Lecture 7:
Image Transformations IV:
The Discrete Wavelet Transform

Contents

Motivation

Pros and Cons of Previous Methods

- representation in the spatial domain:
  - optimal spatial localisation
  - no direct access to frequencies or scales
- Fourier transform and discrete cosine transform:
  - optimal resolution with respect to the frequencies
  - no direct access to the localisation of structures
- Laplacian pyramid is a compromise:
  - splits image into frequency bands
  - good localisation at fine scales, bad localisation at coarse scales

  However, it is redundant: requires more space than the original image.

Is there a more compact representation with localisation both in space and frequency ?

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1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32

Motivation (2)

Example: Signal Representation in Another Basis

Represent the signal $\boldsymbol{f} = (6,4,5,1)^\top$ in the following orthonormal basis of $\mathbb{R}^4$ with respect to the Euclidean inner product $\langle \boldsymbol{f}, \boldsymbol{g} \rangle = \boldsymbol{f}^\top \boldsymbol{g}$:

$$\begin{aligned}
\boldsymbol{b}_1 &:= \frac{1}{2}\Big(1,\ 1,\ 1,\ 1\Big)^\top, \\
\boldsymbol{b}_2 &:= \frac{1}{2}\Big(1,\ 1,\ -1,\ -1\Big)^\top, \\
\boldsymbol{b}_3 &:= \frac{1}{\sqrt{2}}\Big(1,\ -1,\ 0,\ 0\Big)^\top, \\
\boldsymbol{b}_4 &:= \frac{1}{\sqrt{2}}\Big(0,\ 0,\ 1,\ -1\Big)^\top.
\end{aligned}$$

Motivation (3)

In the representation $\boldsymbol{f} = \sum_{i=1}^{4} \alpha_i \boldsymbol{b}_i$ the coefficients α_i are given by

$$\begin{aligned}
\alpha_1 &= \boldsymbol{f}^\top \boldsymbol{b}_1 = \frac{1}{2}\Big(6\cdot 1 + 4\cdot 1 + 5\cdot 1 + 1\cdot 1\Big) = 8\,, \\
\alpha_2 &= \boldsymbol{f}^\top \boldsymbol{b}_2 = \frac{1}{2}\Big(6\cdot 1 + 4\cdot 1 - 5\cdot 1 - 1\cdot 1\Big) = 2\,, \\
\alpha_3 &= \boldsymbol{f}^\top \boldsymbol{b}_3 = \frac{1}{\sqrt{2}}\Big(6\cdot 1 - 4\cdot 1 + 5\cdot 0 + 1\cdot 0\Big) = \frac{2}{\sqrt{2}}\,, \\
\alpha_4 &= \boldsymbol{f}^\top \boldsymbol{b}_4 = \frac{1}{\sqrt{2}}\Big(6\cdot 0 + 4\cdot 0 + 5\cdot 1 - 1\cdot 1\Big) = \frac{4}{\sqrt{2}}\,.
\end{aligned}$$

These coefficients have the following interpretations:

- α_1: rescaled average grey value
- $|\alpha_2|$: contribution to low frequencies (without localisation)
- $|\alpha_3|$: high frequency contribution in the left half of the signal
- $|\alpha_4|$: high frequency contribution in the right half of the signal

The 1-D Haar Wavelet

General Idea Behind a Wavelet Basis

- A localised, wave-like function with mean 0 *(mother wavelet, Mutterwavelet)* is scaled and shifted.
- Besides these functions, one additional basis function with non-vanishing mean is needed to represent the average grey value *(scaling function, Skalierungsfunktion)*.


Alfréd Haar (1885–1933) was a Hungarian mathematician who studied in Göttingen. In his Ph.D. thesis that was supervised by David Hilbert he introduced the first wavelet concepts. He made a number of significant contributions to the field of calculus. **Left:** Photo of Alfréd Haar. Source: `http://www-history.mcs.st-andrews.ac.uk/PictDisplay/Haar.html`. **Right:** Photo of Alfréd Haar (left) together with the mathematicians Hermann Minkowski (front) and David Hilbert (right). Source: `http://www.goettingen.de/pics/medien//big_image_12125684243613.jpeg`.

The 1-D Haar Wavelet (3)

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The Continuous Haar Wavelet

- simplest wavelet (Alfréd Haar, 1910)
- uses the mother wavelet

$$\Psi(x) \;:=\; \begin{cases} 1 & \text{for } 0 \le x \le \frac{1}{2}, \\ -1 & \text{for } \frac{1}{2} < x \le 1, \\ 0 & \text{else.} \end{cases}$$

- consider scaled and shifted versions:

$$\Psi_{j,k}(x) \;:=\; \frac{1}{2^{j/2}}\,\Psi\left(\frac{x}{2^j} - k\right) \;=\; \frac{1}{2^{j/2}}\,\Psi\left(\frac{x - k\,2^j}{2^j}\right).$$

$\Psi_{j,k}$ has range $\left[-\frac{1}{2^{j/2}}, \frac{1}{2^{j/2}}\right]$, width 2^j, and starts at $k\,2^j$.
The *scale* level is specified by j, the *shift* by k.
Finer scales correspond to smaller scale levels j.

The 1-D Haar Wavelet (4)

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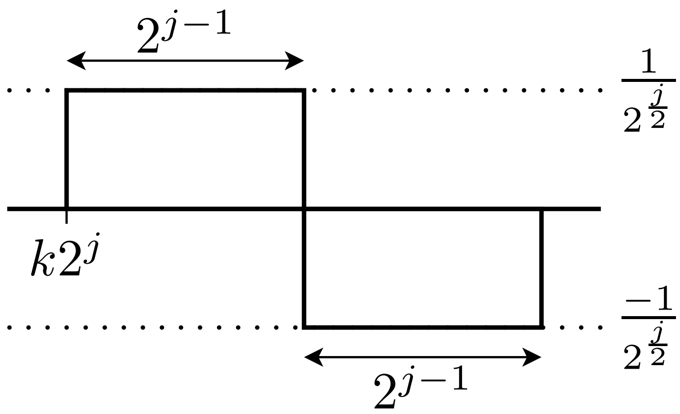


Haar wavelet $\Psi_{j,k}$. Author: M. Mainberger.

The 1-D Haar Wavelet (5)

- The factor $\frac{1}{2^{j/2}}$ guarantees that $\Psi_{j,k}$ has norm 1,

$$\|\Psi_{j,k}\| \;:=\; \sqrt{\langle \Psi_{j,k}, \Psi_{j,k} \rangle} \;=\; 1,$$

in the space of quadratically (Lebesgue) integrable functions

$$L^2(\mathbb{R}) \;:=\; \left\{ f : \mathbb{R} \to \mathbb{R} \;\middle|\; \int_{-\infty}^{\infty} |f(x)|^2 \, dx < \infty \right\}$$

with the inner product

$$\langle f, g \rangle \;:=\; \int_{-\infty}^{\infty} f(x)\, g(x)\, dx.$$

- If j and k are integer numbers, the Haar wavelets $\{\Psi_{j,k}\}$ are even orthonormal:

$$\langle \Psi_{j,k} \,,\, \Psi_{n,m} \rangle \;=\; \begin{cases} 1 & \text{for } (j,k) = (n,m), \\ 0 & \text{else.} \end{cases}$$

The 1-D Haar Wavelet (6)

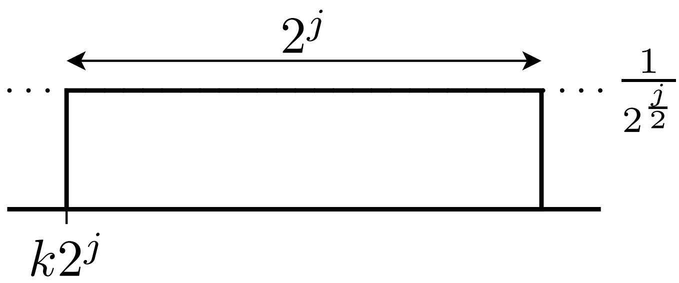
- As scaling function one chooses a box function:

$$\Phi(x) \;:=\; \begin{cases} 1 & \text{for } 0 \le x \le 1, \\ 0 & \text{else.} \end{cases}$$

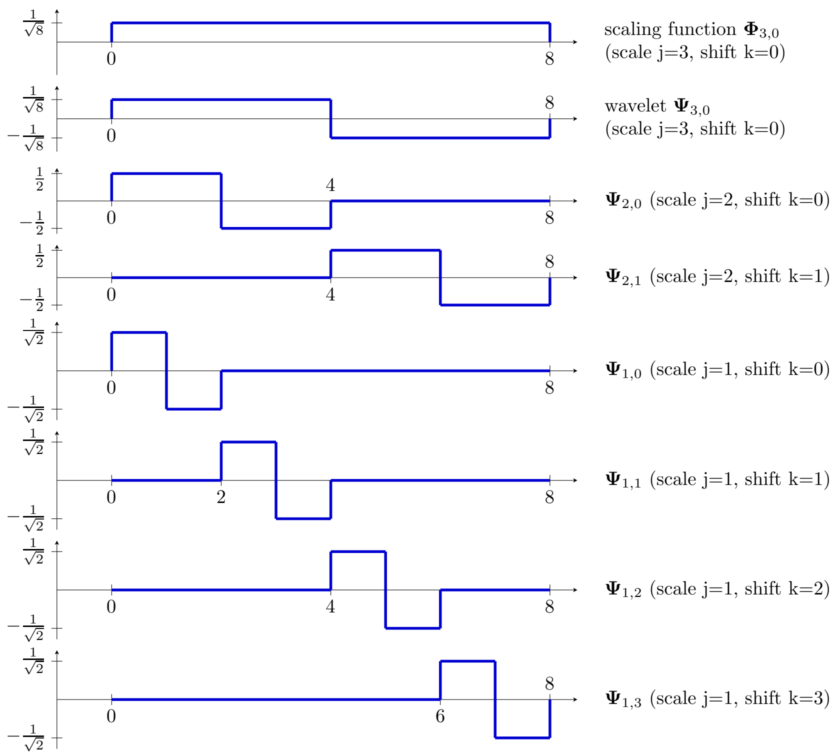
- It may also be scaled and shifted:

$$\Phi_{j,k}(x) \;:=\; \frac{1}{2^{j/2}}\, \Phi\left(\frac{x}{2^j} - k \right) \;=\; \frac{1}{2^{j/2}}\, \Phi\left(\frac{x - k\,2^j}{2^j} \right).$$

$\Phi_{j,k}$ has range $\left[0, \frac{1}{2^{j/2}}\right]$, width 2^j, and starts at $k\,2^j$.



Scaling function $\Phi_{j,k}$. Author: M. Mainberger.



Scaling function and Haar wavelets on the interval $[0, 8] = [0, 2^n]$ with $n = 3$. **From top to bottom:** Scaling function $\Phi_{3,0}$, wavelets $\Psi_{3,0}$, $\Psi_{2,0}$, $\Psi_{2,1}$, $\Psi_{1,0}$, $\Psi_{1,1}$, $\Psi_{1,2}$, $\Psi_{1,3}$. These eight orthonormal functions allow to represent every discrete signal with eight components. Author: T. Schneevoigt.

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The Discrete Haar Wavelet

- The previous model was continuous.
  This helped us to describe scaling and shifting in a more intuitive way.
  However, discrete signals require discrete models.

- For a discrete signal of length $N = 2^n$ one considers the N functions

$\Phi_{n,0}$,	scaling function
$\Psi_{n,0}$,	lowest frequency, unlocalised
$\Psi_{n-1,0}$, $\Psi_{n-1,1}$	second lowest frequency, at 2 locations
$\vdots$ $\ddots$	
$\Psi_{1,0}$, $\Psi_{1,1}$, ... ,$\Psi_{1,2^{n-1}-1}$	highest frequency, at 2^{n-1} locations.

  The example on the previous slide illustrates the case $n = 3$ and $N = 2^n = 8$.

- Sampling at N equidistant grid points $\{\frac{1}{2}, \frac{3}{2}, ..., N-\frac{1}{2}\}$ creates an orthonormal basis of $\mathbb{R}^N$.

- Thus, we can identify the piecewise constant functions $\Psi_{j,k}(x)$ and $\Phi_{j,k}(x)$ with the basis vectors $\boldsymbol{\Psi}_{j,k}$ and $\boldsymbol{\Phi}_{j,k}$.

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Example

- The example on Page 3 was a discrete Haar wavelet transform with $n = 2$:
  We used the basis vectors

$$\boldsymbol{\Phi}_{2,0} = \tfrac{1}{2}\,(1,1,1,1)^\top,$$
$$\boldsymbol{\Psi}_{2,0} = \tfrac{1}{2}\,(1,1,-1,-1)^\top,$$
$$\boldsymbol{\Psi}_{1,0} = \tfrac{1}{\sqrt{2}}\,(1,-1,0,0)^\top, \qquad \boldsymbol{\Psi}_{1,1} = \tfrac{1}{\sqrt{2}}\,(0,0,1,-1)^\top.$$

- The discrete Haar wavelet transform of the signal $\boldsymbol{f} = (6,4,5,1)^\top$ maps the coefficients in the canonical basis to the coefficients in the wavelet basis:

$$\begin{aligned} \boldsymbol{f} &= 6\,\boldsymbol{e}_1 + 4\,\boldsymbol{e}_2 + 5\,\boldsymbol{e}_3 + 1\,\boldsymbol{e}_4 \\ &= 8\,\boldsymbol{\Phi}_{2,0} + 2\,\boldsymbol{\Psi}_{2,0} + \tfrac{2}{\sqrt{2}}\,\boldsymbol{\Psi}_{1,0} + \tfrac{4}{\sqrt{2}}\,\boldsymbol{\Psi}_{1,1} \end{aligned}$$

  where $\boldsymbol{e}_i$ is 1 in its i-th component and 0 elsewhere.

- In practice, the basis vectors $\boldsymbol{\Phi}_{2,0}$, $\boldsymbol{\Psi}_{2,0}$, $\boldsymbol{\Psi}_{1,0}$, and $\boldsymbol{\Psi}_{1,1}$ are known.
  Thus, we only have to store the four coefficients 8, 2, $\frac{2}{\sqrt{2}}$, and $\frac{4}{\sqrt{2}}$.

Widely Used Conventions for the Coefficients

- $c_{j,k} \;:=\; \boldsymbol{f}^\top \boldsymbol{\Phi}_{j,k}$: coefficient of the scaling vector $\boldsymbol{\Phi}_{j,k}$ (c like *coarse*)
- $d_{j,k} \;:=\; \boldsymbol{f}^\top \boldsymbol{\Psi}_{j,k}$: coefficient for the wavelet vector $\boldsymbol{\Psi}_{j,k}$ (d like *detail*)
- The coefficients are stored and transmitted in a coarse-to-fine manner:

  $c_{n,0} \;\;|\;\; d_{n,0} \;\;|\;\; d_{n-1,0} \quad d_{n-1,1} \;\;|\;\; \ldots \;\;|\;\; d_{1,0} \quad \ldots \quad d_{1,2^{n-1}-1}$

  This allows to refine the reconstruction during data transmission.
- These coefficients carry the full information of the Haar wavelet transformation. There is no need to transmit the basis vectors.

The Fast Wavelet Transform

Important Practical Aspect

- naive implementation of the discrete wavelet transform: requires $\mathcal{O}(N^2)$ operations for a signal of length N.
- Are there algorithms with lower complexity?

The Fast Wavelet Transform (2)

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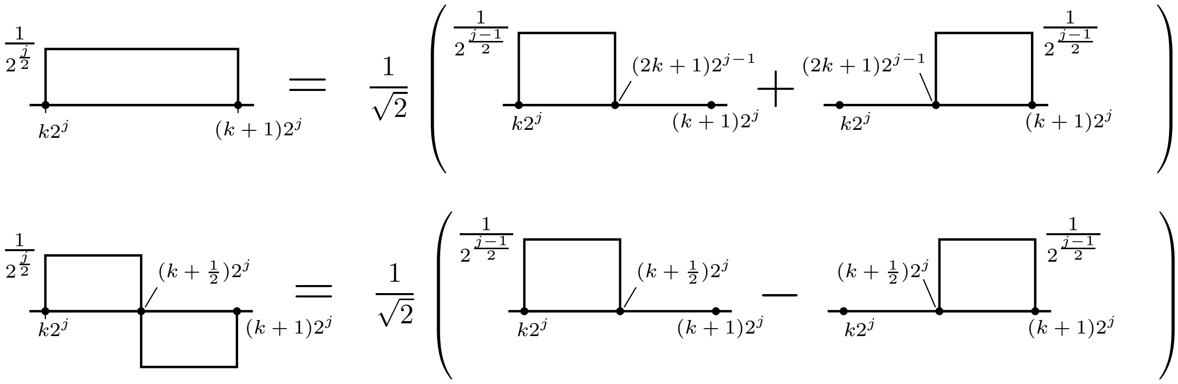
The Fast Wavelet Transform (FWT)



Basic idea behind the Fast Wavelet Transform: The scaling functions and wavelets at the coarser scale j can be expressed by scaling functions at the finer scale $j-1$. **Top:** Visualisation of the formula $\boldsymbol{\Phi}_{j,k} = \frac{1}{\sqrt{2}}(\boldsymbol{\Phi}_{j-1,\,2k} + \boldsymbol{\Phi}_{j-1,\,2k+1})$. **Bottom:** Visualisation of $\boldsymbol{\Psi}_{j,k} = \frac{1}{\sqrt{2}}(\boldsymbol{\Phi}_{j-1,\,2k} - \boldsymbol{\Phi}_{j-1,\,2k+1})$. Author: M. Mainberger.

The Fast Wavelet Transform (3)

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- The definitions of the Haar wavelets and the scaling functions allow to express the vectors with coarser scale j by vectors with finer scale $j-1$:

$$\boldsymbol{\Phi}_{j,k} = \frac{1}{\sqrt{2}}\left(\boldsymbol{\Phi}_{j-1,\,2k} + \boldsymbol{\Phi}_{j-1,\,2k+1}\right) \qquad (k = 0, ..., 2^{n-j}-1),$$

$$\boldsymbol{\Psi}_{j,k} = \frac{1}{\sqrt{2}}\left(\boldsymbol{\Phi}_{j-1,\,2k} - \boldsymbol{\Phi}_{j-1,\,2k+1}\right) \qquad (k = 0, ..., 2^{n-j}-1).$$

  Moreover, at the finest scale $j = 0$ we have

$$f_k = \boldsymbol{f}^\top \boldsymbol{\Phi}_{0,k} \qquad (k = 0, ..., 2^n-1),$$

  since $\boldsymbol{\Phi}_{0,k}$ is identical to the canonical basis vector $\boldsymbol{e}_k$.

- Because of

$$c_{j,k} = \boldsymbol{f}^\top \boldsymbol{\Phi}_{j,k},$$
$$d_{j,k} = \boldsymbol{f}^\top \boldsymbol{\Psi}_{j,k},$$

  these relations for the basis vectors carry over to the coefficients.

The Fast Wavelet Transform (4)

- Thus, for the scales $j = 1, ..., n$ we compute

$$c_{j,k} = \frac{1}{\sqrt{2}}\left(c_{j-1,\,2k} + c_{j-1,\,2k+1}\right) \qquad (k = 0, ..., 2^{n-j}-1),$$

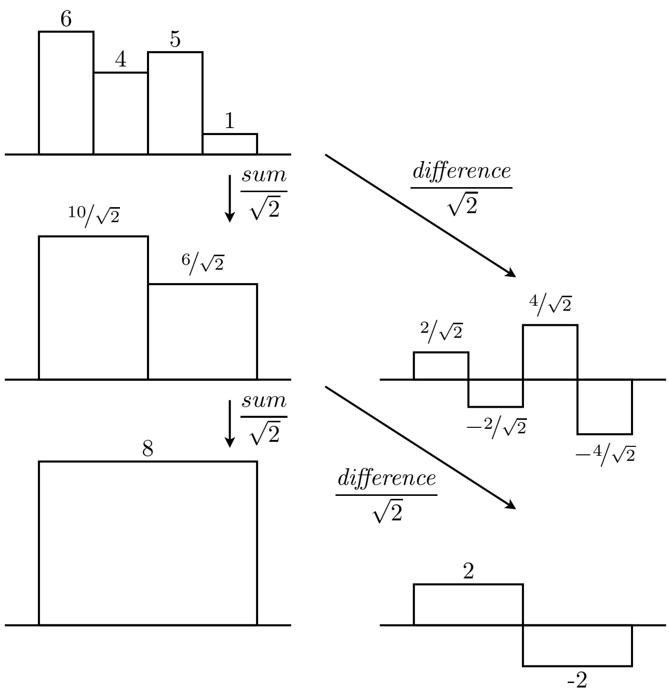
$$d_{j,k} = \frac{1}{\sqrt{2}}\left(c_{j-1,\,2k} - c_{j-1,\,2k+1}\right) \qquad (k = 0, ..., 2^{n-j}-1).$$

  starting from the finest scale $j = 0$ with the initialisation

$$c_{0,k} = f_k \qquad (k = 0, ..., 2^n-1).$$

- This fine-to-coarse algorithm is called *Fast Wavelet Transform (FWT)*.
- For a signal of length $N = 2^n$, one can show that computing the N coefficients $\{c_{n,0},\ d_{n,0}, ...,\ d_{1,2^{n-1}-1}\}$ requires only $\mathcal{O}(N)$ operations.
- Let us now see that the FWT resembles the Laplacian pyramid decomposition from Lecture 6.

The Fast Wavelet Transform (5)



Pyramid-like interpretation of the FWT of the signal $(6,4,5,1)^\top$. The left part resembles the Gaussian pyramid and gives the scaling coefficient $c_{2,0} = 8$. The right part resembles the Laplacian pyramid and yields the wavelet coefficients $d_{1,0} = \frac{2}{\sqrt{2}},\ d_{1,1} = \frac{4}{\sqrt{2}},\ d_{2,0} = 2$. Author: M. Mainberger.

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The Fast Wavelet Transform (6)

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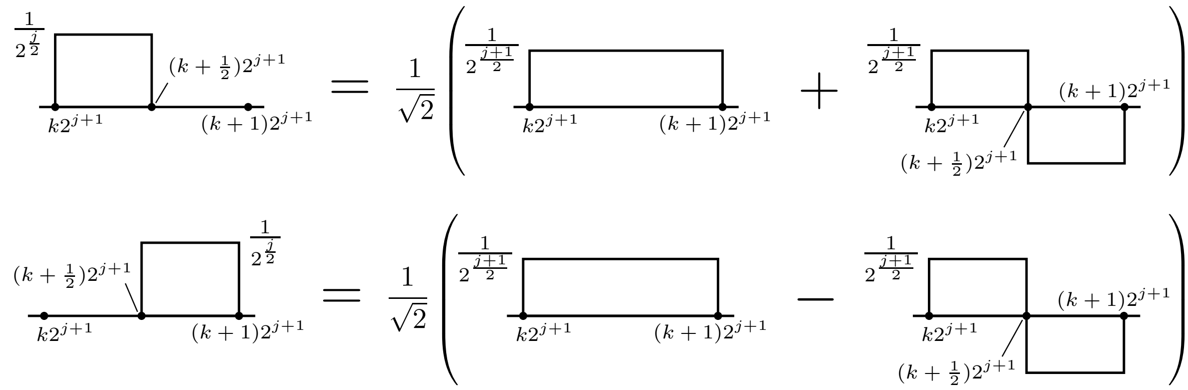
The Inverse Fast Wavelet Transformation



Basic idea behind the Inverse Fast Wavelet Transform: The scaling functions at the finer scale j can be expressed by scaling functions and wavelets at the coarser scale $j+1$. **Top:** Visualisation of the formula $\mathbf{\Phi}_{j,2k} = \frac{1}{\sqrt{2}}(\mathbf{\Phi}_{j+1,k} + \mathbf{\Psi}_{j+1,k})$. **Bottom:** Visualisation of $\mathbf{\Phi}_{j,2k+1} = \frac{1}{\sqrt{2}}(\mathbf{\Phi}_{j+1,k} - \mathbf{\Psi}_{j+1,k})$. Author: M. Mainberger.

The Fast Wavelet Transform (7)

M I A 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32

- Because of

$$\mathbf{\Phi}_{j,2k} = \frac{1}{\sqrt{2}}\left(\mathbf{\Phi}_{j+1,k} + \mathbf{\Psi}_{j+1,k}\right) \qquad (k = 0, ..., 2^{n-j-1}-1),$$

$$\mathbf{\Phi}_{j,2k+1} = \frac{1}{\sqrt{2}}\left(\mathbf{\Phi}_{j+1,k} - \mathbf{\Psi}_{j+1,k}\right) \qquad (k = 0, ..., 2^{n-j-1}-1)$$

  the inverse transformation is as simple as the forward transformation:

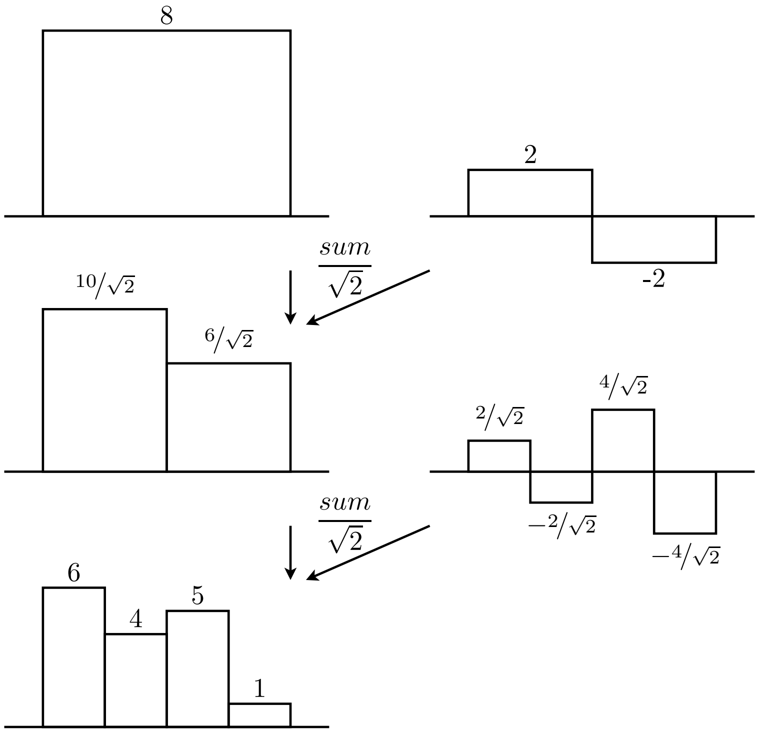
- Proceed in a coarse-to-fine manner from $j = n-1$ to $j = 0$ and compute

$$c_{j,2k} = \frac{1}{\sqrt{2}}\left(c_{j+1,k} + d_{j+1,k}\right) \qquad (k = 0, ..., 2^{n-j-1}-1),$$

$$c_{j,2k+1} = \frac{1}{\sqrt{2}}\left(c_{j+1,k} - d_{j+1,k}\right) \qquad (k = 0, ..., 2^{n-j-1}-1).$$

- Then the reconstructed signal $\boldsymbol{f}$ is given by $f_k = c_{0,k}$ for $k = 0, ..., 2^n-1$.

- resembles the reconstruction of the Gaussian pyramid and the original signal from the Laplacian pyramid



Reconstruction of the original signal $(6,4,5,1)^\top$ starting from the scaling coefficient $c_{2,0} = 8$ and the wavelet coefficients $d_{2,0} = 2,\ d_{1,0} = \frac{2}{\sqrt{2}},\ d_{1,1} = \frac{4}{\sqrt{2}}$. Author: M. Mainberger.

Wavelets versus Pyramids and Fourier Representations

- The pyramid-like algorithm behind the FWT implies the following:
  The discrete wavelet coefficients can be computed in optimal complexity: $\mathcal{O}(N)$.
  (in contrast to FFT: $\mathcal{O}(N \log_2 N)$)

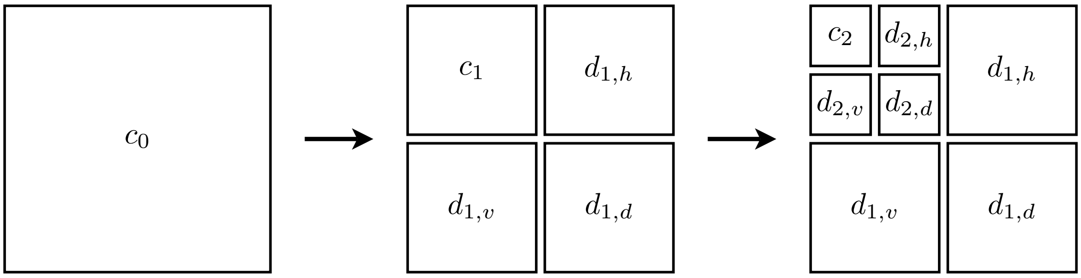
- Pyramids and the discrete wavelet transform are not shift invariant!

- Unlike pyramids, discrete wavelet representations have no redundancy:
  A signal of length N is represented by N coefficients.

The Two-Dimensional Wavelet Transform

Frequently Used (So-Called Nonstandard Decomposition)

- Start with computing the wavelet decomposition on a *single* level, first in x direction then in y direction.
- Perform the next decomposition only in the quadrant that contains the low-frequent parts (scaling coefficients) from both directions.
- Proceed until a single pixel is reached.



Distribution of the coefficients within the first two decomposition steps. Author: M. Mainberger.

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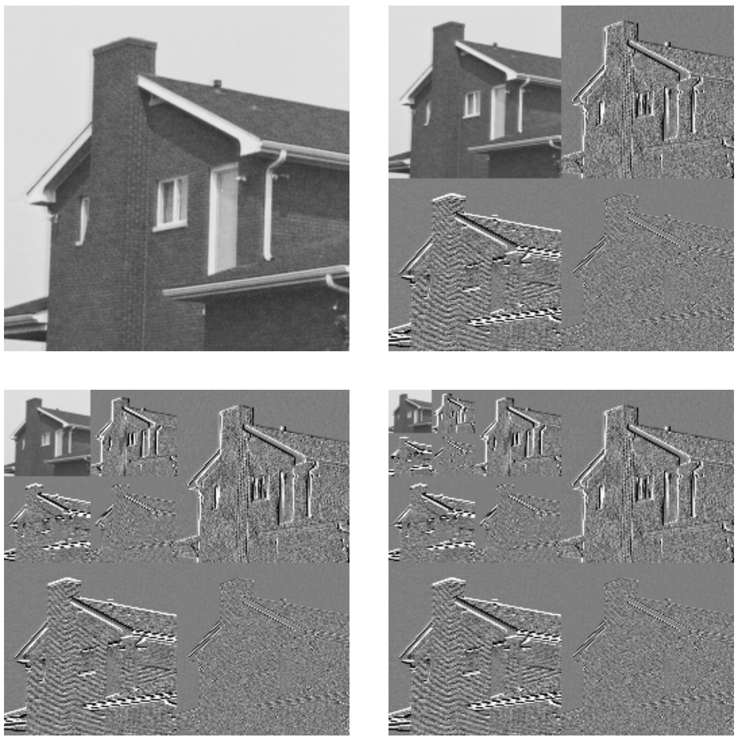
original, 256×256 pixels — decomposition step 1



decomposition step 2 — decomposition step 3

Two-dimensional nonstandard wavelet decomposition. Author: J. Weickert.

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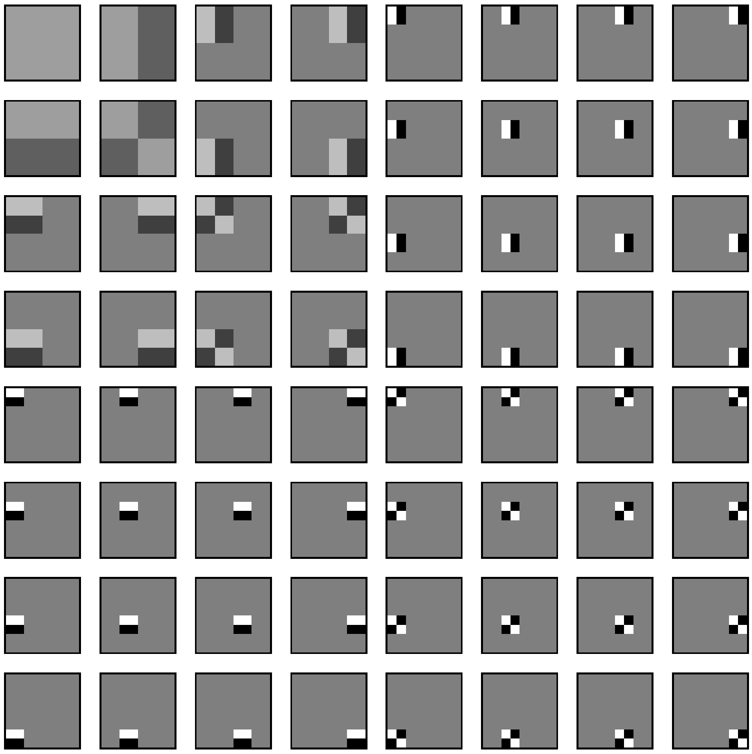

Illustration of the 64 basis vectors of the 2-D nonstandard wavelet decomposition for 8×8 images. Author: T. Schneevoigt.

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1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32

Application to Data Compression

- Images are often piecewise smooth.
- Wavelets are well localised, in particular the high-frequent ones.
- Thus, the wavelet coefficients inside each segment have small magnitude.
- They can be cancelled without severe visual degradations.
- Only a few wavelet coeffients are large in magnitude.
  They represent important structures such as edges and should be kept.
- Cancelling small wavelet coefficients is a powerful compression strategy.
  It has entered modern compression standards such as JPEG 2000.
- However, never cancel the scaling coefficient:
  It determines the average grey value.

Application to Data Compression (2)

original, 256×256 pixels

66.37 % removed

90.34 % removed

96.68 % removed

Removal of the Haar wavelet coefficients that are smallest in magnitude. Author: J. Weickert.

Summary

Summary

- Wavelets provide a signal representation that is localised in space and frequency.
- The Haar wavelet is the simplest wavelet.
- The fast wavelet transform (FWT) is similar to the Laplacian pyramid.
  It has linear complexity.
- In higher dimensions one often uses the so-called nonstandard decomposition.
- Data compression constitutes the most important wavelet application.
- In general, wavelets are neither shift invariant nor invariant under rotations.

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1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32

References

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  (Chapter 7 deals with wavelets.)
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  (contains a well-readable introduction to Haar wavelets)
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  (fairly simple introduction to wavelet concepts; in German)
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  (one of the most comprehensive books on wavelets)
- A. Haar: Zur Theorie der orthogonalen Funktionensysteme. *Mathematische Annalen*, Vol. 69, pp. 331–371, 1910.
  (introduced the Haar wavelet)

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1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32

This concludes our desert ride. The skills we have learned during this expedition will be highly useful in the subsequent lectures. Photo by Hendrik Dacquin (`https://de.wikipedia.org/wiki/Oase`).