$K = (O, \mathcal{B})$ reference frame

$O \in E^n$

$\mathcal{B}$ basis of V^n

We define $[P]_K = [\underbrace{\overrightarrow{OP}}_{\text{"}\vec{r}_P\text{"}}]_{\mathcal{B}}$

<u>Notations:</u> $\varphi : V_1 \to V_2$ linear map

$\mathcal{B}_1, \mathcal{B}_2$ are basis of V_1, V_2

$$M_{\mathcal{B}_2\mathcal{B}_1}(\varphi) = [\varphi]_{\mathcal{B}_1,\mathcal{B}_2} = \left([\varphi(v_1)]_{\mathcal{B}_2} \cdots\cdots [\varphi(v_n)]_{\mathcal{B}_2} \right)$$

V, $\mathcal{B}, \mathcal{B}'$ bases of V

The base change matrix from $\mathcal{B}$ to $\mathcal{B}'$ is

$$M_{\mathcal{B}',\mathcal{B}} = [id]_{\mathcal{B},\mathcal{B}'}.$$

We use it as follows:

$\forall v \in V: \ [v]_{\mathcal{B}'} = M_{\mathcal{B}'\mathcal{B}} \cdot [v]_{\mathcal{B}}$

$$\left.\begin{array}{l} K = (O, B) \\ K' = (O', B') \end{array}\right\}$$ - reference frames for $\mathbb{E}^n$

We will denote: $M_{K',K} = M_{B',B}$

We want $[P]_{K'}$ in terms of $[P]_K$

$$\boxed{[P]_{K'}} = [\overrightarrow{O'P}]_{B'} = M_{K',K} \cdot [\overrightarrow{O'P}]_B =$$

$$= M_{K',K} \cdot [\overrightarrow{OP} - \overrightarrow{OO'}]_B = M_{K',K} \cdot [\overrightarrow{OP}]_K - M_{K',K} \cdot [\overrightarrow{OO'}]_K$$

$$= \boxed{M_{K',K} \cdot \left([P]_K - [O']_K \right)}$$ - 1st var of formula

$$M_{K',K} \cdot [\overrightarrow{OO'}]_K = [\overrightarrow{OO'}]_{K'} = -[\overrightarrow{O'O}]_{K'} = -[O]_{K'}$$

$$\boxed{[P]_{K'} = M_{K',K} \cdot [P]_K + [O]_{K'}}$$ - 2nd var of formula

1.16. $K = (O, \vec{i}, \vec{j})$

$K' = (O', \vec{i}', \vec{j}')$

$$[O']_K = \begin{pmatrix} 7 \\ -1 \end{pmatrix}$$

$$\vec{i}' = -2\vec{i} + \vec{j}$$
$$\vec{j}' = \vec{i} + 2\vec{j}$$

Determine the base-change matrix from K to K' and the coordinates of the points A, B, C in K'

$[A]_K = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $[B]_K = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$ $[C]_K = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

Determine $M_{K,K'}$ and check the previous result

$(M_{K,K'}^{-1} = M_{K',K})$

$$M_{K,K'} = [id]_{K',K} = \begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix}$$

$$[A]_{K'} = [\vec{O'A}]_{K'} = [\vec{OA} + \vec{O'O}]_{K'} = [\vec{OA}]_{K'} + [O]_{K'} =$$

$$= M_{K',K} \cdot [\vec{OA}]_K + M_{K',K} \cdot \underbrace{[\vec{O'O}]_K}_{= -[OO']_K}$$

$$M_{K',K} = M_{K,K'}^{-1} = \begin{pmatrix} -\frac{2}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{2}{5} \end{pmatrix}$$

$$\Rightarrow [A]_{K'} = \begin{pmatrix} -\frac{2}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{2}{5} \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} -\frac{2}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{2}{5} \end{pmatrix} \begin{pmatrix} 7 \\ -1 \end{pmatrix} =$$

$$= \begin{pmatrix} -\frac{2}{5} + \frac{2}{5} \\ \frac{1}{5} + \frac{4}{5} \end{pmatrix} + \begin{pmatrix} -\frac{14}{5} - \frac{1}{5} \\ \frac{7}{5} - \frac{2}{5} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$$

$$[B]_{K'} = [\vec{O'B}]_{K'} = [\vec{OB} - \vec{OO'}]_{K'} =$$

$$= M_{K',K} \left([B]_K - [O']_K \right) =$$

$$= M_{K',K} \left(\begin{pmatrix} 2 \\ 4 \end{pmatrix} - \begin{pmatrix} 7 \\ -1 \end{pmatrix} \right) = \begin{pmatrix} -\frac{2}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{2}{5} \end{pmatrix} \begin{pmatrix} -5 \\ 5 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

$$[C]_{K'} = [\overrightarrow{O'C}]_{K'} = [\overrightarrow{O'O} + \overrightarrow{OC}]_{K'} =$$

$$= [\overrightarrow{OC}]_{K'} - [\overrightarrow{OO'}]_{K'} = M_{K',K}\left([C]_K - [O']_K\right) =$$

$$= \begin{pmatrix} -\frac{2}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{2}{5} \end{pmatrix} \begin{pmatrix} 3-7 \\ 1+1 \end{pmatrix} = \begin{pmatrix} -\frac{2}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{2}{5} \end{pmatrix} \begin{pmatrix} -4 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

Verification (for practice)

$$[A]_K = [\overrightarrow{OA}]_K = [\overrightarrow{OO'} + \overrightarrow{O'A}]_K = [\overrightarrow{O'A}]_K - [\overrightarrow{O'O}]_K =$$

$$= M_{K,K'}\left([A]_{K'} - [O]_{K'}\right)$$

$$[O]_{K'} = [\overrightarrow{O'O}]_{K'} = -[\overrightarrow{OO'}]_{K'} = -\left(M_{K',K} \cdot [O']_K\right) =$$

$$= -\left(\begin{pmatrix} -\frac{2}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{2}{5} \end{pmatrix} \begin{pmatrix} 7 \\ -1 \end{pmatrix}\right) = -\begin{pmatrix} -\frac{14}{5} - \frac{1}{5} \\ \frac{7}{5} - \frac{2}{5} \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$$

$$[A]_K = \begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix} \left(\begin{pmatrix} 3 \\ 0 \end{pmatrix} - \begin{pmatrix} 3 \\ -1 \end{pmatrix} \right) = \begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$[B]_K = [\vec{OB}]_K = [\vec{O'B} - \vec{O'O}]_K = [\vec{O'B}]_K - [\vec{O'O}]_K =$$

$$= M_{K,K'}\left([B]_{K'} - [O]_{K'}\right) = \begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix}\left(\begin{pmatrix} 3 \\ 1 \end{pmatrix} - \begin{pmatrix} 3 \\ -1 \end{pmatrix}\right) =$$

$$= \begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

$$[C]_K = [\vec{OC}]_K = [\vec{O'C} - \vec{O'O}]_K =$$

$$= M_{K,K'}\left([C]_{K'} - [O]_{K'}\right) = \begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix}\left(\begin{pmatrix} 2 \\ 0 \end{pmatrix} - \begin{pmatrix} 3 \\ -1 \end{pmatrix}\right) =$$

$$= \begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

1.17. Consider the tetrahedron $ABCD$ and the coordinate systems:

$K_A = (A, \vec{AB}, \vec{AC}, \vec{AD})$

$K_A' = (A, \vec{AB}, \vec{AD}, \vec{AC})$

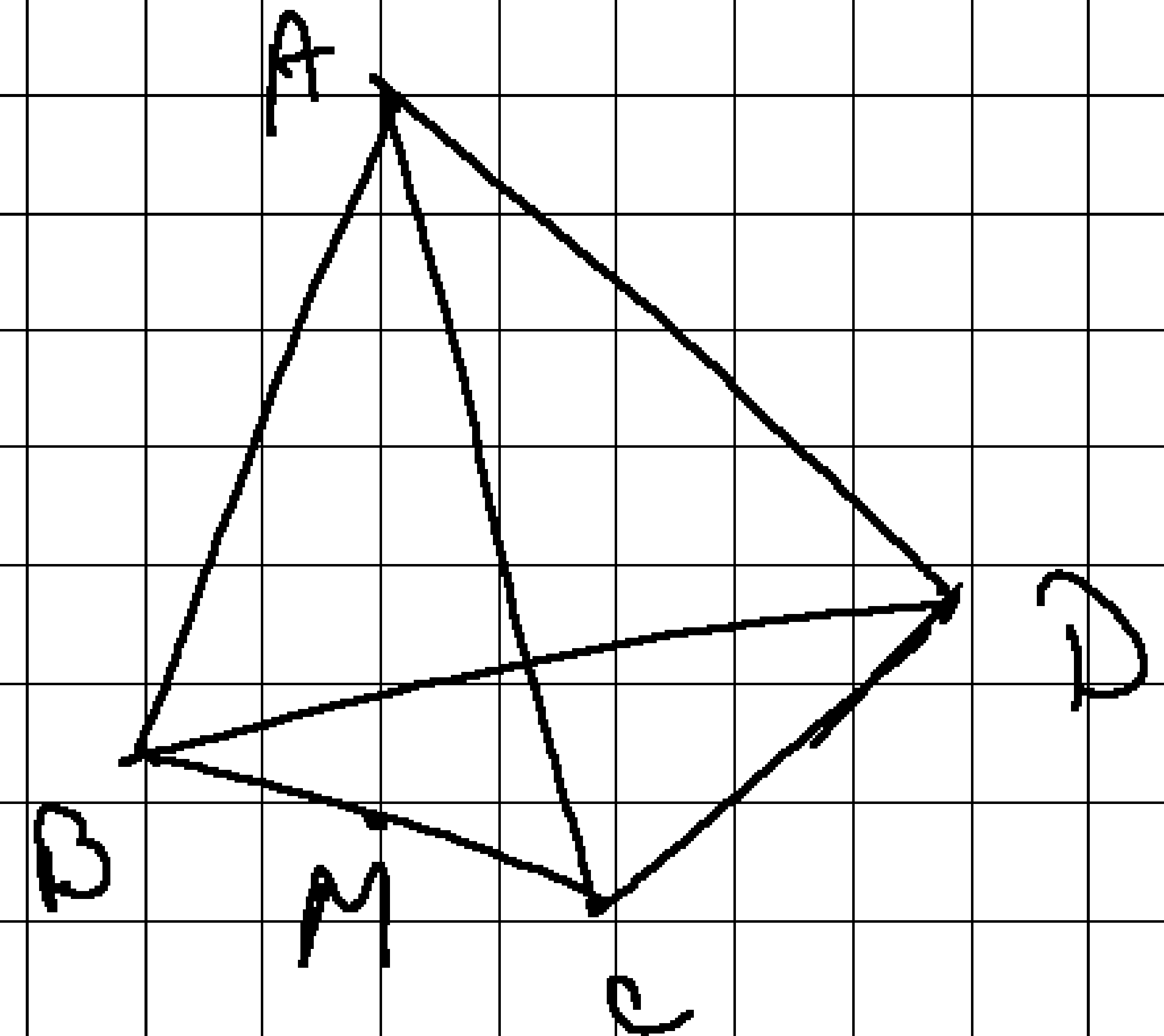
$K_B = (B, \vec{BA}, \vec{BC}, \vec{BD})$

Midpoint of $[BC]$

a) find the coordinates of A, B, C, D, M in the three coordinate systems

b) the base change matrix from K_A to K_A'

c) — " — — " — from K_B to K_A



$$[A]_{K_A} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$[B]_{K_A} = [\overrightarrow{AB}]_{K_A} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$[C]_{K_A} = [\overrightarrow{AC}]_{K_A} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$[D]_{K_A} = [\overrightarrow{AD}]_{K_A} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Midpoint $[BC]$

$$\overrightarrow{BM} = \frac{1}{2} \overrightarrow{BC}$$

$$BC = \overrightarrow{BA} + \overrightarrow{AC} = -\overrightarrow{AB} + \overrightarrow{AC}$$

$$\overrightarrow{BM} = -\frac{1}{2}\overrightarrow{AB} + \frac{1}{2}\overrightarrow{AC}$$

$$\overrightarrow{AM} = \overrightarrow{BM} - \overrightarrow{BA} = -\frac{1}{2}\overrightarrow{AB} + \overrightarrow{AB} + \frac{1}{2}\overrightarrow{AC} = \frac{1}{2}\overrightarrow{AB} + \frac{1}{2}\overrightarrow{AC}$$

$$[M]_{K_A} = [\overrightarrow{AM}]_{K_A} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix}$$

$$[A]_{K'_A} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$[B]_{K'_A} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$[C]_{K'_A} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$[D]_{K'_A} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$[M]_{K'_A} = \begin{pmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{pmatrix}$$

$$[A]_{K_B} = [\vec{BA}] = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$[B]_{K_B} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$[C]_{K_B} = [\vec{BC}] = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$[D]_{K_B} = [\vec{BD}] = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$[M]_{K_B} = \frac{\vec{BB} + \vec{BC}}{2} = \left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right) \cdot \frac{1}{2}$$

$$= \begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \end{pmatrix}$$

b) $$M_{K'_A K_A} = [\vec{AB}]_{K'_A} \cdot [\vec{AC}]_{K'_A} \cdot [\vec{AD}]_{K'_A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$M_{K_A K_B} = [\vec{BA}]_{K_A} \; [\vec{BC}]_{K_A} \cdot [\vec{BD}]_{K_A} =$$

$$= \begin{pmatrix} -1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$