Example Solutions for Classroom Assignment 6 (C6)

Problem 1: (Linear Filters)

(a) Given is the one-dimensional binomial mask

$$\frac{1}{4}\ \boxed{1 \mid 2 \mid 1}$$

Note that binomial kernels approximate Gaussians and are thus lowpass filters. The stencil of a two-dimensional separable binomial filter that is based on the given one-dimensional mask can be determined as:

$$\frac{1}{4^2}\ \begin{array}{|c|c|c|}\hline 1 & 2 & 1 \\ \hline 2 & 4 & 2 \\ \hline 1 & 2 & 1 \\ \hline \end{array}$$

This is a two-dimensional lowpass filter.

(b) A corresponding highpass filter can be constructed by taking the difference between the identity and that lowpass filter:

$$\begin{array}{|c|c|c|}\hline 0 & 0 & 0 \\ \hline 0 & 1 & 0 \\ \hline 0 & 0 & 0 \\ \hline \end{array} \quad - \quad \frac{1}{16}\ \begin{array}{|c|c|c|}\hline 1 & 2 & 1 \\ \hline 2 & 4 & 2 \\ \hline 1 & 2 & 1 \\ \hline \end{array} \quad = \quad \frac{1}{16}\ \begin{array}{|c|c|c|}\hline -1 & -2 & -1 \\ \hline -2 & 12 & -2 \\ \hline -1 & -2 & -1 \\ \hline \end{array}$$

Problem 2 (1-D Derivative Filters)

(a) The finite difference approximation should use four values $f_{i-1}, \ldots, f_{i+2}$. Therefore we need to determine four coefficients $\alpha_{-1}, \ldots, \alpha_2$. To uniquely identify these four coefficients, we need four equations. If we use more equations, we can not guarantee that the derived system of equation has a solution, as we assume a higher consistency order than the actual approximation offers. If we use less equations, some of the coefficients can only be determined in dependance of other coefficients, i.e. are not uniquely determined anymore. This leads in general to a lower consistency order.

In order to determine the coefficients of the derivative mask, we have to perform a Taylor expansion for all neighbours of f_i that are relevant for our approximation filter. This yields

$$\begin{aligned}
f_{i-1} &= f_i - \frac{1}{1}hf_i' + \frac{1}{2}h^2 f_i'' - \frac{1}{6}h^3 f_i''' + \frac{1}{24}h^4 f_i^{(4)} + O(h^5) \\
f_i &= f_i \\
f_{i+1} &= f_i + \frac{1}{1}hf_i' + \frac{1}{2}h^2 f_i'' + \frac{1}{6}h^3 f_i''' + \frac{1}{24}h^4 f_i^{(4)} + O(h^5) \\
f_{i+2} &= f_i + \frac{2}{1}hf_i' + \frac{4}{2}h^2 f_i'' + \frac{8}{6}h^3 f_i''' + \frac{16}{24}h^4 f_i^{(4)} + O(h^5)
\end{aligned}$$

Since we are interested in computing an approximation to the first derivative, we have to choose the parameters $\alpha_{-1}, \ldots, \alpha_2$ in such a way that the following holds:

$$\begin{aligned}
0f_i + 1f_i' + 0f_i'' + 0f_i''' &\stackrel{!}{=} \alpha_{-1}f_{i-1} + \alpha_0 f_i + \alpha_1 f_{i+1} + \alpha_2 f_{i+2} \\
&\approx \quad (\alpha_{-1} + \alpha_0 + \alpha_1 + \alpha_2) f_i \\
&\quad + (-\alpha_{-1} + \alpha_1 + 2\alpha_2) h f_i' \\
&\quad + (\alpha_{-1} + \alpha_1 + 4\alpha_2) \frac{1}{2} h^2 f_i'' \\
&\quad + (-\alpha_{-1} + \alpha_1 + 8\alpha_2) \frac{1}{6} h^3 f_i''' \, .
\end{aligned}$$

This leads to the linear system

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 2 \\ 1 & 0 & 1 & 4 \\ -1 & 0 & 1 & 8 \end{pmatrix} \begin{pmatrix} \alpha_{-1} \\ \alpha_0 \\ \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{h} \\ 0 \\ 0 \end{pmatrix} .$$

(b) Plugging the given coefficients into the Taylor expansion gives

$$\begin{aligned}
&-\frac{1}{6h}f_{i+2}+\frac{1}{h}f_{i+1}-\frac{1}{2h}f_i-\frac{1}{3}f_{i-1}\\
=\;&\underbrace{(-2-3+6-1)}_{=0}\frac{1}{6}\frac{1}{h}f_i\\
&+\underbrace{(2-6-2)\frac{1}{6}\frac{h}{h}}_{=1}f_i'\\
&+\underbrace{(-2+6-4)}_{=0}\frac{1}{6}\frac{1}{2}\frac{h^2}{h}f_i''\\
&+\underbrace{(2+6-8)}_{=0}\frac{1}{6}\frac{1}{6}\frac{h^3}{h}f_i'''\\
&+\underbrace{(-2+6-16)}_{=-12\neq 0}\frac{1}{6}\frac{1}{24}\frac{h^4}{h}f_i^{(4)}\\
&+O(h^4)\\
=\;&f_i'-\frac{h^3}{12}f_i^{(4)}+O(h^4)
\end{aligned}$$

which shows that the order of consistency of the approximation is 3.

(c) If a derivative of order d is approximated with n points, we have (w.l.o.g.) n coefficients $\{\alpha_0, \ldots, \alpha_{n-1}\}$ such that

$$\begin{aligned} &\alpha_0 f_i + \alpha_1 f_{i+1} + \cdots + \alpha_{n-1} f_{i+(n-1)} \\ &= \beta_0 f_i + \cdots + \beta_d h^d f_i^{(d)} + \cdots + \beta_{n-1} h^{n-1} f_i^{(n-1)} + \beta_n h^n f_i^{(n)} + \cdots , \end{aligned}$$

where it is assumed that $d \leq n-1$. The values β_k $(k = 0, 1, \ldots)$ are linear combinations of the coefficients α_i $(i = 0, \ldots, n-1)$ (See previous a) and b)). Since we are approximating the $d{-}th$ derivative of f_i, we know that $\forall i$, $\alpha_i \sim \frac{1}{h^d}$ (the expression $x \sim y$ means that x is proportional to y). Thus, it must hold that $\beta_d = \frac{1}{h^d}$, and $\beta_k = \frac{0}{h^d}$, $\forall k \in \{0, \ldots, n-1\} \setminus \{d\}$. The approximation error depends on the values β_k for $k \geq n$. In particular, the lower bound of the order of consistency is obtained when $\beta_n \sim \frac{1}{h^d} \neq 0$. Without loss of generality, let $\beta_k := \frac{\tilde{\beta}_k}{h^d}$ for $k \geq n$. In this case, the approximation reads

$$\begin{aligned} &\alpha_0 f_i + \alpha_1 f_{i+1} + \cdots + \alpha_{n-1} f_{i+(n-1)} \\ &= f_i^{(d)} + \tilde{\beta}_n \frac{h^n}{h^d} f_i^{(n)} + \tilde{\beta}_{n+1} \frac{h^{n+1}}{h^d} f_i^{(n+1)} + \ldots \\ &= f_i^{(d)} + O(h^{n-d}) , \end{aligned}$$

which shows that the order of consistency is at least $n - d$.