1. Find the second-order Taylor-polynomial

a) $f(x,y) = \sin(x+2y)$ at $(0,0)$

Theorem 10.4.

$$f(x) = f(x_0) + \nabla f(x_0)\cdot \underbrace{(x-x_0)}_{\text{column vector}} + \frac{1}{2}\underbrace{(x-x_0)^t}_{\text{row vector}} \cdot$$

$$\cdot H(x_0)(x-x_0) + \underbrace{R(x-x_0)}_{\text{Remainder}}$$

$$\frac{R(x-x_0)}{\|x-x_0\|^2} \xrightarrow{x\to x_0} 0$$

$(x_0, y_0) = (0,0)$

$$\nabla f(x,y) = \left(\frac{\partial f}{\partial x}(x,y), \frac{\partial f}{\partial y}(x,y)\right) =$$

$$= \left(\cos(x+2y), \cos(x+2y)\cdot 2\right)$$

$$\nabla f(0,0) = (\cos 0, \cos 0 \cdot 2) = (1,2)$$

$$H(x,y) = \begin{pmatrix} \frac{\partial^2 f}{\partial^2 x}(x,y), & \frac{\partial^2 f}{\partial x\partial y}(x,y) \\ \frac{\partial^2 f}{\partial y\partial x}(x,y), & \frac{\partial^2 f}{\partial^2 y}(x,y) \end{pmatrix} =$$

$$= \begin{pmatrix} -\sin(x+2y) & -2\sin(x+2y) \\ -2\sin(x+2y) & -4\sin(x+2y) \end{pmatrix}$$

$$H(0,0) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$f(x,y) = f(0,0) + \nabla f(0,0) \cdot \big((x,y) - (0,0)\big) + \frac{1}{2}(x,y) \cdot H(0,0)$$

$\cdot (x,y) + R \rightarrow$ exercise does not ask for R so we only calculate the polynomial

$$= 0 + (1,2)(x,y) + 0 + R = x + 2y + R$$

b) $f(x,y) = e^{x+y}$ at $(0,0)$ and $(1,-1)$

$$f(0,0) = e^0 = 1$$

$$\nabla f(x,y) = (e^{x+y}, e^{x+y})$$

$$\nabla f(0,0) = (e^0, e^0) = (1,1)$$

$$H(x) = \begin{pmatrix} e^{x+y} & e^{x+y} \\ e^{x+y} & e^{x+y} \end{pmatrix} \qquad H(0,0) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$T_f(x,y) = 1 + (1,1)(x,y) + \frac{1}{2}(x,y)\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} =$$

$$= 1 + x + y + \frac{1}{2}(x+y \quad x+y)\begin{pmatrix} x \\ y \end{pmatrix} =$$

$$= x+y+1+\frac{1}{2}x(x+y)+\frac{1}{2}y(x+y) =$$

$$= x+y+1+\frac{1}{2}x^2+xy+\frac{1}{2}y^2 \quad \text{at } (0,0)$$

II at $(1,-1)$

$$f(1,-1) = e^0 = 1$$

$$\nabla f(1,-1) = (e^0, e^0) = (1,1)$$

$$H(1,-1) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$T_f(x,y) = 1 + (1,1)(x-1, y+1) + \frac{1}{2}(x-1, y+1)\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x-1 \\ y+1 \end{pmatrix}$$

$$= 1 + x \cancel{-1} + y \cancel{+1} + \frac{1}{2}(x-1)^2 + \frac{1}{2}(y+1)^2 + (x-1)(y+1)$$

$$= 1 + x + y + \frac{1}{2}x^2 \cancel{-x} + \cancel{\frac{1}{2}} + \frac{1}{2}y^2 \cancel{+y} + \cancel{\frac{1}{2}} + xy + \cancel{x} - \cancel{y} - \cancel{1}$$

$$= 1 + x + y + \frac{1}{2}x^2 + \frac{1}{2}y^2 + xy$$

2. Compute the Hessian matrix and its eigenvalues

a) $f(x,y) = (y-1)\cdot e^x + (x-1)e^y$ at $(0,0)$

$$\nabla f(x,y) = \left((y-1)e^x + e^y \; , \; e^x + (x-1)e^y\right)$$

$$H(x,y) = \begin{pmatrix} (y-1)e^x & e^x+e^y \\ e^x+e^y & e^y(x-1) \end{pmatrix}$$

$$H(0,0) = \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix}$$

λ is an eigenvalue of $A \Leftrightarrow \det(A - \lambda I_n) = 0$

$$A - \lambda I_2 = \begin{pmatrix} -1-\lambda & 2 \\ 2 & -1-\lambda \end{pmatrix}$$

$$\det(A - \lambda I_2) = (-1-\lambda)^2 - 4 =$$

$$= 1 + 2\lambda + \lambda^2 - 4 =$$

$$= \lambda^2 + 2\lambda - 3 = (\lambda+1)^2 - 4 = (\lambda+3)(\lambda-1) \quad \begin{cases} \lambda_1 = -3 \\ \lambda_2 = 1 \end{cases}$$

3) Find and classify the critical points

a) $f(x,y) = x^3 - 3x + y^2$

(x,y) critical point of f if $\nabla f(x,y) = (0,0)$

(x_0, y_0) is a local min. point if all eigenvalues of $H(x_0,y_0)$ are positive

(x_0,y_0) is a local max. point if all eigenvalues of $H(x_0,y_0)$ are negative

(x_0,y_0) is a saddle point if at least one of the eigenvalues is positive and at least one is negative

(proposition 10.7 - 10.8)

An $n \times n$ matrix A is positive-definite (positive eigenvalues) if $x^t A x > 0$, $\forall x \in (\mathbb{R}^n)^*$

negative-definite (negative eigenvalues) if $x^t A x < 0$, $\forall x \in (\mathbb{R}^n)^*$

indefinite if $\exists x_1, x_2 \in (\mathbb{R}^n)^*$

$x_1^t A x_1 > 0$, $x_2^t A x_2 < 0$

if $\nabla f(x_0, y_0) = (0,0)$ and $H(x_0,y_0)$ is indefinite then (x_0,y_0) is a saddle point (10.6, 10.7, 10.8)

$$\nabla f(x,y) = (3x^2-3,\ 2y) = (0,0) \Rightarrow (x,y) \in \underbrace{\{(1,0),(-1,0)\}}_{\text{critical points}}$$

$$\begin{cases} 3x^2-3=0 \Leftrightarrow x^2-1=0 \Rightarrow x \in \{\pm 1\} \\ 2y=0 \Leftrightarrow y=0 \end{cases}$$

$$H(x,y) = \begin{pmatrix} 6x & 0 \\ 0 & 2 \end{pmatrix}$$

$$H(1,0) = \begin{pmatrix} 6 & 0 \\ 0 & 2 \end{pmatrix}$$

$\lambda_1 = 6$, $\lambda_2 = 2$, both $> 0 \Rightarrow$

$\Rightarrow H(1,0)$ - positive-definite

$\Rightarrow (1,0)$ is a local min. point

$$H(-1,0) = \begin{pmatrix} -6 & 0 \\ 0 & 2 \end{pmatrix}$$

$\lambda_1 = -6$, $\lambda_2 = 2$

$\Rightarrow H(-1,0)$ is indefinite

$\Rightarrow (-1,0)$ is a saddle point

Second method

II $$(x,y)\begin{pmatrix} 6 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 6x^2+2y^2 > 0 \quad \forall (x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$$

$\Rightarrow H(1,0)$ - pos.-def. $\Rightarrow (1,0)$ - local min point

b) $f(x,y) = x^3 + y^3 - 6xy$

$\nabla f(x,y) = (3x^2 - 6y \ , \ 3y^2 - 6x)$

$\nabla f(x_0, y_0) = 0 \Rightarrow \begin{cases} 3x^2 - 6y = 0 & /:3 \\ 3y^2 - 6x = 0 & /:3 \end{cases} \Rightarrow$

$\Rightarrow \begin{cases} x^2 - 2y = 0 \\ y^2 - 2x = 0 \end{cases} \Rightarrow y = \frac{x^2}{2}$

$\left(\frac{x^2}{2}\right)^2 - 2x = 0 \Rightarrow \frac{x^4}{4} - 2x = 0 \Rightarrow x\left(\frac{x^3}{4} - 2\right) = 0 \Rightarrow$

$\Rightarrow \begin{cases} x_1 = 0 \\ x^3 - 8 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = 0 \ , \ y_1 = 0 \\ x_2 = 2 \ , \ y_2 = 2 \end{cases}$

$H(x,y) = \begin{pmatrix} 6x & -6 \\ -6 & 6y \end{pmatrix}$

$H(0,0) = \begin{pmatrix} 0 & -6 \\ -6 & 0 \end{pmatrix}$

$\det(A - \lambda I) = 0 \Rightarrow \begin{vmatrix} -\lambda & -6 \\ -6 & -\lambda \end{vmatrix} = 0 \Rightarrow$

$\Rightarrow \lambda^2 - 36 = 0 \Rightarrow \lambda = \pm 6 \Rightarrow (0,0)$ saddle point

$H(0,0)$ is indefinite

$$H(2,2) = \begin{pmatrix} 12 & -6 \\ -6 & 12 \end{pmatrix}$$

$$\det(A - \lambda i) = \begin{vmatrix} 12-\lambda & -6 \\ -6 & 12-\lambda \end{vmatrix} = (12-\lambda)^2 - 36 = 0 \Rightarrow$$

$$\Rightarrow (12-\lambda)^2 = 36$$

$$\Rightarrow \begin{cases} 12-\lambda = 6 \Rightarrow \lambda_1 = 6 \\ 12-\lambda = -6 \Rightarrow \lambda = 18 \end{cases} \Rightarrow$$

$\Rightarrow (2,2)$ local min. point

4) Let A be a symetric $n \times n$ matrix and $f: \mathbb{R}^n \Rightarrow \mathbb{R}$ $\quad f(X) = \frac{1}{2} X^t \cdot A \cdot X$

Prove that $\nabla f(X) = A \cdot X$ and $H(X) = A$

Taylor expansion:

$$f(x_0 + h) = f(x_0) + \underbrace{\nabla f(x_0) \cdot h}_{\text{linear in } h} + \underbrace{\frac{1}{2} h^t H(x_0) \cdot h}_{\text{quadratic in } h} + R$$

$$f(x_0 + h) = \frac{1}{2}(x_0 + h)^t \cdot A \cdot (x_0 + h) =$$

$$= \frac{1}{2}\left(x_0^t A x_0 + x_0^t A h + h^t A x_0 + h^t A h\right)$$

$$= \underbrace{\frac{1}{2} x_0^t A x_0}_{f(x_0)} + \frac{1}{2} x_0^t A h + \frac{1}{2} h^t A x_0 + \underbrace{\frac{1}{2} h^t A h}_{\text{quadratic}}$$

$$= f(x_0) + \frac{1}{2}\left(\underbrace{x_0^t A h} + \underbrace{h^t A x_0}\right) + \frac{1}{2} h^t A h =$$

= igual

$$\boxed{x_0^t A h} = \langle x_0, Ah \rangle = \langle Ah, x_0 \rangle = (Ah)^t \cdot x_0 =$$

$$(A \cdot B)^t = B^t A^t \qquad\qquad = \boxed{h^t A x_0}$$

$$= f(x_0) + \underbrace{x_0^t A}_{Df(x_0)} h + \frac{1}{2} h^t \underbrace{A}_{H(x_0)} h$$