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October 25, 2018

Abstract

We use the backstepping method to study the stabilization of a 1-D linear transport equation on the interval $(0, L)$, by controlling the scalar amplitude of a piecewise regular function of the space variable in the source term. We prove that if the system is controllable in a periodic Sobolev space of order greater than 1, then the system can be stabilized exponentially in that space and, for any given decay rate, we give an explicit feedback law that achieves that decay rate.

Keywords. Backstepping, transport equation, Fredholm transformations, stabilization, rapid stabilization, internal control.

1 Introduction

We study the linear 1-D hyperbolic equation

$$\begin{cases} y_t + y_x + a(x)y = u(t)\tilde{\varphi}(x), \ x \in [0, L], \\ y(t, 0) = y(t, L), \ \forall t \geq 0, \end{cases} \tag{1}$$

where a is continuous, real-valued, $\tilde{\varphi}$ is a given real-valued function that will have to satisfy certain conditions, and at time t, $y(t, \cdot)$ is the state and $u(t)$ is the control. As the system can be transformed into

$$\begin{cases} \alpha_t + \alpha_x + \mu\alpha = u(t)\varphi(x), \ x \in [0, L], \\ \alpha(t, 0) = \alpha(t, L), \ \forall t \geq 0, \end{cases} \tag{2}$$

through the state transformation

$$\alpha(t, x) := e^{\int_0^x a(s)ds - \mu x} y(x, t),$$

where $\mu = \displaystyle\int_0^L a(s)ds$, and with

$$\varphi(x) := e^{\int_0^x a(s)ds - \mu x} \tilde{\varphi}(x),$$

we will focus on systems of the form (2) in this article.

1.1 Notations and definitions

We note ℓ^2 the space of summable square series $\ell^2(\mathbb{Z})$. To simplify the notations, we will note L^2 the space $L^2(0, L)$ of complex-valued L^2 functions, with its hermitian product

$$\langle f, g \rangle = \int_0^L f(x)\overline{g(x)}dx, \quad \forall f, g \in L^2, \tag{3}$$

and the associated norm $\|\cdot\|$. We also use the following notation

$$e_n(x) = \frac{1}{\sqrt{L}} e^{\frac{2i\pi}{L} nx}, \quad \forall n \in \mathbb{Z}, \tag{4}$$

the usual Hilbert basis for L^2. For a function $f \in L^2$, we will note $(f_n) \in \ell^2$ its coefficients in this basis:

$$f = \sum_{n\in\mathbb{Z}} f_n e_n.$$

Note that with this notation, we have

$$\bar{f} = \sum_{n\in\mathbb{Z}} \overline{f_{-n}} e_n,$$

so that, in particular, if f is real-valued:

$$f_{-n} = \overline{f_n}, \quad \forall n \in \mathbb{Z}.$$

Functions of L^2 can also be seen as L-periodic functions on $\mathbb{R}$, by the usual L-periodic continuation: in this article, for any $f \in L^2$ we will also note f its L-periodic continuation on $\mathbb{R}$.

We will use the following definition of the convolution product on L-periodic functions:

$$f \star g = \sum_{n\in\mathbb{Z}} f_n g_n e_n = \int_0^L f(s) g(\cdot - s) ds \in L^2, \quad \forall f, g \in L^2, \tag{5}$$

where $g(x-s)$ should be understood as the value taken in $x-s$ by the L-periodic continuation of g.

Let us now note $\mathcal{E}$ the space of finite linear combinations of the $(e_n)_{n\in\mathbb{Z}}$. Then, any sequence $(f_n)_{n\in\mathbb{Z}}$ defines an element f of $\mathcal{E}'$:

$$\langle e_n, f\rangle = \overline{f_n}.$$

On this space of linear forms, derivation can be defined by duality:

$$f' = \left(\frac{2i\pi n}{L} f_n\right), \quad \forall f \in \mathcal{E}'.$$

We also define the following spaces:

Definition 1.1. *Let $m \in \mathbb{N}$. We note H^m the usual Sobolev spaces on the interval $(0, L)$, equipped with the Hermitian product*

$$\langle f, g\rangle_m = \int_0^L \partial^m f \overline{\partial^m g}, \quad \forall f, g \in H^m,$$

and the associated norm $\|\cdot\|_m$.

For $m \geq 1$ we also define $H^m_{(pw)}$ the space of piecewise H^m functions, that is, $f \in H^m_{(pw)}$ if there exists a finite number d of points $(\sigma_j)_{1\leq j\leq d} \in [0, L]$ such that, noting $\sigma_0 := 0$ and $\sigma_{d+1} := L$, f is H^m on every $[\sigma_j, \sigma_{j+1}]$ for $0 \leq j \leq d$. This space can be equipped with the norm

$$\|f\|_{m,pw} := \sum_{j=0}^{d} \|f_{|[\sigma_j,\sigma_{j+1}]}\|_{H^m(\sigma_j,\sigma_{j+1})}.$$

For $s > 0$, we also define the periodic Sobolev space H^s_{per} as the subspace of L^2 functions $f = \sum_{n\in\mathbb{Z}} f_n e_n$ such that

$$\sum_{n\in\mathbb{Z}} \left(1 + \left|\frac{2i\pi n}{L}\right|^{2s}\right) |f_n|^2 < \infty.$$

H^s is a Hilbert space, equipped with the Hermitian product

$$\langle f, g\rangle_s = \sum_{n\in\mathbb{Z}} \left(1 + \left|\frac{2i\pi n}{L}\right|^{2s}\right) f_n \overline{g_n}, \quad \forall f, g \in H^s,$$

and the associated norm $\|\cdot\|_s$, as well as the Hilbert basis

$$(e_n^s) := \left(\frac{e_n}{\sqrt{1 + \left|\frac{2i\pi n}{L}\right|^{2s}}}\right).$$

Note that for $m \in \mathbb{N}$, H^m_{per} is a closed subspace of H^m, with the same scalar product and norm, thanks to the Parseval identity. Moreover,

$$H^m_{per} = \left\{f \in H^m, \quad f^{(i)}(0) = f^{(i)}(L), \forall i \in \{0, \cdots, m\}\right\}.$$

1.2 Main result

To stabilize (2), we will be considering linear feedbacks of the form

$$\langle \alpha(t), F\rangle = \sum_{n\in\mathbb{Z}} \overline{F_n}\alpha_n(t) = \int_0^L \bar{F}(s)\alpha(s)ds$$

where $F \in \mathcal{E}'$ and $(F_n) \in \mathbb{C}^{\mathbb{Z}}$ are its Fourier coefficients, and F is real-valued, that is,

$$F_{-n} = \overline{F_n}, \quad \forall n \in \mathbb{Z}.$$

In fact, the integral notation will appear as purely formal, as the (F_n) will have a prescribed growth, so that $F \notin L^2$. The associated closed-loop system now writes

$$\begin{cases} \alpha_t + \alpha_x + \mu\alpha = \langle \alpha(t), F\rangle \varphi(x), \ x \in [0, 1], \\ \qquad \alpha(t, 0) = \alpha(t, L), \ \forall t \geq 0. \end{cases} \tag{6}$$

This is a linear transport equation, which we seek to stabilize with an internal, scalar feedback, given by a real-valued feedback law. This article aims at proving the following class of stabilization results:

Theorem 1.1 (Rapid stabilization in Sobolev norms)**.** *Let $m \geq 1$. Let $\varphi \in H^m_{(pw)} \cap H^{m-1}_{per}$ such that*

$$\frac{c}{\sqrt{1 + \left|\frac{2i\pi n}{L}\right|^{2m}}} \leq |\varphi_n| \leq \frac{C}{\sqrt{1 + \left|\frac{2i\pi n}{L}\right|^{2m}}}, \quad \forall n \in \mathbb{Z}, \tag{7}$$

where $c, C > 0$ are the optimal constants for these inequalities. Then, for every $\lambda \geq 0$ there exists a stationary feedback law F such that for all $\alpha_0 \in H^m_{per}$ the closed-loop system (6) has a solution $\alpha(t)$ which satisfies

$$\|\alpha(t)\|_m \leq \left(\frac{C}{c}\right)^2 e^{(\mu+\lambda)L} e^{-\lambda t} \|\alpha_0\|_m, \quad \forall t \geq 0.$$

The growth restriction on the Fourier coefficients of φ can be written, more intuitively, and for some other constants $c', C' > 0$,

$$\frac{c'}{1 + \left|\frac{2i\pi n}{L}\right|^{m}} \leq |\varphi_n| \leq \frac{C'}{1 + \left|\frac{2i\pi n}{L}\right|^{m}}, \quad \forall n \in \mathbb{Z},$$

and corresponds to the necessary and sufficient condition for the controllability of system (2) in H^m_{per}. Indeed, this system satisfies an observability inequality in H^m_{per} if and only if φ satisfies (7). The controllability of system (2), in turn, will allow us to "shift its poles", using the so-called backstepping method.

On the other hand, the additional regularity $\varphi \in H^m_{(pw)}$ gives us the following equality, by iterated integration by parts on each interval $[\sigma_j, \sigma_{j+1}]$:

$$\varphi_n = \tau_n^\varphi \frac{(-1)^{m-1}}{\left(\frac{2i\pi}{L}n\right)^m} + \frac{(-1)^m}{\left(\frac{2i\pi}{L}n\right)^m} \sum_{j=0}^{d} \left\langle \chi_{[\sigma_j,\sigma_{j+1}]} \partial^m \varphi, e_n \right\rangle, \quad \forall n \in \mathbb{Z}^*, \tag{8}$$

where

$$\tau_n^\varphi := \frac{1}{\sqrt{L}} \left(\partial^{m-1}\varphi(L) - \partial^{m-1}\varphi(0) + \sum_{j=1}^{d} e^{-\frac{2i\pi}{L} n\sigma_j} (\partial^{m-1}\varphi(\sigma_j^-) - \partial^{m-1}\varphi(\sigma_j^+)) \right), \quad \forall n \in \mathbb{Z}^*,$$

and we can set

$$\tau_0^\varphi := 1.$$

Note that, thanks to condition (7), there exists $C_1, C_2 > 0$ such that

$$C_1 \le |\tau_n^\varphi| \le C_2, \quad n \in \mathbb{Z},$$

so that these numbers are the eigenvalues of a diagonal isomorphism of any Sobolev space into itself, which we note τ^φ. Also, note that $\tau_n^\varphi \neq 0$, and thus, $\varphi \notin H^m_{per}$. Finally, note that

$$\left(\sum_{j=0}^{d} \left\langle \chi_{[\sigma_j,\sigma_{j+1}]} \partial^m \varphi, e_n \right\rangle \right) \in \ell^2. \tag{9}$$

1.3 The backstepping method: a finite-dimensional example

Consider the finite-dimensional control system

$$\dot{x} = Ax + Bu(t), \quad x \in \mathbb{C}^n, A \in \mathcal{M}_n(\mathbb{C}), B \in \mathcal{M}_{n,1}(\mathbb{C}). \tag{10}$$

Suppose that (10) is controllable. Then, it is well known (see for example [13]) that for every polynomial $P \in \mathbb{C}[X]$ there exists a feedback $K \in \mathcal{M}_{1,n}(\mathbb{R})$ such that P is the characteristic polynomial of $A + BK$.

This pole-shifting property for controllable systems can be formulated in another way, by trying to invertibly transform system (10) into another system with shifted poles, namely

$$\dot{x} = (A - \lambda I)x + Bv(t), \tag{11}$$

which is asymptotically stable for a large enough λ.

Suppose that $x(t)$ is a solution of system (10) with $u(t) = Kx(t) + v(t)$ for some control function v. Such a transformation T would map (10) into

$$\dot{(Tx)} = T\dot{x} = T(A + BK)x + TBv(t).$$

In order for Tx to be a solution of (11), we need

$$T(A + BK)x + TBv(t) = (A - \lambda I)Tx + Bv(t),$$

hence the conditions

$$\begin{aligned} T(A + BK) &= AT - \lambda T, \\ TB &= B, \end{aligned} \tag{12}$$

for which one has the following theorem (see for example [14], or [7] for a different proof, more adaptable to the context of PDEs):

Theorem 1.2. *There exists a unique pair* (T, K) *satisfying conditions* (12).

The controllability of (10) is crucial here, as it allows to build a basis for the space state, in which T can then be constructed. The Hautus test gives the invertibility of T, and the uniqueness is given by the $TB = B$ condition.

This other approach to pole-shifting, which links controllability to stabilization, can be used in infinite dimensions. In our case, the controllability of (2) will have the same importance: it will also allow us to build some sort of basis for the state space, and find a general form for the backstepping transformation, depending on F.

1.4 Related results

To investigate the stabilization of infinite-dimensional systems, there are three main types of approaches.

The first type of approach relies on abstract methods, such as the Gramian approach and the Riccati equations (see for example [29, 28, 20]). Although quite powerful, it seems that these methods fail to obtain the stabilization of nonlinear systems from the stabilization of their linearized systems.

The second approach relies on Lyapunov functions. Many results on the boundary stabilization of first-order hyperbolic systems, linear and nonlinear, have been obtained using this approach: see for example the book [2], and the recent results in [16, 17]. However, this approach can be limited, as it is sometimes impossible to obtain an arbitrary decay rate using Lyapunov functions (see [13, Remark 12.9, page 318] for a finite dimensional example).

The third approach, which we will be using in this article, is the backstepping method. This name originally refers to a way of designing, in a recursive way, more effective feedback laws, for systems for which one already has a Lyapunov function and a feedback law which globally asymptotically stabilizes the system, see [13, 26] for an overview of the finite-dimensional case, and [6] or [22] for applications to partial differential equations. Another way of applying this approach to partial differential equations was then developed in [3] and [1]: when applied to the discretization of the heat equation, the backstepping approach yielded a change of coordinates which was equivalent to a Volterra transform of the second kind. Backstepping then took yet another successful form, consisting in mapping the system to stable target system, using a Volterra transformation of the second kind (see [21] for a comprehensive introduction to the method):

$$f(t, x) \mapsto f(t, x) - \int_0^x k(x, y) f(t, y) dy.$$

This was used to prove a host of results on the boundary stabilization of partial differential equations: let us cite for example [19] and [25] for the wave equation, [31, 32] for the Korteweg-de Vries, [2, chapter 7] for an application to first-order hyperbolic systems, and also [15], which combines the backstepping method with Lyapunov functions to prove finite-time stabilization in H^2 for a quasilinear 2×2 hyperbolic system.

In some cases, the method was used to obtained stabilization with an internal feedback. This was done in [27] and [30] for parabolic systems, and [33] for first-order hyperbolic systems. The strategy in these works is to first apply a Volterra transformation as usual, which still leaves an unstable source term in the target, and then apply a second invertible transformation to reach a stable target system. Let us note that in the latter reference, the authors study a linear transport equation and get finite-time stabilization. However, their controller takes a different form than ours, and several hypotheses are made on the space component of the controller so that a Volterra transform can be successfully applied to the system. This is in contrast with the method in this article, where the assumption we make on the controller corresponds to the exact null-controllability of the system.

In this paper, we use another application of the backstepping method, which uses another type of linear transformations, namely, Fredholm transformations:

$$f(t, x) \mapsto \int_0^L k(x, y) f(t, y) dy.$$

These are more general than Volterra transformations, but they require more work: indeed, Volterra transformations are always invertible, but the invertibility of a Fredholm transform is harder to check. Even though it is sometimes more involved and technical, the use of a Fredholm transformation proves more effective for certain types of control: for example, in [11] for the Korteweg-de Vries equation and [10] for a Kuramoto-Sivashinsky, the position of the control makes it more appropriate to use a Fredholm transformation. Other boundary stabilization results using a Fredholm transformation can be found in [8] for integro-differential hyperbolic systems, and in [9] for general hyperbolic balance laws. Fredholm transformations have also been used in [7], where the authors prove the rapid stabilization of the Schrödinger equation with an internal feedback.

The backstepping method has the advantage of providing explicit feedback laws, which makes it a powerful tool to prove other related results, such as null-controllability or small-time stabilization (stabilization in an arbitrarily small time). This is done in [12], where the authors give an explicit control to bring a heat equation to 0, then a time-varying, periodic feedback to stabilize the equation in small time. In [32], the author obtains the same kind of results for the Korteweg-de Vries equation.

1.5 Structure of the article

The structure of this article is as follows: in Section 2, after some formal calculations, and using a formal $TB = B$ condition, we build candidates for the backstepping transformation. Using the properties of Riesz bases, we prove that such candidates are indeed invertible, under some conditions on the feedback coefficients (F_n). For consistency, we then determine the feedback law (F_n) such that the corresponding transformation indeed satisfies a weak form of the $TB = B$ condition. Then, in Section 3, we check that the corresponding transformation indeed satisfies an operator equality analogous to (12), making it a valid backstepping transformation. We check the well-posedness of the closed-loop system for the feedback law obtained in Section 2, which allows us to prove the stability result. Finally, Section 4 gives a few remarks on the result, as well as further questions on this stabilization problem.

2 Definition and properties of the transformation

Let $\lambda' > 0$, and $m \geq 1$. Let $\varphi \in H^m \cap H^{m-1}_{per}$ be a real-valued function satisfying (7). We consider the following target system:

$$\begin{cases} z_t + z_x + \lambda' z = 0, \quad x \in (0, L), \\ \qquad z(t,0) = z(t,L), \quad t \geq 0. \end{cases} \tag{13}$$

Then it is well-known that, taking $\alpha_0 \in L^2$, the solution to (13) with initial condition α_0 writes

$$z(t,x) = e^{-\lambda' t}\alpha_0(x-t), \quad \forall (t,x) \in \mathbb{R}^+ \times (0,L).$$

Hence,

Proposition 2.1. *For all $s \geq 0$, the system* (13) *is exponentially stable for $\|\cdot\|_s$, for initial conditions in H^s_{per}.*

2.1 Kernel equations

As mentioned in the introduction, we want to build backstepping transformations T as a kernel operator of the Fredholm type:

$$f(t,x) \mapsto \int_0^L k(x,y) f(t,y) dy.$$

To have an idea of what this kernel looks like, we can do the following formal computation for some Fredholm operator T: first the boundary conditions

$$\left(\int_0^L k(0,y)\alpha(y)dy \right) = \left(\int_0^L k(L,y)\alpha(y)dy \right),$$

then the equation of the target system, for $x \in [0, L]$:

$$\begin{aligned}
0 &= \left(\int_0^L k(x,y)\alpha(y)dy\right)_t + \left(\int_0^L k(x,y)\alpha(y)dy\right)_x + \lambda' \left(\int_0^L k(x,y)\alpha(y)dy\right) \\
&= \left(\int_0^L k(x,y)\alpha_t(y)dy\right) + \left(\int_0^L k_x(x,y)\alpha(y)dy\right) + \lambda' \left(\int_0^L k(x,y)\alpha(y)dy\right) \\
&= \left(\int_0^L k(x,y)(-\alpha_x(y) - \mu\alpha(y) + \langle \alpha, F\rangle \varphi(y))dy\right) + \left(\int_0^L (k_x(x,y) + \lambda' k(x,y))\alpha(y)dy\right) \\
&= \left(\int_0^L k_y(x,y)\alpha(y)dy\right) - (k(x,L)\alpha(L) - k(x,0)\alpha(0)) + \left(\int_0^L k(x,y)\langle \alpha, F\rangle \varphi(y))dy\right) + \\
&\quad \left(\int_0^L (k_x(x,y) + (\lambda' - \mu)k(x,y))\alpha(y)dy\right) \\
&= \left(\int_0^L k(x,y)\left(\int_0^L \bar{F}(s)\alpha(s)ds\right)\varphi(y))dy\right) - (k(x,L)\alpha(L) - k(x,0)\alpha(0)) \\
&\quad + \left(\int_0^L (k_y(x,y) + k_x(x,y) + (\lambda' - \mu)k(x,y))\alpha(y)dy\right) \\
&= \left(\int_0^L \bar{F}(s)\left(\int_0^L k(x,y)\varphi(y)dy\right)\alpha(s)ds\right) - (k(x,L)\alpha(L) - k(x,0)\alpha(0)) \\
&\quad + \left(\int_0^L (k_y(x,y) + k_x(x,y) + (\lambda' - \mu)k(x,y))\alpha(y)dy\right).
\end{aligned}$$

Now, suppose we have the formal $TB = B$ condition

$$\int_0^L k(x,y)\varphi(y)dy = \varphi(x), \quad \forall x \in [0, L].$$

Then, we get, noting $\lambda := \lambda' - \mu$,

$$\left(\int_0^L \big(k_y(x,y) + k_x(x,y) + \lambda k(x,y) + \varphi(x)\bar{F}(y)\big)\alpha(y)dy\right) - (k(x,L)\alpha(L) - k(x,0)\alpha(0)) = 0.$$

Hence the kernel equation:

$$\begin{cases} k_x + k_y + \lambda k = -\varphi(x)\bar{F}(y), \\ \qquad\quad k(0,y) = k(L,y), \\ \qquad\quad k(x,0) = k(x,L), \end{cases} \tag{14}$$

together with the $TB = B$ condition

$$\langle k(x,\cdot), \varphi(\cdot)\rangle = \varphi(x), \quad \forall x \in [0, L]. \tag{15}$$

2.2 Construction of Riesz bases for Sobolev spaces

To study the solution to the kernel equation, we project it along the variable y. Let us write heuristically

$$k(x,y)=\sum_{n\in\mathbb{Z}}k_n(x)e_n(y),$$

so that we get the projected kernel equations

$$k_n'+\lambda_n k_n=-\overline{F_{-n}}\varphi, \tag{16}$$

where

$$\lambda_n=\lambda+\frac{2i\pi}{L}n. \tag{17}$$

Note that

$$\frac{2i\pi p}{L}\frac{1}{\lambda_{n+p}}+\lambda_n\frac{1}{\lambda_{n+p}}=1,\quad \forall n,p\in\mathbb{Z}. \tag{18}$$

Now consider the L^2 function given by

$$\Lambda_n^\lambda(x)=\frac{\sqrt{L}}{1-e^{-\lambda L}}e^{-\lambda_n x},\quad \forall n\in\mathbb{Z},\quad \forall x\in[0,L). \tag{19}$$

Then, for all $m\geq 0$, $\Lambda_n^\lambda\in H^m$, and we have

$$\langle\Lambda_n^\lambda,e_p\rangle=\frac{1}{\sqrt{L}}\int_0^L\frac{\sqrt{L}}{1-e^{-\lambda L}}e^{-\lambda_n x}e^{-\frac{2i\pi p}{L}x}dx=\frac{1}{1-e^{-\lambda L}}\int_0^L e^{-\lambda_{n+p}x}=\frac{1}{\lambda_{n+p}},\quad \forall n,p\in\mathbb{Z},$$

so that, using (18),

$$(\Lambda_n^\lambda)'+\lambda_n\Lambda_n^\lambda=\sum_{p\in\mathbb{Z}}e_p \text{ in } \mathcal{E}'.$$

Remark 2.1. *In $\mathcal{E}'$, $\sum_{p\in\mathbb{Z}}e_p$ is the equivalent of the Dirac comb, or the "Dirac distribution" on the space of functions on $[0,L]$. So, in a sense, Λ_n^λ is the elementary solution of* (16).

Let us now define, in analogy with the elementary solution method,

$$k_{n,\lambda}=-\overline{F_{-n}}\Lambda_n^\lambda\star\varphi\in H^m_{per},\quad \forall n\in\mathbb{Z}. \tag{20}$$

The regularity comes from the definition of the convolution product, (7) and (17), and one can check, using (18), that $k_{n,\lambda}$ is a solution of (16).

The next step to build an invertible transformation is to find conditions under which $(k_{n,\lambda})$ is some sort of basis. More precisely we use the notion of Riesz basis (see [5, Chapter 4])

Definition 2.1. *A Riesz basis in a Hilbert space H is the image of an orthonormal basis of H by an isomorphism.*

Proposition 2.2. *Let H be a Hilbert space. A family of vectors $(f_k)_{k\in\mathbb{N}}\in H$ is a Riesz basis if and only if it is complete (i.e., $\overline{\mathrm{Span}(f_k)}=H$) and there exists constants $C_1,C_2>0$ such that, for any scalar sequence (a_k) with finite support,*

$$C_1\sum|a_k|^2\leq\left\|\sum a_kf_k\right\|_H^2\leq C_2\sum|a_k|^2. \tag{21}$$

Let us now introduce the following growth condition:

Definition 2.2. *Let $s \geq 0$, $(u_n) \in \mathbb{C}^{\mathbb{Z}}$ (or $u \in \mathcal{E}'$) . We say that (u_n) (or u) has s-growth if*

$$c\sqrt{1+\left|\frac{2i\pi n}{L}\right|^{2s}} \leq |u_n| \leq C\sqrt{1+\left|\frac{2i\pi n}{L}\right|^{2s}}, \quad \forall n \in \mathbb{Z}, \tag{22}$$

for some $c, C > 0$. The optimal constants for these inequalities are called growth constants.

Remark 2.2. *The inequalities* (22) *can also be written, more intuitively, and for some other positive constants,*

$$c\left(1+|n|^s\right) \leq |u_n| \leq C\left(1+|n|^s\right), \quad \forall n \in \mathbb{Z}. \tag{23}$$

We can now establish the following Riesz basis properties for the $(k_{n,\lambda})$:

Proposition 2.3. *Let $s \geq 0$. If (F_n) has s-growth, then the family of functions*

$$(k^s_{n,\lambda}) := \left(\frac{k_{n,\lambda}}{\sqrt{1+\left|\frac{2i\pi n}{L}\right|^{2s}}}\right)$$

is a Riesz basis for H^m_{per}.

Proof. We use the characterization of Riesz bases given in Proposition 2.2. First, let us prove the completeness of $(k^s_{n,\lambda})$. Let $f \in H^m_{per}$ be such that

$$\langle f, k^s_{n,\lambda}\rangle_m = 0, \quad \forall n \in \mathbb{Z}.$$

Then for all $n \in \mathbb{Z}$ we get

$$0 = \langle \Lambda^\lambda_n \star \varphi, f\rangle_m = \sum_{p\in\mathbb{Z}} \frac{\overline{f_p}\varphi_p}{\lambda_{n+p}}\left(1+\left|\frac{2i\pi p}{L}\right|^{2m}\right) = \left\langle \Lambda^\lambda_n, \sum_{p\in Z}\left(1+\left|\frac{2i\pi p}{L}\right|^{2m}\right) f_p\overline{\varphi_p}e_p\right\rangle,$$

as, thanks to (7), and using the fact that $f \in H^m_{per}$,

$$\sum_{p\in Z}\left(1+\left|\frac{2i\pi p}{L}\right|^{2m}\right) f_p\overline{\varphi_p}e_p \in L^2.$$

Now, (Λ^λ_n) is a complete family of L^2, as it is a Riesz basis, so that

$$f_p\varphi_p = 0, \quad \forall p \in \mathbb{Z}.$$

Recalling condition (7), this yields

$$f_p = 0, \quad \forall p \in \mathbb{Z},$$

which proves the completeness of $(k^s_{n,\lambda})$.

Now let $I \subset \mathbb{Z}$ be a finite set, and $(a_n) \in \mathbb{C}^I$. Then,

$$\begin{aligned}
\left\|\sum_{n\in I} a_n k^s_{n,\lambda}\right\|^2_m &= \left\|\sum_{n\in I} -a_n\frac{\overline{F_{-n}}}{\sqrt{1+\left|\frac{2i\pi n}{L}\right|^{2s}}}\Lambda^\lambda_n \star \varphi\right\|^2_m \\
&= \left\|\sum_{n\in I} a_n\frac{\overline{F_{-n}}}{\sqrt{1+\left|\frac{2i\pi n}{L}\right|^{2s}}}\sum_{p\in\mathbb{Z}}\frac{\varphi_p}{\lambda_{n+p}}e_p\right\|^2_m \\
&= \left\|\sum_{p\in\mathbb{Z}} \varphi_p\sum_{n\in I}\frac{a_n\overline{F_{-n}}}{\lambda_{n+p}\sqrt{1+\left|\frac{2i\pi n}{L}\right|^{2s}}}e_p\right\|^2_m \\
&= \sum_{p\in\mathbb{Z}}\left(1+\left|\frac{2\pi p}{L}\right|^{2m}\right)|\varphi_p|^2\left|\sum_{n\in I}\frac{a_n\overline{F_{-n}}}{\lambda_{n+p}\sqrt{1+\left|\frac{2i\pi n}{L}\right|^{2s}}}\right|^2.
\end{aligned}$$

Now, using condition (7), we have

$$c^2 \sum_{p\in\mathbb{Z}} \left| \sum_{n\in I} \frac{a_n \overline{F_{-n}}}{\lambda_{n+p}\sqrt{1+\left|\frac{2i\pi n}{L}\right|^{2s}}} \right|^2 \leq \left\| \sum_{n\in I} a_n k^s_{n,\lambda} \right\|_m^2 \leq C^2 \sum_{p\in\mathbb{Z}} \left| \sum_{n\in I} \frac{a_n \overline{F_{-n}}}{\lambda_{n+p}\sqrt{1+\left|\frac{2i\pi n}{L}\right|^{2s}}} \right|^2,$$

where $c, C > 0$ are the decay constants in condition (7).

This last inequality can be rewritten

$$c^2 \left\| \sum_{n\in I} \frac{a_n \overline{F_{-n}}}{\sqrt{1+\left|\frac{2i\pi n}{L}\right|^{2s}}} \Lambda^\lambda_n \right\|^2 \leq \left\| \sum_{n\in I} a_n k^s_{n,\lambda} \right\|_m^2 \leq C^2 \left\| \sum_{n\in I} \frac{a_n \overline{F_{-n}}}{\sqrt{1+\left|\frac{2i\pi n}{L}\right|^{2s}}} \Lambda^\lambda_n \right\|^2,$$

as

$$\Lambda^\lambda_n = \sum_{p\in\mathbb{Z}} \frac{1}{\lambda_{n+p}} e_p.$$

We now use the fact that (Λ^λ_n) is a Riesz basis of L^2: indeed, it is the image of the Hilbert basis (e_n) by the isomorphism

$$\Lambda^\lambda : f \in L^2 \mapsto \frac{\sqrt{L}}{1-e^{-\lambda L}} e^{-\lambda \cdot} f.$$

The norms of Λ^λ and its inverse are rather straightforward to compute using piecewise constant functions, we have

$$|||\Lambda^\lambda||| = \frac{\sqrt{L}}{1-e^{-\lambda L}},$$
$$|||(\Lambda^\lambda)^{-1}||| = \frac{1-e^{-\lambda L}}{\sqrt{L}} e^{\lambda L},$$

so that

$$\frac{1}{|||(\Lambda^\lambda)^{-1}|||^2} \sum_{n\in I} \left| \frac{a_n \overline{F_{-n}}}{\sqrt{1+\left|\frac{2i\pi n}{L}\right|^{2s}}} \right|^2 \leq \left\| \sum_{n\in I} \frac{a_n \overline{F_{-n}}}{\sqrt{1+\left|\frac{2i\pi n}{L}\right|^{2s}}} \Lambda^\lambda_n \right\|^2 \leq |||\Lambda^\lambda|||^2 \sum_{n\in I} \left| \frac{a_n \overline{F_{-n}}}{\sqrt{1+\left|\frac{2i\pi n}{L}\right|^{2s}}} \right|^2,$$

and we finally get, using the fact that (F_n) has s-growth,

$$c^2 C_1^2 \frac{1}{|||(\Lambda^\lambda)^{-1}|||^2} \sum_{n\in I} |a_n|^2 \leq \left\| \sum_{n\in I} a_n k^s_{n,\lambda} \right\|_m^2 \leq C^2 C_2^2 |||\Lambda^\lambda|||^2 \sum_{n\in I} |a_n|^2.$$

where $C_1, C_2 > 0$ are the growth constants of (F_n), so that the constants in the inequalities above are optimal. Hence, using again point 2. of Proposition 2.2, $(k^s_{n,\lambda})$ is a Riesz basis of H^m_{per}. ☐

We now have candidates for the backstepping transformation, under some conditions on F:

Corollary 2.1. *Let $m \in \mathbb{N}^*$, and F such that (F_n) has m-growth, with growth constants $C_1, C_2 > 0$. Define*

$$T^\lambda \alpha := \sum_{n\in\mathbb{Z}} \sqrt{1+\left|\frac{2i\pi n}{L}\right|^{2m}} \alpha_n k^m_{-n,\lambda} \in H^m_{per}, \quad \forall \alpha \in H^m_{per}, \tag{24}$$

where $\alpha = \sum_{n\in\mathbb{Z}} \alpha_n e_n$. Then, $T^\lambda : H^m_{per} \to H^m_{per}$ is an isomorphism. Moreover,

$$|||T^\lambda||| = \frac{C C_2 \sqrt{L}}{1-e^{-\lambda L}},$$
$$|||(T^\lambda)^{-1}||| = \frac{1-e^{-\lambda L}}{c C_1 \sqrt{L}} e^{\lambda L}. \tag{25}$$

Proof. The invertibility of T^λ is clear thanks to the Riesz basis property of $(k^m_{-n,\lambda})$, and (25) comes from the fact that, as mentioned at the end of the proof of Proposition 2.3, all the constants in the inequalities are optimal. $\square$

2.3 Definition of the feedback law

In order to further determine the feedback law, and define our final candidate for the backstepping transformation, the idea is now to return to the $TB = B$ condition (15), as we have used it in the formal computations of section 2.1, in the equation (15). However, in this case, $\varphi \notin H^m_{per}$, and so it is not clear whether $T^\lambda \varphi$ is well-defined.

We can nonetheless obtain a $TB = B$ condition in some weak sense: indeed, let us set

$$\varphi^{(N)} := \sum_{n=-N}^{N} \varphi_n e_n \in H^m_{per}, \quad \forall N \in \mathbb{N}.$$

Then,

$$\varphi^{(N)} \xrightarrow[N\to\infty]{H^{m-1}} \varphi$$

and

$$\begin{aligned} T^\lambda \varphi^{(N)} &= \sum_{n=-N}^{N} -\varphi_n \overline{F_n} \Lambda^\lambda_{-n} \star \varphi \\ &= \sum_{n=-N}^{N} \sum_{p\in\mathbb{Z}} \frac{-\varphi_n \overline{F_n} \varphi_p}{\lambda_{-n+p}} e_p \\ &= \sum_{p\in\mathbb{Z}} \varphi_p \left(\sum_{n=-N}^{N} \frac{-\varphi_n \overline{F_n}}{\lambda_{-n+p}} \right) e_p. \end{aligned}$$

Now, notice that one can apply the Dirichlet convergence theorem for Fourier series (see for example [18]) to $\Lambda^\lambda_p, p \in \mathbb{Z}$ at 0:

$$\sum_{n=-N}^{N} \frac{1}{\lambda_{-n+p}} = \sum_{n=-N}^{N} \frac{1}{\lambda_{n+p}} \xrightarrow[N\to\infty]{} \frac{\Lambda^\lambda_p(0) + \Lambda^\lambda_p(L)}{2} = \frac{\sqrt{L}}{1-e^{-\lambda L}} \frac{1+e^{-\lambda L}}{2}.$$

Let us note

$$K(\lambda) := \frac{2}{\sqrt{L}} \frac{1-e^{-\lambda L}}{1+e^{-\lambda L}},$$

and set

$$F_n := -\frac{K(\lambda)}{\overline{\varphi_n}}, \quad \forall n \in \mathbb{Z}. \tag{26}$$

This defines a feedback law $F \in \mathcal{E}'$ which is real-valued, as φ is real-valued, and which has m-growth thanks to condition (7), so that T^λ is a valid backstepping transformation. Moreover,

$$\langle T^\lambda \varphi^{(N)}, e_p \rangle = \varphi_p K(\lambda) \sum_{n=-N}^{N} \frac{1}{\lambda_{-n+p}} \xrightarrow[N\to\infty]{} \varphi_p, \quad \forall p \in \mathbb{Z}, \tag{27}$$

which corresponds to the $TB = B$ condition in some weak sense.

With this feedback law, the backstepping transformation now writes

$$T^\lambda \alpha = \sum_{n\in\mathbb{Z}} \alpha_n k_{-n,\lambda}, \quad \forall \alpha \in H^m_{per}, \tag{28}$$

and

$$\begin{aligned} |||T^{\lambda}||| &= \frac{CK(\lambda)\sqrt{L}}{c(1-e^{-\lambda L})}, \\ |||(T^{\lambda})^{-1}||| &= \frac{C(1-e^{-\lambda L})}{cK(\lambda)\sqrt{L}} e^{\lambda L}. \end{aligned} \tag{29}$$

2.4 Regularity of the feedback law

Finally, in order to study the well-posedness of the closed-loop system corresponding to (26), we need some information on the regularity of F.

Let us first begin by a general lemma for linear forms with coefficients that have m-growth:

Lemma 2.1. *Let $m \geq 0$, and $G \in \mathcal{E}'$ with m-growth.*

Then, for all $s > 1/2$, G is defined on H^{m+s}_{per}, is continuous for $\|\cdot\|_{m+s}$, but not for $\|\cdot\|_{m+\sigma}$, for $-m \leq \sigma < 1/2$.

In particular, the feedback law $F \in \mathcal{E}'$ defined by (26) defines a linear form on H^{m+1}_{per} which is continuous for $\|\cdot\|_{m+1}$ but not for $\|\cdot\|_{m}$.

Proof. Let $s > 1/2$, and let $\alpha \in H^{m+s}_{per}$. Using the growth conditions (22), we can do the following computations for $\alpha \in H^{m+s}_{per}$:

$$\begin{aligned} \sum_{n\in\mathbb{Z}} |G_n||\alpha_n| &\leq C \sum_{n\in\mathbb{Z}} \sqrt{1+\left|\frac{2i\pi n}{L}\right|^{2m}} |\alpha_n| \\ &\leq C' \sum_{n\in\mathbb{Z}} \frac{1}{1+|n|^s} \sqrt{1+\left|\frac{2i\pi n}{L}\right|^{2m+2s}} |\alpha_n| \\ &\leq C' \left(\sqrt{\sum_{n\in\mathbb{Z}} \frac{1}{(1+|n|^s)^2}}\right) \|\alpha\|_{m+s} \end{aligned}$$

where $C, C' > 0$ are constants that do not depend on α, and where the last inequality is obtained using the Cauchy-Schwarz inequality. Thus G is defined on H^{m+s}_{per} by

$$\langle \alpha, G\rangle := \sum_{n\in\mathbb{Z}} G_n \alpha_n, \quad \forall \alpha \in H^{m+s}_{per},$$

and G is continuous on H^{m+s}_{per}.

On the other hand, let $-m \leq \sigma < 1/2$, and consider, for $N \geq 1$,

$$\gamma^{(N)} := \sum_{|n|\geq N} \frac{1}{\overline{G_n}\,(1+|n|^{1+s})} e_n \in H^{m+s}_{per}.$$

We have

$$\|\gamma^{(N)}\|^2_{m+\sigma} = \sum_{|n|\geq N} \frac{\left(1+\left|\frac{2i\pi n}{L}\right|^{2m+2\sigma}\right)}{|G_n|^2} \frac{1}{(1+|n|^{1+s})^2} \leq C \sum_{|n|\geq N} \frac{1}{1+|n|^{2+2s-2\sigma}}$$

for some constant $C > 0$. Then,

$$\begin{aligned}
|\langle \gamma^{(N)}, G\rangle| &= \sum_{|n|\geq N} \frac{1}{1+|n|^{1+s}} \\
&\geq c \sum_{|n|\geq N} |n|^{1+s-2\sigma} \frac{1}{1+|n|^{2+2s-2\sigma}} \\
&\geq cN^{1+s-2\sigma} \sum_{|n|\geq N} \frac{1}{1+|n|^{2+2s-2\sigma}} \\
&\geq c' N^{1+s-2\sigma} \sqrt{\sum_{|n|\geq N} \frac{1}{1+|n|^{2+2s-2\sigma}}} \|\gamma^{(N)}\|_{m+\sigma}
\end{aligned}$$

for some constants $c, c' > 0$. Now, we know that there exists constants $c'', C'' > 0$ such that

$$\frac{c''}{N^{1+2s-2\sigma}} \leq \sum_{|n|\geq N} \frac{1}{1+|n|^{2+2s-2\sigma}} \leq \frac{C''}{N^{1+2s-2\sigma}},$$

So that

$$N^{1+s-2\sigma} \sqrt{\sum_{|n|\geq N} \frac{1}{1+|n|^{2+2s-2\sigma}}} \geq c'' N^{\frac{1}{2}-\sigma} \xrightarrow[N\to\infty]{} \infty.$$

This proves that G is not continuous for$\|\cdot\|^{m+\sigma}$. □

Let us now give a more precise description of the domain of definition and regularity of F. Recalling the identity (8), we can derive the following identity for F_n:

$$F_n = \frac{K(\lambda)}{\tau_n^\varphi}\left(\frac{2i\pi n}{L}\right)^m - \frac{K(\lambda)}{\tau_n^\varphi}\left(\frac{2i\pi n}{L}\right)^m \frac{\overline{\sum_{j=0}^{d}\left\langle \chi_{[\sigma_j,\sigma_{j+1}]}\partial^m\varphi, e_n\right\rangle}}{\overline{\tau_n^\varphi - \sum_{j=0}^{d}\left\langle \chi_{[\sigma_j,\sigma_{j+1}]}\partial^m\varphi, e_n\right\rangle}}, \quad \forall n \in \mathbb{Z}^*, \tag{30}$$

so that

$$\left(\frac{1}{\left(\frac{2i\pi n}{L}\right)^m}\left(F_n - \frac{K(\lambda)}{\tau_n^\varphi}\left(\frac{2i\pi n}{L}\right)^m\right)\right)_{n\in\mathbb{Z}^*} \in \ell^2. \tag{31}$$

Let us then note

$$h_n := \frac{K(\lambda)}{\tau_n^\varphi}\left(\frac{2i\pi n}{L}\right)^m, \quad \forall n \in \mathbb{Z},$$

and h the associated linear form in $\mathcal{E}'$.

Proposition 2.4. *The linear form h defines the following linear form on $\tau^\varphi(H^{m+1}_{(pw)})$, continuous for $\|\cdot\|_{m+1,pw}$:*

$$\langle \alpha, h\rangle = (-1)^m\sqrt{L}\frac{K(\lambda)}{2}\partial^m\left((\tau^\varphi)^{-1}\alpha\right)(0) + \partial^m\left((\tau^\varphi)^{-1}\alpha\right)(L), \quad \forall \alpha \in H^{m+1}. \tag{32}$$

Moreover, $\tilde{F} := F - h$ is continuous for $\|\cdot\|_m$, so that F is defined on $\tau^\varphi(H^{m+1}_{(pw)}) \cap H^m_{per}$, and is continuous for $\|\cdot\|_{m+1,pw}$, but not $\|\cdot\|_m$.

Proof. It is clear, by definition of H^m_{per}, and using (31), that for $\alpha \in H^m_{per}$, the expression

$$\langle \alpha, F - h\rangle = \sum_{n\in\mathbb{Z}} \alpha_n(\overline{F_n} - \overline{h_n}) = \sum_{n\neq 0} \left(\frac{2i\pi n}{L}\right)^m \alpha_n \frac{1}{\left(\frac{2i\pi n}{L}\right)^m}(\overline{F_n} + \overline{h_n}) + \frac{K(\lambda)\alpha_0}{\varphi_0}$$

defines a continuous linear form on H^m_{per}.

On the other hand, let $\alpha \in \tau^\varphi(H^{m+2}_{(pw)})$, then

$$\sum_{n=-N}^{N} \alpha_n \overline{h_n} = (-1)^m \sqrt{L} K(\lambda) \sum_{n=-N}^{N} \left(\frac{2i\pi n}{L}\right)^m \frac{\alpha_n}{\tau^\varphi_n} \frac{1}{\sqrt{L}}$$

we can use the Dirichlet convergence theorem (see [18]) on $\partial^m\left((\tau^\varphi)^{-1}\alpha\right) \in H^2_{(pw)}$ at 0, so that

$$\begin{aligned}\sum_{n=-N}^{N} \alpha_n \overline{h_n} &= (-1)^m \sqrt{L} K(\lambda) \sum_{n=-N}^{N} \left(\frac{2i\pi n}{L}\right)^m \frac{\alpha_n}{\tau^\varphi_n} \frac{1}{\sqrt{L}} \\ &\xrightarrow[N\to\infty]{} (-1)^m \sqrt{L} \frac{K(\lambda)}{2}\left(\partial^m\left((\tau^\varphi)^{-1}\alpha\right)(0) + \partial^m\left((\tau^\varphi)^{-1}\alpha\right)(L)\right).\end{aligned}$$

Now, we know that $H^{m+2}_{(pw)}$ is dense in $H^{m+1}_{(pw)}$ for the $H^{m+1}_{(pw)}$ norm. As τ^φ is a sum of translations, it is continuous for $\|\cdot\|_{m+1,pw}$, so that $\tau^\varphi(H^{m+2}_{(pw)})$ is dense in $\tau^\varphi(H^{m+1}_{(pw)})$ for $\|\cdot\|_{m+1,pw}$.

Moreover, using the Sobolev inequality for H^1 and L^∞ (see for example [4, Chapter 8, Theorem 8.8]), we get the continuity of h for $\|\cdot\|_{m+1,pw}$, so that we can extend it from $\tau^\varphi(H^{m+2}_{(pw)})$ to $\tau^\varphi(H^{m+1}_{(pw)})$ by density. We also get that h is not continuous for $\|\cdot\|_m$, as $\alpha \in H^m \mapsto \partial^m\alpha(0)$ and $\alpha \in H^m \mapsto \partial^m\alpha(L)$ are not continuous for $\|\cdot\|_m$.

Thus, $F = \tilde{F} + h$ is defined on $\tau^\varphi(H^{m+1}_{(pw)}) \cap H^m_{per}$, is continuous for $\|\cdot\|_{m+1}$ but not for $\|\cdot\|_m$. $\square$

3 Well-posedness and stability of the closed-loop system

Let $m \geq 1$, $\varphi \in H^m_{(pw)} \cap H^{m-1}_{per}$ satisfying growth condition (7). Let the feedback law F be defined by (26).

3.1 Operator equality

Now that we have completely defined the feedback F and the transformation T^λ, let us check that we have indeed built a backstepping tranformation. As in the finite dimensional example of subsection 1.3, this corresponds to the formal operator equality

$$T(A + BK) = (A - \lambda I)T.$$

Let us define the following domain:

$$D_m := \left\{\alpha \in \tau^\varphi(H^{m+1}_{(pw)}) \cap H^m_{per}, \quad -\alpha_x - \mu\alpha + \langle \alpha, F\rangle\varphi \in H^m_{per}\right\}. \tag{33}$$

Notice that, as $\varphi \in H^m_{(pw)}$, the condition $\alpha \in H^{m+1}_{(pw)} \supset \tau^\varphi(H^{m+1}_{(pw)})$ is necessary for $-\alpha_x - \mu\alpha + \langle \alpha, F\rangle\varphi$ to be in H^m_{per} . Let us first check the following property:

Proposition 3.1. *For $m \geq 1$, D_m is dense in H^m_{per} for $\|\cdot\|_m$.*

Proof. It is clear that $H^{m+1}_{per} \subset \tau^\varphi\left(H^{m+1}_{(pw)}\right)$, so that

$$\mathcal{K}_m := \left\{\alpha \in H^{m+1}_{per},\ \langle \alpha, F\rangle = 0\right\} \subset D_m.$$

Now, by Lemma 2.1, as F has m-growth, $\mathcal{K}_m$ is dense in H^{m+1}_{per} for $\|\cdot\|_m$, as the kernel of the linear form F which is not continuous for $\|\cdot\|_m$. As H^{m+1}_{per} is dense in H^m_{per}, then D_m is dense in H^m_{per} for $\|\cdot\|_m$. $\square$

Now, on this dense domain, let us establish the operator equality:

Proposition 3.2.

$$T^\lambda(-\partial_x - \mu I + \langle \cdot, F\rangle\varphi)\alpha = (-\partial_x - \lambda' I)T^\lambda\alpha \quad \textit{in } H^m_{per}, \quad \forall \alpha \in D_m. \tag{34}$$

Proof. First let us rewrite (34) in terms of λ:

$$T^\lambda(-\partial_x + \langle \cdot, F\rangle\varphi)\alpha = (-\partial_x - \lambda I)T^\lambda\alpha \quad \text{in } H^m_{per}, \quad \forall \alpha \in D_m(F).$$

Let $\alpha \in D_m$. By definition of the domain D_m, the left-hand side of (34) is a function of $H^m_{per} \subset \mathcal{E}'$, and by construction of T^λ, the right-hand side of (34) is a function of $H^{m-1}_{per} \subset \mathcal{E}'$. To prove that these functions are equal, it is thus sufficient to prove their equality in $\mathcal{E}'$. Let us then write each term of the equality against e_n for $n \in \mathbb{Z}$. One has

$$\begin{aligned}
\left\langle(-\partial_x - \lambda I)T^\lambda\alpha, e_n\right\rangle &= \left\langle T^\lambda\alpha, \frac{2i\pi n}{L}e_n\right\rangle - \lambda\left\langle T^\lambda\alpha, e_n\right\rangle \\
&= -\lambda_n\langle T^\lambda\alpha, e_n\rangle.
\end{aligned}$$

Let us now prove that

$$\langle T^\lambda(-\alpha_x + \langle\alpha, F\rangle\varphi), e_n\rangle = -\lambda_n\langle T^\lambda\alpha, e_n\rangle, \quad \forall n \in \mathbb{Z}. \tag{35}$$

Now, as we only have $\alpha_x \in H^{m-1}_{per}$, $T^\lambda\alpha_x$ is not defined *a priori*. In order to allow for more computations, let us define

$$\begin{aligned}
\alpha^{(N)} &:= \sum_{n=-N}^{N} \alpha_n e_n, \quad \forall N \in \mathbb{N}, \\
\varphi^{(N)} &:= \sum_{n=-N}^{N} \varphi_n e_n,
\end{aligned}$$

so that we have, by property of the partial Fourier sum of a H^m_{per} function,

$$-\alpha^{(N)}_x + \langle\alpha, F\rangle\varphi^{(N)} \xrightarrow[N\to\infty]{H^m} -\alpha_x + \langle\alpha, F\rangle\varphi,$$

so that in particular,

$$\langle T^\lambda(-\alpha^{(N)}_x + \langle\alpha, F\rangle\varphi^{(N)}), e_n\rangle \xrightarrow[N\to\infty]{} \langle T^\lambda(-\alpha_x + \langle\alpha, F\rangle\varphi), e_n\rangle \tag{36}$$

Let $N \in \mathbb{N}$. We can now write

$$\begin{aligned}
\langle T^\lambda(-\alpha^{(N)}_x + \langle\alpha, F\rangle\varphi^{(N)}), e_n\rangle &= -\langle T^\lambda\alpha^{(N)}_x, e_n\rangle + \langle\alpha, F\rangle\langle T^\lambda\varphi^{(N)}, e_n\rangle \\
&= -\left\langle \sum_{p=-N}^{N} \frac{2i\pi p}{L}\alpha_p k_{-p,\lambda}, e_n\right\rangle + \langle\alpha, F\rangle\langle T^\lambda\varphi^{(N)}, e_n\rangle.
\end{aligned}$$

Now, using (16), we get

$$\frac{2i\pi p}{L}k_{-p,\lambda} = (k_{-p,\lambda})_x + \lambda k_{-p,\lambda} + \overline{F_p}\varphi,$$

so that

$$-T^\lambda \alpha_x^{(N)} = \sum_{p=-N}^{N} \alpha_p \left((k_{-p,\lambda})_x + \lambda k_{-p,\lambda} + \overline{F_p}\varphi\right).$$

Hence

$$-\langle T^\lambda \alpha_x^{(N)}, e_n\rangle = -\left\langle \left(T^\lambda \alpha^{(N)}\right)_x, e_n\right\rangle - \lambda \left\langle T^\lambda \alpha^{(N)}, e_n\right\rangle - \langle \alpha^{(N)}, F\rangle \varphi_n,$$

and finally,

$$\begin{aligned}\langle T^\lambda(-\alpha_x^{(N)} + \langle \alpha, F\rangle \varphi^{(N)}), e_n\rangle = -\lambda_n \left\langle T^\lambda \alpha^{(N)}, e_n\right\rangle &+ \left(\langle \alpha - \alpha^{(N)}, F\rangle\right) \varphi_n \\ &+ \langle \alpha, F\rangle \left(\left\langle T^\lambda \varphi^{(N)} - \varphi, e_n\right\rangle\right).\end{aligned} \tag{37}$$

To deal with the third term of the right-hand side of this equality, recall that we have chosen a feedback law so that the $TB = B$ condition (15) holds. Thus,

$$\left\langle T^\lambda \varphi^{(N)} - \varphi, e_n\right\rangle \xrightarrow[N\to\infty]{} 0. \tag{38}$$

To deal with the second term, recall that F is the sum of a regular part $\tilde{F}$ and a singular part h:

$$\langle \alpha - \alpha^{(N)}, F\rangle = \left\langle \alpha - \alpha^{(N)}, \tilde{F}\right\rangle + \langle \alpha - \alpha^{(N)}, h\rangle.$$

Now, by definition of $\alpha^{(N)}$ and continuity of $\tilde{F}$ for $\|\cdot\|_m$,

$$\left\langle \alpha - \alpha^{(N)}, \tilde{F}\right\rangle \xrightarrow[N\to\infty]{} 0. \tag{39}$$

On the other hand, for all $N \in \mathbb{N}$,

$$\begin{aligned}\langle \alpha^{(N)}, h\rangle &= K(\lambda) \sum_{n=-N}^{N} \frac{\alpha_n}{\tau_n^\varphi} \left(\frac{2i\pi n}{L}\right)^m \\ &= \frac{K(\lambda)}{2} \sum_{n=-N}^{N} \left(\frac{\alpha_n}{\tau_n^\varphi} + (-1)^m \frac{\alpha_{-n}}{\overline{\tau_n^\varphi}}\right) \left(\frac{2i\pi n}{L}\right)^m. \\ &= \frac{K(\lambda)}{2} \partial^{m-1} \tilde{\tau}^\varphi \alpha_x^{(N)}(0),\end{aligned} \tag{40}$$

where

$$\tilde{\tau}^\varphi f = \sum_{n\in\mathbb{Z}} \left(\frac{f_n}{\tau_n^\varphi} + (-1)^{m-1} \frac{f_{-n}}{\overline{\tau_n^\varphi}}\right) e_n, \quad \forall f \in L^2.$$

Now, notice that, by definition of τ^φ and D_m,

$$\tilde{\tau}^\varphi \left(-\alpha_x - \mu\alpha + \langle \alpha, F\rangle \varphi\right) \in H^m_{per}. \tag{41}$$

Moreover, using (8), we have for $n \in \mathbb{Z}^*$:

$$\begin{aligned}\frac{\varphi_n}{\tau_n^\varphi} + (-1)^{m-1} \frac{\varphi_{-n}}{\overline{\tau_n^\varphi}} &= \frac{\varphi_n}{\tau_n^\varphi} + (-1)^{m-1} \overline{\frac{\varphi_n}{\tau_n^\varphi}} \\ &= \frac{(-1)^{m-1} + (-1)^{m-1}(-1)^{m-1}(-1)^m}{\left(\frac{2i\pi}{L} n\right)^m} + \frac{r_n}{\left(\frac{2i\pi}{L} n\right)^m} \\ &= \frac{r_n}{\left(\frac{2i\pi}{L} n\right)^m},\end{aligned}$$

where $r_n \in \ell^2$. Hence, $\tilde{\tau}^\varphi \varphi \in H^m_{per}$. This, together with (41), yields

$$\tilde{\tau}^\varphi \alpha_x \in H^m_{per}.$$

This implies that

$$\tilde{\tau}^\varphi \alpha_x^{(N)} \xrightarrow[N\to\infty]{H^m} \tilde{\tau}^\varphi \alpha_x,$$

as $\tilde{\tau}^\varphi \alpha_x^{(N)}$ is the partial sum of $\tilde{\tau}^\varphi \alpha_x$.

Hence, by continuity of $\alpha \mapsto \partial^{m-1}\alpha(0)$ for $\|\cdot\|_m$, (40) implies that

$$\left\langle \alpha - \alpha^{(N)}, h \right\rangle \xrightarrow[N\to\infty]{} 0. \tag{42}$$

Finally, (37), (38), (39), (42), and the continuity of T^λ yield

$$\langle T^\lambda(-\alpha_x^{(N)} + \langle \alpha, F\rangle \varphi^{(N)}), e_n\rangle \xrightarrow[N\to\infty]{} -\lambda_n \left\langle T^\lambda \alpha, e_n \right\rangle.$$

This, put together with (36), gives (35) by unicity of the limit, which in turn proves (34).

□

Remark 3.1. *When $\varphi \in H^m$, τ^φ is simply $(-1)^{m-1}(\partial^{m-1}\varphi(L) - \partial^{m-1}\varphi(0))Id$, F is defined on $H^{m+1} \cap H^m_{per}$, and $\tilde{\tau}^\varphi \alpha$ is simply the symmetrisation $\alpha + (-1)^m \alpha(L - \cdot)$, which is H^m_{per} if $\alpha \in H^m$.*

3.2 Well-posedness of the closed-loop system

The operator equality we have established in the previous section means that T^λ transforms, if they exist, solutions of the closed-loop system with a well-chosen feedback into solutions of the target system. Let us now check that the closed-loop system in question is indeed well-posed.

Proposition 3.3. *The operator $A + BK := -\partial_x - \mu\alpha + \langle F, \cdot\rangle\varphi$ defined on D_m generates a C^0-semigroup on H^m_{per}.*

Proof. We know from Lemma 3.1 that $A + BK$ is densely defined. We use the Lumer-Phillips theorem (see for example [24, Section 1.4]): if $A + BK$ is densely defined, closed, and if $A + BK$ and its adjoint are both dissipative on their respective domains, then $A + BK$ generates a C^0-semigroup.

Let us prove that $A + BK$ is closed. Let

$$\alpha^{(N)} \in D_m \xrightarrow[N\to\infty]{H^m} \alpha \in H^m_{per}$$

such that

$$(A + BK)\alpha^{(N)} \xrightarrow[N\to\infty]{H^m} \beta \in H^m_{per}$$

i.e.

$$-\alpha_x^{(N)} + \mu\alpha^{(N)} + \langle \alpha^{(N)}, F\rangle \varphi \xrightarrow[N\to\infty]{H^m} \beta. \tag{43}$$

Now, as

$$-\alpha_x^{(N)} \xrightarrow[N\to\infty]{H^{m-1}} -\alpha_x,$$

the first two terms of the left-hand side of (43) converge, which implies that

$$\langle \alpha^{(N)}, F\rangle \varphi \xrightarrow[N\to\infty]{H^{m-1}} \gamma\varphi$$

and even

$$\langle \alpha^{(N)}, F\rangle \varphi \xrightarrow[N\to\infty]{H^m_{(pw)}} \gamma\varphi$$

for some limit $\gamma \in \mathbb{C}$, so that

$$-\alpha_x^{(N)} \xrightarrow[N\to\infty]{H^m_{(pw)}} \beta - \mu\alpha - \gamma\varphi. \tag{44}$$

Now, notice that (8) implies that

$$\varphi = \tau^{\varphi}\left(\sum_{n\in\mathbb{Z}^*} \frac{(-1)^{m-1}}{\left(\frac{2i\pi}{L}n\right)^m} e_n + r\right),$$

where $r \in H^m_{per}$. Moreover, it is well-known that

$$\partial^m \sum_{n\in\mathbb{Z}^*} \frac{(-1)^{m-1}}{\left(\frac{2i\pi}{L}n\right)^m} e_n = \sum_{n\in\mathbb{Z}^*} \frac{(-1)^{m-1}}{\frac{2i\pi}{L}n} e_n$$

is a piecewise constant function, so in particular

$$\sum_{n\in\mathbb{Z}^*} \frac{(-1)^{m-1}}{\left(\frac{2i\pi}{L}n\right)^m} e_n \in H^m_{(pw)}$$

which means that $\varphi \in \tau^{\varphi}\left(H^m_{(pw)}\right) \cap H^{m-1}_{per}$, and thus, by (44),

$$-\alpha_x + \mu\alpha + \gamma\varphi = \beta,$$

$$\alpha \in \tau^{\varphi}\left(H^{m+1}_{(pw)}\right) \cap H^m_{per},$$

$$\alpha^{(N)} \xrightarrow[N\to\infty]{H^{m+1}_{(pw)}} \alpha.$$

Then, using Proposition 2.4, we get, by continuity of F for the $H^{m+1}_{(pw)}$ norm:

$$\langle \alpha, F\rangle = \gamma.$$

Hence, $\alpha \in D_m$ and $(A+BK)\alpha = \beta$, and $A+BK$ is closed.

Finally, to study the behavior of the adjoint, let us consider the norm $\|\cdot\|_{T^\lambda} := \|T^\lambda \cdot\|_m$, which is equivalent and its associated scalar product $\langle\cdot,\cdot\rangle_{T^\lambda} := \langle T^\lambda\cdot, T^\lambda\cdot\rangle_m$. Then, for $\alpha \in D_m, \beta \in H^m_{per}$, we get, using the operator equality (34) of Proposition 3.2:

$$\begin{aligned}
\langle (A+BK)\alpha, \beta\rangle_{T^\lambda} &= \langle T^\lambda(-\alpha_x + \mu\alpha + \langle\alpha, F\rangle\varphi), T^\lambda\beta\rangle_m \\
&= \langle -\partial_x T^\lambda\alpha - \lambda' T^\lambda\alpha, T^\lambda\beta\rangle_m \\
&= \langle T^\lambda\alpha, \partial_x T^\lambda\beta\rangle_m - \lambda'\langle\alpha,\beta\rangle_{T^\lambda},
\end{aligned}$$

so that the adjoint of $(A+BK)$ for this scalar product is defined by

$$\begin{gathered}
D_m^* = \{\beta \in H^m_{per}, \quad T^\lambda\beta \in H^{m+1}\}, \\
(A+BK)^* = (T^\lambda)^{-1}\partial_x\, T^\lambda - \lambda' I \text{ on } D_m^*.
\end{gathered}$$

Thus, for all $\alpha \in D_m, \gamma \in D_m^*$:

$$\langle (A+BK)\alpha, \alpha\rangle_{T^\lambda} = \langle \partial_x T^\lambda\alpha, T^\lambda\alpha\rangle_m - \lambda'\|\alpha\|_{T^\lambda},$$

and

$$\langle (A+BK)^*\gamma, \gamma\rangle_{T^\lambda} = \langle \partial_x T^\lambda\gamma, T^\lambda\gamma\rangle_m - \lambda'\|\gamma\|_{T^\lambda}.$$

Now, as for any $f \in H^{m+1}_{per}$, we have, by integration by parts,

$$\mathfrak{Re}\left(\langle \partial_x f, f\rangle_m\right) = 0,$$

both $A+BK$ and $(A+BK)^*$ are dissipative. $\square$

3.3 Stability of the closed-loop system

We can now prove Theorem 1.1.

Let $m \geq 1$, $m \geq 0$. Let $(F_n) \in \mathbb{C}^{\mathbb{Z}}$ with m-growth and growth constants $C_1, C_2 > 0$. Let us first consider trajectories with initial data in D_m.

System (6) with initial data $\alpha_0 \in D_m$ has a unique solution $\alpha(t) \in C^0(\mathbb{R}^+, D_m) \cap C^1(\mathbb{R}^+, H^m_{per})$, and we have

$$\begin{aligned}
\frac{d}{dt} T^\lambda \alpha &= T^\lambda \frac{d}{dt}\alpha \\
&= T^\lambda(-\alpha_x + \mu\alpha + \langle \alpha, F\rangle \varphi) \\
&= (-\partial_x - \lambda' I) T^\lambda \alpha.
\end{aligned}$$

Thus, $T^\lambda \alpha(t) \in H^m_{per}$ is a solution of the target system (13) and satisfies for $t \geq 0$,

$$\|T^\lambda \alpha(t)\|_m \leq e^{-\lambda' t} \|T^\lambda \alpha_0\|_m. \tag{45}$$

Using (29), we then get, for $t \geq 0$,

$$\begin{aligned}
\|\alpha(t)\|_m &\leq |||(T^\lambda)^{-1}||| \|T^\lambda \alpha(t)\|_m \\
&\leq |||(T^\lambda)^{-1}|||\ e^{-\lambda' t} \|T^\lambda \alpha_0\|_m \\
&\leq |||(T^\lambda)^{-1}||| |||T^\lambda||| e^{-\lambda' t} \|\alpha_0\|_m \\
&\leq \left(\frac{C}{c}\right)^2 e^{\lambda L} e^{-\lambda' t} \|\alpha_0\|_m
\end{aligned}$$

This proves the exponential stability of solutions to system (6) with initial data in D_m. As the constant in this last inequality does not depend on the initial conditions, by density of D_m in H^m_{per}, any solution to system (6) with initial data in H^m_{per} satisfies the last inequality with the same constant.

3.4 Application

Let $m = 1, \lambda > 0$, and let us suppose, to simplify the computations, that $a \equiv 0$. Define

$$\varphi(x) = L - x, \quad \forall x \in (0, L), \tag{46}$$

so that $\varphi \in H^1$ but is not periodic, and satisfies (7), with

$$\begin{aligned}
\varphi_n &= -\frac{iL^{\frac{3}{2}}}{2\pi n}, \quad \forall n \in \mathbb{Z}^*, \\
\varphi_0 &= \frac{L^{\frac{3}{2}}}{2}.
\end{aligned}$$

Then,

$$\langle \alpha, F\rangle = -\frac{2K(\lambda)}{L^{\frac{3}{2}}}\alpha_0 - K(\lambda)\alpha_x(0), \quad \forall \alpha \in H^2 \cap H^1_{per}, \tag{47}$$

and

$$D_1 = \left\{ \alpha \in H^2 \cap H^1_{per}, \quad \frac{2K(\lambda)}{L^{\frac{3}{2}}}\alpha_0 + \left(\frac{1}{L} - K(\lambda)\right)\alpha_x(0) - \frac{1}{L}\alpha_x(L) = 0 \right\},$$

so that

$$\begin{cases} \alpha_t + \alpha_x = \left(-\dfrac{2K(\lambda)}{L^{\frac{3}{2}}}\alpha_0 - K(\lambda)\alpha_x(0)\right)\varphi(x), \ x \in [0, L], \\ \alpha(t, 0) = \alpha(t, L), \ \forall t \geq 0, \end{cases} \tag{48}$$

has a unique solution for initial conditions in D_1.

The backstepping transformation can be written as:

$$\alpha = \frac{\sqrt{L}}{1-e^{-\lambda L}}\left(e^{-\lambda x}\left(-\frac{K(\lambda)}{\sqrt{L}}\alpha_x - \frac{2K(\lambda)}{L^2}\alpha_0\right)\right)\star\varphi, \quad \forall \alpha \in H^1_{per}T^\lambda. \tag{49}$$

Let $\alpha(t) \in D_1$ be the solution of the closed loop system (48) with initial condition $\alpha^0 \in D_1$, and let us note $z(t) := T^\lambda\alpha(t)$, then

$$\begin{aligned}
z_t &= \frac{\sqrt{L}}{1-e^{-\lambda L}}\left(e^{-\lambda x}\left(-\frac{K(\lambda)}{\sqrt{L}}\alpha_{xt} - \frac{2K(\lambda)}{L^2}\alpha_0'\right)\right)\star\varphi. \\
&= \frac{\sqrt{L}}{1-e^{-\lambda L}}\left(e^{-\lambda x}\left(-\frac{K(\lambda)}{\sqrt{L}}(-\alpha_{xx} + \langle \alpha, F\rangle\varphi_x) - \frac{2K(\lambda)}{L^2}\alpha_0'\right)\right)\star\varphi. \\
&= \frac{\sqrt{L}}{1-e^{-\lambda L}}\left(e^{-\lambda x}\left(-\frac{K(\lambda)}{\sqrt{L}}(-\alpha_{xx} - \langle \alpha, F\rangle) - \frac{2K(\lambda)}{L^2}\alpha_0'\right)\right)\star\varphi. \\
z_x &= \frac{\sqrt{L}}{1-e^{-\lambda L}}\left(-e^{-\lambda x}\frac{K(\lambda)}{\sqrt{L}}\alpha_{xx}\right)\star\varphi - \lambda z \\
z_t + z_x + \lambda z &= \frac{\sqrt{L}}{1-e^{-\lambda L}}\left(e^{-\lambda x}\left(\frac{K(\lambda)}{\sqrt{L}}\langle \alpha, F\rangle - \frac{2K(\lambda)}{L^2}\alpha_0'\right)\right)\star\varphi.
\end{aligned}$$

By projecting the closed loop system on e_0, we get

$$\alpha_0' = \langle \alpha, F\rangle\varphi_0 = \langle \alpha, F\rangle\frac{L^{\frac{3}{2}}}{2}$$

so that

$$z_t + z_x + \lambda z = 0.$$

In particular,

$$\frac{d}{dt}\|z\|_1^2 = -2\lambda\|z\|_1^2. \tag{50}$$

Let us now set

$$V(\alpha) := \|z\|_1^2, \quad \forall \alpha \in H^1_{per}.$$

Now, notice that

$$\begin{aligned}
\|T^\lambda\alpha\|_1^2 &= \frac{L}{(1-e^{-\lambda L})^2}\sum_{n\in\mathbb{Z}}\left(1+\left|\frac{2i\pi n}{L}\right|^2\right)|\varphi_n|^2\left|\left\langle e^{-\lambda x}\left(-\frac{K(\lambda)}{\sqrt{L}}\alpha_x - \frac{2K(\lambda)}{L^2}\alpha_0\right), e_n\right\rangle\right|^2 \\
&\geq C\left\|e^{-\lambda x}\left(-\frac{K(\lambda)}{\sqrt{L}}\alpha_x - \frac{2K(\lambda)}{L^2}\alpha_0\right)\right\|^2 \\
&\geq Ce^{2\lambda L}\left\|-\frac{K(\lambda)}{\sqrt{L}}\alpha_x - \frac{2K(\lambda)}{L^2}\alpha_0\right\|^2 \\
&\geq C'K(\lambda)^2e^{2\lambda L}\|\alpha\|_1^2.
\end{aligned}$$

Together with (50), this shows that V is a Lyapunov function, and (48) is exponentially stable.

4 Further remarks and questions

4.1 Controllability and the $TB = B$ condition

In the introduction we have mentioned that the growth constraint on the Fourier coefficients of φ actually corresponds to the exact null controllability condition in some Sobolev space for the control system (2). As we have mentioned in the finite dimensional example, the controllability condition is essential to solve the operator equation: in our case, formal computations lead to a family of functions that turns out to be a Riesz basis precisely thanks to that rate of growth. Moreover, that rate of growth is essential for the compatibility of the $TB = B$ condition and the invertibility of the backstepping transformation. Indeed, as the transformation is constructed formally using a formal $TB = B$ condition, that same $TB = B$ condition fixes the value of the coefficients of F_n, giving them the right rate of growth for T^λ to be an isomorphism.

It should be noted that, while in [7] the $TB = B$ condition is well-defined, in our case, it only holds in a rather weak sense. This is probably because of a lack of regularization, indeed in [7] the backstepping transformation has nice properties, as it can be decomposed in Fredholm form, i.e. as the sum of a isomorphism and a compact operator. Accordingly, the Riesz base in that case is quadratically close to the orthonormal base given by the eigenvectors of the Laplacian operator. That is not the case for our backstepping transformation, as it is closely linked to the operator Λ^λ, which does not have any nice spectral properties.

Nonetheless, it appears that thanks to some information on the regularity of F, a weak sense is sufficient and allows us to prove the operator equality by convergence.

In that spirit, it would be interesting to investigate if a backstepping approach is still valid if the conditions on φ are weakened. For example, if we suppose approximate controllability instead of exact controllability, i.e.

$$\varphi_n \neq 0, \quad \forall n \in \mathbb{Z},$$

then F can still be defined using a weak $TB = B$ condition. However, it seems delicate to prove, in the same direct way as we have done, that T^λ is an isomorphism, as we only get the completeness of the corresponding $(k_{n,\lambda})$, but not the Riesz basis property.

4.2 Null-controllability and finite-time stabilization

As we have mentioned in the introduction, one of the advantages of the backstepping method is that it can provide an explicit expression for feedbacks, thus allowing the construction of explicit controls for null controllability, as well as time-varying feedbacks that stabilize the system in finite time $T > 0$.

The general strategy (as is done in [12], [31]) is to divide the interval $[0, T]$ in smaller intervals $[t_n, t_{n+1}]$, the length of which tends to 0, and on which one applies feedbacks to get exponential stabilization with decay rates λ_n, with $\lambda_n \to \infty$. Then, for well-chosen t_n, λ_n, the trajectory thus obtained reaches 0 in time T. Though this provides an explicit control to steer the system to 0, the norm of the operators applied successively to obtain the control tends to infinity. As such, it does not provide a reasonably regular feedback. However, the previous construction of the control can be used, with some adequate modifications (see [12] and [32]) to design a time-varying, periodic feedback, with some regularity in the state variable, which stabilizes the system in finite time.

Let us first note that, due to the hyperbolic nature of the system, there is a minimal control time, and thus small-time stabilization cannot be expected. Moreover, even for $T > L$, the estimates we have established on the backstepping transforms prevent us from applying the strategy we have described above: indeed, for any sequences $(t_n) \to T$, $\lambda_n \to \infty$, we have

$$\|\alpha(t)\|_m \leq \prod_{k=0}^{n} \left(\frac{C}{c}\right)^{2n} e^{n\mu L} exp\left(\sum_{k=0}^{n} -\lambda_k (t_{k+1} - t_k - L)\right) \|\alpha_0\|_m, \quad \forall t \in [t_n, t_{n+1}],$$

where c, C are the decay constants in (8). Moreover, as $t_{k+1} - t_k \to 0$, we have

$$exp\left(\sum_{k=0}^{n} -\lambda_k(t_{k+1} - t_k - L)\right) \xrightarrow[n\to\infty]{} \infty.$$

Another approach could be to draw from [9] and apply a second transformation to design a more efficient feedback law. It would also be interesting to adapt the strategy in [33], inspired from [27], to our setting.

4.3 Nonlinear systems

Finally, another prospect, having obtained explicit feedbacks that stabilize the linear system, is to investigate the stabilization of nonlinear transport equations. This has been done in [10], where the authors show that the feedback law obtained for the linear Korteweg-de Vries equation also stabilize the nonlinear equation. However, as in [7], the feedback law we have obtained is not continuous in the norm for which the system is stabilize. This would require some nonlinear modifications to the feedback law in order to stabilize the nonlinear system.

Acknowledgements

This work is partially supported by ANR project Finite4SoS (ANR-15-CE23-0007), and the French Corps des Mines.

The author would like to thank Jean-Michel Coron, for bringing the problem to his attention, his constant support, and his valuable remarks, as well as Amaury Hayat and Shengquan Xiang for discussions on this problem.

References

[1] Dejan M. Bosković, Andras Balogh, and Miroslav Krstić. Backstepping in infinite dimension for a class of parabolic distributed parameter systems. *Math. Control Signals Systems*, 16(1):44–75, 2003.

[2] Georges Bastin and Jean-Michel Coron. *Stability and boundary stabilization of 1-D hyperbolic systems*, volume 88 of *Progress in Nonlinear Differential Equations and their Applications*. Birkhäuser/Springer, [Cham], 2016. Subseries in Control.

[3] Andras Balogh and Miroslav Krstić. Infinite Dimensional Backstepping-Style Feedback Transformations for a Heat Equation with an Arbitrary Level of Instability. *European Journal of Control*, 8(2):165–175, 2002.

[4] Haim Brezis. *Functional analysis, Sobolev spaces and partial differential equations*. Universitext. Springer, New York, 2011.

[5] Ole Christensen. *An introduction to frames and Riesz bases*. Applied and Numerical Harmonic Analysis. Birkhäuser/Springer, [Cham], second edition, 2016.

[6] Jean-Michel Coron and Brigitte d'Andréa Novel. Stabilization of a rotating body beam without damping. *IEEE Trans. Automat. Control*, 43(5):608–618, 1998.

[7] Jean-Michel Coron, Ludovick Gagnon, and Morgan Morancey. Rapid stabilization of a linearized bilinear 1-D Schrödinger equation. *J. Math. Pures Appl. (9)*, 115:24–73, 2018.

[8] Jean-Michel Coron, Long Hu, and Guillaume Olive. Stabilization and controllability of first-order integro-differential hyperbolic equations. *J. Funct. Anal.*, 271(12):3554–3587, 2016.

[9] Jean-Michel Coron, Long Hu, and Guillaume Olive. Finite-time boundary stabilization of general linear hyperbolic balance laws via Fredholm backstepping transformation. *Automatica J. IFAC*, 84:95–100, 2017.

[10] Jean-Michel Coron and Qi Lü. Local rapid stabilization for a Korteweg-de Vries equation with a Neumann boundary control on the right. *J. Math. Pures Appl. (9)*, 102(6):1080–1120, 2014.

[11] Jean-Michel Coron and Qi Lü. Fredholm transform and local rapid stabilization for a Kuramoto-Sivashinsky equation. *J. Differential Equations*, 259(8):3683–3729, 2015.

[12] Jean-Michel Coron and Hoai-Minh Nguyen. Null controllability and finite time stabilization for the heat equations with variable coefficients in space in one dimension via backstepping approach. *Arch. Ration. Mech. Anal.*, 225(3):993–1023, 2017.

[13] Jean-Michel Coron. *Control and nonlinearity*, volume 136 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2007.

[14] Jean-Michel Coron. Stabilization of control systems and nonlinearities. In *Proceedings of the 8th International Congress on Industrial and Applied Mathematics*, pages 17–40. Higher Ed. Press, Beijing, 2015.

[15] Jean-Michel Coron, Rafael Vazquez, Miroslav Krstić, and Georges Bastin. Local exponential H^2 stabilization of a 2×2 quasilinear hyperbolic system using backstepping. *SIAM J. Control Optim.*, 51(3):2005–2035, 2013.

[16] Amaury Hayat. Exponential stability of general 1-D quasilinear systems with source terms for the C 1 norm under boundary conditions. preprint, October 2017. `https://hal.archives-ouvertes.fr/hal-01613139`

[17] Amaury Hayat. On boundary stability of inhomogeneous 2×2 1-D hyperbolic systems for the C^1 norm. preprint, July 2018. `https://hal.archives-ouvertes.fr/hal-01790104/`

[18] Jean-Pierre Kahane and Pierre-Gilles Lemarié-Rieusset. *Séries de Fourier et ondelettes.* Cassini, 1998.

[19] Miroslav Krstić, Bao-Zhu Guo, Andras Balogh, and Andrey Smyshlyaev. Output-feedback stabilization of an unstable wave equation. *Automatica J. IFAC*, 44(1):63–74, 2008.

[20] Vilmos Komornik. Rapid boundary stabilization of linear distributed systems. *SIAM J. Control Optim.*, 35(5):1591–1613, 1997.

[21] Miroslav Krstić and Andrey Smyshlyaev. *Boundary control of PDEs*, volume 16 of *Advances in Design and Control*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2008. A course on backstepping designs.

[22] Wei Jiu Liu and Miroslav Krstić. Backstepping boundary control of Burgers' equation with actuator dynamics. *Systems Control Lett.*, 41(4):291–303, 2000.

[23] Wei Jiu Liu and Miroslav Krstić. Boundary feedback stabilization of homogeneous equilibria in unstable fluid mixtures. *Internat. J. Control*, 80(6):982–989, 2007.

[24] Amnon Pazy. *Semigroups of linear operators and applications to partial differential equations*, volume 44 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1983.

[25] Andrey Smyshlyaev, Eduardo Cerpa, and Miroslav Krstić. Boundary stabilization of a 1-D wave equation with in-domain antidamping. *SIAM J. Control Optim.*, 48(6):4014–4031, 2010.

[26] Eduardo D. Sontag. *Mathematical control theory*, volume 6 of *Texts in Applied Mathematics*. Springer-Verlag, New York, second edition, 1998. Deterministic finite-dimensional systems.

[27] Daisuke Tsubakino, Miroslav Krstić, and Shinji Hara. Backstepping control for parabolic pdes with in-domain actuation. *2012 American Control Conference (ACC)*, pages 2226–2231, 2012.

[28] Jose Manuel Urquiza. Rapid exponential feedback stabilization with unbounded control operators. *SIAM J. Control Optim.*, 43(6):2233–2244, 2005.

[29] Ambroise Vest. Rapid stabilization in a semigroup framework. *SIAM J. Control Optim.*, 51(5):4169–4188, 2013.

[30] Frank Woittennek, Siqian Wang, and Torsten Knüppel. Backstepping design for parabolic systems with in-domain actuation and Robin boundary conditions. *IFAC Proceedings Volumes*, 47(3):5175–5180, 2014. 19th IFAC World Congress.

[31] Shengquan Xiang. Null controllability of a linearized Korteweg-de Vries equation by backstepping approach. preprint, February 2017. `https://hal.archives-ouvertes.fr/hal-01468750`

[32] Shengquan Xiang. Small-time local stabilization for a Korteweg–de Vries equation. *Systems Control Lett.*, 111:64–69, 2018.

[33] Xin Yu, Chao Xu, Huacheng Jiang, Arthi Ganesan, and Guojie Zheng. Backstepping synthesis for feedback control of first-order hyperbolic PDEs with spatial-temporal actuation. *Abstr. Appl. Anal.*, pages Art. ID 643640, 13, 2014.