Invariant Extended Kalman Filter for Localization

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1 State representation $\boldsymbol{x}$ and motion input $\boldsymbol{u}$

Let $\mathcal{X}$ represents the robot pose in SE(3).

$$\mathcal{X} = \begin{bmatrix} \boldsymbol{R} & \boldsymbol{t} \\ \boldsymbol{0}^T & 1 \end{bmatrix} \in \mathrm{SE}(3)$$

where $\boldsymbol{R}$ is the rotation matrix in SO(3) and $\boldsymbol{t}$ is the translation vector in $\mathbb{R}^3$.

The motion input (control signal) $\boldsymbol{u}$ is a twist vector in $\mathbb{R}^6$ ($\boldsymbol{u}^\wedge$ is the correspondence element in $\mathfrak{se}(3)$) comprising linear velocity $\boldsymbol{\nu} \in \mathbb{R}^3$ and angular velocity $\boldsymbol{\omega} \in \mathbb{R}^3$, integrated over the sampling time δt.

$$\boldsymbol{u} = \begin{bmatrix} \boldsymbol{\nu} \\ \boldsymbol{\omega} \end{bmatrix}, \ \boldsymbol{\nu} = \begin{bmatrix} \nu_x \\ \nu_y \\ \nu_z \end{bmatrix}, \ \boldsymbol{\omega} = \begin{bmatrix} \omega_\phi \\ \omega_\theta \\ \omega_\psi \end{bmatrix}$$

Additionally, we assume that there are no linear velocities in the lateral and vertical direction and no angular velocities in the roll and pitch rotation, i.e., $\nu_y = \nu_z = \omega_\phi = \omega_\theta = 0$.

The motion input is corrupted by additive Gaussian noise $\boldsymbol{\epsilon}$, with a mean of $\boldsymbol{0}$ and a covariance matrix of $\boldsymbol{Q}$. This noise accounts for possible control error through a value of $\sigma \neq 0$

$$\boldsymbol{u} = \begin{bmatrix} \nu_x \delta t \\ 0 \\ 0 \\ 0 \\ 0 \\ \omega_\psi \delta t \end{bmatrix} + \boldsymbol{\epsilon} \in \mathbb{R}^6$$

$$\boldsymbol{Q} = \begin{bmatrix} \sigma_x^2 \delta t & 0 & 0 & 0 & 0 & 0 \\ 0 & \sigma_y^2 \delta t & 0 & 0 & 0 & 0 \\ 0 & 0 & \sigma_z^2 \delta t & 0 & 0 & 0 \\ 0 & 0 & 0 & \sigma_\phi^2 \delta t & 0 & 0 \\ 0 & 0 & 0 & 0 & \sigma_\theta^2 \delta t & 0 \\ 0 & 0 & 0 & 0 & 0 & \sigma_\psi^2 \delta t \end{bmatrix} \in \mathbb{R}^{6\times 6}$$

Let x and y represent positions in the x- and y-directions, respectively. The variables θ, ν, ω, and α denote the yaw angle, linear velocity, yaw angle rate, and linear acceleration, respectively. The target state $\boldsymbol{x}$ is defined as:

$$\boldsymbol{x} = [x, y, \theta, \nu, \omega, \alpha]^T$$

The motion input is defined as:

$$\boldsymbol{u} = [\omega, \alpha]^T$$

2 The measurement $\boldsymbol{z}$

Consider a scenario where we have both a GPS sensor and an IMU sensor. The GPS sensor provides information about the longitudinal position (x) and lateral position (y). On the other hand, the IMU sensor supplies data regarding the yaw angle (θ), yaw angle rate (ω), and linear acceleration (α). Additionally, the vehicle itself provides its velocity (ν). We can define the measurement vector $\boldsymbol{z}$ as follows:

$$\boldsymbol{z} = [x, y, \theta, \omega, \alpha, \nu]^T$$

This representation captures the key measurements of interest.

3 State transition function $\boldsymbol{f}$ and its derivative

The 2-D kinematic equation for the vehicle state at time t, denoted as $\boldsymbol{x}_t$ and described by kinematic model in discrete time space, can be expressed as follows:

$$\begin{bmatrix} x_t \\ y_t \\ \theta_t \\ \nu_t \\ \omega_t \\ \alpha_t \end{bmatrix} = \begin{bmatrix} x_{t-1} + (\nu_{t-1}\Delta t + \frac{1}{2}\alpha_{t-1}\Delta t^2)\cos(\theta_{t-1} + \frac{1}{2}\omega_{t-1}\Delta t) + \epsilon_x \\ y_{t-1} + (\nu_{t-1}\Delta t + \frac{1}{2}\alpha_{t-1}\Delta t^2)\sin(\theta_{t-1} + \frac{1}{2}\omega_{t-1}\Delta t) + \epsilon_y \\ \theta_{t-1} + \omega_{t-1}\Delta t + \epsilon_\theta \\ \nu_{t-1} + \alpha_{t-1}\Delta t + \epsilon_\nu \\ \omega_{t-1} + \epsilon_\omega \\ \alpha_{t-1} + \epsilon_\alpha \end{bmatrix} \tag{1}$$

The state transition equation (1) can be expressed through the nonlinear function $\boldsymbol{f}$ as follows:

$$\boldsymbol{x}_t = \boldsymbol{f}(\boldsymbol{x}_{t-1}, \boldsymbol{u}_t, \boldsymbol{\epsilon}_t)$$

Here, the vector $\boldsymbol{\epsilon}_t = [\epsilon_x, \epsilon_y, \epsilon_\theta, \epsilon_\nu, \epsilon_\omega, \epsilon_\alpha]^T$ represents the system noise. To simplify our analysis, we assume that system noises ϵ_x, ϵ_y, ϵ_θ, and ϵ_ν are set to zero, making $\boldsymbol{\epsilon}_t$ solely associated with the motion input $\boldsymbol{u}_t$. Furthermore, the nonlinear function $\boldsymbol{f}$ can be approximated through a Taylor expansion around the previously updated state $\boldsymbol{\mu}_{t-1}$, with $\boldsymbol{\epsilon}_t = \boldsymbol{0}$:

$$\boldsymbol{f}(\boldsymbol{x}_{t-1}, \boldsymbol{u}_t, \boldsymbol{\epsilon}_t) \approx \boldsymbol{f}(\boldsymbol{\mu}_{t-1}, \boldsymbol{u}_t, \boldsymbol{0}) + \boldsymbol{F}_t(\boldsymbol{x}_{t-1} - \boldsymbol{\mu}_{t-1}) + \boldsymbol{W}_t\boldsymbol{\epsilon}_t$$

Here, $\boldsymbol{F}_t$ represents the derivative of the state transition function $\boldsymbol{f}$ with respect to state $\boldsymbol{x}$, and $\boldsymbol{W}_t$ represents the derivative of the state transition function $\boldsymbol{f}$ with respect to the noise $\boldsymbol{\epsilon}$. The term $\boldsymbol{W}_t$ essentially denotes the noise gain of the system. To simplify the notation, we define Δl as $\nu_{t-1}\Delta t + \frac{1}{2}\alpha_{t-1}\Delta t^2$ and $\Delta\theta$ as $\theta_{t-1} + \frac{1}{2}\omega_{t-1}\Delta t$. The derivative matrix of the state transition function $\boldsymbol{F}_t$ is denoted as:

$$\begin{aligned} \boldsymbol{F}_t &= \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}} \\ &= \begin{bmatrix} 1 & 0 & -\Delta l\sin(\Delta\theta) & \Delta t\cos(\Delta\theta) & -\frac{1}{2}\Delta t\Delta l\sin(\Delta\theta) & \frac{1}{2}\Delta t^2\cos(\Delta\theta) \\ 0 & 1 & \Delta l\cos(\Delta\theta) & \Delta t\sin(\Delta\theta) & \frac{1}{2}\Delta t\Delta l\cos(\Delta\theta) & \frac{1}{2}\Delta t^2\sin(\Delta\theta) \\ 0 & 0 & 1 & 0 & \Delta t & 0 \\ 0 & 0 & 0 & 1 & 0 & \Delta t \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

Furthermore, $\boldsymbol{W}_t$ can be expressed in matrix form:

$$
\begin{aligned}
\boldsymbol{W}_t &= \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{\epsilon}} \\
&= \begin{bmatrix}
0 & 0 & 0 & 0 & -\frac{1}{2}\Delta t \Delta l \sin(\Delta\theta) & \frac{1}{2}\Delta t^2 \cos(\Delta\theta) \\
0 & 0 & 0 & 0 & \frac{1}{2}\Delta t \Delta l \cos(\Delta\theta) & \frac{1}{2}\Delta t^2 \sin(\Delta\theta) \\
0 & 0 & 0 & 0 & \Delta t & 0 \\
0 & 0 & 0 & 0 & 0 & \Delta t \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\end{aligned}
$$

4 Measurement function $\boldsymbol{h}$ and its derivative

The measurements of the system can be described by a linear relation between the state and the measurement at the current step t. This relation is expressed as:

$$
\begin{bmatrix} z_x \\ z_y \\ z_\theta \\ z_\omega \\ z_\alpha \\ z_\nu \end{bmatrix} = \begin{bmatrix} x_t + \delta_x \\ y_t + \delta_y \\ \theta_t + \delta_\theta \\ \omega_t + \delta_\omega \\ \alpha_t + \delta_\alpha \\ \nu_t + \delta_\nu \end{bmatrix} \tag{2}
$$

We assume that the measurements are associated with measurement noises, and the noises are independent of each other. The above equation (2) can be expressed as the linear function $\boldsymbol{h}$:

$$
\boldsymbol{z}_t = \boldsymbol{h}(\boldsymbol{x}_t, \boldsymbol{\delta}_t)
$$

Here, $\boldsymbol{\delta}_t = [\delta_x, \delta_y, \delta_\theta, \delta_\omega, \delta_\alpha, \delta_\nu]^T$ represents the measurement error. We can apply the same technique to expend the function $\boldsymbol{h}$ at the previously predicted state $\bar{\boldsymbol{\mu}}_t$, with $\boldsymbol{\delta}_t = \mathbf{0}$:

$$
\boldsymbol{h}(\boldsymbol{x}_t, \boldsymbol{\delta}_t) \approx \boldsymbol{h}(\bar{\boldsymbol{\mu}}_t, \mathbf{0}) + \boldsymbol{H}_t(\boldsymbol{x}_t - \bar{\boldsymbol{\mu}}_t) + \boldsymbol{V}_t \boldsymbol{\delta}_t
$$

Here, $\boldsymbol{H}_t$ represents the derivative of the measurement function $\boldsymbol{h}$ with respect to $\boldsymbol{x}$, and $\boldsymbol{V}_t$ represents the derivative of the measurement function $\boldsymbol{h}$ with respect to the noise $\boldsymbol{\delta}$. The term $\boldsymbol{V}_t$ is the noise gain of the measurement. The derivative matrix of the measurement function is denoted as:

$$
\begin{aligned}
\boldsymbol{H}_t &= \frac{\partial \boldsymbol{h}}{\partial \boldsymbol{x}} \\
&= \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0
\end{bmatrix}.
\end{aligned}
$$

In addition, $\boldsymbol{V}_t$ can be expressed in matrix form:

$$
\begin{aligned}
\boldsymbol{V}_t &= \frac{\partial \boldsymbol{h}}{\partial \boldsymbol{\delta}} \\
&= \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\end{aligned}
$$

5 Process noise covariance matrix $\boldsymbol{Q}$

As we mentioned earlier, we assume that the system noise $\boldsymbol{\epsilon}_t$ is associated with the motion input $\boldsymbol{u}_t$. Furthermore, we assume that $\boldsymbol{\epsilon}_t$ follows a normal distribution with a mean of $\mathbf{0}$ and a covariance matrix of $\boldsymbol{Q}_t$. Additionally, we define the covariance matrix $\boldsymbol{Q}_t$ as

$$\boldsymbol{Q}_t = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sigma_\omega^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sigma_\alpha^2 \end{bmatrix}$$

Here, σ_ω and σ_α are the standard deviation of the yaw angle rate and linear acceleration, respectively.

6 Measurement noise covariance matrix $\boldsymbol{R}$

For the measurement error $\boldsymbol{\delta}_t$, we assume it follows a normal distribution with a mean of $\mathbf{0}$ and a covariance matrix of $\boldsymbol{R}_t$. Additionally, we assume that all measurement noises are independent. Consequently, we can disregard any interaction between them, resulting in the following diagonal covariance matrix:

$$\boldsymbol{R}_t = \begin{bmatrix} \tau_x^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & \tau_y^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \tau_\theta^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \tau_\omega^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \tau_\alpha^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & \tau_\nu^2 \end{bmatrix}.$$

Here, τ_x, τ_y, τ_θ, τ_ω, τ_α, and τ_ν represent the standard deviations of x, y, θ, ω, α, and ν respectively.

7 Kalman filter algorithm

Now we are ready to implement extended Kalman filter (EKF) for the vehicle localization. Given the initial state $\boldsymbol{\mu}_0$ and state covariance $\boldsymbol{\Sigma}_0$, the EKF algorithm is summarized as follows:

Prediction:

$$\begin{aligned} &\text{Predicted state estimate:} && \bar{\boldsymbol{\mu}}_t = \boldsymbol{f}_t(\boldsymbol{\mu}_{t-1}, \boldsymbol{u}_t) \\ &\text{Predicted error covariance:} && \bar{\boldsymbol{\Sigma}}_t = \boldsymbol{F}_t \boldsymbol{\Sigma}_{t-1} \boldsymbol{F}_t^T + \boldsymbol{W}_t \boldsymbol{Q}_t \boldsymbol{W}_t^T \end{aligned}$$

Update:

$$\begin{aligned} &\text{Innovation:} && \boldsymbol{y}_t = \boldsymbol{z}_t - \boldsymbol{h}(\bar{\boldsymbol{\mu}}_t) \\ &\text{Innovation covariance:} && \boldsymbol{S}_t = \boldsymbol{H}_t \bar{\boldsymbol{\Sigma}}_t \boldsymbol{H}_t^T + \boldsymbol{V}_t \boldsymbol{R}_t \boldsymbol{V}_t^T \\ &\text{Kalman gain:} && \boldsymbol{K}_t = \bar{\boldsymbol{\Sigma}}_t \boldsymbol{H}_t^T \boldsymbol{S}_t^{-1} \\ &\text{Updated state estimate:} && \boldsymbol{\mu}_t = \bar{\boldsymbol{\mu}}_t + \boldsymbol{K}_t \boldsymbol{y}_t \\ &\text{Updated error covariance:} && \boldsymbol{\Sigma}_t = (\boldsymbol{I} - \boldsymbol{K}_t \boldsymbol{H}_t) \bar{\boldsymbol{\Sigma}}_t \end{aligned}$$

For the updated error covariance, the Joseph formula should be employed for numerical stability. This can be expressed as follows:

$$\boldsymbol{\Sigma}_t = (\boldsymbol{I} - \boldsymbol{K}_t \boldsymbol{H}_t)\bar{\boldsymbol{\Sigma}}_t(\boldsymbol{I} - \boldsymbol{K}_t \boldsymbol{H}_t)^T + \boldsymbol{K}_t \boldsymbol{V}_t \boldsymbol{R}_t \boldsymbol{V}_t^T \boldsymbol{K}_t^T$$

In the case where the measurement model is correct, the Kalman filter utilizes it for updates. Thus, a conditional statement for data association is introduced. The Mahalanobis distance is calculated for the measurement residual to determine if the measurement is suitable for updating:

$$(\boldsymbol{z}_t - \boldsymbol{h}(\bar{\boldsymbol{\mu}}_t))^T \boldsymbol{S}_t^{-1} (\boldsymbol{z}_t - \boldsymbol{h}(\bar{\boldsymbol{\mu}}_t)) \leq D_{th},$$

Here, D_{th} represents a predetermined threshold. One can further show that the quadratic term is actually a χ_m^2 distribution, where m is the number of measurements.