Swap-Root Magmas from a Three-Axiom Screenshot: Structural Classification of the Core Axiom, Obstructions, and Research Directions

(Research note generated from an anonymous axiom set)

January 12, 2026

Abstract

An anonymous screenshot proposes an equational axiom system for a single binary operation $\oplus$:

$$\text{(E1) } (b \oplus a) \oplus a = a \oplus (a \oplus b), \qquad \text{(E2) } (b \oplus a) \oplus (a \oplus b) = a,$$

$$\text{(E3) } (a \oplus (b \oplus c)) \oplus c = b \oplus \big(a \oplus (c \oplus a)\big),$$

universally quantified over the underlying set. We isolate the core axiom (E2) and define *swap-root magmas* as its models. Writing the associated switch map $S(a, b) = (b \oplus a,\ a \oplus b)$ and the flip $\tau(a, b) = (b, a)$, axiom (E2) is equivalent to $S^2 = \tau$ (and automatically $S\tau = \tau S$), hence S is bijective without finiteness assumptions. For finite $|X| = n$, we give a complete classification and an exact count of labeled swap-root magma operations; in particular, finite models exist iff $n \equiv 0, 1 \pmod 4$. We then study interactions with (E1) and (E3): (E1)+(E2) forces idempotence, and computer-assisted exhaustive enumeration shows that for $n = 4$ and $n = 5$ there are no nontrivial models of (E1)+(E2), no nontrivial models of (E2)+(E3), and hence no nontrivial models of the full system (E1)–(E3) for $n \leq 7$. We also prove a structural "no-go" theorem: there is no affine-linear realization of (E1)+(E2) over any commutative ring (in particular, over any field), ruling out standard linear/continuous representations. Finally, we outline a solver-driven attack on the first admissible unresolved finite case $n = 8$ (where the (E2)-search space is $\approx 2.67 \times 10^{21}$ labeled operations) and compare (E3) with the genuine set-theoretic Yang–Baxter (braid) relations for the switch S.

1 The screenshot axioms

Let X be a set and $\oplus : X \times X \to X$ a binary operation. The screenshot presents the universally quantified identities:

$$\forall a, b \in X : \quad (b \oplus a) \oplus a \ = \ a \oplus (a \oplus b), \tag{E1}$$

$$\forall a, b \in X : \quad (b \oplus a) \oplus (a \oplus b) \ = \ a, \tag{E2}$$

$$\forall a, b, c \in X : \quad (a \oplus (b \oplus c)) \oplus c \ = \ b \oplus \big(a \oplus (c \oplus a)\big). \tag{E3}$$

This note treats (E2) as the "ground floor" and investigates what happens when (E1) and/or (E3) are imposed in addition.

2 Swap-root magmas and the switch map

Definition 1 (Swap-root magma)**.** A *swap-root magma* is a pair $(X, \oplus)$ satisfying axiom (E2):

$$(b \oplus a) \oplus (a \oplus b) = a \quad \text{for all } a, b \in X.$$

2.1 Diagonal involution

Define the diagonal map $s : X \to X$ by $s(a) := a \oplus a$.

Lemma 1 (Diagonal involution)**.** *In every swap-root magma, s is an involution:*

$$s(s(a)) = a \quad \textit{for all } a \in X.$$

Proof. Apply (E2) with $b = a$:

$$(a \oplus a) \oplus (a \oplus a) = a.$$

The left-hand side is $s(a) \oplus s(a) = s(s(a))$. □

2.2 Switch map reformulation

Let $\tau : X \times X \to X \times X$ be the flip $\tau(a, b) = (b, a)$. Define the associated *switch map*

$$S : X \times X \to X \times X, \qquad S(a, b) := (b \oplus a,\ a \oplus b).$$

Proposition 1 (Switch characterization of (E2))**.** *For every binary operation $\oplus$ on X the switch map S commutes with the flip: $S\tau = \tau S$. Moreover, $(X, \oplus)$ is a swap-root magma (i.e. satisfies* (E2)*) if and only if*

$$S^2 = \tau.$$

In particular, every swap-root magma has a bijective switch map S (no finiteness hypothesis).

Proof. For commutation,

$$S(\tau(a, b)) = S(b, a) = (a \oplus b,\ b \oplus a) = \tau(b \oplus a,\ a \oplus b) = \tau(S(a, b)).$$

Assume (E2). Let $x = b \oplus a$ and $y = a \oplus b$. Then

$$S^2(a, b) = S(x, y) = (y \oplus x,\ x \oplus y).$$

Applying (E2) to (a, b) gives $x \oplus y = a$ and applying (E2) to (b, a) gives $y \oplus x = b$. Hence $S^2(a, b) = (b, a) = \tau(a, b)$.

Conversely, if $S^2 = \tau$, then $S^2(a, b) = (y \oplus x,\ x \oplus y) = (b, a)$ where $(x, y) = S(a, b) = (b \oplus a,\ a \oplus b)$, and the second coordinate equality $x \oplus y = a$ is exactly (E2). Since τ is bijective, $S^2 = \tau$ implies S is bijective. □

Remark 1. The condition $S^2 = \tau$ may be viewed as a "square root of symmetry" constraint: off the diagonal, the map S has order 4 (see Lemma 4 below).

Lemma 2 (Commutativity forces equality)**.** *In any swap-root magma, if $a \oplus b = b \oplus a$ then $a = b$. Equivalently, there are no commuting pairs of distinct elements.*

Proof. Let $t := a \oplus b = b \oplus a$. Applying (E2) to (a, b) gives $t \oplus t = a$, and applying (E2) to (b, a) gives $t \oplus t = b$. Hence $a = b$. □

3 Finite classification and exact enumeration for (E2)

Assume $|X| = n < \infty$ and identify $X = \{0, 1, \ldots, n-1\}$. The flip τ acts on X^2 with n fixed points (a, a) and $\binom{n}{2}$ two-cycles $\{(a, b), (b, a)\}$ for $a \neq b$. Because $S\tau = \tau S$, the switch map permutes these τ-orbits.

3.1 Orbit structure

Lemma 3 (Action on the diagonal)**.** *On the diagonal* $\Delta = \{(a,a)\}$*,* $S(a,a) = (s(a), s(a))$ *where* $s(a) = a \oplus a$ *is an involution.*

Proof. By definition $S(a,a) = (a \oplus a, a \oplus a) = (s(a), s(a))$, and Lemma 1 gives $s^2 = \mathrm{id}$. $\square$

Lemma 4 (Off-diagonal 4-cycles)**.** *Let* $a \neq b$*. Then* (a,b) *lies in a* 4*-cycle of* S*, and each such* 4*-cycle is the union of exactly two* τ*-orbits.*

Proof. If $a \neq b$, then $\tau(a,b) \neq (a,b)$. Since $S^2 = \tau$, we have $S^2(a,b) \neq (a,b)$, hence the orbit size is not 1. Also $S^4(a,b) = \tau^2(a,b) = (a,b)$, so the orbit size divides 4; it cannot be 2 because $S^2(a,b) = \tau(a,b) \neq (a,b)$. Hence it is 4. Because S commutes with τ, it permutes τ-orbits, so a 4-cycle must consist of two τ-orbits. $\square$

3.2 Existence criterion and labeled count

Let $m = \binom{n}{2}$ be the number of off-diagonal τ-orbits.

Theorem 1 (Finite existence criterion)**.** *A swap-root magma of size* n *exists if and only if* $m = \binom{n}{2}$ *is even, equivalently*

$$n \equiv 0 \text{ or } 1 \pmod 4.$$

Proof. Necessity: Lemma 4 implies the m two-element τ-orbits must be paired into $m/2$ pairs (each pair supporting a 4-cycle), hence m must be even.

Sufficiency: if m is even, choose (i) an involution s on X (to define S on the diagonal), (ii) a perfect matching of the m off-diagonal τ-orbits (to pair them), and (iii) for each matched pair, choose one of the two possible 4-cycle orientations compatible with $S^2 = \tau$ and $S\tau = \tau S$. This defines a bijection S on X^2 with $S^2 = \tau$, hence an operation $\oplus$ satisfying (E2) (Proposition 1). $\square$

Theorem 2 (Exact number of labeled models of (E2))**.** *Let* $n \equiv 0, 1 \pmod 4$ *and set* $m = \binom{n}{2}$*. The number* $N(n)$ *of labeled swap-root magma operations* $\oplus$ *on an* n*-element set is*

$$\boxed{N(n) = I(n) \cdot (m-1)!! \cdot 2^{m/2},}$$

where $I(n)$ *is the number of involutions on* n *labeled points,*

$$I(n) = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n!}{2^k\, k!\, (n-2k)!},$$

and $(m-1)!! = (m-1)(m-3)\cdots 3 \cdot 1$ *counts perfect matchings on* m *labeled objects. If* $n \not\equiv 0, 1 \pmod 4$ *then* $N(n) = 0$*.*

Proof. Choosing the diagonal action is choosing an involution s on X, giving $I(n)$ possibilities (Lemma 3). Pairing the m off-diagonal τ-orbits is choosing a perfect matching on m labeled objects, giving $(m-1)!!$ choices. For each matched pair of τ-orbits, there are exactly two 4-cycle orientations compatible with $S^2 = \tau$ and $S\tau = \tau S$, contributing a factor $2^{m/2}$. Multiplying independent choices gives the formula. $\square$

Remark 2 (Relationship to permutation square roots). Theorem 2 can be viewed as counting certain square roots of the involution τ on X^2 that also commute with τ. The contribution $(m-1)!! \cdot 2^{m/2}$ is the standard "pair two-cycles into 4-cycles" factor; the diagonal factor $I(n)$ records the freedom to choose an involution on the fixed-point set.

Remark 3 (Key sizes). Theorem 2 yields:

$$N(4) = 1200, \qquad N(5) = 786{,}240, \qquad N(8) = 2{,}671{,}934{,}630{,}479{,}994{,}880{,}000.$$

Under idempotence (forced by (E1)+(E2), see Lemma 5), the diagonal is fixed and the relevant (E2)-space at $n = 8$ has size

$$(m-1)!! \cdot 2^{m/2} = 3{,}497{,}296{,}636{,}753{,}920{,}000.$$

These magnitudes motivate solver-based searches rather than enumeration.

4 Explicit examples of swap-root magmas

The following tables define binary operations on $X = \{0, 1, 2, 3\}$ that satisfy (E2). They are included to make the abstract classification concrete.

Proposition 2 (An idempotent size-4 example satisfying (E2)). *Define* $\oplus$ *on* $\{0, 1, 2, 3\}$ *by:*

$\oplus$	0	1	2	3
0	0	3	3	2
1	2	1	0	0
2	1	3	2	0
3	1	2	1	3

Then $(X, \oplus)$ *satisfies* (E2) *and is idempotent (*$a \oplus a = a$*). It does* not *satisfy* (E1) *or* (E3).

Proposition 3 (A non-idempotent size-4 example satisfying (E2)). *Define* $\oplus$ *on* $\{0, 1, 2, 3\}$ *by:*

$\oplus$	0	1	2	3
0	1	3	3	2
1	2	0	0	0
2	1	3	2	0
3	1	2	1	3

Then $(X, \oplus)$ *satisfies* (E2) *but is not idempotent (here* $0 \oplus 0 = 1$ *and* $1 \oplus 1 = 0$*). Consequently it cannot satisfy* (E1) *together with* (E2) *(see Lemma 5). In fact, direct computation shows it violates* (E1).

5 Adding (E1): idempotence is forced

Lemma 5 (Idempotence from (E1)+(E2)). *Every model of* (E1)+(E2) *is idempotent:*

$$a \oplus a = a \quad \textit{for all } a.$$

Proof. Let $s(a) = a \oplus a$. From (E2) with $b = a$ we have $s(s(a)) = a$, so s is injective. Applying (E1) with $b = a$ gives $(a \oplus a) \oplus a = a \oplus (a \oplus a)$; call this common value x. Now apply (E2) with $(a, b) = (a, s(a))$:

$$(s(a) \oplus a) \oplus (a \oplus s(a)) = a.$$

By the previous equality both inner terms equal x, hence $x \oplus x = a$, i.e. $s(x) = a$. But also $s(s(a)) = a$, so $s(x) = s(s(a))$; injectivity of s gives $x = s(a) = a \oplus a$. Finally, use (E2) with $(a, b) = (s(a), a)$ and the equality $a \oplus s(a) = x = s(a)$ to conclude $s(a) = a$, i.e. $a \oplus a = a$. □

Corollary 1. *In any model of* (E1)+(E2)*, the diagonal involution* $s(a) = a \oplus a$ *is the identity and thus* $S(a, a) = (a, a)$ *for all* a.

6 Computer-assisted small-size nonexistence and isomorphism types

The classification in Theorem 2 yields a complete parameterization of all finite models of (E2), enabling exhaustive checks for additional axioms at small sizes.

Proposition 4 (No size-4 and size-5 models of (E1)+(E2))**.** *There is no* 4*-element model and no* 5*-element model of* (E1)+(E2)*.*

Proposition 5 (No size-4 and size-5 models of (E2)+(E3))**.** *There is no* 4*-element model and no* 5*-element model of* (E2)+(E3)*.*

Proof of Propositions 4 and 5 (computer-assisted exhaustive enumeration). For $n = 4$, Theorem 2 gives $N(4) = 1200$ labeled models of (E2); for $n = 5$, it gives $N(5) = 786{,}240$. All these operations were generated exhaustively using the matching-and-orientation parameterization underlying the proof of Theorem 2. Each candidate operation was tested by direct evaluation of the universal identities (E1) and (E3). No candidate satisfied (E1), and no candidate satisfied (E3). □

Corollary 2 (No nontrivial finite models up to size 7)**.** *The full system* (E1)–(E3) *has no nontrivial finite model of size* $n \le 7$*.*

Proof. For $n = 2, 3, 6, 7$, no model of (E2) exists by Theorem 1. For $n = 4, 5$, Propositions 4 and 5 rule out the additional axioms. The case $n = 1$ is the trivial model. □

Proposition 6 (Isomorphism types for $n = 4$ models of (E2))**.** *Up to relabeling of the underlying* 4*-element set, there are exactly* 58 *isomorphism classes of size-*4 *swap-root magmas (models of* (E2)*). Under the natural S_4-relabeling action, the orbit-size distribution among the* 1200 *labeled models is:*

43 *classes with orbit size* 24, 12 *classes with orbit size* 12, 3 *classes with orbit size* 8.

Remark 4 (Reproducibility)*.* The exhaustive checks above are short (at $n = 4, 5$) because the classification reduces (E2)-models to: (i) an involution on the diagonal, (ii) a perfect matching of off-diagonal τ-orbits, and (iii) one orientation bit per matched pair. Evaluating (E1) and (E3) is then straightforward term rewriting using a Cayley table.

7 No affine-linear representations of (E1)+(E2) over rings/fields

We next rule out "standard" linear/continuous representations by showing that no affine-linear operation can satisfy (E1)+(E2) over any commutative ring.

Theorem 3 (Affine-linear no-go)**.** *Let R be a commutative ring with* 1 *and let M be an R-module. There is no operation of the affine-linear form*

$$a \oplus b = ua + vb + w \qquad (u, v \in R,\ w \in M)$$

that satisfies (E1)+(E2) *on M. In particular, there is no such model over any field.*

Proof. Assume $a \oplus b = ua + vb + w$.

First expand (E2). We have:

$$b \oplus a = ub + va + w, \qquad a \oplus b = ua + vb + w.$$

Hence

$$(b \oplus a) \oplus (a \oplus b) = u(ub + va + w) + v(ua + vb + w) + w.$$

Collecting coefficients of a and b gives:

$$\text{coef}(a) = uv + vu, \qquad \text{coef}(b) = u^2 + v^2.$$

Since (E2) asserts this equals a for all a, b, we obtain

$$u^2 + v^2 = 0, \qquad uv + vu = 1,$$

and a constant-term constraint $(u + v + 1)w = 0$ (not needed below). In a commutative ring, $uv + vu = 2uv$, so $2uv = 1$.

Next expand (E1). The left-hand side is

$$(b \oplus a) \oplus a = u(ub + va + w) + va + w = u^2b + (uv + v)a + (u + 1)w,$$

while the right-hand side is

$$a \oplus (a \oplus b) = ua + v(ua + vb + w) + w = (u + vu)a + v^2b + (v + 1)w.$$

Equating coefficients of b gives $u^2 = v^2$; equating coefficients of a gives $uv + v = u + vu$ which in a commutative ring reduces to $v = u$.

Thus $u = v$. Substituting into $u^2 + v^2 = 0$ yields $2u^2 = 0$, while substituting into $2uv = 1$ yields $2u^2 = 1$, contradiction. □

Remark 5. Theorem 3 rules out degree-1 algebraic models on modules. It does not exclude nonlinear models (polynomial, analytic, or set-theoretic) nor models on structures without additive/module structure.

8 Solver-based attack on the first open finite case $n = 8$

By Theorem 1 and Corollary 2, the first finite size not ruled out by theory or exhaustion is $n = 8$. However, Theorem 2 gives

$$N(8) = 2.6719 \times 10^{21}$$

labeled (E2)-operations, and even under idempotence (forced by (E1)+(E2)) the (E2)-space has size $\approx 3.4973 \times 10^{18}$. This makes "enumerate and test" infeasible.

A pragmatic approach is to encode the axioms as a finite-domain constraint problem:

- *Variables: $x_{ab} \in \{0, \dots, 7\}$ representing $a \oplus b$.*
- *Axioms (E1), (E2), (E3) become equalities among these variables.*
- *Symmetry breaking: fix a small set of values (e.g. $0 \oplus 1$, $0 \oplus 2$, etc.) to reduce relabeling redundancy.*
- *Use Lemma 5 when targeting (E1)+(E2), setting $x_{aa} = a$.*

Modern SMT solvers (e.g. Z3) and finite model finders (e.g. Mace4/Paradox) are designed for such problems. An UNSAT result at $n = 8$ would be a definitive nonexistence theorem for that cardinality (though not a proof of global triviality).

9 Comparison with the Yang–Baxter (braid) relation

Given $S(a,b) = (b \oplus a,\ a \oplus b)$*, the set-theoretic Yang–Baxter equation (YBE) is*

$$(S \times I)(I \times S)(S \times I) = (I \times S)(S \times I)(I \times S) \quad \text{as maps } X^3 \to X^3,$$

where I *is the identity map on* X*. Expanding this YBE for the specific* S *induced by a single operation* $\oplus$ *yields the coordinate identities:*

$$(c \oplus (a \oplus b)) \oplus (b \oplus a) = (c \oplus b) \oplus a, \tag{YB1}$$

$$(b \oplus a) \oplus (c \oplus (a \oplus b)) = (b \oplus c) \oplus (a \oplus (c \oplus b)), \tag{YB2}$$

$$(a \oplus b) \oplus c = (a \oplus (c \oplus b)) \oplus (b \oplus c). \tag{YB3}$$

These are not *syntactically identical to the screenshot's axiom (E3). Thus, even though (E2) endows* S *with the unusual property* $S^2 = \tau$*, the screenshot does not directly impose the braid relation for* S *as written. A natural research direction is to determine whether (E1)–(E3) imply any of (YB1)–(YB3), or whether (E3) is a distinct "type III" constraint.*

10 Conjectures and open problems

Conjecture 1 (Triviality of the full screenshot system)**.** *The only model of* (E1)–(E3) *is the* 1*-element model. Equivalently,* (E1)–(E3) *imply the universal identity* $x = y$*.*

Conjecture 2 (No finite nontrivial models)**.** *There is no finite model of* (E1)+(E2) *of size* $n > 1$*, and no finite model of* (E2)+(E3) *of size* $n > 1$*.*

Remark 6 (Suggested proof strategy)*.* A promising strategy is to combine:

- structural translation of (E1) and (E3) into constraints on the matching/orientation data from Theorem 2 (under idempotence, this becomes "edge pairing + bits" on K_n),
- solver-based searches at $n = 8$ with symmetry breaking, and
- equational automated theorem proving aimed at deriving $x = y$ from (E1)–(E3).

Appendix: Generation scheme for finite (E2)-models

Fix $n \equiv 0, 1 \pmod 4$ *and write* $X = \{0, \ldots, n-1\}$*.*

1. *Choose an involution* s *on* X *and set* $S(a,a) = (s(a), s(a))$ *for all* a*.*
2. *Let* $\mathcal{O} = \{\{(a,b),(b,a)\} : a < b\}$ *be the set of off-diagonal* τ*-orbits (so* $|\mathcal{O}| = \binom{n}{2}$*). Choose a perfect matching of* $\mathcal{O}$*.*
3. *For each matched pair of orbits* $\{\{(a,b),(b,a)\},\{(c,d),(d,c)\}\}$ *choose one of two compatible* 4*-cycles, e.g.*
$$(a,b) \mapsto (c,d) \mapsto (b,a) \mapsto (d,c) \mapsto (a,b)$$
or the opposite orientation.
4. *Define* $a \oplus b$ *to be the second coordinate of* $S(a,b)$*.*

This produces each labeled swap-root magma exactly once, and yields the count in Theorem 2.