Example Solutions for Classroom Assignment 4 (C4)

Problem 1 (Discrete Wavelet Transform)

- Total number of vectors:

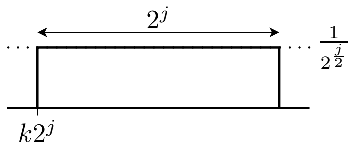
$$1 + (1 + 2 + 4 + \cdots + 2^{n-1}) = 1 + \sum_{i=0}^{n-1} 2^i \stackrel{(1)}{=} 1 + (2^n - 1) = 2^n = N$$

  (1) geometric series

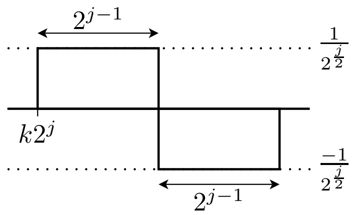
- Different vectors are orthogonal:

  Recall: In the continuous case we have:

  $\Phi_{j,k}$ has width 2^j , height $\dfrac{1}{2^{\frac{j}{2}}}$, and starts at $k2^j$.



  $\Psi_{j,k}$ has width 2^j , height $\dfrac{1}{2^{\frac{j}{2}}}$, and starts at $k2^j$.



  For the discrete setting we sample at N equidistant grid points $\left\{\frac{1}{2}, \frac{3}{2}, \ldots, N - \frac{1}{2}\right\}$. Thus

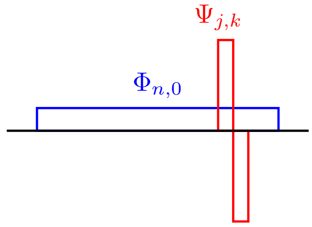
$$\begin{aligned}
\Phi_{n,0} &= ((\Phi_{n,0})_0, (\Phi_{n,0})_1, \ldots, (\Phi_{n,0})_{N-1}) \\
&= \frac{1}{2^{\frac{n}{2}}} (\underbrace{1, 1, \ldots, 1}_{N \text{ times}})^T
\end{aligned}$$

and

$$\begin{aligned}
\Psi_{j,k} &= ((\Psi_{j,k})_0, (\Psi_{j,k})_1, \ldots, (\Psi_{j,k})_{N-1}) \\
&= \frac{1}{2^{\frac{j}{2}}} (\underbrace{0, \ldots, 0}_{\substack{k2^j \\ \text{times}}}, \underbrace{1, \ldots, 1}_{\substack{2^{j-1} \\ \text{times}}}, \underbrace{-1, \ldots, -1}_{\substack{2^{j-1} \\ \text{times}}}, \underbrace{0, \ldots, 0}_{\substack{N-(k+1)2^j \\ \text{times}}})^T
\end{aligned}$$

To show that the given vectors are orthogonal, the inner product between two arbitrary different vectors has to be 0:

- First we consider the inner products regarding the mother wavelet:

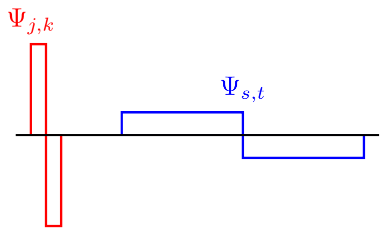


$$
\begin{aligned}
\langle \Phi_{n,0}, \Psi_{j,k} \rangle &= \sum_{i=0}^{N-1} (\Phi_{n,0})_i \cdot (\Psi_{j,k})_i \\
&= \sum_{i=k2^j}^{k2^j+2^j-1} (\Phi_{n,0})_i \cdot (\Psi_{j,k})_i \\
&= \sum_{i=k2^j}^{k2^j+2^{j-1}-1} \frac{1}{2^{\frac{n}{2}}} \cdot \frac{1}{2^{\frac{j}{2}}} + \sum_{i=k2^j+2^{j-1}}^{k2^j+2^j-1} \frac{1}{2^{\frac{n}{2}}} \cdot \frac{-1}{2^{\frac{j}{2}}} \\
&= 2^{j-1} \cdot \frac{1}{2^{\frac{n}{2}}} \cdot \frac{1}{2^{\frac{j}{2}}} + 2^{j-1} \cdot \frac{-1}{2^{\frac{n}{2}}} \cdot \frac{1}{2^{\frac{j}{2}}} \\
&= 0
\end{aligned}
$$

- For $\langle \Psi_{j,k}, \Psi_{s,t} \rangle$, we consider 4 cases: First of all let us assume w.l.o.g. $j \leq s$ (i.e $\Psi_{j,k}$ is more or equally "narrow" than $\Psi_{s,t}$).

  Case 1:

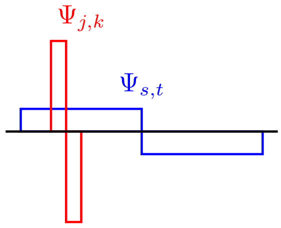
  $k2^j < t2^s$



$$
\langle \Psi_{j,k}, \Psi_{s,t} \rangle = \sum_{i=0}^{N-1} (\Psi_{j,k})_i \cdot (\Psi_{s,t})_i = \sum_{i=0}^{N-1} 0 = 0
$$

Case 2:

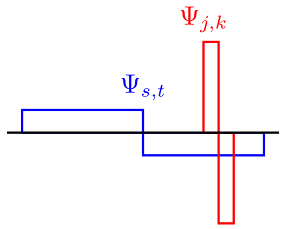
$t2^s \leq k2^j < t2^s + 2^{s-1}$



$$\langle \Psi_{j,k}, \Psi_{s,t} \rangle = \cdots = 2^{j-1} \cdot \frac{1}{2^{\frac{j}{2}}} \cdot \frac{1}{2^{\frac{s}{2}}} + 2^{j-1} \cdot \frac{-1}{2^{\frac{j}{2}}} \cdot \frac{1}{2^{\frac{s}{2}}} = 0$$

Case 3:

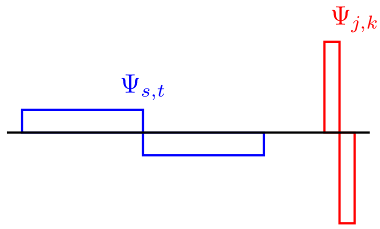
$t2^s + 2^{s-1} \leq k2^j < t2^s + 2^s$



$$\langle \Psi_{j,k}, \Psi_{s,t} \rangle = \cdots = 2^{j-1} \cdot \frac{1}{2^{\frac{j}{2}}} \cdot \frac{-1}{2^{\frac{s}{2}}} + 2^{j-1} \cdot \frac{-1}{2^{\frac{j}{2}}} \cdot \frac{-1}{2^{\frac{s}{2}}} = 0$$

Case 4:

$t2^s + 2^s \leq k2^j$



$$\langle \Psi_{j,k}, \Psi_{s,t} \rangle = \cdots = \sum_{i=0}^{N-1} 0 = 0$$

- The vectors have norm 1

$$|\Phi_{n,0}|^2 = \sum_{i=0}^{N-1} (\Phi_{n,0})_i^2 = 2^n \cdot \left(\frac{1}{2^{\frac{n}{2}}} \right)^2 = \frac{2^n}{2^n} = 1$$
$$\Rightarrow |\Phi_{n,0}| = 1$$
$$|\Psi_{j,k}|^2 = \ldots = 2^{j-1} \cdot \left(\frac{1}{2^{\frac{j}{2}}} \right)^2 + 2^{j-1} \cdot \left(\frac{-1}{2^{\frac{j}{2}}} \right)^2 = 2 \cdot \frac{2^{j-1}}{2^j} = 1$$
$$\Rightarrow |\Psi_{j,k}| = 1$$

So we have proven that the given vectors form an orthonormal basis of $\mathbb{R}^N$ with respect to the inner product.