Chapter 5: 1, 2, 3, 4, 5, 11, 12, 13, 14, 16

$\varphi : \mathbb{A}^n \longrightarrow \mathbb{A}^m$, $n, m \in \mathbb{N}$

affine morphism if:

$$\varphi(\overrightarrow{AB}) = \overrightarrow{\varphi(A)\varphi(B)}$$

$\forall\ \varphi : \mathbb{A}^n \to \mathbb{A}^m$

$$\varphi(p) = Ap + b$$

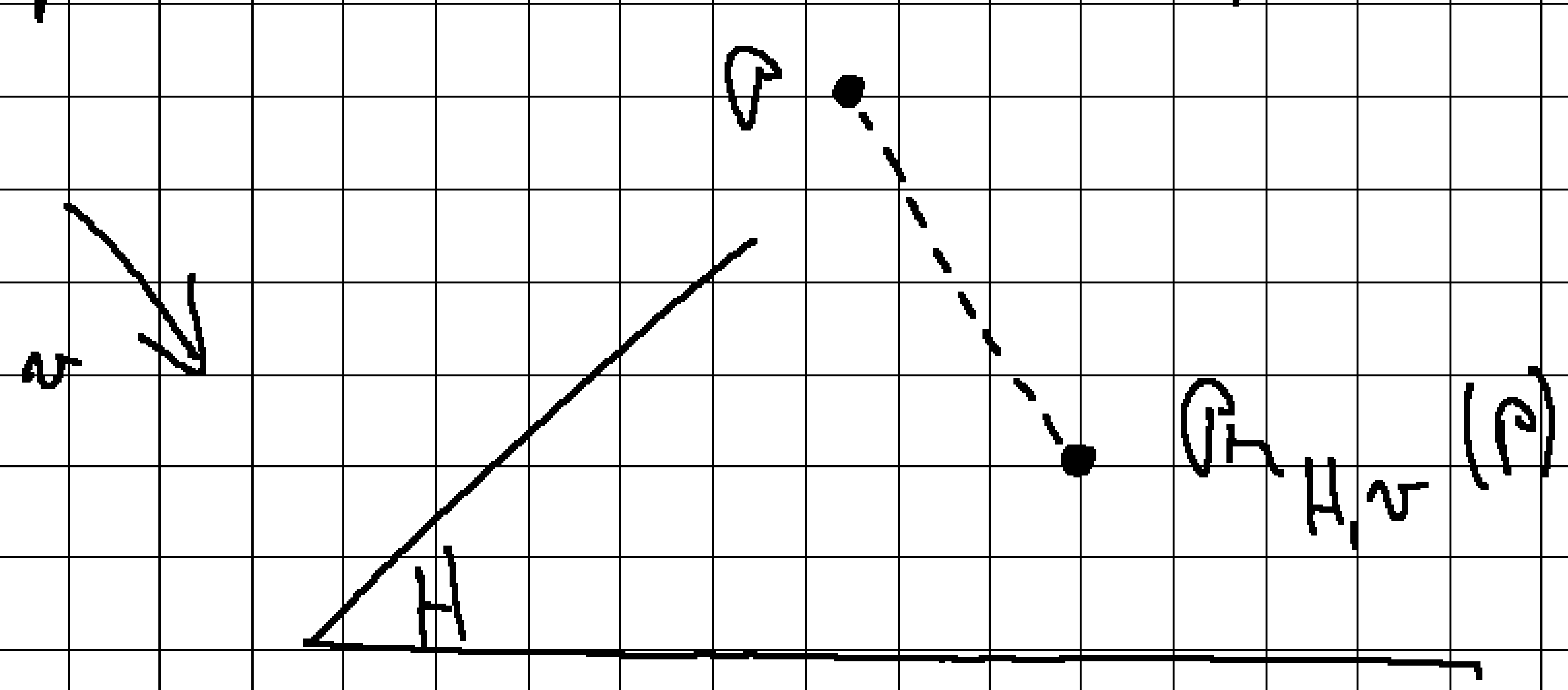
$$A \in \mathcal{M}_{n,m}(\mathbb{R}),\ b \in \mathbb{R}^m$$

$$(\mathrm{lin}\,\varphi)(p) = A \cdot p$$

- projection on a hyperplane

$$H: a_1 x_1 + \dots + a_n x_n + a_{n+1} = 0$$

parallel to $v(v_1, \dots, v_n)$



$$\Pr_{H,v}(p) = \left(I_n - \frac{v \cdot a^T}{v^T \cdot a} \right) \cdot p - \frac{a_{n+1}}{v^T \cdot a} \cdot v =$$

$$= \left(I_n - \frac{v \otimes a}{\underbrace{\langle v, a \rangle}_{\text{dot product}}} \right) \cdot p - \frac{a_{n+1}}{\langle v, a \rangle} \cdot v$$

$$v = (x_1, \dots, x_n)^T$$

$$W = (y_1, \dots, y_n)^T$$

$$v \otimes W = v \cdot W^T = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \begin{pmatrix} y_1 & \dots & y_n \end{pmatrix} =$$

$$= \begin{pmatrix} x_1 y_1 & \dots & x_1 y_n \\ \vdots & & \vdots \\ x_n y_1 & \dots & x_n y_n \end{pmatrix}$$

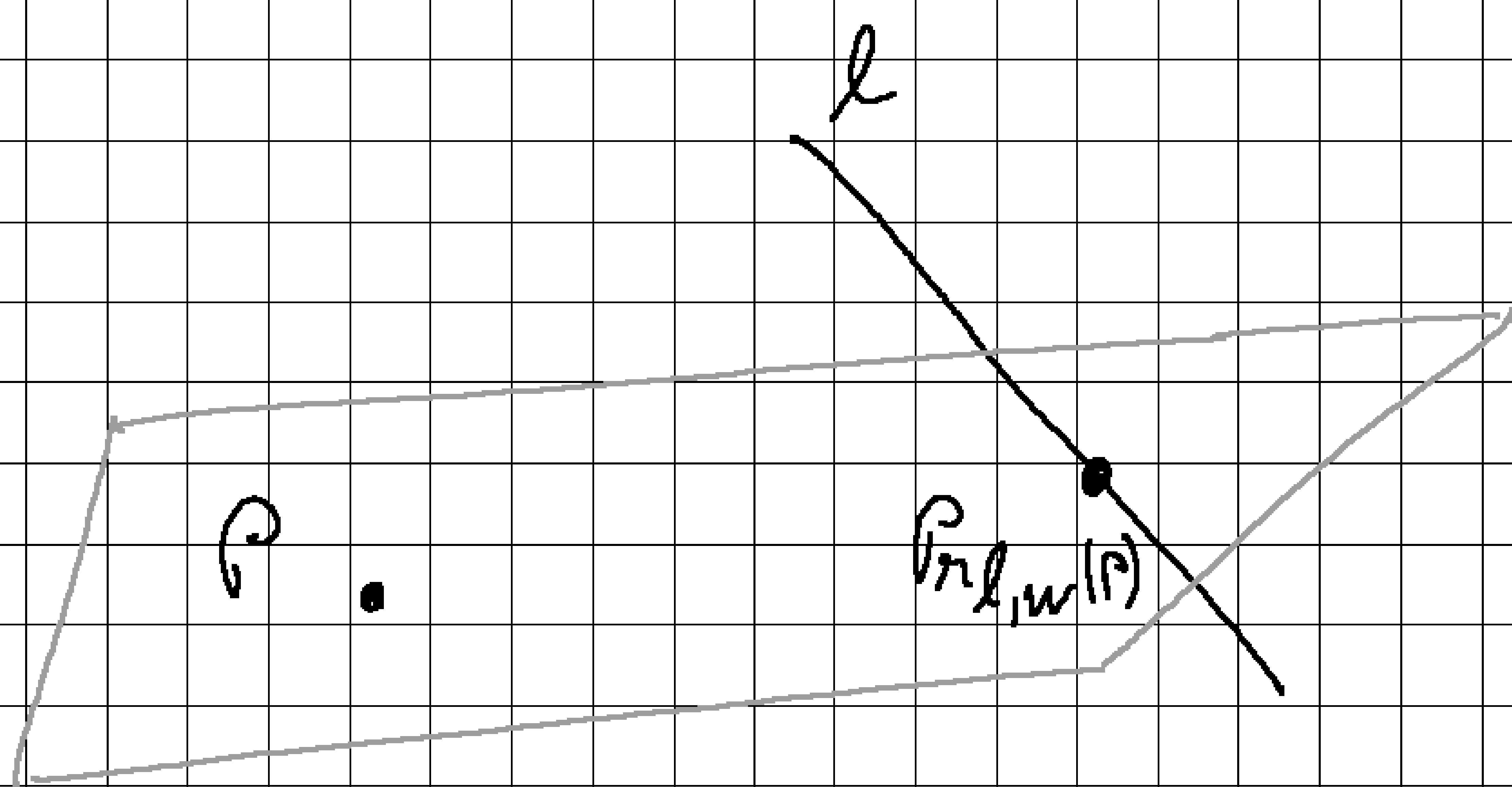
$$\langle v, W \rangle = v^T \cdot W = \begin{pmatrix} x_1 & \dots & x_n \end{pmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} =$$

$$= x_1 y_1 + \dots + x_n y_n$$

- projection on a line parallel to a hyperplane

$$W: a_1x_1 + \dots + a_nx_n + a_{n+1} = 0$$

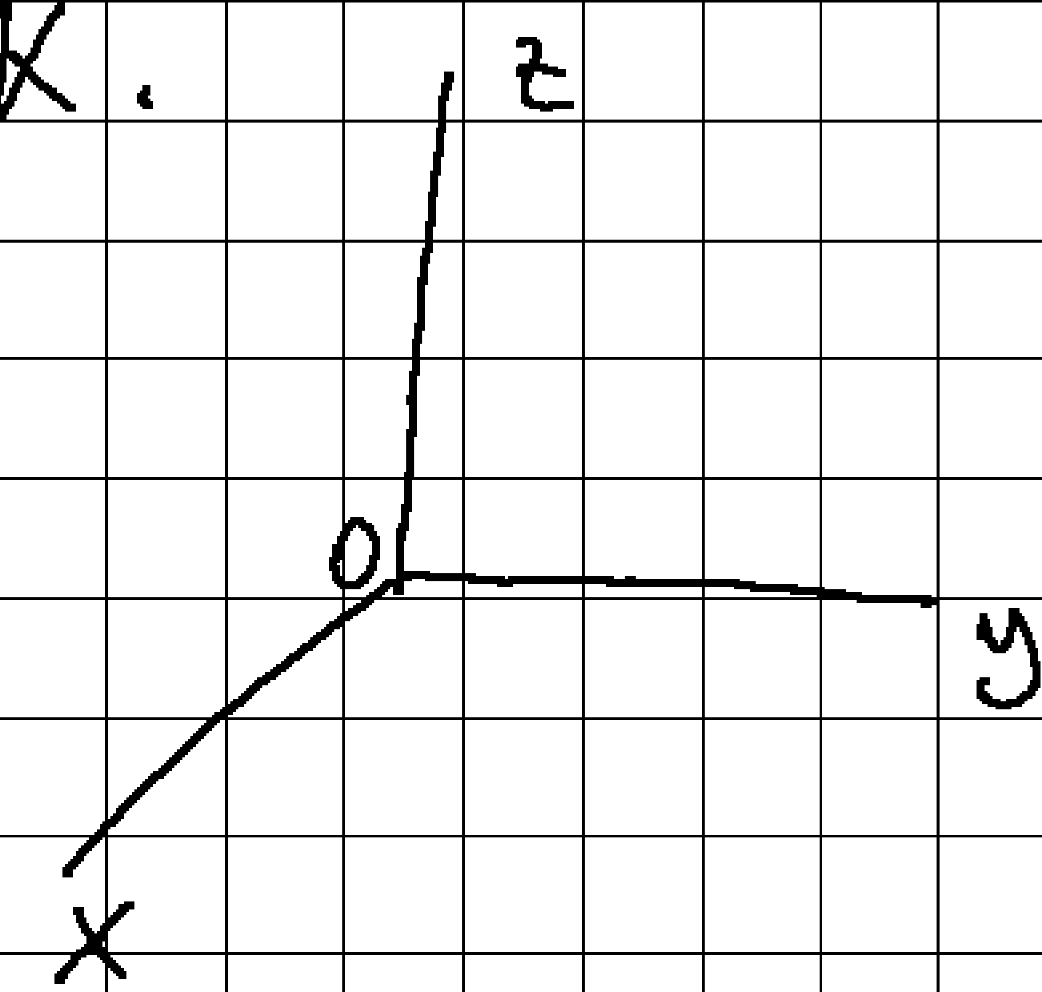
$$Q(q_1, \dots q_n) \in l \;, \; v \in D(l)$$



$$Pr_{l,W} = \frac{v \cdot a^T}{v^T \cdot a} P + \left(i_n - \frac{v \cdot a^T}{v^T \cdot a} \right) \cdot Q$$

5.1. Consider an orthonormal coordinate system K of E^n where $n = 2,3$. Deduce the matrices of.

a) the orthogonal projection on the coordinate axes and the coordinate hyperplanes of K.



$n = 3$

$(xOy): \ z = 0$

$(yOz): \ x = 0$

$(zOx): \ y = 0$

$Ox: \begin{cases} y = 0 \\ z = 0 \end{cases} \Leftrightarrow \begin{cases} x = 0 + 1\lambda \\ y = 0 + 0\lambda \\ z = 0 + 0\lambda \end{cases} \qquad Oy: \begin{cases} x = 0 \\ z = 0 \end{cases} \qquad Oz: \begin{cases} x = 0 \\ y = 0 \end{cases}$

$$Pr_{H,v}(P) = \left(I_n - \frac{v \cdot a^T}{v^T \cdot a} \right) \cdot P - \frac{a_{n+1}}{v^T \cdot a} \cdot v$$

$$Pr_H^{\perp}(P) = \left(I_n - \frac{a \cdot a^T}{a^T \cdot a} \right) \cdot P - \frac{a_{n+1}}{a^T \cdot a} \cdot a$$

$n_{xOy} = (0,0,1)$

$n_{yOz} = (1,0,0)$

$n_{zOx} = (0,1,0)$

$$n_{xoy} \otimes n_{xoy} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\langle n_{xoy}, n_{xoy} \rangle = \langle (0,0,1), (0,0,1) \rangle = 1$$

$$P^{\perp}_{n_{xoy}}(P) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} P$$

$$n_{yoz} \otimes n_{yoz} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$P^{\perp}_{n_{yoz}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} P$$

$$n_{zoy} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$P^{\perp}_{n_{zox}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} P$$

$$Pr_{l,H}(P) = \frac{v \cdot a^T}{v^T \cdot a} \cdot P + \left(I_n - \frac{v \cdot a^T}{v^T \cdot a} \right) \cdot Q$$

orthogonal projection: $v = a$

$$Pr^{\perp}_{l,H}(P) = \frac{a \cdot a^T}{a^T \cdot a} P + \left(I_n - \frac{a \cdot a^T}{a^T \cdot a} \right) \cdot Q$$

Ox:

$a = (1,0,0) \qquad Q(0,0,0) \in l$

$$Pr^{\perp}_{l}(P) = \left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \cdot \frac{1}{1} \right] P + \left[I_n - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right] \cdot Q =$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} P$$

Oy: $a = (0,1,0)$

$$Pr^{\perp}_{l}(P) = \left[\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \cdot 1 \right] P + \left[I_n - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right] \cdot Q$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} P$$

$O_z: \quad a = (0,0,1)$

$$P^{\perp}_{r_\ell}(p) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} p + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} Q = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} p$$

5.4. Determine the orthogonal projection of the point $P(6,-5,5)$ in the plane

$\pi: \ 2x-3y+z-4=0$

by determining the matrix form of the projection.

$$P^{\perp}_{r_\pi} = \left(I_n - \frac{a \otimes a}{\langle a,a\rangle} \right) \cdot p - \frac{a_{n+1}}{\langle a,a\rangle} \cdot a$$

$a = (2,-3,1)$

$$a \otimes a = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 2 & -3 & 1 \end{pmatrix} = \begin{pmatrix} 4 & -6 & 2 \\ -6 & 9 & -3 \\ 2 & -3 & 1 \end{pmatrix}$$

$\langle a,a\rangle = 4+9+1 = 14$

$$P^{\perp}_{r_\pi}(p) = \left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \frac{1}{14} \begin{pmatrix} 4 & -6 & 2 \\ -6 & 9 & -3 \\ 2 & -3 & 1 \end{pmatrix} \right] \begin{pmatrix} 6 \\ -5 \\ 5 \end{pmatrix} - \frac{-4}{14} \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} =$$

$$= \begin{pmatrix} 1-\frac{4}{14} & \frac{6}{14} & -\frac{1}{7} \\ \frac{6}{14} & 1-\frac{9}{14} & \frac{3}{14} \\ -\frac{1}{7} & \frac{3}{14} & 1-\frac{1}{14} \end{pmatrix} \begin{pmatrix} 6 \\ -5 \\ 5 \end{pmatrix} + \frac{2}{7} \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} =$$

$$= \frac{1}{14}\left[\begin{pmatrix}10 & 6 & -2\\ 6 & 5 & 3\\ -2 & 3 & 13\end{pmatrix}\begin{pmatrix}6\\ -5\\ 5\end{pmatrix} + \begin{pmatrix}8\\ -12\\ 4\end{pmatrix}\right] = \frac{1}{14}\left[\begin{pmatrix}20\\ 26\\ 28\end{pmatrix} + \begin{pmatrix}8\\ -12\\ 4\end{pmatrix}\right] =$$

$$= \frac{1}{14}\begin{pmatrix}28\\ 14\\ 32\end{pmatrix}$$

$$Pr_{H,v}(p) = \left(I_n - \frac{v \otimes a}{\langle v, a\rangle}\right)\cdot p - \frac{a_{n+1}}{\langle v, a\rangle}\cdot a$$

$$Ref_{H,v}(p) = 2\, Pr_{H,v}(p) - \underbrace{p}_{I_n \cdot p} =$$

$$= \left(I_n - 2\cdot\frac{v \otimes a}{\langle v, a\rangle}\right)\cdot p - \frac{2a_{n+1}}{\langle v, a\rangle}\cdot a$$

5.12. Hyperplane, $v \in V^n$ $v \nparallel H$. Use the matrix forms to show that.

a) $Pr_{H,v} \circ Pr_{H,v} = Pr_{H,v}$

b) $Ref_{H,v} \circ Ref_{H,v} = id$

a) We have to show that

$$\forall p: \quad Pr_{H,v}\left(Pr_{H,v}(p)\right) = Pr_{H,v}(p)$$

$$\mathrm{Pr}_{H,v}\left(\mathrm{Pr}_{H,v}(p)\right) = \left(I_n - \frac{v \otimes a}{\langle v, a\rangle}\right)\left(\left(I_n - \frac{v \otimes a}{\langle v, a\rangle}\right) p -\right.$$

$$\left. - \frac{a_{n+1}}{\langle v, a\rangle} \cdot a \right) - \frac{a_{n+1}}{\langle v, a\rangle} \cdot a = -2 \frac{a_{n+1}}{\langle v, a\rangle} a + \frac{a_{n+1} (v \otimes a) \cdot a}{\langle v, a\rangle^2} +$$

$$+ \left(I_n - \frac{v \otimes a}{\langle v, a\rangle} \right)^2 p = -2 \frac{a_{n+1}}{\langle v, a\rangle} a + \frac{a_{n+1} (v \otimes a) \cdot a}{\langle v, a\rangle^2} +$$

$$+ \left(I_n - 2 \frac{v \otimes a}{\langle v, a\rangle} + \frac{1}{\langle v, a\rangle^2} \cdot v \underbrace{a^T v}_{\langle v, a\rangle} a^T \right) p =$$

$$= -2 \frac{a_{n+1}}{\langle v, a\rangle} a + \frac{a_{n+1} (v \otimes a) \cdot a}{\langle v, a\rangle^2} + \left(I_n + \frac{-2\, v \otimes a}{\langle v, a\rangle} + \frac{\langle v, a\rangle\, v \otimes a}{\langle v, a\rangle^2} \right) p$$

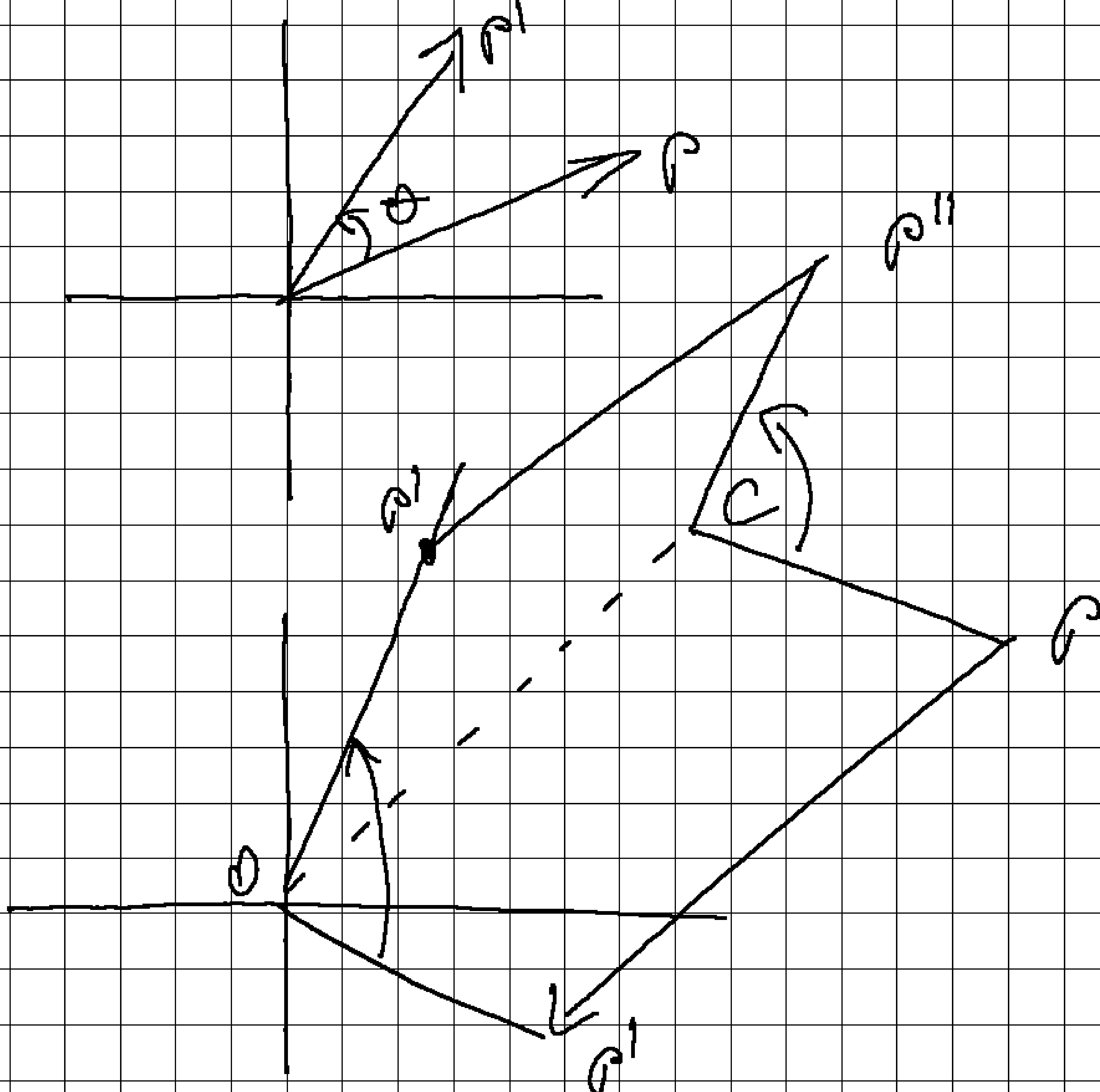
$$= -2 \frac{a_{n+1}}{\langle v, a\rangle} a + \frac{a_{n+1} (v \otimes a) \cdot a}{\langle v, a\rangle^2} + \left(I_n - \frac{v \otimes a}{\langle v, a\rangle} \right) =$$

$$= a_{n+1} \left(\frac{-2\, I_n}{\langle v, a\rangle} + \frac{v \otimes a}{\langle v, a\rangle^2} \right) \cdot a + \left(I_n - \frac{v \otimes a}{\langle v, a\rangle} \right) =$$

$$= a_{n+1} \left(\frac{-2\, I_n \langle v, a\rangle + v \otimes a}{\langle v, a\rangle^2} \right) \cdot a + \left(I_n - \frac{v \otimes a}{\langle v, a\rangle} \right)$$

5.14. $A(1,1)$, $B(4,4)$, $C(2,3)$. Determine the image of ABC under $Rot_{C,\frac{\pi}{2}}$ followed by an orthogonal reflection relative to AB.

$$Rot_{\alpha}(P) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} P$$



$$Rot_{C,\theta}(P) = T_{\vec{OC}} \circ Rot_{\theta} \circ T_{\vec{CO}}(P)$$

$f : \mathbb{A}^n \to \mathbb{A}^m$

$f(p) = A \cdot p + b$

projected coordinates

$\overline{p} = (p, 1)$

$$\overline{f}(p) = \left(\begin{array}{c|c} A & b \\ \hline 0 & 1 \end{array}\right) \cdot \overline{p}$$

projective version of f

$C(x_c, y_c)$

$$T_{\overrightarrow{CO}}(p) = \left(\begin{array}{cc|c} 1 & 0 & -x_c \\ 0 & 1 & -y_c \\ \hline 0 & 0 & 1 \end{array}\right) \cdot p$$

$$Rot_\theta = \left(\begin{array}{cc|c} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ \hline 0 & 0 & 1 \end{array}\right) p$$

$$T_{\overrightarrow{OC}}(p) = \left(\begin{array}{cc|c} 1 & 0 & x_c \\ 0 & 1 & y_c \\ \hline 0 & 0 & 1 \end{array}\right)$$

$$Rot_{C,\theta}(P) = \begin{pmatrix} 1 & 0 & x_c \\ 0 & 1 & y_c \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -x_c \\ 0 & 1 & -y_c \\ 0 & 0 & 1 \end{pmatrix} P$$

$$Rot_{C,\frac{\pi}{2}}(A) = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -2 \\ 1 & 0 & -3 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & -1 & 2 \\ 1 & 0 & 3 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & -1 & 5 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix}$$