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Super-exponential endogenous bubbles in an equilibrium model of fundamentalist and chartist traders

Taisei Kaizoji[a,1], Matthias Leiss[b,*,1], Alexander Saichev[b,c,2], Didier Sornette[b,d]

[a] Graduate School of Arts and Sciences, International Christian University, 3-10-2 Osawa, Mitaka, Tokyo 181-8585, Japan
[b] ETH Zurich, Department of Management, Technology and Economics, Kreuzplatz 5, CH-8032 Zurich, Switzerland
[c] Nizhni Novgorod State University, Department of Mathematics, Russia
[d] Swiss Finance Institute, c/o University of Geneva, 40 Boulevard Du Pont d'Arve, CH 1211 Geneva 4, Switzerland

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ABSTRACT

We introduce a model of super-exponential financial bubbles with two assets (risky and risk-free), in which fundamentalist and chartist traders co-exist. Fundamentalists form expectations on the return and risk of a risky asset and maximize their constant relative risk aversion expected utility with respect to their allocation on the risky asset versus the risk-free asset. Chartists are subjected to social imitation and follow momentum trading. Allowing for random time-varying herding propensity, we are able to reproduce several well-known stylized facts of financial markets such as a fat-tail distribution of returns and volatility clustering. In particular, we observe transient faster-than-exponential bubble growth with approximate log-periodic behavior and give analytical arguments why this follows from our framework. The model accounts well for the behavior of traders and for the price dynamics that developed during the dotcom bubble in 1995–2000. Momentum strategies are shown to be transiently profitable, supporting these strategies as enhancing herding behavior.

1. Introduction

The very existence of financial bubbles has been a controversial and elusive subject. Some have argued that financial bubbles play a huge role in the global economy, affecting hundreds of millions of people (Kindleberger, 1978; Shiller, 2000; Sornette, 2003). Others have basically ignored or refuted their possibility (Fama, 1998). Moreover, until recently, the existence of such bubbles, much less their effects, have been ignored at the policy level. Finally, only after the most recent historical global financial crisis, officials at the highest level of government and academic finance have acknowledged the existence and importance of identifying and understanding bubbles. The President of the Federal Reserve Bank of New York, William C. Dudley, stated in April 2010 "what I am proposing is that we try–try to identify bubbles in real time, try to develop tools to address those bubbles, try to use those tools when appropriate to limit the size of those bubbles and, therefore, try to limit

* Corresponding author. Tel.: +41 446328708.
E-mail addresses: kaizoji@icu.ac.jp (T. Kaizoji), mleiss@ethz.ch (M. Leiss), dsornette@ethz.ch (D. Sornette).
[1] Joint first authors.
[2] Deceased 8 June 2013.

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the damage when those bubbles burst." Such a statement from the New York Fed representing, essentially, the monetary policy of the United States governmental banking system would have been, and, in some circles, still is, unheard of. This, in short, is a bombshell and a wake-up call to academics and practitioners. Dudley exhorts to try to develop tools to address bubbles.

But before acting against bubbles, before even making progress in ex-ante diagnosing bubbles, one needs to define what is a bubble. The problem is that the "econometric detection of asset price bubbles cannot be achieved with a satisfactory degree of certainty. For each paper that finds evidence of bubbles, there is another one that fits the data equally well without allowing for a bubble. We are still unable to distinguish bubbles from time-varying or regime-switching fundamentals, while many small sample econometrics problems of bubble tests remain unresolved." summarizes Gurkaynak (2008) in his review paper.

Let us start with the rather generally accepted stylized fact that, in a period where a bubble is present, the stock return exhibits transient excess return above the long-term historical average, giving rise to what could be termed a "bubble risk premium puzzle". For instance, as we report in the empirical section, the valuation of the Internet stock index went from a reference value 100 in January 1998 to a peak of 1400.06 in March 9, 2000, corresponding to an annualized return of more than 350% ! A year and a half later, the Internet stock valuation was back at its pre-1998 level. Such explosive super-exponential growth has been documented extensively for bubbles in real markets (see for example Sornette et al., 2009; Jiang et al., 2010; Yan et al., 2012) and recently observed in lab experiments (Hüsler et al., 2013). Another stylized fact well represented during the dotcom bubble is the highly intermittent or punctuated growth of the stock prices, with super-exponential accelerations followed by transient corrections, themselves followed by further vigorous rebounds (Johansen and Sornette, 2010; Sornette and Woodard, 2010).

Bubbles are usually followed by crashes, in an often tautological logic resulting from the fact that the existence of a crash is usually taken as the ex-post signature of a bubble, as summarized by Greenspan (2002): "We, at the Federal Reserve... recognized that, despite our suspicions, it was very difficult to definitively identify a bubble until after the fact, that is, when its bursting confirmed its existence...." More optimistically but still controversial, recent systematic econometric studies have shown that it is possible to relate objectively an anomalous transient excess return and the subsequent crash (Sornette, 2003; Johansen and Sornette, 2010). Furthermore, there is another relatively new stream of literature devoted to the early detection of bubbles, which also focuses on the often observed extreme growth of the price mentioned above. Phillips et al. (2011) have employed mildly explosive autoregressive processes of the log-price with an AR coefficient slightly larger than one decreasing toward one over time. This model results in super-exponential growth of the price and has led to bubble tests based on Markov-switching state-space models (Al-Anaswah and Wilfling, 2011; Lammerding et al., 2013), as well as sequential Chow-type and augmented Dickey–Fuller testing procedures for a structural breaks. Such a break could be either the start of a bubble, i.e. a transition from a random walk to a mildly explosive regime (Phillips et al., 2011; Phillips et al., 2013a; Homm and Breitung, 2012) or vice versa its end (Breitung and Kruse, 2013; Breitung, 2014). Both methods rely on the type of indirect stationarity tests initiated by Diba and Grossman (1988) and Hamilton and Whiteman (1985).

Going from econometrics to financial economics, there are several branches dedicated to modeling deviations from fundamental value. One important class of theories is related to *noise traders* (also referred to as positive-feedback investors), a term first introduced by Kyle (1985) and Black (1986) to describe irrational investors. Thereafter, many scholars exploited this concept to extend the standard models by introducing the simplest possible heterogeneity in terms of two interacting populations of rational and noise traders. One can say that the one-representative-agent theory is being progressively replaced by a two-representative-agents theory, analogously to the progress from the one-body to the two-body problems in physics. It has been often explained that markets bubble and crash in the absence of significant shifts in economic fundamentals when herders such as chartists deliberately act against their private information and follow the crowd.

De Long et al. (1990a,b) proposed the first model of market bubbles and crashes which exploits this idea of the possible role of noise traders following positive feedback or momentum investment strategies in the development of bubbles. They showed a possible mechanism for why asset prices may deviate from the fundamentals over long time periods. The key point is that trading between rational arbitrageurs and chartists gives rise to bubble-like price patterns. In their model, rational speculators destabilize prices because their trading triggers positive feedback trading by noise traders. This in turn leads to a positive auto-correlation of returns at short horizons. Eventually, arbitrage by rational speculators will pull the prices back to fundamentals. Their arbitrage trading leads to a negative autocorrelation of returns at longer horizons.

Their work was followed by a number of empirical studies on positive feedback trading. Influential empirical evidence on positive feedback trading came from the works of De Bondt and Thaler (1985), and Jegadeesh and Titman (1993, 2001), which established that stock returns exhibit momentum behavior at intermediate horizons, and reversals at long horizons. That is, strategies which buy stocks that have performed well in the past and sell stocks that have performed poorly in the past generate significant positive returns over 3- to 12-month holding periods. However, stocks that perform poorly in the past perform better over the next 3–5 years than stocks that perform well in the past. Behavioral models that explain the coexistence of intermediate horizon momentum and long horizon reversals in stock returns are proposed by Barberis et al. (1998), Daniel et al. (1998), and Hong and Stein (1999).

The behavior of investors who are driven by group psychology and the aggregate behavioral outcomes, have also been studied using frameworks suggested by Weidlich and Haag (1983), Blume (1993, 1995), Brock (1993), Durlauf (1997, 1999), Kirman (1993), Brock and Durlauf (2001), Aoki and Yoshikawa (2007), Chiarella et al. (2009) and Hommes and Wagener

(2009). Phan et al. (2004) summarize the formalism starting with different implementation of the agents' decision processes whose aggregation is inspired from statistical mechanics to account for social influence in individual decisions. Lux (1995), Lux and Marchesi (1999), Brock and Hommes (1999), Kaizoji (2000, 2010), and Kirman and Teyssiere (2002) have developed related models in which agents' successful forecasts reinforce the forecasts. Such models have been found to generate swings in opinions, regime changes and long memory. An essential feature of these models is that agents are wrong for a fraction of the time but, whenever they are in the majority, they are essentially right by a kind of self-fulfilling prophecy. Thus, they are not systematically irrational (Kirman, 1997). Sornette and Zhou (2006) showed how irrational Bayesian learning added to the Ising model framework reproduces the stylized facts of financial markets. Harras and Sornette (2011) showed how over-learning from lucky runs of random news in the presence of social imitation may lead to endogenous bubbles and crashes.

Here, we follow this modeling path and develop a model of the pricing mechanism and resulting dynamics of two co-existing classes of assets, a risky asset representing for instance the Internet sector during the dotcom bubble and a risk-free asset, in the presence of two types of investors having different opinions concerning the risky asset (Harrison and Kreps, 1978; Scheinkman and Xiong, 2003). The first type of traders is a group of fundamentalists who maximize their expected utility. The second type of traders is a group of "chartists" who trade only the risky asset by using heuristics such as past momentum and social imitation. The chartist traders do not consider the fundamentals, while the fundamentalist investors allocate their wealth based on their expectation of the future returns and risks of the risky asset.

Our framework combines elements from various groundbreaking works. The setup of chartists follows closely Lux and Marchesi (1999), where an opinion index determined by past momentum and social imitation describes the prevailing investment behavior among this group. The description of fundamentalists is related to Brock and Hommes (1999) and to Chiarella et al. (2009). In particular, we employ a utility function with constant relative risk aversion, as this is a realistic choice in a growing economy.

One important ingredient that we introduce here is that we do not allow our agents to switch their investment behavior from rational to noise trading or vice versa. This reflects the empirical fact that many large institutional investors such as pension funds have to follow strict guidelines on how to split their portfolio on assets of different risk classes. In previous models, the occurrence of a bubble was related to a convergence of a large fraction of traders on noise trading, see for example Lux and Marchesi (1999). Instead of strategy switching, we account for the volatility of the imitation propensity of chartists by assuming that it fluctuates randomly around some anchoring value as in Stauffer and Sornette (1999) and Harras et al. (2012). By keeping track of the agents' wealth levels, we are able to explain bubbles only with the transient increasing influence of chartists on the market price during an appreciation of the risky asset. While its price is rising, noise traders believing in momentum tend to invest more in the risky asset and thus become richer, thereby gaining more importance. The chartists' belief is further reinforced by social imitation, which becomes self-fulfilling. This, in turn, has destabilizing effects leading to an increase in the volatility and usually finishes in a crash when the prevailing opinion switches to pessimistic.

Within our simple setup without strategy switching we show theoretically and by simulations that bubbles start with a phase of transient super-exponential growth. As mentioned before, faster-than-exponential growth behavior has recently been picked up by the econometric literature, but to the best of our knowledge it has been rarely discussed in the context of agent-based models. A first instantiation is found in Corcos et al. (2002), in a much simplified model of imitative and contrarian agents. The present model is one of the first in which we can provide a transparent analytical explanation for the existence of a transient faster-than-exponential growth. Moreover, we observe approximate log-periodic behavior during the rise of a bubble, that can result from the nature of the fluctuations of the opinion index. Furthermore, our model reproduces several stylized facts of financial markets. The distribution of returns is fat-tailed. Also, signed returns are characterized by a fast-decaying autocorrelation, while the autocorrelation function for absolute returns has a long memory (volatility clustering). While many of the ingredients and conditions used in our agent-based model may be found in various forms in some previous agent-based models, none have documented explicitly the important transient super-exponential behavior associated with bubbles, nor explained qualitatively or quantitatively the underlying mechanisms and the coexisting salient stylized facts.

The paper is organized as follows: the basic model is presented in Sections 2 and 3 and analyzed theoretically in Section 4. Numerical simulations of the model are performed and the results are discussed in Section 5, together with a discussion of the price dynamics, its returns and momentum strategies during the dotcom bubble from 1998 to 2000. We conclude in Section 6.

2. Set-up of the model of an economy made of fundamentalists and chartists

We consider fixed numbers N_f of fundamentalist and N_c of chartist investors who trade the same risky asset, represented here for simplicity by a single representative risky asset fund. The former diversify between the risky asset and a risk-free asset on the basis of maximizing their constant relative risk aversion expected utility of returns and variance of the risky asset over the next period. The latter use technical and social indicators, such as price momentum and social imitation to allocate their wealth. A dynamically evolving fraction of them buys the risky asset while others stay out of the risky asset and have their wealth invested in the risk-free asset.

In the next Section 2.1, we solve the standard allocation problem for the fundamentalists that determines their demand for the risky asset. Then, in Section 2.2, the general ingredients controlling the dynamics of the demand of chartists are developed.

2.1. Allocation equation for the fundamentalists

The objective of the N_f fundamentalists is assumed to be the maximization at each time t of the expected utility of their expected wealth W^f_{t+1} at the next period, thus following Chiarella et al. (2009) and Hommes and Wagener (2009). To perform this optimization, they select at each time t a portfolio mix of the risky asset and of the risk-free asset that they hold over the period from t to $t+1$. Such one-period ahead optimization strategy can be reconciled with underlying expected utility maximizing stories as given for example in Brock and Hommes (1997, 1998), Chiarella et al. (2009), Boswijk et al. (2007) and Hommes and Wagener (2009).

The fundamentalists are assumed to be identical, so that we can consider the behavior of one representative fundamental trader hereafter. We shall assume that fundamentalists are myopic mean-variance maximizers, which means that only the expected portfolio value and its variance impact their allocation. We denote P_t the price of the risky asset and x^f_t the number of risky assets that the representative fundamentalist holds at instant t. We also assume that the risky asset pays a dividend d_t at each period t. Similarly, P_{ft} and X_{ft} correspond to the price and number of a risk-free asset held by the fundamentalist. The risk-free asset is in perfectly elastic supply and pays a constant return R_f. Thus, at time t, the wealth of the fundamentalist is given by

$$W^f_t = P_t x^f_t + P_{ft} X_{ft}. \tag{1}$$

The wealth of the fundamentalist changes from time t to $t+1$ according to

$$W^f_{t+1} - W^f_t = (P_{t+1} - P_t)x^f_t + (P_{ft+1} - P_{ft})X_{ft} + d_{t+1}x^f_t. \tag{2}$$

This expression takes into account that the wealth at time $t+1$ is determined by the allocation choice at time t and the new values of the risky and the risk-free asset at time $t+1$, which includes the payment of the dividend ($W^f_{t+1} = P_{t+1}x^f_t + P_{ft+1}X_{ft} + d_{t+1}x^f_t$). Let us introduce the variables

$$x^f_t := \frac{P_t x^f_t}{W^f_t}, \quad R_{t+1} := \frac{P_{t+1}}{P_t} - 1, \quad R_f := \frac{P_{ft+1}}{P_{ft}} - 1. \tag{3}$$

They are respectively the fraction x^f_t of the fundamentalist's wealth invested in the risky asset at time t, the discrete time return R_{t+1} per stock of the risky asset from time t to $t+1$ and the risk-free rate of return R_f assumed constant. This allows us to rewrite (2) as giving the total relative wealth variation from t to $t+1$:

$$W^f_{t+1} - W^f_t = W^f_t \left[R_f + x^f_t\left(R_{t+1} - R_f + \frac{d_{t+1}}{P_t}\right)\right] \equiv W^f_t \left[R_f + x^f_t R_{\text{excess},t+1}\right], \tag{4}$$

where we define

$$R_{\text{excess},t+1} = R_{t+1} - R_f + \frac{d_{t+1}}{P_t} \tag{5}$$

as the excess return of capital and dividend gains over the risk-free rate.

The problem of the fundamentalist at time t is to maximize the expected utility of his wealth for the next period by choosing the right proportion of wealth x^f_t to invest in the risky asset,

$$\max_{x^f_t} E_t[U(W^f_{t+1})], \tag{6}$$

where $E_t[.]$ means the expectation of the variable in the bracket performed at time t, i.e. under the knowledge of available information up to and including time t. If we assume the fundamentalist to have constant relative risk aversion, this proportion is constant in time and wealth. This can be shown by employing the explicit utility function $U(W)$ exhibiting constant relative risk aversion γ:

$$U(W) = \begin{cases} \log(W) & \text{for } \gamma = 1, \\ \dfrac{W^{1-\gamma}}{1-\gamma} & \text{for } \gamma \neq 1. \end{cases} \tag{7}$$

Given this utility function and wealth evolution (4), it is easy to see that the maximization condition (6) is independent of W^f_t.

We may obtain an approximate solution for x_t^f in the special case where the wealth does not change much, i.e. in the case of small returns, so that the following expansion becomes approximately valid: $R_f, R_{\text{excess},t+1} \ll 1$.

$$E_t[U(W_{t+1}^f)] = U(W_t^f) + U'(W_t^f)W_t^f(R_f + x_t^f E_t[R_{\text{excess},t+1}]) + \frac{1}{2}U''(W_t^f)W_t^2(x_t^f)^2 \text{Var}_t[R_{\text{excess},t+1}] + \mathcal{O}(R_f^3, R_{\text{excess},t+1}^3). \quad (8)$$

Maximizing this expression with respect to x_t^f gives

$$x_t^f = \frac{1}{\gamma}\frac{E_t[R_{\text{excess},t+1}]}{\text{Var}_t[R_{\text{excess},t+1}]}, \quad (9)$$

where

$$\gamma \equiv -\frac{W_t^f U''(W_t^f)}{U'(W_t^f)}. \quad (10)$$

In expression (9), $E_t[R_{\text{excess},t+1}] \equiv E_t[R_{t+1}] - R_f + E_t[d_{t+1}]/P_t$ represents the total excess expected rate of return of the risky asset from time t to $t+1$ above the risk-free rate. In the following, we assume myopic fundamentalists who do not learn but invest according to fundamental valuation. They expect a steady relative growth rate embodied by a constant total excess rate of return R_{excess}, which is based on the behavior of stock markets in the long run:

$$R_{\text{excess}} := E_t[R_{t+1}] - R_f + \frac{E_t[d_{t+1}]}{P_t} = \text{constant}. \quad (11)$$

We will assume that $R_{\text{excess}} > 0$, so that the risky asset is desirable. The variance $\text{Var}_t[R_{\text{excess},t+1}]$ will be denoted by $\tilde{\sigma}^2$ and is given by

$$\tilde{\sigma}^2 := \text{Var}_t[R_{\text{excess},t+1}] = \sigma^2 + \frac{\text{Var}[d_{t+1}]}{P_t^2}, \quad \sigma^2 := \text{Var}[R_{t+1}]. \quad (12)$$

The expression for $\text{Var}_t[R_{\text{excess},t+1}]$ relies on the absence of correlation between R_{t+1} and d_{t+1}, because the dividend policy is assumed independent of the market price and vice-versa. Modigliani and Miller (1958, 1963) show that this holds true in the case of symmetric information and bounded rationality. Our fundamentalists believe to act in this world and take the quantities R_{t+1} and d_{t+1} as exogenous to the price dynamics developed below, because they reflect the information coming from a fundamental analysis.

In the sequel, we assume that $\tilde{\sigma}^2$ is independent of the price P_t and that $P_t \gg \sqrt{\text{Var}[d_{t+1}]/\sigma^2}$. Thus, $\tilde{\sigma}^2 \simeq \sigma^2$ and $\tilde{\sigma}^2$ is approximately constant, as long as the fundamentalist investors form a non-varying expectation of the volatility of future prices of the risky asset. The assumption that $\tilde{\sigma}^2$ is constant is also made by Chiarella et al. (2009) and in the framework of Boswijk et al. (2007), if investors are assumed to be myopic, i.e. only look at the next period.

Expression (9) then becomes

$$x_t = x := \frac{R_{\text{excess}}}{\gamma\tilde{\sigma}^2}, \quad (13)$$

which is a constant. Note that this is not an ad hoc assumption, but a consequence of constant relative risk aversion and of the stationary nature of the dividend process. In particular, because of the constant relative risk aversion of the fundamentalists, as already mentioned, x is independent of the current wealth W_t^f of the agents. This allows us to treat all fundamentalists as one group with total wealth W_t^f irrespective of the distribution of the agents' individual wealth levels within the group. From here on, we will call W_t^f the wealth of *the fundamentalists*.

The assumption, that the variance $\tilde{\sigma}^2$ given by (12) is constant, implies $\text{Var}[d_t] = (\tilde{\sigma}^2 - \sigma^2)P_{t-1}^2$. Therefore, the flow of dividend d_t follows the stochastic process:

$$d_t = P_{t-1}[r + \sigma_r u_t], \quad (14)$$

where $r := R_{\text{excess}} - \text{E}_{t-1}[R_t] + R_f$, $\sigma_r = \sqrt{\tilde{\sigma}^2 - \sigma^2}$ and u_t forms a series of standard i.i.d. random variables with distribution $N(0, 1)$.

Thus, under the above assumptions, the fundamentalist investors rebalance their portfolio so as to have a constant relative weight exposure to the risky asset. This is equivalent to the traditional portfolio allocation benchmark of 70% bonds and 30% stocks used by many mutual and pension funds. Rewriting expression (2) with the condition of a fixed fraction x invested in the risky asset, the wealth W_t^f at time t of the fundamentalists becomes at $t+1$

$$W_{t+1}^f = (P_{t+1} + d_{t+1})x\frac{W_t^f}{P_t} + (1-x)W_t^f(1+R_f). \quad (15)$$

The excess demand of the risky asset from $t-1$ to t of the group of fundamentalists is defined by

$$\Delta D_t^f := P_t x_t^f - P_t x_{t-1}^f = P_t x_t^f - \frac{P_t}{P_{t-1}} P_{t-1} x_{t-1}^f = x W_t^f \left(1 - \frac{P_t}{P_{t-1}} \frac{W_{t-1}^f}{W_t^f}\right). \tag{16}$$

Expression (2) with definitions (3) gives

$$\frac{P_t}{P_{t-1}} \frac{W_{t-1}^f}{W_t^f} = \frac{P_t}{(P_t + d_t)x + P_{t-1}(1-x)(1+R_f)}. \tag{17}$$

This allows us to rewrite the excess demand ΔD_t^f as

$$\Delta D_t^f = x W_{t-1}^f \left[(1-x)\frac{P_{t-1}(1+R_f) - P_t}{P_{t-1}} + \frac{x d_t}{P_{t-1}}\right], \tag{18}$$

where x is given by expression (13). This last expression can be written, using (14), as

$$\Delta D_t^f = x W_{t-1}^f \left[(1-x)\frac{P_{t-1}(1+R_f) - P_t}{P_{t-1}} + x(r + \sigma_r u_t)\right]. \tag{19}$$

This corresponds to a kind of mean-reversing excess demand, where fundamentalists tend to buy the risky asset when its price is low and vice-versa. But this mean-reversing excess demand is adjusted by taking into account two factors that quantify an abnormal price increase (resp. decrease), which would justify unloading (resp. adding) the risky asset to the fundamentalists' portfolio. First, a price change is compared with the change that would occur if the corresponding wealth was instead invested in the risk-free asset. Second, even if its price decreases, the risky asset may still be attractive if it pays a sufficient dividend to compensate.

In absence of chartists, the market clearing condition $\Delta D_t^f = 0$ leads to

$$P_t = (1+R_f)P_{t-1} + \frac{x}{1-x} d_t. \tag{20}$$

In the simplified case where the dividends d_t are growing at a constant rate $g > 0$ such that $d_t = d_0(1+g)^t$, Eq. (20) solves into

$$P_t = (1+R_f)^t P_0 + \frac{x}{1-x}(1+R_f)^t \frac{d}{R_f - g}, \tag{21}$$

for $g < R_f$, neglecting a term $[(1+g)/(1+R_f)]^t$ compared to 1. One recognizes the Gordon–Shapiro fundamental valuation, price = dividend/$(R_f - g)$, multiplied by a scaling factor taking into account the partitioning of the wealth of the fundamentalists with the condition that a constant fraction is invested in the risky asset.

2.2. Excess demand of the chartists

2.2.1. General framework

We assume that (a) the chartists are characterized by polarized decisions (in or out of the risky asset), (b) they tend to herd and (c) they are trend-followers. A large body of literature indeed documents a lack-of-diversification puzzle (Kelly, 1995; Baxter and Jermann, 1995; Statman, 2004) as well as over-reactions (De Bondt and Thaler, 1987, 1990). There is strong evidence for imitation and herding, even among sophisticated mutual fund managers (Wermers, 1999), and technical analysis and chart trading is ubiquitous.

We account for the observations of lack-of-diversification by assuming that a chartist trader is fully invested either in the risky asset or in the risk-free asset. In contrast to the fundamentalist agents, our chartists have different opinions, which fluctuate stochastically according to laws given below. Due to the probabilistic setup the assumption of an "all or nothing" behavior at the individual level translates into a continuous investment weight of chartists at the group level and is given by the fraction of chartists invested in the risky asset varying smoothly between 0 and 1. The number of chartist investors invested in the risky asset (respectively invested in the risk-free asset) is N_t^+ (respectively N_t^-), and we have

$$N_t^+ + N_t^- \equiv N_c. \tag{22}$$

We do not aim at describing the heterogeneity between chartists, which has been shown to lead to fat-tailed distribution of their wealth as a result of heterogeneous investment decisions (Bouchaud and Mezard, 2000; Klass et al., 2007; Harras and Sornette, 2011). This is not a restriction in so far as we consider their aggregate impact.

Therefore, as for the fundamentalists, we treat the chartists as one group with total wealth W_t^c. The ratio of wealth of the group of chartists invested in the risky asset corresponds to the ratio of bullish investors among the population of chartists. Let us denote this quantity at time t by

$$x_t^c := \frac{N_t^+}{N_c}. \tag{23}$$

Then, the wealth W_t^c of chartists at time t becomes at $t+1$

$$W_{t+1}^c = (P_{t+1} + d_{t+1}) x_t^c \frac{W_t^c}{P_t} + (1 - x_t^c) W_t^c (1 + R_f). \tag{24}$$

The excess demand of the chartists over the time interval $(t-1, t)$ is equal to

$$\Delta D_t^c = x_t^c W_t^c - \frac{P_t}{P_{t-1}} x_{t-1}^c W_{t-1}^c = \tag{25}$$

$$W_{t-1}^c \left[x_t^c (1 - x_{t-1}^c)(1 + R_f) - x_{t-1}^c (1 - x_t^c) \frac{P_t}{P_{t-1}} + x_t^c x_{t-1}^c \frac{d_t}{P_{t-1}} \right]. \tag{26}$$

Let us introduce the opinion index (Lux and Marchesi, 1999)

$$s_t := \frac{N_t^+ - N_t^-}{N_c} \in [-1, 1], \tag{27}$$

which can be interpreted as the aggregate bullish ($s_t > 0$) versus bearish ($s_t < 0$) stance of the chartists with respect to the risky asset. With this definition (27) and with (22), we have

$$\frac{N_t^+}{N_c} = \frac{1}{2}(1 + s_t) = x_t^c, \quad \frac{N_t^-}{N_c} = \frac{1}{2}(1 - s_t) = 1 - x_t^c. \tag{28}$$

Expression (26) with (28) yields

$$\Delta D_t^c = \frac{W_{t-1}^c}{4P_{t-1}} [(1 + s_t)(1 - s_{t-1})(1 + R_f) P_{t-1} - (1 - s_t)(1 + s_{t-1}) P_t + (1 + s_t)(1 + s_{t-1}) d_t]. \tag{29}$$

2.2.2. Master equation for the bullish/bearish chartist trader unbalance s_t

Let us now specify the dynamics of the opinion index s_t. We assume that, at each time step, each chartist trader may change her mind and either sell her risky portfolio if she was previously invested or buy the risky portfolio if she had only the risk-free asset. Again, we assume an all-or-nothing strategy for each chartist trader at each time step. Let p_{t-1}^+ be the probability that any of the N_{t-1}^+ chartists who is currently fully invested in the risky portfolio decides to remove her exposure during the time interval $(t-1, t)$. Analogously, let p_{t-1}^- be the probability that any of the N_{t-1}^- traders who are currently (at time $t-1$) out of the risky market decides to buy it. For a chartist k who owns the risky asset, her specific decision is represented by the random variable $\zeta_k(p^+)$, which takes the value 1 (sell) with probability p^+ and the value 0 (keep the position) with probability $1 - p^+$. Similarly, for a chartist j who does not own the risky asset, her specific decision is represented by the random variable $\xi_j(p^-)$, which takes the value 1 (buy) with probability p^- and the value 0 (remain invested in the risk-free asset) with probability $1 - p^-$. For given p^+ and p^-, the variables $\{\xi_j(p^+)\}$ and $\{\zeta_k(p^-)\}$ are i.i.d..

Aggregating these decisions over all chartists invested in the risky asset at time t, we have

$$N_t^+ = \sum_{k=1}^{N_{t-1}^+} [1 - \zeta_k(p_{t-1}^+)] + \sum_{j=1}^{N_{t-1}^-} \xi_j(p_{t-1}^-). \tag{30}$$

The first term in the r.h.s. of (30) corresponds to all the traders who held the risky asset at $t-1$ and continue to hold it at t. The second term in the r.h.s. of (30) represents the chartists who were holding the risk-free asset at $t-1$ and sold it to buy the risky asset at time t. Similarly,

$$N_t^- = \sum_{k=1}^{N_{t-1}^+} \zeta_k(p_{t-1}^+) + \sum_{j=1}^{N_{t-1}^-} [1 - \xi_j(p_{t-1}^-)]. \tag{31}$$

The opinion index s_t (27) is thus given by

$$s_t = \frac{1}{N_c} \left(\sum_{k=1}^{N_{t-1}^+} [1 - 2\zeta_k(p_{t-1}^+)] + \sum_{j=1}^{N_{t-1}^-} [2\xi_j(p_{t-1}^-) - 1] \right). \tag{32}$$

Using the i.i.d. property of the $\{\xi_j(p)\}$ and $\{\zeta_k(p)\}$ variables allows us to obtain the following exact expression for the mean of s_t:

$$E[s_t] = s_{t-1} + p_{t-1}^-(1 - s_{t-1}) - p_{t-1}^+(1 + s_{t-1}). \tag{33}$$

2.2.3. Influence of herding and momentum on the behavior of chartists

As can be seen from (29) together with (32), the probabilities $p^{\pm}$ embody completely the behavior of the chartists. We assume that $p^{\pm}$ at time $t-1$ are both a function of s_{t-1} (social imitation effect) defined by (27) and of a measure H_t of the price momentum given by

$$H_t = \theta H_{t-1} + (1-\theta)\left(\frac{P_t}{P_{t-1}} - 1\right), \tag{34}$$

which is nothing but the expression for an exponential moving average of the history of past returns. The parameter $0 \le \theta < 1$ controls the length of the memory that chartists keep of past returns, the closer to 1, the longer the memory $\sim 1/(1-\theta)$.

Considering that the probabilities $p^{\pm}$ are functions of s_{t-1} and H_{t-1},

$$p^{\pm}_{t-1} = p^{\pm}(s_{t-1}, H_{t-1}), \tag{35}$$

means that the chartists make their decisions to buy or sell the risky Internet stock based on (i) the majority view held by their group and (ii) the recent capital gains that the risky asset has provided over a time frame $\sim 1/(1-\theta)$. We assume the chartists buy and sell symmetrically with no bias: a strong herding in favor of the risky asset or a strong positive momentum has the same relative effect on the drive to buy (or to sell) than a strong negative sentiment or strong negative momentum on the push to sell (or to buy). This is expressed by the following symmetry relation

$$p^{-}(s, H) = p^{+}(-s, -H). \tag{36}$$

The simplest functions satisfying (36) are the linear expressions[3]

$$p^{-}(s, H) = \frac{1}{2}[p + \kappa \cdot (s + H)], \quad p^{+}(s, H) = \frac{1}{2}[p - \kappa \cdot (s + H)]. \tag{37}$$

This defines two parameters p and κ, chosen sufficiently small such that $p^{-}(s, H)$ and $p^{+}(s, H)$ remain between 0 and 1. The positive parameter p controls the average holding time of the positions in the absence of any other influence. In other words, a position will last typically $\sim 2/p$ time steps in the absence of social imitation and momentum influence. The parameter κ quantifies the strength of social imitation and of momentum trading. Instead of κ, one could use two parameters for the opinion index and momentum, respectively. For the sake of parsimony we will only work with one parameter treating s and H symmetrically. For instance, for $\kappa > 0$, if there is already a majority of agents holding the risky asset and/or if its price has been increasing recently, then the probability for chartists holding the risk-free asset to shift to the risky asset is increased and the probability for the chartists who are already invested to sell their risky asset is decreased. The reverse holds for $\kappa < 0$, which describes "contrarian" traders. In the sequel, we will only consider the case $\kappa > 0$, which describes imitative and trend-following agents. Generalizations to allow for additional heterogeneous beliefs, involving mixtures as well as adaptive imitative and contrarian agents, is left for other communications. In this spirit, let us mention that Corcos et al. (2002) have introduced a simple model of imitative agents who turn contrarian when the proportion of herding agents is too large, which generates chaotic price dynamics.

Putting expressions (37) in (33) yields

$$E[s_t] = (1 + \kappa - p)s_{t-1} + \kappa H_{t-1}. \tag{38}$$

3. Dynamical market equations

3.1. Market clearing condition and price dynamics

The equation for the risky asset price dynamics is obtained from the condition that, in the absence of external supply, the total excess demand summed over the fundamentalist and chartist traders vanishes:

$$\Delta D^f_t + \Delta D^c_t = 0. \tag{39}$$

In other words, the net buy orders of chartists are satisfied by the net sell orders of the fundamentalists, and vice-versa. Substituting in (39) expression (19) for the excess demands ΔD^f_t of the fundamentalists and Eq. (29) for the excess demand ΔD^c_t of the chartists, we obtain the price equation

$$\frac{P_t}{P_{t-1}} = \frac{(1+s_t)((1+R_f)(1-s_{t-1}) + (r+\sigma_r u_t)(1+s_{t-1}))W^c_{t-1} + 4x((1+R_f)(1-x) + (r+\sigma_r u_t)x)W^f_{t-1}}{(1+s_{t-1})(1-s_t)W^c_{t-1} + 4W^f_{t-1}x(1-x)}. \tag{40}$$

[3] Another possibility for the transition probabilities which we have explored but do not elaborate on in this paper is the hyperbolic tangent: $p^{\pm}(s, H) = \frac{1}{2}[1 \mp \kappa/p \tanh(s + H)]$. This corresponds to the Glauber transition rates of an ensemble of spins on a fully connected graph with equal interaction strengths, see for example Harras et al. (2012). However, already the linear probabilities (37) translate into a very nonlinear S-like behavior at the aggregate level, which is quantitatively similar to the nonlinear case.

Expression (40) shows that the price of the risk asset changes as a result of two stochastic driving forces: (i) the dividend-price ratio $(r+\sigma_r u_t)$ and (ii) the time increments of the bullish/bearish chartist unbalance $\{s_t\}$. The impact of $\{s_t\}$ is controlled by the wealth of the group of chartists W^c_{t-1}. As we shall demonstrate below, this becomes particularly important during a bubble where trend-following chartists tend to gain much more than fundamentalists. With the increasing influence of chartists, the market becomes much more prone to self-fulfilling prophecies. Fundamentalist traders are less able to attenuate the irrational exuberance – they simply do not have enough wealth invested in the game.

3.2. Complete set of dynamical equations

Let us put all ingredients of our model together to state concisely all the equations controlling the price dynamics coupled with the opinion forming process of the chartists. As discussed above, the wealth levels of the fundamentalist and chartist traders are also time-dependent and influence the market dynamics. We thus arrive at the following equations.

Dynamics of the chartists opinion index:

$$s_t = \frac{1}{N_c}\left(\sum_{k=1}^{N_c(1+s_{t-1})/2}[1-2\zeta_k(p^+_{t-1})] + \sum_{j=1}^{N_c(1-s_{t-1})/2}[2\xi_j(p^-_{t-1})-1]\right), \tag{41}$$

where $\zeta_k(p^+_{t-1})$ takes the value 1 with probability p^+_{t-1} and the value 0 with probability $1-p^+_{t-1}$, $\xi_j(p^-_{t-1})$ takes the value 1 with probability p^-_{t-1} and the value 0 with probability $1-p^-_{t-1}$, and p^+_{t-1} and p^-_{t-1} are given by expressions (37):

$$\begin{aligned} p^-_{t-1}(s_{t-1},H_{t-1}) &= \frac{1}{2}[p+\kappa\cdot(s_{t-1}+H_{t-1})],\\ p^+_{t-1}(s_{t-1},H_{t-1}) &= \frac{1}{2}[p-\kappa\cdot(s_{t-1}+H_{t-1})]. \end{aligned} \tag{42}$$

Thus, $E[s_t]$ given by expression (38).

Dynamics of the risky asset price:

$$P_t/P_{t-1} = \frac{(1+s_t)((1+R_f)(1-s_{t-1})+(r+\sigma_r u_t)(1+s_{t-1}))W^c_{t-1} + +4x((1+R_f)(1-x)+(r+\sigma_r u_t)x)W^f_{t-1}}{(1+s_{t-1})(1-s_t)W^c_{t-1}+4x(1-x)W^f_{t-1}}. \tag{43}$$

Wealth dynamics of fundamentalists:

$$\frac{W^f_t}{W^f_{t-1}} = x\left(\frac{P_t}{P_{t-1}}+(r+\sigma_r u_t)\right)+(1-x)(1+R_f). \tag{44}$$

Wealth dynamics of chartists:

$$\frac{W^c_t}{W^c_{t-1}} = \frac{1+s_{t-1}}{2}\left(\frac{P_t}{P_{t-1}}+(r+\sigma_r u_t)\right)+\frac{1-s_{t-1}}{2}(1+R_f). \tag{45}$$

Momentum of the risky asset price:

$$H_t = \theta H_{t-1}+(1-\theta)\left(\frac{P_t}{P_{t-1}}-1\right). \tag{46}$$

And u_t forms a series of standard i.i.d. random variables with distribution $N(0, 1)$.

The set of equations (41)–(46) together with the realization of the stochastic dividend process u_t completely specify the model and its dynamics. Eq. (41) describes how chartists form their opinion s_t based on the previous prevalent opinion s_{t-1} and the recent price trend H_t. Fundamentalist traders stick to their choice of investing x in the risky asset. Eq. (43) gives the new market price P_t when excess demands of both groups are matched. Eqs. (44) and (45) describe the evolution of the wealth levels W^f_t and W^c_t for fundamentalist and chartist traders, respectively. There are capital gains and dividend gains from the risky asset, and interest payments by the risk-free asset. The new market price also feeds into the momentum of the risky asset described by Eq. (46).

We have the following flow of causal influences:

1. The recent price trend H_{t-1} and the prevailing opinion s_{t-1} among chartists determine the investment decision of chartists governed by s_t, while fundamentalists invest a constant fraction x of their wealth.
2. Market clearing determines the price P_t based on investment decisions x and s_t, and previous wealth levels W^f_{t-1} and W^c_{t-1} for fundamentalist and chartist traders, respectively.
3. The new wealth levels W^f_t and W^c_t are based on the market price P_t and investment decisions x and s_t.

3.3. Control parameters and their time-scale dependence

The set of equations (41)–(46) depends on the following parameters:

1. x quantifies the constant fraction of wealth that fundamentalists invest in the risky asset.
2. θ fixes the time scale over which chartists estimate price momentum. By construction, $0 \leq \theta < 1$.
3. N_c is the number of chartists that controls the fluctuations of the majority opinion of chartists.
4. p controls the average holding time of the positions of chartists in the absence of any other influence.
5. κ quantifies the strength of social imitation and of momentum trading by chartists.
6. R_f is the rate of return of the risk-free asset.
7. r and σ_r are the mean and standard deviation of the dividend–price ratio.

In order to have an intuitive understanding of the role and size of these parameters, it is useful to discuss how they depend on the time scale over which traders reassess their positions. Until now, we have expressed the time t in units of a unit step 1, which could be taken for instance to be associated with the circadian rhythm, i.e. 1 day. But there is no fundamental reason for this choice and our theory has the same formulation under a change of the time step. Let us call τ the time interval between successive reassessments of the fundamentalists, with τ being measured in a calendar time scale, for instance, in seconds, hours or days.

First, the parameters N_c and N_f are a priori independent of τ, while they may be a function of time t. We neglect this dependence as we are interested in the dynamics over time scales of a few years that are characteristic of bubble regimes. The parameter γ is also independent of τ.

In contrast, the parameters R_f, r and σ_r^2 are functions of τ, as the return of the risk-free asset, the average expected dividend return and its variance depend on the time scale. The simplest and standard dependence of Wiener processes or discrete random walks is $R_f \sim r \sim \sigma_r^2 \sim \tau$. Because of its definition, $x = R_{\text{excess}}/\gamma\tilde{\sigma}^2$, the fraction of wealth x fundamentalists hold is independent of time.

By construction, the parameter θ characterizing the memory of the price momentum influencing the decisions of chartists depends on τ. This can be seen by replacing $t-1$ by $t-\tau$ to make explicit the unit time scale in expression (34), giving

$$\frac{H_t - H_{t-\tau}}{\tau} = \frac{1-\theta}{\tau}\left(\frac{P_t}{P_{t-\tau}} - 1 - H_{t-\tau}\right). \tag{47}$$

Requesting a bona-fide limit for small τ's leads to

$$\frac{1-\theta}{\tau} = \varrho = \text{const}, \tag{48}$$

where the time scale $\mathcal{T}_H := 1/\varrho$ is the true momentum memory. Thus, we have

$$1-\theta = \varrho \cdot \tau, \quad \mathcal{T}_H := \frac{1}{\varrho} = \frac{\tau}{1-\theta}. \tag{49}$$

4. Theoretical analysis and super-exponential bubbles

4.1. Reduction to deterministic equations

It is possible to get an analytical understanding of the solutions of the set of equations (41)–(46) if we reduce them into their deterministic components. The full set including their stochastic contributions will be studied with the help of numerical simulations in the next section.

Taking $u_t \equiv 0$ and replacing s_t by its expectation $E[s_t]$ given by (38), we obtain the following deterministic equations

Dynamics of the chartists opinion index:

$$s_t = (1+\kappa-p)s_{t-1} + \kappa H_{t-1}. \tag{50}$$

Dynamics of the risky asset price:

$$\frac{P_t}{P_{t-1}} = \frac{(1+s_t)((1+R_f)(1-s_{t-1}) + r(1+s_{t-1}))W_{t-1}^c + 4x((1+R_f)(1-x) + rx)W_{t-1}^f}{(1+s_{t-1})(1-s_t)W_{t-1}^c + 4x(1-x)W_{t-1}^f}. \tag{51}$$

Wealth dynamics of fundamentalists:

$$\frac{W_t^f}{W_{t-1}^f} = x\left(\frac{P_t}{P_{t-1}} + r\right) + (1-x)(1+R_f). \tag{52}$$

Wealth dynamics of chartists:

$$\frac{W_t^c}{W_{t-1}^c} = \frac{1+s_{t-1}}{2}\left(\frac{P_t}{P_{t-1}}+r\right)+\frac{1-s_{t-1}}{2}(1+R_f). \quad (53)$$

Momentum of the risky asset price:

$$H_t = \theta H_{t-1} + (1-\theta)\left(\frac{P_t}{P_{t-1}}-1\right). \quad (54)$$

This system of five coupled deterministic equations is non-linear and completely coupled, there is no autonomous subsystem. In particular, the multiplicative price equation is highly non-linear. The wealth equations describe the multiplicative process of capital accumulation depending on the choice of how to split the portfolio on the risky and risk-free asset yielding capital gains, given the dividend gains and the risk-free rate.

4.2. Fixed points and stability analysis

To gain insights into the system of coupled equations, we will consider the stationary case where the wealth levels of fundamentalist and chartist traders only change slowly and remain of roughly the same order of magnitude.[4] This happens when both groups keep their portfolio allocation approximately fixed and their endowments mainly grow due to dividends and risk-free returns. According to (50), we are in the regime $\kappa < p$ and may treat the ratio of wealth levels ν as approximately constant,

$$\nu := \frac{W_t^c}{W_t^f} \simeq \text{const} \sim \mathcal{O}(1). \quad (55)$$

This allows us to decouple the equations for H_t, s_t and P_t from the wealth equations. The fixed points $\{(H^*, s^*)\}$ are determined by the system:

$$H^* = R_f + r\frac{\nu(1+s^*)^2+4x^2}{\nu(1+s^*)(1-s^*)+4x(1-x)}, \quad (56)$$

$$s^* = \frac{\kappa}{p-\kappa}H^*. \quad (57)$$

Since this system is essentially one third-order equation, it can be solved analytically yielding three fixed points. As we will see later, for typical parameter values, there is one solution s^*, $H^* \ll 1$, while the other two lie outside the restricted domain of $[-1, 1]$ for s. An expansion in the small parameters r, $R_f \ll 1$ permits the approximation:

$$H^* = R_f + \frac{\nu+4x^2}{\nu+4x-4x^2}r + \mathcal{O}(r^2, R_f^2), \quad (58)$$

$$s^* = \frac{\kappa}{p-\kappa}\left[R_f + \frac{\nu+4x}{\nu+4x-4x^2}r + \mathcal{O}(r^2, R_f^2)\right]. \quad (59)$$

This fixed point is stable for $\kappa < p$ over a range of the other parameter values and unstable for $\kappa > p$. A deviation from the fixed point due to stochastic fluctuations in the opinion index leads to a price change in the same direction. According to (50), for $\kappa > p$, the opinion index grows transiently exponentially (until its saturation). Since the stability is mainly governed by the relative value of the two parameters κ and p characterizing chartist behavior, we conclude that there is an inherent instability caused by herding and trend following, which is independent of the stochastic dividend process.

4.3. Super-exponential bubbles

It is well-known that many bubbles in financial markets start with a phase of super-exponential growth, see for example Sornette et al. (2009) for oil prices, Jiang et al. (2010) for the Chinese stock market and Yan et al. (2012) for major equity markets. Furthermore, Sornette et al. (2013) discuss various theoretical and empirical questions related to faster-than-exponential growth of asset prices, while Hüsler et al. (2013) document super-exponential bubbles in a controlled experiment in the laboratory.

One of the main findings of this paper is that phases with faster-than-exponential growth of the price are inherent also in the present model. If a bubble is essentially driven by herding and trend following, we may neglect the dividend process and expand the pricing formula (51) in terms of r and R_f:

$$\frac{P_t}{P_{t-1}} = \frac{(1+s_t)(1-s_{t-1})+4x(1-x)W_{t-1}^f/W_{t-1}^c}{(1-s_t)(1+s_{t-1})+4x(1-x)W_{t-1}^f/W_{t-1}^c} + \mathcal{O}(r, R_f). \quad (60)$$

[4] This last condition is necessary for nontrivial dynamics, as both populations remain relevant to the economy.

Again, we focus on the scenario that the ratio of the wealth levels W^f_{t-1} and W^c_{t-1} of the fundamentalist and chartist traders remains approximately constant,

$$\frac{W^c_{t-1}}{W^f_{t-1}} = \frac{W^c_{t_0}}{W^f_{t_0}} = \nu \simeq \text{const.} \tag{61}$$

This is the case if the endowments of both groups grow at the same constant exponential growth rate or, more accurate here, if both grow in the same super-exponential way. Starting with an opinion index s_0 at time $t = t_0$, we can further simplify the price equation to:

$$\frac{P_t}{P_{t-1}} = 1 + b(s_t - s_{t-1}) + \mathcal{O}(r, R_f, (s - s_0)^2), \tag{62}$$

where the constant quantity b is of order 1 provided the initial levels of wealth were of the same order of magnitude:

$$b = \frac{2}{1 + 4x(1-x)\nu - s_0^2} \sim \mathcal{O}(1). \tag{63}$$

Therefore, up to terms of order $\mathcal{O}(r, R_f, (s - s_0)^2)$, the price evolves as

$$\frac{P_t}{P_0} = \prod_{j=1}^{t}[1 + b(s_j - s_{j-1})] \simeq \prod_{j=1}^{t} \mathrm{e}^{b(s_j - s_{j-1})} = \mathrm{e}^{b(s_t - s_0)}. \tag{64}$$

Since s_t grows exponentially with time according to expression (50) for $\kappa > p$, the price P_t grows as an exponential of an exponential of time. In other words, for the regimes when the opinion index grows exponentially ($\kappa > p$), we expect super-exponential bubbles in the price time series. Since our equations are symmetric in the sign of the opinion index s_t, the same mechanism leads also to "negative bubbles" for a negative herding associated with a transition from bullish to bearish behavior for which the price drops also super-exponentially in some cases.

4.4. Time-dependent social impact and bubble dynamics

The strength of herding is arguably regime dependent. In some phases, chartists are prone to herding, while at other times, they are more incoherently disorganized "noise" traders. This captures in our dynamical framework the phenomenon of regime switching (Hamilton, 1989; Lux, 1995; Hamilton and Raj, 2002; Yukalov et al., 2009; Binder and Gross, 2013; Fischer and Seidl, 2013; Kadilli, 2013), where successive phases are characterized by changing values of the herding propensity. In this respect, we follow the model approach of Harras et al. (2012) developed in a similar context and assume that the strength κ of social imitation and momentum influence slowly varies in time. In this way, we incorporate the effects of a changing world on financial markets such as a varying economic and geopolitical climate into the model. More generally, we allow for varying uncertainties influencing the behavior of chartists. As we shall show, this roots the existence of the bubbles documented below in the mechanism of "sweeping of an instability" (Sornette, 1994; Stauffer and Sornette, 1999).

More specifically, we propose that κ undergoes a discretized Ornstein–Uhlenbeck process[5]:

$$\kappa_t - \kappa_{t-1} = \eta(\mu_\kappa - \kappa_{t-1}) + \sigma_\kappa v_t. \tag{65}$$

Here $\eta > 0$ is the mean reversion rate, μ_κ is the mean reversion level and $\sigma_\kappa > 0$ is the step size of the Wiener process realized by the series v_t of standard i.i.d. random variables with distribution $N(0, 1)$.

Our approach is related to how Lux (1995) describes switching between bear and bull markets. While we propose a stochastic process for the strength of social imitation κ, Lux adds a new deterministic term proportional to $\mathrm{d}\log P_t/\mathrm{d}t$ to the transition probabilities, which corresponds to a direct positive feedback.

The interesting case is $\mu_\kappa \lesssim p$, where κ is on average below the critical value p but, due to stochastic fluctuations, may occasionally enter the regime with faster-than-exponential growth $\kappa > p$ described in the previous section. Since an Ornstein–Uhlenbeck process with deterministic initial value is a Gaussian process, its distribution is fully determined by the first and second moments. Starting from an initial value κ_0, the non-stationary mean and covariance are given by

$$E[\kappa_t] = \kappa_0 \mathrm{e}^{-\eta t} + \mu_\kappa(1 - \mathrm{e}^{-\eta t}), \tag{66}$$

$$\mathrm{Cov}[\kappa_s, \kappa_t] = \frac{\sigma_\kappa^2}{2\eta}(\mathrm{e}^{-\eta(t-s)} + \mathrm{e}^{-\eta(t+s)}), \quad s < t. \tag{67}$$

[5] Choosing a confined random walk yields similar results, but the mean reversion is then effectively nonlinear (or threshold based), which is less standard.

Both moments converge such that in the long run κ_t admits the following stationary distribution:

$$\kappa_t \sim N\left(\mu, \frac{\sigma_\kappa}{\sqrt{2\eta}}\right). \tag{68}$$

If, at some time t, the social imitation strength is above the critical value $\kappa_t \equiv \kappa_0 > p$, the time ΔT needed for κ_t to revert to the subcritical regime $\kappa_t < p$ can be estimated from Eq. (66):

$$\Delta T = \frac{1}{\eta}\log\left(\frac{\kappa_0 - \mu_\kappa}{p - \mu_\kappa}\right). \tag{69}$$

Expressions (68) and (69) will allow us to estimate how often the group of chartists will interact in the supercritical regime of the opinion index related to transient faster-than-exponential growth in the price and how long a typical bubble will last.

5. Numerical simulations and qualitative comparison with the dotcom bubble

5.1. Estimation of parameter values

Let us take $\tau = 1$ day and assume a typical memory used by chartists for the estimation of price momentum equal to about 1 month. This amounts approximately to 20 trading days, hence $\mathcal{T}_H \simeq \tau/(1-\theta) = 20$, leading to $\theta = 0.95$.

We calibrate the average dividend–price ratio r and its standard deviation σ_r to the values given by Engstend and Pedersen (2010), which are quite similar for various countries. We set the mean daily dividend–price ratio to $r = 1.6 \times 10^{-4}$ and the daily standard deviation to $\sigma_r = 9.5 \times 10^{-4}$. Furthermore, we assume a constant return of the risk-free asset of annualized 2%, i.e. a daily value of $R_f = 8 \times 10^{-5}$.

Fundamentalists keep 30% of their wealth in the risky asset, that is, $x = 0.3$. The wealth levels W_t^f and W_t^c of fundamentalist and chartist traders evolve dynamically and determine the relative influence of the two groups. We analyze the importance of the initial endowments W_0^f and W_0^c on the stability of the market. We capture this by the parameter $\nu = W_0^c/W_0^f$ and set ν to 1, 2 or 0.5 in three different sets of simulations.[6]

For the parameter p entering in expressions (37), recall that it is equal to twice the probability that during a given day some chartist will buy (or sell) the risky asset. We posit $p = 0.2$, which means that the natural trading frequency of traders in absence of social influence is about 2 weeks. For the parameter κ in (37) describing the strength of social imitation and of momentum trading, we assume that it is close to the parameter p. Specifically, for the Ornstein–Uhlenbeck process given in expression (65), we choose $\mu_\kappa = 0.98p = 0.196$. We set the mean reversion speed η and the step size σ_κ such that (i) the Ornstein–Uhlenbeck process has a standard deviation of $0.1p$ and (ii) a deviation of κ_t two standard deviations above μ_κ in the supercritical regime will revert within $\Delta T = \mathcal{T}_H = 20$:

$$\eta = \frac{1}{\Delta T}\log\left(\frac{\mu_\kappa + 2\cdot 0.1p - \mu_\kappa}{p - \mu_\kappa}\right) = \frac{\log(10)}{20} \simeq 0.11, \tag{70}$$

$$\sigma_\kappa = 0.1p\sqrt{2\eta} \simeq 0.001. \tag{71}$$

Summarizing, the numerical simulations presented in the figures correspond to

$$\theta = 0.95, \quad r = 1.6\times 10^{-4}, \quad \sigma_r = 9.5\times 10^{-4}, \quad R_f = 8\times 10^{-5}, \quad x = 0.3, \tag{72}$$

$$p = 0.2, \quad \mu_\kappa = 0.196, \quad \sigma_\kappa = 0.001, \quad \eta = 0.11, \tag{73}$$

and ν will be varied as $\nu = 0.5$, 1, 2. Furthermore, we run the simulations over 20 trading years, i.e. $T = 5000$.

We can now test our claims from the fixed points analysis in Section 4.2 numerically. Assuming that κ_t will not deviate further than five standard deviations from its mean μ_κ, we find that one fixed point for the opinion index is indeed close to zero, $s^* \sim \mathcal{O}(10^{-3})$, while the other two lie well outside of the domain of definition $[-1, 1]$.

5.2. Results and interpretation

Figs. 1, 4 and 5 show the time dependence of the variables P_t, s_t, κ_t, H_t, W_t^f, W_t^c and the time series of returns that are generated by numerical solutions of the set (41)–(46) for three different parameter values for $\nu = 1$, 2 and 0.5, respectively, of the relative important of chartists compared with fundamentalists in their price impact.

Fig. 1 corresponds to the situation where both groups have equal initial endowments ($\nu = 1$). One can observe a general positive log-price trend biasing upward a fluctuating random walk-like trajectory. The upward drift reflects a combination

[6] Note that this is equivalent to setting the ratio of group sizes $\nu = N_c/N_f$ with the assumption that both groups consist of representative agents with equal initial wealth. In our formulation, N_c has no further importance than controlling the smoothness of the opinion index. Thus it disappears from the deterministic equations (50)–(54). The simulations are run with $N_c = 1000$.

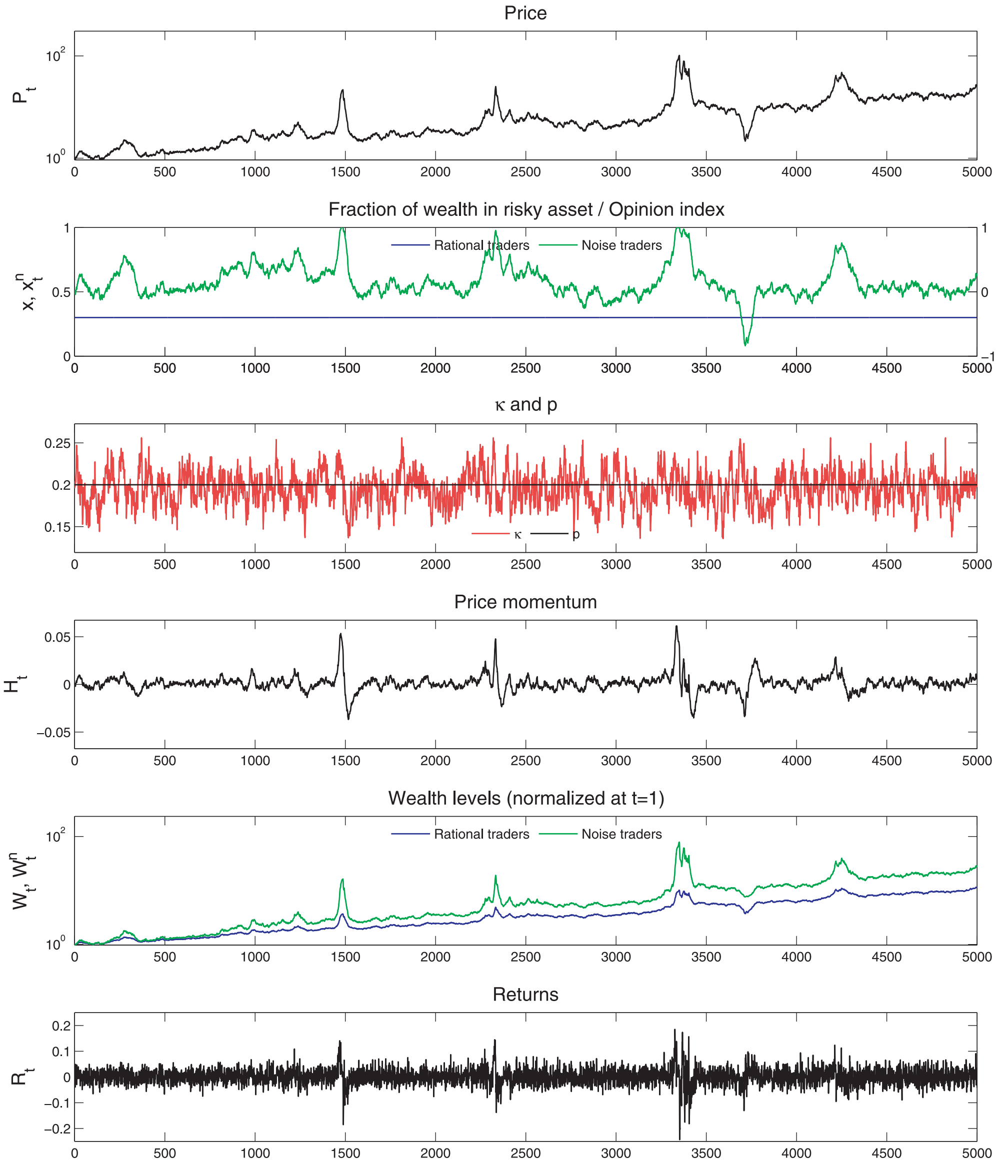

Fig. 1. Time dependence of the variables P_t (log-scale), s_t, κ_t, H_t, W_t^f, W_t^c (both in log-scale) and the time series of returns that are generated by numerical solutions of the set (41)–(46) for the value $\nu = 1$ of the relative important of chartists compared with fundamentalists in their price impact at the origin of time.

of the dividend gains, of the rate of return paid by the risk-free asset as well as a component resulting from the herding behavior of chartists who tend intermittently to push prices in a kind of self-fulfilling prophecy or convention à la Orléan (Boyer and Orléan, 1992; Orléan, 1994; Eymard-Duvernay et al., 2005).

But the most striking aspect of the price dynamics is the occurrence of four clearly identifiable bubbles occurring within the chosen time interval, defined by the transient explosive growth of the price P_t followed by sharp crashes bringing the prices back approximately to pre-bubble levels. As seen from the second panel of Fig. 1 showing the opinion index dynamics of the chartists, the bubbles are essentially driven by the chartist traders. As described in Section 4.3, the start of the growth of herding among chartists feeds the price dynamics, resulting in a larger price momentum (fourth panel), which amplifies herding, enhancing further the bubble growth and so on. One can observe in each bubble that the growth of the opinion index (or equivalently the fraction of wealth invested in the risky asset) precedes and then accompanies the explosive price growth, as predicted by expression (64).

The transient bubbles and their subsequent crashes are associated with clustered volatility and the existence of outliers in the price momentum. During the bubbles, the wealth levels of chartists and of fundamentalists diverge. In the long run, chartists outperform fundamentalists because they tend to invest more in the risky asset, which exhibits higher average returns.

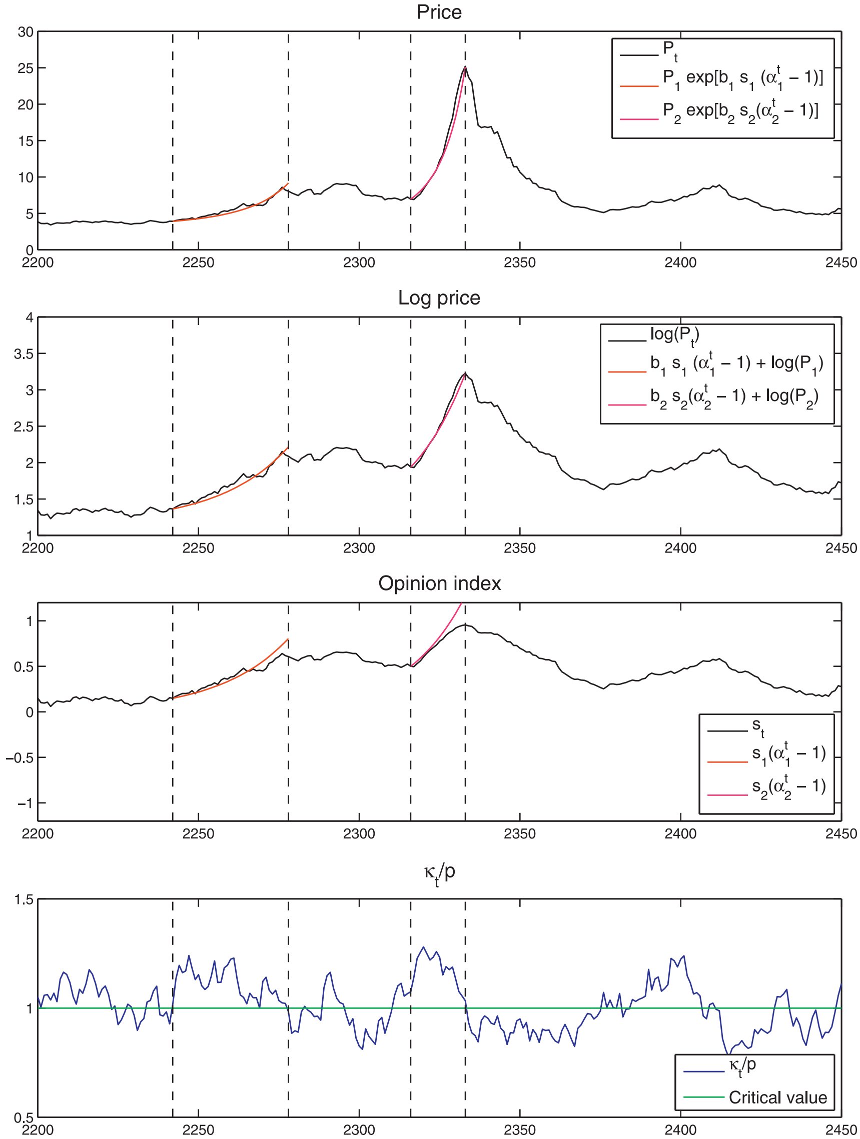

Fig. 2. Zoom of Fig. 1 for the price P_t, log-price $\ln P_t$, opinion index s_t and imitation parameter κ_t in units of the random component strength p as a function of time around bubbles. The panels show the behavior of these variables for typical bubbles, demonstrating the characteristic transient faster-than-exponential growth behavior.

Fig. 2 presents a more detailed analysis of a typical bubble from the time series shown in Fig. 1, demonstrating the characteristic transient faster-than-exponential growth behavior predicted theoretically in Section 4.3. For periods when $\kappa_t > p$, we may approximate the opinion index as exponentially growing:

$$s_t = s_1(\alpha_1^t - 1), \tag{74}$$

where t runs over the growth period $[t_1, t_1 + \Delta T]$, with initial value $s_1 \equiv s_{t_1}$ and where $\alpha_1 > 1$ is an empirical effective multiplicative factor, $\log \alpha_1$ being the effective growth rate of s_t. One can verify that the length ΔT of such a period is compatible with our theoretical prediction (69), which for our chosen parameters gives $\Delta T = 20$. Bubbles with longer lifetimes are easily engineered in our framework by allowing κ to remain close and higher than p for longer times. Our model supports therefore the view that long-lived bubbles may be associated with excess positive sentiments catalyzing a herding propensity that is sustained and self-reinforcing (via the momentum mechanism) over long periods.

Furthermore, the exponential growth in the opinion index results in a faster-than-exponential growth of the price, as can be seen in the log-linear plot of P_t. From expression (64), we deduce

$$\log(P_t) = b_1 s_1(\alpha_1^t - 1) + \log(P_0), \tag{75}$$

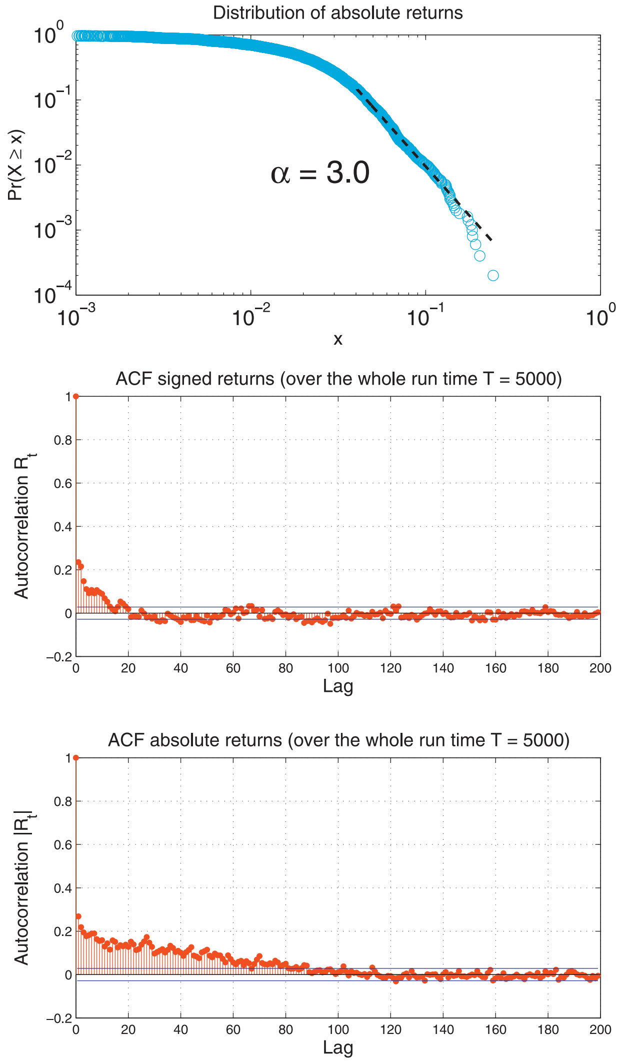

Fig. 3. Top panel: complementary cumulative distribution function of absolute values of the returns in log–log, where the straight dashed line qualifies a fat-tail $p(x)\sim x^{-1-\alpha}$ with exponent $\alpha = 3.0$; middle panel: auto-correlation function of the signed returns R_t; lower panel: autocorrelation function of the absolute values $|R_t|$ of returns. The parameters are the same as in Fig. 1.

where $b_1 = b_{t_1}$, which fits well the transient super-exponential price dynamics. These observations presented in Fig. 2 are in agreement with the theoretical derivation of Section 4.3. It is interesting to note also that the dynamics of κ_t, with its tendency to present a transient oscillatory behavior due to the interplay between rare large excursions with the mean reversal of the constrained random walk associated with the discrete Ornstein–Uhlenbeck process, leads to an approximate log-periodic behavior[7] of the price during its ascendency, which is similar to many observations reported empirically (Sornette, 2003; Johansen and Sornette, 2010; Jiang et al., 2010; Yan et al., 2012; Sornette et al., 2013).

Fig. 3 presents three statistical properties of our generated price time series. Various well-known stylized facts are matched by our model. First, we show the distribution of absolute values of the returns, which has a fat-tail $p(x)\sim x^{-1-\alpha}$ with exponent $\alpha = 3.0$, which is in the range of accepted values in the empirical literature (de Vries, 1994; Pagan, 1996; Guillaume et al., 1997; Gopikrishnan et al., 1998; Jondeau and Rockinger, 1999). Furthermore, signed returns R_t are characterized by a fast-decaying autocorrelation function, which is consistent with an almost absence of arbitrage opportunities in the presence of transaction costs. In contrast, the absolute values $|R_t|$ of returns have an autocorrelation function with longer memory (Ding et al., 1993; Cont, 2007).

Figs. 4 and 5 present the same panels as in Fig. 1 but with $\nu = 2$ and $\nu = 0.5$, respectively. Due to their larger relative weight compared to the case shown in Fig. 1, one can observe in Fig. 4 bubbles with stronger "explosive" trajectories. The wealth

[7] Log-periodicity here refers to transient oscillations with increasing local frequency. Formal mathematical definitions and illustrations can be found in Sornette (1998).

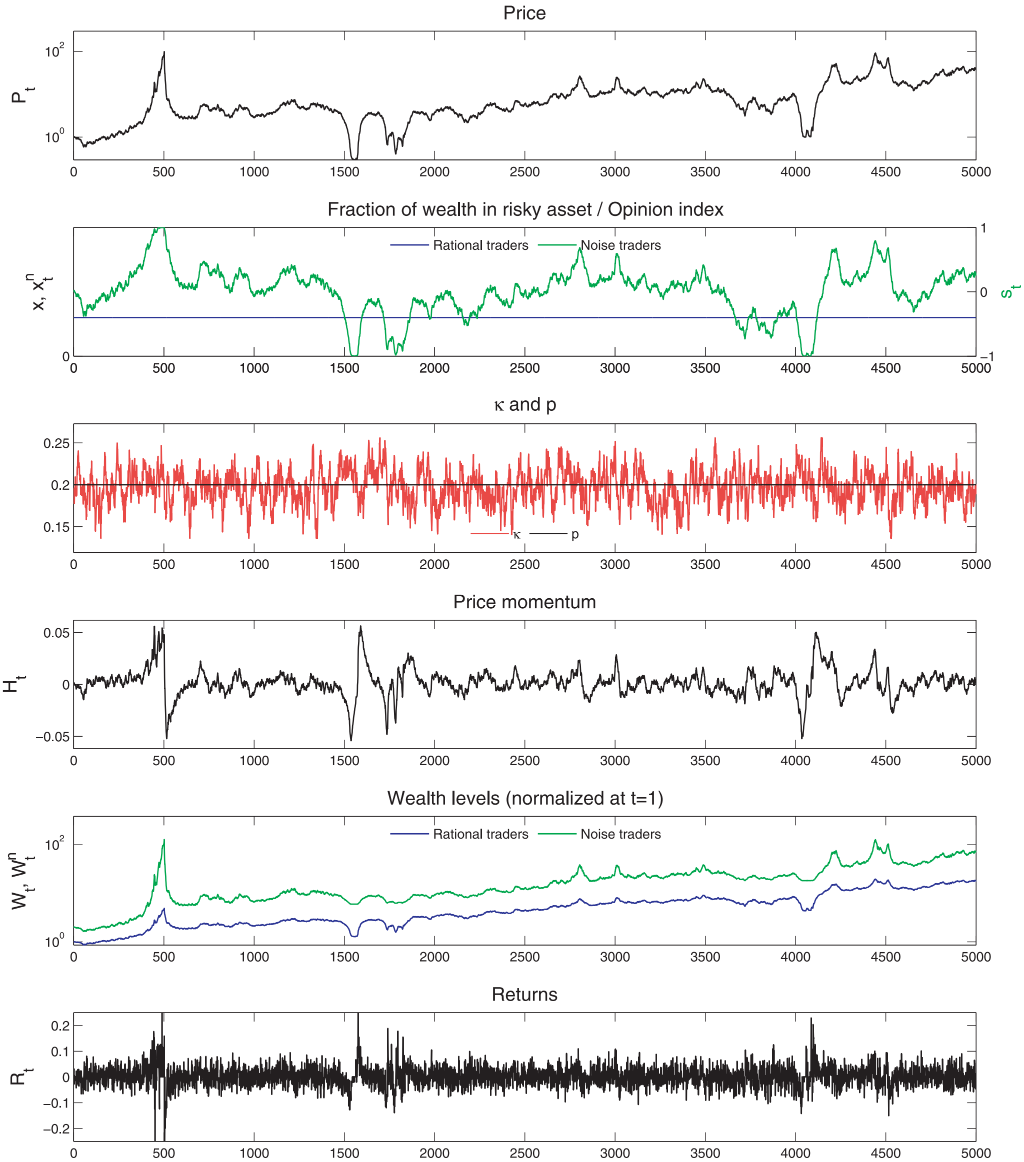

Fig. 4. Same as Fig. 1 for $\nu = 2$.

of chartists fluctuates widely, but amplifies to values that are many times larger than that of fundamentalists. This is due to the self-fulfilling nature of the chartist strategies that impact the price dynamics. In contrast, Fig. 5 with $\nu = 0.5$ shows that the wealth of the fundamentalists remains high for a long transient, even if in the long term the chartists end up dominating the price dynamics. The chartists also transiently over-perform dramatically the fundamentalists during the bubbles. It is informative to observe that, even a minority of chartists ($\nu = 0.5$ shown in Fig. 5) ends up creating bubbles and crashes. Their influence progressively increases and their transient herding behavior becomes intermittently destabilizing.

5.3. Comparison with the dotcom bubble

This section compares the insights obtained from the above theoretical and numerical analyses to empirical evidence on momenta and reversals in the period when the dotcom bubble developed.[8] We study the characteristics of the share prices of Internet-related companies over the period from January 1, 1998 to December 31, 2002, which covers the period

[8] The dotcom bubble (followed by its subsequent crash) is widely believed to be a speculative bubble, as documented by Ofek and Richardson (2003), Brunnermeier and Nagel (2004), and Battalio and Schultz (2006).

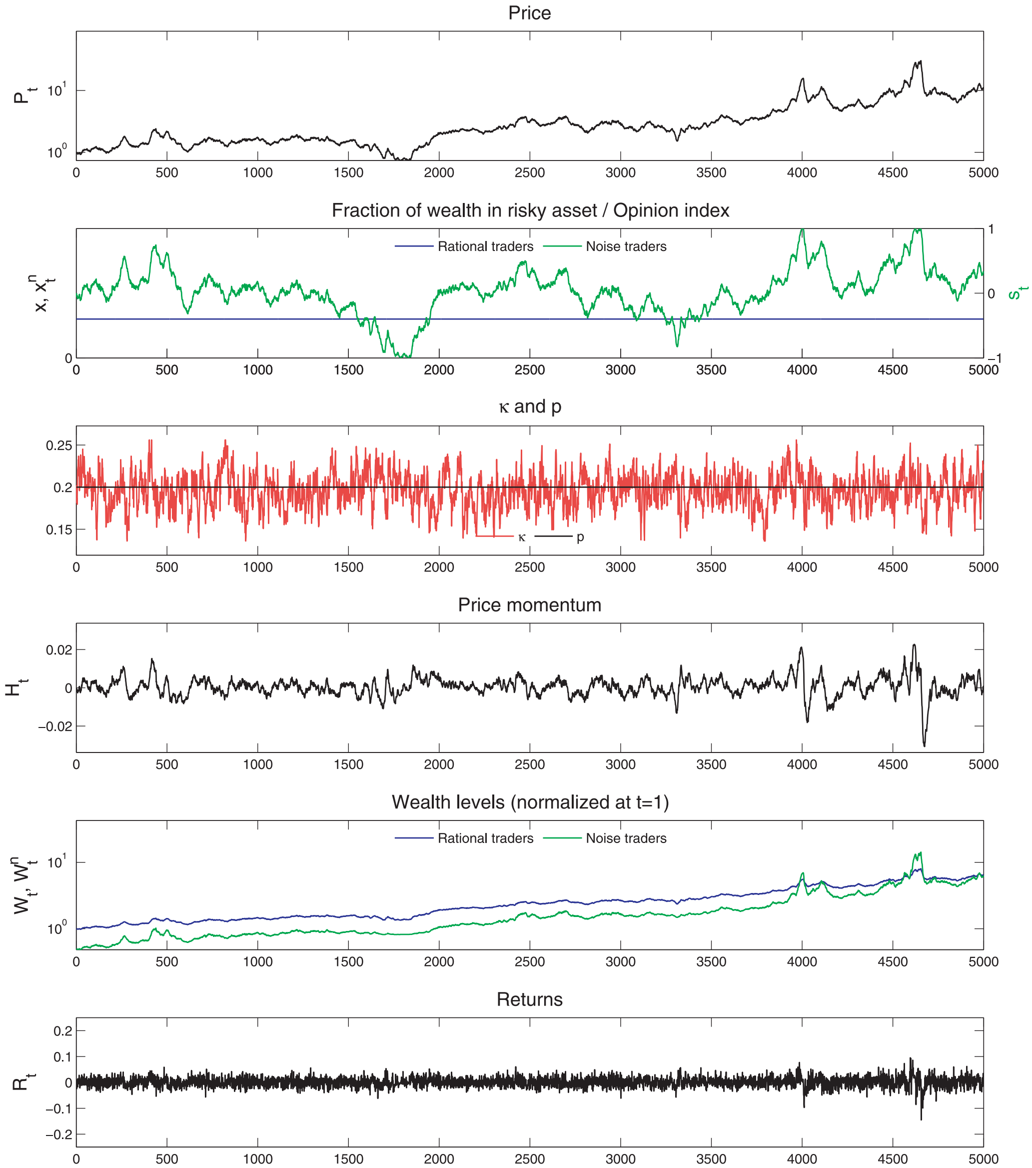

Fig. 5. Same as Fig. 1 for $\nu = 0.5$.

of the development of the dotcom bubble and its collapse. We use the list of 400 companies belonging to the Internet-related sector that has been published by Morgan Stanley and has already been investigated by Ofek and Richardson (2003). The criteria for a company to be included in that list is that it must be considered a "pure" Internet company, i.e. whose commercial goals are associated exclusively to the Internet. This implies that technology companies such as Cisco, Microsoft, and telecommunication firms, notwithstanding their extensive Internet-related businesses, are excluded.

Fig. 6 graphs the index of an equally weighted portfolio of the Internet stocks over the sample period of January 1998 to December 2002. The time evolution of the equally weighted portfolio of the Internet stocks is strikingly different from that shown in Fig. 7 for the index of an equally weighted portfolio of non-Internet stocks over this same period. The two indexes are scaled to be 100 on January 2, 1998. The two figures illustrate clearly the widely held view that a divergence developed over this period between the relative pricing of Internet stocks and the broad market as a whole. In the 2 year period from early 1998 through February 2000, the Internet related sector earned over 1300 percent returns on its public equity while the price index of the non-Internet sectors rose by only 40%. However, these astronomical returns of the Internet stocks had completely evaporated by March, 2001. Note how Fig. 6 is strikingly similar to the dynamics generated by the theoretical model in the bubble regime shown at the end of the top panel of Fig. 5 ($\nu = 0.5$).

We now focus our attention on the profitability of the momentum strategies studied by Jegadeesh and Titman (1993, 2001) and others. Table 1 provides some descriptive statistics about annual returns of the Internet-stock index versus of the

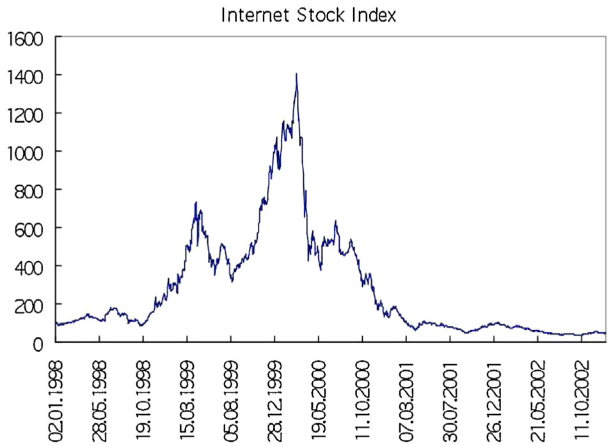

Fig. 6. The equally weighted Internet stock index for the period 1/2/1998–12/31/2002. The index is scaled to be 100 on 1/2/1998.

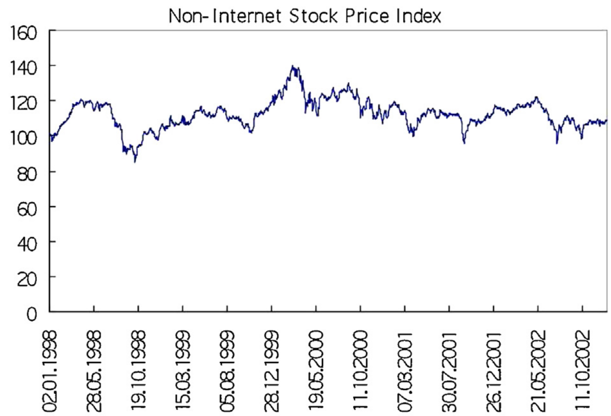

Fig. 7. The equally weighted non-Internet stock index for the period 1/2/1998–12/31/2002. The index is scaled to be 100 on 1/2/1998.

Table 1
Annual returns for Internet and non-Internet stock indices.

Year	1998	1999	2000	2001	2002
Internet stock index (per month)	116.8% (9.7%)	815.6% (68%)	−875.9% (−73%)	−62% (−5.2%)	−48.8% (−4.1%)
Non-Internet stock index (per month)	6.5% (0.5%)	17% (1.4%)	−9% (−0.8%)	3.6% (0.3%)	−9% (−0.7%)

Table 2
Cumulative returns for Internet and non-Internet stock indices.

Year	1998	1999	2000	2001	2002
Internet stock index (per month)	116.8% (9.7%)	932.5% (38.9%)	56.6% (1.6%)	−5.4% (−0.1%)	−54.2% (−0.9%)
Non-Internet stock index (per month)	6.5% (0.5%)	23.1% (1.0%)	−14% (0.4%)	17.6% (0.4%)	8.6% (0.1%)

non-Internet stock index from the beginning of 1998 to the end of 2002. In the 12 months of 1998, the annual cumulative return of the Internet stock index was 117%, while that of the non-Internet stock index was 6.5%. In the 12 months of 1999, the annual cumulative return of the Internet stock index surged to 816%, and that of the non-Internet stock index increased to 16.6%. The Internet stock index clearly outperformed the non-Internet stock index by 800% in 1999. This implies a strong profitability of momentum strategies applied to the Internet stocks over the period of the dotcom bubble. However, after its burst in March 2000, the return of the Internet stocks sharply declined, from 2000 to 2002. In the 12 months of 2000, the annual return of the Internet-stock index fell to −876%, followed by −62% and −49% in 2001 and in 2002, respectively. On the other hand, the annual returns of the non-Internet stock index in the period from 2000 to 2002 remain modest in amplitude at −9%, 3.6% and −9%, respectively. After the bust of the dotcom bubble, the Internet stocks continued to underperform the non-Internet stocks.

Table 2 shows the cumulative returns for the Internet stock index and for the non-Internet stock index in the 5 years from the beginning of 1998 to the end of 2002. The cumulative return of the Internet stock index in the first 24 months of the holding period is 932.5%, but the cumulative returns ends at the net loss of −54.2% over the 5 year holding period. In contrast, the cumulative returns of the non-Internet stock index over the same 5 year holding period is 8.6%.

These figures can be reproduced by our simulations, and are visualized by the extremely good performance of our chartists during the bubble phases, as shown in the fifth panels (from the top) of Figs. 1, 4 and 5.

In summary, these empirical facts constitute strong evidence for the Internet stock for momentum profit at intermediate time scales of about 2 years and reversals at longer time scales of about 5 years. These empirical facts confirm for this specific bubble and crash period the general evidence documented by many researchers (e.g. Jegadeesh and Titman, 1993, 2001). They are consistent with the stylized facts described by the model that predict that the momentum profits will eventually reverse in cycle bubbles and crashes as illustrated above. The qualitative comparison between the empirical data and our simulations suggest that chartists do not need to be a majority, as their superior performance during the bubble make them dominate eventually utterly the investment ecology.

6. Conclusions

We have introduced a model of financial bubbles with two assets (risky and risk-free), in which fundamentalists and chartists co-exist. Fundamentalists form expectations on the return and risk of a risky asset and maximize their constant relative risk aversion expected utility with respect to their portfolio allocation. Chartists are subjected to social imitation and follow momentum trading.

In contrast to various previous models, agents do not switch between investment strategies. By keeping track of their wealth levels, we still observe the formation of endogenous bubbles and match several stylized facts of financial markets such as a fat-tail distribution of returns and volatility clustering. In particular, we observe transient faster-than-exponential bubble growth with approximate log-periodic behavior. Although faster-than-exponential growth at the beginning of a bubble has been found in many econometric studies of bubbles in real markets and recent lab experiments, it has been hardly discussed in the context of agent-based models. Our model is one of the first offering a transparent analytical explanation for this stylized fact.

To the important question of whether and when fundamentalist investors are able to stabilize financial markets by arbitraging chartists, our analysis suggests that chartists may eventually always lead to the creation of bubbles, given sufficient time, if a mechanism exists or some sentiment develops that increase their propensity for herding. Momentum strategies have been shown to be transiently profitable, supporting the hypothesis that these strategies enhance herding behavior.

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