Riemann Surfaces

Hand in 1

1: Set $f(z)=\frac{az+b}{cz+d}$ where it is defined. Note that if $c=0$ then $d\neq 0$ and $f(z)=\frac{a}{d}z+\frac{b}{d}$ is an affine function. If $c\neq 0$, one has $f=f_3\circ f_2\circ f_1$, where

$$f_1(z)=cz+d,$$
$$f_2(z)=\frac{1}{z},$$
$$f_3(z)=\frac{bc-ad}{c}z+\frac{a}{c}.$$

Now if we are able to show that affine functions and f_2 send "circles" to "circles", the composition would assure us that f sends "circles" to "circles". Let us first investigate the case of affine functions. Let us set $g(z)=az+b$, $z=x+iy$, $a=\alpha+i\beta$, $b=\gamma+i\delta$ and $g(z)=u+iv$. Note that we only care about the case where $a\neq 0$. By computation, we obtain the following equation

$$\begin{aligned}u+iv=g(z)&=(\alpha+i\beta)(x+iy)+(\gamma+i\delta)\\&=(\alpha x-\beta y+\gamma)+i(\beta x+\alpha y+\delta)\end{aligned}$$

which is equivalent to the system

$$\begin{cases}u=\alpha x-\beta y+\gamma,\\ v=\beta x+\alpha y+\delta.\end{cases}$$

We can inverse this system by the following computations

$$\begin{aligned}\alpha u+\beta v&=\alpha^2x-\alpha\beta y+\alpha\gamma+\beta^2x+\alpha\beta y+\beta\delta\\&=(\alpha^2+\beta^2)x+\alpha\gamma+\beta\delta\end{aligned}$$
$$\Leftrightarrow$$
$$x=\frac{\alpha}{\alpha^2+\beta^2}(u-\gamma)+\frac{\beta}{\alpha^2+\beta^2}(v-\delta),$$
$$\alpha v+\beta u=(\alpha^2+\beta^2)y+\alpha\delta+\beta\gamma$$
$$\Leftrightarrow$$
$$y=\frac{-\beta}{\alpha^2+\beta^2}(u-\gamma)+\frac{\alpha}{\alpha^2+\beta^2}(v-\delta)$$

and we then get the system

$$\begin{cases}x=\frac{\alpha}{\alpha^2+\beta^2}(u-\gamma)+\frac{\beta}{\alpha^2+\beta^2}(v-\delta);\\ y=\frac{-\beta}{\alpha^2+\beta^2}(u-\gamma)+\frac{\alpha}{\alpha^2+\beta^2}(v-\delta).\end{cases}$$

Remember that we want to show that g send "circles" to "circles". We treat the 2 ~~circles~~ cases separately:

Case 1 $\lambda x + \mu y = \nu$, where $\lambda, \mu, \nu \in \mathbb{R}$ and λ or μ is nonzero (otherwise it would be a line). We then get

$$\begin{aligned}
\nu &= \lambda x + \mu y \\
&= \lambda\left[\frac{\alpha}{\alpha^2+\beta^2}(u-\gamma) + \frac{\beta}{\alpha^2+\beta^2}(v-\delta)\right] + \mu\left[\frac{-\beta}{\alpha^2+\beta^2}(u-\gamma) + \frac{\alpha}{\alpha^2+\beta^2}(v-\delta)\right] \\
&= \frac{\lambda\alpha - \mu\beta}{\alpha^2+\beta^2}u + \frac{\mu\alpha+\lambda\beta}{\alpha^2+\beta^2}v - \frac{\lambda\alpha\gamma + \lambda\beta\delta - \mu\beta\gamma + \mu\alpha\delta}{\alpha^2+\beta^2} \\
&= \frac{\lambda\alpha - \mu\beta}{\alpha^2+\beta^2}u + \frac{\mu\alpha+\lambda\beta}{\alpha^2+\beta^2}v - c,
\end{aligned}$$

$$\Longleftrightarrow$$

$$\frac{\lambda\alpha - \mu\beta}{\alpha^2+\beta^2}u + \frac{\mu\alpha+\lambda\beta}{\alpha^2+\beta^2}v = \nu + c,$$

where $c = \dfrac{\lambda\alpha\gamma + \lambda\beta\delta - \mu\beta\gamma + \mu\alpha\delta}{\alpha^2+\beta^2}$. Notice that this equation of the line.

Case 2 $(x-x_0)^2 + (y-y_0)^2 = R^2$, where $x_0, y_0 \in \mathbb{R}$ and $R>0$. Note that the system of equations above gives us the following one

$$\begin{cases} x_0 = \dfrac{\alpha}{\alpha^2+\beta^2}(u_0-\gamma) + \dfrac{\beta}{\alpha^2+\beta^2}(v_0-\delta), \\ y_0 = \dfrac{-\beta}{\alpha^2+\beta^2}(u_0-\gamma) + \dfrac{\alpha}{\alpha^2+\beta^2}(v_0-\delta). \end{cases}$$

From this we can then compute what shape has the image of this circle

$$\begin{aligned}
R^2 &= (x-x_0)^2 + (y-y_0)^2 \\
&= \left[\frac{\alpha}{|a|^2}(u-\gamma) + \frac{\beta}{|a|^2}(v-\delta) - \frac{\alpha}{|a|^2}(u_0-\gamma) - \frac{\beta}{|a|^2}(v_0-\delta)\right]^2 + \\
&\quad + \left[\frac{-\beta}{|a|^2}(u-\gamma) + \frac{\alpha}{|a|^2}(v-\delta) - \frac{-\beta}{|a|^2}(u_0-\gamma) - \frac{\alpha}{|a|^2}(v_0-\delta)\right]^2 \\
&= \left[\frac{\alpha}{|a|^2}(u-u_0) + \frac{\beta}{|a|^2}(v-v_0)\right]^2 + \left[\frac{-\beta}{|a|^2}(u-u_0) + \frac{\alpha}{|a|^2}(v-v_0)\right]^2 \\
&= \frac{\alpha^2}{|a|^4}(u-u_0)^2 + \frac{2\alpha\beta}{|a|^4}(u-u_0)(v-v_0) + \frac{\beta^2}{|a|^4}(v-v_0)^2 + \frac{\beta^2}{|a|^4}(u-u_0)^2 - \frac{2\alpha\beta}{|a|^4}(u-u_0)(v-v_0) + \\
&\quad + \frac{\alpha^2}{|a|^4}(v-v_0)^2 \\
&= \frac{|a|^2}{|a|^4}(u-u_0)^2 + \frac{|a|^2}{|a|^4}(v-v_0)^2 = \frac{1}{|a|^2}\left[(u-u_0)^2 + (v-v_0)^2\right],
\end{aligned}$$

$$\Longleftrightarrow$$

$$(|a|R)^2 = (u-u_0)^2 + (v-v_0)^2,$$

which is the equation of a circle. So we proved that affine maps send "circles" to "circles".

Next, let us investigate the multiplicative inverse function, i.e. $g(z)=\frac{1}{z}$. As before let us set $z=x+iy$ and $g(z)=u+iv$. We then get the system of equations

$$\begin{cases} u=\frac{x}{|z|}, \\ v=\frac{-y}{|z|}, \end{cases}$$

which we can inverse this system to the following system:

$$\begin{cases} x=\frac{u}{u^2+v^2}, \\ y=\frac{-v}{u^2+v^2}. \end{cases}$$

As before, we distinguish the two cases.

<u>Case 1 $\lambda x+\mu y=\nu$:</u> One has the following equalities.

$$\nu=\lambda x+\mu y=\lambda\frac{u}{u^2+v^2}+\mu\frac{-v}{u^2+v^2}=\frac{1}{u^2+v^2}(\lambda u-\mu v) \qquad (1)$$

$$\Leftrightarrow (u^2+v^2)\nu=\lambda u-\mu v,$$

$$\Leftrightarrow \nu u^2-\lambda u+\nu v^2+\mu v=0,$$

$$\Leftrightarrow u^2-\frac{\lambda}{\nu}u+v^2+\frac{\mu}{\nu}v=0,$$

$$\Leftrightarrow u^2-\frac{\lambda}{\nu}u+\frac{\lambda^2}{4\nu^2}+v^2+\frac{\mu}{\nu}v+\frac{\mu^2}{4\nu^2}=\frac{\lambda^2}{4\nu^2}+\frac{\mu^2}{4\nu^2},$$

$$\Leftrightarrow \left(u-\frac{\lambda}{2\nu}\right)^2+\left(v+\frac{\mu}{2\nu}\right)^2=\left(\frac{\sqrt{\lambda^2+\mu^2}}{2\nu}\right)^2,$$

if we assume that $\nu\neq 0$. ~~This~~ Note that this the equation of a circle.
Now let us treat the case where $\nu=0$. ~~Note~~ From (1), we get the equation

$$\lambda u-\mu\nu=0,$$

which is the equation of a line.

<u>Case 2 $(x-x_0)^2+(y-y_0)^2=R^2$:</u> Note that the the previous system gives the following one

$$\begin{cases} x_0=\frac{u_0}{u_0^2+v_0^2}, \\ y_0=\frac{-v_0}{u_0^2+v_0^2}. \end{cases}$$

This said, we get

$$R^2=(x-x_0)^2+(y-y_0)^2$$

$$=\left(\frac{u}{u^2+v^2}-\frac{u_0}{u_0^2+v_0^2}\right)^2+\left(\frac{-v}{u^2+v^2}-\frac{-v_0}{u^2+v^2}\right)^2$$

$$=\frac{u^2}{u^2+v^2}-2\frac{u_0u}{(u^2+v^2)(u_0^2+v_0^2)}+\frac{u_0^2}{(u_0^2+v_0^2)^2}+\frac{v^2}{(u^2+v^2)^2}-2\frac{vv_0}{(u^2+v^2)(u_0^2+v_0^2)}+\frac{v_0^2}{(u_0^2+v_0^2)^2}$$

$$= \frac{u^2+v^2}{(u^2+v^2)^2} + \frac{u_0^2+v_0^2}{(u_0^2+v_0^2)^2} - 2\,\frac{uu_0 - vv_0}{(u^2+v^2)(u_0^2+v_0^2)} =$$

$$= \frac{1}{u^2+v^2} + \frac{1}{u_0^2+v_0^2} - 2\,\frac{uu_0 - vv_0}{(u^2+v^2)(u_0^2+v_0^2)}$$

$$= \frac{1}{u^2+v^2}\left(1 - 2\,\frac{uu_0 - vv_0}{u_0^2+v_0^2}\right) + \frac{1}{u_0^2+v_0^2},$$

$$\Longleftrightarrow R^2 - \frac{1}{u_0^2+v_0^2} = \frac{1}{u^2+v^2}\left(1 - 2\,\frac{uu_0 - vv_0}{u_0^2+v_0^2}\right).$$

We know face two cases. The first one is $R^2 - \frac{1}{u_0^2+v_0^2} = 0$. In that case, we have

$$1 - 2\,\frac{uu_0 - vv_0}{u_0^2+v_0^2} = 0,$$

$$\Longleftrightarrow \frac{u_0}{u_0^2+v_0^2}\,u - \frac{v_0}{u_0^2+v_0^2}\,v = \frac{1}{2},$$

which is the equation of a line. Now let us suppose $R^2 - \frac{1}{u_0^2+v_0^2} \neq 0$. We then have

$$\boxed{\begin{aligned} \cancel{R^2 - \frac{1}{u_0^2+v_0^2}} &= \cancel{\frac{1}{u^2+v^2}\left(1 - 2\,\frac{uu_0 - vv_0}{u_0^2+v_0^2}\right)} \\ &= \cancel{\frac{1}{u^2+v^2}\,\frac{u_0^2+v_0^2 - 2uu_0 + 2vv_0}{u_0^2+v_0^2}} \\ &= \cancel{\frac{1}{u^2+v^2}} \end{aligned}}$$

$$\begin{aligned} R^2(u_0^2+v_0^2) - 1 &= \frac{1}{u^2+v^2}\left(u_0^2+v_0^2 - 2uu_0 - 2vv_0\right) \\ &= \frac{u^2 - 2uu_0 + u_0^2 + v^2 - 2vv_0 + v_0^2 - u^2 - v^2}{u^2+v^2} \\ &= \frac{(u-u_0)^2 + (v-v_0)^2 - (u^2+v^2)}{u^2+v^2} \\ &= \frac{(u-u_0)^2}{u^2+v^2} + \frac{(v-v_0)^2}{u^2+v^2} - 1 \end{aligned}$$

$$\Longleftrightarrow \quad R^2(u_0^2+v_0^2) = \frac{(u-u_0)^2}{u^2+v^2} + \frac{(v-v_0)^2}{u^2+v^2}$$

$$\begin{aligned} \Longleftrightarrow R^2(u_0^2+v_0^2)(u^2+v^2) &= (u-u_0)^2 + (v-v_0)^2 \\ &= u^2 - 2u_0u + u_0^2 + v^2 - 2v_0v + v_0^2 \end{aligned}$$

$$\Longleftrightarrow u^2\left[1 - R^2(u_0^2+v_0^2)\right] - 2u_0u + u_0^2 + v^2\left[1 - R^2(u_0^2+v_0^2)\right] - 2v_0v + v_0^2 = 0$$

$$\Longleftrightarrow u^2 - \frac{2u_0 u}{1-R^2(u_0^2+v_0^2)} + v^2 - \frac{2v_0 v}{1-R^2(u_0^2+v_0^2)} = -\frac{u_0^2+v_0^2}{1-R^2(u_0^2+v_0^2)}$$

$$\Longleftrightarrow \left(u - \frac{u_0}{1-R^2(u_0^2+v_0^2)}\right)^2 + \left(v - \frac{v_0}{1-R^2(u_0^2+v_0^2)}\right)^2 = \frac{u_0^2+v_0^2}{(1-R^2(u_0^2+v_0^2))^2} - \frac{u_0^2+v_0^2}{1-R^2(u_0^2+v_0^2)}$$

which is the equation of a circle.

2: (a) One point of interest to answer this question is to look at ~~where~~ the image of the boundary of the disc. Since $f(z) = \frac{2}{z-i} - i = \frac{-iz+1}{z-i}$, we know that the image of the boundary must a "circle". Note that

$$f(-i) = 0,$$
$$f(1) = 1,$$
$$f(-1) = -1.$$

and thus we see that the image of the boundary is the real line. Note that we know that since $(-i)(-i) - 1 \neq 0$, we have that f is an ~~holomorphic function~~ automorphism from $\mathbb{C}\setminus\{i\}$ to $\mathbb{C}\setminus\{-i\}$. Note that separating $\mathbb{C}\setminus\{i\}$ int the components

$$\mathbb{C}\setminus\{i\} = \overline{\mathbb{D}}\setminus\{i\} \sqcup (\mathbb{C}\setminus(\overline{\mathbb{D}}\cup\{i\}))$$

we have two path connected s.th. every path from a point of the first component to a point of the second component must go through the boundary of the disc (minus i). Using the fact that f is continuous and bijective, we get a decomposition of $\mathbb{C}\setminus\{-i\}$:

$$\mathbb{C}\setminus\{-i\} = \overline{\mathbb{H}^2}\setminus\{-i\} \sqcup \{z\in\mathbb{C}\setminus\{-i\} \mid Im(z) < 0\}$$

such that $f(\overline{\mathbb{D}}\setminus\{i\}) = \overline{\mathbb{H}^2}\setminus\{-i\}$ and ~~$f(\{z\in\mathbb{C}\setminus\{i\} \mid Im(z)<0\}) =$~~

$$f(\mathbb{C}\setminus(\overline{\mathbb{D}}\cup\{i\})) = \{z\in\mathbb{C}\setminus\{-i\} \mid Im(z) < 0\}.$$

because it is like we ~~separate~~ cut $\mathbb{C}\setminus\{-i\}$ along the real line (minus $-i$).
Since $f(0) = i$, we know that $f(\overline{\mathbb{D}}\setminus\{i\}) = \overline{\mathbb{H}^2}\setminus\{-i\}$ and not to the other component.

(b) As ~~$cc - 1$~~ $1 - c\bar{c} \neq 0$ (since $c < 1$), we see that f is an automorphism from $\mathbb{C}\setminus\{-\frac{1}{\bar{c}}\}$ to $\mathbb{C}\setminus\{\frac{1}{\bar{c}}\}$. We will use the same method as before. Let us first compute the image of the boundary:

$$f(1) = \frac{1+c}{1+\bar{c}}$$
$$f(-1) = \frac{c-1}{(-1)(\bar{c}-1)}$$
$$f(i) = \frac{\alpha + i(\beta+1)}{(1+\beta) + i\alpha}$$

and note that $|f(1)|=|f(-1)|=|f(i)|=1$ (here $c=\alpha+i\beta$) and so the image of the boundary is the boundary itself. Now ~~we ha that~~ Note that ~~eithe~~ the image of the disc is either itself or $(\mathbb{C}\setminus\{+\frac{1}{\bar{c}}\})\setminus\overline{D}$. Since $f(-c)=0$, we conclude that the disc maps to itself.

4: Let us consider the function $g(z)=\dfrac{z-f(0)}{-\overline{f(0)}z+1}$. Note that by a previous exercise, we know that g is an holomorphic function that restricts to a bijection on the disc. Composing the two functions $h=g\circ f$ gives ~~a function~~ than holomorphic function bijective on the disc to itself ~~s the~~. Note that h^{-1} has the same propertis. Note also that ~~t to~~

$$h(0)=g(f(0))=0$$

$$\Rightarrow h^{-1}(0)=0$$

We can then use the Schwarz lemma to say that:

$$|z|\leq|h^{-1}\circ h(z)|\leq|h(z)|\leq|z| \quad \text{for all } z\in D$$

and thus, still according to Schwarz lemma, $h(z)=\lambda z$ for some $|\lambda|=1$. Let us set $\alpha=f(0)$. Then we have that

$$\lambda z=\frac{f(z)-\alpha}{-\bar{\alpha}f(z)+1},$$

$$\Leftrightarrow \lambda z(-\bar{\alpha}f(z)+1)=f(z)-\alpha,$$

$$\Leftrightarrow -\bar{\alpha}\lambda z f(z)+\lambda z=f(z)-\alpha,$$

$$\Leftrightarrow f(z)(-1-\bar{\alpha}\lambda z)=-\alpha-\lambda z,$$

$$\Leftrightarrow f(z)=\frac{\lambda z+\alpha}{\bar{\alpha}\lambda z+1}.$$

From this, one can prove that f has the form ~~as~~ as described in the exercise sheet. In fact, one has

$$f(z)=\frac{\lambda z+\alpha}{\bar{\alpha}\lambda z+1}$$

$$=\frac{\lambda z - \lambda\left(-\frac{\alpha}{\lambda}\right)}{1-\overline{\left(-\frac{\alpha}{\lambda}\right)}z}$$

$$=\frac{\lambda z-\lambda\beta}{1-\bar{\beta}z}$$

where $\beta = -\frac{\alpha}{\lambda}$. Now let $k = 1 - |\alpha|^2$ and set $a = \sqrt{\frac{\lambda}{k}}$. Note that $\overline{a} = \overline{\sqrt{\frac{\lambda}{k}}} = \sqrt{\frac{\overline{\lambda}}{k}} = \frac{1}{\sqrt{\lambda k}}$.

This allows us to write

$$
\begin{aligned}
f(z) &= \frac{\lambda z - \lambda \beta}{1 - \overline{\beta} z} \\
&= \frac{z\sqrt{\lambda} - \beta\sqrt{\lambda}}{-\left(\overline{\beta}/\sqrt{\lambda}\right) z + 1/\sqrt{\lambda}} \\
&= \frac{z\sqrt{\frac{\lambda}{k}} - \beta\sqrt{\frac{\lambda}{k}}}{-\left(\overline{\beta}/\sqrt{\lambda k}\right) + 1/\sqrt{\lambda k}}.
\end{aligned}
$$

Note that this precisely the form $f(z) = \frac{az + b}{\overline{b}z + \overline{a}}$, where $a = \sqrt{\lambda/k}$ and $b = \alpha\sqrt{\lambda/k}$. Moreover, one has

$$
\begin{aligned}
|a|^2 - |b|^2 &= a\overline{a} - b\overline{b} \\
&= 1/k - |\beta|^2/k \\
&= \frac{1 - |\beta|^2}{k} \\
&= 1.
\end{aligned}
$$

Conversly, if $f(z) = \frac{az+b}{\overline{b}z + \overline{a}}$ with $|a|^2 - |b|^2 = 1$, then we note that

$$
\begin{aligned}
f(1) &= \frac{a+b}{\overline{a+b}} \\
f(-1) &= \frac{-a+b}{-\overline{b}+\overline{a}} = (-1)\frac{-a+b}{\overline{-a+b}} \\
f(i) &= \frac{ai+b}{\overline{b}i+\overline{a}}
\end{aligned}
$$

which tells us that f is a biholomorphism from $\mathbb{C}\setminus\{-\frac{\overline{a}}{\overline{b}}\}$ to $\mathbb{C}\setminus\{\frac{a}{\overline{b}}\}$ which sends the unit disc either to itsel, either to its complement minus the boundary. Noting that $-\frac{b}{a} \in D$, we note that $f(-\frac{b}{a}) = 0$ and thus the unit disc is sent to itself. We then get that f is an automorphism.