q_4

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Question 4

4.17. Let $\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3, \mathbf{X}_4$, and $\mathbf{X}_5$ be independent and identically distributed random vectors with mean vector $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$. Find the mean vector and covariance matrices for each of the two linear combinations of random vectors

$$\tfrac{1}{5}\mathbf{X}_1 + \tfrac{1}{5}\mathbf{X}_2 + \tfrac{1}{5}\mathbf{X}_3 + \tfrac{1}{5}\mathbf{X}_4 + \tfrac{1}{5}\mathbf{X}_5$$

and

$$\mathbf{X}_1 - \mathbf{X}_2 + \mathbf{X}_3 - \mathbf{X}_4 + \mathbf{X}_5$$

in terms of $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$. Also, obtain the covariance between the two linear combinations of random vectors.

To solve this we follow the procedure from p. 76 / 114 - 77 / 115

$1/5X_1 + 1/5X_2 + 1/5X_3 + 1/5X_5$

First off all we need to create our **CX** matrix.

$$Z_1 = \begin{bmatrix} 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1/5x_1 + 1/5x_2 + 1/5x_3 + 1/5x_4 + 1/5x_5 \end{bmatrix} = CX$$

Note that 1/5 corresponds to **C**.

The mean vector is gained this way:

$$\mu = \begin{bmatrix} 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \end{bmatrix} \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \\ \mu_5 \end{bmatrix} = \begin{bmatrix} 1/5\mu_1 + 1/5\mu_2 + 1/5\mu_3 + 1/5\mu_4 + 1/5\mu_5 \end{bmatrix} = c\acute{}\mu$$

The variance co-variance matrix is gained this way (p. 76 formula):

$$\Sigma_{Z_1} = \begin{bmatrix} 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \end{bmatrix} \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} & \sigma_{14} & \sigma_{15} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} & \sigma_{24} & \sigma_{25} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} & \sigma_{34} & \sigma_{35} \\ \sigma_{41} & \sigma_{42} & \sigma_{43} & \sigma_{44} & \sigma_{45} \\ \sigma_{51} & \sigma_{52} & \sigma_{53} & \sigma_{45} & \sigma_{55} \end{bmatrix} \begin{bmatrix} 1/5 \\ 1/5 \\ 1/5 \\ 1/5 \\ 1/5 \end{bmatrix} = C\Sigma_X C'$$

Let’s do it for the next linear combination.

$$Z_2 = X_1 - X_2 + X_3 - X_4 + X_5$$

Let’s get **CX**

$$Z_2 = \begin{bmatrix} 1 & -1 & 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} x_1 - x_2 + x_3 - x_4 + x_5 \end{bmatrix} = CX$$

Let us now get μ

$$\mu = \begin{bmatrix} 1 & -1 & 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \\ \mu_5 \end{bmatrix} = \begin{bmatrix} \mu_1 - \mu_2 + \mu_3 - \mu_4 + \mu_5 \end{bmatrix} = c\,'\mu$$

Let us now get Σ_{Z_2}

$$\Sigma_{Z_2} = \begin{bmatrix} 1 & -1 & 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} & \sigma_{14} & \sigma_{15} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} & \sigma_{24} & \sigma_{25} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} & \sigma_{34} & \sigma_{35} \\ \sigma_{41} & \sigma_{42} & \sigma_{43} & \sigma_{44} & \sigma_{45} \\ \sigma_{51} & \sigma_{52} & \sigma_{53} & \sigma_{45} & \sigma_{55} \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 1 \end{bmatrix} = C\Sigma_X C'$$

Lastly we can get the values μ and Σ from both linear combinations

Let us get **CX**

$$Z = \begin{bmatrix} 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \\ 1 & -1 & 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} (1/5x_1 + 1/5x_2 + 1/5x_3 + 1/5x_4 + 1/5x_5) & (x_1 - x_2 + x_3 - x_4 + x_5) \end{bmatrix} = CX$$

Now let us get μ

$$\mu_Z = \begin{bmatrix} 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \\ 1 & -1 & 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \\ \mu_5 \end{bmatrix} = \begin{bmatrix} \mu_1 - \mu_2 + \mu_3 - \mu_4 + \mu_5 \\ 1/5\mu_1 + 1/5\mu_2 + 1/5\mu_3 + 1/5\mu_4 + 1/5\mu_5 \end{bmatrix}$$

Now let us get Σ

$$\Sigma = \begin{bmatrix} 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \\ 1 & -1 & 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} & \sigma_{14} & \sigma_{15} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} & \sigma_{24} & \sigma_{25} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} & \sigma_{34} & \sigma_{35} \\ \sigma_{41} & \sigma_{42} & \sigma_{43} & \sigma_{44} & \sigma_{45} \\ \sigma_{51} & \sigma_{52} & \sigma_{53} & \sigma_{45} & \sigma_{55} \end{bmatrix} \begin{bmatrix} 1/5 & 1 \\ 1/5 & -1 \\ 1/5 & 1 \\ 1/5 & -1 \\ 1/5 & 1 \end{bmatrix} = C\Sigma_X C'$$