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Exercise Sheet 9
Solution

Exercise 34. (5 Points)

In a university far, far away... every year a lot of students take the Artificial Intelligence (AI) lecture. From years and years of experience it is known that there are three (distinct) types of students at the university:

- *H(ard-working)*, who solve all exercises and qualify for the exam,
- *L(azy)*, who get enough points to qualify for the exam, but then stop working on the exercises,
- *U(nfortunate)*, who do not qualify for the exam.

Typically, 90% of the H-students pass the AI exam with a good grade, but only half of the L-students get a good grade. Every student that qualifies for an exam takes it.

a) Last year, 7 out of 10 students belonged to the L category and there were 10% U-students. What was the probability that a student who got a good grade in the AI exam was of $Type\ H$?

b) In another year 400 students took the AI course. We know that 36 H-students got a good grade, and that 80% of those who got a good grade were L-students. What's the probability that a student in that year was a U-student?

c) Assume there exists a second lecture at the university, namely Machine Learning (ML). This year, we expect 35% of the students to be working hard and 27% of the students to be lazy. Overall, 40% of the students get a good grade in ML and 45% of the students get a good grade in AI. Assume the latter two events to be independent. Furthermore, 80% of the students that get a good grade in ML are H-students, and 5% of the students with a good grade in both ML and AI are lazy.

   Are the events of getting a good grade in AI and getting a good grade in ML conditionally independent given the type of student? Justify your answer by checking if the condition from the definition of conditional independence holds.

(Solution)

We know that:

(1) $P(Grade = good \mid Type = H) = 0.9$

(2) $P(Grade = good \mid Type = L) = 0.5$

a) Further, for a) we know that:

(3a) $P(Type = L) = 0.7$

(4a) $P(Type = U) = 0.1$

We look for: $P(Type = H \mid Grade = good)$
Using the Bayes' rule,

$$P(Type = H \mid Grade = good) = \frac{P(Grade = good \mid Type = H)P(Type = H)}{P(Grade = good)}$$

To compute this we need $P(Type = H)$ and $P(Grade = good)$.

$$P(Type = H) = 1 - P(Type = L) - P(Type = U) = 0.2 \qquad /(3a, 4a)$$

$$\begin{aligned} P(Grade = good) &= P(Grade = good \wedge Type = H) + P(Grade = good \wedge Type = L) = \\ &= P(Grade = good \mid Type = H)P(Type = H) + \\ &\quad + P(Grade = good \mid Type = L)P(Type = L) = \\ &= 0.9 * 0.2 + 0.5 * 0.7 = 0.53 \qquad /(1, 2, 3a, 4a) \end{aligned}$$

Therefore

$$P(Type = H \mid Grade = good) = \frac{0.9 * 0.2}{0.53} = 33.96\%.$$

The probability that a student who got a good grade in the AI exam was of *Type H* is 33.96%.

b) (3b) $\#\ students = 400$

(4b) $\#$ *H-students with good Grade* $= 36$

(5b) $P(Type = L \mid Grade = good) = 0.8$

We look for: $P(Type = U)$

$$P(Type = U) = 1 - P(Type = H) - P(Type = L)$$

So we need $P(Type = H)$ and $P(Type = L)$.

Computing $P(Type = H)$:

$$P(Type = H) = \frac{P(Type = H \wedge Grade = good)}{P(Grade = good \mid Type = H)} = \frac{0.09}{0.9} = 0.1 \qquad /(1)$$

since

$$P(Type = H \mid Grade = good) = 1 - P(Type = L \mid Grade = good) = 1 - 0.8 = 0.2, \qquad /(5b)$$

and

$$P(Type = H \wedge Grade = good) = \frac{36}{400} = 0.09. \qquad /(3b, 4b)$$

Computing $P(Type = L)$:

Using the rearranged Bayes' rule:

$$P(Type = L) = \frac{P(Type = L \mid Grade = good)P(Grade = good)}{P(Grade = good \mid Type = L)} = \frac{0.8 * 0.45}{0.5} = 0.72 \qquad /(2, 5b)$$

since

$$P(Grade = good) = \frac{P(Type = H \wedge Grade = good)}{P(Type = H \mid Grade = good)} = \frac{0.09}{0.2} = 0.45$$

Therefore

$$P(Type = U) = 1 - 0.1 - 0.72 = 0.18.$$

c) We use "X = g" to abbreviate $GradeX = good$.

$$\begin{aligned}\mathbf{P}(AI = g, ML = g \mid Type) &= \frac{\mathbf{P}(Type \mid AI = g, ML = g)P(AI = g, ML = g)}{\mathbf{P}(Type)} \\ &= \frac{\mathbf{P}(Type \mid AI = g, ML = g)P(AI = g)P(ML = g)}{\mathbf{P}(Type)}\end{aligned}$$

$$\mathbf{P}(AI = g, ML = g \mid Type = H) = \frac{0.95 \cdot 0.45 \cdot 0.4}{0.35} \approx 0.49$$

We know that $P(AI = g \mid Type = H) = 0.9$

$$\mathbf{P}(ML = g \mid Type) = \frac{\mathbf{P}(Type \mid ML = g)P(ML = g)}{\mathbf{P}(Type)}$$

$$P(AI = g \mid Type = H)P(ML = g \mid Type = H) = 0.9 \cdot \frac{0.8 \cdot 0.4}{0.35} \approx 0.82 \neq 0.49$$

Thus, the two events are not conditionally independent given the variable $Type$.

(/Solution)

Exercise 35. (5 Points)

In our model, the dog likes to bark when it snows, and usually when it barks the cat is scared. Figure 3 shows the bayesian network of the model. The probabilities are given in Figure 1 and Figure 2. The overall probability of snow is 20%.

	snow	¬ snow
dogBarks	0.7	0.1
¬ dogBarks	0.3	0.9

Figure 1: Probability of DogBarks given Snow.

	dogBarks	¬ dogBarks
catScared	0.9	0.4
¬ catScared	0.1	0.6

Figure 2: Probability of CatScared given DogBarks.

Compute the following probabilities:

1. P(snow,dogBarks,catScared)
2. P(not catScared)
3. P(dogBarks ∧ snow)
4. P(snow | catScared)
5. P(catScared | snow)
6. Are the variables Snow and CatScared independent? Justify your answer.
7. Are the variables Snow and CatScared conditionally independent given dogBarks? Justify your answer.

(Solution)

1.

$$P(snow, dogBarks, catScared) = P(snow)P(dogBarks \mid snow)P(catScared \mid dogBarks) =$$
$$= 0.2 * 0.7 * 0.9 = 0.126$$

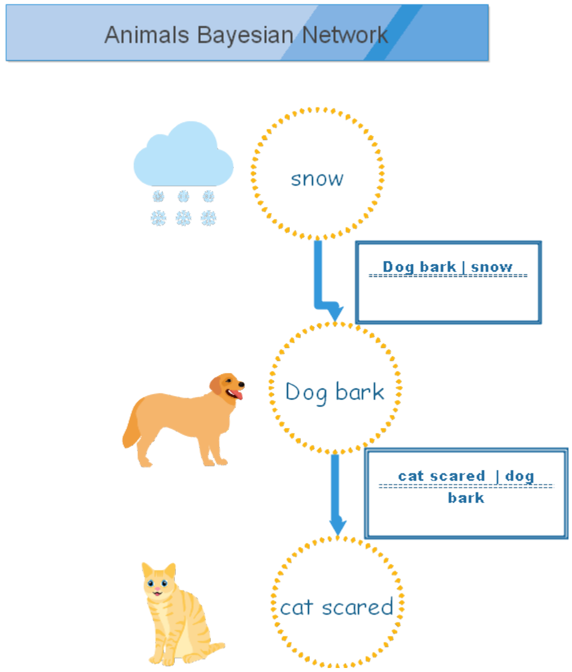

Figure 3: Bayesian Network

2.

$$\begin{aligned}
P(\neg catScared) &= P(\neg catScared \mid dogBarks)P(dogBarks)+ \\
&\quad + P(\neg catScared \mid \neg dogBarks)P(\neg dogBarks) \\
&= 0.1P(dogBarks) + 0.6(1 - P(dogBarks))
\end{aligned}$$

So we compute $P(dogBarks)$ first:

$$\begin{aligned}
P(dogBarks) &= P(dogBarks \mid snow)P(snow)+ \\
&\quad + P(dogBarks \mid \neg snow)P(\neg snow) \\
&= 0.7 * 0.2 + 0.1 * 0.8 = 0.22
\end{aligned}$$

$$P(\neg catScared) = 0.1 * 0.22 + 0.6 * 0.78 = 0.49$$

3.

$$P(dogBarks, snow) = P(snow)P(dogBarks \mid snow) = 0.2 * 0.7 = 0.14$$

4.

$$
\begin{aligned}
P(snow \mid catScared) &= \frac{P(snow, catScared)}{P(catScared)} \\
&= \frac{P(snow, catScared, dogBarks) + P(snow, catScared, \neg dogBarks)}{1 - P(\neg catScared)} \\
&= \frac{0.126 + P(snow)P(\neg dogBarks \mid snow)P(catScared \mid \neg dogBarks)}{0.51} \\
&= \frac{0.126 + 0.2 * 0.3 * 0.4}{0.51} \approx 0.29
\end{aligned}
$$

5.

$$P(catScared \mid snow) = \frac{P(snow, catScared)}{P(snow)} = \frac{0.15}{0.2} = 0.75$$

6. No, since $P(snow) \neq P(snow \mid catScared)$.

7. We need to check if the following holds:

$$P(Snow, catScared \mid dogBarks) = P(Snow \mid dogBarks)P(catScared \mid dogBarks)$$

From what we already computed:

$$P(Snow, catScared \mid dogBarks) = \frac{P(Snow, catScared, dogBarks)}{P(dogBarks)} = \frac{0.126}{0.22} \approx 0.57$$

$$P(Snow \mid dogBarks) = \frac{P(Snow, dogBarks)}{P(dogBarks)} = \frac{0.14}{0.22} \approx 0.64$$

$$
\begin{aligned}
P(snow \mid dogBarks)P(catScared \mid dogBarks) &= \frac{0.14 * 0.9}{0.22} = \frac{0.126}{0.22} \\
&= P(Snow, catScared \mid dogBarks)
\end{aligned}
$$

Therefore Snow and CatScared are conditionally independent given dogBarks

(/Solution)