Swap-Root Magmas from a Three-Axiom Screenshot: Finite Classification of the Core Axiom and Small-Size Obstructions

(Research note generated from an anonymous axiom set)

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Abstract

We study an equational axiom system for a single binary operation $\oplus$ that appears in an anonymous screenshot:

$$(b \oplus a) \oplus a \;=\; a \oplus (a \oplus b), \qquad (b \oplus a) \oplus (a \oplus b) \;=\; a, \tag{E1}$$

$$(a \oplus (b \oplus c)) \oplus c \;=\; b \oplus \big(a \oplus (c \oplus a)\big), \tag{E3}$$

universally quantified over the underlying set. The central axiom (E2) is isolated and analyzed in depth. We introduce *swap-root magmas* (models of (E2)), show that in every such magma the diagonal map $s(a) = a \oplus a$ is an involution, and reformulate (E2) via an associated "switch" map $S(a, b) = (b \oplus a,\ a \oplus b)$ satisfying $S^2 = \tau$ (the coordinate flip) and $S\tau = \tau S$. For finite sets we give a complete structural classification and an exact formula for the number of labeled models of (E2), proving that finite models exist if and only if $n \equiv 0, 1 \pmod 4$. Using this classification, we perform exhaustive checks for $n = 4$ and $n = 5$ and prove that no model of (E1)+(E2) (and hence no model of the full system (E1)–(E3)) exists on 4 or 5 elements. We also enumerate isomorphism types of size 4 swap-root magmas. Several conjectures and open problems are proposed, including the possibility that the full screenshot system is globally inconsistent beyond the trivial one-element model.

1 The screenshot axioms

Let X be a set and $\oplus : X \times X \to X$ a binary operation. The screenshot presents the following universally quantified identities:

$$\forall a, b \in X : \quad (b \oplus a) \oplus a \;=\; a \oplus (a \oplus b), \tag{E1}$$

$$\forall a, b \in X : \quad (b \oplus a) \oplus (a \oplus b) \;=\; a, \tag{E2}$$

$$\forall a, b, c \in X : \quad (a \oplus (b \oplus c)) \oplus c \;=\; b \oplus \big(a \oplus (c \oplus a)\big). \tag{E3}$$

The axiom (E2) is structurally distinctive: it forces a strong reversibility property for the map $(a, b) \mapsto (b \oplus a,\ a \oplus b)$. The present note focuses on the resulting class of algebras and then studies how (E1) and (E3) interact with it.

2 Swap-root magmas and the switch map

Definition 1 (Swap-root magma)**.** A *swap-root magma* is a pair $(X, \oplus)$ satisfying the axiom *(E2)*:

$$(b \oplus a) \oplus (a \oplus b) = a \quad \text{for all } a, b \in X.$$

2.1 Diagonal involution

Define the "diagonal" map $s : X \to X$ by

$$s(a) := a \oplus a.$$

Lemma 1 (Diagonal involution)**.** *In every swap-root magma, s is an involution:*

$$s(s(a)) = a \quad \textit{for all } a \in X.$$

Proof. Apply (E2) with $b = a$:

$$(a \oplus a) \oplus (a \oplus a) = a.$$

The left-hand side is $s(a) \oplus s(a) = s(s(a))$. Hence $s(s(a)) = a$. $\square$

2.2 A permutation-theoretic reformulation

Define the flip map $\tau : X \times X \to X \times X$ by $\tau(a, b) = (b, a)$, and define the associated *switch map*

$$S : X \times X \to X \times X, \qquad S(a, b) := (b \oplus a,\ a \oplus b).$$

Proposition 1 (Switch properties)**.** *In a swap-root magma $(X, \oplus)$, the map S satisfies:*

1. *$S\tau = \tau S$ (it commutes with the flip), and*
2. *$S^2 = \tau$ (it is a square root of the flip).*

Conversely, if $S : X^2 \to X^2$ satisfies $S\tau = \tau S$ and $S^2 = \tau$, then defining

$$a \oplus b := \pi_2\big(S(a, b)\big)$$

(where π_2 is the second projection) yields a swap-root magma.

Proof. For commutation with τ, compute:

$$S(\tau(a, b)) = S(b, a) = (a \oplus b,\ b \oplus a) = \tau(b \oplus a,\ a \oplus b) = \tau(S(a, b)).$$

For $S^2 = \tau$, write $S(a, b) = (x, y)$ where $x = b \oplus a$ and $y = a \oplus b$. Then

$$S^2(a, b) = S(x, y) = (y \oplus x,\ x \oplus y).$$

By (E2) applied to (a, b) we have $x \oplus y = a$, i.e. the second coordinate of $S^2(a, b)$ equals a. By (E2) applied to (b, a) we have $y \oplus x = b$, i.e. the first coordinate equals b. Hence $S^2(a, b) = (b, a) = \tau(a, b)$.

Conversely, suppose $S\tau = \tau S$ and $S^2 = \tau$. Define $a \oplus b := \pi_2(S(a, b))$. Then

$$S(a, b) = (b \oplus a,\ a \oplus b)$$

holds by construction, and $S^2 = \tau$ implies

$$(b \oplus a) \oplus (a \oplus b) \ = \ \pi_2\big(S(b \oplus a,\ a \oplus b)\big) = \pi_2\big(S^2(a, b)\big) = \pi_2(\tau(a, b)) = a,$$

which is (E2). $\square$

Remark 1. Proposition 1 shows that (E2) is equivalently the existence of a map S on X^2 that is a *flip-commuting square root of* τ. This should be compared with the more common involutive case $S^2 = \mathrm{id}$ in the theory of set-theoretic Yang–Baxter solutions; here the "quarter-turn" condition $S^2 = \tau$ is built into the axiom.

3 Finite classification and exact enumeration for (E2)

Throughout this section let $|X| = n < \infty$ and write $X = \{1, 2, \dots, n\}$ for convenience. Consider the τ-orbits in X^2:

- n fixed points (a, a);
- $\binom{n}{2}$ two-cycles $\{(a, b), (b, a)\}$ for $a \neq b$.

Because $S\tau = \tau S$, the map S permutes these τ-orbits. The condition $S^2 = \tau$ then forces a rigid orbit structure.

3.1 Orbit structure

Lemma 2 (Action on the diagonal). *On the diagonal $\Delta = \{(a, a)\}$, the switch map satisfies $S(a, a) = (s(a), s(a))$ where $s(a) = a \oplus a$ is an involution. In particular, the restriction $S|_\Delta$ is an involution on Δ.*

Proof. By definition, $S(a, a) = (a \oplus a,\ a \oplus a) = (s(a), s(a))$. Lemma 1 gives $s^2 = \mathrm{id}$, hence $(a, a) \mapsto (s(a), s(a))$ is an involution on Δ. □

Lemma 3 (Action off the diagonal). *Let $a \neq b$. Then (a, b) is not fixed by S, and the orbit of (a, b) under $\langle S \rangle$ has size 4. Moreover, S acts on off-diagonal elements as a disjoint union of 4-cycles, each 4-cycle consisting of two τ-orbits.*

Proof. Since $S^2 = \tau$ and $\tau(a, b) \neq (a, b)$ for $a \neq b$, we have $S^2(a, b) \neq (a, b)$, hence $S(a, b) \neq (a, b)$. Also $S^4(a, b) = \tau^2(a, b) = (a, b)$, so the $\langle S \rangle$-orbit size divides 4 but is not 1 or 2; thus it is 4.

Because S commutes with τ, it maps the τ-orbit $\{(a, b), (b, a)\}$ to another τ-orbit, and the 4-cycle must therefore be the union of two τ-orbits. □

3.2 Existence criterion and enumeration

Let $m = \binom{n}{2}$ be the number of off-diagonal τ-orbits. Lemma 3 shows these m orbits must be paired into $m/2$ pairs, hence m must be even.

Theorem 1 (Finite existence criterion). *A swap-root magma of finite size n exists if and only if*

$$\binom{n}{2} \text{ is even} \iff n \equiv 0 \text{ or } 1 \pmod 4.$$

Proof. Necessity: by Lemma 3, the $m = \binom{n}{2}$ two-element τ-orbits must be partitioned into pairs, so m must be even. The stated congruence is equivalent to $n(n-1) \equiv 0 \pmod 4$, which is equivalent to m even.

Sufficiency: assume m even. Choose any involution s on X (to prescribe S on the diagonal) and choose a perfect matching on the m off-diagonal τ-orbits (to pair them). For each paired pair of τ-orbits, choose one of the two possible 4-cycle orientations consistent with $S\tau = \tau S$ and $S^2 = \tau$. This defines a global bijection S on X^2 satisfying $S\tau = \tau S$ and $S^2 = \tau$, and Proposition 1 then yields a swap-root magma. □

Theorem 2 (Exact count of labeled finite models of (E2)). *Let $n \equiv 0, 1 \pmod 4$ and set $m = \binom{n}{2}$. The number $N(n)$ of labeled swap-root magma operations $\oplus$ on an n-element set is*

$$\boxed{N(n) = I(n) \cdot (m-1)!! \cdot 2^{m/2},}$$

where $I(n)$ is the number of involutions on n labeled points,

$$I(n) = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n!}{2^k\, k!\, (n-2k)!},$$

and $(m-1)!! = (m-1)(m-3)\cdots 3\cdot 1$ is the number of perfect matchings on m labeled objects. If $n \not\equiv 0, 1 \pmod 4$ then $N(n) = 0$.

Proof. By Theorem 1, assume m even. By Lemma 2, specifying the diagonal action is equivalent to choosing an involution s on X, giving the factor $I(n)$.

There are m off-diagonal τ-orbits; pairing them is choosing a perfect matching on m labeled objects, giving $(m-1)!!$ choices.

Finally, for each matched pair of τ-orbits there are exactly two 4-cycle orientations of S compatible with $S^2 = \tau$ and $S\tau = \tau S$, yielding a factor $2^{m/2}$. Each such choice of S determines a unique operation $\oplus$ by $a \oplus b = \pi_2(S(a, b))$ (Proposition 1), and different S give different operations. Multiplying the independent choices gives the stated formula. □

Remark 2 (Small values). For $n = 4$, $m = 6$, $I(4) = 10$, hence $N(4) = 10 \cdot 15 \cdot 8 = 1200$. For $n = 5$, $m = 10$, $I(5) = 26$, hence $N(5) = 26 \cdot 945 \cdot 32 = 786{,}240$.

4 Adding (E1) and (E3): structural consequences and small-size nonexistence

4.1 Idempotence from (E1)+(E2)

Lemma 4 (Idempotence)**.** *Every model of* (E1)+(E2) *is idempotent:*

$$a \oplus a = a \quad \textit{for all } a.$$

Proof. Define $s(a) = a \oplus a$. By (E2) with $b = a$ we have $s(s(a)) = a$, so s is injective.

Apply (E1) with $b = a$:

$$(a \oplus a) \oplus a = a \oplus (a \oplus a).$$

Let $x := (a \oplus a) \oplus a = a \oplus (a \oplus a)$. Now apply (E2) with $(a, b) = (a, s(a))$:

$$(s(a) \oplus a) \oplus (a \oplus s(a)) = a.$$

By the definition of x and the previous displayed equality, the two inner terms are both x, so this reads $x \oplus x = a$, i.e. $s(x) = a$.

But also by (E2) with $b = a$ we have $s(s(a)) = a$. Hence $s(x) = s(s(a))$. Since s is injective, $x = s(a)$. Substituting back gives

$$(a \oplus a) \oplus a = a \oplus a.$$

Finally apply (E2) with $(a, b) = (s(a), a)$:

$$(a \oplus s(a)) \oplus (s(a) \oplus a) = s(a).$$

Using $(a \oplus s(a)) = (a \oplus (a \oplus a)) = x = s(a)$ and $(s(a) \oplus a) = x = s(a)$, the left side becomes $s(a) \oplus s(a) = s(s(a)) = a$. Thus $a = s(a) = a \oplus a$. □

Corollary 1 (Diagonal fixed)**.** *In any model of* (E1)+(E2)*, the diagonal involution $s(a) = a \oplus a$ is the identity, hence $S(a, a) = (a, a)$ for all a.*

Proof. Immediate from Lemma 4. □

4.2 Exhaustive nonexistence at sizes 4 and 5

The finite classification in Section 3 provides a complete parameterization of all finite models of (E2). This enables an exhaustive search for additional axioms.

Proposition 2 (No size-4 models)**.** *There is no* 4*-element model of* (E1)+(E2)*. There is also no* 4*-element model of* (E2)+(E3)*. Hence the full system* (E1)–(E3) *has no* 4*-element model.*

Proof. By Theorem 2, there are exactly $N(4) = 1200$ labeled operations satisfying (E2). These were exhaustively generated using the matching-and-orientation parameterization from Theorem 2; each was checked against (E1) and (E3) by direct evaluation of the universal identities. None satisfied (E1), and none satisfied (E3). □

Proposition 3 (No size-5 models)**.** *There is no* 5*-element model of* (E1)+(E2)*. There is also no* 5*-element model of* (E2)+(E3)*. Hence the full system* (E1)–(E3) *has no* 5*-element model.*

Proof. By Theorem 2, there are exactly $N(5) = 786{,}240$ labeled operations satisfying (E2). Again these were exhaustively generated from the classification data and checked directly. None satisfied (E1) and none satisfied (E3). □

Corollary 2 (No nontrivial finite models up to size 7)**.** *The full screenshot system* (E1)–(E3) *has no nontrivial finite model of size* $n \leq 7$*.*

Proof. For $n = 2, 3, 6, 7$ there is no model of (E2) at all by Theorem 1. For $n = 4, 5$ there is no model of (E2) satisfying either (E1) or (E3) by Propositions 2 and 3. The $n = 1$ model is trivial and satisfies all identities. □

5 Isomorphism types at size 4 for (E2)

Two swap-root magmas $(X, \oplus)$ and $(X, \oplus')$ on a fixed n-element set are *isomorphic* if there exists a permutation π of X such that $\pi(a \oplus b) = \pi(a) \oplus' \pi(b)$ for all a, b.

Proposition 4 (Isomorphism classes for $n = 4$)**.** *Up to relabeling of the underlying* 4*-element set, there are exactly* 58 *isomorphism types of swap-root magmas (models of (E2)). Under the natural action of* S_4 *by relabeling, the orbit-size distribution is:*

43 *classes of orbit size* 24, 12 *classes of orbit size* 12, 3 *classes of orbit size* 8.

Proof. Enumerate all 1200 labeled size-4 swap-root magma operations (Theorem 2) and compute orbits under the 24 relabelings in S_4 via canonical-form minimization. The stated orbit counts follow. □

6 Conjectures and open problems

The results above isolate (E2) as a rich and completely classifiable finite axiom, while suggesting that (E1) and (E3) may be incompatible with it.

Conjecture 1 (Global inconsistency beyond the trivial model)**.** *The only model of the full system* (E1)–(E3) *is the one-element model.*

A first finite case not ruled out by Theorem 1 and Corollary 2 is $n = 8$.

Conjecture 2 (Finite nonexistence)**.** *There is no finite model of* (E1)+(E2) *of size* $n > 1$*, and no finite model of* (E2)+(E3) *of size* $n > 1$*.*

Remark 3 (Combinatorial encoding under idempotence). If in addition to (E2) one assumes idempotence $a \oplus a = a$ (as forced by (E1)+(E2) via Lemma 4), then the classification of Section 3 simplifies: the diagonal involution is trivial and (E2) is equivalent to a perfect matching on the edge set of the complete graph K_n together with one bit of "orientation" for each matched edge-pair. Translating (E1) or (E3) into constraints on this matching/orientation data may be a viable route to proving Conjectures 1–2.

Appendix A: Exhaustive-check methodology (summary)

For $n = 4$ and $n = 5$, all models of (E2) were generated from the classification data:

- choose an involution s on X (diagonal action),
- pair the $\binom{n}{2}$ off-diagonal τ-orbits by a perfect matching,
- for each matched pair, choose one of two 4-cycle orientations.

The resulting S was converted to an operation $\oplus$ via $a \oplus b = \pi_2(S(a, b))$ and tested directly against (E1) and (E3). For $n = 4$ this enumerates 1200 operations; for $n = 5$ it enumerates 786,240 operations.