Lecture 27:
3-D Reconstruction III:
Shape from Shading

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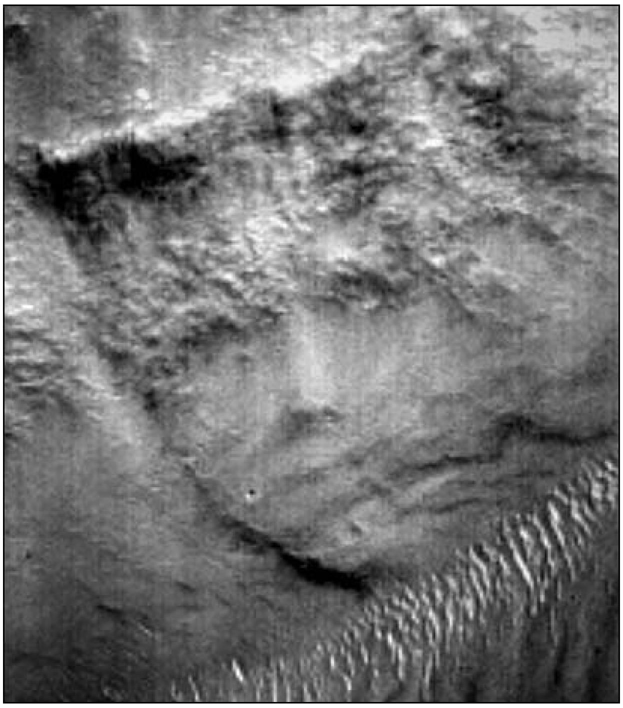
Introduction

$\longrightarrow$

Left: A single 2-D image of the face of Mozart. **Right:** Goal of shape from shading: 3-D reconstruction, if one knows the illumination conditions and the reflectance properties of the surface. Authors: O. Vogel et al.

Introduction (2)



A NASA image of the surface of the planet Mars. Does it really depict a face? Reconstructing the 3-D profile from such a single image requires shape from shading techniques. Source: `http://www.marsfindings.com/marsfind.hain/CrownedFace.jpg`.

Introduction (3)

The Shape from Shading Problem

- An illuminated 3-D surface is photographed.
  Let us assume that we know
  – the position of the light source,
  – the reflectance properties of the surface.
  Is it possible to recover the 3-D scene from a single image?

- important e.g. in astronomy:
  A planet is too far away to use stereo reconstruction techniques.

- a classical computer vision problem:
  Already in 1970, Berthold Horn wrote his Ph.D. thesis on this topic.

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Illumination and Reflectance

Two factors determine the radiation properties of a surface:

- illumination direction:
  position of the light source relative to the surface
- reflectance properties of the surface:
  depend on the optical properties of the material

Bidirectional Reflectance Distribution Function

- Consider a surface element in the X-Y-plane.
  Assume that is illuminated by a single light source.
  The incoming light direction has the polar angles (θ_i, ϕ_i); see slides.
  The corresponding incoming energy is denoted by $E(\theta_i, \phi_i)$.
- The brightness that is emitted by the surface under the polar angles (θ_e, ϕ_e) is denoted by $L(\theta_e, \phi_e)$; see slide on Page 8.
- The *bidirectional reflectance distribution function (BRDF, bidirektionale Reflexionsverteilung)* $f_r(\theta_i, \phi_i, \theta_e, \phi_e)$ decribes the ratio between the emitted brightness $L(\theta_e, \phi_e)$ and the incoming energy $E(\theta_i, \phi_i)$:

$$f_r(\theta_i, \phi_i, \theta_e, \phi_e) \;:=\; \frac{L(\theta_e, \phi_e)}{E(\theta_i, \phi_i)}.$$

- For an isotropic material, the BRDF only depends on the difference angles:

$$f_r(\theta_i, \phi_i, \theta_e, \phi_e) \;=\; \tilde{f}_r(\theta_i - \theta_e,\ \phi_i - \phi_e).$$

- Nevertheless, even such a function of two variables can be rather complicated.
  Note that it depends on the material.

Illumination and Reflectance (3)

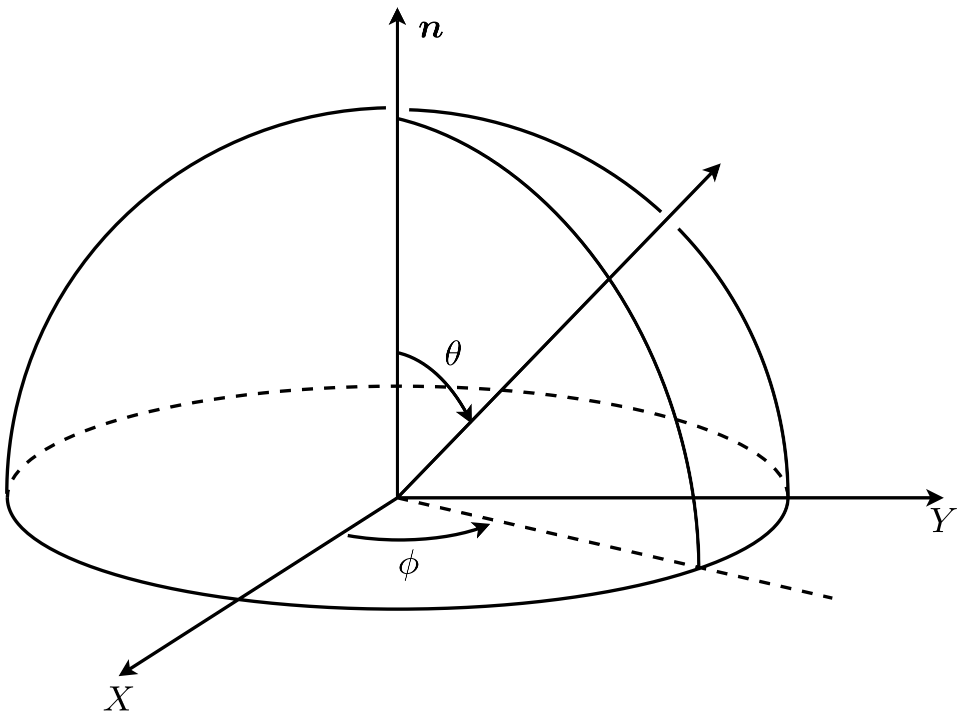

Illustration of 3-D polar coordinates. ϕ is called *azimuth angle,* and θ is the *zenith angle.* Author: M. Mainberger.

Illumination and Reflectance (4)

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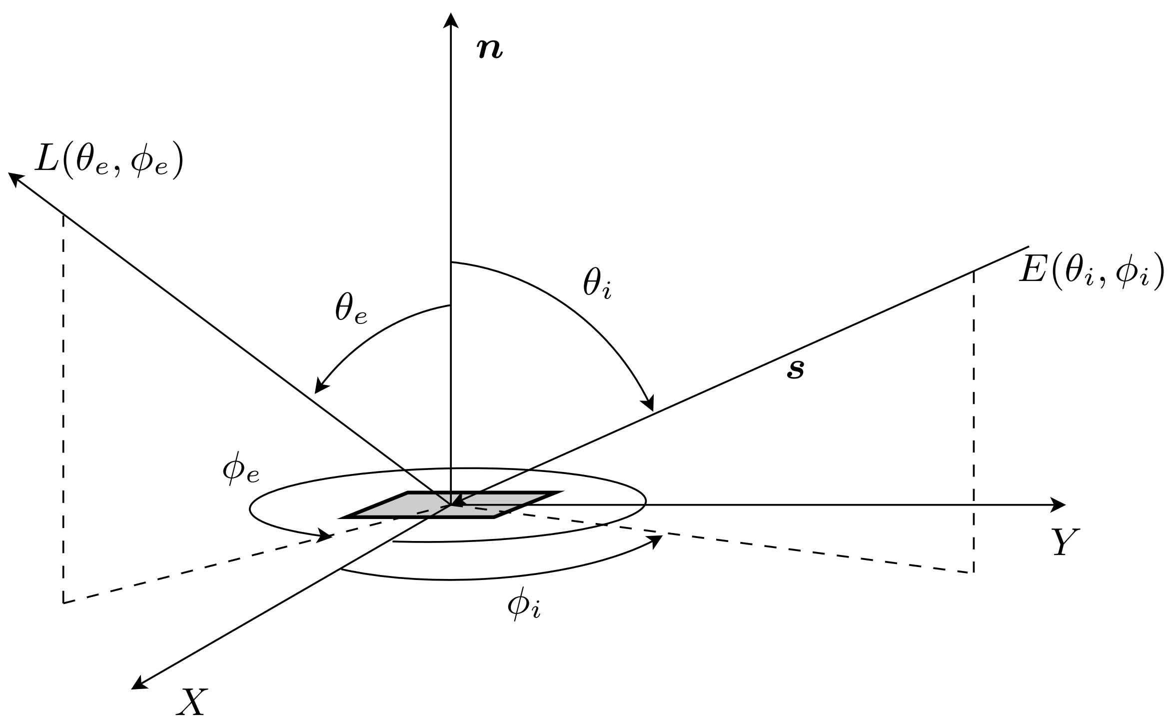

Angles of the BRDF. Author: M. Mainberger.

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Extremal Cases of Reflectance

- *Ideally Reflecting Surface:*
  - requires a perfectly smooth surface
  - reflects the incoming ray in exactly one direction:
    An incoming ray with polar angles (θ_i, ϕ_i) is reflected in direction $(\theta_e, \phi_e) = (\theta_i, \phi_i + \pi)$.
- *Lambertian Surface (Lambert'sche Oberfläche):*
  - frequently used as a model for a rough, unsmooth surface
  - reflects light energy in all directions *(diffuse reflection)*
  - Often one also assumes that the Lambertian surface does not absorb energy.
  - However, we do permit absorption. The nonabsorbed (i.e. reflected) fraction is determined by the *albedo (reflectance coefficient, Albedofaktor)* $\rho \in [0, 1]$.

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Basic Assumptions for Shape from Shading

Lambertian Assumption

- We restrict ourselves to a Lambertian surface.
- reflected radiance R (emitted brightness over all directions) in (X_w, Y_w, Z_w): proportional to the cosine of the angle θ_i between the light source direction $\boldsymbol{s}$ and the surface normal vector $\boldsymbol{n}$:

$$R_{\rho,\boldsymbol{s}}(X_w, Y_w, Z_w) \;=\; \rho \underbrace{|\boldsymbol{s}^\top \boldsymbol{n}|}_{\cos\theta_i} .$$

  The albedo ρ is the corresponding constant of this proportionality.
  Using $|\,.\,|$ guarantees that $\boldsymbol{n}$ lies in the same half space as $\boldsymbol{s}$.

- Thus, $R_{\rho,\boldsymbol{s}}$ is largest if the light source direction $\boldsymbol{s}$ is in the normal direction $\boldsymbol{n}$.
- There exist methods for estimating ρ and $\boldsymbol{s}$, but we do not consider them here: We assume that ρ and $\boldsymbol{s}$ are known.

Basic Assumptions for Shape from Shading (2)

Orthographic Camera Model

- For simplicity, we use the *orthographic camera model* from Lecture 25: We identify the image coordinates (x, y) with the world coordinates (X_w, Y_w).
- Moreover, we identify the reflected radiance $R_{\rho,\boldsymbol{s}}$ in a scene point (X_w, Y_w, Z_w) with the grey value f in the corresponding image point (x, y):

$$f(x, y) \;=\; R_{\rho,\boldsymbol{s}}(X_w, Y_w, Z_w).$$

(image irradiance equation)

This is justified by the Lambertian assumption.

Basic Assumptions for Shape from Shading (3)

Our Goal

- We want the depth map Z_w.
- All that we have is the equation

$$f \;=\; R_{\rho,\boldsymbol{s}} \;=\; \rho\,|\boldsymbol{s}^\top \boldsymbol{n}|\,,$$

where we know the image f, the albedo ρ, and the light source direction $\boldsymbol{s}$.

- Thus, we must recover the depth Z_w from the surface normal $\boldsymbol{n}$.

Our Plan to Achieve it

- We assume that Z_w is a function $Z_w = Z_w(x, y)$.
- We will recover $Z_w(x, y)$ from its partial derivatives $p := \partial_x Z_w$ and $q := \partial_y Z_w\,.$
- To this end, we must relate p and q to $\boldsymbol{n}$.

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Basic Assumptions for Shape from Shading (4)

Relating the Depth Derivatives to the Surface Normal

- We differentiate the surface $(x,\, y,\, Z_w(x,y))^\top$ with respect to x and y. This creates two vectors in the tangential plane of $Z_w(x,y)$:

$$\begin{pmatrix} 1 \\ 0 \\ p \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ q \end{pmatrix}.$$

- Their cross product (Vektorprodukt, Kreuzprodukt)

$$\begin{pmatrix} 1 \\ 0 \\ p \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ q \end{pmatrix} = \begin{pmatrix} -p \\ -q \\ 1 \end{pmatrix}$$

  yields a vector that is orthogonal to both tangential vectors.

- Thus, the unit normal vector is given by

$$\boldsymbol{n} = \frac{1}{\sqrt{1+p^2+q^2}} \begin{pmatrix} -p \\ -q \\ 1 \end{pmatrix}.$$

Basic Assumptions for Shape from Shading (5)

Putting Everything Together

- We plug

$$\boldsymbol{n} = \frac{1}{\sqrt{1+p^2+q^2}} \begin{pmatrix} -p \\ -q \\ 1 \end{pmatrix}$$

  into

$$f = R_{\rho,\boldsymbol{s}} = \rho\,|\boldsymbol{s}^\top \boldsymbol{n}|\,.$$

- This gives our main equation for shape from shading:

$$\boxed{f(x,y) = \frac{\rho}{\sqrt{1+p^2+q^2}} \left| \boldsymbol{s}^\top \begin{pmatrix} -p \\ -q \\ 1 \end{pmatrix} \right|.}$$

- similar situation as in the optic flow problem:
  - one equation with two unknowns (p and q)
  - unique solution $(p(x,y),\, q(x,y))$ requires one additional assumption

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Variational Method of Ikeuchi and Horn (1981)

Assumption

- p and q vary smoothly in space.

Energy Functional

$$E(p,q) \;=\; \int_{\Omega} \left(\big(f(x,y) - R(p,q)\big)^2 + \alpha \, \left(|\boldsymbol{\nabla} p|^2 + |\boldsymbol{\nabla} q|^2\right) \right) \, dx \, dy \, .$$

In the case of a Lambertian surface, we saw that

$$R(p,q) \;=\; \frac{\rho}{\sqrt{1+p^2+q^2}} \left| \boldsymbol{s}^\top \begin{pmatrix} -p \\ -q \\ 1 \end{pmatrix} \right| .$$

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Problem

- The data term does not guarantee strict convexity.
  Thus, the energy functional may have multiple local minima.
- How can we encourage convergence to a global minimum?

Remedy

- Use similar ideas as for stereo reconstruction (Lecture 26):
  - smooth data f on a coarse scale where not many local minima exist
  - reduce smoothing scale gradually
  - use solution of previous scale to initialise iterations at next finer scale

Euler–Lagrange Equations

$$\begin{aligned} \alpha \, \Delta p \;+\; \big(f - R(p,q)\big)\, \partial_p R(p,q) \;&=\; 0 \, , \\ \alpha \, \Delta q \;+\; \big(f - R(p,q)\big)\, \partial_q R(p,q) \;&=\; 0 \, . \end{aligned}$$

Variational Method of Ikeuchi and Horn (3)

Discretisation of the Euler–Lagrange Equations

- cf. Lecture 24 (grid size h):
  usual approximation of the Laplacian with reflecting boundary conditions:

$$\Delta p\,|_i \;\approx\; \frac{1}{h^2} \sum_{j \in \mathcal{N}(i)} (p_j - p_i)$$

$\mathcal{N}(i)$: 4 neighbours of pixel i (boundary pixels: 3 neighbours; corner pixels: 2)

- creates the difference equations

$$0 \;=\; \frac{\alpha}{h^2} \sum_{j \in \mathcal{N}(i)} (p_j - p_i) \;+\; \Big(f_i - R(p_i, q_i)\Big)\, \partial_p R(p_i, q_i),$$

$$0 \;=\; \frac{\alpha}{h^2} \sum_{j \in \mathcal{N}(i)} (q_j - q_i) \;+\; \Big(f_i - R(p_i, q_i)\Big)\, \partial_q R(p_i, q_i)$$

for all pixels $(i = 1, ..., N)$

Variational Method of Ikeuchi and Horn (4)

- Moving the red expressions to the left hand side gives

$$\frac{\alpha}{h^2}\,|\mathcal{N}(i)|\,p_i \;=\; \frac{\alpha}{h^2} \sum_{j \in \mathcal{N}(i)} p_j \;+\; \Big(f_i - R(p_i, q_i)\Big)\, \partial_p R(p_i, q_i)\,,$$

$$\frac{\alpha}{h^2}\,|\mathcal{N}(i)|\,q_i \;=\; \frac{\alpha}{h^2} \sum_{j \in \mathcal{N}(i)} q_j \;+\; \Big(f_i - R(p_i, q_i)\Big)\, \partial_q R(p_i, q_i)\,,$$

where $|\mathcal{N}(i)|$ denotes the number of neighbours of pixel i.

- This inspires a simple fixed point scheme:

$$p_i^{(k+1)} \;=\; \frac{\frac{\alpha}{h^2} \sum\limits_{j \in \mathcal{N}(i)} p_j^{(k)} \;+\; \Big(f_i - R(p_i^{(k)}, q_i^{(k)})\Big)\, \partial_p R(p_i^{(k)}, q_i^{(k)})}{\frac{\alpha}{h^2}\,|\mathcal{N}(i)|}\,,$$

$$q_i^{(k+1)} \;=\; \frac{\frac{\alpha}{h^2} \sum\limits_{j \in \mathcal{N}(i)} q_j^{(k)} \;+\; \Big(f_i - R(p_i^{(k)}, q_i^{(k)})\Big)\, \partial_q R(p_i^{(k)}, q_i^{(k)})}{\frac{\alpha}{h^2}\,|\mathcal{N}(i)|}\,.$$

Integrating the Depth Gradient $(p, q)^\top$

Problem

- We need the depth Z_w rather than its gradient $(p, q)^\top = (\partial_x Z_w,\, \partial_y Z_w)^\top$.
- In general, our approximations $\left(p^{(k+1)},\, q^{(k+1)}\right)^\top$ for $(p, q)^\top$ are not integrable: We cannot expect to find a function $Z(x, y)$ with $\partial_x Z_w = p$ and $\partial_y Z_w = q$.

Remedy: Reprojection (Frankot / Chellappa 1988)

- After each iteration, integrability is enforced: Project the results $p^{(k+1)}$ and $q^{(k+1)}$ onto the closest integrable pair of functions.
- This is done my minimising

$$E\left(Z^{(k+1)}\right) = \int_\Omega \left(\left(\partial_x Z^{(k+1)} - p^{(k+1)}\right)^2 + \left(\partial_y Z^{(k+1)} - q^{(k+1)}\right)^2\right) dx\, dy.$$

(Such "integration" approaches of "gradient" data have led to so-called *gradient domain methods*. They are very popular in computer graphics.)

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- Euler-Lagrange equation for the unknown function $Z^{(k+1)}$:

$$\Delta Z^{(k+1)} = \operatorname{div} \begin{pmatrix} p^{(k+1)} \\ q^{(k+1)} \end{pmatrix}.$$

- can be conveniently solved in the Fourier domain (cf. Lecture 4), where differentiation becomes multiplication with the frequency:

$$\left((i2\pi u)^2 + (i2\pi v)^2\right) \hat{Z}^{(k+1)} = i2\pi u\, \hat{p}^{(k+1)} + i2\pi v\, \hat{q}^{(k+1)}.$$

- Afterwards integrable variants $\left(\tilde{p}^{(k+1)},\, \tilde{q}^{(k+1)}\right)^\top$ of $\left(p^{(k+1)},\, q^{(k+1)}\right)^\top$ are obtained by differentiation:

$$\begin{aligned} \tilde{p}^{(k+1)} &:= \partial_x Z^{(k+1)}, \\ \tilde{q}^{(k+1)} &:= \partial_y Z^{(k+1)}. \end{aligned}$$

They are used in the next iteration step.

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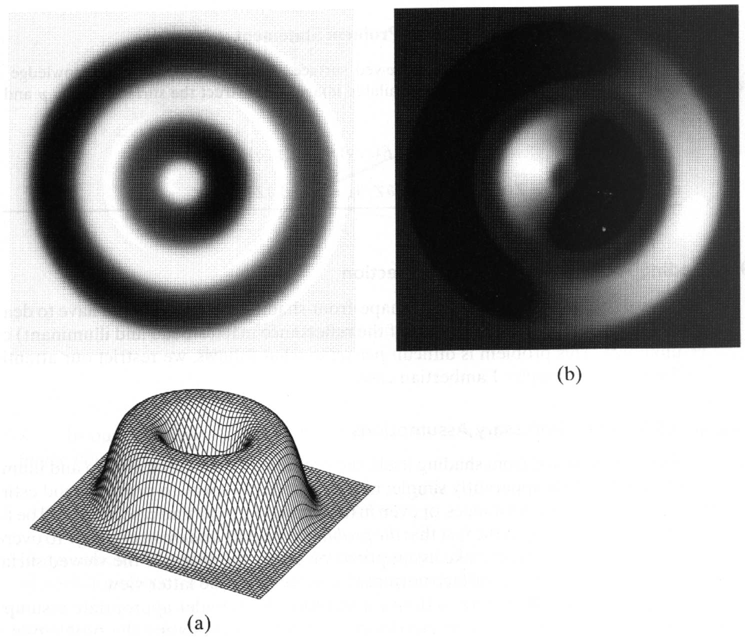
Variational Method of Ikeuchi and Horn (7)

Experiment with Synthetic Data



Top: Two images of the same Lambertian surface, but illuminated from different directions. **Bottom:** Ground truth 3-D model. Authors: E. Trucco, A. Verri.

Variational Method of Ikeuchi and Horn (8)

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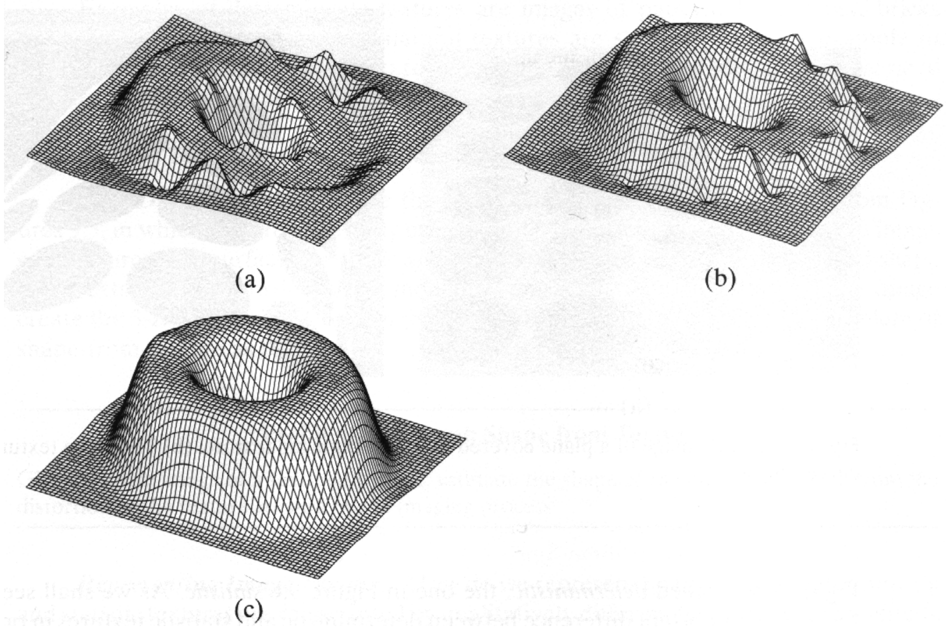
Results with the Frankot–Chellappa Algorithm



Reconstructed surface after (a) 100 iterations, (b) 1000 iterations, (c) 2000 iterations. The asymmetry in (a) und (b) has been caused by the illumination direction. To enforce integrability, the reprojection method of Frankot and Chellappa has been used. Authors: E. Trucco, A. Verri.

Photometric Stereo Reconstruction

- The same object is photographed several times under different illumination conditions, but from the same position.
- Advantage compared to classical stereo reconstruction:
  There is no correspondence problem between the images.
- Advantage compared to usual shape from shading:
  The method is more robust and precise, since errors are averaged.

Energy Functional for m Images $f_1, \ldots, f_m$:

$$E(p,q) \;=\; \int_\Omega \left(\frac{1}{m} \sum_{j=1}^{m} \big(f_j(x,y) - R_j(p,q)\big)^2 + \alpha \left(|\nabla p|^2 + |\nabla q|^2 \right) \right) dx\, dy.$$

If the accuracy of the measurements differs, one can assign different weights to the individual data terms.

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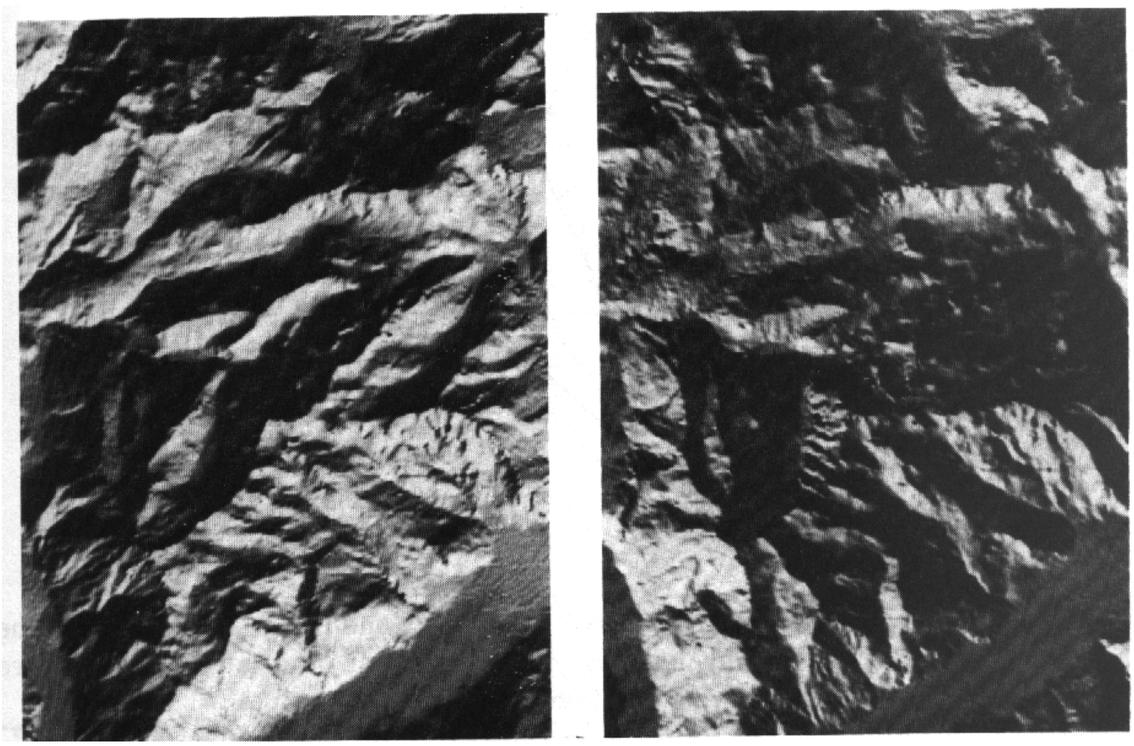
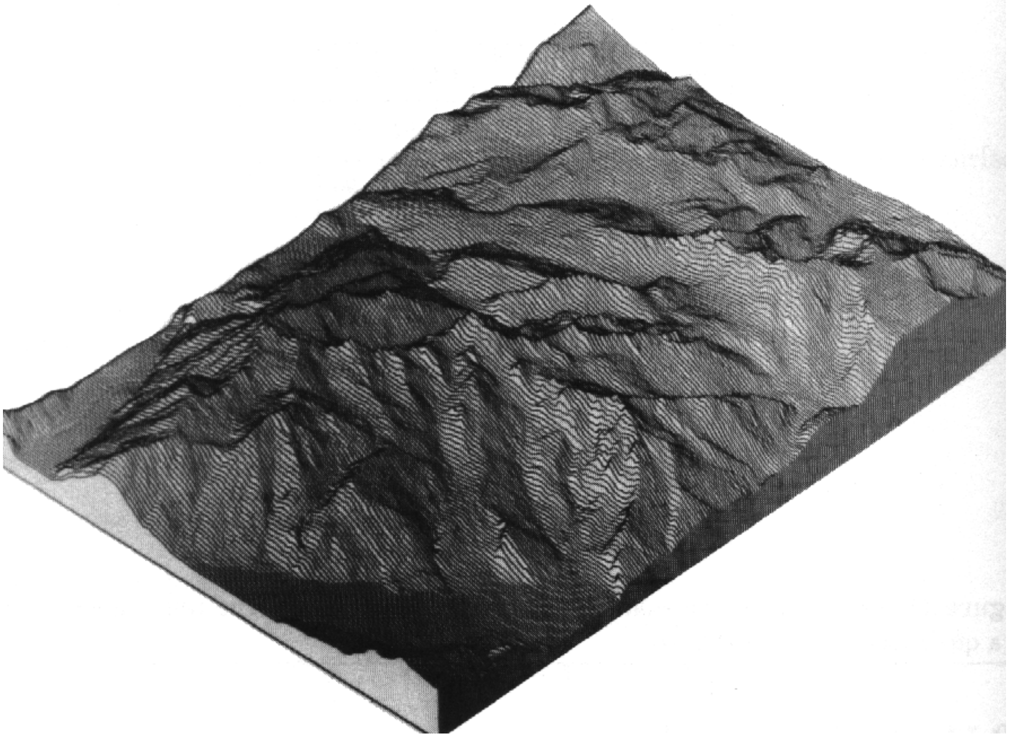


Two photos of the same mountains, but with different illumination. Author: B. K. P. Horn.



Digital terrain model reconstructed from the preceding two images. Author: B. K. P. Horn.

Other Shape from X Methods

- *Shape from Motion, Structure from Motion*
  Disparities between subsequent frames are used in a similar way as for stereo reconstruction in order to estimate the depth map. Important for robotics.

- *Shape from Focus*
  Depth information is recovered from varying the focus setting of the lens system.

- *Shape from Texture*
  Density, size, and orientation of the texture are used for 3-D reconstruction.

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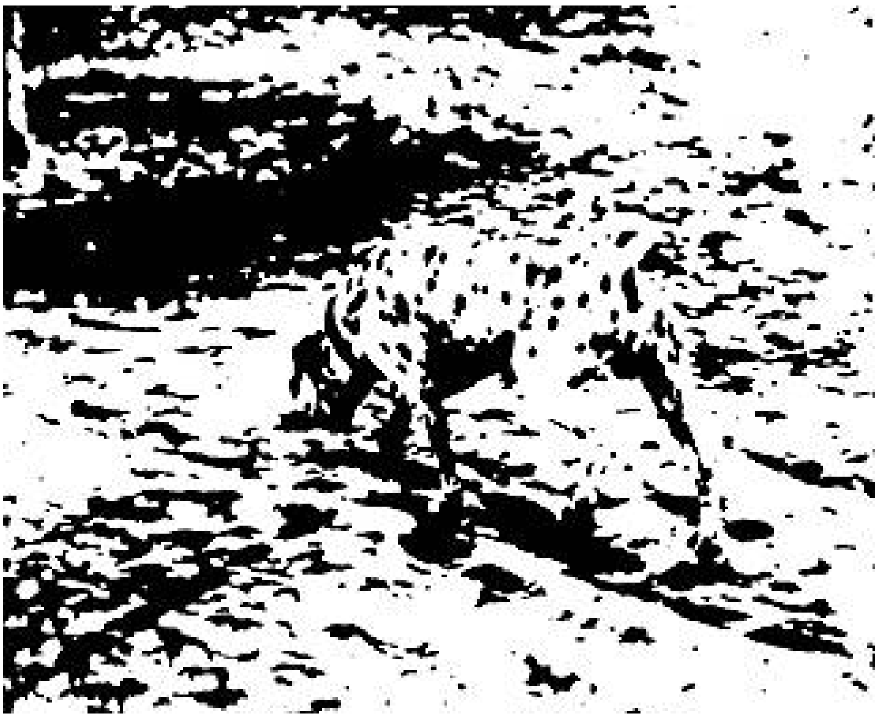
Other Shape from X Methods (2)

A nice challenge for shape from texture: The logo of the theorem prover SPASS. Source: `http://spass.mpi-sb.mpg.de/graphics/bigspasslogo.jpg`.

Other Shape from X Methods (3)



The depicted Dalmatian dog shows the difficulty of the shape from texture problem. Source: `www.unesco.org/courier/1999_06/photoshr/41_2.htm`. Based on a photo by Ronald C. James (1966). So far, there is no convincing computer-based reconstruction of the dog.

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Summary

- Shape from shading aims at recovering a surface from a single 2-D image.
  It assumes that the illumination direction and the reflection properties are known.
- A Lambertian surface emits radiation in all directions.
- It leads to an analytical description between the observed grey value und the surface orientation.
  However, it is not sufficient for recovering the full depth information.
- A variational model incorporates an additional smoothness assumption.
- The energy functional is nonconvex and may have local minima.
  This can be addressed by gradually reducing a Gaussian presmoothing.
- In general, the different partial derivatives of the depth map are nonintegrable.
  This is handled by projecting them to the closest integrable function.
- For photometric stereo reconstructions, one photographs the same object under different illuminations. This can give better results than shape from shading.

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  (Chapter 9)
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  (introduced the variational approach from this lecture)
- R. T. Frankot, R. Chellappa: A method for enforcing integrability in shape from shading algorithms. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, Vol. 10, No. 4, 439–451, 1988.
  (projection to the closest integrable function)
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  (entire edited book on shape from shading)