Lecture 1, 4/3/23

We will begin thinking about homological algebra in a more categorical way. We use "methods of homological algebra," by Gelfond-Manin (Gelfond & Manin?). Here is a brief overview of some things we will see during the course:

- Additive and Abelian categories
- Example: category of sheaves of Abelian groups on a topological space
- Derived category (objects: complexes of objects in a given Abelian category, morphisms: a quasi-isomorphism of complexes becomes an isomorphism of objects)
- Derived functors, $f_*, f^*, f_!, f^!, \mathbb{D}, \cdots$, Hom for sheaves
- Triangulated categories
- t-structures
- Examples: perverse sheaves

Let's remind ourselves of some definitions:

Additive & Abelian categories

Example 0.1. (a) Ab, the category of Abelian groups with group homomorphisms.

- (b) R Mod, the category of R-modules, with R-module homomorphisms as morphisms.
- (c) SAb, PAb, the categories of sheaves of Abelian groups and presheaves of Abelian groups
- (d) Sheaves of modules over a ringed space
- (e) (quasi)-coherent sheaves (ask Zhao)

Definition 0.1. An Abelian category contains the following information:

1. Any hom-set $\operatorname{Hom}_{\mathscr{C}}(X,Y)$ is an Abelian group (+), and the composition of morphisms is bi-additive

In particular:

• Hom_{\mathscr{C}} is a functor $\mathscr{C}^{\circ} \times \mathscr{C} \to \mathsf{Ab}$. We notate the first factor with the $^{\circ}$

- $0 \in \operatorname{Hom}_{\mathscr{C}}(X,Y)$ for any objects X,Y of \mathscr{C}
- **2.** There exists a zero object $0 \in \mathcal{C}$, that is an object such that $\operatorname{Hom}_{\mathscr{C}}(0,0) = 0$.

This gives: $\operatorname{Hom}_{\mathscr{C}}(0,X) = 0$, $\operatorname{Hom}_{\mathscr{C}}(X,0) = 0$ for all objects X of \mathscr{C} .

We know that $\operatorname{Hom}_{\mathscr{C}}(0,0)$ consists of one object. In particular, it must be $\operatorname{Id}_0 = 0$. So

$$(0 \xrightarrow{f} X) = (0 \xrightarrow{\operatorname{Id}_0} 0 \xrightarrow{f} X) = (0 \xrightarrow{0} X)$$

Any two zero objects are isomorphic.

3. For any $X_1, X_2 \in \mathcal{C}$, there exists an object Y and morphisms

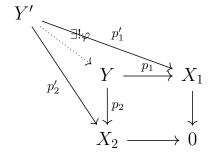
$$X_1 \stackrel{i_1}{\underbrace{\hspace{1cm}}} Y \stackrel{i_2}{\underbrace{\hspace{1cm}}} X_2$$

such that

$$p_1 i_1 = \operatorname{Id}_{X_1}$$

 $p_2 i_2 = \operatorname{Id}_{X_2}$
 $i_1 p_1 + i_2 p_2 = \operatorname{Id}_Y$
 $p_2 i_1 = p_1 i_2 = 0$

Lemma 1. We have cartesian diagram



That is, for any Y', with morphisms p'_1, p'_2 as in the diagram, there is a morphism from Y' to Y making the diagram commute. Similarly, we have co-cartesian diagram

$$Y \leftarrow i_1 \qquad X_1$$

$$\downarrow i_2 \qquad \uparrow \qquad \uparrow$$

$$X_2 \leftarrow 0$$

Proof. We need to construct $\varphi: Y' \to Y$ such that $p'_1 = p_1 \varphi$ and $p'_2 = p_2 \varphi$ Take $\varphi = i_1 p'_1 + i_2 p'_2$. Then

$$p_1 \circ \varphi = \underbrace{p_1 i_1}_{= \mathrm{Id}_{X_1}} p'_1 + \underbrace{p_1 i_2}_{=0} p'_2 = p'_1$$

The uniqueness of φ can be verified as an exercise

Definition 0.2. An <u>additive category</u> is one which satisfies only the first three of these axioms

To state the 4th axiom of Abelian categories, we need more notations:

Definition 0.3. Let A_1, A_2 be objects of \mathscr{C} , and let $\varphi: X \to Y$.

- **1.** A <u>kernel</u> of φ is a morphism $i: Z \to X$ such that
 - (a) $\varphi \circ i = 0$
 - (b) For all $i': Z' \to X$ such that $\varphi \circ i' = 0$, there is a unique $g: Z' \to Z$ such that $i' = i \circ g$.

$$Z \xrightarrow{g} \downarrow_{i'} \downarrow_{0}$$

$$Z \xrightarrow{i} X \xrightarrow{\varphi} Y$$

2. A <u>cokernel</u> is a kernel but with the arrows reversed

Exercise: Verify that 1 is equivalent to the following: for all $Z' \in \mathscr{C}$,

$$0 \longrightarrow \operatorname{Hom}(Z',Z) \xrightarrow{i_*} \operatorname{Hom}(Z',X) \xrightarrow{\varphi_*} \operatorname{Hom}(Z',Y)$$

is exact, and similarly for cokernel

With all that, we are ready for:

For any $\varphi: X \to Y$, there exists a sequence of morphisms

$$K \xrightarrow{k} X \xrightarrow{i} I \xrightarrow{j} Y \xrightarrow{c} K'$$

such that

- (a) $j \circ i = \varphi$
- (b) $k = \ker \varphi, k' = \operatorname{coker} \varphi$
- (c) $I = \operatorname{coker} k = \ker c$

This finishes the definition

Lecture 2, 4/5/23

Sheaves

Here are some examples of sheaves from complex analysis

Example 0.2.

(a) The set of holomorphic functions on $\mathbb{P} = \mathbb{C} \sup\{\infty\}$

For each open subset U of \mathbb{P} , we can consider the ring of holomorphic function $f: U \to \mathbb{C}$, $\mathcal{H}(U)$.

The collection of $\{(f, V) \mid V \text{ open}, f \in \mathcal{H}(V)\}$ is called the sheaf \mathscr{O} of holomorphic functions on \mathbb{P} .

(b) The sheaf of solutions of a linear ODE

Let $U \subseteq \mathbb{P}$ be open, and let $a_i(z) \in \Gamma(U, \mathcal{O})$ (in this context this will wind up meaning $\mathcal{H}(U)$), $i = 0, 1, \ldots, n-1$.

Denote by S the collection of (V, f) such that $V \subseteq U$ is open, and f is holomorphic in V such that

$$Lf \stackrel{\text{def}}{=} \frac{d^n f}{dz^n} + \sum_{i=0}^{n-1} a_i(z) \frac{d^i f}{dz^i} = 0$$

Let $\Gamma(V) = \{ f \in \mathcal{H}(V) \mid Lf = 0 \}$. When V is connected and simply connected, it is a basic result of ODEs that $\Gamma(V) \cong \mathbb{C}^n$

In general, it may have to do with the topology of V. For example, if $U = \mathbb{C} \setminus \{0\}$, $L = \frac{d^2}{dz^2} + \frac{1}{z}\frac{d}{dz}$, the solutions are $c_1 \log(z) + c_2$ for any "branch" of $\log(z)$, but $\Gamma(V) = \{\text{constant}\}$. This is related to the Riemann-Hilbert correspondence (whatever that is)

Definition 0.4.

- (a) A presheaf of sets \mathcal{F} on a topological space Y consists of the following data:
 - A set $\mathcal{F}(U)$ for any open $U \subseteq Y$
 - For any open $V \subseteq U$, a (restriction) map $\gamma_{U,V} : \mathcal{F}(U) \to \mathcal{F}(V)$ such that $\gamma_{V,V} = \mathrm{Id}_{\mathcal{F}(V)}$, and if $W \subseteq V \subseteq U$, then $\gamma_{V,W} \circ \gamma_{U,V} = \gamma_{U,W}$. When there is ambiguity about which sheaf γ belongs to, we further specify with $\gamma^{\mathcal{F}}$
- (b) A presheaf \mathcal{F} is a <u>sheaf</u> if:

• For any open covering $U = \bigcup_{i \in I} U_i$ and $s_i \in \mathcal{F}(U_i)$ such that for all $i, j \in I$,

$$\gamma_{U_i,U_i\cap U_j}(s_i) = \gamma_{U_j,U_i\cap U_j}(s_j)$$

then there exists a unique $s \in \mathcal{F}(U)$ such that $s_i = \gamma_{U,U_i}(s)$ for all i.

(c) A morphism of presheaves $f: \mathcal{F} \to \mathcal{G}$ is a family of maps $f(U): \mathcal{F}(U) \to \mathcal{G}(U)$ for all open $U \subseteq Y$, such that for all open $V \subseteq U$, the diagram

$$\begin{array}{ccc}
\mathcal{F}(U) & \xrightarrow{f(U)} & \mathcal{G}(U) \\
\gamma_{U,V}^{\mathcal{F}} \downarrow & & & \downarrow \gamma_{U,V}^{\mathcal{G}} \\
\mathcal{F}(V) & \xrightarrow{f(V)} & \mathcal{G}(V)
\end{array}$$

commutes.

A morphism of sheaves is a morphism of the underlying presheaves

A presheaf \mathcal{F} of groups/rings/ \mathbb{F} -vector spaces is a presheaf such that $\mathcal{F}(U)$ is a group/ring/ \mathbb{F} -vector space. Then $\gamma_{U,V}$ is a morphism of groups/rings/ \mathbb{F} -vector spaces.

Let \mathcal{F}, \mathcal{G} be two Abelian presheaves (meaning presheaves of Abelian groups), and let $f: \mathcal{F} \to \mathcal{G}$ be a morphism of Abelian presheaves.

Let $K(U) = \ker(f(U)), C(U) = \operatorname{coker}(f(U))$ with natural restrictions.

Definition 0.5. A sequence of presheaves

$$\mathcal{F} \stackrel{f}{\longrightarrow} \mathcal{G} \stackrel{g}{\longrightarrow} \mathcal{H}$$

is $\underline{\text{exact}}$ if, for all open U,

$$\mathcal{F}(U) \xrightarrow{f(U)} \mathcal{G}(U) \xrightarrow{g(U)} \mathcal{H}(U)$$

is exact.

So, presheaves of Abelian groups, with morphisms of Abelian presheaves, is an Abelian category.

How about Abelian sheaves?

Let $f: \mathcal{F} \to \mathcal{G}$ be a morphism of Abelian sheaves. Once again, let $K(U) = \ker(f(U)), C(U) = \operatorname{coker}(f(U)) = \frac{\mathcal{G}(U)}{f(U)(\mathcal{F}(U))}$.

Proposition 1.

- (a) The kernel K is an Abelian sheaf
- (b) The cokernel C is always a presheaf, but might not be a sheaf.

Lecture 3, 4/7/23

Proof.

(a) Let $U = \bigcup U_i$, $s_i \in K(U_i)$ agree on pairwise intersections. As $K(U_i) \hookrightarrow \mathcal{F}(U_i)$, there exists a unique $s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i \in \mathcal{F}(U_i)$ for all i.

Note that $f(s)|_{U_i} = f(s|_{U_i}) = 0$, so $f(s) = 0 \in \mathcal{G}(U)$. Here, we are using the uniqueness of gluing in \mathcal{G} .

Then s " \in "K(U)

- (b) First, we give an example of a cokernel which is a presheaf, but not a sheaf. Let $Y = \mathbb{C} \setminus \{0\}, f : \mathcal{O}_Y \to \mathcal{O}_Y$ given by $\varphi \mapsto \frac{d\varphi}{d\zeta}$.
 - For any $y \in Y$, there exists a neighborhood $V_y \ni y$ such that coker $f(V_y) = 0$. That is, for every $f \in \mathcal{H}(V_y)$, there is a $g \in \mathcal{H}(V_y)$ so that $\frac{dg}{d\zeta} = f$ in V_y (every point admits a simply connected neighborhood)
 - However, coker $f(Y) \cong \mathbb{C}$: $\Psi = \sum_{i=-\infty}^{\infty} a_i \zeta^i = \frac{d\varphi}{d\zeta}$ has a solution iff $a_{-1} = 0$. This is because $\frac{1}{z}$ is defined on Y. (keyword: vanishing cycle and v-filtration).

Hence this violates the uniqueness: any $\overline{\Psi} \in \operatorname{coker} f(Y)$ restricts to $0 \in \operatorname{coker} f(V_y)$. However, the V_y cover Y. So, we have an open cover, with sections of each, which agree on intersections, but which do not have a unique global section to which they all glue. So, this is not a sheaf.

Sheafification

Denote by SAb the additive category of abelian sheaves on a fixed topological space M. We have the inclusion functor $\iota : \mathrm{SAb} \to \mathrm{PAb}$.

Proposition 2. ι admits a left adjoint $s : PAb \to SAb$, i.e.

$$\operatorname{Hom}_{\operatorname{SAb}}(sX,Y) \cong \operatorname{Hom}_{\operatorname{PAb}}(X,\iota Y)$$

and this isomorphism is natural in both X and Y.

Proof. Let

$$s(X)(U) \stackrel{\text{def}}{=} \frac{\{\{(U_i, e_i)\}_{i \in I} \mid U = \bigcup_{i \in I} U_i, e_i \in X(U_i), e_i|_{U_i \cap U_j} = e_j|_{U_i \cap U_j}\}}{s(X)(U) \stackrel{\text{def}}{=} \frac{\{\{(U_i, e_i)\}_{i \in I} \mid U = \bigcup_{i \in I} U_i, e_i \in X(U_i), e_i|_{U_i \cap U_j} = e_j|_{U_i \cap U_j}\}}{s(X)(U) \stackrel{\text{def}}{=} \frac{\{\{(U_i, e_i)\}_{i \in I} \mid U = \bigcup_{i \in I} U_i, e_i \in X(U_i), e_i|_{U_i \cap U_j} = e_j|_{U_i \cap U_j}\}}{s(X)(U) \stackrel{\text{def}}{=} \frac{\{\{(U_i, e_i)\}_{i \in I} \mid U = \bigcup_{i \in I} U_i, e_i \in X(U_i), e_i|_{U_i \cap U_j} = e_j|_{U_i \cap U_j}\}}{s(X)(U) \stackrel{\text{def}}{=} \frac{\{\{(U_i, e_i)\}_{i \in I} \mid U = \bigcup_{i \in I} U_i, e_i \in X(U_i), e_i|_{U_i \cap U_j} = e_j|_{U_i \cap U_j}\}}{s(X)(U) \stackrel{\text{def}}{=} \frac{\{\{(U_i, e_i)\}_{i \in I} \mid U = \bigcup_{i \in I} U_i, e_i \in X(U_i), e_i|_{U_i \cap U_j} = e_j|_{U_i \cap U_j}\}}{s(X)(U) \stackrel{\text{def}}{=} \frac{\{\{(U_i, e_i)\}_{i \in I} \mid U = \bigcup_{i \in I} U_i, e_i \in X(U_i), e_i|_{U_i \cap U_j} = e_j|_{U_i \cap U_j}\}}{s(X)(U) \stackrel{\text{def}}{=} \frac{\{\{(U_i, e_i)\}_{i \in I} \mid U = \bigcup_{i \in I} U_i, e_i \in X(U_i), e_i|_{U_i \cap U_j} = e_j|_{U_i \cap U_j}\}}{s(X)(U) \stackrel{\text{def}}{=} \frac{\{\{(U_i, e_i)\}_{i \in I} \mid U = \bigcup_{i \in I} U_i, e_i \in X(U_i), e_i|_{U_i \cap U_j} = e_j|_{U_i \cap U_j}\}}{s(X)(U) \stackrel{\text{def}}{=} \frac{\{\{(U_i, e_i)\}_{i \in I} \mid U = \bigcup_{i \in I} U_i, e_i \in X(U_i), e_i|_{U_i \cap U_j} = e_j|_{U_i \cap U_j}\}}{s(X)(U) \stackrel{\text{def}}{=} \frac{\{\{(U_i, e_i)\}_{i \in I} \mid U = \bigcup_{i \in I} U_i, e_i \in X(U_i), e_i|_{U_i \cap U_j} = e_j|_{U_i \cap U_j}\}}{s(X)(U) \stackrel{\text{def}}{=} \frac{\{\{(U_i, e_i)\}_{i \in I} \mid U = \bigcup_{i \in I} U_i, e_i \in X(U_i), e_i \in X(U_i), e_i \in X(U_i)\}}{s(X)(U) \stackrel{\text{def}}{=} \frac{\{(U_i, e_i)\}_{i \in I} \mid U = \bigcup_{i \in I} U_i, e_i \in X(U_i), e_i \in X(U_i)$$

Where $\{(U_i, e_i)\} \sim \{(U'_j, e_j)\}$ if there exists $\{U''_k\}$ refining $\{U_i\}, \{U'_j\}$ and $e_i|_{U''_k} = e'_j|_{U''_k}$ for $U''_k \subset U_i \cap U'_j$.

Define $\gamma_{U,V}: sX'(U) \to sX(V)$, for $V \subseteq U$, by

$$\gamma_{U,V}[\{(U_i, e_i)\}] = [\{(U_i \cap V, e_i|_{U_i \cap V})\}]$$

There is a lot to verify; see [GM] 2.5.13

Example 0.3. For coker f from previous example, now $[\overline{\Psi}] = [0]$, since, when we restrict to V_y , $\overline{\Psi}$ becomes 0. So coker f = 0

With this modification, SAb is an abelian category!

Proposition 3. Let $\varphi: X \to Y$ be a morphism of abelian sheaves, and let

$$K \xrightarrow{k} \iota X \xrightarrow{i} I \xrightarrow{j} \iota Y \xrightarrow{c} K'$$

be the canonical decomposition of $\iota(\varphi)$ in (the abelian category!) PAb. Then

$$\underbrace{sK}_{=K} \xrightarrow{-sk} \underbrace{X}_{=s\iota X} \xrightarrow{-si} sI \xrightarrow{sj} \underbrace{Y}_{=s\iota Y} \xrightarrow{-sc} sK'$$

is the canonical decomposition of φ in SAb. In particular, SAb is an abelian category.

Proof. We'll just verify that sK' is indeed the cokernel: Let $Z \in \text{Ob SAb}$. Then there exists an exact sequence

$$0 \longrightarrow \operatorname{Hom}_{\operatorname{PAb}}(K', \iota Z) \longrightarrow \operatorname{Hom}_{\operatorname{PAb}}(Y, \iota Z) \longrightarrow \operatorname{Hom}_{\operatorname{PAb}}(X, \iota Z) \longrightarrow 0$$

(this is equivalent to saying K' is coker φ in PAb). By adjunction,

$$0 \longrightarrow \operatorname{Hom}_{\operatorname{SAb}}(sK', Z) \longrightarrow \operatorname{Hom}_{\operatorname{SAb}}(\underbrace{sY}, Z) \longrightarrow \operatorname{Hom}_{\operatorname{SAb}}(sX, Z) \longrightarrow 0$$

is exact. Other parts are exercises

Definition 0.6. Let A be an abelian group, Y a topological space.

- (a) The constant presheaf \mathbb{A} on Y is $\mathbb{A}(U) = A$ for all open $U \subseteq Y$, and $\gamma_{U,V} = \operatorname{Id}_A$ for any open $V \subseteq U$.
- (b) The constant sheaf A on Y is sA. (Check: for connected U, A(U) = A)

(c) A sheaf \mathcal{F} is <u>locally constant</u> if any point has a neighborhood U such that $\mathcal{F}|_U$ is a constant sheaf. (for open $V \subset U \subset Y$, $\mathcal{F}|_U(V) \stackrel{\text{def}}{=} \mathcal{F}(V)$) (keyword: representation of π_1 and local systems)

Germs and stalks

Definition 0.7. The stalk of a (pre)sheaf at a point y is

$$\mathcal{F}_y \stackrel{\text{def}}{=} \lim_{\stackrel{\longrightarrow}{V \ni y}} \mathcal{F}(V)$$

with the limit taken with respect to inclusion.

Concretely, $\mathcal{F}_y \stackrel{\text{def}}{=} \frac{\{(s,V)|y \in V, s \in \mathcal{F}(V)\}}{\sim}$, where $(s,V) \sim (s',V')$ if there exists a $W \subseteq V \cap V'$ such that $\gamma_{V,W}(s) = \gamma_{V',W}(s')$.

Such an equivalence class is called a germ

Remark: [GM, I.5.5, I.5.6]

If we have an exact sequence of sheaves

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$$

if and only if for any $y \in Y$,

$$0 \longrightarrow \mathcal{F}'_y \longrightarrow \mathcal{F}_y \longrightarrow \mathcal{F}''_y \longrightarrow 0$$

is exact. This can be verified as an extremely important exercise.

Definition 0.8. For $s \in \mathcal{F}(U)$, the support of s, supp s, is the closure of the set of points at which the germ of s is not zero.

Remark:

In the definition of a stalk, we can replace a point y by a closed subset Z of Y.

Lecture 4, 4/10/23

Functors in abelian categories

Definition 0.9.

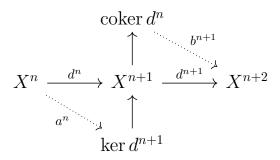
(a) Let $\mathscr{C}, \mathscr{C}'$ be additive categories. A functor $F : \mathscr{C} \to \mathscr{C}'$ is an additive functor if all maps $F : \operatorname{Hom}_{\mathscr{C}}(X,Y) \to \operatorname{Hom}_{\mathscr{C}'}(FX,FY)$ is a homomorphism of abelian groups.

(b) A complex in \mathscr{C} is a sequence

$$X : \xrightarrow{d^{n-1}} X^n \xrightarrow{d^n} X^{n+1} \xrightarrow{d^{n+1}} \cdots$$

with $d^n \circ d^{n-1} = 0$ for all n

(c) Now assume $\mathscr{C}, \mathscr{C}'$ are abelian categories. If we have a complex, then, because $d^n \circ d^{n+1} = 0$, the universal properties of the kernel and cokernel (as well as their existence, which is guaranteed because we are in an abelian category), guarantee us unique maps a^n, b^{n+1} making the diagram commute:



The (n+1)-cohomology of X is

$$H^{n+1}(X^{\cdot}) \stackrel{\text{def}}{=} \operatorname{coker} a^n = \ker b^{n+1}$$

this equality can be verified as an exercise..

- (d) X^{\cdot} is acyclic at X^{b} if $H^{n}(X^{\cdot}) = 0$.
- (e) X is exact/acyclic if it is acyclic at X^n for all n
- (f) An additive functor $F: \mathscr{C} \to \mathscr{C}'$ is exact if it sends a short exact sequence

$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$$

to a short exact sequence

$$0 \longrightarrow FX \longrightarrow FY \longrightarrow FZ \longrightarrow 0$$

It is <u>left exact</u> if

$$0 \longrightarrow FX \longrightarrow FY \longrightarrow FZ$$

is exact, and right exact if

$$FX \longrightarrow FY \longrightarrow FZ \longrightarrow 0$$

is exact.

Example 0.4. Let \mathscr{C} be an abelian category, and consider $\operatorname{Hom}_{\mathscr{C}}(Y,-):\mathscr{C}\to\operatorname{\mathsf{Ab}}$ and $\operatorname{Hom}_{\mathscr{C}}(-,Y):\mathscr{C}^\circ\to\operatorname{\mathsf{Ab}}$, where \mathscr{C}° means the opposite category of \mathscr{C} (this is just saying this functor is contravariant). These two morphisms are both left-exact.

Example 0.5. For a fixed ring R, consider R - Mod, the category of left R-modules, and Y a right R-module (so an object of Mod-R). Then we have a functor

$$Y \otimes_R : R - \text{Mod} \to \mathsf{Ab}$$

which is right-exact.

Proposition 4. Let X be a topological space, and fix an open set $U \subseteq X$. Consider SAb, the category of abelian sheaves on X. The functor SAb \rightarrow Ab given by $\mathcal{F} \mapsto \mathcal{F}(U)$ is an additive functor which is left exact.

Proof. Let $\iota: \mathrm{SAb} \to \mathrm{PAb}$ be the inclusion of sheaves into presheaves. This is left exact, which follows from the fact that the kernel of a morphism of sheaves is again a sheaf. The kernel doesn't need sheafification! Now $\mathrm{PAb} \to \mathrm{Ab} : \mathcal{F} \mapsto \mathcal{F}(U)$ is exact by definition. The composition of a left exact and an exact functor is left exact, so we are done.

From now on, we will always be working in an abelian category unless otherwise stated.

Definition 0.10.

- (a) An object Y is projective if $Hom_{\mathscr{C}}(Y, -)$ is exact.
- (b) An object Y is injective if $Hom_{\mathscr{C}}(-,Y)$ is exact
- (c) A right module-R Y is $\underline{\text{flat}}$ if $Y \otimes_R \text{is exact.}$

Direct images

Definition 0.11. Let $f: M \to N$ be a continuous map of topological spaces, and let \mathcal{F} be a sheaf on M. The <u>direct image</u> $f_{\star}\mathcal{F}$ (written as $f.\mathcal{F}$ in Gelfond-Manin) is defined as follows.

For any open $U \subseteq N$, $f^{-1}(U)$ is open in M. So we simply define

$$f_{\star}\mathcal{F}(U) \stackrel{\mathrm{def}}{=} \mathcal{F}(f^{-1}(U))$$

and restriction for $V \subseteq U$ induced from $\gamma_{f^{-1}(U), f^{-1}(V)}$ Exercise: verify that this is indeed a sheaf!

Proposition 5.

- (a) Let $f: M \to \{1\}$ be the constant map. Then $f_{\star}\mathcal{F} = \Gamma(M, \mathcal{F}) = \mathcal{F}(M)$.
- (b) Let $i: M \to N$ be the inclusion of a closed subspace M of N. Then

$$(i_{\star}\mathcal{F})_x = \begin{cases} \mathcal{F}_x & x \in M \\ 0 & x \notin M \end{cases}$$

Some call this an "extension by zero." (If $i: M \hookrightarrow N$ is the inclusion of an open subset M of N, then $i_*\mathcal{F}$ may have nonzero stalk at $x \in N \setminus M$. Prove this as an exercise!)

(c) $f_*: SAb(M) \to SAb(N)$ is a functor, $(fg)_* = f_*g_*$

Proof.

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Inverse image. Let $f: M \to N$ be continuous.

Definition 0.12. For $\mathcal{F} \in SAb_N$, first define $f_p^{\star}\mathcal{F}$ as a presheaf:

$$U \mapsto \mathcal{F}(f(U)) \stackrel{\text{def}}{=} \lim_{\substack{N \supset V \supset f(U)}} \mathcal{F}(V)$$

where V is open. Then take $f^*\mathcal{F} \stackrel{\text{def}}{=} s(f_p^*\mathcal{F})$

Exercise: for all $x \in M$, $(f^*\mathcal{F})_x = \mathcal{F}_{f(x)}$

Proposition 6. We have an adjunction $f^* \dashv f_*$,

$$\operatorname{Hom}_{\operatorname{SAb}_N}(f^{\star}\mathcal{F},\mathcal{G}) \cong \operatorname{Hom}_{\operatorname{SAb}_N}(\mathcal{F},f_*\mathcal{G})$$

Proof. We know $s \dashv \iota$ by construction. So, we need only show

$$\operatorname{Hom}_{\operatorname{PAb}_{M}}(f_{n}^{\star}\mathcal{F},\mathcal{G}) \cong \operatorname{Hom}_{\operatorname{PAb}_{N}}(\mathcal{F}, f_{\star}\mathcal{G})$$

We establish a functorial morphism: $\mathcal{F} \to f_{\star} f_{p}^{\star} \mathcal{F}$.

Let $\mathcal{G} = f_p^{\star} \mathcal{F}$,

$$\operatorname{Hom}(f_{p}^{\star}\mathcal{F}, f_{p}^{\star}\mathcal{F}) \cong \operatorname{Hom}(\mathcal{F}, f_{\star}f_{p}^{\star}\mathcal{F})$$

Note: for $V \subseteq N$ open, $\mathcal{F}(V) \to f_p^{\star}(f^{-1}(V))$

But V is open in N containing $f(f^{-1}(V))$. So we have restriction

?????????????

Exercise: Check morphism of presheaves.

This is compatible with restrictions and gives us a presheaf morphism $i_{\mathcal{F}}: \mathcal{F} \to f_{\star}f_{p}^{\star}\mathcal{F}$.

This induces the isomorphism in the statement.

$$(\psi: f_p^{\star} \mathcal{F} \to \mathcal{G}) \longrightarrow ((f_{\star} \psi) \circ i_{\mathcal{F}}: \mathcal{F} \to f_{*} \mathcal{G}$$

The other direction uses $f_p^{\star} f_{\star} \mathcal{G} \to \mathcal{G}$

Exercise: construct this and check inverse

Proposition 7. Let \mathscr{C}, \mathscr{D} be Abelian categories. Let $F : \mathscr{C} \to \mathscr{D}$ and $G : \mathscr{D} \to \mathscr{C}$ be adjoint functors, $F \dashv G$.

Theorem 0.1. F is right exact, and G is left exact

Proof. We will just check G is left exact.

Let $0 \longrightarrow Y' \stackrel{f}{\longrightarrow} Y \stackrel{g}{\longrightarrow} Y'' \longrightarrow 0$ be a short exact sequence.

Apply the left exact functor $\operatorname{Hom}_{\mathscr{D}}(Fx,-)$ for all $x\in\mathscr{C}$. We have the diagram

$$0 \longrightarrow \operatorname{Hom}_{\mathscr{D}}(FX,Y') \longrightarrow \operatorname{Hom}_{\mathscr{D}}(FX,Y'')$$

$$\parallel \qquad \qquad \parallel \qquad \qquad \parallel$$

$$0 \longrightarrow \operatorname{Hom}_{\mathscr{C}}(X,GY') \longrightarrow \operatorname{Hom}_{\mathscr{C}}(X,GY) \longrightarrow \operatorname{Hom}_{\mathscr{C}}(X,GY'')$$

This is exact for all $X \in \mathscr{C}$ iff $0 \longrightarrow GY' \longrightarrow GY \longrightarrow GY''$ is exact

Proposition 8. In SAb, f_{\star} is left exact, f^{\star} is exact.

Proof. By exercise, f^* is exact on stalks.

Direct images with compact support

GM, III8.7 - > 8.10

All topological spaces are assumed to be locally compact and first countable, meaning every point has a countable neighborhood basis.

Recall: A morphism of topological spaces is proper if the preimage of compact sets are compact.

Definition 0.13. Let $f: X \to Y$, \mathcal{F} a sheaf on X. Let $U \subseteq Y$ be open. We define

$$F_!\mathcal{F}(U) = \{s \in \Gamma(f^{-1}(U), \mathcal{F}) \mid f : \operatorname{supp}(s) \to U \text{ is proper } \}$$

Definition 0.14. Let $s \in \Gamma(V, \mathcal{G})$. Then $\operatorname{supp}(s) = \overline{\{x \in V \mid \overline{s} \neq 0 \in \mathcal{G}_x\}}, \Gamma(V, \mathcal{G}) \to \mathcal{G}, s \mapsto \overline{s}$

Lemma 2.

- (a) $f_!\mathcal{F}$ is a subsheaf of $f_*\mathcal{F}$
- (b) $f_!$ is a left exact functor.

Proof.

Lecture 6, 4/14/23

Let $f: X \to Y$ be continuous, and \mathcal{F} be a sheaf on X. Recall the definition of $f_!\mathcal{F}$:

$$f_!\mathcal{F}(U) = \{s \in \underbrace{\Gamma(f^{-1}(U), \mathcal{F})}_{=f_\star\mathcal{F}(U)} \mid f : \operatorname{supp}(s) \to U \text{ is proper } \}$$

where supp(s) is the closure of the set of points where \overline{s} , the germ of s, \overline{s} , is not zero. **Theorem 0.2.**

(a) $f_!\mathcal{F}$ is a subsheaf of $f_*\mathcal{F}$.

(b) f! is a left exact functor "direct image with compact support"

Proof.

(a) $f_!\mathcal{F}$ is clearly a subpresheaf of $f_*\mathcal{F}$. Any set of compatible sections of $f_!\mathcal{F}$ glue uniquely to a section of $f_*\mathcal{F}$. This comes down to a topological statement.

Exercise: For (U_i) open subsets of Y, $f_i: V_i \to U_i$ is proper, then $f: \cup V_i \to \cup U_i$ is proper.

$$f^{-1}(K) = \cup f_i^{-1}(V_i \cap K)$$

(b)

Sections with compact support

Consider the special case $f: X \to \{1\}$, the one point space. Then $f_!\mathcal{F}$ is the set of sections $s \in \mathcal{F}(X)$ such that supp(s) is compact.

Denote this by $\Gamma_c(X, \mathcal{F})$

$$X \xrightarrow{f} Y$$

$$\downarrow i \qquad \qquad \uparrow \qquad \qquad \uparrow$$

$$f^{-1}(y) \longrightarrow y$$

Proposition 9. The stalk of $f_!\mathcal{F}$ at $y \in Y$ is isomorphic to

$$\Gamma_C(f^{-1}(y), \underbrace{\mathcal{F}|_{f^{-1}(y)}}_{=i\star\mathcal{F}})$$

Proof. First construct $\varphi: (f_!\mathcal{F})_y \to \Gamma_c(f^{-1}(y), \mathcal{F}|_{f^{-1}(y)}).$

Let $s \in f_!\mathcal{F}_y$ and choose a representative $\tilde{s} \in \Gamma(f^{-1}(U), \mathcal{F})$ with U an open neighborhood of y, and supp $\tilde{s} \to U$ proper.

Then:
$$\tilde{s}|_{f^{-1}(y)}$$
 is in $\Gamma_c(f^{-1}(y), \underbrace{\mathcal{F}|_{f^{-1}(y)}}_{=i^{\star}\mathcal{F}})$

Exercise: $\varphi(s) \stackrel{\text{def}}{=} \tilde{s}|_{f^{-1}(y)}$ only depends on s.

We now show φ is injective. Suppose $\varphi(s) = 0$. Then $\operatorname{supp}(\tilde{s}) \cap f^{-1}(y) = \emptyset$. So $y \notin f(\operatorname{supp} \tilde{s})$. But $f(\operatorname{supp} \tilde{s})$ is closed (proper + locally compact) So s = 0.

To show φ is surjective: choose a local basis $V_i \ni y$ with $\cap V_i = y$. Then $f^{-1}(y) = \bigcap_{i=1}^{n} f^{-1}(U_i)$

Exercise:

Locally compadct implies $\Gamma_c(f^{-1}(y), \mathcal{F}|_{f^{-1}(y)} = \lim_{\to} A_i$ where

$$A_i = \{t \in \Gamma(f^{-1}(U_i), \mathcal{F}) \mid \text{supp } t = K \cap f^{-1}(U_i) \text{ for some compact } K \subseteq X\}$$

Example 0.6. 1. Let $i: U \hookrightarrow X$ be open, \mathcal{F} a sheaf on U.

$$(i_!\mathcal{F})_x = \begin{cases} \mathcal{F}_x, & x \in U \\ 0, & x \in U \end{cases}$$

"extend by 0"

2. $j: V \to X$ proper (in particular, closed embedding), $j_! \mathcal{G} = j_* \mathcal{G}$.

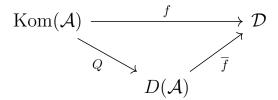
Derived Categories via Localizations

Definition 0.15. Let $f: K^{\cdot} \to L^{\cdot}$ be a morphism of complexes in an Abelian category \mathcal{A} . f is a quasi-isomorphism if the induced morphism $H^n(f): H^n(K^{\cdot}) \to H^n(L^{\cdot})$ is an isomorphism for all n.

Definition 0.16. Let \mathcal{A} be an Abelian category, $\operatorname{Kom}(\mathcal{A})$ the category of complexes in \mathcal{A} . The <u>derived category of \mathcal{A} </u> is a category D(A) and a functor $Q : \operatorname{Kom}(A) \to D(A)$ such that

(a) Q(f) is an isomorphism for any quasi-isomorphism f.

(b) Q is universal in the following sense. Suppose that $g: \mathcal{A} \to \mathcal{D}$ is a functor such that g(f) is an isomorphism for any quasi-isomorphism f. Then there is a unique functor $\overline{f}: D(\mathcal{A}) \to \mathcal{D}$, making the diagram commute:



Theorem 0.3. Every abelian category admits a derived category.

Proof.

Lecture 7, 4/17/23

Homework hint:

Let \mathcal{A} be an abelian category with P a projective generator. We wish to define an equivalence with R-mod, where $R = \operatorname{Hom}_{\mathcal{A}}(P, P)$. Try to construct an equivalence going the other way as follows:

We want a projective resolution of M

$$R^{\oplus J} \longrightarrow R^{\oplus I} \longrightarrow M \longrightarrow 0$$

We want to use this to get a morphism $P^{\oplus J} \to P^{\oplus I}$. You can do this by taking the identity and looking at the image (?).

Now we continue.

Remark: We can define $\mathrm{Kom}^+(\mathcal{A})$ to be all the chain complexes in \mathcal{A} that are bounded on the right. That is, $\mathrm{Kom}^+(\mathcal{A}) = \{K^\cdot \mid K^i = 0 \text{ for } i << 0\}$. $\mathrm{Kom}^-(\mathcal{A})$ is defined similarly. $\mathrm{Kom}^b(\mathcal{A}) = \mathrm{Kom}^+(\mathcal{A}) \cap \mathrm{Kom}^-(\mathcal{A})$.

These are each abelian categories, so we can consider $D^+(\mathcal{A}), D^-(\mathcal{A}), D^b(\mathcal{A})$, but this is for later.

Construction of D(A)

This is similar to "localization of rings"

Definition 0.17. A class of morphism $S \subset \operatorname{Mor} \mathscr{B}$ is said to be localizing if

- (a) S is closed under composition: $\operatorname{Id}_X \in S$ for all $X \in \operatorname{Ob} \mathscr{B}$ and $s \circ t \in S$ for all $s, t \in S$ such that the composition is defined.
- (b) Extension: for all $f \in \text{Mor } \mathcal{B}, s \in S$, there exists $g \in \text{Mor } \mathcal{B}, t \in S$ such that we can make one of the following diagrams commute:

$$\begin{array}{ccc} W & \stackrel{g}{\longrightarrow} Z \\ \downarrow & & \downarrow s \\ X & \stackrel{f}{\longrightarrow} Y \end{array}$$

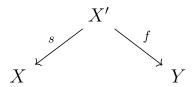
or

$$\begin{array}{ccc} W & \stackrel{g}{\longleftarrow} & Z \\ \downarrow & & \uparrow s \\ X & \stackrel{f}{\longleftarrow} & Y \end{array}$$

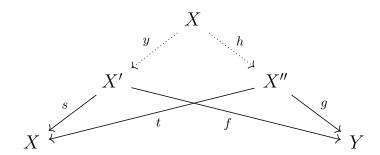
(c) If $f, g \in \text{Mor}(X, Y)$, the existence of $s \in S$ with sf = sg is equivalent to the existence of $t \in S$ such that ft = gt

Definition 0.18. Given a category \mathscr{B} and class of morphisms S which is localizing, we introduce a category $\mathscr{B}[S^{-1}]$, with $\operatorname{Ob}\mathscr{B}[S^{-1}] = \operatorname{Ob}\mathscr{B}$ and

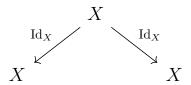
(a) A morphism $X \to Y$ in $\mathscr{B}[S^{-1}]$ is the equivalence class of <u>roofs</u> of the form



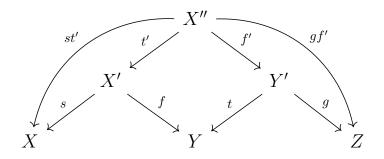
with $s \in S, f \in \text{Mor } \mathcal{B}$. Two roofs are equivalent if there exists a third roof making the following diagram commute:



With $sr = th, fr = gh, y \in S$. Id_X is the class of



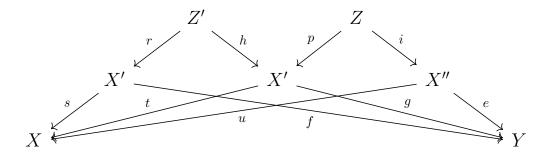
(b) The composition of roofs (s, f) and (t, g) is the class of (st', gf')



$$gt^{-1}fs^{-1} = gf't'^{-1}s^{-1} = (gf)(st')^{-1}$$

Theorem 0.4. This is in fact a well defined equivalence relation

Proof. Symmetry and reflexivity are easy, so we will just show transitivity. Suppose $(s,f) \sim (t,g), (t,g) \sim (u,e)$. We want to show $(s,f) \sim (u,e)$. So we have



 $s, t, u, r, p \in S$ First consider

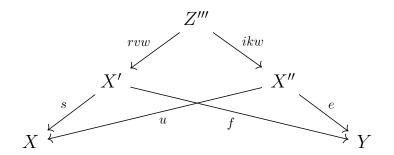
$$Z' \leftarrow W$$

$$sr \downarrow \qquad \qquad \downarrow k$$

$$X \leftarrow_{tp} Z''$$

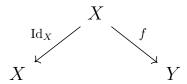
with $sr \in S$. Then thv = srv = tpk, so by c there exists w such that hvw = pkw for

some $w: Z''' \to W$, $w \in S$. We now build the roof



Definition 0.19. The category $\mathscr{B}[S^{-1}]$ is called a <u>localization of \mathscr{B} </u>.

Let $F: \mathscr{B} \to \mathscr{B}[S^{-1}]$ map each object to its equivalence class. We define F(f) to be the equivalence class of the roof



This satisfies the universal property:

Let $T: \mathcal{B} \to \mathcal{D}$ be such that T(s) is an isomorphism for all $s \in S$. Then there exists a unique factorization

