Lecture 1, 4/3/23

We will begin thinking about homological algebra in a more categorical way. We use "methods of homological algebra," by Gelfond-Manin (Gelfond & Manin?). Here is a brief overview of some things we will see during the course:

- Additive and Abelian categories
- Example: category of sheaves of Abelian groups on a topological space
- Derived category (objects: complexes of objects in a given Abelian category, morphisms: a quasi-isomorphism of complexes becomes an isomorphism of objects)
- Derived functors, $f_*, f^*, f_!, f^!, \mathbb{D}, \cdots$, Hom for sheaves
- Triangulated categories
- t-structures
- Examples: perverse sheaves

Let's remind ourselves of some definitions:

Additive & Abelian categories

Example 0.1. (a) Ab, the category of Abelian groups with group homomorphisms.

(b) $R-\text{Mod}$, the category of R-modules, with R-module homomorphisms as morphisms.

(c) SAb, PAb, the categories of sheaves of Abelian groups and presheaves of Abelian groups

(d) Sheaves of modules over a ringed space

(e) (quasi)-coherent sheaves (ask Zhao)

Definition 0.1. An Abelian category contains the following information:

1. Any hom-set $\text{Hom}_{\mathscr{C}}(X, Y)$ is an Abelian group $(+)$, and the composition of morphisms is bi-additive

In particular:

- $\text{Hom}_{\mathscr{C}}$ is a functor $\mathscr{C}^\circ \times \mathscr{C} \to \mathsf{Ab}$. We notate the first factor with the $^\circ$

- $0 \in \mathrm{Hom}_{\mathscr{C}}(X, Y)$ for any objects X, Y of $\mathscr{C}$

2. There exists a zero object $0 \in \mathscr{C}$, that is an object such that $\mathrm{Hom}_{\mathscr{C}}(0,0) = 0$.

This gives: $\mathrm{Hom}_{\mathscr{C}}(0, X) = 0$, $\mathrm{Hom}_{\mathscr{C}}(X, 0) = 0$ for all objects X of $\mathscr{C}$.

We know that $\mathrm{Hom}_{\mathscr{C}}(0,0)$ consists of one object. In particular, it must be $\mathrm{Id}_0 = 0$. So

$$\left(0 \xrightarrow{f} X \right) = \left(0 \xrightarrow{\mathrm{Id}_0} 0 \xrightarrow{f} X \right) = \left(0 \xrightarrow{0} X \right)$$

Any two zero objects are isomorphic.

3. For any $X_1, X_2 \in \mathscr{C}$, there exists an object Y and morphisms

$$\begin{tikzcd} X_1 \arrow[r, bend left, "i_1"] & Y \arrow[l, bend left, "p_1"] \arrow[r, bend right, "p_2"'] & X_2 \arrow[l, bend right, "i_2"'] \end{tikzcd}$$

such that

$$\begin{aligned} p_1 i_1 &= \mathrm{Id}_{X_1} \\ p_2 i_2 &= \mathrm{Id}_{X_2} \\ i_1 p_1 + i_2 p_2 &= \mathrm{Id}_Y \\ p_2 i_1 = p_1 i_2 &= 0 \end{aligned}$$

Lemma 1. *We have cartesian diagram*

$$\begin{tikzcd} Y' \arrow[dr, dotted, "\exists!\varphi"] \arrow[drr, "p_1'"] \arrow[ddr, "p_2'"'] & & \\ & Y \arrow[r, "p_1"] \arrow[d, "p_2"] & X_1 \arrow[d] \\ & X_2 \arrow[r] & 0 \end{tikzcd}$$

That is, for any Y', with morphisms p_1', p_2' as in the diagram, there is a morphism from Y' to Y making the diagram commute. Similarly, we have co-cartesian diagram

$$\begin{tikzcd} Y & X_1 \arrow[l, "i_1"'] \\ X_2 \arrow[u, "i_2"] & 0 \arrow[l] \arrow[u] \end{tikzcd}$$

Proof. We need to construct $\varphi : Y' \to Y$ such that $p_1' = p_1\varphi$ and $p_2' = p_2\varphi$
Take $\varphi = i_1p_1' + i_2p_2'$. Then

$$p_1 \circ \varphi = \underbrace{p_1 i_1}_{=\mathrm{Id}_{X_1}} p_1' + \underbrace{p_1 i_2}_{=0} p_2' = p_1'$$

The uniqueness of φ can be verified as an exercise

∎

Definition 0.2. An additive category is one which satisfies only the first three of these axioms

To state the 4th axiom of Abelian categories, we need more notations:

Definition 0.3. Let A_1, A_2 be objects of $\mathscr{C}$, and let $\varphi : X \to Y$.

1. A kernel of φ is a morphism $i : Z \to X$ such that

(a) $\varphi \circ i = 0$

(b) For all $i' : Z' \to X$ such that $\varphi \circ i' = 0$, there is a unique $g : Z' \to Z$ such that $i' = i \circ g$.

$$\begin{tikzcd} & Z' \arrow[dl, dotted, "g"'] \arrow[d, "i'"] \arrow[dr, "0"] & \\ Z \arrow[r, "i"] & X \arrow[r, "\varphi"] & Y \end{tikzcd}$$

2. A cokernel is a kernel but with the arrows reversed

Exercise: Verify that 1 is equivalent to the following: for all $Z' \in \mathscr{C}$,

$$0 \longrightarrow \mathrm{Hom}(Z', Z) \xrightarrow{i_*} \mathrm{Hom}(Z', X) \xrightarrow{\varphi_*} \mathrm{Hom}(Z', Y)$$

is exact, and similarly for cokernel
With all that, we are ready for:

For any $\varphi : X \to Y$, there exists a sequence of morphisms

$$K \xrightarrow{k} X \xrightarrow{i} I \xrightarrow{j} Y \xrightarrow{c} K'$$

such that

(a) $j \circ i = \varphi$

(b) $k = \ker \varphi, k' = \operatorname{coker} \varphi$

(c) $I = \operatorname{coker} k = \ker c$

This finishes the definition

Lecture 2, 4/5/23

Sheaves

Here are some examples of sheaves from complex analysis

Example 0.2.

(a) The set of holomorphic functions on $\mathbb{P} = \mathbb{C} \sup\{\infty\}$

For each open subset U of $\mathbb{P}$, we can consider the ring of holomorphic function $f : U \to \mathbb{C}$, $\mathscr{H}(U)$.

The collection of $\{(f, V) \mid V \text{ open}, f \in \mathscr{H}(V)\}$ is called the sheaf $\mathscr{O}$ of holomorphic functions on $\mathbb{P}$.

(b) The sheaf of solutions of a linear ODE

Let $U \subseteq \mathbb{P}$ be open, and let $a_i(z) \in \Gamma(U, \mathscr{O})$ (in this context this will wind up meaning $\mathscr{H}(U)$), $i = 0, 1, \ldots, n-1$.

Denote by S the collection of (V, f) such that $V \subseteq U$ is open, and f is holomorphic in V such that

$$Lf \overset{\text{def}}{=} \frac{d^n f}{dz^n} + \sum_{i=0}^{n-1} a_i(z) \frac{d^i f}{dz^i} = 0$$

Let $\Gamma(V) = \{f \in \mathscr{H}(V) \mid Lf = 0\}$. When V is connected and simply connected, it is a basic result of ODEs that $\Gamma(V) \cong \mathbb{C}^n$

In general, it may have to do with the topology of V. For example, if $U = \mathbb{C} \backslash \{0\}$, $L = \frac{d^2}{dz^2} + \frac{1}{z}\frac{d}{dz}$, the solutions are $c_1 \log(z) + c_2$ for any "branch" of $\log(z)$, but $\Gamma(V) = \{\text{constant}\}$. This is related to the Riemann-Hilbert correspondence (whatever that is)

Definition 0.4.

(a) A <u>presheaf of sets $\mathcal{F}$ on a topological space Y</u> consists of the following data:

- A set $\mathcal{F}(U)$ for any open $U \subseteq Y$
- For any open $V \subseteq U$, a (restriction) map $\gamma_{U,V} : \mathcal{F}(U) \to \mathcal{F}(V)$ such that $\gamma_{V,V} = \mathrm{Id}_{\mathcal{F}(V)}$, and if $W \subseteq V \subseteq U$, then $\gamma_{V,W} \circ \gamma_{U,V} = \gamma_{U,W}$. When there is ambiguity about which sheaf γ belongs to, we further specify with $\gamma^{\mathcal{F}}$

(b) A presheaf $\mathcal{F}$ is a <u>sheaf</u> if:

- For any open covering $U = \cup_{i\in I} U_i$ and $s_i \in \mathcal{F}(U_i)$ such that for all $i, j \in I$,

$$\gamma_{U_i, U_i \cap U_j}(s_i) = \gamma_{U_j, U_i \cap U_j}(s_j)$$

  then there exists a unique $s \in \mathcal{F}(U)$ such that $s_i = \gamma_{U,U_i}(s)$ for all i.

(c) A morphism of presheaves $f : \mathcal{F} \to \mathcal{G}$ is a family of maps $f(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ for all open $U \subseteq Y$, such that for all open $V \subseteq U$, the diagram

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "f(U)"] \arrow[d, "\gamma^{\mathcal{F}}_{U,V}"'] & \mathcal{G}(U) \arrow[d, "\gamma^{\mathcal{G}}_{U,V}"] \\ \mathcal{F}(V) \arrow[r, "f(V)"] & \mathcal{G}(V) \end{tikzcd}$$

commutes.

A morphism of sheaves is a morphism of the underlying presheaves

A presheaf $\mathcal{F}$ of groups/rings/$\mathbb{F}$-vector spaces is a presheaf such that $\mathcal{F}(U)$ is a group/ring/$\mathbb{F}$-vector space. Then $\gamma_{U,V}$ is a morphism of groups/rings/$\mathbb{F}$-vector spaces.

Let $\mathcal{F}, \mathcal{G}$ be two Abelian presheaves (meaning presheaves of Abelian groups), and let $f : \mathcal{F} \to \mathcal{G}$ be a morphism of Abelian presheaves.
Let $K(U) = \ker(f(U))$, $C(U) = \operatorname{coker}(f(U))$ with natural restrictions.

Definition 0.5. A sequence of presheaves

$$\mathcal{F} \xrightarrow{f} \mathcal{G} \xrightarrow{g} \mathcal{H}$$

is exact if, for all open U,

$$\mathcal{F}(U) \xrightarrow{f(U)} \mathcal{G}(U) \xrightarrow{g(U)} \mathcal{H}(U)$$

is exact.
So, presheaves of Abelian groups, with morphisms of Abelian presheaves, is an Abelian category.
How about Abelian sheaves?
Let $f : \mathcal{F} \to \mathcal{G}$ be a morphism of Abelian sheaves. Once again, let $K(U) = \ker(f(U)), C(U) = \operatorname{coker}(f(U)) = \frac{\mathcal{G}(U)}{f(U)(\mathcal{F}(U))}$.

Proposition 1.

(a) The kernel K is an Abelian sheaf

(b) The cokernel C is always a presheaf, but might not be a sheaf.

Lecture 3, 4/7/23

Proof.

(a) Let $U = \bigcup U_i$, $s_i \in K(U_i)$ agree on pairwise intersections. As $K(U_i) \hookrightarrow \mathcal{F}(U_i)$, there exists a unique $s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i \in \mathcal{F}(U_i)$ for all i.

Note that $f(s)|_{U_i} = f(s|_{U_i}) = 0$, so $f(s) = 0 \in \mathcal{G}(U)$. Here, we are using the uniqueness of gluing in $\mathcal{G}$.

Then s"$\in$"$K(U)$

(b) First, we give an example of a cokernel which is a presheaf, but not a sheaf. Let $Y = \mathbb{C} \setminus \{0\}$, $f : \mathcal{O}_Y \to \mathcal{O}_Y$ given by $\varphi \mapsto \frac{d\varphi}{d\zeta}$.

- For any $y \in Y$, there exists a neighborhood $V_y \ni y$ such that $\operatorname{coker} f(V_y) = 0$. That is, for every $f \in \mathcal{H}(V_y)$, there is a $g \in \mathcal{H}(V_y)$ so that $\frac{dg}{d\zeta} = f$ in V_y (every point admits a simply connected neighborhood)
- However, $\operatorname{coker} f(Y) \cong \mathbb{C}$: $\Psi = \sum_{i=-\infty}^{\infty} a_i \zeta^i = \frac{d\varphi}{d\zeta}$ has a solution iff $a_{-1} = 0$. This is because $\frac{1}{z}$ is defined on Y. (keyword: vanishing cycle and v-filtration).

  Hence this violates the uniqueness: any $\overline{\Psi} \in \operatorname{coker} f(Y)$ restricts to $0 \in \operatorname{coker} f(V_y)$. However, the V_y cover Y. So, we have an open cover, with sections of each, which agree on intersections, but which do not have a unique global section to which they all glue. So, this is not a sheaf.

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Sheafification

Denote by SAb the additive category of abelian sheaves on a fixed topological space M. We have the inclusion functor $\iota : \mathrm{SAb} \to \mathrm{PAb}$.

Proposition 2. *ι admits a left adjoint $s : \mathrm{PAb} \to \mathrm{SAb}$, i.e.*

$$\mathrm{Hom}_{\mathrm{SAb}}(sX, Y) \cong \mathrm{Hom}_{\mathrm{PAb}}(X, \iota Y)$$

and this isomorphism is natural in both X and Y.

Proof. Let

$$s(X)(U) \overset{\text{def}}{=} \frac{\{\{(U_i, e_i)\}_{i \in I} \mid U = \bigcup_{i \in I} U_i, e_i \in X(U_i), e_i|_{U_i \cap U_j} = e_j|_{U_i \cap U_j}\}}{\sim}$$

Where $\{(U_i, e_i)\} \sim \{(U'_j, e_j)\}$ if there exists $\{U''_k\}$ refining $\{U_i\}, \{U'_j\}$ and $e_i|_{U''_k} = e'_j|_{U''_k}$ for $U''_k \subset U_i \cap U'_j$.
Define $\gamma_{U,V} : sX(U) \to sX(V)$, for $V \subseteq U$, by

$$\gamma_{U,V}[\{(U_i, e_i)\}] = [\{(U_i \cap V, e_i|_{U_i \cap V})\}]$$

There is a lot to verify; see [GM] 2.5.13

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Example 0.3. For coker f from previous example, now $[\overline{\Psi}] = [0]$, since, when we restrict to V_y, $\overline{\Psi}$ becomes 0. So coker $f = 0$

With this modification, SAb is an abelian category!

Proposition 3. *Let $\varphi : X \to Y$ be a morphism of abelian sheaves, and let*

$$K \xrightarrow{\ k\ } \iota X \xrightarrow{\ i\ } I \xrightarrow{\ j\ } \iota Y \xrightarrow{\ c\ } K'$$

be the canonical decomposition of $\iota(\varphi)$ in (the abelian category!) PAb*. Then*

$$\underbrace{sK}_{=K} \xrightarrow{\ sk\ } \underbrace{X}_{=s\iota X} \xrightarrow{\ si\ } sI \xrightarrow{\ sj\ } \underbrace{Y}_{=s\iota Y} \xrightarrow{\ sc\ } sK'$$

is the canonical decomposition of φ in SAb*. In particular,* SAb *is an abelian category.*

Proof. We'll just verify that sK' is indeed the cokernel:
Let $Z \in \operatorname{Ob} \mathrm{SAb}$. Then there exists an exact sequence

$$0 \longrightarrow \mathrm{Hom}_{\mathrm{PAb}}(K', \iota Z) \longrightarrow \mathrm{Hom}_{\mathrm{PAb}}(Y, \iota Z) \longrightarrow \mathrm{Hom}_{\mathrm{PAb}}(X, \iota Z) \longrightarrow 0$$

(this is equivalent to saying K' is coker φ in PAb). By adjunction,

$$0 \longrightarrow \mathrm{Hom}_{\mathrm{SAb}}(sK', Z) \longrightarrow \mathrm{Hom}_{SAb}(\underbrace{sY}_{=Y}, Z) \longrightarrow \mathrm{Hom}_{\mathrm{SAb}}(sX, Z) \longrightarrow 0$$

is exact. Other parts are exercises

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Definition 0.6. Let A be an abelian group, Y a topological space.

(a) The <u>constant presheaf $\mathbb{A}$ on Y</u> is $\mathbb{A}(U) = A$ for all open $U \subseteq Y$, and $\gamma_{U,V} = \mathrm{Id}_A$ for any open $V \subseteq U$.

(b) The <u>constant sheaf $\mathcal{A}$ on Y</u> is $s\mathbb{A}$.
(Check: for connected U, $\mathcal{A}(U) = A$)

(c) A sheaf $\mathcal{F}$ is locally constant if any point has a neighborhood U such that $\mathcal{F}|_U$ is a constant sheaf. (for open $V \subset U \subset Y$, $\mathcal{F}|_U(V) \overset{\text{def}}{=} \mathcal{F}(V)$) (keyword: representation of π_1 and local systems)

Germs and stalks

Definition 0.7. The stalk of a (pre)sheaf at a point y is

$$\mathcal{F}_y \overset{\text{def}}{=} \varinjlim_{V \ni y} \mathcal{F}(V)$$

with the limit taken with respect to inclusion.
Concretely, $\mathcal{F}_y \overset{\text{def}}{=} \frac{\{(s,V) | y \in V, s \in \mathcal{F}(V)\}}{\sim}$, where $(s,V) \sim (s',V')$ if there exists a $W \subseteq V \cap V'$ such that $\gamma_{V,W}(s) = \gamma_{V',W}(s')$.
Such an equivalence class is called a germ
Remark:[GM, I.5.5, I.5.6]
If we have an exact sequence of sheaves

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$$

if and only if for any $y \in Y$,

$$0 \longrightarrow \mathcal{F}'_y \longrightarrow \mathcal{F}_y \longrightarrow \mathcal{F}''_y \longrightarrow 0$$

is exact. This can be verified as an extremely important exercise.

Definition 0.8. For $s \in \mathcal{F}(U)$, the support of s, $\operatorname{supp} s$, is the closure of the set of points at which the germ of s is not zero.
Remark:
In the definition of a stalk, we can replace a point y by a closed subset Z of Y.

Lecture 4, 4/10/23

Functors in abelian categories

Definition 0.9.

(a) Let $\mathscr{C}, \mathscr{C}'$ be additive categories. A functor $F : \mathscr{C} \to \mathscr{C}'$ is an additive functor if all maps $F : \operatorname{Hom}_{\mathscr{C}}(X,Y) \to \operatorname{Hom}_{\mathscr{C}'}(FX, FY)$ is a homomorphism of abelian groups.

(b) A complex in $\mathscr{C}$ is a sequence

$$X^{\cdot} : \quad \cdot \xrightarrow{d^{n-1}} X^n \xrightarrow{d^n} X^{n+1} \xrightarrow{d^{n+1}} \cdots$$

with $d^n \circ d^{n-1} = 0$ for all n

(c) Now assume $\mathscr{C}, \mathscr{C}'$ are abelian categories. If we have a complex, then, because $d^n \circ d^{n+1} = 0$, the universal properties of the kernel and cokernel (as well as their existence, which is guaranteed because we are in an abelian category), guarantee us unique maps a^n, b^{n+1} making the diagram commute:

$$\begin{tikzcd}
& \operatorname{coker} d^n \arrow[dr, dotted, "b^{n+1}"] & \\
X^n \arrow[r, "d^n"] \arrow[dr, dotted, "a^n"'] & X^{n+1} \arrow[u] \arrow[r, "d^{n+1}"] & X^{n+2} \\
& \ker d^{n+1} \arrow[u] &
\end{tikzcd}$$

The $(n+1)$-cohomology of $X^{\cdot}$ is

$$H^{n+1}(X^{\cdot}) \overset{\text{def}}{=} \operatorname{coker} a^n = \ker b^{n+1}$$

this equality can be verified as an exercise..

(d) $X^{\cdot}$ is acyclic at X^b if $H^n(X^{\cdot}) = 0$.

(e) $X^{\cdot}$ is exact/acyclic if it is acyclic at X^n for all n

(f) An additive functor $F : \mathscr{C} \to \mathscr{C}'$ is exact if it sends a short exact sequence

$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$$

to a short exact sequence

$$0 \longrightarrow FX \longrightarrow FY \longrightarrow FZ \longrightarrow 0$$

It is left exact if

$$0 \longrightarrow FX \longrightarrow FY \longrightarrow FZ$$

is exact, and right exact if

$$FX \longrightarrow FY \longrightarrow FZ \longrightarrow 0$$

is exact.

Example 0.4. Let $\mathscr{C}$ be an abelian category, and consider $\mathrm{Hom}_{\mathscr{C}}(Y,-) : \mathscr{C} \to \mathsf{Ab}$ and $\mathrm{Hom}_{\mathscr{C}}(-,Y) : \mathscr{C}^{\circ} \to \mathsf{Ab}$, where $\mathscr{C}^{\circ}$ means the opposite category of $\mathscr{C}$ (this is just saying this functor is contravariant). These two morphisms are both left-exact.

Example 0.5. For a fixed ring R, consider $R-\mathrm{Mod}$, the category of left R-modules, and Y a right R-module (so an object of Mod-R). Then we have a functor

$$Y \otimes_R : R - \mathrm{Mod} \to \mathsf{Ab}$$

which is right-exact.

Proposition 4. *Let X be a topological space, and fix an open set $U \subseteq X$. Consider* SAb*, the category of abelian sheaves on X. The functor* SAb $\to$ *Ab given by $\mathcal{F} \mapsto \mathcal{F}(U)$ is an additive functor which is left exact.*

Proof. Let $\iota : \mathrm{SAb} \to \mathrm{PAb}$ be the inclusion of sheaves into presheaves. This is left exact, which follows from the fact that the kernel of a morphism of sheaves is again a sheaf. The kernel doesn't need sheafification! Now $\mathrm{PAb} \to \mathsf{Ab} : \mathcal{F} \mapsto \mathcal{F}(U)$ is exact by definition. The composition of a left exact and an exact functor is left exact, so we are done. ∎

From now on, we will always be working in an abelian category unless otherwise stated.

Definition 0.10.

(a) An object Y is projective if $\mathrm{Hom}_{\mathscr{C}}(Y,-)$ is exact.

(b) An object Y is injective if $\mathrm{Hom}_{\mathscr{C}}(-,Y)$ is exact

(c) A right module-R Y is flat if $Y \otimes_R -$ is exact.

Direct images

Definition 0.11. Let $f : M \to N$ be a continuous map of topological spaces, and let $\mathcal{F}$ be a sheaf on M. The direct image $f_\star\mathcal{F}$ (written as $f.\mathcal{F}$ in Gelfond-Manin) is defined as follows.
For any open $U \subseteq N$, $f^{-1}(U)$ is open in M. So we simply define

$$f_\star\mathcal{F}(U) \overset{\text{def}}{=} \mathcal{F}(f^{-1}(U))$$

and restriction for $V \subseteq U$ induced from $\gamma_{f^{-1}(U), f^{-1}(V)}$
Exercise: verify that this is indeed a sheaf!

Proposition 5.

(a) Let $f : M \to \{1\}$ *be the constant map. Then* $f_\star \mathcal{F} = \Gamma(M, \mathcal{F}) = \mathcal{F}(M)$.

(b) Let $i : M \to N$ *be the inclusion of a closed subspace* M *of* N*. Then*

$$(i_\star \mathcal{F})_x = \begin{cases} \mathcal{F}_x & x \in M \\ 0 & x \notin M \end{cases}$$

Some call this an "extension by zero." (If $i : M \hookrightarrow N$ *is the inclusion of an open subset* M *of* N*, then* $i_* \mathcal{F}$ *may have nonzero stalk at* $x \in N \setminus M$*. Prove this as an exercise!)*

(c) $f_* : \mathrm{SAb}(M) \to \mathrm{SAb}(N)$ *is a functor,* $(fg)_* = f_* g_*$

Proof. ∎

Lecture 5, 4/12/23

Inverse image. Let $f : M \to N$ *be continuous.*

Definition 0.12. For $\mathcal{F} \in \mathrm{SAb}_N$, first define $f_p^\star \mathcal{F}$ as a presheaf:

$$U \mapsto \mathcal{F}(f(U)) \stackrel{\text{def}}{=} \varinjlim_{N \supset V \supset f(U)} \mathcal{F}(V)$$

where V is open. Then take $f^\star \mathcal{F} \stackrel{\text{def}}{=} s(f_p^\star \mathcal{F})$
Exercise: for all $x \in M$, $(f^\star \mathcal{F})_x = \mathcal{F}_{f(x)}$

Proposition 6. *We have an adjunction* $f^\star \dashv f_\star$,

$$\mathrm{Hom}_{\mathrm{SAb}_M}(f^\star \mathcal{F}, \mathcal{G}) \cong \mathrm{Hom}_{\mathrm{SAb}_N}(\mathcal{F}, f_* \mathcal{G})$$

Proof. We know $s \dashv \iota$ by construction. So, we need only show

$$\mathrm{Hom}_{\mathrm{PAb}_M}(f_p^\star \mathcal{F}, \mathcal{G}) \cong \mathrm{Hom}_{\mathrm{PAb}_N}(\mathcal{F}, f_\star \mathcal{G})$$

We establish a functorial morphism: $\mathcal{F} \to f_\star f_p^\star \mathcal{F}$.
Let $\mathcal{G} = f_p^\star \mathcal{F}$,

$$\mathrm{Hom}(f_p^\star \mathcal{F}, f_p^\star \mathcal{F}) \cong \mathrm{Hom}(\mathcal{F}, f_\star f_p^\star \mathcal{F})$$

Note: for $V \subseteq N$ open, $\mathcal{F}(V) \to f_p^\star(f^{-1}(V))$
But V is open in N containing $f(f^{-1}(V))$. So we have restriction
?????????????
Exercise: Check morphism of presheaves.

This is compatible with restrictions and gives us a presheaf morphism $i_{\mathcal{F}} : \mathcal{F} \to f_\star f_p^\star \mathcal{F}$.
This induces the isomorphism in the statement.

$$(\psi : f_p^\star \mathcal{F} \to \mathcal{G}) \longrightarrow ((f_\star \psi) \circ i_{\mathcal{F}} : \mathcal{F} \to f_* \mathcal{G}$$

The other direction uses $f_p^\star f_\star \mathcal{G} \to \mathcal{G}$
Exercise: construct this and check inverse

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Proposition 7. *Let $\mathscr{C}, \mathscr{D}$ be Abelian categories. Let $F : \mathscr{C} \to \mathscr{D}$ and $G : \mathscr{D} \to \mathscr{C}$ be adjoint functors, $F \dashv G$.*

Theorem 0.1. *F is right exact, and G is left exact*

Proof. We will just check G is left exact.
Let $0 \longrightarrow Y' \xrightarrow{f} Y \xrightarrow{g} Y'' \longrightarrow 0$ be a short exact sequence.
Apply the left exact functor $\mathrm{Hom}_{\mathscr{D}}(Fx, -)$ for all $x \in \mathscr{C}$. We have the diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathrm{Hom}_{\mathscr{D}} & \longrightarrow & \mathrm{Hom}_{\mathscr{D}}(FX, Y') & \longrightarrow & \mathrm{Hom}_{\mathscr{D}}(FX, Y'') \\
 & & \| & & \| & & \| \\
0 & \longrightarrow & \mathrm{Hom}_{\mathscr{C}}(X, GY') & \longrightarrow & \mathrm{Hom}_{\mathscr{C}}(X, GY) & \longrightarrow & \mathrm{Hom}_{\mathscr{C}}(X, GY'')
\end{array}$$

This is exact for all $X \in \mathscr{C}$ iff $0 \longrightarrow GY' \longrightarrow GY \longrightarrow GY''$ is exact

Proposition 8. *In* SAb, $f_\star$ *is left exact,* $f^\star$ *is exact.*

Proof. By exercise, $f^\star$ is exact on stalks.

Direct images with compact support

GM, $III8.7- > 8.10$
All topological spaces are assumed to be locally compact and first countable, meaning every point has a countable neighborhood basis.
Recall: A morphism of topological spaces is proper if the preimage of compact sets are compact.

Definition 0.13. Let $f : X \to Y$, $\mathcal{F}$ a sheaf on X. Let $U \subseteq Y$ be open. We define

$$F_! \mathcal{F}(U) = \{s \in \Gamma(f^{-1}(U), \mathcal{F}) \mid f : \mathrm{supp}(s) \to U \text{ is proper }\}$$

Definition 0.14. Let $s \in \Gamma(V, \mathcal{G})$. Then $\mathrm{supp}(s) = \overline{\{x \in V \mid \overline{s} \neq 0 \in \mathcal{G}_x\}}$, $\Gamma(V, \mathcal{G}) \to \mathcal{G}$, $s \mapsto \overline{s}$

Lemma 2.

(a) $f_!\mathcal{F}$ is a subsheaf of $f_*\mathcal{F}$

(b) $f_!$ is a left exact functor.

Proof. ■

Lecture 6, 4/14/23

Let $f : X \to Y$ be continuous, and $\mathcal{F}$ be a sheaf on X. Recall the definition of $f_!\mathcal{F}$:

$$f_!\mathcal{F}(U) = \{s \in \underbrace{\Gamma(f^{-1}(U), \mathcal{F})}_{=f_\star\mathcal{F}(U)} \mid f : \text{supp}(s) \to U \text{ is proper } \}$$

where supp(s) is the closure of the set of points where $\overline{s}$, the germ of s, $\overline{s}$, is not zero.

Theorem 0.2.

(a) $f_!\mathcal{F}$ is a subsheaf of $f_*\mathcal{F}$.

(b) $f_!$ is a left exact functor "direct image with compact support"

Proof.

(a) $f_!\mathcal{F}$ is clearly a subpresheaf of $f_\star\mathcal{F}$. Any set of compatible sections of $f_!\mathcal{F}$ glue uniquely to a section of $f_\star\mathcal{F}$. This comes down to a topological statement.

Exercise: For (U_i) open subsets of Y, $f_i : V_i \to U_i$ is proper, then $f : \cup V_i \to \cup U_i$ is proper.

$$f^{-1}(K) = \cup f_i^{-1}(V_i \cap K)$$

(b)

■

Sections with compact support

Consider the special case $f : X \to \{1\}$, the one point space. Then $f_!\mathcal{F}$ is the set of sections $s \in \mathcal{F}(X)$ such that supp(s) is compact.

Denote this by $\Gamma_c(X, \mathcal{F})$

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ f^{-1}(y) \arrow[u, hook, "i"] \arrow[r] & y \arrow[u, hook] \end{tikzcd}$$

Proposition 9. *The stalk of $f_!\mathcal{F}$ at $y \in Y$ is isomorphic to*

$$\Gamma_C(f^{-1}(y), \underbrace{\mathcal{F}|_{f^{-1}(y)}}_{=i^\star\mathcal{F}})$$

Proof. First construct $\varphi : (f_!\mathcal{F})_y \to \Gamma_c(f^{-1}(y), \mathcal{F}|_{f^{-1}(y)})$.
Let $s \in f_!\mathcal{F}_y$ and choose a representative $\tilde{s} \in \Gamma(f^{-1}(U), \mathcal{F})$ with U an open neighborhood of y, and $\operatorname{supp}\tilde{s} \to U$ proper.
Then: $\tilde{s}|_{f^{-1}(y)}$ is in $\Gamma_c(f^{-1}(y), \underbrace{\mathcal{F}|_{f^{-1}(y)}}_{=i^\star\mathcal{F}})$
Exercise: $\varphi(s) \stackrel{\text{def}}{=} \tilde{s}|_{f^{-1}(y)}$ only depends on s.
We now show φ is injective. Suppose $\varphi(s) = 0$. Then $\operatorname{supp}(\tilde{s}) \cap f^{-1}(y) = \varnothing$. So $y \notin f(\operatorname{supp}\tilde{s})$. But $f(\operatorname{supp}\tilde{s})$ is closed (proper + locally compact)
So $s = 0$.
To show φ is surjective: choose a local basis $V_i \ni y$ with $\cap V_i = y$. Then $f^{-1}(y) = \cap f^{-1}(U_i)$
Exercise:
Locally compadct implies $\Gamma_c(f^{-1}(y), \mathcal{F}|_{f^{-1}(y)} = \lim_{\rightarrow} A_i$ where

$$A_i = \{t \in \Gamma(f^{-1}(U_i), \mathcal{F}) \mid \operatorname{supp} t = K \cap f^{-1}(U_i) \text{ for some compact } K \subseteq X\}$$

■

Example 0.6. 1. Let $i : U \hookrightarrow X$ be open, $\mathcal{F}$ a sheaf on U.

$$(i_!\mathcal{F})_x = \begin{cases} \mathcal{F}_x, & x \in U \\ 0, & x \in U \end{cases}$$

"extend by 0"

2. $j : V \to X$ proper (in particular, closed embedding), $j_!\mathcal{G} = j_*\mathcal{G}$.

Derived Categories via Localizations

Definition 0.15. Let $f : K^\cdot \to L^\cdot$ be a morphism of complexes in an Abelian category $\mathcal{A}$. f is a <u>quasi-isomorphism</u> if the induced morphism $H^n(f) : H^n(K^\cdot) \to H^n(L^\cdot)$ is an isomorphism for all n.

Definition 0.16. Let $\mathcal{A}$ be an Abelian category, $\operatorname{Kom}(\mathcal{A})$ the category of complexes in $\mathcal{A}$. The <u>derived category of $\mathcal{A}$</u> is a category $D(A)$ and a functor $Q : \operatorname{Kom}(A) \to D(A)$ such that

(a) $Q(f)$ is an isomorphism for any quasi-isomorphism f.

(b) Q is universal in the following sense. Suppose that $g : \mathcal{A} \to \mathcal{D}$ is a functor such that $g(f)$ is an isomorphism for any quasi-isomorphism f. Then there is a unique functor $\overline{f} : D(\mathcal{A}) \to \mathcal{D}$, making the diagram commute:

$$\begin{tikzcd} \mathrm{Kom}(\mathcal{A}) \arrow[rr, "f"] \arrow[dr, "Q"'] & & \mathcal{D} \\ & D(\mathcal{A}) \arrow[ur, "\overline{f}"'] & \end{tikzcd}$$

Theorem 0.3. *Every abelian category admits a derived category.*

Proof. ∎

Lecture 7, 4/17/23

Homework hint:
Let $\mathcal{A}$ be an abelian category with P a projective generator. We wish to define an equivalence with R-mod, where $R = \mathrm{Hom}_{\mathcal{A}}(P, P)$. Try to construct an equivalence going the other way as follows:
We want a projective resolution of M

$$R^{\oplus J} \longrightarrow R^{\oplus I} \longrightarrow M \longrightarrow 0$$

We want to use this to get a morphism $P^{\oplus J} \to P^{\oplus I}$. You can do this by taking the identity and looking at the image (?).

Now we continue.
Remark: We can define $\mathrm{Kom}^+(\mathcal{A})$ to be all the chain complexes in $\mathcal{A}$ that are bounded on the right. That is, $\mathrm{Kom}^+(\mathcal{A}) = \{K^\cdot \mid K^i = 0 \text{ for } i << 0\}$. $\mathrm{Kom}^-(\mathcal{A})$ is defined similarly. $\mathrm{Kom}^b(\mathcal{A}) = \mathrm{Kom}^+(\mathcal{A}) \cap \mathrm{Kom}^-(\mathcal{A})$.
These are each abelian categories, so we can consider $D^+(\mathcal{A}), D^-(\mathcal{A}), D^b(\mathcal{A})$, but this is for later.

Construction of $D(\mathcal{A})$

This is similar to "localization of rings"

Definition 0.17. A class of morphism $S \subset \mathrm{Mor}\,\mathscr{B}$ is said to be localizing if

(a) S is closed under composition: $\mathrm{Id}_X \in S$ for all $X \in \mathrm{Ob}\,\mathscr{B}$ and $s \circ t \in S$ for all $s, t \in S$ such that the composition is defined.

(b) Extension: for all $f \in \mathrm{Mor}\,\mathscr{B}$, $s \in S$, there exists $g \in \mathrm{Mor}\,\mathscr{B}, t \in S$ such that we can make one of the following diagrams commute:

$$\begin{tikzcd}
W \arrow[r, dotted, "g"] \arrow[d, dotted, "t"'] & Z \arrow[d, "s"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

or

$$\begin{tikzcd}
W & Z \arrow[l, dotted, "g"'] \\
X \arrow[u, dotted, "t"] & Y \arrow[l, "f"] \arrow[u, "s"']
\end{tikzcd}$$

(c) If $f, g \in \mathrm{Mor}(X, Y)$, the existence of $s \in S$ with $sf = sg$ is equivalent to the existence of $t \in S$ such that $ft = gt$

Definition 0.18. Given a category $\mathscr{B}$ and class of morphisms S which is localizing, we introduce a category $\mathscr{B}[S^{-1}]$, with $\mathrm{Ob}\,\mathscr{B}[S^{-1}] = \mathrm{Ob}\,\mathscr{B}$ and

(a) A morphism $X \to Y$ in $\mathscr{B}[S^{-1}]$ is the equivalence class of roofs of the form

$$\begin{tikzcd}
& X' \arrow[dl, "s"'] \arrow[dr, "f"] & \\
X & & Y
\end{tikzcd}$$

with $s \in S, f \in \mathrm{Mor}\,\mathscr{B}$. Two roofs are equivalent if there exists a third roof making the following diagram commute:

$$\begin{tikzcd}
& & X \arrow[dl, dotted, "y"'] \arrow[dr, dotted, "h"] & & \\
& X' \arrow[dl, "s"'] \arrow[drrr, "f", pos=0.7] & & X'' \arrow[dlll, "t"', pos=0.7] \arrow[dr, "g"] & \\
X & & & & Y
\end{tikzcd}$$

With $sr = th, fr = gh, y \in S$. Id_X is the class of

$$\begin{tikzcd}
& X \arrow[dl, "\mathrm{Id}_X"'] \arrow[dr, "\mathrm{Id}_X"] & \\
X & & X
\end{tikzcd}$$

(b) The composition of roofs (s, f) and (t, g) is the class of (st', gf')

$$\begin{tikzcd}
& & X'' \arrow[dl, "t'"'] \arrow[dr, "f'"] \arrow[ddll, bend right, "st'"'] \arrow[ddrr, bend left, "gf'"] & & \\
& X' \arrow[dl, "s"'] \arrow[dr, "f"] & & Y' \arrow[dl, "t"'] \arrow[dr, "g"] & \\
X & & Y & & Z
\end{tikzcd}$$

$$gt^{-1}fs^{-1} = gf't'^{-1}s^{-1} = (gf)(st')^{-1}$$

Theorem 0.4. *This is in fact a well defined equivalence relation*

Proof. Symmetry and reflexivity are easy, so we will just show transitivity. Suppose $(s, f) \sim (t, g), (t, g) \sim (u, e)$. We want to show $(s, f) \sim (u, e)$.
So we have

$$\begin{tikzcd}
& & Z' \arrow[dl, "r"'] \arrow[dr, "h"] & & Z \arrow[dl, "p"'] \arrow[dr, "i"] & & \\
& X' \arrow[dl, "s"'] \arrow[drrrrr, "f"] & & X' \arrow[dlll, "t"'] \arrow[drrr, "g"] & & X'' \arrow[dlllll, "u"'] \arrow[dr, "e"] & \\
X & & & & & & Y
\end{tikzcd}$$

$s, t, u, r, p \in S$ First consider

$$\begin{tikzcd}
Z' \arrow[d, "sr"'] & W \arrow[l, dotted, "v"'] \arrow[d, dotted, "k"] \\
X & Z'' \arrow[l, "tp"]
\end{tikzcd}$$

with $sr \in S$. Then $thv = srv = tpk$, so by c there exists w such that $hvw = pkw$ for some $w : Z''' \to W$, $w \in S$. We now build the roof

$$\begin{tikzcd}
& & Z''' \arrow[dl, "rvw"'] \arrow[dr, "ikw"] & & \\
& X' \arrow[dl, "s"'] \arrow[drrr, "f"' near end] & & X'' \arrow[dr, "e"] \arrow[dlll, "u" near end] & \\
X & & & & Y
\end{tikzcd}$$

$\blacksquare$

Definition 0.19. The category $\mathscr{B}[S^{-1}]$ is called a <u>localization of $\mathscr{B}$</u>.
Let $F : \mathscr{B} \to \mathscr{B}[S^{-1}]$ map each object to its equivalence class. We define $F(f)$ to be the equivalence class of the roof

$$\begin{tikzcd}
& X \arrow[dl, "\mathrm{Id}_X"'] \arrow[dr, "f"] & \\
X & & Y
\end{tikzcd}$$

This satisfies the universal property:
Let $T : \mathscr{B} \to \mathscr{D}$ be such that $T(s)$ is an isomorphism for all $s \in S$. Then there exists a unique factorization

$$\begin{tikzcd}
\mathscr{B} \arrow[rr, "T"] \arrow[dr, "F"'] & & \mathscr{D} \\
& \mathscr{B}[S^{-1}] \arrow[ur, dotted, "\exists!"'] &
\end{tikzcd}$$