Internal stabilization of transport systems

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INRIA Cage

Control theory in 1 slide

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Examples: gmaps itinerary, parallel parking...

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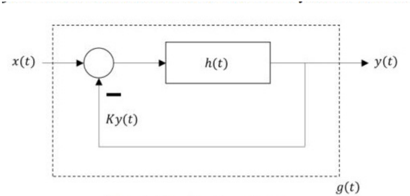
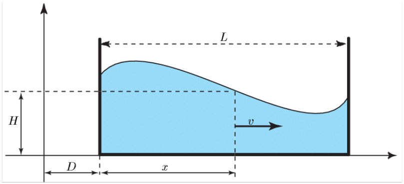


Fig. 1. Feedback system

Example: the water tank

$$\begin{cases} H_t + (HV)_x = 0, \\ V_t + \left(gH + \dfrac{V^2}{2}\right)_x = \underbrace{-u(t)}_{\text{acceleration}}, \\ V(t,0) = V(t,L) = 0, \quad \forall t \geq 0. \end{cases}$$



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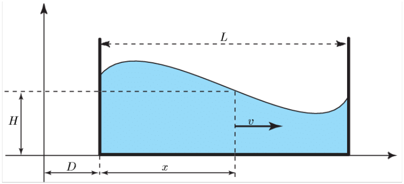

Linearised around $(H^\gamma, V^\gamma) := (H_0 - \gamma x, 0)$ (constant acceleration):

$$\begin{cases} h_t + h^\gamma (V)_x = 0, \\ v_t + g\,(h)_x = -u(t), \\ v(t,0) = v(t,L) = 0, \quad \forall t \geq 0. \end{cases}$$

Controllable. Stabilizable?

An even simpler model

$$\begin{cases} \alpha_t + \alpha_x = u(t)\varphi(x), \ x \in [0, L], \\ \alpha(t, 0) = \alpha(t, L), \ \forall t \geq 0, \end{cases}$$

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Controllable in H^m_{per} if

$$\frac{c}{\sqrt{1 + \left|\frac{2i\pi n}{L}\right|^{2m}}} \leq |\varphi_n| \leq \frac{C}{\sqrt{1 + \left|\frac{2i\pi n}{L}\right|^{2m}}}, \quad \forall n \in \mathbb{Z},$$

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$$\varphi \in H^{m-1}_{per} \cap H^m_{(pw)} \quad (m \geq 1)$$

Results

Theorem (Rapid stabilization in Sobolev norms)

Let $m \geq 1$. If the system is controllable in H^m_{per} and φ has extra piecewise regularity, then the system can be stabilized exponentially for any decay rate.

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Theorem (Finite-time stabilization in Sobolev norms)

Under the same conditions, there exists a feedback law that stabilizes the system in finite time $T = L$.

Stabilization of hyperbolic systems

Approaches to solve a stabilization problem:

- Gramian approach (abstract), Riccati equations...
- Lyapunov functionals: find a feedback that allows for a (exponentially) decreasing energy functional

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- Backstepping
  Your system with feedback $K \xrightarrow{\text{Invertible T}}$ A *stable* target system

Summary

The Krstic parable

Unstable heat equation:

$$\begin{cases} u_t - u_{xx} = \lambda u, \\ u(0) = 0, \quad u(1) = U(t). \end{cases} \tag{1}$$

Transformation (Volterra):

$$w(t,x) = u(t,x) - \int_0^x k(x,y)u(t,y)dy$$

Exponentially stable target system:

$$\begin{cases} w_t - w_{xx} = 0, \\ w(0) = 0, \quad w(1) = 0. \end{cases} \tag{2}$$

Control design: $U(t) = \int_0^1 k(1,y)u(t,y)dy.$

Kernel equations

T is a kernel operator: $f \mapsto f - \int_0^x k(x,y) f(y) dy$.

Target equation $\xrightarrow{\text{Formal computations (IBP...)}}$ PDE for $k(x,y)$.

Kernel equations on $\mathcal{T} := \{0 \leq y \leq x \leq 1\}$:

$$\begin{cases} k_{xx} - k_{yy} = \lambda k, \\ k(x,0) = 0, \\ k(x,x) = -\lambda \dfrac{x}{2} \end{cases} \tag{3}$$

Solving the kernel equation

Wave equation with special boundary conditions.
Variable change:

$$\xi = x + y, \quad \eta = x - y$$

New equation on new domain $\mathcal{T}'$:

$$\begin{cases} 4G_{\xi\eta}(\xi,\eta) = \lambda G(\xi,\eta), \\ G(\xi,\xi) = 0, \\ G(\xi,0) = -\lambda\dfrac{\xi}{4}. \end{cases} \tag{4}$$

Idea: integral equation, iterative scheme, exact solution.

Inverse transformation

$$k(x,y) = -\lambda y \frac{I_1\left(\sqrt{\lambda(x^2-y^2)}\right)}{\sqrt{\lambda(x^2-y^2)}}$$

Good regularity: inverse can be searched as

$$u(t,x) = w(t,x) + \int_0^x l(x,y)w(t,y)dy$$

Almost the same computations as before:

$$l(x,y,\lambda) = k(x,y,-\lambda).$$

Remarks

- k is regular: formal computations actually valid.
- Inverse fairly easy to find.
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- More general transformations?
- Other types of control (internal...)

Summary

Classical pole-shifting

Consider the finite-dimensional **controllable** control system

$$\dot{x} = Ax + Bu(t), \quad x \in \mathbb{C}^n, A \in \mathcal{M}_n(\mathbb{C}), B \in \mathcal{M}_{n,1}(\mathbb{C}).$$

Kalman condition: $rank\{A^n B \mid n = 0, \cdots, n-1\} = n$.

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Poleshifting: $\forall P, \ \exists K \in \mathcal{M}_{1,n}(\mathbb{C}), \quad \chi(A + BK) = P$.
Idea: Brunovski normal form

Brunovski form for PDEs?

D.L. Russell, *Canonical forms and spectral determination for a class of hyperbolic distributed parameter control systems*, JMAA 62, 1978.

$$\frac{\partial}{\partial t}\begin{pmatrix} u \\ v \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{\partial}{\partial x}\begin{pmatrix} u \\ v \end{pmatrix} - A(x)\begin{pmatrix} u \\ v \end{pmatrix} = g(x)u(t) \qquad (5)$$

Canonical form: time-delay system

$$\zeta(t+2) = e^{2\alpha}\zeta(t) + \int_0^2 \overline{p(2-s)}\zeta(t+s)ds + u(t) \qquad (6)$$

Works for *bounded* feedback laws!

Summary

Finite-dimensional backstepping

Another way of shifting poles: map

$$\dot{x} = Ax + B(Kx + v(t))$$

into the stable system

$$\dot{x} = (A - \lambda I)x + Bv(t).$$

The mapping T should be invertible and satisfy

$$T(A + BK) = AT - \lambda T,$$
$$TB = B.$$

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"Backstepping equations"

Finite-dimensional backstepping

Proposition

If the system (14) ***is controllable****, then there exists a unique pair* (T, K) *satisfying conditions* (17)

Proof in Brunovski form.

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- Structural condition for Brunovski normal form (initialization of iterative proof)

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- Sets a *nice form* of the problem.

K is a parameter of T.

From finite-dimension to PDEs

Suppose A is diagonalizable, with eigenvectors and eigenvalues (e_i, λ_i), $\lambda \neq \lambda_i, \forall i$.

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$$((A - \lambda I)T - TA)e_i = BKe_i, \quad \forall i$$

$$Te_i = (Ke_i)(A - (\lambda + \lambda_i)I)^{-1}B.$$

1. **Basis property**: $f_i := ((A - (\lambda + \lambda_i)I)^{-1}B)$ is a basis.

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2. **Definition of** (T, K)

$$B^*T^*f_i = B^*f_i \to (Ke_i) = \frac{B^*f_i}{B^*e_i}.$$

Controllability: $B^*e_i \neq 0$.

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3. **Invertibility of** T Also with controllability.

Summary

Our system

Linear feedbacks:

$$\langle \alpha(t), F \rangle = \sum_{n \in \mathbb{Z}} \overline{F_n} \alpha_n(t) = \int_0^L \bar{F}(s) \alpha(s) ds$$

Closed-loop system:

$$\begin{cases} \alpha_t + \alpha_x = \langle \alpha(t), F \rangle \varphi(x), \ x \in [0, L], \\ \alpha(t, 0) = \alpha(t, L), \ \forall t \geq 0. \end{cases}$$

Target system:

$$\begin{cases} z_t + z_x + \lambda z = 0, \quad x \in (0, L), \\ z(t, 0) = z(t, L), \quad t \geq 0. \end{cases}$$

Kernel equations

T is a kernel operator: $f \mapsto \int_0^L k(x,y)f(y)dy$.

Operator equation $\xrightarrow{\text{Formal computations (IBP...)}}$ PDE for $k(x,y)$.

$(A - \lambda I)T - TA$ $= TBK$	$\begin{cases} k_x + k_y + \lambda k + \int_0^L k(x,s)\varphi(s)ds\bar{K}(y) = 0, \\ k(0,y) = k(L,y), \\ k(x,0) = k(x,L). \end{cases}$

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$$k_n = -\overline{K_{-n}} \underbrace{\frac{L}{1-e^{-\lambda L}} e^{-\lambda x} e_{-n} \star \varphi}_{\text{Riesz basis of } H^m_{per}}$$

Controllability gives a basis property!

Invertibility and feedback

$$T\alpha = \sum_{n\in\mathbb{Z}} \alpha_n T e_n, \quad \alpha \in H^m_{per}$$

Invertible iff $|K_n| \sim n^m$ $(n^m \alpha_n \in \ell^2)$.

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$$\boxed{TB = B}$$

$$TB = B \to b_i(Ke_i) = \tilde{b_i}.$$

Controllability:

$$b_i \neq 0 \to Ke_i = \frac{\tilde{b_i}}{b_i}$$

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Controllability:

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But...$\varphi \notin H^m_{per}$. $T\varphi$?

Weak condition:

$$\varphi^{(N)} \xrightarrow[N\to\infty]{H^{m-1}_{per}} \varphi, \quad T\varphi^{(N)} \rightharpoonup \varphi$$

iff $K_n := -\dfrac{2}{L\overline{\varphi_n}} \dfrac{1-e^{-\lambda L}}{1+e^{-\lambda L}} \sim n^m$

Dirichlet convergence theorem

Almost done...

- **Kernel equations** Derived formally using the $TB = B$ condition!

$$\left\{\begin{array}{l}\textbf{Basis property}\\ \textbf{Definition of } (T, K) \\ \textbf{Invertibility of } T\end{array}\right. \quad \rightarrow \text{weak } TB = B!$$

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- **Kernel equations** Derived formally using the $TB = B$ condition!

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- **Operator equality** $T(A + BK) = AT - \lambda T$ on

$$D(A+BK) := \left\{\alpha \in H^{m+1} \cap H^m_{per}, \quad -\alpha_x + \langle \alpha, F\rangle \varphi \in H^m_{per}\right\}.$$

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- **Well-posedness of the closed-loop system.** Lumer-Phillips theorem (study the regularity of the feedback law).

Summary

A naive idea

Recall

$$K_n^{\lambda} := -\frac{2}{L\overline{\varphi_n}}\frac{1-e^{-\lambda L}}{1+e^{-\lambda L}}$$

$$-\frac{2}{L}\frac{1-e^{-\lambda L}}{1+e^{-\lambda L}} \xrightarrow[\lambda\to\infty]{} -2/L$$

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$$K^{\lambda} \xrightarrow[\lambda\to\infty]{} K^{\infty} := -\frac{2}{\overline{\varphi_n}L}, \quad \forall n \in \mathbb{Z}.$$

New feedback law K^{∞}?

Going to the limit

Sketch of proof:

- Give an expression of the semigroup for $\lambda > 0$:

$$S^\lambda(t)\alpha = -\frac{L}{2}\varphi \star \left(\chi_{[0,t]} e^{-\lambda L}\alpha \star \widetilde{K}^\infty(\cdot - t + L) \right.$$
$$\left. + \chi_{[t,L]}\alpha \star \widetilde{K}^\infty(\cdot - t)\right).$$

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- Study the new semigroup (well-defined? Infinitesimal generator?):

$$S^\infty(t)\alpha := -\frac{L}{2}\varphi\star\Big(\chi_{[t,L]}\alpha \star \widetilde{K}^\infty(\cdot - t)\Big)$$

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- Study the new semigroup (well-defined? Infinitesimal generator?):

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- This semigroup satisfies

$$S^\infty(t)\alpha^0 = 0, \quad \forall t \geq L, \forall \alpha^0 \in H^m_{per}.$$

Going to the limit

Sketch of proof:

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- Study the new semigroup (well-defined? Infinitesimal generator?):

$$S^{\infty}(t)\alpha := -\frac{L}{2}\varphi\star\Big(\chi_{[t,L]}\alpha \star \widetilde{K}^{\infty}(\cdot - t)\Big) \overset{?}{\Leftrightarrow} \alpha_t + \alpha_x = \langle \alpha, K^{\infty}\rangle\varphi$$

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- Works thanks to **exact** controllability.