Image Processing and Computer Vision (IPCV)

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Example Solutions for Classroom Assignment 6 (C6)

Problem 1: (Linear Filters)

(a) Given is the one-dimensional binomial mask

$$\frac{1}{16}\begin{array}{|c|c|c|c|c|}\hline 1 & 4 & 6 & 4 & 1 \\ \hline\end{array}$$

Note that binomial kernels approximate Gaussians and are thus lowpass filters. The stencil of a two-dimensional separable binomial filter that is based on the given one-dimensional mask can be determined as:

$$\frac{1}{16^2}\begin{array}{|c|c|c|c|c|}\hline 1 & 4 & 6 & 4 & 1 \\ \hline 4 & 16 & 24 & 16 & 4 \\ \hline 6 & 24 & 36 & 24 & 6 \\ \hline 4 & 16 & 24 & 16 & 4 \\ \hline 1 & 4 & 6 & 4 & 1 \\ \hline\end{array}$$

This is a two-dimensional lowpass filter.

(b) A corresponding highpass filter can be constructed by taking the difference between the identity and that lowpass filter:

$$\begin{array}{|c|c|c|c|c|}\hline 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ \hline\end{array} \; - \; \frac{1}{256}\begin{array}{|c|c|c|c|c|}\hline 1 & 4 & 6 & 4 & 1 \\ \hline 4 & 16 & 24 & 16 & 4 \\ \hline 6 & 24 & 36 & 24 & 6 \\ \hline 4 & 16 & 24 & 16 & 4 \\ \hline 1 & 4 & 6 & 4 & 1 \\ \hline\end{array}$$

$$= \frac{1}{256}\begin{array}{|c|c|c|c|c|}\hline -1 & -4 & -6 & -4 & -1 \\ \hline -4 & -16 & -24 & -16 & -4 \\ \hline -6 & -24 & 220 & -24 & -6 \\ \hline -4 & -16 & -24 & -16 & -4 \\ \hline -1 & -4 & -6 & -4 & -1 \\ \hline\end{array}$$

(c) Now we want to design a bandpass filter by means of another binomial filter based on

$$\frac{1}{4}\begin{array}{|c|c|c|}\hline 1 & 2 & 1 \\ \hline\end{array}$$

The stencil of the two-dimensional binomial filter based on that mask can be determined as:

$$\frac{1}{4^2}\begin{array}{|c|c|c|}\hline 1 & 2 & 1 \\ \hline 2 & 4 & 2 \\ \hline 1 & 2 & 1 \\ \hline\end{array}$$

This is again a lowpass filter. The bandpass filter is obtained by subtracting the "narrow" lowpass filter from the "broad" lowpass:

$$\frac{1}{16}\begin{array}{|c|c|c|c|c|}\hline 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 2 & 1 & 0 \\ \hline 0 & 2 & 4 & 2 & 0 \\ \hline 0 & 1 & 2 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ \hline\end{array} \quad - \quad \frac{1}{256}\begin{array}{|c|c|c|c|c|}\hline 1 & 4 & 6 & 4 & 1 \\ \hline 4 & 16 & 24 & 16 & 4 \\ \hline 6 & 24 & 36 & 24 & 6 \\ \hline 4 & 16 & 24 & 16 & 4 \\ \hline 1 & 4 & 6 & 4 & 1 \\ \hline\end{array}$$

$$= \quad \frac{1}{256}\begin{array}{|c|c|c|c|c|}\hline -1 & -4 & -6 & -4 & -1 \\ \hline -4 & 0 & 8 & 0 & -4 \\ \hline -6 & 8 & 28 & 8 & -6 \\ \hline -4 & 0 & 8 & 0 & -4 \\ \hline -1 & -4 & -6 & -4 & -1 \\ \hline\end{array}$$

Please note, that in this case, the terms "broad" and "narrow" refer to their respective representation in the Fourier domain, where these kernels approximate Gaussians with inverted variances (as compared to the spatial domain). Because of the correspondences between binomial kernels and Gaussians found in lecture 11, the "narrow" filter is really the one with the larger kernel.

Problem 2 (Derivative Filters)

(a) From the exponent k of the factor $\frac{1}{h^k}$ it is immediately clear that if the approximation is consistent, it has to approximate a derivative of order $k = 1$. This is due to the fact that the coefficient in the Taylor series that corresponds to the k-th order derivative contains a factor h^k which has to be compensated for in order to get a consistent approximation. Naturally, this does not prove already, that the approximation is consistent. The corresponding proof is a by-product of part (b).

(b) In order to compute the order of approximation, we need to compute the Taylor-expansions of $f_{i-2}, \ldots, f_{i+4}$ in terms of the derivatives of $f(x)$ evaluated at the point i (which we abbreviate as f_i, f_i', f_i''). This yields:

$$\begin{aligned}
f_{i-2} &= f_i - \frac{2}{1}hf_i' + \frac{4}{2}h^2 f_i'' - \frac{8}{6}h^3 f_i''' + \frac{16}{24}h^4 f_i^{(4)} - \frac{32}{120}h^5 f_i^{(5)} + O(h^6) \\
f_{i-1} &= f_i - \frac{1}{1}hf_i' + \frac{1}{2}h^2 f_i'' - \frac{1}{6}h^3 f_i''' + \frac{1}{24}h^4 f_i^{(4)} - \frac{1}{120}h^5 f_i^{(5)} + O(h^6) \\
f_i &= f_i \\
f_{i+1} &= f_i + \frac{1}{1}hf_i' + \frac{1}{2}h^2 f_i'' + \frac{1}{6}h^3 f_i''' + \frac{1}{24}h^4 f_i^{(4)} + \frac{1}{120}h^5 f_i^{(5)} + O(h^6) \\
f_{i+2} &= f_i + \frac{2}{1}hf_i' + \frac{4}{2}h^2 f_i'' + \frac{8}{6}h^3 f_i''' + \frac{16}{24}h^4 f_i^{(4)} + \frac{32}{120}h^5 f_i^{(5)} + O(h^6)
\end{aligned}$$

Note that the Taylor series for each point f_{i+k}, $k \in \mathbb{Z}$ are always the same, no matter how the weights for the finite difference approximation look like. Thus, we have to compute the expansions for $f_i, \ldots, f_{i+2}$, which are needed to determine the order of consistency for both approximations, only once.

Plugging the Taylor expansion into the first approximation formula with the larger stencil gives us

$$
\begin{aligned}
&\frac{1}{12h}f_{i-2}-\frac{8}{12h}f_{i-1}+0f_i+\frac{8}{12h}f_{i+1}-\frac{1}{12h}f_{i+2}\\
=\;& \underbrace{(1-8+0+8-1)}_{=0}\frac{1}{12}\frac{1}{h}f_i\\
&+\underbrace{(-2+8+8-2)}_{=1}\frac{1}{12}\frac{h}{h}f_i'\\
&+\underbrace{(4-8+8-4)}_{=0}\frac{1}{12}\frac{1}{2}\frac{h^2}{h}f_i''\\
&+\underbrace{(-8+8+8-8)}_{=0}\frac{1}{12}\frac{1}{6}\frac{h^3}{h}f_i'''\\
&+\underbrace{(16-8+8-16)}_{=0}\frac{1}{12}\frac{1}{24}\frac{h^4}{h}f_i^{(4)}\\
&+\underbrace{(-32+8+8-32)}_{=-48\neq 0}\frac{1}{12}\frac{1}{120}\frac{h^5}{h}f_i^{(5)}\\
&+O(h^5)\\
=\;& f_i'-\frac{h^4}{30}f_i^{(5)}+O(h^5)\\
=\;& f_i'+O(h^4)\,,
\end{aligned}
$$

which shows that the order of consistency of the approximation is 4. For the second approximation we get in the same way

$$\frac{1}{2h}\left(-3f_i+4f_{i+1}-f_{i+2}\right)=0f_i+1f_i'+0f_i''-\frac{1}{3}h^2f_i'''+\mathcal{O}\left(h^3\right)$$

and thus we have an approximation of order 2.

(c) The order of approximation is significantly higher for the first approximation, but it also uses a filter mask of size 4 (in a neighbourhood of size 5) instead of size 3, i.e. there is more computational effort involved. However, the second approximation has two additional drawbacks. First, it is possible to achieve the same order of approximation with smaller filter masks and second, the filter mask has a very evident bias to the "right" (i.e. only indices $\geq i$ are represented in the mask). It only makes sense to use this second approximation if a specific application would only allow one-sided approximations.