Ex3: Using the semantic tableaux method, decide whether the following logical consequences hold or not. If a logical consequence does not hold find an anti-model of it.

5) $\underbrace{p \wedge (q \to r)}_{U_1}, \underbrace{q \vee r}_{U_2} \models \underbrace{p \to (q \to r)}_{V}$

Theoretical results

① $\models U$ iff $\neg U$ has a closed sem. tabl.

② $U_1, \dots U_n \models V$ iff $U_1 \wedge \dots \wedge U_n \wedge \neg V$ has a closed sem. tableau

$U_1 \wedge U_2 \wedge \neg V$ (1) ✓

| α rule for (1)

U_1: $p \wedge (q \to r)$ (2) ✓

|

U_2: $q \wedge r$ (3) ✓

|

$\neg V$ $\neg(p \to (q \to r))$ (4) ✓

| α rule for (2)

p

|

$q \to r$ (5) ✓

|

| α rule for (4)

p

|

$\neg(q \to r)$ (6) ✓

| α rule for (6)

q

|

$\neg r$

β rule for (3)

q $\qquad$ r ⊗

β (4)

$\neg q$ ⊗ $\qquad$ r ⊗

$U_1 \wedge U_2 \wedge \neg V$ has a closed semantic tableaux with 3 closed branches: $(q, \neg q)$, $(r, \neg r)$, $(r, \neg r)$ so $U_1, U_2 \models V$

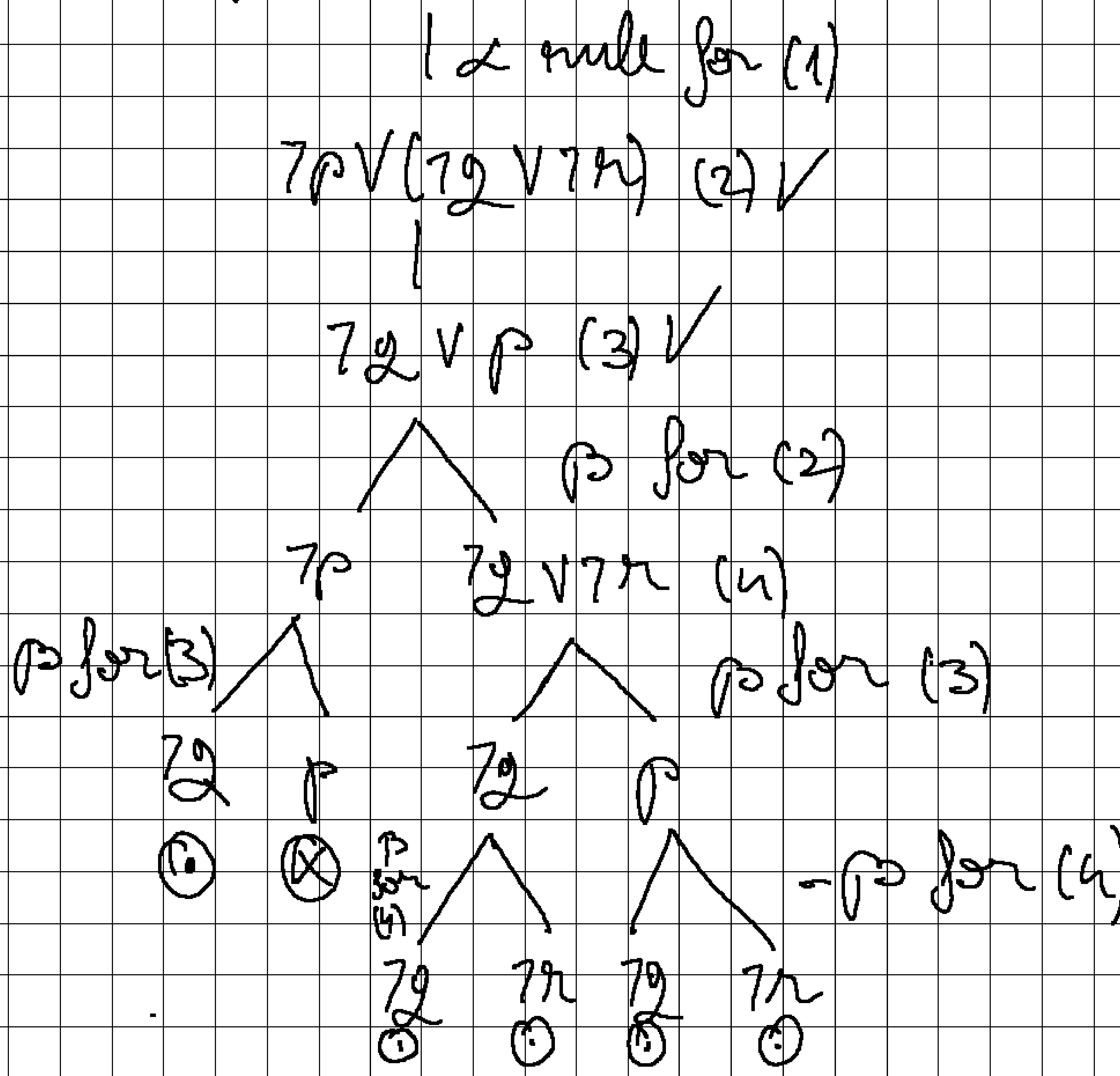
4.5. Find the anti-models using the semantic tableaux method:

$$U_5 = \neg p \vee (\neg q \vee \neg r) \rightarrow q \wedge \neg p$$

(3) The open branches of the sem. tab. of U provide the models of U

(4) The open branches of the sem. tab. of $\neg U$ provide the models of $\neg U$ which are anti-models of U

$$\neg U_5 = \neg(\neg p \vee (\neg q \vee \neg r) \rightarrow q \wedge \neg p) \quad (1) ✓$$



$$DNF(\neg\mathcal{U}_{\sigma}) = (\neg q \wedge \neg p) \vee (\underbrace{p \wedge \neg p}_{F}) \vee \neg q \vee$$

$$\vee (\neg q \wedge \neg r) \vee (\neg q \wedge p) \vee (\neg r \wedge p) \equiv$$

$$\equiv \neg q \vee (\neg r \wedge p)$$

Absorbtion Law

$u \vee (u \wedge v) \equiv u$

$u \wedge (u \vee v) \equiv u$

Cube $\neg q \equiv T$ provides 4 models for $\neg\mathcal{U}_{\sigma}$

$i_{1,2,3,4} : \{p, q, r\} \Rightarrow \{T, F\}$

$i_1(p) = T$ $\quad i_{1,2,3,4}(q) = F \quad$ $i_1(r) = T$

$i_2(p) = T$ $\quad$ $i_2(r) = F$

$i_3(p) = F$ $\quad$ $i_3(r) = T$

$i_4(p) = F$ $\quad$ $i_4(r) = F$

Cube $\neg r \wedge p$ provides 2 models

$i_5, i_6 : \{p, q, r\} \Rightarrow \{T, F\}$

$i_5(p) = T \quad i_5(q) = T \quad i_5(r) = F$

$i_6(p) = T \quad i_6(q) = F \quad i_6(r) = F$

$i_2 = i_6$

$$i_1(\neg U_5) = i_2(\neg U_5) = i_3(\neg U_5) = i_4(\neg U_5) = i_5(\neg U_5) = T$$

anti-models of U_5:

$$i_1(U_5) = i_2(U_5) = i_3(U_5) = i_4(U_5) = i_5(U_5) = F$$

5. Check whether C is a logical consequence of the hypotheses using the semantic tableaux method

H_1: All hummingbirds are richly colored

H_2: No large birds live on honey

H_3: Birds that do not live on honey are dull in color.

Conclusion: C. All hummingbirds are small

$H_1: (\forall x)(hb(x) \to rc(x))$

$H_2: (\neg\exists x)(\neg sb(x) \wedge lh(x)) \equiv (\forall x)(sb(x) \vee \neg lh(x))$

$H_3: (\forall x)(\neg lh \to \neg rc(x))$

$C: (\forall x)(hb(x) \to sb(x))$

$$\boxed{H_1, H_2, H_3 \models C \ ?}$$

$H_1 \wedge H_2 \wedge H_3 \wedge \neg C$ (1)

| α for (1)

H_1: $(\forall x)(hb(x) \to rc(x))$ (2) ✓

H_2: $(\forall x)(sb(x) \vee \neg lh(x))$ (3) ✓

H_3: $(\forall x)(\neg lh(x) \to \neg rc(x))$ (4) ✓

$\neg C$: $\neg(\forall x)(hb(x) \to sb(x)) \equiv (\exists x)\neg(hb(x) \to sb(x))$ (5) ✓

| δ for (5), a - new. const.

$\neg(hb(a) \to sb(a))$ (6)

| α for 6

$hb(a)$

$\neg sb(a)$

| γ for (2), $[x \leftarrow a]$

$hb(a) \to rc(a)$ (7)

(2)

β for (7)

$\neg hb(a)$ $rc(a)$

$\otimes$

γ for (3), $[x \leftarrow a]$

$sb(a) \lor \neg lh(a)$ (8)

(3)

β for (8)

$sb(a)$ $\quad$ $\neg lh(a)$

$\otimes$

γ for (4), $[x \leftarrow a]$

$\neg lh(a) \Rightarrow \neg rc(a)$ (9)

(4)

β for 9

$lh(a)$ $\quad$ $\neg rc(a)$

$\otimes$ $\quad$ $\otimes$

$H_1 \land H_2 \land H_3 \land \neg C$ has a closed sem. tableaux with 4 closed branches $(hb(a), \neg hb(a))$, $(sb(a), \neg sb(a))$, $(\neg lh(a), lh(a))$, $(rc(a), \neg rc(a))$

so $H_1, H_2, H_3 \models C$

7.5. $U_5: (\exists x)(A(x) \to B(x)) \to ((\forall x)A(x) \to (\exists x)B(x))$

$\neg U_5: \neg\big((\exists x)(A(x) \to B(x)) \to ((\forall x)A(x) \to (\exists x)B(x))\big)$ (1) ✓

| α rule for (1)

$(\exists x)(A(x) \to B(x))$ (2) ✓

|

$\neg((\forall x)A(x) \to (\exists x)B(x))$ (3) ✓

| α rule for (3)

$(\forall x)A(x)$ (4) ✓

|

$(\forall x)\neg B(x) \equiv \neg(\exists x)B(x)$ (5) ✓

| δ for (2) a - new const.

$A(a) \to B(a)$ (6) ✓

| γ for (4), $[x \leftarrow a]$

$A(a)$

|

(4)

| γ for (5), $[x \leftarrow a]$

$\neg B(a)$

|

(5)

|

| β for (6)

$\neg A(a)$ $\qquad$ $B(a)$

$\otimes$ $\qquad$ $\otimes$

$\neg \mathcal{U}_5$ has a closed sem. tab. with 2 closed branches $(A(a), \neg A(a))$, $(B(a), \neg B(a))$

so $\neg \mathcal{U}_5$ is inconsistent and $\models \mathcal{U}_5$

$\mathcal{U}_5: ((\exists x) A(x) \to (\exists x) B(x)) \to (\forall x)(A(x) \to B(x))$

$\neg \mathcal{U}_5: \neg(((\exists x) A(x) \to (\exists x) B(x)) \to (\forall x)(A(x) \to B(x)))$ (1) ✓

| α-rule for (1)

$(\exists x) A(x) \to (\exists x) B(x)$ (2)

|

$(\exists x) \neg(A(x) \to B(x)) \equiv \neg(\forall x)(A(x) \to B(x))$ (3) ✓

| δ (3)
| a new const.

$\neg(A(a) \to B(a))$ (4)

| α for (4)

$A(a)$

|

$\neg(B(a))$

β for (2)

(5) $(\forall x)\neg A(x) \equiv \neg(\exists x)A(x)$

(6) $(\exists x)B(x)$

γ for $[x \leftarrow a]$

$\neg(A(a))$

(5) copy

$\otimes$

δ for (6)
b new const.

$B(b)$

$\boxed{\cdot}$

$\neg U_5$ has a finite complete and open sem. tabl., so $\neg U_5$ is consistent and $\not\models U_5$

The sem. tabl. has a closed branch $(A(a), \neg A(a))$ and an open branch $A(a) \wedge \neg B(a) \wedge B(b)$
The open branch provides models of $\neg U_5$ which are anti-models of U_5

$A(a) \wedge \neg B(a) \wedge B(b) \equiv T$

$i = \langle D, m \rangle$, $D = \{a, b\}$, i-generic model

$m(A)(a) = T$ $\quad$ $m(A)(b) = F$

$m(B)(a) = F$ $\quad$ $m(B)(b) = T$

$V^{i}(\neg U_5) = T$, $V^{i}(U_5) = F$

Based on the generic model we build a concrete model

$i_1 = \langle D_1, m_1 \rangle$, $D_1 = \{3, 4\}$

$m(A)(x) =$ "x is prime"

$m(B)(x) =$ "x is a perfect square"

$V^{i_1}(\neg U_5) = T$, $V^{i_1}(U_5) = T$

1.5. Using general resolution prove that the following formulas are theorems.

$U_5 = A \vee (B \to C) \to (A \vee B) \to (A \vee C)$ $\equiv$ we apply the normaliz alg.

(T2) $\vdash U$ iff $CNF(\neg U) \vdash_{Res} \square$

$A \to B \equiv \neg A \vee B$

$\neg U_5 = \neg [A \vee (B \overset{1}{\to} C) \overset{2}{\to} ((A \vee B) \overset{3}{\to} (A \vee C))]$

$\overset{\text{replace } 1,3}{\equiv} \neg [A \vee (\neg B \vee C) \to (\neg (A \vee B) \vee A \vee C)]$

$\overset{\text{replace } 3}{\equiv} \neg [\neg (A \vee \neg B \vee C) \vee \neg (A \vee B) \vee A \vee C]$

de Morgan's law

$$\equiv \underbrace{(A \vee \neg B \vee C)}_{C_1} \wedge \underbrace{(A \vee B)}_{C_2} \wedge \underbrace{\neg A}_{C_3} \wedge \underbrace{\neg C}_{C_4}$$ CNF with 4 clauses

Resolution:

$C_1 = \underline{p} \vee q$; $C_2 = \underline{\neg p} \vee r$

$C_3 = Res_p(C_1, C_2) = q \vee r$

$C_4 = p$ $\qquad C_5 = \neg p$

$C_6 = Res_p(C_4, C_5) = \square$

(T1) S is inconsistent iff $S \vdash_{Res} \square$

(T3) $U_1, \dots, U_m \vdash V$ iff

$CNF(U_1 \wedge \dots \wedge U_m \wedge \neg V) \vdash_{Res} \square$

$S = \{C_1, C_2, C_3, C_4\}$, $\quad S \vdash_{Res} \square$

$C_5 = Res_A(C_1, C_3) = \neg B \vee C$

$C_6 = Res_B(C_2, C_5) = A \vee C$

$C_7 = Res_C(C_4, C_5) = \neg B$

$C_8 = Res_C(C_4, C_6) = A$

$C_9 = Res_A(C_3, C_8) = \square$

$CNF(\neg U_5) \vdash_{res} \square$ so $(T_2) \Rightarrow \vdash U_5$

3. $H_1: L \wedge \neg G \to M \overset{\text{replace} \to}{\equiv} \neg(L \wedge \neg G) \vee M \overset{\text{De Morgan Law}}{\equiv} \underbrace{\neg L \vee G \vee M}_{C_1}$

$H_2: J \to L \equiv \neg J \vee L = C_2$

$H_3: J_t \to J \equiv \neg J_t \vee J = C_4$

$H_4: G_s \wedge \neg G = C_4 \wedge C_5$

$H_5: J_t = C_6$

$H_1, H_2, H_3, H_4, H_5 \overset{?}{\vdash} C \qquad C: M; \neg C = \neg M = C_7$

L: Lucy will go to the party

G: George will go to the party

M: Mary will go to the party

J: John will go to the party

J_t: John is in town

G_s: George is sick

$S = \{ C_1, \ldots, C_7 \}$, $S \vdash_{Res} \square$

$C_8 = Res_G(C_1, C_5) = \neg L \vee M$

$C_9 = Res_{Jt}(C_3, C_6) = J$

$C_{10} = Res_J(C_2, C_9) = L$

$C_{11} = Res_L(C_8, C_{10}) = M$

$C_{12} = Res_M(C_{10}, C_7) = \square$

$CNF(H_1 \wedge H_2 \wedge H_3 \wedge H_4 \wedge H_5 \wedge \neg C) \vdash_{Res} \square$

$\Rightarrow T_3: H_1, H_2, H_3, H_4, H_5 \vdash C$

4. Build a linear refutation from the following set of clauses.

$S_5 = \{ \underbrace{p \vee \neg r}_{C_1}, \underbrace{\neg q}_{C_2}, \underbrace{\neg p \vee \neg r}_{C_3}, \underbrace{q \vee r}_{C_4} \}$

Top clause

Side clauses

central clauses

$\boxed{C_1 = p \vee \neg r}$ $\boxed{C_3 = \neg p \vee \neg r}$, C_4

$\boxed{C_5 = \neg r}$ $\boxed{C_4 = q \vee r}$

$\boxed{C_6 = q}$ $\boxed{C_2 = \neg q}$

$$\boxed{C_7 = \square}$$

$$S_4 \vdash_{Res}^{Linear} \square \Rightarrow S_4 \text{ - inconsistent}$$

5. Prove the consistency of the following sets of clauses using linear resolution

$$S_5 = \{ \underbrace{p \vee q}_{C_1}, \underbrace{r \vee p \vee \neg q}_{C_2}, \underbrace{\neg r \vee \neg p}_{C_3} \}$$

I

$\boxed{C_1 = p \vee q}$ $\boxed{C_2 = r \vee p \vee \neg q}$, C_3

$\boxed{C_4 = p \vee r}$ $\boxed{C_3 = \neg r \vee \neg p}$, C_3

$\boxed{C_5 = r \vee \neg r \equiv T}$ process blocked

II $\boxed{C_1 = p \vee q}$ $\boxed{C_2 = r \vee p \vee \neg q}$, C_3

$\boxed{C_4 = p \vee r}$ $\boxed{C_3 = \neg r \vee \neg p}$, C_3

$\boxed{C_6 = p \vee \neg p \equiv T}$ process blocked

III $\boxed{C_1 = p \vee q}$ C_2 , $\boxed{C_3 = \neg p \vee \neg r}$

$\boxed{C_7 = q \vee \neg r}$ $\boxed{C_2 = r \vee p \vee \neg q}$, C_2

$\boxed{C_8 = \neg r \vee r \vee p \equiv T}$ process blocked

IV

$\boxed{C_1 = p \vee q}$ C_2 , $\boxed{C_3 = \neg p \vee \neg r}$

$\boxed{C_7 = q \vee \neg r}$ $\boxed{C_2 = r \vee p \vee \neg q}$, C_2

$\boxed{C_9 = q \vee \neg q \vee p}$ process blocked

Conclusion: After a complete search using the backtracking alg. without the derivation of the empty clause we conclude that the set is consistent