Image Processing and Computer Vision (IPCV)

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Example Solutions for Homework Assignment 2 (H2)

Problem 1 (Properties of the continuous Fourier transform)

Verify that the following properties of the continuous Fourier transform are true.

Linearity: $\mathcal{F}[a \cdot f(x) + b \cdot g(x)](u) = a \cdot \mathcal{F}[f](u) + b \cdot \mathcal{F}[g](u)$

$$
\begin{aligned}
\mathcal{F}[a \cdot f(x) + b \cdot g(x)](u) &= \int_{\mathbb{R}} \left(a \cdot f(x) + b \cdot g(x)\right) e^{-i2\pi ux} \mathrm{d}x \\
&= a \int_{\mathbb{R}} f(x) e^{-i2\pi ux} \mathrm{d}x + b \int_{\mathbb{R}} g(x) e^{-i2\pi ux} \mathrm{d}x \\
&= a \cdot \mathcal{F}[f](u) + b \cdot \mathcal{F}[g](u)
\end{aligned}
$$

Spatial Shift: $\mathcal{F}[f(x-a)](u) = e^{-i2\pi ua} \cdot \mathcal{F}[f](u)$

$$
\begin{aligned}
\mathcal{F}[f(x-a)](u) &= \int_{\mathbb{R}} f(x-a) e^{-i2\pi ux} \mathrm{d}x \quad t \leftrightarrow x-a \\
&= \int_{\mathbb{R}} f(t) e^{-i2\pi u(t+a)} \mathrm{d}t \\
&= e^{-i2\pi ua} \int_{\mathbb{R}} f(t) e^{-i2\pi ut} \mathrm{d}t \\
&= e^{-i2\pi ua} \mathcal{F}[f](u)
\end{aligned}
$$

Frequency Shift: $\mathcal{F}[f(x) \cdot e^{-i2\pi u_0 x}](u) = \mathcal{F}[f](u+u_0)$

$$
\begin{aligned}
\mathcal{F}[f(x) \cdot e^{-i2\pi u_0 x}](u) &= \int_{\mathbb{R}} f(x) e^{-i2\pi u_0 x} e^{-i2\pi ux} \mathrm{d}x \\
&= \int_{\mathbb{R}} f(x) e^{-i2\pi (u_0+u)x} \mathrm{d}x \\
&= \mathcal{F}[f](u_0+u)
\end{aligned}
$$

Scaling: $\mathcal{F}[f(ax)](u) = \frac{1}{|a|} \cdot \mathcal{F}[f](\frac{u}{a})$

$$\begin{aligned}\mathcal{F}[f(ax)](u) &= \int_{\mathbb{R}} f(ax)e^{-i2\pi ux}\mathrm{d}x \quad t \leftrightarrow ax \\ &= \int_{\mathbb{R}} f(t)e^{-i2\pi u\frac{t}{a}}\frac{1}{|a|}\mathrm{d}t \\ &= \frac{1}{|a|}\mathcal{F}[f]\left(\frac{u}{a}\right)\end{aligned}$$

Please note that here, we assume that $a \neq 0$. Furthermore, in the substitution step, we implicitly do a case distinction that results in the absolute value $\frac{1}{|a|}$ as an additional factor. For $a > 0$, the substitution is straightforward since $\frac{dt}{dx} = a \Leftrightarrow dx = \frac{1}{a}dt = \frac{1}{|a|}dt$ and the integral limits stay the same. For $a < 0$, one naturally also gets $\frac{1}{a}$ as an additional factor from the substitution, however the integral limits are flipped, which can be reversed by multiplying the integrand with -1, thus leading to $-\frac{1}{a} = \frac{1}{|a|}$ for $a < 0$.

Convolution: $\mathcal{F}[(f * g)(x)](u) = \mathcal{F}[f](u) \cdot \mathcal{F}[g](u)$

$$\begin{aligned}\mathcal{F}[(f * g)(x)](u) &= \int_{\mathbb{R}} (f * g)(x)e^{-i2\pi ux}\mathrm{d}x \\ &= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(\xi)g(x-\xi)\mathrm{d}\xi\right) e^{-i2\pi ux}\mathrm{d}x \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} f(\xi)g(x-\xi)e^{-i2\pi ux}\mathrm{d}x\mathrm{d}\xi \\ &= \int_{\mathbb{R}} f(\xi)e^{-i2\pi u\xi}\left(\int_{\mathbb{R}} g(x-\xi)e^{-i2\pi u(x-\xi)}\mathrm{d}x\right)\mathrm{d}\xi \quad t \leftrightarrow x-\xi \\ &= \int_{\mathbb{R}} f(\xi)e^{-i2\pi u\xi}\left(\int_{\mathbb{R}} g(t)e^{-i2\pi ut}\mathrm{d}t\right)\mathrm{d}\xi \\ &= \int_{\mathbb{R}} f(\xi)e^{-i2\pi u\xi}\mathrm{d}\xi\mathcal{F}[g](u) \\ &= \mathcal{F}[f](u) \cdot \mathcal{F}[g](u)\end{aligned}$$

Derivative: $\mathcal{F}[f'](u) = i2\pi u \cdot \mathcal{F}[f](u)$

We know that $f(x) = \mathcal{F}^{-1}[\mathcal{F}[f](u)](x)$, therefore

$$f(x) = \int_{\mathbb{R}} \mathcal{F}[f](u) \cdot e^{i2\pi ux}\,du\,.$$

Taking the derivative w.r.t. x on both sides yields

$$f'(x) = \int_{\mathbb{R}} i2\pi u \cdot \mathcal{F}[f](u) \cdot e^{i2\pi ux}\, du = \mathcal{F}^{-1}[i2\pi u \cdot \mathcal{F}[f](u)](x)\,.$$

Finally, we take the Fourier transform on both sides and obtain

$$\mathcal{F}[f'](u) = \mathcal{F}[\mathcal{F}^{-1}[i2\pi u \cdot \mathcal{F}[f](u)](x)](u) = i2\pi u \cdot \mathcal{F}[f](u)\,.$$

An alternative approach is by using a partial integration, where you can use the hint to get rid of the boundary expression.

Problem 2 (Continuous Fourier Transform)

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(\frac{-x^2}{2\sigma^2}\right)$$

(a) The Fourier transform $\hat{f}(u) = \mathcal{F}[f]$ of f can be computed as:

$$\begin{aligned}
\hat{f}(u) &= \int_{-\infty}^{\infty} f(x)\exp(-i2\pi ux)dx \\
&= \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} \exp\left(\frac{-x^2}{2\sigma^2}\right) \exp(-i2\pi ux)dx \\
&= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(\frac{-x^2}{2\sigma^2} - i2\pi ux\right) dx \\
&\overset{(i)}{=} \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\pi\left(\frac{x^2}{2\pi\sigma^2} + i2ux - u^2 2\pi\sigma^2 + u^2 2\pi\sigma^2\right)\right) dx \\
&= \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-2\pi^2u^2\sigma^2\right) \cdot \int_{-\infty}^{\infty} \exp\left(-\pi\left(\left(\frac{x}{\sqrt{2\pi}\sigma}\right)^2 + i2ux + (i\sqrt{2\pi}u\sigma)^2\right)\right) dx \\
&= \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-2\pi^2u^2\sigma^2\right) \cdot \int_{-\infty}^{\infty} \exp\left(-\pi\left(\frac{x}{\sqrt{2\pi}\sigma} + i\sqrt{2\pi}u\sigma\right)^2\right) dx
\end{aligned}$$

i. completing the square (quadratische Ergänzung)

Substitute $\frac{x}{\sqrt{2\pi}\sigma} + i\sqrt{2\pi}u\sigma =: z$

$$\begin{aligned}
\hat{f}(u) &= \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-2\pi^2u^2\sigma^2\right) \cdot \int_{-\infty}^{\infty} \sqrt{2\pi}\sigma \exp\left(-\pi z^2\right) dz \\
&= \exp\left(-2\pi^2u^2\sigma^2\right) \cdot \int_{-\infty}^{\infty} \exp\left(-\pi z^2\right) dz
\end{aligned}$$

Using the hint

$$\int_{-\infty}^{\infty} \exp\left(-\pi x^2\right) dx = 1,$$

we finally see that the Fourier transform of a Gaussian is a Gaussian-like function with inverse variance (compare with Slide 27 of Lecture 4):

$$\hat{f}(u) = \exp\left(\frac{-(2\pi u)^2}{2\sigma^{-2}}\right)$$

(b) Finally, we are in the position to compute the Fourier spectrum of the Gaussian. Since the Fourier transform $\hat{f}(u)$ is real-valued in the case of the Gaussian, it is identical to the Fourier spectrum $|\hat{f}(u)|$:

$$\left|\hat{f}(u)\right| = \exp\left(\frac{-(2\pi u)^2}{2\sigma^{-2}}\right) .$$

Problem 3 (Symmetry and Antisymmetry of the Fourier Transform)

(a) The Fourier transform of f is given by:

$$\begin{aligned}\hat{f}(u) &= \int_{-\infty}^{\infty} f(x) e^{-i2\pi ux} dx \\ &= \int_{-\infty}^{\infty} f(x) \left(\cos(-2\pi ux) + i \sin(-2\pi ux)\right) dx \\ &= \int_{-\infty}^{\infty} f(x) \cos(-2\pi ux) dx + i \int_{-\infty}^{\infty} f(x) \sin(-2\pi ux) dx.\end{aligned}$$

Here we have used Euler's formula. If f is real then both of the integrals are real and the Fourier transform of f has the real part:

$$Re\left(\hat{f}(u)\right) = \int_{-\infty}^{\infty} f(x) \cos(-2\pi ux) dx$$

and the imaginary part:

$$Im\left(\hat{f}(u)\right) = \int_{-\infty}^{\infty} f(x) \sin(-2\pi ux) dx.$$

We can now verify that:

$$\begin{aligned}
Re\left(\hat{f}(-u)\right) &= \int_{-\infty}^{\infty} f(x)\cos(2\pi ux)dx \\
&= \int_{-\infty}^{\infty} f(x)\frac{e^{2i\pi ux} + e^{-2i\pi ux}}{2}dx \\
&= \int_{-\infty}^{\infty} f(x)\frac{e^{2i\pi(-u)x} + e^{-2i\pi(-u)x}}{2}dx \\
&= \int_{-\infty}^{\infty} f(x)\cos(-2\pi ux)dx \\
&= Re\left(\hat{f}(u)\right)
\end{aligned}$$

i.e. the real part of $\hat{f}(u)$ is symmetric around the origin. Identically we can verify that

$$\begin{aligned}
Im\left(\hat{f}(-u)\right) &= \int_{-\infty}^{\infty} f(x)\sin(2\pi ux)dx \\
&= \int_{-\infty}^{\infty} f(x)\frac{e^{2i\pi ux} - e^{-2i\pi ux}}{2}dx \\
&= -\int_{-\infty}^{\infty} f(x)\frac{e^{2i\pi(-u)x} - e^{-2i\pi(-u)x}}{2}dx \\
&= -\int_{-\infty}^{\infty} f(x)\sin(-2\pi ux)dx \\
&= -Im\left(\hat{f}(u)\right)
\end{aligned}$$

i.e. the imaginary part of $\hat{f}(u)$ is antisymmetric around the origin.

(b) Our frequency u is representing the contribution of the function

$$b_u(x) = e^{2i\pi ux}$$

in the spatial domain. If we concider the frequency $-u$ we get:

$$\begin{aligned}
b_{-u}(x) &= e^{-2i\pi ux} \\
&= \cos(-2\pi ux) + i\sin(-2\pi ux) \\
&= \frac{e^{-i2\pi ux} + e^{2i\pi ux}}{2} + i\frac{e^{-2i\pi ux} - e^{2i\pi ux}}{2} \\
&= \frac{e^{2i\pi ux} + e^{-2i\pi ux}}{2} - i\frac{e^{2i\pi ux} - e^{-2i\pi ux}}{2} \\
&= \cos(2\pi ux) - i\sin(2\pi ux) \\
&= \cos(2\pi ux) + i\sin(\pi + 2\pi ux)
\end{aligned}$$

We now see the negative frequencies have the same real part as the positive frequencies. For the imaginary part, we both see that its sign has changed and that it can be viewed as a shifted positive frequency.

Problem 4 (Colour Spaces)

(a) The conversion from RGB to YCbCr images requires to supplement the following code:

```
/* computes the YCbCr values */
for (i=1;i<=nx;i++)
  for (j=1;j<=ny;j++)
      {
        u_YCbCr[0][i][j]  =   0.299 * u_RGB[0][i][j]
                            + 0.587 * u_RGB[1][i][j]
                            + 0.114 * u_RGB[2][i][j];
        u_YCbCr[1][i][j] = 127.5 + (- 0.169 * u_RGB[0][i][j]
                                    - 0.331 * u_RGB[1][i][j]
                                    + 0.500 * u_RGB[2][i][j]);
        u_YCbCr[2][i][j] = 127.5 + (  0.500 * u_RGB[0][i][j]
                                    - 0.419 * u_RGB[1][i][j]
                                    - 0.081 * u_RGB[2][i][j]);
      }
```

 

 

Table 1: Compressed variants of the image `kodim14.ppm`. *(a) Top left:* Image for $S = 1$ (original image). *(b) Top right:* $S = 2$. *(c) Bottom left:* $S = 4$. *(d) Bottom right:* $S = 8$.

(b) The compressed variants of the image `kodim14.ppm` for subsampling factors of $S = 1, 2, 4$ and 8 are depicted in Tab 1. As one can see, the variants for $S = 2$ and $S = 4$ provide still a quite good quality. This is due to the fact that the details (vests, waves and rocks) are preserved, since the Y-channel that contains these details is not compressed (downsampled). In the case of $S = 8$, however, slight block artifacts become visible. This is verified by the images in Tab. 2 that depict a zoom of one of the vest regions. Here also for $S = 4$ the loss of quality becomes obvious.

(c) While the original RGB image requires $3 \cdot 8 = 24$ bpp to store the information, the requirements of the compressed (subsampled) images is given by

$$v = 8\left(1 + \frac{1}{S^2} + \frac{1}{S^2}\right) .$$

While the Y channels remain uncompressed, the Cb- and Cr-channel are reduced in both dimensions by a factor of S. Thus, 12 bpp for $S = 2$, 9 bpp for $S = 4$, and 8.25 bpp for $S = 8$ are needed. This is in the order of the memory consumption of grey value images.

Table 2: Zoom into the compressed variants of the image `kodim14.ppm`. *(a) Top left:* Image for $S = 1$ (original image). *(b) Top right:* $S = 2$. *(c) Bottom left:* $S = 4$. *(d) Bottom right:* $S = 8$.