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Exercise Sheet 3.
Solution

Exercise 9: CNF. (1 Point)

Transform the following formulas to CNF. To do so, follow the steps from the lecture (Chapter 5 slide 21) specifying which steps you are applying and giving the intermediate results.

Note: Simplify the resulting formulas where possible.

(a) $\big(P \vee (Q \leftrightarrow R)\big) \wedge \neg(Q \rightarrow R)$

(b) $\neg(P \leftrightarrow Q) \rightarrow (Q \leftrightarrow R)$

(Solution)

(a)
1. Eliminate $\leftrightarrow$: $\big(P \vee ((Q \rightarrow R) \wedge (R \rightarrow Q))\big) \wedge \neg(Q \rightarrow R)$
2. Eliminate $\rightarrow$: $\big(P \vee ((\neg Q \vee R) \wedge (\neg R \vee Q))\big) \wedge \neg(\neg Q \vee R)$
3. Move $\neg$ inwards: $\big(P \vee ((\neg Q \vee R) \wedge (\neg R \vee Q))\big) \wedge (Q \wedge \neg R)$
4. Distribute $\vee$ over $\wedge$: $(P \vee \neg Q \vee R) \wedge (P \vee \neg R \vee Q) \wedge Q \wedge \neg R$

CNF: $(P \vee \neg Q \vee R) \wedge (P \vee \neg R \vee Q) \wedge Q \wedge \neg R$

(b)
1. Eliminate $\leftrightarrow$: $\neg\big((P \rightarrow Q) \wedge (Q \rightarrow P)\big) \rightarrow \big((Q \rightarrow R) \wedge (R \rightarrow Q)\big)$
2. Eliminate $\rightarrow$: $\big((\neg P \vee Q) \wedge (\neg Q \vee P)\big) \vee \big((\neg Q \vee R) \wedge (\neg R \vee Q)\big)$
3. Move $\neg$ inwards: *no steps required*
4. Distribute $\vee$ over $\wedge$: $(\neg P \vee Q \vee \neg Q \vee R) \wedge (\neg P \vee Q \vee \neg R \vee Q) \wedge (\neg Q \vee P \vee \neg Q \vee R) \wedge (\neg Q \vee P \vee \neg R \vee Q)$
5. *After simplification*: $(\neg P \vee Q \vee \neg R) \wedge (P \vee \neg Q \vee R)$

CNF: $(\neg P \vee Q \vee \neg R) \wedge (P \vee \neg Q \vee R)$

(/Solution)

Exercise 10: Resolution. (3 Points)

For the three formulas below, use resolution to prove that the formulas are unsatisfiable. To do so, first **give a set of clauses Δ that is equivalent to the formula** and second, **use resolution to prove that it is unsatisfiable**. Write the resolution process in the form of a tree for easier readability.

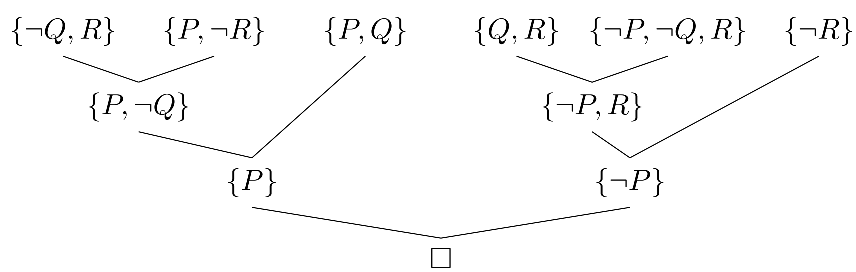
(a) $\phi_1 = (\neg Q \vee R) \wedge (P \vee \neg R) \wedge (P \vee Q) \wedge (Q \vee R) \wedge (\neg P \vee \neg Q \vee R) \wedge \neg R$

(b) $\phi_2 = (P \vee R) \wedge (\neg R \vee Q) \wedge (\neg Q \vee P) \wedge (Q \vee \neg P) \wedge (\neg Q \vee \neg P)$

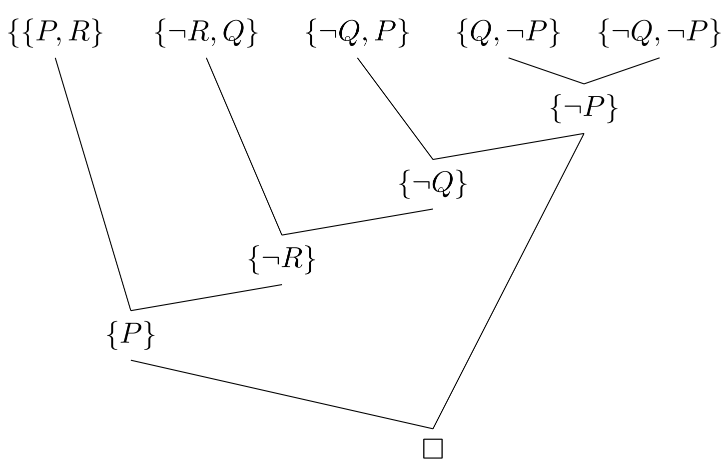
(c) $\phi_3 = (\neg P \vee R \vee S) \wedge (\neg R \vee \neg Q) \wedge (P \vee S \vee \neg Q) \wedge (\neg S \vee \neg Q) \wedge (\neg P \vee Q) \wedge (P \vee Q)$

(Solution)

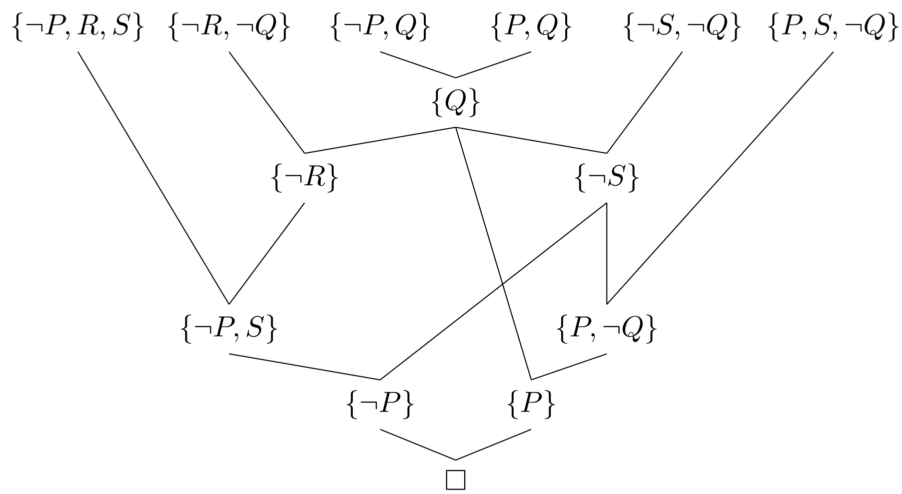
(a) $\Delta_1 = \big\{\{\neg Q, R\}, \{P, \neg R\}, \{P, Q\}, \{Q, R\}, \{\neg P, \neg Q, R\}, \{\neg R\}\big\}$



(b) $\Delta_2 = \big\{\{P, R\}, \{\neg R, Q\}, \{\neg Q, P\}, \{Q, \neg P\}, \{\neg Q, \neg P\}\big\}$



(c) $\Delta_3 = \big\{\{\neg P, R, S\}, \{\neg R, \neg Q\}, \{P, S, \neg Q\}, \{\neg S, \neg Q\}, \{\neg P, Q\}, \{P, Q\}\big\}$



(/Solution)

Exercise 11: DPLL. (3 Points)

For each of the following formulas, **use the DPLL procedure to determine whether it is satisfiable or unsatisfiable**. Transform each formula ϕ_i into an equivalent set of clauses Δ_i. Give a complete trace of the algorithm, **showing the simplified set of clauses for each recursive call of the DPLL function**. Assume that DPLL selects variables in alphabetical order (i.e., $A, B, C, D, E, \ldots$), and that the **splitting rule first attempts the value False (F) and then the value True (T)**.

(a) $\phi_1 = (\neg A \vee B \vee C) \wedge (\neg B \vee \neg C) \wedge (\neg A \vee \neg C \vee \neg D) \wedge (C \vee \neg D) \wedge (A \vee D) \wedge (A \vee \neg C \vee \neg D)$

(b) $\phi_2 = (\neg A \vee \neg B \vee C \vee \neg E) \wedge (\neg A \vee \neg B \vee C \vee E) \wedge (A \leftrightarrow B) \wedge (B \vee D) \wedge (B \vee C \vee \neg D) \wedge (\neg C)$

(Solution)

(a) $\Delta_1 = \big\{\{\neg A, B, C\}, \{\neg B, \neg C\}, \{\neg A, \neg C, \neg D\}, \{C, \neg D\}, \{A, D\}, \{A, \neg C, \neg D\}\big\}$

1. Splitting rule:

   1a. $A \mapsto F$
   $\big\{\{\neg B, \neg C\}, \{C, \neg D\}, \{D\}, \{\neg C, \neg D\}\big\}$

2a. Unit propagation: $D \mapsto T$
$\{\{\neg B, \neg C\}, \{C\}, \{\neg C\}\}$

3a. Unit propagation: $C \mapsto T$
$\{\{\neg B\}, \square\}$

1b. $A \mapsto T$
$\{\{B, C\}, \{\neg B, \neg C\}, \{\neg C, \neg D\}, \{C, \neg D\}\}$

2b. Splitting rule:

1ba. $B \mapsto F$
$\{\{C\}, \{\neg C, \neg D\}, \{C, \neg D\}\}$

2ba. Unit propagation: $C \mapsto T$
$\{\{\neg D\}\}$

3ba. Unit propagation: $D \mapsto F$
$\{\}$

Satisfying assignment: $A, \neg B, C, \neg D$

(b) $\Delta_2 = \{\{\neg A, \neg B, C, \neg E\}, \{\neg A, \neg B, C, E\}, \{\neg A, B\}, \{\neg B, A\}, \{B, D\}, \{B, C, \neg D\}, \{\neg C\}\}$

1. Unit propagation: $C \mapsto F$
$\{\{\neg A, \neg B, \neg E\}, \{\neg A, \neg B, E\}, \{\neg A, B\}, \{\neg B, A\}, \{B, D\}, \{B, \neg D\}\}$
2. Splitting rule:

1a. $A \mapsto F$
$\{\{\neg B\}, \{B, D\}, \{B, \neg D\}\}$

2a. Unit propagation: $B \mapsto F$
$\{\{D\}, \{\neg D\}\}$

3a. Unit propagation: $D \mapsto T$
$\{\square\}$

1b. $A \mapsto T$
$\{\{\neg B, \neg E\}, \{\neg B, E\}, \{B\}, \{B, D\}, \{B, \neg D\}\}$

2b. Unit propagation: $B \mapsto T$
$\{\{\neg E\}, \{E\}, \}$

2b. Unit propagation: $E \mapsto F$
$\{\square\}$

There is no satisfying assignment.

(/Solution)

Exercise 12: CDCL. (3 Points)

Perform DPLL with clause learning as explained in slides 44-46 of Chapter 6 on the following clause set (give the full DPLL trace). Start by using the splitting rule and assign the value F to A. For the next splitting rule, assign T to B. If you encounter a case where two or more different unit propagation rules are applicable choose the one which gets assigned to T (**remember: if a new unit clause is learned, start assigning this clause**). Whenever you encounter a conflict, draw the corresponding implication graph as well as its conflict graphs, and mention which clause can be learned with the clause learning method. Draw vertices of choice literals as boxes and implied literals with circles. Then use this information and continue with the DPLL procedure, backtracking the last choice as specified in slide 44 of Chapter 6. Do this until the clause set is proven to be satisfiable or unsatisfiable.

$$\begin{aligned}\Delta =\big\{&\{A,B,C,D\},\{\neg A,\neg B\},\{\neg B,\neg C\},\{\neg A,\neg D\},\{A,\neg D\},\{C,\neg D\},\\ &\{B,\neg C\},\{\neg B,C\},\{\neg A,C,D\}\big\}\end{aligned}$$

(Solution)

1. Splitting rule:

   1a. $A \mapsto F$
   $\big\{\{B,C,D\},\{\neg B,\neg C\},\{\neg D\},\{C,\neg D\},\{B,\neg C\},\{\neg B,C\}\big\}$

   2a. Unit propagation: $D \mapsto F$
   $\big\{\{B,C\},\{\neg B,\neg C\},\{B,\neg C\},\{\neg B,C\}\big\}$

   3a. Splitting rule:

      1aa. $B \mapsto T$
      $\big\{\{\neg C\},\{C\}\big\}$

      2aa. Unit propagation: $C \mapsto T$
      $\{\Box\}$

      $\rightarrow$ Learned clause: $\neg B$ (see Figure 1)

      i. add $\neg B$ to Δ

      ii. Go back to last splitting rule ($B \mapsto T$)

      iii. Continue: $\big\{\{B,C\},\{\neg B,\neg C\},\{B,\neg C\},\{\neg B,C\},\{\neg B\}\big\}$

1ab. Unit propagation: $B \mapsto F$

$\{\{C\}, \{\neg C\}\}$

2ab. Unit propagation: $C \mapsto T$

$\{\Box\}$

$\rightarrow$ Learned clause: A (see Figure 1)

i. add A to Δ

ii. Go back to last splitting rule ($A \mapsto F$)

iii. Continue: $\{\{A, B, C, D\}, \{\neg A, \neg B\}, \{\neg B, \neg C\}, \{\neg A, \neg D\}, \{A, \neg D\}, \{C, \neg D\}, \{B, \neg C\}, \{\neg B, C\}, \{\neg A, C, D\}, \{A\}, \{\neg B\}\}$

1b. Unit propagation: $A \mapsto T$

$\{\{\neg B\}, \{\neg B, \neg C\}, \{\neg D\}, \{C, \neg D\}, \{B, \neg C\}, \{\neg B, C\}, \{C, D\}\}$

2b. Unit propagation: $B \mapsto F$

$\{\{\neg D\}, \{C, \neg D\}, \{\neg C\}, \{C, D\}\}$

2b. Unit propagation: $C \mapsto F$

$\{\{\neg D\}, \{D\}\}$

3b. Unit propagation: $D \mapsto T$

$\{\Box\}$

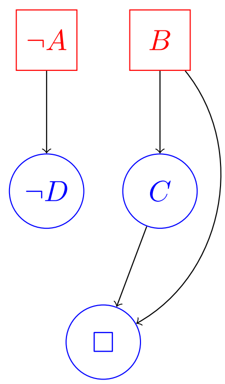
There is no satisfying assignment.

(/Solution)

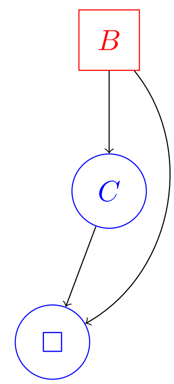
Exercise 13: Contraposition theorem. (**2 Bonus Points**)

Prove the contraposition theorem: $\mathbf{KB} \cup \{\varphi\} \models \neg\psi$ iff $\mathbf{KB} \cup \{\psi\} \models \neg\varphi$.

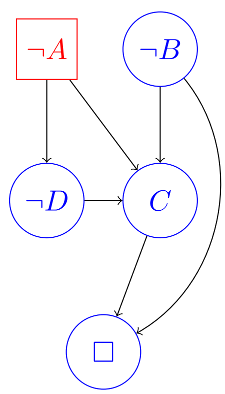
Note: You **have to prove any other theorem used as part of your proof** unless it appears in the lecture slides.



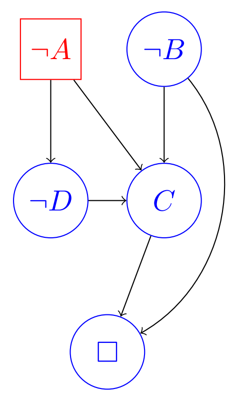
(a) Implication graph in 2aa.



(b) Conflict graph in 2aa.



(c) Implication graph in 2ab.



(d) Conflict graph in 2ab.

Figure 1: Implication and conflict graphs for Exercise 12. Choice literals are indicated by boxes, implied literals by circles.