2. a) Find the sol of form $X_k = a \cdot 3^k$ of the $X_{k+1} = 2 \cdot X_k + 3^k$

$$a \cdot 3^{k+1} = 2 \cdot a \cdot 3^k + 3^k$$

$$3a \cdot 3^k = 3^k (2a+1)$$

$$3a = 2a+1 \Rightarrow a = 1$$

$$\Rightarrow x_k^p = 3^k \quad \text{(particular sol)}$$

b) Find the general sol

$$x_{k+1} - 2x_k = 0$$

$$r^1 - 2r^0 = 0$$

$$r = 2, \quad x_k^h = r^k$$

$$\Rightarrow x_k^h = c \cdot 2^k \quad \text{- hom sol}$$

$$\Rightarrow x_k = c \cdot 2^k + 3^k \quad \text{- gen. sol}$$

c) Sol of ivp $\begin{cases} x_{k+1} = 2x_k + 3^k \\ x_0 = 0 \end{cases}$

$$x_0 = 0 \Leftrightarrow c \cdot 2^0 + 3^0 = 0$$

$$c + 1 = 0$$

$$c = -1$$

$$\Rightarrow x_k = 3^k - 2^k$$

3. Find the sol of form $x_k = ak+b$ of the

$x_{k+1} = -5x_k - k \quad , k \geq 0 \quad , a,b \in \mathbb{R}$

$a(k+1) + b = -5(ak+b) - k$

$ak + a + b = -5ak - k - 5b \Rightarrow$

$\Rightarrow \begin{cases} a = -5a - 1 \\ a + b = -5b \end{cases} \Rightarrow \begin{cases} a = -\frac{1}{6} \\ b = \frac{1}{36} \end{cases}$

$\Rightarrow x_k^p = -\frac{1}{6} \cdot k + \frac{1}{36}$

b) Find the general sol.

$x_{k+1} + 5x_k = 0$

$r^{k+1} + 5r^k = 0 \Rightarrow r = -5$

$x_k^h = c \cdot (-5)^k$

$\Rightarrow x_k = x_k^p + x_k^h = -\frac{1}{6}k + \frac{1}{36} + c \cdot (-5)^k \quad , c \in \mathbb{R}$

c) IVP $\begin{cases} x_{k+1} = -5x_k - k \\ x_0 = -1 \end{cases}$

$y_k = c \cdot (-7)^k - \frac{1}{6}k + \frac{1}{36}$

$\Rightarrow x_0 = c + \frac{1}{36} \Rightarrow c + \frac{1}{36} = -1 \Rightarrow c = -\frac{37}{36} \Rightarrow$

$\Rightarrow x_k = -\frac{37}{36} \cdot (-7)^k - \frac{1}{6}k + \frac{1}{36}$ - sol of IVP

4) Find the general sol of:

a) $x_{k+2} - 6x_{k+1} + 9x_k = 0$

The char. eq: $r^2 - 6r + 9 = 0$

$\Delta = 0 \Rightarrow r_1 = r_2 = 3$

$x_k^1 = 3^k$

$x_k^2 = k \cdot 3^k$

gen sol: $x_k = c_1 \cdot 3^k + c_2 \cdot k \cdot 3^k$

b) $x_{k+2} - 2x_{k+1} + x_k = 0$

The char. eq: $r^2 - 2r + 1 = 0$

$\Delta = 0 \Rightarrow r_1 = r_2 = 1$

$x_k^1 = 1$

$x_k^2 = k$

$x_k = c_1 + c_2 \cdot k, \; c_1, c_2 \in \mathbb{R}$

c) $x_{k+2} + x_{k+1} + x_k = 0$

$$r^2 + r + 1 = 0 \Rightarrow \Delta = 1 - 4 = -3$$

$$\Rightarrow r_{1,2} = \frac{-1 \pm i\sqrt{3}}{2}$$

$$z = \rho(\cos\theta + i\sin\theta)$$

$$\rho = \sqrt{\left(-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)} = 1$$

$$\left.\begin{array}{l} \cos\theta = -\dfrac{1}{2} \\ \sin\theta = \dfrac{\sqrt{3}}{2} \end{array}\right\} \Rightarrow \theta = \frac{2\pi}{3}$$

$$z^k = \left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)^k = \underbrace{\cos\frac{2\pi k}{3}}_{x_k^1} + i\,\underbrace{\sin\frac{2\pi k}{3}}_{x^k}$$

$$x_k = c_1 \cdot \cos\frac{2\pi k}{3} + c_2 \cdot \sin\frac{2\pi k}{3}$$

5. Find the expression of the Fibonacci sequence

$$x_{k+2} = x_{k+1} + x_k \quad , \quad x_0 = 0 \quad , \quad x_1 = 1$$

$$r^{k+2} = r^{k+1} + r^k$$

$$r^2 = r + 1$$

$$r^2 - r - 1 = 0$$

$$r_{1,2} = \frac{1 \pm \sqrt{5}}{2}$$

$$\Rightarrow x_k = C_1 \cdot r_1^k + C_2 \cdot r_2^k$$

$$x_0 = 0 \Rightarrow \quad C_1 \cdot r_1^0 + C_2 \cdot r_2^0 = 0$$

$$C_1 + C_2 = 0 \quad \Rightarrow C_1 = -C_2 \quad , \quad C_2 = -C_1$$

$$x_1 = 1 \Rightarrow \quad C_1 \cdot r_1 + C_2 \cdot r_2 = 1$$

$$C_1 \cdot \frac{1+\sqrt{5}}{2} + C_2 \cdot \frac{1-\sqrt{5}}{2} = 1$$

$$C_1 \cdot \frac{1+\sqrt{5}}{2} - C_1 \frac{1-\sqrt{5}}{2} = 1$$

$$C_1 \left(\frac{1+\sqrt{5} - 1 + \sqrt{5}}{2} \right) = 1$$

$$C_1 \cdot \sqrt{5} = 1 \quad \Rightarrow C_1 = \frac{1}{\sqrt{5}} \qquad C_2 = -\frac{1}{\sqrt{5}}$$

$$x_k = \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2}\right)^k - \left(\frac{1-\sqrt{5}}{2}\right)^k \right)$$

6. Find the linear homogeneous difference eq of minimal order that has the solution $(x_k)_{k\geq 0}$ such that:

a) $x_k = \frac{7}{2^k} - \frac{2}{3^k}$, $k \geq 0$

$\Rightarrow r = \frac{1}{2}$, $r = \frac{1}{3}$

$$\left(r - \frac{1}{2}\right)\left(r - \frac{1}{3}\right) = 0$$

$$r^2 - \left(\frac{1}{2} + \frac{1}{3}\right) r + \frac{1}{6} = 0$$

$$r^2 - \frac{5}{6} r + \frac{1}{6} = 0$$

$$x_{k+2} - \frac{5}{6} x_{k+1} + \frac{1}{6} x_k = 0$$

b) $x_k = 7 \operatorname{Re}(i^k) - 2 \operatorname{Im}(i^k)$, $k \geq 0$

$$i^k = \begin{cases} 1 : k \% 4 = 0 \\ i : k \% 4 = 1 \\ -1 : k \% 4 = 2 \\ -i : k \% 4 = 3 \end{cases}$$

$$\operatorname{Re}(i^k) = \begin{cases} 1 : k \% 4 = 0 \\ 0 : k \% 4 = 1 \\ -1 : k \% 4 = 2 \\ 0 : k \% 4 = 3 \end{cases} \qquad \operatorname{Im}(i^k) = \begin{cases} 0 : k \% 4 = 0 \\ 1 : k \% 4 = 1 \\ 0 : k \% 4 = 2 \\ -1 : k \% 4 = 3 \end{cases}$$

$$i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$$

$$i^k = \cos\frac{k\pi}{2} + i\sin\frac{k\pi}{2}$$

$$z = \rho(\cos\theta + i\sin\theta)$$

$$z^n = \rho(\cos(n\theta) + i\sin(n\theta))$$

$$Re(i^k) = \cos\frac{k\pi}{2} \Rightarrow x_k^1$$

$$Im(i^k) = \sin\frac{k\pi}{2} \Rightarrow x_k^2$$

$$\Rightarrow x_k = c_1\cos\frac{k\pi}{2} + c_2\sin\frac{k\pi}{2}$$

$$r = \pm i$$

$$(r-i)(r+i) = 0$$

$$r^2+1 = 0$$

$$x_{k+2} + x_k = 0$$