# Homotopical Algebra
# Hand in 2

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## Notation :

From now on in this document, we will use the followind notation. For $c' \xrightarrow{f} c \in \operatorname{Ob} \mathcal{C}/c$, we will note $(c', f)$ and $(c'', h, c') \in \mathcal{C}((c'', g), (c', f))$ for every morphisms from $(c'', g)$ to $(c', f)$

$$\begin{tikzcd} c'' \arrow[rr, "h"] \arrow[dr, "g"'] & & c' \arrow[dl, "f"] \\ & c & \end{tikzcd}$$

This may be a confusing notation since it does not contain the information of $f$ or $g$, but it will be made clear from context.

## Exercise 7 :

### Part (a) :

Let $d \in \operatorname{Ob} \mathscr{D}$ and let us define the functor $\bar{G} : \mathscr{D}/d \longrightarrow \mathcal{C}/G(d)$. We set $\bar{G}(d', f) = (G(d'), G(f))$ on objects of $\mathscr{D}/d$ and $\bar{G}((d'', h, d')) = (G(d''), G(h), G(d'))$ for the a morphism from $(d'', g)$ to$(d', f)$. Observe that $(G(d''), G(h), G(d'))$ is indeed a morphism since we have that the commutativity of the diagram

$$\begin{tikzcd} d'' \arrow[rr, "h"] \arrow[dr, "g"'] & & d' \arrow[dl, "f"] \\ & d & \end{tikzcd}$$

implies the commutativity of the diagram

$$\begin{tikzcd} G(d'') \arrow[rr, "G(h)"] \arrow[dr, "G(g)"'] & & G(d') \arrow[dl, "G(f)"] \\ & G(d) & \end{tikzcd}$$

Now we need to prove that this defines a functor. Let us compute the image of $\operatorname{Id}_{(d',f)}$ for $(d', f) \in \operatorname{Ob} \mathscr{D}/d$. We have

$$\bar{G}(\operatorname{Id}_{(d',f)}) = (G(d'), G(\operatorname{Id}_{d'}), G(d')) = (G(d'), \operatorname{Id}_{G(d')}, G(d')) = \operatorname{Id}_{\bar{G}(d',f)}.$$

Now let us compute the image of a composition. For $(d''', i, d'')$ and $(d'', j, d')$, we have

$$\begin{aligned} \bar{G}((d'', j, d')(d''', i, d'')) =& \bar{G}(d''', ji, d') \\ =& (G(d'''), G(ji), G(d')) \\ =& (G(d'''), G(j)G(i), G(d')) \\ =& (G(d''), G(j), G(d'))(G(d'''), G(i), G(d'')) \\ =& \bar{G}(d'', j, d')\bar{G}(d''', i, d''). \end{aligned}$$

This allows to say that $\bar{G}$ is a functor.

### Part (b) :

As we know that the pair $(F, G)$ is a pair of adjoint functors, we know there exists a natural isomorphism

$$\tau : \mathcal{C}(-, G(-)) \Longrightarrow \mathscr{D}(F(-), -).$$

We set the following functor $\bar{F} : \mathcal{C}/G(d) \longrightarrow \mathscr{D}/d$ the following way. On the objects, we set $\bar{F}(c, f) = (F(c), \tau_{(c,d)}(f))$ and on morphisms $\bar{F}(c', h, c) = (F(c'), F(h), F(c))$ for a morphism form $(c, f)$ to $(c', g)$. We need now to prove that $(F(c'), F(h), F(c))$ is indeed a morphism from $(F(c'), \tau_{(c',d)}(g))$ to $(F(c), \tau_{(c,d)}(f))$, i.e. we need to prove that the triangle

$$\begin{tikzcd} F(c') \arrow[rr, "F(h)"] \arrow[dr, "\tau_{(c',d)}(g)"'] & & F(c) \arrow[dl, "\tau_{(c,d)}(f)"] \\ & d & \end{tikzcd}$$

Note first that since $(c', h, c)$ is a morphism from $(c, f)$ to $(c', g)$, we have $fh = g$. The fact that $\tau$ is a natural transformation gives us that

$$\begin{aligned} \mathscr{D}(F(-), -)(h^{op}, \mathrm{Id}_d)\tau_{(c,d)}(f) &= \mathrm{Id}_d \tau_{(c,d)}(f) F(h) = \tau_{(c,d)}(f) F(h) \\ \tau_{(c',d)} \mathcal{C}(-, G(-))(h^{op}, \mathrm{Id}_d)(f) &= \tau_{(c',d)}(G(\mathrm{Id}_d) f h) = \tau_{(c',d)}(fh) = \tau_{(c'd)}(g) \end{aligned}$$

those two ligns are equal and so we conclude that the last triangle commutes.
Now we need to prove that $\bar{F}$ is a functor. First consider $\mathrm{Id}_{(c,f)}$ and let us compute its image. We have

$$\bar{F}(\mathrm{Id}_{(c,f)}) = (F(c), F(\mathrm{Id}_{F(c)}, F(c))) = \mathrm{Id}_{F(c)} = \mathrm{Id}_{\bar{F}(c,f)}.$$

Let us compute the image of a composition. Suppose, we have $(c'', i, c')$ and $(c', j, c)$. Then

$$\begin{aligned} \bar{F}((c', j, c)(c'', i, c')) &= \bar{F}(c'', ji, c) \\ &= (F(c''), F(ji), F(c)) \\ &= (F(c''), F(j)F(i), F(c)) \\ &= (F(c''), F(j), F(c'))(F(c'))(F(c'), F(i), F(c)) \\ &= \bar{F}(c'', j, c')\bar{F}(c', i, c) \end{aligned}$$

and thus we conclude that $\bar{F}$ is a functor. What is left is to prove that the pair $(\bar{F}, \bar{G})$ is a pair of adjoint functors. Let $(c, f) \in \mathrm{Ob}\, \mathcal{C}/G(d)$ and $(d', g) \in \mathrm{Ob}\, \mathscr{D}$. We would like to find a map from $\mathcal{C}/G(d)((c, f), (G(d'), G(g)))$ to $\mathscr{D}/d((F(c), \tau_{(c,d)}(f)), (d', g))$, that is if we have the commutative triangle

$$\begin{tikzcd} c \arrow[rr, "h"] \arrow[dr, "f"'] & & G(d') \arrow[dl, "G(g)"] \\ & G(d) & \end{tikzcd}$$

we would like to send $h$ to a map $\bar{h}$ such that the triangle

$$\begin{tikzcd} F(c) \arrow[rr, "\bar{h}"] \arrow[dr, "\tau_{(c,d)}(f)"'] & & d' \arrow[dl, "g"] \\ & d & \end{tikzcd}$$

commutes. Recall that since $\tau$ is a natural isomorphism, the square

$$\begin{tikzcd} \mathcal{C}(G(d'), G(d')) \arrow[r, "\tau_{(G(d'),d')}"] \arrow[d, "\bar{\mathcal{C}}(h^{op},g)"'] & \mathscr{D}(FG(d'), d') \arrow[d, "\bar{\mathscr{D}}(h^{op},g)"] \\ \mathcal{C}(c', G(d)) \arrow[r, "\tau_{(c,d)}"'] & \mathscr{D}(F(c'), d) \end{tikzcd}$$

commutes, where $\bar{\mathcal{C}} = \mathcal{C}(-, G(-))$ and $\bar{\mathscr{D}} = \mathscr{D}(F(-), -)$. Note that we then have that

$$\begin{aligned} \tau_{(c,d)}(f) =& \tau_{(c,d)}(G(g)h) = \tau_{(c,d)}\bar{\mathcal{C}}(h^{op}, g)(\mathrm{Id}_{G(d')}) \\ =& \bar{\mathscr{D}}(h^{op}, g)\tau_{(G(d'),d')}(\mathrm{Id}_{G(d')}) \\ =& g\tau_{(G(d'),d')}(\mathrm{Id}_{G(d')})F(h) \end{aligned}$$

and so our guess is $\bar{h} = \tau' F(h)$, where $\tau' = \tau_{(G(d'),(d'))}$. Let us denote the family of maps obtained by $\theta_{((c,f),(d',g)}$ and let us denote by $\theta$ the natural transformation given by this family.