Homework

1. Prove using the ε-definition that:

$$\lim_{n\to\infty} \frac{n+1}{2n+3} = \frac{1}{2}$$

$x_1, x_2, x_3, \dots, x \in \mathbb{R}$

$(x_n)_{n\in\mathbb{N}^*}$ sequence has limit $L \in \mathbb{R} \overset{def}{\Leftrightarrow}$

$$\overset{def}{\Leftrightarrow} \forall \varepsilon > 0 \ \exists N_\varepsilon \in \mathbb{N} : |x_n - L| < \varepsilon \ \forall n \geq N_\varepsilon$$

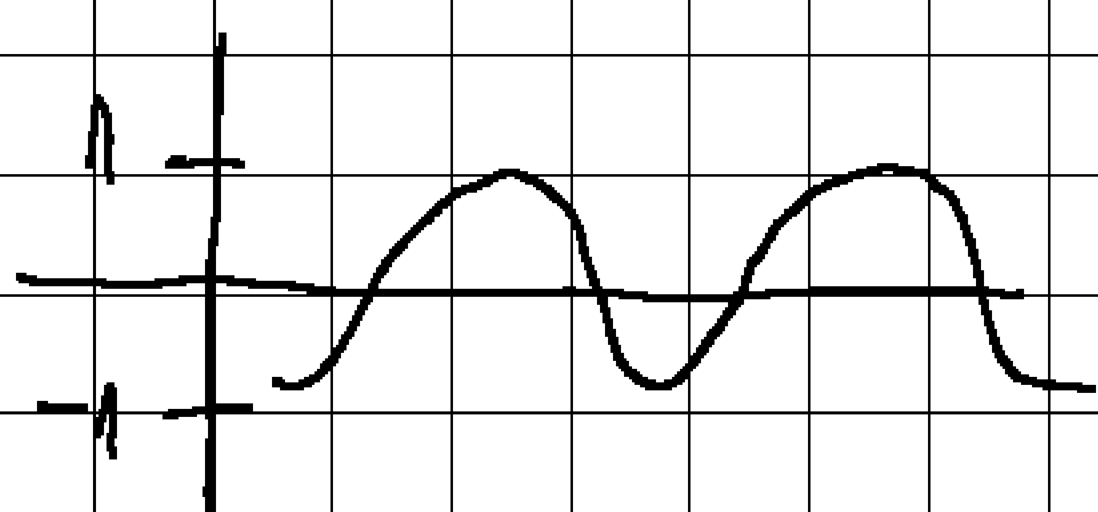
$$x_n = \frac{n+1}{2n+3}, \quad L = \frac{1}{2} \ \forall \varepsilon \ ?\exists N_\varepsilon \in \mathbb{N} : \left|\frac{n+1}{2n+3} - \frac{1}{2}\right| < \varepsilon \Leftrightarrow$$

$$\Leftrightarrow \frac{n+1}{2n+1} < \varepsilon + \frac{1}{2}$$

2. Study if the sequence (x_n) is bounded monotone, and convergent.

c) $x_n = \frac{\sin(n)}{n}$

$$-1 \leq \sin(n) \leq 1 \ /:n \Rightarrow -\frac{1}{n} \leq \frac{\sin(n)}{n} \leq \frac{1}{n} \ /\lim_{n\to\infty}$$



$$\underbrace{\lim_{n\to\infty} \frac{-1}{\infty}}_{=0} \leq \lim_{n\to\infty} \frac{\sin(n)}{n} \leq \underbrace{\lim_{n\to\infty} \frac{1}{n}}_{=0}$$

$$\Rightarrow 0 \leq \lim_{n\to\infty} \frac{\sin(n)}{n} \leq 0$$

$$\Rightarrow \lim_{n\to\infty} \frac{\sin(n)}{n} = 0$$

$\left.\begin{array}{l} x_n \text{ is decreasing} \\ x_n > 0 \Rightarrow x_n \text{ is boundeo from below by } 0 \end{array}\right\} \Rightarrow$

$\Rightarrow x_n$ is convergent

3) Find the limit

d) $\lim\limits_{n\to\infty} \sqrt[n]{1+2+\ldots+n} = \lim\limits_{n\to\infty} \sqrt[n]{\frac{n(n+1)}{2}} =$

$= \lim\limits_{n\to\infty} \sqrt[n]{\frac{n^2+n}{2}} = \lim\limits_{n\to\infty} \sqrt[n]{\frac{n^2\left(1+\frac{1}{n}\right)}{2}} =$

$= \lim\limits_{n\to\infty} \sqrt[n]{\frac{n^2}{2}} \cdot \sqrt[n]{1+\frac{1}{n}} = \underbrace{\lim\limits_{n\to\infty} \left(\frac{n^2}{2}\right)^{\frac{1}{n}}}_{\to 1} \cdot \underbrace{\lim\limits_{n\to\infty} \sqrt[n]{1+\frac{1}{n}}}_{\to 1} =$

$= 1 \cdot 1 = 1$

5. Find the limit

c) $\lim_{n\to\infty}\left(\frac{\ln(n+1)}{\ln n}\right)^n = \left(\frac{\infty}{\infty}\right)^\infty = \lim_{n\to\infty}\left(\frac{\ln[n(1+\frac{1}{n})]}{\ln n}\right)^n =$

$= \lim_{n\to\infty}\left(\frac{\ln n + \ln(1+\frac{1}{n})}{\ln n}\right)^n = \lim_{n\to\infty}\left(1+\frac{\ln(1+\frac{1}{n})}{\ln n}\right)^n =$

$= e^{\lim_{n\to\infty}\frac{n\cdot\ln(1+\frac{1}{n})}{\ln n}} = e^{\lim_{n\to\infty}\frac{\ln(1+\frac{1}{n})^n}{\ln n}} = e^0 = 1$

6. Prove that the sequence (x_n) given by $x_n = 1+\frac{1}{2}+\ldots+\frac{1}{n}-\ln n$ is decreasing and bounded, hence convergent.

Monotonicity: $x_{n+1} \le x_n$, $\forall n\in\mathbb{N}$

$x_{n+1}-x_n = \left(1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n+1}-\ln(n+1)\right)-\left(1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}-\ln(n)\right)$

$\left.\begin{array}{l}\ln(n+1) > \ln(n)\\ \frac{1}{n+1} < \frac{1}{n}\\ \text{the other terms are equal}\end{array}\right\} \Rightarrow x_{n+1}-x_n = \underbrace{\left(\frac{1}{n+1}-\ln(n+1)\right)-\left(\frac{1}{n}-\ln(n)\right)}_{<0}$

$\Rightarrow x_n$ is decreasing (1)

x_n bounded:

lower bound: $\ln(x+1) > \ln(n)$, $\forall n\in\mathbb{N} \Rightarrow$

$\Rightarrow -\ln(n+1) < -\ln(n) \Rightarrow -\ln(n+1)$ is a lower bound for x_n

upper bound: $\frac{1}{n} > 0$, $\forall n\in\mathbb{N} \Rightarrow \frac{1}{n}$ is an upper bound for x_n

$\Rightarrow x_n$ is bounded (2)

(1) && (2) $\Rightarrow x_n$ is convergent

8. Find the limit

b) $$\lim_{n\to\infty} \frac{n^n}{1+2^2+3^3+\ldots+n^n} =$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{a_{n+1}-a_n}{b_{n+1}-b_n}$$

$$= \lim_{n\to\infty} \frac{(n+1)^{n+1} - n^n}{\cancel{1}+\cancel{2^2}+\cancel{3^3}+\ldots+(n+1)^{n+1} - \cancel{1} - \cancel{2^2} - \cancel{3^3} - \ldots - \cancel{n^n}} =$$

$$= \lim_{n\to\infty} \frac{(n+1)^{n+1} - n^n}{(n+1)^{n+1}} = \lim_{n\to\infty} \frac{\cancel{(n+1)^{n+1}}}{\cancel{(n+1)^{n+1}}} - \lim_{n\to\infty} \frac{n^n}{(n+1)^{n+1}} =$$

$$= 1 - \lim_{n\to\infty} \frac{n^n}{(n+1)^{n+1}} = -1 - \lim_{n\to\infty} \left(\frac{n^n}{(n+1)^n \cdot (n+1)} \right) =$$

$$= 1 - \lim_{n\to\infty} \left(\left(\frac{n}{n+1}\right)^n \cdot \frac{1}{n+1} \right) = 1 - \left(e^{-1} \cdot 0 \right) = 1$$

10. Study the convergence and find the limit

b) $x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$, $x_1 = 1$, $a > 1$

Monotonicity:

$$x_{n+1} - x_n = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right) - \overset{2/}{x_n} =$$

$$= \frac{x^n}{2} + \frac{a}{2x_n} - \frac{2x_n}{2} = \frac{1}{2}\left(\frac{a}{x_n} - x_n\right)$$

$$\frac{x+y}{2} \geq \sqrt{xy}, \quad x = x_n, \quad y = \frac{a}{x_n}$$

$$\frac{x_n + \frac{a}{x_n}}{2} \geq \sqrt{x_n \cdot \frac{a}{x_n}} = \sqrt{a}$$

$$x_n \geq \sqrt{a} \Rightarrow \frac{a}{x_n} - x_n \leq \frac{a}{\sqrt{a}} - \sqrt{a} = 0$$

$\Rightarrow x_{n+1} - x_n \leq 0 \Rightarrow$ $\left.\begin{array}{l}\text{is decreasing} \\ \text{and bounded below} \\ \text{by } \sqrt{a}\end{array}\right\} \Rightarrow$

$\Rightarrow$ is convergent

$$L = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \frac{1}{2}\left(x_n + \frac{a}{x_n}\right) \sim \lim_{n\to\infty} x_n = l$$

$$\Rightarrow x_{n+1} = \frac{1}{2}\left(x_n + a \cdot \frac{1}{x_n}\right) \Big| \cdot \lim_{n\to\infty}$$

$$l = \frac{1}{2}\left(\overset{l}{l} + a \cdot \frac{1}{l}\right)$$

$$l = \frac{1}{2}\left(\frac{l^2+a}{l}\right) \quad / \cdot 2l$$

$$2l^2 = l^2 + a \quad / -l^2 \Rightarrow l^2 = a \Rightarrow \left.\begin{array}{l} l = \sqrt{a} \\ a > 1 \end{array}\right\} \Rightarrow l \text{ is finite}$$