Image Processing and Computer Vision (IPCV)

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Example Solutions for Classroom Assignment 8 (C8)

Problem 1 (Bilateral Filtering and NL Means)

Given is a filter of the form

$$u_i = \sum_{j=1}^{N} q_{i,j} f_j \qquad \forall i \in \{1, \dots, N\}$$

or in matrix notation $\quad \boldsymbol{u} = Q\boldsymbol{f}$

where $\boldsymbol{f}$ is a discrete signal of length N and $\boldsymbol{u}$ is its filtered version.

The two properties given as assumptions in the different subtask can be formalised as:

Unit row sums:	$\forall i \in \{1, \dots, N\}:$	$\sum_{j=1}^{N} q_{i,j} = 1$
Unit column sums:	$\forall j \in \{1, \dots, N\}:$	$\sum_{i=1}^{N} q_{i,j} = 1$

(a) We show that the average grey level is preserved, provided Q has unit column sums:

$\forall i \in \{1, \dots, N\}:$

$$\frac{1}{N}\sum_{i=1}^{N} u_i \overset{\text{def } u_i}{=} \frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N} q_{i,j} f_j \overset{\text{commuta-tivity}}{=} \frac{1}{N}\sum_{j=1}^{N}\sum_{i=1}^{N} q_{i,j} f_j$$

$$\overset{\text{distribu-tivity}}{=} \frac{1}{N}\sum_{j=1}^{N} f_j \underbrace{\sum_{i=1}^{N} q_{i,j}}_{=1} \overset{\text{unit column sums}}{=} \frac{1}{N}\sum_{i=1}^{N} f_i$$

(b) We want to show that the maximum-minimum principle holds, provided Q has unit row sums and $q_{i,j} \geq 0$.

$\forall i \in \{1, \dots, N\}:$

$$
\begin{aligned}
\max_k f_k &\overset{\substack{\text{unit row}\\\text{sums}}}{=} \max_k f_k \underbrace{\sum_{j=1}^{N} q_{i,j}}_{=1} &&\overset{\substack{\text{distribu-}\\\text{tivity}}}{=} \sum_{j=1}^{N} q_{i,j} \max_k f_k \\
&\overset{q_{i,j} \geq 0}{\geq} \sum_{j=1}^{N} q_{i,j} f_j &&\overset{\text{def } u_i}{=} u_i \\
&\overset{\text{def } u_i}{=} \sum_{j=1}^{N} q_{i,j} f_j &&\overset{q_{i,j} \geq 0}{\geq} \sum_{j=1}^{N} q_{i,j} \min_k f_k \\
&\overset{\substack{\text{distribu-}\\\text{tivity}}}{=} \min_k f_k \underbrace{\sum_{j=1}^{N} q_{i,j}}_{=1} &&\overset{\substack{\text{unit row}\\\text{sums}}}{=} \min_k f_k
\end{aligned}
$$

Thus $\qquad \max_j f_j \geq u_i \geq \min_j f_j.$

(c) **Unit row sums property is satisfied for bilateral filtering and NL means**

For the sake of simplicity let us define a unifying model for both filter types:

$$u_i = \frac{\sum_{j=1}^{N} a_{i,j} f_j}{\sum_{j=1}^{N} a_{i,j}}$$

Bilateral filtering is defined as

$$u_i = \frac{\sum_{j=1}^{N} g(|f_i - f_j|^2) w(|x_i - x_j|^2) f_j}{\sum_{j=1}^{N} g(|f_i - f_j|^2) w(|x_i - x_j|^2)}.$$

thus we have

$$a_{i,j} := g(|f_i - f_j|^2) w(|x_i - x_j|^2).$$

NL-means are defined as

$$u_i = \frac{\sum_{j=1}^{N} g(|f(\mathcal{N}_i) - f(\mathcal{N}_j)|) f_j}{\sum_{j=1}^{N} g(|f(\mathcal{N}_i) - f(\mathcal{N}_j)|)}.$$

As for bilateral filtering we define

$$a_{i,j} := g(|f(\mathcal{N}_i) - f(\mathcal{N}_j)|).$$

We can now prove the unit row sum property for both filters at once:
$\forall i \in \{1, \dots, N\}$:

$$\sum_{j=1}^{N} q_{i,j} = \sum_{j=1}^{N} \frac{a_{i,j}}{\sum_{k=1}^{N} a_{i,k}} = \frac{\sum_{j=1}^{N} a_{i,j}}{\sum_{j=1}^{N} a_{i,j}} = 1$$

Hence the unit row sums property is satisfied.

Unit column sums property is not satisfied for bilateral filtering

For bilateral filtering we have

$$u_i = \frac{\sum_{j=1}^{N} a_{i,j} f_j}{\sum_{j=1}^{N} a_{i,j}} = \sum_{j=1}^{N} \frac{a_{i,j}}{\sum_{k=1}^{N} a_{i,k}} f_j =: \sum_{j=1}^{N} q_{i,j} f_j.$$

To show that the unit column sums property is not satisfied we consider a counter example:
Consider the 1-D signal $\boldsymbol{f} = (0, 0, 1)$.

For $j = 1$ we get:

$$\begin{aligned}\sum_{i=1}^{3} q_{i,1} &= q_{1,1} + q_{2,1} + q_{3,1} \\ &= \frac{a_{1,1}}{\sum_{k=1}^{3} a_{1,k}} + \frac{a_{2,1}}{\sum_{k=1}^{3} a_{2,k}} + \frac{a_{3,1}}{\sum_{k=1}^{3} a_{3,k}} \\ &= \frac{a_{1,1}}{a_{1,1} + a_{1,2} + a_{1,3}} + \frac{a_{2,1}}{a_{2,1} + a_{2,2} + a_{2,3}} + \frac{a_{3,1}}{a_{3,1} + a_{3,2} + a_{3,3}}\end{aligned}$$

As weight functions g and w usually a Gaussian is used. So let us define $w(x^2) = g(x^2) := \exp(-x^2)$.
We get

$$\begin{aligned}a_{1,1} = a_{2,2} = a_{3,3} &= \exp(-0^2)\exp(-0^2) = 1 \\ a_{1,2} = a_{2,1} &= \exp(-0^2)\exp(-1^2) = e^{-1} \\ a_{1,3} = a_{3,1} &= \exp(-1^2)\exp(-2^2) = e^{-5} \\ a_{2,3} = a_{3,2} &= \exp(-1^2)\exp(-1^2) = e^{-2}\end{aligned}$$

and

$$\sum_{i=1}^{3} q_{i,1} = \frac{1}{1 + e^{-1} + e^{-5}} + \frac{e^{-1}}{e^{-1} + 1 + e^{-2}} + \frac{e^{-5}}{e^{-5} + e^{-2} + 1} \approx 0.98 \neq 1$$

So obviously, in general we do not have unit column sums for bilateral filtering.

Unit column sums property is not satisfied for NL-means:

Again, we have

$$u_i = \frac{\sum_{j=1}^{N} a_{i,j} f_j}{\sum_{j=1}^{N} a_{i,j}} = \sum_{j=1}^{N} \frac{a_{i,j}}{\sum_{k=1}^{N} a_{i,k}} f_j =: \sum_{j=1}^{N} q_{i,j} f_j.$$

To show that the unit column sums property is not satisfied we consider again a counter example:
Consider the 1-D signal $\boldsymbol{f} = (0,0,1)$ and assume reflecting boundary conditions. Furthermore we choose a patch size of 3. We get

$$f(\mathcal{N}_1) = (0,0,0)^T \qquad f(\mathcal{N}_2) = (0,0,1)^T \qquad f(\mathcal{N}_3) = (0,1,1)^T$$

For $j = 1$ we get:

$$\begin{aligned}\sum_{i=1}^{3} q_{i,1} &= q_{1,1} + q_{2,1} + q_{3,1} \\ &= \frac{a_{1,1}}{\sum_{k=1}^{3} a_{1,k}} + \frac{a_{2,1}}{\sum_{k=1}^{3} a_{2,k}} + \frac{a_{3,1}}{\sum_{k=1}^{3} a_{3,k}} \\ &= \frac{a_{1,1}}{a_{1,1} + a_{1,2} + a_{1,3}} + \frac{a_{2,1}}{a_{2,1} + a_{2,2} + a_{2,3}} + \frac{a_{3,1}}{a_{3,1} + a_{3,2} + a_{3,3}}\end{aligned}$$

As weight function g usually a Gaussian is used. So let us define $g(x) := \exp(-x^2)$. We get

$$\begin{aligned}a_{1,1} = a_{2,2} = a_{3,3} &= \exp(-0^2) = 1 \\ a_{1,2} = a_{2,1} &= \exp(-1^2) = e^{-1} \\ a_{1,3} = a_{3,1} &= \exp(-2^2) = e^{-4} \\ a_{2,3} = a_{3,2} &= \exp(-1^2) = e^{-1}\end{aligned}$$

and

$$\sum_{i=1}^{3} q_{i,1} = \frac{1}{1 + e^{-1} + e^{-4}} + \frac{e^{-1}}{e^{-1} + 1 + e^{-1}} + \frac{e^{-4}}{e^{-4} + e^{-1} + 1} \approx 0.95 \neq 1$$

Thus, in general, we do not have unit column sums for NL-means either.