Lecture 1, 4/4/23

We use the following two books:

1. Linear Analysis, by B. Bolobás

2. Analysis, by E. Lieb and M. Loss

We begin with metric spaces

Definition 0.1. Let X be a nonempty set and let $\rho : X \times X \to [0, \infty)$. Then $\rho(x, y)$ is called a metric on X if

(i) $\rho(x, y) \geq 0$ for all $x, y \in X$ and $\rho(x, y) = 0$ iff $x = y$.

(ii) $\rho(x, y) = \rho(y, x)$ for all $x, y \in X$

(iii) $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$ for any $x, y, z \in X$. This is called the triangle inequality

ρ is also called a distance. As in, $\rho(x, y)$ is the distance between x and y.

Example 0.1. Let $X = \mathbb{R}^n$, and define $\rho(x, y) = \left(\sum_{i=1}^n (x_i - y_i)^2\right)^{\frac{1}{2}}$. We can in fact replace 2 in this expression with any real $r \geq 1$, or with ∞ (in which case we just take the maximum)

Example 0.2. Let $X = C[a, b]$, the set of continuous $f : [a, b] \to \mathbb{R}$, and define $\rho(x, y) = \max_{x \in [a,b]} |f(x) - g(x)|$

Definition 0.2. Let (X, ρ) be a metric space. For all $x \in X$ and $r > 0$, we defined the open ball centered at x and having radius r as

$$B_r(x) \overset{\text{def}}{=} \{y \in X \mid \rho(x, y) < r\}$$

The closed ball is

$$\overline{B_r}(x) \overset{\text{def}}{=} \{y \in X \mid \rho(x, y) \leq r\}$$

Definition 0.3. Let (X, ρ) be a metric space and let $A \subseteq X$. Then $a \in A$ is

(i) an interior point of A if there is some $r > 0$ such that $B_r(a) \subseteq A$

(ii) The set of all interior points of A is called the interior of A and is denoted by $\operatorname{int} A$, or A°

(iii) A set A is said to be *open* if $A = A^\circ$

Example 0.3. Let $X = \mathbb{R}^3$, $A = \{(x, y, 0) \mid x, y \in \mathbb{R}\}$. We can see that $A^\circ = \varnothing$.

Proposition 1. *For any x, r, $B_r(x)$ is open.*

Proof. Let $y \in B_r(x)$. Let $r_1 = r - \rho(x, y) > 0$.
Consider $z \in B_{r_1}(y)$. By the triangle inequality, $\rho(x, z) \leq \rho(z, y) + \rho(y, x) < r_1 + \rho(x, y) = r$. So $z \in B_r(x)$, so $B_{r_1}(y) \subseteq B_r(x)$, so y is an interior point. y was arbitrary, so we are done.

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Definition 0.4. $A \subseteq X$ is closed if $A^c = X \setminus A$ is open.

Definition 0.5. The point $x \in X$ is a limit point of A if there exists a sequence $\{x_n\}_{n=1}^\infty \subseteq A$ such that $\rho(x, x_n) \to 0$ as $n \to \infty$

Definition 0.6. Let $\{x_n\} \subseteq X$, $x \in X$. Then we we say x_n converges to x, or $x_n \to x$, if $\rho(x_n, x) \to 0$ as $n \to \infty$. In this case, $\{x_n\}_{n=1}^\infty$ is said to be convergent, with limit x.

Theorem 0.1. *If a limit of a sequence $\{x_n\} \subseteq X$ exists, then it is unique.*

Proof. Think

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Definition 0.7. A sequence $\{x_n\}_{n=1}^\infty \subseteq X$ is called a Cauchy sequence if $\rho(x_n, x_m) \to 0$ as $n, m \to \infty$. That is, for any $\varepsilon > 0$, there exists an $N \in \mathbb{N}$, such that if $n, m \geq N$, then $\rho(x_n, x_m) < \varepsilon$

Theorem 0.2. *Any convergent sequence is Cauchy*

Proof. Think

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Definition 0.8. A metric space (X, ρ) is called complete if every Cauchy sequence converges to some point in X. A metric space which is not complete is called incomplete.

Example 0.4. $X = \mathbb{R}^n$ or $X = C[a, b]$ with the metrics above are complete.

Example 0.5. $\mathbb{Q}$ is incomplete.

Definition 0.9. Let (X, ρ) and $(Y, \tilde{\rho})$ be metric spaces. Then X and Y are isometric if there exist a bijection $f : X \to Y$ such that $\rho(x_1, x_2) = \tilde{\rho}(f(x_1), f(x_2))$ for all $x_1, x_2 \in X$.

Definition 0.10. Let (X, ρ) be a metric space and let $A, B \subseteq X$. Then we say that A is dense in B if $B \subseteq \overline{A}$, where $\overline{A} = \{\text{all limit points of } A\}$

Definition 0.11. Let (X, ρ) and $(\tilde{X}, \tilde{\rho})$ be metric spaces. Then $(\tilde{X}, \tilde{\rho})$ is a completion of $(X,$ if

(i) $X \subseteq \tilde{X}$, and $\tilde{\rho}(x, y) = \rho(x, y)$ for any $x, y \in X$

(ii) X is dense in $\tilde{X}$ in the $\tilde{\rho}$ metric

(iii) $(\tilde{X}, \tilde{\rho})$ is complete

Theorem 0.3. *Any incomplete metric space (X, ρ) admits a completion which is unique up to isometry.*

Proof. Think

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Theorem 0.4. *(The nested ball theorem)*
Let (X, ρ) be a complete metric space, and let $\overline{B_n} = \overline{B_{r_n}}(x_n) \subseteq X$ be a sequence of nested closed balls (meaning $\overline{B_{n+1}} \subseteq \overline{B_n}$) such that $r_n \to 0$. Then $\bigcap_{n=1}^{\infty} \overline{B_n} \neq \varnothing$.

Proof. Consider the centers $\{x_n\}_{n=1}^{\infty} \subseteq X$.

Claim. *$\{x_n\}$ is Cauchy*

Proof. If $m \geq n$, then $\overline{B_m} \subseteq \overline{B_m}$, so $x_m \in \overline{B_m}$, so $\rho(x_m, x_n) \leq r_n$, so $\rho(x_n, x_m) \to 0$ as $m, n \to \infty$.

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