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Finite-time internal stabilization of a linear 1-D transport equation

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Abstract

We consider a 1-D linear transport equation on the interval $(0, L)$, with an internal scalar control. We prove that if the system is controllable in a periodic Sobolev space of order greater than 1, then the system can be stabilized in finite time, and we give an explicit feedback law.

1 Introduction

We study the linear 1-D hyperbolic equation

$$\begin{cases} y_t + y_x + a(x)y = u(t)\tilde{\varphi}(x), & x \in [0, L], \\ y(t, 0) = y(t, L), & \forall t \geq 0, \end{cases} \quad (1)$$

where a is continuous, real-valued, $\tilde{\varphi}$ is a given real-valued function of space, and at time t , $y(t, \cdot)$ is the state and $u(t)$ is the control. As in [38], the system can be transformed into

$$\begin{cases} \alpha_t + \alpha_x + \mu\alpha = u(t)\varphi(x), & x \in [0, L], \\ \alpha(t, 0) = \alpha(t, L), & \forall t \geq 0, \end{cases} \quad (2)$$

through the state transformation

$$\alpha(t, x) := e^{\int_0^x a(s)ds - \mu x} y(t, x),$$

where $\mu = \int_0^L a(s)ds$, and with

$$\varphi(x) := e^{\int_0^x a(s)ds - \mu x} \tilde{\varphi}(x),$$

so that we focus on systems of the form (2) in this article.

1.1 Notations and definitions

We note ℓ^2 the space of summable square series $\ell^2(\mathbb{Z}, \mathbb{C})$. To simplify the notations, we will note L^2 the space $L^2(0, L)$ of complex-valued L^2 functions on the interval $(0, L)$, with its hermitian product

$$\langle f, g \rangle = \int_0^L f(x) \overline{g(x)} dx, \quad \forall f, g \in L^2, \quad (3)$$

and the associated norm $\|\cdot\|$. Functions of L^2 can also be seen as L -periodic functions on \mathbb{R} , by the usual L -periodic continuation: in this article, for any $f \in L^2$ we will also note f its L -periodic continuation on \mathbb{R} .

We also use the following notation

$$e_n(x) = \frac{1}{\sqrt{L}} e^{\frac{2i\pi}{L} nx}, \quad \forall n \in \mathbb{Z}, \quad (4)$$

the usual Hilbert basis for L^2 . For a function $f \in L^2$, we will note $(f_n)_{n \in \mathbb{Z}} \in \ell^2$ its coefficients in this basis:

$$f = \sum_{n \in \mathbb{Z}} f_n e_n.$$

Note that with this notation, we have

$$\bar{f} = \sum_{n \in \mathbb{Z}} \overline{f_{-n}} e_n,$$

so that, in particular, if f is real-valued:

$$f_{-n} = \overline{f_n}, \quad \forall n \in \mathbb{Z}.$$

We will use the following definition of the convolution product on L -periodic functions:

$$f \star g = \sum_{n \in \mathbb{Z}} f_n g_n e_n = \int_0^L f(s) g(\cdot - s) ds \in L^2, \quad \forall f, g \in L^2, \quad (5)$$

where $g(x - s)$ should be understood as the value taken in $x - s$ by the L -periodic continuation of g .

Let us now note \mathcal{E} the space of finite linear combinations of the $(e_n)_{n \in \mathbb{Z}}$. Then, any sequence $(f_n)_{n \in \mathbb{Z}}$ defines an element f of \mathcal{E}' :

$$\langle e_n, f \rangle = \overline{f_n}, \quad \forall n \in \mathbb{Z}. \quad (6)$$

On this space of linear forms, we can extend our previous definition of convolution:

$$\langle e_n, f \star g \rangle = \overline{f_n g_n} \quad (7)$$

derivation can be defined by duality from (6):

$$f' = \left(\frac{2i\pi n}{L} f_n \right), \quad \forall f \in \mathcal{E}'. \quad (8)$$

We also define the following spaces:

Definition 1.1. Let $m \in \mathbb{N}$. We note H^m the usual Sobolev spaces on the interval $(0, L)$, equipped with the Hermitian product

$$\langle f, g \rangle_m = \int_0^L f \bar{g} + \partial^m f \overline{\partial^m g}, \quad \forall f, g \in H^m,$$

and the associated norm $\|\cdot\|_m$.

For $m \geq 1$ we also define $H_{(pw)}^m$ the space of piecewise H^m functions, that is, $f \in H_{(pw)}^m$ if there exists a finite number d of points $(\sigma_j)_{1 \leq j \leq d} \in [0, L]$ such that, noting $\sigma_0 := 0$ and $\sigma_{d+1} := L$, f is H^m on every $[\sigma_j, \sigma_{j+1}]$ for $0 \leq j \leq d$. This space can be equipped with the norm

$$\|f\|_{m,pw} := \sum_{j=0}^d \|f|_{[\sigma_j, \sigma_{j+1}]}\|_{H^m(\sigma_j, \sigma_{j+1})}.$$

Note that this expression does not depend on the choice of the (σ_j) , and thus is well-defined. For $s > 0$, we also define the periodic Sobolev space H_{per}^s as the subspace of L^2 functions $f = \sum_{n \in \mathbb{Z}} f_n e_n$ such that

$$\sum_{n \in \mathbb{Z}} \left(1 + \left| \frac{2i\pi n}{L} \right|^{2s} \right) |f_n|^2 < \infty.$$

H^s is a Hilbert space, equipped with the Hermitian product

$$\langle f, g \rangle_s = \sum_{n \in \mathbb{Z}} \left(1 + \left| \frac{2i\pi n}{L} \right|^{2s} \right) f_n \overline{g_n}, \quad \forall f, g \in H^s,$$

and the associated norm $\|\cdot\|_s$, as well as the Hilbert basis

$$(e_n^s) := \left(\frac{e_n}{\sqrt{1 + \left| \frac{2i\pi n}{L} \right|^{2s}}} \right).$$

Note that for $m \in \mathbb{N}$, H_{per}^m is a closed subspace of H^m , with the same scalar product and norm, thanks to the Parseval identity. Moreover,

$$H_{per}^m = \left\{ f \in H^m, \quad f^{(i)}(0) = f^{(i)}(L), \forall i \in \{0, \dots, m\} \right\}.$$

1.2 Main result

To stabilize (2), we will be considering linear feedbacks, that is, formally,

$$\langle \alpha(t), F \rangle = \sum_{n \in \mathbb{Z}} \overline{F_n} \alpha_n(t) = \int_0^L \bar{F}(s) \alpha(s) ds$$

where $F \in \mathcal{E}'$ and $(F_n)_{n \in \mathbb{Z}} \in \mathbb{C}^{\mathbb{Z}}$ are its Fourier coefficients, and F is “real-valued”:

$$F_{-n} = \overline{F_n}, \quad \forall n \in \mathbb{Z}.$$

In fact, the integral notation will appear as purely formal, as the $(F_n)_{n \in \mathbb{Z}}$ will have a prescribed growth, so that $F \notin L^2$. The associated closed-loop system now writes

$$\begin{cases} \alpha_t + \alpha_x + \mu \alpha = \langle \alpha(t), F \rangle \varphi(x), & x \in [0, L], \\ \alpha(t, 0) = \alpha(t, L), & \forall t \geq 0. \end{cases} \quad (9)$$

This is a linear transport equation, which we seek to stabilize with an internal, scalar feedback, given by a real-valued feedback law. In [38], we proved the following theorem for system (2):

Theorem 1.1 (Rapid stabilization in Sobolev norms). *Let $m \geq 1$. Let $\varphi \in H_{(pw)}^m \cap H_{per}^{m-1}$ such that*

$$\frac{c}{\sqrt{1 + \left| \frac{2i\pi n}{L} \right|^{2m}}} \leq |\varphi_n| \leq \frac{C}{\sqrt{1 + \left| \frac{2i\pi n}{L} \right|^{2m}}}, \quad \forall n \in \mathbb{Z}, \quad (10)$$

where $c, C > 0$. Then, for every $\lambda \geq 0$ there exists a stationary feedback law F such that for all $\alpha_0 \in H_{per}^m$ the closed-loop system (9) has a solution $\alpha(t)$ which satisfies

$$\|\alpha(t)\|_m \leq \left(\frac{C}{c} \right)^2 e^{(\mu+\lambda)L} e^{-\lambda t} \|\alpha_0\|_m, \quad \forall t \geq 0.$$

If c and C are sharp for (10), then the above estimate is sharp.

Now, notice that, for $\lambda > 0$, the corresponding feedback law obtained using the backstepping method is the linear form $F^{\lambda-\mu} \in \mathcal{E}'$ defined by

$$F_n^{\lambda-\mu} := -\frac{K(\lambda-\mu)}{\overline{\varphi_n}}, \quad \forall n \in \mathbb{Z}, \quad (11)$$

where

$$K(\lambda - \mu) := \frac{2}{L} \frac{1 - e^{-(\lambda - \mu)L}}{1 + e^{-(\lambda - \mu)L}} \xrightarrow{\lambda \rightarrow \infty} \frac{2}{L}, \quad (12)$$

so that

$$F^\lambda \xrightarrow[\lambda \rightarrow \infty]{\mathcal{E}'} F^\infty \quad (13)$$

where

$$F_n^\infty := -\frac{2}{\varphi_n L}, \quad \forall n \in \mathbb{Z}. \quad (14)$$

Moreover, when $\lambda \rightarrow \infty$, the stability estimate in Theorem 1.1 becomes, for $t > L$,

$$\|\alpha(t)\|_m = 0.$$

This would suggest that taking the limit feedback F^∞ could result in finite-time stabilization of (2). This is indeed the case, and in this article we will prove the following theorem:

Theorem 1.2 (Finite-time stabilization in Sobolev norms). *Let $m \geq 1$. Let $\varphi \in H_{(pw)}^m \cap H_{per}^{m-1}$ satisfying (10) for some $c, C > 0$. Then, if the feedback law is defined by (14), for any initial data $\alpha_0 \in H_{per}^m$ the corresponding closed-loop system (9) has a solution $\alpha(t)$ which satisfies*

$$\|\alpha(t)\|_m = 0, \quad \forall t \geq L.$$

1.3 Related results

To investigate the stabilization of infinite-dimensional systems, there are three main types of approaches.

The first type of approach relies on abstract methods, such as the Gramian approach and the Riccati equations (see for example [33, 32, 21]). Although quite powerful, it seems that these methods fail to obtain the stabilization of nonlinear systems from the stabilization of their linearized systems.

The second approach relies on Lyapunov functions. Many results on the boundary stabilization of first-order hyperbolic systems, linear and nonlinear, have been obtained using this approach: see for example the book [2], and the recent results in [17, 18]. However, this approach can be limited, as it is sometimes impossible to obtain an arbitrary decay rate using Lyapunov functions (see [14, Remark 12.9, page 318] for a finite dimensional example).

The third approach is the backstepping method. This name originally refers to a way of designing, in a recursive way, more effective feedback laws, for systems for which one already has a Lyapunov function and a feedback law which globally asymptotically stabilizes the system, see [14, 30] for an overview of the finite-dimensional case, and [7] or [23] for applications to partial differential equations. Another way of applying this approach to partial differential equations was then developed in [3] and [1]: when applied to the discretization of the heat equation, the backstepping approach yielded a change of coordinates which was equivalent to a Volterra transformation of the second kind, a tool that was already used for pole displacement on hyperbolic systems in [29], although the ideas already appear in [29]. Backstepping then took yet another successful form, consisting in mapping the system to stable target system, using a Volterra transformation of the second kind (see [22] for a comprehensive introduction to the method):

$$f(t, x) \mapsto f(t, x) - \int_0^x k(x, y) f(t, y) dy.$$

This was used to prove a host of results on the boundary stabilization of partial differential equations: let us cite for example [20] and [27] for the wave equation, [35, 36] for the Korteweg-de Vries, [2, chapter 7] for an application to first-order hyperbolic systems, and also [16], which combines the backstepping method with Lyapunov functions to prove finite-time stabilization in H^2 for a quasilinear 2×2 hyperbolic system.

In some cases, the method was used to obtain stabilization with an internal feedback. This was done in [31] and [34] for parabolic systems, and [37] for first-order hyperbolic systems. The strategy in these

works is to first apply a Volterra transformation as usual, which still leaves an unstable source term in the target, and then apply a second invertible transformation to reach a stable target system. Let us note that in the latter reference, the authors study a linear transport equation and get finite-time stabilization. However, their controller takes a different form than ours, and several hypotheses are made on the space component of the controller so that a Volterra transformation can be successfully applied to the system. This is in contrast with our result, where the assumption we make on the controller simply corresponds to the exact null-controllability of the system.

More recently, the backstepping approach has been applied, using another type of linear transformations, namely, Fredholm transformations:

$$f(t, x) \mapsto \int_0^L k(x, y) f(t, y) dy.$$

These are more general than Volterra transformations, but they require more work: indeed, Volterra transformations are always invertible, but the invertibility of a Fredholm transformation is harder to check. Even though it is sometimes more involved and technical, the use of a Fredholm transformation proves more effective for certain types of control: for example, in [12] for the Korteweg-de Vries equation and [11] for a Kuramoto-Sivashinsky, the position of the control makes it more appropriate to use a Fredholm transformation. Other boundary stabilization results using a Fredholm transformation can be found in [9] for integro-differential hyperbolic systems, and in [10] for general hyperbolic balance laws. Fredholm transformations have also been used in [8], where the authors prove the rapid stabilization of the Schrödinger equation with an internal feedback.

The backstepping method has the advantage of providing explicit feedback laws, which makes it a powerful tool to prove other related results, such as null-controllability or small-time stabilization (stabilization in an arbitrarily small time). This is done in [13], where the authors give an explicit control to bring a heat equation to 0, then a time-varying, periodic feedback to stabilize the equation in small time. In [36], the author obtains the same kind of results for the Korteweg-de Vries equation. In this article, we use the explicit feedback laws obtained by the backstepping method in [38] to design an explicit feedback law that achieves finite-time stabilization.

1.4 Structure of the article

In Section 2, we derive an expression for the exponentially stable semigroup corresponding to the explicit feedback laws obtained in [38] for exponential stabilization. Then, in Section 3, we study the semigroup obtained when $\lambda \rightarrow \infty$. In particular, we derive its infinitesimal generator and prove that it corresponds to a closed-loop system which goes to 0 in finite time, which yields a feedback law achieving stabilization in finite time. Finally, Section 4 is devoted to some comments on the result, and on further questions.

2 The exponentially stable semigroup

We recall some specifics of Theorem 1.1, which can be found in more detail in [38].

2.1 Backstepping transformation

To prove Theorem 1.1, the backstepping method was used. This method consists in mapping our system into a stable target system, here

$$\begin{cases} z_t + z_x + \lambda' z = 0, & x \in (0, L), \\ z(t, 0) = z(t, L), & t \geq 0, \end{cases} \quad (15)$$

with $\lambda' > 0$. To find an invertible transformation that does this, the idea is to write it as a Fredholm operator:

$$T : \alpha(t, x) \mapsto \int_0^L k(x, y) \alpha(t, y) dy$$

so that the mapping condition becomes a partial differential equation in k (the kernel equation). This equation contains non-local terms, which are resolved by adding a constraint to the kernel equation, called the $TB = B$ condition:

$$\int_0^L k(x, y) \varphi(t, y) dy = \varphi(x), \quad \forall x \in [0, L].$$

From this kernel equation, conditions on F for the invertibility of T can be derived. Then, using a weak version of the $TB = B$ condition, a suitable feedback is computed, so that a candidate for the backstepping transformation can be derived:

$$T^\lambda \alpha = \sum_{n \in \mathbb{Z}} \alpha_n \overline{F_n^\lambda} \Lambda_{-n}^\lambda \star \varphi, \quad \forall \alpha \in H_{per}^m, \quad (16)$$

where $\lambda := \lambda' - \mu$, $\overline{F_n^\lambda}$ is defined by (11), and

$$\Lambda_n^\lambda(x) := \frac{\sqrt{L}}{1 - e^{-\lambda L}} e^{-\lambda_n x} = \Lambda(x) e_{-n}(x), \quad \forall n \in \mathbb{Z}, \quad \forall x \in [0, L], \quad (17)$$

where

$$\lambda_n = \lambda + \frac{2i\pi n}{L}, \quad \forall n \in \mathbb{Z}, \quad (18)$$

and where Λ is the L -periodic function defined by

$$\Lambda(x) = \frac{L}{1 - e^{-\lambda L}} e^{-\lambda x}, \quad \forall x \in [0, L].$$

2.2 Well-posedness of the closed-loop system

Now that a candidate for the backstepping transformation has been determined, it must be proved that is indeed a backstepping transformation, and that the closed-loop with the feedback defined above is well-posed. We first define the domains

$$D_m^\lambda := \left\{ \alpha \in \tau^\varphi(H_{(pw)}^{m+1}) \cap H_{per}^m, \quad -\alpha_x - \mu\alpha + \langle \alpha, F \rangle \varphi \in H_{per}^m \right\} \quad (19)$$

where τ^φ is the diagonal operator defined by the eigenvalues

$$\tau_0^\varphi := 1$$

$$\tau_n^\varphi := \frac{1}{\sqrt{L}} \left(\partial^{m-1} \varphi(L) - \partial^{m-1} \varphi(0) + \sum_{j=1}^d e^{-\frac{2i\pi}{L} n \sigma_j} (\partial^{m-1} \varphi(\sigma_j^-) - \partial^{m-1} \varphi(\sigma_j^+)) \right), \quad \forall n \in \mathbb{Z}^*.$$

In [38], we investigate the regularity of the feedback law, using the controllability condition (10). This helps to prove that the associated operator

$$A + BK := -\partial_x - \mu I + \langle \cdot, F^\lambda \rangle \varphi$$

is densely defined and closed. Finally, to check that the mapping property between systems (9) and (15) is verified, one proves the operator equality

$$T^\lambda (-\partial_x + \langle \cdot, F^\lambda \rangle \varphi) \alpha = (-\partial_x - \lambda I) T^\lambda \alpha \quad \text{in } H_{per}^m, \quad \forall \alpha \in D_m^\lambda. \quad (20)$$

This operator equality helps prove the dissipativity of $A + BK$ and its adjoint. Then the Lumer-Phillips theorem implies that $A + BK$ generates a semigroup $S^\lambda(t)$ of contractions for the norm $\|T^\lambda \cdot\|_m$ on D_m^λ , hence the well-posedness of (9), and the exponential stability then follows from the operator equality.

2.3 Expression of the semigroup

To study what happens when $\lambda \rightarrow \infty$, let us first derive an expression for $S^\lambda(t)$.

First, we derive from (16) and (17) the following expression for the backstepping transformation:

$$T^\lambda \alpha = \varphi \star \left(\Lambda(\alpha \star \tilde{F}^\lambda) \right), \quad \forall \alpha \in H_{per}^m, \quad (21)$$

where $\tilde{F}^\lambda \in \mathcal{E}'$ is defined by:

$$\langle e_n, \tilde{F}^\lambda \rangle = F_n^\lambda, \quad n \in \mathbb{Z}.$$

Then, by definition of F^λ , it follows that

$$(T^\lambda)^{-1} \alpha = \frac{1}{K(\lambda)^2} \varphi \star \left(\frac{1}{\Lambda} (\alpha \star \tilde{F}^\lambda) \right), \quad \forall \alpha \in H_{per}^m. \quad (22)$$

Now, recall that for all initial data $z_0 \in H_{per}^m$, the solution of system (15) can be written

$$z(t, x) = e^{-\lambda' t} z_0(x - t), \quad \forall (t, x) \in \mathbb{R}^+ \times (0, L). \quad (23)$$

Thus, by the mapping property of T^λ (see subsection 2.1 of [38]), for all initial data $\alpha^0 \in D_m^\lambda$, the solution of system (9) can be written:

$$\alpha(t, x) = (T^\lambda)^{-1} e^{-\lambda' t} (T^\lambda \alpha^0)(x - t), \quad \forall (t, x) \in \mathbb{R}^+ \times (0, L). \quad (24)$$

Now, notice that convolution and translation commute, so for $(t, x) \in \mathbb{R}^+ \times (0, L)$, we get, using (21)

$$\begin{aligned} \alpha(t, x) &= \frac{1}{K(\lambda)^2} \varphi \star \left(\frac{1}{\Lambda} \left(\tilde{F}^\lambda \star \left(e^{-\lambda' t} \left(\varphi \star \left(\Lambda(\alpha^0 \star \tilde{F}^\lambda) \right) (\cdot - t) \right) \right) \right) \right) \\ &= -\frac{e^{-\lambda' t}}{K(\lambda)} \varphi \star \left(\frac{1}{\Lambda} \left(\left(\Lambda(\alpha^0 \star \tilde{F}^\lambda) \right) (\cdot - t) \right) \right) \\ &= -\frac{e^{-\lambda' t}}{K(\lambda)} \varphi \star \left(\chi_{[0, t]} e^{\lambda(t-L)} \alpha^0 \star \tilde{F}^\lambda(x - t + L) + \chi_{[t, L]} e^{\lambda t} \alpha^0 \star \tilde{F}^\lambda(x - t) \right) \\ &= -\frac{e^{-\mu t}}{K(\lambda)} \varphi \star \left(\chi_{[0, t]} e^{-\lambda L} \alpha^0 \star \tilde{F}^\lambda(x - t + L) + \chi_{[t, L]} \alpha^0 \star \tilde{F}^\lambda(x - t) \right). \end{aligned}$$

This expression is derived for $\alpha_0 \in D_m^\lambda$, but it actually defines a semigroup on all of H_{per}^m . Note that this semigroup is the extension of the semigroup of contractions generated by $A + BK$ on D_m^λ (see section 2.2):

$$S^\lambda(t) \alpha^0 = -\frac{e^{-\mu t}}{K(\lambda)} \varphi \star \left(\chi_{[0, t]} e^{-\lambda L} \alpha^0 \star \tilde{F}^\lambda(x - t + L) + \chi_{[t, L]} \alpha^0 \star \tilde{F}^\lambda(x - t) \right), \quad \forall t \geq 0, \forall \alpha^0 \in H_{per}^m. \quad (25)$$

3 The limit semigroup

Now, notice that, from (12) and (14),

$$F^\lambda = \frac{LK(\lambda)}{2} F^\infty,$$

so that we can write

$$S^\lambda(t) \alpha = -\frac{Le^{-\mu t}}{2} \varphi \star \left(\chi_{[0, t]} e^{-\lambda L} \alpha \star \widetilde{F^\infty}(x - t + L) + \chi_{[t, L]} \alpha \star \widetilde{F^\infty}(x - t) \right), \quad \forall t \geq 0, \forall \alpha \in H_{per}^m.$$

Then it is clear from the above that

$$S^\lambda(t) \alpha \xrightarrow{\lambda \rightarrow \infty} S^\infty(t) \alpha := -\frac{Le^{-\mu t}}{2} \varphi \star \left(\chi_{[t, L]} \alpha \star \widetilde{F^\infty}(x - t) \right), \quad \forall t \geq 0, \forall \alpha \in H_{per}^m.$$

Hence we have defined a new semigroup $S^\infty(t)$ on H_{per}^m , which we now study in order to establish Theorem 1.2.

3.1 A useful semigroup

Consider the semigroup given by

$$S_0(t)\alpha = e^{-\mu t}\chi_{[t,L]}\alpha(x-t), \quad \forall t \geq 0, \forall \alpha \in L^2.$$

This is actually a contraction semigroup, the infinitesimal generator of which is given by

$$\begin{aligned} D(A_0) &= \{\alpha \in H^1, \quad \alpha(0) = 0\} \\ A_0 &= -\partial_x - \mu I \end{aligned} \tag{26}$$

where the derivative is to be understood as the usual derivative of a Sobolev function, not as the derivative in \mathcal{E}' . Note that this semigroup is associated to the following transport equation:

$$\begin{cases} y_t + y_x + \mu y = 0, & x \in [0, L], \\ y(t, 0) = 0, & \forall t \geq 0, \end{cases} \tag{27}$$

and that in particular

$$S_0(t)\alpha = 0, \quad \forall t \geq L, \forall \alpha \in L^2. \tag{28}$$

3.2 Infinitesimal generator

Now let us compute the infinitesimal generator of S^∞ . First, notice that

$$S^\infty(t)\alpha = -\frac{L}{2}\varphi \star S_0(t) \left(\alpha \star \widetilde{F^\infty} \right). \tag{29}$$

Now, let us define the following domain, in the same spirit as in section 2.1:

$$D_m^\infty := \left\{ \alpha \in \tau^\varphi(H_{(pw)}^{m+1}) \cap H_{per}^m, \quad -\alpha_x - \mu\alpha + \langle \alpha, F^\infty \rangle \varphi \in H_{per}^m \right\}. \tag{30}$$

This domain is dense in H_{per}^m , as it contains the following dense subspace (see [38, Proposition 3.1]):

$$\left\{ \alpha \in H_{per}^{m+1}, \quad \langle \alpha, F \rangle = 0 \right\}.$$

Let us now prove that on this domain, S^∞ has an infinitesimal generator. For $\alpha \in D_m^\infty$, we have

$$r := \widetilde{F^\infty} \star (-\alpha_x - \mu\alpha + \langle \alpha, F^\infty \rangle \varphi) \in L^2. \tag{31}$$

Thus, taking the Fourier coefficients, we get:

$$\alpha_n \overline{F_n^\infty} = -\frac{r_n + \mu\alpha_n \overline{F_n^\infty}}{\left(\frac{2i\pi n}{L}\right)} + i\frac{\langle \alpha, F^\infty \rangle}{\pi n}, \quad \forall n \neq 0.$$

Now, note that

$$\sum_{n \in \mathbb{Z}^*} i\frac{\langle \alpha, F^\infty \rangle}{\pi n} e_n(x) = \frac{2}{L} \langle \alpha, F^\infty \rangle \left(\frac{x}{\sqrt{L}} - \frac{\sqrt{L}}{2} \right),$$

so that

$$\alpha \star \widetilde{F^\infty} = \tilde{r} + \frac{2}{L} \langle \alpha, F^\infty \rangle \left(\frac{x}{\sqrt{L}} - \frac{\sqrt{L}}{2} \right), \tag{32}$$

where

$$\tilde{r} = \frac{\alpha_0 F_0^\infty}{\sqrt{L}} - \sum_{n \in \mathbb{Z}^*} \frac{r_n + \mu\alpha_n \overline{F_n^\infty}}{\left(\frac{2i\pi n}{L}\right)} e_n \in H_{per}^1. \tag{33}$$

Hence, $\alpha \star \widetilde{F^\infty} \in H^1$, and, from (32) and (33) we get

$$(\alpha \star \widetilde{F^\infty})_x = - \left(r - \frac{r_0}{\sqrt{L}} + \mu \alpha \star \widetilde{F^\infty} - \mu \frac{\alpha_0 F_0^\infty}{\sqrt{L}} \right) + \frac{2}{L\sqrt{L}} \langle \alpha, F^\infty \rangle.$$

Now, by (31), (14) and by definition of the convolution product,

$$r_0 = -\mu F_0^\infty \alpha_0 - \frac{2\langle \alpha, F^\infty \rangle}{L},$$

so that, again by (31),

$$(\alpha \star \widetilde{F^\infty})_x = -r + \mu \alpha \star \widetilde{F^\infty} = -\widetilde{F^\infty} \star (-\alpha_x + \langle \alpha, F^\infty \rangle \varphi) \quad \text{in } L^2. \quad (34)$$

On the other hand, we know, by the Dirichlet convergence theorem (see [19]) applied to $\alpha \star \widetilde{F^\infty} \in H^1$ at point 0, that

$$\frac{\alpha \star \widetilde{F^\infty}(0) + \alpha \star \widetilde{F^\infty}(L)}{2} = \sum_{n \in \mathbb{Z}} \frac{\alpha_n \overline{F_n^\infty}}{\sqrt{L}} = \frac{\langle \alpha, F^\infty \rangle}{\sqrt{L}}.$$

On the other hand, by (32),

$$(\alpha \star \widetilde{F^\infty} - \tilde{r})(0) = -\frac{\langle \alpha, F^\infty \rangle}{\sqrt{L}} = -(\alpha \star \widetilde{F^\infty} - \tilde{r})(L), \quad (35)$$

thus, as \tilde{r} is periodic,

$$\tilde{r}(0) = \frac{\alpha \star \widetilde{F^\infty}(0) + \alpha \star \widetilde{F^\infty}(L)}{2} = \frac{\langle \alpha, F^\infty \rangle}{\sqrt{L}}. \quad (36)$$

From (35) and (36), we get

$$\alpha \star \widetilde{F^\infty}(0) = \tilde{r}(0) - \frac{\langle \alpha, F^\infty \rangle}{\sqrt{L}} = 0,$$

so that $\alpha \star \widetilde{F^\infty} \in D(A_0)$.

We can now compute the infinitesimal generator of S^∞ : let $\alpha \in D_m^\infty$. Then, thanks to the above, $\alpha \star \widetilde{F^\infty} \in D(A_0)$, which means in particular that

$$\frac{S_0(t)(\alpha \star \widetilde{F^\infty}) - (\alpha \star \widetilde{F^\infty})}{t} \xrightarrow[t \rightarrow 0^+]{L^2} -(\alpha \star \widetilde{F^\infty})_x - \mu(\alpha \star \widetilde{F^\infty}).$$

This, together with (29) and (10), implies that

$$\frac{S^\infty(t)\alpha - \alpha}{t} = -\frac{L}{2}\varphi \star \left(\frac{S_0(t)(\alpha \star \widetilde{F^\infty}) - (\alpha \star \widetilde{F^\infty})}{t} \right) \xrightarrow[t \rightarrow 0^+]{H^m} \frac{L}{2}\varphi \star \left((\alpha \star \widetilde{F^\infty})_x + \mu(\alpha \star \widetilde{F^\infty}) \right).$$

By (34), we have

$$\varphi \star \left((\alpha \star \widetilde{F^\infty})_x + \mu(\alpha \star \widetilde{F^\infty}) \right) = \varphi \star (-\widetilde{F^\infty}) \star (-\alpha_x - \mu\alpha + \langle \alpha, F^\infty \rangle \varphi) = \frac{2}{L} (-\alpha_x - \mu\alpha + \langle \alpha, F^\infty \rangle \varphi)$$

so that, finally,

$$\frac{S^\infty(t)\alpha - \alpha}{t} \xrightarrow[t \rightarrow 0^+]{H^m} -\alpha_x - \mu\alpha + \langle \alpha, F^\infty \rangle \varphi.$$

This, together with (30), means that the infinitesimal generator of $S^\infty(t)$ can be given by the domain D_m^∞ and the unbounded operator $-\partial_x - \mu I + \langle \cdot, F^\infty \rangle \varphi$. Hence, $S^\infty(t)$ corresponds to the closed loop system

$$\begin{cases} \alpha_t + \alpha_x + \mu\alpha = \langle \alpha(t), F^\infty \rangle \varphi(x), & x \in [0, L], \\ \alpha(t, 0) = \alpha(t, L), & \forall t \geq 0. \end{cases} \quad (37)$$

which is well-posed. Moreover, by (28) and (29),

$$S^\infty(t)\alpha^0 = 0, \quad \forall t \geq L, \forall \alpha^0 \in H_{per}^m,$$

which proves Theorem 1.2.

4 Comments and further questions

4.1 Backstepping and finite-time stabilization

As we have mentioned in the introduction, one of the advantages of the backstepping method is that it can provide explicit feedback laws for exponential stabilization. This allows the construction of explicit controls for null controllability ([35, 13]) as well as time-varying feedbacks that stabilize the system in finite time $T > 0$ ([36, 13]).

The general strategy in these articles is to divide the interval $[0, T]$ in smaller intervals $[t_n, t_{n+1}]$ on which the feedback corresponding to some $\lambda_n > 0$ is applied. The idea is then to choose the t_n so that the length of the intervals $[t_n, t_{n+1}]$ tends to 0 fast enough to compensate the growth of the norm of the feedback law as $\lambda_n \rightarrow \infty$. Building from this, the authors design a time-varying feedback law that stabilizes the system in finite-time.

Here, the feedback is stationary, and we do not need to define it piecewise: indeed, the norm of the feedback law F^λ is bounded when $\lambda \rightarrow \infty$. This comes from the fact that we used a special type of convergence to define the feedback law, using the $TB = B$ condition. Indeed, in [38], we set

$$\varphi^{(N)} := \sum_{n=-N}^N \varphi_n e_n \in H_{per}^m, \quad \forall N \in \mathbb{N}.$$

Then,

$$\begin{aligned} T^\lambda \varphi^{(N)} &= \sum_{n=-N}^N -\varphi_n \overline{F_n^\lambda} \Lambda_{-n}^\lambda \star \varphi \\ &= \sum_{n=-N}^N \sum_{p \in \mathbb{Z}} \frac{-\varphi_n \overline{F_n^\lambda} \varphi_p}{\lambda_{-n+p}} e_p \\ &= \sum_{p \in \mathbb{Z}} \varphi_p \left(\sum_{n=-N}^N \frac{-\varphi_n \overline{F_n^\lambda}}{\lambda_{-n+p}} \right) e_p. \end{aligned}$$

and F^λ is defined by

$$\frac{1}{-\varphi_n \overline{F_n^\lambda}} = \lim_{N \rightarrow \infty} \sum_{n=-N}^N \frac{1}{\lambda_{-n+p}}$$

in order to have

$$\langle T^\lambda \varphi^{(N)}, e_n \rangle \xrightarrow{N \rightarrow \infty} \varphi_n, \quad n \in \mathbb{Z}.$$

Now, if the convergence of the right-hand side had been absolute, the limit would have gone to 0 when $\lambda \rightarrow \infty$. However, here the sum converges in a special way due to the Dirichlet convergence theorem (see for example [19]), which is why it remains positive (and thus F^λ remains bounded) when $\lambda \rightarrow \infty$.

Hence, a weaker $TB = B$ condition seems to allow for better behavior of the feedback law when $\lambda \rightarrow \infty$.

4.2 Regularity of the feedback law

A remarkable point of this application of the backstepping method, both for rapid and finite-time stabilization, is that the feedback law is not regular on the state space: indeed, it is continuous for $\|\cdot\|_{m+1}$ but not for $\|\cdot\|_m$.

On the other hand, it seems that a continuous feedback law would have a more restricted action on the eigenvalues of the system. Indeed, in [29] it is proved that if the sequence of complex numbers $(\rho_n)_{n \in \mathbb{Z}}$ satisfies

$$\left(\left| \frac{\rho_n - \frac{2i\pi n}{L}}{\varphi_n} \right| \right) \in \ell^2, \quad (38)$$

then there exists a *bounded* feedback law such that the resulting closed-loop system has eigenvalues $(\rho_n)_{n \in \mathbb{Z}}$. It is clear that (38) does not allow for a uniform pole-shifting as we have done in [38]. But even though (38) is not a necessary condition, subsequent works such as [28, 6, 26] turn to unbounded feedback laws, as they are proved to allow for more eigenvalue displacement, and in particular uniform pole-shifting. *A fortiori*, the stronger notion of finite-time stabilization, in which case the operator associated to the closed-loop system has an empty spectrum (see for example [28, Theorem 3 and comments]), probably requires an unbounded feedback law.

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