Example Solutions for Classroom Assignment 2 (C2)

Problem 1 (Colour Spaces)

We construct the formula for transformation in the same way as for YC_bC_r in the lecture. The luma component, Z, an explicit formula in terms of R, G and B is already given:

$$Z = \frac{1}{3}(R+G+B)$$

For the channel C_r, we have the same direction as in the lecture, red-cyan, but the definition of Z influences the transformation formula for C_r. Since we want to move in a linear fashion from red to cyan in an interval from $[0,255]$ (deviation is shifted by 127.5), for $(R,G,B) = (255,0,0)$ (red) we have to fulfil

$$127.5 + a \cdot 255 + b \cdot 0 + c \cdot 0 = 255$$

and for $(R,G,B) = (0,255,255)$ (cyan) we want to have

$$127.5 + a \cdot 0 + b \cdot 255 + c \cdot 255 = 0.$$

Thus, $a = \frac{1}{2}$ is uniquely defined. For the remaining unknowns we seemingly have only one equation, but we also have to incorporate the ratio between G and B in the formula for the luma channel. Since all colours are equally weighted there, we get $b = c = -\frac{1}{4}$.

Treating the green-cyan direction in the same way yields the transformation formula in the matrix-vector notation from Slide 26 of Lecture 3:

$$\begin{pmatrix} Z \\ C_r \\ C_g \end{pmatrix} = \begin{pmatrix} 0 \\ 127.5 \\ 127.5 \end{pmatrix} + \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{2} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{1}{2} & -\frac{1}{4} \end{pmatrix} \begin{pmatrix} f_r \\ f_g \\ f_b \end{pmatrix}$$

Note that we can also identify cyan with 255 in the C_r channel and red with 0 which might be a more intuitive representation of the direction red-cyan, but this notation is consitent with the definition from the Lecture.

We can also compute the transformation formula for other YC_bC_r-like spaces using the general formulas

$$\begin{aligned} Z &= rR + gG + (1-r-g)B \\ C_r &= 127.5 + \frac{1}{2}\frac{R-Z}{1-r} \\ C_g &= 127.5 + \frac{1}{2}\frac{G-Z}{1-g} \end{aligned}$$

We fix the weight for two of the RGB-channels (in this example R and B) and determine by this the weight of the third channel. The deviation of a color channel in respect to the luma channel (e.g. $R - Z$) is normalised to the interval $[-127.5, 127.5]$ (e.g. by multiplication with $\frac{1}{2(1-r)}$) and shifted to the interval $[0, 255]$. The colour space in this exercise results from the choice $r = g = \frac{1}{3}$.

Remark: In the literature, one usually distinguishes a colour space with continuous chroma channels (YP_bP_r) and its discrete counter part (YC_bC_r). For simplicity reasons, we don't make this notational distinction here.

Problem 2 (Continuous Fourier Transform)

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(\frac{-x^2}{2\sigma^2}\right)$$

(a) The Fourier transform $\hat{f}(u) = \mathcal{F}[f]$ of f can be computed as:

$$\begin{aligned}
\hat{f}(u) &= \int_{-\infty}^{\infty} f(x) \exp(-i2\pi ux) dx \\
&= \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} \exp\left(\frac{-x^2}{2\sigma^2}\right) \exp(-i2\pi ux) dx \\
&= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(\frac{-x^2}{2\sigma^2} - i2\pi ux\right) dx \\
&\overset{(i)}{=} \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\pi\left(\frac{x^2}{2\pi\sigma^2} + i2ux - u^2 2\pi\sigma^2 + u^2 2\pi\sigma^2\right)\right) dx \\
&= \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-2\pi^2 u^2 \sigma^2\right) \cdot \int_{-\infty}^{\infty} \exp\left(-\pi\left(\left(\frac{x}{\sqrt{2\pi}\sigma}\right)^2 + i2ux + (i\sqrt{2\pi}u\sigma)^2\right)\right) dx \\
&= \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-2\pi^2 u^2 \sigma^2\right) \cdot \int_{-\infty}^{\infty} \exp\left(-\pi\left(\frac{x}{\sqrt{2\pi}\sigma} + i\sqrt{2\pi}u\sigma\right)^2\right) dx
\end{aligned}$$

i. completing the square (quadratische Ergänzung)

Substitute $\frac{x}{\sqrt{2\pi}\sigma} + i\sqrt{2\pi}u\sigma =: z$

$$\begin{aligned}
\hat{f}(u) &= \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-2\pi^2 u^2 \sigma^2\right) \cdot \int_{-\infty}^{\infty} \sqrt{2\pi}\sigma \exp\left(-\pi z^2\right) dz \\
&= \exp\left(-2\pi^2 u^2 \sigma^2\right) \cdot \int_{-\infty}^{\infty} \exp\left(-\pi z^2\right) dz
\end{aligned}$$

Using the hint

$$\int_{-\infty}^{\infty} \exp\left(-\pi x^2\right) dx = 1,$$

we finally see that the Fourier transform of a Gaussian is a Gaussian-like function with inverse variance (compare with Slide 27 of Lecture 4):

$$\hat{f}(u) = \exp\left(\frac{-(2\pi u)^2}{2\sigma^{-2}}\right)$$

(b) Finally, we are in the position to compute the Fourier spectrum of the Gaussian. Since the Fourier transform $\hat{f}(u)$ is real-valued in the case of the Gaussian, it is identical to the Fourier spectrum $|\hat{f}(u)|$:

$$\left|\hat{f}(u)\right| = \exp\left(\frac{-(2\pi u)^2}{2\sigma^{-2}}\right) .$$
