Kronecker Capelli:

(S) compatible $\Leftrightarrow$ rank M_S = rank $\overline{M_S}$

Rouché Theorem:

(S) compatible $\Leftrightarrow$ all characteristic minor are 0

8.2,3

(iii) $\begin{cases} x+y+z=3 \\ x-y+z=1 \\ 2x-y+2z=3 \\ x+z=4 \end{cases}$

(Solve using Kronecker-Capelli/Rouché)

$$A=\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & -1 & 2 \\ 1 & 0 & 1 \end{pmatrix} \qquad \overline{A}=\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & -1 & 1 & 1 \\ 2 & -1 & 2 & 3 \\ 1 & 0 & 1 & 4 \end{array}\right)$$

We can see that rank $A<3$

$\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} \neq 0 \Rightarrow$ rank$(A)=2$

$\begin{vmatrix} 1 & 1 & 3 \\ -1 & 1 & 1 \\ -1 & 2 & 3 \end{vmatrix} = 3-6-1+3-2+3=0$

$$\begin{vmatrix} 1 & 1 & 3 \\ 1 & -1 & 1 \\ 1 & 0 & 4 \end{vmatrix} = -4 \neq 0 \Rightarrow \operatorname{rank} \bar{A} \geq 3 \Rightarrow$$ the system is incompatible

Don't use this methods at the exam

Check page before

8.9. Solve the following systems using the Gauss and Gauss-Jordan methods.

$$\begin{cases} 2x + 2y + 3z = 3 \\ x - y = 1 \\ -x + 2y + z = 2 \end{cases}$$

$$\left(\begin{array}{ccc|c} 2 & 2 & 3 & 3 \\ 1 & -1 & 0 & 1 \\ -1 & 2 & 1 & 2 \end{array}\right) \underset{\sim}{L_1 \leftrightarrow L_2} \left(\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 2 & 2 & 3 & 3 \\ -1 & 2 & 1 & 2 \end{array}\right)$$

$$\underset{L_3 \leftarrow L_3 + L_1}{\overset{L_2 \leftarrow L_2 - 2L_1}{\sim}} \left(\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 0 & 4 & 3 & 1 \\ 0 & 1 & 1 & 3 \end{array}\right) \underset{\sim}{L_2 \leftrightarrow L_3} \left(\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 0 & 1 & 1 & 3 \\ 0 & 4 & 3 & 1 \end{array}\right)$$

$$\underset{\sim}{L_3 \leftarrow L_3 - 4L_2} \left(\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & -1 & -11 \end{array}\right)$$

This is where we stop. If we're doing Gauss, then we revert to the start.

$$\begin{cases} x-y=1 \\ y+z=3 \\ -z=-11 \end{cases} \Rightarrow z=11,\ y=-8,\ x=-7$$

If we're doing Gauss-Jordan, we continue with the transformations.

$$\left(\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & -1 & -11 \end{array}\right) \underset{L_3 \leftarrow -L_3}{\sim} \left(\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 11 \end{array}\right) \underset{L_2 \leftarrow L_2 - L_3}{\sim}$$

$$\sim \left(\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 0 & 1 & 0 & -8 \\ 0 & 0 & 1 & 11 \end{array}\right) \underset{L_1 \leftarrow L_1 + L_2}{\sim} \left(\begin{array}{ccc|c} 1 & 0 & 0 & -7 \\ 0 & 1 & 0 & -8 \\ 0 & 0 & 1 & 11 \end{array}\right) \Rightarrow$$

$$\Rightarrow \begin{array}{l} x=-7 \\ y=-8 \\ z=11 \end{array}$$

(iii) $\begin{cases} x+y+z=3 \\ x-y+z=1 \\ 2x-y+2z=3 \\ x+z=4 \end{cases}$

$$\overline{M} = \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & -1 & 1 & 1 \\ 2 & -1 & 2 & 3 \\ 1 & 0 & 1 & 4 \end{array}\right) \underset{L_4 \leftarrow L_4 - L_1}{\overset{\substack{L_2 \leftarrow L_2 - L_1 \\ L_3 \leftarrow L_3 - 2L_1}}{\sim}} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & -2 & 0 & -2 \\ 0 & -3 & 0 & -3 \\ 0 & -1 & 0 & 1 \end{array}\right) \overset{L_2 \leftarrow -L_4}{\sim}$$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & -3 & 0 & -3 \\ 0 & -2 & 0 & -2 \end{array}\right) \underset{L_4 \leftarrow L_3 + 2L_2}{\overset{L_3 \leftarrow L_3 + 3L_2}{\sim}} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & -6 \\ 0 & 0 & 0 & -4 \end{array}\right)$$

$\Rightarrow$ the system is incompatible

6. $\begin{cases} 2x_1 + x_2 + x_3 + x_4 = 1 \\ x_1 + 2x_2 - x_3 - 4x_4 = 2 \\ x_1 + 5x_2 - 4x_3 + 11x_4 = \lambda \end{cases}$ $\lambda \in \mathbb{R}$

$$\bar{A} = \left(\begin{array}{cccc|c} 2 & 1 & 1 & 1 & 1 \\ 1 & 2 & -1 & -4 & 2 \\ 1 & 5 & -4 & 11 & \lambda \end{array}\right) \overset{L_1 \leftrightarrow L_2}{\sim} \left(\begin{array}{cccc|c} 1 & 2 & -1 & -4 & 1 \\ 2 & 1 & 1 & 1 & 2 \\ 1 & 5 & -4 & 11 & \lambda \end{array}\right)$$

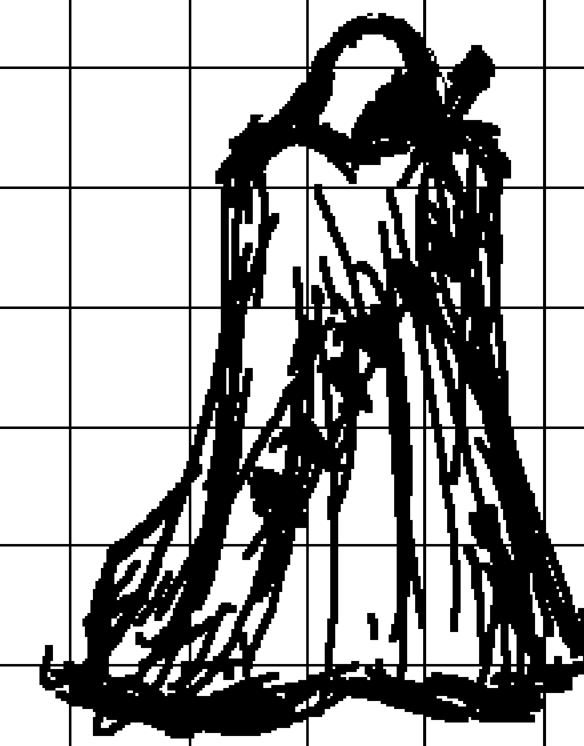
$$\underset{L_3 \leftarrow L_3 - L_1}{\overset{L_2 \leftarrow L_2 - 2L_1}{\sim}} \left(\begin{array}{cccc|c} 1 & 2 & -1 & 4 & 2 \\ 0 & -3 & 3 & -7 & -3 \\ 0 & 3 & -3 & 7 & (\lambda-2) \end{array}\right) \overset{L_3 \leftarrow L_3 + L_2}{\sim}$$

$$\sim \left(\begin{array}{cccc|c} 1 & 2 & -1 & 4 & 2 \\ 0 & -3 & 3 & -7 & -3 \\ 0 & 0 & 0 & 0 & (\lambda-5) \end{array}\right)$$

if $\lambda \neq 5 \Rightarrow$ incompatible system

if $\lambda = 5 \Rightarrow \left(\begin{array}{cccc|c} 1 & 2 & -1 & 4 & 2 \\ 0 & -3 & 3 & -7 & -3 \end{array}\right)$

$$\begin{cases} x_1 + 2x_2 - 3x_3 + 4x_4 = 2 \\ \quad -3x_2 + 3x_3 - 7x_4 = -3 \end{cases}$$



$$\left.\begin{aligned} x_3 &= \alpha \\ x_4 &= \beta \end{aligned}\right\} \Rightarrow \begin{aligned} x_1 + 2x_2 &= 2 + \alpha - 4\beta \\ -3x_2 &= -3 - 3\alpha + 7\beta \end{aligned}$$

$$\Rightarrow \quad x_2 = 1 + \alpha - \frac{7}{3}\beta$$

$$\Rightarrow \quad x_1 = 2 + \alpha - 4\beta - 2 - 2\alpha + \frac{14}{3}\beta$$

$$x_1 = -\alpha + \frac{2}{3}\beta$$

7. $\begin{cases} ax + y + z = 1 \\ x + ay + z = a \\ x + y + az = a^2 \end{cases} \quad (a \in \mathbb{R})$

$$\bar{M} = \left(\begin{array}{ccc|c} a & 1 & 1 & 1 \\ 1 & a & 1 & a \\ 1 & 1 & a & a^2 \end{array}\right) \overset{L_1 \leftrightarrow L_2}{\sim} \left(\begin{array}{ccc|c} 1 & a & 1 & a \\ a & 1 & 1 & 1 \\ 1 & 1 & a & a^2 \end{array}\right) \sim$$

$$\underset{L_3 \leftarrow L_3 - L_1}{\overset{L_2 \leftarrow L_2 - aL_1}{\sim}} \left(\begin{array}{ccc|c} 1 & a & 1 & a \\ a & 1-a^2 & 1-a & 1-a^2 \\ 0 & 1-a & a-1 & a(a-1) \end{array}\right) \overset{L_2 \leftrightarrow L_3}{\sim}$$

$$\left(\begin{array}{ccc|c} 1 & a & 1 & a \\ 0 & 1-a & a-1 & a(a-1) \\ 0 & 1-a^2 & 1-a & 1-a^2 \end{array}\right)$$

if $a=1 \Rightarrow \left.\begin{array}{l} x+y+z\ 1 \\ y=\alpha \\ z=\beta \end{array}\right| \Rightarrow x=1-\alpha-\beta$

if $x\neq 1$: $\bar{M} \underset{L_2 \leftarrow L_2 \cdot \frac{1}{1-a}}{\sim} \left(\begin{array}{ccc|c} 1 & a & 1 & a \\ 0 & 1 & -1 & -a \\ 0 & 1-a^2 & 1-a & (1-a)^2 \end{array}\right) \sim$

$\underset{L_3 \leftarrow L_3-(1-a^2)\cdot L_2}{\sim} \left(\begin{array}{ccc|c} 1 & a & 1 & a \\ 0 & 1 & -1 & -a \\ 0 & 0 & 2-a-a^2 & 1-a^2+a-a^3 \end{array}\right)$

if $2-a-a^2=0$ $\begin{cases} a=1 \\ a=-2 \end{cases}$ $\left.\right\}$ $\overset{a\neq 1}{\Rightarrow}$ $a=-2 \Rightarrow \left(\begin{array}{ccc|c} 1 & -2 & 1 & -2 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 3 \end{array}\right)$

$\Rightarrow$ the system is incompatible

if $2-a-a^2\neq 0 \Rightarrow \begin{cases} x+ay+z=a \\ y-z=-a \\ (2-a-a^2)z=1-a^2+a-a^3 \end{cases} \Rightarrow$

$\Rightarrow \begin{cases} y=a+\dfrac{1-a^2+a-a^3}{2-a-a^2} \\ z=\dfrac{1-a^2+a-a^3}{2-a-a^2} \end{cases}$