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Theoretical Exercise Sheet 4.

Solutions due Tuesday, June 2, 23:59 uploaded in the AI CMS.
Total points of the sheet: 20

Exercise 1: Predicate Logic Basics — 3 Points

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1. For each of the following sentences, write down a **predicate logic formula that has the same meaning**. You may only use the following predicates: Course(x), Student(x), Takes(x, y), Likes(x, y), and Equals(x, y).
   (i) Every course is taken by at least one student who likes the course.
   (ii) Every student who takes all courses likes at least one course.
2. For each of the following predicate logic formulas, write down an **English sentence with the same meaning.**
   (i) $\exists x\,[\text{Student}(x) \wedge \forall y\,[\text{Course}(y) \rightarrow (\text{Takes}(x, y) \wedge \neg\text{Likes}(x, y))]]$
   (ii) $\exists x\,[\text{Course}(x) \wedge \forall y\,[\text{Course}(y) \rightarrow ((\forall z\,[\text{Student}(z) \rightarrow \text{Likes}(z, y)]) \leftrightarrow \text{Equals}(x, y))]]$

Solution:

1. (i) $\forall x\,[\text{Course}(x) \rightarrow \exists y\,[\text{Student(y)} \wedge \text{Takes(y, x)} \wedge \text{Likes(y, x)}]]$
   (ii) $\forall x\,[(\text{Student}(x) \wedge \forall y\,[\text{Course}(y) \rightarrow \text{Takes}(x, y)]) \rightarrow \exists z\,[\text{Course}(z) \wedge \text{Likes}(x, z)]]$
2. (i) There is a student who takes all courses but likes none of them.
   (ii) There is exactly one course that all students like.

Exercise 2: Normal Forms — 6 Points

Transform the following predicate logic formulas into Clausal Normal Form. Write down the results of all intermediate steps, specifying which steps you are applying and giving the intermediate results.

Note: Simplify the formulas where possible.

1. $\forall x\,[A(x) \to \exists x \forall y\,[B(x,y) \to C(x)]]$
2. $\neg\forall x \exists y \forall z\,[(A(z) \leftrightarrow B(y,x)) \vee \forall x\,[C(x) \wedge D(y,z)]]$

Solution:

1. $\forall x\,[A(x) \to \exists x \forall y\,[B(x,y) \to C(x)]]$

   (i) Eliminate $\leftrightarrow$: nothing to do

   (ii) Eliminate $\to$:
   $$\forall x\,[\neg A(x) \vee \exists x \forall y\,[\neg B(x,y) \vee C(x)]]$$

   (iii) Move $\neg$ inwards: nothing to do.

   (iv) Move quantifiers outwards to get Prenex Normal Form, renaming when needed:
   $$\forall x \exists z \forall y\,[\neg A(x) \vee (\neg B(z,y) \vee C(z))]$$

   (v) Eliminate existential quantifiers to get Skolem Normal Form:
   $$\forall x \forall y\,[\neg A(x) \vee \neg B(f(x),y) \vee C(f(x))]$$

   (vi) Transform to CNF: nothing to do.

   (vii) Write as set of clauses:
   $$\{\{\neg A(x), \neg B(f(x),y), C(f(x))\}\}$$

   (viii) Standardize variables apart to get Clausal Normal Form: nothing to do.

2. $\neg\forall x \exists y \forall z\,[(A(z) \leftrightarrow B(y,x)) \vee \forall x\,[C(x) \wedge D(y,z)]]$

   (a) Eliminate $\leftrightarrow$:
   $$\neg\forall x \exists y \forall z\,[(A(z) \to B(y,x)) \wedge (B(y,x) \to A(z)) \vee \forall x\,[C(x) \wedge D(y,z)]]$$

   (b) Eliminate $\to$:
   $$\neg\forall x \exists y \forall z\,[(\neg A(z) \vee B(y,x)) \wedge (\neg B(y,x) \vee A(z)) \vee \forall x\,[C(x) \wedge D(y,z)]]$$

   (c) Move $\neg$ inwards:
   $$\exists x \forall y \exists z\,[\neg((\neg A(z) \vee B(y,x)) \wedge (\neg B(y,x) \vee A(z)) \vee \forall x\,[C(x) \wedge D(y,z)])]$$
   $$\exists x \forall y \exists z\,[((A(z) \wedge \neg B(y,x)) \vee (B(y,x) \wedge \neg A(z))) \wedge \neg\forall x\,[C(x) \wedge D(y,z)]]$$
   $$\exists x \forall y \exists z\,[((A(z) \wedge \neg B(y,x)) \vee (B(y,x) \wedge \neg A(z))) \wedge \exists x\,[\neg C(x) \vee \neg D(y,z))]]$$

   (d) Move quantifiers outwards to get Prenex Normal Form, renaming when needed:
   $$\exists x \forall y \exists z \exists u\,[((A(z) \wedge \neg B(y,x)) \vee (B(y,x) \wedge \neg A(z))) \wedge (\neg C(u) \vee \neg D(y,z)))]$$

(e) Eliminate existential quantifiers to get Skolem Normal Form:

$$\forall y \exists z \exists u \, [((A(z) \wedge \neg B(y,a)) \vee (B(y,a) \wedge \neg A(z))) \wedge (\neg C(u) \vee \neg D(y,z)))]$$
$$\forall y \exists u \, [((A(f(y)) \wedge \neg B(y,a)) \vee (B(y,a) \wedge \neg A(f(y)))) \wedge (\neg C(u) \vee \neg D(y,f(y))))]$$
$$\forall y \, [((A(f(y)) \wedge \neg B(y,a)) \vee (B(y,a) \wedge \neg A(f(y)))) \wedge (\neg C(g(y)) \vee \neg D(y,f(y))))]$$

(f) Transform to CNF:

$$\forall y \, [(A(f(y)) \vee B(y,a)) \wedge (\neg B(y,a) \vee \neg A(f(y))) \wedge (\neg C(g(y)) \vee \neg D(y,f(y))))]$$

(g) Write as set of clauses:

$$\{\{A(f(y)), B(y,a))\}, \{\neg B(y,a), \neg A(f(y))\}, \{\neg C(g(y)), \neg D(y,f(y))\}\}$$

(h) Standardize variables apart to get Clausal Normal Form:

$$\{\{A(f(y_1)), B(y_1,a))\}, \{\neg B(y_2,a), \neg A(f(y_2))\}, \{\neg C(g(y_3)), \neg D(y_3,f(y_3))\}\}$$

Exercise 3: Herbrand Expansion — 3 Points

Show that the set θ of PL1 formulas below is unsatisfiable. Perform the following steps:

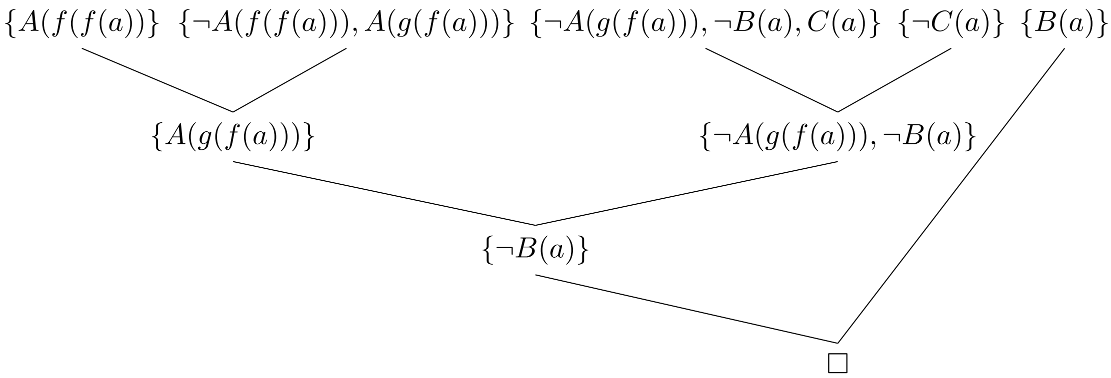
1. Transform θ into a set θ^* of PL1 formulas in Skolem Normal Form.
2. Specify the set of constant and function symbols $CF(\theta^*)$.
3. Specify the Herbrand universe $HU(\theta^*)$.
4. Specify the Herbrand expansion $HE(\theta^*)$.
5. Apply propositional resolution to $HE(\theta^*)$ to derive unsatisfiability of θ^*. Do so by drawing a resolution tree.

$$\begin{aligned}\theta = \{&\forall x\,[A(g(f(x))) \rightarrow \neg B(x)] \vee C(a), \forall y\,[A(f(y))],\\ &\forall z\,[A(f(z)) \rightarrow A(g(z))], B(a) \wedge \neg C(a)\}\end{aligned}$$

Hint: The Herbrand universe and thus the Herbrand expansion are infinite sets, hence you cannot enumerate all of their elements. Instead, make clear how their elements are formed using basic set notation. For propositional resolution over this infinite Herbrand expansion just choose an appropriate finite subset to derive $\Box$. 5 elements are sufficient.

Solution:

1. $\theta^* = \{\forall x\,[\neg A(g(f(x)))\vee\neg B(x)\vee C(a)], \forall y\,[A(f(y))], \forall z\,[\neg A(f(z))\vee A(g(z))], B(a), \neg C(a)\}$
2. $CF(\theta^*) = \{f, g, a\}$.
3. $HU(\theta^*) = \{a, h_1(\ldots(h_n(a)))\,|\,n \in \mathbb{N}, \forall i \in \{1,\ldots,n\} : h_i \in \{f, g\}\}$
4. $$\begin{aligned}HE(\theta^*) =& \{\neg A(g(f(x))) \vee \neg B(x) \vee C(a)\,|\,x \in HU(\theta^*)\} \cup\\ &\{A(f(y))\,|\,y \in HU(\theta^*)\} \cup\\ &\{\neg A(f(z)) \vee A(g(z))\,|\,z \in HU(\theta^*)\} \cup\\ &\{B(a)\} \cup \{\neg C(a)\}\end{aligned}$$
5. Here is one resolution tree. There are several possible solutions.



Exercise 4: Unification 4.5 Points

Run the unification algorithm from the lecture on each of the given sets. Assume v, w, x, y, z to be *variables*, and a, b, c to be *constants*. **For each step, write down the set to unify, the disagreement set, and the updated set of substitutions, i.e., $T_0, D_0, s_1, T_1, D_1, \dots$ until the algorithm terminates** (cf. Chapter 6, slide 26-27). In particular, if the algorithm does not output a failure, list the *entire* content of the substitutions $s_1, \dots, s_n$.

(i) $\{P(x, a, f(a)), P(f(a), a, x)\}$

(ii) $\{P(f(x, a, b)), P(f(b, y, g(z)))\}$

(iii) $\{P(z, x), P(f(x), z)\}$

(iv) $\{P(x, f(b, y)), P(g(a), v), P(w, f(z, y))\}$

Solution:

(i) $\{P(x, a, f(a)), P(f(a), a, x)\}$

$$\begin{aligned} D_0 &= \{x, f(a)\} \\ s_1 &= \left\{ \frac{x}{f(a)} \right\} \\ T_1 &= \{P(f(a), a, f(a))\} \end{aligned}$$

T_1 is a singleton, so stop and output $s_1 = \{\frac{x}{f(a)}\}$.

(ii) $\{P(f(x,a,b)), P(f(b,y,g(z)))\}$

$$\begin{aligned} D_0 &= \{x, b\} \\ s_1 &= \left\{\frac{x}{b}\right\} \\ T_1 &= \{P(f(b,a,b)), P(f(b,y,g(z)))\} \end{aligned}$$

$$\begin{aligned} D_2 &= \{y, a\} \\ s_2 &= s_1\left\{\frac{y}{a}\right\} = \left\{\frac{x}{b}, \frac{y}{a}\right\} \\ T_2 &= \{P(f(b,a,b)), P(f(b,a,g(z)))\} \end{aligned}$$

$$D_3 = \{b, g(z)\}$$

No variable is contained in D_3. Stop and output "failure".

(iii) $\{P(z,x), P(f(x),z)\}$

$$\begin{aligned} D_0 &= \{z, f(x)\} \\ s_1 &= \left\{\frac{z}{f(x)}\right\} \\ T_1 &= \{P((f(x),x), P(f(x),f(x))\} \end{aligned}$$

$$D_1 = \{x, f(x)\}$$

There is no pair of a term t and a variable x such that x is not contained in t. Stop and output "failure".

(iv) $\{P(x, f(b, y)), P(g(a), v), P(w, f(z, y))\}$

$$\begin{aligned}
D_0 &= \{x, g(a), w\}\\
s_1 &= \left\{\frac{x}{g(a)}\right\}\\
T_1 &= \{P(g(a), f(b, y)), P(g(a), v), P(w, f(z, y))\}\\
\\
D_1 &= \{g(a), w\}\\
s_2 &= s_1\left\{\frac{w}{g(a)}\right\} = \left\{\frac{x}{g(a)}, \frac{w}{g(a)}\right\}\\
T_2 &= \{P(g(a), f(b, y)), P(g(a), v), P(g(a), f(z, y))\}\\
\\
D_2 &= \{f(b, y), v, f(z, y)\}\\
s_3 &= s_2\left\{\frac{v}{f(b, y)}\right\} = \left\{\frac{x}{g(a)}, \frac{w}{g(a)}, \frac{v}{f(b, y)}\right\}\\
T_3 &= \{P(g(a), f(b, y)), P(g(a), f(b, y)), P(g(a), f(z, y))\}\\
&= \{P(g(a), f(b, y)), P(g(a), f(z, y))\}\\
\\
D_3 &= \{b, z\}\\
s_4 &= s_3\left\{\frac{z}{b}\right\} = \left\{\frac{x}{g(a)}, \frac{w}{g(a)}, \frac{v}{f(b, y)}, \frac{z}{b}\right\}\\
T_4 &= \{P(g(a), f(b, y))\}
\end{aligned}$$

T_4 is a singleton, so stop and output $s_4 = \left\{\frac{x}{g(a)}, \frac{w}{g(a)}, \frac{v}{f(b,y)}, \frac{z}{b}\right\}$.

Exercise 5: PL1 Resolution 3.5 Points

Consider the following set of predicate logic formulas in Skolem Normal Form:

1. Every university mascot is an animal.
   $\varphi_1 = \forall x[\neg Mascot(x) \vee Animal(x)]$
2. Steffi the owl or Kasper the clown are university mascots.
   $\varphi_2 = Mascot(Steffi) \vee Mascot(Kasper)$
3. Every animal is liked by some student.
   $\varphi_3 = \forall x[\neg Animal(x) \vee (Student(f(x)) \wedge Likes(f(x), x))]$
4. No student likes Kasper the clown.
   $\varphi_4 = \forall x[\neg Student(x) \vee \neg Likes(x, Kasper)]$

We want to prove by contradiction that Steffi is an animal. Therefore, we add to our set of clauses:

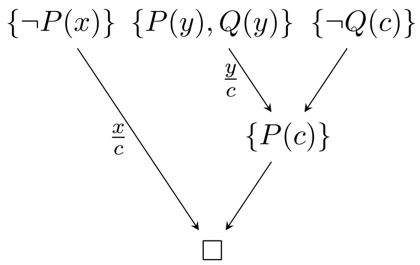
5. Steffi the owl is not an animal.
   $\varphi_5 = \neg Animal(Steffi)$

Perform the following operations:

(i) Write down the **set of clauses Δ corresponding to the Clausal Normal Forms** of these formulas.

(ii) Use **binary PL1 resolution** to show that Δ is unsatisfiable. Also specify the unifiers when applying a resolution step.
Note: If you need to use a clause more than once, make a copy of the clause and rename the variables to avoid name collisions.

Depict the resolution process in form of a tree for easier readability. Label edges with the corresponding unifying substitution:



Solution:

(i) The only formula that is not in CNF is φ_3. This can be transformed into:
$\varphi_3 = \forall x[\neg Animal(x) \vee Student(f(x))] \wedge \forall x[\neg Animal(x) \vee Likes(f(x), x))]$

The clauses for each formula are:

$$\begin{aligned}
\varphi_1 &= \{\neg Mascot(x), Animal(x)\} \\
\varphi_2 &= \{Mascot(Steffi), Mascot(Kasper)\} \\
\varphi_{3,1} &= \{\neg Animal(x), Student(f(x))\} \\
\varphi_{3,2} &= \{\neg Animal(x), Likes(f(x), x))\} \\
\varphi_4 &= \{\neg Student(x), \neg Likes(x, Kasper)\} \\
\varphi_5 &= \{\neg Animal(Steffi)\}
\end{aligned}$$

When creating Δ, do not forget to standardize the variables apart!

$$\begin{aligned}
\varphi_1 &= \{\neg Mascot(x), Animal(x)\} \\
\varphi_2 &= \{Mascot(Steffi), Mascot(Kasper)\} \\
\varphi_{3,1} &= \{\neg Animal(y), Student(f(y))\} \\
\varphi_{3,2} &= \{\neg Animal(z), Likes(f(z), z))\} \\
\varphi_4 &= \{\neg Student(w), \neg Likes(w, Kasper)\} \\
\varphi_5 &= \{\neg Animal(Steffi)\}
\end{aligned}$$

(ii) Figure 1 shows a suitable resolution tree.

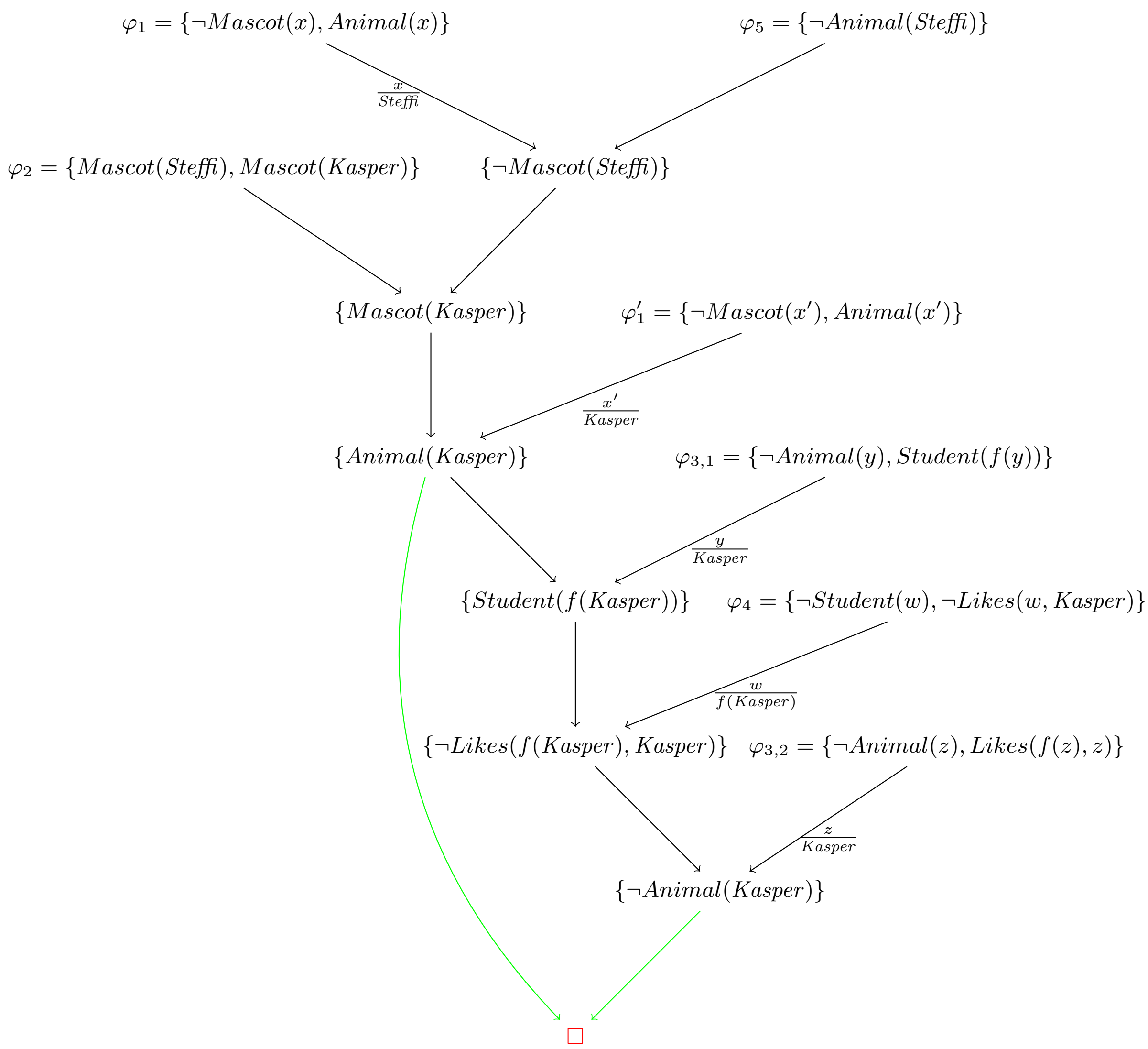


Figure 1: Solution of Exercise 19 (b)

Submission Instructions

Solutions need to be packaged into a `.zip` file and uploaded in the AI CMS. The `.zip` file has to contain a single folder with name:
`AI2020_TE4_mat1_mat2_mat3` where `mat1, mat2, mat3` are the matriculation numbers of the students who submit together. This folder must contain the following files:

- `authors.txt` listing the names and matriculation numbers of all students who submit together. Use one line per student and no spaces: Name;Matriculation number.
- The `.pdf` file containing your solutions.

Do not add any other folder or sub folder, this means place all files directly into `AI2020_TE4_mat1_mat2_mat3`. Do not place any file outside of `AI2020_TE4_mat1_mat2_mat3`.

Only one student of each group needs to do the submission! Remember that this sheet can be submitted in groups of up to three members (all members of the group must however be assigned to the same tutorial). If you are still looking for submission partners, it is recommended to ask in the forum of the CMS in the category "Student Room".