From chapter 3: 26, 29, 30, 31, 32, 33, 35, 37, 38, 39, 42

(We work with respect to an orthonormal system)

$l: \quad ax + by + c = 0$

$\Rightarrow \vec{v}(a,b)$ normal vector of l (in 2D)

$\pi: \quad ax + by + cz + d = 0$

$\Rightarrow \vec{v}(a,b,c)$ normal vector of π (in 3D)

3.26 The point $A(3,-2)$ is the vertex of a square and $M(1,1)$ is the intersection point of the diagonals. Determine Cartesian equations for the sides of the square.

$$\begin{cases} \dfrac{x_A + x_C}{2} = 1 \\ \dfrac{y_A + y_C}{2} = 1 \end{cases}$$

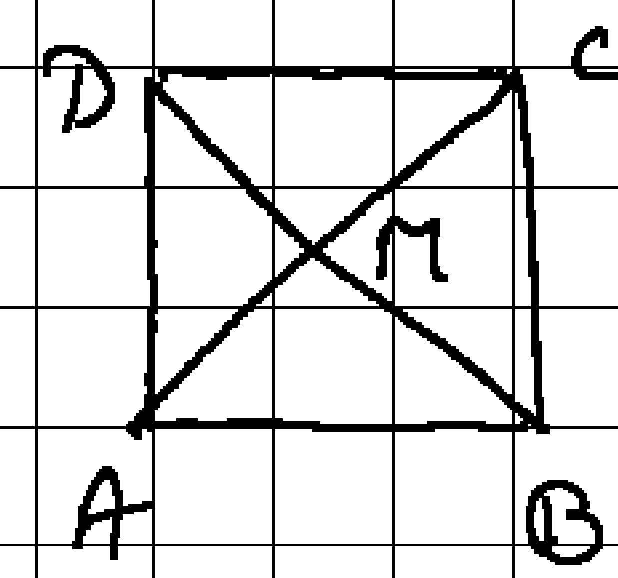
$\Rightarrow C(-1,4)$

$$\begin{aligned} AC^2 &= (x_C - x_A)^2 + (y_C - y_A)^2 \\ &= 4^2 + 6^2 \\ &= 52 \end{aligned}$$

$\Rightarrow BD^2 = 52 \iff (x_B - x_D)^2 + (y_B - y_D)^2 = 52$

$\iff x_B^2 + x_D^2 - 2x_Bx_D + y_B^2 - 2y_By_D + y_D^2 = 52$

$$MA^2 = \left(\frac{1}{2}AC\right)^2$$
$$= \frac{52}{4}$$
$$= 13$$



$$\Rightarrow MB^2 = 13$$

$$m_{AC} = \frac{y_C - y_A}{x_C - x_A} = \frac{4+2}{-1-3} = -\frac{6}{4} = -\frac{3}{2}$$

$$\Rightarrow m_{BD} = \frac{2}{3}$$

$$\Rightarrow BD: \quad y-1 = \frac{2}{3}(x-1)$$
$$3y - 3 = 2x - 2$$
$$2x - 3y + 1 = 0$$

$$B, D: \begin{cases} 2x - 3y + 1 = 0 \\ (x-1)^2 + (y-1)^2 = 13 \end{cases}$$

$$\Leftrightarrow \begin{cases} 2x - 3y + 1 = 0 \Rightarrow y = \frac{2}{3}x + \frac{1}{3} \\ x^2 + 1 - 2x + y^2 + 1 - 2y = 13 \end{cases}$$

$$x^2 + 1 - 2x + \left(\frac{2}{3}x + \frac{1}{3}\right)^2 + 1 - 2\left(\frac{2}{3}x + \frac{1}{3}\right) = 13$$
$$x^2 - 2x + 1 + \frac{4}{9}x^2 + \frac{4}{9}x + \frac{1}{9} + 1 - \frac{4}{3}x - \frac{2}{3} = 13$$
$$\frac{13}{9}x^2 - \frac{26}{9}x + \frac{13}{9} = 13 \Leftrightarrow x^2 - 2x + 1 = 9$$
$$\Rightarrow \quad x = \{4, -2\}$$

$\Rightarrow \quad y_1 = 3 \qquad y_2 = -1$

$\Rightarrow B(4,3) \ , \quad D(-2,-1)$

or

$B(4,3) \ , \ D(-2,-1)$

$$m_{AB} = \frac{y_B - y_A}{x_B - x_A} = \frac{3+2}{4-3} = 5$$

$AB: \quad y+2 = 5(x-3)$

$\qquad 5x - y + 17 = 0$

$CD: \quad y-4 = 5(x+1)$

$\qquad 5x + y - 9 = 0$

$m_{BC} = -\frac{1}{5}$

$BC: \quad y-4 = -\frac{1}{5}(x+1)$

$\qquad 5y - 20 = -x - 1$

$\qquad x + 5y - 19 = 0$

$AD: \quad y+2 = -\frac{1}{5}(x-3)$

$\qquad 5y + 10 = -x + 3$

$\qquad x + 5y + 7 = 0$

3.30. $A(2,1,0)$

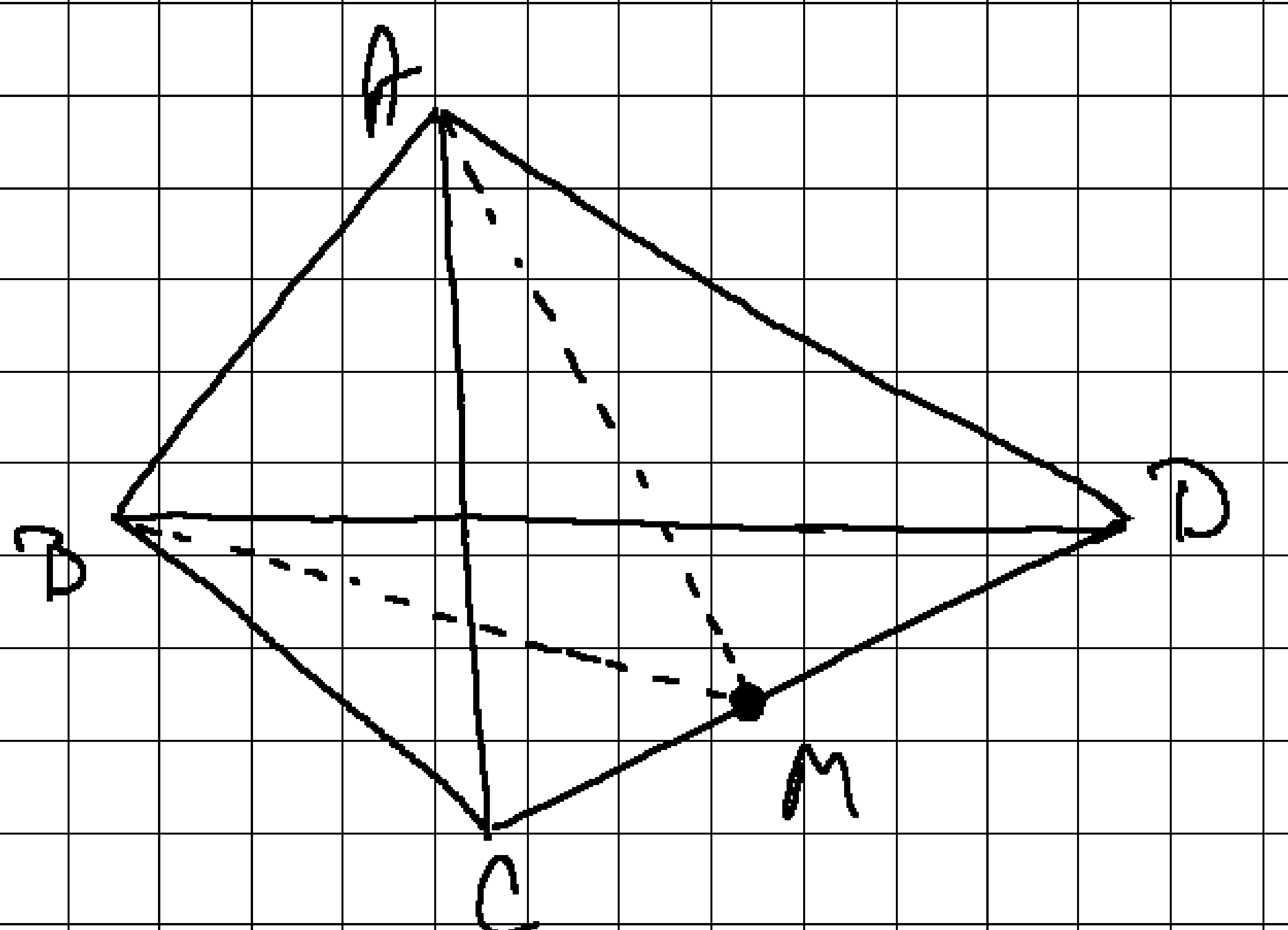
$B(1,3,5)$

$C(6,3,4)$

$D(0,-7,8)$

Verticies of a tetrahedron

Determine a Cartesian equation for the plane containing [AB] and the midpoint of [CD]



$M\left(\frac{x_C+x_D}{2}, \frac{y_C+y_D}{2}, \frac{z_C+z_D}{2}\right)$

$M(3,-2,6)$

$$\begin{vmatrix} x & y & z & 1 \\ x_A & y_A & z_A & 1 \\ x_B & y_B & z_B & 1 \\ x_M & y_M & z_M & 1 \end{vmatrix} = 0 \Leftrightarrow \begin{vmatrix} x & y & z & 1 \\ 2 & 4 & 0 & 1 \\ 1 & 3 & 5 & 1 \\ 3 & -2 & 6 & 1 \end{vmatrix} = 0$$

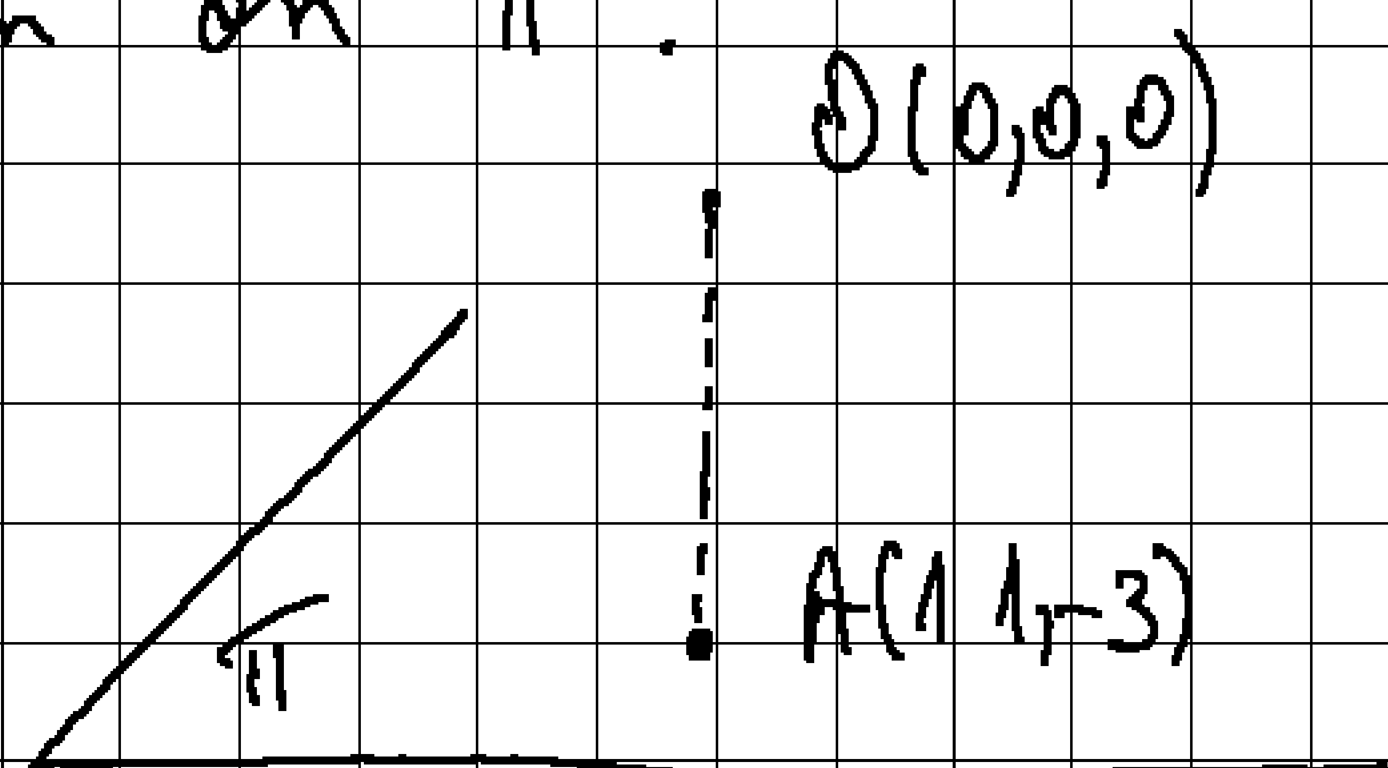
$$\Leftrightarrow x\cdot(-1)^2\cdot\begin{vmatrix} 4 & 0 & 1 \\ 3 & 5 & 1 \\ -2 & 6 & 1 \end{vmatrix} + y\cdot(-1)^3\cdot\begin{vmatrix} 2 & 0 & 1 \\ 1 & 5 & 1 \\ 3 & 6 & 1 \end{vmatrix} +$$

$$+\ z\cdot(-1)^4\cdot\begin{vmatrix}2 & 1 & 1\\ 1 & 3 & 1\\ 3 & -2 & 1\end{vmatrix} + (-1)^5\cdot\begin{vmatrix}2 & 1 & 0\\ 1 & 3 & 5\\ 3 & -2 & 6\end{vmatrix} = 0 \iff$$

$$\iff x(5+18+10-6) - y(10+6-15-12) + z(6+3-2-9+4-1) -$$
$$-(36+15+20-6) = 0$$

$$\iff 27x + 11y + z - 65 = 0$$

3.32. Determine a Cartesian equation of the plane π if $A(1,-1,3)$ is the orthogonal projection of the origin on π.



$OA \perp \pi$

$\vec{OA} \perp \pi$

$\Rightarrow \vec{OA}$ is a normal vector of π.

$$ax + by + cz + d = 0$$

$$1\cdot x + (-1)\cdot y + 3z + d = 0$$

$$A \in \pi \Rightarrow \quad 1\cdot 1 + 1 + 9 + d = 0 \Rightarrow d = -11$$

$$\Rightarrow x - y + 3z - 11 = 0$$

3.33. Determine the distance between the planes

$\pi_1: \ x-2y-2z+7=0$

$\pi_2: \ 2x-4y-4z+17=0$

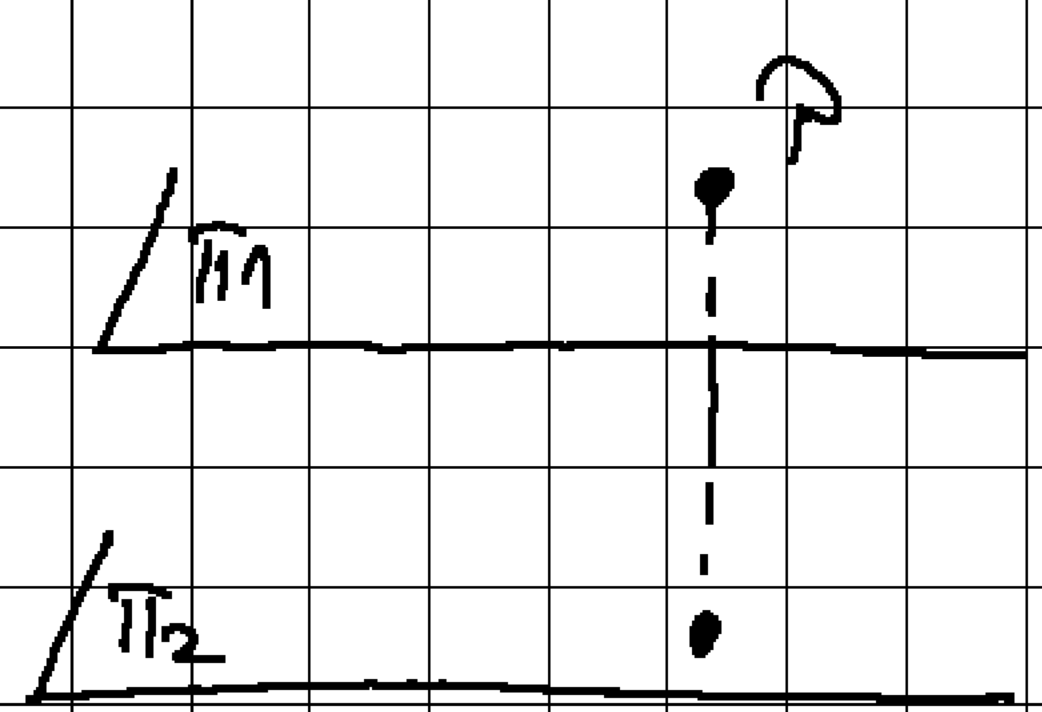
$\pi_2: \ x-2y-2z+\frac{17}{2}=0$

$\vec{n}(1,-2,-2)$ is a normal vector for π_1 and π_2

$\Rightarrow \pi_1 \parallel \pi_2$

$\Rightarrow \text{dist}(\pi_1,\pi_2)=\text{dist}(P,\pi_2)$

$P \in \pi_1$



Let $P(-5,1,0) \in \pi_1$

$$d(P,\pi_2)=\frac{\left|-5-2+\frac{17}{2}\right|}{\sqrt{1+4+4}}=\frac{-7+\frac{17}{2}}{3}=\frac{3}{6}=\frac{1}{2}$$

$\Rightarrow d(\pi_1,\pi_2)=\frac{1}{2}$

3.37. Determine the values a and c for which the line $l: \begin{cases} 3x-2y+z+3=0 \\ 4x-3y+4z+1=0 \end{cases}$ is perpendicular to the plane $ax+8y+cz+2=0$

$$\begin{cases} 3x-2y+z+3=0 & \Big| \cdot \frac{4}{3} \\ 4x-3y+4z+1=0 & \ominus \end{cases} \Leftrightarrow 0+\frac{1}{3}y-\frac{8}{3}z+3=0$$

$$\Leftrightarrow \begin{cases} 3x-2y+\frac{1}{8}y+\frac{9}{8}+3=0 \\ z=\frac{1}{8}y+\frac{9}{8} \end{cases} \Leftrightarrow \begin{cases} 3x-\frac{15}{8}\alpha+\frac{33}{8}=0 \\ z=\frac{1}{8}\alpha+\frac{9}{8} \\ y=\alpha \end{cases}$$

$$\Leftrightarrow \begin{cases} x=\frac{5}{8}\alpha-\frac{11}{8} \\ z=\frac{1}{8}\alpha+\frac{9}{8} \\ y=\alpha \end{cases}$$

$$D(l)=\left\langle \left(\frac{5}{8}, 1, \frac{1}{8}\right) \right\rangle = \langle (5,8,1) \rangle$$

$$l \perp \pi \Leftrightarrow (5,8,1) \parallel \vec{n}_\pi$$

$$\vec{n}_\pi\ (a,8,c)$$

$$k \cdot \vec{n}_\pi = (5,8,1) \Rightarrow \begin{cases} k \cdot a = 5 \Rightarrow a=5 \\ k \cdot 8 = 8 \Rightarrow k=1 \\ k \cdot c = 1 \Rightarrow c=1 \end{cases}$$