SAARLAND UNIVERSITY | ARTIFICIAL INTELLIGENCE

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Theoretical Exercise Sheet 3.

Solutions due Tuesday, May 26, 23:59 uploaded in the AI CMS.

Total points of the sheet: 33

Exercise 1: Who is guilty? 6 Points

Anna, Ben and Carl are being questioned by the police. They all tell the truth and these are the statements they make:

- *Anna*: If one of Ben and Carl is guilty, then so is the other.
- *Ben*: If Carl is guilty, then so am I.
- *Carl*: Ben and I are not both guilty.

Draw a truth table for the given scenario including subformulas for all statements. Use A for denoting that Anna is guilty. Can you identify who (if any) of the three is guilty?

Solution:

We abbreviate $(B \Leftrightarrow C) \wedge (C \Rightarrow B) \wedge \neg(B \wedge C)$ with φ.

A	B	C	$B \Leftrightarrow C$	$C \Rightarrow B$	$\neg(B \wedge C)$	φ
T	T	T	T	T	F	F
T	T	F	F	T	T	F
T	F	T	F	F	T	F
T	F	F	T	T	T	T
F	T	T	T	T	F	F
F	T	F	F	T	T	F
F	F	T	F	F	T	F
F	F	F	T	T	T	T

We can conclude that neither Ben nor Carl is guilty as φ evaluates to false if B or C is true. Even more, φ is propositionally equivalent to $\neg B \wedge \neg C$. We cannot determine whether Anna is guilty, which is not surprising as none of the statements says anything about Anna and A is hence not included in φ.

Total points: 6

Exercise 2: Finding Logical Formulas — 6 Points

Consider the following truth table for the proposition φ.

A	B	C	φ
T	T	T	T
T	T	F	T
T	F	T	F
T	F	F	F
F	T	T	T
F	T	F	F
F	F	T	T
F	F	F	F

Provide a DNF for φ. Additionally, briefly describe a general method for constructing a DNF corresponding to an arbitrary truth table.

Solution:

$(A \wedge B \wedge C) \vee (A \wedge B \wedge \neg C) \vee (\neg A \wedge B \wedge C) \vee (\neg A \wedge \neg B \wedge C)$

Assume we are given a truth table for a proposition φ in the variables $P_1, \ldots, P_n$ and we want to construct a DNF propositionally equivalent to φ. For each row where φ is true we construct a conjunction of n literals representing the entries of that row. I.e., we conjoin P_i if the value of P_i is true and $\neg P_i$ if its value is false. Consequently, the disjunction of these conjunctions yields a DNF for φ.

Total points: 6

Exercise 3: CNF & DNF Transformation — 8 points

Transform the following formulas to CNF and DNF. To do so, follow the steps from the lecture specifying which steps you are applying and giving the intermediate results. Apply equivalences 1-5 first to arrive at an intermediate formula ψ. Then apply the corresponding distributivity laws on ψ to compute the CNF/DNF.

Note: You may simplify by collapsing duplicate literals.

(a) $(\neg P \lor Q) \Rightarrow (Q \Rightarrow R)$

(b) $(P \lor \neg Q) \Rightarrow (P \Leftrightarrow R)$

Equivalences from the slides:

1. $\neg\neg\phi \equiv \phi$
2. $\neg(\phi \land \psi) \equiv (\neg\phi \lor \neg\psi)$ (de Morgan)
3. $\neg(\phi \lor \psi) \equiv (\neg\phi \land \neg\psi)$ (de Morgan)
4. $(\phi \Leftrightarrow \psi) \equiv [(\phi \Rightarrow \psi) \land (\psi \Rightarrow \phi)]$
5. $(\phi \Rightarrow \psi) \equiv (\neg\phi \lor \psi)$
6. $\phi \land (\psi \lor \chi) \equiv (\phi \land \psi) \lor (\phi \land \chi)$ (distributivity)
7. $\phi \lor (\psi \land \chi) \equiv (\phi \lor \psi) \land (\phi \lor \chi)$ (distributivity)

Solution:

(a) Original formula: $(\neg P \lor Q) \Rightarrow (Q \Rightarrow R)$

1. *Eliminate* $\Rightarrow$*:* $\neg(\neg P \lor Q) \lor (\neg Q \lor R)$
2. *Move* $\neg$ *inwards:* $(\neg\neg P \land \neg Q) \lor (\neg Q \lor R)$
3. *Remove Double Negation:* $(P \land \neg Q) \lor (\neg Q \lor R)$

Continue with CNF:

4. *Distribute* $\lor$ *over* $\land$*:* $(P \lor (\neg Q \lor R)) \land (\underline{\neg Q} \lor (\underline{\neg Q} \lor R))$

- *After simplification:* $(P \lor \neg Q \lor R) \land (\neg Q \lor R)$

Continue with DNF:

The formula is already in Disjunctive Normal Form.

(b) *Original formula:* $(P \vee \neg Q) \Rightarrow (P \Leftrightarrow R)$

1. *Eliminate* $\Leftrightarrow$*:* $(P \vee \neg Q) \Rightarrow [(P \Rightarrow R) \wedge (R \Rightarrow P)]$
2. *Eliminate* $\Rightarrow$*:* $\neg(P \vee \neg Q) \vee [(\neg P \vee R) \wedge (\neg R \vee P)]$
3. *Move* $\neg$ *inwards:* $(\neg P \wedge \neg\neg Q) \vee [(\neg P \vee R) \wedge (\neg R \vee P)]$
4. *Remove Double Negation:* $(\neg P \wedge Q) \vee [(\neg P \vee R) \wedge (\neg R \vee P)]$

Continue with CNF:

5. *Distribute* $\vee$ *over* $\wedge$*:* $(\underline{\neg P} \vee \underline{\neg P} \vee R) \wedge (\neg P \vee \neg R \vee P) \wedge (Q \vee \neg P \vee R) \wedge (Q \vee \neg R \vee P)$

\- *After simplification:* $(\neg P \vee R) \wedge (Q \vee \neg P \vee R) \wedge (Q \vee \neg R \vee P)$

Continue with DNF:

5. *Distribute* $\wedge$ *over* $\vee$*:* $(\neg P \wedge Q) \vee [(\neg P \wedge \neg R) \vee (\underline{\neg P} \wedge \underline{P}) \vee (\underline{R} \wedge \underline{\neg R}) \vee (R \wedge P)]$

\- *After simplification:* $(\neg P \wedge Q) \vee (\neg P \wedge \neg R) \vee (R \wedge P)$

Total points: 8

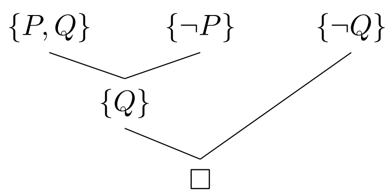
Exercise 4: Resolution Proof 4 Points

For the formula below, use resolution to prove that the formula is unsatisfiable. To do so, first **give a set of clauses Δ that is equivalent to the formula** and second, **use resolution to prove that it is unsatisfiable**.

$$\phi = (\neg Q \vee R) \wedge (Q \vee \neg S) \wedge (P \vee Q \vee S) \wedge \neg P \wedge (P \vee \neg Q \vee \neg R)$$

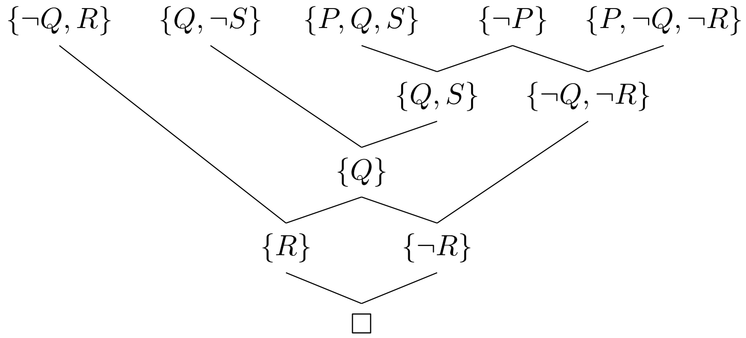
You may (but do not have to) write the resolution process in the form of a tree for easier readability:

$\Delta = \{\{P, Q\}, \{\neg P\}, \{\neg Q\}\}$



Solution:

$\Delta = \big\{\{\neg Q, R\}, \{Q, \neg S\}, \{P, Q, S\}, \{\neg P\}, \{P, \neg Q, \neg R\}\big\}$



Total points: 4

Exercise 5: DPLL 9 Points

For each of the following formulas, **use the DPLL procedure to determine whether ϕ is satisfiable or unsatisfiable**. Give a complete trace of the algorithm, **showing the simplified formula for each recursive call of the DPLL function**. Assume that DPLL selects variables in alphabetical order (i.e. A before B before $C \dots$), and that the **splitting rule first attempts the value True (T) and then the value False (F)**.

Example DPLL trace:

$$\big\{\{\neg P, Q\}, \{\neg P, \neg Q\}\big\}$$

1. Splitting rule: P

1a. $P \mapsto T$

$\big\{\{Q\}, \{\neg Q\}\big\}$

2a. UP rule: $Q \mapsto T$

$\{\square\}$

1b. $P \mapsto F$

$\{\}$

✓. All clauses have been satisfied.

Satisfying partial assignment: $\neg P$

(a) $\phi_1 = (\neg A \vee B) \wedge (C \vee D \vee \neg E) \wedge (\neg A \vee \neg D) \wedge (\neg B \vee \neg C) \wedge (\neg A \vee E) \wedge (A \vee B)$

(b) $\phi_2 = (A \vee B \vee C) \wedge (A \vee \neg B) \wedge (\neg A \vee \neg B) \wedge (\neg A \vee B) \wedge (\neg C \vee D) \wedge (D \vee \neg E)$
$\wedge (\neg D \vee \neg E) \wedge (\neg D \vee E)$

Solution:
a) DPLL trace:

$$\big\{\{\neg A, B\}, \{C, D, \neg E\}, \{\neg A, \neg D\}, \{\neg B, \neg C\}, \{\neg A, E\}, \{A, B\}\big\}$$

1. Splitting rule: A

1a. $A \mapsto T$

$\big\{\{B\}, \{C, D, \neg E\},$
$\{\neg D\}, \{\neg B, \neg C\}, \{E\}\big\}$

2a. UP rule: $B \mapsto T$

$\big\{\{C, D, \neg E\}, \{\neg D\}, \{\neg C\}, \{E\}\big\}$

3a. UP rule: $C \mapsto F$

$\big\{\{D, \neg E\}, \{\neg D\}, \{E\}\big\}$

4a. UP rule: $D \mapsto F$

$\big\{\{\neg E\}, \{E\}\big\}$

5a. UP rule: $E \mapsto T$

$\{\Box\}$

1b. $A \mapsto F$

$\big\{\{C, D, \neg E\}, \{\neg B, \neg C\}, \{B\}\big\}$

2b. UP rule: $B \mapsto T$

$\big\{\{C, D, \neg E\}, \{\neg C\}\big\}$

3b. UP rule: $C \mapsto F$

$\big\{\{D, \neg E\}\big\}$

4b. Splitting rule: D

1ba. $D \mapsto T$

$\{\}$

Stop. All clauses have been satisfied.

Satisfying (partial) assignment: $\neg A, B, \neg C, D$

b) DPLL trace:

$$\{\{A, B, C\}, \{A, \neg B\}, \{\neg A, \neg B\}, \{\neg A, B\}, \{\neg C, D\}, \{D, \neg E\}, \{\neg D, \neg E\}, \{\neg D, E\}\}$$

1. Splitting rule:

   1b. $A \mapsto T$
   $\{\{\neg B\}, \{B\}, \{\neg C, D\}, \{D, \neg E\}, \{\neg D, \neg E\}, \{\neg D, E\}\}$

   2b. UP rule: $B \mapsto F$
   $\{\Box, \{\neg C, D\}, \{D, \neg E\}, \{\neg D, \neg E\}, \{\neg D, E\}\}$

1. Splitting rule:

   1a. $A \mapsto F$
   $\{\{B, C\}, \{\neg B\}, \{\neg C, D\}, \{D, \neg E\}, \{\neg D, \neg E\}, \{\neg D, E\}\}$

   2a. UP rule: $B \mapsto F$
   $\{\{C\}, \{\neg C, D\}, \{D, \neg E\}, \{\neg D, \neg E\}, \{\neg D, E\}\}$

   3a. UP rule: $C \mapsto T$
   $\{\{D\}, \{D, \neg E\}, \{\neg D, \neg E\}, \{\neg D, E\}\}$

   4a. UP rule: $D \mapsto T$
   $\{\{\neg E\}, \{E\}\}$

   5a. UP rule: $E \mapsto F$
   $\{\Box\}$

There is no satisfying assignment.

Total points: 9

Submission Instructions

Solutions need to be packaged into a `.zip` file and uploaded in the AI CMS. The `.zip` file has to contain a single folder with name:
`AI2020_TE3_mat1_mat2_mat3` where `mat1, mat2, mat3` are the matriculation numbers of the students who submit together. This folder must contain the following files:

- `authors.txt` listing the names and matriculation numbers of all students who submit together. Use one line per student and no spaces: Name;Matriculation number.
- The `.pdf` file containing your solutions.

Do not add any other folder or sub folder, this means place all files directly into `AI2020_TE3_mat1_mat2_mat3`. Do not place any file outside of `AI2020_TE3_mat1_mat2_mat3`.

Only one student of each group needs to do the submission! Remember that this sheet can be submitted in groups of up to three members (all members of the group must however be assigned to the same tutorial).