Study if the following series are convergent or divergent

b) $$\sum_{n\geq 1} \frac{1\cdot 3\cdot\ldots\cdot(2n-1)}{2\cdot 4\cdot\ldots\cdot 2n}\cdot\frac{1}{n^2} =$$

$$1\cdot 3\cdot\ldots\cdot(2n-1) = \frac{1\cdot 2\cdot 3\cdot\ldots\cdot 2n}{2\cdot 4\cdot\ldots\cdot 2n} = \frac{(2n)!}{2\cdot n!}$$

$$2\cdot 4\cdot\ldots\cdot 2n = 2\cdot n!$$

$$= \sum_{n\geq 1} \frac{\frac{(2n)!}{2\cdot n!}}{2\cdot n!} = \sum_{n\geq 1}\frac{(2n)!}{(2\cdot n!)^2}$$

Ratio test:

$$\frac{a_{n+1}}{a_n} = \frac{\frac{(2(n+1))!}{(2(n+1)!)^2}}{\frac{2n!}{(2\cdot n!)^2}} = \frac{(2(n+1))!}{(2(n+1)!)^2}\cdot\frac{(2n!)^2}{(2n)!}$$

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\left(\frac{(2(n+1))!}{(2(n+1)!)^2}\cdot\frac{(2n!)^2}{(2n)!}\right) =$$

$$= \lim_{n\to\infty}\frac{(2n+2)(2n+1)\cdot\cancel{(2n)!}}{(2(n+1)!)^2}\cdot\frac{(2n!)^2}{\cancel{(2n)!}} =$$

$$= \lim_{n\to\infty}\frac{(2n+2)(2n+1)}{(2(n+1)\cdot n!)^2}\cdot(2n!)^2 =$$

$$\lim_{n\to\infty} \frac{(2n+2)(2n+1)}{(n+1)^2 \cdot \cancel{(2n!)^2}} \cdot \cancel{(2n!)^2} =$$

$$= \lim_{n\to\infty} \frac{(2n+2)(2n+1)}{n^2+2n+1} = \lim_{n\to\infty} \frac{4n^2+6n+2}{n^2+2n+1} =$$

$$= \lim_{n\to\infty} \frac{\cancel{n^2}\left(4+\frac{6}{n}+\frac{2}{n^2}\right)}{\cancel{n^2}\left(1+\frac{2}{n}+\frac{1}{n^2}\right)} = 4$$

$$\Rightarrow L > 1 \Rightarrow \sum_{n\geq 1} x_n \text{ is divergent}$$

6. Find the radius of convergence and the convergence set for each of the following series:

c) $\sum_{n\geq 1} \frac{n x^n}{2^n}$

Ratio test: $\quad x_n = \frac{n x^n}{2^n}$

$$\frac{x_{n+1}}{x_n} = \frac{(n+1)\cdot \cancel{x^{n+1}}}{\cancel{2^{n+1}}} \cdot \frac{\cancel{2^n}}{n \cancel{x^n}} = \frac{(n+1)\cdot x}{2\cdot n}$$

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \frac{(n+1)\cdot x}{2\cdot n} = \lim_{n\to\infty} \frac{\cancel{n}\left(1+\frac{1}{n}\right)\cdot x}{2\cdot \cancel{n}} = \frac{x}{2}$$

$\frac{|x|}{2} < 1 \Rightarrow |x| < 2 \Rightarrow$ then the series is conv

$\Rightarrow$ the radius of convergence is $R = 2$

$\exists\, R \in (0,\infty)$, the series is conv. for $|x-c| < R$

the series is diverg. for $|x-c| > R$

check $|x| = 2 \Rightarrow x \in \{-2, 2\}$

$x = -2 \Rightarrow \sum_{n\geq 1} \frac{n \cdot \cancel{(-2)^n}}{\cancel{2^n}} = \sum_{n \geq 1} (-n) = \lim_{n\to\infty} (-n) = -\infty$

$\Rightarrow$ is divergent

$x = 2 \Rightarrow \sum_{n\geq 1} \frac{n \cdot \cancel{2^n}}{\cancel{2^n}} = \sum_{n\geq 1} n = \lim_{n\to\infty} n = \infty$

$\Rightarrow$ is divergent

$\Rightarrow$ The conv. set is $(-2, 2)$