Example Solutions for Classroom Assignment 9 (C9)

Problem 1 (Convexity of a Discrete Energy Function)

We want to analyse the convexity of

$$E(\boldsymbol{u}) := \frac{1}{2}\sum_{k=1}^{N}(u_k - f_k)^2 + \frac{\alpha}{2}\sum_{k=1}^{N-1}(u_{k+1} - u_k)^2$$

with $\boldsymbol{u}, \boldsymbol{f} \in \mathbb{R}^N$ and $\alpha > 0$. We recall the definition of convexity:

A function $g : \mathbb{R}^n \to \mathbb{R}$ is called convex if for any $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ and $t \in [0,1]$ the following inequality holds

$$g(t\boldsymbol{x} + (1-t)\boldsymbol{y}) \leqslant tg(\boldsymbol{x}) + (1-t)g(\boldsymbol{y})$$

If for any $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ with $\boldsymbol{x} \neq \boldsymbol{y}$ and $t \in (0,1)$ we even have

$$g(t\boldsymbol{x} + (1-t)\boldsymbol{y}) < tg(\boldsymbol{x}) + (1-t)g(\boldsymbol{y})$$

then the function is called strictly convex. For functions from $\mathbb{R}$ to $\mathbb{R}$ the above definition of strict convexity is equivalent to requiring that the second derivative is strictly positive everywhere. Therefore, linear operations are clearly convex but not strictly convex. Furthermore, $f(x) = x^2$ is a strictly convex function because for $x \neq y$ and $0 < t < 1$ we have

$$\begin{aligned}
& (tx + (1-t)y)^2 < tx^2 + (1-t)y^2 \\
\Leftrightarrow\ & t^2x^2 + 2t(1-t)xy + (1-t)^2y^2 < tx^2 + (1-t)y^2 \\
\Leftrightarrow\ & 0 < t(1-t)x^2 + t(1-t)y^2 - 2t(1-t)xy \\
\Leftrightarrow\ & 0 < t(1-t)(x-y)^2
\end{aligned}$$

where the last inequality is always fulfilled under the above assumptions. From this it follows that the squared norm in $\mathbb{R}^n$ is also strictly convex since

$$\begin{aligned}
\|t\boldsymbol{x} + (1-t)\boldsymbol{y}\|^2 &= \sum_{i=1}^{n}(tx_i + (1-t)y_i)^2 \\
&< \sum_{i=1}^{n} tx_i^2 + (1-t)y_i^2 \\
&= t\left(\sum_{i=1}^{n} x_i^2\right) + (1-t)\left(\sum_{i=1}^{n} y_i^2\right) \\
&= t\|\boldsymbol{x}\|^2 + (1-t)\|\boldsymbol{y}\|^2
\end{aligned}$$

Using these results it is now possible to show that the above energy is strictly convex. First let us now rewrite the energy in a more comfortable form.

$$E(\boldsymbol{u}) = \frac{1}{2}\|\boldsymbol{u} - \boldsymbol{f}\|^2 + \frac{\alpha}{2}\|D\boldsymbol{u}\|^2$$

where $D \in \mathbb{R}^{(N-1)\times N}$ is a matrix with the following structure

$$D = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & -1 & 1 & 0 & 0 & \dots & 0 \\ & & & \dots & & & \\ 0 & & \dots & 0 & -1 & 1 & 0 \\ 0 & & \dots & & 0 & -1 & 1 \end{pmatrix}$$

Now let $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n$, $\boldsymbol{u} \neq \boldsymbol{v}$ and $0 < \beta < 1$, then we have

$$\begin{aligned} E(\beta\boldsymbol{u} + (1-\beta)\boldsymbol{v}) &= \frac{1}{2}\|\beta\boldsymbol{u} + (1-\beta)\boldsymbol{v} - \boldsymbol{f}\|^2 + \frac{\alpha}{2}\|D(\beta\boldsymbol{u} + (1-\beta)\boldsymbol{v})\|^2 \\ &= \frac{1}{2}\|\beta(\boldsymbol{u} - \boldsymbol{f}) + (1-\beta)(\boldsymbol{v} - \boldsymbol{f})\|^2 \\ &\quad + \frac{\alpha}{2}\|\beta D\boldsymbol{u} + (1-\beta)D\boldsymbol{v}\|^2 \\ &< \frac{1}{2}\beta\|\boldsymbol{u} - \boldsymbol{f}\|^2 + \frac{1}{2}(1-\beta)\|\boldsymbol{v} - \boldsymbol{f}\|^2 \\ &\quad + \frac{\alpha}{2}\|\beta D\boldsymbol{u} + (1-\beta)D\boldsymbol{v}\|^2 \\ &\leqslant \frac{1}{2}\beta\|\boldsymbol{u} - \boldsymbol{f}\|^2 + \frac{1}{2}(1-\beta)\|\boldsymbol{v} - \boldsymbol{f}\|^2 \\ &\quad + \frac{\alpha}{2}\left(\beta\|D\boldsymbol{u}\|^2 + (1-\beta)\|D\boldsymbol{v}\|^2\right) \\ &= \beta E(\boldsymbol{u}) + (1-\beta)E(\boldsymbol{v}) \end{aligned}$$

Thus, the energy is strictly convex. Note that the first inequality is always strict because if $\boldsymbol{u} \neq \boldsymbol{v}$ then we also have $\boldsymbol{u} - \boldsymbol{f} \neq \boldsymbol{v} - \boldsymbol{f}$ but the second inequality must not necessarily be strict. If both $\boldsymbol{u}$ and $\boldsymbol{v}$ are in the nullspace of D, then $D\boldsymbol{u} = D\boldsymbol{v} = 0$ and we cannot have a strict inequality. Concerning possible minimisers, we observe that the energy is bounded below and coercive (e.g. $\|\boldsymbol{v}\| \to \infty$ implies $E(\boldsymbol{v}) \to \infty$ as well.) Thus, there exists a closed and bounded level set which contains the minimiser. Finally, because of the strict convexity of the energy, we can even state that this minimum is unique.

Remark: The coercivity of E is really necessary because functions like the exponential function are also strictly convex and bounded below but they fail to *have a minimum. They possess an infimum to which one can come arbitrarily close, but which one cannot attain. Here, the coercivity assures that there exists a compact level set that contains the infimum. Since E is a continuous function, it must attain its minimum in this compact level set (Extreme value theorem or Theorem of Weierstrass/Bolzano).*

Problem 2 (Euler-Lagrange Equations)

Given is the 3-D energy functional

$$E(u) = \int\limits_{\Omega} \left(\frac{(u-f)^2}{2} + \alpha\sqrt{\epsilon + |\boldsymbol{\nabla} u|^2} \right) \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

The Euler-Lagrange equation in 3-D looks as follows:

$$F_u - \frac{\partial}{\partial x} F_{u_x} - \frac{\partial}{\partial y} F_{u_y} - \frac{\partial}{\partial z} F_{u_z} = 0$$

where we have in this case:

$$\begin{aligned} F(x,y,u,u_x,u_y,u_z) &= \frac{(u-f)^2}{2} + \alpha\sqrt{\epsilon + |\boldsymbol{\nabla} u|^2} \\ &= \frac{(u-f)^2}{2} + \alpha\sqrt{\epsilon + u_x^2 + u_y^2 + u_z^2} \end{aligned}$$

Let us compute the derivatives in the Euler-Lagrange equation:

$$\begin{aligned} F_u &= u - f \\ F_{u_x} &= \alpha \frac{2u_x}{2\sqrt{\epsilon + u_x^2 + u_y^2 + u_y^2}} = \alpha \frac{u_x}{\sqrt{\epsilon + u_x^2 + u_y^2 + u_z^2}} \\ F_{u_y} &= \alpha \frac{2u_y}{2\sqrt{\epsilon + u_x^2 + u_y^2 + u_z^2}} = \alpha \frac{u_y}{\sqrt{\epsilon + u_x^2 + u_y^2 + u_z^2}} \\ F_{u_z} &= \alpha \frac{2u_z}{2\sqrt{\epsilon + u_x^2 + u_y^2 + u_z^2}} = \alpha \frac{u_z}{\sqrt{\epsilon + u_x^2 + u_y^2 + u_z^2}} \end{aligned}$$

$$\begin{aligned} \frac{\partial}{\partial x} F_{u_x} &= \alpha \frac{u_{xx}\sqrt{\epsilon + u_x^2 + u_y^2 + u_z^2} - u_x \frac{u_x u_{xx} + u_y u_{yx} + u_z u_{zx}}{\sqrt{\epsilon + u_x^2 + u_y^2 + u_z^2}}}{(\epsilon + u_x^2 + u_y^2 + u_z^2)} \\ &= \alpha \frac{u_{xx}(\epsilon + u_x^2 + u_y^2 + u_z^2) - u_x^2 u_{xx} - u_x u_y u_{yx} - u_x u_z u_{zx}}{(\epsilon + u_x^2 + u_y^2 + u_z^2)^{\frac{3}{2}}} \end{aligned}$$

$$= \alpha \frac{u_{xx}(\epsilon + u_y^2 + u_z^2) - u_x u_y u_{xy} - u_x u_z u_{xz}}{(\epsilon + u_x^2 + u_y^2 + u_z^2)^{\frac{3}{2}}}$$

$$\frac{\partial}{\partial y} F_{u_y} = \alpha \frac{u_{yy}(\epsilon + u_x^2 + u_z^2) - u_y u_x u_{xy} - u_y u_z u_{zy}}{(\epsilon + u_x^2 + u_y^2 + u_z^2)^{\frac{3}{2}}}$$

$$\frac{\partial}{\partial z} F_{u_z} = \alpha \frac{u_{zz}(\epsilon + u_x^2 + u_y^2) - u_z u_x u_{xz} - u_z u_y u_{yz}}{(\epsilon + u_x^2 + u_y^2 + u_z^2)^{\frac{3}{2}}}$$

The resulting Euler-Lagrange equation is:

$$\begin{aligned} & u - f - \alpha \frac{u_{xx}(\epsilon + u_y^2 + u_z^2) - u_x u_y u_{xy} - u_x u_z u_{xz}}{(\epsilon + u_x^2 + u_y^2 + u_z^2)^{\frac{3}{2}}} \\ & - \alpha \frac{u_{yy}(\epsilon + u_x^2 + u_z^2) - u_y u_x u_{xy} - u_y u_z u_{zy}}{(\epsilon + u_x^2 + u_y^2 + u_z^2)^{\frac{3}{2}}} \\ & - \alpha \frac{u_{zz}(\epsilon + u_x^2 + u_y^2) - u_z u_x u_{xz} - u_z u_y u_{yz}}{(\epsilon + u_x^2 + u_y^2 + u_z^2)^{\frac{3}{2}}} & = 0 \end{aligned}$$

$$\begin{aligned} \Leftrightarrow \; & 2(u - f) - \alpha \frac{u_{xx}(\epsilon + u_y^2 + u_z^2) + u_{yy}(\epsilon + u_x^2 + u_z^2) + u_{zz}(\epsilon + u_x^2 + u_y^2)}{(\epsilon + u_x^2 + u_y^2 + u_z^2)^{\frac{3}{2}}} \\ & - 2 \frac{u_x u_y u_{xy} + u_x u_z u_{xz} + u_z u_y u_{yz}}{(\epsilon + u_x^2 + u_y^2 + u_z^2)^{\frac{3}{2}}} & = 0 \end{aligned}$$