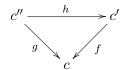
# Homotopical Algebra Hand in 2

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### **Notation:**

From now on in this document, we will use the followind notation. For  $c' \xrightarrow{f} c \in \text{Ob } \mathcal{C}/c$ , we will note (c', f) and  $(c'', h, c') \in \mathcal{C}((c'', g), (c', f))$  for every morphisms from (c'', g) to (c', f)

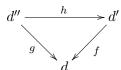


This may be a confusing notation since it does not contain the information of f or g, but it will be made clear from context.

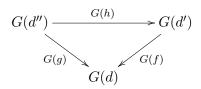
#### Exercise 7:

# Part (a) :

Let  $d \in \text{Ob } \mathscr{D}$  and let us define the functor  $\bar{G} : \mathscr{D}/d \longrightarrow \mathscr{C}/G(d)$ . We set  $\bar{G}(d',f) = (G(d'),G(f))$  on objects of  $\mathscr{D}/d$  and  $\bar{G}((d'',h,d')) = (G(d''),G(h),G(d'))$  for the a morphism from (d'',g) to (d',f). Observe that (G(d''),G(h),G(d')) is indeed a morphism since we have that the commutativity of the diagram



implies the commutativity of the diagram



Now we need to prove that this defines a functor. Let us compute the image of  $\mathrm{Id}_{(d',f)}$  for  $(d',f) \in \mathrm{Ob}\,\mathscr{D}/d$ . We have

$$\bar{G}(\mathrm{Id}_{(d',f)}) = (G(d'), G(\mathrm{Id}_{d'}), G(d')) = (G(d'), \mathrm{Id}_{G(d')}, G(d')) = \mathrm{Id}_{\bar{G}(d',f)}.$$

Now let us compute the image of a composition. For (d''', i, d'') and (d'', j, d'), we have

$$\begin{split} \bar{G}((d'',j,d')(d''',i,d'')) = & \bar{G}(d''',ji,d') \\ = & (G(d'''),G(ji),G(d')) \\ = & (G(d'''),G(j)G(i),G(d')) \\ = & (G(d''),G(j),G(d'))(G(d'''),G(i),G(d'')) \\ = & \bar{G}(d'',j,d')\bar{G}(d''',i,d''). \end{split}$$

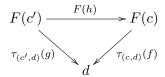
This allows to say that  $\bar{G}$  is a functor.

## Part (b):

As we know that the pair (F,G) is a pair of adjoint functors, we know there exists a natural isomorphism

$$\tau: \mathcal{C}(-,G(-)) \Longrightarrow \mathscr{D}(F(-),-).$$

We set the following functor  $\bar{F}: \mathcal{C}/G(d) \longrightarrow \mathcal{D}/d$  the following way. On the objects, we set  $\bar{F}(c,f) = (F(c), \tau_{(c,d)}(f))$  and on morphisms  $\bar{F}(c',h,c) = (F(c'),F(h),F(c))$  for a morphism form (c,f) to (c',g). We need now to prove that (F(c'),F(h),F(c)) is indeed a morphism from  $(F(c'),\tau_{(c',d)}(g))$  to  $(F(c),\tau_{(c,d)}(f))$ , i.e. we need to prove that the triangle



Note first that since (c', h, c) is a morphism from (c, f) to (c', g), we have fh = g. The fact that  $\tau$  is a natural transformation gives us that

$$\mathcal{D}(F(-), -)(h^{op}, \mathrm{Id}_d)\tau_{(c,d)}(f) = \mathrm{Id}_d\tau_{(c,d)}(f)F(h) = \tau_{(c,d)}(f)F(h)$$

$$\tau_{(c',d)}\mathcal{C}(-, G(-))(h^{op}, \mathrm{Id}_d)(f) = \tau_{(c',d)}(G(\mathrm{Id}_d)fh) = \tau_{(c',d)}(fh) = \tau_{(c',d)}(g)$$

those two ligns are equal and so we conclude that the last triangle commutes.

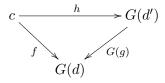
Now we need to prove that  $\bar{F}$  is a functor. First consider  $\mathrm{Id}_{(c,f)}$  and let us compute its image. We have

$$\bar{F}(\mathrm{Id}_{(c,f)}) = (F(c), F(\mathrm{Id}_{F(c)}, F(c))) = \mathrm{Id}_{F(c)} = \mathrm{Id}_{\bar{F}(c,f)}.$$

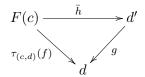
Let us compute the image of a composition. Suppose, we have (c'', i, c') and (c', j, c). Then

$$\bar{F}((c',j,c)(c'',i,c')) = \bar{F}(c'',ji,c) 
= (F(c''), F(ji), F(c)) 
= (F(c''), F(j)F(i), F(c)) 
= (F(c''), F(j), F(c'))(F(c'), F(i), F(c)) 
= \bar{F}(c'',j,c')\bar{F}(c',i,c)$$

and thus we conclude that  $\bar{F}$  is a functor. What is left is to prove that the pair  $(\bar{F}, \bar{G})$  is a pair of adjoint functors. Let  $(c, f) \in \mathrm{Ob}\,\mathcal{C}/G(d)$  and  $(d', g) \in \mathrm{Ob}\,\mathcal{D}$ . We would like to find a map from  $\mathcal{C}/G(d)((c, f), (G(d'), G(g)))$  to  $\mathscr{D}/d((F(c), \tau_{(c,d)}(f)), (d', g))$ , that is if we have the commutative triangle



we would like to send h to a map  $\bar{h}$  such that the triangle



commutes. Recall that since  $\tau$  is a natural isomorphism, the square

$$\begin{array}{c|c} \mathcal{C}(G(d'),G(d')) \xrightarrow{\tau_{(G(d'),d')}} & \mathscr{D}(FG(d'),d') \\ \hline \bar{\mathcal{C}}(h^{op},g) & & & \bar{\mathcal{D}}(h^{op},g) \\ & & \mathcal{C}(c',G(d)) \xrightarrow{\tau_{(c,d)}} & \mathscr{D}(F(c'),d) \end{array}$$

commutes, where  $\bar{\mathcal{C}} = \mathcal{C}(-, G(-))$  and  $\bar{\mathscr{D}} = \mathscr{D}(F(-), -)$ . Note that we then have that

$$\tau_{(c,d)}(f) = \tau_{(c,d)}(G(g)h) = \tau_{(c,d)}\bar{\mathcal{C}}(h^{op}, g)(\operatorname{Id}_{G(d')})$$

$$= \bar{\mathcal{D}}(h^{op}, g)\tau_{(G(d'),d')}(\operatorname{Id}_{G(d')})$$

$$= g\tau_{(G(d'),d')}(\operatorname{Id}_{G(d')})F(h)$$

and so our guess is  $\bar{h} = \tau' F(h)$ , where  $\tau' = \tau_{(G(d'),(d'))}$ . Let us denote the family of maps obtained by  $\theta_{((c,f),(d',g))}$  and let us denote by  $\theta$  the natural transformation given by this family.