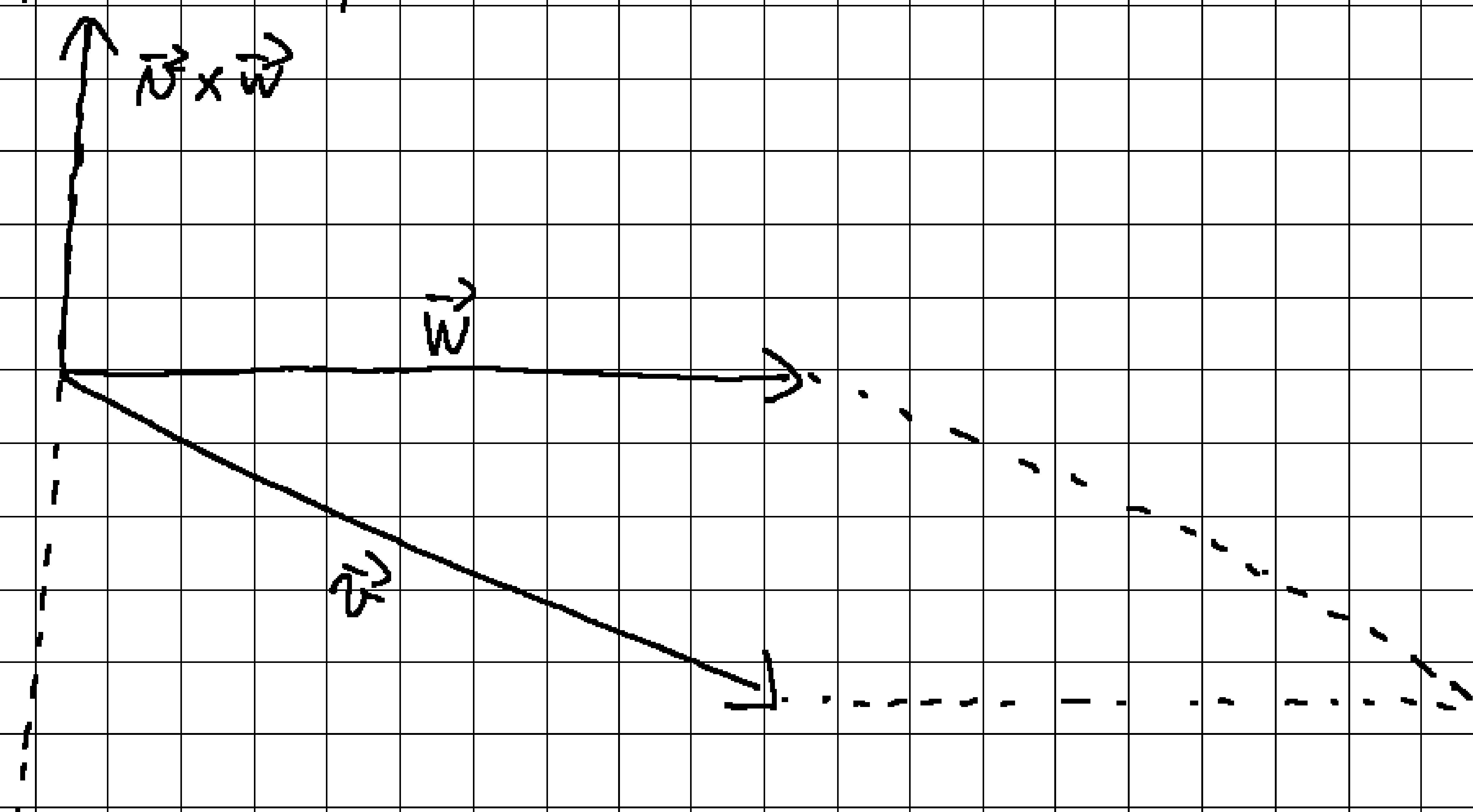
Chapter 4: 2, 3, 4, 10a, 11a, 13, 16, 17

$\vec{v}, \vec{w} \in V^3$



$$|\vec{v} \times \vec{w}| = |\vec{v}| \cdot |\vec{w}| \cdot \sin(\widehat{\vec{v}, \vec{w}})$$

Properties:

- bilinear

$$(\alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2) \times \vec{w} = \alpha_1(\vec{v}_1 \times \vec{w}) + \alpha_2(v_2 \times \vec{w})$$

(same for the other value)

- Skew symmetry

$$\vec{v} \times \vec{w} = -\vec{w} \times \vec{v}$$

- if $\vec{v}, \vec{w}$ lin. dep. $\Rightarrow \vec{v} \times \vec{w} = \vec{0}$

if we work with a right oriented orthonormal system.

$$\vec{v} = (x_{\vec{v}}, y_{\vec{v}}, z_{\vec{v}})$$

$$\vec{w} = (x_{\vec{w}}, y_{\vec{w}}, z_{\vec{w}})$$

$$\vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x_{\vec{v}} & y_{\vec{v}} & z_{\vec{v}} \\ x_{\vec{w}} & y_{\vec{w}} & z_{\vec{w}} \end{vmatrix}$$

i, j, k right oriented:

$$\vec{k} = \vec{i} \times \vec{j} = -\vec{j} \times \vec{i}$$

$$\vec{i} = \vec{j} \times \vec{k} = -\vec{k} \times \vec{j}$$

$$\vec{j} = \vec{k} \times \vec{i} = -\vec{i} \times \vec{k}$$

4.2. With respect to a right oriented orthonormal basis of V^3 consider the vectors: $a(3,-1,-2)$

$b(1,2,-1)$

Calculate: $a \times b$, $(2a+b) \times b$ and $(2a+b) \times (2a-b)$

$$a \times b = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & -1 & -2 \\ 1 & 2 & -1 \end{vmatrix} = \vec{i} + 6\vec{k} - 2\vec{j} + \vec{k} + 4\vec{i} + 3\vec{j} =$$

$$= 5\vec{i} + 7\vec{k} + \vec{j}$$

$$(2a+b) \times b = \left(2(3,-1,-2) + (1,2,-1)\right) \times (1,2,-1) =$$

$$= (7,0,-5) \times (1,2,-1) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 7 & 0 & -5 \\ 1 & 2 & -1 \end{vmatrix} = 10\vec{i} + 2\vec{j} + 14\vec{k}$$

$(2a+b) \times (2a-b) =$

$(2a-b) = (6,-2,-4) - (1,2,-1) = (5,-4,-3)$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 7 & 0 & -5 \\ 5 & -4 & -3 \end{vmatrix} = -20\vec{i} - 4\vec{j} - 28\vec{k}$$

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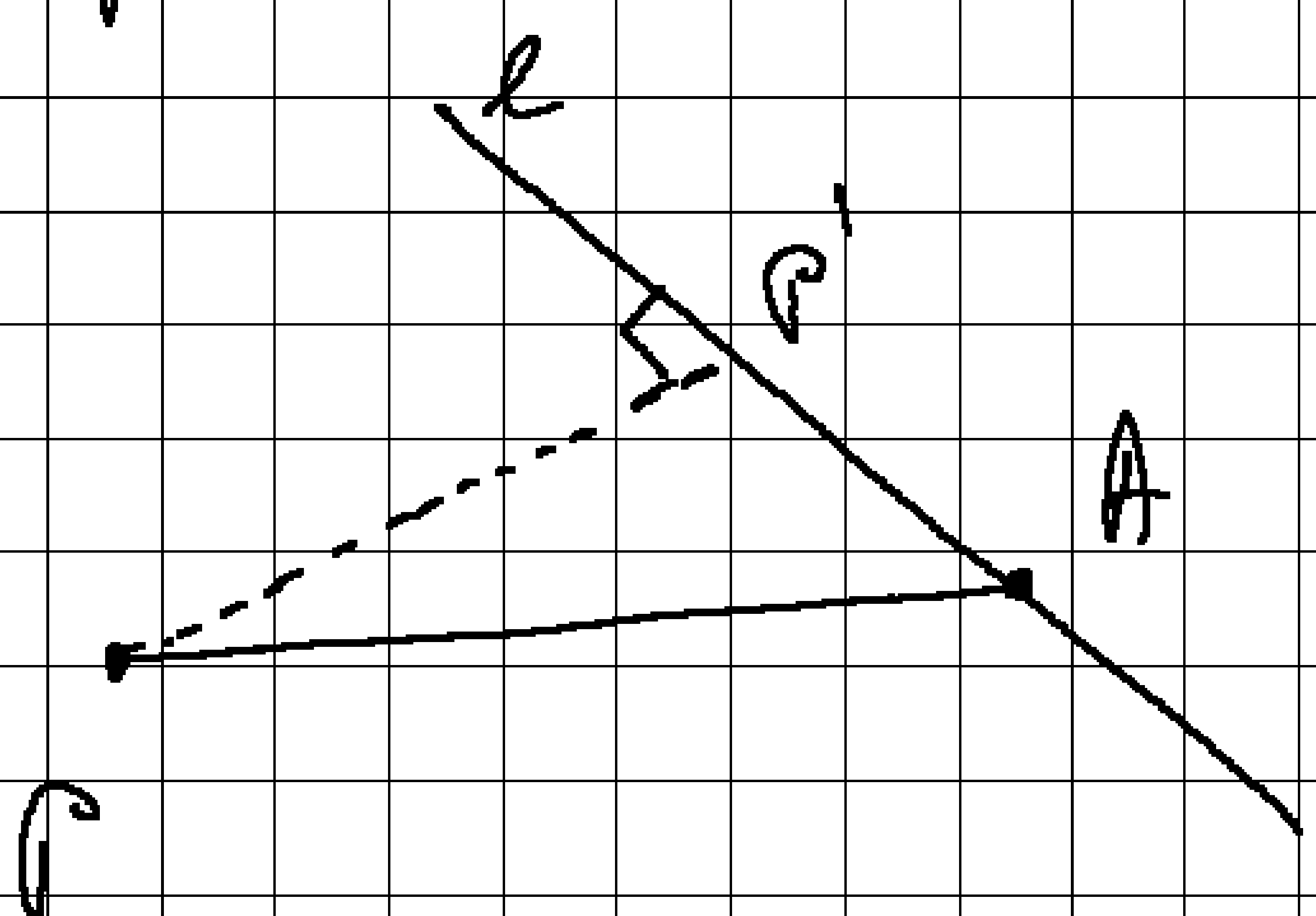
l line in $\mathbb{E}^3$

P point in $\mathbb{E}^3$ $\qquad P \notin l$



choose $A \in l$

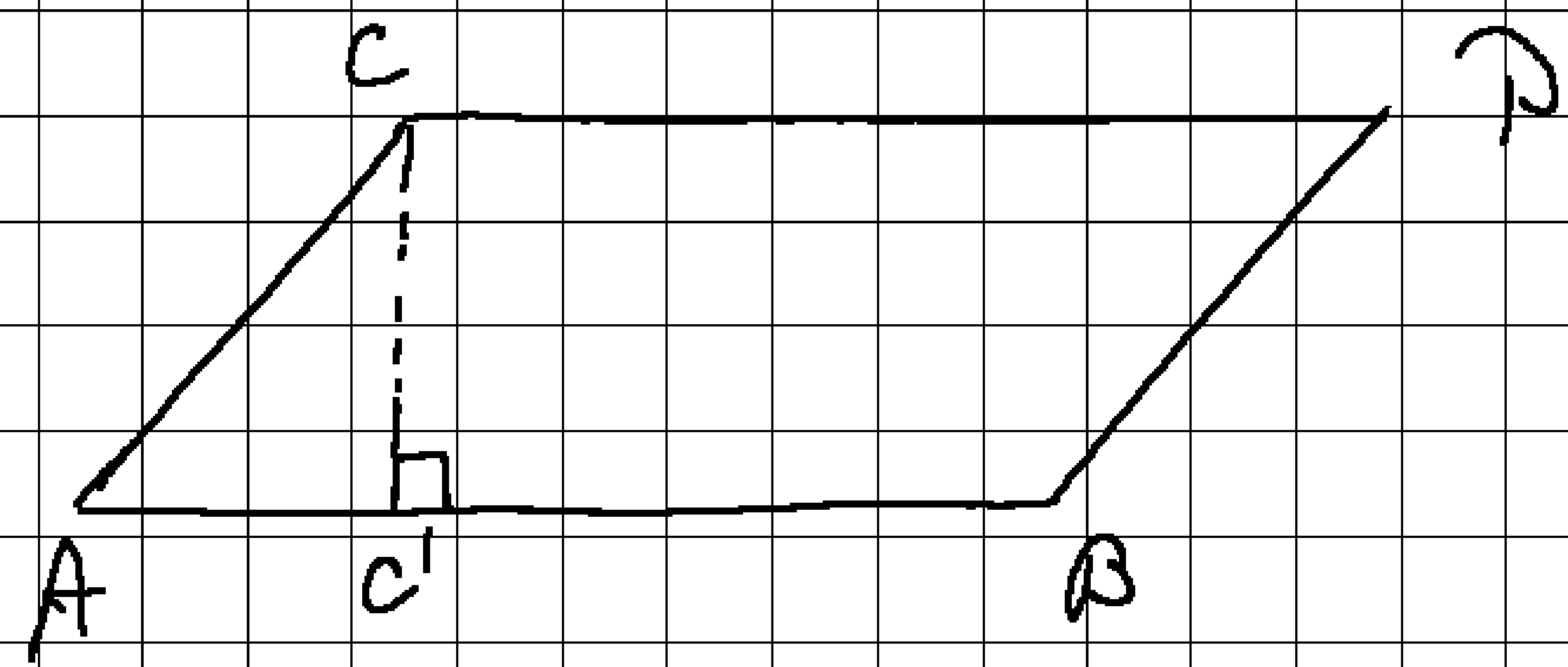
$PP' = PA \cdot \sin(\widehat{PAP'})$

$$|\vec{PA} \times \vec{P'A}| = |\vec{PA}| \cdot |\vec{P'A}| \cdot \sin(\widehat{PAP'}) = |\vec{PA}| \cdot PP'$$

$$\Rightarrow \operatorname{dist}(P,l) = \frac{|\vec{PA} \times \vec{P'A}|}{|\vec{P'A}|} =$$

$$= \frac{|\vec{PA} \times \vec{v}|}{|\vec{v}|} \quad \forall\, v \in D(l)$$

4.3. Determine the distances between opposite sides of the parallelogram ABDC spanned by the vectors $\vec{AB}(6,0,1)$ and $\vec{AC}\left(\frac{3}{2},2,1\right)$. (right oriented orthonormal system)



$$\text{dist}(AB, CD) = \text{dist}(C, AB) = \frac{|\vec{AC} \times AB|}{|\vec{AB}|}$$

$$\vec{AC} \times \vec{AB} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{3}{2} & 2 & 1 \\ 6 & 0 & 1 \end{vmatrix} = 2\vec{i} + \frac{9}{2}\vec{j} - 12\vec{k} = \left(2, \frac{9}{2}, -12\right)$$

$$|\vec{AC} \times \vec{AB}| = \sqrt{4 + \left(\frac{9}{2}\right)^2 + (-12)^2} = \sqrt{\frac{592+81}{4}} = \frac{\sqrt{673}}{2}$$

$$|\vec{AB}| = \sqrt{36+0+1} = \sqrt{37}$$

$$\text{dist}(AB, CD) = CC' = \frac{\sqrt{673}}{2\sqrt{37}}$$

4.4. $a(2,3,-1)$ $b(1,-1,3)$ (right oriented orthonormal basis)

a) Determine the vector subspace $\langle \vec{a}, \vec{b} \rangle^{\perp} = \langle \vec{a} \times \vec{b} \rangle$

b) $\vec{p} \in V^3$ which is $\perp$ to $\vec{a}$ and $\vec{b}$ and for which $\vec{p} \cdot (2\vec{i} - 3\vec{j} + 4\vec{k}) = 51$

$S \subseteq V^3$

$S^{\perp} = \{ v \in V^3 \mid \vec{v} \cdot \vec{w} = 0, \forall \vec{w} \in S \}$

a) $$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 3 & -1 \\ 1 & -1 & 3 \end{vmatrix} = 8\vec{i} - 7\vec{j} - 5\vec{k}$$

$$\langle \vec{a}, \vec{b} \rangle^{\perp} = \langle (8,-7,-5) \rangle$$

b) Let $\vec{p} = c(8,-7,-5)$

$$(8c\vec{i} - 7c\vec{j} - 5c\vec{k}) \cdot (2\vec{i} - 3\vec{j} + 4\vec{k}) = 16c + 21c - 20c = 17c$$

$\Rightarrow c = 3$

$\Rightarrow p = (24, -21, -15)$

4.10. a) $(\vec{i}, \vec{j}, \vec{k})$ right oriented orthonormal basis

$\vec{w} = -\vec{i} + 3\vec{j} + \vec{k}$. Determine the matrices of the linear maps

$$\varphi : V^3 \to V^3 \qquad \vec{v} \longmapsto \vec{w} \times \vec{v}$$

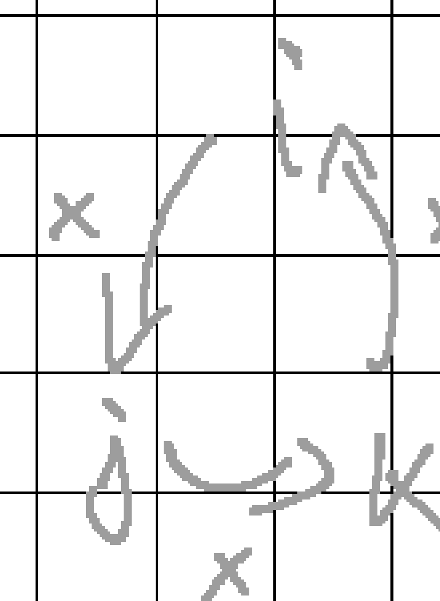
$$\psi : V^3 \to V^3 \qquad \vec{v} \longmapsto \vec{v} \times \vec{w}$$

$$M_b(\varphi) = ?$$

$$\varphi(\vec{i}) = \vec{w} \times \vec{i} = (-\vec{i} + 3\vec{j} + \vec{k}) \times \vec{i} =$$

$$= -\vec{i} \times \vec{i} + 3\vec{j} \times \vec{i} + \vec{k} \times \vec{i} =$$

$$= \vec{0} - 3\vec{k} + \vec{j}$$



$$[\varphi(\vec{i})]_b = \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix}$$

$$\varphi(\vec{j}) = \vec{w} \times \vec{j} = (-\vec{i} + 3\vec{j} + \vec{k}) \times \vec{j} = -\vec{i} \times \vec{j} + 3\vec{j} \times \vec{j} + \vec{k} \times \vec{j} =$$

$$= -\vec{k} + \vec{0} - \vec{i}$$

$$\Rightarrow [\varphi(\vec{j})]_b = \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix}$$

$$\varphi(\vec{k}) = \vec{w} \times \vec{k} = (-\vec{i} + 3\vec{j} + \vec{k}) \times k =$$

$$= -\vec{i} \times k + 3\vec{j} \times k + \vec{k} \times k = \vec{j} + 3\vec{i} + \vec{0}$$

$$[\varphi(k)]_b = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$$

$$M_b(\varphi) = \begin{pmatrix} 0 & -1 & 3 \\ 1 & 0 & 1 \\ -3 & -1 & 0 \end{pmatrix}$$

$$M_b(\psi) = \begin{pmatrix} 0 & 1 & -3 \\ -1 & 0 & -1 \\ 3 & 1 & 0 \end{pmatrix}$$

4.11. a) Prove the Grassmann identity

$v_1, v_2, v_3 \in V^3$

$$(\vec{v}_1 \times \vec{v}_2) \times \vec{v}_3 = \begin{vmatrix} \vec{v}_2 & \vec{v}_1 \\ \vec{v}_2 \cdot \vec{v}_3 & \vec{v}_2 \cdot \vec{v}_1 \end{vmatrix} =$$

$$= (\vec{v}_1 \cdot \vec{v}_3)\, \vec{v}_2 - (\vec{v}_2 \cdot \vec{v}_3)\, \vec{v}_1$$

Let $v_i (x_i, y_i, z_i)$ $\quad i = \overline{1,3}$

$$(\vec{v}_1 \times \vec{v}_2) \times \vec{v}_3 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x_{\vec{v}_1 \times \vec{v}_2} & y_{\vec{v}_1 \times \vec{v}_2} & z_{\vec{v}_1 \times \vec{v}_2} \\ x_3 & y_3 & z_3 \end{vmatrix}$$

$$\vec{v}_1 \times \vec{v}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix}$$

$$\Rightarrow (v_1 \times v_2) \times v_3 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix} & \begin{vmatrix} z_1 & x_1 \\ z_2 & x_2 \end{vmatrix} & \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} \\ x_3 & y_3 & z_3 \end{vmatrix} =$$

$$= \vec{i}\left(z_3 \begin{vmatrix} z_1 & x_1 \\ z_2 & x_2 \end{vmatrix} - y_3 \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} \right) + \vec{j}(\ldots) + \vec{k}(\ldots)$$

$$x_{(v_1 \times v_2) \times v_3} = z_3(z_1 x_2 - x_1 z_2) - y_3(x_1 y_2 - x_2 y_1) =$$

$$= x_2(z_3 z_1 + y_3 y_1) - x_1(z_3 z_2 + y_2 y_3) =$$

$$= x_2 \; x_1 x_3 + y_1 y_3 + z_1 z_3) - x_1(x_2 x_3 + y_2 y_3 + z_2 z_3) =$$

$$= (\vec{v}_1 \bullet \vec{v}_3) \cdot x_{\vec{v}_2} - (\vec{v}_2 \bullet \vec{v}_3)\, x_{\vec{v}_1}$$

$$\Rightarrow x_{(\vec{v}_1 \times \vec{v}_2) \times \vec{v}_3} = x_{(\vec{v}_1 \cdot \vec{v}_3)\vec{v}_2 - (\vec{v}_2 \cdot \vec{v}_3) \cdot \vec{v}_1}$$

Prove the same for y and z

4.16. The volume of a tetrahedron ABCD is 5.

$A(2,1,-1)$, $B(3,0,1)$, $C(2,-1,3)$, $D \in Oy$

Determine the coordinates of D

$$Vol(ABCD) = \frac{1}{6}[\vec{AB}, \vec{AC}, \vec{AD}]$$

$$[\vec{v_1}, \vec{v_2}, \vec{v_3}] = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = (\vec{v_1} \times \vec{v_2}) \cdot \vec{v_3}$$

$$D \in Oy \Rightarrow D(0, a, 0)$$

$$\Rightarrow \frac{1}{6} \begin{vmatrix} 1 & -1 & 2 \\ 0 & -2 & 4 \\ -2 & a-1 & 1 \end{vmatrix} = 5 \Leftrightarrow$$

$$\Leftrightarrow -2 + \cancel{8} - \cancel{8} - 4(a-1) = 30$$

$$\Leftrightarrow -4a - 2 + 4 = 30$$

$$\Leftrightarrow 4a = -28$$

$$a = -7$$