Image Processing and Computer Vision (IPCV)

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Example Solutions for Classroom Assignment 8 (C8)

Problem 1 (Bilateral Filtering and NL Means)

Given is a filter of the form

$$u_i = \sum_{j=1}^{N} q_{i,j} f_j \qquad \forall i \in \{1, \dots, N\}$$

or in matrix notation $\quad \boldsymbol{u} = Q\boldsymbol{f}$

where $\boldsymbol{f}$ is a discrete signal of length N and $\boldsymbol{u}$ is its filtered version.

The two properties given as assumptions in the different subtask can be formalised as:

Unit row sums:	$\forall i \in \{1, \dots, N\}$:	$\sum_{j=1}^{N} q_{i,j} = 1$
Unit column sums:	$\forall j \in \{1, \dots, N\}$:	$\sum_{i=1}^{N} q_{i,j} = 1$

(a) We show that the average grey level is preserved, provided all columns of Q have sum 1:

$\forall i \in \{1, \dots, N\}$:

$$\frac{1}{N}\sum_{i=1}^{N} u_i \overset{\text{def } u_i}{=} \frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N} q_{i,j} f_j \overset{\text{commuta-tivity}}{=} \frac{1}{N}\sum_{j=1}^{N}\sum_{i=1}^{N} q_{i,j} f_j$$

$$\overset{\text{distribu-tivity}}{=} \frac{1}{N}\sum_{j=1}^{N} f_j \underbrace{\sum_{i=1}^{N} q_{i,j}}_{=1} \overset{\text{column sum 1}}{=} \frac{1}{N}\sum_{i=1}^{N} f_i$$

(b) We want to show that the maximum-minimum principle holds, provided all rows of Q have sum 1 and $q_{i,j} \geq 0$.

$\forall i \in \{1, \ldots, N\}:$

$$\begin{aligned}
\max_k f_k \quad &\overset{\substack{\text{row}\\ \text{sum 1}}}{=} \quad \max_k f_k \underbrace{\sum_{j=1}^N q_{i,j}}_{=1} \quad \overset{\substack{\text{distribu-}\\ \text{tivity}}}{=} \quad \sum_{j=1}^N q_{i,j} \max_k f_k \\
&\overset{q_{i,j} \geq 0}{\geq} \quad \sum_{j=1}^N q_{i,j} f_j \quad \overset{\text{def } u_i}{=} \quad u_i \\
&\overset{\text{def } u_i}{=} \quad \sum_{j=1}^N q_{i,j} f_j \quad \overset{q_{i,j} \geq 0}{\geq} \quad \sum_{j=1}^N q_{i,j} \min_k f_k \\
&\overset{\substack{\text{distribu-}\\ \text{tivity}}}{=} \quad \min_k f_k \underbrace{\sum_{j=1}^N q_{i,j}}_{=1} \quad \overset{\substack{\text{row}\\ \text{sum 1}}}{=} \quad \min_k f_k
\end{aligned}$$

Thus $\qquad \max_j f_j \geq u_i \geq \min_j f_j.$

(c) **Unit row sums property is satisfied for bilateral filtering and NL means**

For the sake of simplicity let us define a unifying model for both filter types:

$$u_i = \frac{\sum_{j=1}^N a_{i,j} f_j}{\sum_{j=1}^N a_{i,j}}$$

Bilateral filtering is defined as

$$u_i = \frac{\sum_{j=1}^N g(|f_i - f_j|) w(|x_i - x_j|) f_j}{\sum_{j=1}^N g(|f_i - f_j|) w(|x_i - x_j|)}.$$

thus we have

$$a_{i,j} := g(|f_i - f_j|) w(|x_i - x_j|).$$

NL-means are defined as

$$u_i = \frac{\sum_{j=1}^N g(|f(\mathcal{N}_i) - f(\mathcal{N}_j)|) f_j}{\sum_{j=1}^N g(|f(\mathcal{N}_i) - f(\mathcal{N}_j)|)}.$$

As for bilateral filtering we define

$$a_{i,j} := g(|f(\mathcal{N}_i) - f(\mathcal{N}_j)|).$$

We can now prove that the rows have sum 1 for both filters at once:
$\forall i \in \{1, \dots, N\}$:

$$\sum_{j=1}^{N} q_{i,j} = \sum_{j=1}^{N} \frac{a_{i,j}}{\sum_{k=1}^{N} a_{i,k}} = \frac{\sum_{j=1}^{N} a_{i,j}}{\sum_{j=1}^{N} a_{i,j}} = 1$$

Hence the rows have sum 1.

(d) **The column do not have sum 1 for bilateral filtering**

For bilateral filtering we have

$$u_i = \frac{\sum_{j=1}^{N} a_{i,j} f_j}{\sum_{j=1}^{N} a_{i,j}} = \sum_{j=1}^{N} \frac{a_{i,j}}{\sum_{k=1}^{N} a_{i,k}} f_j =: \sum_{j=1}^{N} q_{i,j} f_j.$$

To show that the columns do not always have sum 1 we consider a counter example:
Consider the 1-D signal $\boldsymbol{f} = (1, 0, 0)$.

For $j = 1$ we get:

$$\begin{aligned}
\sum_{i=1}^{3} q_{i,1} &= q_{1,1} + q_{2,1} + q_{3,1} \\
&= \frac{a_{1,1}}{\sum_{k=1}^{3} a_{1,k}} + \frac{a_{2,1}}{\sum_{k=1}^{3} a_{2,k}} + \frac{a_{3,1}}{\sum_{k=1}^{3} a_{3,k}} \\
&= \frac{a_{1,1}}{a_{1,1} + a_{1,2} + a_{1,3}} + \frac{a_{2,1}}{a_{2,1} + a_{2,2} + a_{2,3}} + \frac{a_{3,1}}{a_{3,1} + a_{3,2} + a_{3,3}}
\end{aligned}$$

As weight functions g and w we use the one proposed in the assigment. So let us define

$$w(x) = g(x) := \frac{1}{1 + x^2}.$$

We get

$$\begin{aligned}
a_{1,1} = a_{2,2} = a_{3,3} &= \frac{1}{1+0^2} \frac{1}{1+0^2} = 1 \\
a_{1,2} = a_{2,1} &= \frac{1}{1+1^2} \frac{1}{1+1^2} = \frac{1}{4} \\
a_{1,3} = a_{3,1} &= \frac{1}{1+1^2} \frac{1}{1+2^2} = \frac{1}{10} \\
a_{2,3} = a_{3,2} &= \frac{1}{1+0^2} \frac{1}{1+1^2} = \frac{1}{2}
\end{aligned}$$

and

$$\sum_{i=1}^{3} q_{i,1} = \frac{1}{1+\frac{1}{4}+\frac{1}{10}} + \frac{\frac{1}{4}}{\frac{1}{4}+1+\frac{1}{2}} + \frac{\frac{1}{10}}{\frac{1}{10}+\frac{1}{2}+1}$$
$$= \frac{20}{20+5+2} + \frac{1}{1+4+2} + \frac{1}{1+5+10}$$
$$= \frac{20 \cdot 16 \cdot 7 + 27 \cdot 16 + 27 \cdot 7}{27 \cdot 16 \cdot 7}.$$

Since the denominator is an even number the column sum can only be an integer if the numerator is also an even number. Since both $20 \cdot 16 \cdot 7$ and $27 \cdot 16$ are even and $27 \cdot 7$ is odd, the numerator is an odd number and the column sum can not be an integer. In particular the column sum is not 1.

We could also have evaluated the column sum to be

$$\frac{20 \cdot 16 \cdot 7 + 27 \cdot 16 + 27 \cdot 7}{27 \cdot 16 \cdot 7} = \frac{2861}{3024} \neq 1.$$

So obviously, in general the column do not have sum 1 for bilateral filtering.

Unit column sums property is not satisfied for NL-means:

Again, we have

$$u_i = \frac{\sum_{j=1}^{N} a_{i,j} f_j}{\sum_{j=1}^{N} a_{i,j}} = \sum_{j=1}^{N} \frac{a_{i,j}}{\sum_{k=1}^{N} a_{i,k}} f_j =: \sum_{j=1}^{N} q_{i,j} f_j.$$

To show that the column do not always have sum 1 we consider the 1-D signal $\boldsymbol{f} = (1, 0, 0)$ and assume reflecting boundary conditions. Furthermore we choose a patch size of 3. We get

$$f(\mathcal{N}_1) = (1,1,0)^T \qquad f(\mathcal{N}_2) = (1,0,0)^T \qquad f(\mathcal{N}_3) = (0,0,0)^T$$

For $j = 1$ we get:

$$\sum_{i=1}^{3} q_{i,1} = q_{1,1} + q_{2,1} + q_{3,1}$$
$$= \frac{a_{1,1}}{\sum_{k=1}^{3} a_{1,k}} + \frac{a_{2,1}}{\sum_{k=1}^{3} a_{2,k}} + \frac{a_{3,1}}{\sum_{k=1}^{3} a_{3,k}}$$
$$= \frac{a_{1,1}}{a_{1,1} + a_{1,2} + a_{1,3}} + \frac{a_{2,1}}{a_{2,1} + a_{2,2} + a_{2,3}} + \frac{a_{3,1}}{a_{3,1} + a_{3,2} + a_{3,3}}$$

We use again the weighting function $g(x)$ from the assignment. We get

$$\begin{aligned}
a_{1,1} = a_{2,2} = a_{3,3} &= \frac{1}{1+0^2} = 1 \\
a_{1,2} = a_{2,1} &= \frac{1}{1+1^2} = \frac{1}{2} \\
a_{1,3} = a_{3,1} &= \frac{1}{1+\left(\sqrt{2}\right)^2} = \frac{1}{3} \\
a_{2,3} = a_{3,2} &= \frac{1}{1+1^2} = \frac{1}{2}
\end{aligned}$$

and

$$\begin{aligned}
\sum_{i=1}^{3} q_{i,1} &= \frac{1}{1+\frac{1}{2}+\frac{1}{3}} + \frac{\frac{1}{2}}{\frac{1}{2}+1+\frac{1}{2}} + \frac{\frac{1}{3}}{\frac{1}{3}+\frac{1}{2}+1} \\
&= \frac{6}{6+3+2} + \frac{1}{1+2+1} + \frac{2}{2+3+6} \\
&= \frac{8}{11} + \frac{1}{4} = \frac{43}{44} \neq 1.
\end{aligned}$$

Thus, in general, the columns do not have sum 1 for NL-means either.