# Lecture 1, 4/3/23

We will begin thinking about homological algebra in a more categorical way. We use "methods of homological algebra," by Gelfond-Manin (Gelfond & Manin?). Here is a brief overview of some things we will see during the course:

- Additive and Abelian categories
- Example: category of sheaves of Abelian groups on a topological space
- Derived category (objects: complexes of objects in a given Abelian category, morphisms: a quasi-isomorphism of complexes becomes an isomorphism of objects)
- Derived functors,  $f_*, f^*, f_!, f^!, \mathbb{D}, \cdots$ , Hom for sheaves
- Triangulated categories
- t-structures
- Examples: perverse sheaves

Let's remind ourselves of some definitions:

### Additive & Abelian categories

**Example 0.1.** (a) Ab, the category of Abelian groups with group homomorphisms.

- (b) R Mod, the category of R-modules, with R-module homomorphisms as morphisms.
- (c) SAb, PAb, the categories of sheaves of Abelian groups and presheaves of Abelian groups
- (d) Sheaves of modules over a ringed space
- (e) (quasi)-coherent sheaves (ask Zhao)

**Definition 0.1.** An Abelian category contains the following information:

1. Any hom-set  $\operatorname{Hom}_{\mathscr{C}}(X,Y)$  is an Abelian group (+), and the composition of morphisms is bi-additive

In particular:

• Hom<sub>\mathscr{C}</sub> is a functor  $\mathscr{C}^{\circ} \times \mathscr{C} \to \mathsf{Ab}$ . We notate the first factor with the  $^{\circ}$ 

- $0 \in \operatorname{Hom}_{\mathscr{C}}(X,Y)$  for any objects X,Y of  $\mathscr{C}$
- **2.** There exists a zero object  $0 \in \mathcal{C}$ , that is an object such that  $\text{Hom}_{\mathcal{C}}(0,0) = 0$ .

This gives:  $\operatorname{Hom}_{\mathscr{C}}(0,X) = 0$ ,  $\operatorname{Hom}_{\mathscr{C}}(X,0) = 0$  for all objects X of  $\mathscr{C}$ .

We know that  $\operatorname{Hom}_{\mathscr{C}}(0,0)$  consists of one object. In particular, it must be  $\operatorname{Id}_0 = 0$ . So

$$(0 \xrightarrow{f} X) = (0 \xrightarrow{\operatorname{Id}_0} 0 \xrightarrow{f} X) = (0 \xrightarrow{0} X)$$

Any two zero objects are isomorphic.

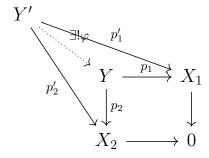
**3.** For any  $X_1, X_2 \in \mathcal{C}$ , there exists an object Y and morphisms

$$X_1 \stackrel{i_1}{\underbrace{\hspace{1cm}}} Y \stackrel{i_2}{\underbrace{\hspace{1cm}}} X_2$$

such that

$$p_1 i_1 = \operatorname{Id}_{X_1}$$
  
 $p_2 i_2 = \operatorname{Id}_{X_2}$   
 $i_1 p_1 + i_2 p_2 = \operatorname{Id}_Y$   
 $p_2 i_1 = p_1 i_2 = 0$ 

#### Lemma 1. We have cartesian diagram



That is, for any Y', with morphisms  $p'_1, p'_2$  as in the diagram, there is a morphism from Y' to Y making the diagram commute. Similarly, we have co-cartesian diagram

$$Y \stackrel{i_1}{\longleftarrow} X_1$$

$$\downarrow i_2 \qquad \qquad \uparrow$$

$$X_2 \longleftarrow 0$$

*Proof.* We need to construct  $\varphi: Y' \to Y$  such that  $p'_1 = p_1 \varphi$  and  $p'_2 = p_2 \varphi$  Take  $\varphi = i_1 p'_1 + i_2 p'_2$ . Then

$$p_1 \circ \varphi = \underbrace{p_1 i_1}_{= \mathrm{Id}_{X_1}} p'_1 + \underbrace{p_1 i_2}_{=0} p'_2 = p'_1$$

The uniqueness of  $\varphi$  can be verified as an exercise

**Definition 0.2.** An <u>additive category</u> is one which satisfies only the first three of these axioms

To state the 4th axiom of Abelian categories, we need more notations:

**Definition 0.3.** Let  $A_1, A_2$  be objects of  $\mathscr{C}$ , and let  $\varphi: X \to Y$ .

- **1.** A <u>kernel</u> of  $\varphi$  is a morphism  $i: Z \to X$  such that
  - (a)  $\varphi \circ i = 0$
  - (b) For all  $i': Z' \to X$  such that  $\varphi \circ i' = 0$ , there is a unique  $g: Z' \to Z$  such that  $i' = i \circ g$ .

$$Z \xrightarrow{g} \downarrow_{i'} \downarrow_{0}$$

$$Z \xrightarrow{i} X \xrightarrow{\varphi} Y$$

2. A <u>cokernel</u> is a kernel but with the arrows reversed

Exercise: Verify that 1 is equivalent to the following: for all  $Z' \in \mathscr{C}$ ,

$$0 \longrightarrow \operatorname{Hom}(Z',Z) \xrightarrow{i_*} \operatorname{Hom}(Z',X) \xrightarrow{\varphi_*} \operatorname{Hom}(Z',Y)$$

is exact, and similarly for cokernel

With all that, we are ready for:

For any  $\varphi: X \to Y$ , there exists a sequence of morphisms

$$K \xrightarrow{k} X \xrightarrow{i} I \xrightarrow{j} Y \xrightarrow{c} K'$$

such that

- (a)  $j \circ i = \varphi$
- (b)  $k = \ker \varphi, k' = \operatorname{coker} \varphi$
- (c)  $I = \operatorname{coker} k = \ker c$

This finishes the definition

# Lecture 2, 4/5/23

#### Sheaves

Here are some examples of sheaves from complex analysis

#### Example 0.2.

(a) The set of holomorphic functions on  $\mathbb{P} = \mathbb{C} \sup\{\infty\}$ 

For each open subset U of  $\mathbb{P}$ , we can consider the ring of holomorphic function  $f: U \to \mathbb{C}$ ,  $\mathcal{H}(U)$ .

The collection of  $\{(f, V) \mid V \text{ open}, f \in \mathcal{H}(V)\}$  is called the sheaf  $\mathscr{O}$  of holomorphic functions on  $\mathbb{P}$ .

(b) The sheaf of solutions of a linear ODE

Let  $U \subseteq \mathbb{P}$  be open, and let  $a_i(z) \in \Gamma(U, \mathcal{O})$  (in this context this will wind up meaning  $\mathcal{H}(U)$ ),  $i = 0, 1, \ldots, n-1$ .

Denote by S the collection of (V, f) such that  $V \subseteq U$  is open, and f is holomorphic in V such that

$$Lf \stackrel{\text{def}}{=} \frac{d^n f}{dz^n} + \sum_{i=0}^{n-1} a_i(z) \frac{d^i f}{dz^i} = 0$$

Let  $\Gamma(V) = \{ f \in \mathcal{H}(V) \mid Lf = 0 \}$ . When V is connected and simply connected, it is a basic result of ODEs that  $\Gamma(V) \cong \mathbb{C}^n$ 

In general, it may have to do with the topology of V. For example, if  $U = \mathbb{C} \setminus \{0\}$ ,  $L = \frac{d^2}{dz^2} + \frac{1}{z}\frac{d}{dz}$ , the solutions are  $c_1 \log(z) + c_2$  for any "branch" of  $\log(z)$ , but  $\Gamma(V) = \{\text{constant}\}$ . This is related to the Riemann-Hilbert correspondence (whatever that is)

#### Definition 0.4.

- (a) A presheaf of sets  $\mathcal{F}$  on a topological space Y consists of the following data:
  - A set  $\mathcal{F}(U)$  for any open  $U \subseteq Y$
  - For any open  $V \subseteq U$ , a (restriction) map  $\gamma_{U,V} : \mathcal{F}(U) \to \mathcal{F}(V)$  such that  $\gamma_{V,V} = \mathrm{Id}_{\mathcal{F}(V)}$ , and if  $W \subseteq V \subseteq U$ , then  $\gamma_{V,W} \circ \gamma_{U,V} = \gamma_{U,W}$ . When there is ambiguity about which sheaf  $\gamma$  belongs to, we further specify with  $\gamma^{\mathcal{F}}$
- (b) A presheaf  $\mathcal{F}$  is a <u>sheaf</u> if:

• For any open covering  $U = \bigcup_{i \in I} U_i$  and  $s_i \in \mathcal{F}(U_i)$  such that for all  $i, j \in I$ ,

$$\gamma_{U_i,U_i\cap U_j}(s_i) = \gamma_{U_j,U_i\cap U_j}(s_j)$$

then there exists a unique  $s \in \mathcal{F}(U)$  such that  $s_i = \gamma_{U,U_i}(s)$  for all i.

(c) A morphism of presheaves  $f: \mathcal{F} \to \mathcal{G}$  is a family of maps  $f(U): \mathcal{F}(U) \to \mathcal{G}(U)$  for all open  $U \subseteq Y$ , such that for all open  $V \subseteq U$ , the diagram

$$\mathcal{F}(U) \xrightarrow{f(U)} \mathcal{G}(U) 
\gamma_{U,V}^{\mathcal{F}} \downarrow \qquad \qquad \downarrow \gamma_{U,V}^{\mathcal{G}} 
\mathcal{F}(V) \xrightarrow{f(V)} \mathcal{G}(V)$$

commutes.

A morphism of sheaves is a morphism of the underlying presheaves

A presheaf  $\mathcal{F}$  of groups/rings/ $\mathbb{F}$ -vector spaces is a presheaf such that  $\mathcal{F}(U)$  is a group/ring/ $\mathbb{F}$ -vector space. Then  $\gamma_{U,V}$  is a morphism of groups/rings/ $\mathbb{F}$ -vector spaces.

Let  $\mathcal{F}, \mathcal{G}$  be two Abelian presheaves (meaning presheaves of Abelian groups), and let  $f: \mathcal{F} \to \mathcal{G}$  be a morphism of Abelian presheaves.

Let  $K(U) = \ker(f(U)), C(U) = \operatorname{coker}(f(U))$  with natural restrictions.

**Definition 0.5.** A sequence of presheaves

$$\mathcal{F} \stackrel{f}{\longrightarrow} \mathcal{G} \stackrel{g}{\longrightarrow} \mathcal{H}$$

is  $\underline{\text{exact}}$  if, for all open U,

$$\mathcal{F}(U) \xrightarrow{f(U)} \mathcal{G}(U) \xrightarrow{g(U)} \mathcal{H}(U)$$

is exact.

So, presheaves of Abelian groups, with morphisms of Abelian presheaves, is an Abelian category.

How about Abelian sheaves?

Let  $f: \mathcal{F} \to \mathcal{G}$  be a morphism of Abelian sheaves. Once again, let  $K(U) = \ker(f(U)), C(U) = \operatorname{coker}(f(U)) = \frac{\mathcal{G}(U)}{f(U)(\mathcal{F}(U))}$ .

### Proposition 1.

- (a) The kernel K is an Abelian sheaf
- (b) The cokernel C is always a presheaf, but might not be a sheaf.

# Lecture 3, 4/7/23

Proof.

(a) Let  $U = \bigcup U_i$ ,  $s_i \in K(U_i)$  agree on pairwise intersections. As  $K(U_i) \hookrightarrow \mathcal{F}(U_i)$ , there exists a unique  $s \in \mathcal{F}(U)$  such that  $s|_{U_i} = s_i \in \mathcal{F}(U_i)$  for all i.

Note that  $f(s)|_{U_i} = f(s|_{U_i}) = 0$ , so  $f(s) = 0 \in \mathcal{G}(U)$ . Here, we are using the uniqueness of gluing in  $\mathcal{G}$ .

Then s " $\in$  "K(U)

- (b) First, we give an example of a cokernel which is a presheaf, but not a sheaf. Let  $Y = \mathbb{C} \setminus \{0\}, f : \mathcal{O}_Y \to \mathcal{O}_Y$  given by  $\varphi \mapsto \frac{d\varphi}{d\zeta}$ .
  - For any  $y \in Y$ , there exists a neighborhood  $V_y \ni y$  such that coker  $f(V_y) = 0$ . That is, for every  $f \in \mathcal{H}(V_y)$ , there is a  $g \in \mathcal{H}(V_y)$  so that  $\frac{dg}{d\zeta} = f$  in  $V_y$  (every point admits a simply connected neighborhood)
  - However, coker  $f(Y) \cong \mathbb{C}$ :  $\Psi = \sum_{i=-\infty}^{\infty} a_i \zeta^i = \frac{d\varphi}{d\zeta}$  has a solution iff  $a_{-1} = 0$ . This is because  $\frac{1}{z}$  is defined on Y. (keyword: vanishing cycle and v-filtration).

Hence this violates the uniqueness: any  $\overline{\Psi} \in \operatorname{coker} f(Y)$  restricts to  $0 \in \operatorname{coker} f(V_y)$ . However, the  $V_y$  cover Y. So, we have an open cover, with sections of each, which agree on intersections, but which do not have a unique global section to which they all glue. So, this is not a sheaf.

### Sheafification

Denote by SAb the additive category of abelian sheaves on a fixed topological space M. We have the inclusion functor  $\iota : \mathrm{SAb} \to \mathrm{PAb}$ .

**Proposition 2.**  $\iota$  admits a left adjoint  $s : PAb \to SAb$ , i.e.

$$\operatorname{Hom}_{\operatorname{SAb}}(sX,Y) \cong \operatorname{Hom}_{\operatorname{PAb}}(X,\iota Y)$$

and this isomorphism is natural in both X and Y.

*Proof.* Let

$$s(X)(U) \stackrel{\text{def}}{=} \frac{\{\{(U_i, e_i)\}_{i \in I} \mid U = \bigcup_{i \in I} U_i, e_i \in X(U_i), e_i|_{U_i \cap U_j} = e_j|_{U_i \cap U_j}\}}{s(X)(U) \stackrel{\text{def}}{=} \frac{\{\{(U_i, e_i)\}_{i \in I} \mid U = \bigcup_{i \in I} U_i, e_i \in X(U_i), e_i|_{U_i \cap U_j} = e_j|_{U_i \cap U_j}\}}{s(X)(U) \stackrel{\text{def}}{=} \frac{\{\{(U_i, e_i)\}_{i \in I} \mid U = \bigcup_{i \in I} U_i, e_i \in X(U_i), e_i|_{U_i \cap U_j} = e_j|_{U_i \cap U_j}\}}{s(X)(U) \stackrel{\text{def}}{=} \frac{\{\{(U_i, e_i)\}_{i \in I} \mid U = \bigcup_{i \in I} U_i, e_i \in X(U_i), e_i|_{U_i \cap U_j} = e_j|_{U_i \cap U_j}\}}{s(X)(U) \stackrel{\text{def}}{=} \frac{\{\{(U_i, e_i)\}_{i \in I} \mid U = \bigcup_{i \in I} U_i, e_i \in X(U_i), e_i|_{U_i \cap U_j} = e_j|_{U_i \cap U_j}\}}{s(X)(U) \stackrel{\text{def}}{=} \frac{\{\{(U_i, e_i)\}_{i \in I} \mid U = \bigcup_{i \in I} U_i, e_i \in X(U_i), e_i|_{U_i \cap U_j} = e_j|_{U_i \cap U_j}\}}{s(X)(U) \stackrel{\text{def}}{=} \frac{\{\{(U_i, e_i)\}_{i \in I} \mid U = \bigcup_{i \in I} U_i, e_i \in X(U_i), e_i|_{U_i \cap U_j} = e_j|_{U_i \cap U_j}\}}{s(X)(U) \stackrel{\text{def}}{=} \frac{\{\{(U_i, e_i)\}_{i \in I} \mid U = \bigcup_{i \in I} U_i, e_i \in X(U_i), e_i|_{U_i \cap U_j} = e_j|_{U_i \cap U_j}\}}{s(X)(U) \stackrel{\text{def}}{=} \frac{\{\{(U_i, e_i)\}_{i \in I} \mid U = \bigcup_{i \in I} U_i, e_i \in X(U_i), e_i|_{U_i \cap U_j} = e_j|_{U_i \cap U_j}\}}{s(X)(U) \stackrel{\text{def}}{=} \frac{\{\{(U_i, e_i)\}_{i \in I} \mid U = \bigcup_{i \in I} U_i, e_i \in X(U_i), e_i|_{U_i \cap U_j} = e_j|_{U_i \cap U_j}\}}{s(X)(U) \stackrel{\text{def}}{=} \frac{\{\{(U_i, e_i)\}_{i \in I} \mid U = \bigcup_{i \in I} U_i, e_i \in X(U_i), e_i|_{U_i \cap U_j} = e_j|_{U_i \cap U_j}\}}{s(X)(U) \stackrel{\text{def}}{=} \frac{\{\{(U_i, e_i)\}_{i \in I} \mid U = \bigcup_{i \in I} U_i, e_i \in X(U_i), e_i \in X(U_i), e_i \in X(U_i), e_i \in X(U_i)\}}$$

Where  $\{(U_i, e_i)\} \sim \{(U'_j, e_j)\}$  if there exists  $\{U''_k\}$  refining  $\{U_i\}, \{U'_j\}$  and  $e_i|_{U''_k} = e'_j|_{U''_k}$  for  $U''_k \subset U_i \cap U'_j$ .

Define  $\gamma_{U,V}: sX(U) \to sX(V)$ , for  $V \subseteq U$ , by

$$\gamma_{U,V}[\{(U_i, e_i)\}] = [\{(U_i \cap V, e_i|_{U_i \cap V})\}]$$

There is a lot to verify; see [GM] 2.5.13

**Example 0.3.** For coker f from previous example, now  $[\overline{\Psi}] = [0]$ , since, when we restrict to  $V_y$ ,  $\overline{\Psi}$  becomes 0. So coker f = 0

With this modification, SAb is an abelian category!

**Proposition 3.** Let  $\varphi: X \to Y$  be a morphism of abelian sheaves, and let

$$K \xrightarrow{k} \iota X \xrightarrow{i} I \xrightarrow{j} \iota Y \xrightarrow{c} K'$$

be the canonical decomposition of  $\iota(\varphi)$  in (the abelian category!) PAb. Then

$$\underbrace{sK}_{=K} \xrightarrow{-sk} \underbrace{X}_{=s\iota X} \xrightarrow{-si} sI \xrightarrow{-sj} \underbrace{Y}_{=s\iota Y} \xrightarrow{-sc} sK'$$

is the canonical decomposition of  $\varphi$  in SAb. In particular, SAb is an abelian category.

*Proof.* We'll just verify that sK' is indeed the cokernel: Let  $Z \in \text{Ob SAb}$ . Then there exists an exact sequence

$$0 \longrightarrow \operatorname{Hom}_{\operatorname{PAb}}(K', \iota Z) \longrightarrow \operatorname{Hom}_{\operatorname{PAb}}(Y, \iota Z) \longrightarrow \operatorname{Hom}_{\operatorname{PAb}}(X, \iota Z) \longrightarrow 0$$

(this is equivalent to saying K' is coker  $\varphi$  in PAb). By adjunction,

$$0 \longrightarrow \operatorname{Hom}_{\operatorname{SAb}}(sK', Z) \longrightarrow \operatorname{Hom}_{\operatorname{SAb}}(\underbrace{sY}, Z) \longrightarrow \operatorname{Hom}_{\operatorname{SAb}}(sX, Z) \longrightarrow 0$$

is exact. Other parts are exercises

**Definition 0.6.** Let A be an abelian group, Y a topological space.

- (a) The constant presheaf  $\mathbb{A}$  on Y is  $\mathbb{A}(U) = A$  for all open  $U \subseteq Y$ , and  $\gamma_{U,V} = \operatorname{Id}_A$  for any open  $V \subseteq U$ .
- (b) The constant sheaf A on Y is sA. (Check: for connected U, A(U) = A)

(c) A sheaf  $\mathcal{F}$  is <u>locally constant</u> if any point has a neighborhood U such that  $\mathcal{F}|_U$  is a constant sheaf. (for open  $V \subset U \subset Y$ ,  $\mathcal{F}|_U(V) \stackrel{\text{def}}{=} \mathcal{F}(V)$ ) (keyword: representation of  $\pi_1$  and local systems)

#### Germs and stalks

**Definition 0.7.** The stalk of a (pre)sheaf at a point y is

$$\mathcal{F}_y \stackrel{\text{def}}{=} \lim_{\stackrel{\longrightarrow}{V \ni y}} \mathcal{F}(V)$$

with the limit taken with respect to inclusion.

Concretely,  $\mathcal{F}_y \stackrel{\text{def}}{=} \frac{\{(s,V)|y \in V, s \in \mathcal{F}(V)\}}{\sim}$ , where  $(s,V) \sim (s',V')$  if there exists a  $W \subseteq V \cap V'$  such that  $\gamma_{V,W}(s) = \gamma_{V',W}(s')$ .

Such an equivalence class is called a germ

Remark: [GM, I.5.5, I.5.6]

If we have an exact sequence of sheaves

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$$

if and only if for any  $y \in Y$ ,

$$0 \longrightarrow \mathcal{F}'_y \longrightarrow \mathcal{F}_y \longrightarrow \mathcal{F}''_y \longrightarrow 0$$

is exact. This can be verified as an extremely important exercise.

**Definition 0.8.** For  $s \in \mathcal{F}(U)$ , the <u>support of s</u>, supp s, is the closure of the set of points at which the germ of s is not zero.

Remark:

In the definition of a stalk, we can replace a point y by a closed subset Z of Y.

## Lecture 4, 4/10/23

### Functors in abelian categories

Definition 0.9.

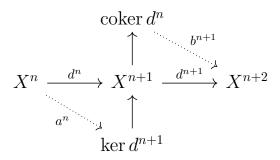
(a) Let  $\mathscr{C}, \mathscr{C}'$  be additive categories. A functor  $F : \mathscr{C} \to \mathscr{C}'$  is an additive functor if all maps  $F : \operatorname{Hom}_{\mathscr{C}}(X,Y) \to \operatorname{Hom}_{\mathscr{C}'}(FX,FY)$  is a homomorphism of abelian groups.

(b) A complex in  $\mathscr{C}$  is a sequence

$$X : \xrightarrow{d^{n-1}} X^n \xrightarrow{d^n} X^{n+1} \xrightarrow{d^{n+1}} \cdots$$

with  $d^n \circ d^{n-1} = 0$  for all n

(c) Now assume  $\mathscr{C}, \mathscr{C}'$  are abelian categories. If we have a complex, then, because  $d^n \circ d^{n+1} = 0$ , the universal properties of the kernel and cokernel (as well as their existence, which is guaranteed because we are in an abelian category), guarantee us unique maps  $a^n, b^{n+1}$  making the diagram commute:



The (n+1)-cohomology of X is

$$H^{n+1}(X^{\cdot}) \stackrel{\text{def}}{=} \operatorname{coker} a^n = \ker b^{n+1}$$

this equality can be verified as an exercise..

- (d)  $X^{\cdot}$  is acyclic at  $X^{b}$  if  $H^{n}(X^{\cdot}) = 0$ .
- (e) X is exact/acyclic if it is acyclic at  $X^n$  for all n
- (f) An additive functor  $F: \mathscr{C} \to \mathscr{C}'$  is exact if it sends a short exact sequence

$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$$

to a short exact sequence

$$0 \longrightarrow FX \longrightarrow FY \longrightarrow FZ \longrightarrow 0$$

It is <u>left exact</u> if

$$0 \longrightarrow FX \longrightarrow FY \longrightarrow FZ$$

is exact, and right exact if

$$FX \longrightarrow FY \longrightarrow FZ \longrightarrow 0$$

is exact.

**Example 0.4.** Let  $\mathscr{C}$  be an abelian category, and consider  $\operatorname{Hom}_{\mathscr{C}}(Y,-):\mathscr{C}\to\operatorname{\mathsf{Ab}}$  and  $\operatorname{Hom}_{\mathscr{C}}(-,Y):\mathscr{C}^\circ\to\operatorname{\mathsf{Ab}}$ , where  $\mathscr{C}^\circ$  means the opposite category of  $\mathscr{C}$  (this is just saying this functor is contravariant). These two morphisms are both left-exact.

**Example 0.5.** For a fixed ring R, consider R - Mod, the category of left R-modules, and Y a right R-module (so an object of Mod-R). Then we have a functor

$$Y \otimes_R : R - \text{Mod} \to \mathsf{Ab}$$

which is right-exact.

**Proposition 4.** Let X be a topological space, and fix an open set  $U \subseteq X$ . Consider SAb, the category of abelian sheaves on X. The functor SAb  $\rightarrow$  Ab given by  $\mathcal{F} \mapsto \mathcal{F}(U)$  is an additive functor which is left exact.

*Proof.* Let  $\iota: \mathrm{SAb} \to \mathrm{PAb}$  be the inclusion of sheaves into presheaves. This is left exact, which follows from the fact that the kernel of a morphism of sheaves is again a sheaf. The kernel doesn't need sheafification! Now  $\mathrm{PAb} \to \mathrm{Ab} : \mathcal{F} \mapsto \mathcal{F}(U)$  is exact by definition. The composition of a left exact and an exact functor is left exact, so we are done.

From now on, we will always be working in an abelian category unless otherwise stated.

#### Definition 0.10.

- (a) An object Y is projective if  $\operatorname{Hom}_{\mathscr{C}}(Y, -)$  is exact.
- (b) An object Y is injective if  $Hom_{\mathscr{C}}(-,Y)$  is exact
- (c) A right module-R Y is  $\underline{\text{flat}}$  if  $Y \otimes_R \text{is exact.}$

### Direct images

**Definition 0.11.** Let  $f: M \to N$  be a continuous map of topological spaces, and let  $\mathcal{F}$  be a sheaf on M. The <u>direct image</u>  $f_{\star}\mathcal{F}$  (written as  $f.\mathcal{F}$  in Gelfond-Manin) is defined as follows.

For any open  $U \subseteq N$ ,  $f^{-1}(U)$  is open in M. So we simply define

$$f_{\star}\mathcal{F}(U) \stackrel{\mathrm{def}}{=} \mathcal{F}(f^{-1}(U))$$

and restriction for  $V \subseteq U$  induced from  $\gamma_{f^{-1}(U),f^{-1}(V)}$ Exercise: verify that this is indeed a sheaf!

### Proposition 5.

- (a) Let  $f: M \to \{1\}$  be the constant map. Then  $f_{\star}\mathcal{F} = \Gamma(M, \mathcal{F}) = \mathcal{F}(M)$ .
- (b) Let  $i: M \to N$  be the inclusion of a closed subspace M of N. Then

$$(i_{\star}\mathcal{F})_x = \begin{cases} \mathcal{F}_x & x \in M \\ 0 & x \notin M \end{cases}$$

Some call this an "extension by zero." (If  $i: M \hookrightarrow N$  is the inclusion of an open subset M of N, then  $i_*\mathcal{F}$  may have nonzero stalk at  $x \in N \setminus M$ . Prove this as an exercise!)

(c)  $f_*: SAb(M) \to SAb(N)$  is a functor,  $(fg)_* = f_*g_*$ 

Proof.

# Lecture 5, 4/12/23

Inverse image. Let  $f: M \to N$  be continuous.

**Definition 0.12.** For  $\mathcal{F} \in SAb_N$ , first define  $f_p^{\star}\mathcal{F}$  as a presheaf:

$$U \mapsto \mathcal{F}(f(U)) \stackrel{\text{def}}{=} \lim_{\substack{N \supset V \supset f(U)}} \mathcal{F}(V)$$

where V is open. Then take  $f^*\mathcal{F} \stackrel{\text{def}}{=} s(f_p^*\mathcal{F})$ Exercise: for all  $x \in M, (f^*\mathcal{F})_x = \mathcal{F}_{f(x)}$ 

**Proposition 6.** We have an adjunction  $f^* \dashv f_*$ ,

$$\operatorname{Hom}_{\operatorname{SAb}_N}(f^{\star}\mathcal{F},\mathcal{G}) \cong \operatorname{Hom}_{\operatorname{SAb}_N}(\mathcal{F},f_*\mathcal{G})$$

*Proof.* We know  $s \dashv \iota$  by construction. So, we need only show

$$\operatorname{Hom}_{\operatorname{PAb}_{M}}(f_{n}^{\star}\mathcal{F},\mathcal{G}) \cong \operatorname{Hom}_{\operatorname{PAb}_{N}}(\mathcal{F}, f_{\star}\mathcal{G})$$

We establish a functorial morphism:  $\mathcal{F} \to f_{\star} f_{p}^{\star} \mathcal{F}$ . Let  $\mathcal{G} = f_{p}^{\star} \mathcal{F}$ ,

$$\operatorname{Hom}(f_{p}^{\star}\mathcal{F}, f_{p}^{\star}\mathcal{F}) \cong \operatorname{Hom}(\mathcal{F}, f_{\star}f_{p}^{\star}\mathcal{F})$$

Note: for  $V \subseteq N$  open,  $\mathcal{F}(V) \to f_p^{\star}(f^{-1}(V))$ 

But V is open in N containing  $f(\hat{f}^{-1}(V))$ . So we have restriction

?????????????

Exercise: Check morphism of presheaves.

This is compatible with restrictions and gives us a presheaf morphism  $i_{\mathcal{F}}: \mathcal{F} \to f_{\star}f_{p}^{\star}\mathcal{F}$ .

This induces the isomorphism in the statement.

$$(\psi: f_p^{\star} \mathcal{F} \to \mathcal{G}) \longrightarrow ((f_{\star} \psi) \circ i_{\mathcal{F}}: \mathcal{F} \to f_{*} \mathcal{G}$$

The other direction uses  $f_p^{\star} f_{\star} \mathcal{G} \to \mathcal{G}$ 

Exercise: construct this and check inverse

**Proposition 7.** Let  $\mathscr{C}, \mathscr{D}$  be Abelian categories. Let  $F : \mathscr{C} \to \mathscr{D}$  and  $G : \mathscr{D} \to \mathscr{C}$  be adjoint functors,  $F \dashv G$ .

**Theorem 0.1.** F is right exact, and G is left exact

*Proof.* We will just check G is left exact.

Let  $0 \longrightarrow Y' \stackrel{f}{\longrightarrow} Y \stackrel{g}{\longrightarrow} Y'' \longrightarrow 0$  be a short exact sequence.

Apply the left exact functor  $\operatorname{Hom}_{\mathscr{D}}(Fx,-)$  for all  $x\in\mathscr{C}$ . We have the diagram

$$0 \longrightarrow \operatorname{Hom}_{\mathscr{D}}(FX,Y') \longrightarrow \operatorname{Hom}_{\mathscr{D}}(FX,Y'')$$

$$\parallel \qquad \qquad \parallel \qquad \qquad \parallel$$

$$0 \longrightarrow \operatorname{Hom}_{\mathscr{C}}(X,GY') \longrightarrow \operatorname{Hom}_{\mathscr{C}}(X,GY) \longrightarrow \operatorname{Hom}_{\mathscr{C}}(X,GY'')$$

This is exact for all  $X \in \mathscr{C}$  iff  $0 \longrightarrow GY' \longrightarrow GY \longrightarrow GY''$  is exact

**Proposition 8.** In SAb,  $f_{\star}$  is left exact,  $f^{\star}$  is exact.

*Proof.* By exercise,  $f^*$  is exact on stalks.

### Direct images with compact support

GM, III8.7 - > 8.10

All topological spaces are assumed to be locally compact and first countable, meaning every point has a countable neighborhood basis.

Recall: A morphism of topological spaces is proper if the preimage of compact sets are compact.

**Definition 0.13.** Let  $f: X \to Y$ ,  $\mathcal{F}$  a sheaf on X. Let  $U \subseteq Y$  be open. We define

$$F_!\mathcal{F}(U) = \{s \in \Gamma(f^{-1}(U), \mathcal{F}) \mid f : \operatorname{supp}(s) \to U \text{ is proper } \}$$

**Definition 0.14.** Let  $s \in \Gamma(V, \mathcal{G})$ . Then  $\operatorname{supp}(s) = \overline{\{x \in V \mid \overline{s} \neq 0 \in \mathcal{G}_x\}}, \Gamma(V, \mathcal{G}) \to \mathcal{G}, s \mapsto \overline{s}$ 

Lemma 2.

- (a)  $f_!\mathcal{F}$  is a subsheaf of  $f_*\mathcal{F}$
- (b)  $f_!$  is a left exact functor.

Proof.

# Lecture 6, 4/14/23

Let  $f: X \to Y$  be continuous, and  $\mathcal{F}$  be a sheaf on X. Recall the definition of  $f_!\mathcal{F}$ :

$$f_!\mathcal{F}(U) = \{s \in \underbrace{\Gamma(f^{-1}(U), \mathcal{F})}_{=f_\star\mathcal{F}(U)} \mid f : \operatorname{supp}(s) \to U \text{ is proper } \}$$

where supp(s) is the closure of the set of points where  $\overline{s}$ , the germ of s,  $\overline{s}$ , is not zero. Theorem 0.2.

- (a)  $f_!\mathcal{F}$  is a subsheaf of  $f_*\mathcal{F}$ .
- (b) f! is a left exact functor "direct image with compact support"

Proof.

(a)  $f_!\mathcal{F}$  is clearly a subpresheaf of  $f_*\mathcal{F}$ . Any set of compatible sections of  $f_!\mathcal{F}$  glue uniquely to a section of  $f_*\mathcal{F}$ . This comes down to a topological statement.

Exercise: For  $(U_i)$  open subsets of Y,  $f_i: V_i \to U_i$  is proper, then  $f: \cup V_i \to \cup U_i$  is proper.

$$f^{-1}(K) = \cup f_i^{-1}(V_i \cap K)$$

(b)

### Sections with compact support

Consider the special case  $f: X \to \{1\}$ , the one point space. Then  $f_!\mathcal{F}$  is the set of sections  $s \in \mathcal{F}(X)$  such that supp(s) is compact.

Denote this by  $\Gamma_c(X, \mathcal{F})$ 

$$X \xrightarrow{f} Y$$

$$\downarrow i \qquad \qquad \uparrow \qquad \qquad \uparrow$$

$$f^{-1}(y) \longrightarrow y$$

**Proposition 9.** The stalk of  $f_!\mathcal{F}$  at  $y \in Y$  is isomorphic to

$$\Gamma_C(f^{-1}(y), \underbrace{\mathcal{F}|_{f^{-1}(y)}}_{=i^*\mathcal{F}})$$

*Proof.* First construct  $\varphi: (f_!\mathcal{F})_y \to \Gamma_c(f^{-1}(y), \mathcal{F}|_{f^{-1}(y)}).$ 

Let  $s \in f_!\mathcal{F}_y$  and choose a representative  $\tilde{s} \in \Gamma(f^{-1}(U), \mathcal{F})$  with U an open neighborhood of y, and supp  $\tilde{s} \to U$  proper.

Then: 
$$\tilde{s}|_{f^{-1}(y)}$$
 is in  $\Gamma_c(f^{-1}(y), \underbrace{\mathcal{F}|_{f^{-1}(y)}}_{=i^{\star}\mathcal{F}})$ 

Exercise:  $\varphi(s) \stackrel{\text{def}}{=} \tilde{s}|_{f^{-1}(y)}$  only depends on s.

We now show  $\varphi$  is injective. Suppose  $\varphi(s) = 0$ . Then  $\operatorname{supp}(\tilde{s}) \cap f^{-1}(y) = \emptyset$ . So  $y \notin f(\operatorname{supp} \tilde{s})$ . But  $f(\operatorname{supp} \tilde{s})$  is closed (proper + locally compact) So s = 0.

To show  $\varphi$  is surjective: choose a local basis  $V_i \ni y$  with  $\cap V_i = y$ . Then  $f^{-1}(y) = \bigcap_{i=1}^{n} f^{-1}(U_i)$ 

Exercise:

Locally compadct implies  $\Gamma_c(f^{-1}(y), \mathcal{F}|_{f^{-1}(y)} = \lim_{\to} A_i$  where

$$A_i = \{t \in \Gamma(f^{-1}(U_i), \mathcal{F}) \mid \text{supp } t = K \cap f^{-1}(U_i) \text{ for some compact } K \subseteq X\}$$

**Example 0.6. 1.** Let  $i: U \hookrightarrow X$  be open,  $\mathcal{F}$  a sheaf on U.

$$(i_!\mathcal{F})_x = \begin{cases} \mathcal{F}_x, & x \in U \\ 0, & x \in U \end{cases}$$

"extend by 0"

**2.**  $j: V \to X$  proper (in particular, closed embedding),  $j_! \mathcal{G} = j_* \mathcal{G}$ .

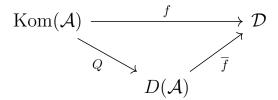
### Derived Categories via Localizations

**Definition 0.15.** Let  $f: K^{\cdot} \to L^{\cdot}$  be a morphism of complexes in an Abelian category  $\mathcal{A}$ . f is a quasi-isomorphism if the induced morphism  $H^n(f): H^n(K^{\cdot}) \to H^n(L^{\cdot})$  is an isomorphism for all n.

**Definition 0.16.** Let  $\mathcal{A}$  be an Abelian category,  $\operatorname{Kom}(\mathcal{A})$  the category of complexes in  $\mathcal{A}$ . The <u>derived category of  $\mathcal{A}$ </u> is a category D(A) and a functor  $Q : \operatorname{Kom}(A) \to D(A)$  such that

(a) Q(f) is an isomorphism for any quasi-isomorphism f.

(b) Q is universal in the following sense. Suppose that  $g: \mathcal{A} \to \mathcal{D}$  is a functor such that g(f) is an isomorphism for any quasi-isomorphism f. Then there is a unique functor  $\overline{f}: D(\mathcal{A}) \to \mathcal{D}$ , making the diagram commute:



**Theorem 0.3.** Every abelian category admits a derived category.

Proof.

# Lecture 7, 4/17/23

Homework hint:

Let  $\mathcal{A}$  be an abelian category with P a projective generator. We wish to define an equivalence with R-mod, where  $R = \operatorname{Hom}_{\mathcal{A}}(P, P)$ . Try to construct an equivalence going the other way as follows:

We want a projective resolution of M

$$R^{\oplus J} \longrightarrow R^{\oplus I} \longrightarrow M \longrightarrow 0$$

We want to use this to get a morphism  $P^{\oplus J} \to P^{\oplus I}$ . You can do this by taking the identity and looking at the image (?).

Now we continue.

Remark: We can define  $\mathrm{Kom}^+(\mathcal{A})$  to be all the chain complexes in  $\mathcal{A}$  that are bounded on the right. That is,  $\mathrm{Kom}^+(\mathcal{A}) = \{K^{\cdot} \mid K^i = 0 \text{ for } i << 0\}$ .  $\mathrm{Kom}^-(\mathcal{A})$  is defined similarly.  $\mathrm{Kom}^b(\mathcal{A}) = \mathrm{Kom}^+(\mathcal{A}) \cap \mathrm{Kom}^-(\mathcal{A})$ .

These are each abelian categories, so we can consider  $D^+(\mathcal{A}), D^-(\mathcal{A}), D^b(\mathcal{A})$ , but this is for later.

# Construction of D(A)

This is similar to "localization of rings"

**Definition 0.17.** A class of morphism  $S \subset \operatorname{Mor} \mathscr{B}$  is said to be localizing if

- (a) S is closed under composition:  $\operatorname{Id}_X \in S$  for all  $X \in \operatorname{Ob} \mathscr{B}$  and  $s \circ t \in S$  for all  $s, t \in S$  such that the composition is defined.
- (b) Extension: for all  $f \in \text{Mor } \mathcal{B}, s \in S$ , there exists  $g \in \text{Mor } \mathcal{B}, t \in S$  such that we can make one of the following diagrams commute:

$$\begin{array}{ccc} W & \stackrel{g}{\longrightarrow} Z \\ \downarrow & & \downarrow s \\ X & \stackrel{f}{\longrightarrow} Y \end{array}$$

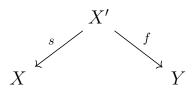
or

$$\begin{array}{ccc} W & \stackrel{g}{\longleftarrow} & Z \\ \downarrow & & \uparrow s \\ X & \stackrel{f}{\longleftarrow} & Y \end{array}$$

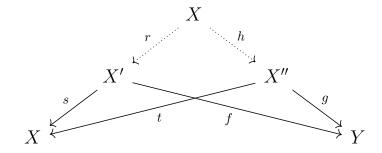
(c) If  $f, g \in \text{Mor}(X, Y)$ , the existence of  $s \in S$  with sf = sg is equivalent to the existence of  $t \in S$  such that ft = gt

**Definition 0.18.** Given a category  $\mathscr{B}$  and class of morphisms S which is localizing, we introduce a category  $\mathscr{B}[S^{-1}]$ , with  $\operatorname{Ob}\mathscr{B}[S^{-1}] = \operatorname{Ob}\mathscr{B}$  and

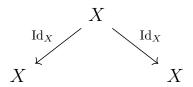
(a) A morphism  $X \to Y$  in  $\mathscr{B}[S^{-1}]$  is the equivalence class of <u>roofs</u> of the form



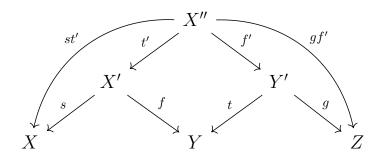
with  $s \in S, f \in \text{Mor } \mathcal{B}$ . Two roofs are equivalent if there exists a third roof making the following diagram commute:



With  $sr = th, fr = gh, r \in S$ . Id<sub>X</sub> is the class of



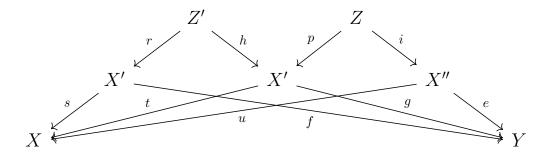
(b) The composition of roofs (s, f) and (t, g) is the class of (st', gf')



$$gt^{-1}fs^{-1} = gf't'^{-1}s^{-1} = (gf)(st')^{-1}$$

**Theorem 0.4.** This is in fact a well defined equivalence relation

*Proof.* Symmetry and reflexivity are easy, so we will just show transitivity. Suppose  $(s,f) \sim (t,g), (t,g) \sim (u,e)$ . We want to show  $(s,f) \sim (u,e)$ . So we have



 $s, t, u, r, p \in S$  First consider

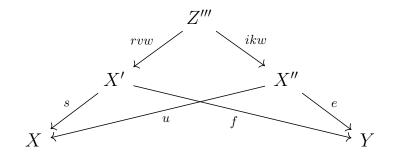
$$Z' \longleftarrow W$$

$$sr \downarrow \qquad \qquad \downarrow k$$

$$X \longleftarrow p \qquad Z''$$

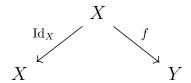
""" with  $sr \in S$ . Then thv = srv = tpk, so by c there exists w such that hvw = pkw

for some  $w: Z''' \to W$ ,  $w \in S$ . We now build the roof



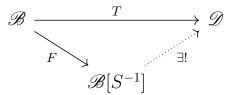
**Definition 0.19.** The category  $\mathscr{B}[S^{-1}]$  is called a <u>localization of  $\mathscr{B}$ </u>.

Let  $F: \mathscr{B} \to \mathscr{B}[S^{-1}]$  map each object to its equivalence class. We define F(f) to be the equivalence class of the roof



This satisfies the universal property:

Let  $T: \mathcal{B} \to \mathcal{D}$  be such that T(s) is an isomorphism for all  $s \in S$ . Then there exists a unique factorization



### Lecture 8

**Proposition 10.** Let  $T: \mathcal{B} \to \mathcal{D}$  be a functor such that T(s) is an isomorphism for any localizing class  $s \circ f$  S of  $\mathcal{B}$ . Then T uniquely factors through

$$\mathscr{B} \xrightarrow{T} \mathscr{D}$$

$$\mathscr{B}[S^{-1}]$$

*Proof.* To construct  $G: \mathcal{B} \to \mathcal{D}$  such that  $T = G \circ F$ , consider

$$\underbrace{G(x)}_{\in \mathrm{Obj}\mathscr{B}[S^{-1}]} = \underbrace{T(x)}^{\in \mathrm{Obj}\mathscr{B}}$$

But by definition  $\text{Obj}\mathscr{B}[S^{-1}] = \text{Obj}\mathscr{B}$ .  $G([S, F]) = T(f) \circ \underbrace{T(s)^{-1}}$ 

Show it is well defined and unique as an exercise.

We have an issue: the class of quasi-isomorphisms in Kom(A) is <u>not</u> a localizing class.

**Definition 0.20.** Fix n. For  $K^{\cdot} = (K^{\cdot}, d_{K}^{\cdot})$ , define a complex  $K[n]^{\cdot}$  by  $(K[n])^{i} = K^{n+i}$ ,  $d_{K[n]} = (-1)^{n} d_{K}$ .

This is the shift to the left by n map.

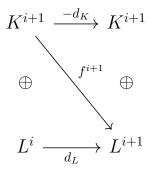
For  $F: K^{\cdot} \to L^{\cdot}$ , let  $F[n]: K[n]^{\cdot} \to L[n]^{\cdot}$  coincide with F componentwise.

The translation functor  $T^n: \mathrm{Kom}(\mathcal{A}) \to \mathrm{Kom}(\mathcal{A})$  is an autoequivalence.

**Definition 0.21.** Let  $F: K^{\cdot} \to L^{\cdot}$ . The mapping cone of F is the complex

$$C(f): C(f)^{i} = K[1]^{i} \oplus L^{i}$$
$$d_{C(f)}^{i}(k^{i+1}, \ell^{i}) = (-d_{K}k^{i+1}, f(k^{i+1}) + d_{L}(\ell^{i}))$$

Check that  $d^2 = 0$ .



**Example 0.7.** If F is a morphism of "0-comple, i.e.  $F: K^0 \to L^0$ , then C(f) is the complex

$$\cdots \longrightarrow 0 \longrightarrow \underbrace{K^0}_{\deg -1} \stackrel{f}{\longrightarrow} \underbrace{L^0}_{\deg 0} \longrightarrow 0 \longrightarrow \cdots$$

In particular,  $H^{-1}(C(f)) = \ker f$ ,  $H^{0}(C(f)) = \operatorname{coker} f$ .

**Definition 0.22.** The mapping cylinder Cyl(f) is

$$\operatorname{Cyl}(f) = K \oplus K[1] \oplus L$$

$$K^{i} \xrightarrow{-\operatorname{Id}} K^{i+1}$$

$$K^{i+1} \xrightarrow{-d_{K}} K^{i+2}$$

$$L^{i} \xrightarrow{d_{L}} L^{i+1}$$

**Lemma 3.** The following diagram commutes and has exact rows

$$0 \longrightarrow L^{\cdot} \longrightarrow C(f) \longrightarrow K[1]^{\cdot} \longrightarrow 0$$

$$\downarrow^{\alpha} \qquad \qquad \parallel$$

$$0 \longrightarrow K^{\cdot} \longrightarrow \operatorname{Cyl}(f) \longrightarrow C(f) \longrightarrow 0$$

$$\downarrow^{\beta}$$

$$K^{\cdot} \longrightarrow L^{\cdot}$$

it's functorial in F. The morphisms  $\alpha, \beta$  are quasi-isomorphism,  $\beta \alpha = \mathrm{Id}_L$ ,  $\alpha \beta$  is homotopic to  $\mathrm{Id}_{\mathrm{Cyl}(F)}$ 

*Proof.*  $\alpha$  is the "inclusion"

$$\beta: (K^{i}, K^{i+1}, L^{i}) \xrightarrow{d_{\text{Cyl}(F)}} \text{nothing?}$$

$$\downarrow^{\beta^{i}} \qquad \qquad \downarrow^{\beta^{i+1}}$$

$$(F(K^{i}) + L^{i}) \xrightarrow{d_{L}} \text{nothing again?}$$

$$k^{i}: \operatorname{Cyl}(F)^{i} \to \operatorname{Cyl}(f)^{i-1}$$
  
 $(K^{i}, K^{i+1}, L^{i}) \mapsto (0, K^{i}, 0)$   
 $\alpha\beta = \operatorname{Id} + dk + kd$ 

**Definition 0.23.** Let  $f, g: K^{\cdot} \to L^{\cdot}$  be chain maps. We say f and g are homotopic if there exists k such that f - g = kd + dk, where

$$K^{i-1} \xrightarrow{k^{i}} K^{i} \xrightarrow{K^{i+1}} K^{i+1}$$

$$L^{i-1} \xrightarrow{k^{i}} L^{i} \xrightarrow{K^{i+1}} L^{i+1}$$

## Lecture 9, 4/21/23

**Definition 0.24.** Let  $\mathcal{A}$  be an abelian category. The <u>homotopy category</u>  $K(\mathcal{A})$  is the category whose objects are objects of  $Kom(\mathcal{A})$ , and whose morphisms are homotopy equivalence classes of maps in  $Kom(\mathcal{A})$ .

Recall that homotopy equigvalent morphisms induce the same morphisms on cohomology, so it makes sense to talk about quasi-isomorphisms.

**Theorem 0.5.** The class of quasi-isomorphisms in K(A) is localizing.

*Proof.* We'll give a more conceptual proof next week, based on two more results:

- K(A) is a triangulated category
- quasi-isomorphisms are "obtained by cohomological functors"

It is obvious that this class of morphisms is closed under composition. Let  $f: K^{\cdot} \to L^{\cdot}$  be a quasi-isomorphism. Then for any  $g: M^{\cdot} \to L^{\cdot}$ , there is a complex  $N^{\cdot}$ , a morphism h, and a quasi-isomorphism k making the diagram commute:

$$\begin{array}{ccc}
N & \xrightarrow{k} & M \\
\downarrow h & & \downarrow g \\
K & \xrightarrow{f} & L
\end{array}$$

This comes from the following diagram:

this commutes up to homotopy. The other part can be similarly verified. We won't prove the last part.

**Definition 0.25.**  $D(A) = K(A)[\{\text{quasi-iso}\}^{-1}]$ Note that D(A) is an additive category: GET NOTES FROM JOEL

### Triangulated categories ( $\triangle$ -cat)

**Definition 0.26.** Let  $\mathscr{D}$  be an additive category. A structure of  $\triangle$ -cat on  $\mathscr{D}$  is data a)&b) satisfying TR1-TR4:

a) Additive autoequivalence  $T: \mathcal{D} \to \mathcal{D}$  called the translation  $(X[n] \stackrel{\text{def}}{=} T^n(x), f[n] \stackrel{\text{def}}{=} T^n(f))$ 

Say a triangle in  $\mathscr{D}$  is  $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$  and a morphism of triangles

$$\begin{array}{cccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X[1] \\ \downarrow^f & & \downarrow^g & & \downarrow^k & & \downarrow^{f(1)} \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & X'[1] \end{array}$$

b) A class of distinguished triangles  $(d - \triangle)$ 

#### $\underline{\text{TR1}}$

(a) 
$$X \xrightarrow{\operatorname{Id}_X} X \longrightarrow 0 \longrightarrow X[1]$$
 is  $d - \triangle$ 

- (b) Any  $\triangle$  iso to a  $d \triangle$  is a  $d \triangle$
- (c) Any morphism  $X \xrightarrow{u} Y$  can be completed to a  $d \triangle$

#### TR2

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1] \text{ is a } d - \triangle \text{ iff } Y \xrightarrow{v} Z \xrightarrow{w} X[1] \xrightarrow{-u(1)} Y[1]$$
 is a  $d - \triangle$ 

TR3

Assume we have two  $d - \triangle' s$  and f, g:

Then there exists h, not necessarily unique, which makes this a morphism of  $d-\triangle$  TR4 "Octahedron" GET FROM JOEL