Example Solutions for Classroom Assignment 13 (C13)

Problem 1 (Homogeneous Coordinates)

We first want to remark that all projective coordinates that are a multiple of each other, i.e. where $\tilde{\boldsymbol{a}} = \lambda \tilde{\boldsymbol{b}}$ for $\lambda \neq 0$, denote the same point. This reflects the depth ambiguity.

(a) If $\boldsymbol{m}_1$ lies on $\boldsymbol{l}_1$ the following holds:

$$a_1 x_1 + b_1 y_1 + c_1 = 0 \tag{1}$$

Thus we have:

$$\widetilde{\boldsymbol{m}}_1^\top \boldsymbol{l}_1 = (x_1, x_2, 1)^\top \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} = x_1 \cdot a_1 + y_1 \cdot b_1 + 1 \cdot c_1 \overset{(1)}{=} 0 \, .$$

(b) Since the intersection point $\boldsymbol{m}_1$ lies on $\boldsymbol{l}_1$ as well as on $\boldsymbol{l}_2$, we know from (a) that $\widetilde{\boldsymbol{m}}_1^\top \boldsymbol{l}_1 = 0$ and $\widetilde{\boldsymbol{m}}_1^\top \boldsymbol{l}_2 = 0$ holds. Furthermore we know, that the inner product between two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is 0 iff $\boldsymbol{a} \perp \boldsymbol{b}$. Thus we have:

$$\widetilde{\boldsymbol{m}}_1 \perp \boldsymbol{l}_1 \quad \text{and} \quad \widetilde{\boldsymbol{m}}_1 \perp \boldsymbol{l}_2$$

That means $\widetilde{\boldsymbol{m}}_1$ is a vector which is orthogonal to the vectors $\boldsymbol{l}_1$ and $\boldsymbol{l}_2$ at the same time. The vectors $\boldsymbol{l}_1$ and $\boldsymbol{l}_2$ are linearly independent since $\boldsymbol{l}_1 \nparallel \boldsymbol{l}_2$. Hence, the vector $\widetilde{\boldsymbol{m}}_1$ is up to a scalar factor λ uniquely given by the crossproduct of $\boldsymbol{l}_1$ and $\boldsymbol{l}_2$:

$$\begin{aligned} \widetilde{\boldsymbol{m}}_1 &= \lambda(\boldsymbol{l}_1 \times \boldsymbol{l}_2) \\ \Leftrightarrow \quad \frac{1}{\lambda}\widetilde{\boldsymbol{m}}_1 &= \boldsymbol{l}_1 \times \boldsymbol{l}_2 \end{aligned}$$

Due to the depth ambiguity this yields

$$\widetilde{\boldsymbol{m}}_1 = \boldsymbol{l}_1 \times \boldsymbol{l}_2$$

Alternatively:

If $\boldsymbol{m}_1$ is the intersection point of $\boldsymbol{l}_1$ and $\boldsymbol{l}_2$ we know that following system of equations holds:

$$\begin{aligned} a_1 x_1 + b_1 y_1 + c_1 = 0 \\ a_2 x_1 + b_2 y_1 + c_2 = 0 \end{aligned}$$

Solving it for x_1 and x_2 gives:

$$x_1 = \frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1} \quad \text{and} \quad y_1 = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$$

Note that $a_1b_2 - a_2b_1 \neq 0$ due to $\boldsymbol{l}_1 \nparallel \boldsymbol{l}_2$. So

$$\widetilde{\boldsymbol{m}}_1 = \begin{pmatrix} \dfrac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1} \\ \dfrac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1} \\ 1 \end{pmatrix} = \lambda \begin{pmatrix} b_1c_2 - b_2c_1 \\ a_1c_2 - a_2c_1 \\ a_1b_2 - a_2b_1 \end{pmatrix} = \lambda(\boldsymbol{l}_1 \times \boldsymbol{l}_2)$$

with $\lambda = \dfrac{1}{a_1b_2 - a_2b_1}$.

So $\dfrac{1}{\lambda}\widetilde{\boldsymbol{m}}_1 = \boldsymbol{l}_1 \times \boldsymbol{l}_2$ and the depth ambiguity yield

$$\widetilde{\boldsymbol{m}}_1 = \boldsymbol{l}_1 \times \boldsymbol{l}_2$$

(c) Since both points, $\boldsymbol{m}_1$ as well as $\boldsymbol{m}_2$ lie on $\boldsymbol{l}_1$ we know from (a) that $\widetilde{\boldsymbol{m}}_1^\top \boldsymbol{l}_1 = 0$ and $\widetilde{\boldsymbol{m}}_2^\top \boldsymbol{l}_1 = 0$ holds. We get again:

$$\widetilde{\boldsymbol{m}}_1 \perp \boldsymbol{l}_1 \quad \text{and} \quad \widetilde{\boldsymbol{m}}_2 \perp \boldsymbol{l}_1$$

The vectors $\widetilde{\boldsymbol{m}}_1$ and $\widetilde{\boldsymbol{m}}_2$ are linearly independent since $\boldsymbol{m}_1 \neq \boldsymbol{m}_2$. So the vector $\boldsymbol{l}_1$ is up to a scalar factor λ uniquely given by the crossproduct of $\widetilde{\boldsymbol{m}}_1$ and $\widetilde{\boldsymbol{m}}_2$:

$$\begin{aligned} \boldsymbol{l}_1 &= \lambda(\widetilde{\boldsymbol{m}}_1 \times \widetilde{\boldsymbol{m}}_2) \\ &= (\lambda\widetilde{\boldsymbol{m}}_1) \times \widetilde{\boldsymbol{m}}_2 \\ &= \widetilde{\boldsymbol{m}}_1 \times (\lambda\widetilde{\boldsymbol{m}}_2) \end{aligned}$$

The depth ambiguity yields

$$\boldsymbol{l}_1 = \widetilde{\boldsymbol{m}}_1 \times \widetilde{\boldsymbol{m}}_2$$

Alternatively:

We want to show, that $\boldsymbol{l}_1$ is given by:

$$\boldsymbol{l}_1 = \widetilde{\boldsymbol{m}}_1 \times \widetilde{\boldsymbol{m}}_2 = \begin{pmatrix} x_1 \\ y_1 \\ 1 \end{pmatrix} \times \begin{pmatrix} x_2 \\ y_2 \\ 1 \end{pmatrix} = \begin{pmatrix} y_1 - y_2 \\ x_2 - x_1 \\ x_1y_2 - x_2y_1 \end{pmatrix}$$

i.e.

$$(y_1 - y_2)x + (x_2 - x_1)y + (x_1y_2 - x_2y_1) = 0$$

Since a line is uniquely defined by two points it suffices to show, that the points $\boldsymbol{m}_1$ and $\boldsymbol{m}_2$ lie on $\boldsymbol{l}_1$.

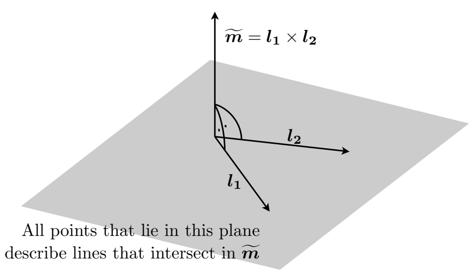
Plugging in $\boldsymbol{m}_1$:

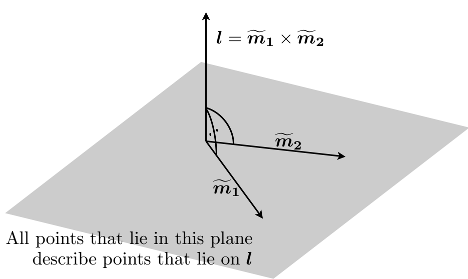
$$\begin{aligned} & (y_1 - y_2)x_1 + (x_2 - x_1)y_1 + (x_1y_2 - x_2y_1) \\ = \; & y_1x_1 - y_2x_1 + x_2y_1 - x_1y_1 + x_1y_2 - x_2y_1 \\ = \; & (y_1x_1 - x_1y_1) + (x_1y_2 - y_2x_1) + (x_2y_1 - x_2y_1) \\ = \; & 0 \end{aligned}$$

Plugging in $\boldsymbol{m}_2$:

$$\begin{aligned} & (y_1 - y_2)x_2 + (x_2 - x_1)y_2 + (x_1y_2 - x_2y_1) \\ = \; & y_1x_2 - y_2x_2 + x_2y_2 - x_1y_2 + x_1y_2 - x_2y_1 \\ = \; & (y_1x_2 - x_2y_1) + (x_2y_2 - y_2x_2) + (x_1y_2 - x_1y_2) \\ = \; & 0 \end{aligned}$$

Last but not least consider the following to pictures which summarise what we have learned in this assignment:





Problem 2 (Rotation Matrices in 3-D)

By three angles Φ, Θ and Ψ, the so-called *Euler angles*, we are able to describe the orientation of a 3-D-object in space using a single rotation matrix. To this end we state three matrices that describe rotations around the z, the x and again the z axis and concatenate them afterwards.

Rotation by Ψ degrees around the z-axis:

$$\boldsymbol{R}_z(\Psi) := \begin{pmatrix} \cos(\Psi) & -\sin(\Psi) & 0 \\ \sin(\Psi) & \cos(\Psi) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Rotation by Θ degrees around the x-axis:

$$\boldsymbol{R}_x(\Theta) := \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\Theta) & -\sin(\Theta) \\ 0 & \sin(\Theta) & \cos(\Theta) \end{pmatrix}$$

Rotation by Φ degrees around the z-axis:

$$\boldsymbol{R}_z(\Phi) := \begin{pmatrix} \cos(\Phi) & -\sin(\Phi) & 0 \\ \sin(\Phi) & \cos(\Phi) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Let $\boldsymbol{p} = (p_1, p_2, p_3)^\top$ be an object point. To get the new orientation we first rotate Ψ degrees around the original z-axis, i.e we compute $\boldsymbol{R}_z(\Psi)\boldsymbol{p}$. This is followed by a rotation around the *new* x-axis by an angle of Θ. That means we have to use the basistransformed rotation matrix $\boldsymbol{R}'_x(\Theta) := \boldsymbol{R}_z(\Psi)\boldsymbol{R}_x(\Theta)\boldsymbol{R}_z^{-1}(\Psi)$:

$$\begin{aligned} \boldsymbol{R}'_x(\Theta)\boldsymbol{R}_z(\Psi)\boldsymbol{p} &= \boldsymbol{R}_z(\Psi)\boldsymbol{R}_x(\Theta)\boldsymbol{R}_z^{-1}(\Psi)\boldsymbol{R}_z(\Psi)\boldsymbol{p} \\ &= \boldsymbol{R}_z(\Psi)\boldsymbol{R}_x(\Theta)\boldsymbol{p} \quad . \end{aligned}$$

Finally we rotate Φ degrees around the z-axis that was created by the previous rotations. Again we have to use the basistransformed rotation matrix $\boldsymbol{R}'_z(\Phi) := \boldsymbol{R}_z(\Psi)\boldsymbol{R}_x(\Theta)\boldsymbol{R}_z(\Phi)(\boldsymbol{R}_z(\Psi)\boldsymbol{R}_x(\Theta))^{-1}$:

$$\begin{aligned} \boldsymbol{R}'_z(\Phi)\boldsymbol{R}_z(\Psi)\boldsymbol{R}_x(\Theta)\boldsymbol{p} &= \boldsymbol{R}_z(\Psi)\boldsymbol{R}_x(\Theta)\boldsymbol{R}_z(\Phi)(\boldsymbol{R}_z(\Psi)\boldsymbol{R}_x(\Theta))^{-1}\boldsymbol{R}_z(\Psi)\boldsymbol{R}_x(\Theta)\boldsymbol{p} \\ &= \boldsymbol{R}_z(\Psi)\boldsymbol{R}_x(\Theta)\boldsymbol{R}_z(\Phi)\boldsymbol{R}_x^{-1}(\Theta)\boldsymbol{R}_z^{-1}(\Psi)\boldsymbol{R}_z(\Psi)\boldsymbol{R}_x(\Theta)\boldsymbol{p} \\ &= \boldsymbol{R}_z(\Psi)\boldsymbol{R}_x(\Theta)\boldsymbol{R}_z(\Phi)\boldsymbol{R}_x^{-1}(\Theta)\boldsymbol{R}_x(\Theta)\boldsymbol{p} \\ &= \boldsymbol{R}_z(\Psi)\boldsymbol{R}_x(\Theta)\boldsymbol{R}_z(\Phi)\boldsymbol{p} \quad . \end{aligned}$$

So the full concatenated rotation matrix $\boldsymbol{M}_{zxz}$ is then given as

$$\boldsymbol{M}_{zxz} = \boldsymbol{R}_z(\Psi)\boldsymbol{R}_x(\Theta)\boldsymbol{R}_z(\Phi) =$$

$$\begin{pmatrix} \cos(\Psi)\cos(\Phi) - \sin(\Psi)\cos(\Theta)\sin(\Phi) & -\cos(\Psi)\sin(\Phi) - \sin(\Psi)\cos(\Theta)\cos(\Phi) & \sin(\Psi)\sin(\Theta) \\ \sin(\Psi)\cos(\Phi) + \cos(\Psi)\cos(\Theta)\sin(\Phi) & \cos(\Psi)\cos(\Theta)\cos(\Phi) - \sin(\Psi)\sin(\Phi) & -\cos(\Psi)\sin(\Theta) \\ \sin(\Theta)\sin(\Phi) & \sin(\Theta)\cos(\Phi) & \cos(\Theta) \end{pmatrix} .$$

As we rotate first around the z-axis, then around the x-axis and afterwards again around the z-axis, this definition of the Euler angles is called z-x-z convention. It is one of several common conventions; others are for example the x-y-z and z-y-x convention.

Note that $\boldsymbol{M}_{zxz} \neq \boldsymbol{R}_z(\Phi)\boldsymbol{R}_x(\Theta)\boldsymbol{R}_z(\Psi)$, which would describe a rotation by Ψ degrees around the original z-axis followed by a rotation of Θ degrees around the original x-axis followed by a rotation of Φ degrees around the original z-axis.