Lecture 5:
Image Transformations II:
Sampling Theorem and Discrete Fourier Transform

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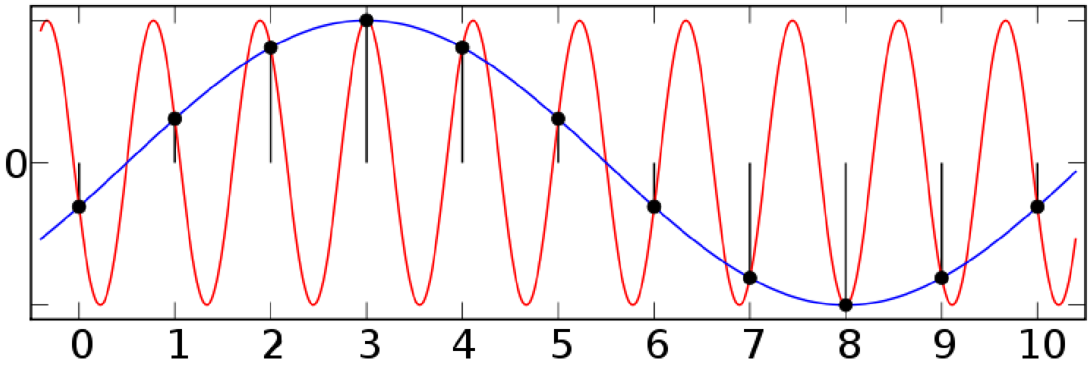
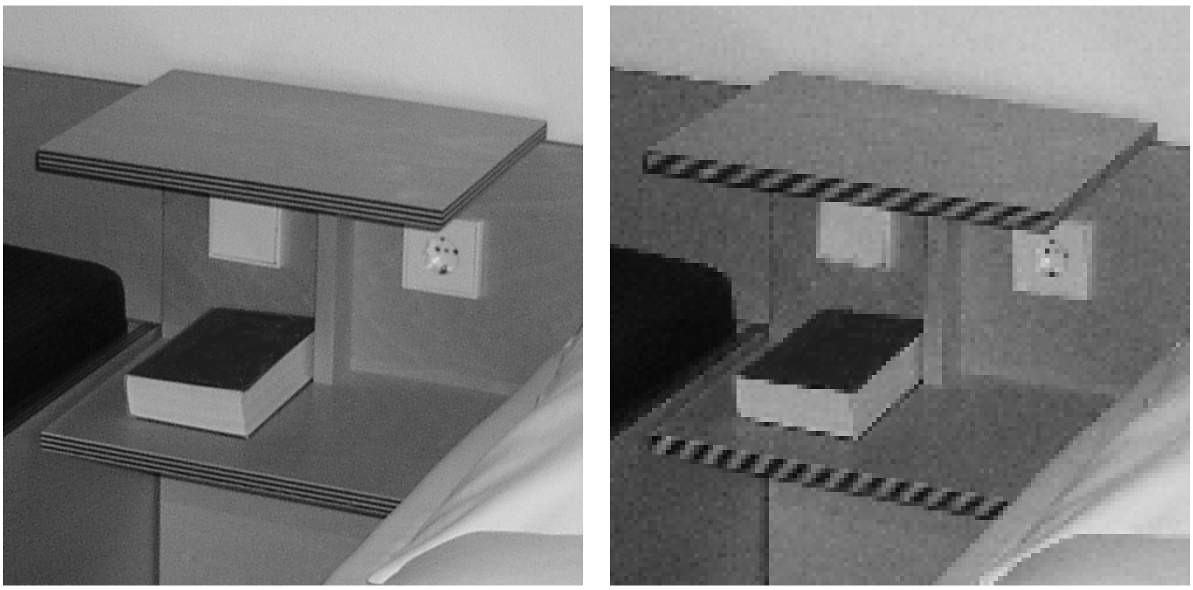
Motivation

- We have seen that the Fourier transform is a useful tool for
  - representing images by their frequencies,
  - expressing convolutions and derivatives in terms of multiplications.
- Our considerations were based on the continuous Fourier transform.
  It requires continuous and infinitely extended images.
- In practice, digital images are sampled and have a finite extent.
- Is there a Fourier-based theory that allows to understand this sampling process and its difficulties ?
- Is there a discrete analogue of the continuous Fourier transform that works on a finite domain ?
- Can it be computed in an efficient way ?

Towards the Discrete Setting: Sampling Theorem



Aliasing effect in 1D. If the sampling rate is too low, high frequent signal components (red) are observed as low frequent artifacts (blue). Source: Wikipedia.



Aliasing effect in 2D. **Left:** Original image, 496×496 pixels. **Right:** Downsampled with xv to 124×124 pixels. Author: J. Weickert.

Aliasing Effect

- If a high-frequent signal is sampled too coarsely, low-frequent artifacts arise.
- This is called *aliasing*, for images sometimes also *Moiré effect*.
- can be observed quite often, e.g.
  - if the resolution of a scanner is too low,
  - when using inappropriate programs for downsampling (such as `xv`),
  - when some internet browsers automatically scale down large images.
- Is there a theory that tells us how fine we have to sample ?

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Sampling Theorem (Abtasttheorem)

(Whittaker 1915, Nyquist 1928, Kotelnikov 1933, Shannon 1949)

- Let a continuous signal f be *band-limited,* i.e. there exists a highest frequency W:

$$\hat{f}(u) \;=\; 0 \quad \text{for } |u| > W.$$

- In order to sample a band-limited signal without aliasing artifacts, one must sample the highest frequency more than twice per period.

  For the sampling distance h this means

$$h \;<\; h_{max} \;:=\; \frac{1}{2W}.$$

Towards the Discrete Setting: Sampling Theorem (5)

Remarks

- The critical sampling frequency $2W$, below which aliasing starts, is called *Nyquist frequency*.
- If the sampling theorem is obeyed, it is even possible to reconstruct the continuous signal $f(x)$ exactly (!) from its discrete samples $\{f(kh) \mid k \in \mathbb{Z}\}$:

  The *Whittaker–Shannon interpolation formula* states that in this case

$$f(x) \;=\; \sum_{k=-\infty}^{\infty} f(kh)\,\mathrm{sinc}\left(\frac{\pi}{h}(x-kh)\right).$$

  We will understand this formula better in Lecture 9.

- For images, the sampling theorem must hold in x- and in y-direction.

Towards the Discrete Setting: Sampling Theorem (6)

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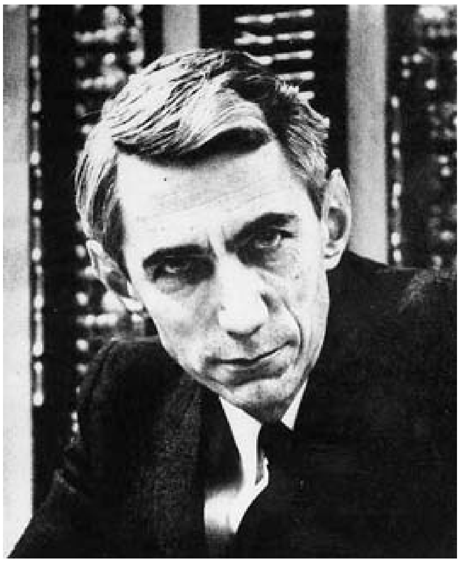
Three researchers who have contributed substantially to the discovery of the sampling theorem. **Left:** The British mathematician Sir Edmund Taylor Whittaker (1873–1956). **Middle:** The Swedish–American electrical engineer Harry Nyquist (1889–1976). **Right:** The Russian electrical engineer Vladimir Kotelnikov (1908–2005). Sources: `http://www.npgprints.com/`, `http://fineartamerica.com`, and `http://en.wikipedia.org/wiki/Harry_Nyquist`.

Towards the Discrete Setting: Sampling Theorem (7)



The American mathematician, electrical engineer and cryptographer Claude E. Shannon (1916–2001) is regarded as the founder of information theory. He has devised the first chess-playing programmes, and he is also one of the pioneers of the sampling theorem. The right image shows his machine that simulates a maze-solving electromechanical mouse. Sources: `http://www-gap.dcs.st-and.ac.uk/~history/Mathematicians/Shannon.html` and `http://afflictor.com/tag/claude-shannon/`.

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Towards the Discrete Setting: Sampling Theorem (8)

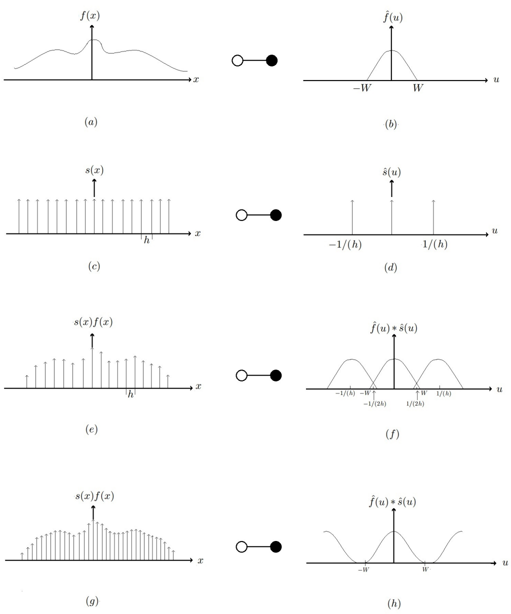

Illustration of the sampling theorem. **(a)** Band-limited function. **(b)** Its Fourier transform. **(c)** Delta comb. **(d)** The Fourier transform of the delta comb is a delta comb with reciprocal grid distance. **(e)** Sampling a band-limited function is multiplication with a delta comb in the spatial domain. **(f)** In the Fourier domain this gives convolution of the Fourier transforms of (b) and (d). Overlapping frequency bands from different periods create aliasing. **(g)** Reduction of the sampling distance. **(h)** In the Fourier domain the frequency bands do no longer overlap. No aliasing effects arise. Author: A. Goswani.

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33 34 35

How Can One Avoid Aliasing ?

- Best solution, if possible:
  - Use a sufficiently high sampling rate.
  - This allows to represent also high frequencies adequately.
- Second best solution:
  - Suppress high frequencies by smoothing your image *before* downsampling.
  - Example of such a smoothing filter:
    Gaussian convolution with a sufficiently large σ:
    at least half the size of the coarser grid.
  - Formally this does not eliminate all high frequencies:
    However, it reduces them substantially such that no artifacts remain visible.

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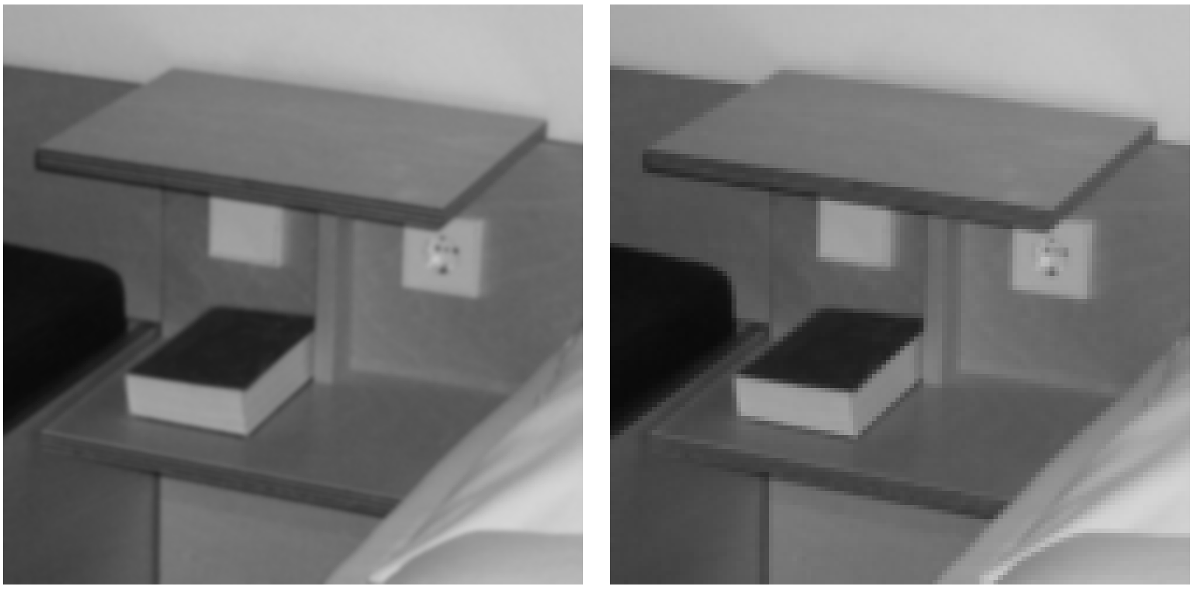

Avoidance of aliasing. **Left:** Original image (496×496 pixels) after suppression of high frequencies by Gaussian convolution with $\sigma = 2$. **Right:** Downsampling with xv to 124×124 pixels does not create visible aliasing effects in this case. Author: J. Weickert.

Discrete Fourier Transform in 1-D

Goals

- discrete analogue to the continuous Fourier transform
- should deal with *sampled* signals of *finite* extent
- signal with M values is decomposed into M frequency components

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1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33 34 35

Reminder from Lecture 4

- Consider a continuous signal $f : \mathbb{R} \to \mathbb{R}$ with infinite extent.
  Its continuous Fourier transform is given by

$$\hat{f}(u) = \int_{-\infty}^{\infty} f(x)\, e^{-i2\pi ux}\, dx \qquad (u \in \mathbb{R})$$

  with $i^2 = -1$.

- The corresponding inverse continuous Fourier transform is defined as

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(u)\, e^{i2\pi ux}\, du \qquad (x \in \mathbb{R}).$$

Discrete Fourier Transform in 1-D (3)

Definition

- Consider a discrete signal $\boldsymbol{f} = (f_0, ..., f_{M-1})^\top$ with finite extent.
  Its *discrete Fourier transform (DFT)* is defined as

$$\hat{f}_p \;:=\; \frac{1}{\sqrt{M}} \sum_{m=0}^{M-1} f_m \exp\left(-\frac{i2\pi pm}{M}\right) \qquad (p = 0, ..., M-1)$$

  with $i^2 = -1$.

- The corresponding *inverse discrete Fourier transform* is given by

$$f_m \;=\; \frac{1}{\sqrt{M}} \sum_{p=0}^{M-1} \hat{f}_p \exp\left(\frac{i2\pi pm}{M}\right) \qquad (m = 0, ..., M-1)$$

Discrete Fourier Transform in 1-D (4)

Interpretation as Change of Basis

- Hermitian inner product of two vectors $\boldsymbol{f} = (f_i)_{i=0}^{M-1}$ and $\boldsymbol{g} = (g_i)_{i=0}^{M-1}$ in $\mathbb{C}^M$:

$$\langle \boldsymbol{f}, \boldsymbol{g} \rangle \;:=\; \sum_{m=0}^{M-1} f_m \, \bar{g}_m \,.$$

- One orthonormal basis of $(\mathbb{C}^M, \langle \cdot, \cdot \rangle)$ is given by the M vectors

$$\boldsymbol{b}_p \;:=\; \tfrac{1}{\sqrt{M}} \left(\exp\left(\tfrac{i2\pi p0}{M}\right), \exp\left(\tfrac{i2\pi p1}{M}\right), ..., \exp\left(\tfrac{i2\pi p(M-1)}{M}\right)\right)^\top$$
$$(p = 0, ..., M-1)\,.$$

- Representing a vector $\boldsymbol{f}$ in this *discrete Fourier basis* yields

$$\boldsymbol{f} \;=\; \sum_{p=0}^{M-1} \langle \boldsymbol{f}, \boldsymbol{b}_p \rangle \, \boldsymbol{b}_p.$$

- The DFT computes the Fourier coefficients $\hat{f}_p := \langle \boldsymbol{f}, \boldsymbol{b}_p \rangle$ for $p = 0, ..., M-1$.
  The inverse DFT reconstructs the signal from these coefficients via $\boldsymbol{f} = \sum_{p=0}^{M-1} \hat{f}_p \, \boldsymbol{b}_p$.

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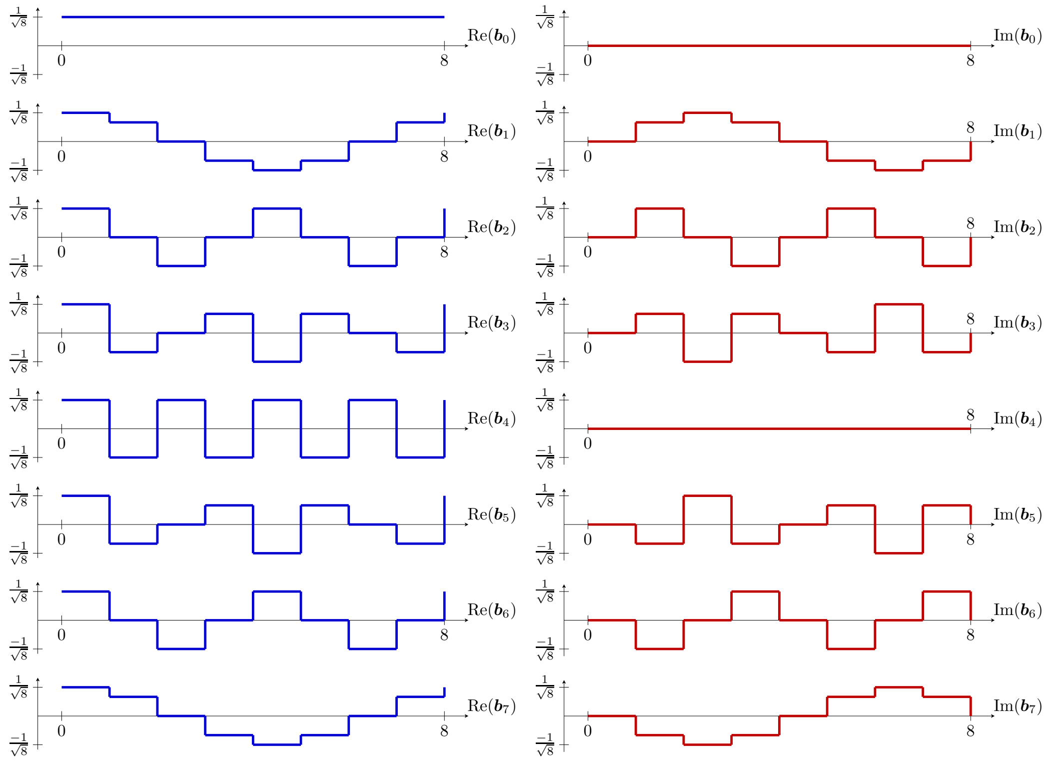

Signal representation of the basis vectors of the DFT for $M = 8$. **Left:** Real part (cosine). **Right:** Imaginary part (sine). Author: T. Schneevoigt.

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Example: DFT of $\boldsymbol{f} = (6, 4, 5, 1)^\top$

For $M = 4$, the Fourier basis vectors are given by

$$
\begin{aligned}
\boldsymbol{b}_0 &= \tfrac{1}{2}\left(e^{i0\frac{\pi}{2}0},\, e^{i0\frac{\pi}{2}1},\, e^{i0\frac{\pi}{2}2},\, e^{i0\frac{\pi}{2}3}\right)^\top &= \tfrac{1}{2}\left(1,\, 1,\, 1,\, 1\right)^\top, \\
\boldsymbol{b}_1 &= \tfrac{1}{2}\left(e^{i1\frac{\pi}{2}0},\, e^{i1\frac{\pi}{2}1},\, e^{i1\frac{\pi}{2}2},\, e^{i1\frac{\pi}{2}3}\right)^\top &= \tfrac{1}{2}\left(1,\, i,\, -1,\, -i\right)^\top, \\
\boldsymbol{b}_2 &= \tfrac{1}{2}\left(e^{i2\frac{\pi}{2}0},\, e^{i2\frac{\pi}{2}1},\, e^{i2\frac{\pi}{2}2},\, e^{i2\frac{\pi}{2}3}\right)^\top &= \tfrac{1}{2}\left(1,\, -1,\, 1,\, -1\right)^\top, \\
\boldsymbol{b}_3 &= \tfrac{1}{2}\left(e^{i3\frac{\pi}{2}0},\, e^{i3\frac{\pi}{2}1},\, e^{i3\frac{\pi}{2}2},\, e^{i3\frac{\pi}{2}3}\right)^\top &= \tfrac{1}{2}\left(1,\, -i,\, -1,\, i\right)^\top.
\end{aligned}
$$

You can check that they are orthonormal w.r.t. the Hermitian inner product

$$
\langle \boldsymbol{f}, \boldsymbol{g} \rangle \;:=\; \boldsymbol{f}^\top \bar{\boldsymbol{g}} \;=\; \sum_{m=0}^{3} f_m\, \bar{g}_m\,.
$$

Do not forget to take the complex conjugate in the second argument!

With this inner product, the Fourier coefficients of $\boldsymbol{f} = (6,4,5,1)^\top$ are given by

$$\hat{f}_0 = \langle \boldsymbol{f}, \boldsymbol{b}_0 \rangle = \boldsymbol{f}^\top \bar{\boldsymbol{b}}_0 = \tfrac{1}{2}\Big(6\cdot 1 + 4\cdot 1 + 5\cdot 1 + 1\cdot 1\Big) = 8\,,$$

$$\hat{f}_1 = \langle \boldsymbol{f}, \boldsymbol{b}_1 \rangle = \boldsymbol{f}^\top \bar{\boldsymbol{b}}_1 = \tfrac{1}{2}\Big(6\cdot 1 + 4\cdot(-i) + 5\cdot(-1) + 1\cdot i\Big) = \tfrac{1}{2} - \tfrac{3}{2}\,i\,,$$

$$\hat{f}_2 = \langle \boldsymbol{f}, \boldsymbol{b}_2 \rangle = \boldsymbol{f}^\top \bar{\boldsymbol{b}}_2 = \tfrac{1}{2}\Big(6\cdot 1 + 4\cdot(-1) + 5\cdot 1 + 1\cdot(-1)\Big) = 3\,,$$

$$\hat{f}_3 = \langle \boldsymbol{f}, \boldsymbol{b}_3 \rangle = \boldsymbol{f}^\top \bar{\boldsymbol{b}}_3 = \tfrac{1}{2}\Big(6\cdot 1 + 4\cdot i + 5\cdot(-1) + 1\cdot(-i)\Big) = \tfrac{1}{2} + \tfrac{3}{2}\,i\,.$$

This is the discrete Fourier transform. It transforms the coefficients $(f_0, ..., f_3)^\top$ in the canonical basis to the coefficients $(\hat{f}_0, ..., \hat{f}_3)^\top$ in the Fourier basis.

By plugging in, one can check that

$$\boldsymbol{f} = \hat{f}_0\,\boldsymbol{b}_0 + \hat{f}_1\,\boldsymbol{b}_1 + \hat{f}_2\,\boldsymbol{b}_2 + \hat{f}_3\,\boldsymbol{b}_3\,.$$

This is the inverse discrete Fourier transform. It maps the coefficients $(\hat{f}_0, ..., \hat{f}_3)^\top$ in the Fourier basis to the coefficients $(f_0, ..., f_3)^\top$ in the canonical basis.

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1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33 34 35

Remarks

The previous example illustrates some general properties of the DFT:

- $\hat{f}_0 = \frac{1}{\sqrt{M}} \sum\limits_{m=0}^{M-1} f_m$ is $\sqrt{M}$ times the average grey value $\frac{1}{M} \sum\limits_{m=0}^{M-1} f_m\,.$
- $\mathrm{Re}(\hat{f}_p)$ is an even function in p with respect to the index $M/2$:

$$\mathrm{Re}(\hat{f}_1) = \mathrm{Re}(\hat{f}_3)$$

- $\mathrm{Im}(\hat{f}_p)$ is an odd function in p with respect to the index $M/2$:

$$\mathrm{Im}(\hat{f}_1) = -\mathrm{Im}(\hat{f}_3).$$

It vanishes for $p = M/2$:

$$\mathrm{Im}(\hat{f}_2) = 0.$$

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Discrete Fourier Transform in 2-D

- Consider a discrete, image $\boldsymbol{f} = (f_{m,n})$ with $m = 0, ..., M-1$ and $n = 0, ..., N-1$. Its *discrete Fourier transform* is given by

$$\hat{f}_{p,q} \quad := \quad \frac{1}{\sqrt{MN}} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f_{m,n} \exp\left(-\frac{i2\pi pm}{M}\right) \exp\left(-\frac{i2\pi qn}{N}\right)$$

$$(p = 0, ..., M-1; \quad q = 0, ..., N-1)\,.$$

- The corresponding *discrete inverse Fourier transform* is defined as

$$f_{n,m} \quad = \quad \frac{1}{\sqrt{MN}} \sum_{p=0}^{M-1} \sum_{q=0}^{N-1} \hat{f}_{p,q} \exp\left(\frac{i2\pi pm}{M}\right) \exp\left(\frac{i2\pi qn}{N}\right)$$

$$(n = 0, ..., M-1; \quad m = 0, ..., N-1)\,.$$

In higher dimensions, the DFT is defined in an analogue way.
Just like the continuous FT, the DFT is separable.

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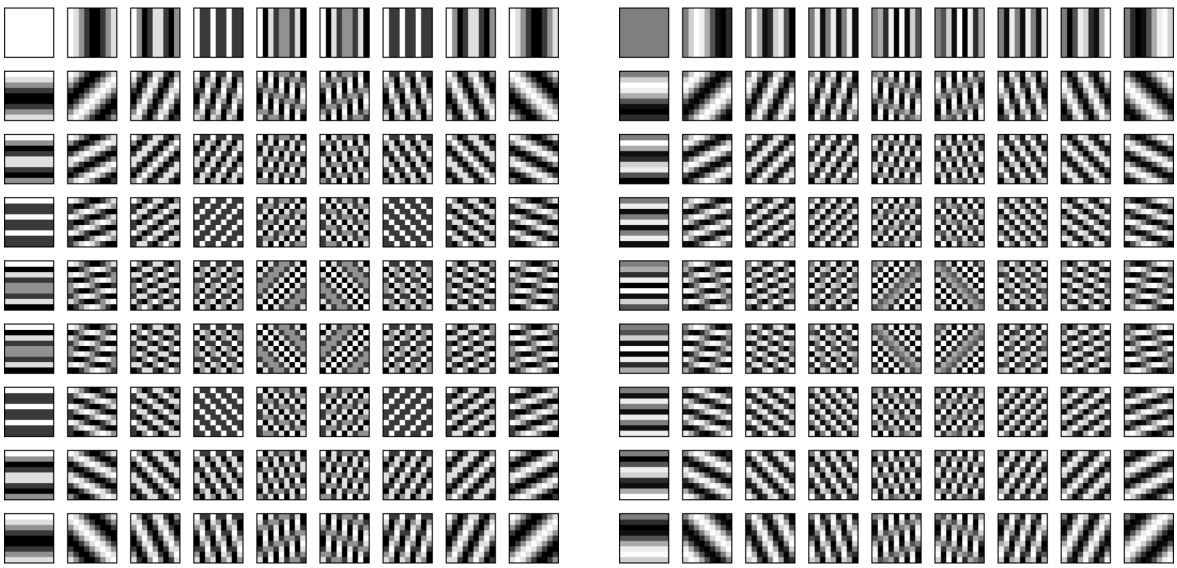


The 81 basis vectors of the DFT for $M = N = 9$. **Left:** Real part (cosine). **Right:** Imaginary part (sine). Author: T. Schneevoigt.

Properties of the Discrete Fourier Transform

Many important properties of the continuous FT carry over to the discrete FT:

- linearity
- shift theorem (when signal is extended periodically)
- convolution theorem

Some properties, however, can only be approximated on a discrete grid:

- scaling theorem
- rotation invariance

Often one uses the continuous FT for designing filters,
and the discrete FT for implementing them.

Translation of the Fourier Spectrum

- **Problem:** The DFT yields segment with frequencies in $[0, M-1] \times [0, N-1]$.
  It would be nice to shift the origin of the spectrum to the centre $(\frac{M}{2}, \frac{N}{2})$.
  Then the DFT looks more similar to the continuous FT.
- discrete shift theorem gives transform pairs

$$f_{m-m_0,\,n-n_0} \quad \circ\!\!-\!\!\bullet \quad \hat{f}_{p,q} \exp\left(-\frac{i2\pi p m_0}{M}\right) \exp\left(-\frac{i2\pi q n_0}{N}\right)$$

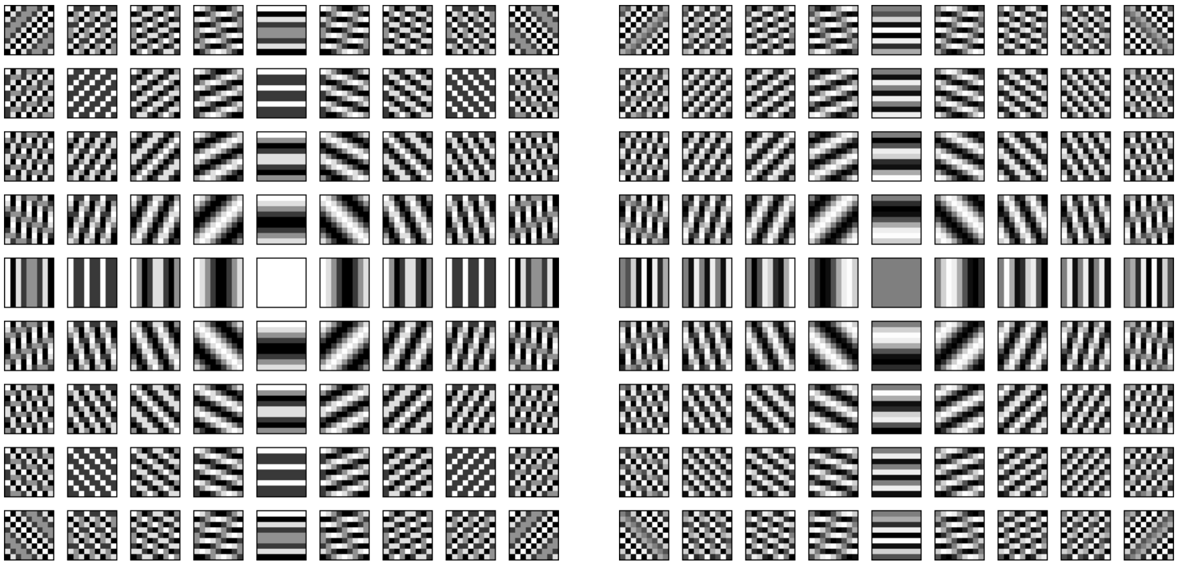
$$f_{m,n} \exp\left(\frac{i2\pi p_0 m}{M}\right) \exp\left(\frac{i2\pi q_0 n}{N}\right) \quad \circ\!\!-\!\!\bullet \quad \hat{f}_{p-p_0,\,q-q_0}$$

- With $p_0 = \frac{M}{2}$ and $q_0 = \frac{N}{2}$, one replaces the image $f_{m,n}$ by

$$f_{m,n} \exp\left(\frac{i2\pi M m}{2M}\right) \exp\left(\frac{i2\pi N n}{2N}\right) = f_{m,n}\, e^{i\pi(m+n)} = f_{m,n}(-1)^{m+n}.$$

- Thus, all one has to do is to multiply $\boldsymbol{f}$ with a checkerboard-like sign pattern.

Properties of the Discrete Fourier Transform (3)



The 81 basis vectors of the shifted DFT for $M = N = 9$. The shift moves the low frequencies to the centre. **Left:** Real part (cosine). **Right:** Imaginary part (sine). Author: T. Schneevoigt.

Properties of the Discrete Fourier Transform (4)

Logarithmic Scaling of the Fourier Spectrum

- **Problem:** The range of the Fourier spectrum covers many orders of magnitude.
- Thus, for visualisation purposes one often uses a logarithmic transformation:

$$D_{p,q} \;=\; c \ln \left(1 + |\hat{f}_{p,q}|\right) .$$

  Adding 1 ensures that the result of the logarithm is nonnegative.

- Usually c is chosen such that

$$\max_{p,q} D_{p,q} = 255.$$

  This allows a convenient visualisation of the result, since its range is in $[0, 255]$.

- *This logarithmic transformation is so common that often people simply forget to tell you that they have used it.*

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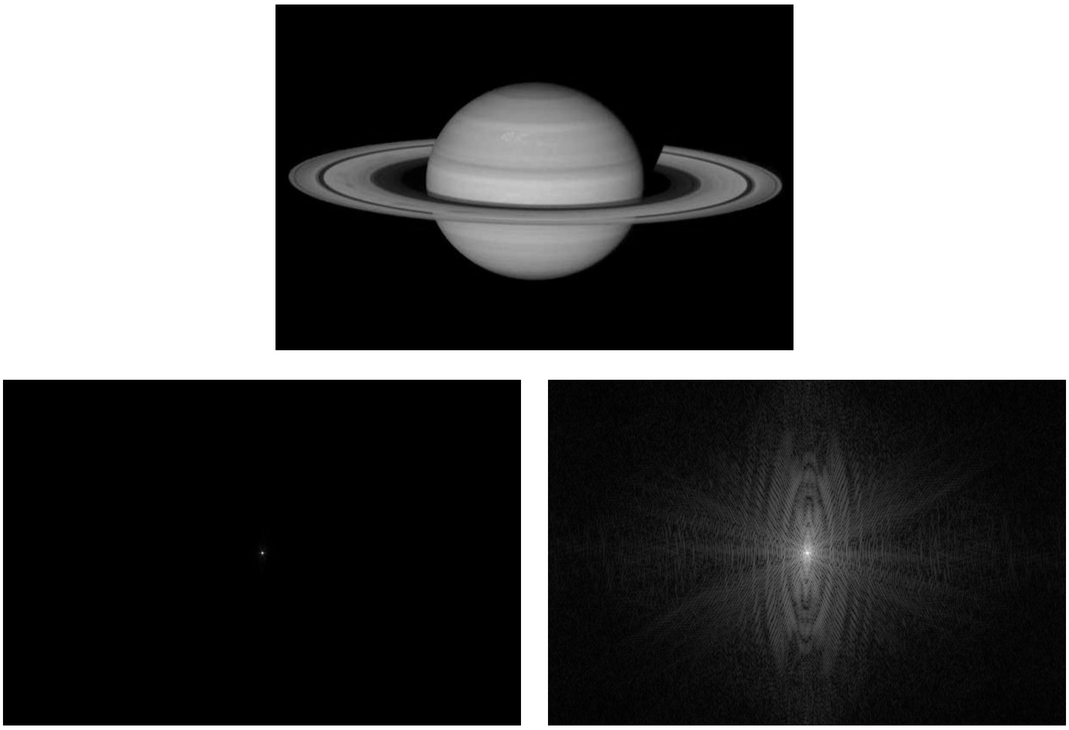

Top: Image of the planet Saturn, 600×400 pixels (`www.androidworld.com/Saturn.jpeg`). **Bottom left:** Fourier spectrum scaled to $[0, 255]$. **Bottom right:** Fourier spectrum after logarithmic scaling with $\max D_{p,q} = 255$. Author: J. Weickert.

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Boundary Artifacts

- Fundamental difference between the continuous and the discrete FT:
  For the DFT, the signal $\boldsymbol{f}$ has *finite* extent: $f_0, \ldots, f_{M-1}$.

- The periodicity of the complex exponential function automatically creates a *periodic continuation* of the image in its Fourier and its spatial representation (see the definitions on Page 21):

$$
\begin{aligned}
\hat{f}_{p,q} &= \hat{f}_{p+kM,\, q+\ell N} \qquad (k, \ell \in \mathbb{Z}), \\
f_{n,m} &= f_{n+kM,\, m+\ell N} \qquad (k, \ell \in \mathbb{Z}).
\end{aligned}
$$

  This can create undesired boundary artifacts.

- **Example 1:**
  Discontinuities at periodically extended boundaries create high-frequent Fourier components in x- and y-direction.

- **Example 2:**
  Wraparound errors in connection with convolutions:
  Grey values near the right boundary perturb grey values at the left boundary.

Boundary Artifacts (2)

How can These Artifacts be Handled ?

- **Fatalism**

  Do nothing and trust only your convolution results far away from the boundaries.
  Disadvantage: For large convolution kernels, not many results are trustworthy.
  Not recommended.

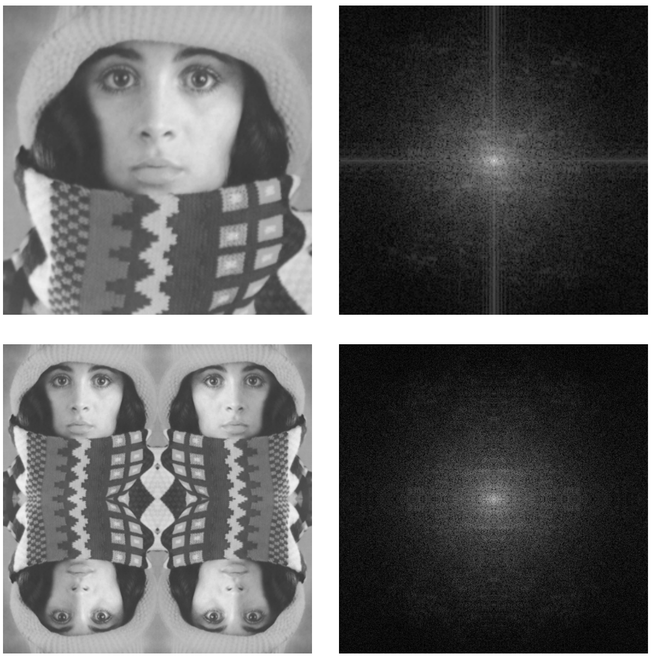
- **Zero Padding**

  Supplement a layer of zeroes at the boundaries whose thickness respects the size of the convolution kernel.
  Disadvantage: Also zeroes can spoil your signal !
  Not recommended.

- **Mirror Image at Boundaries**

  cleanest solution
  Disadvantage: The computational load increases substantially by doubling the signal length in each dimension.
  Recommended.

Boundary Artifacts (3)



Top left: Original image, 256×256 pixels. **Top right:** Its logarithmic DFT spectrum, size 256×256. Note the horizontal and vertical artifacts due to discontinuities in the periodic extension. **Bottom left:** Mirrored image extension, 512×512 pixels. It does not have discontinuities at the image boundaries. **Bottom right:** Its logarithmic DFT spectrum has size 512×512 and does not show artifacts. Author: J. Weickert.

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The Fast Fourier Transform (FFT)

(Gauß 1805; Cooley/Tukey 1965)

- A litteral implementation of 1-D DFT of a signal of length M is quite expensive: M^2 (complex-valued) multiplications and $M^2 - M$ (complex-valued) additions
- Basic idea behind the *Fast Fourier Transform (FFT):* divide-and-conquer.
  - split problem of size M into two subproblems of size $\frac{M}{2}$
  - continue until size 1 is reached
- Advantages:
  - very efficient: $\mathcal{O}(M \log_2 M)$ operations
  - available in many numerical packages (see e.g. `www.fftw.org`)
- Disadvantage:
  - standard FFT requires signals of size $M = 2^k$
- For 2-D images, one exploits the separability of the DFT:
  - hardly additional memory requirements
  - well-suited for parallel computing

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Left: James W. Cooley (1926–2016) was an applied mathematician who has popularised the FFT in 1965, in a joint publication with John Wilder Tukey. He perfomed the indexing and implementation work. Source: `http://www.ieeeghn.org/wiki/index.php/Oral-History:James_W._Cooley`. **Middle:** The mathematician John Wilder Tukey (1915–2000) was one of the pioneers of robust statistics (leading also to median filtering), and he has coined the words *bit* and *software*. Source: `http://www-history.mcs.st-and.ac.uk/PictDisplay/Tukey.html`. **Right:** Carl Friedrich Gauß (1777–1855) is generally considered as one of the most brilliant mathematicians of all times. Already in 1805, Gauß used the FFT for some of his astronomical computations. Source: `http://de.wikipedia.org/wiki/Bild:Carl_Friedrich_Gauss.jpg`.

The Inverse DFT

- For computing the inverse DFT, no second algorithm is needed:
  - Just replace the Fourier coefficients $\hat{f}_p$ by their complex conjugates $\bar{\hat{f}}_p$.
  - Apply the DFT to these numbers.

- Explanation: Computing the inverse DFT

$$f_m = \frac{1}{\sqrt{M}} \sum_{p=0}^{M-1} \hat{f}_p \exp\left(\frac{i2\pi pm}{M}\right)$$

and taking its complex conjugate gives

$$\bar{f}_m = \frac{1}{\sqrt{M}} \sum_{p=0}^{M-1} \bar{\hat{f}}_p \exp\left(-\frac{i2\pi pm}{M}\right).$$

Note that $\bar{f}_m = f_m$ for real-valued images.

Summary

- A frequency must be sampled more than twice per period in order to avoid aliasing.
- The discrete Fourier transform (DFT) decomposes a discrete signal of size M into M frequency components.
- similar properties as continuous FT:
  complex-valued, linear, separable, shift theorem, convolution theorem
- main difference: finite signal size introduces periodic signal extension.
  This can create problems such as wraparound errors (remedy: mirroring).
- The Fast Fourier Transform (FFT) allows efficient computation of the DFT.
  In 1D its complexity is $\mathcal{O}(M \log_2 M)$.

References

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  (an entire book on aliasing!)
- I. Amidror: *The Theory of the Moiré Phenomenon. Volume II: Aperiodic Layers*. Springer, Dordrecht, 2007.
  (further extensions on aliasing)
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  (An entertaining article on the past, the presence and the future of the FFT.)