Hand in 1

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Exercise 6

Suppose we have a natural transformation $\tau : F \to G$. We want to associate to τ a functor $H_\tau : \mathcal{C} \times [1] \to \mathscr{D}$ in a meaningful way. In fact we want that this association to remind us of the notion of homotopy in regular homotopy theory.
We propose the follwing definition. For an object $(c, x) \in \mathrm{Ob}\,\mathcal{C} \times [1]$ and an arrow $(f, p) \in \mathrm{Ob}\,\mathcal{C} \times [1]((c, x), (c', x'))$ we set their image by H_τ :

$$H_\tau(c, x) = \begin{cases} F(c) & \text{if } x = 0, \\ G(c) & \text{if } x = 1 \end{cases}$$

and

$$H_\tau(f, p) = \begin{cases} F(f) & \text{if } p = 0 \leq 0, \\ \tau_{c'} F(f) = G(f)\tau_c & \text{if } p = 0 \leq 1, \\ G(f) & \text{if } p = 1 \leq 1. \end{cases}$$

Note that the equalitiy in the second line is due to the fact that τ is a natural transformation.
First, we need to prove that H_τ is a functor. Let us consider the identity $\mathrm{Id}_{(c,x)} = (\mathrm{Id}_c, x \leq x)$. We want to prove that $H_\tau(\mathrm{Id}_{(c,x)}) = \mathrm{Id}_{H_\tau(c,x)}$. We distinguish two cases :

- <u>If $x = 0$</u> : Then we get that $H_\tau(\mathrm{Id}_{(c,0)}) = H_\tau(Id_c, 0 \leq 0) = F(\mathrm{Id}_c) = \mathrm{Id}_{F(c)} = \mathrm{Id}_{H_\tau(c,0)}$.
- <u>If $x = 1$</u> : Then we get that $H_\tau(\mathrm{Id}_{(c,1)}) = H_\tau(Id_c, 1 \leq 1) = G(\mathrm{Id}_c) = \mathrm{Id}_{G(c)} = \mathrm{Id}_{H_\tau(c,1)}$.

Note that the third equalities are given by the fact that F and G are functors. Secondly, we need to prove that H_τ respects composition. Suppose we have $(c, x) \xrightarrow{(f,p)} (c', x') \xrightarrow{(f',p')} (c'', x'')$. We consider the following cases :

- <u>If $p = p' = 0 \leq 0$</u> : We have
$$H_\tau((f', 0 \leq 0)(f, 0 \leq 0)) = H_\tau(f'f, 0 \leq 0) = F(f'f) = F(f')F(f) = H_\tau(f', 0 \leq 0)H_\tau(f, 0 \leq 0).$$
- <u>If $p = p' = 1 \leq 0$</u> : We have
$$H_\tau((f', 1 \leq 1)(f, 1 \leq 1)) = H_\tau(f'f, 1 \leq 1) = F(f'f) = F(f')F(f) = H_\tau(f', 1 \leq 1)H_\tau(f, 1 \leq 1).$$
- <u>If $p = 0 \leq 0$ and $p' = 0 \leq 1$</u> : We have
$$H_\tau((f', 0 \leq 1)(f, 0 \leq 0)) = H_\tau(f'f, 0 \leq 1) = \tau_{c''} F(f'f) = (\tau_{c''} F(f'))F(f) = H_\tau(f', 0 \leq 1)H_\tau(f, 0 \leq 0).$$
- <u>If $p = 0 \leq 1$ and $p' = 1 \leq 1$</u> : We have
$$H_\tau((f', 1 \leq 1)(f, 0 \leq 1)) = H_\tau(f'f, 0 \leq 1) = G(f'f)\tau_c = G(f')(G(f)\tau_c) = H_\tau(f', 1 \leq 1)H_\tau(f, 0 \leq 1).$$

(Note that the third and fourth cases are actually the same). This proves that H_τ is a functor. Now, thirdly, we need to prove that H_τ respects the wanted commutative property. We have that

$$\begin{cases} H_\tau \circ \mathrm{Id}_{\mathcal{C}} \times \{0\}(c) = H_\tau(c, 0) = F(c), \\ H_\tau \circ \mathrm{Id}_{\mathcal{C}} \times \{1\}(c) = H_\tau(c, 1) = G(c), \\ H_\tau \circ \mathrm{Id}_{\mathcal{C}} \times \{0\}(f) = H_\tau(f, 0 \leq 0) = F(f), \\ H_\tau \circ \mathrm{Id}_{\mathcal{C}} \times \{1\}(f) = H_\tau(f, 1 \leq 1) = G(f), \end{cases}$$

for each $c \in \mathrm{Ob}\,\mathcal{C}$ and $f \in \mathrm{Mor}\,\mathcal{C}$. So the commutativity criterion is respected.

Now, let $H \in \mathrm{Fun}_{F,G}(\mathcal{C} \times [1], \mathscr{D})$. We want to associate to H a natural transformation τ_H from F to G. The natural thing to do seems to set $(\tau_H)_c = H(\mathrm{Id}_c, 0 \le 1)$ for each $c \in \mathrm{Ob}\,\mathcal{C}$. Note that the commutative properties of elements of $\mathrm{Fun}_{F,G}(\mathcal{C} \times [1], \mathscr{D})$ gives that the domain and the codomain of $(\tau_H)_c$ are

$$\begin{aligned}
\mathrm{dom}(\tau_H)_c &= \mathrm{dom}\, H(\mathrm{Id}_c, 0 \le 1) = \mathrm{dom}\{H(\mathrm{Id}_c, 0 \le 1) H(\mathrm{Id}_c, 0 \le 0)\} \\
&= \mathrm{dom}\, H(\mathrm{Id}_c, 0 \le 0) = \mathrm{dom}\, F(\mathrm{Id}_c) = \mathrm{dom}\, \mathrm{Id}_{F(c)} = F(c), \\
\mathrm{cod}(\tau_H)_c &= \mathrm{cod}\, H(\mathrm{Id}_c, 0 \le 1) = \mathrm{cod}\{H(\mathrm{Id}_c, 1 \le 1) H(\mathrm{Id}_c, 0 \le 1)\} \\
&= \mathrm{cod}\, H(\mathrm{Id}_c, 1 \le 1) = \mathrm{cod}\, G(\mathrm{Id}_c) = \mathrm{cod}\, \mathrm{Id}_{G(c)} = G(c)
\end{aligned}$$

and so $(\tau_H)_c$ is indeed a morphism from $F(c)$ to $G(c)$. We then need to show that this defines a natural transformation. Let $c, d \in \mathrm{Ob}\,\mathcal{C}$ and $f \in \mathcal{C}(c, d)$. We need to show that

$$\begin{tikzcd}
F(c) \arrow[r, "(\tau_H)_c"] \arrow[d, "F(f)"'] & G(c) \arrow[d, "G(f)"] \\
F(d) \arrow[r, "(\tau_H)_d"'] & G(d)
\end{tikzcd}$$

commutes. This said we have that

$$G(f)(\tau_H)_c = H(f, 1 \le 1) H(\mathrm{Id}_c, 0 \le 1) = H(f, 0 \le 1) = H(\mathrm{Id}_d, 0 \le 1) H(f, 0 \le 0) = (\tau_H)_d F(f)$$

and thus the diagram commutes and we have that τ_H is a natural transformation.

Lastly, we need to prove that these associations are inverses of each other, i.e. $H_{\tau_H} = H$ for each $H \in \mathrm{Fun}_{F,G}(\mathcal{C} \times [1], \mathscr{D})$ and $\tau_{H_\tau} = \tau$ for each $\tau \in \mathrm{Nat}(F, G)$. For $H \in \mathrm{Fun}_{F,G}(\mathcal{C} \times [1], \mathscr{D})$, $(c, x), (d, y) \in \mathrm{Ob}\,\mathcal{C} \times [1]$ and $(f, p) \in \mathrm{Ob}\,\mathcal{C} \times [1]((c, x), (d, y))$, we have that

$$H_{\tau_H}(c, x) = \begin{cases} F(c) = H(c, 0) & \text{if } x = 0, \\ G(c) = H(c, 1) & \text{if } x = 1. \end{cases}$$

$$H_{\tau_H}(f, p) = \begin{cases} F(f) = H(f, 0 \le 0) & \text{if } p = 0 \le 0, \\ (\tau_H)_d F(f) = H(\mathrm{Id}_d, 0 \le 1) H(f, 0 \le 0) = H(f, 0 \le 1) & \text{if } p = 0 \le 1, \\ G(f) = H(f, 1 \le 1) & \text{if } p = 1 \le 1. \end{cases}$$

We clearly see that H_{τ_H} is equal to H. Lastly, let $\tau \in \mathrm{Nat}(F, G)$. Then

$$(\tau_{H_\tau})_c = H_\tau(\mathrm{Id}_c, 0 \le 1) = \tau_c$$

for each $c \in \mathrm{Ob}\,\mathcal{C}$ and thus we get that τ_{H_τ} is equal to τ. We can then conclude that those assignations are inverses to each other and define bijective maps between $\mathrm{Nat}(F, G)$ and $\mathrm{Fun}_{F,G}(\mathcal{C} \times [1], \mathscr{D})$.