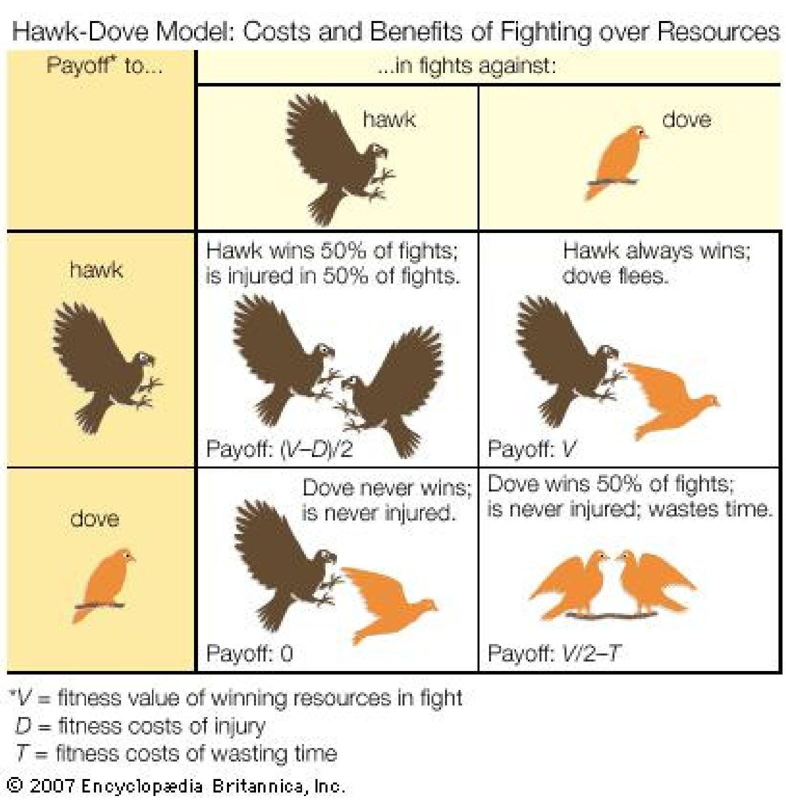
Olga Ibáñez Solé
MA in Bioinformatics and Modeling
Université Libre de Bruxelles

Assignment 1: Game theory

The Hawk-Dove game

The Hawk-Dove game is a coordination game that aims to model the resolution of conflicts in the animal kingdom. There are two players that can act either as a hawk or as a dove, meaning that they can either escalate inmediately and fight or take some time to display before fighting. When both agents play hawk, they have 50% of chance of being injured (-D/2) and 50% chance of wining (V/2). When one of them plays hawk and the other one plays dove, the former wins (V). When two doves fight each other, none of them is injured and one of them wins after a period of mutual displays (-T).



Olga Ibáñez Solé
MA in Bioinformatics and Modeling
Université Libre de Bruxelles

<u>Mixed strategy Nash equilibria</u>

In order to calculate the mixed strategy Nash equilibria of this game, let us define the two agents as Player 1 and Player 2, which can behave play hawk or dove with a probability of p and (1-p) and q and (1-q) respectively.

		Hawk	Dove
		q	1-q
Hawk	p	$\frac{(V-D)}{2}$ $\frac{(V-D)}{2}$	0 V
Dove	1-p	V 0	$\frac{V}{2}-T$ $\frac{V}{2}-T$

The utility function for player 1 is derived as follows:

$$u_1(p)=p\cdot[\frac{q\cdot V-D}{2}+(1-q)\cdot V]+(1-p)\cdot[\frac{(1-q)\cdot V}{2}-T]$$

Which can be simplified to:

$$u_1(p)=p\cdot[\frac{-q\cdot(V+D)}{2}+V]+(1-p)\cdot[q\cdot(T-\frac{V}{2})+\frac{V}{2}-T]$$

We can see that there are two main terms, the first one is multiplying p and corresponds to the payoff that player 1 gets when she plays hawk, and the second one is multiplying (1-p) and corresponds to the payoff that she obtains when she plays dove. If the former were strictly greater than the latter, then player 1 would want to play the pure strategy hawk (p = 1). If the latter were strictly greater, then she would want to play the pure strategy dove (p = 0). If both terms were equal, then she would be able to freely chose between both strategy. We represent this using the inequation below:

Olga Ibáñez Solé
MA in Bioinformatics and Modeling
Université Libre de Bruxelles

$$\frac{-q\cdot(V+D)}{2}+V>q\cdot(T-\frac{V}{2})+\frac{V}{2}-T$$

We solve this inequation:

$$q<\frac{T+\frac{V}{2}}{T+\frac{D}{2}}$$

As the game is symmetric, the same reasoning can be applied to the best response analysis for player 2.

Utility function: $u_2(q)=q\cdot[\frac{-p\cdot(V+D)}{2}+V]+(1-q)\cdot[p\cdot(T-\frac{V}{2})+\frac{V}{2}-T]$

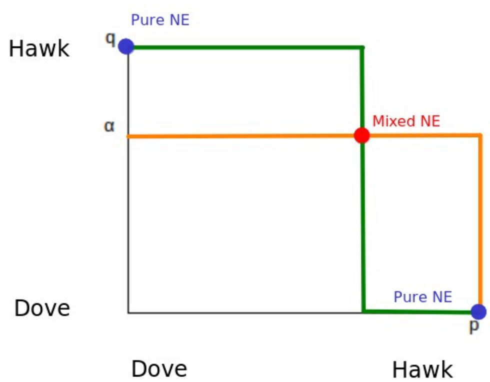
Inequation: $\frac{-p\cdot(V+D)}{2}+V>p\cdot(T-\frac{V}{2})+\frac{V}{2}-T$; $p<\frac{T+\frac{V}{2}}{T+\frac{D}{2}}$

For the sake of clarity, hereafter I will only refer to the best response analysis of player 1. The inequation for player 1 will be true or false depending on the values of V and D. Note that $0 \leq p \leq 1$ and $0 \leq q \leq 1$.

- If **V > D**: as q cannot be greater than 1 and α of the inequation (hereafter named α) will always be greater than 1, the inequation is always true. Therefore, the best strategy will be the <u>pure strategy hawk</u>. Since D represents the cost of injury and V represents the value of the resources obtained in the victory, the biological meaning of this is that when the victory outweighs the costs of injuries, it is always worth escalating and fighting.

- If **V ≤ D**: both p and α range between 0 and 1, and the inequation will be true or false, depending on the specific values of both. If player 2 is playing a mixed strategy with a q that is greater than α, then player 1 may prefer to play dove. However, if player 2 is playing a lot of dove (q is smaller than α), then player 1 might want to play hawk. The meaning of this is that, if player 2 is playing a lot of hawk and the cost of injuries is greater than the benefit player 1 could obtain from the victory, player 1 might prefer to play dove and avoid being injured. In fact, when she plays dove against player 2, she knows that she will loose most of the times and obtain a zero payoff, but if she played hawk instead, she would risk being injured and getting a negative payoff. When q is equal to α, any mixed strategy will be equally suitable for player 1, since the payoff of playing hawk and dove will be the same. In the limit case, in which V = D, player 1 may prefer to play hawk except for the case in which player 2 is playing the pure strategy hawk. If this happens, player 1 may be able to freely choose any combination of hawk and dove, since both strategies will lead her to the same payoff values.

The whole picture can be seen in the diagram below, in which the mixed strategy Nash equilibrium (red), and the pure Nash equilibriums (blue) are shown, together with the critical value of α.



Which social dilemma?

We describe a situation in which two players are playing against each other in of the three social dilemmas: the prisoner’s dilemma, the stag-hunt game or the snowdrift game. In each game, player A needs to decide if he will cooperate (C) or defect (D), but he does not know which play he is actually playing. The three games are equally likely. Player B knows which social dilemma they are playing in.

Bayesian game analysis enables us to determine the Nash equilibria of a game in which one of the players is not fully aware of environmental aspects that may be relevant for her choice of action. In this case, we consider that player A does not know which social dilemma she is playing in, so she needs to form a belief about what kind of actions player B will take when they are playing each type of game.

Olga Ibáñez Solé
MA in Bioinformatics and Modeling
Université Libre de Bruxelles

Prisonners dilemma

	C	D
C	2,2	0,5
D	5,0	1,1

Stag-Hunt game

	C	D
C	5,5	0,2
D	2,0	1,1

Snowdrift game

	C	D
C	2,2	1,5
D	5,1	0,0

In order to calculate the Nash equilibria, we create a matrix that contains the expected payoff for all possible combinations of player B types. We calculate the expected payoff by combining the payoffs for each player in each different game. As the games are equally likely, we only have to calculate the average payoff.

player A, player B	**CCC**	**CCD**	**CDC**	**CDD**	**DCC**	**DCD**	**DDC**	**DDD**
C	3, 3	8/3, 4	4/3, 2	1, 3	7/3, 4	2, 5	2/3, 3	1/3, 4
D	4, 1/3	7/3, 0	11/3, 2/3	2, 1/3	8/3, 2/3	1, 1/3	7/3, 1	2/3, 2/3

We perform the best response analysis for each player (player A in green, player B in blue). The yellow cells correspond to Nash equilibria, where the best response of A to the strategy of B is the same as the best response of B to the strategy of A.

player A, player B	**CCC**	**CCD**	**CDC**	**CDD**	**DCC**	**DCD**	**DDC**	**DDD**
C	3, 3	**8/3**, 4	4/3, 2	1, 3	7/3, 4	**2**, **5**	2/3, 3	1/3, 4
D	**4**, 1/3	7/3, 0	**11/3**, 2/3	**2**, 1/3	**8/3**, 2/3	1, 1/3	**7/3**, **1**	**2/3**, 2/3

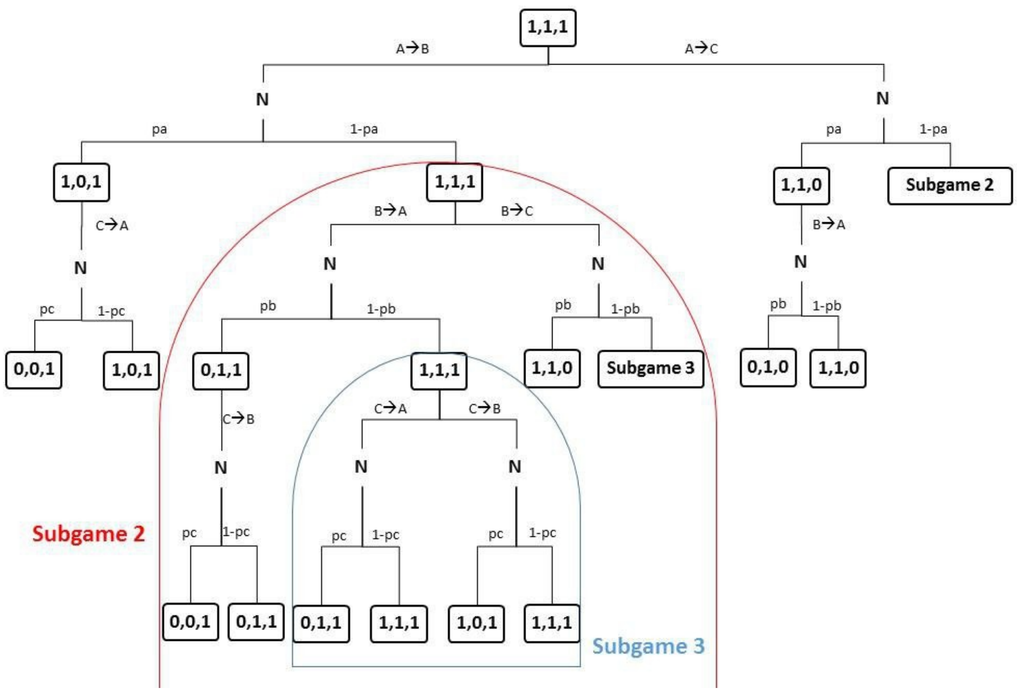
Therefore, there are two pure Nash equilibria in this game:

- {C, DCD}
- {D, DDC}

Sequential truel

In this game, there are three players, A, B and C, each of them holds a gun containing a single bullet. Each of the players get to shoot once: first player A shoots, then player B (if still alive), and finally player C (if still alive).

We denote p_i the probability that player i hits her intendet target. We assume that $0 < p_i < 1$. We modeled this situation using an extensive game and obtained the following decision tree:



We can note that once player A has shot and misses her target, no matter whether she aimed to shoot at B or at C, the following subgame (subgame 2) will be exactly the same in both cases.

Olga Ibáñez Solé
MA in Bioinformatics and Modeling
Université Libre de Bruxelles

In order to study the Nash equilibria of the game, we had to create a 2x2x2 matrix whose cells contain the payoffs for each of the players in every combination of the possible actions of the three players. As it is a 3D matrix, it appears here as two 2x2 matrices. The first one corresponds to the front slice of the matrix, the second one to the back slice of the matrix. In order to make the matrix more understandable, below are shown slices of the matrix corresponding to a given action of each player. The colors (blue for A, green for B and red for C) match those used in the matrix.

In the matrix, the letters in black with the grey background correspond to the player that is taking the action (shooting either one or another player). Players that are shown in color are the ones being shot by the players shown in black. For example, the first cell in the matrix represents a situation in which player A shoots B, player B shoots A and player C shoots A.

The three payoffs shown in each cell of the matrix correspond to those of players A (blue), B (green) and C (red). The payoffs that are underlined are the best responses. We can observe that in cells 4, 6 and 8 (shown in pink) the payoffs for the three responses are underlined, that is, they are the best response to the other players' actions, which means that those cells correspond to the Nash equilibria of the game.

The two cells in pale orange (cells 1 and 2) correspond to Nash equilibria that are dependent on the relative values of p_b and p_c. If p_b is greater than p_c, cell 1 will correspond to a Nash equilibrium. If p_b is smaller than p_c, then cell 2 will correspond to a Nash equilibrium.

Olga Ibáñez Solé
MA in Bioinformatics and Modeling
Université Libre de Bruxelles

C: A

A \ B	A	C
B	1 $p_A(1-p_C) + (1-p_A)(1-p_B)(1-p_C)$ if $p_B>p_C$ $1-p_A$ 1	2 $p_A(1-p_C) + (1-p_A)p_B + (1-p_A)(1-p_B)(1-p_C)$ $1-p_A$ $p_A +(1-p_A)(1-p_B)$
C	3 $p_A(1-p_B) +(1-p_A)(1-p_B)(1-p_C)$ if $p_C>p_B$ 1 $1-p_A$	4 $p_A + (1-p_A)p_B + (1-p_A)(1-p_B)(1-p_C)$ 1 $(1-p_A)(1-p_B)$

C: B

A \ B	A	C
B	5 $p_A + (1-p_A)(1-p_B)$ $(1-p_A)(1-p_C)$ 1	6 1 $(1-p_A)p_B + (1-p_A)(1-p_B)(1-p_C)$ $p_A + (1-p_A)(1-p_B)$
C	7 $1-p_B$ $p_A + (1-p_A)(1-p_C)$ $1-p_A$	8 1 $p_A + (1-p_A)p_B + (1-p_A)(1-p_B)(1-p_C)$ $(1-p_A)(1-p_B)$

The process of building the matrix, doing the best response analysis and identifying the Nash equilibria and the conditions for Nash equilibria was a bit laborious and the details are not shown in the report, but here are some interesting remarks about the process and the matrix we obtained:

- For a given player, there is always a combination of the other players' actions that lead to her having a payoff of 1. A payoff of 1 represents a situation in which none of the other players shot her, so the probability that she survives is 1. For player A, this happens when player B shoots C and player C shoots B (cells 6 and 8. In other words, the B and C shoot each other and A always survives.

- We can note in the pairs of cells to be compared in the best response analysis (the ones where the other players' actions are kept constant and only that of C changes) have the same value for the payoff of C. This means that, once the previous two players have either used their single bullet or died, C can take no action that will change her probability of survival. The survival of C is determined by the time she gets to shoot someone and does not depend on her choice of actions.

- While most comparisons in the best response analysis are very straightforward, some others need a little bit of analysis. For example, when we compare the payoff for A in cells 5 and 7, we might not see at first sight that $p_a + (1-p_a)(1-p_b)$ is greater than $1-p_b$. We can deduce that this is true by testing the corresponding inequation:

$$p_a + (1-p_a)(1-p_b) > 1-p_b$$

$$p_a + 1 - p_a - p_b + p_a p_b > 1-p_b$$

$$p_a p_b > 0$$

  We know that this is true no matter which values p_a and p_b take, because we defined at the beginning that the probability of hitting a target is non zero ($0 < p_i < 1$).

- In the cells 1 and 2 we have two conditioned Nash equilibriums. The conditions make sense because both B and C will shoot at A, so the best for A to increase her probability of survival is to shoot first at the most skilled shooter. This means that if p_b is greater than p_c, A might want to shoot at B. In other words, shooting at C cannot be part of the best strategy for A.

Once we obtained the Nash equilibria, we could see wheter they were subgame perfect or not. We checked that by coming back to the tree and identifying the Nash strategies corresponding to the Nash equilibria. If we go to subgame 3, we can see that player C needs to choose between shooting at A or B. Both correspond to Nash equilibria according to our matrix. We conclude that both are subgame perfect Nash equilibria because there are no inconsistencies between the Nash equilibria predicted for the whole game and the preferences of C in subgame 3. C may decide to shoot either A or B and both are equally profitable for her, because her probability of surviving is the same in both cases. Had we noticed that one of the actions were more profitable for C than the other action, this would mean that the Nash equilibria predicted for the whole game would not be consistent with C's preferences at subgame 3, so that specific Nash equilibria would not be subgame perfect.

The statement that "weakness is strength" can be argued in the following terms: if C is not very good at shooting and p_b is greater than p_c, then A may prefer to shoot at B in order to avoid the danger posed by her. By having a very poor p_c, C protects herself from being the player that poses most danger to A and B and thus decreases the likelihood of her being shot.