$$X_1, \dots, X_n \overset{iid}{\sim} \text{Geom}(\theta)$$

$$\theta \sim \text{Beta}(a, b)$$

a.) $$p(x_{1:n} \mid \theta) = \prod_{i=1}^{n} \theta (1-\theta)^{x_i} = \theta^n (1-\theta)^{\sum_{i=1}^{n} x_i}$$

b.) $$p(\theta \mid x_{1:n}) \propto \theta^n (1-\theta)^{\sum_{i=1}^{n} x_i} \theta^{a-1} (1-\theta)^{b-1}$$

$$= \theta^{n+a-1} (1-\theta)^{\sum_{i=1}^{n} x_i + b - 1}$$

$$\Rightarrow \quad \theta \mid x_{1:n} \sim \text{Beta}\left(n + a, \sum_{i=1}^{n} x_i + b\right)$$

c.) $$p(x_{1:n}) = \int p(x_{1:n} \mid \theta) p(\theta) \, d\theta$$

$$= \int_0^1 \theta^n (1-\theta)^{\sum x_i} \frac{1}{B(a,b)} \theta^{a-1} (1-\theta)^{b-1} \, d\theta$$

$$= \frac{1}{B(a,b)} = \int_0^1 \theta^{n+a-1} (1-\theta)^{\sum x_i + b - 1} \, d\theta$$

$$= \frac{B(a_n, b_n)}{B(a,b)} = \int_0^1 \frac{\theta^{a_n - 1} (1-\theta)^{b_n - 1}}{B(a_n, b_n)} \, d\theta$$

$a_n = n + a$

$b_n = \sum x_i + b$

$$= \frac{B(a_n, b_n)}{B(a, b)}$$

$$= \frac{\Gamma(a_n)\Gamma(b_n)}{\Gamma(a_n + b_n)} \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}$$

$$= \frac{\Gamma(n+a)\Gamma\left(\sum x_i + b\right)}{\Gamma(n+a+b+\sum x_i)} \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}$$

8.) posterior predictive

$$p(x_{n+1} \mid x_{1:n}) = \int_0^1 p(x_{n+1} \mid \theta)\, p(\theta \mid x_{1:n})\, d\theta$$

$$= \int_0^1 \theta^{(1-x_{n+1})} \times \text{Beta}(\theta \mid a_n, b_n)\, d\theta$$

$$= \int_0^1 \theta (1-\theta)^{1-x_{n+1}} \times \frac{1}{B(a_n, b_n)} \theta^{a_n - 1} (1-\theta)^{b_n - 1}\, d\theta$$

$a_n = n + a$

$$= \int_0^1 \frac{\theta^{a_n + 1 - 1} (1-\theta)^{1 - x_{n+1} + \sum x_i - 1}}{B(a_n, b_n)}\, d\theta$$

Kernel of $\text{Beta}(n + a + 1, \sum x_i + x_{n+1} + 1)$

$$= \int_0^1 \frac{\theta^{n+a} (1-\theta)^{\sum x_i + x_{n+1}}}{B(n+a, \sum_{i=1}^{n} x_i + b)}\, d\theta$$

$$= \int_0^1 \frac{\theta^{n+a} (1-\theta)^{\sum x_i + x_{n+1}}}{B(n+a, \sum x_i + b)} \cdot \frac{B(n+a+1, \sum x_i + x_{n+1} + 1)}{B(n+a+1, \sum x_i + x_{n+1} + 1)}\, d\theta$$

$$= \frac{B(n+a+1, \sum x_i + x_{n+1} + 1)}{B(n+a, \sum x_i + b)}$$

parts c + d are repeat. so we can cut one of these for the exam.