Lecture 1, 4/3/23

We will begin thinking about homological algebra in a more categorical way. We use "methods of homological algebra," by Gelfond-Manin (Gelfond & Manin?). Here is a brief overview of some things we will see during the course:

- Additive and Abelian categories
- Example: category of sheaves of Abelian groups on a topological space
- Derived category (objects: complexes of objects in a given Abelian category, morphisms: a quasi-isomorphism of complexes becomes an isomorphism of objects)
- Derived functors, $f_*, f^*, f_!, f^!, \mathbb{D}, \cdots$, Hom for sheaves
- Triangulated categories
- t-structures
- Examples: perverse sheaves

Let's remind ourselves of some definitions:

Additive & Abelian categories

Example 0.1. (a) Ab, the category of Abelian groups with group homomorphisms.

(b) $R-\text{Mod}$, the category of R-modules, with R-module homomorphisms as morphisms.

(c) SAB, PAB, the categories of sheaves of Abelian groups and presheaves of Abelian groups

(d) Sheaves of modules over a ringed space

(e) (quasi)-coherent sheaves (ask Zhao)

Definition 0.1. An Abelian category contains the following information:

1. Any hom-set $\text{Hom}_{\mathscr{C}}(X,Y)$ is an Abelian group $(+)$, and the composition of morphisms is bi-additive

In particular:

- $\text{Hom}_{\mathscr{C}}$ is a functor $\mathscr{C}^\circ \times \mathscr{C} \to \mathsf{Ab}$. We notate the first factor with the $^\circ$

- $0 \in \mathrm{Hom}_{\mathscr{C}}(X,Y)$ for any objects X, Y of $\mathscr{C}$

2. There exists a zero object $0 \in \mathscr{C}$, that is an object such that $\mathrm{Hom}_{\mathscr{C}}(0,0) = 0$.

This gives: $\mathrm{Hom}_{\mathscr{C}}(0,X) = 0$, $\mathrm{Hom}_{\mathscr{C}}(X,0) = 0$ for all objects X of $\mathscr{C}$.

We know that $\mathrm{Hom}_{\mathscr{C}}(0,0)$ consists of one object. In particular, it must be $\mathrm{Id}_0 = 0$. So

$$\left(\ 0 \xrightarrow{f} X\ \right) = \left(\ 0 \xrightarrow{\mathrm{Id}_0} 0 \xrightarrow{f} X\ \right) = \left(\ 0 \xrightarrow{0} X\ \right)$$

Any two zero objects are isomorphic.

3. For any $X_1, X_2 \in \mathscr{C}$, there exists an object Y and morphisms

$$\begin{tikzcd} X_1 \arrow[r, bend left, "i_1"] & Y \arrow[l, bend left, "p_1"] \arrow[r, bend right, "p_2"'] & X_2 \arrow[l, bend right, "i_2"'] \end{tikzcd}$$

such that

$$\begin{aligned} p_1 i_1 &= \mathrm{Id}_{X_1} \\ p_2 i_2 &= \mathrm{Id}_{X_2} \\ i_1 p_1 + i_2 p_2 &= \mathrm{Id}_Y \\ p_2 i_1 = p_1 i_2 &= 0 \end{aligned}$$

Lemma 1. *We have cartesian diagram*

$$\begin{tikzcd} Y' \arrow[dr, dotted, "\exists!\varphi"] \arrow[drr, bend left, "p_1'"] \arrow[ddr, bend right, "p_2'"'] & & \\ & Y \arrow[r, "p_1"] \arrow[d, "p_2"] & X_1 \arrow[d] \\ & X_2 \arrow[r] & 0 \end{tikzcd}$$

That is, for any Y', with morphisms p_1', p_2' as in the diagram, there is a morphism from Y' to Y making the diagram commute. Similarly, we have co-cartesian diagram

$$\begin{tikzcd} Y & X_1 \arrow[l, "i_1"'] \\ X_2 \arrow[u, "i_2"] & 0 \arrow[l] \arrow[u] \end{tikzcd}$$

Proof. We need to construct $\varphi : Y' \to Y$ such that $p_1' = p_1\varphi$ and $p_2' = p_2\varphi$
Take $\varphi = i_1p_1' + i_2p_2'$. Then

$$p_1 \circ \varphi = \underbrace{p_1 i_1}_{=\mathrm{Id}_{X_1}} p_1' + \underbrace{p_1 i_2}_{=0} p_2' = p_1'$$

The uniqueness of φ can be verified as an exercise

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Definition 0.2. An <u>additive category</u> is one which satisfies only the first three of these axioms

To state the 4th axiom of Abelian categories, we need more notations:

Definition 0.3. Let A_1, A_2 be objects of $\mathscr{C}$, and let $\varphi : X \to Y$.

1. A <u>kernel</u> of φ is a morphism $i : Z \to X$ such that

(a) $\varphi \circ i = 0$

(b) For all $i' : Z' \to X$ such that $\varphi \circ i' = 0$, there is a unique $g : Z' \to Z$ such that $i' = i \circ g$.

$$\begin{tikzcd} & Z' \arrow[dl, dotted, "g"'] \arrow[d, "i'"] \arrow[dr, "0"] & \\ Z \arrow[r, "i"] & X \arrow[r, "\varphi"] & Y \end{tikzcd}$$

2. A <u>cokernel</u> is a kernel but with the arrows reversed

Exercise: Verify that 1 is equivalent to the following: for all $Z' \in \mathscr{C}$,

$$0 \longrightarrow \mathrm{Hom}(Z', Z) \xrightarrow{i_*} \mathrm{Hom}(Z', X) \xrightarrow{\varphi_*} \mathrm{Hom}(Z', Y)$$

is exact, and similarly for cokernel
With all that, we are ready for:

For any $\varphi : X \to Y$, there exists a sequence of morphisms

$$K \xrightarrow{k} X \xrightarrow{i} I \xrightarrow{j} Y \xrightarrow{c} K'$$

such that

(a) $j \circ i = \varphi$

(b) $k = \ker \varphi, k' = \operatorname{coker} \varphi$

(c) $I = \operatorname{coker} k = \ker c$

This finishes the definition