Lecture 1, 4/4/23

We use the following two books:

1. Linear Analysis, by B. Bolobás

2. Analysis, by E. Lieb and M. Loss

We begin with metric spaces

Definition 0.1. Let X be a nonempty set and let $\rho : X \times X \to [0, \infty)$. Then $\rho(x, y)$ is called <u>a metric on X</u> if

(i) $\rho(x, y) \geq 0$ for all $x, y \in X$ and $\rho(x, y) = 0$ iff $x = y$.

(ii) $\rho(x, y) = \rho(y, x)$ for all $x, y \in X$

(iii) $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$ for any $x, y, z \in X$. This is called the <u>triangle inequality</u>

ρ is also called a <u>distance</u>. As in, $\rho(x, y)$ is the <u>distance between x and y</u>.

Example 0.1. Let $X = \mathbb{R}^n$, and define $\rho(x, y) = \left(\sum_{i=1}^{n}(x_i - y_i)^2\right)^{\frac{1}{2}}$. We can in fact replace 2 in this expression with any real $r \geq 1$, or with ∞ (in which case we just take the maximum)

Example 0.2. Let $X = C[a, b]$, the set of continuous $f : [a, b] \to \mathbb{R}$, and define $\rho(x, y) = \max_{x \in [a,b]} |f(x) - g(x)|$

Definition 0.2. Let (X, ρ) be a metric space. For all $x \in X$ and $r > 0$, we defined the <u>open ball centered at x and having radius r</u> as

$$B_r(x) \stackrel{\text{def}}{=} \{y \in X \mid \rho(x, y) < r\}$$

The closed ball is

$$\overline{B_r}(x) \stackrel{\text{def}}{=} \{y \in X \mid \rho(x, y) \leq r\}$$

Definition 0.3. Let (X, ρ) be a metric space and let $A \subseteq X$. Then $a \in A$ is

(i) an <u>interior point</u> of A if there is some $r > 0$ such that $B_r(a) \subseteq A$

(ii) The set of all interior points of A is called the <u>interior of A</u> and is denoted by $\operatorname{int} A$, or A°

(iii) A set A is said to be <u>*open*</u> if $A = A^\circ$

Example 0.3. Let $X = \mathbb{R}^3$, $A = \{(x, y, 0) \mid x, y \in \mathbb{R}\}$. We can see that $A^\circ = \varnothing$.

Proposition 1. *For any x, r, $B_r(x)$ is open.*

Proof. Let $y \in B_r(x)$. Let $r_1 = r - \rho(x, y) > 0$.
Consider $z \in B_{r_1}(y)$. By the triangle inequality, $\rho(x, z) \leq \rho(z, y) + \rho(y, x) < r_1 + \rho(x, y) = r$. So $z \in B_r(x)$, so $B_{r_1}(y) \subseteq B_r(x)$, so y is an interior point. y was arbitrary, so we are done.

■

Definition 0.4. $A \subseteq X$ is closed if $A^c = X \setminus A$ is open.

Definition 0.5. The point $x \in X$ is a limit point of A if there exists a sequence $\{x_n\}_{n=1}^\infty \subseteq A$ such that $\rho(x, x_n) \to 0$ as $n \to \infty$

Definition 0.6. Let $\{x_n\} \subseteq X$, $x \in X$. Then we we say x_n converges to x, or $x_n \to x$, if $\rho(x_n, x) \to 0$ as $n \to \infty$. In this case, $\{x_n\}_{n=1}^\infty$ is said to be convergent, with limit x.

Theorem 0.1. *If a limit of a sequence $\{x_n\} \subseteq X$ exists, then it is unique.*

Proof. Think

■

Definition 0.7. A sequence $\{x_n\}_{n=1}^\infty \subseteq X$ is called a Cauchy sequence if $\rho(x_n, x_m) \to 0$ as $n, m \to \infty$. That is, for any $\varepsilon > 0$, there exists an $N \in \mathbb{N}$, such that if $n, m \geq N$, then $\rho(x_n, x_m) < \varepsilon$

Theorem 0.2. *Any convergent sequence is Cauchy*

Proof. Think

■

Definition 0.8. A metric space (X, ρ) is called complete if every Cauchy sequence converges to some point in X. A metric space which is not complete is called incomplete.

Example 0.4. $X = \mathbb{R}^n$ or $X = C[a, b]$ with the metrics above are complete.

Example 0.5. $\mathbb{Q}$ is incomplete.

Definition 0.9. Let (X, ρ) and $(Y, \tilde{\rho})$ be metric spaces. Then X and Y are isometric if there exist a bijection $f : X \to Y$ such that $\rho(x_1, x_2) = \tilde{\rho}(f(x_1), f(x_2))$ for all $x_1, x_2 \in X$.

Definition 0.10. Let (X, ρ) be a metric space and let $A, B \subseteq X$. Then we say that A is dense in B if $B \subseteq \overline{A}$, where $\overline{A} = \{\text{all limit points of } A\}$

Definition 0.11. Let (X, ρ) and $(\tilde{X}, \tilde{\rho})$ be metric spaces. Then $(\tilde{X}, \tilde{\rho})$ is a completion of $(X,$
if

(i) $X \subseteq \tilde{X}$, and $\tilde{\rho}(x, y) = \rho(x, y)$ for any $x, y \in X$

(ii) X is dense in $\tilde{X}$ in the $\tilde{\rho}$ metric

(iii) $(\tilde{X}, \tilde{\rho})$ is complete

Theorem 0.3. *Any incomplete metric space (X, ρ) admits a completion which is unique up to isometry.*

Proof. Think

■

Theorem 0.4. *(The nested ball theorem)*
Let (X, ρ) be a complete metric space, and let $\overline{B_n} = \overline{B_{r_n}}(x_n) \subseteq X$ be a sequence of nested closed balls (meaning $\overline{B_{n+1}} \subseteq \overline{B_n}$) such that $r_n \to 0$. Then $\bigcap_{n=1}^{\infty} \overline{B_n} \neq \varnothing$.

Proof. Consider the centers $\{x_n\}_{n=1}^{\infty} \subseteq X$.

Claim. *$\{x_n\}$ is Cauchy*

Proof. If $m \geq n$, then $\overline{B_m} \subseteq \overline{B_m}$, so $x_m \in \overline{B_m}$, so $\rho(x_m, x_n) \leq r_n$, so $\rho(x_n, x_m) \to 0$ as $m, n \to \infty$.

■

Lecture 2, 4/6/23

Definition 0.12. Let (X, ρ) be a metric space, and let $A \subseteq X$. Then A is <u>nowhere dense</u> if $\text{int}(\overline{A}) = \varnothing$

Definition 0.13. Let (X, ρ) be a metric space, and A a set. $A \subseteq X$ is <u>of Baire first category</u> if $A = \bigcup_{n=1}^{\infty} A_n$, where A_n are nowhere dense. Otherwise, A is <u>of Baire second category</u>

Theorem 0.5. *(Baire Category Theorem)*
A complete space is of Baire second category.

Proof. Towards contradiction, assume $X = \bigcup_{n=1}^{\infty} A_n$, where $A_n \subseteq X$ are nowhere dense.
Let $B_1 = B(X_1, 1)$ be a ball in X.
Since A_1 is nowhere dense, there exists $\overline{B}_2 = \overline{B}(x_2, r_2) \subseteq \overline{B}_1$ such that $\overline{B}_2 \cap A_1 = \varnothing$.
Without loss of generality, assume $r_2 < \frac{1}{2}$. Now there exists $\overline{B}_3 = \overline{B}(x_3, r_3) \subseteq \overline{B}_2$ such that $\overline{B}_2 \cap A_1 = \varnothing$.
Without loss of generality, assume $r_3 < \frac{1}{3}$.
At the kth step, there exists $\overline{B}_{k+1} = \overline{B}(x_{k+1}, r_{k+1}) \subseteq \overline{B}_k$ such that $\overline{B}_{k+1} \cap A_k = \varnothing$, $r_{k+1} \leq \frac{1}{k+1}$.
By the nested balls theorem, $\bigcap_{n=1}^{\infty} \overline{B}_n = \{x\}$. By construction, $x \notin A_n$ for all $n \in \mathbb{N}$.
So $X \neq \bigcup_{n=1}^{\infty} A_n$, a contradiction.

■

Definition 0.14. Let (X, ρ) be a metric space and let $A \subseteq X$. A collection $\{U_\alpha\}_{\alpha \in A}$ of open subsets of X is an open cover of A if $A \subseteq \bigcup_{\alpha \in A} U_\alpha$
A set $K \subseteq X$ is called compact if any open cover of K has a finite subcover.
Equivalently, $K \subseteq X$ is compact if any sequence $\{x_n\} \subseteq K$ has a limit point $x \in K$.

Theorem 0.6. *(Nested compact set theorem)*
Let (X, ρ) be a metric space and let $\{K_n\}_{n=1}^\infty$ be a sequence of nonempty and nested compact sets. Then $\bigcap_{n=1}^\infty K_n \neq \varnothing$.

Proof. Consider $\{x_n\}_{n=1}^\infty$ with $x_n \in K_n$.
Note for all $n, x_n \in K_1$. Thus there exists a subsequence $\{x_{n_k}\}$ converging to some $x \in K_1$.
We claim that $x \in \bigcap_{n=1}^\infty K_n$.
Fix $m \in \mathbb{N}$. $x_{n_m}, x_{n_{m+1}}, x_{n_{m+2}}, \cdots \in K_m$.
The only limit point is x, thus $x \in K_m$.

■

Definition 0.15. Let (X, ρ) be a metric space and let $A \subseteq X$. Then A is bounded if $A \subseteq B(x, r)$ for some $x \in X, r > 0$.

Theorem 0.7. *A compact set in (X, ρ) is closed and bounded.*

Proof. think

■

Normed Spaces

Definition 0.16. Let X be a vector space. A function $\|\cdot\| : X \to \mathbb{R}$ is called a norm on X if

(i) $\|x\| \geq 0$ for all $x \in X$. Further, $\|x\| = 0 \iff x = 0$

(ii) $\|\alpha x\| = |\alpha| \|x\|$ for all $x \in X, \alpha \in \mathbb{C}$.

(iii) $\|x + y\| \leq \|x\| + \|y\|$

$(X, \|\cdot\|)$ is called a normed space
Remark:
If one defines $\rho(x, y) = \|x - y\|$, then (X, ρ) is a metric space.

Lecture 3, 4/11/23

Return to metric spaces

Definition 0.17. Let (X, d) be a metric space. Let $T : X \to X$ be a mapping. Then T is said to be a contraction if there exists $\alpha \in (0, 1)$ such that $\rho(Tx, Ty) \leq \alpha\rho(x, y)$ for all $x, y \in X$.

Theorem 0.8. *(Contraction mapping principle)*
Let (X, ρ) be a complete metric space and let $T : X \to X$ be a contraction. Then T has a unique fixed point, i.e. there exists a unique $x \in X$ such that $Tx = x$.

Proof. Proven in 221A

■

Claim. $(0, 1) \neq \bigcup_{n=1}^{\infty} [a_n, b_n]$, *$[a_n, b_n]$ are disjoint.*

Proof. Assume $(0, 1) = \bigcup_{n=1}^{\infty} [a_n, b_n]$
We claim the set $X = \bigcup_{n=1}^{\infty}\{a_n, b_n\} \cup \{0\} \cup \{1\}$ is closed. Thus X is a complete metric space. Next, we claim that $\{a_n\}, \{b_n\}, \{0\}, \{1\}$
Alternative proof:
Assume $(0, 1) = \bigcup_{n=1}^{\infty} [a_n, b_n]$.
We construct a function $f : (0, 1) \to \mathbb{R}$ continuous that takes countably many values.
blah blah blah

■

Return to normed spaces

Definition 0.18. A complete normed space is called a Banach Space

Definition 0.19. Let X, Y be normed spaces on $K = \mathbb{R}$ or $\mathbb{C}$. A mapping $T : X \to Y$ is linear if $T(\alpha x_1 + \beta x_2) = \alpha Tx_1 + \beta Tx_2$ for all $x_1, x_2 \in X$ and all $\alpha, \beta \in K$.

Definition 0.20. Let X, Y be normed spaces on $K = \mathbb{R}$ or $\mathbb{C}$. Let $T : X \to Y$ be linear. Then

1. T is bounded if there exists $M > 0$ such that for all $x \in X$, $\|Tx\| \leq M\|x\|$

2. The operator norm is

$$\|T\| \stackrel{\text{def}}{=} \sup_{x \in X, x \neq 0} \frac{\|Tx\|}{\|x\|}$$

If T is bounded, then $\|T\| \leq M$

Definition 0.21. Let X and Y be normed spaces on $K = \mathbb{R}$ or $\mathbb{C}$. Let $T : X \to Y$ be linear. T is continuous at $x_0 \in X$ if $x \to x_0$ implies $Tx \to Tx_0$.

Theorem 0.9. *Let X, Y be normed spaces on $K = \mathbb{R}$ or $\mathbb{C}$. Let $T : X \to Y$ be linear. Then the following are equivalent:*

1. *T is continuous at some $x_0 \in X$.*

2. *T is continuous at 0*

3. *T is continuous on X*

4. *T is bounded*

5. *T is Lipschitz*

Proof.

$(3) \Rightarrow (1), (2)$

This is obvious

$(1) \Rightarrow (3)$

Assume T is continuous at some x_0. We want to show continuity at y_0.
Suppose $(y_n) \to y$. Define $x_n = y_n - y_0 + x_0$.
Note $x_n \to x_0$. So $Tx_n \to Tx_0$. Thus

$$\|Ty_n - Ty_0\| = \|Tx_n + Ty_0 - Tx_0 - Ty_0\| \to 0$$

Letting $x_0 = 0$, we get $(2) \Rightarrow (3)$

$(4) \Rightarrow (2)$

Suppose $\|Tx\| \leq M\|x\|$ for all $x \in X$.
Then as $x \to 0$, $\|Tx - T0\| = \|Tx\| \leq M\|x\| \to M\|0\| = 0$.

$(2) \Rightarrow (4)$

For all $\varepsilon > 0$, there exists $\delta > 0$ such that $\|x\| < \delta \implies \|Tx\| < \varepsilon$.
Choose $\varepsilon = 1$. Then there exists $\delta > 0$ such that $\|x\| < \delta \Rightarrow \|Tx\| < 1$
For any $x \in X, x \neq 0 \implies Tx = \frac{\|x\|}{\delta} T(\frac{x}{\|x\|}\delta)$
Set $\overline{x} = \frac{x}{\|x\|}\delta$.
Thus $\|Tx\| = \frac{\|x\|}{\delta}\|T\overline{x}\| \leq \frac{\|x\|}{\delta} = \frac{1}{\delta}\|x\|$
So $\|Tx\| \leq \frac{1}{\delta}\|x\|$ for all $x \in X$, i.e. $M = \frac{1}{\delta}$.

Lecture 4, 4/13/23

(4) $\Rightarrow$ (5)

$\|Tx_1 - Tx_2\| = \|T(x_1 - x_2)\| \leq M\|x_1 - x_2\|$. Thus T is Lipshitz.
Clearly, (5) $\Rightarrow$ (3).

$\blacksquare$

Definition 0.22. For X, Y normed spaces, the set of all bounded linear operators $T : X \to Y$ is denoted by $B(X, Y)$.

Theorem 0.10. *Let X, Y be normed spaces and let $T \in B(X, Y)$. Then*

$$\|T\| = \sup_{\|x\| \leq 1} \|Tx\| = \sup_{\|x\|=1} \|Tx\| = \sup_{\|x\|<1} \|Tx\|$$

Proof. $\|T\| = \sup_{x \in X, x \neq 0} \frac{\|Tx\|}{\|x\|} = \sup_{x \in X, x \neq 0} \|T(\frac{x}{\|x\|})\|$.
So, letting $\overline{x} = \frac{x}{\|x\|}$, $\|T\| = \sup_{x \in X, x \neq 0} \|T\overline{x}\| \leq \sup_{\|x\|=1} \|Tx\|$
Similarly, $\|T\| \leq \sup_{\|x\| \leq 1} \|Tx\|$
$\|T\| = \sup_{x \in X, x \neq 0} \frac{\|Tx\|}{\|x\|} \geq \sup_{x \in X, \|x\|=1} \frac{\|Tx\|}{\|x\|}$, so we get equality.
We could prove $<$ using limits.

Theorem 0.11. *Let X and Y be normed spaces and let $T : X \to Y$ be linear. Then T is bounded iff T maps bounded sets to bounded sets.*

Proof. Later

$\blacksquare$

Note: $\|Tx\| \leq \|T\|\|x\|$. Also, a convergent sequence in a metric space is bounded.

Theorem 0.12. *Let X and Y be normed spaces. Then $B(X, Y)$ is a normed space endowed with the operator norm $\|T\|$. Moreover, if Y is Banach, then $B(X, Y)$ is Banach.*

Proof. $B(X, Y)$ is a vector space:
Choose $\alpha, \beta \in K, T_1, T_2 \in B(X, Y)$. Then $\alpha T_1 + \beta T_2$ is linear and continuous. Hence, $\alpha T_1 + \beta T_2 \in B(X, Y)$.
Now we show that $(B(X, Y), \|T\|)$ is a normed space.

1. $\|T\| \geq 0$ for all $T \in B(X, Y)$ clearly, and $\|T\| = 0$ means $Tx = 0$ for any $x \in X$, so $T = 0$.

2. $\|\alpha T\| = |\alpha|\|T\|$

3. Triangle inequality: Choose $x \in B(X, Y)$. Then

$$\begin{aligned} \|(T_1 + T_2)x\| &= \|T_1x + T_2x\| \\ &\leq \|T_1x\| + \|T_2x\| \\ &\leq \|T_1\|\|x\| + \|T_2\|\|x\| \\ &= \|x\|(\|T_1\| + \|T_2\|) \end{aligned}$$

Thus, $\|T_1+T_2\| = \sup_{x\in X, x\neq 0}\|(T_1+T_2)x\| \leq \sup_{\|x\|\leq 1}(\|T_1\|+\|T_2\|)\|x\| \leq \|T_1\|+\|T_2\|$

Now assume Y is a Banach space.
Let $\{T_n\} \subseteq B(X, Y)$ be a Cauchy sequence.
We construct an operator $T : X \to Y$ as follows.
For all $x \in X$, $\{T_nx\}$ is a Cauchy sequence in Y.
Since Y is complete, $\{T_nx\}$ converges to some Tx
We want to show T is linear and bounded.
For all $n \in \mathbb{N}$, $\alpha, \beta \in K$, $x_1, x_2 \in X, T_n(\alpha x_1 + \beta x_2) = \alpha T x_1 + \beta T x_2$.
Thus $T(\alpha x_1 + \beta x_2) = \alpha T x_1 + \beta T x_2$
Now we show T is bounded.
Since a Cauchy sequence in a metric space is bounded, $\|T_n\| \leq M$ for all $n \in \mathbb{N}$.
For all $x \in \overline{B(0, 1)}, (T_nx) \to Tx$.
Thus $\|Tx\| = \lim_{n\to\infty}\|T_nx\| \leq \lim_{n\to\infty} M\|x\| \leq M$.
Hence $\|T\| \leq M$.

Finally, we show $(T_n) \to T$.
Choose $\varepsilon > 0$. There exists N such that for all $n \geq N$, for all $k \in \mathbb{N}$, $\|T_n - T_{n+k}\| < \varepsilon$.
For all $x \in \overline{B(0, 1)}$, $\|T_nx - T_{n+k}x\| \leq \varepsilon\|x\|$.
Fix $n \geq N$, and let $k \to \infty$ to get $\|T_nx - Tx\| \leq \varepsilon\|x\|$. Thus, $\|T_n - T\| \leq \varepsilon$ for all $n \in \mathbb{N}$¿
So $(T_n) \to T$ in $B(X, Y)$

■

Definition 0.23. Let X be a normed space over a field $K = \mathbb{R}$ or $\mathbb{C}$. Then $B(X, K) = X^\star$ is called <u>the dual space of X</u>. $T \in X^\star$ is called a <u>functional</u>.
Special case: $X = L^p(\Omega)$, we will characterize $X^\star$.
When $1 \leq p < \infty$, then $(L^p(\Omega))^\star = L^q(\Omega)$, where $\frac{1}{p} + \frac{1}{q} = 1$ (when $p = 1, q = \infty$)

Theorem 0.13. *Let X be a normed space and Y a Banach space. Assume $A \subseteq X$ is a dense subspace of X and $T \in B(A, Y)$. Then T admits a unique extension $\overline{T} \in B(X, Y)$. Moreover, $\|\overline{T}\| = \|T\|$.*

Proof. Chose $x \in X \setminus A$.
There exists $\{a_n\} \in A$ converging to x because A is dense. Define $Tx = \lim_{n\to\infty} Ta_n$.

First we show this limit exists.
Note $\{Ta_n\}$ is Cauchy, since $\|Ta_n - Ta_m\| \leq \|T\|\|a_n - a_m\| \to 0$.
Since Y is Banach, $\{Ta_n\}$ conveges.
Now we show the limit does not depend on choice of sequence $\{a_n\}$.
Assume $\{b_n\} \to X$. Then

$$\|Ta_n - Tb_n\| = \|T\|\|a_n - b_n\| \leq \|T\|(\|a_n - x\| + \|b_n - x\|) \to 0$$

Thus $\lim_{n\to\infty} Ta_n = \lim_{n\to\infty} Tb_n$
Next we show $\overline{T}$ is linear.

Choose $\alpha, \beta \in K, x_1, x_2 \in X$.
Let $\{a_n\} \to x_1, \{b_n\} \to x_2$ where $\{a_n\}, \{b_n\} \subseteq A$.
$\alpha a_n \to \alpha x_1, \beta b_n \to \beta x_2$
Thus $\alpha a_n + \beta b_n \to \alpha x_1 + \beta x_2$
So

$$\overline{T}(\alpha x_1 + \beta x_2) = \lim_{n\to\infty} T(\alpha x_1 + \beta x_2) = lim_{n\to\infty} T(\alpha x_n) + \lim_{n\to\infty} T(\beta b_n) = \alpha \overline{T} x_1 + \beta \overline{T} x_2$$

■

Lecture 5, 4/18/23

Theorem 0.14. *Let X be a normed space, and Y a Banach space. Assume $A \subseteq X$ is a dense subspace of X and $T \in B(A, Y)$. Then T admits a unit extension $\overline{T} \in B(X, Y)$. Moreover, $\|\overline{T}\| = \|T\|$*

Proof. If $x \notin A$, choose $a_n \in A$ such that $a_n \to x$. Define $\overline{T}x = \lim_{n\to\infty} Ta_n$. Then $\{Ta_n\}$ is Cauchby, and Y is Banach.

1. $\overline{T}$ is linear

2. $\overline{T}$ is bounded, with $\|\overline{T}\| \leq \|T\|$.

For any $x \in B(0,1)$, there exists $a_n \in B(0,1)$ such that $a_n \to x, a_n \in A$.

$$\begin{aligned}
\|\overline{T}x\| &\leq \|\overline{T}x - \overline{T}a_n\| + \|Ta_n\| \\
&\leq \|\overline{T}x - Ta_n\| + \|T\|\|a_n\| \\
&\leq \|\overline{T}x - Ta_n\| + \|T\|
\end{aligned}$$

As $n \to \infty$, we get $\|\overline{T}x\| \leq \|T\|$ for all $x \in B(0,1)$. Thus $\|T\| = \sup_{x\in B(0,1)} \|\overline{T}x\| \leq \|T\|$

■

Theorem 0.15. *Let X be a normed space, and let $A = \{x_1, \dots, x_n\}$, where $\{x_i\}_{i=1}^n$ is linearly independent. Then*

1. *A is closed*

2. *Let $a_k = \sum_{i=1}^n \alpha_i x_i \in A$, and $a_k \to x \in X$. Then by 1, $x \in \langle A \rangle$, i.e. $x = \sum_{i=1}^n \alpha_i x_i$. Then $\alpha_i^k \to \alpha_i$ for $i = 1, 2, \dots, n$.*

(Convergence in A is equivalent to convergence of coordinates).

Proof. First, assume X is a Euclidean space. Define $T : \mathbb{R}^n \to \langle A \rangle$ given by $T(c_1, \dots, c_n) = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n$.
Note that

- T is linear on $\mathbb{R}^n$.
- T is continuous:

$$\begin{aligned} \|T(c_1, \dots, c_n)\| &= \|c_1 x_1 + \cdots + c_n x_n\| \\ &\le |c_1|\|x_1\| + \cdots + |c_n|\|x_n\| \\ &\le |c| \max_i \|x_i\| \end{aligned}$$

$$\begin{aligned} \|T(c_1, \dots, c_n) - T(b_1, \dots, b_n)\| &= \|T(c_1 - b_1, \dots, c_n - b_n)\| \\ &\le \sum_{i=1}^n |c_i - b_i|\|x_i\| \\ &\le n|c - b|(\max_i \|x_i\|) \\ &\le |c - b| \left(\sum_{i=1}^n \|x_i\|^2 \right)^{\frac{1}{2}} \end{aligned}$$

Thus T is continuous.

$S^{n-1} = \partial B(0,1) = \{(c_1, \dots, c_n) \mid \|c\| = 1\}$.
S^{n-1} is compact, so $f(c) = \|Tc\|, c \in S^{n-1}$ attains its minimal value on S^{n-1}.
$|\|x\| - \|y\|| \le \|x - y\|$ for all $x, y \in X$.
$\min_{c \in S^{n-1}} \|Tc\| = \|Tc_0\|$, where $c_0 = (c_1^\circ, \dots, c_n^\circ) \in S^{n-1}$
I claim now that $\|Tc_0\| > 0$.
$Tc_0 = c_0^1 x_1 + \cdots + c_0^n x_n \neq 0$, so $\|Tc_0\| > 0$.
$r = \|Tc_0\| = \min_{c \in S^{n-1}} \|Tc\| > 0$

$\|Tc\| \geq r > 0$ for all $c \in S^{n-1}$.
For all $b \in \mathbb{R}^n, b \neq 0$,

$$\begin{aligned}
\|Tb\| &= \||b|T(\frac{b}{|b|})\| \\
&= |b|\|T(\frac{b}{|b|})\| \\
&\geq r|b|
\end{aligned}$$

Thus $\|Tb\| \geq rb$ for all $b \in \mathbb{R}^n$.
Assume now $a_k = \sum_{i=1}^n \alpha_i^k x_i \to x \in X$.
$a_k = T(\alpha_1^k, \ldots, \alpha_n^k), x = \sum_{i=1}^n \alpha_i x_i$.
Since $a_k \to a$, $\|a_k\| \leq M$.
So $|(\alpha_1^k, \ldots, \alpha_n^k)| \leq \frac{1}{r}\|T(\alpha_1^k, \ldots, \alpha_n^k)\| \leq \frac{M}{r}$
By Bolzano-Weierstrass, there exists $(\alpha_1^{k_\ell}, \ldots, \alpha_n^{k_\ell}) \to (\alpha_1, \ldots, \alpha_n)$ as $\ell \to \infty$
Since T is continuous,

$$\alpha_{k_\ell} = T(\alpha_1^{k_\ell}, \ldots, \alpha_n^{k_\ell}) = \alpha_1 x_1 + \cdots + \alpha_n x_n \in \langle A \rangle$$

■

Theorem 0.16 (Uniform Boundedness Principle, Banach-Steinhaus Theorem)**.** *Let X be a banach space, and let Y be a normed space. Let $\{T_\alpha\}_{\alpha \in A} \subseteq B(X,Y)$ be a family such that the set $\{T_\alpha x\}_{\alpha \in \triangle}$ is bounded for all $x \in X$. Then $\{T_\alpha\}$ is bounded in $B(X,Y)$, i.e. there exists $M > 0$ such that $\|T_\alpha\| \leq M$ for all $\alpha \in \triangle$*

Proof. Consider the sets $A_n = \{x \in X \mid \|T_\alpha x\| \leq n \forall \alpha \in \triangle\} \subseteq X$

1. $\bigcup_{n=1}^\infty A_n = X$.

2. Each A_n is closed in X.

Let $x_k \in A_n, x_k \to x \in X$.

We want to show $x \in A_n$.

$\forall \alpha \in \triangle, \|T_\alpha x_k\| \leq n, k = 1, 2, \ldots$

As $k \to \infty$, we have that $\|T_\alpha x\| \leq n$

So $x \in A_n$.

By Baire Category Theorem, there exists some A_N not nowhere dense.

$\text{int}\,\overline{A_n} = \text{int}\,A_n \neq \varnothing$.

So there exists $x_0 \in X, r > 0$ such that $B(x_0, r) \subseteq A_N$.

Thus $\overline{B}(x_0, \frac{r}{2}) \subseteq A_N$. Let $R = \frac{r}{2}$

$\|T_\alpha(\overline{B}(x_0, R)\| \leq N \forall \alpha \in \triangle$
Let $x \in X, \|x\| < R$. $x = x_0 + x - x_0$
$x_0 + x \in \overline{B}(x_0, R)$.

$$\begin{aligned}\|T_\alpha x\| &= \|T_\alpha(x + x_0) - T_\alpha x_0\| \\ &\leq \|T_\alpha(x + x_0)\| + \|T_\alpha x_0\| \\ &\leq 2N\end{aligned}$$

$\forall x \in \overline{B}(0, R), \forall \alpha \in \triangle, \|T_\alpha x\| \leq 2N$
$\forall x \in \overline{B}(0, 1), \forall \alpha \in \triangle, \|T_\alpha x\| = \|\frac{1}{R}T_\alpha(Rx)\| = \frac{1}{R}\|T_\alpha(Rx)\| \leq \frac{1}{R}2N = \frac{2N}{R} = M$
$\forall x \in \overline{B}(0, 1), \forall \alpha \in \triangle, \|T_\alpha x\| \leq M \implies \|T_\alpha\| \leq M$

Lecture 6, 4/20/23

Open Mapping Theorem

Definition 0.24. Let X and Y be normed spaces and let $A : X \to Y$. Then A is open if it maps open sets in X to open sets in Y.

Theorem 0.17. *(Open Mapping Theorem)*
Let X and Y be Banach spaces and let $T \in B(X, Y)$ be onto. Then T is open.

Proof. omitted from these notes ∎

Theorem 0.18. *(Bounded Inverse Theorem)*
Let X and Y be Banach spaces and let $T \in B(X, Y)$ be a bijection. Then $T^{-1} \in B(Y, X)$.

Proof. omitted from these notes ∎

Composition of Operators

Assume X, Y, Z are noremd spaces.
Assume $T \in B(X, Y), S \in B(Y, Z)$.
Then $S \circ T : X \to Z$ satisfies the properties

1. $S \circ T \in B(X, Z)$

2. $\|S \circ T\| \leq \|T\| \cdot \|S\|$

Linearity:

$$\begin{aligned} S \circ T(\alpha x_1 + \beta x_2) &= S(T(\alpha x_1 + \beta x_2)) \\ &= S(\alpha T x_1 + \beta T x_2) \\ &= \alpha (S \circ T) x_1 + \beta (S \circ T) x_2 \end{aligned}$$

Boundedness: choose $x \in X$. Then

$$\begin{aligned} \|(S \circ T)x\| &= \|S(T(x))\| \\ &\leq \|S\|\|Tx\| \\ &\leq \|S\|\|T\|\|x\| \end{aligned}$$

Product space and its norm

Let X and Y be normed vector spaces. Then $X \times Y = \{(x, y) \mid x \in X, y \in Y\}$ is a vector space.

Theorem 0.19. *$X \times Y$ becomes a normed space with the norm $\|(x, y)\| = \|x\| + \|y\|$. Moreover, if X and Y are Banach, then $X \times Y$ is Banach.*

Proof. omitted

■

Lecture 7

Theorem 0.20. *(Product space as a normed space)*
Let X and Y be normed space. Then

1. *$(X \times Y, \|(\cdot, \cdot)\|)$ is a normed space.*

2. *Convergence in $X \times Y$ is equivalent to convergence in coordinates.*

3. *If in addition X and Y are equivalent, then $(X \times Y, \|(\cdot, \cdot)\|)$ is complete.*

Theorem 0.21. *(Closed graph theorem)*
Let X, Y be Banach spaces and let $A : X \to Y$ be linear. Then A is continuous (bounded) iff the graph of A is closed in $X \times Y$.

Proof. ■

Definition 0.25. Given X and Y sets and an operator $A : X \to Y$, we define the graph of A to be the set $\text{Gr}\{(x, Ax) \mid x \in X\} \subseteq X \times Y$.
Assume now that X, Y are normed spaces. What does it mean for $\text{Gr}\, A \subseteq X \times Y$ to be closed?
$\text{Gr}\, A$ is closed if it contains all its limit points.
Assume $(x_n, Ax_n) \to (x, y)$.
Thus $x_n \to x, Ax_n \to y$ We reguire that $(x, y) \in \text{Gr}\, A$, so $y = Ax$.
In sum, whenever $x_n \to X$ and $Ax_n \to y$, $y = Ax$.

The Hahn-Banach Theorems

Definition 0.26. Let X be a vector spaces over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$.

(a) A map $q : X \to \mathbb{R}$ is called a quasi-seminorm if

  (i) $q(x + y) \leq q(x) + q(y)$ for all $x, y \in X$ (triangle inequality)

  (ii) $q(tx) = tq(x)$ for all $t > 0, x \in X$.

(b) A map $P : X \to |R$ is called a seminorm if P is a quasi-seminorm and in addition $P(\alpha x) = |\alpha| P(x)$ for all $\alpha \in K, x \in X$.

Theorem 0.22. *(Hahn-Banach, $\mathbb{R}$-version)*
Let X be a real vector space and let $Y \subseteq X$ be a subspace. Assume $\phi : Y \to \mathbb{R}$ is a linear function and $q : X \to \mathbb{R}$ is a quasi-seminorm such that $\phi(y) \leq q(y)$ for all $y \in Y$. Then there exists a linear extension ψ of ϕ onto X such that it is dominated by q in X.

(i) $\psi : X \to \mathbb{R}$ *is linear*

(ii) $\psi(y) = \phi(y)$ *for all* $y \in Y$

(iii) $\psi(x) \leq q(x)$ *for all* $x \in X$

Theorem 0.23. *($\mathbb{C}$-version, first version)*
Let X be a vector space on $\mathbb{C}$ and let $Y \subseteq X$ be a subspace. Assume $\phi : Y \to \mathbb{C}$ is a $\mathbb{C}$-linear map and $q : X \to \mathbb{R}$ is a quasi-seminorm such that $\text{Re}\, \phi(y) \leq q(y)$ for all $y \in Y$. Then there exists a linear map $\psi : X \to \mathbb{C}$ such that

(i) $\psi(y) = \phi(y)$ *for all* $y \in Y$

(ii) $\text{Re}\, \psi(x) \leq q(x)$ *for all* $x \in X$.

Lemma 1. *Let X be a $\mathbb{C}$-vector space and let $\phi : X \to \mathbb{C}$ be given by $\phi(x) = u(x) + iv(x)$, where $u, v : X \to \mathbb{R}$. $u(x) = \operatorname{Re}\phi(x)$, $v(x) = \operatorname{Im}\phi(x)$. Then ϕ is $\mathbb{C}$-linear iff u is $\mathbb{R}$-linear and $v(x) = -u(ix)$.*

Proof. omitted

■

Lecture 8, 5/2/23

Definition 0.27. $f : D \to D$ is <u>convex</u> if for all $\lambda \in [0,1]$, $x, y \in D$, $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$.
Recall a quasi-seminorm is convex, since $q(x+y) \leq q(x) + q(y), q(tx) = tq(x)$ for all $t > 0$.

Theorem 0.24. *(Hahn-Banach)*
Let X be a vector space on $K \in \{\mathbb{C}, \mathbb{R}\}$ and let $Y \subseteq X$ be a nonempty subspace. Assume $\phi : Y \to K$ is a K-linear map and $q : X \to \mathbb{R}$ is a seminorm such that $|\phi(y)| \leq q(y)$ for all $y \in Y$. Then there eixsts a linear map $\psi : X \to K$ such that

(i) $\psi(y) = \phi(y)$ for all $y \in Y$.

(ii) $|\psi(x)| \leq q(x)$ for all $x \in X$.

Proof. omitted

■

Definition 0.28. A Banach space X is called <u>reflexive</u> if $X^{**} = \{F_x\}_{x \in X}$, where F_x is the evaluation operator, $f \mapsto f(x)$.

Lecture 9, 5/4/23

Let X be a normed space. For all $x \in X$, $F_x \in (X^*)^*$, $F_x(f) = f(x)$, for all $f \in X^*$. Then $F : X \to \{F_x \mid x \in X\} \subseteq (X^*)^*$

1. The mapping F is an isometry

Definition 0.29. A Banach space X is called <u>reflexive</u> if $(X^*)^* = \{F_x \mid x \in X\}$.
One identifies the subspace $\{F_x \mid x \in X\}$ with X. One sets $X \subseteq (X^*)^*$.
We will prove later that $(L^p(\mathbb{R}^n))^* = L^q(\mathbb{R}^n)$ for all $1 < p < \infty$ where $q > 0$ and $\frac{1}{q} + \frac{1}{q} = 1$. In this case q is called the <u>dual exponent</u> of p.
Then $((L^p(\mathbb{R}^n))^*)^* = (L^q(\mathbb{R}^n))^* = L^p(\mathbb{R}^n)$, so $L^p(\mathbb{R}^n)$ is reflexive.

Geometric Hahn-Banach theorems, separation of convex sets

Definition 0.30. Let X be a vector space.

1. A set $C \subseteq X$ is called convex if $\lambda x + (1-\lambda)y \in C$ for all $\lambda \in [0,1]$, whenever $x, y \in C$.

2. A set $A \subseteq X$ is called absorbing if for every $x \in X$, there exists $t > 0$ such that $tx \in A$. This is equivalent to $X = \bigcup_{t>0} t \cdot C$

Example 0.6. For X a normed space, then any open or closed ball is convex.

Example 0.7. Any open set containing the origin is absorbing.

Definition 0.31. Let X be a real vector space and let $C \subseteq X$ be a convex and absorbing set containing the origin. So $0 \in C$, $X = \bigcup_{t>0} t \cdot C$.
For every $x \in X$, define $M_c(x) = \inf\{t > 0 \mid x \in t \cdot C\}$. So $M_c : X \to [0, \infty)$. This is called the Minkowski functional of C

Theorem 0.25. *Under the conditions of the previous definition, M_c is a quasi-seminorm.*

Proof.

Claim. *Define $I_C(x) = \{t > 0 \mid x \in t \cdot C\} \subseteq \mathbb{R}$ for all $x \in X$. $I_c(x)$ has the following properties*

1. $I_C(\lambda x) = \lambda \cdot I_C(x)$ *for all* $x \in X, \lambda > 0$.

2. $I_C(x) + I_C(y) \subseteq I_C(x+y)$ *for all* $x, y \in X$.

Proof.

1. Let's prove that $I_C(\lambda x) \leq \lambda \cdot I_C(x)$ for all $\lambda > 0$ and all $x \in X$.

Then $I_C(x) = I_C(\frac{1}{\lambda} \cdot (\lambda x)) \subseteq \frac{1}{\lambda} \cdot I_C(\lambda x)$

So $\lambda \cdot I_C(x) \subseteq I_C(\lambda x)$.

Let $t \in I_C(\lambda x)$. Then $\lambda x \in t \cdot C$, $x \in \frac{t}{\lambda} \cdot C$. So $\frac{t}{\lambda} \in I_C(x)$, $t \in \lambda \cdot I_C(x)$, so $I_C(\lambda x) \subseteq \lambda I_C(x)$

2. Let $T \in I_C(x)$ and $s \in I_C(y)$. We need to prove $s + t \in I_C(x+y)$. We know $x \in t \cdot C$, and $y \in s \cdot C$.

For some $t, s > 0, c_1, c_2 \in C$, $x = t \cdot c_1, y = s \cdot c_2$. Consider the convex combination $\frac{t}{t+s} \cdot c_1 + \frac{s}{t+s} \cdot c_2 \in C$. So $\frac{1}{t+s}(x+y) \in C$.

Thus $t + s \in I_C(x+y)$

■

So $M_C(\lambda \cdot X) = \lambda \cdot M_C(x)$ for all $\lambda > 0, x \in X$.
Suppose $A + B \subseteq D$. Then $\inf A + \inf B \geq \inf D$.
So $M_C(x+y) \leq M_C(x) + M_C(y)$ for all $x, y \in X$.

■

Theorem 0.26. *Under the conditions of the definition, one has*

$$\{x \in X \mid M_C(x) < 1\} \subseteq C \subseteq \{x \in X \mid M_C(x) \leq 1\}$$

Proof.

1. $C \subseteq \{x \in X \mid M_C(x) \leq 1\}$. Assume $x \in C$. Then $1 \cdot x \in C$. So $1 \in I_C(x)$, so $\inf I_C(x) \leq 1$, so $M_C(x) \leq 1$

2. $\{x \in X \mid M_C(x) < 1\} \subseteq C$. Let $x \in \{x \in X \mid M_C(x) < 1\}$. $M_C(x) < 1$, so $\inf I_C(x) < 1$.

So there exists $t < 1$ such that $x \in t \cdot C$.

So $t \in I_C(x)$. $x = t \cdot c, c \in C$. So $x = t \cdot c + (1-t) \cdot 0 \in C$ because C is convex.

■

Theorem 0.27. *Let X be a real normed space and let $C \subseteq X$ be an open, convex set containing the origin.*
Then $C = \{x \in X \mid M_C(x) < 1\}$

Proof. By previous theorem, we have $\{x \in X \mid M_C(x) < 1\} \subseteq X$.
We need to prove that if $M_C(x) = 1$, then $x \notin C$.
Assume $x \in C$. There exists $\overline{B}_\delta(x) \subseteq C$, $\delta > 0$.
$y = x\frac{\|x\|+\delta}{\|x\|} \in \overline{B}_\delta(x)$, $\|y - x\| = \|\frac{\delta x}{\|x\|}\| = \delta$
So $\|y\| = \|x\| + \delta > \|x\|$
So $t = \frac{\|x\|}{\|x\|+\delta} < 1$, so $x \in t \cdot C$, so $x = t \cdot y, y \in C$.
So $M_C(x) \leq t < 1$

■

Lecture 10, 5/9/23

Theorem 0.28. *(Separation of a convex set from a point)*
Let X be a real normed space, let $C \subseteq X$ be an open convex set and let $x_0 \notin C$. There exists a linear continuous functional $\varphi : X \to \mathbb{R}$ such that

1. $\varphi(x_0) = 1$

2. $\varphi(c) < 1$ *for all* $c \in C$.

In Euclidean space, $\varphi(x) = \sum_{i=1}^{n} a_i x_i$.
$\varphi(x) = 1$ *- hyperplane.*

Proof. C open and $0 \in C \Rightarrow C$ is absorbing.
Thus the Minkowski functional of C, $M_C(x)$, is a quasi-seminorm.
Theorem 3 gives $C = \{x \in X \mid M_C(x) < 1\}$.
Consider the subspace $Y = \{tx_0 \mid t \in \mathbb{R}\} \subseteq X$.
Define $\phi : Y \to \mathbb{R}$ linear functional as $\phi(t \cdot x_0) = t$, for all $t \in \mathbb{R}$, or for all $tx_0 \in Y$.
Claim 1: $\phi(y) \leq M_c(y)$ for all $y \in Y$.
Let $y = tx_0$. We want to show that $t = \phi(y) \leq M_c(tx_0)$.
If $y < 0$, this is obvious, since $M_C(y) \geq 0$ for all $y \in Y$.
Now assume $t > 0$.
M_C is a quasi-seminorm, so $M_C(tx_0) = tM_C(x_0)$.
Since $x_0 \notin C = \{x \in X \mid M_C(x) < 1\}$, we have $M_C(x_0) \geq 1$.
So $M_C(tx_0) \geq t$.
By Hahn-Banach, we can extend ϕ to X as a linear functional that $\leq M_C$.
Further, $\phi : X \to \mathbb{R}$ is linear, $\phi(x_0) = 1$.
I'm not finishing this

■

Theorem 0.29. *(Separation of two convex sets)*
Let X be a real normed space and let A, B be disjoint convex sets with A open. There exists a continuous linear functional $\phi : X \to \mathbb{R}$ such that $\phi(a) < \alpha \leq \phi(b)$ for all $a \in A, b \in B$, where $\alpha = \inf_{b \in B} \phi(b)$

Proof. ■

Theorem 0.30. *(Separation of Two convex sets)*
Let X be a complex normed space and let A, B be disjoint convex sets with A open. There exists a continuous linear funcitonal $\phi : X \to \mathbb{C}$ such that $\operatorname{Re}\phi(a) < \alpha \leq \operatorname{Re}\phi(b)$ for all $a \in A, b \in B$, where $\alpha = \inf_{b \in B} \operatorname{Re}\phi(b)$

L^p spaces

Definition 0.32. Assume Ω is an open set, $p \in [1, \infty]$. For a measurable (Lebesgue-measurable) function $f : \Omega \to \mathbb{C}$, define

$$\|f\|_{L^p(\Omega)} = \begin{cases} \left(\int_\Omega |f(x)|^p \, dx\right)^{\frac{1}{p}} & p \in [1, \infty) \\ \operatorname{esssup}_{x \in \Omega} |f(x)| & p = \infty \end{cases}$$

We define $\operatorname{esssup}_{x \in \Omega} |f(x)| = \inf\{M : |\{x \in \Omega : |f(x)| > M\}| = 0\}$

Here $|A|$ = Lebesgue measure of A.
Define the set $L^p(\Omega) = \{f : \Omega \to \mathbb{C} : f$ measurable, $\|f\|_{L^p(\Omega)} < \infty$.

Theorem 0.31. *$L^p(\Omega)$ is a banach space with norm $\|f\|_{L^p(\Omega)}$ (modulo almost everywhere agreement of functions).*

Proof. ∎

General setting: X a nonempty set, μ a measure on X. Choose $1 \leq p \leq \infty$. For $f : X \to \overline{\mathbb{R}}$ μ-measurable, define

$$\|f\|_{L^p(X,\mu)} = \begin{cases} \left(\int_X |f(x)|^p \, dx\right)^{\frac{1}{p}} \ 1 \leq p < \infty \\ \operatorname{esssup}_{x \in X} |f(x)| \qquad\qquad p = \infty \end{cases}$$

Definition 0.33. $\operatorname{esssup}_{x \in X} |f(x)| = \inf\{M : \mu\{x \in X : |f(x)| > M\} = 0\}$

Proposition 2. *Denote $M_0 = \operatorname{esssup}_{x \in X} |f(x)|$. Then $\mu\{|f(x)| > M_0\} = 0$.*

Proof. ∎

Definition 0.34. Define for $1 \leq p \leq \infty$,

$$\underline{L^p(X,\mu)} = L^p(X) \stackrel{\text{def}}{=} \{f : X \to \overline{\mathbb{R}}, f \mu - \text{measurable }, \|f\|_p < \infty\}$$

For a function f, denote by $\hat{f}$ the equivalence class of f under μ-almost everywhere equivalence.
From now on, we identify two functions that agree μ-almost everywhere in X.

Theorem 0.32. *$\{\hat{f}\}$ with the norm $\|\cdot\|_p$ is a Banach space, called the Lebesgue space.*

Lemma 2. *(Young's Inequality)*
Assume $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$ for all $a, b \geq 0$. Equivalently, $a^\lambda b^{1-\lambda} \leq \lambda a + (1-\lambda)b$ for all $\lambda \in [0,1]$, $a, b \geq 0$. Convention is to take $0^0 = 0$, or $\lambda \in (0,1)$.

Theorem 0.33. *If $\phi \in C^2(\mathbb{R})$, then ϕ is convex iff $\phi''(t) \geq 0$.*

Proof. ∎

Lecture 11, 5/16/23

Definition 0.35. Let X be a normed space and let $(x_n) \subseteq X, x \in X$. (x_n) converges weakly to x if $\phi(x_n) \to \phi(x)$ for all $\phi \in X^*$.
Note strong convergence implies weak convergence.

Definition 0.36. Let X be a normed space and let $(f_n) \subseteq X^*$, $f \in X^*$. (f_n) converges weak-* to f if $f_n(x) \to f(x)$ for all $x \in X$.
If X is reflexive, the above two definitions are equivalent.

Theorem 0.34. *9Alaoglu's Theorem)*
Let X be a normed space. The closed unit ball in X^* is weak- compact.*