Image Processing and Computer Vision (IPCV)

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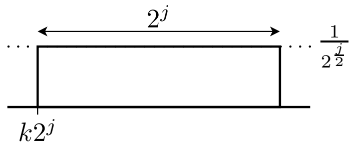
Example Solutions for Classroom Assignment 4 (C4)

Problem 1 (Discrete Wavelet Transform)

(a) **Recall:** In the continuous case we have:

$\Phi_{j,k}$ has width 2^j , height $\dfrac{1}{2^{\frac{j}{2}}}$,
and starts at $k2^j$.



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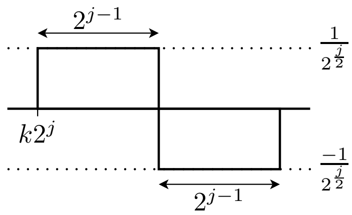
and starts at $k2^j$.



For the discrete setting we sample at N equidistant grid points $\left\{\frac{1}{2}, \frac{3}{2}, \dots, N - \frac{1}{2}\right\}$. Thus

$$\begin{aligned}
\Phi_{n,0} &= ((\Phi_{n,0})_0, (\Phi_{n,0})_1, \dots, (\Phi_{n,0})_{N-1})) \\
&= \frac{1}{2^{\frac{n}{2}}}(\underbrace{1, 1, \dots, 1}_{N \text{ times}})^T
\end{aligned}$$

and

$$\begin{aligned}
\Psi_{j,k} &= ((\Psi_{j,k})_0, (\Psi_{j,k})_1, \dots, (\Psi_{j,k})_{N-1})) \\
&= \frac{1}{2^{\frac{j}{2}}}(\underbrace{0, \dots, 0}_{\substack{k2^j \\ \text{times}}}, \underbrace{1, \dots, 1}_{\substack{2^{j-1} \\ \text{times}}}, \underbrace{-1, \dots, -1}_{\substack{2^{j-1} \\ \text{times}}}, \underbrace{0, \dots, 0}_{\substack{N-(k+1)2^j \\ \text{times}}})^T
\end{aligned}$$

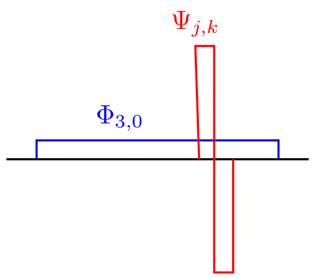
That means that the vectors for $N = 8$ are given by

$$\Phi_{3,0} = \frac{1}{\sqrt{8}}(1,1,1,1,1,1,1,1)^\top$$
$$\Psi_{3,0} = \frac{1}{\sqrt{8}}(1,1,1,1,-1,-1,-1,-1)^\top$$
$$\Psi_{2,0} = \frac{1}{2}(1,1,-1,-1,0,0,0,0)^\top$$
$$\Psi_{2,1} = \frac{1}{2}(0,0,0,0,1,1,-1,-1)^\top$$
$$\Psi_{1,0} = \frac{1}{\sqrt{2}}(1,-1,0,0,0,0,0,0)^\top$$
$$\Psi_{1,1} = \frac{1}{\sqrt{2}}(0,0,1,-1,0,0,0,0)^\top$$
$$\Psi_{1,2} = \frac{1}{\sqrt{2}}(0,0,0,0,1,-1,0,0)^\top$$
$$\Psi_{1,3} = \frac{1}{\sqrt{2}}(0,0,0,0,0,0,1,-1)^\top$$

(b) It is obvious that the total number of vectors is 8.

To show that the given vectors are orthogonal, the inner product between two arbitrary different vectors has to be 0:

- First we consider the inner products regarding the mother wavelet:

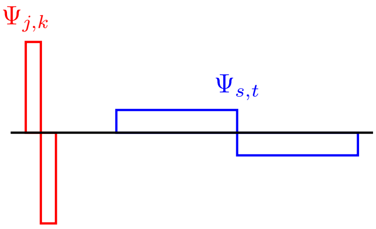


$$
\begin{aligned}
\langle \Phi_{3,0}, \Psi_{j,k} \rangle &= \sum_{i=0}^{7} (\Phi_{3,0})_i \cdot (\Psi_{j,k})_i \\
&= \sum_{i=k2^j}^{k2^j+2^j-1} (\Phi_{3,0})_i \cdot (\Psi_{j,k})_i \\
&= \sum_{i=k2^j}^{k2^j+2^{j-1}-1} \frac{1}{\sqrt{8}} \cdot \frac{1}{2^{\frac{j}{2}}} + \sum_{i=k2^j+2^{j-1}}^{k2^j+2^j-1} \frac{1}{\sqrt{8}} \cdot \frac{-1}{2^{\frac{j}{2}}} \\
&= 2^{j-1} \cdot \frac{1}{\sqrt{8}} \cdot \frac{1}{2^{\frac{j}{2}}} + 2^{j-1} \cdot \frac{1}{\sqrt{8}} \cdot \frac{-1}{2^{\frac{j}{2}}} \\
&= 0
\end{aligned}
$$

- For $\langle \Psi_{j,k}, \Psi_{s,t} \rangle$, we consider 4 cases: First of all let us assume w.l.o.g. $j \leq s$ (i.e $\Psi_{j,k}$ is more or equally "narrow" than $\Psi_{s,t}$).

  Case 1:

  $k2^j + 2^j < t2^s$

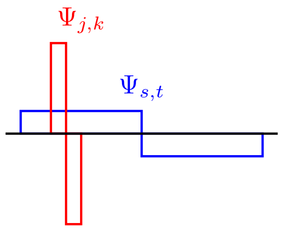
  

  $$\langle \Psi_{j,k}, \Psi_{s,t} \rangle = \sum_{i=0}^{7} (\Psi_{j,k})_i \cdot (\Psi_{s,t})_i = \sum_{i=0}^{7} 0 = 0$$

  We are in this case for:
  $\langle \Psi_{2,0}, \Psi_{2,1} \rangle$, $\langle \Psi_{1,0}, \Psi_{2,1} \rangle$, $\langle \Psi_{1,1}, \Psi_{2,1} \rangle$, $\langle \Psi_{1,0}, \Psi_{1,1} \rangle$, $\langle \Psi_{1,0}, \Psi_{1,2} \rangle$, $\langle \Psi_{1,0}, \Psi_{1,3} \rangle$, $\langle \Psi_{1,1}, \Psi_{1,2} \rangle$, $\langle \Psi_{1,2}, \Psi_{1,3} \rangle$ and $\langle \Psi_{1,2}, \Psi_{1,3} \rangle$

  Case 2:

  $t2^s \leq k2^j < t2^s + 2^{s-1}$

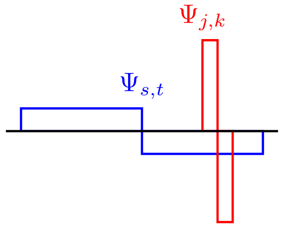
  

  $$\langle \Psi_{j,k}, \Psi_{s,t} \rangle = \cdots = 2^{j-1} \cdot \frac{1}{2^{\frac{j}{2}}} \cdot \frac{1}{2^{\frac{s}{2}}} + 2^{j-1} \cdot \frac{-1}{2^{\frac{j}{2}}} \cdot \frac{1}{2^{\frac{s}{2}}} = 0$$

We are in this case for :
$\langle \Psi_{1,0}, \Psi_{3,0} \rangle$, $\langle \Psi_{1,1}, \Psi_{3,0} \rangle$, $\langle \Psi_{2,0}, \Psi_{3,0} \rangle$, $\langle \Psi_{1,0}, \Psi_{2,0} \rangle$ and $\langle \Psi_{1,2}, \Psi_{2,1} \rangle$.

Case 3:

$t2^s + 2^{s-1} \leq k2^j < t2^s + 2^s$

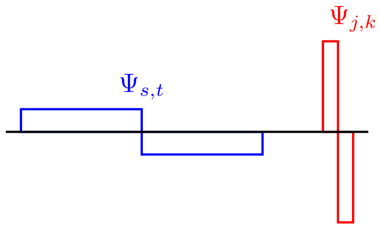


$$\langle \Psi_{j,k}, \Psi_{s,t} \rangle = \cdots = 2^{j-1} \cdot \frac{1}{2^{\frac{j}{2}}} \cdot \frac{-1}{2^{\frac{s}{2}}} + 2^{j-1} \cdot \frac{-1}{2^{\frac{j}{2}}} \cdot \frac{-1}{2^{\frac{s}{2}}} = 0$$

We are in this case for :
$\langle \Psi_{1,2}, \Psi_{3,0} \rangle$, $\langle \Psi_{1,3}, \Psi_{3,0} \rangle$, $\langle \Psi_{2,1}, \Psi_{3,0} \rangle$, $\langle \Psi_{1,1}, \Psi_{2,0} \rangle$ and $\langle \Psi_{1,3}, \Psi_{2,1} \rangle$.

Case 4:

$t2^s + 2^s \leq k2^j$



$$\langle \Psi_{j,k}, \Psi_{s,t} \rangle = \cdots = \sum_{i=0}^{7} 0 = 0$$

We are in this case for :
$\langle \Psi_{1,2}, \Psi_{2,0} \rangle$ and $\langle \Psi_{1,3}, \Psi_{2,0} \rangle$.

Note that these four cases show all possible outcomes, as either the non-zero entries of the wavelets don't overlap due to the shift (Case 1 and 4), or the non-zero entries of the smaller wavelet lie completely in the range of the wider wavelet where all entries are constant due to the scale (Case 2 and 3).

- The Euclidian norm of the vectors is 1

$$\begin{aligned}
|\Phi_{3,0}|^2 &= \sum_{i=0}^{7} (\Phi_{3,0})_i^2 = 2^3 \cdot \left(\frac{1}{\sqrt{8}}\right)^2 = \frac{8}{8} = 1 \\
&\Rightarrow |\Phi_{3,0}| = 1 \\
|\Psi_{j,k}|^2 = \ldots &= 2^{j-1} \cdot \left(\frac{1}{2^{\frac{j}{2}}}\right)^2 + 2^{j-1} \cdot \left(\frac{-1}{2^{\frac{j}{2}}}\right)^2 = 2 \cdot \frac{2^{j-1}}{2^j} = 1 \\
&\Rightarrow |\Psi_{j,k}| = 1
\end{aligned}$$

So we have proven that the given vectors form an orthonormal basis of $\mathbb{R}^8$ with respect to the inner product.

Problem 2 (Information Content)
Assume we are being sent the two words, one letter at a time. The entropy of the word tells us how much information we gain when we receive one letter of it. Since the two words have the same length, the one with higher entropy contains more information.

The entropy of "SHANNON" is:

$$\begin{aligned}
H =& - p_S \operatorname{ld} p_S - p_H \operatorname{ld} p_H - p_A \operatorname{ld} p_A - p_N \operatorname{ld} p_N - p_O \operatorname{ld} p_O \\
=& - \frac{1}{7} \operatorname{ld} \frac{1}{7} - \frac{1}{7} \operatorname{ld} \frac{1}{7} - \frac{1}{7} \operatorname{ld} \frac{1}{7} - \frac{3}{7} \operatorname{ld} \frac{3}{7} - \frac{1}{7} \operatorname{ld} \frac{1}{7} \\
=& - \frac{4}{7} \operatorname{ld} \frac{1}{7} - \frac{3}{7} \operatorname{ld} \frac{3}{7} \\
\approx& \, 2.13\,.
\end{aligned}$$

The entropy of "ENTROPY" is:

$$\begin{aligned}
H =& - p_E \operatorname{ld} p_E - p_N \operatorname{ld} p_N - p_T \operatorname{ld} p_T - p_R \operatorname{ld} p_R - p_O \operatorname{ld} p_O - p_P \operatorname{ld} p_P - p_Y \operatorname{ld} p_Y \\
=& - \frac{1}{7} \operatorname{ld} \frac{1}{7} - \frac{1}{7} \operatorname{ld} \frac{1}{7} - \frac{1}{7} \operatorname{ld} \frac{1}{7} - \frac{1}{7} \operatorname{ld} \frac{1}{7} - \frac{1}{7} \operatorname{ld} \frac{1}{7} - \frac{1}{7} \operatorname{ld} \frac{1}{7} - \frac{1}{7} \operatorname{ld} \frac{1}{7} \\
=& - \frac{7}{7} \operatorname{ld} \frac{1}{7} \\
\approx& \, 2.81\,.
\end{aligned}$$

We now see that "ENTROPY" has higher entropy than "SHANNON" and therefore contains more information.