Image Processing and Computer Vision
Joachim Weickert, Summer Term 2019

Lecture 4:
Image Transformations I:
Continuous Fourier Transform

Contents

1. Motivation
2. Complex Numbers
3. Prerequisites from Linear Algebra
4. Continuous Fourier Transform in 1-D
5. Continuous Fourier Transform in 2-D
6. Properties

M I A
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33 34 35 36 37 38 39

Motivation (1)

This lecture is the first of a series of four lectures that take you to a desert ride. At first glance, the desert looks dry and dusty, but you will see structures much clearer. Your math knowledge is your survival kit. Photo by Daniela Borchert (`http://de.wikipedia.org/wiki/Namib`).

M I A
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33 34 35 36 37 38 39

Motivation

- Transformations are useful for analysing the properties of images and processing them in an efficient way.
- For analysing audio signals, a decomposion into their frequencies is very natural.
- Similar things can be done with images, which can be seen as 2-D signals:
  - They can be decomposed into their frequency content in x- and y-direction.
  - This is useful for many things, e.g. filter design and fast convolution algorithms.
- The transformation that represents signals in terms of their frequencies is called Fourier transform.
- It can be regarded as a change of basis with trigonometric functions as basis functions.
- To understand this, we first have to remember some facts about complex numbers and representations of vectors in orthonormal bases.

Joseph Fourier (1768–1830) did not only discover the so-called Fourier transform, he also introduced the diffusion equation, made archeological discoveries in Egypt, and acted as a prefect in Grenoble. Source: `http://de.wikipedia.org/wiki/Joseph_Fourier`.

M I A
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33 34 35 36 37 38 39

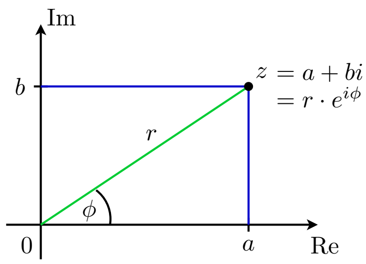
Complex Numbers

- The set $\mathbb{C}$ of complex numbers extends the set $\mathbb{R}$ of real numbers.
  It allows to compute the square root of a negative number.

- A complex number $z = a + bi \in \mathbb{C}$ consists of two parts:
  a *real part* $\mathrm{Re}(z) := a \in \mathbb{R}$ and an *imaginary part* $\mathrm{Im}(z) := b \in \mathbb{R}$.
  The number i denotes the *imaginary unit*, i.e. $i^2 = -1$.

- The *conjugate complex number* of $z = a + bi$ is defined as $\bar{z} := a - bi$.

- $a + bi$ may also be identified with a vector $(a, b)^\top \in \mathbb{R}^2$.
  This is its so-called *Cartesian representation*.



A complex number in Cartesian and polar form (see later). Author: M. Mainberger.

M I A
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33 34 35 36 37 38 39

- Addition and subtraction of complex numbers is done componentwise:

$$
\begin{aligned}
(a+bi) + (c+di) &= (a+c) + (b+d)\,i\,,\\
(a+bi) - (c+di) &= (a-c) + (b-d)\,i\,.
\end{aligned}
$$

- Multiplication uses $i^2 = -1$:

$$
\begin{aligned}
(a+bi)\cdot(c+di) &= ac + adi + bci + bdi^2\\
&= (ac - bd) + (ad + bc)\,i\,.
\end{aligned}
$$

- Division expands the fraction by the complex conjugate of the denominator:

$$
\frac{a+bi}{c+di} = \frac{(a+bi)\cdot(c-di)}{(c+di)\cdot(c-di)} = \frac{ac+bd}{c^2+d^2} + \frac{bc-ad}{c^2+d^2}\,i\,.
$$

- The *norm (modulus, magnitude)* of a complex number $z = a + ib$ is given by

$$
|z| := \sqrt{z\bar{z}} = \sqrt{(a+bi)\,(a-bi)} = \sqrt{a^2 - b^2\,i^2} = \sqrt{a^2+b^2}\,.
$$

  Note that $\sqrt{zz}$ would not give the desired result $\sqrt{a^2+b^2}$.

M I A
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33 34 35 36 37 38 39

Complex Numbers (3)

M I A
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33 34 35 36 37 38 39

- With the power series representations

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}, \qquad \cos x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}, \qquad \sin x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$$

one obtains *Euler's formula*

$$e^{i\phi} \;=\; \cos\phi \;+\; i\sin\phi\,.$$

- Euler's formula implies that $e^{i\phi}$ lies on the unit circle,

$$|e^{i\phi}| \;=\; \sqrt{\cos^2\phi + \sin^2\phi} \;=\; 1 \qquad \forall\,\phi,$$

and is 2π-periodic:

$$e^{i(\phi+2k\pi)} \;=\; e^{i\phi} \qquad \forall\, k \in \mathbb{Z}.$$

- With $\sin(-\phi) = -\sin\phi$ and $\cos(-\phi) = \cos\phi$, we obtain the useful relations

$$\cos\phi \;=\; \frac{e^{i\phi}+e^{-i\phi}}{2}, \qquad \sin\phi \;=\; \frac{e^{i\phi}-e^{-i\phi}}{2i}.$$

Complex Numbers (4)

M I A
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33 34 35 36 37 38 39

- One may also express any nonzero complex number $z = a + bi$ in its *polar form*

$$z \;=\; r\,e^{i\phi}$$

with radius $r := |z| = \sqrt{a^2+b^2}$ and *argument* (angle between z and real axis)

$$\phi \;:=\; \arg(z) \;:=\; \begin{cases} \arccos\left(\frac{a}{r}\right) & \text{if } b \geq 0, \\ -\arccos\left(\frac{a}{r}\right) & \text{if } b < 0. \end{cases}$$

This representation gives a polar angle $\phi \in (-\pi, \pi]$.
(Note that the $\arccos$ function yields values in $[0, \pi]$.)

- The polar form is convenient for multiplications,

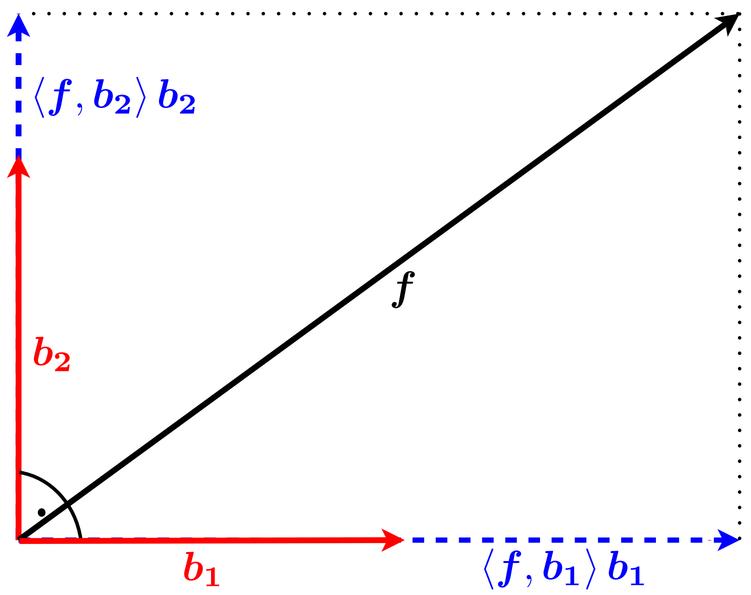
$$z_1 z_2 \;=\; |z_1|\,|z_2|\,e^{i(\phi_1+\phi_2)},$$

and for raising a complex number to some power p:

$$z^p \;=\; |z|^p\,e^{ip\phi}.$$

In polar form, the complex conjugate of $z \;=\; r\,e^{i\phi}$ is given by $\bar{z} \;=\; r\,e^{-i\phi}$.

Prerequisites from Linear Algebra



A vector $\boldsymbol{f} \in \mathbb{R}^2$ can be represented in an arbitrary orthonormal (orthogonal with norm 1) basis $\{\boldsymbol{b}_1, \boldsymbol{b}_2\}$ of $\mathbb{R}^2$ by the formula $\boldsymbol{f} = \langle \boldsymbol{f}, \boldsymbol{b_1} \rangle \, \boldsymbol{b_1} + \langle \boldsymbol{f}, \boldsymbol{b_2} \rangle \, \boldsymbol{b_2}$. This insight can be generalised. It will be very useful for us in this and the next three lectures. Author: P. Peter.

M I A
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33 34 35 36 37 38 39

- Representing a vector $\boldsymbol{f} \in \mathbb{R}^N$ in an orthonormal basis $\{\boldsymbol{b_1}, ..., \boldsymbol{b_N}\}$ of $\mathbb{R}^N$ gives

$$\boldsymbol{f} \;=\; \sum_{k=1}^{N} \langle \boldsymbol{f}, \boldsymbol{b_k} \rangle \, \boldsymbol{b_k}$$

where $\langle ., . \rangle$ denotes the *Euclidean inner product (euklidisches Skalarprodukt):*

$$\langle \boldsymbol{f}, \boldsymbol{g} \rangle \;:=\; \sum_{j=1}^{N} f_j \, g_j .$$

The coefficient $\langle \boldsymbol{f}, \boldsymbol{b_k} \rangle$ quantifies the projection of $\boldsymbol{f}$ onto the basis vector $\boldsymbol{b_k}$:
It tells us to which extent the vector $\boldsymbol{b_k}$ contributes to $\boldsymbol{f}$.

- For a complex vector $\boldsymbol{f} \in \mathbb{C}^N$ one uses the *Hermitian inner product*

$$\langle \boldsymbol{f}, \boldsymbol{g} \rangle \;:=\; \sum_{j=1}^{N} f_j \, \bar{g}_j$$

where $\bar{g}_j$ is the complex conjugate of g_j.
The complex conjugation allows e.g. to define the norm of $\boldsymbol{f}$ via $|\boldsymbol{f}| := \sqrt{\langle \boldsymbol{f}, \boldsymbol{f} \rangle}$.

- An N-dimensional vector $\boldsymbol{f} \in \mathbb{R}^N$ has N components $\{f_j \mid j = 1, ..., N\}$.
  A function $f : \mathbb{R} \to \mathbb{R}$ has infinitely many values $\{f(x) \mid x \in \mathbb{R}\}$.
  In this sense it can be interpreted as an "infinite-dimensional vector".

- Representing $f : \mathbb{R} \to \mathbb{R}$ with an infinite set of orthonormal basis functions $\{b_u \,|\, u \in \mathbb{R}\}$ yields

$$f \;=\; \int_{\mathbb{R}} \langle f, b_u \rangle \, b_u \, du$$

  with the inner product

$$\langle f, g \rangle \;:=\; \int_{\mathbb{R}} f(x)\, g(x)\, dx.$$

  The coefficient $\langle f, b_u \rangle$ quantifies the projection of f onto the basis function b_u.
  It measures to which extent b_u contributes to f.

- For complex-valued functions $f : \mathbb{R} \to \mathbb{C}$ one uses the inner product

$$\langle f, g \rangle \;:=\; \int_{\mathbb{R}} f(x)\, \bar{g}(x)\, dx.$$

  Note the complex conjugation.

M I A
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33 34 35 36 37 38 39

Continuous Fourier Transform in 1-D

Goals

- decompose a signal into its frequency components
- compute convolutions in a highly efficient way
- have an ideal tool for analysing and designing linear shift-invariant filters (later)

Basic Intuition

- express a 1-D signal $f : \mathbb{R} \to \mathbb{R}$ with a specific orthonormal basis $\{b_u \mid u \in \mathbb{R}\}$:

$$f \;=\; \int_{\mathbb{R}} \langle f, b_u \rangle \, b_u \, du$$

- choose the basis functions $\{b_u \mid u \in \mathbb{R}\}$ such that they represent all frequencies u
- coefficient $\langle f, b_u \rangle$ measures to which extent a frequency u contributes to f

M I A
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33 34 35 36 37 38 39

Continuous Fourier Transform in 1-D (2)

How to Choose the Basis Functions

- To represent a function f in terms of its frequencies, a natural idea would be to use trigonometric functions of cosine and sine type:

$$\begin{aligned} c_u(x) &= \cos(2\pi u x), \\ s_u(x) &= \sin(2\pi u x), \end{aligned}$$

  where u denotes the frequency (number of oscillations within x-interval $[0,1]$).

- Since we are lazy, we combine $c_u(x)$ and $s_u(x)$ with Euler's formula: We use them as real and imaginary part of the complex-valued function

$$b_u(x) \;=\; c_u(x) + i\, s_u(x) \;=\; e^{2\pi i u x}.$$

- One can show that $\{b_u \,|\, u \in \mathbb{R}\}$ is an orthonormal set of basis functions for 1-D signals, if we use the the complex-valued inner product:

$$\langle f, g \rangle \;:=\; \int_{\mathbb{R}} f(x)\, \bar{g}(x)\, dx.$$

Continuous Fourier Transform in 1-D (3)

The Fourier Transform

- The Fourier transform yields the coefficient $\hat{f}(u) := \langle f, b_u \rangle$ for each frequency u. This coefficient measures the contribution of a frequency u to the signal f.

- The *Fourier transform (FT)* of a 1-D function $f : \mathbb{R} \to \mathbb{R}$ is defined as

$$\boxed{\hat{f}(u) \;:=\; \mathcal{F}[f](u) \;:=\; \int_{-\infty}^{\infty} f(x)\, e^{-i2\pi u x}\, dx\,.}$$

- The minus sign comes from the complex conjugation of $b_u(x) = e^{2\pi i u x}$.

- $\{\hat{f}(u) \,|\, u \in \mathbb{R}\}$ is the representation of the function $\{f(x) \,|\, x \in \mathbb{R}\}$ in the *frequency domain (Fourier domain).*

M I A
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33 34 35 36 37 38 39

Remarks on the Complex-Valuedness

- Note that the Fourier transform is complex-valued:
  - $\mathrm{Re}(\hat{f}(u))$ measures the contribution of the function $c_u(x) = \cos(2\pi ux)$ to $f(x)$. It vanishes for odd signals $f(x)$, i.e. $f(x) = -f(-x)$, since the cosine is even.
  - $\mathrm{Im}(\hat{f}(u))$ measures the contribution of the function $s_u(x) = \sin(2\pi ux)$ to $f(x)$. It vanishes for even signals $f(x)$, i.e. $f(x) = f(-x)$, since the sine is odd.
- The polar form of $\hat{f}(u)$ gives useful insights:
  - The magnitude $|\hat{f}(u)|$ is called *Fourier spectrum (Fourierspektrum)*. It expresses the total importance of the frequency u within the signal f.
  - The angle $\phi(u) = \arg\left(\hat{f}(u)\right)$ is called the *phase angle (Phasenwinkel)*. It characterises the phase shift relative to a cosine function:
    - $\phi(u) = 0$ corresponds to the pure cosine function $c_u(x) = \cos(2\pi ux)$.
    - $\phi(u) = \frac{\pi}{2}$ corresponds to the pure sine function $s_u(x) = \sin(2\pi ux)$.
    - An arbitrary angle $\phi(u)$ corresponds to the function $t_{u,\phi}(x) = \cos(2\pi ux - \phi)$.
- Often one is only interested in the Fourier spectrum $|\hat{f}(u)|$. Also the so-called *power spectrum (Powerspektrum)* $|\hat{f}(u)|^2$ is popular.

The Inverse Fourier Transform

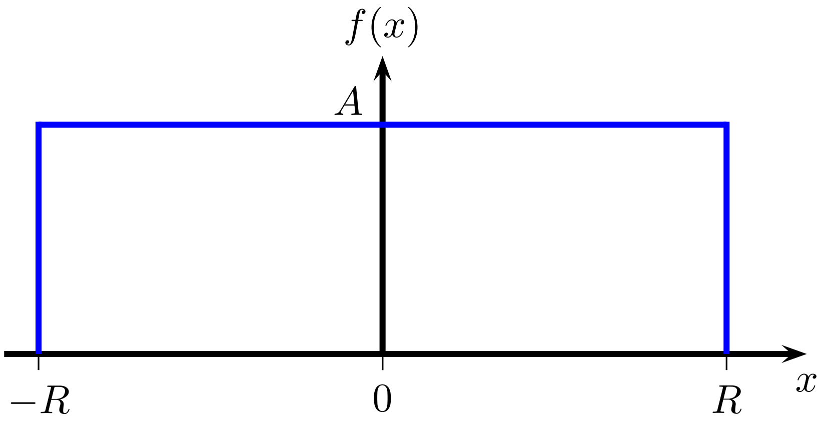
- The goal of the inverse Fourier transform is to synthesise the signal f from its Fourier coefficients $\hat{f}(u) = \langle f, b_u \rangle$.
- Thus, we have to use the formula $f = \int_{\mathbb{R}} \langle f, b_u \rangle \, b_u \, du$ with $b_u = e^{2\pi iux}$.
- The *inverse 1-D Fourier transform* of $\hat{f}(u)$ is defined as

$$\boxed{f(x) \;=\; \mathcal{F}^{-1}[\hat{f}](x) \;:=\; \int_{-\infty}^{\infty} \hat{f}(u)\, e^{i2\pi ux}\, du\,.}$$

- Note that there is no minus sign here.

M I A
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33 34 35 36 37 38 39

Example: Fourier Transform of a Box Function



A box function. Author: M. Mainberger.

M I A
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33 34 35 36 37 38 39

- The Fourier transform of this box function is given by

$$
\begin{aligned}
\hat{f}(u) &= \int_{-\infty}^{\infty} f(x)\, e^{-i2\pi ux}\, dx \\
&= \int_{-R}^{R} A\, e^{-i2\pi ux}\, dx \\
&= A \left[\frac{-1}{i2\pi u}\, e^{-i2\pi ux} \right]_{-R}^{R} \\
&= \frac{-A}{i2\pi u} \left(e^{-i2\pi uR} - e^{i2\pi uR} \right) \\
&= \frac{A}{i2\pi u} \left(e^{i2\pi uR} - e^{-i2\pi uR} \right) \\
&= \frac{A}{i2\pi u}\, 2i\, \sin(2\pi uR) \\
&= \frac{A}{\pi u}\, \sin(2\pi uR).
\end{aligned}
$$

M I A
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33 34 35 36 37 38 39

Continuous Fourier Transform in 1-D (8)

M I A
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33 34 35 36 37 38 39

- Note that $\hat{f}(u)$ is real-valued, since $f(x)$ is an even function.
- The Fourier spectrum is given by

$$
\begin{aligned}
|\hat{f}(u)| &= \left|\frac{A}{\pi u}\right| |\sin(2\pi u R)| \\
&= 2RA \left|\frac{\sin(2\pi u R)}{2\pi u R}\right| \\
&= 2RA\,|\operatorname{sinc}(2\pi u R)|
\end{aligned}
$$

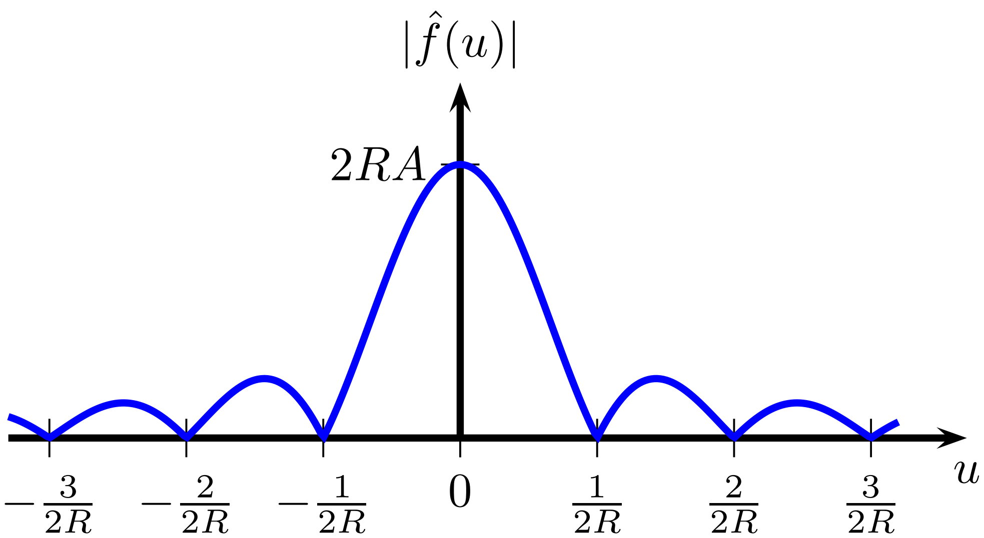
with the so-called *sinc function*

$$
\operatorname{sinc}(x) := \frac{\sin(x)}{x}.
$$

- The sinc function satisfies $\operatorname{sinc}(0) = 1$. It becomes 0 in $x = k\pi$ with $k \in \mathbb{Z} \setminus \{0\}$.
- Thus, $|\hat{f}(u)|$ satisfies $|\hat{f}(0)| = 2RA$ and becomes 0 in $u = \frac{k}{2R}$ with $k \in \mathbb{Z} \setminus \{0\}$.

Continuous Fourier Transform in 1-D (9)

M I A
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33 34 35 36 37 38 39



Fourier spectrum $|\hat{f}(u)|$ of the box function. Author: M. Mainberger.

Remark

While the box function $f(x)$ has finite extent in the spatial domain,
its Fourier transform $\hat{f}(u)$ has infinite extent in the frequency domain.

M I A 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33 34 35 36 37 38 39

Continuous Fourier Transform in 2-D

Definition

- The *Fourier transform (FT)* of a 2-D function $f(x,y)$ is defined as

$$\hat{f}(u,v) \;:=\; \mathcal{F}[f](u,v) \;:=\; \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, e^{-i2\pi(ux+vy)}\, dx\, dy$$

  where u and v are the frequencies in x- and y-direction.

- The *inverse 2-D Fourier transform* is given by

$$f(x,y) \;=\; \mathcal{F}^{-1}[\hat{f}](x,y) \;:=\; \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \hat{f}(u,v)\, e^{i2\pi(ux+vy)}\, du\, dv.$$

M I A 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33 34 35 36 37 38 39

Don't Fear High Dimensions!

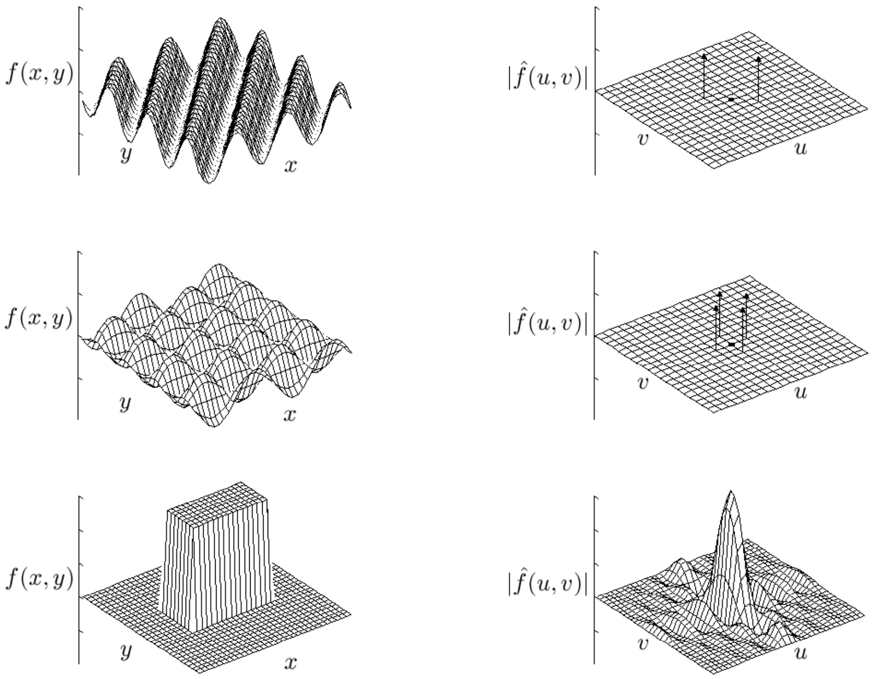
- In higher dimensions the definition proceeds in the same way.
- Because of

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, e^{-i2\pi(ux+vy)}\, dx\, dy = \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(x,y)\, e^{-i2\pi ux}\, dx\right) e^{-i2\pi vy}\, dy$$

  it follows that the Fourier transform is *separable:*

  - First compute the Fourier transform in x-direction.
    Then apply the Fourier transform in y-direction to this result.
  - An m-dimensional Fourier transform is computed via a sequence of m one-dimensional transforms.
  - This is computationally very nice and greatly reduces the workload.

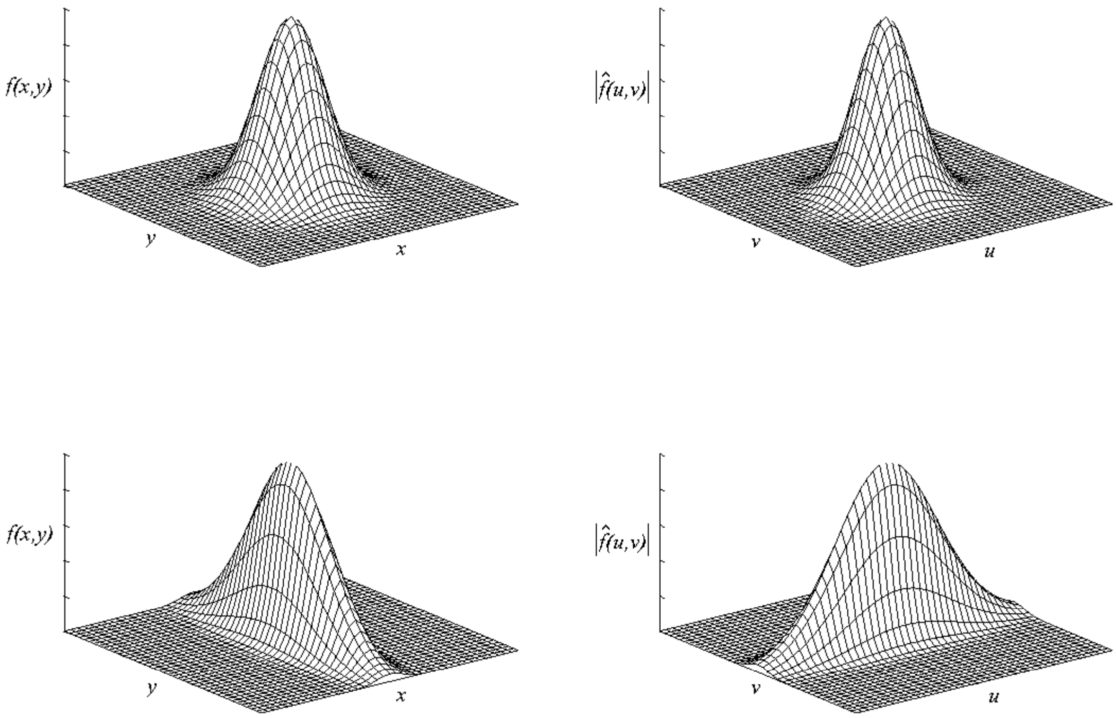
Continuous Fourier Transform in 2-D (3)



Fourier spectra of some 2-D functions. Author: N. Khan.

M I A
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33 34 35 36 37 38 39

Continuous Fourier Transform in 2-D (4)



Further 2-D Fourier spectra. Author: N. Khan.

M I A
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33 34 35 36 37 38 39

Properties of the Continuous Fourier Transform

- **Linearity**

  Let f and g be functions and $a, b \in \mathbb{R}$.
  Then the Fourier transform satisfies the superposition principle:

$$\mathcal{F}[af + bg] \;=\; a\mathcal{F}[f] + b\mathcal{F}[g]\,.$$

- **Similarity Theorem**

$$\mathcal{F}[f(ax, by)](u, v) \;=\; \frac{1}{|ab|}\,\mathcal{F}[f]\left(\frac{u}{a}, \frac{v}{b}\right) \qquad \forall\, a, b \in \mathbb{R} \setminus \{0\}.$$

  Elongation in the spatial domain gives shortening in the Fourier domain:
  Both domains are reciprocal.

M I A
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33 34 35 36 37 38 39

- **Differentiation**

$$\mathcal{F}\left[\frac{\partial^{n+m} f}{\partial x^n \partial y^m}\right] \;=\; (i2\pi u)^n\,(i2\pi v)^m\,\mathcal{F}[f](u, v).$$

  Differentiation in the spatial domain gives multiplication with the frequency in the Fourier domain. Thus, high frequent components (e.g. noise) are amplified!

- **Shift Theorem**

$$\mathcal{F}[f(x - x_0,\, y - y_0)](u, v) \;=\; e^{-i2\pi(ux_0 + vy_0)}\,\mathcal{F}[f](u, v)$$

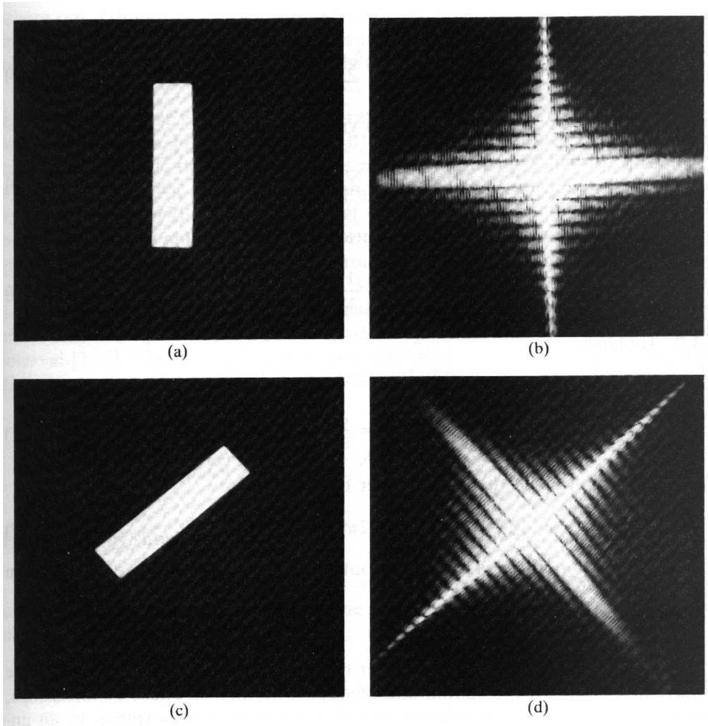
  Shift in the spatial domain rotates the phase angle in the Fourier domain.
  The Fourier *spectrum*, however, is not affected, since $\left|e^{-i2\pi(ux_0+vy_0)}\right| = 1$.
  In this sense the FT is shift-invariant.

- **Rotation Invariance**

  If the image is rotated, its FT is rotated by the same angle.

M I A
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33 34 35 36 37 38 39

Properties of the Continuous Fourier Transform (3)



Rotation invariance of the Fourier transform. **(a) Top left:** Original image. **(b) Top right:** Its Fourier spectrum. **(c) Bottom left:** Rotated image. **(d) Bottom right:** Its Fourier spectrum. Authors: R. C. Gonzalez, R. E. Woods.

M I A
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33 34 35 36 37 38 39

Properties of the Continuous Fourier Transform (4)

- **Convolution Theorem**

  The convolution of two functions $f(x, y)$ and $g(x, y)$ is given by (cf. Lecture 2)

$$(f * g)(x, y) \;:=\; \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x-x', y-y')\, g(x', y')\, dx'\, dy'.$$

  It will be fundamental for all shift-invariant linear filters (Lecture 11). Take e.g.

$$g(x, y) \;:=\; \begin{cases} \frac{1}{\pi r^2} & \text{for } x^2 + y^2 \le r^2, \\ 0 & \text{else.} \end{cases}$$

  Then $f * g$ smoothes the image f by averaging all grey values within a neighbourhood of radius r. Computing this integral is expensive if r is large.

  However, convolution is easily computed as multiplication in the Fourier domain:

$$\mathcal{F}[f * g] \;=\; \mathcal{F}[f] \cdot \mathcal{F}[g]$$

  Afterwards the results must be transformed back to the spatial domain.

M I A
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33 34 35 36 37 38 39

Properties of the Continuous Fourier Transform (5)

M I A
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33 34 35 36 37 38 39

- **Fourier Transform of the Product of Two Functions**

  Multiplication of two functions in the spatial domain becomes convolution in the Fourier domain:

  $$\mathcal{F}[f \cdot g] \;=\; \mathcal{F}[f] * \mathcal{F}[g].$$

  This is the reciprocal convolution theorem.
  Computationally this is not advantageous, but will help us to understand phenomena such as sampling (more details in Lecture 5).

- **Fourier Transform of a Gaussian**

  gives a Gaussian-like function with reciprocal variance. In 2D:

  $$f(x,y) \;:=\; \frac{1}{2\pi\sigma^2}\exp\left(\frac{-(x^2+y^2)}{2\sigma^2}\right) \;\Longrightarrow\; \hat{f}(u,v) \;=\; \exp\left(\frac{-(2\pi)^2(u^2+v^2)}{2\sigma^{-2}}\right).$$

  However, the Gaussian is not the only function that is invariant under the FT:

Properties of the Continuous Fourier Transform (6)

M I A
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33 34 35 36 37 38 39

- **Fourier Transform of a Delta Comb**

  A (continuous) *delta pulse* δ is a model for an infinitely sharp peak centred in 0.
  The integral of a function f times a delta pulse δ evaluates f in 0,

  $$\int_{-\infty}^{\infty} f(x)\,\delta(x)\,dx \;=\; f(0),$$

  while a shifted delta pulse $\delta(.-x_0)$ yields $f(x_0)$:

  $$\int_{-\infty}^{\infty} f(x)\,\delta(\underbrace{x-x_0}_{x'})\,dx \;=\; \int_{-\infty}^{\infty} f(x'+x_0)\,\delta(x')\,dx' \;=\; f(x_0).$$

  The FT of an infinitely extended comb of delta pulses with peak distance λ is a delta comb with reciprocal peak distance $1/\lambda$:

  $$g(x) = \sum_{k=-\infty}^{\infty} \delta(x-k\lambda) \quad\Longrightarrow\quad \hat{g}(u) = \sum_{k=-\infty}^{\infty} \delta\left(u-\frac{k}{\lambda}\right).$$

  Important when sampling a continuous signal f:
  Sampling f means computing the product of f with a delta comb g.

Summary

- The continuous Fourier transform analyses the frequency content of images.
- It is complex-valued, linear, separable, and invariant under rotations.
- Spatial and Fourier domain are reciprocal w. r. t. localisation and orientation.
- Spatial shifts become phase shifts.
- Differentiation becomes multiplication with the frequency.
- Convolution in one domain becomes multiplication in the other.
- The Fourier transform maps
  - box functions to sinc functions,
  - Gaussians to Gaussians with reciprocal variance,
  - delta combs to delta combs with reciprocal distance.

References

- R. C. Gonzalez, R. E. Woods: *Digital Image Processing*. Prentice Hall, Upper Saddle River, International Edition, 2017.
  (a good textbook with in-depth introduction to the FT)
- T. Butz: *Fouriertransformation für Fußgänger*. Teubner, Stuttgart, 2005.
  (for those who wish to learn just a little bit more, but fear the full story)
- R. Bracewell: *The Fourier Transform and its Applications*. McGraw-Hill, New York, 1986.
  (the classical reference when you want to learn the full story about the FT)

M I
A
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33 34 35 36 37 38 39

Assignment C2 – Classroom Work

Problem 1 (Colour Spaces)

Derive a transformation formula from the RGB colour space to a ZCgCr space, where the luma component Z obeys $Z = \frac{1}{3}R + \frac{1}{3}G + \frac{1}{3}B$. Both colour spaces should use the range $[0,1]^3$.

Hint: You will encounter the problem of solving one equation with two unknowns for the C_g and C_r channels. It is common practice to use the relations from the luma channel in order to find a unique solution in this case.

(This assignment will help you to understand the mechanisms behind the conversion formula from RGB to YCgCr.)

Problem 2 (Continuous Fourier Transform)

Compute the continuous Fourier transform of the function f defined by

$$f(x) = \begin{cases} 0, & (x \leq -3), \\ \frac{x^2+6x+9}{16}, & (-3 < x \leq -1), \\ \frac{6-2x^2}{16}, & (-1 < x \leq 1), \\ \frac{x^2-6x+9}{16}, & (1 < x \leq 3), \\ 0, & (x > 3)\,. \end{cases}$$

Hint: You can compute the transform directly or make your life easier with clever use of the properties of the Fourier transform. Keep in mind what you have already learned about the function f.

(After this assignment, you will appreciate an important theorem for the Fourier transform.)

Assignment H2 – Homework

Problem 1 (Properties of the Continuous Fourier Transform) (1+1+1+2+2+1 points)

Verify that the following properties of the continuous Fourier transform are true.

(a)	**Linearity:**	$\mathcal{F}[a\, f(x) + b\, g(x)](u) = a\,\mathcal{F}[f](u) + b\,\mathcal{F}[g](u)$
(b)	**Spatial Shift:**	$\mathcal{F}[f(x-a)](u) = \exp(-i2\pi u a) \cdot \mathcal{F}[f](u)$
(c)	**Frequency Shift:**	$\mathcal{F}[f(x) \cdot \exp(-i2\pi u_0 x)](u) = \mathcal{F}[f](u + u_0)$
(d)	**Scaling:**	$\mathcal{F}[f(ax)](u) = \frac{1}{\lvert a\rvert} \cdot \mathcal{F}[f](\frac{u}{a})$
(e)	**Convolution:**	$\mathcal{F}[(f * g)(x)](u) = \mathcal{F}[f](u) \cdot \mathcal{F}[g](u)$
(f)	**Differentiation:**	$\mathcal{F}[f'](u) = i2\pi u \cdot \mathcal{F}[f](u)$

Hint: You can assume that the product $f(x)\exp(-i2\pi u x)$ vanishes if $|x| \to \infty$.

(The correct application of these properties can simplify the computation of Fourier transforms enormously.)

Problem 2 (Continuous Fourier Transform of a Gaussian) (5+1 points)

Consider the following 1-D Gaussian:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(\frac{-x^2}{2\sigma^2}\right).$$

(a) Show that its Fourier transform $\hat{f}(u) = \mathcal{F}[f]$ is given by a Gaussian-like function with reciprocal variance.

(b) Compute the corresponding Fourier spectrum $|\hat{f}(u)|$.

You may use the following fact without proof:

$$\int_{-\infty}^{\infty} \exp\left(-\pi x^2\right) dx \;=\; 1.$$

(Since the Gaussian is omnipresent, it is important to understand its Fourier transforms.)

Assignment H2 (3)

Problem 3 (Symmetry and Antisymmetry of the Fourier Transform) (2+2 points)

(a) Consider a 1-D real signal f. Show that its Fourier transform $\hat{f}(u) = \mathcal{F}[f]$ has a real part that is symmectric and an imaginary part that is antisymmetric.
Hint: Use Euler's formula.

(b) The Fourier transform decomposes a signal into frequencies. What is the meaning of a negative frequency?

(This assignment answers two of the most frequent questions of students who try to understand the Fourier transform.)

Assignment H2 (4)

M I A
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33 34 35 36 37 38 39

Problem 4 (Colour Spaces) (2+2+2 points)

Please download the archive `Ex02.tar.gz` from the webpage

`http://www.mia.uni-saarland.de/Teaching/ipcv19.shtml`

into your own directory. You can unpack it with the command `tar xvzf Ex02.tar.gz`.

The file `YCbCr.c` implements a program that converts an RGB image into a YCbCr image, subsamples the chroma channels C_b and C_r and transforms the resulting image back to RGB.

(a) Supplement the routine `RGB_to_YCbCr` with the missing code, so that it transforms an RGB into a YCbCr image. The program can then be compiled with the command

```
gcc -O2 -o YCbCr YCbCr.c -lm
```

(b) The integer subsampling factor S allows to reduce the resolution of the chroma channels C_b and C_r by a factor S in each dimension. Use the test image `kodim14.ppm` and compute results for $S = 1, 2, 4,$ and 8. Compare your results visually. What can you observe ?

(c) How many bits per pixel are required to store the YCbCr images directly for $S = 2, 4,$ and 8, if each channel uses 8 bits? Remember that an RGB image typically uses 24 bits per pixel.

Assignment H2 (5)

Submission

Please remember that up to three people from the same tutorial group can work and submit their results together. The theoretical Problems 1, 2, and 3 have to be submitted in handwritten form into the mailbox of your tutorial group **before** the lecture. The mailboxes can be found in Building E2.5 on the ground floor under the stairs. For the practical Problem 4 please submit your files as follows: Rename the main directory Ex02 to Ex02_<your_name> and use the command

```
tar czvf Ex02_<your_name>.tar.gz Ex02_<your_name>
```

to pack the data. The directory that you pack and submit should contain the following files:

- the source code for YCbCr.c with implemented conversion from RGB to YCbCr,
- the results for kodim14.ppm for $S = 2, 4$ and 8,
- a text file README that contains answers to all questions of problem 4 as well as information on all people working together for this assignment.

Please make sure that only your final version of the programs and images are included. Do **not** submit any additional files, especially no executables. Submit the file via e-mail to your tutor via the address:

```
ipcv-xx@mia.uni-saarland.de
```

where xx is either t1, t2, t3, t4, t5, w1, w2, w3 or w4 depending on your tutorial group.

Deadline for submission: Friday, April 26, 10 am (before the lecture)