Localization EKF

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1 State representation $\boldsymbol{x}$ and motion input $\boldsymbol{u}$

Let x and y represent positions in the x- and y-directions, respectively. The variables θ, ν, ω, and α denote the yaw angle, linear velocity, yaw angle rate, and linear acceleration, respectively. The target state $\boldsymbol{x}$ is defined as:

$$\boldsymbol{x} = [x, y, \theta, \nu, \omega, \alpha]^T$$

The motion input is defined as:

$$\boldsymbol{u} = [\omega, \alpha]^T$$

2 The measurement $\boldsymbol{z}$

Suppose we have a GPS sensor and IMU sensor. The GPS sensor provides the longitudinal position x and lateral position y. The IMU sensor provides yaw angle θ, yaw angle rate ω, and linear acceleration α. The measurement $\boldsymbol{z}$ is defined as:

$$\boldsymbol{z} = [x, y, \theta, \omega, \alpha]^T$$

3 State transition function $\boldsymbol{f}$ and its derivative

The 2-D kinematic equation for the vehicle state at time t, denoted as $\boldsymbol{x}_t$ and described by kinematic model in discrete time space, can be expressed as follows:

$$\begin{bmatrix} x_t \\ y_t \\ \theta_t \\ \nu_t \\ \omega_t \\ \alpha_t \end{bmatrix} = \begin{bmatrix} x_{t-1} + (\nu_{t-1}\Delta t + \frac{1}{2}\alpha_{t-1}\Delta t^2)\cos(\theta_{t-1} + \frac{1}{2}\omega_{t-1}\Delta t) + \epsilon_x \\ y_{t-1} + (\nu_{t-1}\Delta t + \frac{1}{2}\alpha_{t-1}\Delta t^2)\sin(\theta_{t-1} + \frac{1}{2}\omega_{t-1}\Delta t) + \epsilon_y \\ \theta_{t-1} + \omega_{t-1}\Delta t + \epsilon_\theta \\ \nu_{t-1} + \alpha_{t-1}\Delta t + \epsilon_\nu \\ \omega_{t-1} + \epsilon_\omega \\ \alpha_{t-1} + \epsilon_\alpha \end{bmatrix} \tag{1}$$

We assume that the noises originate from the motion input. The above equation (1) can be expressed as the non-linear function $\boldsymbol{f}$:

$$\boldsymbol{x}_t = \boldsymbol{f}(\boldsymbol{x}_{t-1}, \boldsymbol{u}_t, \boldsymbol{\epsilon}_t)$$

Here, the vector $\boldsymbol{\epsilon}_t = [\epsilon_x, \epsilon_y, \epsilon_\theta, \epsilon_\nu, \epsilon_\omega, \epsilon_\alpha]^T$ represents the system noise. We make the assumption that ϵ_x, ϵ_y, ϵ_θ, and ϵ_ν are all set to zero, ensuring that $\boldsymbol{\epsilon}_t$ is solely associated with the motion input $\boldsymbol{u}_t$. Additionally, the non-linear function $\boldsymbol{f}$ can be approximated using a Taylor expansion at the previously updated state $\boldsymbol{\mu}_{t-1}$, with $\boldsymbol{\epsilon}_t = \mathbf{0}$:

$$\boldsymbol{f}(\boldsymbol{x}_{t-1}, \boldsymbol{u}_t, \boldsymbol{\epsilon}_t) \approx \boldsymbol{f}(\boldsymbol{\mu}_{t-1}, \boldsymbol{u}_t, \mathbf{0}) + \boldsymbol{F}_t(\boldsymbol{x}_{t-1} - \boldsymbol{\mu}_{t-1}) + \boldsymbol{W}_t\boldsymbol{\epsilon}_t$$

Here, $\boldsymbol{F}_t$ represents the derivative of the state transition function $\boldsymbol{f}$ with respect to state $\boldsymbol{x}$, and $\boldsymbol{W}_t$ represents the derivative of the state transition function $\boldsymbol{f}$ with respect to the noise $\boldsymbol{\epsilon}$. The term $\boldsymbol{W}_t$ essentially denotes the noise gain of the system. To simplify the notation, we define Δl as $\nu_{t-1}\Delta t + \frac{1}{2}\alpha_{t-1}\Delta t^2$ and $\Delta\theta$ as $\theta_{t-1} + \frac{1}{2}\omega_{t-1}\Delta t$. The derivative matrix of the state transition function $\boldsymbol{F}_t$ is denoted as:

$$\begin{aligned}
\boldsymbol{F}_t &= \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}} \\
&= \begin{bmatrix}
1 & 0 & -\Delta l \sin(\Delta\theta) & \Delta t \cos(\Delta\theta) & -\frac{1}{2}\Delta t \Delta l \sin(\Delta\theta) & \frac{1}{2}\Delta t^2 \cos(\Delta\theta) \\
0 & 1 & \Delta l \cos(\Delta\theta) & \Delta t \sin(\Delta\theta) & \frac{1}{2}\Delta t \Delta l \cos(\Delta\theta) & \frac{1}{2}\Delta t^2 \sin(\Delta\theta) \\
0 & 0 & 1 & 0 & \Delta t & 0 \\
0 & 0 & 0 & 1 & 0 & \Delta t \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\end{aligned}$$

Furthermore, $\boldsymbol{W}_t$ can be expressed in matrix form:

$$\begin{aligned}
\boldsymbol{W}_t &= \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{\epsilon}} \\
&= \begin{bmatrix}
0 & 0 & 0 & 0 & -\frac{1}{2}\Delta t \Delta l \sin(\Delta\theta) & \frac{1}{2}\Delta t^2 \cos(\Delta\theta) \\
0 & 0 & 0 & 0 & \frac{1}{2}\Delta t \Delta l \cos(\Delta\theta) & \frac{1}{2}\Delta t^2 \sin(\Delta\theta) \\
0 & 0 & 0 & 0 & \Delta t & 0 \\
0 & 0 & 0 & 0 & 0 & \Delta t \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\end{aligned}$$

4 Measurement function $\boldsymbol{h}$ and its derivative

The measurements of the system can be described by a linear relation between the state and the measurement at the current step t. This relation is expressed as:

$$\begin{bmatrix} z_x \\ z_y \\ z_\theta \\ z_\omega \\ z_\alpha \end{bmatrix} = \begin{bmatrix} x_t + \delta_x \\ y_t + \delta_y \\ \theta_t + \delta_\theta \\ \omega_t + \delta_\omega \\ \alpha_t + \delta_\alpha \end{bmatrix} \tag{2}$$

We assume that the measurements are associated with measurement noises, and the noises are independent of each other. The above equation (2) can be expressed as the linear function $\boldsymbol{h}$:

$$\boldsymbol{z}_t = \boldsymbol{h}(\boldsymbol{x}_t, \boldsymbol{\delta}_t)$$

Here, $\boldsymbol{\delta}_t = [\delta_x, \delta_y, \delta_\theta, \delta_\omega, \delta_\alpha]^T$ represents the measurement error. We can apply the same technique to expend the function $\boldsymbol{h}$ at the previously predicted state $\bar{\boldsymbol{\mu}}_t$, with $\boldsymbol{\delta}_t = \mathbf{0}$:

$$\boldsymbol{h}(\boldsymbol{x}_t, \boldsymbol{\delta}_t) \approx \boldsymbol{h}(\bar{\boldsymbol{\mu}}_t, \mathbf{0}) + \boldsymbol{H}_t(\boldsymbol{x}_t - \bar{\boldsymbol{\mu}}_t) + \boldsymbol{V}_t\boldsymbol{\delta}_t$$

Here, $\boldsymbol{H}_t$ represents the derivative of the measurement function $\boldsymbol{h}$ with respect to $\boldsymbol{x}$, and $\boldsymbol{V}_t$ represents the derivative of the measurement function $\boldsymbol{h}$ with respect to the noise $\boldsymbol{\delta}$. The term

$\boldsymbol{V}_t$ is the noise gain of the measurement. The derivative matrix of the measurment function is denoted as:

$$
\begin{aligned}
\boldsymbol{H}_t &= \frac{\partial \boldsymbol{h}}{\partial \boldsymbol{x}} \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.
\end{aligned}
$$

In addition, $\boldsymbol{V}_t$ can be expressed in matrix form:

$$
\begin{aligned}
\boldsymbol{V}_t &= \frac{\partial \boldsymbol{h}}{\partial \boldsymbol{\delta}} \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

5 Process noise covariance matrix $\boldsymbol{Q}$

As we mentioned earlier, we assume that the system noise $\boldsymbol{\epsilon}_t$ is associated with the motion input $\boldsymbol{u}_t$. Furthermore, we assume that $\boldsymbol{\epsilon}_t$ follows a normal distribution with a mean of $\mathbf{0}$ and a covariance matrix of $\boldsymbol{Q}_t$. Additionally, we define the covariance matrix $\boldsymbol{Q}_t$ as

$$
\boldsymbol{Q}_t = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sigma_\omega^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sigma_\alpha^2 \end{bmatrix}
$$

Here, σ_ω and σ_α are the standard deviation of the yaw angle rate and linear acceleration, respectively.

6 Measurement noise covariance matrix $\boldsymbol{R}$

For the measurement error $\boldsymbol{\delta}_t$, we assume it follows a normal distribution with a mean of $\mathbf{0}$ and a covariance matrix of $\boldsymbol{R}_t$. Additionally, we assume that all measurement noises are independent. Consequently, we can disregard any interaction between them, resulting in the following diagonal covariance matrix:

$$
\boldsymbol{R}_t = \begin{bmatrix} \tau_x^2 & 0 & 0 & 0 & 0 \\ 0 & \tau_y^2 & 0 & 0 & 0 \\ 0 & 0 & \tau_\theta^2 & 0 & 0 \\ 0 & 0 & 0 & \tau_\omega^2 & 0 \\ 0 & 0 & 0 & 0 & \tau_\alpha^2 \end{bmatrix}.
$$

Here, τ_x, τ_y, τ_θ, τ_ω, and τ_α represent the standard deviations of x, y, θ, ω, and α, respectively.

7 Kalman filter algorithm

Now we are ready to implement extended Kalman filter (EKF) for the vehicle localization. Given the initial state $\boldsymbol{\mu}_0$ and state covariance $\boldsymbol{\Sigma}_0$, the EKF algorithm is summarized as follows:

Prediction:

$$\begin{array}{ll} \text{Predicted state estimate:} & \bar{\boldsymbol{\mu}}_t = \boldsymbol{f}_t(\boldsymbol{\mu}_{t-1}, \boldsymbol{u}_t) \\ \text{Predicted error covariance:} & \bar{\boldsymbol{\Sigma}}_t = \boldsymbol{F}_t \boldsymbol{\Sigma}_{t-1} \boldsymbol{F}_t^T + \boldsymbol{W}_t \boldsymbol{Q}_t \boldsymbol{W}_t^T \end{array}$$

Update:

$$\begin{array}{ll} \text{Innovation:} & \boldsymbol{y}_t = \boldsymbol{z}_t - \boldsymbol{h}(\bar{\boldsymbol{\mu}}_t) \\ \text{Innovation covariance:} & \boldsymbol{S}_t = \boldsymbol{H}_t \bar{\boldsymbol{\Sigma}}_t \boldsymbol{H}_t^T + \boldsymbol{V}_t \boldsymbol{R}_t \boldsymbol{V}_t^T \\ \text{Kalman gain:} & \boldsymbol{K}_t = \bar{\boldsymbol{\Sigma}}_t \boldsymbol{H}_t^T \boldsymbol{S}_t^{-1} \\ \text{Updated state estimate:} & \boldsymbol{\mu}_t = \bar{\boldsymbol{\mu}}_t + \boldsymbol{K}_t \boldsymbol{y}_t \\ \text{Updated error covariance:} & \boldsymbol{\Sigma}_t = (\boldsymbol{I} - \boldsymbol{K}_t \boldsymbol{H}_t) \bar{\boldsymbol{\Sigma}}_t \end{array}$$

For the updated error covariance, the Joseph formula should be employed for numerical stability. This can be expressed as follows:

$$\boldsymbol{\Sigma}_t = (\boldsymbol{I} - \boldsymbol{K}_t \boldsymbol{H}_t) \bar{\boldsymbol{\Sigma}}_t (\boldsymbol{I} - \boldsymbol{K}_t \boldsymbol{H}_t)^T + \boldsymbol{K}_t \boldsymbol{V}_t \boldsymbol{R}_t \boldsymbol{V}_t^T \boldsymbol{K}_t^T$$

In the case where the measurement model is correct, the Kalman filter utilizes it for updates. Thus, a conditional statement for data association is introduced. The Mahalanobis distance is calculated for the measurement residual to determine if the measurement is suitable for updating:

$$(\boldsymbol{z}_t - \boldsymbol{h}(\bar{\boldsymbol{\mu}}_t))^T \boldsymbol{S}_t^{-1} (\boldsymbol{z}_t - \boldsymbol{h}(\bar{\boldsymbol{\mu}}_t)) \leq D_{th},$$

Here, D_{th} represents a predetermined threshold. One can further show that the quadratic term is actually a χ_m^2 distribution, where m is the number of measurements.