Lecture 1, 4/3/23

We will begin thinking about homological algebra in a more categorical way. We use "methods of homological algebra," by Gelfond-Manin (Gelfond & Manin?). Here is a brief overview of some things we will see during the course:

- Additive and Abelian categories
- Example: category of sheaves of Abelian groups on a topological space
- Derived category (objects: complexes of objects in a given Abelian category, morphisms: a quasi-isomorphism of complexes becomes an isomorphism of objects)
- Derived functors, $f_*, f^*, f_!, f^!, \mathbb{D}, \cdots$, Hom for sheaves
- Triangulated categories
- t-structures
- Examples: perverse sheaves

Let's remind ourselves of some definitions:

Additive & Abelian categories

Example 0.1. (a) Ab, the category of Abelian groups with group homomorphisms.

(b) $R-\text{Mod}$, the category of R-modules, with R-module homomorphisms as morphisms.

(c) SAb, PAb, the categories of sheaves of Abelian groups and presheaves of Abelian groups

(d) Sheaves of modules over a ringed space

(e) (quasi)-coherent sheaves (ask Zhao)

Definition 0.1. An <u>Abelian category</u> contains the following information:

1. Any hom-set $\text{Hom}_{\mathscr{C}}(X, Y)$ is an Abelian group $(+)$, and the composition of morphisms is bi-additive

In particular:

- $\text{Hom}_{\mathscr{C}}$ is a functor $\mathscr{C}^\circ \times \mathscr{C} \to \mathsf{Ab}$. We notate the first factor with the $^\circ$

- $0 \in \mathrm{Hom}_{\mathscr{C}}(X, Y)$ for any objects X, Y of $\mathscr{C}$

2. There exists a zero object $0 \in \mathscr{C}$, that is an object such that $\mathrm{Hom}_{\mathscr{C}}(0,0) = 0$.

This gives: $\mathrm{Hom}_{\mathscr{C}}(0, X) = 0$, $\mathrm{Hom}_{\mathscr{C}}(X, 0) = 0$ for all objects X of $\mathscr{C}$.

We know that $\mathrm{Hom}_{\mathscr{C}}(0,0)$ consists of one object. In particular, it must be $\mathrm{Id}_0 = 0$. So

$$\left(0 \xrightarrow{f} X \right) = \left(0 \xrightarrow{\mathrm{Id}_0} 0 \xrightarrow{f} X \right) = \left(0 \xrightarrow{0} X \right)$$

Any two zero objects are isomorphic.

3. For any $X_1, X_2 \in \mathscr{C}$, there exists an object Y and morphisms

$$\begin{tikzcd} X_1 \arrow[r, bend left, "i_1"] & Y \arrow[l, bend left, "p_1"] \arrow[r, bend right, "p_2"'] & X_2 \arrow[l, bend right, "i_2"'] \end{tikzcd}$$

such that

$$\begin{aligned} p_1 i_1 &= \mathrm{Id}_{X_1} \\ p_2 i_2 &= \mathrm{Id}_{X_2} \\ i_1 p_1 + i_2 p_2 &= \mathrm{Id}_Y \\ p_2 i_1 = p_1 i_2 &= 0 \end{aligned}$$

Lemma 1. *We have cartesian diagram*

$$\begin{tikzcd} Y' \arrow[drr, bend left, "p_1'"] \arrow[ddr, bend right, "p_2'"'] \arrow[dr, dotted, "\exists!\varphi"] & & \\ & Y \arrow[r, "p_1"] \arrow[d, "p_2"] & X_1 \arrow[d] \\ & X_2 \arrow[r] & 0 \end{tikzcd}$$

That is, for any Y', with morphisms p_1', p_2' as in the diagram, there is a morphism from Y' to Y making the diagram commute. Similarly, we have co-cartesian diagram

$$\begin{tikzcd} Y & X_1 \arrow[l, "i_1"'] \\ X_2 \arrow[u, "i_2"] & 0 \arrow[l] \arrow[u] \end{tikzcd}$$

Proof. We need to construct $\varphi : Y' \to Y$ such that $p_1' = p_1\varphi$ and $p_2' = p_2\varphi$
Take $\varphi = i_1p_1' + i_2p_2'$. Then

$$p_1 \circ \varphi = \underbrace{p_1 i_1}_{=\mathrm{Id}_{X_1}} p_1' + \underbrace{p_1 i_2}_{=0} p_2' = p_1'$$

The uniqueness of φ can be verified as an exercise

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Definition 0.2. An <u>additive category</u> is one which satisfies only the first three of these axioms

To state the 4th axiom of Abelian categories, we need more notations:

Definition 0.3. Let A_1, A_2 be objects of $\mathscr{C}$, and let $\varphi : X \to Y$.

1. A <u>kernel</u> of φ is a morphism $i : Z \to X$ such that

(a) $\varphi \circ i = 0$

(b) For all $i' : Z' \to X$ such that $\varphi \circ i' = 0$, there is a unique $g : Z' \to Z$ such that $i' = i \circ g$.

$$\begin{tikzcd} & Z' \arrow[dl, dotted, "g"'] \arrow[d, "i'"] \arrow[dr, "0"] & \\ Z \arrow[r, "i"] & X \arrow[r, "\varphi"] & Y \end{tikzcd}$$

2. A <u>cokernel</u> is a kernel but with the arrows reversed

Exercise: Verify that 1 is equivalent to the following: for all $Z' \in \mathscr{C}$,

$$0 \longrightarrow \mathrm{Hom}(Z', Z) \xrightarrow{i_*} \mathrm{Hom}(Z', X) \xrightarrow{\varphi_*} \mathrm{Hom}(Z', Y)$$

is exact, and similarly for cokernel
With all that, we are ready for:

For any $\varphi : X \to Y$, there exists a sequence of morphisms

$$K \xrightarrow{k} X \xrightarrow{i} I \xrightarrow{j} Y \xrightarrow{c} K'$$

such that

(a) $j \circ i = \varphi$

(b) $k = \ker\varphi, k' = \operatorname{coker}\varphi$

(c) $I = \operatorname{coker} k = \ker c$

This finishes the definition

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Sheaves

Here are some examples of sheaves from complex analysis

Example 0.2.

(a) The set of holomorphic functions on $\mathbb{P} = \mathbb{C}\sup\{\infty\}$

For each open subset U of $\mathbb{P}$, we can consider the ring of holomorphic function $f : U \to \mathbb{C}$, $\mathscr{H}(U)$.

The collection of $\{(f, V) \mid V \text{ open}, f \in \mathscr{H}(V)\}$ is called the sheaf $\mathscr{O}$ of holomorphic functions on $\mathbb{P}$.

(b) The sheaf of solutions of a linear ODE

Let $U \subseteq \mathbb{P}$ be open, and let $a_i(z) \in \Gamma(U, \mathscr{O})$ (in this context this will wind up meaning $\mathscr{H}(U)$), $i = 0, 1, \ldots, n-1$.

Denote by S the collection of (V, f) such that $V \subseteq U$ is open, and f is holomorphic in V such that

$$Lf \stackrel{\text{def}}{=} \frac{d^n f}{dz^n} + \sum_{i=0}^{n-1} a_i(z) \frac{d^i f}{dz^i} = 0$$

Let $\Gamma(V) = \{f \in \mathscr{H}(V) \mid Lf = 0\}$. When V is connected and simply connected, it is a basic result of ODEs that $\Gamma(V) \cong \mathbb{C}^n$

In general, it may have to do with the topology of V. For example, if $U = \mathbb{C}\backslash\{0\}$, $L = \frac{d^2}{dz^2} + \frac{1}{z}\frac{d}{dz}$, the solutions are $c_1 \log(z) + c_2$ for any "branch" of $\log(z)$, but $\Gamma(V) = \{\text{constant}\}$. This is related to the Riemann-Hilbert correspondence (whatever that is)

Definition 0.4.

(a) A presheaf of sets $\mathcal{F}$ on a topological space Y consists of the following data:

- A set $\mathcal{F}(U)$ for any open $U \subseteq Y$
- For any open $V \subseteq U$, a (restriction) map $\gamma_{U,V} : \mathcal{F}(U) \to \mathcal{F}(V)$ such that $\gamma_{V,V} = \mathrm{Id}_{\mathcal{F}(V)}$, and if $W \subseteq V \subseteq U$, then $\gamma_{V,W} \circ \gamma_{U,V} = \gamma_{U,W}$. When there is ambiguity about which sheaf γ belongs to, we further specify with $\gamma^{\mathcal{F}}$

(b) A presheaf $\mathcal{F}$ is a sheaf if:

- For any open covering $U = \cup_{i \in I} U_i$ and $s_i \in \mathcal{F}(U_i)$ such that for all $i, j \in I$,

$$\gamma_{U_i, U_i \cap U_j}(s_i) = \gamma_{U_j, U_i \cap U_j}(s_j)$$

then there exists a unique $s \in \mathcal{F}(U)$ such that $s_i = \gamma_{U,U_i}(s)$ for all i.

(c) A morphism of presheaves $f : \mathcal{F} \to \mathcal{G}$ is a family of maps $f(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ for all open $U \subseteq Y$, such that for all open $V \subseteq U$, the diagram

$$\begin{tikzcd}
\mathcal{F}(U) \arrow[r, "f(U)"] \arrow[d, "\gamma^{\mathcal{F}}_{U,V}"'] & \mathcal{G}(U) \arrow[d, "\gamma^{\mathcal{G}}_{U,V}"] \\
\mathcal{F}(V) \arrow[r, "f(V)"] & \mathcal{G}(V)
\end{tikzcd}$$

commutes.

A morphism of sheaves is a morphism of the underlying presheaves

A presheaf $\mathcal{F}$ of groups/rings/$\mathbb{F}$-vector spaces is a presheaf such that $\mathcal{F}(U)$ is a group/ring/$\mathbb{F}$-vector space. Then $\gamma_{U,V}$ is a morphism of groups/rings/$\mathbb{F}$-vector spaces.

Let $\mathcal{F}, \mathcal{G}$ be two Abelian presheaves (meaning presheaves of Abelian groups), and let $f : \mathcal{F} \to \mathcal{G}$ be a morphism of Abelian presheaves.
Let $K(U) = \ker(f(U))$, $C(U) = \operatorname{coker}(f(U))$ with natural restrictions.

Definition 0.5. A sequence of presheaves

$$\mathcal{F} \xrightarrow{f} \mathcal{G} \xrightarrow{g} \mathcal{H}$$

is exact if, for all open U,

$$\mathcal{F}(U) \xrightarrow{f(U)} \mathcal{G}(U) \xrightarrow{g(U)} \mathcal{H}(U)$$

is exact.
So, presheaves of Abelian groups, with morphisms of Abelian presheaves, is an Abelian category.
How about Abelian sheaves?
Let $f : \mathcal{F} \to \mathcal{G}$ be a morphism of Abelian sheaves. Once again, let $K(U) = \ker(f(U)), C(U) = \operatorname{coker}(f(U)) = \frac{\mathcal{G}(U)}{f(U)(\mathcal{F}(U))}$.

Proposition 1.

(a) The kernel K is an Abelian sheaf

(b) The cokernel C is always a presheaf, but might not be a sheaf.

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Proof.

(a) Let $U = \bigcup U_i$, $s_i \in K(U_i)$ agree on pairwise intersections. As $K(U_i) \hookrightarrow \mathcal{F}(U_i)$, there exists a unique $s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i \in \mathcal{F}(U_i)$ for all i.

Note that $f(s)|_{U_i} = f(s|_{U_i}) = 0$, so $f(s) = 0 \in \mathcal{G}(U)$. Here, we are using the uniqueness of gluing in $\mathcal{G}$.

Then s"$\in$"$K(U)$

(b) First, we give an example of a cokernel which is a presheaf, but not a sheaf. Let $Y = \mathbb{C} \setminus \{0\}$, $f : \mathcal{O}_Y \to \mathcal{O}_Y$ given by $\varphi \mapsto \frac{d\varphi}{d\zeta}$.

- For any $y \in Y$, there exists a neighborhood $V_y \ni y$ such that $\operatorname{coker} f(V_y) = 0$. That is, for every $f \in \mathcal{H}(V_y)$, there is a $g \in \mathcal{H}(V_y)$ so that $\frac{dg}{d\zeta} = f$ in V_y (every point admits a simply connected neighborhood)
- However, $\operatorname{coker} f(Y) \cong \mathbb{C}$: $\Psi = \sum_{i=-\infty}^{\infty} a_i \zeta^i = \frac{d\varphi}{d\zeta}$ has a solution iff $a_{-1} = 0$. This is because $\frac{1}{z}$ is defined on Y. (keyword: vanishing cycle and v-filtration).

  Hence this violates the uniqueness: any $\overline{\Psi} \in \operatorname{coker} f(Y)$ restricts to $0 \in \operatorname{coker} f(V_y)$. However, the V_y cover Y. So, we have an open cover, with sections of each, which agree on intersections, but which do not have a unique global section to which they all glue. So, this is not a sheaf.

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Sheafification

Denote by SAb the additive category of abelian sheaves on a fixed topological space M. We have the inclusion functor $\iota : \mathrm{SAb} \to \mathrm{PAb}$.

Proposition 2. *ι admits a left adjoint $s : \mathrm{PAb} \to \mathrm{SAb}$, i.e.*

$$\mathrm{Hom}_{\mathrm{SAb}}(sX, Y) \cong \mathrm{Hom}_{\mathrm{PAb}}(X, \iota Y)$$

and this isomorphism is natural in both X and Y.

Proof. Let

$$s(X)(U) \stackrel{\text{def}}{=} \frac{\{(\{U_i\}_{i \in I}, e_i) \mid U = \bigcup_{i \in I} U_i, e_i \in X(U_i), e_i|_{U_i \cap U_j} = e_j|_{U_i \cap U_j}\}}{\sim}$$

Where $(\{U_i\}, e_i) \sim (\{U'_j\}, e_j)$ if there exists $\{U''_k\}$ refining $\{U_i\}, \{U'_j\}$ and $e_i|_{U''_k} = e'_j|_{U''_k}$ for $U''_k \subset U_i \cap U'_j$.
Define $\gamma_{U,V} : sX(U) \to sX(V)$, for $V \subseteq U$, by

$$\gamma_{U,V}[(\{U_i\}, e_i)] = [\{U_i \cap V\}, e_i|_{U_i \cap V}]$$

There is a lot to verify; see [GM] 2.5.13

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Example 0.3. For coker f from previous example, now $[\overline{\Psi}] = [0]$, since, when we restrict to V_y, $\overline{\Psi}$ becomes 0. So coker $f = 0$

With this modification, SAb is an abelian category!

Proposition 3. *Let $\varphi : X \to Y$ be a morphism of abelian sheaves, and let*

$$K \xrightarrow{k} \iota X \xrightarrow{i} I \xrightarrow{j} \iota Y \xrightarrow{c} K'$$

be the canonical decomposition of $\iota(\varphi)$ in (the abelian category!) PAb*. Then*

$$sK(= K) \xrightarrow{sk} X(= s\iota X) \xrightarrow{si} sI \xrightarrow{sj} Y(= s\iota Y) \xrightarrow{sc} sK'$$

is the canonical decomposition of φ in SAb*. In particular,* SAb *is an abelian category.*

Proof. We'll just verify that sK' is indeed the cokernel:
Let $Z \in \operatorname{Ob} \mathrm{SAb}$. Then there exists an exact sequence

$$0 \longrightarrow \mathrm{Hom}_{\mathrm{PAb}}(K', \iota Z) \longrightarrow \mathrm{Hom}_{\mathrm{PAb}}(Y, \iota Z) \longrightarrow \mathrm{Hom}_{\mathrm{PAb}}(X, \iota Z) \longrightarrow 0$$

(this is equivalent to saying K' is coker φ in PAb). By adjunction,

$$0 \longrightarrow \mathrm{Hom}_{\mathrm{SAb}}(sK', Z) \longrightarrow \mathrm{Hom}_{SAb}(\underbrace{sY}_{=Y}, Z) \longrightarrow \mathrm{Hom}_{\mathrm{SAb}}(sX, Z) \longrightarrow 0$$

is exact. Other parts are exercises

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Definition 0.6. Let A be an abelian group, Y a topological space.

(a) The constant presheaf $\mathbb{A}$ on Y is $\mathbb{A}(U) = A$ for all open $U \subseteq Y$, and $\gamma_{U,V} = \mathrm{Id}_A$ for any open $V \subseteq U$.

(b) The constant sheaf $\mathcal{A}$ on Y is $s\mathbb{A}$.
(Check: for connected U, $\mathcal{A}(U) = A$)

(c) A sheaf $\mathcal{F}$ is locally constant if any point has a neighborhood U such that $\mathcal{F}|_U$ is a constant sheaf. (for open $V \subset U \subset Y$, $\mathcal{F}|_U(V) \overset{\text{def}}{=} \mathcal{F}(V)$) (keyword: representation of π_1 and local systems)

Germs and stalks

Definition 0.7. The stalk of a (pre)sheaf at a point y is

$$\mathcal{F}_y \stackrel{\text{def}}{=} \varinjlim_{V \ni y} \mathcal{F}(V)$$

with the limit taken with respect to inclusion.
Concretely, $\mathcal{F}_y \stackrel{\text{def}}{=} \frac{\{(s,V) | y \in V, s \in \mathcal{F}(V)\}}{\sim}$, where $(s, V) \sim (s', V')$ if there exists a $W \subseteq V \cap V'$ such that $\gamma_{V,W}(s) = \gamma_{V',W}(s')$.
Such an equivalence class is called a germ
Remark (GM, I.5.5, I.5.6). If we have an exact sequence of sheaves

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$$

if and only if for any $y \in Y$,

$$0 \longrightarrow \mathcal{F}'_y \longrightarrow \mathcal{F}_y \longrightarrow \mathcal{F}''_y \longrightarrow 0$$

is exact. This can be verified as an extremely important exercise.

Definition 0.8. For $s \in \mathcal{F}(U)$, the support of s, $\operatorname{supp} s$, is the closure of the set of points at which the germ of s is not zero.

Remark. In the definition of a stalk, we can replace a point y by a closed subset Z of Y.