Hand in 1

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Exercise 6

Suppose we have a natural transformation $\tau : F \to G$. We want to associate to τ a functor $H_\tau : \mathcal{C} \times [1] \to \mathscr{D}$ in a meaningful way. In fact we want that this association to remind us of the notion of homotopy in regular homotopy theory.
We propose the follwing definition. For an object $(c, x) \in \operatorname{Ob} \mathcal{C} \times [1]$ and an arrow $(f, p) \in \operatorname{Ob} \mathcal{C} \times [1]((c, x), (c', x'))$ we set their image by H_τ :

$$H_\tau(c, x) = \begin{cases} F(c) & \text{if } x = 0, \\ G(c) & \text{if } x = 1 \end{cases}$$

and

$$H_\tau(f, p) = \begin{cases} F(f) & \text{if } p = 0 \leq 0, \\ \tau_{c'} F(f) = G(f) \tau_c & \text{if } p = 0 \leq 1, \\ G(f) & \text{if } p = 1 \leq 1. \end{cases}$$

Note that the equalitiy in the second line is due to the fact that τ is a natural transformation. First, we need to prove that H_τ is a functor.
Let us consider the identity $\operatorname{Id}_{(c,x)}$. We want to prove that $H_\tau(\operatorname{Id}_{(c,x)}) = \operatorname{Id}_{H_\tau(c,x)}$. We distinguish two cases :