Lecture 1, 4/3/23

We will begin thinking about homological algebra in a more categorical way. We use "methods of homological algebra," by Gelfond-Manin (Gelfond & Manin?). Here is a brief overview of some things we will see during the course:

- Additive and Abelian categories
- Example: category of sheaves of Abelian groups on a topological space
- Derived category (objects: complexes of objects in a given Abelian category, morphisms: a quasi-isomorphism of complexes becomes an isomorphism of objects)
- Derived functors, $f_*, f^*, f_!, f^!, \mathbb{D}, \cdots$, Hom for sheaves
- Triangulated categories
- t-structures
- Examples: perverse sheaves

Let's remind ourselves of some definitions:

Additive & Abelian categories

Example 0.1. (a) Ab, the category of Abelian groups with group homomorphisms.

- (b) R Mod, the category of R-modules, with R-module homomorphisms as morphisms.
- (c) SAB, PAB, the categories of sheaves of Abelian groups and presheaves of Abelian groups
- (d) Sheaves of modules over a ringed space
- (e) (quasi)-coherent sheaves (ask Zhao)

Definition 0.1. An Abelian category contains the following information:

1. Any hom-set $\operatorname{Hom}_{\mathscr{C}}(X,Y)$ is an Abelian group (+), and the composition of morphisms is bi-additive

In particular:

• Hom_{\mathscr{C}} is a functor $\mathscr{C}^{\circ} \times \mathscr{C} \to \mathsf{Ab}$. We notate the first factor with the $^{\circ}$

- $0 \in \operatorname{Hom}_{\mathscr{C}}(X,Y)$ for any objects X,Y of \mathscr{C}
- **2.** There exists a zero object $0 \in \mathcal{C}$, that is an object such that $\operatorname{Hom}_{\mathscr{C}}(0,0) = 0$.

This gives: $\operatorname{Hom}_{\mathscr{C}}(0,X)=0$, $\operatorname{Hom}_{\mathscr{C}}(X,0)=0$ for all objects X of \mathscr{C} .

We know that $\text{Hom}_{\mathscr{C}}(0,0)$ consists of one object. In particular, it must be $\text{Id}_0 = 0$. So

$$(0 \xrightarrow{f} X) = (0 \xrightarrow{\operatorname{Id}_0} 0 \xrightarrow{f} X) = (0 \xrightarrow{0} X)$$

Any two zero objects are isomorphic.

3. For any $X_1, X_2 \in \mathcal{C}$, there exists an object Y and morphisms

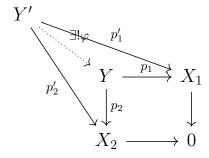
$$X_1 \stackrel{i_1}{\underbrace{\hspace{1cm}}} Y \stackrel{i_2}{\underbrace{\hspace{1cm}}} X_2$$

such that

$$p_1 i_1 = \operatorname{Id}_{X_1}$$

 $p_2 i_2 = \operatorname{Id}_{X_2}$
 $i_1 p_1 + i_2 p_2 = \operatorname{Id}_Y$
 $p_2 i_1 = p_1 i_2 = 0$

Lemma 1. We have cartesian diagram



That is, for any Y', with morphisms p'_1, p'_2 as in the diagram, there is a morphism from Y' to Y making the diagram commute. Similarly, we have co-cartesian diagram

$$Y \stackrel{i_1}{\longleftarrow} X_1$$

$$\downarrow i_2 \qquad \qquad \uparrow$$

$$X_2 \longleftarrow 0$$

Proof. We need to construct $\varphi: Y' \to Y$ such that $p'_1 = p_1 \varphi$ and $p'_2 = p_2 \varphi$ Take $\varphi = i_1 p'_1 + i_2 p'_2$. Then

$$p_1 \circ \varphi = \underbrace{p_1 i_1}_{= \mathrm{Id}_{X_1}} p'_1 + \underbrace{p_1 i_2}_{=0} p'_2 = p'_1$$

The uniqueness of φ can be verified as an exercise

Definition 0.2. An <u>additive category</u> is one which satisfies only the first three of these axioms

To state the 4th axiom of Abelian categories, we need more notations:

Definition 0.3. Let A_1, A_2 be objects of \mathscr{C} , and let $\varphi : X \to Y$.

- **1.** A <u>kernel</u> of φ is a morphism $i: Z \to X$ such that
 - (a) $\varphi \circ i = 0$
 - (b) For all $i': Z' \to X$ such that $\varphi \circ i' = 0$, there is a unique $g: Z' \to Z$ such that $i' = i \circ g$.

$$Z \xrightarrow{g} \downarrow_{i'} \downarrow_{0}$$

$$Z \xrightarrow{i} X \xrightarrow{\varphi} Y$$

2. A <u>cokernel</u> is a kernel but with the arrows reversed

Exercise: Verify that 1 is equivalent to the following: for all $Z' \in \mathscr{C}$,

$$0 \longrightarrow \operatorname{Hom}(Z',Z) \xrightarrow{i_*} \operatorname{Hom}(Z',X) \xrightarrow{\varphi_*} \operatorname{Hom}(Z',Y)$$

is exact, and similarly for cokernel

With all that, we are ready for:

For any $\varphi: X \to Y$, there exists a sequence of morphisms

$$K \xrightarrow{k} X \xrightarrow{i} I \xrightarrow{j} Y \xrightarrow{c} K'$$

such that

- (a) $j \circ i = \varphi$
- (b) $k = \ker \varphi, k' = \operatorname{coker} \varphi$
- (c) $I = \operatorname{coker} k = \ker c$

This finishes the definition