Introduction to direct limits

23 septembre 2020

After the introduction of the local cohomology, our next goal is to link those cohomologies to other more familiar objects like Ext groups. For that, we'll need a tool coming directly from category theory : The direct limit.

Definition 0.0.1. Let $(I, \leq)$ be a P.O. set. We view it as a category whose objects are the elements of I with a unique morphism from i to j whenever $i \leq j$. The morphism will also be denoted $i \leq j$. Note that the composition is well defined ,since $(I, \leq)$ is a P.O. set, and that $i \leq i = \mathrm{Id}_i$.
Let $\mathcal{A}$ be a category. An *I-diagram in $\mathcal{A}$* is a covariant functor $\Phi : I \longrightarrow \mathcal{A}$. We often write A_i for $\Phi(i)$ and $\varphi_{i,j}$ for $\Phi(i \leq j)$. Since Φ is a covariant functor, we have that $\Phi(i \leq i) = \mathrm{Id}_{A_i}$. We call the morphisms the *structure morphisms* of the I-diagram

Definition 0.0.2. Let $\mathcal{A}$ be a category and Φ be an I-diagram in $\mathcal{A}$. The *direct limit of* Φ is an object A of $\mathcal{A}$ together with a morphims $\varphi_i : A_i \longrightarrow A$ for each $i \in I$ such that the diagram

$$\begin{tikzcd}
A_i \arrow[d, "\varphi_{i,j}"'] \arrow[drr, "\varphi_i"] & & \\
A_j \arrow[rr, "\varphi_j"'] & & A
\end{tikzcd}$$

commutes for all $i \leq j$. Moreover, A should be universal with respect to these properties. In other words, if A' is an object of $\mathcal{A}$ with morphisms $\psi_i : A_i \longrightarrow A'$ satisfying the commutative diagrams as above, there's a unique morphism $\theta \in \mathcal{A}(A, A')$ such that for each i the diagram below commutes :

$$\begin{tikzcd}
A_i \arrow[dr, "\varphi_i"'] \arrow[drr, "\psi_i"] & & \\
& A \arrow[r, dotted, "\theta"'] & A'
\end{tikzcd} .$$

The direct limit is denoted $\varinjlim_I \Phi$ or $\varinjlim_I A_i$ or just $\varinjlim A_i$ when it is clear that it is an I-diagram.

Proposition 0.0.3. *Direct limits, when they exist, are unique up to unique isomorphism compatible with the structure morphisms φ_i.*

Démonstration. Let Φ be an I-diagram where I is a P.O. set. now suppose that $A, A' = \varinjlim_I \Phi$. Consider the families of morphisms $(\theta_i)_{i \in I}$ and $(\psi_i)_{i \in I}$ such that the diagrams

$$\begin{tikzcd}
A_i \arrow[d, "\varphi_{i,j}"'] \arrow[dr, "\theta_i"] & \\
A_j \arrow[r, "\theta_j"'] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
A_i \arrow[d, "\varphi_{i,j}"'] \arrow[dr, "\psi_i"] & \\
A_j \arrow[r, "\psi_j"'] & A'
\end{tikzcd}$$

commutes for each pair $i \leq j$. By the universal property of limits, we have unique morphisms Σ_1 and Σ_2 such that the diagrams

$$\begin{tikzcd}
& A_i \arrow[dl, "\theta_i"'] \arrow[dr, "\psi_i"] & \\
A \arrow[rr, "\Sigma_1"'] & & A'
\end{tikzcd}
\qquad
\begin{tikzcd}
& A_i \arrow[dl, "\psi_i"'] \arrow[dr, "\theta_i"] & \\
A' \arrow[rr, "\Sigma_2"'] & & A
\end{tikzcd}$$

commute for each $i \in I$. Since these morphisms are unique for this property, showing that they're isomorphisms suffice. By these commutativities, we get that

$$\Sigma_1 \circ \Sigma_2 \psi_i = \Sigma_1 \circ \theta_i = \psi_i.$$

This implies that

$$\begin{tikzcd} & A_i \arrow[dl, "\psi_i"'] \arrow[dr, "\psi_i"] & \\ A' \arrow[rr, "\Sigma_1 \circ \Sigma_2"'] & & A' \end{tikzcd}$$

commutes for each $i \in I$. But since the diagram

$$\begin{tikzcd} & A_i \arrow[dl, "\psi_i"'] \arrow[dr, "\psi_i"] & \\ A' \arrow[rr, "\mathrm{Id}_{A'}"'] & & A' \end{tikzcd}$$

also commutes, the universal property of direct limits tells us that

$$\Sigma_1 \circ \Sigma_2 = \mathrm{Id}_{A'}.$$

By the same argument, we can show that

$$\Sigma_2 \circ \Sigma_1 = \mathrm{Id}_A.$$

This shows that Σ_1 and Σ_2 are isomorphisms. $\square$

Example 0.0.4. Consider a family of R-modules $\{M_j\}_{j\in\mathbb{N}}$ such that $M_j \subseteq M_{j+1}$. Here we can consider it as a $\mathbb{N}$-diagram with the structure morphisms $\iota_{i,j} : M_i \longrightarrow M_j$ are just the inclusion morphisms.

We set $M = \bigcup_{j\in\mathbb{N}} M_j$ and $\iota_i : M_i \longrightarrow M$ to be the inclusion. M is clearly an R-module and ι_i is a morphism for each i. We can compute the composition to see that the diagram

$$\begin{tikzcd} M_i \arrow[d, "\iota_{i,j}"'] \arrow[dr, "\iota_i"] & \\ M_j \arrow[r, "\iota_j"'] & M \end{tikzcd}$$

commutes. To prove that indeed $\varinjlim_j M_j = M$ we still need to prove that it satisfies the universal property.

Suppose M' is R-module and $\psi_i : M_i \longrightarrow M'$ morphisms for each i compatible with the structure morphisms. We define $\theta : M \longrightarrow M'$ the following way :

$$m \in M \implies m \in M_t \text{ for some } t \in \mathbb{N}. \text{ We then set } \theta(m) = \psi_t(m)$$

The fact that θ is well defined comes directly from the fact that the ψ_i are compatible with the structure morphisms that are just the inclusions. Indeed, suppose $m \in M$ is such that $m \in M_i$ and $m \in M_j$, where $i \leq j$. Then, since the diagram

$$\begin{tikzcd} M_i \arrow[d, "\iota_{i,j}"'] \arrow[dr, "\psi_i"] & \\ M_j \arrow[r, "\psi_j"'] & M' \end{tikzcd}$$

commutes, we get that

$$\psi_j(m) = \psi_j \circ \iota_{i,j}(m) = \psi_i(m).$$

Also we get that since the ψ_i are morphisms, θ also is one. By definition of θ, we see that the diagram

$$\begin{tikzcd} M_i \arrow[dr, "\iota_i"'] \arrow[drr, "\psi_i"] & & \\ & M \arrow[r, dotted, "\theta"'] & M' \end{tikzcd}$$

commutes for each i. Also θ is unique for this property since if we have $\theta' : M \longrightarrow M'$ that respects the desired properties, we get that

$$\text{For } m \in M_i, \theta'(m) = \theta' \circ \iota_i(m) = \psi_i(m) = \theta(m).$$

Thus we do get that

$$\varinjlim_{\mathbb{N}} M_n = \bigcup_{n\in\mathbb{N}} M_n$$

This example tells us that direct limits of some systems are really simple objects.

Remark 0.0.5. Note that by taking an increasing chain of R-modules and considering it as a $\mathbb{Z}$-diagram $\{M_n\}_{n\in\mathbb{Z}}$ also with the structure morphisms to be the inclusions, we get that

$$\varinjlim_{\mathbb{Z}} M_n = \bigcup_{n\in\mathbb{Z}} M_n$$

Proposition 0.0.6. *Let M be a R-module and x be an element of R. We consider the following $\mathbb{N}$-diagram :*

1. *$M_i = M$ for each $i \in \mathbb{N}$*
2. *For $i \leq j$ we define $\varphi_{i,j} = [\cdot x^{j-i}]$*

Then $\varinjlim_{\mathbb{N}} M_t = M_x$.

Démonstration. We set $\varphi_j = \left[\cdot \frac{1}{x^j}\right]$ and check the compatibility. We see clearly that the diagram

$$\begin{tikzcd} M_i = M \arrow[d, "{[\cdot x^{j-i}]}"'] \arrow[dr, "{\left[\cdot \frac{1}{x^i}\right]}"] & \\ M_j \arrow[r, "{\left[\cdot \frac{1}{x^j}\right]}"'] & M' \end{tikzcd}$$

commutes once we describe explicitly what the transformations are. We still have to verify the universal property to declare that M_x is the direct limit of the system.
Suppose we have an R-module M' and a family of morphism $\psi_i : M_i \longrightarrow M'$ such that for every $i \leq j$ the diagram :

$$\begin{tikzcd} M_i = M \arrow[d, "{[\cdot x^{j-i}]}"'] \arrow[dr, "\psi_i"] & \\ M_j = M \arrow[r, "{[\cdot x^j]}"'] & M' \end{tikzcd}$$

commutes. Then we define $\Psi : M_x \longrightarrow M'$ the following way :

$$\text{For } \frac{m}{x^t} \in M_x, \Psi\left(\frac{m}{x^t}\right) = \psi_t(m).$$

First, we need to check that Ψ is a well defined function. Suppose that $\frac{m}{x^j} = \frac{n}{x^k}$. By definition of the localization, we get that there is an natural integer t such that

$$x^t(x^k m - x^j n) = 0.$$

That said, we get that

$$\Psi\left(\frac{m}{x^j}\right) = \psi_j(m) = \psi_{j+k+t}(x^{k+t}m) = \psi_{j+k+t}(x^{j+t}n) = \Psi\left(\frac{n}{x^k}\right)$$
$$\Leftrightarrow f_{j+k+t}(x^t(x^k m - x^j n)) = f_{j+k+t}(0) = 0$$

which confirms that Ψ is well defined. Now we just need to show that it's a morphism. Here we suppose that $j \leq k$.

$$\begin{aligned}
\Psi\left(\frac{m}{x^j} + \frac{n}{x^k}\right) &= \Psi\left(\frac{x^{k-j}m + n}{x^k}\right) \\
&= \psi_k(x^{k-j}m + n) \\
&= \psi_k(x^{k-j}m) + \psi_k(n) \\
&= \psi_j(m) + \psi_k(n) \\
&= \Psi\left(\frac{m}{x^j}\right) + \Psi\left(\frac{n}{x^k}\right).
\end{aligned}$$

This gives us that Ψ is morphism and for $j \in \mathbb{N}$ we have

$$\text{For } m_j \in M_j, \Psi \circ \left[\cdot \frac{1}{x^j}\right](m_j) = \Psi\left(\frac{m_j}{x^j}\right) = \psi_j(m_j)$$

which tells us that the diagram

$$\begin{tikzcd}
M_j \arrow[dr, "{\left[\cdot \frac{1}{x^j}\right]}"'] \arrow[drr, "\psi_i"] & & \\
& M \arrow[r, dotted, "\Psi"'] & M'
\end{tikzcd}$$

commutes. Also the unicity of Ψ for the desired property is pretty obvious.

$\square$

Proposition 0.0.7. *Let $(I, \leq)$ be a P.O. set and Φ be an I-diagram in the category of R-modules. Let E be the submodule of $\bigoplus_I A_i$ spanned by*

$$\{g \in \bigoplus A_i \mid \text{for some } i \leq j, g(j) = -\varphi_{i,j}(g(i)) \text{ and } g(t) = 0, \text{ for } t \neq i, j\}.$$

Then $\varinjlim_I A_i = \bigoplus A_i/E$ with the morphisms $\varphi_j = \pi \circ \iota_j$, where $\iota_j : A_j \longrightarrow \bigoplus A_i$ is the standard embedding and $\pi : \bigoplus A_i \longrightarrow \bigoplus A_i/E$ is the canonical projection.

Démonstration. Let $i \leq j$ and set for $a_i \in A_i$, set $g = \iota_j \circ \varphi_{i,j}(a_i)$ and $g' = \iota_i(a_i)$. Then

$$\varphi_j \circ \varphi_{i,j}(a_i) = [g] = [g'] = \varphi_i(a_i) \Leftrightarrow g' - g \in E.$$

But from the definition of g and g' we get that

$$\begin{cases} (g' - g)(j) = g'(j) - g(j) = 0 - \varphi_{i,j}(a_i) = -\varphi_{i,j}(a_i) = -\varphi_{i,j}((g' - g)(i)) \\ (g' - g)(t) = g'(t) - g(t) = 0 - 0 = 0 \text{ if } t \neq i, j \end{cases}$$

which tells us exactly that $(g' - g) \in E$ and that the diagram

$$\begin{tikzcd}
A_i \arrow[d, "\varphi_{i,j}"'] \arrow[dr, "\varphi_i"] & \\
A_j \arrow[r, "\varphi_j"'] & \bigoplus A_i/E
\end{tikzcd}$$

commutes. Now suppose we have an R-module A with morphisms $f_i : A_i \longrightarrow A$ that are compatible with the system. We then set $\Psi : \bigoplus A_i/E \longrightarrow A$ the following way :

$$\text{For } [g] \in \bigoplus A_i/E,\ \Psi([g]) = \sum_I f_i(g(i)).$$

Note that since $g \in \bigoplus A_i$, the sum is finite. We still have to check that Ψ is a well defined function. Suppose $[g] = [g']$. Then we get that

$$\Psi([g]) = \sum f_i(g(i)) = \sum f(g'(i)) = \Psi([g']) \Leftrightarrow \sum f_i((g-g')(i)) = 0.$$

Since $[g] = [g']$, we have some data on $(g - g')$, namely

$$\begin{cases} (g-g')(k) = -\varphi_{j,k}((g-g')(j) \text{ for some } j \leq k \\ (g-g')(t) = 0 \text{ if } t \neq j,k \end{cases}$$

which tells us that

$$\begin{aligned} \sum (f_i((g-g')(i)) &= f_j((g-g')(j)) + f_k((g-g')(k) \\ &= f_j((g-g')(j)) + f_k(-\varphi_{j,k}((g-g')(j))) \\ &= f_j((g-g')(j)) - f_j((g-g')(j)) \\ &= 0 \end{aligned}$$

and that Ψ is well defined. Also it is pretty clear that Ψ is a morphism. We also get that for $j \in I$, we have that

$$\text{For } a_j \in A_j, \Psi \circ \varphi_j(a_j) = \sum_{i \in I} f_i(\iota_j(a_j)(i)) = f_j(a_j)$$

which tells us that the diagram

$$\begin{tikzcd} A_j \arrow[dr, "\varphi_j"'] \arrow[drr, "f_j"] & & \\ & \bigoplus A_i \arrow[r, dotted, "\Psi"'] & A \end{tikzcd}$$

commutes. The unicity of Ψ is also pretty obvious. □

Remark 0.0.8. This proposition is important as it gives us as a fact that every I-diagram in the category of R-modules admit a direct limit for every P.O. set $(I, \leq)$ and a way to compute it.

Definition 0.0.9. Let $(I, \leq)$ be a P.O. set and R be a ring. We define the *category of I-diagrams in the category of R-modules*, denoted $\mathfrak{Dir}_I^R$, whose objects are the I-diagrams in the category of R-modules and the morphisms are just the natural transformations from an I-diagram to an other, i.e for two I-diagrams Φ and Φ' a morphism is given by a family of morphisms $\nu = \{\nu_i : A_i \longrightarrow A'_i\}_{i \in I}$ such that for each pair $i \leq j$ the diagram

$$\begin{tikzcd} A_i \arrow[r, "\nu_i"] \arrow[d, "\varphi_{i,j}"'] & A'_i \arrow[d, "\varphi'_{i,j}"] \\ A_j \arrow[r, "\nu_j"'] & A'_j \end{tikzcd}$$

commutes.

Definition 0.0.10. Let $(I, \leq)$ be a P.O. set and R be a ring. A *chain complex of I-diagrams in the category of R-modules* is a sequence of I-diagrams together with morphims of I-diagrams $\Phi_\bullet = \left(\Phi_{(n)}, \nu_{(n)} : \Phi_{(n)} \longrightarrow \Phi_{(n-1)}\right)_{n \in \mathbb{Z}}$ such that $\nu_n \circ \nu_{n+1} = 0$ for each $n \in \mathbb{Z}$. We define *cochain* *complexes of I-diagrams in the category of R-modules* an analogous way with increasing indices. uch a sequence is said to be *exact* if the sequences

$$\cdots \xrightarrow{\nu^i_{(n+2)}} \Phi_{(n+1)} \xrightarrow{\nu^i_{(n+1)}} \Phi_{(n)} \xrightarrow{\nu^i_{(n)}} \Phi_{(n-1)} \xrightarrow{\nu^i_{(n-1)}} \cdots$$

are exact for each $i \in I$.

Theorem 0.0.11. *Let R be a ring, $\mathcal{R}$ be the category of R-modules and $(I, \leq)$ be a P.O. set. Then we have that $\varinjlim : \mathfrak{Dir}^R_I \longrightarrow \mathcal{R}$ is a left exact additive covariant functor.*

Definition 0.0.12. Let $(I, \leq)$ be a P.O. set. We say that I is *filtered* if for each $i, j \in I$, there exists $k \in I$ such that $i, j \leq k$. We can also use the word *directed* instead of filtered.

Lemma 0.0.13. *Let I be a filtered P.O. set. Let R be a ring and $\mathcal{R}$ be the category of R-modules. Let Φ be an I-diagram in $\mathcal{R}$. Let a be an element in $\bigoplus_I A_i$ and $[a]$ its image in $\varinjlim \Phi$. Then :*

1. *There exists $i \in I$ in I and an element $a_i \in A_i$ such that $\varphi_i(a_i) = [a]$.*
2. *Write $a = (a_i) \in \bigoplus_I A_i$. Then $[a] = 0$ if and only if there exists an index t such that $a_i = 0$ when $i \nleq t$ and $\sum_{j \leq t} \varphi_{j,t}(a_j) = 0$.*

Démonstration. See Lemma 4.32 from [**?**] □

Theorem 0.0.14. *Let I be a filtered P.O. set, R a ring, and $\mathcal{R}$ be the category of R-modules. Then the functor $\varinjlim : \mathfrak{Dir}^R_I \longrightarrow \mathcal{R}$ is exact*

Démonstration. See Theorem 4.33 from [**?**] for a proof. □

Corollary 0.0.15. *Let*

$$\cdots \xrightarrow{\nu^{(n-2)}} \Phi^{(n-1)} \xrightarrow{\nu^{(n-1)}} \Phi^{(n)} \xrightarrow{\nu^{(n)}} \Phi^{(n+1)} \xrightarrow{\nu^{(n+1)}} \cdots$$

be an exact sequence of I-diagrams. Then the sequence

$$\cdots \xrightarrow{\varinjlim \nu^{(n-2)}} \varinjlim \Phi^{(n-1)} \xrightarrow{\varinjlim \nu^{(n-1)}} \varinjlim \Phi^{(n)} \xrightarrow{\varinjlim \nu^{(n)}} \varinjlim \Phi^{(n+1)} \xrightarrow{\varinjlim \nu^{(n+1)}} \cdots$$

is an exact sequence of R-modules.

Démonstration. This a direct consequence of the last theorem. □

Definition 0.0.16. Let

$$\cdots \xrightarrow{\nu^{(n-2)}} \Phi^{(n-1)} \xrightarrow{\nu^{(n-1)}} \Phi^{(n)} \xrightarrow{\nu^{(n)}} \Phi^{(n+1)} \xrightarrow{\nu^{(n+1)}} \cdots$$

be a sequence of I-diagrams. For $n \in \mathbb{Z}$ and $i \leq j$, we define

$$H^{(n)}_j = \operatorname{Ker} \nu^{(n)} / \operatorname{Im} \nu^{(n-1)}$$

and maps $[\varphi_{i,j}] : H^{(n)}_i \longrightarrow H^{(n)}_j$ the following way :

$$\text{For } [a] \in H^{(n)}_i,\ \left[\varphi^{(n)}_{i,j}\right]([a]) = \left[\varphi^{(n)}_{i,j}(a)\right].$$

Proposition 0.0.17. *The previously defined $\left[\varphi^{(n)}_{i,j}\right]$ are morphisms of R-modules.*

Démonstration. First we need to show that for $a \in \operatorname{Ker} \nu_i^{(n)}$, we have that $\varphi_{i,j}^{(n)}(a) \in \operatorname{Ker} \nu_i^{(n)}$. By the definition of morphisms of I-diagrams, we that the following diagram

$$\begin{tikzcd} \Phi^{(n)}(i) \arrow[r, "\varphi_{i,j}^{(n)}"] \arrow[d, "\nu_i^{(n)}"'] & \Phi^{(n)}(j) \arrow[d, "\nu_j^{(n)}"] \\ \Phi^{(n+1)}(i) \arrow[r, "\varphi_{i,j}^{(n+1)}"'] & \Phi^{(n+1)}(j) \end{tikzcd}$$

commutes.Thus we get that

$$\nu_j^{(n)} \circ \varphi_{i,j}^{(n)}(a) = \varphi_{i,j}^{(n+1)} \circ \nu_i^{(n)}(a) = \varphi_{i,j}^{(n+1)}(0) = 0$$

and hence we get that $\varphi_{i,j}^{(n)}(a) \in \operatorname{Ker} \nu_j^{(n)}$. Next we need to prove that for $b \in \operatorname{Im} \nu^{(n-1)}$, $\left[\varphi_{i,j}^{(n)}(b)\right] = 0$. We write $b = \nu_i^{(n-1)}(a)$ for $a \in \Phi^{(n)}(i)$. Again by the definition of I-diagrams, we have that the diagram

$$\begin{tikzcd} \Phi^{(n-1)}(i) \arrow[r, "\nu_i^{(n-1)}"] \arrow[d, "\varphi_{i,j}^{(n-1)}"'] & \Phi^{(n)}(i) \arrow[d, "\varphi_{i,j}^{(n)}"] \\ \Phi^{(n-1)}(j) \arrow[r, "\nu_j^{(n-1)}"'] & \Phi^{(n)}(j) \end{tikzcd}$$

commutes. Thus we get that

$$\varphi_{i,j}^{(n)}(b) = \varphi_{i,j}^{(n)} \circ \nu_i^{(n-1)}(a) = \nu_j^{(n-1)} \circ \varphi_{i,j}^{(n-1)}(a) \in \operatorname{Im} \nu_j^{(n-1)}$$

and hence $\left[\varphi_{i,j}^{(n)}(b)\right] = 0$. This proves that $\left[\varphi_{i,j}^{(n)}\right]$ is a well defined map. The fact that it's a morphism comes directly from the fact that $\varphi_{i,j}^{(n)}$ is already one. □

Proposition 0.0.18. *The previously defined $H_i^{(n)}$ together with the morphisms $\left[\varphi_{i,j}^{(n)}\right]$ form an I-diagram in the category of R-modules.*

Démonstration. This is easily seen when computing that for $i \leq j \leq k$ we get that

$$\left[\varphi_{j,k}^{(n)}\right] \circ \left[\varphi_{i,j}^{(n)}\right] = \left[\varphi_{j,k}^{(n)} \circ \varphi_{i,j}^{(n)}\right] = \left[\varphi_{i,k}^{(n)}\right].$$

□

Theorem 0.0.19. *Let $\Phi^\bullet = \left(\Phi^{(n)},\ \nu^{(n)} : \Phi^{(n)} \longrightarrow \Phi^{(n+1)}\right)_{n\in\mathbb{Z}}$ be a cochain complex of I-diagrams in the category of R-modules. We set*

$$\varinjlim \Phi^\bullet = \left(\varinjlim \Phi^{(n)}, \varinjlim \nu^{(n)}\right)_{n\in\mathbb{Z}}$$

which is a complex of R-modules.
Then we get that for $n \in \mathbb{Z}$ we have

$$\varinjlim H_i^{(n)} \cong H^n\left(\varinjlim \Phi\right) = \operatorname{Ker} \varinjlim \nu^{(n)} / \operatorname{Im} \varinjlim \nu^{(n-1)}.$$

Démonstration. For $i \in I$, let $\varphi_i : \Phi_i^{(n)} \longrightarrow \varinjlim \Phi^{(n)}$ be the morphisms that come with the direct limit. We define, for each $i \in I$, $\left[\varphi_i^{(n)}\right] : H_i^{(n)} \longrightarrow H^n\left(\varinjlim \Phi\right)$ the following way :

$$\text{For } [a] \in H_i^{(n)},\ \left[\varphi_i^{(n)}\right]([a]) = \left[\varphi_i^{(n)}(a)\right].$$

We need to show that it's a well defined morphism. The first step is to prove that

$$a \in \operatorname{Ker} \nu_i^{(n)} \implies \varphi_i^{(n)}(a) \in \operatorname{Ker} \varinjlim \nu^{(n)}.$$

To achieve that we use the fact that the diagram

$$\begin{tikzcd}
\Phi^{(n)}(i) \arrow[r, "\varphi_i^{(n)}"] \arrow[d, "\nu_i^{(n)}"'] & \varinjlim \Phi^{(n)} \arrow[d, "\varinjlim \nu^{(n)}"] \\
\Phi^{(n+1)}(i) \arrow[r, "\varphi_i^{(n+1)}"'] & \varinjlim \Phi^{(n+1)}
\end{tikzcd}$$

commutes. This gives us that for $a \in \operatorname{Ker} \nu_i^{(n)}$

$$\varinjlim \nu^{(n)} \circ \varphi_i^{(n)}(a) = \varphi_i^{(n+1)} \circ \nu_i^{(n)}(a) = \varphi_i^{(n+1)}(0) = 0 \implies \varphi_i^{(n)}(a) \in \operatorname{Ker} \varinjlim \nu^{(n)}.$$

Let $b \in \operatorname{Im} \nu_i^{(n-1)}$. Next we need to show that

$$b \in \operatorname{Im} \nu_i^{(n-1)} \implies \varphi_i^{(n)}(b) \in \operatorname{Im} \varinjlim \nu^{(n-1)}.$$

Setting $b = \nu_i^{(n-1)}(a)$ for $a \in \Phi^{(n-1)}(i)$, we use the fact that the following diagram

$$\begin{tikzcd}
\Phi^{(n-1)}(i) \arrow[r, "\nu_i^{(n-1)}"] \arrow[d, "\varphi_i^{(n-1)}"'] & \Phi^{(n)}(i) \arrow[d, "\varphi_i^{(n)}"] \\
\varinjlim \Phi^{(n-1)} \arrow[r, "\varinjlim \nu^{(n-1)}"'] & \varinjlim \Phi^{(n)}
\end{tikzcd}$$

commutes and we can say that

$$\varphi_i^{(n)}(b) = \varphi_i^{(n)} \circ \nu_i^{(n-1)}(a) = \varinjlim \nu^{(n-1)} \circ \varphi_i^{(n-1)}(a) \in \operatorname{Im} \varinjlim \nu^{(n-1)}.$$

This show us that the $\left[\varphi_i^{(n)}\right]$ are well defined maps. The fact that they're morphisms comes directly from the fact that the $\varphi_i^{(n)}$ are already morphisms themselves. Also by computing we easily see that for each pair $i \leq j$ the diagram

$$\begin{tikzcd}
H_i^{(n)} \arrow[d, "\left[\varphi_{i,j}^{(n)}\right]"'] \arrow[dr, "\left[\varphi_i^{(n)}\right]"] & \\
H_j^{(n)} \arrow[r, "\left[\varphi_j^{(n)}\right]"'] & H^n\left(\varinjlim \Phi\right)
\end{tikzcd}$$

comutes. By the unicity of direct limits, we get that $H^n\left(\varinjlim \Phi\right) \cong \varinjlim H_i^{(n)}$. □

This theorem allows us to state an important corollary.

Corollary 0.0.20. *Let*

$$\cdots \longrightarrow M^{\bullet_{n-1}} \longrightarrow M^{\bullet_n} \longrightarrow M^{\bullet_{n+1}} \longrightarrow \cdots$$

be cochain complex of cochain complexes of R-modules. Then for $j \in \mathbb{Z}$ we have that

$$\varinjlim_n H^j(M^{\bullet_n}) \cong H^j(\varinjlim_n M^{\bullet_n}).$$

Démonstration. Noting that cochain complexes of R-modules can be seen as $\mathbb{Z}$-diagrams in the category of R-modules, the result is directly given by the previous theorem. $\square$

Proposition 0.0.21. *Let M be an R-module. Let $\{N_t\}_{t\in\mathbb{Z}}$ and $\{K_t\}_{t\in\mathbb{Z}}$ be two chains of increasing submodules of M such that for each $t \in \mathbb{Z}$, there exists natural integers c_t and d_t such that*

$$N_t \subseteq K_{t+c_t} \text{ and } K_t \subseteq N_{t+d_t}.$$

By considering the two families as a $\mathbb{Z}$-diagram with the structure morphisms being the inclusions, we get a functorial isomorphism

$$\varinjlim_t K_t \cong \varinjlim_t N_t.$$

Démonstration. By the Remark 0.0.5, we just have

$$\varinjlim_t K_t \cong \bigcup_{t\in\mathbb{Z}} K_t = \bigcup_{t\in\mathbb{Z}} N_t \cong \varinjlim_t N_t.$$

$\square$

Remark 0.0.22. This proof is also valid in a more general case when considering a cofinal system on an Abelian category and replacing the inclusions by monomorphisms.

Corollary 0.0.23. *Let $\mathfrak{a} \subset R$ be an ideal and let $\{\mathfrak{a}_t\}_{t\in\mathbb{N}}$ be a decreasing chain of ideals such that for each t, there exists natural integers c and d such that*

$$\mathfrak{a}^{t+c} \subseteq \mathfrak{a}_t \text{ and } \mathfrak{a}_{t+d} \subseteq \mathfrak{a}^t.$$

Then we have a functorial isomorphism such that

$$\varinjlim_t \operatorname{Ext}^j_R(R/\mathfrak{a}_t, M) \cong H^j_{\mathfrak{a}}(M)$$

Démonstration. We use the fact that for each $t, j \in \mathbb{N}$ we have

$$\mathfrak{a}^{t+c} \subseteq \mathfrak{a}_t \text{ and } \mathfrak{a}_{t+d} \subseteq \mathfrak{a}^t \implies \begin{cases} \operatorname{Hom}_R(R/\mathfrak{a}_t, I^j) \subseteq \operatorname{Hom}_R(R/\mathfrak{a}^{t+c}, I^j) \\ \operatorname{Hom}_R(R/\mathfrak{a}^t, I^j) \subseteq \operatorname{Hom}_R(R/\mathfrak{a}_{t+d}, I^j) \end{cases}$$

where $I^\bullet$ is an injective resolution of M. By the last proposition, this tells us that

$$\varinjlim_t \operatorname{Hom}_R(R/\mathfrak{a}^t, I^\bullet) \cong \varinjlim_t \operatorname{Hom}_r(R/\mathfrak{a}_t, I^\bullet).$$

By the previous proposition, we finally get that

$$\begin{aligned} \varinjlim_t \operatorname{Ext}^j_R(R/\mathfrak{a}^t, M) &= \varinjlim_t H^j(\operatorname{Hom}_R(R/\mathfrak{a}^t, I^\bullet) \\ &\cong H^j(\varinjlim_t \operatorname{Hom}_R(R/\mathfrak{a}_t, I^\bullet) \\ &\cong \varinjlim_t H^j(\operatorname{Hom}_R(R/\mathfrak{a}_t, I^\bullet) \\ &= \varinjlim_t \operatorname{Ext}^j_R(R/\mathfrak{a}_t, M). \end{aligned}$$

$\square$

Proposition 0.0.24. *Let M be an R-module. Let $\{N_t\}_{t\in\mathbb{Z}}$ and $\{K_t\}_{t\in\mathbb{Z}}$ be two chains of increasing submodules of M such that for each $t \in \mathbb{Z}$, there exists natural integers c_t and d_t such that*

$$N_t \subseteq K_{t+c_t} \text{ and } K_t \subseteq N_{t+d_t}.$$

By considering the two families as a $\mathbb{Z}$-diagram with the structure morphisms being the inclusions, we get a functorial isomorphism

$$\varinjlim_t K_t \cong \varinjlim_t N_t$$

Démonstration. See Proposition 0.0.21 for the proof. □

Corollary 0.0.25. *Let $\mathfrak{a} \subset R$ be an ideal and let $\{\mathfrak{a}_t\}_{t\in\mathbb{N}}$ be a decreasing chain of ideals such that for each t, there exists natural integers c and d such that*

$$\mathfrak{a}^{t+c} \subseteq \mathfrak{a}_t \text{ and } \mathfrak{a}_{t+d} \subseteq \mathfrak{a}^t.$$

Then we have a functorial isomorphism such that

$$\varinjlim_t \operatorname{Ext}^j_R(R/\mathfrak{a}_t, M) \cong H^j_{\mathfrak{a}}(M)$$

Démonstration. See Corollary 0.0.23 for the proof. □