1. Find the tangent plane to the unit sphere $x^2+y^2+z^2=1$ at an arbitrary point (x_0, y_0, z_0)

$f(x,y,z) = x^2+y^2+z^2-1$

$\nabla f = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \rangle = \langle 2x, 2y, 2z \rangle$

$\nabla f \cdot \langle x-x_0, y-y_0, z-z_0 \rangle = 0 \Leftrightarrow$

$\Leftrightarrow \langle 2x_0, 2y_0, 2z_0 \rangle \cdot \langle x-x_0, y-y_0, z-z_0 \rangle = 0 \Leftrightarrow$

$\Leftrightarrow 2x_0(x-x_0) + 2y_0(y-y_0) + 2z_0(z-z_0) = 0 \Leftrightarrow$

$\Leftrightarrow 2x_0x - 2x_0^2 + 2y_0y - 2y_0^2 + 2z_0z - 2z_0^2 = 0 \quad | :2$

$\Leftrightarrow x_0x - x_0^2 + y_0y - y_0^2 + z_0z - z_0^2 = 0$

$\Leftrightarrow x_0^2 + y_0^2 + z_0^2 - x_0x - y_0y - z_0z = 0$

2. $f: \mathbb{R}^2 \to \mathbb{R}, \quad f(x,y) = \frac{1}{2}(x^2+by^2), \quad b>0$

$(x_{k+1}, y_{k+1}) = (x_k, y_k) - s_k \nabla f(x_k, y_k), \quad s_k > 0$

a) $s_k = ?$ using exact line search such tha it minimizes the function:

$$\varphi(s_k) = f(x_{k+1}, y_{k+1}) = f((x_k, y_k) - s_k \nabla f(x_k, y_k))$$

$\varphi'(s_k) = 0$

$$\varphi(s_k) = \frac{1}{2}\left(\left(x_k - s_k \nabla f(x_k, y_k) \right)^2 + b\left(y_k - s_k \nabla f(x_k, y_k) \right)^2 \right)$$

$$\frac{\partial \varphi}{\partial s_k} = 0 \iff \nabla f(x_k, y_k) \cdot \nabla f(x_{k+1}, y_{k+1}) = 0$$

$$(x_{k+1}, y_{k+1}) = (x_k, y_k) - s_k \nabla f(x_k, y_k)$$

$$\nabla f = \begin{bmatrix} x \\ by \end{bmatrix}$$

$$\nabla f(x_k, y_k) \cdot \nabla f(x_{k+1}, y_{k+1}) = 0$$

$$\mathrm{I} \quad \nabla f(x_k, y_k) = \begin{bmatrix} x_k \\ by_k \end{bmatrix}$$

$$\mathrm{II} \quad \nabla f(x_{k+1}, y_{k+1}) = \begin{bmatrix} x_{k+1} \\ by_{k+1} \end{bmatrix}$$

$$\Rightarrow$$

$$\Rightarrow \begin{bmatrix} x_k \\ by_k \end{bmatrix} \cdot \begin{bmatrix} x_{k+1} \\ by_{k+1} \end{bmatrix} = 0 \iff x_k x_{k+1} + b^2 y_k y_{k+1} = 0$$

$$x_k(x_k - s_k x_k) + b^2 y_k (y_k - s_k b y_k) = 0$$

$$x_k^2 - S_k x_k^2 + b^2 y_k^2 - S_k b^3 y_k^2 = 0$$

$$x_k^2 (1-S_k) + b^2 y_k^2 (1 - S_k b) = 0$$

$$(1-S_k) + b^2 (1 - S_k b) = 0$$

$$1 - S_k + b^2 - S_k b^3 = 0$$

$$S_k (b^3 - 1) = b^2 - 1$$

$$S_k = \frac{b^2-1}{b^3-1}$$

b) As b gets smaller the contours become more elongated and the convergence of the gradient descent algorithm may be slower.
For large values of b, the contours of the quadratic function are more circular, as b decreases, the contour becomes elongated along the y-axis.
For smaller values of b the gradient descent may take longer to converge to the minimum