1) Taylor polynomial

$$f(x) = \underbrace{f(x_0) + f'(x_0)(x-x_0) + \dots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n}_{T_n(x)\text{ - Taylor pol.}} + \frac{f^{(n+1)}(c)}{(n+1)!}(x-c)^{n+1}$$

c between x and x_0

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} \dots \quad \forall x \in \mathbb{R}$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \dots$$

a) Prove that $(\sin x)' = \cos x$ and $(\cos x)' = -\sin x$

$$(\sin x)' = \sum_{i=0}^{\infty} \left(\frac{(-1)^n}{(2n+1)!} x^{2n+1} \right)' = \sum_{i=0}^{\infty} \left[\frac{(-1)^n}{(2n+1)!} \cdot (2n+1) \cdot x^{2n} \right] =$$

$$= \sum_{i=0}^{\infty} \left[\frac{(-1)^n}{(2n)!} \cdot x^{2n} \right] = \cos x$$

$$\cos' x = \sum_{i=0}^{\infty} \left(\frac{(-1)^n}{(2n)!} \cdot x^{2n} \right)' = \sum_{i=1}^{\infty} \frac{(-1)^n}{(2n)!} \cdot 2n \cdot x^{2n-1}$$

$$= \sum_{i=0}^{\infty} \left[\frac{(-1)^n}{(2n-1)!} x^{2n-1} \right] = (-\sin x) = \sum_{i=0}^{\infty} \left[\frac{(-1)^{n-1}}{(2n)!} \cdot x^{2n+1} \right]$$

b) Deduce that $\frac{x-x^3}{6} < \sin x < x \quad \forall x > 0$

I $\sin x < x$, $x > 0$

$\Leftrightarrow x - \sin x > 0$

Let $f:(0,\infty) \Rightarrow \mathbb{R}$, $f(x) = x - \sin x$

$f'(x) = 1 - \cos x \geq 0 \Rightarrow f$ is ascending

$\Rightarrow f(x) > 0$, $\lim_{x \searrow 0} f(x) = 0$

$1 - \cos x = 0 \Leftrightarrow x \in \{k\pi \mid k \in \mathbb{R}\}$

$\sin 2k\pi < 2k\pi$, $k > 1$ (equality for $x = 0$)

II $x - \frac{x^3}{6} < \sin x$

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \dots$

$\cancel{x} - \cancel{\frac{x^3}{6}} < \cancel{x} - \cancel{\frac{x^3}{3!}} + \frac{x^5}{5!} - \frac{x^7}{7!} \dots$

$\Leftrightarrow \frac{x^5}{5!} - \frac{x^7}{7!} \dots > 0$ we use $\sum \frac{f^{(n)}(x_0)}{n!} \cdot (x-x_0)^n$

$\sin x = \underbrace{0 + x - 0}_{T_n(x)} + \underbrace{\frac{-\cos(c)}{3!} \cdot x^3}_{R_n(x)} \in \left[-\frac{x^3}{6}, \frac{x^3}{6}\right]$

$$\sin x = x - \frac{x^3}{6} \cdot \underbrace{\cos c}_{\in[1,1]} \geq x - \frac{x^3}{6}$$

$$c \in (\overset{x_0}{0}, x) \Rightarrow c \neq 0 \Rightarrow \cos c \neq 1 \Rightarrow$$

$$\Rightarrow \quad x - \frac{x^3}{6} \cdot \cos c > x - \frac{x^3}{6}$$

$$\mathrm{I}, \mathrm{II} \Rightarrow \quad x - \frac{x^3}{6} < \sin x < x$$

c) $e^{ix} = \cos x + i \sin x$

$$e^x = 1 + x + \frac{x^2}{2} + \dots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$e^{ix} = \sum_{n=0}^{\infty} \frac{(ix)^n}{n!} = i \underbrace{\sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^{2n+1}}{(2n+1)!}}_{\sin x} + \underbrace{\sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^{2n}}{(2n)!}}_{\cos x}$$

$$e^{ix} = i \sin x + \cos x$$

2) a) Prove the generalized binomial expansion

$$(1+x)^{\alpha} = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^{k} \qquad \alpha \in \mathbb{R}, \ |x| < 1$$

$$\binom{\alpha}{k} = \frac{\alpha(\alpha-1)\cdot \ldots \cdot(\alpha-k+1)}{k!}$$

$$\binom{\alpha}{0} = 1$$

$$f(x) = (1+x)^{\alpha}$$

$$f'(x) = \alpha\,(1+x)^{\alpha-1}$$

$$f^{(n)}(x) = \alpha\cdot(\alpha-1)\cdot \ldots (\alpha-n+1)\cdot(1+x)^{\alpha-n}$$

$$f^{(n)}(0) = \alpha\cdot(\alpha-1)\cdot \ldots \cdot(\alpha-n+1)$$

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}\cdot x^{k} = \sum_{k=0}^{\infty} \frac{\alpha\cdot(\alpha-1)\cdot \ldots \cdot(\alpha-n+1)}{k!}\cdot x^{k} =$$

$$= \sum_{k=0}^{\infty} \binom{\alpha}{k} x^{k} = (1+x)^{\alpha} = f(x)$$

b) Find the first 4 terms in the binomial expansion of $\sqrt{x+1}$

$$(1+x)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k$$

$$\alpha = \frac{1}{2} \Rightarrow \sqrt{1+x} = \sum_{k=0}^{\infty} \binom{\frac{1}{2}}{k} x^k =$$

$$= \binom{\frac{1}{2}}{0} x^0 + \binom{\frac{1}{2}}{1} \cdot x + \binom{\frac{1}{2}}{2} \cdot x^2 + \binom{\frac{1}{2}}{3} x^3 + \dots =$$

$$= 1 + \frac{\frac{1}{2}}{1!} \cdot x + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!} \cdot x^2 + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!} x^3 + \dots$$

$$= 1 + \frac{1}{2} x - \frac{1}{8} x^2 + \frac{-\frac{1}{4} \cdot \left(-\cancel{\frac{3}{2}}\right)}{\cancel{6}_{\,2}} x^3 + \dots =$$

$$= 1 + \frac{1}{2} x - \frac{1}{8} x^2 + \frac{1}{16} \cdot x^3 + \dots$$

for $\frac{1}{\sqrt{x+1}} = (x+1)^{-\frac{1}{2}}$ (same approach)

3. Find the MacLaurin series (Taylor series around 0) and its radius of convergence

a) a^x, $a>0$

Let $f: \mathbb{R} \to \mathbb{R}$, $f(x) = a^x$

$f'(x) = a^x \ln a$

$f''(x) = a^x (\ln a)^2$ $\qquad f^{(n)}(0) = a^0 \cdot (\ln a)^n = (\ln a)^n$

$\vdots$

$f^{(n)}(x) = a^x \cdot (\ln a)^n$

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} \cdot x^k = \sum_{k=0}^{\infty} \underbrace{\boxed{\frac{(\ln a)^k}{k!}}}_{b_k} \cdot x^k$$

$$\frac{b_{n+1}}{b_n} = \left| \frac{\cancel{(\ln a)^{n+1}}}{\cancel{(n+1)!}} \cdot \frac{\cancel{n!}}{\cancel{(\ln a)^n}} \right| = \left| \frac{\ln a}{n+1} \right|$$

$$\lim_{n\to\infty} \left| \frac{\ln a}{n+1} \right| = \ln a \underbrace{\lim_{n\to\infty} \left| \frac{1}{n+1} \right|}_{=0} = 0 = L$$

$R = \frac{1}{L} = \frac{1}{0} = \infty$ (radius of convergence)

b) $(1+x)\ln(1+x)$, $x \in (-1, \infty)$

$$f'(x) = \ln(1+x) + (1+x)\cdot\frac{1}{1+x} = \ln(1+x) + 1$$

$$f''(x) = \frac{1}{1+x}$$

$$f'''(x) = \left((1+x)^{-1}\right)' = -1\,(1+x)^{-2}$$

$\vdots$

$$f^{(n)}(x) = (-1)^n\,(n-2)!\cdot(1+x)^{-n+1}$$

$$f^{(n)}(0) = (-1)^n\,(n-2)!$$

$$f(x) = \sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{k!}\cdot x^k = f(0) + f'(0)\cdot x +$$

$$+\sum_{k=2}^{\infty}\frac{(-1)^k}{k!}\cdot(k-2)!\cdot x^k = x + \sum_{k=2}^{\infty}\frac{(-1)^k}{k(k-1)}\cdot x^k$$

$$\frac{b_{n+1}}{b_n} = \left|\frac{(-1)^{k+1}}{(k+1)k}\cdot\frac{k(k-1)}{(-1)^k}\right| = \left|\frac{k-1}{k+1}\right|$$

$$\lim_{k\to\infty}\left|\frac{k-1}{k+1}\right| = 1 \qquad R = 1 \Rightarrow$$ The series is convergent on $(-1,1)$

c) $\sin^2 x$

$f(x) = \sin^2 x$

$f'(x) = 2\sin x \cos x = \sin 2x$

$f''(x) = 2 \cdot \cos 2x$

$f'''(x) = -4 \sin 2x$

$f^{(4)}(x) = (-1) \cdot 2^3 \cdot \cos 2x$

$\vdots$

$f^{(4n)}(x) = (-1) \cdot 2^{(4n-1)} \cdot \cos 2x$

$f^{(4n+2)}(x) = 2^{(4n+1)} \cdot \cos 2x$

$$f(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \cdot 2^{2n-1}}{(2n)!} \cdot x^{2n}$$

$\frac{b_{n+1}}{b_n} = \ldots$ $\Rightarrow$ calculate the radius