Image Processing and Computer Vision (IPCV)

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Solutions to Self-Test Problems

Problem 1 (Fourier and Wavelet Transform)

(a) The basis vectors are given by

$$\begin{aligned}
\boldsymbol{b}_0 &= \frac{1}{2}\left(e^{i0\frac{\pi}{2}0},\, e^{i0\frac{\pi}{2}1},\, e^{i0\frac{\pi}{2}2},\, e^{i0\frac{\pi}{2}3}\right)^\top &= \frac{1}{2}\Big(1,\, 1,\, 1,\, 1\Big)^\top,\\
\boldsymbol{b}_1 &= \frac{1}{2}\left(e^{i1\frac{\pi}{2}0},\, e^{i1\frac{\pi}{2}1},\, e^{i1\frac{\pi}{2}2},\, e^{i1\frac{\pi}{2}3}\right)^\top &= \frac{1}{2}\Big(1,\, i,\, -1,\, -i\Big)^\top,\\
\boldsymbol{b}_2 &= \frac{1}{2}\left(e^{i2\frac{\pi}{2}0},\, e^{i2\frac{\pi}{2}1},\, e^{i2\frac{\pi}{2}2},\, e^{i2\frac{\pi}{2}3}\right)^\top &= \frac{1}{2}\Big(1,\, -1,\, 1,\, -1\Big)^\top,\\
\boldsymbol{b}_3 &= \frac{1}{2}\left(e^{i3\frac{\pi}{2}0},\, e^{i3\frac{\pi}{2}1},\, e^{i3\frac{\pi}{2}2},\, e^{i3\frac{\pi}{2}3}\right)^\top &= \frac{1}{2}\Big(1,\, -i,\, -1,\, i\Big)^\top.
\end{aligned}$$

We compute

$$\begin{aligned}
\hat{f}_0 &= \langle \boldsymbol{f}, \boldsymbol{b}_1 \rangle = \frac{1}{2}(7+1+2+7) = \frac{17}{2}\\
\hat{f}_1 &= \langle \boldsymbol{f}, \boldsymbol{b}_2 \rangle = \frac{1}{2}(7-i-2+7i) = \frac{1}{2}(5+6i)\\
\hat{f}_2 &= \langle \boldsymbol{f}, \boldsymbol{b}_3 \rangle = \frac{1}{2}(7-1+2-7) = \frac{1}{2}\\
\hat{f}_3 &= \langle \boldsymbol{f}, \boldsymbol{b}_4 \rangle = \frac{1}{2}(7+i-2-7i) = \frac{1}{2}(5-6i),
\end{aligned}$$

where $\langle \cdot, \cdot \rangle$ denotes the Hermitian inner product. Due to the Hermitian inner product we need to make sure we use the complex conjugate of $\boldsymbol{b}_i$ in the calculations. Thus, the discrete Fourier transform of $\boldsymbol{f}$ is

$$\hat{\boldsymbol{f}} = \frac{1}{2}(17, 5+6i, \frac{1}{2}, 5-6i)^\top.$$

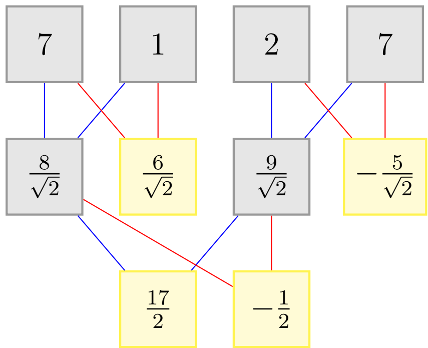
The highest frequency is represented by the coefficient $\hat{f}_2$. Elimination and back-transformation yields

$$\begin{aligned}
\tilde{\boldsymbol{f}} &= \frac{1}{4}\left(17(1,1,1,1)^\top + (5+6i)(1,i,-1,-i)^\top + (5-6i)(1,-i,-1,i)^\top\right)\\
&= \frac{1}{4}\big((17,17,17,17)^\top + (5+6i, -6+5i, -5-6i, 6-5i)^\top\\
&\qquad + (5-6i, -6-5i, -5+6i, 6+5i)^\top\big)\\
&= \frac{1}{4}(27,5,7,29)^\top = (6.75, 1.25, 1.75, 7.25)^\top
\end{aligned}$$

(b) The basis vectors of the discrete Haar wavelet transform are

$$
\begin{aligned}
\Psi_{2,0} &= \frac{1}{2}(1,1,1,1)^\top\,, \\
\Phi_{2,0} &= \frac{1}{2}(1,1,-1,-1)^\top\,, \\
\Phi_{1,0} &= \frac{1}{\sqrt{2}}(1,-1,0,0)^\top\,, \\
\Phi_{1,1} &= \frac{1}{\sqrt{2}}(0,0,1,-1)^\top\,.
\end{aligned}
$$

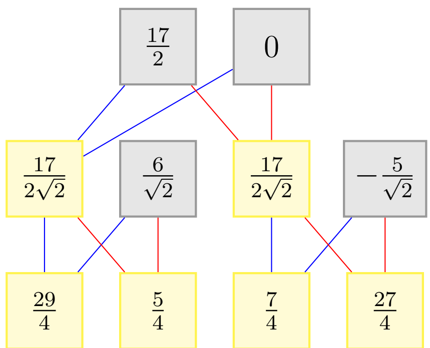
(c) We use the fast wavelet transform and compute



Thus, the wavelet decomposition of $\boldsymbol{f}$ is

$$
\overline{\boldsymbol{f}} = \left(\frac{17}{2}, -\frac{1}{2}, \frac{6}{\sqrt{2}}, -\frac{5}{\sqrt{2}}\right)^\top
$$

We set $d_{2,0} = 0$ and perform the backtransformation:



We obtain

$$
\tilde{\boldsymbol{f}} = (7.25, 1.25, 1.75, 6.75)^\top
$$

Problem 2 (Quartic B-Spline Interpolation)

(a) From the lecture we know the synthesis function $\beta_3(x)$ and $\beta_0(x)$:

$$\beta_3(x) = \begin{cases} 0 & x \le -2 \\ \frac{1}{6}(2+x)^3 & -2 < x \le -1 \\ \frac{2}{3} - x^2 - \frac{1}{2}x^3 & -1 < x \le 0 \\ \frac{2}{3} - x^2 + \frac{1}{2}x^3 & 0 < x < 1 \\ \frac{1}{6}(2-x)^3 & 1 \le x < 2 \\ 0 & 2 \le x \end{cases} \qquad \beta_0(x) = \begin{cases} 1, & \text{for } |x| < \frac{1}{2}\,, \\ \frac{1}{2}, & \text{for } |x| = \frac{1}{2}\,, \\ 0, & \text{else} \end{cases}$$

The synthesis function $\beta_4(x)$ can be computed by convolving $\beta_3(x)$ with $\beta_0(x)$:

$$\begin{aligned} \beta_4(x) &= \beta_3(x) * \beta_0(x) = \int_{-\infty}^{\infty} \beta_0(x-z)\beta_3(z)\,\mathrm{d}z = \int_{x-\frac{1}{2}}^{x+\frac{1}{2}} \beta_3(z)\,\mathrm{d}z \\ &= \begin{cases} 0 & x \le -\frac{5}{2} \\ \int_{-2}^{x+\frac{1}{2}} \frac{1}{6}(2+z)^3\,\mathrm{d}z & -\frac{5}{2} < x \le -\frac{3}{2} \\ \int_{x-\frac{1}{2}}^{-1} \frac{1}{6}(2+z)^3\,\mathrm{d}z + \int_{-1}^{x+\frac{1}{2}} \frac{2}{3} - z^2 - \frac{1}{2}z^3\,\mathrm{d}z & -\frac{3}{2} < x \le -\frac{1}{2} \\ \int_{x-\frac{1}{2}}^{0} \frac{2}{3} - z^2 - \frac{1}{2}z^3\,\mathrm{d}z + \int_{0}^{x+\frac{1}{2}} \frac{2}{3} - z^2 + \frac{1}{2}z^3\,\mathrm{d}z & -\frac{1}{2} < x < \frac{1}{2} \\ \int_{x-\frac{1}{2}}^{1} \frac{2}{3} - z^2 + \frac{1}{2}z^3\,\mathrm{d}z + \int_{1}^{x+\frac{1}{2}} \frac{1}{6}(2-z)^3\,\mathrm{d}z & \frac{1}{2} \le x < \frac{3}{2} \\ \int_{x-\frac{1}{2}}^{2} \frac{1}{6}(2-z)^3\,\mathrm{d}z & \frac{3}{2} \le x < \frac{5}{2} \\ 0 & \frac{5}{2} \le x \end{cases} \\ &= \begin{cases} 0 & x \le -\frac{5}{2} \\ \left[\frac{1}{24}(2+z)^4\right]_{-2}^{x+\frac{1}{2}} & -\frac{5}{2} < x \le -\frac{3}{2} \\ \left[\frac{1}{24}(2+z)^4\right]_{x-\frac{1}{2}}^{-1} + \left[\frac{2}{3}z - \frac{1}{3}z^3 - \frac{1}{8}z^4\right]_{-1}^{x+\frac{1}{2}} & -\frac{3}{2} < x \le -\frac{1}{2} \\ \left[\frac{2}{3}z - \frac{1}{3}z^3 - \frac{1}{8}z^4\right]_{x-\frac{1}{2}}^{0} + \left[\frac{2}{3}z - \frac{1}{3}z^3 + \frac{1}{8}z^4\right]_{0}^{x+\frac{1}{2}} & -\frac{1}{2} < x < \frac{1}{2} \\ \left[\frac{2}{3}z - \frac{1}{3}z^3 + \frac{1}{8}z^4\right]_{x-\frac{1}{2}}^{1} + \left[-\frac{1}{24}(2-z)^4\right]_{1}^{x+\frac{1}{2}} & \frac{1}{2} \le x < \frac{3}{2} \\ \left[-\frac{1}{24}(2-z)^4\right]_{x-\frac{1}{2}}^{2} & \frac{3}{2} \le x < \frac{5}{2} \\ 0 & \frac{5}{2} \le x \end{cases} \end{aligned}$$

$$
= \begin{cases}
0 & x \le -\frac{5}{2} \\
\frac{1}{24}(\frac{5}{2}+x)^4 & -\frac{5}{2} < x \le -\frac{3}{2} \\
\frac{1}{24} - \frac{1}{24}(\frac{3}{2}+x)^4 & \\
\quad +\frac{2}{3}(x+\frac{1}{2}) - \frac{1}{3}(x+\frac{1}{2})^3 - \frac{1}{8}(x+\frac{1}{2})^4 + \frac{2}{3} - \frac{1}{3} + \frac{1}{8} & -\frac{3}{2} < x \le -\frac{1}{2} \\
-\frac{2}{3}(x-\frac{1}{2}) + \frac{1}{3}(x-\frac{1}{2})^3 + \frac{1}{8}(x-\frac{1}{2})^4 & \\
\quad +\frac{2}{3}(x+\frac{1}{2}) - \frac{1}{3}(x+\frac{1}{2})^3 + \frac{1}{8}(x+\frac{1}{2})^4 & -\frac{1}{2} < x < \frac{1}{2} \\
\frac{2}{3} - \frac{1}{3} + \frac{1}{8} - \frac{2}{3}(x-\frac{1}{2}) + \frac{1}{3}(x-\frac{1}{2})^3 - \frac{1}{8}(x-\frac{1}{2})^4 & \\
\quad -\frac{1}{24}(\frac{3}{2}-x)^4 + \frac{1}{24} & \frac{1}{2} \le x < \frac{3}{2} \\
\frac{1}{24}(\frac{5}{2}-x)^4 & \frac{3}{2} \le x < \frac{5}{2} \\
0 & \frac{5}{2} \le x
\end{cases}
$$

(b) To get the linear system of equations we evaluate β_4 at integer values k:

$$
\beta_4(k) = \begin{cases}
0 & k < -2 \\
\frac{1}{384} & k = -2 \\
\frac{76}{384} & k = -1 \\
\frac{230}{384} & k = 0 \\
\frac{76}{384} & k = 1 \\
\frac{1}{384} & k = 2 \\
0 & k > 2
\end{cases}
= \begin{cases}
0 & k < -2 \\
\frac{1}{384} & k = -2 \\
\frac{19}{96} & k = -1 \\
\frac{115}{192} & k = 0 \\
\frac{19}{96} & k = 1 \\
\frac{1}{384} & k = 2 \\
0 & k > 2
\end{cases}
$$

The linear system of equations can now be determined as

$$
\begin{pmatrix}
\frac{115}{192} & \frac{19}{96} & \frac{1}{384} & & & & \\
\frac{19}{96} & \frac{115}{192} & \frac{19}{96} & \frac{1}{384} & & & \\
\frac{1}{384} & \frac{19}{96} & \frac{115}{192} & \frac{19}{96} & \frac{1}{384} & & \\
\ddots & \ddots & \ddots & \ddots & \ddots & & \\
& \frac{1}{384} & \frac{19}{96} & \frac{115}{192} & \frac{19}{96} & \frac{1}{384} & \\
& & \frac{1}{384} & \frac{19}{96} & \frac{115}{192} & \frac{19}{96} & \\
& & & \frac{1}{384} & \frac{19}{96} & \frac{115}{192} &
\end{pmatrix}
\begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_{N-2} \\ c_{N-1} \\ c_N \end{pmatrix}
=
\begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ \vdots \\ f_{N-2} \\ f_{N-1} \\ f_N \end{pmatrix},
$$

where empty entries in the system matrix are 0.

(c) The linear system of equations for the cubic case was considered in the lecture. We see, that for quartic splines pentadiagonal system matrix has to be solved, while the one for cubic splines is tridiagonal. Thus quartic B-splines need more computational effort for determining the coefficients than cubic ones.

Problem 3 (Morphological Operations)

The correspondence is

(A):	White top hat
(B):	Black top hat
(C):	Erosion
(D):	Opening
(E):	Closing
(F):	Dilation

Problem 4 (Variational Problems)

(a) First, let us compute the Euler-Lagrange equations of the Energy functional

$$E(w) = \frac{1}{2}\int_\Omega \left((w-f)^2 + \alpha|\boldsymbol{\nabla} w|^2\right) dx\, dy.$$

We have

$$F(w, w_x, w_y, x, y) = \frac{1}{2}\left((w-f)^2 + \alpha|\boldsymbol{\nabla} w|^2\right)$$

which yields

$$\begin{aligned} F_w &= (w-f) \\ F_{w_x} &= \alpha w_x \\ F_{w_y} &= \alpha w_y \end{aligned}$$

and finally we get

$$F_w - \partial_x F_{w_x} - \partial_y F_{w_y} = (w-f) - \alpha(w_{xx} + w_{yy})$$

as the left side of the Euler-Lagrange equation. We are now interested in the Fourier transform of w, thus we simply compute the Fourier transform of the Euler-Lagrange equation. Using the differential rule and the linearity of the Fourier transform, we obtain

$$\begin{aligned} & (\hat{w} - \hat{f}) - \alpha((2\pi iu)^2\hat{w} + (2\pi iv)^2\hat{w}) = 0 \\ \Leftrightarrow\ & \left(1 - \alpha((2\pi iu)^2 + (2\pi iv)^2)\right)\hat{w} = \hat{f} \\ \Leftrightarrow\ & \hat{w} = \frac{\hat{f}}{1 + \alpha 4\pi^2(u^2+v^2)} \end{aligned}$$

This is exactly what we wanted to show. Since we have $\alpha > 0$, the denominator is ≥ 1 and increases with larger values of u and v. Thus, for higher frequencies

u and v, the scaling factor $0 < \beta := \frac{1}{1+\alpha 4\pi^2(u^2+v^2)} \leq 1$ with $\hat{w} = \beta \hat{f}$ scales down $\hat{f}$, i.e. we have a lowpass filter. This is of course exactly what we expect from a smoothing filter.

(b) The Euler-Lagrange equation that defines constraints for a minimiser w is given by

$$h * (h * w - f) + Kw = 0.$$

Since we should show that w can also be obtained by applying the Wiener filter and the only definition we have of this filter uses the Fourier domain, we compute the Fourier transform of the Euler-Lagrange equation. Here we can use the convolution theorem and the linearity of the Fourier transform to obtain

$$\begin{aligned}
& \hat{h} \cdot (\hat{h} \cdot \hat{w} - \hat{f}) + K\hat{w} = 0 \\
\Leftrightarrow\ & (\hat{h}^2 + K)\hat{w} - \hat{f}\hat{h} = 0 \\
\Leftrightarrow\ & \hat{w} = \frac{\hat{h}\hat{f}}{\hat{h}^2 + K} \\
\Leftrightarrow\ & \frac{1}{\hat{h}} \cdot \frac{\hat{h}^2\hat{f}}{\hat{h}^2 + K}
\end{aligned}$$

Problem 5 (Stereo)

(a) The Hessian H_f is given by the second order derivatives of f:

$$H_f = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} \quad .$$

The assumption that corresponding object points in the two frames have equal entries in the Hessian is therefore expressed by the four equations

$$\begin{aligned} f_{l_{xx}}(x,y) &= f_{r_{xx}}(x+u,y) \\ f_{l_{xy}}(x,y) &= f_{r_{xy}}(x+u,y) \\ f_{l_{yx}}(x,y) &= f_{r_{yx}}(x+u,y) \\ f_{l_{yy}}(x,y) &= f_{r_{yy}}(x+u,y) \end{aligned}$$

where u denotes the disparity. Since the disparities can be large, a Taylor linearisation is here inappropriate.

(b) The data term for the energy functional is obtained by taking the sum over the squared constraints stated in (a). The smoothness term penalises the squared gradient norm $|\nabla u|^2 = u_x^2 + u_y^2$ as usual.

$$\begin{aligned} E(u) = \int_\Omega \Big(\; & \big(f_{r_{xx}}(x+u,y) - f_{l_{xx}}(x,y)\big)^2 \\ & + \big(f_{r_{xy}}(x+u,y) - f_{l_{xy}}(x,y)\big)^2 \\ & + \big(f_{r_{yx}}(x+u,y) - f_{l_{yx}}(x,y)\big)^2 \\ & + \big(f_{r_{yy}}(x+u,y) - f_{l_{yy}}(x,y)\big)^2 \; + \alpha |\nabla u|^2 \Big) \, \mathrm{d}x\mathrm{d}y \; . \end{aligned}$$

(c) The integrand in the energy functional is

$$\begin{aligned} F(x,y,u,u_x,u_y) \;=\; & \big(f_{r_{xx}}(x+u,y) - f_{l_{xx}}(x,y)\big)^2 \\ & + \big(f_{r_{xy}}(x+u,y) - f_{l_{xy}}(x,y)\big)^2 \\ & + \big(f_{r_{yx}}(x+u,y) - f_{l_{yx}}(x,y)\big)^2 \\ & + \big(f_{r_{yy}}(x+u,y) - f_{l_{yy}}(x,y)\big)^2 \; + \alpha |\nabla u|^2 \; . \end{aligned}$$

Inserting this in the usual Euler–Lagrange formalism we find successively

$$\begin{aligned}
F_u &= 2\big(f_{r_{xx}}(x+u,y) - f_{l_{xx}}(x,y)\big)\ \partial_x f_{r_{xx}}(x+u,y) \\
&\quad +2\big(f_{r_{xy}}(x+u,y) - f_{l_{xy}}(x,y)\big)\ \partial_x f_{r_{xy}}(x+u,y) \\
&\quad +2\big(f_{r_{yx}}(x+u,y) - f_{l_{yx}}(x,y)\big)\ \partial_x f_{r_{yx}}(x+u,y) \\
&\quad +2\big(f_{r_{yy}}(x+u,y) - f_{l_{yy}}(x,y)\big)\ \partial_x f_{r_{yy}}(x+u,y) \\
F_{u_x} &= 2\alpha u_x \\
F_{u_y} &= 2\alpha u_y \\
\partial_x F_{u_x} &= 2\alpha u_{xx} \\
\partial_y F_{u_y} &= 2\alpha u_{yy}
\end{aligned}$$

such that the Euler–Lagrange equation reads

$$\begin{aligned}
& 0 = F_u - \partial_x F_{u_x} - \partial_y F_{u_y} \\
\Leftrightarrow\ & 0 = 2\big(f_{r_{xx}}(x+u,y) - f_{l_{xx}}(x,y)\big)\ \partial_x f_{r_{xx}}(x+u,y) \\
&\quad +2\big(f_{r_{xy}}(x+u,y) - f_{l_{xy}}(x,y)\big)\ \partial_x f_{r_{xy}}(x+u,y) \\
&\quad +2\big(f_{r_{yx}}(x+u,y) - f_{l_{yx}}(x,y)\big)\ \partial_x f_{r_{yx}}(x+u,y) \\
&\quad +2\big(f_{r_{yy}}(x+u,y) - f_{l_{yy}}(x,y)\big)\ \partial_x f_{r_{yy}}(x+u,y) - 2\alpha u_{xx} - 2\alpha u_{yy} \\
\Leftrightarrow\ & 0 = \big(f_{r_{xx}}(x+u,y) - f_{l_{xx}}(x,y)\big)\ \partial_x f_{r_{xx}}(x+u,y) \\
&\quad +\big(f_{r_{xy}}(x+u,y) - f_{l_{xy}}(x,y)\big)\ \partial_x f_{r_{xy}}(x+u,y) \\
&\quad +\big(f_{r_{yx}}(x+u,y) - f_{l_{yx}}(x,y)\big)\ \partial_x f_{r_{yx}}(x+u,y) \\
&\quad +\big(f_{r_{yy}}(x+u,y) - f_{l_{yy}}(x,y)\big)\ \partial_x f_{r_{yy}}(x+u,y) - 2\alpha\Delta u
\end{aligned}$$

Problem 6 (Camera Geometry)

(a) First of all we compute the extrinsic matrix, which should describe a rotation around the y-axis by α degrees:

$$\boldsymbol{A}^{\text{ext}} := \begin{pmatrix} r_{11} & r_{12} & r_{13} & t_1 \\ r_{21} & r_{22} & r_{23} & t_2 \\ r_{31} & r_{32} & r_{33} & t_3 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos\alpha & 0 & \sin\alpha & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\alpha & 0 & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Now let us determine the intrinsic matrix. The following information is given:

- Principal point: $(u_0, v_0)^\top = (3, 2)^\top$
- Pixel dimensions: $k_u = k_v = 1$
- Orthogonal coordinate system: $\Theta = 90°$

So the intrinsic matrix is:

$$\boldsymbol{H} := \begin{pmatrix} k_u & -k_u \cot\Theta & u_0 \\ 0 & \frac{k_v}{\sin\Theta} & v_0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

(b) Using the focal distance $f = 2$, the full projection matrix can now be determined:

$$\begin{aligned}
\boldsymbol{H} \begin{pmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} A_{\text{ext}} &= \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \cos\alpha & 0 & \sin\alpha & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\alpha & 0 & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 2 & 0 & 3 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \cos\alpha & 0 & \sin\alpha & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\alpha & 0 & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 2\cos\alpha - 3\sin\alpha & 0 & 2\sin\alpha + 3\cos\alpha & 0 \\ -2\sin\alpha & 2 & 2\cos\alpha & 0 \\ -\sin\alpha & 0 & \cos\alpha & 0 \end{pmatrix} .
\end{aligned}$$