1) Using the ε-definition prove that

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$$

$x_1, x_2, x_3, \dots\dots, x \in \mathbb{R}$

$(x_n)_{n\in\mathbb{N}^*}$ sequence has limit $L \in \mathbb{R} \overset{def}{\Leftrightarrow}$

$\overset{def}{\Leftrightarrow} \forall \varepsilon > 0 \; \exists \; N_\varepsilon \in \mathbb{N} : |x_n - L| < \varepsilon \; \forall n \geq N_\varepsilon$

$x_n = \frac{1}{\sqrt{n}} \quad L = 0 \quad \forall \varepsilon \quad ? \exists N_\varepsilon \in \mathbb{N} : \left| \frac{1}{\sqrt{n}} - 0 \right| < \varepsilon \Leftrightarrow$

Let $N_\varepsilon = \left[\frac{1}{\varepsilon^2}\right] + 1$

$\Leftrightarrow \frac{1}{\sqrt{n}} < \varepsilon \Leftrightarrow n > \frac{1}{\varepsilon^2}$

2) Study the boundedness, monotonicity and convergence

a) $x_n = \sqrt{n+1} - \sqrt{n}$

(x_n) increasing $\Leftrightarrow x_{n+1} \geq x_n$

(x_n) decreasing $\Leftrightarrow x_{n+1} \leq x_n$

$\forall n \in \mathbb{N}^*$

$x_{n+1} - x_n = \sqrt{n+2} - \sqrt{n+1} - \sqrt{n+1} + \sqrt{n} =$

$= \sqrt{n+2} - 2\sqrt{n+1} + \sqrt{n}$

$$\sqrt{n+2} - 2\sqrt{n+1} + \sqrt{n} \overset{?}{<} 0$$

$$\sqrt{n+2} + \sqrt{n} < 2\sqrt{n+1} \quad / ()^2$$

$$n+2 + 2\sqrt{n+2}\cdot\sqrt{n} + n < 4(n+1)$$

$$2\sqrt{n^2+2n} < 2n+2$$

$$\sqrt{n^2+2n} < n+1 \quad ()^2$$

$$\cancel{n^2}+\cancel{2n} < \cancel{n^2}+\cancel{2n}+1 \Leftrightarrow 0 < 1 \quad {}_n\top$$

$\Rightarrow$ x is decreasing

$x_n > 0 \Rightarrow x_n$ is bounded from below by 0 $\Big\} \Rightarrow$

$\Rightarrow x_n$ is convergent

extra: find the limit

$$x_n = \frac{\sqrt{n+1}+\sqrt{n}}{} \sqrt{n+1} - \sqrt{n} \Rightarrow \frac{n+1-n}{\sqrt{n+1}+\sqrt{n}} \to 0$$

b) $x_n = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \dots + \frac{1}{n(n+1)} =$

$$= \frac{2-1}{1\cdot 2} + \frac{3-2}{2\cdot 3} + \dots + \frac{(n+1)-n}{n(n+1)} =$$

$$= \frac{\cancel{2}}{1\cdot \cancel{2}} - \cancel{\frac{1}{1\cdot 2}} + \cancel{\frac{3}{2\cdot 3}} - \cancel{\frac{2}{2\cdot 3}} + \dots + \cancel{\frac{1}{n}} - \frac{1}{n+1} =$$

$$= 1 - \frac{1}{n+1} = \frac{n}{n+1}$$

$$\left.\begin{array}{l} n \in \mathbb{N}^* \Rightarrow \frac{n}{n+1} > 0 \\ n < n+1 \Rightarrow \frac{n}{n+1} < 1 \end{array}\right\} \Rightarrow \left.\begin{array}{l} x_n \in (0,1) \text{ - bounded} \\ x_n \text{ - increasing} \end{array}\right\} \Rightarrow$$

$\Rightarrow x_n$ - convergent

$$\lim_{n\to\infty} x_n = 1$$

d) $x_n = \frac{2^n}{n!}$

$$\frac{x_{n+1}}{x_n} = \frac{\frac{2^{n+1}}{(n+1)!}}{\frac{2^n}{n!}} = \frac{n!}{(n+1)!} \cdot \frac{2^{n+1}}{2^n} = \frac{1}{n+1} \cdot 2 =$$

$$= \frac{2}{n+1} \leq 1 \Rightarrow x_n \text{ is decreasing}$$

$$\left.\begin{array}{l} 2^n > 0 \\ n! > 0 \end{array}\right\} \Rightarrow x_n > 0 \ \forall n \in \mathbb{N}^*$$

$\Rightarrow x_n$ is convergent

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} < 1 \Rightarrow \lim_{n\to\infty} x_n = 0$$

3) Find the limit

a) $\sqrt{n}\left(\sqrt{n+1}-\sqrt{n}\right)$

$$\lim_{n\to\infty}\sqrt{n}\left(\sqrt{n+1}-\sqrt{n}\right)=\lim_{n\to\infty}\frac{\sqrt{n}\left(\sqrt{n+1}-\sqrt{n}\right)\left(\sqrt{n+1}+\sqrt{n}\right)}{\sqrt{n+1}+\sqrt{n}}=$$

$$=\lim_{n\to\infty}\frac{\sqrt{n}\left(\cancel{n}+1-\cancel{n}\right)}{\sqrt{n+1}+\sqrt{n}}=\lim_{n\to\infty}\frac{\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}=$$

$$=\lim_{n\to\infty}\frac{\sqrt{n}}{\sqrt{n}\left(\sqrt{1+\frac{1}{n}}+1\right)}=\frac{1}{1+1}=\frac{1}{2}$$

b) $\left(a_1^n+a_2^n+\dots+a_k^n\right)^{\frac{1}{n}}$, $a_i>0$ $n\to\infty$, $k\in\mathbb{N}^*$

$$\left(2^n+3^n\right)^{\frac{1}{n}}$$

$$\left[3^{\cancel{n}}\left(\frac{2^n}{3^n}+1\right)\right]^{\cancel{\frac{1}{n}}}=3\left(\left(\frac{2}{3}\right)^n+1\right)\xrightarrow{n\to\infty}3$$

We assume $a_k\geq a_{k-1}\geq\dots\geq a_1$

$$\Rightarrow a_k=\left(a_k^n\right)^{\frac{1}{n}}\leq\underbrace{\left(a_1^n+a_2^n+\dots+a_k^n\right)^{\frac{1}{n}}}_{x_n}\leq\left(k\cdot a_k^n\right)^{\frac{1}{n}}$$

$$\left(k\cdot a_k^n\right)^{\frac{1}{n}}=\underbrace{k^{\frac{1}{n}}\cdot a_k}\xrightarrow{\text{tends to}}a_k$$

$$\lim_{n\to\infty}k^{\frac{1}{n}}\cdot a_k=a_k\Rightarrow$$

$$\Rightarrow\lim_{n\to\infty}x_n=a_k=\max\left(a_1,\dots,a_k\right)$$

c) $\sqrt[n]{n}$

$$\lim_{n\to\infty} \sqrt[n]{n} = \lim_{n\to\infty} e^{\ln \sqrt[n]{n}} = \lim_{n\to\infty} e^{\frac{1}{n}\ln n} =$$

$$= \lim_{n\to\infty} e^{\frac{\ln n}{n}} = e^{\lim\limits_{n\to\infty} \frac{\ln n}{n}} = e^{0} = 1$$

$n < e^{n}, \ \forall n \in \mathbb{N}$

$\ln n < n, \ \forall n \in \mathbb{N}$

$\frac{\ln n}{n} < 1$

$$0 \le \frac{\ln n}{n} \le \frac{2 \ln \sqrt{n}}{n} < \frac{2\sqrt{n}}{n} \xrightarrow{\text{tends to}} 0$$

$$\Rightarrow \frac{\ln n}{n} \to 0$$

e) $$\lim_{n\to\infty} \frac{2^{n} + (-1)^{n}}{3^{n}} = \lim_{n\to\infty} \left(\frac{2^{n}}{3^{n}} + \frac{(-1)^{n}}{3^{n}} \right) =$$

$$= \lim_{n\to\infty} \left[\left(\frac{2}{3}\right)^{n} \right] = 0$$

f) $\lim\limits_{n\to\infty} \dfrac{(a\cdot n+1)^2}{4n^2-2n+1}$, $a\in\mathbb{R}$

$$= \lim_{n\to\infty} \frac{a^2n^2+2an+1}{4n^2-2n+1} \overset{\left(\frac{\infty}{\infty}\right)}{=} \lim_{n\to\infty} \frac{n^2\left(a^2+\frac{2a}{n}+\frac{1}{n^2}\right)}{n^2\left(4-\frac{2}{n}+\frac{1}{n^2}\right)} = \frac{a^2}{4}$$

4) Prove that the sequence $e_n = \left(1+\frac{1}{n}\right)^n$ is increasing and bounded.

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k \cdot b^{n-k} \qquad \binom{n}{k} = \frac{n!}{k!(n-k)!} \text{ n choose k}$$

$$(1+x)^n = 1+\binom{n}{1}\cdot x+\binom{n}{2}\cdot x^2+\dots+\binom{n}{k}\cdot x^k+\dots+x^n$$

$$e_n = \Big(1+\underbrace{\tfrac{1}{n}}_{x}\Big)^n = 1+1+\binom{n}{2}\frac{1}{n^2}+\dots+\binom{n}{k}\cdot\frac{1}{n^k}+\dots+\frac{1}{n^n} =$$

$$= 1+1+\frac{n(n-1)}{2}\cdot\frac{1}{n^2}+\dots+\frac{n(n-1)\dots(n-k+1)}{k!}\cdot\frac{1}{n^k}+$$

$$\dots+\frac{1}{n^n} =$$

$$= 2+\frac{1}{2!}\cdot\underbrace{\left(1-\frac{1}{n}\right)}_{<1}+\dots+\frac{1}{k!}\underbrace{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots}_{<1}$$

$$\underbrace{\cdot\left(1-\frac{k-1}{n}\right)}_{<1}+\dots+\frac{1}{n^n} < 2+\frac{1}{2!}+\dots+\frac{1}{k!}+\dots+\frac{1}{n^n} <$$

$$< 2 + \frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^{n-1}}$$

$$< 1$$

$$< 3$$

$$k! > 2^{k-1}$$

$$n^n = n \cdot n \cdot n \dots > 1 \cdot 2 \cdot \dots \cdot n$$

$$\frac{1}{n^n} < \frac{1}{n!} < \frac{1}{2^{n-1}}$$

$\Rightarrow e_n < 3 \quad (1)$

$$e_n = 1 + 1 + \frac{1}{2!} \cdot \left(1 - \frac{1}{n}\right) + \dots + \frac{1}{k!}\left(1 - \frac{1}{n}\right) \cdot \ldots \cdot \left(1 - \frac{k-1}{n}\right) + \dots + \frac{1}{n^n}$$

$$\wedge$$

$$e_{n+1} = 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{n+1}\right) + \dots + \frac{1}{k!}\left(1 - \frac{1}{n+1}\right) \cdot \ldots \cdot \left(1 - \frac{k-1}{n+1}\right) + \dots + \frac{1}{(n+1)^{n+1}}$$

$\Rightarrow e_{n+1} > e_n \Rightarrow (e_n)$ is increasing (2)

$(1), (2) \Rightarrow e_n$ is convergent

7) a) $\lim\limits_{n\to\infty}\left(\frac{2n+1}{2n-1}\right)^n = \lim\limits_{n\to\infty}\left(1+\frac{2n+1}{2n-1}-1\right)^n =$

$= \lim\limits_{n\to\infty}\left(1+\frac{2n+1-2n+1}{2n-1}\right)^n = \lim\limits_{n\to\infty}\left[\left(1+\frac{2}{2n-1}\right)^{\frac{2n-1}{2}}\right]^{\frac{2}{2n-1}\cdot n}$

$= e^{\lim\limits_{n\to\infty}\frac{2n}{2n-1}} = e^{\lim\limits_{n\to\infty}\frac{2n}{n(2-\underbrace{\frac{1}{n}}_{\to 0})}} = e$