

# Localization EKF

Yi-Chen Zhang

January 3, 2024

## 1 State representation $\mathbf{x}$ and motion input $\mathbf{u}$

Let  $x$  and  $y$  represent positions in the  $x$ - and  $y$ -directions, respectively. The variables  $\theta$ ,  $\nu$ ,  $\omega$ , and  $\alpha$  denote the yaw angle, linear velocity, yaw angle rate, and linear acceleration, respectively. The target state  $\mathbf{x}$  is defined as:

$$\mathbf{x} = [x, y, \theta, \nu, \omega, \alpha]^T$$

The motion input is defined as:

$$\mathbf{u} = [\omega, \alpha]^T$$

## 2 The measurement $\mathbf{z}$

Consider a scenario where we have both a GPS sensor and an IMU sensor. The GPS sensor provides information about the longitudinal position ( $x$ ) and lateral position ( $y$ ). On the other hand, the IMU sensor supplies data regarding the yaw angle ( $\theta$ ), yaw angle rate ( $\omega$ ), and linear acceleration ( $\alpha$ ). Additionally, the vehicle itself provides its velocity ( $\nu$ ). We can define the measurement vector  $\mathbf{z}$  as follows:

$$\mathbf{z} = [x, y, \theta, \omega, \alpha, \nu]^T$$

This representation captures the key measurements of interest.

## 3 State transition function $\mathbf{f}$ and its derivative

The 2-D kinematic equation for the vehicle state at time  $t$ , denoted as  $\mathbf{x}_t$  and described by kinematic model in discrete time space, can be expressed as follows:

$$\begin{bmatrix} x_t \\ y_t \\ \theta_t \\ \nu_t \\ \omega_t \\ \alpha_t \end{bmatrix} = \begin{bmatrix} x_{t-1} + (\nu_{t-1}\Delta t + \frac{1}{2}\alpha_{t-1}\Delta t^2) \cos(\theta_{t-1} + \frac{1}{2}\omega_{t-1}\Delta t) + \epsilon_x \\ y_{t-1} + (\nu_{t-1}\Delta t + \frac{1}{2}\alpha_{t-1}\Delta t^2) \sin(\theta_{t-1} + \frac{1}{2}\omega_{t-1}\Delta t) + \epsilon_y \\ \theta_{t-1} + \omega_{t-1}\Delta t + \epsilon_\theta \\ \nu_{t-1} + \alpha_{t-1}\Delta t + \epsilon_\nu \\ \omega_{t-1} + \epsilon_\omega \\ \alpha_{t-1} + \epsilon_\alpha \end{bmatrix} \quad (1)$$

The state transition equation (1) can be expressed through the nonlinear function  $\mathbf{f}$  as follows:

$$\mathbf{x}_t = \mathbf{f}(\mathbf{x}_{t-1}, \mathbf{u}_t, \boldsymbol{\epsilon}_t)$$

Here, the vector  $\boldsymbol{\epsilon}_t = [\epsilon_x, \epsilon_y, \epsilon_\theta, \epsilon_\nu, \epsilon_\omega, \epsilon_\alpha]^T$  represents the system noise. To simplify our analysis, we assume that system noises  $\epsilon_x$ ,  $\epsilon_y$ ,  $\epsilon_\theta$ , and  $\epsilon_\nu$  are set to zero, making  $\boldsymbol{\epsilon}_t$  solely associated with

the motion input  $\mathbf{u}_t$ . Furthermore, the nonlinear function  $\mathbf{f}$  can be approximated through a Taylor expansion around the previously updated state  $\boldsymbol{\mu}_{t-1}$ , with  $\boldsymbol{\epsilon}_t = \mathbf{0}$ :

$$\mathbf{f}(\mathbf{x}_{t-1}, \mathbf{u}_t, \boldsymbol{\epsilon}_t) \approx \mathbf{f}(\boldsymbol{\mu}_{t-1}, \mathbf{u}_t, \mathbf{0}) + \mathbf{F}_t(\mathbf{x}_{t-1} - \boldsymbol{\mu}_{t-1}) + \mathbf{W}_t\boldsymbol{\epsilon}_t$$

Here,  $\mathbf{F}_t$  represents the derivative of the state transition function  $\mathbf{f}$  with respect to state  $\mathbf{x}$ , and  $\mathbf{W}_t$  represents the derivative of the state transition function  $\mathbf{f}$  with respect to the noise  $\boldsymbol{\epsilon}$ . The term  $\mathbf{W}_t$  essentially denotes the noise gain of the system. To simplify the notation, we define  $\Delta l$  as  $\nu_{t-1}\Delta t + \frac{1}{2}\alpha_{t-1}\Delta t^2$  and  $\Delta\theta$  as  $\theta_{t-1} + \frac{1}{2}\omega_{t-1}\Delta t$ . The derivative matrix of the state transition function  $\mathbf{F}_t$  is denoted as:

$$\begin{aligned} \mathbf{F}_t &= \frac{\partial \mathbf{f}}{\partial \mathbf{x}} \\ &= \begin{bmatrix} 1 & 0 & -\Delta l \sin(\Delta\theta) & \Delta t \cos(\Delta\theta) & -\frac{1}{2}\Delta t \Delta l \sin(\Delta\theta) & \frac{1}{2}\Delta t^2 \cos(\Delta\theta) \\ 0 & 1 & \Delta l \cos(\Delta\theta) & \Delta t \sin(\Delta\theta) & \frac{1}{2}\Delta t \Delta l \cos(\Delta\theta) & \frac{1}{2}\Delta t^2 \sin(\Delta\theta) \\ 0 & 0 & 1 & 0 & \Delta t & 0 \\ 0 & 0 & 0 & 1 & 0 & \Delta t \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

Furthermore,  $\mathbf{W}_t$  can be expressed in matrix form:

$$\begin{aligned} \mathbf{W}_t &= \frac{\partial \mathbf{f}}{\partial \boldsymbol{\epsilon}} \\ &= \begin{bmatrix} 0 & 0 & 0 & 0 & -\frac{1}{2}\Delta t \Delta l \sin(\Delta\theta) & \frac{1}{2}\Delta t^2 \cos(\Delta\theta) \\ 0 & 0 & 0 & 0 & \frac{1}{2}\Delta t \Delta l \cos(\Delta\theta) & \frac{1}{2}\Delta t^2 \sin(\Delta\theta) \\ 0 & 0 & 0 & 0 & \Delta t & 0 \\ 0 & 0 & 0 & 0 & 0 & \Delta t \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

## 4 Measurement function $\mathbf{h}$ and its derivative

The measurements of the system can be described by a linear relation between the state and the measurement at the current step  $t$ . This relation is expressed as:

$$\begin{bmatrix} z_x \\ z_y \\ z_\theta \\ z_\omega \\ z_\alpha \\ z_\nu \end{bmatrix} = \begin{bmatrix} x_t + \delta_x \\ y_t + \delta_y \\ \theta_t + \delta_\theta \\ \omega_t + \delta_\omega \\ \alpha_t + \delta_\alpha \\ \nu_t + \delta_\nu \end{bmatrix} \quad (2)$$

We assume that the measurements are associated with measurement noises, and the noises are independent of each other. The above equation (2) can be expressed as the linear function  $\mathbf{h}$ :

$$\mathbf{z}_t = \mathbf{h}(\mathbf{x}_t, \boldsymbol{\delta}_t)$$

Here,  $\boldsymbol{\delta}_t = [\delta_x, \delta_y, \delta_\theta, \delta_\omega, \delta_\alpha, \delta_\nu]^T$  represents the measurement error. We can apply the same technique to expand the function  $\mathbf{h}$  at the previously predicted state  $\bar{\boldsymbol{\mu}}_t$ , with  $\boldsymbol{\delta}_t = \mathbf{0}$ :

$$\mathbf{h}(\mathbf{x}_t, \boldsymbol{\delta}_t) \approx \mathbf{h}(\bar{\boldsymbol{\mu}}_t, \mathbf{0}) + \mathbf{H}_t(\mathbf{x}_t - \bar{\boldsymbol{\mu}}_t) + \mathbf{V}_t\boldsymbol{\delta}_t$$

Here,  $\mathbf{H}_t$  represents the derivative of the measurement function  $\mathbf{h}$  with respect to  $\mathbf{x}$ , and  $\mathbf{V}_t$  represents the derivative of the measurement function  $\mathbf{h}$  with respect to the noise  $\boldsymbol{\delta}$ . The term  $\mathbf{V}_t$  is the noise gain of the measurement. The derivative matrix of the measurement function is denoted as:

$$\begin{aligned}\mathbf{H}_t &= \frac{\partial \mathbf{h}}{\partial \mathbf{x}} \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}.\end{aligned}$$

In addition,  $\mathbf{V}_t$  can be expressed in matrix form:

$$\begin{aligned}\mathbf{V}_t &= \frac{\partial \mathbf{h}}{\partial \boldsymbol{\delta}} \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}\end{aligned}$$

## 5 Process noise covariance matrix $\mathbf{Q}$

As we mentioned earlier, we assume that the system noise  $\boldsymbol{\epsilon}_t$  is associated with the motion input  $\mathbf{u}_t$ . Furthermore, we assume that  $\boldsymbol{\epsilon}_t$  follows a normal distribution with a mean of  $\mathbf{0}$  and a covariance matrix of  $\mathbf{Q}_t$ . Additionally, we define the covariance matrix  $\mathbf{Q}_t$  as

$$\mathbf{Q}_t = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sigma_\omega^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sigma_\alpha^2 \end{bmatrix}$$

Here,  $\sigma_\omega$  and  $\sigma_\alpha$  are the standard deviation of the yaw angle rate and linear acceleration, respectively.

## 6 Measurement noise covariance matrix $\mathbf{R}$

For the measurement error  $\boldsymbol{\delta}_t$ , we assume it follows a normal distribution with a mean of  $\mathbf{0}$  and a covariance matrix of  $\mathbf{R}_t$ . Additionally, we assume that all measurement noises are independent. Consequently, we can disregard any interaction between them, resulting in the following diagonal covariance matrix:

$$\mathbf{R}_t = \begin{bmatrix} \tau_x^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & \tau_y^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \tau_\theta^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \tau_\omega^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \tau_\alpha^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & \tau_\nu^2 \end{bmatrix}.$$

Here,  $\tau_x$ ,  $\tau_y$ ,  $\tau_\theta$ ,  $\tau_\omega$ ,  $\tau_\alpha$ , and  $\tau_\nu$  represent the standard deviations of  $x$ ,  $y$ ,  $\theta$ ,  $\omega$ ,  $\alpha$ , and  $\nu$  respectively.

## 7 Kalman filter algorithm

Now we are ready to implement extended Kalman filter (EKF) for the vehicle localization. Given the initial state  $\boldsymbol{\mu}_0$  and state covariance  $\boldsymbol{\Sigma}_0$ , the EKF algorithm is summarized as follows:

### Prediction:

$$\begin{aligned} \text{Predicted state estimate:} \quad & \bar{\boldsymbol{\mu}}_t = \mathbf{f}_t(\boldsymbol{\mu}_{t-1}, \mathbf{u}_t) \\ \text{Predicted error covariance:} \quad & \bar{\boldsymbol{\Sigma}}_t = \mathbf{F}_t \boldsymbol{\Sigma}_{t-1} \mathbf{F}_t^T + \mathbf{W}_t \mathbf{Q}_t \mathbf{W}_t^T \end{aligned}$$

### Update:

$$\begin{aligned} \text{Innovation:} \quad & \mathbf{y}_t = \mathbf{z}_t - \mathbf{h}(\bar{\boldsymbol{\mu}}_t) \\ \text{Innovation covariance:} \quad & \mathbf{S}_t = \mathbf{H}_t \bar{\boldsymbol{\Sigma}}_t \mathbf{H}_t^T + \mathbf{V}_t \mathbf{R}_t \mathbf{V}_t^T \\ \text{Kalman gain:} \quad & \mathbf{K}_t = \bar{\boldsymbol{\Sigma}}_t \mathbf{H}_t^T \mathbf{S}_t^{-1} \\ \text{Updated state estimate:} \quad & \boldsymbol{\mu}_t = \bar{\boldsymbol{\mu}}_t + \mathbf{K}_t \mathbf{y}_t \\ \text{Updated error covariance:} \quad & \boldsymbol{\Sigma}_t = (\mathbf{I} - \mathbf{K}_t \mathbf{H}_t) \bar{\boldsymbol{\Sigma}}_t \end{aligned}$$

For the updated error covariance, the Joseph formula should be employed for numerical stability. This can be expressed as follows:

$$\boldsymbol{\Sigma}_t = (\mathbf{I} - \mathbf{K}_t \mathbf{H}_t) \bar{\boldsymbol{\Sigma}}_t (\mathbf{I} - \mathbf{K}_t \mathbf{H}_t)^T + \mathbf{K}_t \mathbf{V}_t \mathbf{R}_t \mathbf{V}_t^T \mathbf{K}_t^T$$

In the case where the measurement model is correct, the Kalman filter utilizes it for updates. Thus, a conditional statement for data association is introduced. The Mahalanobis distance is calculated for the measurement residual to determine if the measurement is suitable for updating:

$$(\mathbf{z}_t - \mathbf{h}(\bar{\boldsymbol{\mu}}_t))^T \mathbf{S}_t^{-1} (\mathbf{z}_t - \mathbf{h}(\bar{\boldsymbol{\mu}}_t)) \leq D_{th},$$

Here,  $D_{th}$  represents a predetermined threshold. One can further show that the quadratic term is actually a  $\chi_m^2$  distribution, where  $m$  is the number of measurements.