Homework

8. Let $a, b \in \mathbb{R}$. Prove that there exist neighborhoods $U \in V(a)$ and $V \in V(b)$ s.t. $U \cap V \in V(x)$

Let $a = 2$ $b = 5$

$U = [1,3] \in \mathcal{V}(a)$

$V = [4,5] \in \mathcal{V}(b)$

$U \cap V = [1,3] \cap [4,5] = \phi \Rightarrow$

$\Rightarrow \exists$ $U \in V(a)$ and $V \in \mathcal{V}(b)$

so that

$U \cap V = \phi$

10. Let $A = (0,1) \cap \mathbb{Q}$. Show that $\inf A = 0$, $\sup A = 1$, $\operatorname{int} A = \emptyset$ and $\operatorname{cl} A = [0,1]$

$\inf A = 0$ because:

$0 < a, \forall a \in A \Rightarrow 0$ is a lower bound for A

$\sup A = 1$ because

$1 > a, \forall a \in A \Rightarrow 1$ is an upper bound for A

$\operatorname{int} A = \emptyset$ because:

$\operatorname{int} A = \mathring{A} = \{x \in \mathbb{R} \mid \nexists \varepsilon > 0 : (x-\varepsilon, x+\varepsilon) \subseteq A\}$

$\Rightarrow \operatorname{int} A = \emptyset$

$\operatorname{cl} A = [0,1]$ because:

$\operatorname{cl} A = \overline{A} = \{x \in \mathbb{R} \mid \forall V \in \mathcal{V}(x), V \cap A \neq \emptyset\}$

$\Rightarrow \operatorname{cl} A = [0,1]$

5. Let $a, b \in \mathbb{R}$ with $a > 0$.
if S is nonempty and bounded above,
prove that:

$$\sup_{x \in S}(ax+b) = a \sup(S) + b$$

S is bounded from above $\Rightarrow \exists\, \alpha \in \mathbb{R} : \alpha = \sup(S)$
$\Rightarrow \exists\, \beta \in \mathbb{R} : \beta = \sup(ax+b)$

$\beta \geq ax+b \;,\; \forall x \in S$

$\alpha \in ub(S)$ $\qquad$ α is the least upper bound

$\beta \leq k \;,\; k \in ub(S)$

$\alpha \geq x \,\Big|\cdot a \;,\; \forall x \in S \Leftrightarrow a \cdot \alpha \geq ax \,\Big|^{+b} \Leftrightarrow a \cdot \alpha + b \geq ax+b$

$a \cdot \alpha + b \in \underset{x \in S}{ub}(ax+b)$

$$\Rightarrow \sup_{x \in S}(ax+b) = a \sup(S) + b$$

Homework

8. Let $a, b \in \mathbb{R}$. Prove that there exists neighborhoods $U \in V(a)$ and $V \in V(b)$ s.t. $U \cap V \in V(x)$

Let $a = 3$ $b = 6$

$U = [2,3] \in \mathcal{V}(a)$

$V = [5,6] \in \mathcal{V}(b)$

$U \cap V = [1,3] \cap [4,5] = \emptyset \Rightarrow$

$\Rightarrow \exists\, U \in V(a)$ and $V \in \mathcal{V}(b)$ so that $U \cap V = \emptyset$

5. Let $a, b \in \mathbb{R}$ with $a \geq 0$
If S is nonempty and bounded above, prove that:

$$\sup_{x \in S}(ax + b) = a \sup(S) + b$$

S is bounded from above $\Rightarrow \exists\, \alpha \in \mathbb{R} : \alpha = \sup(S)$

$\Rightarrow \exists\, \beta \in \mathbb{R} : \beta = \sup(ax + b)$

$\beta \geq ax + b, \ \forall x \in S$

$\alpha \in ub(S)$ α is the best upper bound

$\beta \leq k, \ k \in ub(S)$

$\alpha \geq x \mid \cdot a \; \forall x \in S \Leftrightarrow a \cdot \alpha \geq ax \mid + b \Leftrightarrow$

$\Leftrightarrow a \cdot \alpha + b \geq ax + b \qquad a \cdot \alpha + b \in \underset{x \in S}{ub}(ax+b)$

$\Rightarrow \sup_{x \in S}(ax+b) = a \sup(S) + b$

10. Let $A = (0,1) \cap \mathbb{Q}$. Show that $\inf A = 0$, $\sup A = 1$, $\operatorname{int} A = \emptyset$ and $\operatorname{cl} A = [0,1]$

$\inf A = 0$ because:

$0 < a, \forall a \in A \Rightarrow 0$ is a lowerbound for A

$\sup A = 1$ because

$1 > a, \forall a \in A \Rightarrow 1$ is an upperbound for A

$\operatorname{int} A = \emptyset$ because

$\operatorname{int} A = \mathring{A} = \{ x \in \mathbb{R} \mid \nexists \, \varepsilon > 0 : (x-\varepsilon, x+\varepsilon) \subseteq A \}$

$\Rightarrow \operatorname{int} A = \emptyset$

$\operatorname{cl} A = [0,1]$ because

$\operatorname{cl} A = \overline{A} = \{ x \in \mathbb{R} \mid \forall V \in \mathcal{V}(x), V \cap A \neq \emptyset \}$

$\Rightarrow \operatorname{cl} A = [0,1]$