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Exercise Sheet 4.
Solution

Exercise 14. (2 Points)

Prove that an arbitrary formula ϕ in CNF is valid if and only if each clause contains an atom P and its negation $\neg P$.

Exercise 15. (2 Points)

(a) For each of the following sentences, write down a **predicate logic formula that has the same meaning**. You may only use the following predicates: $Course(x)$, $Student(x)$, $Takes(x, y)$, $Likes(x, y)$, and $Equals(x, y)$.

- There is at least one course that all students take, but not all students like.
- Every student who takes all courses likes at most one course.

(b) For each of the following predicate logic formulas, write down an **English sentence with the same meaning.**

- $\forall x[\text{Family}(x) \to \exists y(\text{Member}(x, y) \land \text{Likes}(y, Elvis))]$
- $\forall x\big[(\text{Boy}(x) \land \exists y(\text{RedCar}(y) \land \text{Owns}(x, y)) \to \big(\exists z(\text{Grandfather}(z, x) \land \text{Cool}(z)) \lor \text{Rich}(x)\big)\big]$

(Solution)

(a)
- $\exists x[\text{Course}(x) \land \forall y(\text{Student}(y) \to \text{Takes}(x, y)) \land \exists z(\text{Student}(z) \land \neg\text{Likes}(z, x))]$
- $\forall x\big[(\text{Student}(x) \land \forall y(\text{Course}(y) \to \text{Takes}(x, y))) \to (\forall c_1 \forall c_2(\text{Course}(c_1) \land \text{Course}(c_2) \land \text{Likes}(x, c_1) \land \text{Likes}(x, c_2)) \to \text{Equals}(c_1, c_2))\big]$

(b)
- At least one member of every family likes Elvis.
- Every boy who owns a red car is either rich or has a cool grandfather.

(/Solution)

Exercise 16. (4 Points)

Transform all of the following predicate logic formulas into clausal normal form. Write down the results of all intermediate steps (cf. Chapter 7, slides 38–39), specifying which steps are you applying and giving the intermediate results.

Note: Simplify the final formulas where possible.

(a) $\varphi_1 = \exists x[A(x) \rightarrow \forall x \forall y[B(x) \wedge C(x,y)]]$

(b) $\varphi_2 = \forall x \exists y[[A(x,y) \wedge B(y)] \rightarrow \forall z(C(y,z) \vee D(x))]$

(c) $\varphi_3 = \forall x \forall y \neg \forall z[A(x,y) \leftrightarrow (B(y) \wedge D(y,z))]$

(Solution)

(a) $\exists x[A(x) \rightarrow \forall x \forall y[B(x) \wedge C(x,y)]]$

1. Eliminate $\leftrightarrow$: nothing to do
2. Eliminate $\rightarrow$:
$$\exists x[\neg A(x) \vee (\forall x \forall y[B(x) \wedge C(x,y)])]$$
3. Move $\neg$ inwards: nothing to do
4. Move quantifiers outwards to get prenex normal form, renaming when needed:
$$\exists x \forall y \forall z[\neg A(x) \vee (B(y) \wedge C(y,z))]$$
5. Eliminate existential quantifiers to get Skolem normal form:
$$\forall y \forall z[\neg A(a) \vee (B(y) \wedge C(y,z))]$$
6. Transform to CNF:
$$\forall y \forall z[(\neg A(a) \vee B(y)) \wedge (\neg A(a) \vee C(y,z))]$$
7. Write as set of clauses:
$$\{\{\neg A(a), B(y)\}, \{\neg A(a), C(y,z)\}\}$$
8. Standardize variables apart to get clausal normal form:
$$\{\{\neg A(a), B(y)\}, \{\neg A(a), C(x,z)\}\}$$

(b) $\forall x \exists y[(A(x,y) \wedge B(y)) \rightarrow \forall z[C(y,z) \vee D(x)]]$

1. Eliminate $\leftrightarrow$: nothing to do.

2. Eliminate $\rightarrow$:

$$\forall x \exists y[\neg(A(x,y) \wedge B(y)) \vee \forall z[C(y,z) \vee D(x)]]$$

3. Move $\neg$ inwards:

$$\forall x \exists y[(\neg A(x,y) \vee \neg B(y)) \vee \forall z[C(y,z) \vee D(x)]]$$

4. Move quantifiers outwards to get prenex normal form, renaming when needed:

$$\forall x \exists y \forall z[(\neg A(x,y) \vee \neg B(y)) \vee (C(y,z) \vee D(x))]$$

5. Eliminate existential quantifiers to get Skolem normal form:

$$\forall x \forall z[(\neg A(x,f(x)) \vee \neg B(f(x)) \vee (C(f(x),z) \vee D(x))]$$

6. Transform to CNF: nothing to do.
7. Write as set of clauses:

$$\{\{\neg A(x,f(x)), \neg B(f(x)), C(f(x),z), D(x)\}\}$$

8. Standardize variables apart to get clausal normal form: nothing to do.

(c) $\forall x \forall y \neg \forall z[A(x,y) \leftrightarrow (B(y) \wedge D(y,z))]$

1. Eliminate $\leftrightarrow$:

$$\forall x \forall y \neg \forall z[(A(x,y) \rightarrow (B(y) \wedge D(y,z))) \wedge ((B(y) \wedge D(y,z)) \rightarrow A(x,y))]$$

2. Eliminate $\rightarrow$:

$$\forall x \forall y \neg \forall z[(\neg A(x,y) \vee (B(y) \wedge D(y,z))) \wedge (\neg(B(y) \wedge D(y,z)) \vee A(x,y))]$$

3. Move $\neg$ inwards:

$$\forall x \forall y \exists z \neg[(\neg A(x,y) \vee (B(y) \wedge D(y,z))) \wedge (\neg(B(y) \wedge D(y,z)) \vee A(x,y))]$$

$$\forall x \forall y \exists z[\neg(\neg A(x,y) \vee (B(y) \wedge D(y,z))) \vee \neg(\neg(B(y) \wedge D(y,z)) \vee A(x,y))]$$

$$\forall x \forall y \exists z[(A(x,y) \wedge (\neg B(y) \vee \neg D(y,z))) \vee ((B(y) \wedge D(y,z)) \wedge \neg A(x,y))]$$

4. Move quantifiers outwards to get prenex normal form, renaming when needed: nothing to do.
5. Eliminate existential quantifiers to get Skolem normal form:

$$\forall x \forall y[(A(x,y) \wedge (\neg B(y) \vee \neg D(y,f(x,y)))) \vee ((B(y) \wedge D(y,f(x,y))) \wedge \neg A(x,y))]$$

6. Transform to CNF:

$$\forall x \forall y[((A(x,y) \vee B(y)) \wedge (A(x,y) \vee D(y,f(x,y))) \wedge (\neg B(y) \vee \neg D(y,f(x,y)) \vee \neg A(x,y)))]$$

7. Write as set of clauses:

$$\{\{A(x,y), B(y)\}, \{A(x,y), D(y, f(x,y))\}, \{\neg B(y) \vee \neg D(y, f(x,y)) \vee \neg A(x,y)\}\}$$

8. Standardize variables apart to get clausal normal form:

$$\{\{A(x,y), B(y)\}, \{A(z,u), D(u, f(z,u))\}, \{\neg B(s) \vee \neg D(s, f(t,s)) \vee \neg A(t,s)\}\}$$

(/Solution)

Exercise 17. (2 Points)

Given the knowledge base KB, use resolution on the Herbrand expansion of the formula θ^* that you use to show the claim to deduce that both Max and Anna like candy (cf. Chapter 8, slide 11).

KB=$\{\forall x[(\text{Child}(x) \vee \text{SweetTooth}(x)) \rightarrow \text{LikesCandy}(x)], \text{Child}(Max), \text{SweetTooth}(Anna)\}$

To do so, you need transform KB into Skolem normal form. Also, provide the Herbrand universe $HU(\theta^*)$ and the Herbrand expansion $HE(\theta^*)$.

(Solution)

$$\theta := KB \cup \{\neg\text{LikesCandy}(Anna) \vee \neg\text{LikesCandy}(Max)\}$$

is unsatisfiable iff

$$KB \models \{\text{LikesCandy}(Anna) \wedge \text{LikesCandy}(Max)\}$$

Skolem normal form θ^* of θ:

$$\{\forall x[\neg\text{Child}(x) \vee \text{LikesCandy}(x)], \forall x[\neg\text{SweetTooth}(x) \vee \text{LikesCandy}(x)]$$

$$\text{Child}(Max), \text{SweetTooth}(Anna), \neg\text{LikesCandy}(Anna) \vee \neg\text{LikesCandy}(Max)\}$$

Herbrand universe:

$$HU(\theta^*) = \{Max, Anna\}$$

Herbrand expansion:

$$HE(\theta^*) = \{\neg\text{Child}(Max) \vee \text{LikesCandy}(Max), \neg\text{SweetTooth}(Max) \vee \text{LikesCandy}(Max),$$

$$\neg\text{Child}(Anna) \vee \text{LikesCandy}(Anna), \neg\text{SweetTooth}(Anna) \vee \text{LikesCandy}(Anna), \text{Child}(Max),$$

$$\text{SweetTooth}(Anna), \neg\text{LikesCandy}(Anna) \vee \neg\text{LikesCandy}(Max)\}$$

Resolution:

$$\{\neg\text{Child}(Max), \text{LikesCandy}(Max)\} \qquad \{\neg\text{LikesCandy}(Anna), \neg\text{LikesCandy}(Max)\}$$

$$\{\neg\text{Child}(Max), \neg\text{LikesCandy}(Anna)\} \qquad \{\text{Child}(Max)\}$$

$$\{\neg\text{LikesCandy}(Anna)\} \qquad \{\neg\text{SweetTooth}(Anna), \text{LikesCandy}(Anna)\}$$

$$\{\neg\text{SweetTooth}(Anna)\} \qquad \{\text{SweetTooth}(Anna)\}$$

$$\square$$

Therefore, we can conclude LikesCandy(Max) and LikesCandy($Anna$).

(/Solution)