
# Algorithms and Complexity

## Introduction to complexity analysis

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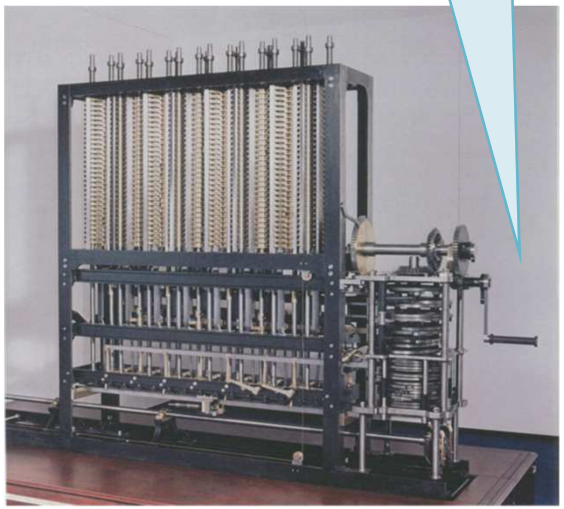
# Complexity Analysis

- Target
  - Predict performance
  - Compare algorithms
  - Provide guarantees

How many times do you have to turn the crank?

*" As soon as an Analytic Engine exists, it will necessarily guide the future course of the science. Whenever any result is sought by its aid, the question will arise. By what course of calculation can these results be arrived at by the machine in the shortest time? "*

Charles Babbage (1864)



# Complexity Analysis

- The challenge
  - Will my program be able to solve large practical problem?

Why is my program so slow?
Why does it run out of memory?

  - "Client gets poor performance because programmer did not understand performance characteristics"

# Complexity Analysis

- Modelling the problem
  - Given an algorithm (or a program written in a specific language)
  - Forecast of the resources the algorithm required to be executed
  - Type of resources
    - Time
    - Memory
- We should be able to prove that
  - A lower complexity may compensate hardware efficiency

# Complexity Analysis

- To really understand programs behavior we have to develop a mathematical model
- This model is usually based on the assumption the program runs on a traditional architecture
  - Sequential and single-processor model
- The model has to be
  - Independent on the hardware (CPU, memory, etc.)
  - Independent of the input data of a particular instance of the problem
    - We may eventually analyze best, average, and worst cases

# Complexity Analysis

- Our model will depend on the size **n** of the problem
- Examples
  - Number of bits of the operands for an integer multiplication
  - Number of data to sort for a sorting algorithm
  - Etc.
- Our analysis should give indications on the
  - Execution time → T(n)
  - Memory occupation → S(n)

Time and Space complexity

# Execution Time Analysis

- Donal Knuth (late '60)
  - T(n) = «number of operations» · «operation cost»
  - Thus we must
    - Evaluates the frequency of all operations
    - Evaluates the cost of each operations

Program dependent

Hardware and software dependent

# A Simple Counting Problem

- Write a program able to
  - Read an integer value **n**
  - Print-out the number **sum** of ordered couples (i, j) such that the two following conditions hold
    - i and j are integer values
    - $1 \leq i \leq j \leq n$
- Example
  - Input: **n** = 4
  - Generated couples
    - (1,1)(1,2)(1,3)(1,4) (2,2)(2,3)(2,4) (3,3)(3,4) (4,4)
  - Output: **sum** = 10

# Algorithm 1: Brute-force

```
int count_ver1 (int n) {
  int i, j, sum;

  sum = 0;

  for (i=1; i<=n; i++) {

    for (j=i; j<=n; j++) {

      sum++;

    }
  }

  return sum;
}
```

It generates all pairs:
$1 \le i \le j \le n$

It counts-them up

It returns the result

Observe that the cycle
for (i=S; i<E, i++)
performs
E-S iterations AND E-S+1 checks

# Algorithm 1: Brute-force

```
int count_ver1 (int n) {
  int i, j, sum;

  sum = 0;

  for (i=1; i<=n; i++) {

    for (j=i; j<=n; j++) {

      sum++;

    }
  }

  return sum;
}
```

1

$1 + (n+1) + n$

$$\sum_{i=1}^{n} [1 + (n-i+2) + (n-i+1])$$

$$\sum_{i=1}^{n} (n-i+1)$$

1

We can evaluate the exact number of operartions performed

We suppose ALL operations have the same constant cost (unit cost)

# Algorithm 1: Brute-force

$$T(n) = 4 + 2n + \sum_{i=1}^{n} (5 + 3n - 3i)$$

$$1$$

$$1 + (n+1) + n$$

$$\sum_{i=1}^{n} [1 + (n - i + 2) + (n - i + 1])$$

$$\sum_{i=1}^{n} (n - i + 1)$$

$$1$$

$$T(n) = 4 + 2n + \underbrace{\sum_{i=1}^{n} (5)}_{5n} + \underbrace{\sum_{i=1}^{n} (3n)}_{3n^2} - \sum_{i=1}^{n} (3i)$$

$$T(n) = 4 + 7n + 3n^2 - 3\sum_{i=1}^{n} i$$

Finite arithmetic progression

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

$$T(n) = 4 + 7n + 3n^2 - 3\frac{n(n+1)}{2}$$

$$\boldsymbol{T(n) = 1.5n^2 + 5.5n + 4}$$

Quadratic behaviour

# Algorithm 2

```
int count_ver1 (int n) {
  int i, j, sum;
  sum = 0;
  for (i=1; i<=n; i++) {
    for (j=i; j<=n; j++) {
      sum++;
    }
  }
  return sum;
}
```

It generates all pairs:
$1 \le i \le j \le n$

```
int count_ver2 (int n) {
  int i, sum;

  sum = 0;

  for (i=1; i<=n; i++) {
    sum = sum + (n-i+1);
  }

  return sum;
}
```

# Algorithm 2

```
int count_ver2 (int n) {
  int i, sum;

  sum = 0;

  for (i=1; i<=n; i++) {
    sum = sum + (n-i+1);
  }

  return sum;
}
```

$1$

$1 + (n + 1) + n$

$$\sum_{i=1}^{n} (4)$$

$1$

# Algorithm 2

$$T(n) = 1 + 1 + 1 + n + n + 4n$$

$$\boldsymbol{T(n) = 6n + 4}$$

1

$1 + (n + 1) + n$

$$\sum_{i=1}^{n}(4) = 4n$$

1

Linear behaviour

# Algorithm 3

```
int count_ver2 (int n) {
  int i, sum;

  sum = 0;

  for (i=1; i<=n; i++) {
    sum = sum + (n-i+1);
  }

  return sum;
}
```

- The for cycle computes
  - $\sum_{i=1}^{n}(n-i+1) =$

$= n^2 + n - \sum_{i=1}^{n} i =$

$= n(n+1) - \frac{n(n+1)}{2}$

$= \frac{n(n+1)}{2}$

# Algorithm 3

- The for cycle computes
  - $\sum_{i=1}^{n}(n-i+1)=\frac{n(n+1)}{2}$
  - Which can be used to substitute the entire cycle

```
int count_ver2 (int n) {
  int i, sum;
  sum = 0;
  for (i=1; i<=n; i++) {
    sum = sum + (n-i+1);
  }
  return sum;
}
```

It generates all pairs:
1 ≤ i ≤ j ≤ n

```
int count_ver3 (int n) {
  return n * (n+1) / 2;
}
```

# Algorithm 3

- The for cycle computes
  - $\sum_{i=1}^{n}(n-i+1) = \frac{n(n+1)}{2}$
  - Which can be used to substitute the entire cycle

```
int count_ver3 (int n) {
  return n * (n+1) / 2;
}
```

4

$\boldsymbol{T(n) = 4}$

Constant behaviour

# Summary

| Algorithm | T(N) | Order of T(N) |
|---|---|---|
| Version 1 | $1.5n^2 + 5.5n + 4$ | $n^2$ |
| Version 2 | $6n + 4$ | $n$ |
| Version 3 | $4$ | constant |

Algorithm Classification

| Asymptotic behavior | Algorithm Class |
|---|---|
| 1 | Constant |
| log n | Logarithmic |
| n | Linear |
| n log n | Linearithmic |
| $n^2$ | Quadratic |
| $n^3$ | Cubic |
| $2^n$ | Exponential |

Complexity grows much faster than the input size

# Summary

A lower complexity may really compensate hardware efficiency !

- Hypothesis
  - 1 operation = 1 nsec = $10^{-9}$ sec

Wall-clock (elapsed) time

| Asymptotic behavior | $10^3$ | $10^4$ | $10^5$ | $10^6$ | $10^7$ |
|---|---|---|---|---|---|
| n | 1μs | 10μs | 100μs | 1ms | 10ms |
| 20 n | 20μs | 200μs | 2ms | 20ms | 200ms |
| n log n | 9.96μs | 132μs | 1.66ms | 19.9ms | 232ms |
| 20 n log n | 199μs | 2.7ms | 32ms | 398ms | 4.6sec |
| $n^2$ | 1ms | 100ms | 10s | 17min | 1.2day |
| 20 $n^2$ | 20ms | 2s | 3.3min | 5.6h | 23day |
| $n^3$ | 1s | 17min | 12day | 32years | 32 millenium |

# Some more examples

- Discrete Fourier Transform
  - Decomposition of a N-sample waveform into periodic components
  - Applications: DVD, JPEG, astrophysics, ….
  - Trivial algorithm: Quadratic ($n^2$)
  - FFT (Fast Fourier Transform): Linearitmic ($n \cdot \log n$)
- Simulation of N bodies
  - Simulates gravity interaction among n bodies
  - Trivial algorithm: Quadratic ($n^2$)
  - Barnes-Hut algorithm: Linearitmic ($n \cdot \log n$)

# Asymptotic Analysis

- Goal
  - Guess an upper-bound for T(n) for an algorithm on n data in the worst possible case
  - Asymptotic
    - For small n, complexity is irrelevant
    - Understand behaviour for $n \rightarrow \infty$

"Order or growth" classification is very important

# Asymptotic Analysis

- Three main analysis
  - Worst case
  - Average case
  - Best case
- Why worst-case analysis?
  - Conservative guess
    - Avoid complex hyphoteses on data
  - Worst case is very frequent
  - Average (and best) case
    - Either it coincides with the worst case
    - It is not definable, unless we resort to complex hypotheses on data

Running time is going somewhere in between

Design for the worst case

# Tilde Notation

- Estimate running time (or memory) as a function of input size n
  - Analyze to "within a constant factor"
- Ignore lower order terms
  - When N is large, terms are negligible
  - When N is small, terms are not negligible but we do not care about them
- Definition

$$f(n) \sim g(n) \Leftrightarrow \lim_{n \to \infty} \frac{f(n)}{g(n)} = 1$$

# Tilde Notation

- Examples
  - $\frac{1}{6}n^3 + 2n + 16 \sim \frac{1}{6}n^3$
  - $\frac{1}{6}n^3 + 100n^{4/3} + 16 \sim \frac{1}{6}n^3$
  - $\frac{1}{6}n^3 + \frac{5}{12}n^2 + 16 \sim \frac{1}{6}n^3$

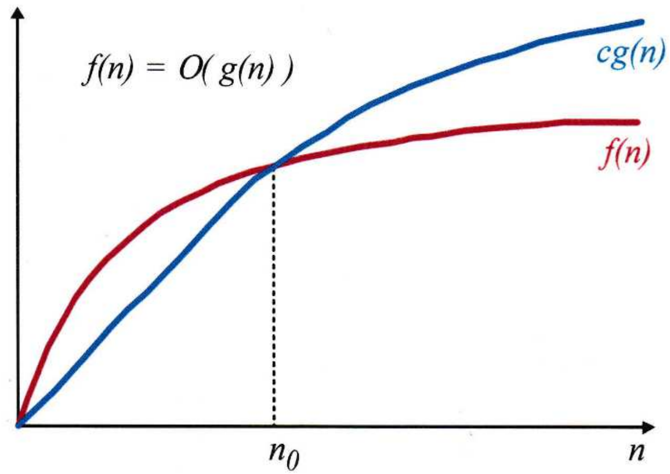
# O Asymptotic Notation

- Definition

$$f(n) = O(g(n)) \Leftrightarrow \exists\ c>0,\ \exists\ n_0>0 \text{ such that } \forall n \geq n_0$$
$$0 \leq f(n) \leq cg(n)$$

g(n) = loose upper bound for f(n)

Big-Oh Notation
Develops upper bounds



# O Asymptotic Notation

- Examples
  - $T(n) = 3n+2 = O(n)$
    - $c=4$ and $n_0=2$
  - $T(n) = 10n^2+4n+2 = O(n^2)$
    - $c=11$ and $n_0=5$

## Theorem

- If $T(n) = a_m n^m + \ldots. + a_1 n + a_0$
  - Then $T(n) = O(n^m)$

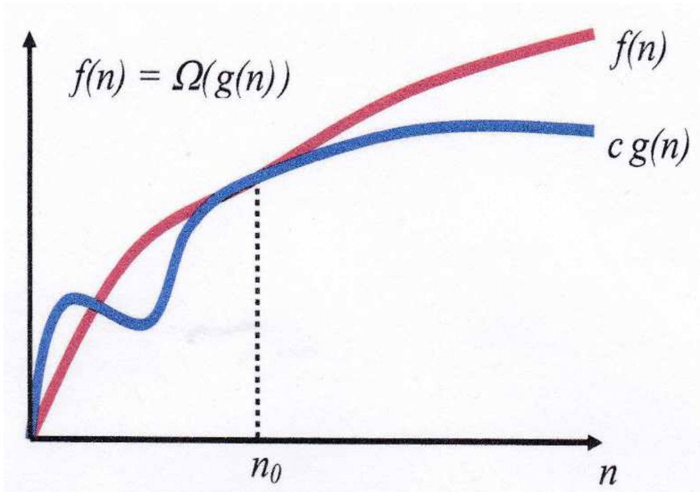
# Ω Asymptotic Notation

- Definition

$$f(n) = \Omega(g(n)) \Leftrightarrow \exists\ c>0,\ \exists\ n_0>0 \text{ such that } \forall n \geq n_0$$
$$0 \leq c\ g(n) \leq f(n)$$

g(n) = loose lower bound for f(n)

Big-Omega Notation Develops lower bounds



# Ω Asymptotic Notation

- Examples
  - $T(n) = 3n+3 = \Omega(n)$
    - $c=3$ and $n_0=1$
  - $T(n) = 10n^2+4n+2 = \Omega(n^2)$
    - $c=1$ and $n_0=1$

## Theorem

- If $T(n) = a_m n^m + .... + a_1 n + a_0$
  - Then $T(n) = \Omega(n^m)$

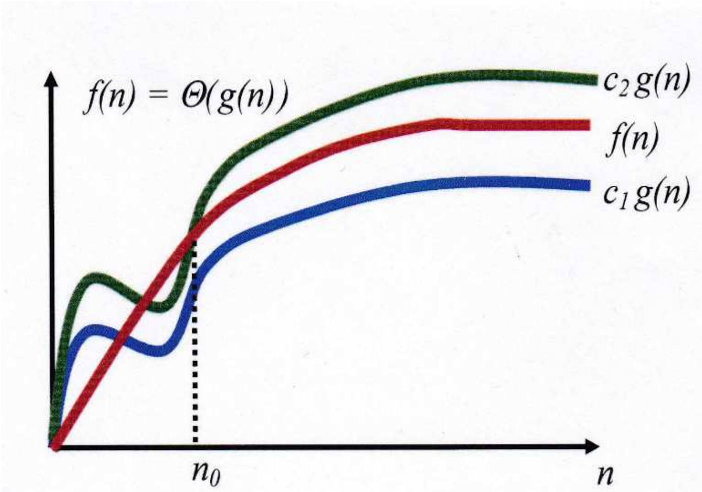
# Θ Asymptotic Notation

- Definition

$$f(n) = \Theta(g(n)) \Leftrightarrow \exists\ c_1, c_2 > 0,\ \exists\ n_0 > 0 \text{ such that } \forall n \geq n_0$$
$$0 \leq c_1\ g(n) \leq f(n) \leq c_2\ g(n)$$

g(n) = tight asymptotic bound for f(n)

Big-Theta Notation
Classify algorithms
Asymptotic order of growth



# $\Theta$ Asymptotic Notation

- Examples
  - $T(n) = 3n+2 = \Theta(n)$
    - $c1=3$, $c2=4$ and $n_0=2$
  - $T(n) = 3n+2 \neq \Theta(n^2)$
  - $T(n) = 10n^2+4n+2 \neq \Theta(n)$

## Theorem

- If $T(n) = a_m n^m + .... + a_1 n + a_0$
  - Then $T(n) = \Theta(n^m)$

# Theorems

- Given two functions f(n) and g(n)
  - $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0 \Rightarrow$ f(n) = O(g(n))
  - $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \infty \Rightarrow$ f(n) = Ω(g(n))
  - $\lim_{n\to\infty} \frac{f(n)}{g(n)} = const \Rightarrow$ f(n) = Θ(g(n))
  - f(n) = Θ(g(n)) ⇔ g(n) = Θ(f(n))
  - f(n) = Θ(g(n)) ⇔

    f(n) = O(g(n)) and f(n) = Ω(g(n))
  - etc.

# Memory Occupation Analysis

| Basics Objects | Size |
|---|---|
| Bit | 0 or 1 |
| Byte | 8 bits |
| 1KByte | $2^{10}$ Byt2 (1 thousand) |
| 1MByte | $2^{20}$ Bytes (1 million) |
| 1GByte | $2^{30}$ Byte (1 billion) |

| C scalar Type | sizeof(type) |
|---|---|
| char | 1 byte |
| int | 4 Bytes |
| float | 4 Bytes |
| double | 8 Bytes |
| etc. | etc. |

Padding may be used, i.e., each object uses a multiple of 4/8 bytes

# Memory Occupation

- Memory occupation from aggregate types may be computed starting from scalar types

```
int vet[N];

struct type {
  char id[N];
  int i;
  float x;
};
```

N · sizeof (int)

N · sizeof (char) + sizeof (int) + sizeof (float) plus padding

- Total memory S(N) usage can be computed based on those considerations