

# Homework 5

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9/23/2021

1. (15 points, 5 points each) Hoff, 3.12 (Jeffrey's prior).

Jeffrey's prior is as follows:  $p_J(\theta) \propto \sqrt{I(\theta)}$  where  $I(\theta) = -E \left[ \delta^2 \log \frac{p(Y|\theta)}{\delta\theta^2} \middle| \theta \right]$  is the fisher information.

- a) Let  $Y \sim \text{binomial}(n, \theta)$ . Obtain Jeffreys' prior distribution  $p_J(\theta)$  for this model.

$$\begin{aligned}
 p_J(\theta) &\propto \sqrt{-E \left[ \delta^2 \log \frac{p(Y|\theta)}{\delta\theta^2} \middle| \theta \right]} \\
 p_J(\theta) &\propto \sqrt{-E \left[ \frac{\delta^2}{\delta\theta^2} \log p(Y|\theta) \right]} \\
 &\sqrt{-E \left[ \frac{\delta^2}{\delta\theta^2} \log \left( \binom{n}{y} \theta^y (1-\theta)^{n-y} \right) \right]} \\
 &\sqrt{-E \left[ \frac{\delta^2}{\delta\theta^2} \log \left( \binom{n}{y} \right) + y \log(\theta) + (n-y) \log(1-\theta) \right]} \\
 &\sqrt{-E \left[ \frac{\delta}{\delta\theta} \left( \frac{y}{\theta} - \frac{n-y}{1-\theta} \right) \right]} \\
 &\sqrt{-E \left[ \frac{-y}{\theta^2} - \frac{n-y}{(1-\theta)^2} \right]} \\
 &\sqrt{-E \left[ \frac{-y}{\theta^2} \right] + E \left[ \frac{n-y}{(1-\theta)^2} \right]} \\
 &\sqrt{\frac{1}{\theta^2} E[y] + \frac{1}{(1-\theta)^2} E[n-y]}
 \end{aligned}$$

$$\begin{aligned}
& \sqrt{\frac{n\theta}{\theta^2} + \frac{n-n\theta}{(1-\theta)^2}} \\
& \sqrt{\frac{n}{\theta} + \frac{n}{1-\theta}} \\
& \sqrt{\frac{n-n\theta}{\theta-\theta^2} + \frac{n\theta}{\theta-\theta^2}} \\
& p_J(\theta) \propto \sqrt{\frac{n}{\theta-\theta^2}}
\end{aligned}$$

b) Reparameterize the binomial sampling model with  $\psi = \log \frac{\theta}{1-\theta}$  so that  $p(y|\psi) = \binom{n}{y} e^{\psi y} (1 + e^\psi)^{-n}$

$$\begin{aligned}
p_J(\psi) & \propto \sqrt{-E \left[ \frac{\delta^2}{\delta\psi^2} \log p(y|\psi) \middle| \psi \right]} \\
& \sqrt{-E \left[ \frac{\delta^2}{\delta\psi^2} \log \left( \binom{n}{y} e^{\psi y} (1 + e^\psi)^{-n} \right) \right]} \\
& \sqrt{-E \left[ \frac{\delta^2}{\delta\psi^2} \log \binom{n}{y} + \log(e^{\psi y}) + \log((1 + e^\psi)^{-n}) \right]} \\
& \sqrt{-E \left[ \frac{\delta^2}{\delta\psi^2} \log \binom{n}{y} + \psi y - n \log(1 + e^\psi) \right]} \\
& \sqrt{-E \left[ \frac{\delta}{\delta\psi} y - \frac{n}{1 + e^\psi} e^\psi \right]} \\
& \sqrt{-E \left[ \frac{\delta}{\delta\psi} y - \frac{ne^\psi}{1 + e^\psi} \right]} \\
& \sqrt{-E \left[ - \left( \frac{ne^\psi(1 + e^\psi) - ne^\psi(e^\psi)}{(1 + e^\psi)^2} \right) \right]} \\
& \sqrt{-E \left[ - \left( \frac{ne^\psi + ne^{2\psi} - ne^{2\psi}}{(1 + e^\psi)^2} \right) \right]} \\
& \sqrt{-E \left[ - \left( \frac{ne^\psi}{(1 + e^\psi)^2} \right) \right]} \\
& \sqrt{- \left( - \left( \frac{ne^\psi}{(1 + e^\psi)^2} \right) \right)}
\end{aligned}$$

$$\sqrt{\frac{ne^\psi}{(1+e^\psi)^2}}$$

$$p_J(\psi) \propto \frac{\sqrt{ne^\psi}}{1+e^\psi}$$

c) Take the prior distribution from a) and apply the change of variables formula to it to obtain the induced prior density on  $\psi$

Change of variables:  $p_\psi(\psi) = p_\theta(h(\psi)) \left| \frac{dh}{d\psi} \right|$  where  $\theta = h(\psi)$

$$\psi = \log\left(\frac{\theta}{1-\theta}\right)$$

$$\frac{\theta}{1-\theta} = e^\psi$$

$$\theta = e^\psi - \theta e^\psi$$

$$\theta + \theta e^\psi = e^\psi$$

$$\theta(1+e^\psi) = e^\psi$$

$$\theta = \frac{e^\psi}{1+e^\psi}$$

$$h(\psi) = \frac{e^\psi}{1+e^\psi}$$

$$p_J(\psi) = p_J(h(\psi)) \left| \frac{dh}{d\psi} \right|$$

$$p_J(\psi) = \sqrt{\frac{n}{h(\psi) - h(\psi)^2}} \left| \frac{d}{d\psi} \left( \frac{e^\psi}{1+e^\psi} \right) \right|$$

$$\sqrt{\frac{n}{h(\psi)(1-h(\psi))}} \left| \frac{e^\psi(1+e^\psi) - e^{2\psi}}{(1+e^\psi)^2} \right|$$

$$\sqrt{\frac{n}{h(\psi)(1-h(\psi))}} \left| \frac{e^\psi + e^{2\psi} - e^{2\psi}}{(1+e^\psi)^2} \right|$$

$$\sqrt{\frac{n}{\left(\frac{e^\psi}{1+e^\psi}\right)\left(1 - \left(\frac{e^\psi}{1+e^\psi}\right)\right)}} \left| \frac{e^\psi}{(1+e^\psi)^2} \right|$$

$$\sqrt{\frac{n}{\left(\frac{e^\psi}{1+e^\psi}\right)\left(\frac{1+e^\psi - e^\psi}{1+e^\psi}\right)}} \left| \frac{e^\psi}{(1+e^\psi)^2} \right|$$

$$\begin{aligned}
& \sqrt{\frac{n}{\left(\frac{e^\psi}{1+e^\psi}\right)\left(\frac{1}{1+e^\psi}\right)}} \left| \frac{e^\psi}{(1+e^\psi)^2} \right| \\
& \sqrt{\frac{n}{\left(\frac{e^\psi}{(1+e^\psi)^2}\right)}} \left| \frac{e^\psi}{(1+e^\psi)^2} \right| \\
& \sqrt{\frac{n\left(\frac{e^\psi}{(1+e^\psi)^2}\right)^2}{\left(\frac{e^\psi}{(1+e^\psi)^2}\right)}} \\
& \sqrt{n\left(\frac{e^\psi}{(1+e^\psi)^2}\right)} \\
& \sqrt{\left(\frac{ne^\psi}{(1+e^\psi)^2}\right)} \\
& p_J(\psi) = \frac{\sqrt{ne^\psi}}{1+e^\psi}
\end{aligned}$$

2. Please refer to lab 5 and complete tasks 4-5.

According to the rejection sampling approach sample from  $f(x)$  using the  $\text{Unif}(0,1)$  pdf as an enveloping function. In order to do this, we write a general rejection sampling function that also allows us to plot the histograms for any simulation size. Finally, our function also allows us to look at task 4 quite easily.

```

set.seed(1)
x <- seq(0, 1, 10^-2) #from task 1
fx <- function(x) sin(pi * x)^2 #from task 1

sim_fun <- function(f, envelope = "unif", par1 = 0, par2 = 1, n = 10^2, plot = TRUE){

  r_envelope <- match.fun(paste0("r", envelope))
  d_envelope <- match.fun(paste0("d", envelope))
  proposal <- r_envelope(n, par1, par2)
  density_ratio <- f(proposal) / d_envelope(proposal, par1, par2)
  samples <- proposal[runif(n) < density_ratio] #for each x_i in the proposal,
#compare f(x_i)/g(x_i) with draw from uniform, intuition being that if the
#density ratio at that potential point x_i > random draw from a unif[0,1] dist,
#then accept samples keeps all proposals that were accepted
  acceptance_ratio <- length(samples) / n
  if (plot) {
    hist(samples, probability = TRUE,

```

```

    main = paste0("Histogram of ",
                  n, " samples from ",
                  envelope, "(", par1, ",", par2,
                  ").\n Acceptance ratio: ",
                  round(acceptance_ratio,2)),
    cex.main = 0.75)
  }
  list(x = samples, acceptance_ratio = acceptance_ratio)
}

par(mfrow = c(2,2), mar = rep(4, 4))
unif_1 <- sim_fun(fx, envelope = "unif", par1 = 0, par2 = 1, n = 10^2)
unif_2 <- sim_fun(fx, envelope = "unif", par1 = 0, par2 = 1, n = 10^5)
unif_3 <- sim_fun(fx, envelope = "beta", par1 = 2, par2 = 2, n = 10^2)
unif_3 <- sim_fun(fx, envelope = "beta", par1 = 2, par2 = 2, n = 10^5)

```

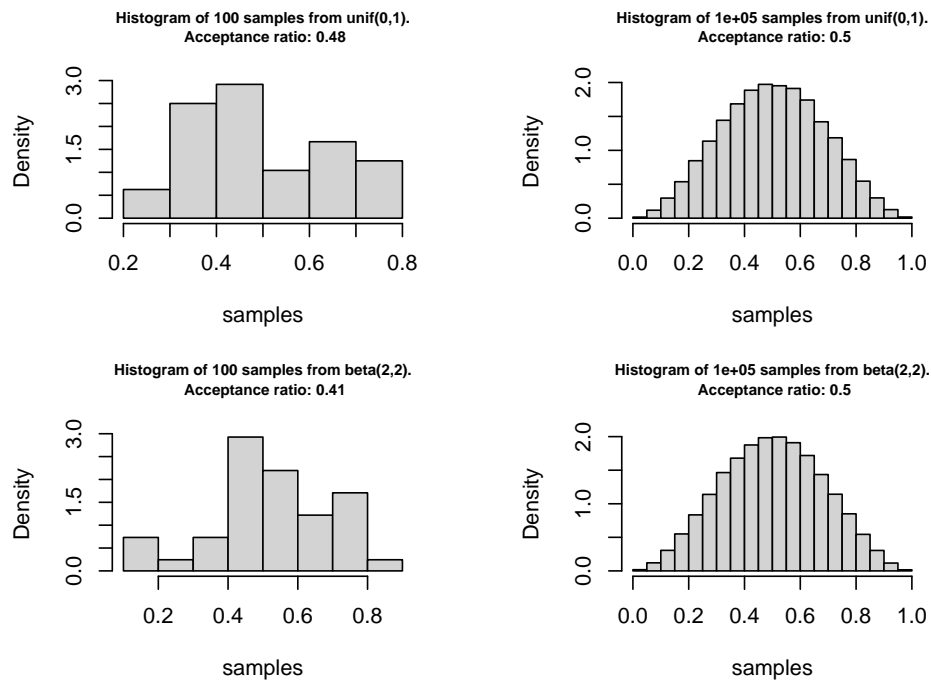


Figure 2: Comparison of the output of the rejection sampling for 100 versus 100,000 simulations with uniform and beta distributions as envelope functions.

```

par(mfrow = c(1,1))

```

When using the uniform distribution as an enveloping function, the acceptance ratio in a sample of 100 was 0.48, and was 0.50 in a sample of 10,000. The distribution of samples appeared to be random in the smaller sample, but approximately normally distributed in the larger one.

When using the beta distribution as an enveloping function, the acceptance ratio in a sample of 100 was 0.41, and was 0.50 in a sample of 10,000. The distribution of samples appeared to be random in the smaller sample, but just as when a uniform enveloping function was used, approximately normally distributed in the larger one. Since the acceptance ratio for a larger amount of samples is more accurate, a beta enveloping function appears to be work about equally as well as a uniform enveloping function.

## Task 5

The  $\text{unif}(0,1)$  and  $\text{beta}(2,2)$  distributions have acceptance ratios that are roughly equal (at about 0.5) one there are many samples, and as such I wouldn't recommend either one over the other. If I were to try to find a better enveloping function, I would use a normal distribution because  $\sin(\pi x^2)$  is shaped similarly to a bell curve in that it contains inflection points.