Hand in 1

Christophe Marciot

17 septembre 2020

1 Introduction

In this document, we will consider sheaves on a fixed topological space X if not precised otherwise.

1.1 Notation

Let $\mathscr{F}$ be a sheaf. For $p \in X$, we will use two different notations for the elements of $\mathscr{F}_p$. The first one is s_p. The second one is $[(s, U)]$. This means that we have an neighbourhood of p, U, and an element $s \in \mathscr{F}(U)$. Note that in the first notation, the existence of such a U is implicit. That is why we will use the second notation when we need to explicitly describe U. Also we sometimes will write $[(s, U)]_p$ when it is not clear in what stalk we work.

1.2 Preliminary result

Proposition 1.2.1. *Let $\varphi : \mathscr{F} \longrightarrow \mathscr{G}$ be a morphism of sheaves. Then for each $s \in \mathscr{F}(U)$ and $p \in U$, we have*

$$(\varphi_U(s))_p = \varphi_p(s_p).$$

Démonstration. Note that $s_p = [(s, U)]_p$ and so $\varphi_p(s_p) = [(\varphi_U(s), U)]$ and $(\varphi_U(s))_p = [(\varphi_U(s), U)]_p$ by definition. $\square$

2 Homeworks

2.1 Exercise 1.2., part (a)

First, let us look at the kernel part. Let $[(s, U)] \in \operatorname{Ker}(\varphi_p)$. Then we know that $[(\varphi_U(s), U)] = 0$ in $\mathscr{G}_p$. in other words, we get that there exists $V_p \subset U$ neighbourhood of p such that $\varphi_U(s)|_{V_p} = 0$ in $\mathscr{G}(V_p)$. Note that since φ is a morphism, we have that

$$\varphi_{V_p}(s|_{V_p}) = \varphi_U(s)|_{V_p} = 0$$

and thus $s|_{V_p} \in (\operatorname{Ker}\varphi)(V_p)$. We then get that $[(s|_{V_p}, V_p)] = [(s, U)] \in (\operatorname{Ker}\varphi)_p$. We then conclude that $\operatorname{Ker}(\varphi_p) \subset (\operatorname{Ker}\varphi)_p$. Now let $[(s, U)] \in (\operatorname{Ker}\varphi)_p$. Note that $s \in (\operatorname{Ker}\varphi)(U) = \operatorname{Ker}\varphi_U$. We then observe that

$$\varphi_p(s_p) = (\varphi_U(s))_p = 0 \text{ in } \mathscr{G}_p$$

adn thus $[(s, U)] \in \operatorname{Ker}\varphi_p$. This allows us to conclude that $\operatorname{Ker}\varphi_p = (\operatorname{Ker}\varphi)_p$.
Now let us look at the image sheaf. One has

$$\begin{aligned}
[(t, V)] \in \operatorname{im}\varphi_p &\iff \exists[(s, U)] \in \mathscr{F}_p \text{ s.th. } \varphi_p(s_p) = t_p \\
&\iff \exists[(s, U)] \in \mathscr{F}_p \text{ s.th. } [(\varphi_U(s), U)] = [(t, V)] \\
&\iff \exists[(s, U)] \in \mathscr{F}_p \text{ s.th. } \exists W \subset U \cap V \text{ a neighbourhood of } p \text{ s.th. } \varphi_U(s)|_W = t|_W \\
&\iff \exists[(s, U)] \in \mathscr{F}_p \text{ s.th. } \exists W \subset U \cap V \text{ a neighbourhood of } p \text{ s.th. } \varphi_W(s|_W) = t|_W \\
&\iff \exists W \subset V \text{ a neighbourhood of } p \text{ s.th. } t|_W \in \operatorname{im}\varphi_W \\
&\iff \exists W \subset V \text{ a neighbourhood of } p \text{ s.th. } [(t, V)] = [(t, W)] \in (\operatorname{im}\varphi)_p
\end{aligned}$$

2.2 Exercise 1.2., part (b)

First let us investigate the injectivity. We recall that φ is injective if the kernel sheaf is trivial. Suppose that φ is injective. As $\operatorname{Ker}\varphi = 0$, we have

$$\operatorname{Ker}\varphi_p = (\operatorname{Ker}\varphi)_p = 0_p = 0, \forall p \in X$$

and so φ_p is injective for all $p \in X$. Now suppose that φ_p is injective for all $p \in X$. Let $U \subset X$ be an open set and let $s \in \operatorname{Ker}\varphi(U)$. Then $s_p \in (\operatorname{Ker}\varphi)_p = \operatorname{Ker}\varphi_p$ for each $p \in U$ and thus $s_p = 0$ for each $p \in U$. Note that this means that

$$\forall p \in U \exists V_p \subset U \text{ a neighbourhood of } p \text{ s.th. } s|_{V_p} = 0.$$

Since the family $\{V_p\}_{p\in U}$ is an open cover of U and $\mathscr{F}$ is a sheaf, we get that $s = 0$. We then conclude that the assertion holds.

Next let us investigate surjectivity. Recall that φ is surjective if $\operatorname{im}\varphi = \mathscr{G}$. Suppose that φ is surjective. Then

$$\operatorname{im}\varphi_p = (\operatorname{im}\varphi)_p = \mathscr{G}_p, \forall p \in X$$

and thus φ_p is surjective for each $p \in X$. Now suppose that φ_p is surjective for each $p \in X$. Let us consider the inclusion morphism $\iota : \operatorname{im}\varphi \longrightarrow \mathscr{G}$. This morphisms at a given stalk is the inclusion map of the subgroup $(\operatorname{im}\varphi)_p$ into the group $\mathscr{G}_p$. Since $(\operatorname{im}\varphi)_p = \operatorname{im}\varphi_p = \mathscr{G}_p$, because of the surjectivity of φ_p, we get that ι_p is an isomorphism for each $p \in X$. Thus we get that ι is an isomorphism which means that $\operatorname{im}\varphi = \mathscr{G}$.

2.3 Exercise 1.2., part (c)

We have that

$$\begin{aligned}
\cdots \longrightarrow \mathscr{F}^{i-1} \xrightarrow{\varphi^{i-1}} \mathscr{F}^{i} \xrightarrow{\varphi^{i}} \mathscr{F}^{i+1} \longrightarrow \cdots \text{ is an exact sequence of sheaves}\\
\iff \operatorname{Ker}\varphi^i = \operatorname{im}\varphi^{i-1}, \forall i \in \mathbb{Z}\\
\iff \varphi^{i-1} : \mathscr{F}^{i-1} \longrightarrow \operatorname{Ker}\varphi^i \text{ is surjective}, \forall i \in \mathbb{Z}\\
\iff (\varphi^{i-1})_p : (\mathscr{F}^{i-1})_p \longrightarrow (\operatorname{Ker}\varphi^i)_p \text{ is surjective}, \forall p \in X, \forall i \in \mathbb{Z}\\
\iff (\operatorname{im}\varphi^{i-1})_p = (\operatorname{Ker}\varphi^i)_p, \forall p \in X, \forall i \in \mathbb{Z}\\
\iff \operatorname{im}\varphi_p^{i-1} = \operatorname{Ker}\varphi_p^i, \forall p \in X, \forall i \in \mathbb{Z}
\end{aligned}$$

$$\iff \cdots \longrightarrow \mathscr{F}_p^{i-1} \xrightarrow{\varphi_p^{i-1}} \mathscr{F}_p^{i} \xrightarrow{\varphi_p^{i}} \mathscr{F}_p^{i+1} \longrightarrow \cdots \text{ is an exact sequence of groups}, \forall p \in X$$