1. Study the limit at (0,0): $(x,y) \to (0,0)$

a) $\frac{x^2-y^2}{x^2+y^2}$ $\boxed{y = m \cdot x}$

$$\lim_{\substack{(x,y)\to(0,0)\\ y=x}} \frac{x^2-y^2}{x^2+y^2} = \lim_{x\to 0} \frac{x^2-x^2}{x^2+x^2} = 0$$

$$\lim_{\substack{(x,y)\to(0,0)\\ y=2x}} \frac{x^2-y^2}{x^2+y^2} = \lim_{x\to 0} \frac{x^2-4x^2}{x^2+4x^2} = -\frac{3}{5}$$

$$\Rightarrow \nexists \lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$$

b) $\frac{x+y}{x^2+y^2}$

if $y=0 \Rightarrow \lim_{\substack{(x,y)\to(0,0)\\ y=0}} \frac{x+y}{x^2+y^2} = \lim_{x\to 0} \frac{x+0}{x^2+0} = \lim_{x\to 0} \frac{1}{x} = \pm\infty \Rightarrow$

$\Rightarrow \lim_{x\to 0} \frac{1}{x}$ does not exist $\Rightarrow \lim_{(x,y)\to(0,0)} \frac{x+y}{x^2+y^2}$ does not exist

second method

II if we chose $y = m\cdot x \Rightarrow$ the limit depends on $m \Rightarrow \nexists$

c) $\dfrac{x^3+y^3}{x^2+y^2}$

$$\lim_{\substack{(x,y)\to(0,0)\\ y=m\cdot x}} \frac{x^3+y^3}{x^2+y^2} = \lim_{(x,y)\to(0,0)} \frac{x^3+(m\cdot x)^3}{x^2+(m\cdot x)^2} = \lim_{x\to 0} \frac{\cancel{x^3}(1+m^3)}{\cancel{x^2}(1+m^2)} =$$

$$= \lim_{x\to 0} \frac{x(1+m^3)}{1+m^2} = 0$$

$$\left|\frac{x^3+y^3}{x^2+y^2}\right| \leq \frac{|x^3+y^3|}{x^2+y^2} \leq \frac{|x^3|+|y^3|}{x^2+y^2} \leq \frac{|x|\cdot x^2+|y|\cdot y^2}{x^2+y^2} \leq$$

$\boxed{|a+b| \leq |a|+|b| \text{ triangle inequality}}$

$$\leq \frac{|x|(x^2+y^2)+|y|(x^2+y^2)}{x^2+y^2} \leq$$

$$\leq |x|+|y| \longrightarrow 0 \text{ when } (x,y)\to(0,0)$$

$$\Rightarrow \lim \frac{x^3+y^3}{x^2+y^2} = 0$$

d) $\dfrac{\sin x - \sin y}{x-y}$

$$\lim_{\substack{(x,y)\to(0,0)\\ y=-x}} \frac{\sin x - \sin y}{x-y} = \lim_{x\to 0} \frac{\sin x + \sin x}{2x} = \lim_{x\to 0} \frac{\cancel{2}\sin x}{\cancel{2}x} = 1$$

$$\lim_{(x,y)\to(0,0)} \frac{\sin x - \sin y}{x-y} = \lim_{(x,y)\to(0,0)} \frac{\sin x - \sin y}{x\cdot y} \cdot \frac{x\cdot y}{x-y} =$$

$$= \lim_{(x,y)\to(0,0)} \left(\overset{\to 1}{\frac{\sin x}{x\cdot y}} - \overset{\to 1}{\frac{\sin y}{x\cdot y}} \right) \frac{x\cdot y}{x-y} =$$

$$= \lim_{(x,y)\to(0,0)} \left({}^{x/}\frac{1}{y} - {}^{y}\frac{1}{x} \right) \cdot \frac{x\cdot y}{x-y} = \lim_{(x,y)\to(0,0)} \frac{x-y}{x\cdot y} \cdot \frac{x\cdot y}{x-y} = 1$$

or

$$\frac{\sin x - \sin y}{x-y} = \frac{2\cdot \overset{\to 0}{\cos\frac{x+y}{2}} \cdot \overset{\to 1}{\sin\frac{x-y}{2}}}{\frac{x-y}{2}\cdot 2} \overset{(x,y)\to(0,0)}{=} 1$$

2. Compute the partial derivatives:

d) $f(x,y,z) = x^2 \cdot y \cdot z + y \cdot e^z$

$$\frac{df}{dx}(x,y,z) \underset{\underline{\hspace{2cm}}}{\overset{\text{x-variable}}{\underline{\text{y,z-constants}}}} \frac{d}{dx}\left(x^2yz + ye^z\right) =$$

$$= 2x \cdot yz + 0 = 2xyz$$

$$\frac{df}{dy}(x,y,z) \underset{\underline{\hspace{2cm}}}{\overset{\text{y variable}}{\underline{\text{x,z-constants}}}} x^2 \cdot 1 \cdot z + 1 \cdot e^z = x^2 \cdot z + e^z$$

$$\frac{df}{dz}(x,y,z) \underset{\underline{\hspace{2cm}}}{\overset{\text{z-variable}}{\underline{\text{x,y-constants}}}} x^2 \cdot y + y \cdot e^z$$

$$\frac{d^2f}{dx\,dy}(x,y,z) = \frac{d}{dx}\left(\frac{df}{dy}(x,y,z)\right) = \frac{d}{dx}\left(x^2z + e^z\right) =$$

$= 2xz$ (this is how to do it if we need the 2^{nd} order partial derivative)

a) $f(x,y) = e^{-(x^2+y^2)}$

$$\frac{df}{dx}(x,y) \xrightarrow[\text{y - constant}]{\text{x - variable}} \frac{d}{dx} e^{-(x^2+y^2)} = e^{-(x^2+y^2)} \frac{df}{dx}(x^2+y^2)$$

$$= e^{-(x^2+y^2)} \cdot (-2x)$$

$$\frac{df}{dy} \xrightarrow[\text{x-constant}]{\text{y - variable}} \frac{d}{dy} e^{-(x^2+y^2)} = e^{-(x^2+y^2)} \cdot (-2y)$$

$\Rightarrow$ Thus $\exists$ everywhere

b) $f(x,y) = \cos x \cos y - \sin x \sin y = \cos(x+y)$

$$\frac{df(x,y)}{dx} = -\sin x \cos y - \cos x \sin y = -\sin(x+y)$$

$$\frac{df(x,y)}{dy} = -\cos x \sin y - \sin x \cos y = -\sin(x+y)$$

3) $f: \mathbb{R}^2 \to \mathbb{R}$ $\quad f(x,y) = x \cdot y$. Using the definition prove that $Df(x_0,y_0) = (y_0, x_0) = \nabla f(x_0,y_0) =$

$$= \left(\frac{df}{dx}(x_0,y_0) \,,\, \frac{df}{dy}(x_0,y_0) \right)$$

Def 9.6. $\displaystyle \lim_{x \to x_0} \frac{\| f(x) - f(x_0) - Df(x_0) \cdot (x - x_0) \|}{\| x - x_0 \|} = 0 \quad , x \in \mathbb{R}^n$

$$L = \lim_{(x,y) \to (x_0,y_0)} \frac{| f(x,y) - f(x_0,y_0) - Df(x_0,y_0) \cdot (x - x_0, y - y_0) |}{\sqrt{(x-x_0)^2 + (y-y_0)^2}} =$$

$$= \lim_{(x,y) \to (x_0,y_0)} \frac{x \cdot y - x_0 y_0 - (y_0, x_0) \cdot (x - x_0, y - y_0)}{\sqrt{(x-x_0)^2 + (y-y_0)^2}} =$$

$$= \lim_{(x,y) \to (x_0,y_0)} \frac{| xy - \cancel{x_0 y_0} - (y_0 x - y_0 x + x_0 y - \cancel{x_0 y}) |}{\sqrt{(x-x_0)^2 + (y-y_0)^2}}$$

$$u = x - x_0$$
$$v = y - y_0$$

$$= \lim_{(x,y) \to (x_0,y_0)} \frac{| (y - y_0)(x - x_0) |}{\sqrt{(x-x_0)^2 + (y-y_0)^2}} =$$

$$= \lim_{(u,v)\to(0,0)} \frac{|u\cdot v|}{\sqrt{u^2+v^2}} = 0 \Rightarrow Df(x_0,y_0) = (y_0, x_0)$$

$$\frac{|u\cdot v|}{\sqrt{u^2+v^2}} = \frac{\sqrt{u^2}\;\;\sqrt{v^2}}{\sqrt{u^2+v^2}} \le \frac{\cancel{\sqrt{u^2+v^2}}\cdot\sqrt{v^2+u^2}}{\cancel{\sqrt{u^2+v^2}}} =$$

$$= \sqrt{u^2+v^2} \xrightarrow{(u,v)\to(0,0)} 0$$

4. Prove that: $f(x,y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}}, & (x,y)\neq(0,0) \\ 0, & (x,y)=(0,0) \end{cases}$

is continuous, has partial derivatives, but it's not differentiable at the origin

f. is continuous at $(0,0) \Leftrightarrow \lim_{(x,y)\to(0,0)} f(x,y) = f(0,0)$

$$\left.\begin{aligned} &\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} = 0 \text{ (see ex. 3)} \\ &f(0,0)=0 \end{aligned}\right\} \Rightarrow f \text{ is continuous at origin}$$

$$\frac{df}{dx}(0,0) = \lim_{x\to 0} \frac{f(x,0)-f(0,0)}{x-0} = \lim_{x\to 0} \frac{\frac{x\cdot 0}{\sqrt{x^2}} - 0}{x} = 0$$

$$\frac{df}{dy}(0,0) = \lim_{y\to 0} \frac{f(0,y)-f(0,0)}{0-y} = \lim_{y\to 0} \frac{\frac{0y}{\sqrt{y^2}} - 0}{-y} = 0$$

$\Rightarrow$ it has partial derivatives

$$\lim_{(x,y)\to(0,0)} \frac{|f(x,y) - f(0,0) - Df(0,0)\cdot(x,y)|}{\sqrt{x^2+y^2}} =$$

$$= \lim_{(x,y)\to(0,0)} \frac{\left|\frac{xy}{\sqrt{x^2+y^2}} - 0 - (0,0)\cdot(x,y)\right|}{\sqrt{x^2+y^2}} = \lim_{(x,y)\to(0,0)} \frac{|xy|}{x^2y^2}$$

limit does not exist $\left(\begin{array}{l} y = x \Rightarrow \lim \to \frac{1}{2} \\ y = 0 \Rightarrow \lim \to 0 \end{array} \right.$

$\Rightarrow f$ is not differentiable at the origin