<u>Ex *:</u> Let $f: \mathbb{R}^2 \to \mathbb{R}^2$ linear maps so that

Ex for midterm. that is not in the Seminar list of pb.

$f(1,2) = (3,5)$

$f(3,1) = (0,2)$

Find f, so $\forall (x,y) \in \mathbb{R}^2$ find $f(x,y)$

$(1,2)$ and $(3,1)$ are linearly independent, $\dim \mathbb{R}^2 = 2$

$\Rightarrow (1,2)$ and $(3,1)$ form a basis of $\mathbb{R}^2$

$\forall (x,y)\ \exists!\ \alpha, \beta \in \mathbb{R}$ so that

$(x,y) = \alpha(1,2) + \beta(3,1)$

$$f(x,y) = f(\alpha(1,2) + \beta(3,1))$$

$$= f(\alpha(1,2)) + f(\beta(3,1))$$

$$= \alpha f(1,2) + \beta f(3,1)$$

$$= \alpha(3,5) + \beta(0,2)$$

$(x,y) = (\alpha + 3\beta, 2\alpha + \beta)$

$$\begin{cases} \alpha + 3\beta = x & / \cdot (-2) \\ 2\alpha + \beta = y \end{cases}$$

$$\begin{cases} -2\alpha - 6\beta = -2x \\ 2\alpha + \beta = y \end{cases} \Big|_{+}$$

$$-5\beta = -2x + y$$

$$\beta = \frac{2}{5}x - \frac{1}{5}y$$

$$\alpha = \frac{1}{2}(y - \beta)$$

$$= \frac{1}{2}\left(y - \frac{2}{5}x + \frac{1}{5}y\right)$$

$$= -\frac{2}{10}x + \frac{3}{5}y$$

$$f(x,y) = \left(-\frac{1}{5}x + \frac{3}{5}y\right)(3,5) + \left(\frac{2}{5}x - \frac{1}{5}y\right)(0,2)$$

$$= \left(-\frac{3}{5}x + \frac{9}{5}y, -x + 3y\right) + \left(0, \frac{4}{5}x - \frac{2}{5}y\right)$$

$$= \left(-\frac{3}{5}x + \frac{9}{5}y, -\frac{1}{5}x + \frac{13}{5}y\right)$$

4,6. $f: \mathbb{R}^3 \to \mathbb{R}^2$

$f(x,y,z) = (y, -x)$

Determine a basis and the dimension for $\ker f$ and $\operatorname{Im} f$. Complete these bases to bases of the ambient spaces.

$\ker f = \{(x,y,z) \mid f(x,y,z) = 0\}$

$f(x,y,z) = 0 \Leftrightarrow (y, -x) = (0,0)$

$\Rightarrow \begin{cases} y = 0 \\ -x = 0 \Rightarrow x = 0 \end{cases}$

$\ker f = \{(0,0,z) \mid z \in \mathbb{R}\} = \langle (0,0,1) \rangle$

$(\underbrace{(0,0,1)}_{=v_1}) \Rightarrow$ lin. indep. $\Rightarrow$ is a basis for $\ker f$

$v_2, v_3 \in \mathbb{R}^3$ s.t. v_1, v_2, v_3 are lin. indep.

First we chose $v_2 \in \mathbb{R}^3 - \langle v_1 \rangle$

$v_2 = (1,0,0)$

Secondly we chose $v_3 \in \mathbb{R}^3 - \langle v_1, v_2 \rangle$

$\langle v_1, v_2 \rangle = \{a v_1, b v_2 \mid a, b \in \mathbb{R}\}$

$\langle v_1, v_2 \rangle = \{ (b, 0, a) \mid a, b \in \mathbb{R} \}$

$v_3 = (0, -1, 0) \notin \langle v_1, v_2 \rangle$

$\left.\begin{array}{l} \Rightarrow v_1, v_2, v_3 \text{ are lin. indep.} \\ \text{The } \dim \mathbb{R}^3 = 3 \end{array}\right\} \Rightarrow v_1, v_2, v_3$ is a basis for $\mathbb{R}^3$

$\operatorname{Im} f = \{ (y, -x) \mid x, y \in \mathbb{R} \} = \{ y(1,0) + x(0,-1) \mid x, y \in \mathbb{R} \} =$

$= \langle (1,0), (0,-1) \rangle$

$\left.\begin{array}{l} \text{Let } A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ \det A = -1 - 0 = -1 \neq 0 \end{array}\right\} \Rightarrow \text{Rank} = 2$

$\Rightarrow$ lin. indep. $\Rightarrow$ is a basis $\Rightarrow \dim(\operatorname{Im} f) = 2$

1st dimension theorem (rank-nullity theorem)

$f : V \to W$ linear. indep

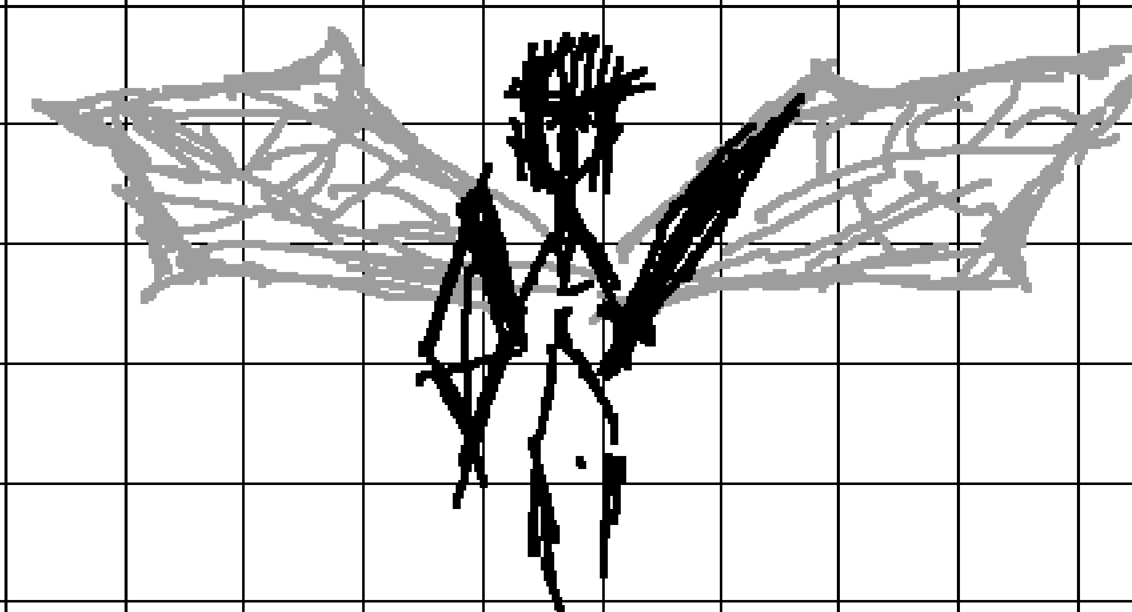
$\Rightarrow \dim V = \underbrace{\dim \operatorname{Im} f}_{= \operatorname{rank}(f)} + \underbrace{\dim \operatorname{Ker} f}_{= \operatorname{null}(f)}$

We could have gotten from here directly that $\dim \operatorname{Im} f = 2$

7. Determine a complement for the following subspaces:

(i) $A = \{(x,y,z) \in \mathbb{R}^3 \mid x + 2y + 3z = 0\}$

(ii) $B = \{ax + bx^3 \mid a, b \in \mathbb{R}\}$ in $\mathbb{R}_3[x]$



(i) $x = -2y - 3z$

$A = \{(-2y-3z, y, z) \mid y, z \in \mathbb{R}\} = \langle (-2,1,0), (-3,0,1) \rangle$

$$M = \begin{pmatrix} -2 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix}$$

$$\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0 \Rightarrow \text{rank } M = 2$$

$\Rightarrow$ lin. indep.

$\Rightarrow \left(\underbrace{(-2,1,0)}_{=v_1}, \underbrace{(-3,0,1)}_{=v_2} \right)$ is a basis of A

find $v_3 \in \underbrace{\mathbb{R}^3 \setminus \langle v_1, v_2 \rangle}_{=\mathbb{R}^3 \setminus A}$

$(1,1,1) \notin A \Rightarrow v_1, v_2, v_3$ is a basis for $\mathbb{R}^3$

$<(1,1,1)>$ is the compliment of A in $\mathbb{R}^3$

(ii) $B = <x, x^3>$

$ax + bx^3 = 0 \Rightarrow a = b = 0 \Rightarrow$

$\Rightarrow x, x^3$ are lin. indep. $\Rightarrow$

$\Rightarrow (x, x^3)$ is a basis for B

$\dim \mathbb{R}_3[x] = 4$

We need to add 2 more vectors

$v_1 = x \quad , v_2 = x^3$

We need to find $v_3 \in \mathbb{R}_3[x] \setminus <v_1, v_2>$

Choose $v_3 = x^2$

We need to find $v_4 \in \mathbb{R}_3[x] \setminus <v_1, v_2, v_3>$

$<v_1, v_2, v_3> = \{ av_1 + bv_2 + cv_3 \mid a, b, c \in \mathbb{R} \}$

$= \{ ax + cx^2 + bx^3 \mid a, b, c \in \mathbb{R} \}$

Choose $v_4 = 1 \notin <v_1, v_2, v_3>$

v_1, v_2, v_3, v_4 is a basis for $\mathbb{R}_3[x]$

Complement of B in $\mathbb{R}_3[x]$ is $<v_3, v_4> = \{ a + bx^2 \mid a, b \in \mathbb{R} \}$

9. $S=\{(x,y,z)\in\mathbb{R}^3 \mid x=0\}$

$T=\langle(0,1,1),(1,1,0)\rangle$

Determine $S\cap T$ and show that $S+T=\mathbb{R}^3$

$S\cap T=\left\{v=(x,y,z) \;\middle|\; \begin{matrix} v\in S \\ v\in T \end{matrix}\right\}=$

$=\{a(0,1,1)+b(1,1,0) \mid b=0\}=\{a(0,1,1) \mid a\in\mathbb{R}\}$

$S\cap T=\langle(0,1,1)\rangle$

$\Rightarrow \dim(S\cap T)=1$

2nd dimension theorem:

$S,T\leq_K V$

Theorem:

$\dim(S+T)=\dim S+\dim T-(\dim(S\cap T))$

For our problem:

$\dim S=2$, because $S=\langle(0,1,0),(0,0,1)\rangle$ and $(0,1,0),(0,0,1)$ lin. indep.

$\dim T = 2$

$(0,1,1)$ and $(1,1,0)$ lin. indep.

$\dim(S+T) = \dim S + \dim T - \dim(S \cap T) = 2+2-1=3$

$$\left.\begin{array}{l} \dim(S+T) = \dim \mathbb{R}^3 = 3 \\ \quad S+T = \mathbb{R}^3 \end{array}\right\} \Rightarrow S+T = \mathbb{R}^3$$