Technical University of Denmark

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Written examination, December 10, 2019

Course title:	Mathematical Software Programming
Course number:	02635
Aids allowed:	All aids allowed
Exam duration:	4 hours
Weighting:	80/100

Final exam
Mathematical Software Programming

This exam contains a total of 16 questions: 12 multiple choice questions (questions 1–12) and 4 programming questions (questions 13–16). Your exam answers to the multiple choice questions must be submitted electronically as a **PDF document**, and you code must be attached (e.g., as a **ZIP file**).

1. (4 points) The unit round-off for double precision floating point numbers is $u = 2^{-53}$.
   (a) The unit round-off u can be represented as a double precision floating point number.
      A. True
      B. False

   (b) The number $1 + u$ can be represented as a double precision floating point number.
      A. True
      B. False

2. (4 points) Consider the real-valued function $f(x) = ax^2 + bx + c$ where $a, b, c, x \in \mathbb{R}$.
   (a) What is the relative condition number of f?
      A. $c_f(x) = \frac{|ax^2+bx+c|}{|2ax+b|}$
      B. $c_f(x) = \frac{|2ax+b|}{|ax^2+bx+c|}$
      C. $c_f(x) = \frac{|2ax^2+bx|}{|ax^2+bx+c|}$
      D. $c_f(x) = \frac{|ax^2+bx+c|}{|2ax^2+bx|}$

   (b) Suppose $a = 1$, $b = 4$, and $c = 4$. For what values of x is the problem of evaluating $f(x)$ ill-posed?
      A. When $x = 0$.
      B. When $x = -1$.
      C. When $x = -2$.
      D. When $x \in \{-2, 0\}$.

3. (6 points) Let $f(x) = \log(1+x)$ where x is a positive real number.

   (a) The relative condition number of f is given by

   A. $c_f(x) = \left|\frac{1}{x\log(1+x)}\right|$

   B. $c_f(x) = \left|\frac{1}{(1+x)\log(1+x)}\right|$

   C. $c_f(x) = \left|\frac{1+x}{x\log(1+x)}\right|$

   D. $c_f(x) = \left|\frac{x}{(1+x)\log(1+x)}\right|$

   (b) What can be said about the relative condition number of f when $|x|$ is close to 0?

   A. $c_f(x)$ is close to 0

   B. $c_f(x)$ is close to 1

   C. $c_f(x)$ is much larger than 1

   (c) Consider the following implementation of the function $f(x) = \log(1+x)$:

```c
#include <math.h>
double fun(double x) {
    return log(1.0+x);
}
```

   This implementation may result in a large relative error when $|x|$ is close to 0. Why?

   A. The condition number of $\log(x)$ is large when x is close to 1.

   B. The sum $1+x$ is prone to catastrophic cancellation when $|x|$ is close to 0.

   C. The sum $1+x$ may underflow when $|x|$ is close to 0.

4. (4 points) The Chebyshev polynomials of the first kind can be defined recursively as

$$\begin{aligned}
T_0(x) &= 1 \\
T_1(x) &= x \\
T_k(x) &= 2xT_{k-1}(x) - T_{k-2}(x), \qquad k = 2, 3, \ldots
\end{aligned}$$

(a) What kind of recursion is this?

A. single recursion

B. multiple recursion

(b) What is the complexity associated with the following implementation?

```
double T(double x, unsigned int k) {
   if (k==0) return 1.0;
   if (k==1) return x;
   return 2*x*T(x,k-1)-T(x,k-2);
}
```

A. $O(k)$ space and $O(k)$ time

B. $O(2^k)$ space and $O(k)$ time

C. $O(k)$ space and $O(2^k)$ time

D. $O(2^k)$ space and $O(2^k)$ time

5. (4 points) Suppose the variable `A` represents a two-dimensional array of size $m \times n$. Consider the following code segment:

```
for (int i=0; i<m; i++) {
   for (int j=0;j<n; j++) {
      y[i] += A[i][j]*x[j];
   }
}
```

(a) References to `y[i]` are temporally local.

A. True

B. False

(b) References to the array represented by `x` are spacially local.

A. True

B. False

6. (6 points) The theoretical improvement in speed of execution of a task executed on p processors can be expressed as

$$S(p) = \frac{T(1)}{T(p)} = \frac{fT(1) + (1-f)T(1)}{(f/p)T(1) + (1-f)T(1)}$$

where $T(p)$ is the execution time on p processors (real time) and f is the so-called parallel fraction of the task. For example, if 50% of a task can be parallelized, then $f = 0.5$.

Suppose that a specific task consists of two subtasks. The first subtask retrieves data from a file; it takes 20 seconds to execute on a given system, and it cannot be parallelized. The second subtask processes a large number of data records; it takes 60 seconds to execute sequentially. Each data record can be processed independently, so the second subtask is easily parallelized.

(a) What is the parallel fraction of the task consisting of the two subtasks?

A. $f = 1/4$
B. $f = 1/3$
C. $f = 3/4$
D. $f = 4/5$

(b) What is the theoretical speedup for $p = 9$ processors?

A. 3
B. 4
C. 6
D. 9

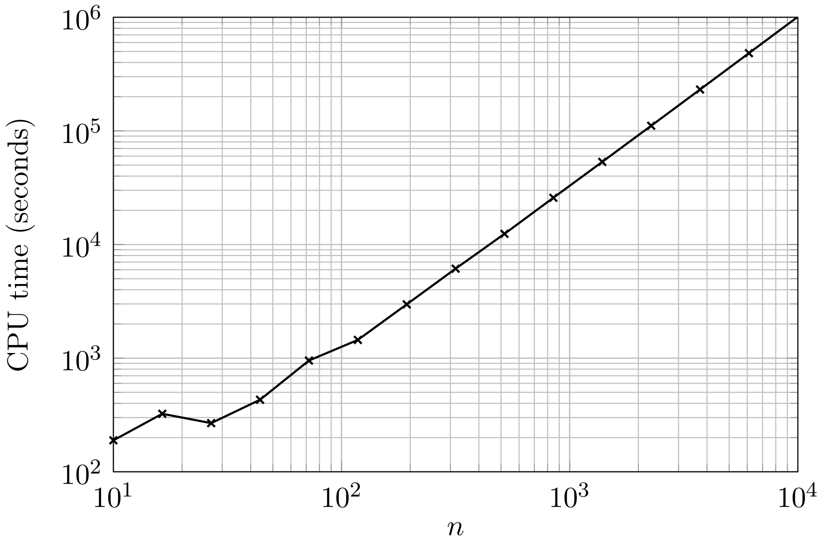
7. (2 points) Memory leaks are possible in C but not in C++.

A. True
B. False

8. (2 points) The operator :: in C++ (e.g. `std::cout`) is referred to as the

A. member access operator
B. scope resolution operator
C. namespace operator
D. function operator

9. (4 points) The following plot shows the CPU time required by some algorithm to solve a certain problem as a function of its dimension n.



What is the time complexity of the algorithm?

A. $O(n)$

B. $O(n^{3/2})$

C. $O(n^2)$

D. $O(2^n)$

10. (4 points) Consider the following function:

```
double * ones(size_t n) {
    double *arr = malloc(n*sizeof(*arr));
    if (arr != NULL) {
        for (size_t k=0; k<n; k++)
            arr[k] = 1.0;
    }
    return arr;
}
```

(a) What kind of memory allocation is used to allocate what `arr` points to?

A. Automatic memory allocation

B. Dynamic memory allocation

(b) What kind of memory allocation is used to allocate the pointer `arr`?

A. Automatic memory allocation

B. Dynamic memory allocation

11. (4 points) Suppose the matrix

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \end{bmatrix}$$

is stored as a *one-dimensional*, *row-major* `double` array of length 12.

The elements of an array of double-precision floating-point numbers can be scaled using the CBLAS routine `cblas_dscal()` which has the following prototype:

```
void cblas_dscal(
    const int n,            // number of elements
    const double a,         // scalar a
    double * x,             // pointer to first element of array
    const int incx          // stride
);
```

Now suppose `pA` is a pointer to A_{11} (i.e., `pA` is a `double *`).

(a) Which of the following calls to `cblas_dscal()` scales the elements A_{22}, A_{23}, and A_{24} by -1?

A. `cblas_dscal(3,-1.0,pA+4,1)`
B. `cblas_dscal(3,-1.0,pA+4,3)`
C. `cblas_dscal(3,-1.0,pA+4,4)`
D. `cblas_dscal(3,-1.0,pA+5,1)`
E. `cblas_dscal(3,-1.0,pA+5,3)`
F. `cblas_dscal(3,-1.0,pA+5,4)`

(b) Consider the following `cblas_dscal()` call:

```
cblas_dscal(4,2.0,pA+1,2);
```

Which elements of A are being scaled by a factor of 2?

A. $A_{11}, A_{13}, A_{21}, A_{23}$
B. $A_{12}, A_{14}, A_{22}, A_{24}$
C. $A_{21}, A_{12}, A_{32}, A_{23}$
D. $A_{11}, A_{31}, A_{22}, A_{13}$

12. (2 points) Cache memory is slower than main memory.

A. True
B. False

13. (8 points) Implement a function that performs the matrix-vector multiplication

$$y \leftarrow A^T x + y$$

where A is a sparse matrix of size $m \times n$ in triplet form, and $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^n$ are vectors.

Requirements

- Your function must have the following prototype:

```c
void sparse_triplet_mv_trans(
    const struct sparse_triplet *A,
    const double *x,
    double *y);
```

  You may assume that all inputs are valid.

- The sparse matrix A should be stored using the sparse triplet format represented by the following data structure:

```c
/* Structure representing a sparse matrix in triplet form */
struct sparse_triplet {
   size_t m;   /* number of rows     */
   size_t n;   /* number of columns  */
   size_t nnz; /* number of nonzeros */
   size_t * I; /* pointer to array with row indices    */
   size_t * J; /* pointer to array with column indices */
   double * V; /* pointer to array with values         */
};
```

  The header file `sparse_triplet.h` in the ZIP file attached to the exam defines the `sparse_triplet` data structure. You do not need to include this header file when you submit your solution.

- Use the template `sparse_triplet_mv_trans.c` for your implementation. The template is included in the ZIP file attached to the exam.

14. (8 points) The Poisson distribution is a discrete probability distribution with probability mass function

$$f(k;\lambda) = \frac{\lambda^k}{k!}e^{-\lambda}, \qquad k \in \mathbb{N}_0,$$

where the parameter $\lambda > 0$ is the so-called rate. Implement a function that evaluates $f(k;\lambda)$.

Requirements

- Your function must have the following prototype:

```
double poisson_pmf(unsigned long k, double lambda);
```

- The function should return `NAN` if λ is not positive.
- Use the template `poisson_pmf.c` for your implementation. The template is included in the ZIP file attached to the exam.

15. (8 points) The centering matrix of order n is given by

$$C = I - \frac{1}{n}\mathbf{1}\mathbf{1}^T$$

where I is the $n \times n$ identity matrix and $\mathbf{1}$ is the vector of length n whose entries are all equal to 1. For example, the centering matrix of order 3 is the matrix

$$C = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix}.$$

Implement a function that computes $x \leftarrow Cx$, i.e., the function should overwrite a vector $x \in \mathbb{R}^n$ by the matrix-vector product Cx.

Requirements

- The function must have the following prototype:

```
int dcemv(int n, double * x);
```

  The input `n` is the order of the matrix C, and `x` is a pointer to the first elements of an array of length `n` that represent the vector x.
- The function should return the value `-1` in case of an error or invalid input, and otherwise it should return the value `0`.
- Use the template `dcemv.c` for your implementation. The template is included in the ZIP file attached to the exam.

16. (10 points) Consider the system of equations

$$\begin{bmatrix} c & -s \\ s & c \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} r \\ 0 \end{bmatrix} \tag{1}$$

where $r \geq 0$, $c = \cos(\theta)$, and $s = \sin(\theta)$ for some $\theta \in [0, 2\pi]$. It is easy to check that a, b must satisfy

$$\begin{bmatrix} a \\ b \end{bmatrix} = r \begin{bmatrix} c \\ -s \end{bmatrix},$$

and hence $r = \sqrt{a^2 + b^2}$. It follows that if $r > 0$, then $c = a/r$ and $s = -b/r$ (the angle θ can be chosen arbitrarily when $r = 0$, e.g., $c = 1$ and $s = 0$).

(a) Implement a function that takes a and b as inputs and computes c and s such that a and b satisfies (1).

Requirements

- Your function must have the following prototype:

```c
void rotg(double a, double b, double * c, double * s);
```

  The inputs `c` and `s` point to the locations where c and s should be stored. You may assume that all inputs are valid.
- Use the template `rotg.c` for your implementation. The template is included in the ZIP file attached to the exam.

(b) Implement a function that applies the transformation

$$\begin{bmatrix} x_i \\ y_i \end{bmatrix} \leftarrow \begin{bmatrix} c & -s \\ s & c \end{bmatrix} \begin{bmatrix} x_i \\ y_i \end{bmatrix}, \qquad i = 1, \ldots, n,$$

where $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ are vectors of length n, and $c = \cos(\theta)$ and $s = \sin(\theta)$ for some $\theta \in [0, 2\pi]$.

Requirements

- Your function must have the following prototype:

```c
int rot(double c, double s, int n, double * x, double * y);
```

  The input `n` is the length of the vectors x and y. The inputs `x` and `y` are pointers to the first element of the arrays that represent x and y, respectively.
- The function should return the value `-1` in case of an invalid input, and otherwise it should return the value `0`.
- Use the template `rot.c` for your implementation. The template is included in the ZIP file attached to the exam.