# Lecture 14:
# Nonlinear Filters I:
# Morphology and Median Filters

**Contents**









# What are Morphological Filters?

## Mathematical Morphology

- analyses the shape of objects in an image
- was founded by Jean Serra and George Matheron around 1965 at the Ecole Normale Supérieure des Mines in Fontainebleau near Paris
- one of the most successful classes of image analysis methods
- numerous applications:
  cell biology, medical image analysis, geostatistics, remote sensing, ...
- provides a nonlinear shift invarinat alternative to linear shift invariant filters, while still showing some structural similarities to them
  (e.g. lowpass-, highpass-, bandpass-like behaviour)

The building in Fontainebleau where mathematical morphology was born is now called Centre for Mathematical Morphology (CMM). Source: `http://cmm.ensmp.fr/presentation.html`.

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# The Building Blocks: Dilation and Erosion

- *Dilation (in German: Dilatation)* replaces the grey value of a continuous image $f(x,y)$ by its supremum within a mask $B$:

  $$(f \oplus B)\,(x,y) \;:=\; \sup\,\{f(x{-}x',\, y{-}y') \mid (x',y'){\in}B\}.$$

  *Erosion (in German: Erosion)* uses the infimum instead:

  $$(f \ominus B)\,(x,y) \;:=\; \inf\,\{f(x{+}x',\, y{+}y') \mid (x',y'){\in}B\}.$$

  For discrete images: maximum / minimum instead of supremum / infimum.

- The mask $B$ is called *structuring element (Strukturelement)*.
  It describes the relevant neighbourhood structure.
  If nothing else is specified, its centre serves as its reference point.

- Often convex structuring elements are used
  (i.e. any line connecting two points of the structuring element lies in the element):
  e.g. discs or squares.

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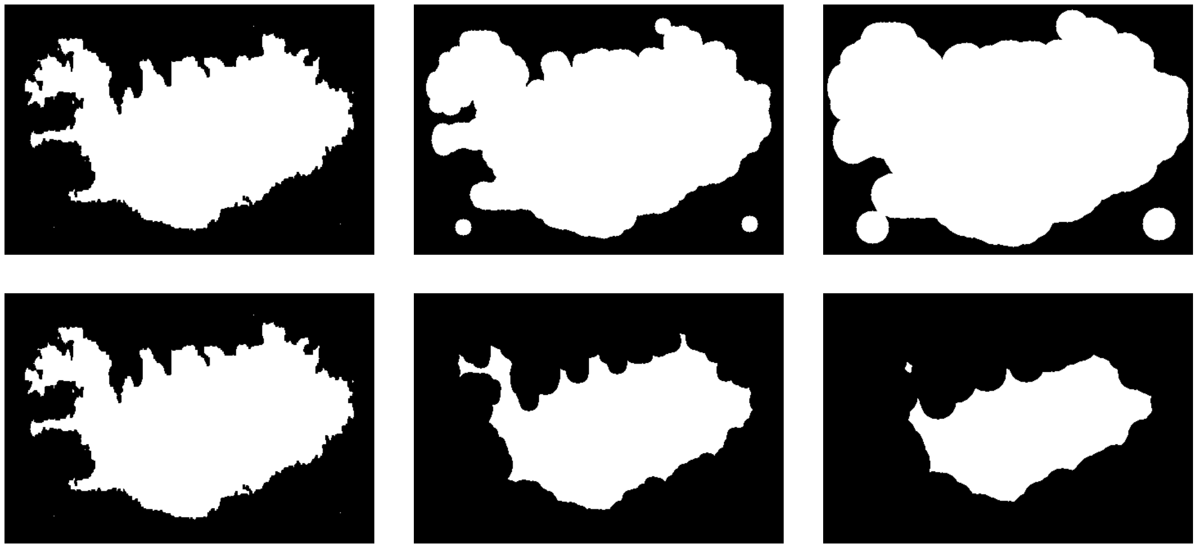
## The Building Blocks: Dilation and Erosion (2)

original, $452 \times 305$ pixels | dilation, radius $10$ pixels | dilation, radius $20$ pixels



original, $452 \times 305$ pixels | erosion, radius $10$ pixels | erosion, radius $20$ pixels

Dilation / erosion of a binary image of Iceland with a disc. Author: J. Weickert.



## The Building Blocks: Dilation and Erosion (3)

### Interpretation for Binary Images with Discs as Structuring Elements

- for bright objects and dark background:
  - Dilation propagates the object contour in outer normal direction.
  - Erosion moves it in inner normal direction.
- analogies from nature:
  - flame propagation of a prairie fire
  - Huygens principle for wave propagation

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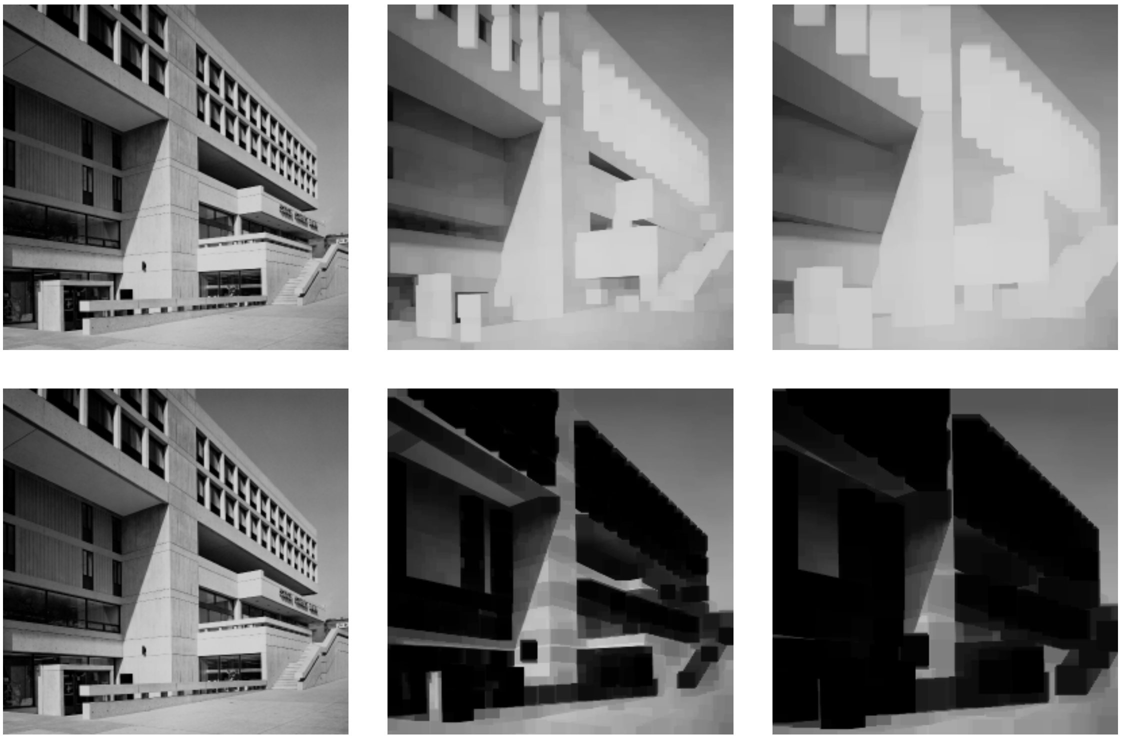
## The Building Blocks: Dilation and Erosion (4)




Dilation / erosion of a greyscale image with a square. Author: J. Weickert.



## The Building Blocks: Dilation and Erosion (5)

### Algorithms for Convex Structuring Elements

- Convex structuring elements are *scalable:*
  One dilation / erosion with structuring element $nB$ is equivalent to
  $n$ dilations / erosions with structuring element $B$.

- implementing large structuring elements from a cascade of small ones is clever:
  - offers higher efficiency for dimensions $\geq 2$
  - Example:
    5 dilations with a disk of radius $2$ is faster than 1 dilation with radius $10$.

### Algorithms for Rectangular Structuring Elements

- Rectangular structuring elements are separable:
  It is sufficient to implement an efficient 1-D algorithm.

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# Morphological Lowpass Filters: Opening and Closing

## Closing (Schließung)

- Goal: simplify image structure while avoiding the expansion effects of dilation
- perform erosion after dilation:

$$f \bullet B \;:=\; (f \oplus B) \ominus B$$

- closes gaps in bright structures by removing dark details
- $f \bullet B$ can be brighter than $f$.
- acts like a morphological lowpass filter due to its simplifying effect



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## Opening (Öffnung)

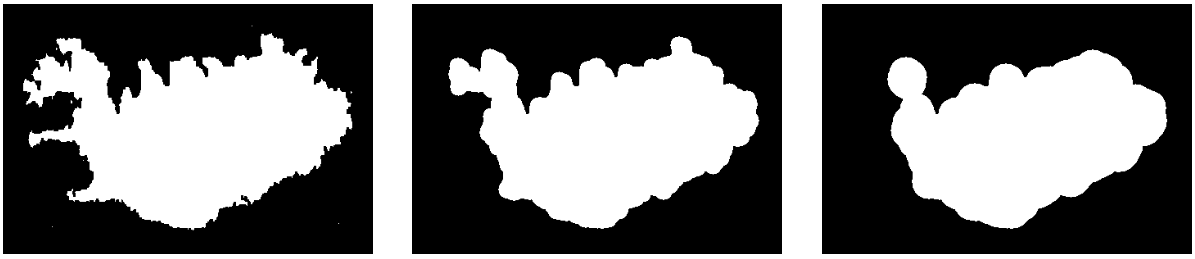
- Goal: simplify image structure while avoiding the shrinkage effects of erosion
- perform dilation after erosion:

$$f \circ B \;:=\; (f \ominus B) \oplus B$$

- closes gaps in dark structures by removing bright details
- $f \circ B$ can be darker than $f$.
- morphological lowpass filter, too

## Morphological Lowpass Filters: Opening and Closing (3)

original, $452 \times 305$ pixels | closing, radius $10$ pixels | closing, radius $20$ pixels



original, $452 \times 305$ pixels | opening, radius $10$ pixels | opening, radius $20$ pixels

Closing / opening of a binary image of Iceland with a disc. Author: J. Weickert.



## Morphological Lowpass Filters: Opening and Closing (4)

original, $256 \times 256$ pixels | closing, length $11$ pixels | closing, length $21$ pixels

original, $256 \times 256$ pixels | opening, length $11$ pixels | opening, length $21$ pixels

Closing / opening of a greyscale image with a square. Author: J. Weickert.

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### Important Difference to Dilation / Erosion

- A second opening or closing with the same structuring element does not alter the image any more *(idempotency):*

$$
\begin{aligned}
(f \circ B) \circ B &= f \circ B, \\
(f \bullet B) \bullet B &= f \bullet B.
\end{aligned}
$$

- Openings and closings are so-called *sieve operations (Sieboperationen):* Structures that pass the sieve (the filter) once also pass it a second time.

- Thus, one cannot obtain an opening / closing with a structuring element of size $nB$ by performing $n$ openings / closings with structuring element $B$.



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# Morphological Highpass Filters: Top Hats

Top hats are morphological tools for extracting fine-scale details.

### White Top Hat (Weißer Zylinderhut)

- The opening $f \circ B$ removes bright details.

- Thus, one can extract these small bright structures by computing

$$\mathrm{WTH}(f) := f - (f \circ B).$$

- This definition guarantees nonnegative results.

- can be seen as morphological highpass filter:
  difference between original image and a morphological lowpass

## Morphological Highpass Filters: Top Hats (2)



### Black Top Hat (Schwarzer Zylinderhut)

- The closing $f \bullet B$ removes dark details.
- Thus, one can extract these small dark structures by computing

$$\mathrm{BTH}(f) \;:=\; (f \bullet B) \;-\; f.$$

- This definition guarantees nonnegative results.

### Selfdual Top Hat (Selbstdualer Zylinderhut)

- One can extract both bright and dark details by computing

$$\rho(f) \;:=\; \mathrm{WTH}(f) \;+\; \mathrm{BTH}(f) \;=\; (f \bullet B) \;-\; (f \circ B).$$

- $\rho(f)$ is invariant with respect to greyscale inversion.

## Morphological Highpass Filters: Top Hats (3)

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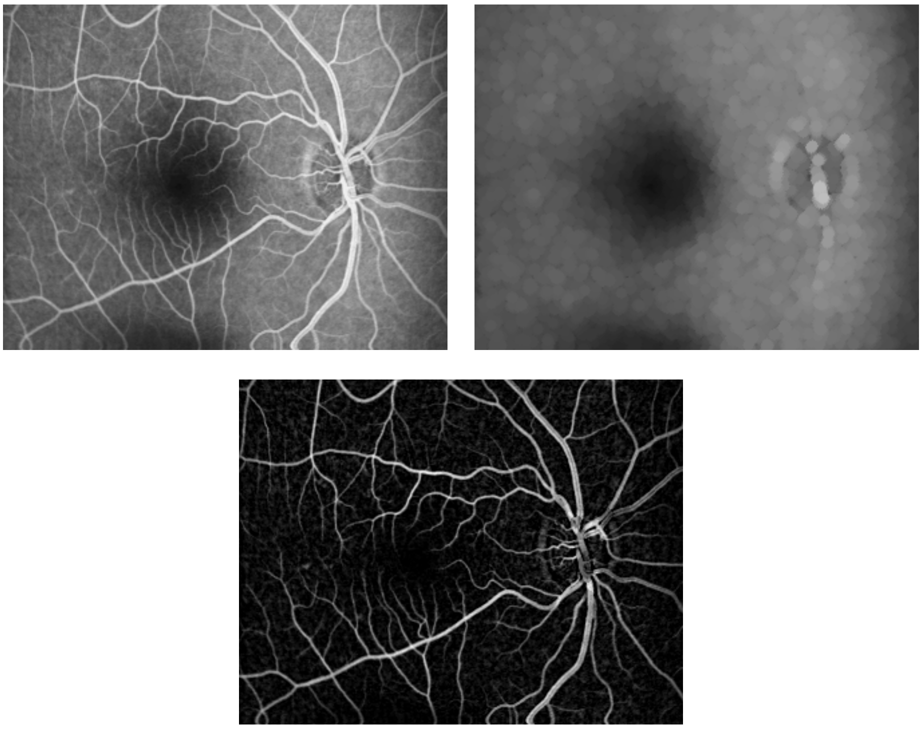
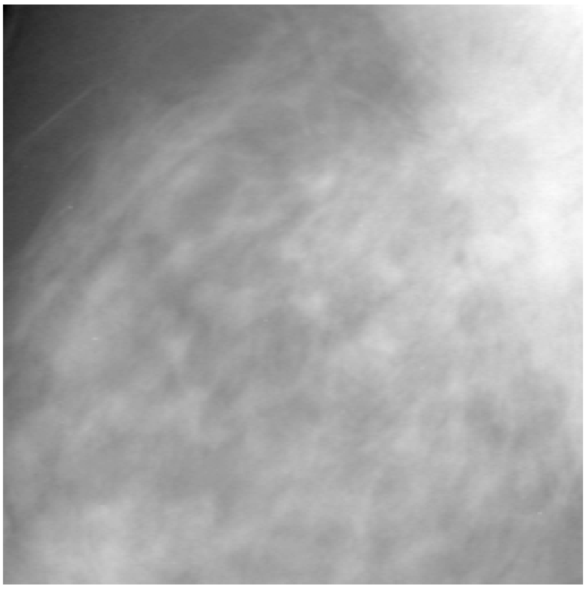
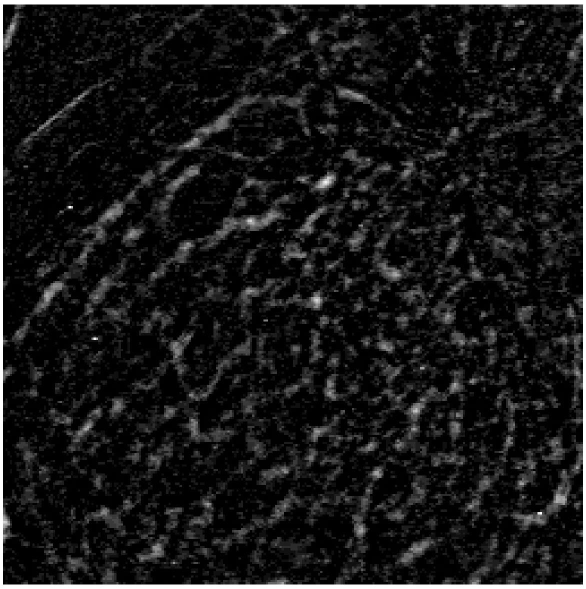


**Top left:** Image of the background of an eye. **Top right:** An opening with a disc of radius $4$ pixels removes all vessels. **Bottom:** The corresponding white top hat extracts these vessels. For better visualisation, the result has been affinely rescaled to $[0, 255]$. Author: J. Weickert.





**Left:** Mammogram, $256 \times 256$ pixels. **Right:** White top hat with a disc of radius $4$ pixels as structuring element. Afterwards the result is affinely rescaled to the interval $[0, 255]$. Microcalcifications and star-shaped structures become better visible. Author: J. Weickert.

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# Morphological Bandpass Filters: Granulometries

- One can compute a chain of openings (or closings) with structuring elements of increasing size.
  This removes small-, middle- and coarse-scale structures step by step.

- gives rise to a morphological image decomposition into structures of different size:
  *granulometry (Granulometrie)*

- Each image in a granulometry is the difference between subsequent images in a chain of openings or closings with increasing size of the structuring elements.

- can be seen as morphological bandpass filters:
  differences of morphological lowpass filters

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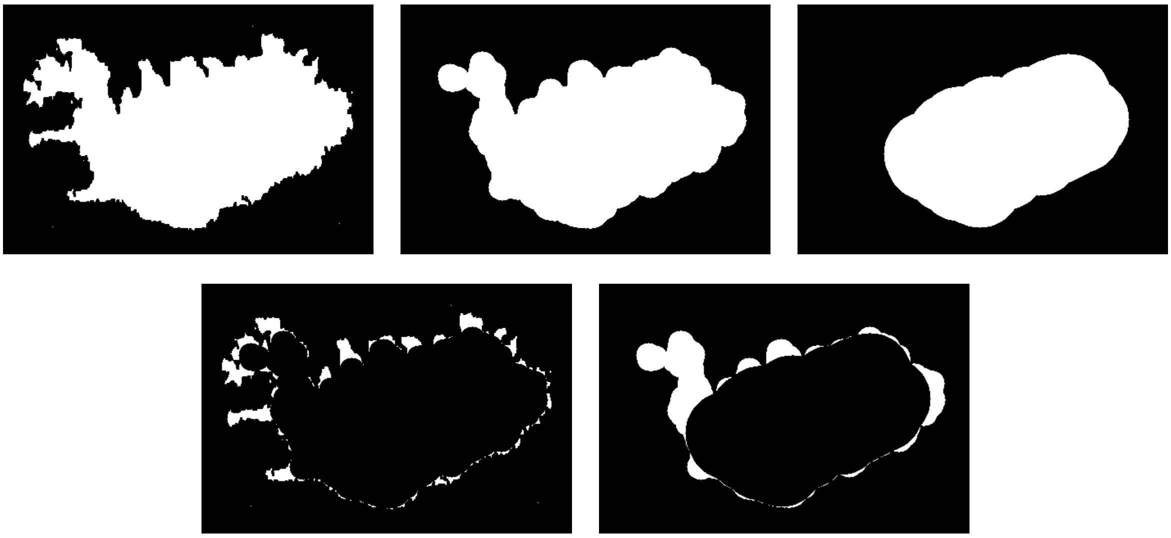
## Morphological Bandpass Filters: Granulometries (2)

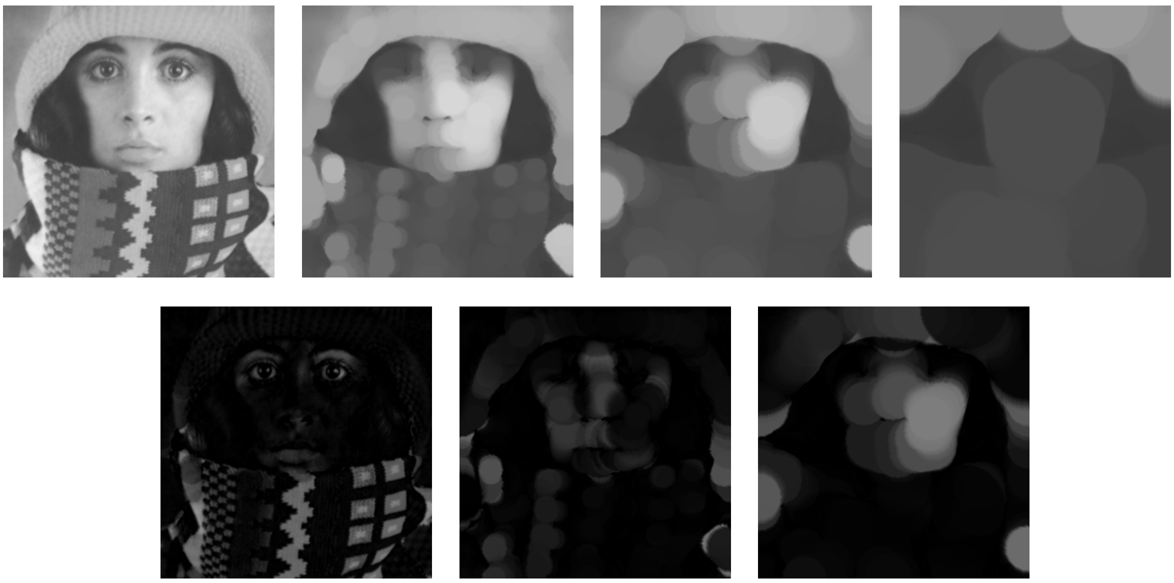




**Top row, from left to right:** Opening of a binary image of Iceland ($452 \times 305$ pixels) with discs of radius $15$ and $50$ pixels. **Bottom row, from left to right:** The corresponding granulometry describes the differences between subsequent images in the opening chain. Author: J. Weickert.

## Morphological Bandpass Filters: Granulometries (3)

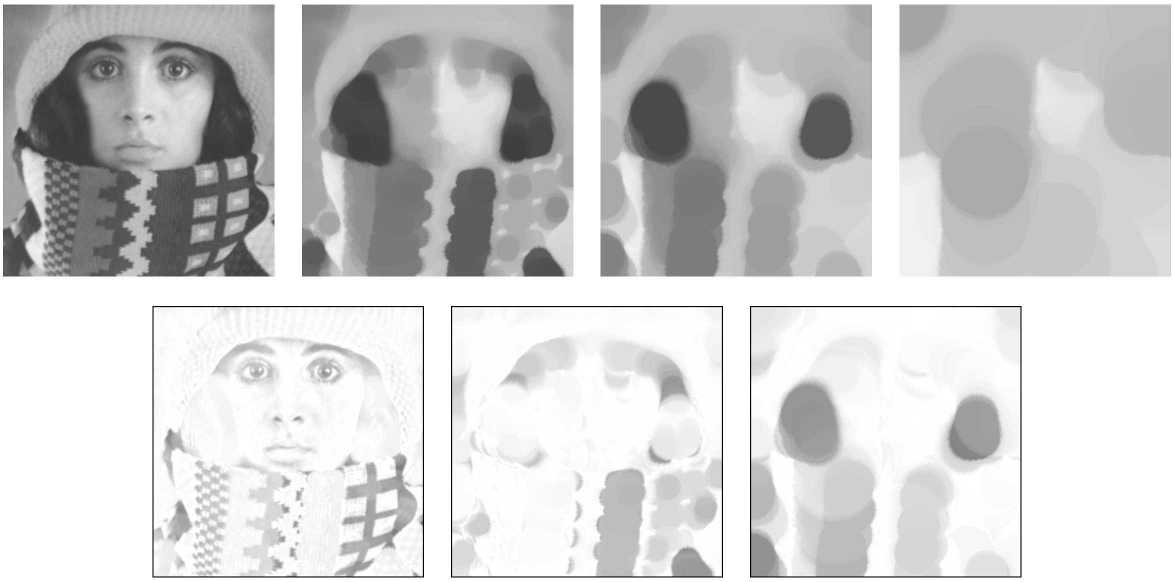




**Top row, from left to right:** Opening of a greyscale image ($256 \times 256$ pixels) with discs of radius $10$, $20$, and $40$ pixels. **Bottom row, from left to right:** The granulometry based on this opening chain is a morphological bandpass decomposition w.r.t. bright structures. Author: J. Weickert.

## Morphological Bandpass Filters: Granulometries (4)





**Top row, from left to right:** Closing of a greyscale image ($256 \times 256$ pixels) with discs of radius $10$, $20$, and $40$ pixels. **Bottom row, from left to right:** The corresponding granulometry describes the differences between subsequent images in the closing chain. Since these differences are nonpositive, the bottom row displays the greyscale range $[-255, 0]$. This granulometry based on a closing chain is a morphological bandpass decomposition w.r.t. dark structures. Author: J. Weickert.

## Median Filters (1)



# Median Filters (Medianfilter)

- So far, all our morphological filters were based on dilation / erosion operations. They compute the largest / smallest element within the neighbourhood.
- Let us now investigate a filter class that exploits another important element in the ordering of the grey values: the median.
- Goal: Remove noise that creates impulse-like outliers (e.g. salt-and-pepper noise, cf. Lecture 2).
- Consider all grey values within a $(2m+1) \times (2m+1)$ mask.
- Create an ordering with increasing grey value.
  If the same grey value appears several times, it is also counted several times.
- As the filtered pixel, choose the *median* of this set, i.e. the value in the middle of this ordering (not the mean).
- Median filters belong to the class of *rank order filters (Rangordnungsfilter).*

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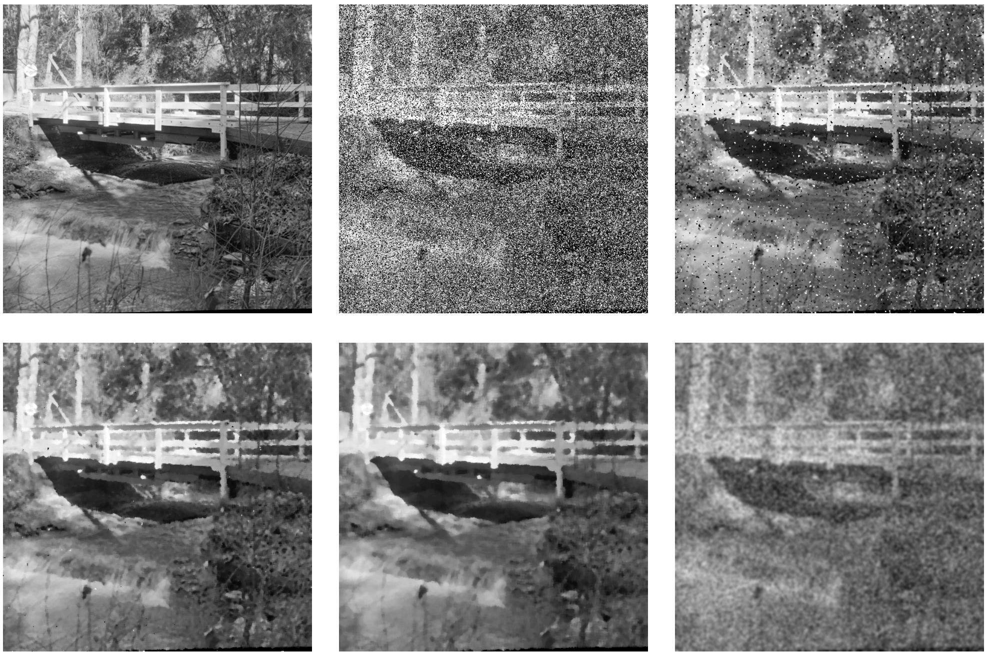


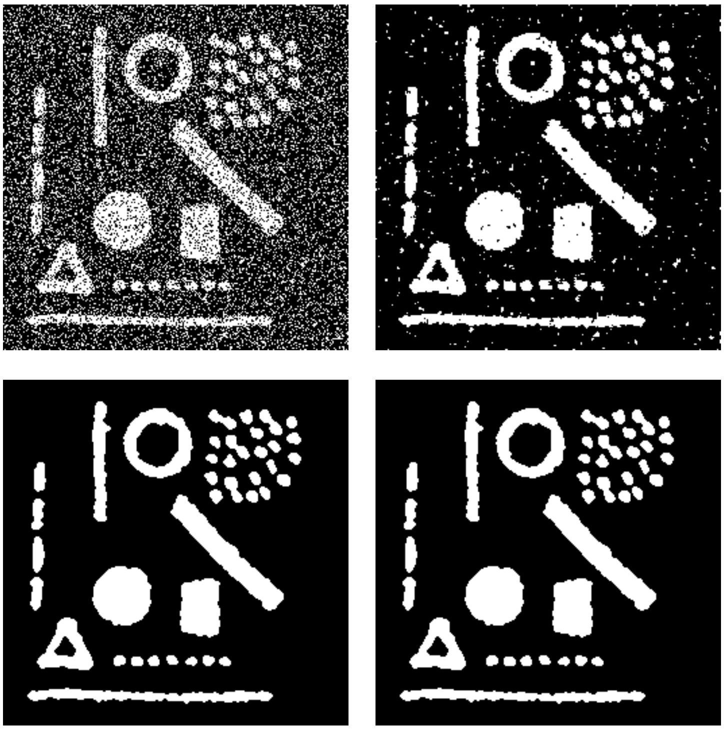
**(a) Top left:** Original image, $512 \times 512$ pixels. **(b) Top middle:** With 40 % salt-and-pepper noise. $\mathrm{PSNR} = 9.22$ dB. **(c) Top right:** Median filtering of (b) with a $3 \times 3$ mask, $\mathrm{PSNR} = 18.08$ dB. **(d) Bottom left:** $5 \times 5$ median, $\mathrm{PSNR} = 22.47$ dB. **(e) Bottom middle:** $7 \times 7$ median, $\mathrm{PSNR} = 21.99$ dB. **(f) Bottom right:** Gaussian smoothing, $\sigma = 3$, $\mathrm{PSNR} = 18.23$ dB. Author: J. Weickert.



1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33

## Properties of Median Filters

- appropriate for removing impulse noise:
  A single outlier does not affect the median of a set.
- preserves (straight or almost straight) edges better than linear filters
- leads to roundings of corners (why?)
- Often it converges after a few iterations to a so-called *root signal.*
- Example of an unstable exception:
  - consider infinite 1-D signal $(..., 0, 1, 0, 1, 0, 1, ...)$
  - iterated median filter of size $3$ swaps zeros and ones in each iteration
- Median filters are not separable!
- Sorting can become rather time consuming for large masks.
- Remedy: By shifting the $(2m+1) \times (2m+1)$ mask by one pixel, one has to exchange only $(2m+1)$ grey values in the sorted list.



**Top left:** Test image with 40 % salt-and-pepper noise. **Top right:** Filtered with a $3 \times 3$ median, 1 iteration. **Bottom left:** 16 iterations. **Bottom right:** Using 256 iterations instead does not improve the image any further. Author: J. Weickert.



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# Summary

- Morphological filters are useful for shape analysis.
- By replacing a grey value by its maximum or minimum within a neighbourhood, dilation and erosion are obtained.
- Sequential combinations of erosion und dilation create openings and closings. They act as morphological lowpass filters.
- Top hats result from computing differences between closing, original image, and opening. They act as morphological highpass filters.
- Granulometries are differences between openings (or closings) with structuring elements of increasing size. They are morphological bandpass filters.
- Median filters replace each grey value by its median within a $(2m+1) \times (2m+1)$ mask. They are well-suited for edge-preserving removal of impulse noise.

# References


- P. Soille: *Morphological Image Analysis*. Second Edition. Springer, Berlin, 1999.
  *(Well readable text book on morphology. Not much theory though.)*
- R. Fisher, S. Perkins, A. Walker, E. Wolfart: *Median Filter*.
  (http://www.dai.ed.ac.uk/HIPR2/median.htm).
  *(useful internet resource on median filtering)*






# Assignment C7 – Classroom Work

**Problem 1 (Tensor Analysis)**

Determine the eigenvalues and corresponding eigenvectors of the tensor $\boldsymbol{\nabla} u \boldsymbol{\nabla} u^\top$.

**Problem 2 (Mean and Median Filtering)**

Consider the following image $\boldsymbol{f}$:

| | | | | |
|---|---|---|---|---|
| 1 | 1 | 7 | 1 | 1 |
| 1 | 0 | 1 | 0 | 7 |
| 6 | 6 | 0 | 6 | 1 |
| 6 | 7 | 6 | 7 | 1 |
| 6 | 6 | 6 | 6 | 0 |
| 6 | 0 | 6 | 0 | 1 |

Assume reflecting boundary conditions and apply the following filters to $\boldsymbol{f}$:

(a) Mean filtering (i.e. using a normalised box filter) within a mask of size $3 \times 3$.

(b) Median filtering within a mask of size $3 \times 3$.

What are your findings?

# Assignment H7 – Homework

**Problem 1: (Structure Tensor Analysis)** (4+2 points)

(a) Let $\boldsymbol{J} \in \mathbb{R}^{n \times n}$ be a symmetric $n \times n$ matrix with real components. We consider its corresponding quadratic form:

$$E : \mathbb{R}^n \to \mathbb{R}, \quad E(\boldsymbol{v}) := \boldsymbol{v}^\top \boldsymbol{J} \boldsymbol{v} \,.$$

Show that among all vectors $\boldsymbol{v} \in \mathbb{R}^n$ with $|\boldsymbol{v}| = 1$, the function value $E(\boldsymbol{v})$ is minimal for the eigenvector of $\boldsymbol{J}$ corresponding to its smallest eigenvalue. What can we say about the sign of $E$, if $\boldsymbol{J}$ is positive definite?

(b) Let $\boldsymbol{J}_\rho$ denote the structure tensor as defined in Lecture 13. If we set the outer scale $\rho = 0$, can the tensor still be used for corner detection? Justify your answer.

**Remark:**
*This problem shows how to solve total least squares problems that are usually formulated in terms of the minimisation of quadratic forms. This will also be useful in the context of motion estimation.*





**Problem 2 (Morphology)** (4+2 points)

Using a structuring element $B$ and the elementary operations dilation and erosion, one can define the following operations:

- $A_B(f) := (f \oplus B) - f$
- $B_B(f) := f - (f \ominus B)$
- $C_B(f) := (f \oplus B) - (f \ominus B)$
- $D_B(f) := A_B(f) - B_B(f)$

(a) Apply these operations to the 1-D signal $\boldsymbol{f} = (\ldots, 1, 1, 1, 1, 0, 0, 0, 0, \ldots)^\top$ which is continued by repeating $1$ to the left and $0$ to the right. Use a symmetric structuring element of size $3$.

(b) State for each of the operations $A_B$, $B_B$, $C_B$, and $D_B$ a linear shift-invariant filter from the lecture that produces similar results.

## Assignment H7 (3)



Please download the required files from the webpage

`http://www.mia.uni-saarland.de/Teaching/ipcv19.shtml`

into your own directory. You can unpack them with the command `tar xvzf Ex07.tar.gz`.

**Problem 3 (Edge and Corner Detection)** (3+3 points)

Your task in this assignment is to implement and experiment with a Canny edge detector and corner detection based on structure tensors.

(a) Supplement the routine `getderivatives` in `canny.c` with code that implements the Sobel operator. Compile the program with the command

```
gcc -O2 -o canny canny.c -lm
```

Find suitable parameters for `objects.pgm` and `pruebab1.pgm`. How and why do they differ?

(b) Add code to the routine `struct_tensor` within `corner.c` that computes the structure tensor. The routine `cornerness` is supposed to compute the structure tensor's determinant as a criterion for corner detection. Supplement the missing code.
Please note: The grey values of the image `u[i][j]` have the index range `i=1,...,nx` and `j=1,...,ny`. You may use the routine `dummies` to create dummy boundary values by mirroring.
To compile the final program, use

```
gcc -O2 -o corner corner.c -lm
```

Find suitable values for the noise and integration scale in order to detect corners in `tree.pgm`, `stairs.pgm` and `acros.pgm`.

## Assignment H7 (4)



**Problem 4: (Morphological Operations)** (3+3 points)

The program `morphology.c` has subroutines for erosion and dilation with a square as structuring element.

(a) Use the finished subroutines for dilation and erosion to complete the subroutines `closing`, `opening`, `white_top_hat`, `black_top_hat`, and `selfdual_top_hat`, such that they perform the corresponding operations. Note that the index range of the image `u[i][j]` is given by `i=1,...,nx` and `j=1,...,ny`.

(b) To compile the final program, use

```
gcc -O2 -o morphology morphology.c -lm
```

With the completed program, try to solve the following problems:

- remove windows and doors (`house.pgm`),
- create university owls at night (`owl.pgm`),
- separate image structures from their background (`fabric.pgm` & `angiogram.pgm`).

If a greyscale image has a poor dynamics, one can normalise it with `xv`: Do a right click on the image, click on the `Windows` button, go to `Color Editor` and click on `Norm`.

## Assignment H7 (5)

### Submission

Please submit the theoretical Problems 1 and 2 in handwritten form before the lecture. For the practical Problems 3 and 4 submit files as follows: Rename the main directory Ex07 to Ex07_<your_name> and use the command

```
tar czvf Ex07_<your_name>.tar.gz Ex07_<your_name>
```

to pack the data. The directory that you pack and submit should contain the following files:

- the source code for canny.c and corner.c;
- the edge and corner maps for the images of Problem 3(a) and (b);
- the source code for morphology.c;
- the four processed images from Problem 4(b);
- a text file README that contains the answer to the question in Problem 3(a); the parameters for the edge and corner detection used in Problem 3(a) and (b); the operators that you have used for the tasks in Problem 4(b); information on all people working together for this assignment.

Submit the file via e-mail to your tutor via the address:

```
ipcv-xx@mia.uni-saarland.de
```

where xx is either t1, t2, t3, t4, t5, w1, w2, w3 or w4 depending on your tutorial group.

**Deadline for submission:** Friday, May 31, 10 am (before the lecture)