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Exercise Sheet 5.

Solutions due Tuesday, **June 7**, 16:00 – 16:15, in the lecture hall.[1]

Exercise 17. (2 Points)

Proof that an arbitrary formula ϕ in CNF is valid if and only if each clause contains an atom P and its negation $\neg P$.

Exercise 18. (3.5 Points)

Perform DPLL with clause learning as explained in slide 44 of Chapter 10 on the following clause set (give the full DPLL trace):

$$\big\{\{A, B, C\}, \{C, D\}, \{\neg B, \neg D\}, \{\neg A, B, \neg C, \neg D\}, \{B, C, \neg D\}, \{\neg C, D\}\big\}$$

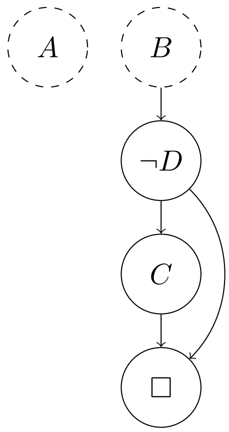
During the execution of DPLL, you **have to use** the following tie breaking choices in every application of the splitting rule:

- choose the smallest predicate according to the lexicographical ordering that has not been assigned to a value, so far (i.e. first assign a value to A, then B, and so on);
- assign first T and then F.

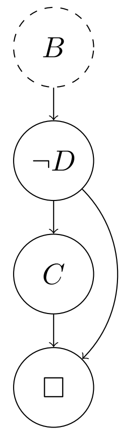
Whenever you encounter a conflict, draw the corresponding implication graph as well as its conflict graph, and mention which clause can be learned from the conflict graph. Then use this information and continue with the DPLL procedure. Do this until the clause set is proven to be satisfiable or unsatisfiable. If Δ is satisfiable, provide an interpretation I that satisfies Δ.

(Solution)

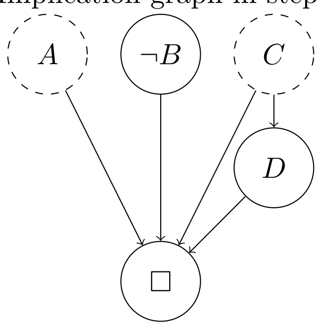
[1] Solutions in paper form only, and solution submission only at the stated time at the stated place. At most 3 authors per solution. All authors must be in the same tutorial group. All sheets of your solution must be stapled together. At the top of the first sheet, you must write the names of the authors and the name of your tutor. Your solution must be placed into the correct box for your tutorial group. If you don't comply with these rules, 3 points will be subtracted from your score for this sheet.



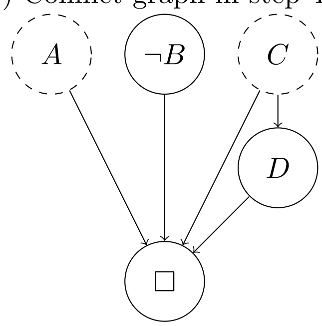
(a) Implication graph in step 4aa



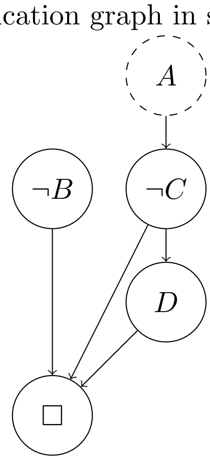
(b) Conflict graph in step 4aa



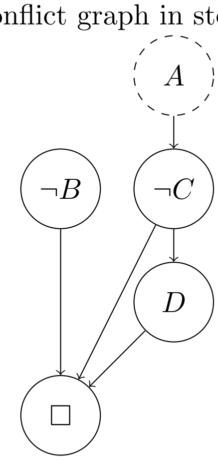
(c) Implication graph in step 4aba



(d) Conflict graph in step 4aba



(e) Implication graph in step 4abb



(f) Conflict graph in step 4abb

Figure 1: Implication and conflict graphs of exercise 18. Dashed nodes correspond to choice literals, solid nodes to implied literals.

1a. Splitting rule: $A \to T$
$\{\{C, D\}, \{\neg B, \neg D\}, \{B, \neg C, \neg D\}, \{B, C, \neg D\}, \{\neg C, D\}\}$

2aa. Splitting rule: $B \to T$
$\{\{C, D\}, \{\neg D\}, \{\neg C, D\}\}$

3aa. Unit propagation: $D \to F$
$\{\{C\}, \{\neg C\}\}$

4aa. Unit propagation: $C \to T$
$\{\Box\}$
Implication graph and conflict graph are shown in Figure 1. We learn the conjunction $C_0 = \{\neg B\}$:

1. We add C_0 to Δ.
2. We go back to last choice literal (which was B).
3. We set $B \to F$ as implied literal.
4. We continue with unit propagation.

2ab. Unit propagation $B \to F$
$\{\{C, D\}, \{\neg C, \neg D\}, \{C, \neg D\}, \{\neg C, D\}\}$

3aba. Splitting rule: $C \to T$
$\{\{\neg D\}, \{D\}\}$

4aba. Unit propagation: $D \to T$
$\{\Box\}$
Implication graph and conflict graph are shown in Figure 1. We learn the conjunction $C_1 = \{\neg A, \neg C\}$:

1. We add C_1 to Δ.
2. We go back to last choice literal (which was C).
3. We set $C \to F$ as implied literal.
4. We continue with unit propagation.

3abb. Unit propagation: $C \to F$
$\{\{D\}, \{\neg D\}\}$

4abb. Unit propagation: $D \to T$
$\{\Box\}$
Implication graph and conflict graph are shown in Figure 1. We learn the conjunction $C_2 = \{\neg A\}$:

1. We add C_2 to Δ.
2. We go back to last choice literal (which was A).
3. We set $A \rightarrow F$ as implied literal.
4. We continue with unit propagation.

1b. Unit propagation: $A \rightarrow F$
$\big\{\{B, C\}, \{C, D\}, \{\neg B, \neg D\}, \{B, C, \neg D\}, \{\neg C, D\}, \{\neg B\}\big\}$ ($\{\neg B\}$ was learned in 4aa)

2b. Unit propagation: $B \rightarrow F$
$\big\{\{C\}, \{C, D\}, \{C, \neg D\}, \{\neg C, D\}\big\}$

3b. Unit propagation: $C \rightarrow T$
$\big\{\{D\}\big\}$

4b. Unit propagation: $D \rightarrow T$
$\{\}$

Satisfying interpretation I: $A \rightarrow F$, $B \rightarrow F$, $C \rightarrow T$, $D \rightarrow T$.
(/Solution)

Exercise 19. (2 Points)

(a) For each of the following sentences, write down a predicate logic formula with the same intuitive meaning.

- Every student that works hard solves at least one exercise.
- No car that is more than 10 years old will be repaired if it is completely broken.

(b) For each of the following predicate logic formulas, write down an English sentence with the same intuitive meaning.

- $\exists x \exists y \forall z[(Exercise(z) \land In(z, Sheet05) \land IsHard(z)) \implies (Equals(z,x) \lor Equals(z,y))]$
- $\forall x \forall y[(Student(x) \land Student(y)) \implies (Helps(x,y) \Leftrightarrow (Friend(x,y) \lor \exists z[Student(z) \land Friend(x,z) \land Friend(z,y)]))]$

(Solution)

(a)
- $\forall x[(Student(x) \land WorksHard(x)) \implies \exists y[Exercise(y) \land Solves(x,y)]]$
- $\forall x[Car(x) \land \mathit{MoreThan10YearsOld}(x) \land \mathit{CompletelyBroken}(x) \implies \neg \mathit{Repaired}(x)]$ or alternatively: $\forall x[Car(x) \land \exists y[Age(x,y) \land y > 10] \land \mathit{CompletelyBroken}(x) \implies \neg \mathit{Repaired}(x)]$

(b)
- There are at most two hard exercises on Sheet05.
- A student helps another if and only if they are friends or they have a common friend that is also a student.

(/Solution)

Exercise 20. (2.5 Points)

Transform all of the following predicate logic formulas into clausal normal form. Write down the results of all intermediate steps (cf. Chapter 11, slides 38–39). For steps that do not change anything, just mention this; there is no need to write the same formula twice.

(a) $\varphi_1 = \exists x \exists y[\neg(A(x) \land B(y)) \land D(y,x)]$

(b) $\varphi_2 = \forall x[(\exists y[F(y) \land G(y,x)] \land \neg F(x)) \to \neg H(x)]$

(c) $\varphi_3 = \forall x[K(x) \to (\exists y[L(y) \wedge M(x,y)] \vee \forall y[(N(y) \wedge O(x,y)) \to P(x,y)])]$

(Solution)

(a) $\exists x \exists y[A(x) \wedge B(y) \wedge D(y,x)]$

1. Eliminate $\leftrightarrow$ and $\to$, move $\neg$ inwards, move quantifiers outwards:

$$\exists x \exists y[(\neg A(x) \vee \neg B(y)) \wedge D(y,x)]$$

2. Eliminate existential quantifiers to get Skolem normal form:

$$(\neg A(a) \vee \neg B(b)) \wedge D(b,a)$$

3. Transform to CNF: nothing to do.
4. Write as set of clauses:

$$\{\{\neg A(a), \neg B(b)\}, \{D(b,a)\}\}$$

5. Standardize variables apart: nothing to do; already in clausal normal form.

(b) $\forall x[(\exists y[F(y) \wedge G(y,x)] \wedge \neg F(x)) \to \neg H(x)]$

1. Eliminate $\leftrightarrow$: nothing to do.
2. Eliminate $\to$:

$$\forall x[\neg(\exists y[F(y) \wedge G(y,x)] \wedge \neg F(x)) \vee \neg H(x)]$$

3. Move $\neg$ inwards:

$$\begin{aligned} &\forall x[\neg \exists y[F(y) \wedge G(y,x)] \vee F(x) \vee \neg H(x)] \\ \equiv\ &\forall x[\forall y[\neg F(y) \vee \neg G(y,x)] \vee F(x) \vee \neg H(x)] \end{aligned}$$

4. Move quantifiers outwards to get prenex normal form:

$$\forall x \forall y[\neg F(y) \vee \neg G(y,x) \vee F(x) \vee \neg H(x)]$$

5. Eliminate existential quantifiers to get Skolem normal form: nothing to do.
6. Transform to CNF: nothing to do.
7. Write as set of clauses:

$$\{\{\neg F(y), \neg G(y,x), F(x), \neg H(x)\}\}$$

8. Standardize variables apart: nothing to do; already in clausal normal form.

(c) $\forall x[K(x) \to (\exists y[L(y) \wedge M(x,y)] \vee \forall y[(N(y) \wedge O(x,y)) \to P(x,y)])]$

1. Eliminate equivalences $\leftrightarrow$: nothing to do.
2. Eliminate implications $\to$:

$$\forall x[\neg K(x) \vee (\exists y[L(y) \wedge M(x,y)] \vee \forall y[\neg(N(y) \wedge O(x,y)) \vee P(x,y)])]$$

3. Move negation $\neg$ inwards:

$$\forall x[\neg K(x) \vee (\exists y[L(y) \wedge M(x,y)] \vee \forall y[(\neg N(y) \vee \neg O(x,y)) \vee P(x,y)])]$$

4. Move quantifiers outwards to get prenex normal form, renaming when needed:

$$\forall x \exists y \forall z[\neg K(x) \vee ([L(y) \wedge M(x,y)] \vee [(\neg N(z) \vee \neg O(x,z)) \vee P(x,z)])]$$

5. Eliminate existential quantifiers to get Skolem normal form:

$$\forall x \forall z[\neg K(x) \vee ([L(f(x)) \wedge M(x,f(x))] \vee [(\neg N(z) \vee \neg O(x,z)) \vee P(x,z)])]$$

6. Transform to CNF:

$$\begin{gathered}\forall x \forall z[(L(f(x)) \vee \neg K(x) \vee \neg N(z) \vee \neg O(x,z) \vee P(x,z)) \wedge \\ (M(x,f(x)) \vee \neg K(x) \vee \neg N(z) \vee \neg O(x,z) \vee P(x,z))]\end{gathered}$$

7. Write as set of clauses:

$$\begin{gathered}\{\{L(f(x)), \neg K(x), \neg N(z), \neg O(x,z), P(x,z)\}, \\ \{M(x,f(x)), \neg K(x), \neg N(z), \neg O(x,z), P(x,z)\}\}\end{gathered}$$

8. Standardize variables apart to get clausal normal form:

$$\begin{gathered}\{\{L(f(x)), \neg K(x), \neg N(z), \neg O(x,z), P(x,z)\}, \\ \{M(y,f(y)), \neg K(y), \neg N(v), \neg O(y,v), P(y,v)\}\}\end{gathered}$$

(/Solution)