$$f(x) = \begin{cases} \frac{k}{x^4}, & \text{for } x \geq 1 \\ 0, & \text{for } x < 1 \end{cases}$$

a) $k = ?$, $x \in [1, \infty)$

f - probability density $\Leftrightarrow$

1. $f(x) \geq 0$, for $x \in \mathbb{R}$

2. $\int_{\mathbb{R}} f(x)\,dx = 1$

1. $\Rightarrow$ $k \geq 0$

2. $\Rightarrow$ $\int_{\mathbb{R}} f(x)\,dx = \int_{-\infty}^{1} 0\,dx + \int_{1}^{\infty} \frac{k}{x^4}\,dx = 0 + k \cdot \left(-\frac{1}{3x^3}\right)\Big|_{1}^{\infty} =$

$= +k \cdot \frac{1}{3} = \frac{k}{3}$

2 $\Rightarrow$ $\int_{1}^{\infty} f(x)\,dx = 1 \Rightarrow \frac{k}{3} = 1 \Rightarrow k = 3$

b) $F(x) = \int_{-\infty}^{x} f(t)\,dt$

Case I: $x < 1 \Rightarrow F(x) = 0$

Case II: $x \geq 1 \Rightarrow F(x) = \int_{1}^{x} f(t)\,dt = \int_{1}^{x} \frac{3}{t^4}\,dt =$

$= -3 \cdot \frac{1}{t^3} \Big|_{1}^{x} = -\frac{3}{3x^3} + \frac{3}{3} = -\frac{1}{x^3} + 1 = 1 - \frac{1}{x^3}$

$$F(x) = \begin{cases} 0, & x < 1 \\ 1 - \frac{1}{x^3}, & x \geq 1 \end{cases}$$

c) $P(X > 2) = 1 - P(X \leq 2) = 1 - F(2) = \cancel{1} - \cancel{1} + \frac{1}{2^3} = \frac{1}{8}$

2. $P(s < X < s+h) = \int_s^{s+h} \frac{1}{b-a}\, dy =$

$= \frac{1}{b-a} y \Big|_s^{s+h} = \frac{1}{b-a}(s+h-s) = \frac{h}{b-a}$

$P(t < X < t+h) = \frac{h}{b-a}$

a) $X \sim U(4{:}50,\ 5{:}10)$

b) just as likely, both intervals have the same length

3. time between breakdowns has $\exp\left(\frac{1}{5}\right)$

$T =$ time until special maintenance is needed

3 $\exp\left(\frac{1}{5}\right)$ random var., T is going to be their sum

$\Rightarrow T \sim \text{gamma}(3,5)$

a) $P(T \leq 9) = F(9) = \text{gamcdf}(9,3,5)$

b) $P(T > 16 \mid T > 12) = \frac{P(T > 16)}{P(T > 12)} = \frac{1 - P(T \leq 16)}{1 - P(T \leq 12)} =$

$= \frac{1 - \text{gamcdf}(16,3,5)}{1 - \text{gamcdf}(12,4,5)}$

4. $f_{(X,Y)}(x,y) = \frac{1}{16}x^3y^3, \; x,y \in [0,2]$

$= 0, \; x$ or $y \notin [0,2]$

a) $f_X(x) = \int_{\mathbb{R}} f(x,y)\,dy$

Case 1: $x \notin [0,2] \Rightarrow f_X(x) = 0$

Case 2: $x \in [0,2] \Rightarrow f_X(x) = \int_0^2 \frac{1}{16}x^3y^3\,dy = \frac{x^3}{16} \cdot \frac{y^4}{4}\Big|_0^2 =$

$= \frac{x^3}{16} \cdot 4 = \frac{x^3}{4}$

$$f_X(x) = \begin{cases} \frac{x^3}{4}, & x \in [0,2] \\ 0, & x \notin [0,2] \end{cases}$$

$$f_Y(y) = \begin{cases} \frac{y^3}{4}, & y \in [0,2] \\ 0, & y \notin [0,2] \end{cases}$$

b) $f_{XY}(x,y) = f_X(x) \cdot f_Y(y) \Rightarrow X, Y$ independent

Case I: $x,y \in [0,2] \Rightarrow f_X(x) \cdot f_Y(y) = \frac{x^3y^3}{16} = f_{XY}(xy)$ $\boxed{T}$

Case II: otherwise $\Rightarrow f_X(x) \cdot f_Y(y) = 0 = f_{XY}(x\,y)$ $\boxed{T}$

$\Rightarrow X$ and Y independent

c) Find $P(X \leq 1)$

$$P(X \leq 1) = F_X(1) = \int_0^1 f_X(x)\,dx = \int_0^1 \frac{x^3}{4}\,dx = \frac{1}{4}\cdot\frac{x^4}{4}\Big|_0^1 = \frac{1}{16}$$

5. $f_X(x) = \frac{1}{4} x e^{-\frac{x}{2}}, \quad x \geq 0$

$Y = \frac{1}{2}X + 2$

$f_Y = ?$

$Y = G(X)$

$$f_Y(y) = \frac{f_X(g^{-1}(y))}{|g'(g^{-1}(y))|}, \quad y \in g(\mathbb{R})$$

$g: \mathbb{R} \to \mathbb{R}$

$g(x) = \frac{1}{2}x + 2$

$g'(x) = \frac{1}{2} \neq 0$

g- strictly monotone

$g(\mathbb{R}) = \mathbb{R}$

$$\frac{1}{2}x + 2 = y$$

$$x = 2y - 4$$

$$g^{-1}(y) = 2y - 4$$

$$2y - 4 \geq 0$$

$$f_y(y) = \frac{f_x(2y-4)}{\left|\frac{1}{2}\right|} \underset{y \geq 2}{=\!=} \frac{\frac{1}{4}(2y-4)\, e^{-\frac{(2y-4)}{2}}}{\frac{1}{2}} =$$

$$= \frac{1}{2} \cdot 2(y-2) \cdot e^{-\frac{2(y-2)}{2}} = (y-2) \cdot e^{-(y-2)}$$

$$f_y(y) = \begin{cases} (y-2) \cdot e^{-(y-2)}, & y \geq 2 \\ 0, & y < 2 \end{cases}$$