MultiVariate Normal Distribution

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How table 4.2 is built?

Table 4.2 Critical Points for the Q–Q Plot Correlation Coefficient Test for Normality

Sample size n	Significance levels α .01	.05	.10
5	.8299	.8788	.9032
10	.8801	.9198	.9351
15	.9126	.9389	.9503
20	.9269	.9508	.9604
25	.9410	.9591	.9665
30	.9479	.9652	.9715
35	.9538	.9682	.9740
40	.9599	.9726	.9771
45	.9632	.9749	.9792
50	.9671	.9768	.9809
55	.9695	.9787	.9822
60	.9720	.9801	.9836
75	.9771	.9838	.9866
100	.9822	.9873	.9895
150	.9879	.9913	.9928
200	.9905	.9931	.9942
300	.9935	.9953	.9960

1. Generate multiple datasets with N(0,1) for n=100
2. Make Q-Q plots for each of the dataset and derive r_Q respectively
3. Collect all the r_Q and find the critical value for given significant level 0.05.

Bivariate normal distribution $\boldsymbol{p = 2}$

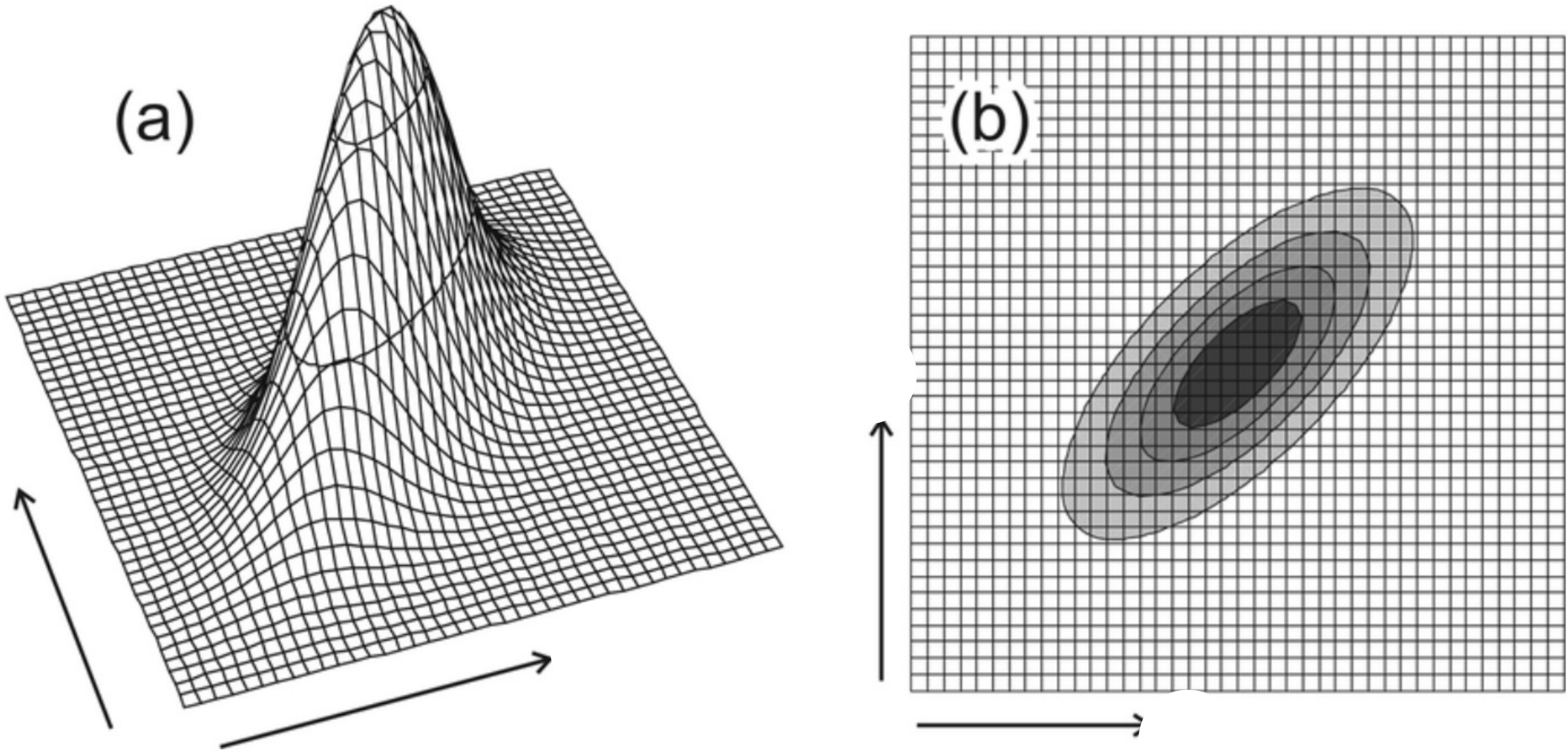
$$\boldsymbol{X} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$$

$$\boldsymbol{\mu} = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}$$

$$\Sigma = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}$$

$$\frac{1}{2\pi\sqrt{\sigma_1^2\sigma_2^2(1-\rho^2)}} \times$$

$$\exp\left\{-\frac{1}{2(1-\rho^2)}\left[\frac{(x_1-\mu_1)^2}{\sigma_1^2} + \frac{(x_2-\mu_2)^2}{\sigma_2^2} - 2\rho\left(\frac{x_1-\mu_1}{\sigma_1}\right)\left(\frac{x_2-\mu_2}{\sigma_2}\right)\right]\right\}$$



Towards general: vector and matrix form

$$\boxed{\frac{1}{2\pi\sqrt{\sigma_1^2\sigma_2^2(1-\rho^2)}} \times \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\frac{(x_1-\mu_1)^2}{\sigma_1^2}+\frac{(x_2-\mu_2)^2}{\sigma_2^2}-2\rho\left(\frac{x_1-\mu_1}{\sigma_1}\right)\left(\frac{x_2-\mu_2}{\sigma_2}\right)\right]\right\}}$$

$$\boldsymbol{\mu} = \begin{pmatrix}\mu_1\\ \mu_2\end{pmatrix} \qquad |\Sigma| = \sigma_1^2\sigma_2^2(1-\rho^2)$$

$$\Sigma = \begin{pmatrix}\sigma_1^2 & \rho\sigma_1\sigma_2\\ \rho\sigma_1\sigma_2 & \sigma_2^2\end{pmatrix} \qquad \Sigma^{-1} = \frac{1}{\sigma_1^2\sigma_2^2(1-\rho^2)}\begin{pmatrix}\sigma_2^2 & -\rho\sigma_1\sigma_2\\ -\rho\sigma_1\sigma_2 & \sigma_1^2\end{pmatrix}$$

$$\boxed{\frac{1}{(2\pi)^{2/2}|\Sigma|^{1/2}} \exp\{-(\boldsymbol{x}-\boldsymbol{\mu})'\Sigma^{-1}(\boldsymbol{x}-\boldsymbol{\mu})/2\}}$$

?

For general $\boldsymbol{p} \geq \mathbf{2}$, joint PDF (4-4) $\boldsymbol{X} \sim N_p(\boldsymbol{\mu}, \Sigma)$

$$\frac{1}{(2\pi)^{2/2}|\Sigma|^{1/2}} \exp\{-(\boldsymbol{x}-\boldsymbol{\mu})'\Sigma^{-1}(\boldsymbol{x}-\boldsymbol{\mu})/2\}$$

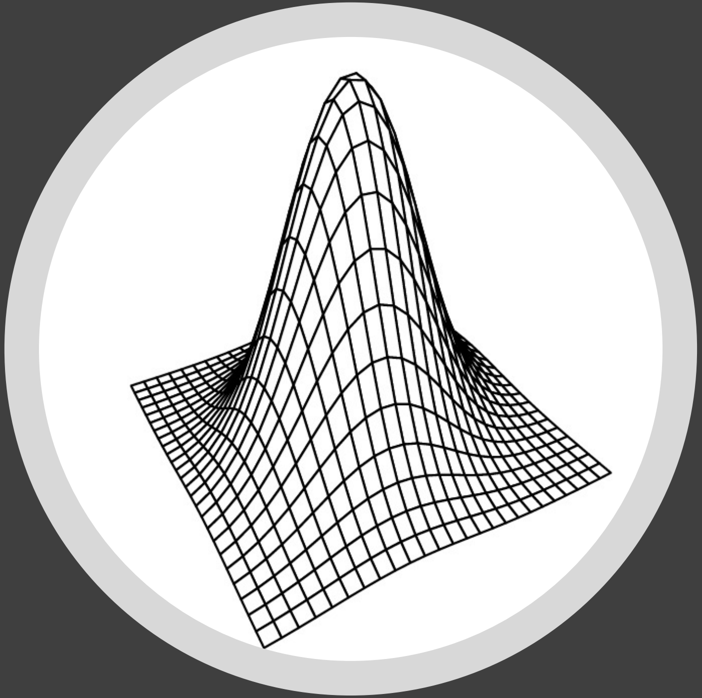
$$\downarrow$$

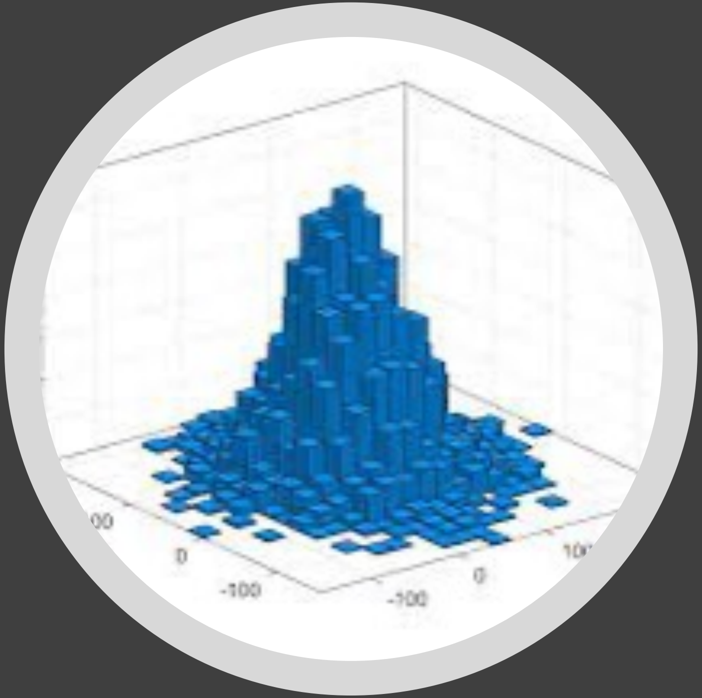
$$\frac{1}{(2\pi)^{p/2}|\Sigma|^{1/2}} \exp\{-(\boldsymbol{x}-\boldsymbol{\mu})'\Sigma^{-1}(\boldsymbol{x}-\boldsymbol{\mu})/2\}$$

This is why we need the vectors!

Consider $\boldsymbol{X} = (X_1, X_2, \ldots, X_p)'$ and $\boldsymbol{x} = (x_1, x_2, \ldots, x_p)'$

$E(\boldsymbol{X}) = \boldsymbol{\mu}$ and $Cov(\boldsymbol{X}) = \Sigma$.





<u>In practice:</u> *Is my data normally distributed*?

Example: radiation data with door open+closed (t4-1.dat; t4-5.dat)

Quadratic form (4-8)

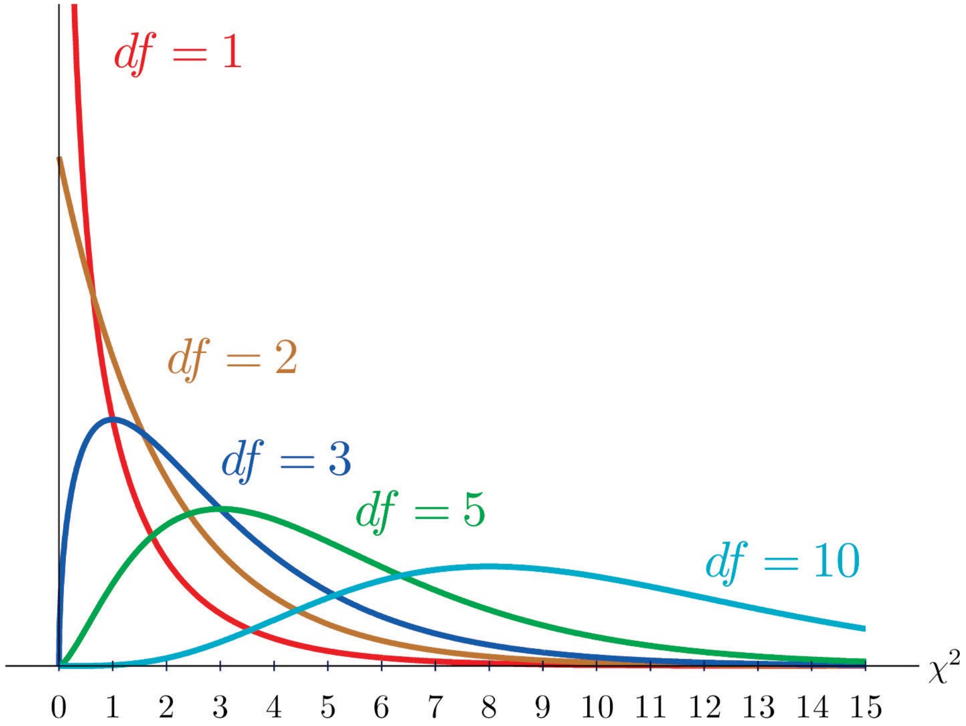
Quadratic form

$$\boldsymbol{X} \sim N_p(\boldsymbol{\mu}, \Sigma) \Rightarrow \frac{1}{(2\pi)^{p/2}|\Sigma|^{1/2}} \exp\{-\boxed{(\boldsymbol{x}-\boldsymbol{\mu})'\Sigma^{-1}(\boldsymbol{x}-\boldsymbol{\mu})}/2\}$$

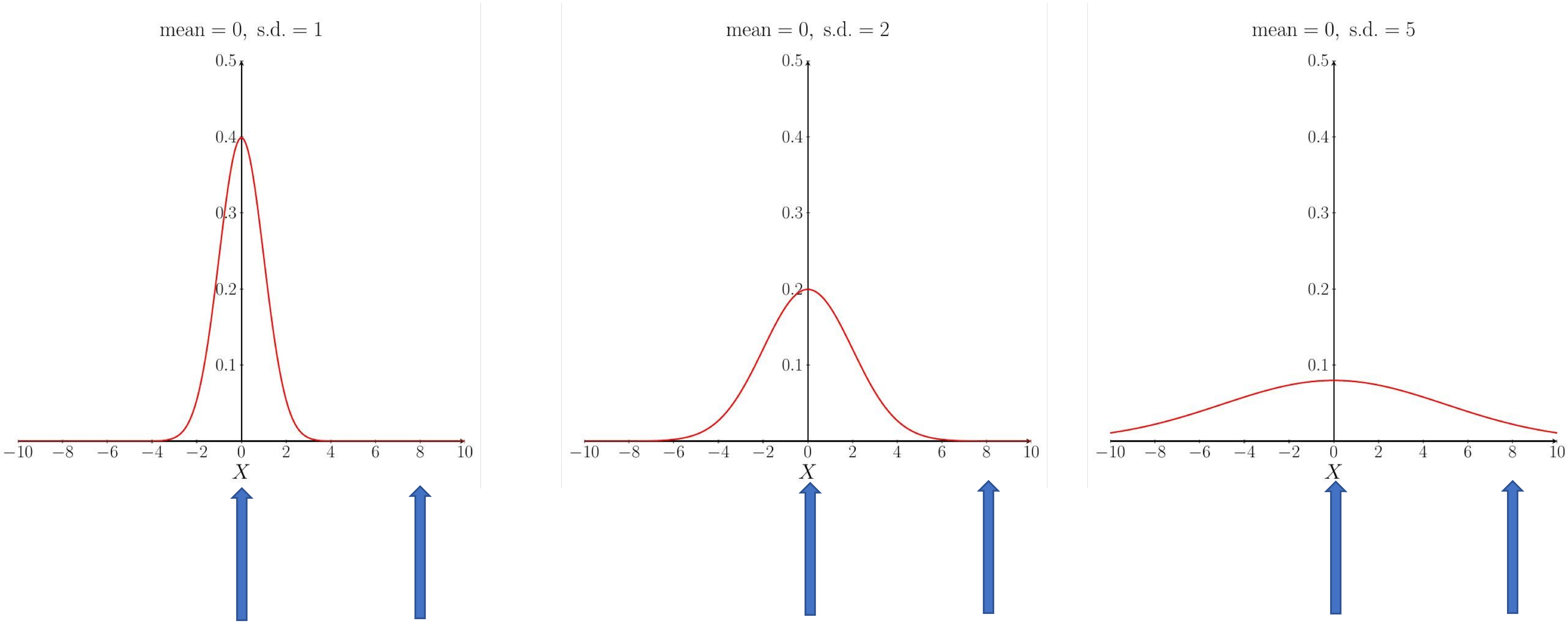
p-dimension

1-dimension

- (4-8) Assume $\boldsymbol{X} \sim N_p(\boldsymbol{\mu}, \Sigma)$, we have $(\boldsymbol{X}-\boldsymbol{\mu})'\Sigma^{-1}(\boldsymbol{X}-\boldsymbol{\mu}) \sim \chi_p^2$

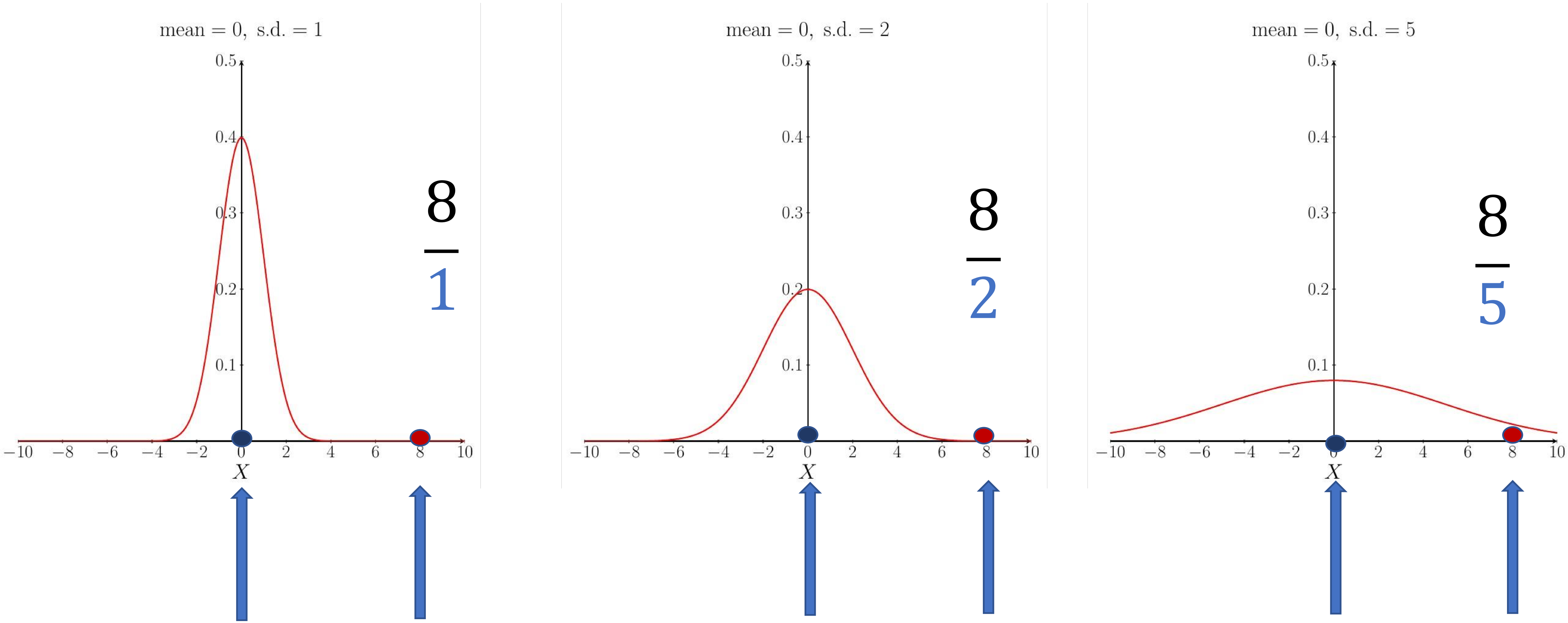


Statistical/Mahalanobis distance



Statistical/Mahalanobis distance

Bigger variability, smaller difference



Statistical/Mahalanobis distance

- (2-17, 4-3) The quadratic form $(\boldsymbol{x}-\boldsymbol{\mu})'\Sigma^{-1}(\boldsymbol{x}-\boldsymbol{\mu})$ is referred to as squared statistical/Mahalanobis distance. R cmd `mahalanobis()`

From $\boldsymbol{x}$ *to* $\boldsymbol{\mu}$

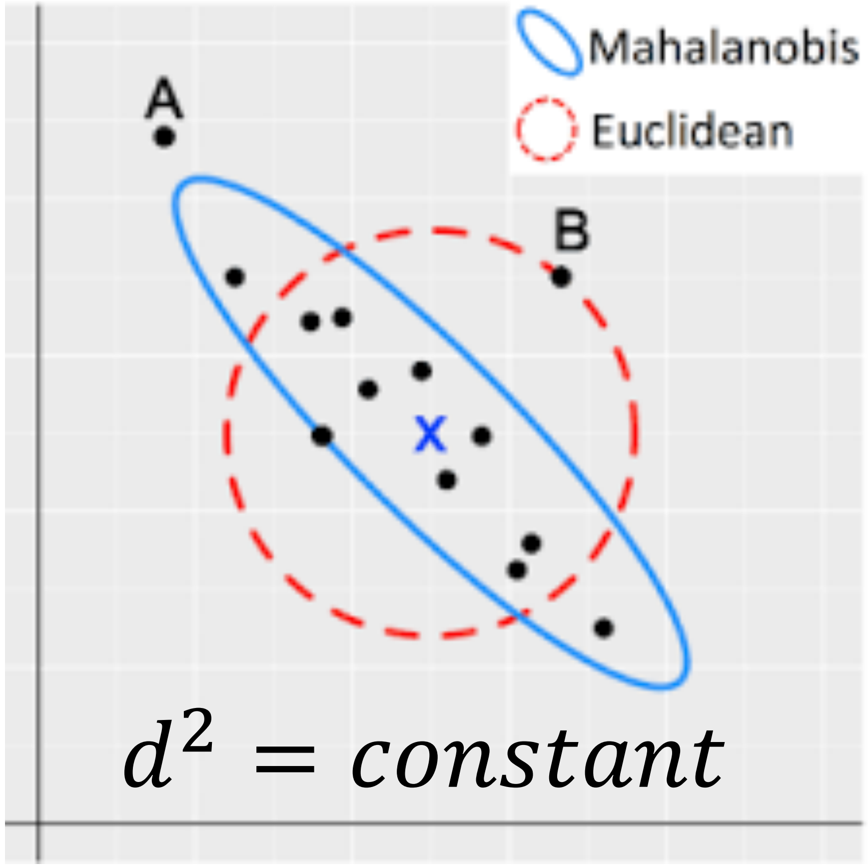
$$d_E(\boldsymbol{x},\boldsymbol{y}) = \sqrt{(\boldsymbol{x}-\boldsymbol{y})^T \cdot (\boldsymbol{x}-\boldsymbol{y})}$$

$$d_M(\boldsymbol{x},\boldsymbol{y}) = \sqrt{(\boldsymbol{x}-\boldsymbol{y})^T \cdot S^{-1} \cdot (\boldsymbol{x}-\boldsymbol{y})}$$

$$= \sqrt{\begin{bmatrix} x_1-y_1 & x_2-y_2 \end{bmatrix} \begin{bmatrix} \frac{1}{\sigma_1^2} & 0 \\ 0 & \frac{1}{\sigma_2^2} \end{bmatrix} \begin{bmatrix} x_1-y_1 \\ x_2-y_2 \end{bmatrix}}$$

$$= \sqrt{\begin{bmatrix} \frac{x_1-y_1}{\sigma_1^2} & \frac{x_2-y_2}{\sigma_2^2} \end{bmatrix} \begin{bmatrix} x_1-y_1 \\ x_2-y_2 \end{bmatrix}}$$

$$= \sqrt{\frac{(x_1-y_1)^2}{\sigma_1^2} + \frac{(x_2-y_2)^2}{\sigma_2^2}}$$



- Result (4.7) The solid ellipsoid of $\boldsymbol{x}$ values satisfying

$$(\boldsymbol{x}-\boldsymbol{\mu})'\Sigma^{-1}(\boldsymbol{x}-\boldsymbol{\mu}) \leq c^2 = \chi_p^2(\alpha)$$

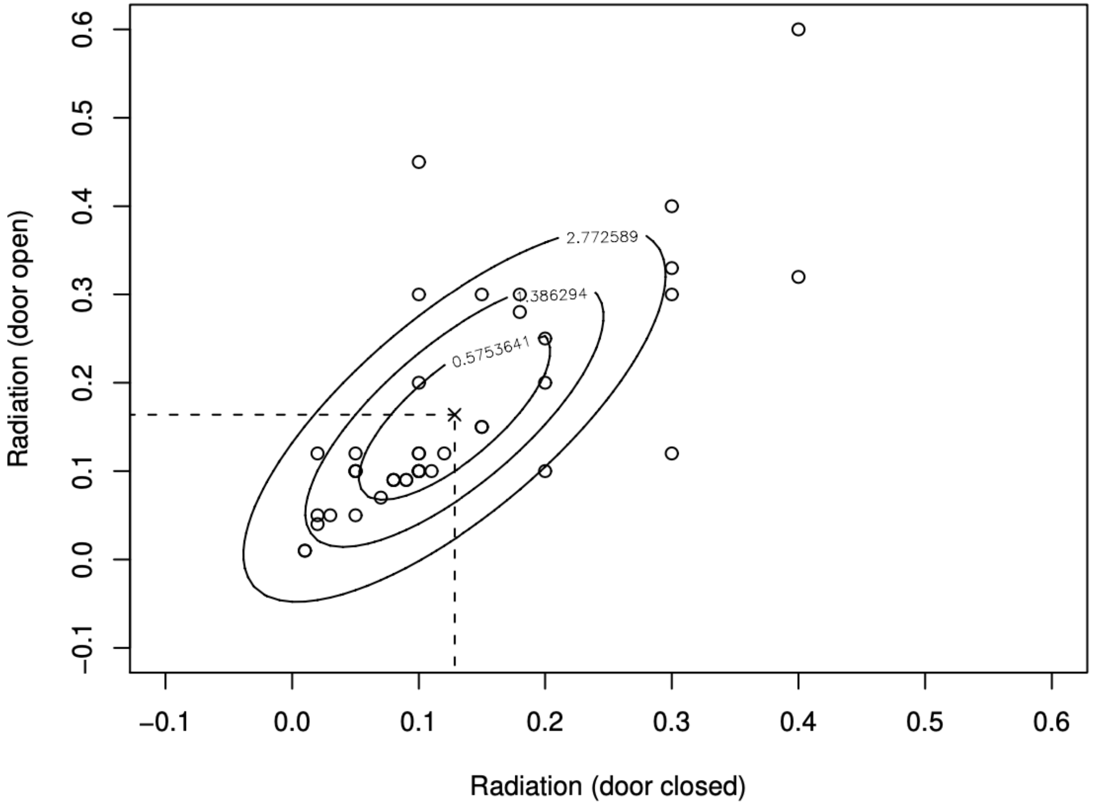
$qchisq(\alpha, p)$

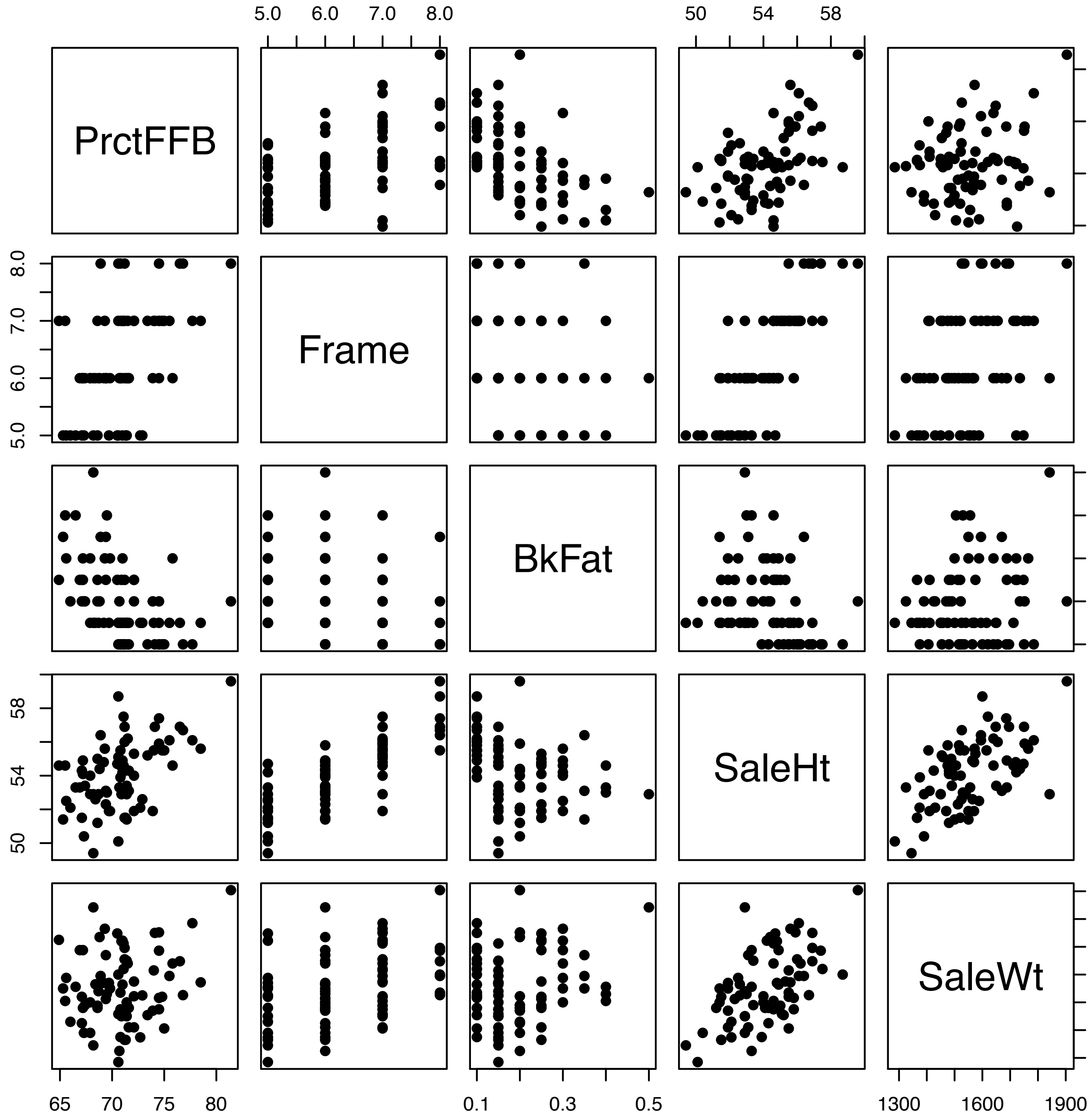
has probability $1-\alpha$.

$1-\alpha$	Observed count	Expected count
0.25	17	10.5
0.50	29	21
0.75	33	31.5

Expected number of observations versus data. Note

n = 42





PrctFFB
Frame
BkFat
SaleHt
SaleWt
5.0 6.0 7.0 8.0
50 54 58
65 70 75 80
0.1 0.3 0.5
1300 1600 1900

Q-Q plot, again

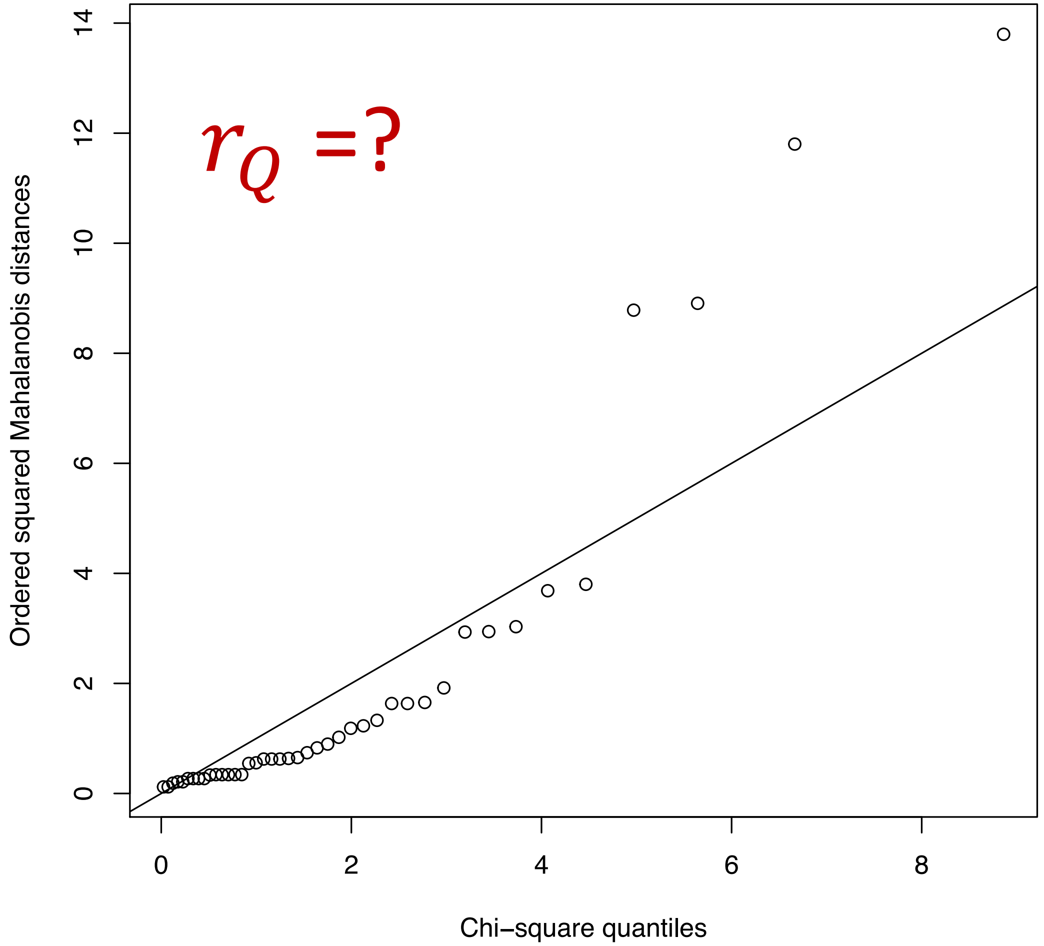
Dimension?

- Result (4.7) Assume $\boldsymbol{X} \sim N_p(\boldsymbol{\mu}, \Sigma)$, we have $\boxed{(\boldsymbol{X} - \boldsymbol{\mu})'\Sigma^{-1}(\boldsymbol{X} - \boldsymbol{\mu})} \sim \chi_p^2$

Q-Q plot, again

Dimension?

- Result (4.7) Assume $\boldsymbol{X} \sim N_p(\boldsymbol{\mu}, \Sigma)$, we have $\boxed{(\boldsymbol{X}-\boldsymbol{\mu})'\Sigma^{-1}(\boldsymbol{X}-\boldsymbol{\mu})} \sim \chi^2_p$



Can we use Table 4.2 again?

Table 4.2 Critical Points for the Q-Q Plot Correlation Coefficient Test for Normality

Sample size n	Significance levels α .01	.05	.10
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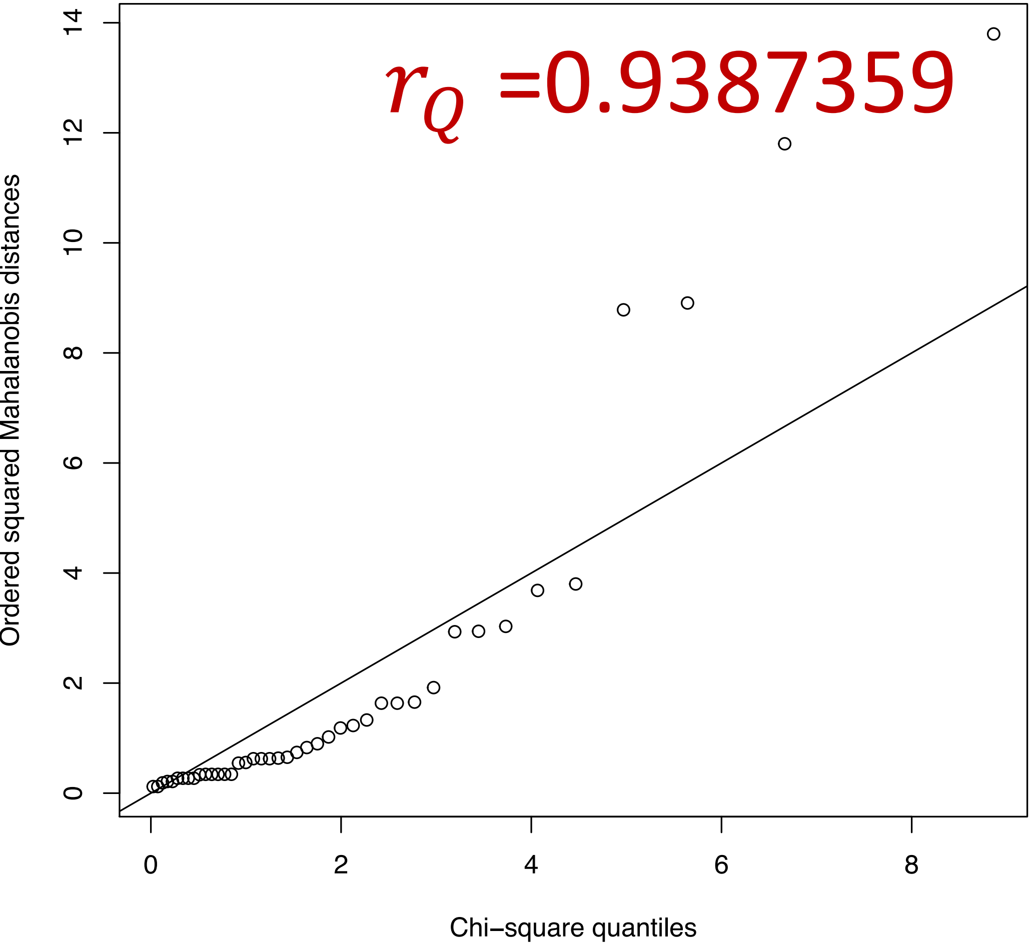
Q-Q plot, again

- Result (4.7) Assume $\boldsymbol{X} \sim N_p(\boldsymbol{\mu}, \Sigma)$, we have $\boxed{(\boldsymbol{X}-\boldsymbol{\mu})'\Sigma^{-1}(\boldsymbol{X}-\boldsymbol{\mu})} \sim \chi^2_p$



1. Generate multiple datasets with χ^2_p for n=42
2. Make Q-Q plots for each of the dataset and derive r_Q respectively
3. Collect all the r_Q and find the critical value for some given significant level.

FindCrikChi2update.R

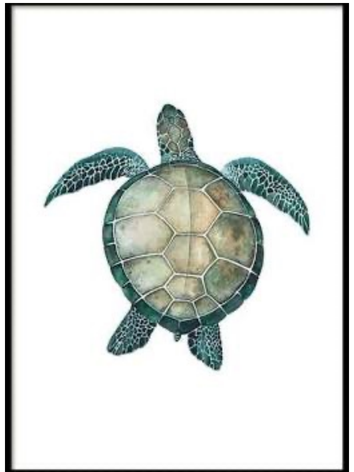
```
> source("FindCrikChi2.R")
> result1[[2]]
[1] 0.9948543
```

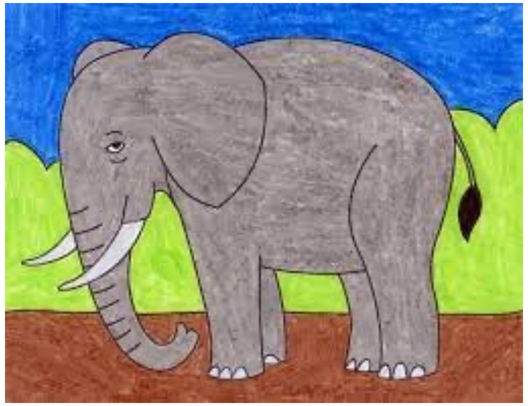
Conclusion ?

Is *quadratic form* enough for assessing normality?

- Result (4.7) Assume $\boldsymbol{X} \sim N_p(\boldsymbol{\mu}, \Sigma)$, we have $\boxed{(\boldsymbol{X}-\boldsymbol{\mu})'\Sigma^{-1}(\boldsymbol{X}-\boldsymbol{\mu})} \sim \chi_p^2$

This animal is a turtle. **This is an animal has 4 legs.**





Check MVN, continued

- (Result 4.4) Any subset of a MVN distributed random vector is normally distributed.

Example 4.5 (The distribution of a subset of a normal random vector)

If $\mathbf{X}$ is distributed as $N_5(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, find the distribution of $\begin{bmatrix} X_2 \\ X_4 \end{bmatrix}$. We set

$$\mathbf{X}_1 = \begin{bmatrix} X_2 \\ X_4 \end{bmatrix}, \qquad \boldsymbol{\mu}_1 = \begin{bmatrix} \mu_2 \\ \mu_4 \end{bmatrix}, \qquad \boldsymbol{\Sigma}_{11} = \begin{bmatrix} \sigma_{22} & \sigma_{24} \\ \sigma_{24} & \sigma_{44} \end{bmatrix}$$

In summary, basic track of checking MVN

- Test univariate normality for each marginal distribution with QQ-plot.
- Test bivariate normality for each pair of attributes. For example, a matrix of scatterplots and QQ-plot based on $(\boldsymbol{X}-\boldsymbol{\mu})'\Sigma^{-1}(\boldsymbol{X}-\boldsymbol{\mu}) \sim \chi_2^2$
- Test over all MVN using QQ-plot based on $(\boldsymbol{X}-\boldsymbol{\mu})'\Sigma^{-1}(\boldsymbol{X}-\boldsymbol{\mu}) \sim \chi_p^2$
- Linear pattern in QQ-plot can be evaluated through hypothesis test.

Advanced track: well, it is by taking care of all the possible subsets of the attributes…or PCA

Other useful properties of MVN

- (Result 4.3) Let $\boldsymbol{A}$ be a $(q \times p)$ numeric matrix, then

$$\boldsymbol{AX} \sim N_q(\boldsymbol{A\mu}, \boldsymbol{A}\Sigma\boldsymbol{A}')$$

- Exercise 2 Find the mean vector and the total variance of $\boldsymbol{AX}$.
  Given that $\boldsymbol{A} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix}$ and $\boldsymbol{X} = (X_1, X_2, X_3)'$.

  Further we know $\boldsymbol{X} \sim N_3(\boldsymbol{\mu}, \Sigma)$, where $\boldsymbol{\mu} = (1, 2, 1)'$ and

$$\Sigma = \begin{pmatrix} 3 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix}$$

Other useful properties of MVN

Result 4.5.

(a) If $\underset{(q_1\times 1)}{\mathbf{X}_1}$ and $\underset{(q_2\times 1)}{\mathbf{X}_2}$ are independent, then $\text{Cov}(\mathbf{X}_1, \mathbf{X}_2) = \mathbf{0}$, a $q_1 \times q_2$ matrix of zeros.

(b) If $\begin{bmatrix} \mathbf{X}_1 \\ \hdashline \mathbf{X}_2 \end{bmatrix}$ is $N_{q_1+q_2}\left(\begin{bmatrix} \boldsymbol{\mu}_1 \\ \hdashline \boldsymbol{\mu}_2 \end{bmatrix}, \left[\begin{array}{c:c} \boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12} \\ \hdashline \boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_{22} \end{array}\right]\right)$, then $\mathbf{X}_1$ and $\mathbf{X}_2$ are independent if and only if $\boldsymbol{\Sigma}_{12} = \mathbf{0}$.

(c) If $\mathbf{X}_1$ and $\mathbf{X}_2$ are independent and are distributed as $N_{q_1}(\boldsymbol{\mu}_1, \boldsymbol{\Sigma}_{11})$ and $N_{q_2}(\boldsymbol{\mu}_2, \boldsymbol{\Sigma}_{22})$, respectively, then $\begin{bmatrix} \mathbf{X}_1 \\ \hdashline \mathbf{X}_2 \end{bmatrix}$ has the multivariate normal distribution

$$N_{q_1+q_2}\left(\begin{bmatrix} \boldsymbol{\mu}_1 \\ \hdashline \boldsymbol{\mu}_2 \end{bmatrix}, \left[\begin{array}{c:c} \boldsymbol{\Sigma}_{11} & \mathbf{0} \\ \hdashline \mathbf{0}' & \boldsymbol{\Sigma}_{22} \end{array}\right]\right)$$