Example Solutions for Homework Assignment 2 (H2)

Problem 1 (Properties of the continuous Fourier transform)

Verify that the following properties of the continuous Fourier transform are true.

Linearity: $\mathcal{F}[a \cdot f(x) + b \cdot g(x)](u) = a \cdot \mathcal{F}[f](u) + b \cdot \mathcal{F}[g](u)$

$$\begin{aligned}
\mathcal{F}[a \cdot f(x) + b \cdot g(x)](u) &= \int_{\mathbb{R}} (a \cdot f(x) + b \cdot g(x)) \exp(-i2\pi ux)\mathrm{d}x \\
&= a \int_{\mathbb{R}} f(x) \exp(-i2\pi ux)\mathrm{d}x + b \int_{\mathbb{R}} g(x) \exp(-i2\pi ux)\mathrm{d}x \\
&= a \cdot \mathcal{F}[f](u) + b \cdot \mathcal{F}[g](u)
\end{aligned}$$

Spatial Shift: $\mathcal{F}[f(x-a)](u) = \exp(-i2\pi ua) \cdot \mathcal{F}[f](u)$

$$\begin{aligned}
\mathcal{F}[f(x-a)](u) &= \int_{\mathbb{R}} f(x-a) \exp(-i2\pi ux)\mathrm{d}x \quad t \leftrightarrow x-a \\
&= \int_{\mathbb{R}} f(t) \exp(-i2\pi u(t+a))\mathrm{d}t \\
&= \exp(-i2\pi ua) \int_{\mathbb{R}} f(t) \exp(-i2\pi ut)\mathrm{d}t \\
&= \exp(-i2\pi ua)\mathcal{F}[f](u)
\end{aligned}$$

Frequency Shift: $\mathcal{F}[f(x) \cdot \exp(-i2\pi u_0 x)](u) = \mathcal{F}[f](u+u_0)$

$$\begin{aligned}
\mathcal{F}[f(x) \cdot \exp(-i2\pi u_0 x)](u) &= \int_{\mathbb{R}} f(x) \exp(-i2\pi u_0 x) \exp(-i2\pi ux)\mathrm{d}x \\
&= \int_{\mathbb{R}} f(x) \exp(-i2\pi (u_0+u)x)\mathrm{d}x \\
&= \mathcal{F}[f](u_0+u)
\end{aligned}$$

Scaling: $\mathcal{F}[f(ax)](u) = \frac{1}{|a|} \cdot \mathcal{F}[f](\frac{u}{a})$

$$\begin{aligned}
\mathcal{F}[f(ax)](u) &= \int_{\mathbb{R}} f(ax) \exp(-i2\pi ux)\mathrm{d}x \quad t \leftrightarrow ax \\
&= \int_{\mathbb{R}} f(t) \exp\left(-i2\pi u \frac{t}{a}\right) \frac{1}{|a|}\mathrm{d}t \\
&= \frac{1}{|a|}\mathcal{F}[f]\left(\frac{u}{a}\right)
\end{aligned}$$

Convolution: $\mathcal{F}[f * g(nx)](u) = \mathcal{F}[f](u) \cdot \mathcal{F}[g](u)$

$$\begin{aligned}
\mathcal{F}[f * g(x)](u) &= \int_{\mathbb{R}} (f * g)(x) \exp(-i2\pi ux) \mathrm{d}x \\
&= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(\xi) g(x-\xi) \mathrm{d}\xi \right) \exp(-i2\pi ux) \mathrm{d}x \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} f(\xi) g(x-\xi) \exp(-i2\pi ux) \mathrm{d}x \mathrm{d}\xi \\
&= \int_{\mathbb{R}} f(\xi) \exp(-i2\pi u\xi) \left(\int_{\mathbb{R}} g(x-\xi) \exp(-i2\pi u(x-\xi)) \mathrm{d}x \right) \mathrm{d}\xi \\
&= \int_{\mathbb{R}} f(\xi) \exp(-i2\pi u\xi) \left(\int_{\mathbb{R}} g(t) \exp(-i2\pi ut) \mathrm{d}t \right) \mathrm{d}\xi \\
&= \int_{\mathbb{R}} f(\xi) \exp(-i2\pi u\xi) \mathrm{d}\xi \mathcal{F}[g](u) \\
&= \mathcal{F}[f](u) \cdot \mathcal{F}[g](u)
\end{aligned}$$

Problem 2 (Continuous Fourier Transform of a Hat Function)

Let us first discuss how this exercise can be solved efficiently. We know from the first home work, that the convolution of a box function with itself yields a hat function. In particular, the hat function f from this exercise corresponds to a convolution $f(x) = g(x) * g(x)$ where g is the box function from H1, Problem 1:

$$f(x) := \begin{cases} \frac{1}{2} & -1 \leq x \leq 1 \\ 0 & \text{else} \end{cases}$$

From the lecture we know what form the Fourier transform of a box function h with height A on the interval $[-R, R]$ is:

$$\mathcal{F}[h](u) = \frac{A}{2\pi u} \sin(2\pi uR)$$

In the case of our box function g we have $A = \frac{1}{2}$ and $R = 1$, thus yielding

$$\mathcal{F}[h](u) = \frac{1}{2\pi u} \sin(2\pi u) = \operatorname{sinc}(2\pi u)$$

Finally, we apply the convolution theorem and get:

$$\mathcal{F}[f](u) = \mathcal{F}[g * g](u) = (\mathcal{F}[g](u))^2 = \operatorname{sinc}^2(2\pi u)$$

For the sake of completeness, we also demonstrate how to solve the exercise in the more tedious, straightforward way by computing $\mathcal{F}[f](u)$ directly:

$$
\begin{aligned}
&\mathcal{F}[f](u)\\
=&\underbrace{\int_{-\infty}^{-2} 0\cdot\exp(-i2\pi ux)dx}_{=0}+\int_{-2}^{2}\frac{1}{4}(2-|x|)\cdot\exp(-i2\pi ux)dx+\underbrace{\int_{2}^{\infty} 0\cdot\exp(-i2\pi ux)dx}_{=0}\\
=&\int_{-2}^{2}\frac{1}{4}(2-|x|)\cdot(\cos(2\pi ux)-i\sin(2\pi ux))dx\\
=&\underbrace{\frac{1}{2}\int_{-2}^{2}\cos(2\pi ux)dx}_{\text{even function}}\underbrace{-\frac{1}{2}\int_{-2}^{2}i\sin(2\pi ux)dx}_{=0\ \text{(odd function)}}\\
&\underbrace{-\frac{1}{4}\int_{-2}^{2}|x|\cos(2\pi ux)dx}_{\text{even function}}\underbrace{+\frac{1}{4}\int_{-2}^{2}|x|i\sin(2\pi ux)dx}_{=0\ \text{(odd function)}}\\
=&2\cdot\frac{1}{2}\int_{0}^{2}\cos(2\pi ux)dx-2\cdot\frac{1}{4}\int_{0}^{2}x\cos(2\pi ux)dx
\end{aligned}
$$

Applying the product rule for integrals to the right part of that term gives:

$$
\begin{aligned}
=&\int_{0}^{2}\cos(2\pi ux)dx-\frac{1}{2}\left[x\frac{\sin(2\pi ux)}{2\pi u}\right]_0^2+\frac{1}{2}\int_0^2\frac{\sin(2\pi ux)}{2\pi u}dx\\
=&\left[\frac{\sin(2\pi ux)}{2\pi u}\right]_0^2-\frac{1}{2}\left[x\frac{\sin(2\pi ux)}{2\pi u}\right]_0^2+\frac{1}{2}\left[\frac{-\cos(2\pi ux)}{(2\pi u)^2}\right]_0^2\\
=&\left(\frac{\sin(4\pi u)}{2\pi u}-\frac{\sin(0)}{2\pi u}\right)-\frac{1}{2}\left(2\frac{\sin(4\pi u)}{2\pi u}-\frac{\sin(0)}{2\pi u}\right)+\frac{1}{2}\left(-\frac{\cos(4\pi u)}{(2\pi u)^2}+\frac{\cos(0)}{(2\pi u)^2}\right)\\
=&\frac{1-\cos(4\pi u)}{2(2\pi u)^2}
\end{aligned}
$$

Using the addition theorems we finally get:

$$
\begin{aligned}
=&\frac{\cos^2(2\pi u)+\sin^2(2\pi u)-\cos(2\pi u+2\pi u)}{2(2\pi u)^2}\\
=&\frac{\cos^2(2\pi u)+\sin^2(2\pi u)-(\cos^2(2\pi u)-\sin^2(2\pi u))}{2(2\pi u)^2}\\
=&\left(\frac{\sin(2\pi u)}{2\pi u}\right)^2=\operatorname{sinc}^2(2\pi u)
\end{aligned}
$$

This exercise demonstrates the importance of the convolution theorem.

Problem 3 (Continuous Fourier Transform of a Discrete Filter)

Computing the Fourier transform of g gives:

$$\begin{aligned}
\mathcal{F}[g](u) &\overset{(i)}{=} \frac{1}{h^4}\left(\mathcal{F}[f(x+2h)](u) - 4\mathcal{F}[f(x+h)](u) + 6\mathcal{F}[f(x)](u) - 4\mathcal{F}[f(x-h)](u) + \mathcal{F}[f(x-2h)](u)\right) \\
&\overset{(ii)}{=} \frac{1}{h^4}\left[\exp(4\pi i h u) - 4\exp(2\pi i h u) - 4\exp(-2\pi i h u) + \exp(-4\pi i h u) + 6\right]\mathcal{F}[f](u) \\
&\overset{(iii)}{=} \frac{1}{h^4}\left[2\cos(4\pi h u) - 8\cos(2\pi h u) + 6\right]\mathcal{F}[f](u) \\
&= \frac{2}{h^4}\left[\cos(4\pi h u) - 4\cos(2\pi h u) + 3\right]\mathcal{F}[f](u)
\end{aligned}$$

i. Linearity

ii. Shift-Theorem

iii. since $\cos(\phi) = (\frac{e^{i\phi}+e^{-i\phi}}{2})$

So we see that the Fourier transform of the function g is essentially the Fourier transform of the original signal multiplied by a combination of trigonometric functions. This result will help us to understand the lowpass effect of derivative filters later on.

Problem 4 (Colour Spaces)

(a) The conversion from RGB to YCbCr images requires to supplement the following code:

```
for (i=1;i<=nx;i++)
  for (j=1;j<=ny;j++)
  {
    Y[i][j] =   0.299 * R[i][j]
              + 0.587 * G[i][j]
              + 0.114 * B[i][j];
    Cb[i][j] = 127.5 + (- 0.169 * R[i][j]
                        - 0.331 * G[i][j]
                        + 0.500 * B[i][j]);
    Cr[i][j] = 127.5 + ( 0.500 * R[i][j]
                       - 0.419 * G[i][j]
                       - 0.081 * B[i][j]);
  }
```

(b) The compressed variants of the image `baboon.ppm` for subsampling factors of $S = 1, 2, 4$ and 8 are depicted in Tab 1. As one can see, the variants for $S = 2$ and $S = 4$ provide still a quite good quality. This is due to the fact that the details (hairs, patterns) are preserved, since the Y-channel that contains these details is not compressed (downsampled). In the case of $S = 8$, however, slight block artifacts become visible. This is verified by the images in Tab. 2 that depict a zoom of the nose region. Here also for $S = 2$ and $S = 4$ the loss of quality becomes obvious.

Table 1: Compressed variants of the image `baboon.ppm`. *(a) Top left:* Image for $S = 1$ (original image). *(b) Top right:* $S = 2$. *(c) Bottom left:* $S = 4$. *(d) Bottom right:* $S = 8$.

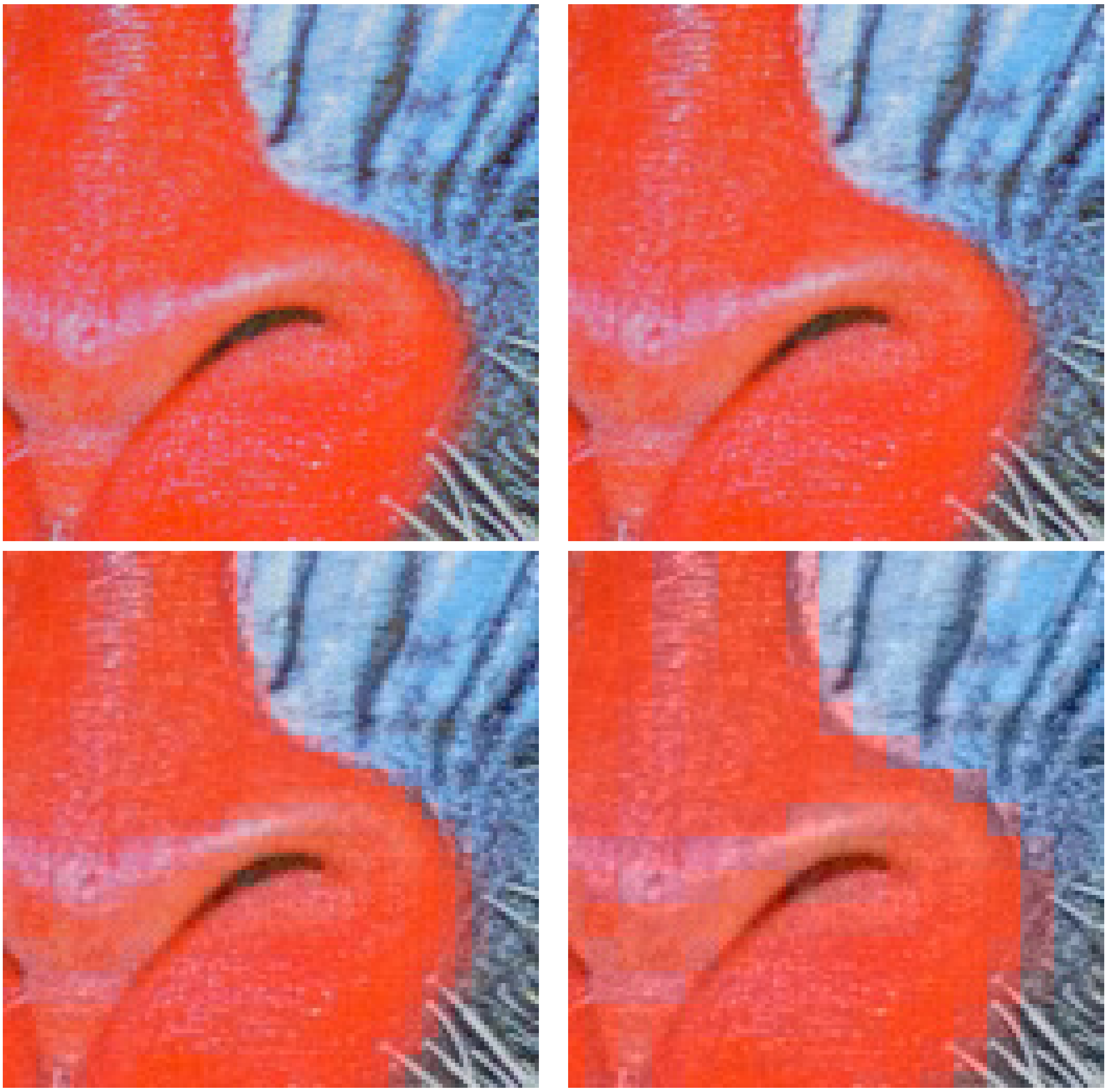

Table 2: Zoom into the compressed variants of the image `baboon.ppm`. *(a) Top left:* Image for $S = 1$ (original image). *(b) Top right:* $S = 2$. *(c) Bottom left:* $S = 4$. *(d) Bottom right:* $S = 8$.

(c) While the original RGB image requires 24 bpp to store the information, the requirements of the compressed (subsampled) images is given by

$$v = 8\left(1 + \frac{1}{S^2} + \frac{1}{S^2}\right) .$$

While the Y channels remain uncompressed, the Cb- and Cr-channel are reduced in both dimensions by a factor of S. Thus, 12 bpp for $S = 2$, 9 bpp for $S = 4$, and 8.25 bpp for $S = 8$ are needed. This is in the order of the memory consumption of grey value images.
