Homework 02

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Due 9/3/2021

1. Please refer to lab 2 and complete tasks 3—5.

```
# set a seed
set.seed(123)
# create the observed data
obs.data <- rbinom(n = 100, size = 1, prob = 0.01)
# inspect the observed data
head(obs.data)
```

```
## [1] 0 0 0 0 0 0
```

```
tail(obs.data)
```

```
## [1] 0 0 0 0 0 0
```

```
length(obs.data)
```

```
## [1] 100
```

Task 3

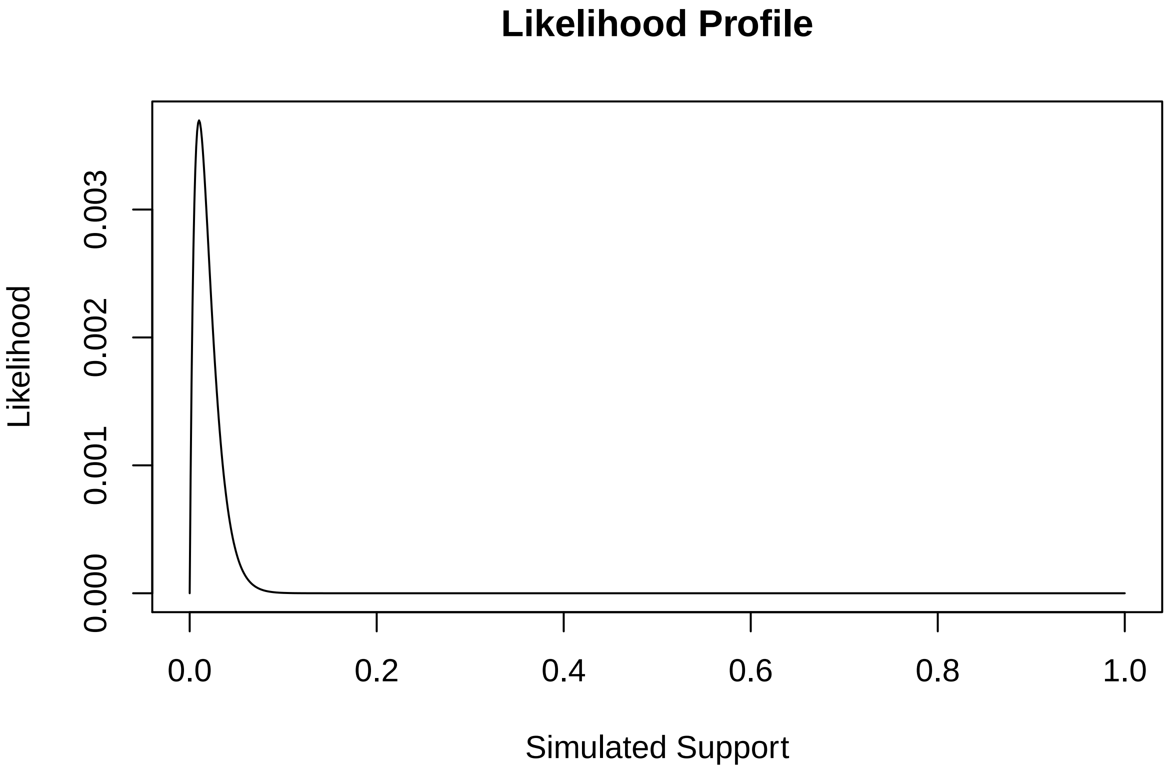
Write a function that takes as its inputs that data you simulated (or any data of the same type) and a sequence of θ values of length 1000 and produces Likelihood values based on the Binomial Likelihood. Plot your sequence and its corresponding Likelihood function.

The likelihood function is given below. Since this is a probability and is only valid over the interval from $[0, 1]$ we generate a sequence over that interval of length 1000.

You have a rough sketch of what you should do for this part of the assignment. Try this out in lab on your own.

```
### Bernoulli LH Function ###
# Input: obs.data, theta
# Output: bernoulli likelihood

bernoulliLH <- function(obs.data, theta){
  N <- length(obs.data)
  x <- sum (obs.data)
  LH <- (theta^x)*((1-theta)^{N-x})
  return(LH)
}
### Plot LH for a grid of theta values ###
# Create the grid #
theta.sim <- seq(from = 0, to = 1, length.out = 1000)
```

```r
# Store the LH values
sim.LH <- bernoulliLH(obs.data, theta = theta.sim)
# Create the Plot
plot(theta.sim, sim.LH, type = "l", main = "Likelihood Profile",
     xlab = "Simulated Support", ylab = "Likelihood")
```



Task 4 (To be completed for homework)

Write a function that takes as its inputs prior parameters `a` and `b` for the Beta-Bernoulli model and the observed data, and produces the posterior parameters you need for the model. **Generate and print** the posterior parameters for a non-informative prior i.e. (a,b) = (1,1) and for an informative case (a,b) = (3,1)}.

```r
betberPos <- function(a, b, obs.data){
  n <- length(obs.data)
  x <- sum (obs.data)

  prior <- c((x+a), (n-x+b))
  return( prior)
}

print(betberPos(1,1,obs.data))
```

```
## [1]   2 100
```

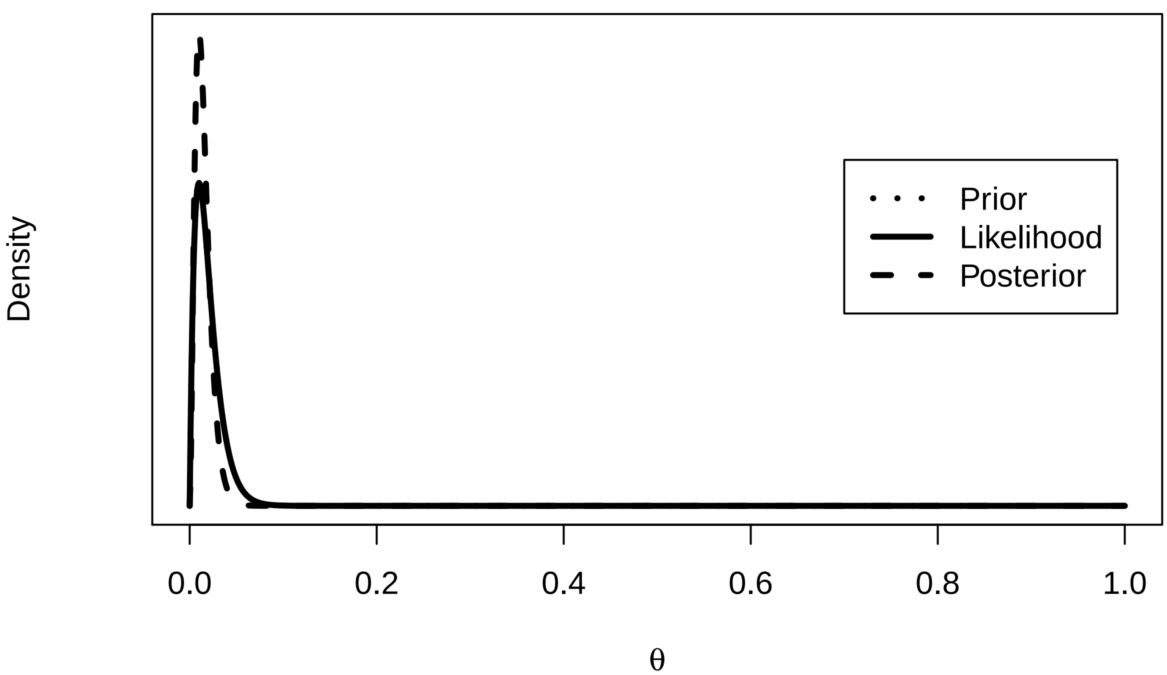
```r
print(betberPos(3,1,obs.data))
```

```
## [1]   4 100
```

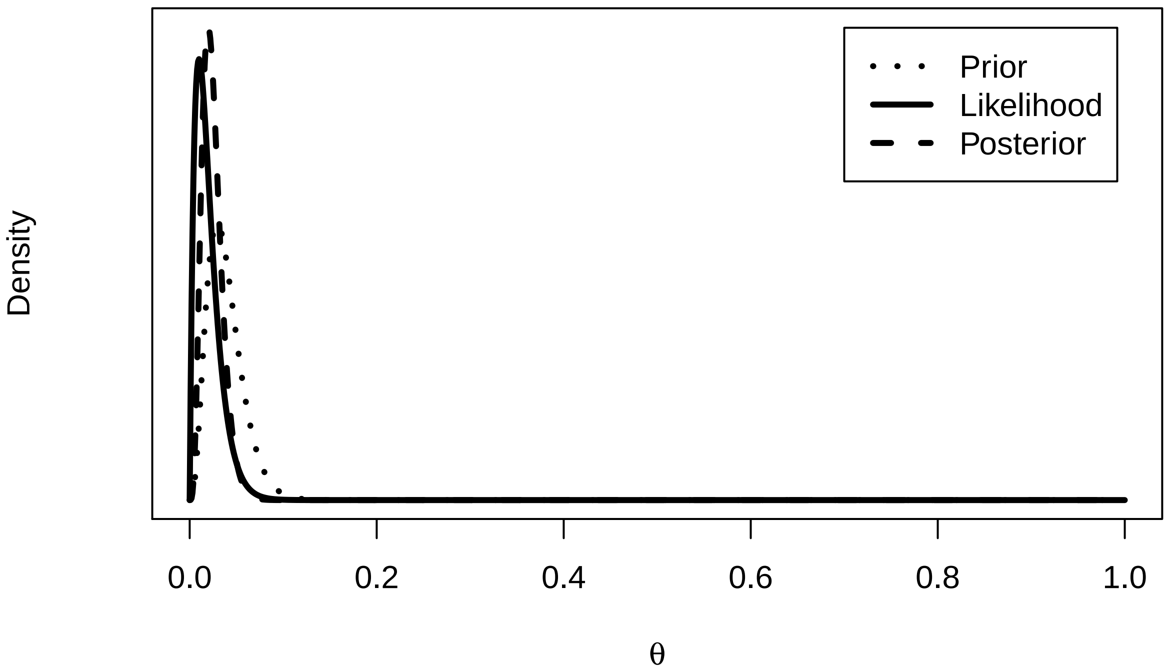
Task 5 (To be completed for homework)

Create two plots, one for the informative and one for the non-informative case to show the posterior distribution and superimpose the prior distributions on each along with the likelihood. What do you see? Remember to turn the y-axis ticks off since superimposing may make the scale non-sense.

```
distPlot <- function(p1,p2){
  n <- length(obs.data)
  x <- sum (obs.data)
  params <- betberPos (p1,p2,obs.data)
  a <- params[1]
  b <- params[2]


  like = dbeta(theta.sim, x+1, n-x+1)
  prior = dbeta(theta.sim, a, b)
  post = dbeta(theta.sim, x+a, n-x+b)


  plot(theta.sim,post,type="l",ylab="Density", yaxt = 'n',lty=2,lwd=3,
  xlab = expression(theta))
  lines(theta.sim,like,lty=1,lwd=3)
  lines(theta.sim,prior,lty=3,lwd=3)
  legend(0.7,40,c("Prior","Likelihood","Posterior"),
  lty=c(3,1,2),lwd=c(3,3,3))
} # yaxt from: https://stackoverflow.com/questions/10393076/suppress-ticks-in-plot-in-r

distPlot(1,1)
```




```
distPlot(3,1)
```



2. {*The Exponential-Gamma Model*}

We write $X \sim \text{Exp}(\theta)$ to indicate that X has the Exponential distribution, that is, its p.d.f. is

$$p(x|\theta) = \text{Exp}(x|\theta) = \theta \exp(-\theta x)\mathbb{1}(x > 0).$$

The Exponential distribution has some special properties that make it a good model for certain applications. It has been used to model the time between events (such as neuron spikes, website hits, neutrinos captured in a detector), extreme values such as maximum daily rainfall over a period of one year, or the amount of time until a product fails (lightbulbs are a standard example).

Suppose you have data $x_1, \ldots, x_n$ which you are modeling as i.i.d. observations from an Exponential distribution, and suppose that your prior is $\boldsymbol{\theta} \sim \text{Gamma}(a, b)$, that is,

$$p(\theta) = \text{Gamma}(\theta|a, b) = \frac{b^a}{\Gamma(a)}\theta^{a-1}\exp(-b\theta)\mathbb{1}(\theta > 0).$$

a. Derive the formula for the posterior density, $p(\theta|x_{1:n})$. Give the form of the posterior in terms of one of the most common distributions (Bernoulli, Beta, Exponential, or Gamma).

By Baye's Theorem:

$$p(\theta|x_{1:n}) = \frac{p(x_{1:n}|\theta)p(\theta)}{p(x_{1:n})} \propto p(x_{1:n}|\theta)p(\theta)$$

$$p(x_{1:n}|\theta) = \prod_{i=1}^{n} \theta e^{-\theta x_i}$$

$$p(x_{1:n}|\theta) = \theta^n e^{-\theta \sum x_i}$$

$$p(\theta|x_{1:n}) = \theta^n e^{-\theta \sum x_i} \frac{b^a}{\Gamma(a)}\theta^{a-1}e^{-b\theta}$$

$$\theta^n e^{-\theta \sum x_i} \frac{b^a}{\Gamma(a)}\theta^{a-1}e^{-b\theta} = \frac{b^a}{\Gamma(a)}\theta^{n+a-1}e^{-\theta \sum x_i - b\theta}$$

$$\frac{b^a}{\Gamma(a)}\theta^{n+a-1}e^{-\theta\sum x_i - b\theta} = \frac{b^a}{\Gamma(a)}\theta^{n+a-1}e^{\sum x_i + b(-\theta)}$$

$$p(\theta|x_{1:n}) = \text{Gamma}(a+n, \sum x_i + b)$$

b. Why is the posterior distribution a *proper* density or probability distribution function?

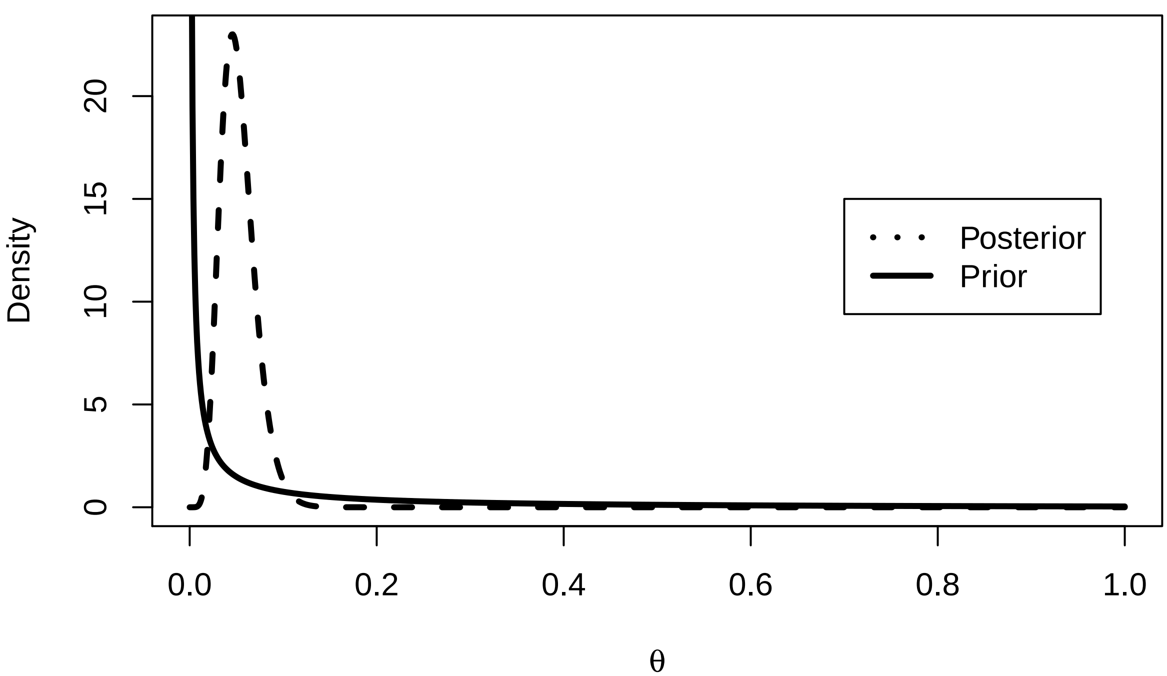
The posterior distribution is a Gamma distribution, which is a proper probability distribution function with an area under the curve of 1.

c. Now, suppose you are measuring the number of seconds between lightning strikes during a storm, your prior is Gamma(0.1, 1.0), and your data is

$$(x_1, \ldots, x_8) = (20.9, 69.7, 3.6, 21.8, 21.4, 0.4, 6.7, 10.0).$$

Plot the prior and posterior p.d.f.s. (Be sure to make your plots on a scale that allows you to clearly see the important features.)

```r
obs.data <- c(20.9, 69.7, 3.6, 21.8, 21.4, 0.4, 6.7, 10.0)

x <- sum (obs.data)
n <- length (obs.data)

a <- 0.1
b <- 1.0

prior = dgamma(theta.sim, a, b)
post = dgamma(theta.sim, a+n, x+b)


plot(theta.sim,post,type="l",ylab="Density",lty=2,lwd=3,
xlab = expression(theta))
lines(theta.sim,prior,lty=1,lwd=3)
legend(0.7,15,c("Posterior", "Prior"),
lty=c(3,1,2),lwd=c(3,3,3))
```



3.

{*Priors, Posteriors, Predictive Distributions (Hoff, 3.9)*}

An unknown quantity

$$Y \mid \theta$$

has a Galenshore(a, θ) distribution if its density is given by

$$p(y \mid \theta) = \frac{2}{\Gamma(a)} \theta^{2a} y^{2a-1} e^{-\theta^2 y^2}$$

for $y > 0, \theta > 0, a > 0$. Assume for now that a is known and θ is unknown and a random variable. For this density,

$$E[Y] = \frac{\Gamma(a + 1/2)}{\theta \Gamma(a)}$$

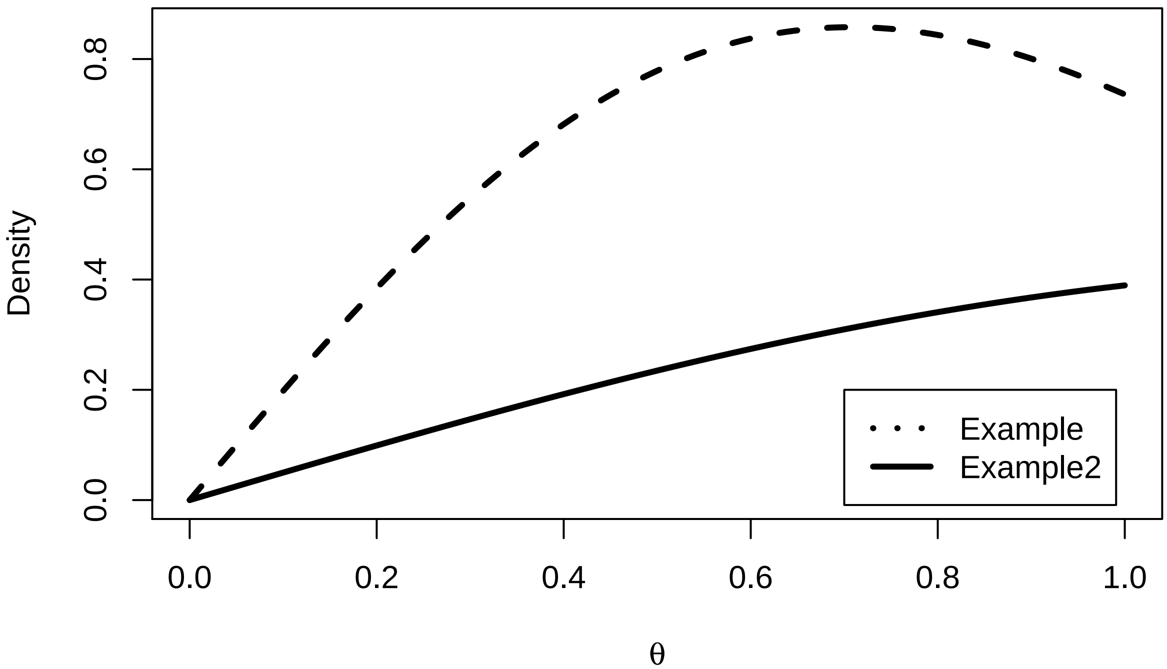
and

$$E[Y^2] = \frac{a}{\theta^2}.$$

a. Identify a class of conjugate prior densities for θ. Assume the prior parameters are c and d. That is, state the distribution that θ should have with parameters c, d such that the resulting posterior is conjugate. Plot a few members of this class of densities.

If the prior is a Galenshore *distribution, then the posterior distribution should also be a Galenshore distribution, and therefore conjugate (since the likelihood is a Galenshore distribution as well).*

```
galenshore <- function(th,c,d){
  e <- exp(1)
  y <- (2*((d^(2*c))*(th^(2*c-1))*e^((-d^2)*(th^2))))/gamma(c)
  return(y)
}

example = galenshore(theta.sim, 1, 1)
example2 = galenshore(theta.sim, 1, 0.5)

  plot(theta.sim,example,type="l",ylab="Density",lty=2,lwd=3,
  xlab = expression(theta))
  lines(theta.sim,example2,lty=1,lwd=3)
  legend(0.7,0.2,c("Example", "Example2"),
  lty=c(3,1,2),lwd=c(3,3,3))
```



b. Let $Y_1, \ldots, Y_n \overset{iid}{\sim}$ Galenshore(a, θ). Find the posterior distribution of $\theta \mid y_{1:n}$ using a prior from your conjugate class.

$$p(\theta \mid y_{1:n}) \propto p(y_{1:n} \mid \theta)p(\theta)$$

$$p(y_{1:n} \mid \theta)p(\theta) = (\frac{2}{\Gamma(a)})^n \ \theta^{2an}(\prod_{i=1}^{n} y_i^{2a-1})e^{-\theta^2 \sum_{i=1}^n y_i^2} * \frac{2}{\Gamma(c)} \ d^{2c}\theta^{2c-1}e^{-d^2\theta^2}$$

$$\propto \theta^{2an}e^{-\theta^2 \sum_{i=1}^n y_i^2} * \theta^{2c-1}e^{-d^2\theta^2}$$

$$p(\theta \mid y_{1:n}) = \theta^{2an+2c-1}e^{-d^2\theta^2-\theta^2 \sum_{i=1}^n y_i^2}$$

$$p(\theta \mid y_{1:n}) = \theta^{2(an+c)-1}e^{-\theta(d^2+\sum_{i=1}^n y_i^2)}$$

c. Show that

$$\frac{p(\theta_a \mid y_{1:n})}{p(\theta_b \mid y_{1:n})} = \left(\frac{\theta_a}{\theta_b}\right)^{2(an+c)-1} e^{(\theta_b^2-\theta_a^2)(d^2+\sum y_i^2)},$$

where

$$\theta_a, \theta_b \sim \textit{Galenshore}(c, d).$$

Identify a sufficient statistic.

$$\frac{p(\theta_a \mid y_{1:n})}{p(\theta_b \mid y_{1:n})} = \frac{\theta_a^{2(an+c)-1}e^{-\theta_a(d^2+\sum_{i=1}^n y_i^2)}}{\theta_b^{2(an+c)-1}e^{-\theta_b(d^2+\sum_{i=1}^n y_i^2)}}$$

$$\frac{p(\theta_a \mid y_{1:n})}{p(\theta_b \mid y_{1:n})} = \left(\frac{\theta_a}{\theta_b}\right)^{2(an+c-1)} e^{(\theta_b^2-\theta_a^2)(d^2+\sum_{i=1}^n y_i^2)}$$

$$\sum_{i=1}^{n} y_i^2$$

is a sufficient statistic for this problem because it is the only information from the sample that is necessary to calculate the answer.

d. Determine

$$E[\theta \mid y_{1:n}]$$

.

$$E[\theta \mid y_{1:n}] = \int \theta * \theta^{2(an+c)-1} e^{-\theta(d^2+\sum_{i=1}^{n} y_i^2)} d\theta$$

$$E[\theta \mid y_{1:n}] = \int \theta^{2(an+c)} e^{-\theta(d^2+\sum_{i=1}^{n} y_i^2)} d\theta$$

e. Show that the form of the posterior predictive density

$$p(y_{n+1} \mid y_{1:n}) = \frac{2y_{n+1}^{2a-1}\Gamma(an+a+c)}{\Gamma(a)\Gamma(an+c)} \frac{(d^2+\sum y_i^2)^{an+c}}{(d^2+\sum y_i^2+y_{n+1}^2)^{(an+a+c)}}.$$

$$p(y_{n+1} \mid y_{1:n}) = \int_\theta p(y_n+1 \mid \theta, y_{1:n}) p(\theta \mid y_{1:n})$$

$$= \int_\theta p(y_{1:n} \mid \theta) p(\theta \mid y_{1:n}) d\theta$$

$$= p(y \mid \theta) = \frac{2}{\Gamma(a)}\ \theta^{2a} y_{n+1}^{2a-1} e^{-\theta^2 y_{n+1}^2} *$$