Taylor polynomial:

- $P(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \dots + a_n(x-x_0)^n$

  $P(x_0) = a_0 = f(x_0)$

- $P'(x) = a_1 + 2a_2(x-x_0) + \dots + na_n(x-x_0)^{n-1}$

  $P'(x_0) = a_1 = f'(x_0)$

- $P''(x) = 2a_2 + \dots + n(n-1)a_n(x-x_0)^{n-2}$

  $P''(x_0) = 2a_2 = f''(x_0) \Rightarrow a_2 = \frac{1}{2} f''(x_0)$

- $P^{(k)}(x_0) = k! \cdot a_k \quad , \quad k = \overline{1,n}$

  $= f^{(k)}(x_0)$

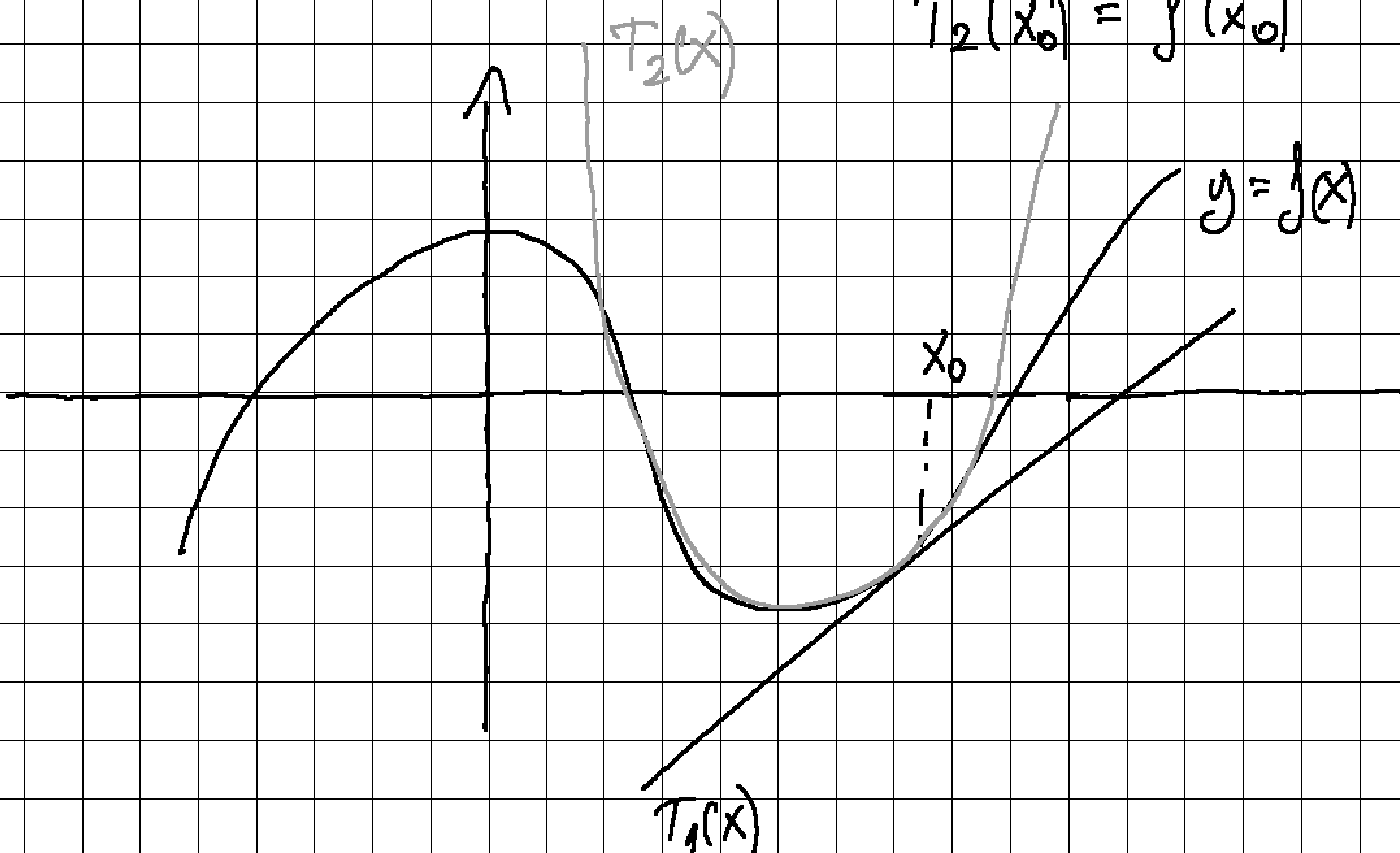
$\Rightarrow a_k = \dfrac{f^{(k)}(x_0)}{k!} \quad , \quad k = \overline{1,n}$

Definition 1

$$T_n(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2}(x-x_0)^2 + \dots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

$$T_1(x) = f(x_0) + f'(x_0)(x-x_0) \quad , \quad T_1(x_0) = f(x_0)$$

$$T_2'(x_0) = f'(x_0)$$



$$T_2(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{1}{2} f'(x_0)(x-x_0)^2$$

$$T_2(x_0) = f(x_0) \quad , \quad T_2'(x_0) = f'(x_0) \quad , \quad T_2''(x_0) = f''(x_0)$$

Ex: $f(x) = e^x$, $x_0 = 0$

$$f'(x) = f''(x) = \ldots = f^{(n)}(x) = e^x$$

$$f'(0) = f''(0) = \ldots = f^{(n)}(0) = 1$$

$$T_n(x) = 1 + x + \frac{x^2}{2} + \ldots + \frac{x^n}{n!} = \sum_{k=0}^{n} \frac{x^k}{k!}$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \ldots + \frac{x^n}{n!} + \ldots$$

Ex: $f(x) = e^x$, $x_0 = 0$

Taylor series: $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$

Ratio test: $\left|\frac{x_{n+1}}{x_n}\right| = \left|\frac{x}{n+1}\right| \to 0$ as $n \to \infty$

$\Rightarrow$ the series converges $\forall x \in \mathbb{R}$

- $T_n(x) = \displaystyle\sum_{k=0}^{n} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2} + \ldots + \frac{x^n}{n!}$

$R_n(x) = \dfrac{e^c}{(n+1)!} x^{n+1} = a_n$, $\left|\dfrac{a_{n+1}}{a_n}\right| = \left|\dfrac{x}{n+2}\right| \to 0 < 1 \Rightarrow$

$\Rightarrow a_n \to 0$

For any $x \in \mathbb{R}$

$R_n(x) \to 0$, as $n \to \infty$

$\Rightarrow T_n(x) \to f(x)$, as $n \to \infty$

$$\Rightarrow \underbrace{\sum_{n=0}^{\infty} \frac{x^n}{n!}}_{= \lim_{n \to \infty} T_n(x)} = f(x) = e^x$$

$$(e^x)' = \left(\sum_{n=0}^{\infty} \frac{x^n}{n!}\right)' = \sum_{n=0}^{\infty} \left(\frac{x^n}{n!}\right)' =$$

$$= \left(1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} + \dots\right)' = 1 + x + \frac{x^2}{2} + \dots + \frac{x^{n-1}}{(n-1)!} + \dots$$

$$= e^x$$

Example 8 (lecture):

$f(x) = \sin x \quad , \quad x_0 = 0$

$f'(x) = \cos x \quad , \quad f'(0) = 1$

$f''(x) = -\sin x \quad , \quad f''(0) = 0$

$f'''(x) = -\cos x \quad , \quad f'''(0) = -1$

$f^{(4)}(x) = \sin x \quad , \quad f^{(4)}(0) = 0$

$f^{(2m+1)}(0) = (-1)^m \qquad , \quad f^{(2m)}(0) = 0$

Taylor series:

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = \sum_{m=0}^{\infty} \frac{f^{(2m+1)}(0)}{(2m+1)!} x^{2m+1}$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} \cdot x^{2m+1}$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

$$\sin(-x) = -x + \frac{x^3}{3!} - \frac{x^5}{5!} + \dots = -\sin(x)$$

$$f(x) = \cos x \quad , \quad x_0 = 0$$

$$f'(x) = -\sin x \quad , \quad f'(0) = 0$$

$$f''(x) = -\cos x \quad , \quad f''(0) = -1$$

$$f'''(x) = \sin x \quad , \quad f'''(0) = 0$$

$$f^{(4)}(x) = \cos x \quad , \quad f^{(4)}(0) = 1$$

$$f^{(2m+1)}(0) = 0 \quad , \quad f^{(2m)} = (-1)^m$$

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} \cdot x^k = \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m)!} x^{2m} = 1 - \frac{x^2}{2} + \frac{x^4}{4!} + \ldots$$

$$\cos(-x) = \cos x$$

Ex: $\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} \cdot x^n = \sum_{n=0}^{\infty} \frac{0}{n!} x^n = 0$

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

$$f'(x) = e^{-\frac{1}{x^2}} \left(-\frac{1}{x^2}\right)' = e^{-\frac{1}{x^2}} \cdot \frac{2}{x^3}$$

$$\lim_{x \to 0} e^{-\frac{1}{x^2}} \cdot \frac{2}{x^3} \overset{t=\frac{1}{x}}{=} \lim_{t \to \infty} e^{-t^2} 2t^3 = \lim_{t \to \infty} \frac{2t^3}{e^{t^2}} = 0$$

$$\Rightarrow f'(0) = 0$$

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} \cdot x^k = 0$$

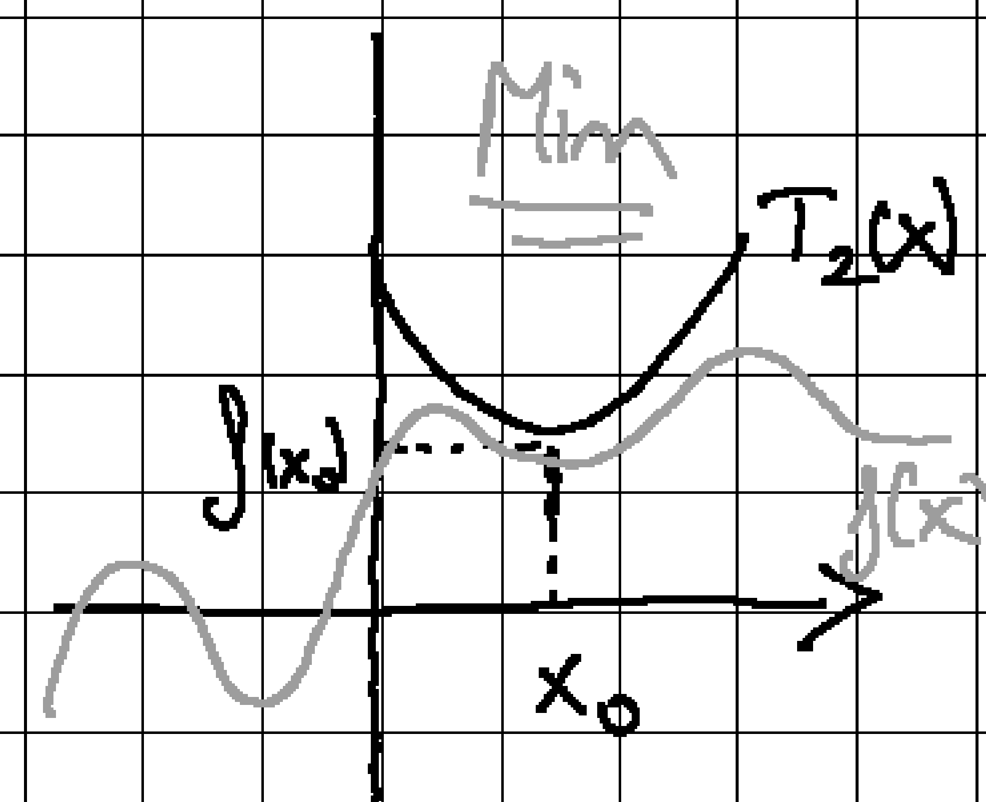
$R_n(x) = f(x) - T_n(x) = f(x) \not\to 0$ as $n \to \infty$

$$\Rightarrow f(x) \neq \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} \cdot x^k$$

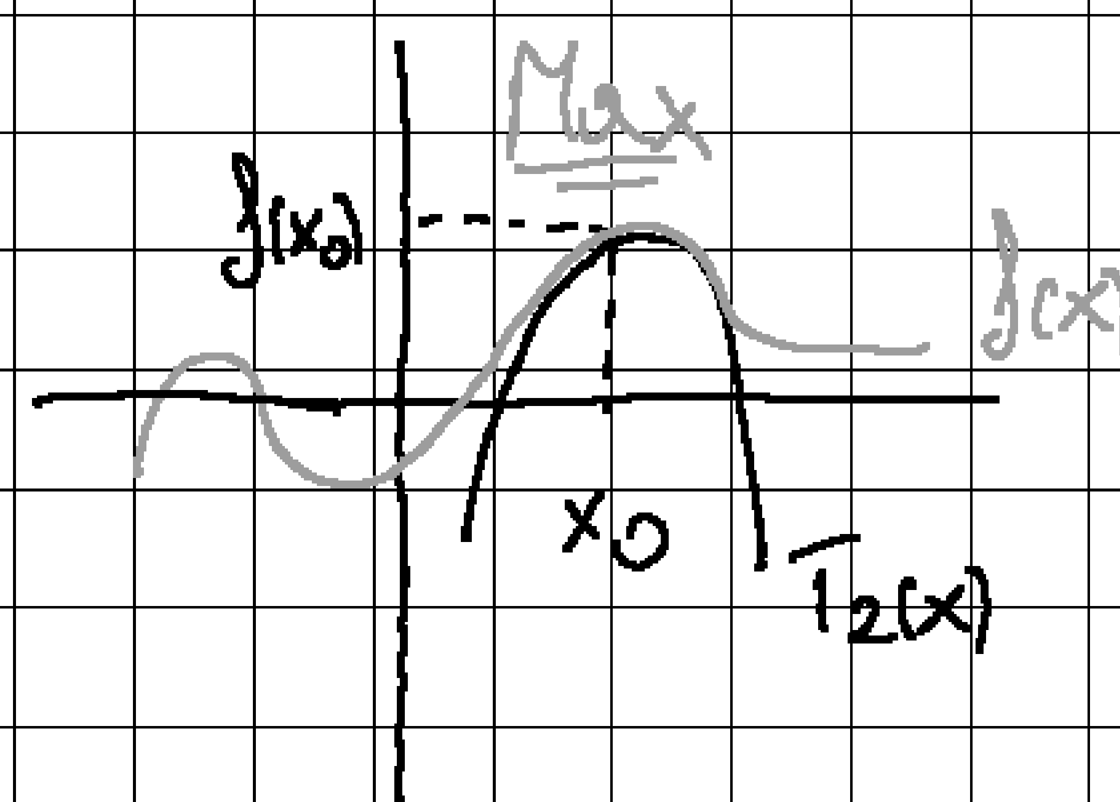
Example 10 (lecture):

$$T_2(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{1}{2} f''(x_0)(x-x_0)^2$$

$$= f(x_0) + \frac{1}{2} f''(x_0)(x-x_0)^2$$



I $\quad f''(x_0) > 0 \Rightarrow T_2(x) > f(x_0)$



II $\quad f''(x_0) < 0 \Rightarrow T_2(x) < f(x_0)$

$$f(x) = T_2(x) + \underbrace{R_2(x)}_{\text{small when } x \simeq x_0}$$