Series:

Ex: $1+\frac{1}{2}+\frac{1}{2^2}+\dots=2$

$\frac{1}{4}+\frac{1}{4^2}+\dots=\frac{1}{3}$

Ex: $\sum_{n=0}^{\infty} q^n = 1+q+q^2+\dots = \frac{1}{1-q}$ if $|q|<1$

$$1+q+q^2+\dots+q^n = \frac{q^{n+1}-1}{q-1} = \frac{1-\underbrace{q^{n+1}}_{\to 0}}{1-q} \xrightarrow{\text{converges to}} \frac{1}{1-q}$$

if $|q|<1$

Ex: $S_n = 1+\frac{1}{2}+\dots+\frac{1}{n}$

$S_{2n}-S_n > \frac{1}{2}$ (Lecture 2)

$\Rightarrow$ S_n does not converge

$\Rightarrow$ $1+\frac{1}{2}+\dots+\frac{1}{n}+\dots=\infty$

Ex: $\sum_{n=0}^{\infty} \frac{1}{n!} = 1+\frac{1}{1!}+\frac{1}{2!}+\dots = e$ (proof in the lecture)

Prop: $\sum_{n=1}^{\infty} x_n$ converges $\Longrightarrow \lim_{n\to\infty} x_n = 0$

Proof: $\sum_{n=1}^{\infty} x_n$ converges $\Leftrightarrow$ (S_n) converges

when $S_n = x_1 + x_2 + \ldots + x_n$

$$\lim_{n\to\infty} \underbrace{(S_n - S_{n+1})}_{=x_n} = 0$$

Consequence: $\lim_{n\to\infty} x_n \neq 0 \Longrightarrow \sum_{n=1}^{\infty} x_n$ does not converge

Ex: $\sum_{n\geq 1} \frac{n+2}{n+1}$ does not converge

Prop: $x_n > 0$, $\sum_{n\geq 0} x_n$ converges $\Leftrightarrow \exists\, x_n$ bounded

Proof: $S_n = x_1 + \ldots + x_n$, $S_{n+1} > S_n$;

(S_n) converges $\Rightarrow$ (S_n) bounded

Theorem 8 (check lecture)

Proof: $x_n \le y_n$, $\forall n \ge n_0$

- $\sum_{n=1}^{\infty} y_n$ converges $\Rightarrow \sum_{n=1}^{\infty} y_n$ bounded

$\Rightarrow \sum_{n=1}^{\infty} x_n$ bounded

$\Rightarrow \sum_{n=1}^{\infty} x_n$ convergent

- $\sum_{n=1}^{\infty} x_n = \infty \Rightarrow \sum_{n=1}^{\infty} y_n = \infty$

Ex: (Telescoping series)

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$$

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

$$\sum_{n=1}^{N} \frac{1}{n(n+1)} = 1 - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}} + \dots + \cancel{\frac{1}{N}} - \frac{1}{N+1} = 1 - \frac{1}{N+1}$$

$$\lim_{n \to \infty} \sum_{n=1}^{N} \frac{1}{n(n+1)} = 1 - \lim_{n \to \infty} \frac{1}{N+1} = 1$$

Ex: $\sum_{n=1}^{\infty} \frac{1}{n^2}$

$n(n+1) > n^2 \Rightarrow \frac{1}{n(n+1)} < \frac{1}{n^2}$ does not work

$n(n+1) < n^2 \Rightarrow \frac{1}{n^2} < \frac{1}{n(n+1)}$ works ✓✓

if $\sum_{n\geq 2} \frac{1}{n(n+1)}$ converges, the $\sum_{n\geq 2} \frac{1}{n^2}$ converges

$\lim_{n\to\infty} \frac{x_n}{y_n} \in (0,\infty) \Rightarrow \sum_{n\geq 2} x_n$ has the same nature $\sum_{n\geq 2} y_n$

$x_n = \frac{1}{n(n+1)}$, $y_n = \frac{1}{n^2}$,

$\lim_{n\to\infty} \frac{x_n}{y_n} = \lim_{n\to\infty} \frac{n^2}{n(n+1)} = 1 \Rightarrow \sum \frac{1}{n^2}$ convergent

Recall: $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$

Ex: $\sum_{n=1}^{\infty} \frac{1}{n^p}$, $p \leq 1$

$p \leq 1$, $n^p \leq n$, $\frac{1}{n} \leq \frac{1}{n^p}$

$$\underbrace{\sum_{n=1}^{\infty} \frac{1}{n}}_{\infty} \leq \underbrace{\sum_{n=1}^{\infty} \frac{1}{n^p}}_{\infty}$$

$p = \frac{1}{2} \qquad \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \infty$

Theorem 13 (check lecture)

<u>Ratio test:</u> $x_n \geq 0$

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = l \begin{cases} l < 1 \Rightarrow \text{convergent series} \\ l > 1 \Rightarrow \text{divergent series} \end{cases}$$

<u>Proof:</u> idea: $x_{n+1} \simeq x_n \cdot l$

$x_n \simeq l^n$

$\sum x_n \simeq$ geometric series with ratio l

- <u>Let</u> $l < 1$

Let $\varepsilon > 0$ s.t. $l + \varepsilon < 1$

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = l \Rightarrow \exists N \in \mathbb{N} \text{ s.t.}$$

$$\frac{x_{n+1}}{x_n} - l < \varepsilon, \forall n \geq N$$

$$x_{n+1} < x_n \cdot \underbrace{(l+\varepsilon)}_{=q}, \forall n \geq N$$

$x_{n+1} < x_n \cdot q \,,\ \forall\, n \geq N$ and $q < 1$

$$x_n < x_{n-1} \cdot q < x_{n-2}\, q^2 < \dots < x_N \cdot q^{n-N}$$

$$x_n < q^n \cdot \underbrace{\frac{x_N}{q^N}}_{\text{constant}} \Rightarrow x_n < \overset{\text{constant}}{C} \cdot q^n$$

$$q < 1 \Rightarrow \sum_{n\geq 1} q^n \text{ converges} \xrightarrow{\text{comparison test}} \sum_{n \geq 1} x_n \text{ converges}$$

Recall Ratio Test for sequences

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} < 1 \Rightarrow \lim_{n\to\infty} x_n = 0$$

Ex: $\displaystyle \sum_{n=1}^{\infty} \frac{n^2}{2^n}$

$$\frac{n^2}{2^n} \to 0 \Rightarrow \text{the series could converge}$$

$$\frac{x_{n+1}}{x_n} = \frac{(n+1)^2}{2^{n+1}} \cdot \frac{2^n}{n^2} = \frac{(n+1)^2}{n^2} \cdot \frac{1}{2} \longrightarrow \frac{1}{2}$$

$\Downarrow$

the series converges

Ex: $\sum_{n\geq 0} \frac{x^n}{n!}$, $x \in \mathbb{R}$

Ratio test: $\frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} = \frac{x}{n+1} \longrightarrow 0$

$\Rightarrow$ • $\lim_{n\to\infty} \frac{x^n}{n!} = 0$

• $\sum_{n\geq 0} \frac{x^n}{n!}$ converges

Theorem 15 (Root test)

$\lim \sqrt[n]{x_n} = l$
- $l < 1 \Rightarrow \sum x_n$ convergent
- $l > 1 \Rightarrow \sum x_n$ divergent

idea: $x_n \simeq l^n$

$\sum x_n \simeq$ geometric series with ratio l

Theorem 16 (Cauchy condensation test)

$x_{n+1} \le x_n$, $x_n > 0$

$$\sum_{n=1}^{\infty} x_n \text{ has the same nature as } \sum_{n=0}^{\infty} 2^n x_{2^n}$$

Proof: Let $S_n = x_1 + x_2 + \dots + x_n$, $S_{n+1} > S_n$

$$T_n = x_1 + 2x_2 + 4x_4 + \dots + 2^n x_{2^n}$$

We want to prove that (S_n) bounded /convergent $\Leftrightarrow$ (T_n) bounded /convergent

Let $n \in \mathbb{N}$ then $\exists\, k \in \mathbb{N}$ s.t. $2^k \le n \le 2^{k+1} - 1$

- $S_n = x_1 + x_2 + \dots + x_n \ge x_1 + x_2 + \dots + x_{2^k}$

$x_2 + x_3 \le 2x_2$

$$= x_1 + x_2 + (x_3 + x_4) + (x_5 + x_6 + x_7 + x_8) + \dots + (x_{2^{k-1}+1} + \dots + x_{2^k})$$

$$\ge x_1 + x_2 + 2x_4 + \dots + 2^{k-1} \cdot x_{2^k}$$

$\Rightarrow \boxed{S_n \ge \frac{1}{2} \cdot T_k}$

$$\ge \frac{x_1}{2} + \frac{1}{2}(x_1 + 2x_2 + 4x_4 + \dots + 2^k x_{2^k}) \ge \frac{x_1}{2} + \frac{1}{2} \cdot T_k \ge \frac{T_k}{2}$$

- $S_n = x_1 + x_2 + \dots + x_n \overset{x_n > 0}{\le} x_1 + x_2 + \dots + x_{2^{k+1}-1}$

$$\le x_1 + (x_2 + x_3) + (x_4 + x_5 + x_6 + x_7) + \dots + x_{2^{k-1}} + \dots + x_{2^{k+1}-1} \le$$

$$\le x_1 + 2x_2 + 4x_4 + \dots + 2^{k-1} x_{2^{k-1}} = T_{k-1}$$

$$\boxed{\Rightarrow S_n \le T_{k-1}}$$

$$\boxed{\frac{1}{2} T_k \le S_n \le T_{k-1}}$$

$\Rightarrow (S_n)$ bounded $\Leftrightarrow (T_k)$ is bounded

Ex: $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges iff $p > 1$

Proof: Condensation test

$\sum \frac{1}{n^p}$ has the same nature as $\sum 2^n \cdot \frac{1}{2^{n \cdot p}} =$

$$= \sum \left(\frac{2}{2^p}\right)^n$$

$$q = \frac{2}{2^p} = 2^{1-p} < 1 \Leftrightarrow 1-p < 0 \Leftrightarrow 1 < p$$

convergence $\Leftrightarrow q < 1 \Leftrightarrow p > 1$