3.17, 3.34, 3.36, 3.40, 3.42, 4.13, 4.18

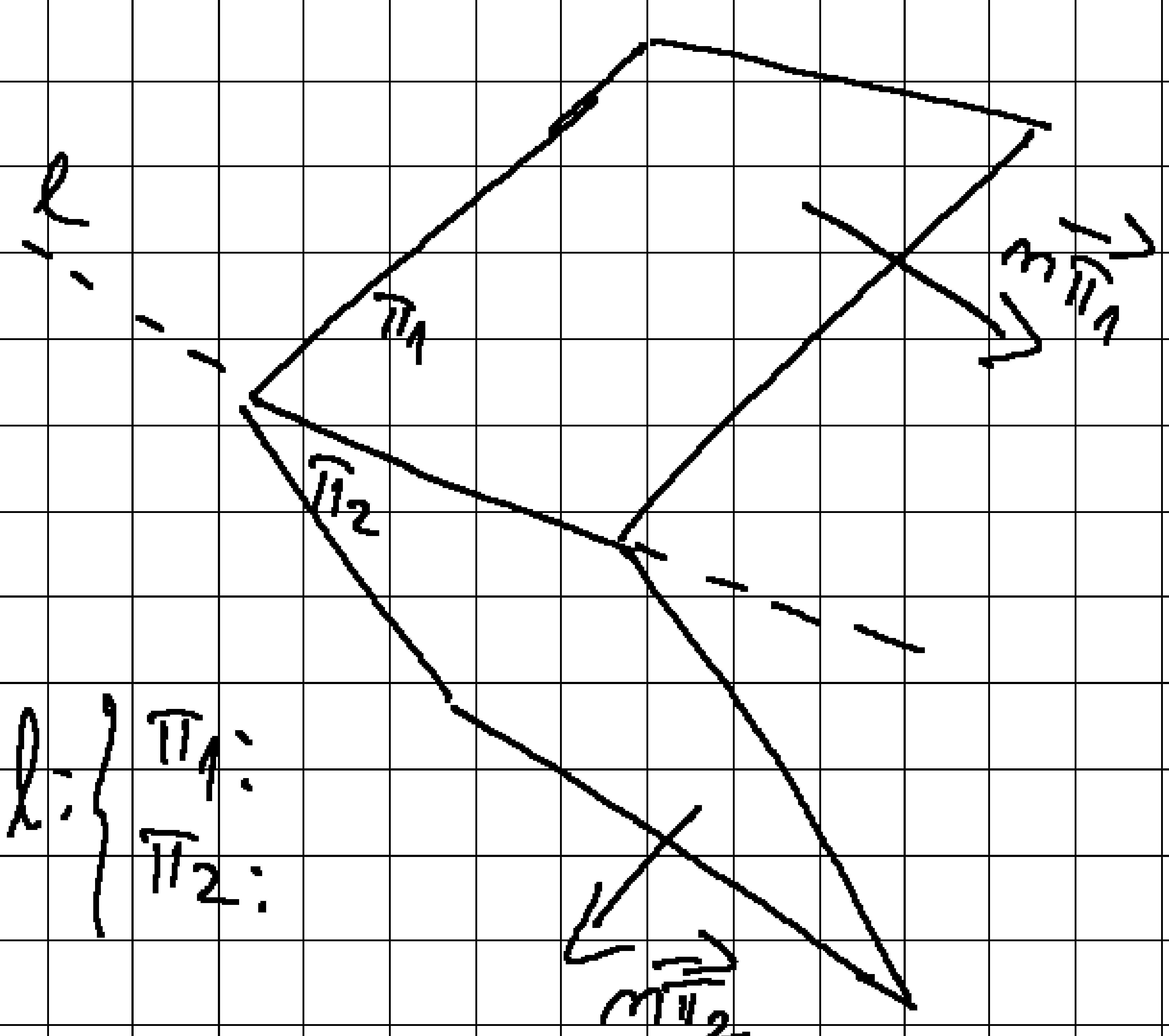
3.34. Solve exercise 2.16 using normal vectors

2.16. Show that the pairwise intersections of the planes are parallel lines:

$\pi_1: \ 3x+y+z-5=0$

$\pi_2: \ 2x+y+3z+2=0$

$\pi_3: \ 5x+2y+4z+1=0$



$l: \begin{cases} \pi_1: \\ \pi_2: \end{cases}$

$\vec{n}_{\pi_1} \times \vec{n}_{\pi_2} \in D(l)$

in fact:

$D(l) = < \vec{n}_{\pi_1} \times \vec{n}_{\pi_2}$

$$l: \begin{cases} A_1x + B_1y + C_1z + D_1 = 0 \\ A_2x + B_2y + C_2y + D_2 = 0 \end{cases}$$

$\vec{n}_{\pi_1}(3,1,1)$ $\quad$ $l_3 : \pi_1 \cap \pi_2$

$\vec{n}_{\pi_2}(2,1,3)$ $\quad$ $l_1 : \pi_2 \cap \pi_3$

$\vec{n}_{\pi_3}(5,2,4)$ $\quad$ $l_2 : \pi_3 \cap \pi_1$

$$\vec{n}_{\pi_1} \times \vec{n}_{\pi_2} = \begin{vmatrix} i & j & k \\ 3 & 1 & 1 \\ 2 & 1 & 3 \end{vmatrix} = 2i - 7j + k$$

$$\vec{n}_{\pi_1} \times \vec{n}_{\pi_3} = \begin{vmatrix} i & j & k \\ 3 & 1 & 1 \\ 5 & 2 & 4 \end{vmatrix} = 2i - 7j + k$$

$$\vec{n}_{\pi_2} \times \vec{n}_{\pi_3} = \begin{vmatrix} i & j & k \\ 2 & 1 & 3 \\ 5 & 2 & 4 \end{vmatrix} = -2i + 7j - k$$

$$\langle \vec{n}_{\pi_1} \times \vec{n}_{\pi_2} \rangle = D(l_3) = \langle (2,-7,1) \rangle$$

$$\langle \vec{n}_{\pi_1} \times n_{\pi_3} \rangle = D(l_2) = \langle (2,-7,1) \rangle$$

$$\langle n_{\pi_2} \times n_{\pi_3} \rangle = D(l_1) = \langle (-2,7,-1) \rangle$$

$$\Rightarrow D(l_1) = D(l_2) = D(l_3) \Rightarrow l_1 \parallel l_2 \parallel l_3$$

3.36. Determine the angles between the planes:

$\Pi_1 : x - \sqrt{2}y + z + 1 = 0$ and $\Pi_2 : x + \sqrt{2}y - z + 3 = 0$

$\vec{n}_{\Pi_1}(1, -\sqrt{2}, 1)$ $\quad$ $\vec{n}_{\Pi_2}(1, \sqrt{2}, -1)$

$$\vec{n}_{\Pi_1} \cdot \vec{n}_{\Pi_2} = |\vec{n}_{\Pi_1}| \cdot |\vec{n}_{\Pi_2}| \cdot \cos(\vec{n}_{\Pi_1}, \vec{n}_{\Pi_2})$$

$$|\vec{n}_{\Pi_1}| = \sqrt{1+2+1} = 2$$

$$|n_{\Pi_2}| = \sqrt{1+2+1} = 2$$

$$\vec{n}_{\Pi_1} \cdot \vec{n}_{\Pi_2} = 1 - 2 - 1 = -2$$

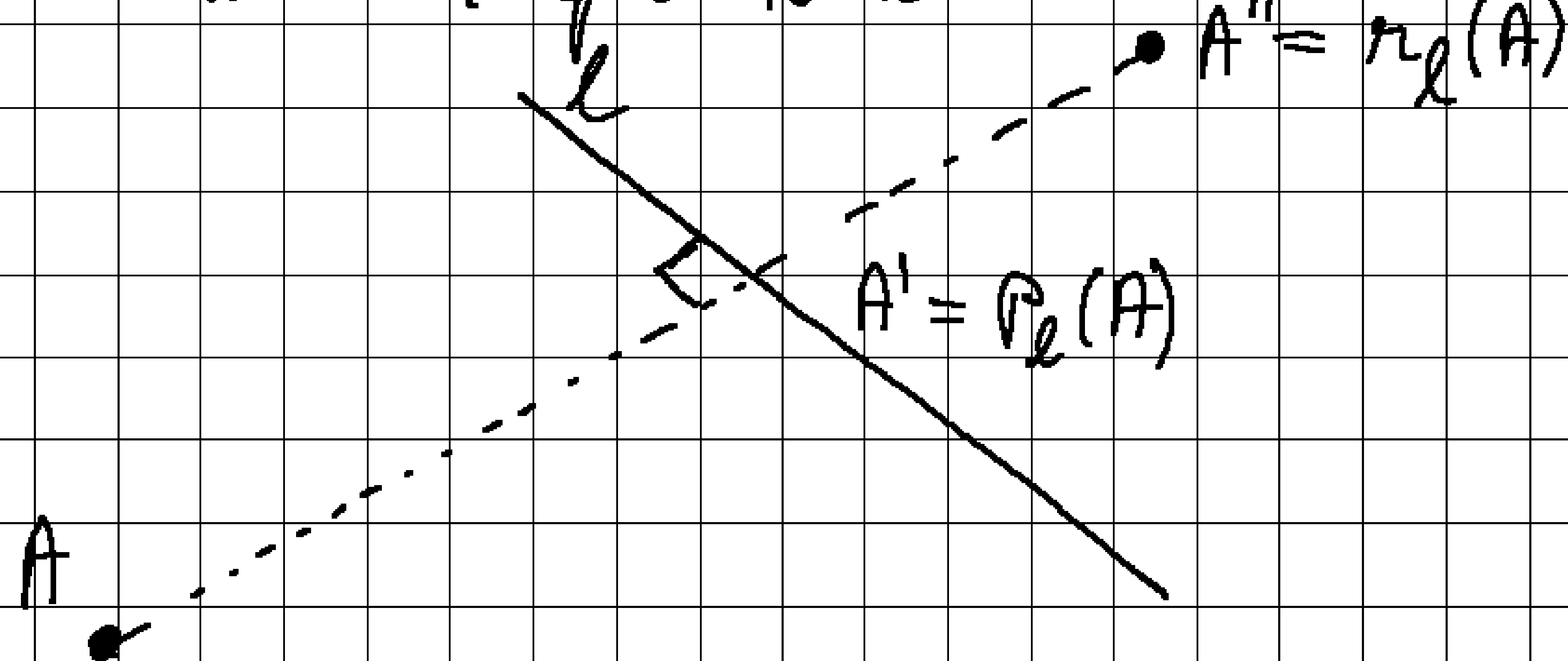
$$\cos(\vec{n}_{\Pi_1}, \vec{n}_{\Pi_2}) = \frac{\vec{n}_{\Pi_1} \cdot \vec{n}_{\Pi_2}}{|n_{\Pi_1}||n_{\Pi_2}|} = \frac{-2}{4} = -\frac{1}{2} \Leftrightarrow$$

$$\Leftrightarrow \sphericalangle(\vec{n}_{\Pi_1}, \vec{n}_{\Pi_2}) = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$$

3.40. $A(1,3,5)$, line l

$$l: \begin{cases} \pi_1: 2x+y+z-1=0 \\ \pi_2: 3x+y+2z-3=0 \end{cases}$$

Determine the orthogonal projection and reflection of A with respect to l



To find $A' = p_l(A)$ we will first construct the plane $\pi \perp l$ and $\pi \ni A$

Afterwards we will have $\pi \cap l = \{A'\}$

$\vec{n}_{\pi_1}(2,1,1)$

$\vec{n}_{\pi_2}(3,1,2)$

$$\vec{n}_{\pi_1} \times \vec{n}_{\pi_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & 1 \\ 3 & 1 & 2 \end{vmatrix} = \vec{i} - \vec{j} - \vec{k} \Rightarrow \vec{v}(1,-1,-1)$$

$$\pi_1: (x-1)-(y-3)-(z-5)=0$$

$$\Leftrightarrow x-y-z+7=0$$

$$A': \begin{cases} x-y-z+7=0 \\ 2x+y+z-1=0 \\ 3x+y+2z-3=0 \end{cases}$$

$(1)+(2) \Rightarrow \quad 3x+6=0 \quad \Rightarrow x=-2$

$$\begin{cases} -2-y-z+7=0 \\ -6+y+2z-3=0 \end{cases}$$

$\Rightarrow \quad z=4 \quad , y=1$

$\Rightarrow A'(-2,1,4)$

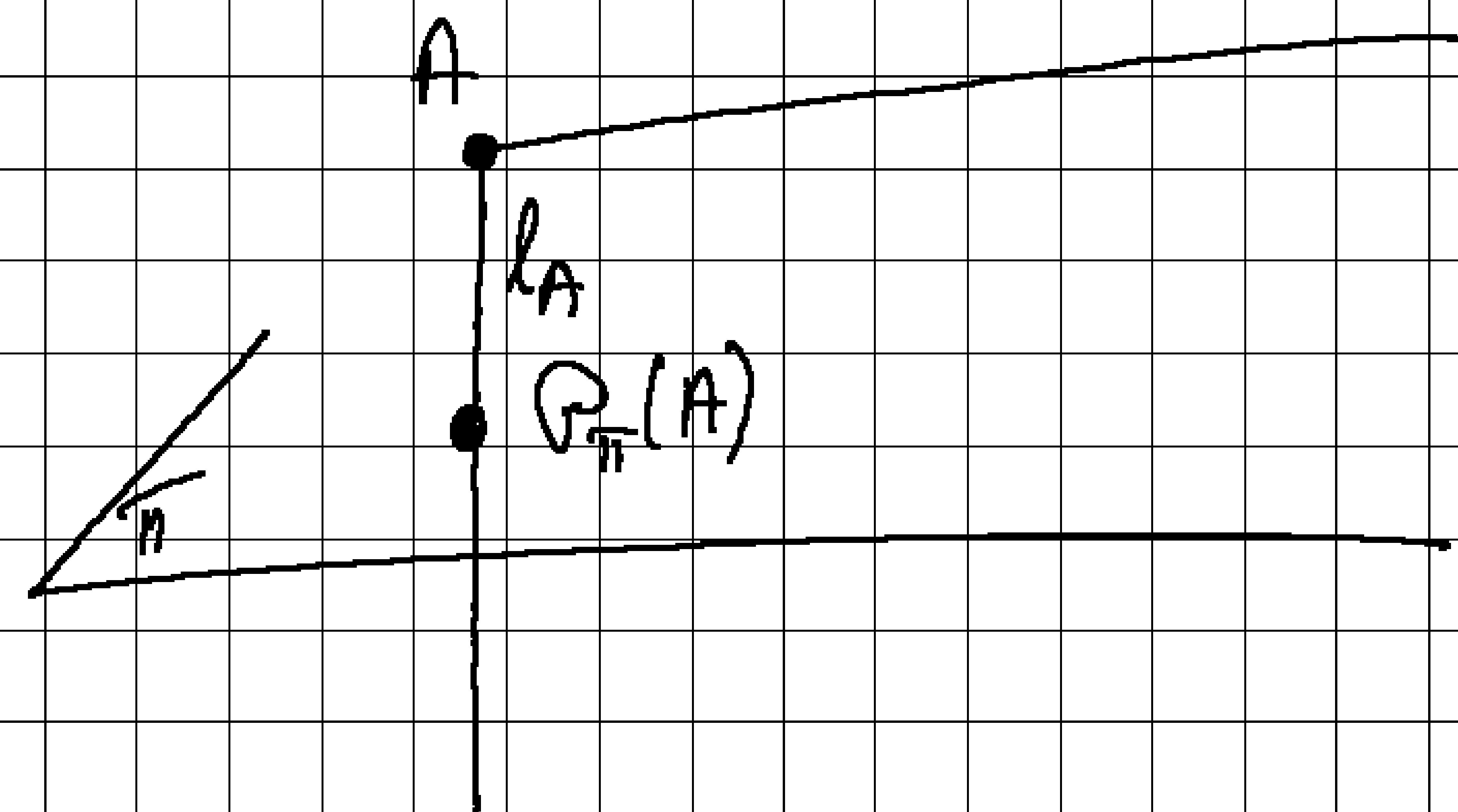
b) $\quad A'': \begin{cases} \frac{1+x}{2}=-2 \\ \frac{3+y}{2}=1 \\ \frac{5+z}{2}=4 \end{cases}$

$\Rightarrow A''(-5,-1,3)$

3.42. Determine the orthogonal projection of the line

$\ell: (\overbrace{2x-y-1=0}^{\pi_1}) \wedge (\overbrace{x+y-z+1=0}^{\pi_2})$

on the plane $\pi: x+2y-z=0$



$$\begin{cases} 2x-y-1=0 \Rightarrow 2x-y=1 \\ x+y-z+1=0 \Rightarrow x-y=\alpha-1 \end{cases} \bigg| \oplus \Rightarrow \quad 3x=\alpha$$

$$\Rightarrow x = \frac{\alpha}{3}$$

Let $z=\alpha$, $\alpha \in \mathbb{R}$

$y = \alpha - 1 - \frac{\alpha}{3} = \frac{2\alpha-3}{3}$

$z = \alpha$

$A(\alpha) = \left(\frac{\alpha}{3}, \frac{2\alpha-3}{3}, \alpha\right)$

$n_\pi(1,2,-1)$

$$l_A: \begin{cases} x = \frac{\alpha}{3} + 1\lambda \\ y = \frac{2\alpha-3}{3} + 2\lambda \\ z = \alpha - \lambda \end{cases}$$

$$x+2y-z=0 \Leftrightarrow \frac{\alpha}{3}+1\lambda+2\left(\frac{2\alpha-3}{3}+2\lambda\right)-\alpha-\lambda=0$$

$$\Leftrightarrow \frac{\alpha}{3}+\lambda+\frac{4\alpha-6}{3}+4\lambda-\alpha+\lambda=0$$

$$\Leftrightarrow \frac{5\alpha-6}{3}\overset{3}{-}\alpha+6\lambda=0 \Leftrightarrow \frac{2\alpha-6}{3}=-6\lambda \Rightarrow$$

$$\Rightarrow \lambda=\frac{6-2\alpha}{18} \Rightarrow \lambda=\frac{3-\alpha}{9}$$

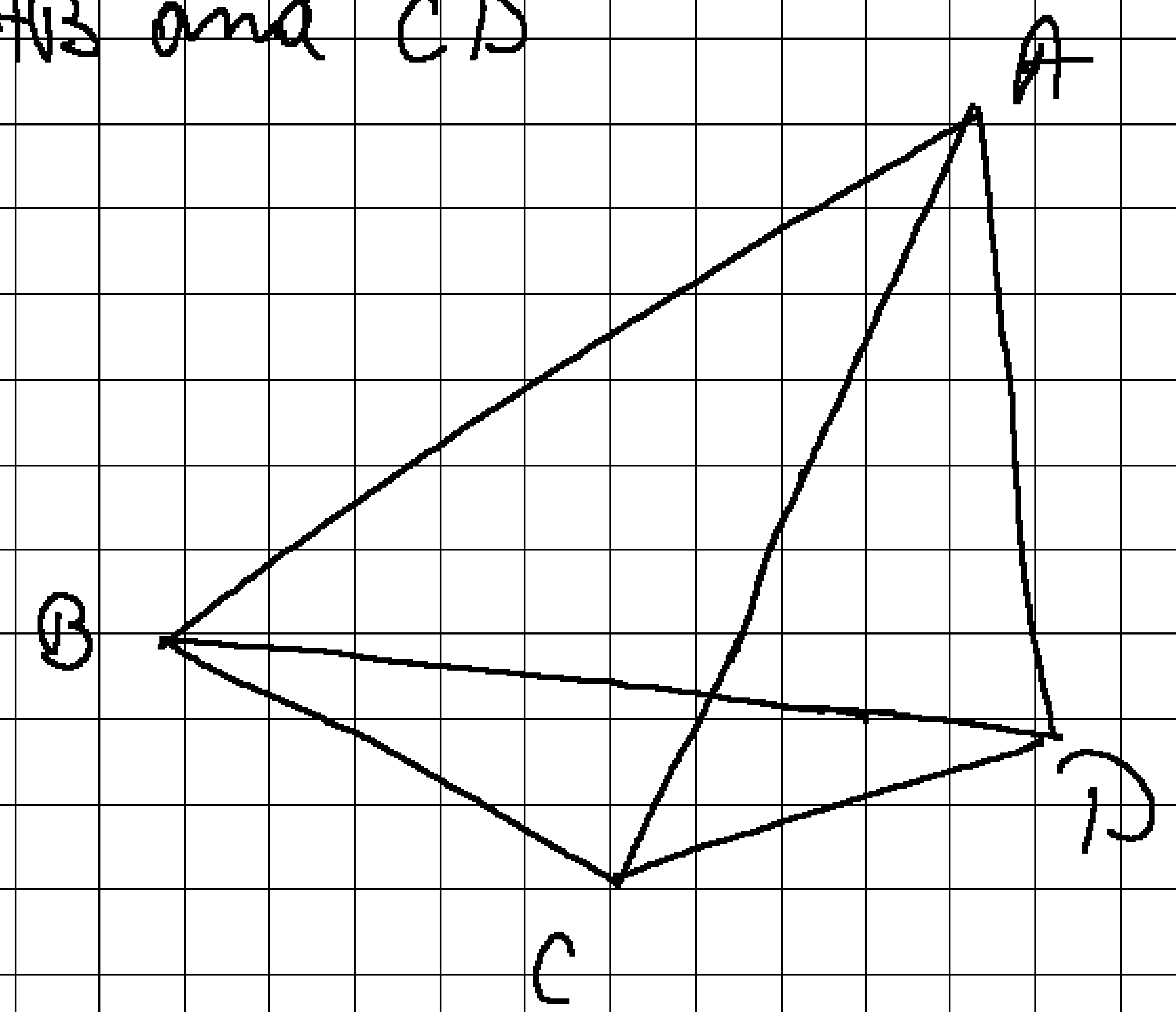
$$P_{\pi}(A)\begin{cases} x=\frac{\alpha}{3}+\frac{3-\alpha}{9}=\frac{3+2\alpha}{9} \\ y=\frac{2\alpha-3}{3}+2\cdot\frac{3-\alpha}{9}=\frac{4\alpha+15}{9} \\ z=\alpha-\frac{3-\alpha}{9}=\frac{10\alpha-3}{9} \end{cases}$$

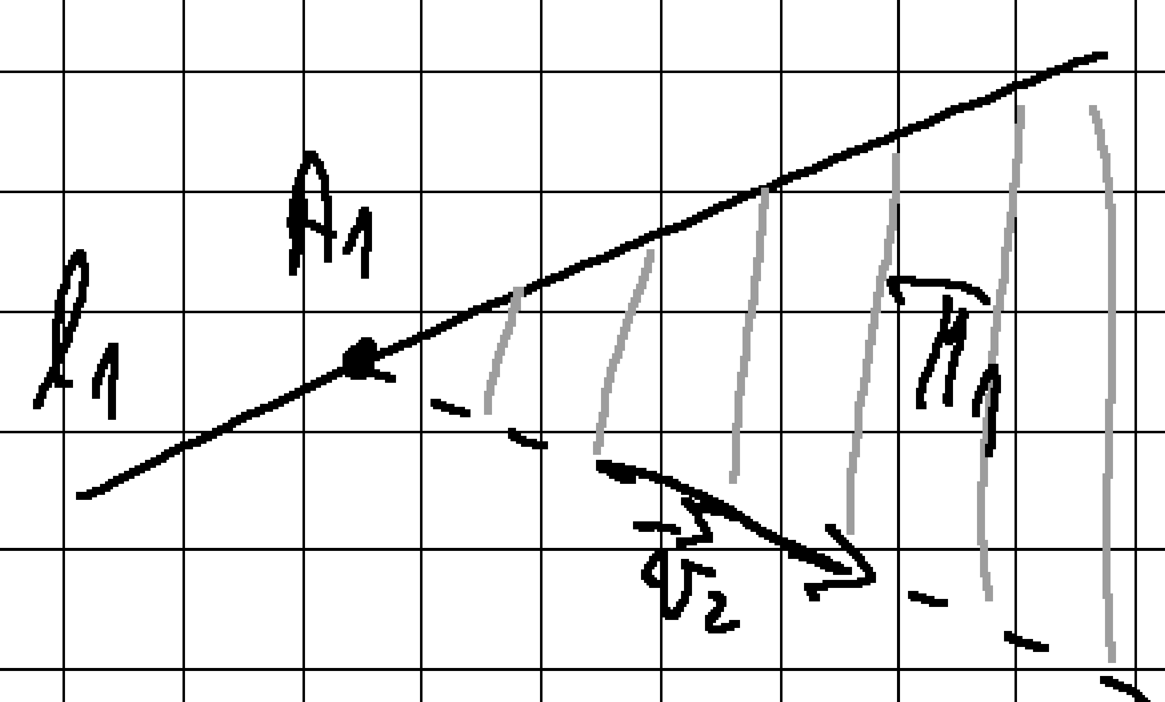
$P_{\pi}(\ell)$ is a line and $\overset{\rightarrow\text{direction}}{D}(P_{\pi}(\ell))=\left\langle\left(\frac{2}{9},\frac{4}{9},\frac{10}{9}\right)\right\rangle$

4.18. For the tetrahedron ABCD in exercise 4.15.

$A(2,-1,1)$, $B(5,5,4)$, $C(3,2,-1)$, $D(4,1,3)$

Determine the common perpendicular of the sides AB and CD

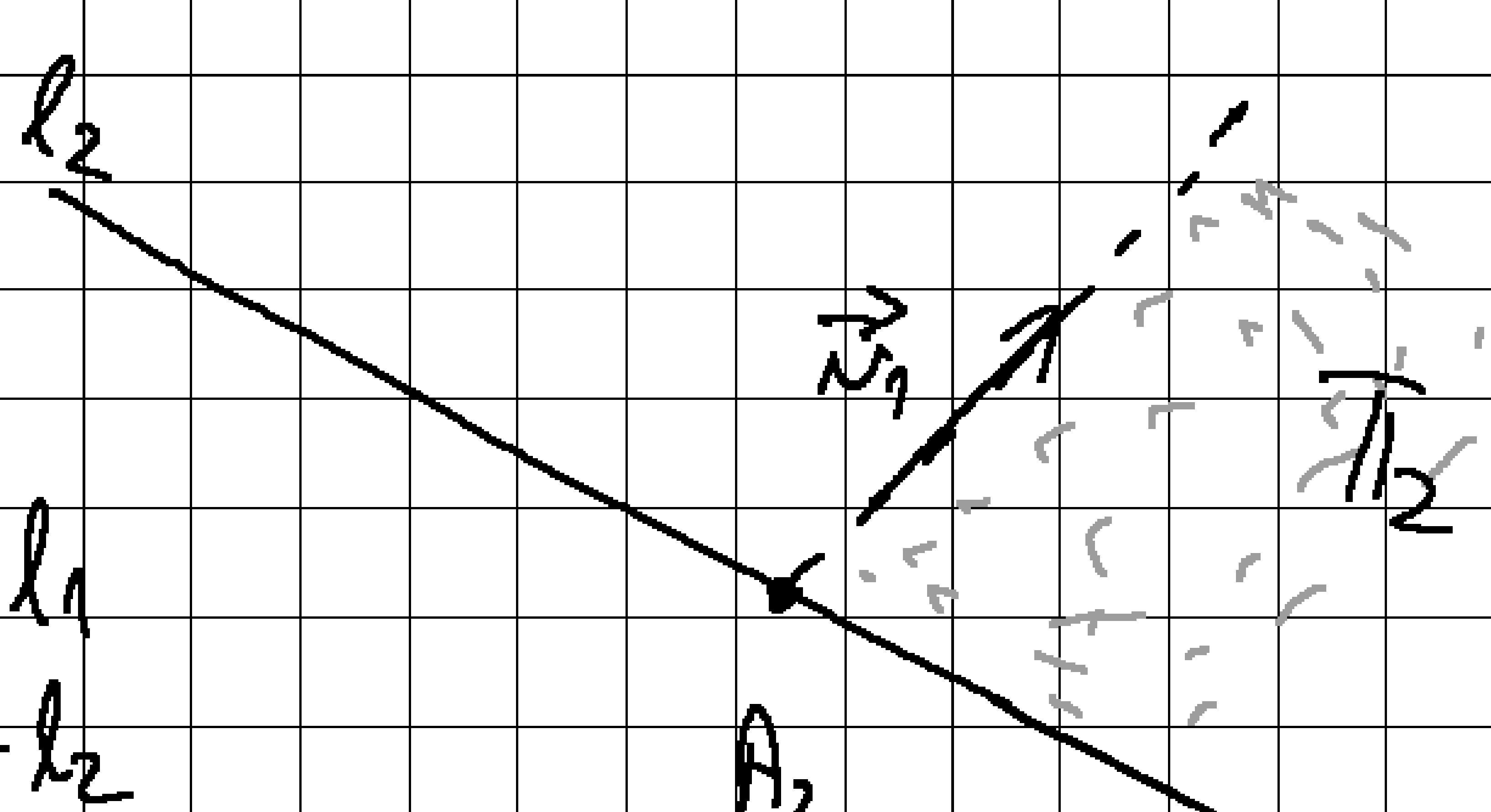




$\vec{v}_1 \times \vec{v}_2$ is the normal vector of π_1 and π_2 where

π_1 given by $A_1, \vec{v}_1, \vec{v}_2$

π_2 given by $A_2, \vec{v}_1, \vec{v}_2$



$A_1 \in l_1$

$A_2 \in l_2$

$\vec{v}_1 \in D(l_1)$

$\vec{v}_2 \in D(l_2)$

Let $\mathcal{P}_1$ be the plane given by $A_1, \vec{v_1}, \vec{v_1}\times\vec{v_2}$

$\mathcal{P}_2$ be the plane given by $A_2, \vec{v_2}, \vec{v_1}\times\vec{v_2}$

$$\vec{n}_{\mathcal{P}_1} = \vec{v_1}\times(\vec{v_1}\times\vec{v_2}) = \begin{vmatrix} \vec{v_1} & \vec{v_2} \\ \vec{v_1}\cdot\vec{v_1} & \vec{v_1}\cdot\vec{v_2} \end{vmatrix} = \begin{vmatrix} \vec{v_1} & \vec{v_2} \\ |\vec{v_1}|^2 & \vec{v_1}\cdot\vec{v_2} \end{vmatrix} =$$

$$= -|v_1|^2\cdot\vec{v_2} + (\vec{v_1}\cdot\vec{v_2})\cdot\vec{v_1}$$

$$\vec{n}_{\mathcal{P}_2} = \vec{v_2}\times(\vec{v_1}\times\vec{v_2}) = \begin{vmatrix} \vec{v_1} & \vec{v_2} \\ \vec{v_2}\cdot\vec{v_1} & \vec{v_2}\cdot\vec{v_2} \end{vmatrix} = |\vec{v_2}|^2\cdot\vec{v_1} - (\vec{v_1}\cdot\vec{v_2})\cdot\vec{v_2}$$

$$\Rightarrow \vec{n}_{\pi_1} \not\parallel \vec{n}_{\pi_2} \Rightarrow \pi_1 \not\parallel \pi_2$$

because $\dfrac{|\vec{v_2}|^2}{\vec{v_1}\cdot\vec{v_2}} \neq \dfrac{|\vec{v_2}|^2}{-(\vec{v_1}\cdot\vec{v_2})}$

$$\Rightarrow \exists\, \ell = \pi_1 \cap \pi_2$$

This is the common perpendicular and we see that $D(\ell) = \langle \vec{v_1}\times\vec{v_2} \rangle$

$$\vec{v_1}\times\vec{v_2} = \begin{vmatrix} i & j & k \\ 1 & 2 & 1 \\ 1 & -1 & 4 \end{vmatrix} = 9i - 3j - 3k$$

$$\mathcal{P}_1: \begin{vmatrix} x-2 & y+1 & z-1 \\ 1 & 2 & 1 \\ 3 & -1 & -1 \end{vmatrix} = 0$$

$$-(x-2)+4(y+1)-7(z-1)=0$$

$$\rho_1: -x+4y-7z+13=0$$

$$\rho_2: \begin{vmatrix} x-3 & y-2 & z+1 \\ 1 & -1 & 4 \\ 3 & -1 & -1 \end{vmatrix}=0$$

$$\rho_2: 5(x-3)+13(y-2)+2(z+1)=0$$

$$5x+13y+2z-39=0$$

So the common perpendicular of AB and CD is:

$$l: \begin{cases} -x+4y-7z+13=0 \\ 5x+13y+2z-39=0 \end{cases}$$