V K-vector space:

- $(V,+)$ abelian group
- $(K,+,\cdot)$ field (usually $K=\mathbb{R}$)
- $\bullet : K \times V \longrightarrow V$
  $(k,v) \longrightarrow kv$
- $\forall\ x,y \in V,\ \forall\ \alpha \in K$:
  $\alpha\cdot(x+y) = \alpha x + \alpha y$
- $\forall\ x \in V,\ \forall\ \alpha,\beta \in K$
- $(\alpha+\beta)\cdot x = \alpha x + \beta x$
- $(\alpha\cdot\beta)\cdot x = \alpha\cdot(\beta\cdot x)$
- $1\cdot x = x,\ \forall x \in V$

4.4. Let $V = \{x \in \mathbb{R} \mid x > 0\}$ define the operation:

$$\forall\ x,y \in V,\ \forall\ k \in \mathbb{R}:$$

$$x \perp y = xy$$

$$k \top x = x^k$$

Show that $(V,\perp)$ is an $\mathbb{R}$-vector space

$(V, \perp) \stackrel{?}{-}$ abelian group

Let $x, y \in V$, $x > 0$, $y > 0 \Rightarrow xy > 0 \Rightarrow x \perp y \in V$

$x \cdot y = y \cdot x$, $\forall x, y \in V \Rightarrow$ „$\perp$" com.

Let $x, y, z \in V$, $(x \perp y) \perp z = x \perp (y \perp z)$;

$(x \perp y) \perp z = (x \cdot y) \perp z = (x \cdot y) \cdot z = x \perp (y \cdot z) =$

$= x \perp (y \perp z) \Rightarrow$ „$\perp$" - associative

Neutral element:

$1 \cdot x = x$, $1 \in V$, $\forall x \in V \Rightarrow 1 \perp x = x \perp 1 = x$, $\forall x \in V$

$\forall x \in V$, $\exists x^{-1} \in V$, $x^{-1} = \frac{1}{x} = x$

$x \perp \frac{1}{x} = \frac{1}{x} \perp x = 1 \Rightarrow (V, \perp)$ - abelian group

Let $x \in V \Rightarrow x > 0$
Let $k \in \mathbb{R}$
$\Big\} \Rightarrow x^k > 0 \Rightarrow$ „T" is well defined

Let $x, y \in V$ and $\alpha \in \mathbb{R}$

$\alpha T (x \perp y) = \alpha T (x \cdot y) = (x \cdot y)^{\alpha} = x^{\alpha} \cdot y^{\alpha}$

$\forall x, y \in V$, $\alpha \in \mathbb{R}$

$(\alpha T x) \perp (\alpha T y) = x^{\alpha} \perp y^{\alpha} = x^{\alpha} \cdot y^{\alpha}$

$\Rightarrow \alpha T (x \perp y) = (\alpha T x) \perp (\alpha T y)$, $\forall x, y \in V$, $\alpha \in \mathbb{R}$

$$(\alpha+\beta) T x \overset{?}{=} (\alpha T x) \perp (\beta T x)$$

$$\left.\begin{array}{l} (\alpha+\beta) T x = x^{\alpha+\beta} \\ (\alpha T x) \perp (\beta T x) = x^{\alpha} \perp x^{\beta} = x^{\alpha} \cdot x^{\beta} = x^{\alpha+\beta} \end{array}\right\} \Rightarrow$$

$$\Rightarrow (\alpha+\beta) T x = (\alpha T x) \perp (\beta T x), \ \forall \alpha, \beta \in \mathbb{R}, \ x \in V$$

$$\left.\begin{array}{l} (\alpha \cdot \beta) T x = x^{\alpha \cdot \beta} \\ \alpha T (\beta T x) = \alpha T x^{\beta} = (x^{\beta})^{\alpha} = x^{\alpha \cdot \beta} \end{array}\right\} \Rightarrow (\alpha\beta) x = \alpha(\beta x)$$

$$1 T x = x^{1} = x, \ \forall x \in V$$

$\Rightarrow (V, \perp, T)$ is an $\mathbb{R}$ vector space

Charac. theorem for subspaces:

V K vector space, $S \subseteq V$

$S \leq_K V \iff$ (i) $S \neq \emptyset$

S is a K subspace of V

(ii) $\forall\, x, y \in S$

$x + y \in S$

(iii) $\forall\, k \in K, \forall\, x \in S$

$k \cdot x \in S$

shorter approach $\Leftarrow$

(i) $S \neq \emptyset$

(ii) $\forall\, k_1, k_2 \in K, \forall\, x, y \in S$

$k_1 x + k_2 y \in S$

4.7. Which one of the following sets are subspaces of $\mathbb{R}^3$?

Exer in the midterm

(i) $A = \{(x, y, z) \in \mathbb{R}^3 \mid x = 0\}$;

(ii) $B = \{(x, y, z) \in \mathbb{R}^3 \mid x = 0 \text{ or } z = 0\}$

(iii) $C = \{(x, y, z) \in \mathbb{R}^3 \mid x \in \mathbb{Z}\}$

(iv) $D = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 0\}$

(v) $E = \{(x,y,z) \in \mathbb{R}^3 \mid x+y+z = 1\}$

(vi) $F = \{(x,y,z) \in \mathbb{R}^3 \mid x = y = z\}$

(i) $A = \{(0,y,z)\}$

let $y_1, z_1, y_2, z_2 \in \mathbb{R}$, $v_1 = (0, y_1, z_1)$, $v_2 = (0, y_2, z_2)$

(i) $S \neq \emptyset$, $(0,1,2) \in A$

(ii) $v_1 + v_2 \overset{?}{\in} A$, $v_1 + v_2 = (\underbrace{0+0}_{0}, y_1+y_2, z_1+z_2) \in A$

(iii) $\forall k \in \mathbb{R}$, $\forall v \in A$, $v = (0,y,z)$

$k \cdot v \overset{?}{\in} A$, $kv = (\underbrace{0 \cdot k}_{=0}, ky, kz) \in A$

$\Rightarrow A \leq_K \mathbb{R}^3$

(ii) $B = \{(x,y,z) \in \mathbb{R}^3 \mid x = 0 \text{ or } z = 0\}$

let $v_1 = (0,3,3)$, $v_2 (3,0,0)$, $v_1, v_2 \in B$

$v_1 + v_2 = (3,3,3)$ but $v_1 + v_2 \notin B$

(iii) $C = \{(x,y,z) \in \mathbb{R}^3 \mid x \in \mathbb{Z}\}$

If we take $k = \frac{1}{5}$ and we have $v_1 = (2,3,4) \in C$

$$k \cdot v_1 = \left(\frac{2}{5}, \frac{3}{5}, \frac{4}{5}\right)$$

$$\frac{2}{5} \notin \mathbb{Z} \Rightarrow C \not\leq \mathbb{R}^3$$

(iv) $D = \{(x,y,z) \in \mathbb{R}^3 \mid x+y+z = 0\}$

i) $D \neq \emptyset$, bcs. $(0,0,0) \in D$

ii) Let $v_1 = (x_1, y_1, z_1)$, $v_2 = (x_2, y_2, z_2)$

$v_1 + v_2 \in D$

$$v_1 + v_2 = (x_1 + x_2, y_1 + y_2, z_1 + z_2)$$

$$x_1 + x_2 + y_1 + y_2 + z_1 + z_2 = \underbrace{x_1 + y_1 + z_1}_{=0} + \underbrace{x_2 + y_2 + z_2}_{=0} = 0$$

iii) Let $k \in \mathbb{R}$ $\quad k_1 \cdot v_1 = (x_2 \cdot k, y_2 \cdot k, z_2 \cdot k)$

$$= x_1 \cdot k + y_1 \cdot k + z_1 \cdot k = k \cdot (\underbrace{x_1 + x_2 + x_3}_{=0}) = k \cdot 0 = 0$$

(V) $E = \{(x,y,z) \in \mathbb{R}^3 \mid x+y+z=1\}$

$v_1 = (1,-1,1) \in E$

$k = 5$

$k \cdot v_1 = (5,5,5)$

$5-5+5 \neq 1$

$k \cdot v_1 \notin E \Rightarrow E \not\leq \mathbb{R}^3$

(VI) $F = \{(x,y,z) \in \mathbb{R}^3 \mid x=y=z\}$

$v_1 = (2,2,2) \in F \Rightarrow F \neq \emptyset$

$v_1 = (x_1, y_1, z_1)$ $\quad$ $v_1 + v_2 = (x_1+x_2, y_1+y_2, z_1+z_2) \in F$

$v_2 = (x_2, y_2, z_2)$

$$\left.\begin{array}{l} x_1 = y_1 = z_1 \\ x_2 = y_1 = z_1 \end{array}\right\} \Rightarrow x_1+x_2 = y_1+y_2 = z_1+z_2$$

Let $k \in \mathbb{R}$

$$\left.\begin{array}{l} k \cdot v_1 = (x_1 \cdot k, y_1 \cdot k, z_1 \cdot k) \\ x_1 = y_1 = z_1 \end{array}\right\} \Rightarrow x_1 \cdot k = y_1 \cdot k = z_1 \cdot k$$

$\Rightarrow F \leq \mathbb{R}^3$

Ex 10: (generalized)

Linear homogeneous system:

$$(S)\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}\cdot x_n = 0 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = 0 \end{cases}$$

$$\underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}}_{=A} \cdot \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ solution of $(S) \Leftrightarrow A\cdot X = 0$

i) X_1, X_2 solutions $\Rightarrow \forall \alpha, \beta$: $\alpha X_1 + \beta X_2$ solutions

1.8. Which of the following sets are subspaces

(i) $[-1,1] \leq_{\mathbb{R}} \mathbb{R}$?

(ii) $D^1 = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \leq 1\} \leq_{\mathbb{R}} \mathbb{R}^2$?

(iii) $T_2(\mathbb{Q}) := \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \;\middle|\; a,b,c \in \mathbb{Q} \right\}$

$\leq_{\mathbb{Q}} M_2(\mathbb{Q})$?

$\leq_{\mathbb{Q}} M_2(\mathbb{R})$?

(iv) $\{ f : \mathbb{R} \to \mathbb{R} \mid f \text{ continuous} \}$

$\leq_{\mathbb{R}} \underbrace{\mathbb{R}^{\mathbb{R}}}_{= \{ f : \mathbb{R} \to \mathbb{R} \}}$?

(i) Let $a = 1$, $b = 1$, $a, b \in [-1,1]$

$a+b = 2 \notin [-1,1] \quad \not\leq_{\mathbb{R}} \mathbb{R}$

(ii) Let $v_1 = (1, 0)$, $v_2 = (0,1)$, $v_1, v_2 \in D'$

$v_1 + v_2 = (1+0, 0+1) = (1,1) \notin D'$

$\Rightarrow D' \not\leq_{\mathbb{R}} \mathbb{R}^2$

(iii) for $\leq_{\mathbb{R}} M_2(\mathbb{R})$

if we take $k = \sqrt{2} \in \mathbb{R}$

Let $A = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} \in T_2(\mathbb{Q})$

$k \cdot A = \begin{pmatrix} \sqrt{2} & 2\sqrt{2} \\ 0 & 3\sqrt{2} \end{pmatrix} \notin T_2(\mathbb{Q})$

$\Rightarrow T_2(\mathbb{Q}) \not\leq_{\mathbb{R}} M_2(\mathbb{R})$

Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in T_2(\mathbb{Q}) \Rightarrow T_2(\mathbb{Q}) \neq \emptyset$

Let $A_1, A_2 \in T_2(\mathbb{Q})$

Let $k_1, k_2 \in \mathbb{Q}$

$k_1 A_1 + k_2 A_2 \in T_2(\mathbb{Q})$

$A_1 = \begin{pmatrix} a_1 & b_1 \\ 0 & c_1 \end{pmatrix}$

$A_2 = \begin{pmatrix} a_2, & b_2 \\ 0, & c_2 \end{pmatrix}$

$k_1 A_1 + k_2 \cdot A_2 = \begin{pmatrix} k_1 a_1 + k_2 a_2 & k_1 b_1 + k_2 b_2 \\ 0 & k_1 c_1 + k_2 c_2 \end{pmatrix} \in T_2(\mathbb{Q})$

$\Rightarrow T_2(\mathbb{Q}) \leq_{\mathbb{Q}} M_2(\mathbb{Q})$