V K - vector space

$X \subseteq V$

$$\overset{= \mathrm{span}(X)}{\langle X \rangle} := \bigcap_{\substack{S \leq_K V \\ S \supseteq X}} S$$

(chevrons)

Subspace of V generated by X (span of X)

$$\langle X \rangle = \left\{ \sum_{i=1}^{n} a_i \cdot x_i \;\middle|\; \begin{matrix} n \in \mathbb{N} \\ x_i \in X \\ a_i \in K \end{matrix} \right\}$$

$\hookrightarrow$ set of all linear combinations of elements from X.

if $X = \{x_1, x_2, \dots, x_n\}$

$$\langle X \rangle = \left\{ \sum_{i=1}^{n} a_i \cdot x_i \;\middle|\; \begin{matrix} x_i \in X \\ a_i \in K \end{matrix} \right\}$$

1) Determine the following generated subspaces

(i) $\langle 1, X, X^2 \rangle \leq_{\mathbb{R}} \mathbb{R}[X]$

(ii) $\left\langle \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\rangle \leq_{\mathbb{R}} \mathcal{M}_2(\mathbb{R})$

(iii) $\langle (1,0,2), (0,-1,1) \rangle \leq_{\mathbb{R}} \mathbb{R}^3$

(i) $\langle 1, X, X^2 \rangle = \{ a_1 \cdot 1 + a_2 \cdot X + a_3 \cdot X^2 \mid a_1, a_2, a_3 \in K \}$

$= \{ f \in \mathbb{R}[X] \mid \deg f \leq 2 \} = \mathbb{R}_2[X]$

degree

(ii) $\left\langle \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\rangle =$

$= \left\{ a_1 \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + a_2 \cdot \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + a_3 \cdot \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \middle| a_1, a_2, a_3 \in K \right\} =$

$= \left\{ \begin{pmatrix} a_1 & a_2 \\ 0 & a_3 \end{pmatrix} \middle| a_1, a_2, a_3 \in K \right\} = T_2(\mathbb{R})$

(iii) $\langle(1,0,2),(0,-1,1)\rangle = \{a_1\cdot(1,0,2)+a_2\cdot(0,-1,1) \mid a_1,a_2\in\mathbb{R}\}$

$= \{(a_1, -a_2, 2a_1+a_2) \mid a_1,a_2\in\mathbb{R}\}$

$x = a_1$

$y = -a_2$

$z = 2a_1+a_2$

$\;\;= 2x-y$

$\langle(1,0,2),(0,-1,1)\rangle = \{(x,y,z) \mid z=2x-y,\ x,y,z\in\mathbb{R}\}$

2. Consider the subspaces of $\mathbb{R}^3$

$A = \{(x,y,z)\in\mathbb{R}^3 \mid x=0\}$

$B = \{(x,y,z)\in\mathbb{R}^3 \mid x+y+z=0\}$

$C = \{(x,y,z)\in\mathbb{R}^3 \mid x=y=z\}$

$D = \left\{\begin{pmatrix} a & b \\ c & d\end{pmatrix}\in M_2(\mathbb{R}) \;\middle|\; \begin{array}{l} a+d=0 \\ b=0\end{array}\right\}$

Write A,B,C,D as generated subspaces with a minimal number of generators

(i) $x=0,\ y,z\in\mathbb{R}$

$A=\{(0,y,z)\in\mathbb{R}^3 \mid y,z\in\mathbb{R}\}=$

$=\{(0,y,0)+(0,0,z) \mid y,z\in\mathbb{R}\}=\{y\cdot(0,1,0)+z(0,0,1) \mid y,z\in\mathbb{R}\}$

$=<(0,1,0),(0,0,1)>$ (1)

$(0,1,0)\notin<(0,0,1)>$ because if $(0,1,0)\in<(0,0,1)>$

then $\exists\ \alpha\in\mathbb{R}$ s.t. $\alpha\cdot(0,0,1)=(0,1,0) \Rightarrow \alpha=0 \Rightarrow$

$\Rightarrow (0,1,0)=(0,0,0)$ False

(1) is the minimal set

(ii) $x+y+z=0 \Leftrightarrow x=-y-z$

$B=\{(-y-z,y,z) \mid y,z\in\mathbb{R}\}=\{y(-1,1,0)+z(-1,0,1) \mid y,z\in\mathbb{R}\}$

$=<(-1,1,0),(-1,0,1)>$

$(-1,1,0)\notin<-1,0,1>$ because $\nexists\ \alpha\in\mathbb{R}$ s.t.

$\alpha(-1,0,1)=(-1,1,0)$

(iii) $C=\{(x,x,x) \mid x\in\mathbb{R}\}=\{x(1,1,1) \mid x\in\mathbb{R}\}=$

$=<(-1,1,0),(-1,0,1)>$

(iv) $a+d=0 \Rightarrow d=-a$

$$P=\left\{\begin{pmatrix} a & 0 \\ c & -a\end{pmatrix} \;\middle|\; a,c\in\mathbb{R}\right\}=$$

$$=\left\{\begin{pmatrix} a & 0 \\ 0 & -a\end{pmatrix}+\begin{pmatrix} 0 & 0 \\ c & 0\end{pmatrix} \;\middle|\; a,c\in\mathbb{R}\right\}=$$

$$=\left\{a\begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}+c\begin{pmatrix} 0 & 0 \\ 1 & 0\end{pmatrix} \;\middle|\; a,c\in\mathbb{R}\right\}=$$

$$=\left\langle \begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0\end{pmatrix}\right\rangle \text{ minimal}$$

V_1, V_2 K - vector space

$f: V_1 \to V_2$ is a K-homomorphism of vector spaces (K-linear map) if:

- $\forall\, v_1, v_2 \in V: \; f(v_1+v_2)=f(v_1)+f(v_2)$
- $\forall\, \alpha\in K, \forall\, v\in V$

$$f(\alpha v)=\alpha f(v)$$

shorter version

$$\forall\, \alpha_1,\alpha_2\in K, \forall\, v_1,v_2\in V$$

$$f(\alpha_1 v_1+\alpha_2 v_2)=\alpha_1 f(v_1)+\alpha_2 f(v_2)$$

6. $f, g : \mathbb{R}^2 \to \mathbb{R}^2$, $h : \mathbb{R}^3 \to \mathbb{R}^3$

for Midterm

$f(x,y) = (x+y\ , x-y)$

$g(x,y) = (2x-y\ , 4x-2y)$

$h(x,y,z) = (x-y,\ y-z,\ z-x)$

Show that they are linear maps

9. Determine the kernel and image of the linear maps at 6.

$$\mathrm{Ker}\, f = \{ v \in V_1 \mid f(v) = 0_{V_2} \} \leq_K V_1$$

$$\mathrm{Im} f = \{ f(v) \mid v \in V_1 \} \leq_K V_2$$

$v_1, v_2 \in \mathbb{R}^2$; $v_1 = (x_1, y_1)$

$v_2 = (x_2, y_2)$

$$f(v_1+v_2) \overset{?}{=} f(v_1) + f(v_2)$$

$$f(v_1+v_2) = f((x_1,y_1)+(x_2,y_2)) = f(x_1+x_2\ , y_1+y_2) =$$

$$= (x_1+x_2+y_1+y_2\ , x_1+x_2-y_1-y_2) \quad (1)$$

$$f(v_1)+f(v_2) = f(x_1,y_1) + f(x_2,y_2) =$$

$$= (x_1+x_2+y_1+y_2\ , x_1+x_2-y_1-y_2) \quad (2)$$

$$(1) = (2)$$

$\alpha \in \mathbb{R}$ and $v \in \mathbb{R}^2$

$$v = (x, y)$$

$$f(\alpha v) \stackrel{?}{=} \alpha f(v)$$

$$f(\alpha v) = f(\alpha(x,y)) = f(\alpha x, \alpha y) = (\alpha x + \alpha y, \alpha x - \alpha y) \quad (3)$$

$$\alpha f(v) = \alpha f(x,y) = \alpha(x+y, x-y) = (\alpha x + \alpha y, \alpha x - \alpha y) \quad (4)$$

$$(3) = (4)$$

$$\Rightarrow f \in \mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^2, \mathbb{R}^2)$$

$$\ker f = \{ (x,y) \in \mathbb{R} \mid \forall\ f(x,y) = (0,0) \}$$

$$\left\{ \begin{array}{l} x + y = 0 \\ x - y = 0 \end{array} \right| \oplus$$

$$2x = 0 \Rightarrow x = 0 \Rightarrow y = 0 \Rightarrow (0,0) \in \ker f$$

$$\ker f = \{ (0,0) \}$$

$$\mathrm{Im} f = \{ f(x,y) \mid x, y \in \mathbb{R} \} = \{ (x+y, x-y) \mid x, y \in \mathbb{R} \} =$$

$$= \{ x(1,1) + y(1,-1) \mid x, y \in \mathbb{R} \} \Leftrightarrow \langle (1,1), (1,-1) \rangle$$

$$\ker h = \{ v \in \mathbb{R}^3 \mid h(v) = (0,0,0) \}$$

$$v = (x,y,z) \in \mathbb{R}^3$$

$$h(v) = (x-y,\ y-z,\ z-x) = (0,0,0)$$

$$\begin{cases} x-y=0 \\ y-z=0 \\ z-x=0 \end{cases} \Leftrightarrow \begin{cases} x=y \\ x-z=0 \\ z-x=0 \end{cases} \Leftrightarrow \begin{cases} x=y \\ x=z \\ z \in \mathbb{R} \end{cases}$$

$$\ker h = \{ (x,x,x) \mid x \in \mathbb{R} \} = \langle (1,1,1) \rangle$$

$$\operatorname{Im} h = \{ f(v) \mid v \in \mathbb{R}^3 \} =$$

$$= \{ (x-y, y-z, z-x) \mid x,y,z \in \mathbb{R} \} =$$

$$= \{ x(1,0,-1) + y(-1,1,0) + z(0,-1,1) \} =$$

$$= \langle (1,0,-1), (-1,1,0), (0,-1,1) \rangle$$

$$(1,0,-1) + (0,-1,1) = (1,-1,0) = (-1)\cdot(-1,1,0) =$$

$$= \langle (1,0,-1), (0,-1,1) \rangle$$