Introduction to direct limits

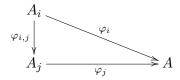
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After the introduction of the local cohomology, our next goal is to link those cohomologies to other more familiar objects like Ext groups. For that, we'll need a tool coming directly from category theory: The direct limit.

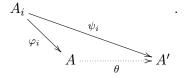
Definition 0.0.1. Let (I, \leq) be a P.O. set. We view it as a category whose objects are the elements of I with a unique morphism from i to j whenever $i \leq j$. The morphism will also be denoted $i \leq j$. Note that the composition is well defined ,since (I, \leq) is a P.O. set, and that $i \leq i = \mathrm{Id}_i$.

Let \mathcal{A} be a category. An *I-diagram in* \mathcal{A} is a covariant functor $\Phi: I \longrightarrow \mathcal{A}$. We often write A_i for $\Phi(i)$ and $\varphi_{i,j}$ for $\Phi(i \leq j)$. Since Φ is a covariant functor, we have that $\Phi(i \leq i) = \operatorname{Id}_{A_i}$. We call the morphisms the *structure morphisms* of the *I*-diagram

Definition 0.0.2. Let \mathcal{A} be a category and Φ be an I-diagram in \mathcal{A} . The direct limit of Φ is an object A of \mathcal{A} together with a morphims $\varphi_i : A_i \longrightarrow A$ for each $i \in I$ such that the diagram



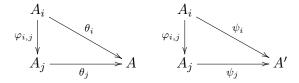
commutes for all $i \leq j$. Moreover, A should be universal with respect to these properties. In other words, if A' is an object of A with morphisms $\psi_i : A_i \longrightarrow A'$ satisfying the commutative diagrams as above, there's a unique morphism $\theta \in \mathcal{A}(A, A')$ such that for each i the diagram below commutes:



The direct limit is denoted $\lim_{\longrightarrow I} \Phi$ or $\lim_{\longrightarrow I} A_i$ or just $\lim_{\longrightarrow} A_i$ when it is clear that it is an *I*-diagram.

Proposition 0.0.3. Direct limits, when they exist, are unique up to unique isomorphism compatible with the structure morphisms φ_i .

Démonstration. Let Φ be an I-diagram where I is a P.O. set. now suppose that $A, A' = \lim_{\longrightarrow I} \Phi$. Consider the families of morphisms $(\theta_i)_{i \in I}$ and $(\psi_i)_{i \in I}$ such that the diagrams



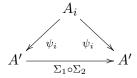
commutes for each pair $i \leq j$. By the universal property of limits, we have unique morphisms Σ_1 and Σ_2 such that the diagrams



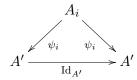
commute for each $i \in I$. Since these morphisms are unique for this property, showing that they're isomorphisms suffice. By these commutativities, we get that

$$\Sigma_1 \circ \Sigma_2 \psi_i = \Sigma_1 \circ \theta_i = \psi_i.$$

This implies that



commutes for each $i \in I$. But since the diagram



also commutes, the universal property of direct limits tells us that

$$\Sigma_1 \circ \Sigma_2 = \operatorname{Id}_{A'}$$
.

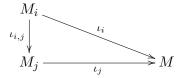
By the same argument, we can show that

$$\Sigma_2 \circ \Sigma_1 = \mathrm{Id}_A$$
.

This shows that Σ_1 and Σ_2 are isomorphisms.

Example 0.0.4. Consider a family of R-modules $\{M_j\}_{j\in\mathbb{N}}$ such that $M_j\subseteq M_{j+1}$. Here we can consider it as a \mathbb{N} -diagram with the structure morphisms $\iota_{i,j}:M_i\longrightarrow M_j$ are just the inclusion morphisms.

We set $M = \bigcup_{j \in \mathbb{N}} M_j$ and $\iota_i : M_i \longrightarrow M$ to be the inclusion. M is clearly an R-module and ι_i is a morphism for each i. We can compute the composition to see that the diagram

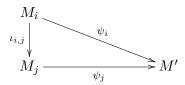


commutes. To prove that indeed $\lim_{\longrightarrow j} M_j = M$ we still need to prove that it satisfies the universal property.

Suppose M' is R-module and $\psi_i: M_i \longrightarrow M'$ morphisms for each i compatible with the structure morphisms. We define $\theta: M \longrightarrow M'$ the following way:

$$m \in M \implies m \in M_t$$
 for some $t \in \mathbb{N}$. We then set $\theta(m) = \psi_t(m)$

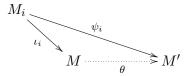
The fact that θ is well defined comes directly from the fact that the ψ_i are compatible with the structure morphisms that are just the inclusions. Indeed, suppose $m \in M$ is such that $m \in M_i$ and $m \in M_i$, where $i \leq j$. Then, since the diagram



commutes, we get that

$$\psi_j(m) = \psi_j \circ \iota_{i,j}(m) = \psi_i(m).$$

Also we get that since the ψ_i are morphisms, θ also is one. By definition of θ , we see that the diagram



commutes for each i. Also θ is unique for this property since if we have $\theta': M \longrightarrow M'$ that respects the desired properties, we get that

For
$$m \in M_i$$
, $\theta'(m) = \theta' \circ \iota_i(m) = \psi_i(m) = \theta(m)$.

Thus we do get that

$$\lim_{n \to \mathbb{N}} M_n = \bigcup_{n \in \mathbb{N}} M_n$$

This example tells us that direct limits of some systems are really simple objects.

Remark 0.0.5. Note that by taking an increasing chain of R-modules and considering it as a \mathbb{Z} -diagram $\{M_n\}_{n\in\mathbb{Z}}$ also with the structure morphisms to be the inclusions, we get that

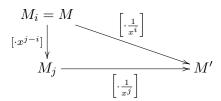
$$\lim_{n \to \mathbb{Z}} M_n = \bigcup_{n \in \mathbb{Z}} M_n$$

Proposition 0.0.6. Let M be a R-module and x be an element of R. We consider the following \mathbb{N} -diagram:

- 1. $M_i = M$ for each $i \in \mathbb{N}$
- 2. For $i \leq j$ we define $\varphi_{i,j} = [\cdot x^{j-i}]$

Then $\lim_{t\to\infty} M_t = M_x$.

Démonstration. We set $\varphi_j = \left[\cdot \frac{1}{x^j} \right]$ and check the compatibility. We see clearly that the diagram



commutes once we describe explicitly what the transformations are. We still have to verify the universal property to declare that M_x is the direct limit of the system.

Suppose we have an R-module M' and a family of morphism $\psi_i: M_i \longrightarrow M'$ such that for every $i \leq j$ the diagram :

$$M_{i} = M$$

$$[\cdot x^{j-i}] \downarrow \qquad \qquad \psi_{i}$$

$$M_{j} = M \xrightarrow{[\cdot x^{j}]} M'$$

commutes. Then we define $\Psi: M_x \longrightarrow M'$ the following way:

For
$$\frac{m}{x^t} \in M_x, \Psi\left(\frac{m}{x^t}\right) = \psi_t(m)$$
.

First, we need to check that Ψ is a well defined function. Suppose that $\frac{m}{x^j} = \frac{n}{x^k}$. By definition of the localization, we get that there is an natural integer t such that

$$x^t(x^k m - x^j n) = 0.$$

That said, we get that

$$\Psi\left(\frac{m}{x^j}\right) = \psi_j(m) = \psi_{j+k+t}(x^{k+t}m) = \psi_{j+k+t}(x^{j+t}n) = \Psi\left(\frac{n}{x^k}\right)$$
$$\Leftrightarrow f_{j+k+t}(x^t(x^km - x^jn)) = f_{j+k+t}(0) = 0$$

which confirms that Ψ is well defined. Now we just need to show that it's a morphism. Here we suppose that $j \leq k$.

$$\Psi\left(\frac{m}{x^{j}} + \frac{n}{x^{k}}\right) = \Psi\left(\frac{x^{k-j}m + n}{x^{k}}\right)$$

$$= \psi_{k}(x^{k-j}m + n)$$

$$= \psi_{k}(x^{k-j}m) + \psi_{k}(n)$$

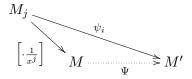
$$= \psi_{j}(m) + \psi_{k}(n)$$

$$= \Psi\left(\frac{m}{x^{j}}\right) + \Psi\left(\frac{n}{x^{k}}\right).$$

This gives us that Ψ is morphism and for $j \in \mathbb{N}$ we have

For
$$m_j \in M_j$$
, $\Psi \circ \left[\cdot \frac{1}{x^j} \right] (m_j) = \Psi \left(\frac{m_j}{x^j} \right) = \psi_j(m_j)$

which tells us that the diagram



commutes. Also the unicity of Ψ for the desired property is pretty obvious.

Proposition 0.0.7. Let (I, \leq) be a P.O. set and Φ be an I-diagram in the category of R-modules. Let E be the submodule of $\bigoplus_{I} A_i$ spanned by

$$\{g \in \bigoplus A_i \mid \text{ for some } i \leq j, g(j) = -\varphi_{i,j}(g(i)) \text{ and } g(t) = 0, \text{ for } t \neq i, j\}.$$

Then $\lim_{\longrightarrow I} A_i = \bigoplus A_i/E$ with the morphisms $\varphi_j = \pi \circ \iota_j$, where $\iota_j : A_j \longrightarrow \bigoplus A_i$ is the standard embedding and $\pi : \bigoplus A_i \longrightarrow \bigoplus A_i/E$ is the canonical projection.

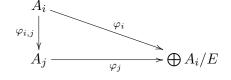
Démonstration. Let $i \leq j$ and set for $a_i \in A_i$, set $g = \iota_j \circ \varphi_{i,j}(a_i)$ and $g' = \iota_i(a_i)$. Then

$$\varphi_j \circ \varphi_{i,j}(a_i) = [g] = [g'] = \varphi_i(a_i) \Leftrightarrow g' - g \in E.$$

But from the definition of g and g' we get that

$$\begin{cases} (g'-g)(j) = g'(j) - g(j) = 0 - \varphi_{i,j}(a_i) = -\varphi_{i,j}(a_i) = -\varphi_{i,j}((g'-g)(i)) \\ (g'-g)(t) = g'(t) - g(t) = 0 - 0 = 0 \text{ if } t \neq i, j \end{cases}$$

which tells us exactly that $(g'-g) \in E$ and that the diagram



commutes. Now suppose we have an R-module A with morphisms $f_i:A_i\longrightarrow A$ that are compatible with the system. We then set $\Psi:\bigoplus A_i/E\longrightarrow A$ the following way:

For
$$[g] \in \bigoplus A_i/E$$
, $\Psi([g]) = \sum_I f_i(g(i))$.

Note that since $g \in \bigoplus A_i$, the sum is finite. We still have to check that Ψ is a well defined function. Suppose [g] = [g']. Then we get that

$$\Psi([g]) = \sum f_i(g(i)) = \sum f(g'(i)) = \Psi([g']) \Leftrightarrow \sum f_i((g - g')(i)) = 0.$$

Since [g] = [g'], we have some data on (g - g'), namely

$$\begin{cases} (g-g')(k) = -\varphi_{j,k}((g-g')(j) \text{ for some } j \le k \\ (g-g')(t) = 0 \text{ if } t \ne j, k \end{cases}$$

which tells us that

$$\sum (f_i((g-g')(i)) = f_j((g-g')(j)) + f_k((g-g')(k))$$

$$= f_j((g-g')(j)) + f_k(-\varphi_{j,k}((g-g')(j)))$$

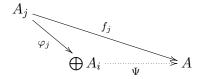
$$= f_j((g-g')(j)) - f_j((g-g')(j))$$

$$= 0$$

and that Ψ is well defined. Also it is pretty clear that Ψ is a morphism. We also get that for $j \in I$, we have that

For
$$a_j \in A_j$$
, $\Psi \circ \varphi_j(a_j) = \sum_{i \in I} f_i(\iota_j(a_j)(i)) = f_j(a_j)$

which tells us that the diagram



commutes. The unicity of Ψ is also pretty obvious.

Remark 0.0.8. This proposition is important as it gives us as a fact that every I-diagram in the category of R-modules admit a direct limit for every P.O. set (I, \leq) and a way to compute it.

Definition 0.0.9. Let (I, \leq) be a P.O. set and R be a ring. We define the category of I-diagrams in the category of R-modules, denoted \mathfrak{Dir}_I^R , whose objects are the I-diagrams in the category of R-modules and the morphisms are just the natural transformations from an I-diagram to an other, i.e for two I-diagrams Φ and Φ' a morphism is given by a family of morphisms $\nu = \{\nu_i : A_i \longrightarrow A_i'\}_{i \in I}$ such that for each pair $i \leq j$ the diagram

$$A_{i} \xrightarrow{\nu_{i}} A'_{i}$$

$$\varphi_{i,j} \downarrow \qquad \qquad \downarrow \varphi'_{i,j}$$

$$A_{j} \xrightarrow{\nu_{i}} A'_{j}$$

commutes.

Definition 0.0.10. Let (I, \leq) be a P.O. set and R be a ring. A chain complex of I-diagrams in the category of R-modules is a sequence of I-diagrams together with morphims of I-diagrams $\Phi_{\bullet} = (\Phi_{(n)}, \nu_{(n)} : \Phi_{(n)} \longrightarrow \Phi_{(n-1)})_{n \in \mathbb{Z}}$ such that $\nu_n \circ \nu_{n+1} = 0$ for each $n \in \mathbb{Z}$. We define cochain

complexes of I-diagrams in the category of R-modules an analogous way with increasing indices. uch a sequence is said to be exact if the sequences

$$\cdots \xrightarrow{\nu_{(n+2)}^i} \Phi_{(n+1)} \xrightarrow{\nu_{(n+1)}^i} \Phi_{(n)} \xrightarrow{\nu_{(n)}^i} \Phi_{(n-1)} \xrightarrow{\nu_{(n-1)}^i} \cdots$$

are exact for each $i \in I$.

Theorem 0.0.11. Let R be a ring, \mathcal{R} be the category of R-modules and (I, \leq) be a P.O. set. Then we have that $\varinjlim : \mathfrak{Dir}_I^R \longrightarrow \mathcal{R}$ is a left exact additive covariant functor.

Definition 0.0.12. Let (I, \leq) be a P.O. set. We say that I is *filtered* if for each $i, j \in I$, there exists $k \in I$ such that $i, j \leq k$. We can also use the word *directed* instead of filtered.

Lemma 0.0.13. Let I be a filtered P.O. set. Let R be a ring and R be the category of R-modules. Let Φ be an I-diagram in R. Let A be an element in A and A is image in A.

- 1. There exists $i \in I$ in I and an element $a_i \in A_i$ such that $\varphi_i(a_i) = [a]$.
- 2. Write $a = (a_i) \in \bigoplus_I A_i$. Then [a] = 0 if and only if there exists an index t such that $a_i = 0$ when $i \nleq t$ and $\sum_{j \leq t} \varphi_{j,t}(a_j) = 0$.

Démonstration. See Lemma 4.32 from [?]

Theorem 0.0.14. Let I be a filtered P.O. set, R a ring, and R be the category of R-modules. Then the functor $\varinjlim : \mathfrak{Dir}_I^R \longrightarrow \mathcal{R}$ is exact

Démonstration. See Theorem 4.33 from [?] for a proof.

Corollary 0.0.15. Let

$$\cdots \xrightarrow{\nu^{(n-2)}} \Phi^{(n-1)} \xrightarrow{\nu^{(n-1)}} \Phi^{(n)} \xrightarrow{\nu^{(n)}} \Phi^{(n+1)} \xrightarrow{\nu^{(n+1)}} \cdots$$

be an exact sequence of I-diagrams. Then the sequence

$$\cdots \xrightarrow{\lim \nu^{(n-2)}} \lim_{\longrightarrow} \Phi^{(n-1)} \xrightarrow{\lim \nu^{(n-1)}} \lim_{\longrightarrow} \Phi^{(n)} \xrightarrow{\lim \nu^{(n)}} \lim_{\longrightarrow} \Phi^{(n+1)} \xrightarrow{\lim \nu^{(n+1)}} \cdots$$

is an exact sequence of R-modules.

Démonstration. This a direct consequence of the last theorem.

Definition 0.0.16. Let

$$\cdots \xrightarrow{\nu^{(n-2)}} \Phi^{(n-1)} \xrightarrow{\nu^{(n-1)}} \Phi^{(n)} \xrightarrow{\nu^{(n)}} \Phi^{(n+1)} \xrightarrow{\nu^{(n+1)}} \cdots$$

be a sequence of *I*-diagrams. For $n \in \mathbb{Z}$ and $i \leq j$, we define

$$H_i^{(n)} = \text{Ker } \nu^{(n)} / \text{Im } \nu^{(n-1)}$$

and maps $[\varphi_{i,j}]: H_i^{(n)} \longrightarrow H_j^{(n)}$ the following way :

For
$$[a] \in H_i^{(n)}, \ \left[\varphi_{i,j}^{(n)}\right]([a]) = \left[\varphi_{i,j}^{(n)}(a)\right].$$

Proposition 0.0.17. The previously defined $\left[\varphi_{i,j}^{(n)}\right]$ are morphisms of R-modules.

Démonstration. First we need to show that for $a \in \operatorname{Ker} \nu_i^{(n)}$, we have that $\varphi_{i,j}^{(n)}(a) \in \operatorname{Ker} \nu_i^{(n)}$. By the definition of morphisms of *I*-diagrams, we that the following diagram

$$\Phi^{(n)}(i) \xrightarrow{\varphi_{i,j}^{(n)}} \Phi^{(n)}(j)$$

$$\downarrow^{\nu_i^{(n)}} \qquad \qquad \downarrow^{\nu_j^{(n)}}$$

$$\Phi^{(n+1)}(i) \xrightarrow{\varphi_{i,j}^{(n+1)}} \Phi^{(n+1)}(j)$$

commutes. Thus we get that

$$\nu_i^{(n)} \circ \varphi_{i,j}^{(n)}(a) = \varphi_{i,j}^{(n+1)} \circ \nu_i^{(n)}(a) = \varphi_{i,j}^{(n+1)}(0) = 0$$

and hence we get that $\varphi_{i,j}^{(n)}(a) \in \operatorname{Ker} \nu_j^{(n)}$. Next we need to prove that for $b \in \operatorname{Im} \nu^{(n-1)}$, $\left[\varphi_{i,j}^{(n)}(b)\right] = 0$. We write $b = \nu_i^{(n-1)}(a)$ for $a \in \Phi^{(n)}(i)$. Again by the definition of *I*-diagrams, we have that the diagram

$$\begin{split} & \Phi^{(n-1)}(i) \xrightarrow{\nu_i^{(n-1)}} & \Phi^{(n)}(i) \\ & \varphi_{i,j}^{(n-1)} \middle\downarrow & & & & \varphi_{i,j}^{(n)} \\ & \Phi^{(n-1)}(j) \xrightarrow{\nu_i^{(n-1)}} & & \Phi^{(n)}(j) \end{split}$$

commutes. Thus we get that

$$\varphi_{i,j}^{(n)}(b) = \varphi_{i,j}^{(n)} \circ \nu_i^{(n-1)}(a) = \nu_j^{(n-1)} \circ \varphi_{i,j}^{(n-1)}(a) \in \operatorname{Im} \nu_j^{(n-1)}$$

and hence $\left[\varphi_{i,j}^{(n)}(b)\right]=0$. This proves that $\left[\varphi_{i,j}^{(n)}\right]$ is a well defined map. The fact that it's a morphism comes directly from the fact that $\varphi_{i,j}^{(n)}$ is already one.

Proposition 0.0.18. The previously defined $H_i^{(n)}$ together with the morphisms $\left[\varphi_{i,j}^{(n)}\right]$ form an I-diagram in the category of R-modules.

Démonstration. This is easily seen when computing that for $i \leq j \leq k$ we get that

$$\left[\varphi_{j,k}^{(n)}\right]\circ\left[\varphi_{i,j}^{(n)}\right]=\left[\varphi_{j,k}^{(n)}\circ\varphi_{i,j}^{(n)}\right]=\left[\varphi_{i,k}^{(n)}\right].$$

Theorem 0.0.19. Let $\Phi^{\bullet} = (\Phi^{(n)}, \ \nu^{(n)} : \Phi^{(n)} \longrightarrow \Phi^{(n+1)})_{n \in \mathbb{Z}}$ be a cochain complex of *I-diagrams in the category of R-modules. We set*

$$\lim_{\longrightarrow} \Phi^{\bullet} = \left(\lim_{\longrightarrow} \Phi^{(n)}, \lim_{\longrightarrow} \nu^{(n)}\right)_{n \in \mathbb{Z}}$$

which is a complex of R-modules.

Then we get that for $n \in \mathbb{Z}$ we have

$$\lim_{\longrightarrow} H_i^{(n)} \cong H^n\Big(\lim_{\longrightarrow} \Phi\Big) = \operatorname{Ker} \lim_{\longrightarrow} \nu^{(n)} / \operatorname{Im} \lim_{\longrightarrow} \nu^{(n-1)}.$$

Démonstration. For $i \in I$, let $\varphi_i : \Phi_i^{(n)} \longrightarrow \varinjlim \Phi^{(n)}$ be the morphisms that come with the direct limit. We define, for each $i \in I$, $\left[\varphi_i^{(n)}\right] : H_i^{(n)} \longrightarrow H^n\left(\varinjlim \Phi\right)$ the following way :

For
$$[a] \in H_i^{(n)}, \ \left[\varphi_i^{(n)}\right]([a]) = \left[\varphi_i^{(n)}(a)\right].$$

We need to show that it's a well defined morphism. The first step is to prove that

$$a \in \operatorname{Ker} \nu_i^{(n)} \implies \varphi_i^{(n)}(a) \in \operatorname{Ker} \lim \nu^{(n)}.$$

To achieve that we use the fact that the diagram

$$\begin{array}{c|c} \Phi^{(n)}(i) & \xrightarrow{\varphi_i^{(n)}} & \lim \Phi^{(n)} \\ \nu_i^{(n)} & & \lim \Phi^{(n)} \\ \hline \downarrow^{\lim \nu^{(n)}} \\ \Phi^{(n+1)}(i) & \xrightarrow{\varphi_i^{(n+1)}} & \lim \Phi^{(n+1)} \end{array}$$

commutes. This gives us that for $a \in \operatorname{Ker} \nu_i^{(n)}$

$$\lim_{i \to \infty} \nu^{(n)} \circ \varphi_i^{(n)}(a) = \varphi_i^{(n+1)} \circ \nu_i^{(n)}(a) = \varphi_i^{(n+1)}(0) = 0 \implies \varphi_i^{(n)}(a) \in \operatorname{Ker} \lim_{i \to \infty} \nu^{(n)}.$$

Let $b \in \operatorname{Im} \nu_i^{(n-1)}$. Next we need to show that

$$b \in \operatorname{Im} \nu_i^{(n-1)} \implies \varphi_i^{(n)}(b) \in \operatorname{Im} \lim_{\longrightarrow} \nu^{(n-1)}.$$

Setting $b = \nu_i^{(n-1)}(a)$ for $a \in \Phi^{(n-1)}(i)$, we use the fact that the following diagram

$$\Phi^{(n-1)}(i) \xrightarrow{\nu_i^{(n-1)}} \Phi^{(n)}(i)$$

$$\varphi_i^{(n-1)} \qquad \qquad \downarrow \varphi_i^{(n)}$$

$$\lim_{\longrightarrow} \Phi^{(n-1)} \xrightarrow{\lim_{\longrightarrow} \nu^{(n-1)}} \lim_{\longrightarrow} \Phi^{(n)}$$

commutes and we can say that

$$\varphi_i^{(n)}(b) = \varphi_i^{(n)} \circ \nu_i^{(n-1)}(a) = \lim_{\longrightarrow} \nu^{(n-1)} \circ \varphi_i^{(n-1)}(a) \in \operatorname{Im} \lim_{\longrightarrow} \nu^{(n-1)}.$$

This show us that the $\left[\varphi_i^{(n)}\right]$ are well defined maps. The fact that they're morphisms comes directly from the fact that the $\varphi_i^{(n)}$ are already morphisms themselves. Also by computing we easily see that for each pair $i \leq j$ the diagram

$$H_{i}^{(n)} \xrightarrow{\left[\varphi_{i,j}^{(n)}\right]} H_{j}^{(n)} \xrightarrow{\left[\varphi_{j}^{(n)}\right]} H^{n}\left(\underset{\longrightarrow}{\lim} \Phi\right)$$

comutes. By the unicity of direct limits, we get that $H^n\left(\varinjlim \Phi\right) \cong \varinjlim H_i^{(n)}$.

This theorem allows us to state an important corollary.

Corollary 0.0.20. Let

$$\cdots \longrightarrow M^{\bullet_{n-1}} \longrightarrow M^{\bullet_n} \longrightarrow M^{\bullet_{n+1}} \longrightarrow \cdots$$

be cochain complex of cochain complexes of R-modules. Then for $j \in \mathbb{Z}$ we have that

$$\lim_{n \to \infty} H^j(M^{\bullet_n}) \cong H^j(\lim_{n \to \infty} M^{\bullet_n}).$$

Démonstration. Noting that cochain complexes of R-modules can be seen as \mathbb{Z} -diagrams in the category of R-modules, the result is directly given by the previous theorem.

Proposition 0.0.21. Let M be an R-module. Let $\{N_t\}_{t\in\mathbb{Z}}$ and $\{K_t\}_{t\in\mathbb{Z}}$ be two chains of increasing submodules of M such that for each $t\in\mathbb{Z}$, there exists natural integers c_t and d_t such that

$$N_t \subseteq K_{t+c_t}$$
 and $K_t \subseteq N_{t+d_t}$.

By considering the two families as a \mathbb{Z} -diagram with the structure morphisms being the inclusions, we get a functorial isomorphism

$$\lim_{t \to t} K_t \cong \lim_{t \to t} N_t.$$

Démonstration. By the Remark 0.0.5, we just have

$$\lim_{t \to t} K_t \cong \bigcup_{t \in \mathbb{Z}} K_t = \bigcup_{t \in \mathbb{Z}} N_t \cong \lim_{t \to t} N_t.$$

Remark 0.0.22. This proof is also valid in a more general case when considering a cofinal system on an Abelian category and replacing the inclusions by monomorphisms.

Corollary 0.0.23. Let $\mathfrak{a} \subset R$ be an ideal and let $\{\mathfrak{a}_t\}_{t\in\mathbb{N}}$ be a decreasing chain of ideals such that for each t, there exists natural integers c and d such that

$$\mathfrak{a}^{t+c} \subseteq \mathfrak{a}_t$$
 and $\mathfrak{a}_{t+d} \subseteq \mathfrak{a}^t$.

Then we have a functorial isomorphism such that

$$\lim_{M \to t} \operatorname{Ext}_{R}^{j}(R/\mathfrak{a}_{t}, M) \cong H_{\mathfrak{a}}^{j}(M)$$

Démonstration. We use the fact that for each $t, j \in \mathbb{N}$ we have

$$\mathfrak{a}^{t+c} \subseteq \mathfrak{a}_t \text{ and } \mathfrak{a}_{t+d} \subseteq \mathfrak{a}^t \implies \begin{cases} \operatorname{Hom}_R(R/\mathfrak{a}_t, I^j) \subseteq \operatorname{Hom}_R(R/\mathfrak{a}^{t+c}, I^j) \\ \operatorname{Hom}_R(R/\mathfrak{a}^t, I^j) \subseteq \operatorname{Hom}_R(R/\mathfrak{a}_{t+d}, I^j) \end{cases}$$

where I^{\bullet} is an injective resolution of M. By the last proposition, this tells us that

$$\lim_{t \to t} \operatorname{Hom}_{R}(R/\mathfrak{a}^{t}, I^{\bullet}) \cong \lim_{t \to t} \operatorname{Hom}_{r}(R/\mathfrak{a}_{t}, I^{\bullet}).$$

By the previous proposition, we finally get that

$$\lim_{\longrightarrow t} \operatorname{Ext}_{R}^{j}(R/\mathfrak{a}^{t}, M) = \lim_{\longrightarrow t} H^{j}(\operatorname{Hom}_{R}(R/\mathfrak{a}^{t}, I^{\bullet}))$$

$$\cong H^{j}(\lim_{\longrightarrow t} \operatorname{Hom}_{R}(R/\mathfrak{a}_{t}, I^{\bullet}))$$

$$\cong \lim_{\longrightarrow t} H^{j}(\operatorname{Hom}_{R}(R/\mathfrak{a}_{t}, I^{\bullet}))$$

$$= \lim_{\longrightarrow t} \operatorname{Ext}_{R}^{j}(R/\mathfrak{a}_{t}, M).$$

Proposition 0.0.24. Let M be an R-module. Let $\{N_t\}_{t\in\mathbb{Z}}$ and $\{K_t\}_{t\in\mathbb{Z}}$ be two chains of increasing submodules of M such that for each $t\in\mathbb{Z}$, there exists natural integers c_t and d_t such that

$$N_t \subseteq K_{t+c_t}$$
 and $K_t \subseteq N_{t+d_t}$.

By considering the two families as a \mathbb{Z} -diagram with the structure morphisms being the inclusions, we get a functorial isomorphism

$$\lim_{t \to t} K_t \cong \lim_{t \to t} N_t$$

 $D\'{e}monstration.$ See Proposition 0.0.21 for the proof.

Corollary 0.0.25. Let $\mathfrak{a} \subset R$ be an ideal and let $\{\mathfrak{a}_t\}_{t\in\mathbb{N}}$ be a decreasing chain of ideals such that for each t, there exists natural integers c and d such that

$$\mathfrak{a}^{t+c} \subseteq \mathfrak{a}_t$$
 and $\mathfrak{a}_{t+d} \subseteq \mathfrak{a}^t$.

Then we have a functorial isomorphism such that

$$\lim_{\longrightarrow t}\operatorname{Ext}_R^j(R/\mathfrak{a}_t,M)\cong H^j_{\mathfrak{a}}(M)$$

 $D\acute{e}monstration.$ See Corollary 0.0.23 for the proof.