extra variables. However, as a crude approximation (which is expected to be reasonable for sufficiently short delays) we can make a first-order Taylor expansion, to write

$$\mathrm{d}\mathbf{x} = A\mathbf{x}\,\mathrm{d}t + B[\mathbf{u}(t) - \tau\dot{\mathbf{u}}(t)]\mathrm{d}t + E\,\mathrm{d}\mathbf{v}_\mathrm{p}(t). \tag{6.132}$$

Now we define new variables as follows:

$$\mathbf{u}'(t) = -\tau\dot{\mathbf{u}}(t), \tag{6.133}$$

$$\mathbf{x}' = \mathbf{u}(t), \tag{6.134}$$

such that $\mathbf{u}'$ is to be considered the new control variable and $\mathbf{x}'$ an extra system variable. Thus the system Langevin equation would be replaced by the pair of equations

$$\mathrm{d}\mathbf{x} = [A\mathbf{x} + B\mathbf{x}']\mathrm{d}t + B\mathbf{u}'(t)\mathrm{d}t + E\,\mathrm{d}\mathbf{v}_\mathrm{p}(t), \tag{6.135}$$

$$\mathrm{d}\mathbf{x}' = B'\mathbf{u}'(t)\mathrm{d}t, \tag{6.136}$$

where $B' = -(1/\tau)I$. Note that there is no noise in the equation of $\mathbf{x}'$, so the observer will have no uncertainty about these variables.

[2] Actually, it would be even more natural to put absolute bounds on $\mathbf{u}$, rather than mean-square bounds. However, such non-quadratic bounds cannot be treated within the LQG framework.

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If there were no costs assigned with either $\mathbf{x}'$ or $\mathbf{u}'$ then the above procedure would be nullified, since one could always choose $\mathbf{u}'$ such that

$$\mathbf{u}'(t) = K\langle \mathbf{x} \rangle_{\mathrm{c}}(t) - \mathbf{x}', \tag{6.137}$$

which would lead to the usual equation for LQG feedback with no delay. But note that this equation can be rewritten as

$$\tau \dot{\mathbf{x}}' = \mathbf{x}' - K\langle \mathbf{x} \rangle_{\mathrm{c}}(t). \tag{6.138}$$

This is an unstable equation for $\mathbf{x}'$, so, as long as some suitable finite cost is assigned to $\mathbf{x}'$ and/or $\mathbf{u}'$, the choice (6.137) would be ruled out. Costs on the control $\mathbf{u}$ in the original problem translate into a corresponding cost on $\mathbf{x}'$ in the new formulation, while a cost placed on $\mathbf{u}'$ would reflect limitations on how fast the control signal $\mathbf{u}$ can be modified. In practice there is considerable arbitrariness in how the cost functions are assigned.

6.4.5 Markovian feedback

General principles. As discussed above, for an additive cost function, state-based feedback (where $\mathbf{u}(t)$ is a function of the conditional state $\wp_{\mathrm{c}}(\check{\mathbf{x}}; t)$) is optimal. In this section we consider a different (and hence non-optimal) sort of feedback: Markovian feedback. By this we mean that the system input $\mathbf{u}(t)$ is a function of the *just-recorded* current $\mathbf{y}(t)$. We have already considered such feedback in the quantum setting in the preceding chapter, in Sections 5.4 and 5.5. This sort of feedback is not so commonly considered in the classical setting, but we will see that for linear systems much of the analysis of optimal feedback also applies to Markovian feedback.

The name 'Markovian feedback' is appropriate because it leads to Markovian evolution of the system, described by a Markovian Langevin equation. For example, for the linear system we have been considering, the control law is

$$\mathbf{u}(t) = L\mathbf{y}(t), \tag{6.139}$$

with L a matrix that could be time-dependent, but for strict Markovicity would not be. The Langevin equation for the system configuration is then

$$\mathrm{d}\mathbf{x} = A\mathbf{x}\,\mathrm{d}t + BL\mathbf{y}\,\mathrm{d}t + E\,\mathrm{d}\mathbf{v}_{\mathrm{p}}(t) \tag{6.140}$$

$$= (A + BLC)\mathbf{x}\,\mathrm{d}t + BL\,\mathrm{d}\mathbf{v}_{\mathrm{m}} + E\,\mathrm{d}\mathbf{v}_{\mathrm{p}}. \tag{6.141}$$

Note that for Markovian feedback it is not necessary to assume or derive Eq. (6.139); any function of the instantaneous current $\mathbf{y}(t)$ that is not linear is not well defined. That is, if one wishes to have Markovian system dynamics then one can only consider what engineers call *proportional feedback*. It should be noted that $\mathbf{y}(t)$ has unbounded variation, so Markovian control is no less onerous than optimal control with unbounded $K(t)$, as occurs for zero control cost, $Q \to 0$. In both cases this is an idealization, since in any physical realisation

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both the measured current and the controller response would roll off in some way at high frequency.

The motivation for considering Markovian feedback is that it is much simpler than optimal feedback. Optimal feedback requires processing or filtering the current $\mathbf{y}(t)$ in an optimal way to determine the state $\wp_c(\check{\mathbf{x}})$ (or, in the LQG case, just its mean $\langle \mathbf{x} \rangle_c(t)$). Markovian feedback is much simpler to implement experimentally. One notable example of the use of Markovian feedback is the feedback-cooling of a single electron in a harmonic trap (in the classical regime) [DOG03], which we discuss below. Markovian feedback is also much simpler to describe theoretically, since it requires only a model of the system instead of a model of the system plus the estimator and the actuator.

The simplicity of Markovian feedback can be seen in that Eq. (6.141) can be turned directly into an OUE. The moment equations are as in Eqs. (6.50) and (6.51) but with drift and diffusion matrices

$$A' = A + BLC, \tag{6.142}$$

$$D' = D + BLL^\top B^\top + BL\Gamma + \Gamma^\top L^\top B^\top. \tag{6.143}$$

Exercise 6.19 *Derive these.*

Recall that the stationary covariance matrix satisfies

$$A' V_{ss} + V_{ss} A'^\top + D' = 0. \tag{6.144}$$

As for an asymptotic LQG problem with no control costs, the aim of the feedback would be to minimize $\mathrm{tr}[PV_{ss}]$ for some PSD matrix P. If B is full row rank and (C, A) is detectable, then by the definition of detectability it is possible to choose an L such that A' is strictly stable. It might be thought that the optimal Markovian feedback would have $\|L\|$ large in order to make the eigenvalues of A' as negative as possible. However, this is not the case in general, because L also affects the diffusion term, and if $\|L\| \to \infty$ then so does D' (quadratically) so that $\|V_{ss}\| \to \infty$ also. Thus there is in general an optimal value for L, to which we return after the following experimental example.

Experimental example: cooling a one-electron oscillator. The existence of an optimal feedback strength for Markovian feedback (in contrast to the case for state-based feedback) is well illustrated in the recent experiment performed at Harvard [DOG03]. Their system was a single electron in a harmonic trap of frequency $\omega = 2\pi \times 65$ MHz, coupled electromagnetically to an external circuit. This induces a current through a resistor, which dissipates energy, causing damping of the electron's motion at rate $\gamma \approx 2\pi \times 8.4$ Hz. Because the resistor is at finite temperature $T \approx 5.2$ K, the coupling also introduces thermal noise into the electron's motion, so that it comes to thermal equilibrium at temperature T. The damping rate γ is seven orders of magnitude smaller than the oscillation frequency, so a secular approximation (explained below) is extremely well justified.

We can describe the motion by the complex amplitude

$$\alpha(t) = \mathrm{e}^{\mathrm{i}\omega t}[x(t) - \mathrm{i}p(t)]. \tag{6.145}$$

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Here (for convenience) x is the electron *momentum* divided by $\sqrt{2m}$ and p is the electron *position* divided by $\sqrt{2/(m\omega^2)}$, where m is the electron mass. The complex exponential in the definition of α removes the fast oscillation from its evolution, so that it should obey the Langevin equation

$$\mathrm{d}\alpha = -\frac{\gamma}{2}\alpha\,\mathrm{d}t + \sqrt{\gamma T}[\mathrm{d}v_1(t) + \mathrm{i}\,\mathrm{d}v_2(t)]/\sqrt{2}, \tag{6.146}$$

where $\mathrm{d}v_1$ and $\mathrm{d}v_2$ are independent Wiener increments. This equation ensures that $\wp_{\mathrm{ss}}(\alpha)$ is a Gaussian that is independent of the phase of α, has a mean of zero, and has

$$\mathrm{E}_{\mathrm{ss}}[|\alpha|^2] = \mathrm{E}_{\mathrm{ss}}[x^2 + p^2] = 2\mathrm{E}_{\mathrm{ss}}[x^2] = T. \tag{6.147}$$

This is as required by the equipartition theorem since $|\alpha|^2$ equals the total energy (we are using units for which $k_{\mathrm{B}} \equiv 1$). For cooling the electron we wish to reduce this mean $|\alpha|^2$.

From Eq. (6.146), the steady-state rate of energy loss from the electron due to the damping (which is balanced by energy gain from the noisy environment) is

$$P = \gamma \mathrm{E}_{\mathrm{ss}}[|\alpha|^2] = 2\gamma \mathrm{E}_{\mathrm{ss}}[x^2]. \tag{6.148}$$

This can be identified with I^2R, the power dissipated in the resistor, so if $I \propto x$ we must have

$$I = \sqrt{2\gamma/R}\,x. \tag{6.149}$$

The voltage drop across the resistor is $V = V_{\mathrm{J}} + IR$, where V_{J} is Johnson (also known as Nyquist) noise. Taking this noise to be white, as is the usual approximation, we have in differential form [Gar85],

$$V\,\mathrm{d}t = \sqrt{2\gamma R}\,x\,\mathrm{d}t + \sqrt{2TR}\,\mathrm{d}v_{\mathrm{J}}. \tag{6.150}$$

But it is this voltage that drives the motion of the electron, giving a term so that the equation for x is

$$\mathrm{d}x = -\omega p\,\mathrm{d}t - \beta V\,\mathrm{d}t \tag{6.151}$$

for some coupling β.

The damping of α at rate $\gamma/2$ arises from this coupling of the electron to the resistor. To derive this we must obtain a damping term $-\gamma x\,\mathrm{d}t$ in Eq. (6.151), which requires that $\beta = \sqrt{\gamma/(2R)}$. This gives

$$\mathrm{d}x = -\omega p\,\mathrm{d}t - \gamma x\,\mathrm{d}t - \sqrt{\gamma T}\,\mathrm{d}v_{\mathrm{J}}, \tag{6.152}$$

together with the position equation $\mathrm{d}p = \omega x\,\mathrm{d}t$. On turning these into an equation for α and making a secular approximation by dropping terms rotating at frequency ω or 2ω, we do indeed end up with Eq. (6.146). Note that the secular approximation does not allow us to drop terms like $\mathrm{e}^{\mathrm{i}\omega t}\mathrm{d}v_{\mathrm{J}}(t)$, because white noise by assumption fluctuates faster than any frequency. Instead, this term gives the complex noise $[\mathrm{d}v_1(t) + \mathrm{i}\,\mathrm{d}v_2(t)]/\sqrt{2}$, as can be verified by making the secular approximation on the *correlation function* for the Gaussian noise process $\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}v_{\mathrm{J}}(t)$.

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Exercise 6.20 *Verify this derivation of Eq. (6.146), and show that it has the steady-state solution with the moments (6.147).*

Note also that the equilibrium temperature T can be obtained by *ignoring the free evolution* (that is, deleting the $-\omega p\,\mathrm{d}t$ term in Eq. (6.152)), calculating $\mathrm{E}_{\mathrm{ss}}[x^2]$ and defining

$$T = 2\mathrm{E}_{\mathrm{ss}}[x^2]. \tag{6.153}$$

That is, the same expression holds as when the $-\omega p\,\mathrm{d}t$ is retained and the secular approximation made as in Eq. (6.147). This happy coincidence will be used later to simplify our description.

From Eq. (6.152), if the voltage were directly measured, the measurement noise would be perfectly correlated with the process noise. For the purpose of feedback, it is necessary to amplify this voltage. In practice this introduces noise into the fed-back signal. Thus, it is better to model the measured voltage as $\varepsilon/\sqrt{\gamma/(2R)}$, where

$$\varepsilon\,\mathrm{d}t = \gamma x\,\mathrm{d}t + \sqrt{\gamma T}\,\mathrm{d}v_{\mathrm{J}} + \sqrt{\gamma T_g}\,\mathrm{d}v_g \tag{6.154}$$

Here the temperature T_g (as used in Ref. [DOG03]) scales the amplifier noise $\mathrm{d}v_g$, and is, in their experiment, much less than T.

In this experiment it was easy to apply feedback using a second plate to which the electron motion also couples. Because the relevant system time-scale $\gamma^{-1} \sim 0.02\,\mathrm{s}$ is so long, the Markovian approximation (ignoring feedback delay time and bandwidth constraints) is excellent. It is not necessary to feed back fast compared with the oscillation frequency – as long as the phase of the fed-back signal is matched to that of the system the effect will be the same as if there were no time delay. As we will show, the effect of the feedback is simply to modify the damping rate and noise power in the system. But, rather than deal with an equation for two real variables (or an equation for a complex variable), we can instead ignore the $-\omega p\,\mathrm{d}t$ term in our equation for $\mathrm{d}x$. This is because the equation with feedback is of the same form as Eq. (6.152), so the same argument as was presented there applies here too. That is, this procedure leads to the same results as are obtained more carefully by applying the secular approximation to the full equations of motion. Thus, after introducing a feedback gain g, the system can be modelled by the following equation for one real variable:

$$\mathrm{d}x = -\gamma x\,\mathrm{d}t - \sqrt{\gamma T}\,\mathrm{d}v_{\mathrm{J}} + g\varepsilon\,\mathrm{d}t \tag{6.155}$$

$$= -(1-g)[\gamma x\,\mathrm{d}t + \sqrt{\gamma T}\,\mathrm{d}v_{\mathrm{J}}] + g\sqrt{\gamma T_g}\,\mathrm{d}v_g. \tag{6.156}$$

This gives a new system damping rate $\gamma_e = \gamma(1-g)$ and a new equilibrium system temperature

$$T_e \equiv 2\mathrm{E}_{\mathrm{ss}}[x^2] = T(1-g) + \frac{g^2 T_g}{1-g}. \tag{6.157}$$

For $T_g \ll T$, the new temperature T_e decreases linearly as g increases towards unity until a turning point at $g \simeq 1 - \sqrt{T_g/T}$, after which it increases rapidly with g. The minimal T_e,

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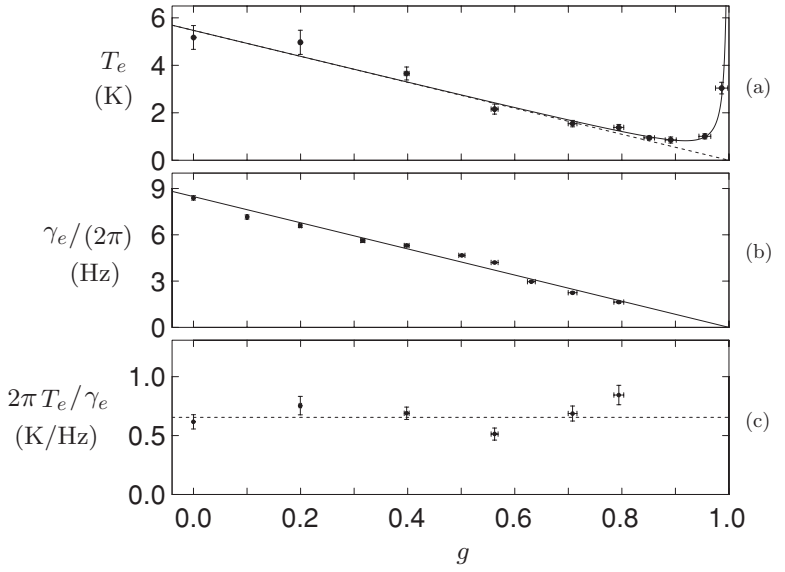

Fig. 6.6 Experimental results for cooling of a single electron by Markovian feedback [DOG03]. T_e is the equilibrium temperature, while γ_e is the measured damping rate of the electron's energy. The lines or curves are the theoretical predictions. Note the existence of an optimal gain g. Figure 5 adapted with permission from B. D'Urso *et al.*, *Phys. Rev. Lett.* **90**, 043001, (2003). Copyrighted by the American Physical Society.

at the optimal gain value, is $T_e \simeq 2\sqrt{T_g T}$. All of this was seen clearly in the experiment, with $T_g \approx 0.04\,\mathrm{K}$ giving a minimum $T_e \approx 0.85\,\mathrm{K}$, a six-fold reduction in temperature. This is shown in Fig. 6.6. The full expression (not given in Ref. [DOG03]) for the minimum temperature with feedback is

$$T_e = 2\left(\sqrt{T T_g + T_g^2} - T_g\right). \tag{6.158}$$

It is interesting to re-examine this system from the viewpoint of conditional dynamics. Ignoring the $-\omega p\,\mathrm{d}t$ term in Eq. (6.152), we have a one-dimensional system, so all of the matrices become scalars. From this equation and Eq. (6.154) it is easy to identify the following:

$$y(t) = [\gamma(T + T_g)]^{-1/2}\varepsilon(t), \tag{6.159}$$

$$A = -\gamma, \tag{6.160}$$

$$C = [\gamma/(T + T_g)]^{1/2}, \tag{6.161}$$

$$\Gamma = -T[\gamma/(T + T_g)]^{1/2}, \tag{6.162}$$

$$\tilde{A} = A - C\Gamma = -\gamma T_g/(T + T_g), \tag{6.163}$$

$$D = \gamma T, \tag{6.164}$$

$$\tilde{E}^2 = D - \Gamma^2 = \gamma T T_g/(T + T_g). \tag{6.165}$$

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It is trivial to verify that this system satisfies the conditions for there to exist a stabilizing solution W for the stationary conditioned variance equation

$$\tilde{A}W + W\tilde{A} + \tilde{E}\tilde{E}^\top = WC^\top CW. \tag{6.166}$$

By substituting into the above we find

$$W^2 + 2T_g W - TT_g = 0, \tag{6.167}$$

giving

$$W = -T_g + \sqrt{T_g^2 + TT_g}. \tag{6.168}$$

Remarkably, this expression, multiplied by two, is identical to the above expression for the minimum temperature (6.158). This identity is no coincidence. Recall that in steady state

$$T = 2\mathrm{E}_{\mathrm{ss}}[x^2] = 2W + 2\mathrm{E}_{\mathrm{ss}}[\langle x\rangle_{\mathrm{c}}^2]. \tag{6.169}$$

Thus $2W$ is a lower bound for the temperature T_e of Eq. (6.157). At the optimal value of feedback gain, the feedback exactly cancels out the noise in the equation for the conditional mean. We saw the same phenomenon for the one-dimensional quantum system considered in Section 5.6. Thus with optimal Markovian feedback $\langle x\rangle_{\mathrm{c}}(t) = 0$ in steady state, and $\mathrm{E}_{\mathrm{ss}}[x^2] = W$. Rather than show this explicitly, we show now that this can be done for any linear system that has a stabilizing solution W.

Understanding Markovian feedback by conditioning. Recall that we can write the conditional mean equation for an arbitrary linear system as

$$\mathrm{d}\langle\mathbf{x}\rangle_{\mathrm{c}} = \left[A - V_{\mathrm{c}}(t)C^\top C - \Gamma^\top C\right]\langle\mathbf{x}\rangle_{\mathrm{c}}\,\mathrm{d}t + B\mathbf{u}(t)\mathrm{d}t + \left[V_{\mathrm{c}}(t)C^\top + \Gamma^\top\right]\mathbf{y}(t)\mathrm{d}t. \tag{6.170}$$

Now, if we add Markovian feedback as defined above then the equation for the covariance matrix is of course unaffected, and that of the mean becomes

$$\mathrm{d}\langle\mathbf{x}\rangle_{\mathrm{c}} = \left[A - V_{\mathrm{c}}(t)C^\top C - \Gamma^\top C\right]\langle\mathbf{x}\rangle_{\mathrm{c}}\mathrm{d}t + \left[V_{\mathrm{c}}(t)C^\top + \Gamma^\top + BL\right]\mathbf{y}(t)\mathrm{d}t. \tag{6.171}$$

Now let us assume that B is such that for all times there exists an L satisfying

$$BL(t) = -V_{\mathrm{c}}(t)C^\top - \Gamma^\top. \tag{6.172}$$

(Obviously that will be the case if B is full row rank.) Then, for this choice of L, the equation for the conditioned mean is simply

$$\mathrm{d}\langle\mathbf{x}\rangle_{\mathrm{c}} = \left[A - V_{\mathrm{c}}(t)C^\top C - \Gamma^\top C\right]\langle\mathbf{x}\rangle_{\mathrm{c}}\,\mathrm{d}t. \tag{6.173}$$

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This is a *deterministic* equation. All noise in the conditional mean has been cancelled out by the feedback, so the unconditioned variance and conditioned variance are equal:

$$V(t) = V_{\mathrm{c}}(t), \tag{6.174}$$

$$\dot{V}_{\mathrm{c}} = AV_{\mathrm{c}} + V_{\mathrm{c}}A^{\top} + D - (V_{\mathrm{c}}C^{\top} + \Gamma^{\top})(CV_{\mathrm{c}} + \Gamma). \tag{6.175}$$

Strictly, the feedback scheme is Markovian only if L is time-independent. This makes sense if we are concerned with asymptotic problems, for which we should choose

$$BL = -WC^{\top} - \Gamma^{\top} = -\mathrm{F}^{\top}, \tag{6.176}$$

where W is the stationary conditioned covariance matrix, which we have assumed to be stabilizing. Such an L will exist provided that $\mathrm{rank}[B] = \mathrm{rank}[B\ \mathrm{F}^{\top}]$. That is, provided that the system is pacifiable (see Section 6.4.4). Then there is a unique (observer-independent) long-time behaviour for $\langle \mathbf{x} \rangle_{\mathrm{c}}$, governed by

$$\mathrm{d}\langle \mathbf{x} \rangle_{\mathrm{c}} = M\langle \mathbf{x} \rangle_{\mathrm{c}}\,\mathrm{d}t, \tag{6.177}$$

where $M = A - WC^{\top}C - \Gamma^{\top}C$. But this M is precisely the matrix defined earlier in Section 6.4.3: it is the matrix that is strictly stable iff W is a stabilizing solution. Therefore in the long-time limit $\langle \mathbf{x} \rangle_{\mathrm{c}} = \mathbf{0}$ and

$$V = W. \tag{6.178}$$

In this limit the current is pure noise:

$$\mathbf{y}\,\mathrm{d}t = \mathrm{d}\mathbf{w}(t). \tag{6.179}$$

This may be a useful fact experimentally for fine-tuning the feedback to achieve the desired result when the system parameters are not known exactly, as shown in the experiment [BRW+06] discussed in Section 5.8.2.

To reiterate, under the conditions

1. there exists a stabilizing solution W to Eq. (6.175)
2. there are no control costs ($Q \to 0$)
3. the system is pacifiable: $\mathrm{rank}[B] = \mathrm{rank}[B\ \mathrm{F}^{\top}]$, where $\mathrm{F} = CW + \Gamma$
4. the system costs apply only to the steady state ($t_1 \to \infty$)

the optimal Markovian feedback scheme is strictly stable and performs precisely as well as does the optimal state-based feedback scheme.

In fact, we can prove that the above Markovian feedback algorithm can be derived as a limit of the optimal feedback algorithm that arises when P is positive definite and $Q \to 0$. As discussed in Section 6.4.4, in this case the optimal control is such that BK acts as an infinite positive matrix on the column space of $\mathrm{F}^{\top}$. Recall that for optimal feedback the conditioned mean obeys

$$\mathrm{d}\langle \mathbf{x} \rangle_{\mathrm{c}}(t) = (M - BK)\langle \mathbf{x} \rangle_{\mathrm{c}}(t) + \mathrm{F}^{\top}\mathbf{y}(t), \tag{6.180}$$

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because $\mathbf{u}(t) = -K\langle\mathbf{x}\rangle_c(t)$. Taking the eigenvalues of BK to positive infinity, the solution to Eq. (6.180) is simply

$$B\mathbf{u}(t) = -\mathrm{F}^\top\mathbf{y}(t), \tag{6.181}$$

which is exactly as derived for Markovian feedback.

Exercise 6.21 *Derive Eq. (6.181).*
Hint: *Solve Eq. (6.180) by use of the Laplace transform, and expand in powers of s (the Laplace variable). Then take the $Q \to 0$ limit to eliminate all terms involving non-zero powers of s.*

The most important message is that the optimal Markovian scheme is intimately connected with the conditioned state $\wp_c(\check{\mathbf{x}}; t)$. It might be thought therefore that distinguishing Markovian feedback from state-based feedback is a false dichotomy. However, there is still an important distinction between the two: Markovian feedback can be described by Langevin equations for the system configuration $\mathbf{x}$; state-based feedback requires the addition of (at minimum) the configuration of the estimator, $\langle\mathbf{x}\rangle_c$. Moreover, we will see in Section 6.6.7 that if the system is not pacifiable then Markovian feedback is generally inferior.

6.5 General quantum systems

This section mirrors Section 6.3 but for quantum systems. That is, we introduce some new notation and terminology, then consider the diffusive unravelling of general Markovian master equations, and finally make some general remarks on optimal feedback control.

6.5.1 Notation and terminology

In quantum mechanics the system configuration $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ is represented not by real numbers, but rather by self-adjoint linear operators on a Hilbert space. We write $\mathfrak{L}(\mathbb{H})$ for the set of all linear operators on a Hilbert space $\mathbb{H}$, and $\mathfrak{D}(\mathbb{H}) \subset \mathfrak{L}(\mathbb{H})$ for all self-adjoint operators. Thus $\forall\, k,\ \hat{x}_k \in \mathfrak{D}(\mathbb{H})$. To obtain $\{\lambda(\hat{x})\} = \mathbb{R}$ it is necessary to consider an infinite-dimensional Hilbert space. As in the classical case, we assume that $\hat{\mathbf{x}}$ is a complete set of observables in the sense that any operator in $\mathfrak{D}(\mathbb{H})$ can be expressed as a function of $\hat{\mathbf{x}}$, but that this is not so if any element $\hat{x}_k$ of $\hat{\mathbf{x}}$ is omitted.

We denote the set of state matrices as $\mathfrak{S}(\mathbb{H}) \subset \mathfrak{D}(\mathbb{H})$. For an observable x with $\{\lambda(\hat{x})\} = \mathbb{R}$, the quantum state ρ defines a function

$$\wp(\check{x}) = \mathrm{Tr}[\rho\delta(\check{x} - \hat{x})], \tag{6.182}$$

which is the probability distribution for x. Note that a multi-dimensional $\delta^{(n)}(\check{\mathbf{x}} - \hat{\mathbf{x}})$ cannot be defined as an operator in $\mathfrak{D}(\mathbb{H})$ because of the non-commutativity of the elements of $\hat{\mathbf{x}}$.

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We will also introduce a new notation for the general Lindblad master equation:

$$\hbar\dot{\rho} = -\mathrm{i}[\hat{H}, \rho] + \mathcal{D}[\hat{\mathbf{c}}]\rho. \tag{6.183}$$

Here $\hat{H} = \hat{H}^\dagger$ as before, we have defined a vector of operators $\hat{\mathbf{c}} = (\hat{c}_1, \ldots, \hat{c}_L)^\top$, and

$$\mathcal{D}[\hat{\mathbf{c}}] \equiv \sum_{l=1}^{L} \mathcal{D}[\hat{c}_l]. \tag{6.184}$$

Note that we have introduced Planck's constant $\hbar \approx 10^{-34}\,\mathrm{J\,s}$ on the left-hand side of Eq. (6.183). This is simply a matter of redefinition of units and is necessary in order to connect quantum operators with their classical counterparts. For example, in this case it is necessary if $\hat{H}$ is to correspond to the classical Hamiltonian function. We will see later in Section 6.6 that keeping $\hbar$ in the formulae, rather than setting $\hbar = 1$ as we have been doing, is useful for keeping track of what is distinctively quantum about quantum control of linear systems.

Equation (6.183) can be derived by generalizing the system–bath coupling introduced in Section 3.11 by introducing L independent baths. Then the Itô stochastic differential equation for the unitary operator that generates the evolution of the system and bath observables obeys

$$\hbar\,\mathrm{d}\hat{U}(t, t_0) = -\Big[\mathrm{d}t(\hat{\mathbf{c}}^\dagger\hat{\mathbf{c}}/2 + \mathrm{i}\hat{H}) + (\hat{\mathbf{c}}^\dagger\,\mathrm{d}\hat{\mathbf{B}}_{z:=-t} - \mathrm{d}\hat{\mathbf{B}}^\dagger_{z:=-t}\,\hat{\mathbf{c}})\Big]\hat{U}(t, t_0). \tag{6.185}$$

Note here that $\hat{\mathbf{c}}^\dagger$ means the transpose of the vector as well as the Hermitian adjoint of the operators. The vector of operators $\mathrm{d}\hat{\mathbf{B}}_{z:=-t}$ has all second-order moments equal to zero except for

$$\mathrm{d}\hat{\mathbf{B}}_{z:=-t}\,\mathrm{d}\hat{\mathbf{B}}^\dagger_{z:=-t} = I\hbar\,\mathrm{d}t, \tag{6.186}$$

where I is the $L \times L$ identity. Note the appearance of $\hbar$ here also.

6.5.2 The Belavkin equation

The master equation (6.183) may be thought of as a quantum analogue of a general Fokker–Planck equation. If we restrict ourselves to diffusive unravellings (see Section 4.5.2), the resultant stochastic master equation can be considered an analogue of the Kushner–Stratonovich equation. This form of the stochastic master equation is sometimes called the Belavkin equation, since diffusive unravellings were first formulated by Belavkin [Bel88] in the mathematical physics literature. (He called it a quantum filtering equation.)

We have already presented a general diffusive unravelling in Section 4.5.2. We present it here again for two reasons. First, we allow for inefficient detection, which we ignored in Section 4.5.2. Second, we use different notation in order to be consistent with the convention in this chapter of capital letters for matrices and small bold letters (usually lower case) for

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vectors. The most general Belavkin equation compatible with Eq. (6.183) is

$$\hbar\,\mathrm{d}\rho_{\mathrm{c}} = \mathrm{d}t\,\mathcal{D}[\hat{\mathbf{c}}]\rho_{\mathrm{c}} + \mathcal{H}[-\mathrm{i}\hat{H}\,\mathrm{d}t + \mathrm{d}\mathbf{z}^{\dagger}(t)\hat{\mathbf{c}}]\rho_{\mathrm{c}}. \tag{6.187}$$

Here we are defining $\mathrm{d}\mathbf{z} = (\mathrm{d}z_1, \ldots, \mathrm{d}z_L)^{\top}$, a vector of infinitesimal complex Wiener increments. Like $\mathrm{d}w$, these are c-number innovations and $\mathrm{d}\mathbf{z}^{\dagger}$ simply means $(\mathrm{d}\mathbf{z}^{*})^{\top}$. Recall that $\mathcal{H}$ is the nonlinear superoperator defined in Eq. (4.24).

The innovations vector $\mathrm{d}\mathbf{z}$ satisfies $\mathrm{E}[\mathrm{d}\mathbf{z}] = 0$, and has the correlations

$$\mathrm{d}\mathbf{z}\,\mathrm{d}\mathbf{z}^{\dagger} = \hbar\mathrm{H}\,\mathrm{d}t, \qquad \mathrm{d}\mathbf{z}\,\mathrm{d}\mathbf{z}^{\top} = \hbar\Upsilon\,\mathrm{d}t, \tag{6.188}$$

where Υ is a complex *symmetric* matrix. Here H (capital η) allows for inefficient detection. The set of allowed Hs is

$$\mathfrak{H} = \left\{\mathrm{H} = \mathrm{diag}(\eta_1, \ddots, \eta_L)\colon \forall\, l,\, \eta_l \in [0,1]\right\}. \tag{6.189}$$

Here η_l can be interpreted as the efficiency of monitoring the lth output channel. This allows for conditional evolution that does not preserve the purity of states when $\mathrm{H} \neq I$.

It is convenient to combine Υ and H in an *unravelling matrix*

$$U = U(\mathrm{H}, \Upsilon) \equiv \frac{1}{2}\begin{pmatrix} \mathrm{H} + \mathrm{Re}[\Upsilon] & \mathrm{Im}[\Upsilon] \\ \mathrm{Im}[\Upsilon] & \mathrm{H} - \mathrm{Re}[\Upsilon] \end{pmatrix}. \tag{6.190}$$

The set $\mathfrak{U}$ of valid Us can then be defined by

$$\mathfrak{U} = \left\{U(\mathrm{H}, \Upsilon)\colon \Upsilon = \Upsilon^{\top},\, \mathrm{H} \in \mathfrak{H},\, U(\mathrm{H}, \Upsilon) \geq 0\right\}. \tag{6.191}$$

Note the requirement that U be PSD.

The output upon which the conditioned state of Eq. (6.187) is conditioned can be written as a vector of complex currents

$$\mathbf{J}^{\top}\,\mathrm{d}t = \left\langle \hat{\mathbf{c}}^{\top}\mathrm{H} + \hat{\mathbf{c}}^{\dagger}\Upsilon \right\rangle_{\mathrm{c}}\mathrm{d}t + \mathrm{d}\mathbf{z}^{\top}. \tag{6.192}$$

In the Heisenberg picture, the output is represented by the following operator:

$$\begin{aligned}\hat{\mathbf{J}}^{\top}\mathrm{d}t \;=\; & \mathrm{d}\hat{\mathbf{B}}_{\mathrm{out}}^{\top}\,\mathrm{H} + \mathrm{d}\hat{\mathbf{B}}_{\mathrm{out}}^{\dagger}\,\Upsilon + \mathrm{d}\hat{\mathbf{A}}^{\dagger}\sqrt{\mathrm{H} - \Upsilon^{*}\mathrm{H}^{-1}\Upsilon} \\ & + \mathrm{d}\hat{\mathbf{V}}^{\top}\sqrt{\mathrm{H}(I - \mathrm{H})} + \mathrm{d}\hat{\mathbf{V}}^{\dagger}\sqrt{\mathrm{H}^{-1} - I}\;\Upsilon.\end{aligned} \tag{6.193}$$

This is the generalization of Eq. (4.210) to allow for inefficent detection, with *two* vectors of ancillary annihilation operators $\mathrm{d}\hat{\mathbf{A}}$ and $\mathrm{d}\hat{\mathbf{V}}$ understood to act on other (ancillary) baths in the vacuum state. Thus, for example,

$$[\mathrm{d}\hat{A}_l, \mathrm{d}\hat{A}_j^{\dagger}] = \hbar\delta_{lj}\,\mathrm{d}t. \tag{6.194}$$

Exercise 6.22 *Show that all of the components of* $\hat{\mathbf{J}}$ *and* $\hat{\mathbf{J}}^{\dagger}$ *commute with one another, as required.*

Note that the restriction on Υ enforced by the requirement $U \geq 0$ ensures that the appearances of the matrix inverse H^{-1} in Eq. (6.193) do not cause problems even if H is not

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positive definite. This restriction also implies that all of the matrices under the radical signs in Eq. (6.193) are PSD, so that the square roots here can be unambiguously defined. Finally, note also that, for efficient monitoring ($\mathrm{H} = I$), the ancillary operators $\mathrm{d}\hat{\mathbf{V}}$ and $\mathrm{d}\hat{\mathbf{V}}^\dagger$ are not needed, but $\mathrm{d}\hat{\mathbf{A}}$ still is, as in Eq. (4.210).

6.5.3 Optimal feedback control

The discussion of optimal feedback control for classical systems in Section 6.3.3 applies to quantum systems with essentially no changes. To re-establish the notation, we will use $\mathbf{y}(t)$ to represent the result of the monitoring rather than $\mathbf{J}(t)$ as above. In the Heisenberg picture, this will be an operator $\hat{\mathbf{y}}(t)$ just as $\hat{\mathbf{J}}(t)$ is in Eq. (6.193). Similarly, the feedback signal $\mathbf{u}(t)$, a functional of $\mathbf{y}(t)$, will be an operator $\hat{\mathbf{u}}(t)$ in the Heisenberg picture. Given the restrictions on quantum dynamics, we cannot postulate arbitrary terms in the evolution equation for the system configuration dependent upon $\mathbf{u}(t)$. Rather we must work within the structure of quantum dynamics, for example by postulating a feedback Hamiltonian that is a function of $\mathbf{u}(t)$:

$$\hat{H}_{\mathrm{fb}}(t) = \hat{F}\big(\mathbf{u}(t), t\big). \tag{6.195}$$

Say the aim of the control is to minimize a cost function that is additive in time. In the Heisenberg picture we can write the minimand as

$$j = \int_{t_0}^{t_1} \big\langle h\big(\hat{\mathbf{x}}, \hat{\mathbf{u}}, t\big)\big\rangle \mathrm{d}t, \tag{6.196}$$

where $\mathbf{x}$ and $\mathbf{u}$ are implicitly time-dependent as usual, and the expectation value is taken using the initial state of the system and bath, including any ancillary baths needed to define the current such as in Eq. (6.193).

In the Schrödinger picture, the current can be treated as a c-number, and we need only the system state. However, this state $\rho_{\mathrm{c}}(t)$ is conditioned and hence stochastic, so we must also take an ensemble average over this stochasticity. Also, the system variables must still be represented by operators, $\hat{\mathbf{x}}$, so that the final expression is

$$j = \int_{t_0}^{t_1} \mathrm{E}\{\mathrm{Tr}[\rho_{\mathrm{c}}(t) h(\hat{\mathbf{x}}, \mathbf{u}, t)]\} \mathrm{d}t. \tag{6.197}$$

As in Section 6.3.3, for an additive cost function the separation principle holds. This was first pointed out by Belavkin [Bel83], who also developed the quantum Bellman equation [Bel83, Bel88, Bel99] (see also Refs. [DHJ$^+$00, Jam04] and Ref. [BvH08] for a rigorous treatment). The quantum separation principle means that the optimal control strategy is quantum-state-based:

$$\mathbf{u}_{\mathrm{opt}} = \mathcal{U}_h\big(\rho_{\mathrm{c}}(t), t\big). \tag{6.198}$$

Even in the Heisenberg picture this equation will hold, but with hats on. The conditional state $\rho_{\mathrm{c}}(t)$ is a functional (or a filter in the broad sense of the word) of the output $\mathbf{y}(t)$.

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So, in the Heisenberg picture, $\hat{\mathbf{y}}(t)$ begets $\hat{\rho}_{\mathrm{c}}(t)$, which begets $\hat{\mathbf{u}}_{\mathrm{opt}}(t)$. Note the distinction between $\hat{\rho}_{\mathrm{c}}(t)$ and $\rho_{\mathrm{c}}(t)$, the latter being a state conditioned upon a c-number measurement record.

6.6 Linear quantum systems

We turn now to linear quantum systems. By linear quantum systems we mean the analogue of linear classical systems as discussed in Section 6.4. This section mirrors that classical presentation, but there are several differences that it is necessary, or at least interesting, to discuss. Thus we include additional subsections on the structure of quantum phase-space and on optimal unravellings.

Before beginning our detailed treatment, we take this opportunity to make the following point. It is commonly stated, see for example [AD01], that quantum control is a *bilinear* control problem, because even if the time-dependent Hamiltonian is linear in the control signal, such as $\hat{F}u(t)$, the equation of motion for the quantum state is not linear in $u(t)$. Rather, taking $\rho(t)$ and $u(t)$ together as describing the control loop, we have the bilinear equation

$$\hbar\dot{\rho} = -\mathrm{i}u(t)[\hat{F}, \rho(t)]. \tag{6.199}$$

However, from our point of view, there is nothing peculiarly quantum about this situation. The classical control problem, expressed in terms of the classical state, has a completely analogous form. For example, consider a classical system, with configuration (q, p), and a classical control Hamiltonian $u(t)F(q, p)$. Then the equation for the classical state is

$$\dot{\wp}(q, p; t) = u(t)\{F(q, p), \wp(q, p; t)\}_{\mathrm{PB}}, \tag{6.200}$$

where $\{\cdot, \cdot\}_{\mathrm{PB}}$ denotes a Poisson bracket [GPS01].

The existence of this bilinear description of classical control of course does not preclude a linear description in terms of the system configuration. Exactly as in the classical case, a quantum system may have linear dynamics (i.e. a suitable set of observables may have linear Heisenberg equations of motion), and the measured current may be a linear function of these observables, with added Gaussian noise (which will be operator-valued). As we will see, a great deal of classical control theory for such linear systems goes over to the quantum case.

6.6.1 Quantum phase-space

In order to obtain linear dynamics we require observables with unbounded spectrum. For example, the position $\hat{q}$ of a particle has $\{\lambda(\hat{q})\} = \mathbb{R}$. Say our system has N such position observables, which all commute. To obtain a complete set of such observables, for each $\hat{q}_m$ we must include a *canonically conjugate* momentum $\hat{p}_m$. It satisfies the canonical

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commutation relation

$$[\hat{p}_m, \hat{q}_m] = -\mathrm{i}\hbar \tag{6.201}$$

with its partner, but commutes with all other positions and momenta. To connect with the classical theory, we write our complete set of observables as

$$\hat{\mathbf{x}}^\top = (\hat{q}_1, \hat{p}_1, \hat{q}_2, \hat{p}_2, \ldots, \hat{q}_N, \hat{p}_N). \tag{6.202}$$

Then the commutation relations they satisfy can be written

$$[\hat{x}_{m'}, \hat{x}_m] = \mathrm{i}\hbar\Sigma_{m',m}, \tag{6.203}$$

where Σ is a $(2N) \times (2N)$ skew-symmetric matrix with the following block-diagonal form:

$$\Sigma = \bigoplus_1^N \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{6.204}$$

This matrix, called the symplectic matrix, is an orthogonal matrix. That is, it satisfies $\Sigma^{-1} = \Sigma^\top = -\Sigma$. This means that $\mathrm{i}\Sigma$ is Hermitian. In this situation, the configuration space is usually called 'phase-space' and the term 'configuration space' is reserved for the space in which $\mathbf{q}$ resides. However, we will not use 'configuration space' with this meaning.

A consequence of the canonical commutation relation is the Schrödinger–Heisenberg uncertainty relation [Sch30], which for any given conjugate pair $(\hat{q}, \hat{p})$ is

$$V_q V_p - C_{qp}^2 \geq (\hbar/2)^2. \tag{6.205}$$

Here the variances are $V_q = \langle(\Delta\hat{q})^2\rangle$ and V_p similarly, while the covariance $C_{qp} = \langle(\Delta\hat{q})(\Delta\hat{p}) + (\Delta\hat{q})(\Delta\hat{p})\rangle/2$. Note the symmetrization necessary in the covariance because of the non-commutation of the deviation terms, defined for an arbitrary observable $\hat{o}$ as $\Delta\hat{o} = \hat{o} - \langle\hat{o}\rangle$. The original Heisenberg uncertainty relation (A.10) [Hei27] is weaker, lacking the term involving the covariance. Using the matrix Σ and the covariance matrix

$$V_{m',m} = V_{m,m'} = \langle\Delta\hat{x}_{m'}\,\Delta\hat{x}_m + \Delta\hat{x}_m\,\Delta\hat{x}_{m'}\rangle/2, \tag{6.206}$$

we can write the Schrödinger–Heisenberg uncertainty relation as the linear matrix inequality [Hol82]

$$V + \mathrm{i}\hbar\Sigma/2 \geq 0. \tag{6.207}$$

This LMI can be derived immediately from Eqs. 6.203 and 6.206, since

$$V + \mathrm{i}\hbar\Sigma/2 = \left\langle(\Delta\hat{\mathbf{x}})(\Delta\hat{\mathbf{x}})^\top\right\rangle, \tag{6.208}$$

and the matrix on the right-hand side is PSD by construction. Since Σ is a real matrix, we can thus also define V by

$$V = \mathrm{Re}\left[\left\langle(\Delta\hat{\mathbf{x}})(\Delta\hat{\mathbf{x}})^\top\right\rangle\right]. \tag{6.209}$$

Recall that this means the real part of each element of the matrix.

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Exercise 6.23 *Show from Eq. (6.207) that V (if finite) must be positive definite.*
Hint: *First show that, if* $\mathbf{r}$ *and* $\mathbf{h}$ *are the real and imaginary parts of an eigenvector of* $V + \mathrm{i}\hbar\Sigma/2$ *with eigenvalue* λ*, then*

$$\begin{pmatrix} V & -\hbar\Sigma/2 \\ \hbar\Sigma/2 & V \end{pmatrix}\begin{pmatrix} \mathbf{r} \\ \mathbf{h} \end{pmatrix} = \lambda\begin{pmatrix} \mathbf{r} \\ \mathbf{h} \end{pmatrix}. \tag{6.210}$$

Then show that this real matrix cannot be positive if V has a zero eigenvalue, using the fact that Σ has full rank.

It is convenient to represent a quantum state for this type of system not as a state matrix ρ but as a Wigner function $W(\check{\mathbf{x}})$ – see Section A.5. This is a pseudo-probability distribution over a classical configuration corresponding to the quantum configuration $\hat{\mathbf{x}}$. It is related to ρ by

$$W(\check{\mathbf{x}}) = \langle \delta_W(\check{\mathbf{x}} - \hat{\mathbf{x}})\rangle = \mathrm{Tr}[\rho\delta_W(\check{\mathbf{x}} - \hat{\mathbf{x}})] \tag{6.211}$$

(*cf.* Eq. (6.27)), where

$$\delta_W(\check{\mathbf{x}} - \hat{\mathbf{x}}) = \int \mathrm{d}^{2N}\mathbf{k}\, \exp\bigl[2\pi\mathrm{i}\mathbf{k}^\top(\check{\mathbf{x}} - \hat{\mathbf{x}})\bigr]. \tag{6.212}$$

If the observables commuted then $\delta_W(\check{\mathbf{x}} - \hat{\mathbf{x}})$ would become a Dirac δ-function $\delta^{(2N)}(\check{\mathbf{x}} - \hat{\mathbf{x}})$. Recall from Section 6.5.1 that the latter object does not exist, which is why the state of a quantum system cannot in general be represented by a true probability distribution over the values of its observables. The Wigner function evades this because $\delta_W(\check{\mathbf{x}} - \hat{\mathbf{x}})$ is not PSD. Thus, for some states, the Wigner function will take negative values for some $\check{\mathbf{x}}$. Nevertheless, it is easy to verify that for any subset of N *commuting* observables the marginal Wigner function is a true probability distribution for those observables. For example,

$$\wp(\check{q}_1, \check{q}_2, \ldots, \check{q}_N) = \int \mathrm{d}^N\check{\mathbf{p}}\, W(\check{q}_1, \check{p}_1, \ldots, \check{q}_N, \check{p}_N) \tag{6.213}$$

is the true probability density for finding the system positions $\mathbf{q}$ to be equal to $\check{\mathbf{q}}$. Moreover, any moments calculated using the Wigner function equal the corresponding *symmetrized* moments of the quantum state. For example, with a two-dimensional phase-space,

$$\int \mathrm{d}\check{q}\,\mathrm{d}\check{p}\;\; W(\check{q}, \check{p})\check{q}^2\check{p} = \mathrm{Tr}\bigl[\rho(\hat{q}^2\hat{p} + 2\hat{q}\,\hat{p}\hat{q} + \hat{p}\hat{q}^2)/4\bigr]. \tag{6.214}$$

Having defined the Wigner function, we can now define a *Gaussian state* to be a state with a Gaussian Wigner function. That is, $W(\check{\mathbf{x}})$ is of the form of Eq. (6.53), with mean vector $\langle\hat{\mathbf{x}}\rangle$ and covariance matrix V as defined in Eq. (6.206). Such a Wigner function is of course positive everywhere, and so has the form of a classical distribution function. Thus, if one restricts one's attention to the observables symmetrized in $\check{\mathbf{x}}$, Gaussian states have a classical analogue. Note that the vacuum state of a bosonic field, which is the bath state we assumed in Section 6.5.2, is a Gaussian state (see Section A.3.3). For Gaussian states, the Schrödinger–Heisenberg uncertainty relation (6.207) is a *sufficient* as well as necessary

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condition on V for it to describe a valid quantum state. This, and the fact that (6.207) is a LMI in V, will be important later.

Exercise 6.24 *Show that the purity $p = \mathrm{Tr}[\rho^2]$ of a state with Wigner function $W(\check{\mathbf{x}})$ is given by*

$$p = (2\pi\hbar)^N \int \mathrm{d}^{2N}\check{\mathbf{x}}[W(\check{\mathbf{x}})]^2. \tag{6.215}$$

Hint: *First generalize Eq. (A.117) to N dimensions.*
Then show that for a Gaussian state this evaluates to

$$p = \det[V]/(\hbar/2)^{2N}. \tag{6.216}$$

6.6.2 Linear unconditional dynamics

Having delineated the structure of quantum phase-space, we can now state the first restriction we require in order to obtain a linear quantum system (in the control theory sense).

(i) The system configuration obeys a linear dynamical equation

$$\mathrm{d}\hat{\mathbf{x}} = A\hat{\mathbf{x}}\,\mathrm{d}t + B\mathbf{u}(t)\mathrm{d}t + E\,\mathrm{d}\hat{\mathbf{v}}_{\mathrm{p}}(t). \tag{6.217}$$

This is precisely the same as the classical condition. Unlike in that case, the restrictions on quantum dynamics mean that the matrices A and E cannot be specified independently. To see this, we must derive Eq. (6.217) from the quantum Langevin equation generated by Eq. (6.185). The QLE for $\hat{\mathbf{x}}$, generalizing that of Eq. (3.172) to multiple baths, is

$$\hbar\,\mathrm{d}\hat{\mathbf{x}} = \mathrm{d}t\big(\mathrm{i}[\hat{H},\hat{\mathbf{x}}] + \hat{\mathbf{c}}^\dagger\hat{\mathbf{x}}\hat{\mathbf{c}} - \{\hat{\mathbf{c}}^\dagger\hat{\mathbf{c}}/2,\hat{\mathbf{x}}\}\big) + [\hat{\mathbf{c}}^\dagger\,\mathrm{d}\hat{\mathbf{B}}_{\mathrm{in}}(t),\hat{\mathbf{x}}] - [\mathrm{d}\hat{\mathbf{B}}_{\mathrm{in}}^\dagger(t)\hat{\mathbf{c}},\hat{\mathbf{x}}]. \tag{6.218}$$

Here $\hat{\mathbf{c}}^\dagger\hat{\mathbf{x}}\hat{\mathbf{c}}$ is to be interpreted as $\sum_l \hat{c}_l^\dagger\hat{\mathbf{x}}\hat{c}_l$, and $\mathrm{d}\hat{\mathbf{B}}_{\mathrm{in}}(t) = \mathrm{d}\hat{\mathbf{B}}_{z:=-t}(t)$ is the vectorial generalization of Eq. (3.167).

We can derive Eq. (6.217) from Eq. (6.218) if we choose the system Hamiltonian to be quadratic in $\hat{\mathbf{x}}$, with the form

$$\hat{H} = \tfrac{1}{2}\hat{\mathbf{x}}^\top G\hat{\mathbf{x}} - \hat{\mathbf{x}}^\top\Sigma B\mathbf{u}(t), \tag{6.219}$$

with G real and symmetric, and the vector of Lindblad operators to be linear in $\hat{\mathbf{x}}$:

$$\hat{\mathbf{c}} = \tilde{C}\hat{\mathbf{x}}. \tag{6.220}$$

In terms of these we then have

$$A = \Sigma(G + \mathrm{Im}[\tilde{C}^\dagger\tilde{C}]), \tag{6.221}$$

$$E\,\mathrm{d}\hat{\mathbf{v}}_{\mathrm{p}}(t) = [-\mathrm{i}\,\mathrm{d}\hat{\mathbf{B}}_{\mathrm{in}}^\dagger(t)\tilde{C}\Sigma + \mathrm{i}\,\mathrm{d}\hat{\mathbf{B}}_{\mathrm{in}}^\top(t)\tilde{C}^*\Sigma]^\top. \tag{6.222}$$

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The second expression can be interpreted as Gaussian quantum process noise, because of the stochastic properties of $d\hat{\mathbf{B}}_{\rm in}(t)$ as defined by Eq. (6.186).

Exercise 6.25 *Derive Eqs. (6.221) and (6.222) from Eq. (6.218).*

We have not specified separately E and $d\hat{\mathbf{v}}_{\rm p}(t)$ because the choice would not be unique. All that is required (for the moment) is that the above expression for $E\,d\hat{\mathbf{v}}_{\rm p}(t)$ gives the correct diffusion matrix:

$$D\,dt = \mathrm{Re}[E\,d\hat{\mathbf{v}}_{\rm p}(t)d\hat{\mathbf{v}}_{\rm p}(t)^{\top}E^{\top}] \tag{6.223}$$

$$= \hbar\Sigma\,\mathrm{Re}[\tilde{C}^{\dagger}\tilde{C}]\Sigma^{\top}dt \tag{6.224}$$

$$= \hbar\Sigma[\bar{C}^{\top}\bar{C}]\Sigma^{\top}dt. \tag{6.225}$$

In Eq. (6.223) we take the real part for the same reason as in Eq. (6.209): to determine the symmetrically ordered moments. In Eq. (6.225) we have introduced a new matrix,

$$\bar{C}^{\top} \equiv (\mathrm{Re}[\tilde{C}^{\top}], \mathrm{Im}[\tilde{C}^{\top}]). \tag{6.226}$$

Using this matrix, we can also write the drift matrix as

$$A = \Sigma(G + \bar{C}^{\top}S\bar{C}), \tag{6.227}$$

where, in terms of the blocks defined by Eq. (6.226),

$$S = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}. \tag{6.228}$$

Exercise 6.26 *Verify that Eq. (6.225) is the correct expression for the diffusion matrix D.*
Hint: *Calculate the moment equations (6.50) and (6.51) for $\langle\hat{\mathbf{x}}\rangle$ and V using the Itô calculus from the quantum Langevin equation (6.218).*

The calculations in the above exercise follow exactly the same form as for the classical Langevin equation. The non-commutativity of the noise operators actually plays no important role here, because of the linearity of the dynamics. Alternatively, the moment equations can be calculated (as in the classical case) directly from the equation for the state:

$$\hbar\dot{\rho} = -\mathrm{i}[\hat{H}, \rho] + \mathcal{D}[\hat{\mathbf{c}}]\rho. \tag{6.229}$$

To make an even closer connection to the classical case, this master equation for the state matrix can be converted into an evolution equation for the Wigner function using the operator correspondences in Section A.5. This evolution equation has precisely the form of the OUE (6.49). Thus, the Wigner function has a Gaussian solution if it has a Gaussian initial state. As explained above, this means that there is a classical analogue to the quantum state, which is precisely the probability distribution that arises from the classical Langevin equation (6.47).

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Fluctuation–dissipation relations. If we restrict ourselves to considering symmetrized moments of the quantum system, then we can go further and say that there is a classical system that is *equivalent* to our quantum system. This is because the linearity of the dynamics means that the quantum configuration ends up being a linear combination of the initial quantum configuration (assumed to have a Gaussian state) and the bath configuration (also assumed to have a Gaussian state, the vacuum). Thus any symmetrized function of the system configuration will be symmetrized in the initial system and bath configuration, and corresponds to a function of a classical random variable with a Gaussian probability distribution.

It should not be thought, however, that there are no quantum–classical differences in the unconditional dynamics of linear systems. As mooted above, the conditions of unitarity place restrictions on quantum evolutions that are not present classically. Specifically, the drift and diffusion matrices, A and D, respectively, cannot be specified independently of one another because both are related to $\tilde{C}$ (or $\bar{C}$). This is despite the fact that, considered on their own, neither A nor D is restricted by quantum mechanics. That is, the drift matrix A is an arbitrary real matrix. To see this, recall that G is arbitrary real symmetric, whereas $\bar{C}^\top S\bar{C}$ is an arbitrary real skew-symmetric matrix, and Σ is invertible. Also, since Σ is orthogonal, the diffusion matrix D is an arbitrary real PSD matrix.

The relation between D and A can be seen by noting that

$$\Sigma^{-1}D\Sigma + \frac{\mathrm{i}\hbar}{2}\left[\Sigma^{-1}A - \left(\Sigma^{-1}A\right)^\top\right] = \hbar\tilde{C}^\dagger\tilde{C} \geq 0. \tag{6.230}$$

Here we have used Eqs. (6.221) and (6.224). Thus

$$D - \mathrm{i}\hbar\left(A\Sigma - \Sigma^\top A^\top\right)/2 \geq 0. \tag{6.231}$$

As well as being a necessary condition, this LMI is also a sufficient condition on D for a given drift matrix A. That is, it guarantees that $V(t) + \mathrm{i}\hbar\Sigma/2 \geq 0$ for all $t > t_0$, provided that it is true at $t = t_0$. This is because the invertibility of Σ allows us to construct a Lindblad master equation explicitly from the above equations given valid A and D matrices.

The LMI (6.231) can be interpreted as a generalized fluctuation–dissipation relation for open quantum systems. A dissipative system is one that loses energy so that the evolution is strictly stable (here we are assuming that the energy is bounded from below; that is, $G \geq 0$). As discussed in Section 6.4.1, this means that the real parts of the eigenvalues of A must be negative. Any strictly stable A must have a non-vanishing value of $A\Sigma - \Sigma^\top A^\top$ and in this case the LMI (6.231) places a lower bound on the fluctuations D about equilibrium. Note that in fact exactly the same argument holds for a strictly unstable system (i.e. one for which all modes are unstable).

By contrast, it is easy to verify that the contribution to A arising from the Hamiltonian $\hat{H}$ places no restriction on D. This is because energy-conserving dynamics cannot give rise to dissipation. To see this, note that $\Sigma^{-1}(\Sigma G)\Sigma = G\Sigma = -(\Sigma G)^\top$. This implies that ΣG has the same eigenvalues as the negative of its transpose, which is to say, the same eigenvalues as the negative of itself. Thus, if λ is an eigenvalue then so is $-\lambda$, and therefore

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it is impossible for all the eignevalues of ΣG to have negative real parts. That is, $A = \Sigma G$ cannot be a strictly stable system.

It might be questioned whether it is appropriate to call Eq. (6.231) a fluctuation–dissipation relation, because in equilibrium thermodynamics this term is used for a relation that precisely specifies (not merely bounds) the strength of the fluctuations for a given linear dissipation [Nyq28, Gar85]. The reason why our relation is weaker is that we have not made the assumption that our system is governed by an evolution equation that will bring it to thermal equilibrium at any particular temperature (including zero temperature). Apart from the Markovicity requirement, we are considering completely general linear evolution. Our formalism can describe the situation of thermal equilibrium, but also a situation of coupling to baths at different temperatures, and even more general situations. Thus, just as the Schrödinger–Heisenberg uncertainty relation can provide only a lower bound on uncertainties in the system observables, our fluctuation–dissipation relation can provide only a lower bound on fluctuations in their evolution.

Stabilizability and controllability. The concepts of stabilizability and controllability for linear quantum systems can be brought over without change from the corresponding classical definitions in Section 6.4.1. However, the term 'controllability' is also used in the context of control of Hamiltonian quantum systems [RSD+95], with a different meaning. To appreciate the relation, let us write the system Hamiltonian, including the control term, as

$$\hat{H} = \hat{H}_0 + \sum_j \hat{H}_j u_j(t), \tag{6.232}$$

where

$$\hat{H}_0 = \tfrac{1}{2}\hat{\mathbf{x}}^\top G \hat{\mathbf{x}}, \tag{6.233}$$

$$\hat{H}_j = -\hat{\mathbf{x}}^\top \Sigma B \mathbf{e}_j, \tag{6.234}$$

where the $\mathbf{e}_j$s are orthonormal vectors such that $\mathbf{u}(t) = \sum_j u_j(t)\mathbf{e}_j$. Note that j in this section is understood to range from 1 to $\omega[B]$, the number of columns of B. From these operators we can form the following quantities:

$$\hat{H}_j = -(\hat{\mathbf{x}}^\top \Sigma) B \mathbf{e}_j, \tag{6.235}$$

$$[\hat{H}_0, \hat{H}_j]/(\mathrm{i}\hbar) = -(\hat{\mathbf{x}}^\top \Sigma) A B \mathbf{e}_j, \tag{6.236}$$

$$[\hat{H}_0, [\hat{H}_0, \hat{H}_j]]/(\mathrm{i}\hbar)^2 = -(\hat{\mathbf{x}}^\top \Sigma) A^2 B \mathbf{e}_j \tag{6.237}$$

and so on. Here we have used $A = \Sigma G$ as appropriate for Hamiltonian systems. The complete set of these operators, plus $\hat{H}_0$, plus real linear combinations thereof, is known as the Lie algebra generated by the operators $\{\hat{H}_0, \hat{H}_1, \hat{H}_2, \ldots, \hat{H}_{\omega[B]}\}$. (See Box 6.2.) It is these operators (divided by $\hbar$ and multiplied by the duration over which they act) which generate the Lie group of unitary operators which can act on the system. Note that there is

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Box 6.2 Groups, Lie groups and Lie algebras

A group is a set $\mathfrak{G}$ with a binary operation '$\cdot$' satisfying the four axioms

1. closure: $\forall A, B \in \mathfrak{G}, A \cdot B \in \mathfrak{G}$
2. associativity: $\forall A, B, C \in \mathfrak{G}, (A \cdot B) \cdot C = A \cdot (B \cdot C)$
3. existence of an identity element $I \in \mathfrak{G}$: $\forall A \in \mathfrak{G}, A \cdot I = I \cdot A = A$
4. existence of an inverse: $\forall A \in \mathfrak{G}, \exists A^{-1} \in \mathfrak{G}$: $A \cdot A^{-1} = A^{-1} \cdot A = I$.

For example, the set of real numbers, with $\cdot$ being addition, forms a group. Also, the set of positive real numbers, with $\cdot$ being multiplication, forms a group. Both of these examples are Abelian groups; that is, $\forall A, B \in \mathfrak{G}, A \cdot B = B \cdot A$. An example of a *non-Abelian* group is the set of unitary matrices in some dimensions $d > 1$, with $\cdot$ being the usual matrix product.

In physics, it is very common to consider groups that are continuous, called *Lie groups*. A common example is a group of matrices (that may be real or complex):

$$\mathfrak{G} = \{\exp(-\mathrm{i}Y) \colon Y \in \mathfrak{g}\}. \tag{6.238}$$

Here $\mathfrak{g}$ is the *Lie algebra* for the Lie group $\mathfrak{G}$. This set is called a Lie algebra because (i) it forms a vector space with a concept of multiplication (in this case, the usual matrix multiplication) that is distributive and (ii) it is closed under a particular sort of binary operation called the *Lie bracket* (in this case, equal to $-\mathrm{i}$ times the commutator $[Y, Z] \equiv YZ - ZY$). Closure means that

$$\forall\, Y, Z \in \mathfrak{g}, -\mathrm{i}[Y, Z] \in \mathfrak{g}. \tag{6.239}$$

The *generator* of a Lie algebra $\mathfrak{g}$ is a set $\mathbb{X}^0 = \{X_k \colon k\}$ of elements of $\mathfrak{g}$ that generate the whole algebra using the Lie bracket. This means the following. We introduce the recursive definition $\mathbb{X}^{n+1} = \mathbb{X}^n \bigcup \{-\mathrm{i}[Y, Z] \colon Y, Z \in \mathbb{X}^n\}$. Then

$$\mathfrak{g} = \mathrm{span}\big(\mathbb{X}^\infty\big), \tag{6.240}$$

where span($\mathbb{S}$) is the set of all real linear combinations of matrices in the set $\mathbb{S}$. In many cases (for example when the X_k are finite-dimensional matrices) the recursive definition will cease to produce distinct sets after some finite number of iterations, so that $\mathbb{X}^\infty = \mathbb{X}^N$ for some N.

no point in considering commutators containing more than one $\hat{H}_j$ ($j \neq 0$) since they will be proportional to a constant.

Now, the criterion for controllability for a linear system is that the controllability matrix (6.66) has full row rank. By inspection, this is equivalent to the condition that, out of the $2N \times \omega(B)$ Hilbert-space operators in the row-vector

$$(\hat{\mathbf{x}}^\top \Sigma)[B \;\; AB \;\; A^2B \;\; \cdots \;\; A^{2N-1}B], \tag{6.241}$$

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$2N$ are linearly independent combinations of the $2N$ canonical phase-space variables $\hat{x}_k$. From the above relations, we can thus see that controllability in the sense appropriate to linear systems can be restated as follows.

> A linear quantum system is controllable iff the Lie algebra generated by $\{\hat{H}_0, \hat{H}_1, \hat{H}_2, \ldots, \hat{H}_{\omega[B]}\}$ includes a complete set of observables.

(See Section 6.5.1 for the definition of a complete set of observables.) The significance of this concept of controllability is that, as defined in Section 6.4.1, the centroid in phase space $\langle\hat{\mathbf{x}}\rangle$ can be arbitrarily displaced as a function of time for a suitable choice of $\mathbf{u}(t)$. Note that this concept of controllability does *not* mean that it is possible to prepare an arbitrary quantum state of the system.

Having formulated our sense of controllability in Lie-algebraic terms, we can now compare this with the other sense used in Refs. [Bro73, HTC83, Alt02, RSD$^+$95] and elsewhere. This sense applies to arbitrary quantum systems, and is much stronger than the notion used above. We consider a Hamiltonian of the form of Eq. (6.232), but with no restrictions on the forms of $\hat{H}_0$ and the $\hat{H}_j$. Then Ref. [RSD$^+$95] defines *operator-controllability* as follows.

> A quantum system is operator-controllable iff the Lie algebra generated by $\{\hat{H}_0, \hat{H}_1, \hat{H}_2, \ldots, \hat{H}_{\omega[B]}\}$ is equal to $\mathfrak{D}(\mathbb{H})$.

(See Section 6.5.1 for the definition of $\mathfrak{D}(\mathbb{H})$.) The significance of this concept of controllability is that any unitary evolution $\hat{U}$ can be realized by some control vector $\mathbf{u}(t)$ over some time interval $[t_0, t_1]$. Hence, operator-controllability means that from an initial pure state it is possible to prepare an arbitrary quantum state of the system.

6.6.3 Linear conditional dynamics

As for the first restriction of Section 6.6.2, the second restriction necessary to obtain a linear quantum system is again identical to the classical case (apart from hats on all of the quantities).

> (ii) The system state is conditioned on the measurement result
>
> $$\hat{\mathbf{y}}\,\mathrm{d}t = C\hat{\mathbf{x}}\,\mathrm{d}t + \mathrm{d}\hat{\mathbf{v}}_{\mathrm{m}}(t). \tag{6.242}$$

This follows automatically from the assumption (6.220) that we have already made, provided that we use a Wiener-process unravelling to condition the system. Once again,

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however, quantum mechanics places restrictions on the matrix C and the correlations of the measurement noise $\mathrm{d}\hat{\mathbf{v}}_{\mathrm{m}}(t)$.

We saw in Section 6.5.2 that the most general output of a quantum system with Wiener noise is a vector of complex currents $\hat{\mathbf{J}}$ defined in Eq. (6.193). This can be turned into a real vector by defining

$$\hat{\mathbf{y}} = T^{+}\begin{pmatrix}\mathrm{Re}\,\hat{\mathbf{J}}\\ \mathrm{Im}\,\hat{\mathbf{J}}\end{pmatrix} = C\hat{\mathbf{x}} + \frac{\mathrm{d}\hat{\mathbf{v}}_{\mathrm{m}}}{\mathrm{d}t}, \tag{6.243}$$

with

$$C = 2T^{\top}\bar{C}/\hbar. \tag{6.244}$$

Here T is, in general, a *non-square* matrix, with $\varepsilon[\bar{C}]$ rows and $\varepsilon[C]$ columns, such that

$$TT^{\top} = \hbar U, \tag{6.245}$$

where U is the unravelling matrix as usual. In Eq. (6.243) T^{+} is the pseudoinverse, or Moore–Penrose inverse [CM91, ZDG96] of T (see Box. 6.1). Note that the numbers of columns of $\bar{C}$ and of C are equal: $\omega[C] = \omega[\bar{C}] = 2N$. The number of rows of C, $\varepsilon[C]$ (also equal to the dimension of $\hat{\mathbf{y}}$), is equal to the rank of U. The number of rows of $\bar{C}$, $\varepsilon[\bar{C}]$ (also equal to twice the dimension of $\hat{\mathbf{c}}$), is equal to the number of rows (or columns) of U. This guarantees that the matrix T exists.

In Eq. (6.243) we have defined

$$\mathrm{d}\hat{\mathbf{v}}_{\mathrm{m}} = T^{+}\begin{pmatrix}\mathrm{Re}\,\delta\hat{\mathbf{J}}\\ \mathrm{Im}\,\delta\hat{\mathbf{J}}\end{pmatrix}\mathrm{d}t, \tag{6.246}$$

where (*cf.* Eq. (6.193))

$$\begin{aligned}\delta\hat{\mathbf{J}}^{\top}\,\mathrm{d}t = {} & \mathrm{d}\hat{\mathbf{B}}_{\mathrm{in}}^{\top}\,\mathrm{H} + \mathrm{d}\hat{\mathbf{B}}_{\mathrm{in}}^{\dagger}\,\Upsilon + \mathrm{d}\hat{\mathbf{A}}^{\dagger}\sqrt{\mathrm{H} - \Upsilon^{*}\mathrm{H}^{-1}\Upsilon}\\ & + \mathrm{d}\hat{\mathbf{V}}^{\top}\sqrt{\mathrm{H}(I-\mathrm{H})} + \mathrm{d}\hat{\mathbf{V}}^{\dagger}\sqrt{\mathrm{H}^{-1} - I}\,\Upsilon.\end{aligned} \tag{6.247}$$

The measurement noise operator $\mathrm{d}\hat{\mathbf{v}}_{\mathrm{m}}(t)$ has the following correlations:

$$\mathrm{d}\hat{\mathbf{v}}_{\mathrm{m}}\,\mathrm{d}\hat{\mathbf{v}}_{\mathrm{m}}^{\top} = I\,\mathrm{d}t, \tag{6.248}$$

$$\mathrm{Re}[E\,\mathrm{d}\hat{\mathbf{v}}_{\mathrm{p}}\,\mathrm{d}\hat{\mathbf{v}}_{\mathrm{m}}^{\top}] = \Gamma^{\top}\mathrm{d}t, \tag{6.249}$$

just as in the classical case, where here

$$\Gamma = -T^{\top}S\bar{C}\Sigma^{\top}. \tag{6.250}$$

Exercise 6.27 *Verify these correlations from the above definitions.*

The evolution conditioned upon measuring $\mathbf{y}(t)$ is the same as in the classical case. That is, the conditioned state is Gaussian, with mean vector and covariance matrix obeying Eqs. (6.85) and (6.86), respectively. In the stochastic equation for the mean, the innovation

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is, as expected,

$$\mathrm{d}\mathbf{w} = \mathrm{d}\hat{\mathbf{v}}_{\mathrm{m}} + C(\hat{\mathbf{x}} - \langle\hat{\mathbf{x}}\rangle_{\mathrm{c}}). \tag{6.251}$$

These quantum Kalman-filter equations can also be derived from the quantum version of the Kushner–Stratonovich equation, Eq. (6.187). Indeed, by using the Wigner function to represent the quantum state, the evolution can be expressed precisely as the Kushner–Stratonovich equation (6.71), involving the matrices A, B, D, C and Γ.

Fluctuation–observation relations. The above analysis shows that, even including measurement, there is a linear classical system with all the same properties as our linear quantum system. As in the case of the unconditioned dynamics, however, the structure of quantum mechanics constrains the possible conditional dynamics. In the unconditioned case this was expressed as a fluctuation–dissipation relation. That is, any dissipation puts lower bounds on the fluctuations. In the present case we can express the constraints as a *fluctuation–observation relation*. That is, for a quantum system, any information gained puts lower bounds on fluctuations in the conjugate variables, which are necessary in order to preserve the uncertainty relations.

Recall that the Riccati equation for the conditioned covariance matrix can be written as

$$\dot{V}_{\mathrm{c}} = \tilde{A}V_{\mathrm{c}} + V_{\mathrm{c}}\tilde{A}^{\top} + \tilde{E}\tilde{E}^{\top} - V_{\mathrm{c}}C^{\top}CV_{\mathrm{c}}, \tag{6.252}$$

where in the quantum case

$$\tilde{A} = A - \Gamma^{\top}C = \Sigma[G + \bar{C}^{\top}S(I - 2U)\bar{C}], \tag{6.253}$$

$$\tilde{E}\tilde{E}^{\top} = D - \Gamma^{\top}\Gamma = \hbar\Sigma\bar{C}^{\top}[I - S^{\top}US]\bar{C}\Sigma^{\top}. \tag{6.254}$$

From Eq. (6.252) we see that $\tilde{E}\tilde{E}^{\top}$ always increases the uncertainty in the system state. This represents fluctuations. By contrast, the term $-V_{\mathrm{c}}C^{\top}CV_{\mathrm{c}}$ always decreases the uncertainty. This represents information gathering, or observation. The fluctuation–observation relation is expressed by the LMI

$$\tilde{E}\tilde{E}^{\top} - \frac{\hbar^2}{4}\Sigma C^{\top}C\Sigma^{\top} \geq 0. \tag{6.255}$$

Exercise 6.28 *Show this, by showing that the left-hand side evaluates to the following matrix which is clearly PSD:*

$$\hbar\Sigma\bar{C}^{\top}\begin{pmatrix} I - \mathrm{H} & 0 \\ 0 & I - \mathrm{H} \end{pmatrix}\bar{C}\Sigma^{\top}. \tag{6.256}$$

The first thing to note about Eq. (6.255) is that it is quantum in origin. If $\hbar$ were zero, there would be no lower bound on the fluctuations. The second thing to notice is that observation of one variable induces fluctuations in the conjugate variable. This follows from the presence of the matrix $\Sigma^{\top}$ that postmultiplies C in Eq. (6.255). It is most easily seen in the case of motion in one dimension ($N = 1$). Say we observe the position q, so

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that

$$y\,\mathrm{d}t = \sqrt{\kappa}\langle\hat{q}\rangle_c\,\mathrm{d}t + \mathrm{d}w, \tag{6.257}$$

where here κ is a scalar expressing the measurement strength. Then Eq. (6.255) says that

$$\tilde{E}\tilde{E}^\top - \frac{\hbar^2}{4}\begin{pmatrix} 0 & 0 \\ 0 & \kappa \end{pmatrix} \geq 0. \tag{6.258}$$

That is, there is a lower bound of $(\hbar\sqrt{\kappa}/2)^2$ on the spectral power of momentum fluctuations. The third thing to note about Eq. (6.255) is that, since $D = \tilde{E}\tilde{E}^\top + \Gamma^\top\Gamma$, our relation implies the *weaker* relation

$$D - \frac{\hbar^2}{4}\Sigma C^\top C\Sigma^\top \geq 0. \tag{6.259}$$

As well as being a necessary condition on $\tilde{E}$ given C, Eq. (6.255) is also a sufficient condition.

Detectability and observability. The definitions of detectability and observability for linear quantum systems replicate those for their classical counterparts – see Section 6.4.2. However, there are some interesting points to make about the quantum case.

First, we can define the notion of *potential detectability*. By this we mean that, given the unconditioned evolution described by A and D, there exists a matrix C such that (C, A) is detectable. Classically this is always the case because C can be specified independently of A and D, so this notion would be trivial, but quantum mechanically there are some evolutions that are not potentially detectable; Hamiltonian evolution is the obvious example.

We can determine which unconditional evolutions are potentially detectable from A and D as follows. First note that from Eq. (6.244) the existence of an unravelling U such that (C, A) is detectable is equivalent to $(\bar{C}, A)$ being detectable. Indeed, $C \propto \bar{C}$ results from the unravelling $U = I/2$, so a system is potentially detectable iff the $U = I/2$ unravelling is detectable. Now, $(\bar{C}, A)$ being detectable is equivalent to $(\bar{C}^\top\bar{C}, A)$ being detectable. But, from Eq. (6.225), $\bar{C}^\top\bar{C} = \Sigma^\top D\Sigma/\hbar$. Since the above arguments, *mutatis mutandis*, also apply for potential observability, we can state the following.

> A quantum system is potentially detectable (observable) iff $(\Sigma^\top D\Sigma, A)$ is detectable (observable).

The second interesting point to make in this section is that, for quantum systems, if $(C, \tilde{A})$ is detectable then $(-\tilde{A}, \tilde{E})$ is stabilizable.[3] Consider for simplicity the case of efficient detection, where $\mathrm{H} = I$. Then the left-hand side of Eq. (6.255) is zero, and we can

[3] Note that in Ref. [WD05] it was incorrectly stated that $(\tilde{A}, \tilde{E})$ was stabilizable, but the conclusion drawn there, discussed in Section 6.6.4, is still correct.

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choose

$$\tilde{E} = \frac{\hbar}{2}\Sigma C^\top. \tag{6.260}$$

Moreover, $\tilde{A} = \Sigma\tilde{G}$, where

$$\tilde{G} = G + \bar{C}^\top \begin{pmatrix} -\mathrm{Im}\,\Upsilon & \mathrm{Re}\,\Upsilon \\ \mathrm{Re}\,\Upsilon & \mathrm{Im}\,\Upsilon \end{pmatrix} \bar{C} \tag{6.261}$$

is a symmetric matrix, so that $\frac{1}{2}\hat{\mathbf{x}}^\top\tilde{G}\hat{\mathbf{x}}$ is a pseudo-Hamiltonian that generates the part of the drift of the system which is independent of the record $\mathbf{y}$. Now, since Σ is invertible, we can replace $\mathbf{x}_\lambda$ by $\Sigma\mathbf{x}_\lambda$ everywhere in the definition (6.87) of detectability. It then follows that $(C, \tilde{A})$ detectable is equivalent to $(C\Sigma^\top, \Sigma\tilde{A}\Sigma^\top)$ detectable (remember that $\Sigma^{-1} = \Sigma^\top$). But $\Sigma\tilde{A}\Sigma^\top = \Sigma^2\tilde{G}\Sigma^\top = -\tilde{G}\Sigma^\top = -\tilde{A}^\top$, while $C\Sigma^\top \propto \tilde{E}^\top$. Thus, by virtue of the detectable–stabilizable duality, we have $(-\tilde{A}, \tilde{E})$ stabilizable. Now for inefficient detection the fluctuations in the system are greater, so $(-\tilde{A}, \tilde{E})$ will also be stabilizable in this case.

Third, we note that, as was the case for controllability, we can give a Lie-algebraic formulation for observability, at least for the non-dissipative case in which $\tilde{C}$ can be taken to be real, so that $\mathrm{Im}[\tilde{C}^\dagger\tilde{C}] = 0$ and $A = \Sigma G$. From Section 6.4.2, observability for the linear system is equivalent to the matrix

$$[C^\top\ A^\top C^\top\ (A^\top)^2 C^\top\ \cdots\ (A^\top)^{2N-1} C^\top] \tag{6.262}$$

having full row rank. Now, for this non-dissipative case (so called because the drift evolution is that of a Hamiltonian system), the method of Section 6.6.2 can be applied to give the following new formulation.

> A non-dissipative linear quantum system is observable iff the Lie algebra generated by $\{\hat{H}_0, \hat{o}_1, \hat{o}_2, \ldots, \hat{o}_{\varepsilon[C]}\}$ includes a complete set of observables.

Here $\hat{H}_0$ is as above, while $\hat{o}_l = \mathbf{e}_l^\top C\hat{\mathbf{x}}$.

This definition does not generalize naturally to other sorts of quantum systems in the way that the definition of controllability does. If $\hat{H}_0$ and $\{\hat{o}_l\}$ are arbitrary operators, then the above Lie algebra does not correspond to the operators the observer obtains information about as the system evolves conditionally. Moreover, unlike in the linear case, the observability of a general system can be enhanced by suitable application of the control Hamiltonians $\{\hat{H}_j\}$. Indeed, Lloyd [Llo00] has defined observability for a general quantum system such that it is achievable iff it is operator-controllable (see Section 6.6.2) and the observer can make at least one nontrivial projective measurement. (His definition of observability is essentially that the observer can measure any observable, and he does not consider continuous monitoring.)

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6.6.4 Stabilizing solutions

As in the classical case, for the purposes of feedback control we are interested in observed systems for which the Riccati equation (6.252) has a stationary solution that is stabilizing. Again, we will denote that solution by W, so we require that

$$-MW_U - W_U M^\top = \tilde{E}\tilde{E}^\top + W_U C^\top C W_U \tag{6.263}$$

has a unique solution such that

$$M \equiv \tilde{A} - W_U C^\top C \tag{6.264}$$

is strictly stable. In the above we have introduced a subscript U to emphasize that the stationary conditioned covariance matrix depends upon the unravelling U, since all of the matrices $\tilde{A}$, $\tilde{E}$ and C depend upon U. We call an unravelling stabilizing if Eq. (6.263) has a stabilizing solution.

As discussed in Section 6.4.3, a solution is stabilizing iff $(C, \tilde{A})$ (or (C, A)) is detectable and condition (6.99) is satisfied. As stated there, the second condition is satisfied if $(-\tilde{A}, \tilde{E})$ is stabilizable. But, as we saw in the preceding section, in the quantum case, this follows automatically from the first condition. That is, *quantum mechanically, the conditions for the existence of a stabilizing solution are weaker than classically.* Detectability of (C, A) is all that we require to guarantee a stabilizing solution.

In the quantum case we can also apply the notion of potential detectability from the preceding section. It can be shown [WD05] that *if the system is potentially detectable then the stabilizing unravellings form a dense subset*[4] *of the set of all unravellings*. Now, for detectable unravellings, the solutions to the algebraic MRE (6.263) are continuous in $\tilde{A}$, $\tilde{E}$ and C [LR91]. But these matrices are continuous in U, and hence W_U is continuous in U. Thus, as long as (i) one restricts oneself to a compact set of W_Us (e.g. a set of bounded W_Us); (ii) one is interested only in continuous functions of W_U; and (iii) the system is potentially detectable, then one can safely assume that any such W_U is a stabilizing solution.

Possible conditional steady states. We showed in the classical case that the only restriction on the possible stationary conditioned covariance matrices that a system described by A and D can have is the LMI

$$AW_U + W_U A^\top + D \geq 0. \tag{6.265}$$

This is also a necessary condition in the quantum case, by exactly the same reasoning, although in this case we also have another necessary condition on the covariance matrix given by the uncertainty relation (6.207), which we repeat here:

$$W_U + \mathrm{i}\hbar\Sigma/2 \geq 0. \tag{6.266}$$

If W_U is the covariance matrix of a pure state, then Eq. (6.265) is also a sufficient condition for W_U to be a realizable stationary conditioned covariance matrix. This can

[4] If a set A is a dense subset of a set B, then for every element of B there is an element of A that is arbitrarily close, by some natural metric.

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be seen from the phenomenon of pure-state steering as discovered by Schrödinger (see Section 3.8.1). This was first pointed out in Ref. [WV01]. Say the system at time t has a covariance matrix W_U corresponding to a pure state. Recall, from Eq. (6.216), that this is true iff

$$\det[W_U] = (\hbar/2)^{2N}. \tag{6.267}$$

Then, if Eq. (6.265) is satisfied, the system at an infinitesimally later time $t + \mathrm{d}t$ will be a mixture of states, all with covariance matrix W_U, and with Gaussian-distributed means, as explained in Section 6.4.3. We call such an ensemble a uniform Gaussian pure-state ensemble. Then, by virtue of the Schrödinger–HJW theorem, there will be some way of monitoring the environment – that part of the bath that has become entangled with the system in the interval $[t, t + \mathrm{d}t)$ – such that the system is randomly collapsed to one of the pure state elements of this ensemble, with the appropriate Gaussian weighting. That is, a pure state with covariance matrix W_U can be reprepared by continuing the monitoring, and therefore W_U must be the stationary conditioned covariance matrix under some monitoring scheme.

Thus we have a necessary condition (Eqs. (6.265) and (6.266)) and a sufficient condition (Eqs. (6.265) and (6.267)) for W_U to be the steady-state covariance matrix of some monitoring scheme.[5] If W_U is such a covariance matrix, then there is an unravelling matrix U that will generate the appropriate matrices $\tilde{A}$, $\tilde{E}$ and C so that W_U is the solution of Eq. (6.263). Moreover, as argued above, as long as W_U is bounded and $(\bar{C}, A)$ is detectable, this W_U can be taken to be stabilizing.

To find an unravelling (which may be non-unique) generating W_U as a stationary conditioned covariance matrix, it is simply necessary to put the U-dependence explicitly in Eq. (6.263). This yields the LME for U:

$$\hbar R^\top U R = D + A W_U + W_U A^\top, \tag{6.268}$$

where $R = 2\bar{C} W_U/\hbar + S\bar{C}\Sigma$. This can be solved efficiently (that is, in a time polynomial in the size of the matrices). It does not matter whether this equation has a non-unique solution U, because in steady state the conditional state and its dynamics will be the same for all U satisfying Eq. (6.268) for a given W_U. This can be seen explicitly as follows. The shape of the conditioned state in the long-time limit is simply W_U. The stochastic dynamics of the mean $\langle \hat{\mathbf{x}} \rangle_\mathrm{c}$ is given by

$$\mathrm{d}\langle \hat{\mathbf{x}} \rangle_\mathrm{c} = [A\langle \hat{\mathbf{x}} \rangle_\mathrm{c} + B\mathbf{u}(t)]\mathrm{d}t + \mathrm{F}^\top \mathrm{d}\mathbf{w}, \tag{6.269}$$

which depends upon U only through the stochastic term. Recall that $\mathrm{F} = C W_U + \Gamma$, which depends on U through C and Γ as well as W_U. However, statistically, all that matters is the

[5] In Ref. [WD05] it was incorrectly stated that Eqs. (6.265) and (6.266) form the necessary and sufficient conditions. However, this does not substantially affect the conclusions of that work, since other constraints ensure that the states under consideration will be pure, as will be explained later.

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covariance of the noise in Eq. (6.269), which, from Eq. (6.263), is given by

$$\mathrm{F}^{\top}\,\mathrm{d}\mathbf{w}\,\mathrm{d}\mathbf{w}^{\top}\,\mathrm{F} = \mathrm{d}t(AW_U + W_U A^{\top} + D), \tag{6.270}$$

which depends on U only through W_U.

Consider the case in which A is strictly stable so that for $\mathbf{u} = \mathbf{0}$ an unconditional steady state exists, with covariance matrix satisfying

$$AV_{\rm ss} + V_{\rm ss}A^{\top} + D = 0. \tag{6.271}$$

As noted in the classical case, there exist states with covariance matrix W satisfying $V_{\rm ss} - W \geq 0$ and yet not satisfying Eq. (6.265). This is also true in the quantum case, even with the added restriction that W correspond to a pure state by satisfying (6.267). That is, there exist uniform Gaussian pure-state ensembles that represent the stationary solution $\rho_{\rm ss}$ of the quantum master equation but cannot be realized by any unravelling. In saying that the uniform Gaussian ensemble represents $\rho_{\rm ss}$ we mean that

$$\rho_{\rm ss} = \int \mathrm{d}^{2N}\langle\hat{\mathbf{x}}\rangle_{\rm c}\, \wp(\langle\hat{\mathbf{x}}\rangle_{\rm c})\rho^{W}_{\langle\hat{\mathbf{x}}\rangle_{\rm c}}, \tag{6.272}$$

where $\rho^{W}_{\langle\hat{\mathbf{x}}\rangle_{\rm c}}$ has the Gaussian Wigner function $W_{\rm c}(\check{\mathbf{x}}) = g(\check{\mathbf{x}};\langle\hat{\mathbf{x}}\rangle_{\rm c}, W)$, and the Gaussian distribution of means is

$$\wp(\langle\hat{\mathbf{x}}\rangle_{\rm c}) = g(\langle\hat{\mathbf{x}}\rangle_{\rm c};\mathbf{0}, V_{\rm ss} - W). \tag{6.273}$$

In saying that the ensemble cannot be realized we mean that there is no way an observer can monitor the output of the system so as to know that the system is in the state $\rho^{W}_{\langle\hat{\mathbf{x}}\rangle_{\rm c}}$, such that W remains fixed in time but $\langle\hat{\mathbf{x}}\rangle_{\rm c}$ varies so as to sample the Gaussian distribution (6.273) over time. On the other hand, there are certainly some ensembles that satisfy both Eq. (6.265) and Eq. (6.267), which thus are physically realizable (PR) in this sense. This existence of some ensembles representing $\rho_{\rm ss}$ that are PR and some that are not is an instance of the *preferred-ensemble fact* discussed in Section 3.8.2.

Example: on-threshold OPO. To illustrate this idea, consider motion in one dimension with a single output channel ($N = L = 1$), described by the master equation

$$\hbar\dot{\rho} = -\mathrm{i}[(\hat{q}\,\hat{p} + \hat{p}\hat{q})/2, \rho] + \mathcal{D}[\hat{q} + \mathrm{i}\hat{p}]\rho, \tag{6.274}$$

where the output arising from the second term may be monitored. This could be realized in quantum optics as a damped cavity (a harmonic oscillator in the rotating frame) containing an on-threshold parametric down-converter, also known as an optical parametric oscillator (OPO). Here p would be the squeezed quadrature and q the anti-squeezed quadrature. The monitoring of the output could be realized by techniques such as homodyne or heterodyne detection.

Exercise 6.29 *Show that in this case we have*

$$G = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \tilde{C} = (1, i) \tag{6.275}$$

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and that the drift and diffusion matrices evaluate to

$$A = \begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix}, \qquad D = \hbar \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{6.276}$$

Since A is not strictly stable, there is no stationary unconditional covariance matrix. However, in the long-time limit

$$V \to \hbar \begin{pmatrix} \infty & 0 \\ 0 & 1/2 \end{pmatrix}, \tag{6.277}$$

and we come to no harm in defining this to be $V_{\rm ss}$. Writing the conditional steady-state covariance matrix as

$$W_U = \frac{\hbar}{2} \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix}, \tag{6.278}$$

the LMIs (6.265) and (6.266) become, respectively,

$$\begin{pmatrix} \alpha & \beta + i \\ \beta - i & \gamma \end{pmatrix} \geq 0, \tag{6.279}$$

$$\begin{pmatrix} 1 & -\beta \\ -\beta & 1 - 2\gamma \end{pmatrix} \geq 0. \tag{6.280}$$

The first of these implies $\alpha > 0$, $\gamma > 0$ and $\alpha\gamma \geq 1 + \beta^2$. The second then implies $\gamma \leq (1 - \beta^2)/2$.

In Fig. 6.7 we show four quantum states $\rho^W_{\langle \hat{\mathbf{x}} \rangle_c}$ that are pure (they saturate Eq. (6.279)) and satisfy $V_{\rm ss} - W \geq 0$. That is, they 'fit inside' $\rho_{\rm ss}$. However, one of them does not satisfy Eq. (6.280). We see the consequence of that when we show the mixed states that these four pure states evolve into after a short time $\tau = 0.2$ in Fig. 6.8. (We obtain this by analytically solving the moment equation (6.51), starting with $\langle \hat{\mathbf{x}} \rangle = \mathbf{0}$ for simplicity.) This clearly shows that, for the initial state that fails to satisfy Eq. (6.280), the mixed state at time τ can no longer be represented by a mixture of the original pure state with random displacements, because the original state does not 'fit inside' the evolved state. The ensemble formed from these states is not physically realizable. We will see later, in Section 6.6.6, how this has consequences in quantum feedback control.

6.6.5 LQG optimal feedback control

We can now consider a cost function (6.197) for the quantum system and controller where h is quadratic, as in Eq. (6.113), but with $\mathbf{x}$ replaced by $\hat{\mathbf{x}}$. This can be justified in the quantum case for the same sorts of reasons as in the classical case; for instance, in linear systems, the free energy is a quadratic function of $\hat{\mathbf{x}}$. The resulting optimization (cost minimization) problem has exactly the same solution as in the classical case, so all of the discussion on LQG control in Section 6.4.4 applies here.

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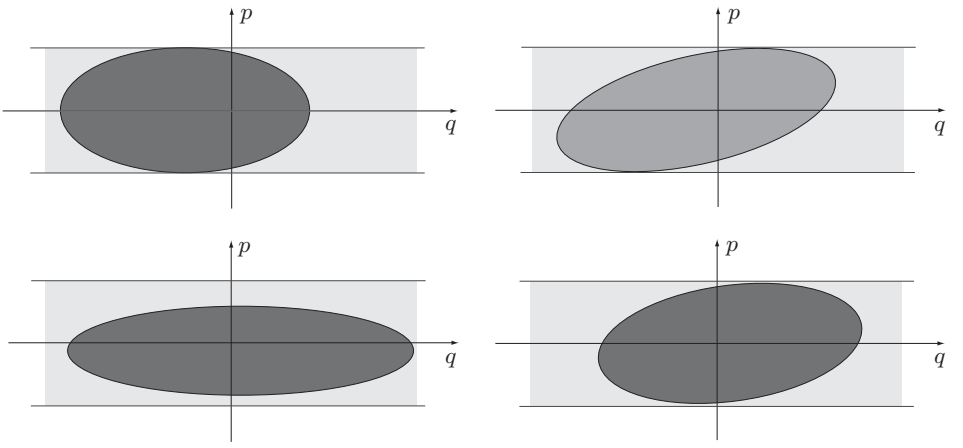

Fig. 6.7 Representation of states in phase-space for the system described in the text. The horizontal and vertical axes are q and p, respectively, and the curves representing the states are one-standard-deviation contours of the Wigner function $W(q, p)$. That is, they are curves defined by the parametric equation $W(q, p) = \mathrm{e}^{-1} W(\bar{q}, \bar{q})$, where $(\bar{q}, \bar{p})^\top$ is the centroid of the state. The stationary unconditioned state has a p-variance of $\hbar/2$ and an unbounded q-variance. A short segment of the Wigner function of this state is represented by the lightly shaded region between the horizontal lines at $p = \pm\sqrt{\hbar/2}$. The ellipses represent pure states, with area $\pi\hbar$. They are possible conditioned states of the system, since they 'fit inside' the stationary state. For states realizable by continuous monitoring of the system output, the centroid of the ellipses wanders stochastically in phase-space, which is indicated in the diagram by the fact that the states are not centred at the origin. The state in the top-right corner is shaded differently from the others because it cannot be physically realized in this way, as Fig. 6.8 demonstrates.

One new feature that arises in the quantum case is the following. Classically, for a minimally disturbing measurement, the stronger the measurement, the better the control. Consider the steady-state case for simplicity. If we say that $C = \sqrt{\kappa} C_1$, with C_1 fixed, then the stationary conditioned covariance matrix W is given by the ARE

$$AW + WA^\top + D = \kappa W {C_1}^\top C_1 W. \tag{6.281}$$

(Note that we have set $\Gamma = 0$ as appropriate for a minimally disturbing measurement.)

Exercise 6.30 *Convince yourself that, for A, D and C_1 fixed, the eigenvalues of W decrease monotonically as κ increases.*

Thus the integrand in the cost function

$$\mathrm{E}_{\mathrm{ss}}[h] = \mathrm{tr}[Y B Q^{-1} B^\top Y W] + \mathrm{tr}[Y D] \tag{6.282}$$

is monotonically decreasing with κ. By contrast, in the quantum case it is not possible to say that D is fixed as κ increases. Rather, for a minimally disturbing quantum measurement, the measurement necessarily contributes to D a term $\hbar^2 \kappa \Sigma {C_1}^\top C_1 \Sigma^\top / 4$, according to the

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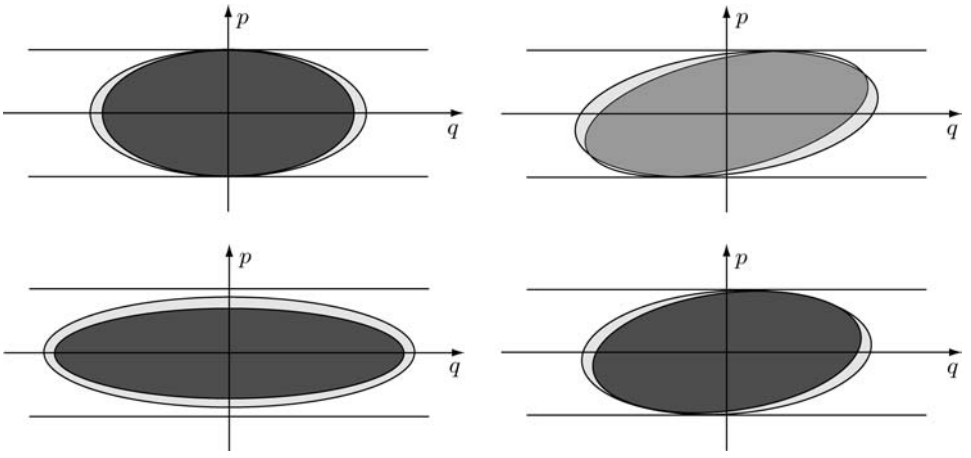

Fig. 6.8 Representation of the four pure states from Fig. 6.7, plus the mixed states they evolve into after a time $\tau = 0.2$. For ease of comparison we have centred each of these states at the origin, and we have omitted the shading for the stationary state, but the other details are the same as for the preceding figure. Note that, apart from the top-right state, the initial states (heavy shading) all 'fit inside' the evolved states (light shading). Hence they are all physically realizable (PR). The top-right initial state is unshaded, as is apparent from the parts that do not 'fit inside' the evolved state (light shading), and so is not PR. (The part that does 'fit inside' appears with medium shading, as in Fig. 6.7). The four initial states that appear here are defined as follows. Top-left: the state with minimum q-variance that fits inside the stationary state. Bottom-left: the state arising from the $U = I/2$ unravelling. Top-right: the state with minimum $(q - p)^2$ that fits inside the stationary state. Bottom-right: the state with minimum $(q - p)^2$ that is PR.

fluctuation–observation relation. Thus with D_0 fixed we have the ARE

$$AW + WA^\top + D_0 + \hbar^2 \kappa \Sigma C_1{}^\top C_1 \Sigma^\top / 4 = \kappa W C_1{}^\top C_1 W. \tag{6.283}$$

Here, the eigenvalues of W are not monotonically decreasing with κ, and neither (in general) is $\mathrm{E}_{\mathrm{ss}}[h]$. The cost may actually monotonically increase with κ, or there may be some optimum κ that minimizes $\mathrm{E}_{\mathrm{ss}}[h]$.

Example: the harmonic oscillator. We can illustrate the above idea, as well as other concepts in LQG control, using the example of the harmonic oscillator with position measurement and controlled by a spatially invariant (but time varying) force. This was considered by Doherty and Jacobs [DJ99], who also discussed a physical realization of this system in cavity QED. We do not assume that the oscillator frequency is much larger than the measurement rate κ, so it is not appropriate to work in the interaction frame. Indeed, we take the oscillator frequency to be unity, and for convenience we will also take the particle mass to be unity. We can model this by choosing

$$G = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \tilde{C} = \sqrt{\kappa}\,(1, 0), \qquad U = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{6.284}$$

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Exercise 6.31 *Show that this gives* $\Gamma = 0$ *and*

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad D = \hbar \begin{pmatrix} 0 & 0 \\ 0 & \kappa \end{pmatrix}, \qquad C_1 = (2, 0)/\sqrt{\hbar}. \tag{6.285}$$

In the above we have assumed that $D_0 = 0$ (that is, that there are no noise sources apart from the measurement back-action). This allows a simple solution to the algebraic MRE (6.283):

$$W = \frac{\hbar}{4\kappa} \begin{pmatrix} \sqrt{2\nu} & \nu \\ \nu & (1+\nu)\sqrt{2\nu} \end{pmatrix}, \tag{6.286}$$

where $\nu = \sqrt{1 + 4\kappa^2} - 1$.

Exercise 6.32 *Show that, as well as solving Eq. (6.283), this* W *saturates the LMI (6.266), and hence corresponds to a pure state.*

When $\kappa \ll 1$, the measurement of position is slow compared with the oscillation of the particle. In this limit, $\nu \to 2\kappa^2$ and $W \to (\hbar/2)I$. That is, the conditioned state is a coherent state of the oscillator, and the conditioned variance in position is $\hbar/2$. In physical units, this (the standard quantum limit for the position variance) is $\hbar/(2m\omega)$.

Now consider feedback control of the oscillator, for the purpose of minimizing the energy in steady state. That is, we choose the cost function $P = I$, and (from the control constraint mentioned above), $B = (0, 1)^\top$, so that Q is just a scalar. Then Eq. (6.117) for Y becomes

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} Y + Y \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = Y \begin{pmatrix} 0 & 0 \\ 0 & Q^{-1} \end{pmatrix} Y. \tag{6.287}$$

Exercise 6.33 *Show that, for* $Q \ll 1$, *this ARE has the approximate solution*

$$Y \simeq \begin{pmatrix} 1 & \sqrt{Q} \\ \sqrt{Q} & \sqrt{Q} \end{pmatrix}. \tag{6.288}$$

Also show that it is a stabilizing solution.
Hint: *For the second part, show first that*

$$BQ^{-1}B^\top Y \simeq -\frac{1}{\sqrt{Q}} \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}. \tag{6.289}$$

Thus the optimal feedback, which adds to the equations of motion the term

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} \langle \hat{q} \rangle \\ \langle \hat{p} \rangle \end{pmatrix}_{\mathrm{fb}} = -\frac{1}{\sqrt{Q}} \begin{pmatrix} 0 \\ \langle \hat{q} \rangle + \langle \hat{p} \rangle \end{pmatrix}, \tag{6.290}$$

is asymptotically stable.

Now for this problem we have

$$\mathrm{F} = CW = \sqrt{\frac{\hbar}{4\kappa}} (\sqrt{2\nu}, \nu). \tag{6.291}$$

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Therefore, under the optimal feedback, the approximate (for $Q \ll 1$) equation for the unconditioned variance (6.124) is

$$\left\{\begin{pmatrix} 0 & 1 \\ Q^{-1/2} & Q^{-1/2} \end{pmatrix}(V_{\rm ss} - W) + \text{m.t.}\right\} = -\frac{\hbar}{4\kappa}\begin{pmatrix} 2\nu & \nu\sqrt{2\nu} \\ \nu\sqrt{2\nu} & \nu^2 \end{pmatrix}. \tag{6.292}$$

In order to counter the largeness of $Q^{-1/2}$, we must have

$$V_{\rm ss} - W = \epsilon\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} + O(Q^{1/2}), \tag{6.293}$$

for some positive constant ϵ.

Exercise 6.34 *Prove this by considering an asymptotic expansion of $V_{\rm ss} - W$ in powers of $Q^{1/2}$.*

On substituting Eq. (6.293) into Eq. (6.292), we find from equating the top-left element of both sides that $\epsilon = \hbar\nu/(4\kappa) + O(Q^{1/2})$. Thus we have finally (remember that $\nu = \sqrt{1+4\kappa^2} - 1$)

$$V_{\rm ss} = \frac{\hbar}{4\kappa}\begin{pmatrix} \nu + \sqrt{2\nu} & 0 \\ 0 & \nu + (1+\nu)\sqrt{2\nu} \end{pmatrix} + O(Q^{1/2}). \tag{6.294}$$

Note that, even though we have set the control cost Q to zero, $V_{\rm ss}$ does not approach W. The 'classical' fluctuations (6.293) are of the same order ($\hbar$) as the quantum noise W. This is because the control constraint, that $B = (0, 1)^\top$, means that the system is not pacifiable. This follows from Eq. (6.127), since rank$[B]$ is one, but rank$[B\ \mathrm{F}^\top]$ is two.

Under this optimal feedback control, the integrand in the cost function (6.126) evaluates to

$$\mathrm{E}_{\rm ss}[h] = \mathrm{tr}[Y B Q^{-1} B^\top Y W] + \mathrm{tr}[Y D] \tag{6.295}$$

$$= \hbar\sqrt{\frac{2\nu}{\kappa}}\left(1 + \sqrt{\nu/2} + \nu/2\right) + \hbar O(Q^{1/2}). \tag{6.296}$$

Considered as a function of κ, this cost has a minimum of $\hbar$ as $\kappa \to 0$ (since then $\nu \to \kappa/2$). That is, the optimal measurement strength is zero, and the cost rises monotonically with κ. Of course, the measurement strength must be non-zero in order for it be possible to stabilize the system at all. Moreover, the time-scale for the conditioned system covariance matrix $V_{\rm c}$ to reach its equilibrium value W is of order κ^{-1}, so in practice κ cannot be set too small. Finally, in a realistic system there will be other sources of noise. That is, D_0 will not be zero. The full solution in that case is considerably more complicated, but it is not difficult to see that in general there will be an optimal non-zero value of κ that depends upon D_0.

Pacifiability revisited. In the limit $\kappa \to 0$ as we have just been considering, $\nu \to 0$ and so $V_{\rm ss} - W \to 0$. That is, the feedback-stabilized system has no excess variance above the conditioned quantum state. This is what we expect for a pacifiable system, but we just showed above that the system is not pacifiable. There are a couple of ways to understand

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this conundrum. First, in the limit $\kappa \to 0$, the matrix F in Eq. (6.291) has one element much larger than the other. Thus to leading order $\mathrm{F} \simeq (\sqrt{\hbar\kappa}, 0)$, so that the measurement-induced noise in the conditioned mean position is much larger than that for the conditioned mean momentum. Using this approximation, the system is pacifiable from the definition (6.127).

The second way to understand how the system is effectively pacifiable in the $\kappa \to 0$ limit is to make a rotating-wave (or secular) approximation. The weak measurement limit is the same as the rapid oscillation limit, so it makes sense to move to a rotating frame at the (unit) oscillator frequency and discard rotating terms. There are many ways to do this: using the Langevin equations (as discussed in Section 6.4.5), the Belavkin equation, or the Kalman filter. Here we do it using the Belavkin equation (the SME), which is

$$\hbar\,\mathrm{d}\rho_{\mathrm{c}} = -\mathrm{i}[(\hat{q}^2 + \hat{p}^2)/2 + u(t)\hat{p},\, \rho_{\mathrm{c}}]\mathrm{d}t + \kappa\,\mathrm{d}t\,\mathcal{D}[\hat{q}]\rho_{\mathrm{c}} + \sqrt{\hbar\kappa}\;\mathrm{d}w(t)\mathcal{H}[\hat{q}]\rho_{\mathrm{c}}. \quad (6.297)$$

On moving to the interaction frame with respect to $\hat{H}_0 = \hbar(\hat{q}^2 + \hat{p}^2)/2$ we have

$$\begin{aligned}\hbar\,\mathrm{d}\rho_{\mathrm{c}} = {} & -\mathrm{i}[u(t)(\hat{p}\cos t - \hat{q}\sin t),\, \rho_{\mathrm{c}}]\mathrm{d}t + \kappa\,\mathrm{d}t\,\mathcal{D}[\hat{q}\cos t + \hat{p}\sin t]\rho_{\mathrm{c}} \\ & + \sqrt{\hbar\kappa}\,\mathrm{d}w(t)\mathcal{H}[\hat{q}\cos t + \hat{p}\sin t]\rho_{\mathrm{c}}. \end{aligned} \quad (6.298)$$

Under the secular approximation, $\mathcal{D}[\hat{q}\cos t + \hat{p}\sin t] \to \frac{1}{2}\mathcal{D}[\hat{q}] + \frac{1}{2}\mathcal{D}[\hat{p}]$. Recall from Section 6.4.5 that we cannot average oscillating terms that multiply $\mathrm{d}w(t)$. Rather, we must consider the average of the correlation functions of $\mathrm{d}w_1(t) = \sqrt{2}\,\mathrm{d}w(t)\cos t$ and $\mathrm{d}w_2(t) = \sqrt{2}\,\mathrm{d}w(t)\sin t$, namely $\mathrm{d}w_i(t)\,\mathrm{d}w_j(t) = \delta_{ij}\,\mathrm{d}t$. Similarly, we cannot assume that $u(t)$ is slowly varying and average over oscillating terms that multiply $u(t)$. Instead we should define $u_1(t) = u(t)\cos t$ and $u_2(t) = -u(t)\sin t$ (and we expect that these will have slowly varying parts). Thus we obtain the approximate SME

$$\begin{aligned}\hbar\,\mathrm{d}\rho_{\mathrm{c}} = {} & -\mathrm{i}[u_1(t)\hat{p} - u_2(t)\hat{q},\, \rho_{\mathrm{c}}]\mathrm{d}t + (\kappa/2)\mathrm{d}t(\mathcal{D}[\hat{q}] + \mathcal{D}[\hat{p}])\rho_{\mathrm{c}} \\ & + \sqrt{\hbar\kappa/2}\;\mathrm{d}w_1\,\mathcal{H}[\hat{q}]\rho_{\mathrm{c}} + \sqrt{\hbar\kappa/2}\;\mathrm{d}w_2\,\mathcal{H}[\hat{p}]\rho_{\mathrm{c}}. \end{aligned} \quad (6.299)$$

Exercise 6.35 *Show that for this system we have* $C = \sqrt{2\kappa/\hbar}\,I$, $A = 0$, $D = (\hbar\kappa/2)I$ *and* $B = I$. *Thus verify that* $W = (\hbar/2)I$ *and that the system is pacifiable.*

6.6.6 Optimal unravellings

The preceding section showed that, as a consequence of the fluctuation–observation relation, quantum feedback control differs from classical feedback control in that it is often the case that it is not optimal to increase the measurement strength without limit. However, even for a given (fixed) measurement strength, there are questions that arise in quantum control that are meaningless classically. In particular, given a linear system with dynamics described by the drift A and diffusion D matrices, what is the optimal way to monitor the bath to minimize some cost function? Classically, the unconditioned evolution described by A and D would not proscribe the measurements that can be made on the system in any way. But for quantum systems the fluctuation–observation relation means that the stationary conditioned

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covariance matrix W_U will be positive definite. Thus the control cost associated with the system will always be non-zero, and will depend upon the unravelling U.

Consider an asymptotic LQG problem. Then the cost to be minimized (by choice of unravelling) is

$$m = \mathrm{E}_{\mathrm{ss}}[h] = \mathrm{tr}[YBQ^{-1}B^{\top}YW_U] + \mathrm{tr}[YD], \tag{6.300}$$

where Y, B, Q and D are constant matrices (independent of the unravelling U). If Y is a stabilizing solution then $YBQ^{-1}B^{\top}Y$ will be positive definite – if it were not then the optimal control could allow the uncertainty in some system modes to grow to infinity. Because of this the optimal solution W_U will always be found to correspond to a pure state, since that of a mixed state would necessarily give a larger value for $\mathrm{tr}[YBQ^{-1}B^{\top}YW_U]$. Thus simply minimizing Eq. (6.300), subject to the condition that W_U correspond to a quantum state, will guarantee that W_U corresponds to pure state. Recall that W_U corresponds to a quantum state provided that it satisfies the LMI

$$W_U + \mathrm{i}\hbar\Sigma/2 \geq 0. \tag{6.301}$$

Recall also from Section 6.6.4 that there is a sufficient condition on a pure-state W_U for it to be physically realizable, namely that it satisfy the second LMI

$$AW_U + W_U A^{\top} + D \geq 0. \tag{6.302}$$

Now the problem of minimizing a linear function (6.300) of a matrix (here W_U) subject to the restriction of one or more LMIs for that matrix is a well-known mathematical problem. Significantly, it can be solved numerically using the efficient technique of semi-definite programming [VB96]. This is a generalization of linear programming and a specialization of convex optimization. Note that here 'efficient' means that the execution time for the semi-definite program scales polynomially in the system size n. As pointed out earlier, an unravelling U that gives any particular permissible W_U can also be found efficiently by solving the linear matrix equation (6.268).

Example: on-threshold OPO. We now illustrate this with an example. Consider the system described in Section 6.6.4, a damped harmonic oscillator at threshold subject to dyne detection (such as homodyne or heterodyne). Since optimal performance will always be obtained for efficient detection, such detection is parameterized by the complex number υ, such that $|\upsilon| \leq 1$, with the unravelling matrix given by

$$U = \frac{1}{2}\begin{pmatrix} 1 + \mathrm{Re}\,\upsilon & \mathrm{Im}\,\upsilon \\ \mathrm{Im}\,\upsilon & 1 - \mathrm{Re}\,\upsilon \end{pmatrix}. \tag{6.303}$$

Homodyne detection of the cavity output corresponds to $\upsilon = \mathrm{e}^{2\mathrm{i}\theta}$, with θ the phase of the measured quadrature,

$$\hat{x}_\theta = \hat{q}\cos\theta - \hat{p}\sin\theta. \tag{6.304}$$

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That is, $\theta = 0$ corresponds to obtaining information only about q, while $\theta = \pi/2$ corresponds to obtaining information only about p. In heterodyne detection information about both quadratures is obtained equally, and $\upsilon = 0$ so that $U = I/2$.

Now let us say that the aim of the feedback control is to produce a stationary state where $q = p$ as nearly as possible. (There is no motivation behind this aim other than to illustrate the technique.) The quadratic cost function to be minimized is thus $\langle(\hat{q} - \hat{p})^2\rangle_{\rm ss}$. That is,

$$P = \begin{pmatrix} 1 & -1 \\ -1 & 1. \end{pmatrix} \tag{6.305}$$

In this optical example it is simple to displace the system in its phase space by application of a coherent driving field. That is, we are justified in taking B to be full row rank, so that the system will be pacifiable.

Any quadratic cost function will be minimized for a pure state, so we may assume that Eq. (6.301) is saturated, with $\alpha\gamma = 1 + \beta^2$. Ignoring any control costs, we have $Q \to 0$. Thus, from Eq. (6.129), the minimum cost m achievable by optimal control is

$$m = \mathrm{E}_{\rm ss}[h] = \mathrm{tr}[P W_U], \tag{6.306}$$

constrained only by $0 < \gamma \leq (1 - \beta^2)/2$. The minimum is found numerically to be $m^\star \approx 1.12\hbar$ at $\beta^\star \approx 0.248$ and $\gamma^\star = [1 - (\beta^\star)^2]/2$. Note that for this simple example we do not need semi-definite programming to find this optimum, but for larger problems it would be necessary.

Having found the optimal $W_U^\star$, we now use Eq. (6.268) to find the optimal unravelling:

$$U^\star = \begin{pmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{pmatrix} \text{ for } \theta \approx 0.278\pi. \tag{6.307}$$

This corresponds to homodyne detection with θ being the phase of the measured quadrature (θ above). Naively, since one wishes to minimize $(q - p)^2$, one might have expected that it would be optimal to obtain information only about $q - p$. That is, from Eq. (6.304), one might have expected the optimal θ to be $\pi/4$. The fact that the optimal θ is different from this points to the nontriviality of the problem of finding the optimal unravelling in general, and hence the usefulness of an efficient numerical technique for achieving it.

6.6.7 Markovian feedback

Recall from Section 6.4.5 that classically, under the conditions that there exists a stabilizing solution W, that there are no control costs, that the system is pacifiable, and that we are interested in steady-state performance only, the optimal control problem can be solved by Markovian feedback. Exactly the same analysis holds in the quantum case. The required feedback Hamiltonian is

$$\hat{H}_{\rm fb}(t) = \hbar\hat{\mathbf{f}}^\top\hat{\mathbf{y}}(t), \tag{6.308}$$

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where

$$\hat{\mathbf{f}}^\top = -\hat{\mathbf{x}}^\top \Sigma B L/\hbar. \tag{6.309}$$

Generalizing the analysis of Section 5.5, the ensemble-average evolution including the feedback is described by the master equation

$$\begin{aligned}\hbar\dot{\rho} &= -\mathrm{i}[\hat{H},\rho] + \mathcal{D}[\hat{\mathbf{c}}]\rho + \hbar\mathcal{D}[\hat{\mathbf{f}}]\rho \\ &\quad + \Big\{\mathrm{i}\big[(\hat{\mathbf{c}}^\top, -\mathrm{i}\hat{\mathbf{c}}^\top)T\rho\hat{\mathbf{f}} + \rho(\hat{\mathbf{c}}^\dagger, \mathrm{i}\hat{\mathbf{c}}^\dagger)T\hat{\mathbf{f}}\big] + \mathrm{H.c.}\Big\}.\end{aligned} \tag{6.310}$$

Remember that the matrix T is defined such that $TT^\top = \hbar U$. Equation (6.310) is not limited to linear systems. That is, it is valid for any $\hat{\mathbf{c}}$ with $\hat{c}_l \in \mathfrak{L}(\mathbb{H})$, any $\hat{H} \in \mathfrak{D}(\mathbb{H})$, any $\hat{\mathbf{f}}$ with $\hat{f}_l \in \mathfrak{D}(\mathbb{H})$ and any $U \in \mathfrak{U}$ given by Eq. (6.190).

Exercise 6.36 *Referring back to Section 5.5, convince yourself of the correctness of Eq. (6.310) and show that it is of the Lindblad form.*

For linear systems, the master equation (6.310) can be turned into an OUE for the Wigner function, as could be done for the original master equation as explained in Section 6.6.2. However, just as for the original evolution (with no feedback), it is easier to calculate the evolution of $\hat{\mathbf{x}}$ in the Heisenberg picture, including the feedback Hamiltonian (6.308). The result is precisely Eq. (6.141), with hats placed on the variables. Thus the classical results for Markovian feedback all hold for the quantum case.

Under the conditions stated at the beginning of this section, it is thus clear that the optimal measurement sensitivity (if it exists) and the optimal unravelling are the same for Markovian feedback as for state-based feedback. The optimal unravelling is found by solving the semi-definite program of minimizing

$$m = \mathrm{E}_{\mathrm{ss}}[h] = \mathrm{tr}[PW_U] \tag{6.311}$$

subject to the LMIs (6.302) and (6.301). Recall that the feedback-modified drift matrix is

$$M = A + BLC = A - W_U C^\top C - \Gamma^\top C. \tag{6.312}$$

For the example considered in the preceding section,

$$\hbar C/2 = -\Gamma = T^\top. \tag{6.313}$$

Thus $M = A - 4W_U U/\hbar + 2U$. For the optimal unravelling (6.307),

$$M^\star \approx \begin{pmatrix} -1.29 & -1.53 \\ 0.32 & -1.62 \end{pmatrix}. \tag{6.314}$$

Exercise 6.37 *Show that this is strictly stable, as it should be.*

Although it is natural to consider these ideal conditions under which state-based and Markovian feedback are equally effective, it is important to note that there are common circumstances for which these conditions do not hold. In particular, there are good reasons why the control matrix B might not have full row rank. If the ps and qs correspond

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to momenta and positions of particles, then it is easy to imagine implementing a time-dependent potential linear in the qs (i.e. a time-dependent but space-invariant force), but not so for a time-dependent Hamiltonian term linear in the ps. In such circumstances state-based feedback may be strictly superior to Markovian feedback.

This can be illustrated by the harmonic oscillator with position measurement, as considered in Section 6.6.5. Say $B = (0, 1)^\top$, describing the situation in which only a position-dependent potential can be controlled. Taking $m = \omega = 1$ as before, the feedback-modified drift matrix is

$$A' = A + BLC \tag{6.315}$$

$$= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\begin{pmatrix} L \end{pmatrix}\begin{pmatrix} C & 0 \end{pmatrix} \tag{6.316}$$

$$= \begin{pmatrix} 0 & 1 \\ -1 + LC & 0. \end{pmatrix}. \tag{6.317}$$

Thus the only effect the feedback can have in this situation is to modify the frequency of the oscillator from unity to $\sqrt{1 - LC}$. It cannot damp the motion of the particle at all.

How do we reconcile this analysis with the experimental result, discussed in Section 5.8.2, demonstrating cooling of an ion using Markovian feedback? The answer lies in the secular approximation, as used in Section 6.6.5 for this sort of system. The rapid ($\nu = 1$ MHz) oscillation of the ion means that the signal in the measured current $y(t)$ also has rapid sinusoidal oscillations. In the experiment the current was filtered through a narrow ($B =$ 30 kHz) band-pass filter centred at the ion's oscillation frequency. This gives rise to two currents – the cosine and the sine components of the original $y(t)$. The innovations in these currents correspond exactly to the two noise terms $\mathrm{d}w_1$ and $\mathrm{d}w_2$ in the SME (6.299) under the secular approximation. As shown in that section, the system in the secular approximation is pacifiable. Moreover, because the bandwidth B was much greater than the characteristic relaxation rate of the ion ($\Gamma = 400$ Hz) it is natural (in this rotating frame) to regard these current components as raw currents $y_1(t)$ and $y_2(t)$ that can be fed back directly, implementing a Markovian feedback algorithm. Thus we see that the limitations of Markovian feedback can sometimes be overcome if one is prepared to be lenient in one's definition of the term 'Markovian'.

6.7 Further reading

6.7.1 Approximations to LQG quantum feedback

No system is exactly linear, hence the LQG control theory discussed here is an idealization. Nevertheless, LQG control theory can be simply adapted to deal with even quite nonlinear systems. The optimal approach with nonlinear systems is to use the full nonlinear filtering equations (the Kushner–Stratonovich equation and Bellman equations classically, or their quantum equivalent). This is often not practical, because of the difficulty of solving these

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nonlinear equations in real time. Hence it is attractive to consider a suboptimal approach based on LQG control. The basic idea is to linearize the system around its mean configuration in phase-space, use LQG theory to control the system and to update one's estimates of the mean vector and covariance matrix for a short time, and then relinearize around the new (approximate) mean configuration. As long as the 'true' (i.e. optimal) conditioned system state remains approximately Gaussian, this procedure works reasonably well. Doherty *et al.* [DHJ$^+$00] demonstrate theoretically that it can be used to control a quantum particle in a double-well potential, forcing it to occupy one or the other well.

A more immediate application for quantum feedback control is in the cooling of oscillators subject to position monitoring. When a rotating-wave approximation can be made, Markovian feedback works well, as discussed in Section 6.6.7. However, for systems with a relatively low oscillation frequency ω, such that the feedback-induced damping rate is comparable to ω, state-based control such as LQG is required. The theory of feedback cooling of nano-mechanical resonators using a simplified version of LQG control was done by Hopkins *et al.* [HJHS03], and recent experiments suggest that it should be possible to implement this scheme [LBCS04].

Another example is the feedback cooling of atoms in a standing wave, as analysed by Steck *et al.* [SJM$^+$04]. In this case the dynamics is nonlinear, and linearization was used to derive approximate equations for the mean vector and covariance matrix as described above. However, in this case the feedback control signal was derived from considering the exact equations, and was a function of both the mean and the covariance (unlike with LQG control, where it is always a function of the mean only). It was shown that the atom could be cooled to within one oscillator quantum of its ground-state energy.

6.7.2 State-based quantum feedback control in finite-dimensional systems

In a series of papers [Kor01b, RK02, ZRK05], Korotkov and co-workers have considered the use of state-based quantum feedback control in a solid-state setting. They show that, by such control, Rabi oscillations of a solid-state qubit may be maintained indefinitely (although imperfectly) even in the presence of environmental noise. The basic idea is to compare the computed phase of the qubit state (as computed from the measurement results) with the time-dependent phase required for the desired Rabi oscillations, and to alter the qubit Hamiltonian in order to reduce the discrepancy. Korotkov has also shown that a more feasible algorithm, which does not involve computing the conditioned state from the measurement record, works almost as well [Kor05]. This approach, in which the observed current is filtered through a simple circuit before being fed back, is more like the current-based feedback considered in Chapter 5.

State-based control of a different two-level system, an atom, has also been considered. Here the measurement record is assumed to arrive from the spontaneous emission of the atom. Markovian feedback in this system was considered first [HHM98, WW01, WWM01]. It was shown that, by controlling the amplitude of a coherent driving field, the atom could be stabilized in almost any pure state (for efficient detection). The exceptions were states

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on the equator of the Bloch sphere, for which the Markovian feedback algorithm produced a completely mixed state in steady state. This deficiency can be overcome using state-based feedback [WMW02]. Moreover, it was proven rigorously (i.e. without reliance on numerical evidence from stochastic simulations) that state-based feedback is superior to Markovian feedback in the presence of imperfections such as inefficient detection or dephasing.

A final application of state-based control is in deterministic Dicke-state preparation. As discussed in Section 5.7, Markovian feedback can (in principle) achieve deterministic spin-squeezing close to the Heisenberg limit. This is so despite the fact that the approximations behind the feedback algorithm [TMW02b], which are based on linearizing about the mean spin vector and treating the two orthogonal spin components as continuous variables, break down in the Heisenberg limit. The breakdown is most extreme when the state collapses to an eigenstate of $\hat{J}_z$ (a Dicke state) with eigenvalue zero. This can be visualized as the equatorial ring around the spin-J Bloch sphere of Fig. 5.4, for which the spin vector has zero mean. Without feedback, the QND measurement alone will eventually collapse the state into a Dicke state, but one that can be neither predicted nor controlled. However, Stockton *et al*. show using stochastic simulations that state-based feedback does allow the deterministic production of a $J_z = 0$ Dicke state in the long-time limit [SvHM04].

Applications of state-based quantum feedback control in quantum information will be considered in Chapter 7.

6.7.3 Beyond state-based control

There are reasons to consider cost functions that are not additive in time. Considering the classical case to start, this means cost functions not of the form of Eq. (6.44) (a time-integral). One reason is found in 'risk-sensitive' control [Whi81, DGKF89], in which small excursions from the desired outcome are tolerated more, large excursions less. Such control tends to be more robust with respect to errors in the equations describing the system dynamics. In such cases it can be shown that $\wp_c(\check{\mathbf{x}}; t)$ is not sufficient to specify the optimal control law. Interestingly, sometimes the optimal control law is a function of a *different* state, $\wp_c'(\check{\mathbf{x}}; t)$. That is, the separation principle still applies, but for a state (a normalized probability distribution) that is *differently* conditioned upon $\{\mathbf{y}(t')\}_{t'=t_0}^{t'=t}$, and so is not an optimal predictor for the properties of the system. An example of a risk-sensitive cost function that yields such a state is an *exponential* of a time-integral.

James [Jam04, Jam05] recently derived a quantum equivalent to this type of control, involving a differently conditioned quantum state $\rho_c'(t)$. Here care must be taken in defining the cost function, because system variables at different times will not commute. Considering the case of no terminal costs for simplicity, James defines the cost function to be $\langle \hat{R}(T) \rangle$, where $\hat{R}(t)$ is the solution of the differential equation

$$\frac{\mathrm{d}\hat{R}}{\mathrm{d}t} = \mu \hat{C}(t) \hat{R}(t) \tag{6.318}$$

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satisfying $\hat{R}(0) = \hat{1}$. Here $\hat{C}(t) = \int_0^t \hat{h}(s)\mathrm{d}s$, where $\hat{h}(t)$ is a function of observables at time t, while $\mu > 0$ is a risk parameter. In the limit $\mu \to 0$, $[\hat{R}(T) - \hat{1}]/\mu \to \hat{C}(T)$, so the problem reduces to the usual ('risk-neutral') sort of control problem.

A useful and elegant example of risk-sensitive control is LEQG [Whi81]. This is akin to the LQG control discussed above (an example of risk-neutral control), in that it involves linear dynamics and Gaussian noise. But, rather than having a cost function that is the expectation of a time-integral of a quadratic function of system and control variables, it has a cost function that is the exponential of a time-integral of a quadratic function. This fits easily in James' formalism, on choosing $\hat{h}(s)$ to be a quadratic function of system observables and control variables (which are also observables in the quantum Langevin treatment [Jam05]). Just as for the LQG case, many results from classical LEQG theory follow over to quantum LEQG theory [Yam06]. This sort of risk-sensitive control is particularly useful because the linear dynamics (in either LQG or LEQG) is typically an approximation to the true dynamics. Because risk-sensitive control avoids large excursions, it can ensure that the system does not leave the regime where linearization is a good approximation. That is, the risk-sensitive nature of the control helps ensure its validity.

A different approach to dealing with uncertainties in the dynamics of systems is the robust estimator approach adopted by Yamamoto [Yam06]. Consider quantum LQG control, but with bounded uncertainties in the matrices A and C. Yamamoto finds a non-optimal linear filter such that the mean square of the estimation error is guaranteed to be within a certain bound. He then shows by example that linear feedback based on this robust observer results in stable behaviour in situations in which both standard (risk-neutral) LQG and (risk-sensitive) LEQG become unstable. Yet another approach to uncertainties in dynamical parameters is to describe them using a probability distribution. One's knowledge of these parameters is then updated simultaneously, and in conjuction, with one's knowledge of the system. The interplay between knowledge about the system and knowledge about its dynamics leads to a surprising range of behaviour under different unravellings. This is investigated for a simple quantum system (resonance fluorescence with an uncertain Rabi frequency) in Ref. [GW01].

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