Example Solutions for Classroom Assignment 12 (C12)

Problem 1 (Optic Flow Constraint in 1-D and 3-D)

(a) For a 1-D signal $f(x,t)$, the grey value constancy assumption can be formulated as

$$f(x+u,t+1) - f(x,t) = 0.$$

If u is small and f varies smoothly, we can perform a linearisation by means of first order Taylor expansion:

$$f(x+u,t+1) \approx f(x,t) + f_x(x,t)u + f_t(x,t).$$

Thus, we obtain the following optic flow constraint:

$$f_x(x,t)u + f_t(x,t) = 0.$$

Since one equation is sufficient to determine one unknown uniquely, no aperture problem exists in the 1-D case. In fact, we can compute u via

$$u = -\frac{f_t}{f_x}.$$

However, there are cases where u cannot be computed: At locations, where $f_x = 0$.

(b) In the case of a 3-D signal $f(x,y,z,t)$, the grey value constancy assumption is given by

$$f(x+u,y+v,z+w,t+1) - f(x,y,z,t) = 0.$$

As in the 1-D case, we can perform a linearisation of this assumptions by means of first order Taylor expansion here. To this end, we assume again that u is small and f only varies smoothly. This give us

$$\begin{aligned} f(x+u,y+v,z+w,t+1) \approx f(x,y,z,t) &+ f_x(x,y,z,t)u \\ &+ f_y(x,y,z,t)v \\ &+ f_z(x,y,z,t)w \\ &+ f_t(x,y,z,t). \end{aligned}$$

The resulting optic flow constraint then reads

$$f_x(x,y,z,t)u + f_y(x,y,z,t)v + f_z(x,y,z,t)w + f_t(x,y,z,t) = 0.$$

This time the aperture problem is present: We have three unknowns but only one equation. Taking a closer look at the previous constraint, one can see that it is actually (save for the missing normalisation) the Hesse normal form of a plane that contains all possible solutions at a point $(x, y, z, t)^\top$. This plane has unit normal vector $\frac{(f_x, f_y, f_z)^\top}{\|(f_x, f_y, f_z)^\top\|}$ and distance $\frac{f_t(x,y,z,t)}{\|(f_x, f_y, f_z)^\top\|}$ from the origin.

Let us now embed this optic flow constraints in a Bigün–like approach. Then, we obtain

$$E(\mathbf{a}) = \int\limits_{B_\rho(x_0,y_0,z_0,t_0)} \Big(f_x a_1 + f_y a_2 + f_z a_3 + f_t a_4\Big)^2 \, dx \, dy \, dz \, dt,$$

where the last component of the minimising vector has to be normalised after the computation to 1. This yields the 3-D optic flow

$$u = \frac{a_1}{a_4}, \qquad v = \frac{a_2}{a_4}, \qquad w = \frac{a_3}{a_4}.$$

As in the original method of Bigün *et al.* the approach can be formulated as a quadratic form. In the 3-D case this form is given by

$$E(\mathbf{a}) = \mathbf{a}^\top \hat{\mathbf{J}}_\rho \, \mathbf{a}\,,$$

with the 3-D spatiotemporal structure tensor

$$\hat{\mathbf{J}}_\rho = \begin{pmatrix} K_\rho * (f_x^2) & K_\rho * (f_x f_y) & K_\rho * (f_x f_z) & K_\rho * (f_x f_t) \\ K_\rho * (f_x f_y) & K_\rho * (f_y^2) & K_\rho * (f_y f_z) & K_\rho * (f_y f_t) \\ K_\rho * (f_x f_z) & K_\rho * (f_y f_z) & K_\rho * (f_z^2) & K_\rho * (f_z f_t) \\ K_\rho * (f_x f_t) & K_\rho * (f_y f_t) & K_\rho * (f_z f_t) & K_\rho * (f_t^2) \end{pmatrix}.$$

Let $\mu_1 \geq \mu_2 \geq \mu_3 \geq \mu_4 \geq 0$ be the eigenvalues of $\hat{\mathbf{J}}_\rho$. Then, we have five different cases that are processed in the following order

- **No eigenvalue is large** ($\mathrm{tr}\hat{\mathbf{J}}_\rho = \hat{j}_{11} + \hat{j}_{22} + \hat{j}_{33} + \hat{j}_{44} \leq \tau_1$): There is a *homogeneous spatiotemporal volume* and thus not sufficient local information to compute the optic flow.
- **All eigenvalues are large** ($\mu_4 \geq \tau_2$): Either the grey value constancy assumption or the assumption of a constant flow is violated.
- **One eigenvalue is large** ($\mu_2 \leq \tau_3$): There is an *edge*, the solution lies on a plane.

- **Two eigenvalues are large** ($\mu_3 \leq \tau_4$): There is a *2-D corner*, the solution lies on a line.
- **Three eigenvalues are large** (the remaining case): There is a *3-D corner*. Then, we have to compute the eigenvector to the smallest eigenvalue of $\hat{\mathbf{J}}_\rho$ and normalise it so that its last component becomes 1.
