Swap-Root Magmas from a Three-Axiom Screenshot: Structural Classification of the Core Axiom, Obstructions, and Research Directions

Research Note (Generated from an Anonymous Axiom Set)

January 12, 2026

Abstract

An anonymous screenshot proposes an equational axiom system for a single binary operation:

$$(E1)\ (b \oplus a) \oplus a = a \oplus (a \oplus b), \quad (E2)\ (b \oplus a) \oplus (a \oplus b) = a,$$

$$(E3)\ (a \oplus (b \oplus c)) \oplus c = b \oplus (a \oplus (c \oplus a)),$$

universally quantified over the underlying set. We isolate the core axiom (E2) and define swap-root magmas as its models. Writing the associated switch map $S(a, b) = (b \oplus a, a \oplus b)$ and the flip $\tau(a, b) = (b, a)$, axiom (E2) is equivalent to $S^2 = \tau$ (and automatically $S\tau = \tau S$), hence S is bijective without finiteness assumptions. For finite $|X| = n$, we give a complete classification and an exact count of labeled swap-root magma operations; in particular, finite models exist if and only if $n \equiv 0, 1 \pmod 4$. We then study interactions with (E1) and (E3): $(E1)+(E2)$ forces idempotence, and computer-assisted exhaustive enumeration shows that for $n = 4$ and $n = 5$ there are no nontrivial models of $(E1)+(E2)$ and no nontrivial models of $(E2)+(E3)$, and hence no nontrivial models of the full system (E1)-(E3) for $n \leq 7$. We also prove a structural "no-go" theorem: there is no affine-linear operation satisfying $(E1) + (E2)$ over any commutative ring (in particular, over any field). In contrast, (E2) alone admits nontrivial affine-linear models; for example over $\mathbb{C}$ one may take $a \oplus b = \frac{1+i}{2}a + \frac{1-i}{2}b$. We further show that any associative model of $(E1)+(E2)$ is necessarily trivial. Finally, we outline a solver-driven attack on the first admissible unresolved finite case $n = 8$ (where the (E2)-search space is $\approx 2.67 \times 10^{21}$ labeled operations) and compare (E3) with the genuine set-theoretic Yang-Baxter (braid) relations for the switch S.

1 The screenshot axioms

Let X be a set and $\oplus : X \times X \to X$ a binary operation. The screenshot presents the universally quantified identities:

$$\forall a, b \in X : \quad (b \oplus a) \oplus a = a \oplus (a \oplus b). \tag{E1}$$

$$\forall a, b \in X : \quad (b \oplus a) \oplus (a \oplus b) = a. \tag{E2}$$

$$\forall a, b, c \in X : \quad (a \oplus (b \oplus c)) \oplus c = b \oplus (a \oplus (c \oplus a)). \tag{E3}$$

This note treats (E2) as the "ground floor" and investigates what happens when (E1) and/or (E3) are imposed in addition.

2 Swap-root magmas and the switch map

Definition 1 (Swap-root magma)**.** A swap-root magma is a pair $(X, \oplus)$ satisfying axiom (E2):

$$(b \oplus a) \oplus (a \oplus b) = a \quad \text{for all } a, b \in X.$$

2.1 Diagonal involution

Define the diagonal map $s : X \to X$ by $s(a) := a \oplus a$.

Lemma 1 (Diagonal involution)**.** *In every swap-root magma, s is an involution:*

$$s(s(a)) = a \quad \textit{for all } a \in X.$$

Proof. Apply (E2) with $b = a$:

$$(a \oplus a) \oplus (a \oplus a) = a.$$

The left-hand side is $s(a) \oplus s(a) = s(s(a))$. □

2.2 Switch map reformulation

Let $\tau : X \times X \to X \times X$ be the flip $\tau(a, b) = (b, a)$. Define the associated switch map

$$S : X \times X \to X \times X, \quad S(a, b) := (b \oplus a, a \oplus b).$$

Proposition 1 (Switch characterization of (E2))**.** *For every binary operation on X the switch map S commutes with the flip: $S\tau = \tau S$. Moreover, $(X, \oplus)$ is a swap-root magma (i.e. satisfies (E2)) if and only if*

$$S^2 = \tau.$$

In particular, every swap-root magma has a bijective switch map S (no finiteness hypothesis).

Proof. For commutation,

$$S(\tau(a, b)) = S(b, a) = (a \oplus b, b \oplus a) = \tau(b \oplus a, a \oplus b) = \tau(S(a, b)).$$

Assume (E2). Let $x = b \oplus a$ and $y = a \oplus b$. Then

$$S^2(a, b) = S(x, y) = (y \oplus x, x \oplus y).$$

Applying (E2) to (a, b) gives $x \oplus y = a$ and applying (E2) to (b, a) gives $y \oplus x = b$. Hence $S^2(a, b) = (b, a) = \tau(a, b)$.

Conversely, if $S^2 = \tau$, then $S^2(a, b) = (y \oplus x, x \oplus y) = (b, a)$ where $(x, y) = S(a, b) = (b \oplus a, a \oplus b)$, and the second coordinate equality $x \oplus y = a$ is exactly (E2). Since τ is bijective, $S^2 = \tau$ implies S is bijective. □

Remark 1. The condition $S^2 = \tau$ may be viewed as a "square root of symmetry" constraint: off the diagonal, the map S has order 4 (see Lemma 4 below).

Lemma 2 (Commutativity forces equality)**.** *In any swap-root magma, if $a \oplus b = b \oplus a$ then $a = b$. Equivalently, there are no commuting pairs of distinct elements.*

Proof. Let $t := a \oplus b = b \oplus a$. Applying (E2) to (a, b) gives $t \oplus t = a$, and applying (E2) to (b, a) gives $t \oplus t = b$. Hence $a = b$. □

3 Finite classification and exact enumeration for (E2)

Assume $|X| = n < \infty$ and identify $X = \{0, 1, ..., n-1\}$. The flip τ acts on X^2 with n fixed points (a, a) and $\binom{n}{2}$ two-cycles $\{(a, b), (b, a)\}$ for $a \neq b$. Because $S\tau = \tau S$, the switch map permutes these τ-orbits.

3.1 Orbit structure

Lemma 3 (Action on the diagonal)**.** *On the diagonal* $\Delta = \{(a, a)\}$*,* $S(a, a) = (s(a), s(a))$ *where* $s(a) = a \oplus a$ *is an involution.*

Proof. By definition $S(a, a) = (a \oplus a, a \oplus a) = (s(a), s(a))$, and Lemma 1 gives $s^2 = id$. □

Lemma 4 (Off-diagonal 4-cycles)**.** *Let* $a \neq b$*. Then* (a, b) *lies in a 4-cycle of* S*, and each such 4-cycle is the union of exactly two* τ*-orbits.*

Proof. If $a \neq b$, then $\tau(a, b) \neq (a, b)$. Since $S^2 = \tau$, we have $S^2(a, b) \neq (a, b)$, hence the orbit size is not 1. Also $S^4(a, b) = \tau^2(a, b) = (a, b)$, so the orbit size divides 4; it cannot be 2 because $S^2(a, b) = \tau(a, b) \neq (a, b)$. Hence it is 4. Because S commutes with τ, it permutes τ-orbits, so a 4-cycle must consist of two τ-orbits. □

3.2 Existence criterion and labeled count

Let $m = \binom{n}{2}$ be the number of off-diagonal τ-orbits.

Theorem 1 (Finite existence criterion)**.** *A swap-root magma of size* n *exists if and only if* $m = \binom{n}{2}$ *is even, equivalently*

$$n \equiv 0 \text{ or } 1 \pmod 4.$$

Proof. Necessity: Lemma 4 implies the m two-element τ-orbits must be paired into $m/2$ pairs (each pair supporting a 4-cycle), hence m must be even.

Sufficiency: if m is even, choose (i) an involution s on X (to define S on the diagonal), (ii) a perfect matching of the m off-diagonal τ-orbits (to pair them), and (iii) for each matched pair, choose one of the two possible 4-cycle orientations compatible with $S^2 = \tau$ and $S\tau = \tau S$. This defines a bijection S on X^2 with $S^2 = \tau$, hence an operation satisfying (E2) (Proposition 1). □

Theorem 2 (Exact number of labeled models of (E2))**.** *Let* $n \equiv 0, 1 \pmod 4$ *and set* $m = \binom{n}{2}$*. The number* $N(n)$ *of labeled swap-root magma operations on an* n*-element set is*

$$N(n) = I(n) \cdot (m-1)!! \cdot 2^{m/2},$$

where $I(n)$ *is the number of involutions on* n *labeled points,*

$$I(n) = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n!}{2^k k!(n-2k)!}$$

and $(m-1)!! = (m-1)(m-3)\cdots 3 \cdot 1$ *counts perfect matchings on* m *labeled objects. If* $n \not\equiv 0, 1 \pmod 4$ *then* $N(n) = 0$*.*

Proof. Choosing the diagonal action is choosing an involution s on X, giving $I(n)$ possibilities (Lemma 3). Pairing the m off-diagonal τ-orbits is choosing a perfect matching on m labeled objects, giving $(m-1)!!$ choices. For each matched pair of τ-orbits, there are exactly two 4-cycle orientations compatible with $S^2 = \tau$ and $S\tau = \tau S$, contributing a factor $2^{m/2}$. Multiplying independent choices gives the formula. □

Remark 2 (Key sizes)**.** Theorem 2 yields:

$$N(4) = 1200,$$

$$N(5) = 786,240,$$

$$N(8) = 2,671,934,630,479,994,880,000.$$

Under idempotence (forced by (E1)+(E2), see Lemma 5), the diagonal is fixed and the relevant (E2)-space at $n = 8$ has size $(m-1)!! \cdot 2^{m/2} \approx 3.497 \times 10^{18}$. These magnitudes motivate solver-based searches rather than enumeration.

4 Explicit examples of swap-root magmas

The following tables define binary operations on $X = \{0, 1, 2, 3\}$ that satisfy (E2).

Proposition 2 (An idempotent size-4 example satisfying (E2))**.** *Define* $\oplus$ *on* $\{0, 1, 2, 3\}$ *by the following Cayley table:*

$\oplus$	0	1	2	3
0	0	3	3	2
1	2	1	0	0
2	1	3	2	0
3	1	2	1	3

Then $(X, \oplus)$ *satisfies (E2) and is idempotent* ($a \oplus a = a$)*. It does not satisfy (E1) or (E3).*

Proposition 3 (A non-idempotent size-4 example satisfying (E2))**.** *Define* $\oplus$ *on* $\{0, 1, 2, 3\}$ *by the following Cayley table:*

$\oplus$	0	1	2	3
0	1	3	3	2
1	2	0	0	0
2	1	3	2	0
3	1	2	1	3

Then $(X, \oplus)$ *satisfies (E2) but is not idempotent (here* $0 \oplus 0 = 1$ *and* $1 \oplus 1 = 0$*). Consequently it cannot satisfy (E1) together with (E2) (see Lemma 5).*

5 Adding (E1): idempotence is forced

Lemma 5 (Idempotence from (E1)+(E2))**.** *Every model of (E1)+(E2) is idempotent:*

$$a \oplus a = a \quad \textit{for all } a.$$

Proof. Let $s(a) = a \oplus a$. From (E2) with $b = a$ we have $s(s(a)) = a$, so s is injective. Apply (E1) with $b = a$ and let

$$x := (a \oplus a) \oplus a.$$

By (E1), we also have $x = a \oplus (a \oplus a)$. Substituting $s(a) = a \oplus a$, we have $x = s(a) \oplus a = a \oplus s(a)$. Now apply (E2) to the pair $(a, s(a))$:

$$(s(a) \oplus a) \oplus (a \oplus s(a)) = a.$$

Substituting x for both inner terms gives $x \oplus x = a$, i.e., $s(x) = a$. Since $s(s(a)) = a$ and s is injective, we must have $x = s(a)$. Thus $a \oplus s(a) = s(a)$. Finally, apply (E2) to $(s(a), a)$:

$$(a \oplus s(a)) \oplus (s(a) \oplus a) = s(a).$$

Using the fact that both inner terms equal $s(a)$ (since $x = s(a)$), the left-hand side becomes $s(a) \oplus s(a)$, which is $s(s(a))$. Thus $s(s(a)) = s(a)$. Since $s(s(a)) = a$, we have $a = s(a)$, i.e., $a \oplus a = a$. □

Corollary 1. *In any model of* (*E*1) + (*E*2)*, the diagonal involution $s(a) = a \oplus a$ is the identity and thus $S(a, a) = (a, a)$ for all a.*

Theorem 3 (Associativity forces triviality)**.** *Let $(X, \oplus)$ satisfy (E1)+(E2). If $\oplus$ is associative, then $|X| = 1$.*

Proof. By Lemma 5, $\oplus$ is idempotent. Assume $\oplus$ is associative. Then for all $a, b \in X$:

$$(b \oplus a) \oplus a = b \oplus (a \oplus a) = b \oplus a,$$

$$a \oplus (a \oplus b) = (a \oplus a) \oplus b = a \oplus b.$$

Thus (E1) implies $b \oplus a = a \oplus b$ for all a, b, i.e., $\oplus$ is commutative. By Lemma 2, commutativity forces equality, hence $a = b$ for all a, b and so $|X| = 1$. □

6 Computer-assisted small-size nonexistence and isomorphism types

The classification in Theorem 2 yields a complete parameterization of all finite models of (E2), enabling exhaustive checks for additional axioms at small sizes.

Proposition 4 (No size-4 and size-5 models of (E1)+(E2))**.** *There is no 4-element model and no 5-element model of (E1)+(E2).*

Proposition 5 (No size-4 and size-5 models of (E2)+(E3))**.** *There is no 4-element model and no 5-element model of (E2)+(E3).*

Proof of Propositions 4 and 5. For $n = 4$, Theorem 2 gives $N(4) = 1200$ labeled models of (E2); for $n = 5$, it gives $N(5) = 786,240$. All these operations were generated exhaustively using the matching-and-orientation parameterization underlying the proof of Theorem 2. Each candidate operation was tested by direct evaluation of the universal identities (E1) and (E3). No candidate satisfied (E1), and no candidate satisfied (E3). □

Corollary 2 (No nontrivial finite models up to size 7)**.** *The full system (E1)-(E3) has no nontrivial finite model of size $n \leq 7$.*

Proof. For $n = 2, 3, 6, 7$, no model of (E2) exists by Theorem 1. For $n = 4, 5$ Propositions 4 and 5 rule out the additional axioms. The case $n = 1$ is the trivial model. $\square$

Proposition 6 (Isomorphism types for $n = 4$ models of (E2))**.** *Up to relabeling of the underlying 4-element set, there are exactly 58 isomorphism classes of size-4 swap-root magmas. Under the natural S_4-relabeling action, the orbit-size distribution among the 1200 labeled models is: 43 classes with orbit size 24, 12 classes with orbit size 12, 3 classes with orbit size 8.*

7 Affine-linear models: (E2) existence and (E1)+(E2) no-go

We first record the exact coefficient constraints for affine-linear realizations of (E2) alone. This yields explicit nontrivial "continuous" examples (e.g. over $\mathbb{C}$). We then show that adding (E1) destroys all affine-linear possibilities over any commutative ring.

Proposition 7 (Affine-linear models of (E2))**.** *Let R be a commutative ring with 1 and let M be an R-module. Fix $u, v \in R$ and $w \in M$, and define*

$$a \oplus b := ua + vb + w.$$

Then $(M, \oplus)$ satisfies (E2) if and only if

$$u^2 + v^2 = 0, \quad 2uv = 1, \quad (u + v + 1)w = 0.$$

Proof. Compute $b \oplus a = ub + va + w$ and $a \oplus b = ua + vb + w$. Then

$$(b \oplus a) \oplus (a \oplus b) = u(ub + va + w) + v(ua + vb + w) + w = (u^2 + v^2)b + (uv + vu)a + (u + v + 1)w.$$

Axiom (E2) requires this to equal a for all $a, b \in M$, hence

$$u^2 + v^2 = 0, \quad uv + vu = 1, \quad (u + v + 1)w = 0.$$

In a commutative ring, $uv + vu = 2uv$, giving the displayed conditions. $\square$

Corollary 3 (A concrete complex affine model of (E2))**.** *On $M = \mathbb{C}$, the operation*

$$a \oplus b := \frac{1+i}{2}a + \frac{1-i}{2}b$$

satisfies (E2).

Proof. Take $u = \frac{1+i}{2}$, $v = \frac{1-i}{2}$ and $w = 0$. Then $u^2 + v^2 = 0$ and $2uv = 1$, so Proposition 7 applies. $\square$

Theorem 4 (Affine-linear no-go for (E1)+(E2))**.** *Let R be a commutative ring with 1 and let M be an R-module. There is no operation of the affine-linear form*

$$a \oplus b = ua + vb + w \quad (u, v \in R, w \in M)$$

that satisfies (E1)+(E2) on M. In particular, there is no such model over any field.

Proof. Assume $a \oplus b = ua + vb + w$ satisfies (E1)+(E2). By Proposition 7, (E2) implies $u^2 + v^2 = 0$ and $2uv = 1$. Next expand (E1). The left-hand side is

$$(b \oplus a) \oplus a = u(ub + va + w) + va + w = u^2 b + (uv + v)a + (u + 1)w,$$

while the right-hand side is

$$a \oplus (a \oplus b) = ua + v(ua + vb + w) + w = v^2 b + (u + vu)a + (v + 1)w.$$

Equating coefficients of b gives $u^2 = v^2$, and equating coefficients of a gives $uv + v = u + vu$, which in a commutative ring reduces to $v = u$. Substituting $u = v$ into $u^2 + v^2 = 0$ yields $2u^2 = 0$, while substituting into $2uv = 1$ yields $2u^2 = 1$, a contradiction. $\square$

8 Solver-based attack on the first open finite case $n = 8$

By Theorem 1 and Corollary 3, the first finite size not ruled out by theory or exhaustion is $n = 8$. However, Theorem 2 gives $N(8) = 2.6719 \times 10^{21}$ labeled (E2)-operations, and even under idempotence the (E2)-space has size $\approx 3.4973 \times 10^{18}$. This makes "enumerate and test" infeasible. A pragmatic approach is to encode the axioms as a finite-domain constraint problem for SMT solvers. An UNSAT result at $n = 8$ would be a definitive nonexistence theorem for that cardinality.

9 Comparison with the Yang-Baxter (braid) relation

Given $S(a, b) = (b \oplus a, a \oplus b)$, the set-theoretic Yang-Baxter equation (YBE) is $(S \times I)(I \times S)(S \times I) = (I \times S)(S \times I)(I \times S)$. Expanding this yields identities not syntactically identical to the screenshot's axiom (E3). Thus, even though (E2) endows S with the unusual property $S^2 = \tau$, the screenshot does not directly impose the braid relation for S.

10 Conjectures and open problems

Conjecture 1. *The only model of (E1)-(E3) is the 1-element model.*

Appendix: Generation scheme for finite (E2)-models

Fix $n \equiv 0, 1 \pmod 4$ and write $X = \{0, ..., n-1\}$.

1. Choose an involution s on X and set $S(a, a) = (s(a), s(a))$ for all a.
2. Let $\mathcal{O} = \{\{(a, b), (b, a)\} : a < b\}$ be the set of off-diagonal τ-orbits (so $|\mathcal{O}| = \binom{n}{2}$). Choose a perfect matching of $\mathcal{O}$.
3. For each matched pair of orbits $\{\{(a, b), (b, a)\}, \{(c, d), (d, c)\}\}$ choose one of two compatible 4-cycles.
4. Define $a \oplus b$ to be the second coordinate of $S(a, b)$.