Image Processing and Computer Vision (IPCV)

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Example Solutions for Classroom Assignment 13 (C13)

Problem 1: (Transformation Matrices)

In Lecture 25, the examples of a rotation around the z-axis and of a translation was given. We have to adapt this for rotations around the x-axis and y-axis.

Even if there was no rotation direction specified in the problem, we want to rotate in the mathematical positive sense, that means counter-clockwise. To understand this, we use a right-handed coordinate system as shown in Fig. 1. All later considerations will also work for a left-handed system, except that all rotation directions are exchanged. To explain a rotation around the z-axis, for instance, we take a look at the $x-y$-plane from the positive z-axis. This is shown in Fig. 2.

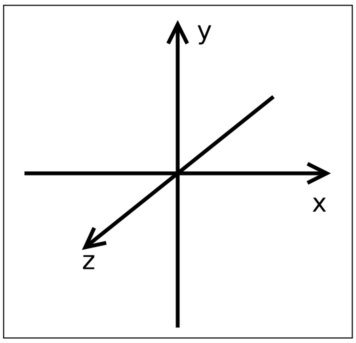

Figure 1: Right-handed coordinate system.

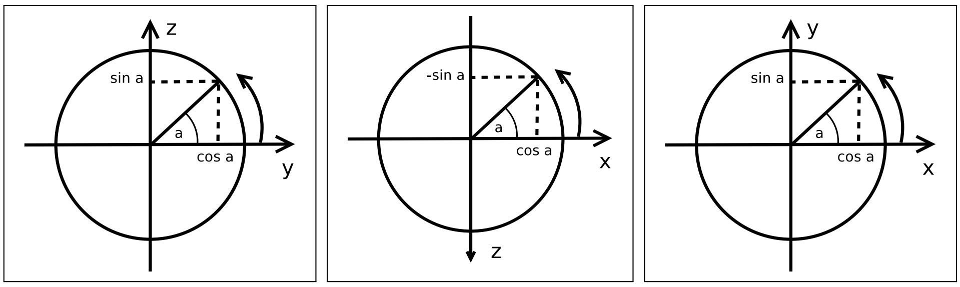

Figure 2: Rotation directions for the three axes. **Left:** Around x-axis. **Middle:** Around y-axis. **Right:** Around z-axis. We see that the y-axis differs from the other two.

We see that for counter-clockwise rotation, we have to distinguish rotations around the y-axis from those around x- and z-axes. We keep in mind that the image of the standard unit vectors under a linear mapping can be found in the columns of the matrix. Together with Fig. 2 this explains why a rotation around the x- and z-axes looks like

$$R_x(\varphi) := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\varphi) & -\sin(\varphi) & 0 \\ 0 & \sin(\varphi) & \cos(\varphi) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and

$$R_z(\varphi) := \begin{pmatrix} \cos(\varphi) & -\sin(\varphi) & 0 & 0 \\ \sin(\varphi) & \cos(\varphi) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

while a rotation around the y-axis is given by

$$R_y(\varphi) := \begin{pmatrix} \cos(\varphi) & 0 & \sin(\varphi) & 0 \\ 0 & 1 & 0 & 0 \\ -\sin(\varphi) & 0 & \cos(\varphi) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Now we want to compute a transformation matrix in homogeneous coordinates which describes a rotation around the x-axis through an angle of 45 degrees followed by a translation with a vector $(2, 4, -1)^\top$ and a rotation around the y-axis through an angle of -60 degrees.

The rotation around the x-axis through an angle of 45 degrees is described by the matrix

$$R_1 := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(45°) & -\sin(45°) & 0 \\ 0 & \sin(45°) & \cos(45°) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad .$$

The translation by a vector $(2, 4, -1)^\top$ is given by the matrix

$$T := \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad .$$

The rotation around the y-axis through an angle of -60 is given by the matrix

$$R_2 := \begin{pmatrix} \cos(-60^\circ) & 0 & \sin(-60^\circ) & 0 \\ 0 & 1 & 0 & 0 \\ -\sin(-60^\circ) & 0 & \cos(-60^\circ) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos(60^\circ) & 0 & -\sin(60^\circ) & 0 \\ 0 & 1 & 0 & 0 \\ \sin(60^\circ) & 0 & \cos(60^\circ) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} .$$

In a matrix-vector product, the vector is always multiplied from the right-hand side to the matrix. That means the rightmost transformation matrix is applied first to the vector, then its left neighbour etc. Transformations are applied from right to left. Thus the overall transformation matrix is then given by the matrix multiplication R_2TR_1:

$$\begin{aligned} R_2TR_1 &= \begin{pmatrix} \cos(60^\circ) & -\sin(45^\circ)\sin(60^\circ) & -\cos(45^\circ)\sin(60^\circ) & 2\cos(60^\circ)+\sin(60^\circ) \\ 0 & \cos(45^\circ) & -\sin(45^\circ) & 4 \\ \sin(60^\circ) & \sin(45^\circ)\cos(60^\circ) & \cos(45^\circ)\cos(60^\circ) & 2\sin(60^\circ)-\cos(60^\circ) \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{4} & -\frac{\sqrt{6}}{4} & 1+\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 4 \\ \frac{1}{\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & \sqrt{2}-\frac{1}{2} \\ 0 & 0 & 0 & 1 \end{pmatrix} . \end{aligned}$$

It is important to respect the order in which the matrices are multiplied as matrix multiplication does not commute.

Problem 2 (Rotation Matrices in 3-D)

By three angles α, β and γ, the so-called *Euler angles*, we are able to describe the orientation of a 3-D-object in space using a single rotation matrix. To this end we state three matrices that describe rotations around the z, the x and again the z axis and concatenate them afterwards.

Rotation by α degrees around the z-axis:

$$\boldsymbol{R}_z(\alpha) := \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) & 0 \\ \sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Rotation by β degrees around the x-axis:

$$\boldsymbol{R}_x(\beta) := \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\beta) & -\sin(\beta) \\ 0 & \sin(\beta) & \cos(\beta) \end{pmatrix}$$

Rotation by γ degrees around the z-axis:

$$\boldsymbol{R}_z(\gamma) := \begin{pmatrix} \cos(\gamma) & -\sin(\gamma) & 0 \\ \sin(\gamma) & \cos(\gamma) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Let $\boldsymbol{p} = (p_1, p_2, p_3)^\top$ be an object point. To get the new orientation we first rotate α degrees around the original z-axis, i.e we compute $\boldsymbol{R}_z(\alpha)\boldsymbol{p}$. This is followed by a rotation around the *new* x-axis by an angle of β. That means we have to use the basistransformed rotation matrix $\boldsymbol{R}'_x(\beta) := \boldsymbol{R}_z(\alpha)\boldsymbol{R}_x(\beta)\boldsymbol{R}_z^{-1}(\alpha)$:

$$\begin{aligned} \boldsymbol{R}'_x(\beta)\boldsymbol{R}_z(\alpha)\boldsymbol{p} &= \boldsymbol{R}_z(\alpha)\boldsymbol{R}_x(\beta)\boldsymbol{R}_z^{-1}(\alpha)\boldsymbol{R}_z(\alpha)\boldsymbol{p} \\ &= \boldsymbol{R}_z(\alpha)\boldsymbol{R}_x(\beta)\boldsymbol{p} \quad . \end{aligned}$$

Finally we rotate γ degrees around the z-axis that was created by the previous rotations. Again we have to use the basistransformed rotation matrix $\boldsymbol{R}'_z(\gamma) := \boldsymbol{R}_z(\alpha)\boldsymbol{R}_x(\beta)\boldsymbol{R}_z(\gamma)(\boldsymbol{R}_z(\alpha)\boldsymbol{R}_x(\beta))^{-1}$:

$$\begin{aligned} \boldsymbol{R}'_z(\gamma)\boldsymbol{R}_z(\alpha)\boldsymbol{R}_x(\beta)\boldsymbol{p} &= \boldsymbol{R}_z(\alpha)\boldsymbol{R}_x(\beta)\boldsymbol{R}_z(\gamma)(\boldsymbol{R}_z(\alpha)\boldsymbol{R}_x(\beta))^{-1}\boldsymbol{R}_z(\alpha)\boldsymbol{R}_x(\beta)\boldsymbol{p} \\ &= \boldsymbol{R}_z(\alpha)\boldsymbol{R}_x(\beta)\boldsymbol{R}_z(\gamma)\boldsymbol{R}_x^{-1}(\beta)\boldsymbol{R}_z^{-1}(\alpha)\boldsymbol{R}_z(\alpha)\boldsymbol{R}_x(\beta)\boldsymbol{p} \\ &= \boldsymbol{R}_z(\alpha)\boldsymbol{R}_x(\beta)\boldsymbol{R}_z(\gamma)\boldsymbol{R}_x^{-1}(\beta)\boldsymbol{R}_x(\beta)\boldsymbol{p} \\ &= \boldsymbol{R}_z(\alpha)\boldsymbol{R}_x(\beta)\boldsymbol{R}_z(\gamma)\boldsymbol{p} \quad . \end{aligned}$$

So the full concatenated rotation matrix $\boldsymbol{M}_{zxz}$ is then given as

$$\boldsymbol{M}_{zxz} = \boldsymbol{R}_z(\alpha)\boldsymbol{R}_x(\beta)\boldsymbol{R}_z(\gamma) =$$

$$\begin{pmatrix} \cos(\alpha)\cos(\gamma) - \sin(\alpha)\cos(\beta)\sin(\gamma) & -\cos(\alpha)\sin(\gamma) - \sin(\alpha)\cos(\beta)\cos(\gamma) & \sin(\alpha)\sin(\beta) \\ \sin(\alpha)\cos(\gamma) + \cos(\alpha)\cos(\beta)\sin(\gamma) & \cos(\alpha)\cos(\beta)\cos(\gamma) - \sin(\alpha)\sin(\gamma) & -\cos(\alpha)\sin(\beta) \\ \sin(\beta)\sin(\gamma) & \sin(\beta)\cos(\gamma) & \cos(\beta) \end{pmatrix}.$$

As we rotate first around the z-axis, then around the x-axis and afterwards again around the z-axis, this definition of the Euler angles is called z-x-z convention. It is one of several common conventions; others are for example the x-y-z and z-y-x convention.

Note that $\boldsymbol{M}_{zxz} \neq \boldsymbol{R}_z(\gamma)\boldsymbol{R}_x(\beta)\boldsymbol{R}_z(\alpha)$, which would describe a rotation by α degrees around the original z-axis followed by a rotation of β degrees around the original x-axis followed by a rotation of γ degrees around the original z-axis.