The Economics of Love

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The Romantic Search for Love

The Realistic Search for Love (??)

People's Park, Shanghai

Finding Love as an Optimal Stopping Problem

- Suppose you are a woman. Imagine you are going to meet n men in your life, but you can only pick one.
- You meet with each man sequentially. Before meeting a man, you do not know good he is. After meeting him, you can rank him against the other guys you have met, and have to decide whether or not to accept him.
- If you accept him, your "search" is stopped and you will not meet with the rest.
- If you reject him, you can go on to meet the next guy, but you will never be able to return to the ones you have already met and rejected.

Finding Love as an Optimal Stopping Problem

Goal

To maximize the probability of finding ***the best*** *candidate*

- Problem: when do you stop?
  - Stopping too soon: you have not yet met the best candidate available.
  - Stopping too late: you have already met (and rejected) the best candidate available.
- Mathematically, this is called an **optimal stopping problem.**

Finding Love as an Optimal Stopping Problem

- Don't stop too early...

  还君明珠双泪垂，恨不相逢未嫁时

- Don't stop too late...

  有花堪折直须折, 莫待无花空折枝

The Optimal Stopping Problem

Assumption

1. *Candidates arrive randomly. Each has a $\frac{1}{n}$ probability of being the best out of the n candidates.*
2. *You do not know the distribution of candidate quality, but after meeting a candidate, you are able to rank him against the candidates you have met before.*

Solution

Consider strategies of the following type: reject the first $k - 1$ candidates, then select the first candidate who is better than all of the previous candidates.

The Optimal Stopping Problem

For $k \geq 2$, if the best candidate arrives on the j^{th}, then

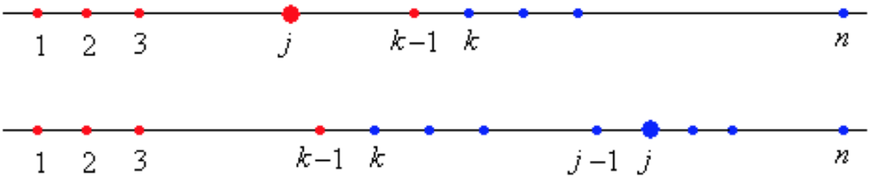
- If $j \leq k-1$,

$$\Pr(\text{choosing } j \text{ if } j \text{ is the best candidate}) = 0$$

- If $k \leq j \leq n$,

$$\begin{aligned} & \Pr(\text{choosing } j \text{ if } j \text{ is the best candidate}) \\ = & \Pr(\text{the best candidate in } \{1, \ldots, j-1\} \in \{1, \ldots, k-1\}) \\ = & \frac{k-1}{j-1} \end{aligned}$$

The Optimal Stopping Problem



If $j \geq k$, then in order for j to be chosen, the best candidate among $1, \ldots, k-1$ must also be the best candidate among $1, \ldots, j-1$

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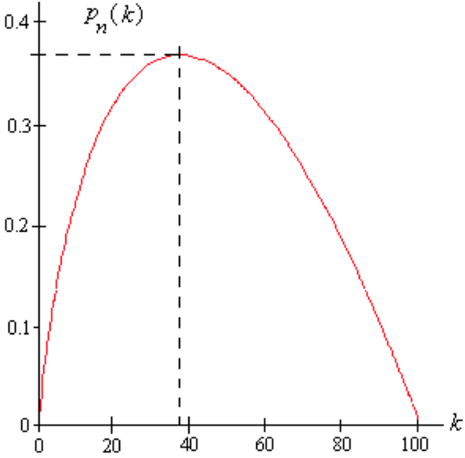
The Optimal Stopping Problem

For $k \geq 2$,

$$\begin{aligned}
p_n(k) &\equiv \Pr(\text{choosing the best candidate}) \\
&= \sum_{j=1}^{n} \{\Pr(j \text{ is the best candidate}) \times \\
&\quad \Pr(\text{choosing } j \text{ if } j \text{ is the best candidate})\} \\
&= \frac{1}{n} \sum_{j=k}^{n} \frac{k-1}{j-1}
\end{aligned}$$

The Optimal Stopping Problem



The Optimal Stopping Problem

$$p_n(k) = \frac{1}{n}\sum_{j=k}^{n}\frac{k-1}{j-1} = \frac{k-1}{n}\sum_{j=k}^{n}\frac{1}{n}\frac{1}{\frac{j-1}{n}}$$
$$\approx \frac{k-1}{n}\int_{\frac{k-1}{n}}^{1}\frac{1}{x}dx = -\frac{k-1}{n}\ln\left(\frac{k-1}{n}\right)$$

When n is large,

$$p_n(k) \to p(y) = -y\ln y, \quad \text{where } y \equiv \frac{k}{n}$$

$\Rightarrow$

$$y^* = \arg\max_y p(y) = \frac{1}{e}$$
$$p(y^*) = -\left(\frac{1}{e}\right)\ln\left(\frac{1}{e}\right) = \frac{1}{e}$$

The Optimal Stopping Problem

Conclusion

Both optimal $\frac{k}{n}$ and optimal probability $= \frac{1}{e} \approx 0.37$

- For large n, it is optimal to reject the first 37% of the candidates and then select the first candidate (if appears) that is better than all of the previous candidates.
- The probability of finding the best candidate using this strategy is also 37%.

This is known as **the 37 percent rule**.

The Optimal Stopping Problem

More general versions of the optimal stopping problem can be used to study:

- How many job candidates should an employer interview for a given position?
- When is it time to introduce a new product to the market?
- When is it optimal to buy or sell an asset?

The Stable Matching Problem

The Stable Matching Problem

- There are n men and n women. Each has a different preference over individuals of the opposite sex[1].
- An individual A is **acceptable** to an individual B of the opposite sex if B prefers A to being unmatched (single).
- A pair (m, w) is called a **blocking pair** if m prefers w to his current partner and w prefers m to her current partner.
- A matching is **unstable** if (1) someone has an unacceptable partner, or (2) there is a blocking pair. Otherwise, the matching is **stable**.

[1] We only consider heterosexual matching problem here.

The Stable Matching Problem

- A man and a woman are **possible** for each other if there is some stable matching in which they are paired.
- A matching is **man-optimal** if it is stable and there is no other stable matching that at least some man prefers.
  - In a man-optimal matching, every man gets his best possible match.
- A matching is **woman-optimal** if it is stable and there is no other stable matching that at least some woman prefers.
  - In a woman-optimal matching, every woman gets her best possible match.

The Stable Matching Problem

Example

- Participants: men m and $m\prime$ and women w and $w\prime$.
- Example 1:
  - m prefers w to w'; m' prefers w' to w.
  - w prefers m to m'; w' prefers m' to m.
  - Only stable matching is $\{(m, w), (m', w')\}$.
- Example 2:
  - m prefers w to w'; m' prefers w' to w.
  - w prefers m' to m; w' prefers m to m'.
  - Stable matchings: $\{(m, w), (m', w')\}$ and $\{(m, w'), (m', w)\}$.
    - First matching is man-optimal (and woman-pessimal).
    - Second matching is woman-optimal (and man-pessimal)
- Conclusion: There are sometimes multiple stable matchings.
- Question: Is there always at least one? How do we find it?

The Gale-Shapley Algorithm

Problem (The stable marriage problem)

Find a stable matching for any dating pool.

Algorithm (The deferred-acceptance (Gale-Shapley) algorithm)

- *Day* 0*: each individual ranks the opposite sex*
- *Day* 1*:*
  - *(a.m.) each man proposes to his top choice*
  - *(p.m.) each woman rejects all but her top suitor*
- *Day* $n + 1$*:*
  - *(a.m.) each man rejected on day* n *proposes to his top remaining choice*
  - *(p.m.) each woman rejects all but her top suitor*
- *When each man is engaged, the algorithm terminates*

The Gale-Shapley Algorithm

Ken	Bev	Cat	Ada
Leo	Ada	Cat	Bev
Max	Ada	Bev	Cat

Ada	Ken	Leo	Max
Bev	Leo	Max	Ken
Cat	Max	Leo	Ken

Ken	$\boxed{\text{Bev}}$	Cat	Ada
Leo	$\boxed{\text{Ada}}$	Cat	Bev
Max	$\boxed{\text{Ada}}$	Bev	Cat

Ada	Ken	$\boxed{\text{Leo}}$	$\boxed{\text{Max}}$
Bev	Leo	Max	$\boxed{\text{Ken}}$
Cat	Max	Leo	Ken

Day 1:

- *Leo & Max propose to Ada. Ken proposes to Bev.*
- *Ada rejects Max.*
- *Ada & Leo and Bev & Ken are engaged.*

The Gale-Shapley Algorithm

Ken	Bev	Cat	Ada
Leo	Ada	Cat	Bev
Max	~~Ada~~	Bev	Cat

Ada	Ken	Leo	~~Max~~
Bev	Leo	Max	Ken
Cat	Max	Leo	Ken

Day 1:

- *Leo & Max propose to Ada. Ken proposes to Bev.*
- *Ada rejects Max.*
- *Ada & Leo and Bev & Ken are engaged.*

The Gale-Shapley Algorithm

Ken	$\boxed{\text{Bev}}$	Cat	Ada
Leo	$\boxed{\text{Ada}}$	Cat	Bev
Max	~~Ada~~	$\boxed{\text{Bev}}$	Cat

Ada	Ken	$\boxed{\text{Leo}}$	Max
Bev	Leo	$\boxed{\text{Max}}$	$\boxed{\text{Ken}}$
Cat	Max	Leo	Ken

Ken	~~Bev~~	Cat	Ada
Leo	$\boxed{\text{Ada}}$	Cat	Bev
Max	~~Ada~~	$\boxed{\text{Bev}}$	Cat

Ada	Ken	$\boxed{\text{Leo}}$	Max
Bev	Leo	$\boxed{\text{Max}}$	~~Ken~~
Cat	Max	Leo	Ken

Day 2:

- *Max proposes to Bev.*
- *Bev rejects Ken.*
- *Ada & Leo and Bev & Max are engaged.*

The Gale-Shapley Algorithm

Ken	~~Bev~~	$\boxed{\text{Cat}}$	Ada
Leo	$\boxed{\text{Ada}}$	Cat	Bev
Max	~~Ada~~	$\boxed{\text{Bev}}$	Cat

Ada	Ken	$\boxed{\text{Leo}}$	Max
Bev	Leo	$\boxed{\text{Max}}$	~~Ken~~
Cat	Max	Leo	$\boxed{\text{Ken}}$

Day 3:

- *Ken proposes to Cat.*
- *It's a match: Ada & Leo, Bev & Max, Cat & Ken.*

The Gale-Shapley Algorithm

Theorem

The Gale-Shapley algorithm produces stable matching.

Theorem

The Gale-Shapley algorithm in which man proposes produces man-optimal (and woman-pessimal) matching. The Gale-Shapley algorithm in which woman proposes produces woman-optimal (and man-pessimal) matching.

- Can individuals improve their results by not proposing or rejecting according to their true preferences?

The Gale-Shapley Algorithm

Definition

A mechanism is **strategy-proof** if it is always in the best interest of the individuals to truthfully report, or act according to, their preferences.

Theorem

The man-proposing Gale-Shapley algorithm is strategy-proof for men, but not for women.

Theorem

There exists no mechanism that always produces stable matching and is strategy-proof for all participants.

The Gale-Shapley Algorithm

Example

Three men (贾宝玉, 许仙, 李逵) and three women (林黛玉, 花木兰, 潘金莲) have the following preferences for each other:

贾宝玉 (Jeremy):	1. Diana;	2. Mulan;	3. Janice
许仙 (Sheldon):	1. Diana;	2. Janice;	3. Mulan
李逵 (Gary):	1. Janice;	2. Mulan;	3. Diana

林黛玉 (Diana):	1. Gary;	2. Jeremy;	3. Sheldon
花木兰 (Mulan):	1. Sheldon;	2. Jeremy;	3. Gary
潘金莲 (Janice):	1. Jeremy;	2. Sheldon;	3. Gary

Find all stable matchings.

The Gale-Shapley Algorithm: Applications

- The Boston and NY School Systems
  - Assigning children to schools
- The National Resident Matching Program (NRMP)
  - Assigning medical school students to hospitals