Image Processing and Computer Vision (IPCV)

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Example Solutions for Classroom Assignment 7 (C7)

Problem 1: (Tensor Analysis)

We want to compute the eigenvalues and corresponding eigenvectors of the tensor $\boldsymbol{\nabla} u \boldsymbol{\nabla} u^\top$

- **Eigenvalues** λ_1, λ_2

  We start with the characteristic polynomial

$$
\begin{aligned}
p(\lambda) &= \det(\boldsymbol{\nabla} u \boldsymbol{\nabla} u^\top - \lambda \boldsymbol{I}) \\
&= \det \left(\begin{pmatrix} u_x \\ u_y \end{pmatrix} (u_x, u_y) - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \right) \\
&= \det \begin{pmatrix} u_x^2 - \lambda & u_x u_y \\ u_x u_y & u_y^2 - \lambda \end{pmatrix} \\
&= (u_x^2 - \lambda)(u_y^2 - \lambda) - u_x^2 u_y^2 \\
&= \lambda^2 - \lambda(u_x^2 + u_y^2)
\end{aligned}
$$

  Now the eigenvalues λ_1 and λ_2 can be computed as the solution of the equation $\lambda^2 - \lambda(u_x^2 + u_y^2) = 0$:

$$
\begin{aligned}
& \lambda^2 - \lambda(u_x^2 + u_y^2) = 0 \\
\Leftrightarrow \quad & \lambda(\lambda - (u_x^2 + u_y^2)) = 0 \\
\Leftrightarrow \quad & \lambda = 0 \quad \vee \quad \lambda - (u_x^2 + u_y^2) = 0 \\
\Rightarrow \quad & \lambda_1 = 0 \quad \text{and} \quad \lambda_2 = u_x^2 + u_y^2 = |\boldsymbol{\nabla} u|^2
\end{aligned}
$$

- **Eigenvectors** $\boldsymbol{v}_1$, $\boldsymbol{v}_2$

  We want to find vectors $\boldsymbol{v}_1$,$\boldsymbol{v}_2 \neq 0$ with $\boldsymbol{\nabla} u \boldsymbol{\nabla} u^\top \boldsymbol{v}_1 = \lambda_1 \boldsymbol{v}_1$ and $\boldsymbol{\nabla} u \boldsymbol{\nabla} u^\top \boldsymbol{v}_2 = \lambda_2 \boldsymbol{v}_2$.

- $\lambda_1 = 0$:

$$\begin{aligned} && \nabla u \underbrace{\nabla u^\top \boldsymbol{v}_1}_{\langle \nabla u, \boldsymbol{v}_1 \rangle} = 0 \\ \Leftrightarrow && \nabla u = 0 \quad \vee \quad \langle \nabla u, \boldsymbol{v}_1 \rangle = 0 \\ \Leftrightarrow && \nabla u = 0 \quad \vee \quad \nabla u \perp \end{aligned}$$

bmv_1

This means by setting $\boldsymbol{v}_1 = (-u_y, u_x) =: \nabla u^\perp$ we found a valid eigenvector.

- $\lambda_2 = |\nabla u|^2$:

$$\begin{aligned} && \nabla u \nabla u^\top \boldsymbol{v}_2 = |\nabla u|^2 \boldsymbol{v}_2 \\ \Leftrightarrow && \nabla u \nabla u^\top \boldsymbol{v}_2 = (\nabla u^\top \nabla u) \boldsymbol{v}_2 \\ \Leftrightarrow && \nabla u \nabla u^\top \boldsymbol{v}_2 = \boldsymbol{v}_2 \nabla u^\top \nabla u \end{aligned}$$

Now it is obvious that this equation holds if $\boldsymbol{v}_2$ is chosen to be ∇u

The eigenvalues of $\nabla u \nabla u^\top$ are 0 and $|\nabla u|^2$ with corresponding eigenvectors $\nabla u^\perp$ and ∇u.

Problem 2 (Mean and Median Filtering)

Given the image

1	7	0	1	1
1	1	1	1	1
7	0	7	1	0
7	7	7	1	1
0	7	0	1	7

which can be interpreted as a noisy version (salt and pepper noise) of the following image

1	1	1	1	1
1	1	1	1	1
7	7	7	1	1
7	7	7	1	1
7	7	7	1	1

- Assuming reflecting boundary conditions we get for mean filtering (i.e. using a normalised box filter) within a mask of size 3×3:

$$\frac{1}{9}\begin{array}{|c|c|c|c|c|}\hline 21 & 19 & 19 & 7 & 9 \\ \hline 26 & 25 & 19 & 13 & 7 \\ \hline 38 & 38 & 26 & 20 & 7 \\ \hline 42 & 42 & 31 & 24 & 19 \\ \hline 35 & 35 & 31 & 25 & 33 \\ \hline \end{array} \approx \begin{array}{|c|c|c|c|c|}\hline 2.33 & 2.11 & 2.11 & 0.77 & 1.00 \\ \hline 2.88 & 2.77 & 1.11 & 1.44 & 0.77 \\ \hline 4.22 & 4.22 & 2.88 & 2.22 & 0.77 \\ \hline 4.66 & 4.66 & 3.44 & 2.66 & 2.11 \\ \hline 3.88 & 3.88 & 3.44 & 2.77 & 3.66 \\ \hline \end{array}$$

  We can observe that the mean filter smoothes the signal by averaging. The noise is removed. However, edges are also smoothed and become thus "blurry". Furthermore, the contrast of the whole image is reduced.

- For median filtering within a mask of size 3×3 we get:

1	1	1	1	1
1	1	1	1	1
7	7	1	1	1
7	7	1	1	1
7	7	1	1	1

  Thus the median filter removes the noise almost perfectly, but in contrast to the mean filter it preserves the edge at the same time. Nevertheless, we also observe, that median filtering leads to roundings of corners and that boundary artifacts can appear due to reflected noise.
