Modern Algebraic Geometry
Hand in 2

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Part (a) :

We have to verify that I_Y checks the gluing properties of a sheaf. If $s \in I_Y(U)$ and $\{V_i\}_{i\in I}$ is an open cover of U sucht that $s|_{V_i} = 0$ for each i then it is clear that $s = 0$ as this property comes from the fact that $\mathcal{O}_X$ is a sheaf. Now if we have $s_i \in I_Y(V_i)$, where $\{V_i\}_{i\in I}$ is again an open cover of U such that $s_i|_{V_{ij}} = s_j|_{V_{ij}}$, we know we can glue back those s_i's to some $s \in \mathcal{O}_X(U)$. To see that s is in fact in $I_Y(U)$, we observe that for $p \in U$, in particular $p \in V_i$ for some $i \in I$, we have

$$s(p) = s|_{V_i}(p) = s_i(p) = 0$$

since s_i vanishes on $U \cap V_i$ and thus $s \in I_Y(U)$. We then conclude that I_Y is a sheaf.

Part (b) :

Here we first begin by a remark. Note that for $f \in \mathcal{O}_X(U)$, then for $p \in U$, one can associate f_p with $f(p)$. Indeed, suppose that for $g \in \mathcal{O}_X(V), p \in V$ such that $f_p = g_p$. Then we get that there exists a neighbourhood of p,$W \subseteq U \cap V$ such that $f|_W = g|_W$ and thus

$$f(p) = f|_W(p) = g|_W(p) = g(p).$$

Now if $f(p) = g(p)$ we get that $(f-g)(p) = 0$. Since $\mathrm{Supp}(f-g)$ is an open set, we know there exists a neighbourhood $U \subseteq \mathrm{Supp}(f-g)$ of p. This said, we get that $f|_U = g|_U$ and thus $f_p = g_p$.
Now let us go back to the problem at hand. Let us define the morphism of sheaves

$$\varphi_U : \mathcal{O}_X(U)/I_Y(U) \longrightarrow (\iota_*\mathcal{O}_Y)(U) = \mathcal{O}(Y \cap U) : \bar{f} \longrightarrow f|_{Y\cap U},$$

where $\bar{f}$ is the class of f in $\mathcal{O}_X(U)/I_Y(U)$. Note that if $\bar{f} = \bar{g}$, then $f = g + r$ with $r \in I_Y(U)$. Thus

$$f|_{Y\cap U} = (g+r)|_{Y\cap U} = g|_{Y\cap U} + r|_{Y\cap U} = g|_{Y\cap U}$$

and thus φ_U is a well defined map. Note that these are homomorphisms of rings. To show that this is a morphism of sheaves, we go to the level of stlaks. Let $p \in X$. we distinguish two cases. If $p \notin Y$, then we have

$$I_{Y,p} = \lim_{\longrightarrow p\in U} I_Y(U).$$

Note that since $p \notin \bar{Y} = Y$, we know there exists a neighbourhood of p,U, such that $Y \cap U = \emptyset$. We then get that for each $p \in V \subseteq U$ open, one has $I_Y(V) = \mathcal{O}_X(V)$ and thus, by te properties of the direct limit, we have that $I_{Y,p} = \mathcal{O}_{X,p}$ and thus

$$(\mathcal{O}_X/I_Y)_p = \mathcal{O}_{X,p}/I_{Y,p} = \mathcal{O}_{X,p}/\mathcal{O}_{X,p} = 0.$$

Also note that for small enough neighbourhoods U of p, we have

$$(\iota_*\mathcal{O}_Y)(U) = \mathcal{O}_Y(Y \cap U) = \mathcal{O}_Y(\emptyset) = 0$$

and thus

$$(\iota_* \mathcal{O}_Y)_p = \varinjlim_{p \in U} (\iota_* \mathcal{O}_Y)(U) = \varinjlim_{p \in U} \mathcal{O}_Y(Y \cap U) = 0.$$

We then deduce that φ_p is an isomorphism. Now suppose that $p \in Y$. Since $p \in Y$, we know that $f(p) = 0$ for all neighbourhoods U of p and $f \in I_Y(U)$ and thus we conclude that $I_{Y,p} = 0$. Thus φ_p becomes

$$\varphi_p : \mathcal{O}_{X,p} \longrightarrow \mathcal{O}_{Y,p} : [(U, f)] \longrightarrow [(Y \cap U, f|_{Y \cap U})]$$

and thus φ_p is an isomorphism. This tells us that φ is an isomorphism.

Part (c) :

Let us consider $Y = \{p, q\}$. Recall that singletons are closed in X and thus Y is closed. We want to show that $\iota_* \mathcal{O}_Y \simeq \iota_* \mathcal{O}_p \oplus \iota_* \mathcal{O}_q$. We set

$$\varphi_U : \mathcal{O}_Y(Y \cap U) \longrightarrow \mathcal{O}_p(\{p\} \cap U) \oplus \mathcal{O}_q(\{q\} \cap U) : f \longrightarrow (f|_{\{p\}}, f|_{\{q\}}),$$

By noting that $\iota_* \mathcal{O}_{Y,x}, \iota_* \mathcal{O}_{p,x}, \iota_* \mathcal{O}_{q,x} = 0$ if $x \neq p, q$, and $\iota_* \mathcal{O}_{Y,p} = \iota_* \mathcal{O}_{p,p}$ and $\iota_* \mathcal{O}_{q,p} = 0$, and $\iota_* \mathcal{O}_{Y,q} = \iota_* \mathcal{O}_{q,q}$ and $\iota_* \mathcal{O}_{p,q} = 0$, we see that φ is an isomorphism of sheaves. Note that φ_U and φ_p are ring homomorphisms for each $U \subseteq X$ open and $p \in X$. Also we note that $\mathcal{O}_X(X) \simeq k$ and $\mathscr{F}(X) \simeq k \oplus k$ and thus the morphism $\varphi_X : \Gamma(X, \mathcal{O}_X) \longrightarrow \Gamma(X, \mathscr{F})$ can not reach the zerodivisors of $k \oplus k$ (elements like $(1, 0)$ for example) and we conclude that φ_X is not surjective.