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Financial crisis: An attempt of mathematical modelling

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ABSTRACT

We attempt to explore the development of a financial crisis mathematically, constructing and analysing a simple qualitative mathematical model of a progressing crisis. This model brings an insight into the evolution of a crisis and enables us to evaluate the likely outcomes of the possible interventions that can slow the crisis' progress or ameliorate its consequences. The model also helps to identify the factors and actions that may enhance the global economic security.

1. Introduction

In this paper we attempt to mathematically explore the development of the ongoing financial crisis. The financial crisis began in July 2007 when a loss of confidence in the value of securitized mortgages in the United States resulted in a liquidity crisis [1,2]. In September 2008, the crisis became visible, as stock markets worldwide crashed and entered a period of high volatility, and a considerable number of banks, mortgage lenders and insurance companies failed in the following weeks. Beginning with failures of large financial institutions in the United States, it rapidly evolved into a global credit crisis, accompanied by deflation [3], resulting in a number of prominent American and European bank failures and declines in various stock indexes, and large reductions in the market value of equities [4] and commodities worldwide [5]. These events show that our economical prosperity is insecure and that the global economy is a very sensitive system that can break down on a global scale due to a disproportionately small factor. This also leads to the question: what can be done to avoid or prevent such events in the future, or at least to reduce their consequences.

In this note we attempt to address these issues, constructing a mathematical model of a financial crisis. In contrast to verbal theories, mathematical modelling, as a tool for exploring natural- and human-created phenomena, gives a precise and explicit connection between a set of assumptions and conclusions. A good mathematical model starts with the smallest possible number of essential assumptions and follows the implications rigorously to their logical conclusions. Thus, a simple and elegant model often has greater intrinsic value than an accurate one that is overloaded with detail; in this aspect mathematical modelling is akin to Ockham's razor.

2. Model

We consider an economy as a "population" of interacting economic agents (thereafter also referred to as players, or units) of size N. These units may be viewed as participants of the economy, even if they are mere home borrowers or small-scale investors. We divide the population into two subpopulations, namely the "healthy" subpopulation of size $x(t)$, and the "activated" subpopulation of size $y(t)$. The members of the second group have financial difficulties or are unable to fulfil their financial obligations. We assume that, apart from belonging to one of these two classes, all the units are the same, and that the population is homogeneous. This assumption may appear too loose for a real economy, where the agents are of very

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different (financial) value. However, in the frames of this model a large financial institution or company can be viewed as a cluster of the basic units; then the financial circumstances of this institution may be described by the ratio of its healthy and activated components.

We assume that there is no influx of new participants into the system. That is equivalent to the assumption that the time scale of a crisis is considerably shorter than the time scale of the steady economical growth. Furthermore, we assume that the activated units can activate the healthy units, with which they are in business relationship, by failing to fulfil their financial obligations. We assume that to be activated, an average healthy unit has to come into contact with α activated units. Then, considering the process of activating as a chemical reaction $x + \alpha y \rightarrow (1+\alpha)y$, we can conclude that the activation occurs at the rate $\beta x y^{\alpha}$ (cf. [6]), where β is a positive activation rate coefficient.

After activation, the unit stays activated and can pass the active state further to healthy units. The activated state lasts for σ time units on average, and then the unit is removed from the population and does not participate in further events. Then, assuming that the active state can be transmitted to the susceptible unit at the rate β, the dynamics of the system can be described by the following differential equations:

$$\dot{x} = -\beta x y^{\alpha},$$
$$\dot{y} = \beta x y^{\alpha} - \frac{1}{\sigma} y. \tag{1}$$

The behaviour of this model is defined by the initial conditions and by three parameters, namely α, β and σ. For a real economy the parameter α, that is an average number of the activated units that are required for activating a healthy unit, must be larger than 1 because, as will be shown later, no economy with $\alpha \le 1$ can sustain. We view the parameter α as a mean value for a large population, and hence its value is not an integer but rather is a real number. For a modern economy $1 < \alpha < 10$, while for the majority of participants $1 < \alpha < 2$. The parameter α is in our control and can be enlarged (e.g. via diversification of investment or supply). The activation rate coefficient β reflects efficiency of an economy: it is proportional to the speed of propagation of information and to the rate of working capital turnover. Estimating this parameter is the most difficult task, as the data for a real crisis are necessary. The parameter σ is an average time during which an activated unit is affecting the others. For a modern economy, σ ranges from a week to a few weeks.

3. Properties of the model

We consider the dynamics of this model in the phase space: every point of the non-negative quadrant of the xy plane corresponds with a state of the economy with x healthy and y activated units (see Fig. 1). It is readily seen that $\dot{x}(t) < 0$ for all $x, y > 0$, and that $\dot{x}(t) = 0$ when $x = 0$ or $y = 0$. That is, the number of the healthy units decreases in the presence of the activated units, and remains constant in the absence of these. The health of economy is reflected by the dynamics of the activated units $y(t)$. It is easy to see that $\dot{y} = 0$ holds on the curve Γ where

$$\beta x y^{\alpha} - \frac{1}{\sigma} y = 0.$$

This curve is shown by the dashed line in Fig. 1. The curve divides the positive quadrant of the xy plane into two regions, which are denoted by A and B in Fig. 1. In the region A, $\dot{y}(t) > 0$ holds, and hence the activated subpopulation steadily grows. This growth continues until eventually a phase trajectory leaves the region A crossing the curve Γ and entering the region B. In the region B, $\dot{y}(t) < 0$, and hence the activated subpopulation decreases to zero (Fig. 1).

It is noteworthy that this behaviour is similar to an epidemic of an infectious disease [7] with the major difference that in the case of epidemics usually $\alpha = 1$ [8]. Nevertheless, this resemblance allows us to view a financial crisis as an epidemic of bankruptcies. This observation links this model in some aspects to the idea of "financial contagion", with the major difference between them that the financial contagion usually implies "the cross-country transmission of shocks or the general cross-country spillover effects", or increase of cross-country correlations [9–19], whereas in our model we are concerned with the relationship of basic agents of an economy assuming that failure of a unit occurs due to failure of its partners to fulfil obligations.

If an economy is initially positioned in the region B, then no rise of the number of the activated players occurs, and the system tends to the state with $y = 0$, where it remains. A crisis of bankruptcies starts only when the system gets into the region A, whereas a healthy economy is positioned in B and may remain there indefinitely. However, if in the xy plane an economy is positioned sufficiently close to the curve Γ, then a stochastic fluctuation can send the system across the boundary Γ into the region A initiating a cascade of financial failures. That is, to prevent a crisis it suffices to keep the system reasonably far from the boundary Γ. The problem is, however, that the boundary Γ tends to the x axis as x grows, as can be seen in Fig. 1. This implies that for a given set of parameters α, β and σ and for any given magnitude of fluctuation there is a population size $x(0)$ such that the fluctuation of this given magnitude will certainly bring the system across the boundary Γ into the crisis region A.

Indeed, the curve Γ is given by the equation

$$y = 1/(\sigma \beta x)^{\frac{1}{\alpha - 1}}. \tag{2}$$

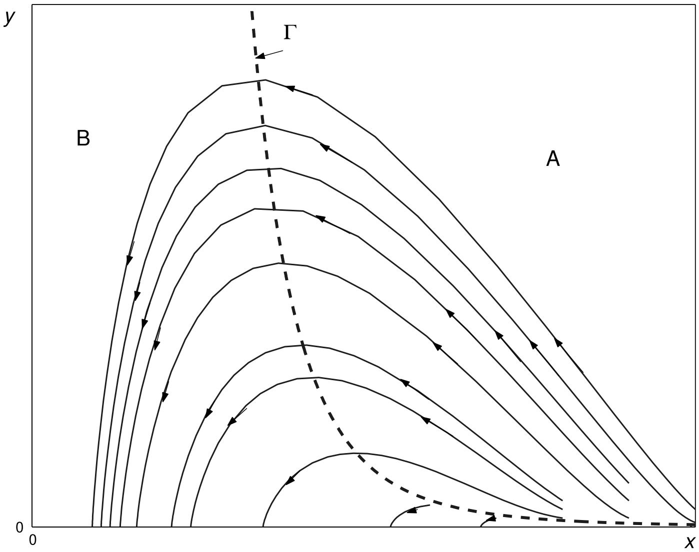

Fig. 1. Schematic phase portrait of the system (1). Here, A is the crisis region, B is the stable economy region, and Γ is the boundary between the two regions.

Assuming that the initial position of the economy is on or near the x axis (that is initially $y(0) = 0$ or very small), we obtain that a fluctuation of a given magnitude Δy certainly shifts an economy of size $x(0)$, that is larger than a critical size $x_{cr} = \left(\sigma\beta\Delta y^{\alpha-1}\right)^{-1}$, into the crisis region. This also implies that if the values of α, β and σ do not depend on the economy size, then a larger economy is more vulnerable. Fig. 1 also gives a general idea what can be done to avoid a crisis. It is easy to see that the higher and further to the right the curve Γ is in the xy plane the safer the economy is. That is, to reduce the likelihood and the consequence of a crisis, it is advisable to take measures that shift the curve Γ higher and to the right in the xy plane. However, that may be a rather difficult and expensive task.

The properties of this model is defined by the initial conditions and by three parameters, namely α, β and σ, where the parameters β and σ are coming as a product, $\rho = \sigma\beta$. For a given value of x (given the economy size), the larger the value of $\rho = \sigma\beta$ is, the lower the curve Γ is placed. As we mentioned earlier, β reflects efficiency of an economy: it is proportional to the speed of propagation of information and to the rate of working capital turnover. Therefore, artificial reduction of this rate is meaningless and hardly possible.

The parameter α is an average number of the activated contacts (the failed partners) that are required to activate a healthy unit. It may be anticipated that the higher the value of α is, the larger the safety margins are. Eq. (2) confirms this deduction. An increase in α can be achieved, for instance, by diversification of investment and supply. However, while the increasing α by all means is advisable, we have to note that increasing the safety margins by increasing α leads to a limited success and may be costly. Fig. 2 shows the safety margin as a function of α for a range of values of $D = \beta\sigma x = \rho x$. It is readily seen that for smaller α the increment of α leads to a remarkable improvement of safety. However, it is obvious from this picture that the safety margin has an upper limit ($y = 1$ in Fig. 2) which does not depends on α; when this limit is approached a further increase of α gives only negligible increase of the safety margin. For the larger values of D (that is for the larger population sizes) the possibility to improve the safety is preserved for larger α. However the general tendency that a large increase can be reached for only comparatively low α holds here as well. Taking into consideration that increasing α beyond some level can be costly, and that the cost grows with α faster than linearly, we come to the conclusion that improving economic safety by mean of increasing α has its limitation and may be a costly option.

Fig. 3 shows why an economy with $\alpha \le 1$ cannot exist. It is obvious from Fig. 3 that in the case $\alpha < 1$ the curve Γ is a parabola of degree $1/(1-\alpha) > 1$, and hence there is no safety margin at all. In this case for any economy size the slightest perturbation causes a crisis. Moreover, it can be seen from Fig. 3 that in this case all the healthy units will be destroyed by a crisis as the population that survive though the crisis in this case is zero. In the case $\alpha = 1$ there also is no safety margin for an economy of size larger than $\rho = \beta\sigma$, and for such an economy a crisis starts at a slightest perturbation. It is obvious that in any case an economy with $\alpha \le 1$ is unsustainable. For a modern economy $1 < \alpha < 10$; however the author believes that in the most cases $1 < \alpha < 2$. Therefore, there is a fair possibility for increasing personal and collective economic safety via diversification of investment or supply.

The parameter σ, that is an average time during which an activated unit is affecting the others, can be the primary means of controlling a crisis: an intervention aimed to reduce this time can slow down the crisis and considerably reduce its consequence. Indeed,

$$x = \frac{1}{\beta\sigma} y^{1-\alpha},$$

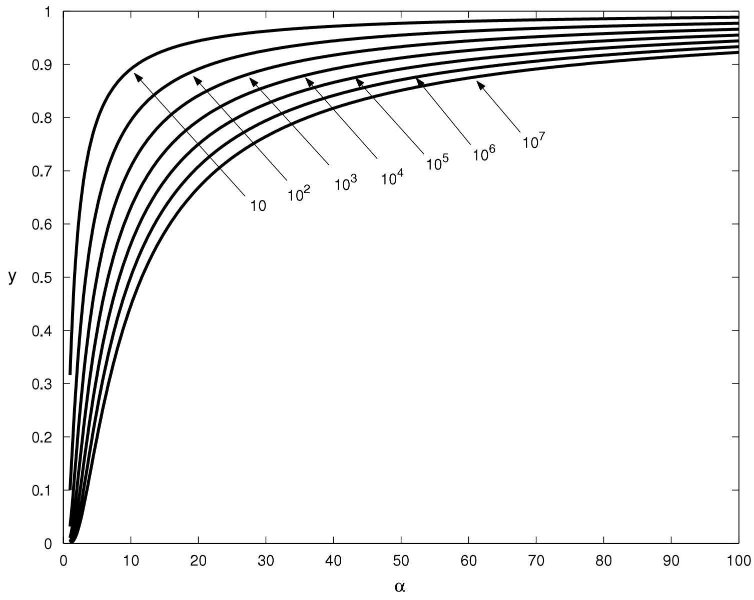

Fig. 2. Safety margin as a function of the parameter α for $D = \beta\sigma x = \rho x$ ranging from 10 to 10^7.

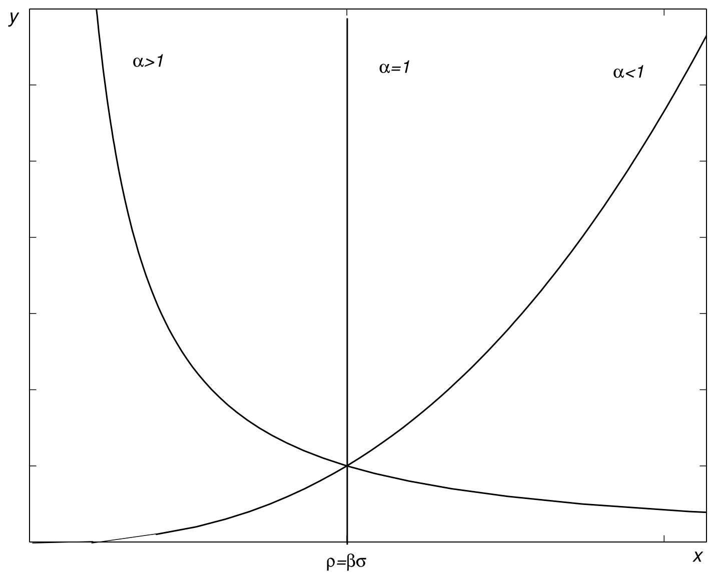

Fig. 3. Curves Γ for $\alpha > 1$, $\alpha = 1$ and $\alpha < 1$, respectively.

and hence decreasing σ is equivalent to the proportionate shifting of the curve Γ to the right. Decreasing σ means that the activated units have to be removed from the system as quickly as possible, and ideally before they actually cause damage to the others. Assuming that the cost of such removal does not depends on the moment of the removal, the most cost effective action is to remove an activated unit immediately after activation and before it causes any further damage. The problem is, however, that the cost of such intervention can be very high. An obvious action that decreases σ is bailing out the hopeless debts [20], and it is obvious that this involved very large expenses. A specific study is needed for every particular case. However, it appears that in the frame of this model an intervention aimed at the reduction of σ is the only option that is available after a crisis has begun. This also indicates that whatever action is chosen it should be swift: to minimize the further damage, a decision to remove the activated players and the action itself should be carried out as quickly as possible.

There is a factor that we have to mention since this particular factor can considerably decrease the safety margin and at the same time moves the economy in xy plane closer to the unsafe boundary. This factor is fraud. A fraudulent company is essentially a parasite that is able to affects (activate) other units while preserving its own activity for longer. In the frame of this model, a fraudulent unit should be classified into the y class, with the difference from the other units of this class in that a fraudulent company belongs to this class from the very beginning of its existence, whereas the rest of the y class units are the consequence of shifting from the x class to the y class as a result of sincere failure or of activation via a contact with y class units. That is, the presence of the malicious units gives a non-zero initial condition for $y(t)$ and places an economy above the x axis. Furthermore, the intention of a fraudulent company is to prolong its existence as long as possible (ideally indefinitely) and to attack as many of other participants as possible. Therefore, for a malicious unit the values of β and σ are considerably larger than these for an average unit. This increases the average values of β and σ of the whole system shifting the curve Γ down in xy plane and hence reducing the safety margin and making the economy more vulnerable. That is, the presence of the malicious units has a dual effect: it moves up the position of the economy on the xy plane, and at the same time it lowers the position of the curve Γ reducing the safety margin. This consideration indicates how dangerous fraudulent companies can be in reality, and how important detecting and removing them in time is.

4. Conclusion

In conclusion the author would like to note that of course this model is a great oversimplification and lacks many features of a real economy. However, it must be kept in consideration that the complexity of a real economy is such that constructing a model of reasonable accuracy, which can be used for quantitative prediction, is a task of tremendous difficulty. A real economy is a tremendously complex system that depends on a very large number of parameters and can be influenced by a vast variety of factors. Many of these factors and parameters are unknown or even in principal unmeasurable. Furthermore, as it may be a priori assumed or concluded from observing the ongoing crisis, and it was shown in this note, a crisis is an unstable process. One hardly can expect that a precise or reasonably accurate quantitative description for such a process is possible at all.

Under such circumstances, constructing and exploiting a simple conceptual model that is aimed at qualitative description rather than at obtaining precise quantitative predictions, is well justified. The author's aim was to construct such a qualitative model that can give an insight into the developments and the outcomes of financial crises. The author's immediate task was to show that such modelling is, in principal, possible and to indicate an avenue for this modelling, and also to select the factors that are of principal importance and affect the dynamics most.

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References

[1] F. Norris, A new kind of bank run tests old safeguards, New York Times (August 10) (2007). http://www.nytimes.com/2007/08/10/business/10liquidity.html.
[2] L. Elliott, Credit crisis — How it all began, Guardian (August 5) (2008). http://www.guardian.co.uk/business/2008/aug/05/northernrock.banking.
[3] V. Bajaj, Stocks are hurt by latest fear: Declining prices, New York Times (November 20) (2008). http://www.nytimes.com/2008/11/20/business/economy/20econ.html.
[4] F. Norris, United panic, New York Times (October 24) (2008). http://norris.blogs.nytimes.com/2008/10/24/united-panic.
[5] A. Evans-Pritchard, Dollar tumbles as huge credit crunch looms. Telegraph.co.uk (Telegraph Media Group Limited) September 7, 2007. http://www.telegraph.co.uk/money/main.jhtml?xml=/money/2007/07/25/cnusecon125.xml.
[6] J. Keener, J. Sneyd, Mathematical Physiology, Springer, New York, 2009.
[7] W.O. Kermack, A.G. McKendrick, A contribution to the mathematical theory of epidemics, Proceedings of the Royal Society of London, Series A 115 (1927) 700–721.
[8] R.M. Anderson, R.M. May, Population biology of infectious diseases: Part I, Nature 280 (1979) 361–367.
[9] M. Dungey, D.N. Tambakis (Eds.), Identifying International Financial Contagion : Progress and Challenges, Oxford University Press, 2005.
[10] World Bank Group. Definitions and Causes of Contagion, Sept 15, 2000. http://www1.worldbank.org/economicpolicy/managing%20volatility/contagion/definitions.html.
[11] T. Baig, I. Goldfajn, Financial market contagion in the Asian crisis, International Monetary Fund Staff Papers 46 (2) (1999) 167–195.
[12] K.J. Forbes, R. Rigobon, No contagion, only interdependence: measuring stock market comovements, Journal of Finance 57 (5) (2002) 2223–2261.
[13] S. Claessens, K. Forbes, International Financial Contagion, Kluwer Academic, New York, 2001.
[14] B. Eichengreen, A. Rose, C. Wyplosz, Contagious currency crises: First tests, Scandinavian Journal of Economics 98 (1996) 463–484.
[15] R. Glick, A.K. Rose, Contagion and trade, Journal of International Money and Finance 18 (1999) 603–617.
[16] P. Masson, Contagion, Journal of International Money and Finance 18 (1999) 587–602.
[17] G.L. Kaminsky, C. Reinhart, On crises, contagion and confusion, Journal of International Economics 51 (2000) 145–168.
[18] F. Allen, D. Gale, Financial contagion, Journal of Political Economy 108 (2000) 1–34.
[19] A.S. Kyle, W. Xiong, Contagion as a wealth effect, Journal of Finance 56 (2001) 1401–1440.
[20] E.L. Andrews, Tracking the Bailout, New York Times (November 25) (2008). http://www.nytimes.com/2008/11/26/business/economy/26fed.html.