Boolean Function Complexity: Problem Set 1

Due Date: 6 May 2019

1. We recall that $\mathsf{L}(n)$ is the smallest number t such that every Boolean function on n variables can be computed by a De Morgan formula of size at most t. Show that $\mathsf{L}(n) \leq 3 \cdot 2^{n-1} - 2$. **(10 points)**

2. Let $\mathcal{B}_2$ be the set of all Boolean functions over 2 variables. Consider circuits where functions in $\mathcal{B}_2$ are allowed as gates, and let $\mathsf{C}_{\mathcal{B}_2}(f)$ denote the minimum number of gates in such a circuit computing f.

   We say that a function $f(x_1, \ldots, x_n)$ depends on the i-th variable if $\exists\, b_1, b_2, \ldots, b_{i-1}, b_{i+1}, \ldots, b_n \in \{0, 1\}$ such that

   $$f(b_1, b_2, \ldots, b_{i-1}, 0, b_{i+1}, \ldots, b_n) \neq f(b_1, b_2, \ldots, b_{i-1}, 1, b_{i+1}, \ldots, b_n).$$

   Let $f(x_1, \ldots, x_n)$ be a function that depends on all n variables. Then, show that $\mathsf{C}_{\mathcal{B}_2}(f) \geq n - 1$. **(10 points)**

3. For each n, define the function $\mathsf{Sum}_n \colon \{0, 1\}^n \to \{0, 1\}^m$ where $m = \lceil \log(n+1) \rceil$, which given an input $x \in \{0, 1\}^n$ outputs the binary expansion of the sum $\sum_{i=1}^{n} x_i$. **(10 + 5 points)**

   (a) Show that $\mathsf{C}_{\mathcal{B}_2}(\mathsf{Sum}_n) \leq 5n$.

   (b) A function is said to be *symmetric* if the output only depends on the number of 1s in the input. For example, OR, AND, MAJ, Parity, etc.
   Show that $\mathsf{C}_{\mathcal{B}_2}(f_n) \leq 5n + o(n)$ for every symmetric function f_n on n variables.

4. Recall the threshold function Th_2^n outputs 1 iff the input has at least 2 ones. Construct a monotone De Morgan formula of size at most $O(n \log n)$ computing Th_2^n. Recall that a monotone formula does not use negations. **(10 points)**

5. For $1 \leq j \leq 2^k$, let $e_j \in \{0, 1\}^{2^k}$ be such that the j-th bit in e_j equals 1 and all other bits are 0. For $x \in \{0, 1\}^k$, define $\mathsf{num}(x) = 1 + \sum_{i=0}^{k-1} x_i 2^i$.

Define the function $\mathsf{Indicator}_k \colon \{0,1\}^k \to \{0,1\}^{2^k}$ which on an input $x \in \{0,1\}^k$ outputs the string $e_{\mathsf{num}(x)}$. (Observe that this function indicates what the given input is.)

We say that a family of circuits is *constant-depth* if the depth of circuits in the family is bounded by a universal constant. That is, the depth is independent of the input length n. When we consider constant-depth circuits, the gates in the circuit, like OR and AND, are allowed to have *unbounded* fan-in. **(10 + 10 points)**

(a) Prove that there exists a *constant-depth* multi-output circuit of size $O(2^k)$ that computes $\mathsf{Indicator}_k$.

(b) Prove that every Boolean function f_n on n variables can be computed by a *constant-depth* circuit of size $O(2^n/n)$.