V, V' K - v.s.

$f: V \to V'$ linear map

$\mathcal{B} = (v_1, \dots, v_n)$ basis for V

$\mathcal{B}' = (v_1', \dots, v_n')$ basis for V'

$$[f]_{\mathcal{B},\mathcal{B}'} = \left([f(v_1)]_{\mathcal{B}'}, [f(v_2)]_{\mathcal{B}'}, \dots, [f(v_n)]_{\mathcal{B}'}\right)$$

We have $\forall v \in V$

$$[f(v)]_{\mathcal{B}'} = [f]_{\mathcal{B},\mathcal{B}'} \cdot [v]_{\mathcal{B}}$$

10.2 $f \in \mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^3, \mathbb{R}^2)$ defined by:

$$f(x,y,z) = (y, -x)$$

$$\mathcal{B} = (v_1, v_2, v_3) = \left((1,1,0), (0,1,1), (1,0,1)\right)$$

$$\mathcal{B}' = (v_1', v_2') = \left((1,1), (1,-2)\right)$$

$$E' = (e_1', e_2') = \left((1,0), (0,1)\right)$$

Find $[f]_{\mathcal{B},E'}$ and $[f]_{\mathcal{B},\mathcal{B}'}$

$$f(v_1) = f(1,1,0) = (1,-1)$$
$$f(v_2) = f(0,1,1) = (1,0)$$
$$f(v_3) = f(1,0,1) = (0,-1)$$

$$(1,-1) = a(1,0) + b(0,1) \Rightarrow a=1,\ b=-1$$
$$\Rightarrow [f(v_1)]_{E'} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad (1)$$

$$(1,0) = a(1,0) + b(0,1) \Rightarrow a=1,\ b=0$$
$$\Rightarrow [f(v_2)]_{E'} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad (2)$$

$$(0,-1) = a(1,0) + b(0,1) \Rightarrow a=0,\ b=-1$$
$$\Rightarrow [f(v_3)]_{E'} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \quad (3)$$

From (1) + (2) + (3) $\Rightarrow [f]_{B,E'} = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & -1 \end{bmatrix}$

$$(1,-1) = a(1,1) + b(1,-2) \Rightarrow a+b=1 \Rightarrow a=1-b \Rightarrow a=\frac{1}{3}$$
$$a-2b=-1 \Rightarrow 1-b-2b=-1$$
$$\Rightarrow 1-3b=-1$$
$$\Rightarrow b=\frac{2}{3}$$
$$\Rightarrow$$

$$\Rightarrow [f(v_1)]_{B,B'} = \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \end{bmatrix}$$

$$(1,0) = a(1,1) + b(1,-2) \Rightarrow \begin{aligned} a+b&=1 \Rightarrow 3b=1 \Rightarrow b=\tfrac{1}{3} \\ a-2b&=0 \Rightarrow a=\tfrac{2}{3} \end{aligned}$$

$$\Rightarrow [f(v_2)]_{B,B'} = \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \end{bmatrix}$$

$$(0,-1) = a(1,1) + b(1,-2) \Rightarrow \left.\begin{aligned} a+b&=0 \\ a-2b&=-1 \end{aligned}\right\} \Rightarrow \begin{aligned} a&=-\tfrac{1}{3} \\ b&=\tfrac{1}{3} \end{aligned}$$

$$\Rightarrow [f(v_3)]_{B,B'} = \begin{bmatrix} -\frac{1}{3} \\ \frac{1}{3} \end{bmatrix}$$

$$[f]_{B,B'} = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

4. $f \in \mathrm{End}_{\mathbb{R}}(\mathbb{R}^4)$

$$[f]_E = \begin{pmatrix} 1 & 1 & -3 & 2 \\ -1 & 1 & 1 & 4 \\ 2 & 1 & -5 & 1 \\ 1 & 2 & -4 & 5 \end{pmatrix}$$

i) Show that $v = (1,4,1,-1) \in \mathrm{Ker} f$ and $v' = (2,-2,4,2) \in \mathrm{Im} f$

ii) Determine a basis and the dimension for $\mathrm{Ker} f$ and $\mathrm{Im} f$

iii) Define f, i.e. $f(x,y,z) = ?$

i) $v \in \mathrm{Ker} f \Leftrightarrow f(v) = 0 \Leftrightarrow [f(v)]_E = 0 \Leftrightarrow$

$\Leftrightarrow [f]_E \cdot [v]_E = 0$

$$[f]_E \cdot [v]_E = \begin{pmatrix} 1 & 1 & -3 & 2 \\ -1 & 1 & 1 & 4 \\ 2 & 1 & -5 & 1 \\ 1 & 2 & -4 & 5 \end{pmatrix} \begin{pmatrix} 1 \\ 4 \\ 1 \\ -1 \end{pmatrix} =$$

$$= \begin{pmatrix} 1+4-3-2 \\ -1+4+1-4 \\ 2+4-5-1 \\ 1+8-4-5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow v \in \mathrm{Ker} f$$

$v' \in \operatorname{Im} f \Leftrightarrow \exists\, u = (x,y,z,t)$ s.t. $f(u) = v' \Leftrightarrow$

$\Leftrightarrow [f]_E \cdot [u]_E = [v']_E \Leftrightarrow$

$$\Leftrightarrow \begin{pmatrix} 1 & 1 & -3 & 2 \\ -1 & 1 & 1 & 4 \\ 2 & 1 & -5 & 1 \\ 1 & 2 & -4 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ 4 \\ 2 \end{pmatrix} \Leftrightarrow$$

$$\left(\begin{array}{cccc|c} 1 & 1 & -3 & 2 & 2 \\ -1 & 1 & 1 & 4 & -2 \\ 2 & 1 & -5 & 1 & 4 \\ 1 & 2 & -4 & 5 & 2 \end{array}\right) \underset{\substack{L_3 \leftarrow L_3 - 2L_1 \\ L_4 \leftarrow L_4 - L_1}}{\overset{L_2 \leftarrow L_2 + L_1}{\sim}} \left(\begin{array}{cccc|c} 1 & 1 & -3 & 2 & 2 \\ 0 & 2 & -2 & 6 & 0 \\ 0 & -1 & 1 & -3 & 0 \\ 0 & 1 & -1 & 3 & 0 \end{array}\right)$$

$$\underset{\substack{L_4 \leftarrow L_4 \\ -2L_2}}{\overset{\substack{L_3 \leftarrow L_3 \\ +\frac{1}{2}L_2}}{\sim}} \left(\begin{array}{cccc|c} 1 & 1 & -3 & 2 & 2 \\ 0 & 2 & -2 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$ is in Ech.-form $\Rightarrow$

$\Rightarrow$ the system is compatible $\Rightarrow v' \in \operatorname{Im} f$

ii) Let $u \in \mathbb{R}$, $u = (x,y,z,t)$

$u \in \operatorname{Ker} f \Leftrightarrow [f]_E \cdot [u]_E = 0$

$$\left(\begin{array}{cccc|c} 1 & 1 & -3 & 2 & 0 \\ -1 & 1 & 1 & 4 & 0 \\ 2 & 1 & -5 & 1 & 0 \\ 1 & 2 & -4 & 5 & 0 \end{array}\right) \begin{array}{l} L_2 \leftarrow L_2 + L_1 \\ L_3 \leftarrow L_3 - 2L_1 \\ L_4 \leftarrow L_4 - L_1 \end{array}$$

$$\sim \left(\begin{array}{cccc|c} 1 & 1 & -3 & 2 & 0 \\ 0 & 2 & -2 & 6 & 0 \\ 0 & -1 & 1 & -3 & 0 \\ 0 & 1 & -1 & 3 & 0 \end{array}\right) \underset{\sim}{L_2 \leftarrow \frac{1}{2} L_2} \left(\begin{array}{cccc|c} 1 & 1 & -3 & 2 & 0 \\ 0 & 1 & -1 & 3 & 0 \\ 0 & -1 & 1 & -3 & 0 \\ 0 & 1 & -1 & 3 & 0 \end{array}\right) \sim$$

$$\begin{array}{l} L_3 \leftarrow L_3 + L_2 \\ \sim \\ L_4 \leftarrow L_4 - L_2 \end{array} \left(\begin{array}{cccc|c} 1 & 1 & -3 & 2 & 0 \\ 0 & 1 & -1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Let $z = \alpha$, $t = \beta$

$$\begin{cases} x + y - 3\alpha + 2\beta = 0 \\ y - \alpha + 3\beta = 0 \end{cases}$$

$\Rightarrow y = \alpha - 3\beta$

$x = 2\alpha + \beta$

$\Rightarrow \operatorname{Ker} f = \{ ((2\alpha + \beta), (\alpha - 3\beta), \alpha, \beta) \mid \alpha, \beta \in \mathbb{R} \}$

$(2\alpha+\beta, \alpha-3\beta, \alpha, \beta) = (2\alpha, \alpha, \alpha, 0) + (\beta, -3\beta, 0, \beta)$

$= \alpha(2,1,1,0) + \beta(1,-3,0,1)$

$\ker f = \langle (2,1,1,0), (1,-3,0,1) \rangle$

$(2,1,1,0), (1,-3,0,1)$ are not proportional, thus linearly independent and they form a basis

$\Rightarrow \dim \ker f = 2$

$\operatorname{Im} f = \{ w = (a,b,c,d) \mid \exists\, u = (x,y,z,t),\ f(u) = w \}$

$= \{ w \mid \exists\, u \in \mathbb{R}^4 : [f]_E \cdot [u]_E = [w]_E \} =$

$= \{ w = (a,b,c,d) \mid \exists\, u = (x,y,z,t) : \begin{pmatrix} 1 & 1 & -3 & 2 \\ -1 & 1 & 1 & 4 \\ 2 & 1 & -5 & 1 \\ 1 & 2 & -4 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$

$$\left(\begin{array}{cccc|c} 1 & 1 & -3 & 2 & a \\ -1 & 1 & 1 & 4 & b \\ 2 & 1 & -5 & 1 & c \\ 1 & 2 & -4 & 5 & d \end{array}\right) \begin{array}{l} L_2 \leftarrow L_2 + L_1 \\ L_3 \leftarrow L_3 - 2L_1 \\ L_4 \leftarrow L_4 - L_1 \end{array} \left(\begin{array}{cccc|c} 1 & 1 & -3 & 2 & a \\ 0 & 2 & -2 & 6 & b+a \\ 0 & -1 & 1 & -3 & c-2a \\ 0 & 1 & -1 & 3 & d-a \end{array}\right)$$

$L_3 \leftarrow L_3 + \frac{1}{2}L_2$

$L_4 \leftarrow L_4 - \frac{1}{2}L_2$

$$\left(\begin{array}{cccc|c} 1 & 1 & -3 & 2 & a \\ 0 & 2 & -2 & 6 & b+a \\ 0 & 0 & 0 & 0 & c-2a+\frac{1}{2}(b+a) \\ 0 & 0 & 0 & 0 & d-a-\frac{1}{2}(b+a) \end{array}\right)$$

System is compatible if

$$\begin{cases} c-2a+\frac{1}{2}(b+a)=0 & (1) \\ d-a-\frac{1}{2}(b+a)=0 & (2) \end{cases}$$

$(2) \quad d = a + \frac{1}{2}(b+a) = \frac{3}{2}a + \frac{1}{2}b$

$(1) \quad c = \frac{3a}{2} + \frac{1}{2}b$

$\operatorname{Im} f = \{ (a, b, \frac{3a}{2} + \frac{1}{2}b, \frac{3}{2}a + \frac{1}{2}b) \mid a, b \in \mathbb{R} \}$

$(a, b, \frac{3}{2}a + \frac{1}{2}b, \frac{3}{2}a + \frac{1}{2}b) = a(1, 0, \frac{3}{2}, \frac{3}{2}) + b(0, 1, \frac{1}{2}, \frac{1}{2})$

$\operatorname{Im} f = \langle (1, 0, \frac{3}{2}, \frac{3}{2}), (0, 1, \frac{1}{2}, \frac{1}{2}) \rangle$

vectors are not proportional $\Rightarrow$ lin indep.

$\Rightarrow$ they form a basis $\Rightarrow \dim \operatorname{Im} f = 2$

(iii) We want to find $f(x,y,z,t)$

$$[f(x,y,z,t)] = [f]_E \cdot [(x,y,z,t)]_E =$$

$$= \begin{pmatrix} 1 & 1 & -3 & 2 \\ -1 & 1 & 1 & 4 \\ 2 & 1 & -5 & 1 \\ 1 & 2 & -4 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} x+y-3z+2t \\ -x+y+z+4t \\ 2x+y-3z+t \\ x+2y-4z+5t \end{pmatrix}$$

$$\Rightarrow f(x,y,z,t) = (x+y-3z+2t)\cdot \ell_1 + (-x+y+z+4t)\cdot \ell_2 +$$

$$+ (2x+y-3z+t)\cdot \ell_3 + (x+2y-4z+5t)\cdot \ell_4 =$$

$$= (x+y-3z+2t,\ -x+y+z+4t,\ 2x+y-3z+t,\ x+2y-4z+5t)$$