Chapter 5:

2. Consider the vector $v(2,1,1) \in V^3$

a) Give the matrix form for the parallel projection on the plane $\pi : z = 0$ parallel to v.

$$P = I - \frac{v \cdot v^T}{v \cdot v}$$

$$v \cdot v = 4+1+1 = 6$$

$$v \cdot v^T = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 2 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 2 & 2 \\ 2 & 1 & 1 \\ 2 & 1 & 1 \end{pmatrix}$$

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \frac{1}{6}\begin{pmatrix} 4 & 2 & 2 \\ 2 & 1 & 1 \\ 2 & 1 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} 1-\frac{4}{6} & 0-\frac{2}{6} & 0-\frac{2}{6} \\ 0-\frac{2}{6} & 1-\frac{1}{6} & 0-\frac{1}{6} \\ 0-\frac{2}{6} & 0-\frac{1}{6} & 1-\frac{1}{6} \end{pmatrix} = \begin{pmatrix} \frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{5}{6} & -\frac{1}{6} \\ -\frac{1}{3} & -\frac{1}{6} & \frac{5}{6} \end{pmatrix}$$

6) Give the matrix form for the parallel reflection in the plane $\pi : z=0$ parallel to v

$$\boxed{R = 2P - i}$$

$$2P = 2\cdot\begin{pmatrix} \frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{5}{6} & -\frac{1}{6} \\ -\frac{1}{3} & -\frac{1}{6} & \frac{5}{6} \end{pmatrix} = \begin{pmatrix} \frac{2}{3} & -\frac{2}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{5}{3} & -\frac{1}{3} \\ -\frac{2}{3} & -\frac{1}{3} & \frac{5}{3} \end{pmatrix}$$

$$R = 2P - i = \begin{pmatrix} \frac{2}{3} & -\frac{2}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{5}{3} & -\frac{1}{3} \\ -\frac{2}{3} & -\frac{1}{3} & \frac{5}{3} \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} -\frac{1}{3} & -\frac{2}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \end{pmatrix}$$

3. Determine the orthogonal projection of the point $A(2,11,-5)$ on the plane $x+4y-3z+7=0$ by determining the matrix form of the projection

$n=(1,4,-3)$ – normal vector of the plane

$$P = I - \frac{n\cdot n^T}{n\cdot n}$$

$$n\cdot n = 1+16+9=26$$

$$n\cdot n^T = \begin{pmatrix}1\\4\\-3\end{pmatrix}\cdot\begin{pmatrix}1 & 4 & -3\end{pmatrix} = \begin{pmatrix}1 & 4 & -3\\4 & 16 & -12\\-3 & -12 & 9\end{pmatrix}$$

$$P = \begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix} - \frac{1}{26}\begin{pmatrix}1 & 4 & -3\\4 & 16 & -12\\-3 & -12 & 9\end{pmatrix} =$$

$$= \begin{pmatrix}1-\frac{1}{26} & -\frac{4}{26} & \frac{3}{26}\\-\frac{4}{26} & 1-\frac{16}{26} & \frac{12}{26}\\\frac{3}{26} & \frac{12}{26} & 1-\frac{9}{26}\end{pmatrix} = \begin{pmatrix}\frac{25}{26} & -\frac{2}{13} & \frac{3}{26}\\-\frac{2}{13} & \frac{5}{13} & \frac{6}{13}\\\frac{3}{26} & \frac{6}{13} & \frac{17}{26}\end{pmatrix}$$

$$A = \begin{pmatrix}2\\11\\-5\end{pmatrix}$$

$$A_{proj} = P\cdot A = \begin{pmatrix}\frac{25}{26} & -\frac{2}{13} & \frac{3}{26}\\-\frac{2}{13} & \frac{5}{13} & \frac{6}{13}\\\frac{3}{26} & \frac{6}{13} & \frac{17}{26}\end{pmatrix}\begin{pmatrix}2\\11\\-5\end{pmatrix}$$

$$= \begin{pmatrix} \frac{25}{26}\cdot 2 - \frac{2}{13}\cdot 11 - \frac{3}{26}\cdot 5 \\ -\frac{2}{13}\cdot 2 + \frac{5}{13}\cdot 11 - \frac{6}{13}\cdot 5 \\ \frac{3}{26}\cdot 2 + \frac{6}{13}\cdot 11 - \frac{17}{26}\cdot 5 \end{pmatrix} = \begin{pmatrix} \frac{50}{26} - {}^{2,}\frac{22}{13} - \frac{15}{26} \\ -\frac{4}{13} + \frac{55}{13} - \frac{30}{13} \\ \frac{6}{26} + {}^{2,}\frac{66}{13} - \frac{85}{26} \end{pmatrix} =$$

$$= \begin{pmatrix} -\frac{9}{26} \\ \frac{21}{13} \\ -\frac{\cancel{13}}{\cancel{26}} \end{pmatrix} = \begin{pmatrix} -\frac{9}{26} \\ \frac{21}{13} \\ -\frac{1}{2} \end{pmatrix}$$

5. Consider an orthonormal coordinate system K of $\mathbb{E}^2$. Starting from the matrix form of the projections and reflections described in this Section 5.1. show that

$$Pr_a^{\perp}(b) = \frac{\langle a,b\rangle}{\langle a,a\rangle}\, a$$

Compare this to the projections described in Section 3.1.

$$a = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

$$\langle a,b\rangle = a_1 b_1 + a_2 b_2$$

$$\langle a,a\rangle = a_1^2 + a_2^2$$

$$P = \frac{a a^T}{a \cdot a}$$

$$a \cdot a^T = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \begin{pmatrix} a_1 & a_2 \end{pmatrix} = \begin{pmatrix} a_1^2 & a_1 a_2 \\ a_2 a_1 & a_2^2 \end{pmatrix}$$

$$P = \frac{1}{a_1^2 + a_2^2} \begin{pmatrix} a_1^2 & a_1 a_2 \\ a_2 a_1 & a_2^2 \end{pmatrix}$$

$$\mathrm{Proj}_a b = \frac{1}{a_1^2 + a_2^2} \begin{pmatrix} a_1^2 & a_1 a_2 \\ a_2 a_1 & a_2^2 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} =$$

$$= \frac{1}{a_1^2 + a_2^2} \begin{pmatrix} a_1^2 b_1 + a_1 a_2 b_2 \\ a_2 a_1 b_1 + a_2^2 b_2 \end{pmatrix} = \frac{1}{a_1^2 + a_2^2} \begin{pmatrix} a_1 (a_1 b_1 + a_2 b_2) \\ a_2 (a_1 b_1 + a_2 b_2) \end{pmatrix} =$$

$$= \frac{a_1 b_1 + a_2 b_2}{a_1^2 + a_2^2} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \frac{\langle a, b \rangle}{\langle a, a \rangle} a$$

11. in $\mathbb{E}^2$, for the lines / hyperplanes

$$\pi: ax+by+c=0$$

$$l: \frac{x-x_0}{v_1} = \frac{y-y_0}{v_2}$$

with $\pi \not\parallel l$, deduce the matrix forms of $Pr_{\pi,l}$ and $Ref_{\pi,l}$

π: $\quad n=(a,b)$ - normal vector

l: $\quad v=(v_1,v_2)$ - direction vector

$$P_{\pi} = I - \frac{n \cdot n^T}{n \cdot n}$$

$$n \cdot n = a^2+b^2$$

$$n \cdot n^T = \begin{pmatrix} a \\ b \end{pmatrix} \begin{pmatrix} a & b \end{pmatrix} = \begin{pmatrix} a^2 & ab \\ ab & b^2 \end{pmatrix}$$

$$P_{\pi} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \frac{1}{a^2+b^2} \begin{pmatrix} a^2 & ab \\ ab & b^2 \end{pmatrix} = \begin{pmatrix} 1-\frac{a^2}{a^2+b^2} & -\frac{ab}{a^2+b^2} \\ -\frac{ab}{a^2+b^2} & 1-\frac{b^2}{a^2+b^2} \end{pmatrix} =$$

$$= \begin{pmatrix} \frac{b^2}{a^2+b^2} & -\frac{ab}{a^2+b^2} \\ -\frac{ab}{a^2+b^2} & \frac{a^2}{a^2+b^2} \end{pmatrix}$$

$$P_\ell = \frac{v \cdot v^T}{v \cdot v} = \frac{1}{v_1^2+v_2^2}\begin{pmatrix} v_1^2 & v_1 v_2 \\ v_1 v_2 & v_2^2 \end{pmatrix}$$

$$v \cdot v = v_1^2 + v_2^2$$

$$v v^T = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\begin{pmatrix} v_1 & v_2 \end{pmatrix} = \begin{pmatrix} v_1^2 & v_1 v_2 \\ v_1 v_2 & v_2^2 \end{pmatrix}$$

$$P_{\Pi,\ell} = P_\Pi P_\ell = \begin{pmatrix} \frac{b^2}{a^2+b^2} & -\frac{ab}{a^2+b^2} \\ -\frac{ab}{a^2+b^2} & \frac{a^2}{a^2+b^2} \end{pmatrix} \cdot \frac{1}{v_1^2+v_2^2} \cdot \begin{pmatrix} v_1^2 & v_1 v_2 \\ v_1 v_2 & v_2^2 \end{pmatrix}$$

$$R_\Pi = 2P_\Pi - I = 2\begin{pmatrix} \frac{b^2}{a^2+b^2} & -\frac{ab}{a^2+b^2} \\ -\frac{ab}{a^2+b^2} & \frac{a^2}{a^2+b^2} \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} \frac{b^2-a^2}{a^2+b^2} & -\frac{2ab}{a^2+b^2} \\ -\frac{2ab}{a^2+b^2} & \frac{a^2-b^2}{a^2+b^2} \end{pmatrix}$$

$$R_\ell = 2P_\ell - I = \frac{2}{v_1^2+v_2^2} \cdot \begin{pmatrix} v_1^2 & v_1 v_2 \\ v_1 v_2 & v_2^2 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} \frac{v_1^2 - v_2^2}{v_1^2+v_2^2} & \frac{2 v_1 v_2}{v_1^2+v_2^2} \\ \frac{2 v_1 v_2}{v_1^2+v_2^2} & \frac{v_2^2 - v_1^2}{v_1^2+v_2^2} \end{pmatrix}$$

$$Ref_{\Pi,\ell} = R_\Pi R_\ell$$

23. Using rotations arround the coordinate axes, give a parametrization of a cylinder with axix $\mathbb{R}\vec{v}$ and diameter $\sqrt{2}$.

$$r = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$$ (radius is half of the diameter)

Standard parametrization of a cylinder with radius r along the z-axis:

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases} \qquad \theta \in [0, 2\pi), \; z \in \mathbb{R}$$

Assume a rotation

$$R_x(\varphi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{pmatrix}$$

$$R_y(\varphi) = \begin{pmatrix} \cos\varphi & 0 & \sin\varphi \\ 0 & 1 & 0 \\ -\sin\varphi & 0 & \cos\varphi \end{pmatrix}$$

$$R_z(\varphi) = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Apply the rotation:

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\Rightarrow \begin{cases} x' = x = \frac{\sqrt{2}}{2}\cos\theta \\ y' = y\cos\varphi - z\sin\varphi = \frac{\sqrt{2}}{2}\sin\varphi\cos\varphi - z\sin\varphi \\ z' = y\sin\varphi + z\cos\varphi = \frac{\sqrt{2}}{2}\sin\varphi\cos\varphi + z\sin\varphi \end{cases}$$

24. Using rotations around the coordinate axes, give a parametrization of a cone containing the line $\ell = \{(0,t,t) : t \in \mathbb{R}\}$ and with axis the z-axis.

Chapter 6:

5. Determine the position of the line
$l: 2x+y-10=0$ relative to the ellipse $\mathcal{E}: \frac{x^2}{9}+\frac{y^2}{4}-1=0$

$$y = -2x+10$$

$$\frac{x^2}{9} + \frac{4x^2-40x+100}{4} - 1 = 0$$

$$\frac{x^2}{9} + x^2 - 10x + 25 - 1 = 0$$

$$\frac{10x^2}{9} - 10x + 24 = 0$$

$$10x^2 - 90x + 216 = 0$$

$$5x^2 - 45x + 108 = 0$$

$\Delta = 2025 - 2160 = -135 < 0 \Rightarrow$ line does not intersect the ellipse.

$\Delta > 0 \Rightarrow$ the line intersects the ellipse at two points (secant)

$\Delta = 0 \Rightarrow$ the line is tangent to the ellipse (one point of contact)

$\Delta < 0 \Rightarrow$ the line does not intersect the ellipse

8. Consider the family of ellipses $\mathcal{E}_a: \frac{x^2}{a^2} + \frac{y^2}{16} = 1$

For what value $a \in \mathbb{R}$ is $\mathcal{E}_a$ tangent to the line $l: x - y + 5 = 0$?

$$y = x + 5$$

$$\frac{x^2}{a^2} + \frac{(x+5)^2}{16} = 1$$

$$\frac{x^2}{a^2} + \frac{x^2 + 10x + 25}{16} = 1$$

$$16x^2 + a^2x^2 + 10a^2x + 25a^2 = 16a^2$$

$$(16 + a^2)x^2 + 10a^2x + 9a^2 = 0$$

$\mathcal{E}_a$ tangent $\Rightarrow \Delta = 0$

$$\Delta = 100a^4 - 4(16 + a^2) \cdot 9a^2$$

$$\Leftrightarrow \Delta = 100a^4 - (64 + 4a^2) \cdot 9a^2$$

$$\Delta = 100a^4 - 576a^2 - 36a^4$$

$$\Delta = 64a^4 - 576a^2$$

$$64a^4 - 576a^2 = 0 \quad | : 64$$

$$a^4 - 9a^2 = 0 \quad | : a^2$$

$$a^2 - 9 = 0 \Rightarrow a^2 = 9 \Rightarrow a = 3$$

we dont take the negative value because lengths cant be negative

9. Consider the family of lines $l_c: \sqrt{5}x - y + c = 0$
For what values $c \in \mathbb{R}$ is l_c tangent to the ellipse: $\mathcal{E}: x^2 + \frac{y^2}{4} = 1$?

$y = \sqrt{5}x + c$

$x^2 + \frac{(\sqrt{5}x+c)^2}{4} = 1 \Leftrightarrow x^2 + \frac{5x^2 + 2\sqrt{5}x \cdot c + c^2}{4} = 1 \Leftrightarrow$

$\Leftrightarrow \quad 4x^2 + 5x^2 + 2\sqrt{5}x \cdot c + c^2 = 4$

$\Leftrightarrow \quad 9x^2 + 2\sqrt{5}x \cdot c + c^2 = 4$

$9x^2 + 2\sqrt{5}x \cdot c + (c^2 - 4)$

$\Delta = 0$ (is tangent)

$\Delta = 20c^2 - 36 \cdot (c^2 - 4)$

$\Delta = 20c^2 - 36c^2 + 144$

$\Delta = -16c^2 + 144$

$-16c^2 + 144 = 0 \Rightarrow 16c^2 = 144 \Rightarrow c^2 = 9$

$\Rightarrow c = \pm 3$

Chapter 7:

6. Consider the rotation $R_{90°}$ of $\mathbb{E}^2$ around the origin and the translation T_v of $\mathbb{E}^2$ with vector $v(1,0)$.

a) Give the algebric form of the isometries R_{90}, T_v, $T_v \circ R_{90}$

$$R_{90} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \Rightarrow R_{90}(x,y) = (-y, x)$$

$$T_v = (x+1, y) \quad , v = (1,0)$$

$$T_v \circ R_{90} = (-y+1, x)$$

b) Determine the equations of the hyperbola $H: \frac{x^2}{4} - \frac{y^2}{9} - 1 = 0$ and the parabola $P: y^2 - 8x = 0$ after transforming them with R_{90} and with $T_v \circ R_{90}$ respectively.

I Hyperbola with R_{90}

$$\frac{(-y)^2}{4} - \frac{x^2}{9} = 1 \Leftrightarrow \frac{y^2}{4} - \frac{x^2}{9} = 1$$

with $T_v \circ R_{90}$ $\Rightarrow \frac{(-y+1)^2}{4} - \frac{x^2}{9} = 1 \Leftrightarrow \frac{y^2 - 2y + 1}{4} - \frac{x^2}{9} = 1$

$$\Rightarrow \frac{y^2}{4} - \frac{y}{2} + \frac{1}{4} - \frac{x^2}{9} = 1 \Leftrightarrow \frac{y^2}{4} - \frac{y}{2} - \frac{x^2}{9} = \frac{3}{4}$$

II Parabola:

with R_{90}: $\quad x^2+8y=0$

with $T_v \circ R_{90}$ $\quad x^2+8(-y+1)=0 \Rightarrow x^2-8y+8=0$

3. Determine the intersection of the ellipsoid

$\mathcal{E}_{2,\sqrt{3},3}: \quad \frac{x^2}{4}+\frac{y^2}{3}+\frac{z^2}{9}=1$ with the line $l: x=y=z$

Write down the equations of the tangent planes in the intersection points.

$$\frac{x^2}{4}+\frac{x^2}{3}+\frac{x^2}{9}=1$$

$$x^2\left(\overset{9)}{\frac{1}{4}}+\overset{12)}{\frac{1}{3}}+\overset{4)}{\frac{1}{9}}\right)=1$$

$$x^2\cdot\frac{25}{36}=1 \Rightarrow x^2=\frac{36}{25} \Rightarrow x=\pm\frac{6}{5}$$

intersection points: $(x,y,z)=\left\{\left(\frac{6}{5},\frac{6}{5},\frac{6}{5}\right),\left(-\frac{6}{5},-\frac{6}{5},-\frac{6}{5}\right)\right\}$

$$F(x,y,z)=\frac{x^2}{4}+\frac{y^2}{3}+\frac{z^2}{9}-1$$

$\Leftarrow$ partial derivatives for x,y,z

$$\nabla F(x,y,z)=\left(\frac{2x}{4},\frac{2y}{3},\frac{2z}{9}\right)$$

$$\left(\frac{6}{5},\frac{6}{5},\frac{6}{5}\right) \Rightarrow \nabla F=\left(\frac{3}{5},\frac{4}{5},\frac{4}{15}\right)$$

$$\left(-\frac{6}{5},-\frac{6}{5},-\frac{6}{5}\right) \Rightarrow \nabla F=\left(-\frac{3}{5},-\frac{4}{5},-\frac{4}{15}\right)$$

I point $\left(\frac{6}{5}, \frac{6}{5}, \frac{6}{5}\right)$ $\nabla F = \left(\frac{3}{5}, \frac{4}{5}, \frac{4}{15}\right)$

$$\frac{3}{5}\left(x-\frac{6}{5}\right) + \frac{4}{5}\left(y-\frac{6}{5}\right) + \frac{4}{15}\left(z-\frac{6}{5}\right) = 0$$

II point $\left(-\frac{6}{5}, -\frac{6}{5}, -\frac{6}{5}\right)$ $\nabla F = \left(-\frac{3}{5}, -\frac{4}{5}, -\frac{4}{15}\right)$

$$-\frac{3}{5}\left(x+\frac{6}{5}\right) - \frac{4}{5}\left(y+\frac{6}{5}\right) - \frac{4}{15}\left(z+\frac{6}{5}\right) = 0$$

5. Determine the points P of the ellipsoid $\mathcal{E}: \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ for which the tangent space $T_P\mathcal{E}$ intersects the coordinate axis in congruent segments

$$F(x,y,z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1$$

$$\nabla F(x,y,z) = \left(\frac{2x}{a^2}, \frac{2y}{b^2}, \frac{2z}{c^2}\right)$$ gives us the normal vector

For a plane with normal vector $n = (A, B, C)$ and passing through (x_0, y_0, z_0): $A(x-x_0) + B(y-y_0) + C(z-z_0) = 0$

At a point (x_0, y_0, z_0):

$$\frac{2x_0}{a^2}(x-x_0) + \frac{2y_0}{b^2}(y-y_0) + \frac{2z_0}{c^2}(z-z_0) = 0 \iff$$

$$\iff \frac{2x_0}{a^2}x + \frac{2y_0}{b^2}y + \frac{2z_0}{c^2}z = 1$$

Find the intersection with the coordinate axes

I x-axis $\Rightarrow$ $y=z=0$

$$\frac{x_0}{a^2}x=1 \Rightarrow x=\frac{a^2}{2x_0}$$

II y-axis $\Rightarrow$ $x=z=0$ $\Rightarrow$ $y=\frac{b^2}{2y_0}$

III z-axis $\Rightarrow$ $x=y=0$ $\Rightarrow$ $z=\frac{c^2}{2z_0}$

intersects in congruent segments $\Rightarrow$

$$\Rightarrow |x|=|y|=|z| \Rightarrow \left|\frac{a^2}{2x_0}\right|=\left|\frac{b^2}{2y_0}\right|=\left|\frac{c^2}{2z_0}\right|=k$$

$$\Rightarrow \begin{cases} x_0=\pm\frac{a^2}{2k} \\ y_0=\pm\frac{b^2}{2k} \\ z_0=\pm\frac{c^2}{2k} \end{cases}$$

$$\left(\frac{\pm\frac{a^2}{2k}}{a}\right)^2+\left(\frac{\pm\frac{b^2}{2k}}{b}\right)^2+\left(\frac{\pm\frac{c^2}{2k}}{c}\right)^2=1$$

$$\Leftrightarrow \left(\pm\frac{a^{\cancel{2}}}{2k}\cdot\frac{1}{\cancel{a}}\right)^2+\left(\pm\frac{b^{\cancel{2}}}{2k}\cdot\frac{1}{\cancel{b}}\right)^2+\left(\pm\frac{c^{\cancel{2}}}{2k}\cdot\frac{1}{\cancel{c}}\right)^2=1$$

$$\Leftrightarrow \frac{a^2+b^2+c^2}{4k^2}=1 \Rightarrow 4k^2=a^2+b^2+c^2 \Rightarrow k=\frac{\sqrt{a^2+b^2+c^2}}{2}$$

$$\Rightarrow P\left(\pm\frac{a^2}{\sqrt{a^2+b^2+c^2}},\ \pm\frac{b^2}{\sqrt{a^2+b^2+c^2}},\ \pm\frac{c^2}{\sqrt{a^2+b^2+c^2}}\right)$$

14. Determine the tangent plane of the hyperboloid

$$H^1_{2,3,1}: \frac{x^2}{4}+\frac{y^2}{9}-\frac{z^2}{1}=1$$

in the point $M(2,3,1)$. Show that the tangent plane intersects the surface in two lines

$$\nabla F = \left(\frac{x}{2}, \frac{2y}{9}, -2z\right)$$

At point $M \Rightarrow \nabla F = \left(1, \frac{2}{3}, -2\right)$

$$\Rightarrow \vec{n} = \left(1, \frac{2}{3}, -2\right)$$

eq. of the tangent plane at M:

$$1(x-2)+\frac{2}{3}(y-3)-2(z-1)=0$$

$$x-2+\frac{2y}{3}-\cancel{2}-2z+\cancel{2}$$

$$x+\frac{2y}{3}-2z=2$$

$$z=\frac{x+\frac{2y}{3}-2}{2}$$

Substitute in the hyperboloid eq to get the intersection

$$\frac{x^2}{4}+\frac{y^2}{9}-\left(\frac{x+\frac{2y}{3}-2}{2}\right)^2=0 \quad \Leftrightarrow$$

$$\Leftrightarrow \frac{x^2}{4}+\frac{y^2}{9}-\frac{x^2+\frac{4xy}{3}+\frac{4y^2}{9}-2x-\frac{4y}{3}+4}{4}=1$$

$$\Leftrightarrow \cancel{\frac{x^2}{4}} + \cancel{\frac{y^2}{9}} - \cancel{\frac{x^2}{4}} - \frac{xy}{3} - \cancel{\frac{y^2}{9}} + \frac{x}{2} + \frac{y}{3} - 1 = 1$$

$$-\frac{xy}{3} + \frac{x}{2} + \frac{y}{3} = 2$$

(multipliers: $2/$, $3/$, $2/$, $6/$)

$$-2xy + 3x + 2y = 12$$

$$3x + 2y - 2xy - 12 = 0$$ (quadratic eq in x and y representing two lines)

Model 2016:

3. a) Determine the coordinates of the foci of the hyperbola $H: \frac{x^2}{9} - \frac{y^2}{4} - 1 = 0$

I Hyperbola opening horizontally

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ Foci located at $(\pm c, 0)$ $c^2 = a^2 + b^2$

II Hyperbola opening vertically:

$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$ Foci located at $(0, \pm c)$ $c^2 = a^2 + b^2$

$$c^2 = 9 + 4 \Rightarrow c = \sqrt{13}$$

$$\Rightarrow (\sqrt{13}, 0) \text{ and } (-\sqrt{13}, 0)$$

if its not in the standard form:

ex: $9x^2-4y^2-36=0$

$9x^2-4y^2=36 \quad /:36$

$\frac{x^2}{4}-\frac{y^2}{9}=1$

b) Determine the focus and the director line of the parabola $P: y^2=16x$

For a parabola in the form $y^2=4ax$ the focus is located at $(a,0)$

$4a=16 \Rightarrow a=4$

$\Rightarrow$ Focus is at $(4,0)$

The directrix/director of the parabola in form $y^2=4ax$ is a vertical line $x=-a$

Director: $x=-4$

Theory:

I Parabola opening to the right

standard form	Focus	Directrix
$y^2 = 4ax$	$(a, 0)$	$x = -a$

I Parabola opening to the left

standard form	Focus	Directrix
$y^2 = -4ax$	$(-a, 0)$	$x = a$

I Parabola opening upwards

standard form	Focus	Directrix
$x^2 = 4ay$	$(0, a)$	$y = -a$

I Parabola opening downwards

standard form	Focus	Directrix
$x^2 = -4ay$	$(0, -a)$	$y = a$

4. Consider the ellipse $\mathcal{E}: x^2+4y^2-20=0$

a) Find the equations of the tangent lines to the ellipse $\mathcal{E}$ having a given angular coefficient $m \in \mathbb{R}$

$$x^2+4y^2=20 \quad /:20$$

$$\frac{x^2}{20}+\frac{y^2}{5}=1$$

line: $y=mx+c$

$$\frac{x^2}{20}+\frac{(mx+c)^2}{5}=1 \Leftrightarrow \frac{x^2}{20}+\frac{m^2x^2+2mxc+c^2}{5}=1$$

$$\Leftrightarrow \frac{x^2}{20}+\frac{{}^{4/}m^2x^2}{5}+\frac{2mxc}{5}+\frac{c^2}{5}=1$$

$$\Leftrightarrow \frac{x^2(1+4m^2)}{20}+\frac{2mxc}{5}+\frac{c^2}{5}=1$$

line tangent to the ellipse $\Rightarrow \Delta=0$

$$x^2(1+4m^2)+8mxc+4c^2-20=0$$

$$\Delta=(8mc)^2-4\cdot(1+4m^2)(4c^2-20) \Leftrightarrow$$

$$\Leftrightarrow \Delta=64m^2c^2-4(4c^2-20+16m^2c^2-80m^2)$$

$$\Leftrightarrow \Delta=\cancel{64m^2c^2}-16c^2+80\cancel{-64m^2c^2}+320m^2$$

$$\Leftrightarrow \Delta=-16c^2+320m^2+80$$

$$\Delta=0 \Rightarrow -16c^2=-320m^2-80 \Leftrightarrow 16c^2=320m^2+80 \quad /:16$$

$$\Leftrightarrow c^2=20m^2+5 \quad \Rightarrow c=\pm\sqrt{20m^2+5}$$

$\Rightarrow y = mx \pm \sqrt{20m^2+5}$

b) Find the equations of the tangent lines to $\mathcal{E}$ which are orthogonal to the line d: $2x-2y-13=0$

$$\frac{x^2}{20}+\frac{y^2}{5}=1$$

$2x-2y-13=0 \Rightarrow 2y = 2x-13 \Rightarrow y = x-\frac{13}{2} \Rightarrow$

$\Rightarrow m=1$ (the slope)

The slope of a line orthogonal to another line with slope m is $-\frac{1}{m}$.

$\Rightarrow m_t = -\frac{1}{1} = -1$

$\Rightarrow y = -x + c$

$$\frac{x^2}{20}+\frac{(-x+c)^2}{5}=1 \Leftrightarrow \frac{x^2}{20}+\frac{x^2-2xc+c^2}{5}=1 \Leftrightarrow$$

$$\Leftrightarrow \frac{x^2}{20}+\frac{x^2}{5}-\frac{2xc}{5}+\frac{c^2}{5}=1 \Leftrightarrow x^2+4x^2-8xc+4c^2=20 \Leftrightarrow$$

$$\Leftrightarrow 5x^2-8xc+4c^2=20$$

tangent to the ellipse $\Rightarrow \Delta=0$

$$\Delta = 64c^2-20(4c^2-20) \Leftrightarrow 64c^2-80c^2+400=0 \Leftrightarrow$$

$$\Leftrightarrow -16c^2=-400 \Rightarrow c^2=25 \Rightarrow c=\pm 5$$

$\Rightarrow \quad y = -x \pm 5$

8. Write the homogenous tranformation matrix of the concatenation (product) of the rotations $R_{\frac{\pi}{4}}(M_0)$ and $R_{\frac{7\pi}{4}}(M_1)$, where $M_0(1,2)$ and $M_1(2,-1)$

Rotation matrix: $R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

Translation matrix

2D: $\begin{pmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{pmatrix}$ 3D: $\begin{pmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{pmatrix}$

inverse translation matrices:

$$T(t_x, t_y)^{-1} = \begin{pmatrix} 1 & 0 & -t_x \\ 0 & 1 & -t_y \\ 0 & 0 & 1 \end{pmatrix}$$

$$T_{M_0} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \qquad T_{M_1} = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$T_{M_0}^{-1} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix} \qquad T_{M_1}^{-1} = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$R_{\frac{\pi}{4}} = \begin{pmatrix} \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} \\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}$$

Homogeneous transformation matrices:

$$H = T \cdot R \cdot T^{-1}$$

$$H_{M_0} = T_{M_0} \cdot R_{\frac{\pi}{4}} \cdot T_{M_0}^{-1} =$$

$$T_{M_0} \cdot R_{\frac{\pi}{4}} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 1 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

$$H_{M_0} = \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 1 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2}+1 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & -\frac{3\sqrt{2}}{2}+2 \\ 0 & 0 & 1 \end{pmatrix}$$

$$H_{M_1} = T_{M_1} \cdot R_{\frac{7\pi}{4}} \cdot T_{M_1}^{-1}$$

$$R_{\frac{7\pi}{4}} = \begin{pmatrix} \cos\frac{7\pi}{4} & -\sin\frac{7\pi}{4} \\ \sin\frac{7\pi}{4} & \cos\frac{7\pi}{4} \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}$$

$$\frac{7\pi}{4} = 2\pi - \frac{\pi}{4}$$

$$\cos\frac{7\pi}{4} = \cos\left(-\frac{\pi}{4}\right) = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$$

$$\sin\frac{7\pi}{4} = \sin\left(-\frac{\pi}{4}\right) = -\sin\frac{\pi}{4} = -\frac{\sqrt{2}}{2}$$

$$T_{M_1} \cdot R_{\frac{7\pi}{4}} = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 2 \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$H_{M_1} = \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 2 \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2}+2 \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & \frac{3\sqrt{2}}{2}+1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$H = H_{M_0} \cdot H_{M_1}$$

Model 2018:

7. Consider the parabola $P: y^2=4x$

a) Determine the focus and the director line of the parabola P.

$4a=4 \Rightarrow a=1 \Rightarrow$ focus is located at $(1,0)$

The director line is: $x=-a \Rightarrow x=-1$

b) Find the equations of the tangent lines to P having a given angular coefficient $m \in \mathbb{R}$

$y=mx+c$

$(mx+c)^2=4x \Leftrightarrow m^2x^2+2mxc+c^2=4x$

$\Leftrightarrow m^2x^2+(2mc-4)x+c^2=0$

tangent line $\Rightarrow \Delta=0$

$\Delta=(2mc-4)^2-4m^2c^2 \Leftrightarrow \Delta=\cancel{4m^2c^2}-16mc+16-\cancel{4m^2c^2}$

$\Leftrightarrow -16mc+16=0 \Rightarrow mc=1 \Rightarrow c=\frac{1}{m}$

$\Rightarrow y=mx+\frac{1}{m}$

c) Find the equations of the tangent lines to $\mathcal{E}$ which pass through the point $P(-4,1)$

The tangent line to a parabola $y^2=4ax$ at a point (x_1,y_1) is given by:

$$yy_1=2a(x+x_1)$$

$$y=2(x-4)$$

$$y^2=4x \Leftrightarrow (2x-8)^2=4x \Leftrightarrow 4x^2-32x+64=4x \Leftrightarrow$$

$$\Leftrightarrow 4x^2-36x+64=0 \quad /:4 \Leftrightarrow$$

$$\Leftrightarrow x^2-9x+16=0$$

$$\Delta = 81-64=17$$

$$x_{1,2}=\frac{9\pm\sqrt{17}}{2}$$

$$\Rightarrow y_{1,2}=2x-8=9\pm\sqrt{17}-8=1\pm\sqrt{17}$$

$$yy_1=2(x+x_1)$$

$$\Rightarrow \begin{cases} y(1+\sqrt{17})=2\left(x+\frac{9+\sqrt{17}}{2}\right) \Leftrightarrow y(1+\sqrt{17})-2x=9+\sqrt{17} \\ y(1-\sqrt{17})=2\left(x+\frac{9-\sqrt{17}}{2}\right) \Leftrightarrow y(1-\sqrt{17})-2x=9-\sqrt{17} \end{cases}$$

Chapter 7:

2. For each of the following matrices A, write down a quadratic equation with associated matrix A and find the matrix $M \in SO(2)$ which diagonalizes A

b) $\begin{pmatrix} 5 & -13 \\ -15 & 5 \end{pmatrix}$

$$P_A(\lambda) = \det(A - \lambda I_n) = \begin{vmatrix} 5-\lambda & -13 \\ -15 & 5-\lambda \end{vmatrix} =$$

$$= (5-\lambda)^2 - 195 = 25 - 10\lambda + \lambda^2 - 195 =$$

$$= \lambda^2 - 10\lambda - 170 \qquad \Delta = 100 + 680 = 780$$

$$\lambda_{1,2} = \frac{10 \pm 2\sqrt{195}}{2} = 5 \pm \sqrt{195}$$

I $\lambda_1 = 5 + \sqrt{195}$

$$\begin{pmatrix} -\sqrt{195} & -13 \\ -15 & -\sqrt{195} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0 \iff$$

$$\begin{cases} -\sqrt{195}x - 13y = 0 \Rightarrow y = -\frac{\sqrt{195}}{13}x \\ -15x - \sqrt{195}y = 0 \end{cases}$$

Let $x = 1 \Rightarrow v_1 = \begin{pmatrix} 1 \\ -\frac{\sqrt{195}}{13} \end{pmatrix}$

II $\lambda = 5 - \sqrt{195}$

$$\begin{pmatrix} \sqrt{195} & -13 \\ -15 & \sqrt{195} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0 \iff$$

$$\iff \begin{cases} \sqrt{195}\,x - 13y = 0 \Rightarrow y = \frac{\sqrt{195}}{13}x \\ -15x + \sqrt{195}\,y = 0 \end{cases}$$

$$\Rightarrow \text{Let } x = 1 \Rightarrow v_2 = \begin{pmatrix} 1 \\ \frac{\sqrt{195}}{13} \end{pmatrix}$$

Normalize v_1, v_2

The norm is: $\sqrt{1^2 + \left(\frac{\sqrt{195}}{13}\right)^2} = \sqrt{1 + \frac{195}{169}} = \frac{\sqrt{364}}{13}$

$$v_1 = \frac{13}{\sqrt{364}} \begin{pmatrix} 1 \\ -\frac{\sqrt{195}}{13} \end{pmatrix}$$

So we have to multiply each v with $\frac{1}{n}$ (where n is the norm)

$$v_2 = \frac{13}{\sqrt{364}} \begin{pmatrix} 1 \\ \frac{\sqrt{195}}{13} \end{pmatrix}$$

$$M_{B',B} = \begin{pmatrix} \frac{13}{\sqrt{364}} & \frac{13}{\sqrt{364}} \\ -\frac{\sqrt{195}}{\sqrt{364}} & \frac{\sqrt{195}}{\sqrt{364}} \end{pmatrix}$$

$$\operatorname{Det} M = \frac{13\sqrt{195}}{364} + \frac{13\sqrt{195}}{364} = \frac{26\sqrt{195}}{364} \simeq 1$$

$$Q: \quad (x, y) \begin{pmatrix} 5 & -13 \\ -15 & 5 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 2 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + 1$$

$$Q: \quad \begin{pmatrix} 5x - 15y & -13x + 5y \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + 2x + 2y + 1 = 0$$

$$Q: \quad 5x^2 - 15xy - 13xy + 5y^2 + 2x + 2y + 1 = 0$$

$$Q: \quad 5x^2 - 28xy + 5y^2 + 2x + 2y + 1 = 0$$

$$Q: \quad \begin{pmatrix} x' & y' \end{pmatrix} \cdot M^T \cdot \begin{pmatrix} 5 & -13 \\ -15 & 5 \end{pmatrix} \cdot M \cdot \begin{pmatrix} x' \\ y' \end{pmatrix} +$$

$$+ \begin{pmatrix} 2 & 2 \end{pmatrix} \cdot M \cdot \begin{pmatrix} x' \\ y' \end{pmatrix} + 1$$

$$Q: \quad \begin{pmatrix} x' & y' \end{pmatrix} \begin{pmatrix} 5 + \sqrt{195} & 0 \\ 0 & 5 - \sqrt{195} \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} +$$

$$+ \begin{pmatrix} 2 & 2 \end{pmatrix} \cdot \begin{pmatrix} \frac{13}{\sqrt{364}} & \frac{13}{\sqrt{364}} \\ -\frac{\sqrt{195}}{\sqrt{364}} & \frac{\sqrt{195}}{\sqrt{364}} \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} + 1$$

$$Q: \quad \begin{pmatrix} x'(5 + \sqrt{195}) & y'(5 - \sqrt{195}) \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} +$$

- - - - - -

c) $\begin{pmatrix} 7 & -2 \\ -2 & \frac{5}{3} \end{pmatrix}$

$P_A(\lambda) = \det(A - \lambda I_n) = \begin{vmatrix} 7-\lambda & -2 \\ -2 & \frac{5}{3}-\lambda \end{vmatrix} =$

$= \lambda^2 - \frac{26\lambda}{3} + \frac{35}{3} - 4 \Rightarrow 3\lambda^2 - 26\lambda + 23 = 0$

$\lambda_{1,2} = \frac{26 \pm \sqrt{400}}{6} = \frac{26 \pm 20}{6}$ $\begin{cases} \lambda_1 = \frac{46}{6} = \frac{23}{3} \\ \lambda_2 = \frac{6}{6} = 1 \end{cases}$

I $\lambda = \frac{23}{3}$

$\begin{pmatrix} 7 - \frac{23}{3} & -2 \\ -2 & \frac{5}{3} - \frac{23}{3} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0 \Leftrightarrow \begin{pmatrix} -\frac{2}{3} & -2 \\ -2 & -6 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$

$\begin{cases} -\frac{2}{3}x - 2y = 0 \\ -2x - 6y = 0 \end{cases} \Rightarrow y = -\frac{x}{3}$

Let $x = 1 \Rightarrow v_1 = \begin{pmatrix} 1 \\ -\frac{1}{3} \end{pmatrix}$

II $\lambda = 1$

$$\begin{pmatrix} 6 & -2 \\ -2 & \frac{2}{3} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$$\begin{cases} 6x - 2y = 0 \Rightarrow y = 3x \\ -2x + \frac{2}{3}y = 0 \end{cases}$$

Let $x = 1 \Rightarrow v_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

The norm for $v_1 \Rightarrow \sqrt{1 + \frac{1}{9}} = \frac{\sqrt{10}}{3}$

$v_2 \Rightarrow \sqrt{1+9} = \sqrt{10}$

$$M = \begin{pmatrix} \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \\ -\frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \end{pmatrix} \qquad D = \begin{pmatrix} \frac{23}{3} & 0 \\ 0 & 1 \end{pmatrix}$$

diagonal matrix

$\det M = 1$

$$Q: (x' \ y') \cdot \underbrace{M^T \cdot \begin{pmatrix} 7 & -2 \\ -2 & \frac{2}{3} \end{pmatrix} \cdot M}_{= D} \cdot \begin{pmatrix} x' \\ y' \end{pmatrix} +$$

$$+ (1 \ 3) \cdot M \cdot \begin{pmatrix} x' \\ y' \end{pmatrix} + 1 = 0$$

$$Q: (x' \ y') \begin{pmatrix} \frac{23}{3} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix} + \begin{pmatrix} 0 & \frac{10}{\sqrt{10}} \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix} + 1 = 0$$

$$Q: \left(\frac{23}{3}x' \quad y'\right)\begin{pmatrix} x' \\ y' \end{pmatrix} + \frac{10}{\sqrt{10}}y' + 1 = 0$$

$$Q: \frac{23}{3}x'^2 + y'^2 + \frac{10}{\sqrt{10}}y' + 1 = 0$$

$$Q: \frac{23}{3}x'^2 + \left(y'^2 + \frac{10}{\sqrt{10}}y' + 5\right) + 1 - 5$$

$$Q: \frac{23}{3}x'^2 + (y' + 5)^2 - 4 = 0$$

$$x'' = \sqrt{\frac{23}{3}}\,x$$

$$y'' = y' + 5$$

$$Q: x''^2 + y''^2 - 4 = 0$$

4. For each of the following equations write down the associated matrix and bring the equation in canonical form.

b) $6xy + x - y = 0$

$$M_Q = \begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix}$$

$$P_{M_Q}(\lambda) = \begin{pmatrix} -\lambda & 3 \\ 3 & -\lambda \end{pmatrix} = \lambda^2 - 9$$

$$\lambda^2 - 9 = 0 \Rightarrow \lambda = \pm 3$$

I $\lambda = 3$

$$\begin{pmatrix} -3 & 6 \\ 6 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$$\begin{cases} -3x + 6y = 0 \;/:3 \Rightarrow x = 2y \Rightarrow x = 2y \\ 6x - 3y = 0 \end{cases}$$

Let $x = 1 \Rightarrow v_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

II $\lambda = -3$

$$\begin{pmatrix} 3 & 6 \\ 6 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$$\begin{cases} 3x + 6y = 0 \Rightarrow x = -2y \\ 6x + 3y = 0 \end{cases}$$

Let $x = 1 \Rightarrow v_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

$\frac{1}{norm}$

$$M = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix}$$

$$Q: \begin{pmatrix} x' & y' \end{pmatrix} M^T \cdot \begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix} \cdot M \cdot \begin{pmatrix} x' \\ y' \end{pmatrix} = 0$$

$$Q: \begin{pmatrix} x' & y' \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & -3 \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} = 0$$

$$Q: \begin{pmatrix} 3x' & -3y' \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} = 0$$

$$Q: 3x'^2 - 3y'^2 = 0 \quad | :3$$

$$Q: x'^2 - y'^2 = 0$$

7.7. Find the canonical equation for each of the following cases:

b) $8y^2 + 6xy - 12x - 26y + 11 = 0$

$$M_q = \begin{pmatrix} 0 & 3 \\ 3 & 8 \end{pmatrix}$$

$$\begin{vmatrix} -\lambda & 3 \\ 3 & 8-\lambda \end{vmatrix} = \lambda^2 - 8\lambda - 9$$

$$\lambda^2 - 8\lambda - 9 = 0$$

$$\lambda_{1,2} = \frac{8 \pm 10}{2} \quad \begin{cases} \lambda_1 = -\frac{2}{2} = -1 \\ \lambda_2 = \frac{18}{2} = 9 \end{cases}$$

I $\lambda = -1$

$$\begin{pmatrix} 1 & 3 \\ 3 & 9 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$$\begin{cases} x + 3y = 0 \Rightarrow y = -\frac{x}{3} \\ 3x + 9y = 0 \end{cases}$$

Let $x = 1 \Rightarrow v_1 = \begin{pmatrix} 1 \\ -\frac{1}{3} \end{pmatrix} \Rightarrow$ norm is $\frac{\sqrt{10}}{3} \Rightarrow \frac{1}{n} = \frac{3}{\sqrt{10}}$

II $\lambda = 9$

$$\begin{pmatrix} -9 & 3 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$$\begin{cases} -9x + 3y = 0 \ | :3 \Rightarrow -3x + y = 0 \Rightarrow y = 3x \\ 3x - y = 0 \end{cases}$$

Let $x = 1 \Rightarrow v_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \Rightarrow$ norm is $\sqrt{10} \Rightarrow \frac{1}{n} = \frac{1}{\sqrt{10}}$

$$M = \begin{pmatrix} \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \\ -\frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \end{pmatrix}$$

$$\det M = \frac{9}{10} + \frac{1}{10} = 1$$

$Q: \begin{pmatrix} x' & y' \end{pmatrix} M^T \cdot M_q \cdot M \begin{pmatrix} x' \\ y' \end{pmatrix} = 0$

$Q: \begin{pmatrix} x' & y' \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 9 \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} = 0$

$Q: \begin{pmatrix} -x' & 9y' \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} = 0$

$Q: -x'^2 - 9y'^2 = 0$

$x'' = x'$

$y'' = \sqrt{9}\, y'$

$Q: -x''^2 - y''^2 = 0$

c) $x^2 - 4xy + y^2 - 6x + 2y + 1 = 0$

$M_q = \begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix}$

$\begin{vmatrix} 1-\lambda & -2 \\ -2 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - 4 = 1 - 2\lambda + \lambda^2 - 4 = \lambda^2 - 2\lambda - 3$

$\lambda^2 - 2\lambda - 3 = 0$

$\lambda_{1,2} = \frac{2 \pm 4}{2} \left\langle \begin{array}{l} \lambda_1 = \frac{6}{2} = 3 \\ \lambda_2 = \frac{-2}{2} = -1 \end{array} \right.$

I $\lambda = 3$

$$\begin{pmatrix} -2 & -2 \\ -2 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$$\begin{cases} -2x - 2y = 0 \quad /:-2 \quad \Rightarrow \quad y = -x \\ -2x - 2y = 0 \end{cases}$$

Let $x = 1 \quad \Rightarrow \quad v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

II $\lambda = -1$

$$\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$$\begin{cases} 2x - 2y = 0 \quad |:2 \quad \Rightarrow \quad y = x \\ -2x + 2y = 0 \end{cases}$$

Let $x = 1 \quad \Rightarrow \quad v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$M = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \qquad \det M = 1$$

$$Q: \begin{pmatrix} x' & y' \end{pmatrix} \cdot M^T \cdot M_Q \cdot M \begin{pmatrix} x' \\ y' \end{pmatrix} = 0$$

$$Q: \begin{pmatrix} x' & y' \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} = 0$$

$$Q: \begin{pmatrix} 3x' & -y' \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} = 0$$

$$Q: 3x'^2 - y'^2 = 0$$

$$x'' = \sqrt{3}x'$$

$$y'' = y'$$

$$Q: x''^2 - y''^2 = 0$$

10. Using the classification of quadrics, decide what surfaces are described by the following equation

c) $x^2 + xy + yz + zx = 1$

$M_Q = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \Rightarrow$ the matrix should be $\begin{pmatrix} 1 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \end{pmatrix}$ because we also have z

$$\begin{vmatrix} 1-\lambda & \frac{1}{2} \\ \frac{1}{2} & -\lambda \end{vmatrix} = (1-\lambda)(-\lambda) - \frac{1}{4} = \lambda^2 - \lambda - \frac{1}{4}$$

$$\lambda^2 - \lambda - \frac{1}{4} = 0 \Rightarrow \lambda_{1,2} = \frac{1 \pm \sqrt{2}}{2}$$

I $\lambda = \frac{1+\sqrt{2}}{2}$

$$\begin{pmatrix} 1-\frac{1+\sqrt{2}}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1+\sqrt{2}}{2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0 \qquad = \frac{1+\sqrt{2}}{2}$$

$$\begin{cases} \frac{1+\sqrt{2}}{2}x + \frac{1}{2}y = 0 \Rightarrow y = -(1+\sqrt{2})x \\ \frac{1}{2}x - \frac{1+\sqrt{2}}{2}y = 0 \end{cases}$$

Let $x=1 \Rightarrow v_1 = \begin{pmatrix} 1 \\ -1-\sqrt{2} \end{pmatrix} \Rightarrow n = \sqrt{1^2+(-1-\sqrt{2})^2} =$

$= \sqrt{1+1+2\sqrt{2}+2} = \sqrt{4+2\sqrt{2}}$

II $\lambda = \frac{1-\sqrt{2}}{2}$

$$\begin{pmatrix} 1-\frac{1-\sqrt{2}}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1-\sqrt{2}}{2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0 \qquad = \frac{1-\sqrt{2}}{2}$$

$$\begin{cases} \frac{1-\sqrt{2}}{2}x + \frac{1}{2}y = 0 \Rightarrow y = -(1-\sqrt{2})x \\ \frac{1}{2}x - \frac{1-\sqrt{2}}{2}y = 0 \end{cases}$$

Let $x=1 \Rightarrow v_2 = \begin{pmatrix} 1 \\ -1+\sqrt{2} \end{pmatrix} \Rightarrow n = \sqrt{1+(-1+\sqrt{2})^2} =$

$= \sqrt{1+1-2\sqrt{2}+2} = \sqrt{4-2\sqrt{2}}$

$$M = \begin{pmatrix} \frac{1}{\sqrt{4+2\sqrt{2}}} & \frac{1}{\sqrt{4-2\sqrt{2}}} \\ \frac{-1-\sqrt{2}}{\sqrt{4+2\sqrt{2}}} & \frac{-1+\sqrt{2}}{\sqrt{4-2\sqrt{2}}} \end{pmatrix} \qquad \det M = 1$$

$$Q: \begin{pmatrix} x' & y' \end{pmatrix} \cdot M^T \cdot M_Q \cdot M \begin{pmatrix} x' \\ y' \end{pmatrix} = 0$$

$$Q: \begin{pmatrix} x' & y' \end{pmatrix} \cdot \begin{pmatrix} \frac{1+\sqrt{2}}{2} & 0 \\ 0 & \frac{1-\sqrt{2}}{2} \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} = 0$$

$$Q: \begin{pmatrix} \frac{1+\sqrt{2}}{2} \cdot x' & \frac{1-\sqrt{2}}{2} \cdot y' \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} = 0$$

$$Q: \frac{1+\sqrt{2}}{2} x'^2 + \frac{1-\sqrt{2}}{2} y'^2 = 0$$

$$x'' = \sqrt{\frac{1+\sqrt{2}}{2}}\, x'$$

$$y'' = \sqrt{\frac{1-\sqrt{2}}{2}}\, y'$$

$$Q: x''^2 + y''^2 = 0$$

Partial 2: Sample:

P1: Give the equation of an ellipse which has focal points on the y-axis symmetric relative to the origin, for which the shortest diameter is 10 and for which the eccentricity is $\frac{12}{13}$

General eq:

with foci on x-axis $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ foci on y-axis $\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$

The foci are located at $(0, \pm c)$ $c=\sqrt{a^2+b^2}$

$d=10 \Rightarrow b=\frac{10}{2}=5$

$e=\frac{12}{13}$ $e=\frac{c}{a}$, $c=\sqrt{a^2-b^2}$

$e=\frac{c}{a} \Leftrightarrow c=e\cdot a$

$c^2=a^2-b^2$ $\Rightarrow a^2=25\cdot\frac{169}{25}$

$(e\cdot a)^2=a^2-25$ $\Rightarrow a=13$

$\frac{144}{169}a^2=a^2-25$ $c=\frac{12}{13}\cdot 13=12$

$a^2-\frac{144}{169}a^2=25$ $\Rightarrow \frac{x^2}{5^2}+\frac{y^2}{13^2}=1 \Leftrightarrow \frac{x^2}{25}+\frac{y^2}{169}=1$

$\frac{25}{169}a^2=25 \Rightarrow$

P2: Determine equations for the tangent lines to the ellipse $\frac{x^2}{2}+y^2=1$ which pass through the point $A(4,1)$

$y = mx + c$

$1 = 4m + c \Rightarrow c = 1-4m$

$\frac{x^2}{2} + (mx+c)^2 - 1 = 0$

$\frac{x^2}{2} + m^2x^2 + 2mxc + c^2 - 1 = 0$

$x^2 + 2m^2x^2 + 4mxc + 2c^2 - 2 = 0$

$(1+2m^2)x^2 + 4mxc + 2c^2 - 2 = 0$

$\Delta = (4mc)^2 - 4(1+2m^2)(2c^2-2)$

$\Delta = 16m^2c^2 - 4(2c^2 - 2 + 4m^2c^2 - 4m^2)$

$\Delta = \cancel{16m^2c^2} - 8c^2 + 8 - \cancel{16m^2c^2} + 16m^2$

$\Delta = 16m^2 - 8c^2 + 8$

tangent $\Rightarrow \Delta = 0 \Rightarrow 16m^2 - 8c^2 + 8 = 0 \quad | :8 \Leftrightarrow$

$\Leftrightarrow 2m^2 - c^2 + 1 = 0 \Leftrightarrow 2m^2 - (1-4m)^2 + 1 = 0$

$\Leftrightarrow 2m^2 - (\cancel{1} - 8m + 16m^2) + \cancel{1} = 0$

$\Leftrightarrow -14m^2 - 8m = 0 \quad |\cdot(-1) \Leftrightarrow 2m(7m+4) = 0$

I $2m>0 \Rightarrow m>0$

$\Rightarrow c = 1-4m = 1$

$\Rightarrow y = 1$

II $7m+4=0 \Rightarrow m = -\frac{4}{7}$

$\Rightarrow c = 1 + \frac{16}{7} = \frac{23}{7}$

$\Rightarrow y = -\frac{4}{7}x + \frac{23}{7}$

P3. Determine equations for one rectilinear generator of the surface

$$x^2 + \frac{y^2}{4} - \frac{z^2}{4} = 1$$

which is parallel to the plane $x+y+z=0$

$d = (d_1, d_2, d_3)$

$d_1 + d_2 + d_3 = 0$

Let $d_1 = d_2 = 1 \Rightarrow d_3 = -2$

$$\begin{cases} x = x_0 + t \\ y = y_0 + t \\ z = z_0 - 2t \end{cases} \Rightarrow (x_0+t)^2 + \frac{(y_0+t)^2}{4} - \frac{(z_0-2t)^2}{4} = 1$$

P_4: Verify that the matrix

$$B = \frac{1}{11}\begin{pmatrix} -2 & 6 & -9 \\ -6 & 7 & 6 \\ 9 & 6 & 2 \end{pmatrix}$$

belongs to $SO(3)$ and determine the cosine of the rotation angle

$$B \cdot B^T = \frac{1}{11}\begin{pmatrix} -2 & 6 & -9 \\ -6 & 7 & 6 \\ 9 & 6 & 2 \end{pmatrix}\begin{pmatrix} -2 & -6 & 9 \\ 6 & 7 & 6 \\ -9 & 6 & 2 \end{pmatrix} \cdot \frac{1}{11} =$$

$$= \frac{1}{121}\begin{pmatrix} 4+36+81 & 12+42-54 & -18+36-18 \\ 12+42-54 & 36+49+36 & -54+42+12 \\ -18+36-18 & -54+42+12 & 81+36+4 \end{pmatrix} =$$

$$= \frac{1}{121}\begin{pmatrix} 121 & 0 & 0 \\ 0 & 121 & 0 \\ 0 & 0 & 121 \end{pmatrix} = I_3 \Rightarrow B \in O(3)$$

$$\det B = \left(\frac{1}{11}\right)^3 \begin{vmatrix} -2 & 6 & -9 \\ -6 & 7 & 6 \\ 9 & 6 & 2 \end{vmatrix} = \frac{1}{1331} \cdot \left(-28+324+324+567+72\right.$$

$$\left.+72\right) = \frac{1331}{1331} = 1 \Rightarrow B \in SO(3)$$

$$Tr(B) = 2\cos\theta + 1 \iff 7 = 2\cos\theta + 1$$

$$\iff \cos\theta = 3$$

Partial 2: Sample 2

Q1: Give the equation of an ellipse containing the points (2,0) and (0,3) and having focal points on one of the coordinate axis equally distanced from the origin.

I Focal points on x axis

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

$(2,0) \Rightarrow \frac{2^2}{a^2}+\frac{0^2}{b^2}=1 \Rightarrow a^2=4$

$(0,3) \Rightarrow \frac{0^2}{a^2}+\frac{3^2}{b^2}=1 \Rightarrow b^2=9$

foci are located at $(\pm c, 0)$

$c^2=a^2-b^2 \Rightarrow c^2=4-9=-5 \Rightarrow$ invalid so the focal point must be on the y-axis

II Focal point on y-axis

$$\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$$

$(2,0) \Rightarrow \frac{2^2}{b^2}+\frac{0^2}{a^2}=1 \Rightarrow b^2=4$

$(0,3) \Rightarrow \frac{0^2}{b^2}+\frac{3^2}{a^2}=1 \Rightarrow a^2=9$

foci are located at $(0, \pm c)$

$c^2 = a^2 - b^2 \Rightarrow c^2 = 9-4 \Rightarrow c = \sqrt{5}$ (valid)

$\Rightarrow$ the eq is $\frac{x^2}{4} + \frac{y^2}{9} = 1$

Q2: Determine equations for the tangent lines to the hyperbola $x^2 - y^2 = 2$ which contain the point $M(2,2)$

at $(2,2) \Rightarrow 2 = 2m + c \Rightarrow c = 2-2m$

$y = mx + c$

$x^2 - (mx+c)^2 = 2 \Leftrightarrow x^2 - m^2x^2 - 2mxc - c^2 - 2 = 0$

$\Leftrightarrow (1-m^2)x^2 - 2mcx - c^2 - 2 = 0$

$\Delta = 4m^2c^2 - 4\cdot(1-m^2)(-c^2-2) \Leftrightarrow$

$\Leftrightarrow \Delta = 4m^2c^2 - 4(-c^2 - 2 + m^2c^2 + 2m^2) \Leftrightarrow$

$\Leftrightarrow \Delta = \cancel{4m^2c^2} + 4c^2 + 8 - \cancel{4m^2c^2} - 8m^2$

$\Leftrightarrow \Delta = 4c^2 + 8 - 8m^2$

$\Delta = 0 \Rightarrow 4c^2 + 8 - 8m^2 = 0$

$\Rightarrow (2-2m)^2 + 8 - 8m^2 = 0 \Leftrightarrow 4 - 8m + \underline{4m^2} + 8 - \underline{8m^2} = 0$

$\Leftrightarrow -4m^2 - 8m + 12 = 0 \quad |: -4 \Leftrightarrow$

$\Leftrightarrow m^2 + 2m - 3 = 0$

$\Delta = 4 + 12 = 16 \qquad m_{1,2} = \frac{-2 \pm 4}{2}$ $\begin{cases} m_1 = \frac{2}{2} = 1 \\ m_2 = \frac{-6}{2} = -3 \end{cases}$

I $m=1 \Rightarrow c = 2-2 = 0$

$\Rightarrow y = x$

II $m=-3 \Rightarrow c = 2-2\cdot(-3) = 8$

$\Rightarrow y = -3x+8$

P3: Consider the surface

$$S: y^2+2xy-2xz+z-2=0$$

given with respect to some coordinate system

a) Determine the intersection of S with the coordinate axis Oy

intersection of S with $Oy \Rightarrow x=z=0$

$\Rightarrow y^2-2=0 \Rightarrow y=\pm\sqrt{2}$

$\Rightarrow$ intersection points $\Rightarrow (0,\sqrt{2},0)$ & $(0,-\sqrt{2},0)$

b) Bring the equation in canonical form. What type of quadric is it? Why?

2D: $M_Q = \begin{pmatrix} x^2 & \frac{xy}{2} \\ \frac{xy}{2} & y^2 \end{pmatrix}$

Use the coefficients of the terms, not the terms themselves

3D: $M_Q = \begin{pmatrix} x^2 & \frac{xy}{2} & \frac{xz}{2} \\ \frac{xy}{2} & y^2 & \frac{yz}{2} \\ \frac{xz}{2} & \frac{yz}{2} & z^2 \end{pmatrix}$

$$M_Q = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

$$\begin{vmatrix} -\lambda & 1 & -1 \\ 1 & 1-\lambda & 0 \\ -1 & 0 & -\lambda \end{vmatrix} = (-\lambda)(1-\lambda)(-\lambda) - 1 + \lambda + \lambda =$$

$$= (-\lambda + \lambda^2)(-\lambda) - 1 + 2\lambda = -\lambda^3 + \lambda^2 + 2\lambda - 1 = 0$$

$$-\lambda^3 + \lambda^2 + 2\lambda - 1 = 0 \qquad \cdots\cdots$$

Method II: Lagrange Method.

$y^2+2xy-2xz+z-2=0 \Leftrightarrow$

$\Leftrightarrow (y^2+2xy+x^2)-(x^2+2xz+z^2)+z^2+z-2=0$

$\Leftrightarrow (y+x)^2-(x+z)^2+z^2+z=2$

$y' = y+x$

$x' = x+z$

$\Leftrightarrow y'^2 - x'^2 + \left(z^2+z+\frac{1}{4}\right) = \overset{\frac{9}{4}}{2+\frac{1}{4}}$

$\Leftrightarrow y'^2 - x'^2 + \left(z+\frac{1}{2}\right)^2 = \frac{9}{4}$

$\Leftrightarrow y'^2 - x'^2 + z'^2 = \frac{9}{4}$ (two terms are positive, one is negative)

$\Rightarrow$ hyperboloid of two sheets

Theory:

1. Ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

All squared terms have the same sign

2. Hyperboloid of one sheet

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$

Two squared terms are positive, one is negative

3. Hyperboloid of two sheets

$$-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

One squared term is positive, the other two are negative

4. Cone: $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0$

Two squared terms are positive, one is negative, but they sum up to 0.

5. Paraboloid (Elliptic): $\frac{x^2}{a^2}+\frac{y^2}{b^2}=z$

All terms are positive, two on the left, one on the right

6. Paraboloid (Hyperbolic) $\frac{x^2}{a^2}-\frac{y^2}{b^2}=z$

One squared term is positive, other negative, third variable isolated on the right.

7. Cylinder: $x^2+y^2=r^2$

r^2 – constant representing the radius

Only two var. appear, third is absent

Q4: Consider the matrix

$$B = \frac{1}{11}\begin{pmatrix} -2 & 6 & -9 \\ -6 & 7 & 6 \\ 9 & 6 & 2 \end{pmatrix}$$

a) Explain why it corresponds to a rotation

$$B \cdot B^T = \frac{1}{11}\begin{pmatrix} -2 & 6 & -9 \\ -6 & 7 & 6 \\ 9 & 6 & 2 \end{pmatrix}\begin{pmatrix} -2 & -6 & 9 \\ 6 & 7 & 6 \\ -9 & 6 & 2 \end{pmatrix} \cdot \frac{1}{11} = I_3 \Rightarrow B \in O(3)$$

(it is orthogonal)

$$\det B = \left(\frac{1}{11}\right)^3 \begin{vmatrix} -2 & 6 & -9 \\ -6 & 7 & 6 \\ 9 & 6 & 2 \end{vmatrix} = 1 \Rightarrow B \in SO(3)$$

(preserves orientation)

b) Determine the cosine of the rotation angle

$$\operatorname{Tr}(B) = 2\cos\theta + 1 \Leftrightarrow 2\cos\theta + 1 = 7 \Rightarrow \cos\theta = 3$$

c) Determine the axis of rotation

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{11}\begin{pmatrix} -2 & 6 & -9 \\ -6 & 7 & 6 \\ 9 & 6 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} \Leftrightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{11}\begin{pmatrix} -2x+6y-9z \\ -6x+7y+6z \\ 9x+6y+2z \end{pmatrix}$$

$$\Leftrightarrow \begin{cases} 11x = -2x+6y-9z \\ 11y = -6x+7y+6z \\ 11z = 9x+6y+2z \end{cases} \Leftrightarrow \begin{cases} 13x = 6y-9z \\ 4y = -6x+6z \\ 9z = 9x+6y \end{cases}$$

$$\begin{cases} 13x = 6y - 9z \\ 4y = -6x + 6z \quad /:2 \\ 9z = 9x + 6y \quad /:3 \end{cases} \Leftrightarrow \begin{cases} 13x = 6y - 9z \\ 2y = -3x + 3z \\ 3z = 3x + 2y \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} 13x = 6y - 9z \Rightarrow 13x = \cancel{6y} - 9x - \cancel{6y} \Rightarrow x = 0 \\ y = \frac{-3x+3z}{2} \Rightarrow y = \frac{3z}{2} \\ z = \frac{3x+2y}{3} \Rightarrow z = \frac{2y}{3} \end{cases}$$

$$Fix(A) = \left\{ \left(0, \alpha, \frac{2\alpha}{3}\right) \right\} \quad \alpha \in \mathbb{R}$$

$$\Rightarrow l : \begin{cases} x = 0 \\ y = \alpha \\ z = \frac{2\alpha}{3} \end{cases}$$ is the axis of rotation

Chapter 8:

17. Determine the intersection of the paraboloid

$$\mathcal{P}^h_{2,\frac{1}{2}} : x^2 - 4y^2 = 4z \quad \text{with the line} \quad l = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} + \left\langle \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} \right\rangle$$

Write down the equations of the tangent planes in the intersection points.

$$l = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} + t \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} \Rightarrow \begin{cases} x = 2 + 2t \\ y = t \\ z = 3 - 2t \end{cases}$$

$$(2+2t)^2 - 4t^2 = 4(3-2t) \Leftrightarrow 4 + 8t + \cancel{4t^2} - \cancel{4t^2} = 12 - 8t \Leftrightarrow$$

$$\Leftrightarrow 16t = 8 \Rightarrow t = \frac{1}{2}$$

$$\Rightarrow \begin{cases} x = 2 + 2 \cdot \frac{1}{2} = 3 \\ y = \frac{1}{2} \\ z = 3 - 2 \cdot \frac{1}{2} = 2 \end{cases} \Rightarrow \text{the intersection point is } \left(3, \frac{1}{2}, 2\right)$$

$$F = x^2 - 4y^2 - 4z$$

$$\nabla F = (2x, -8y, -4)$$

$$2x_0(x - x_0) - 8y_0(y - y_0) - 4(z - z_0) = 0$$

$$6(x-3) - 4\left(y - \frac{1}{2}\right) - 4(z-2) = 0$$

$$6x - 18 - 4y + 2 - 4z + 8 = 0$$

$$6x - 4y - 4z - 8 = 0 \quad /:2 \quad \Rightarrow 3x - 2y - 2z - 4 = 0$$

19. Determine the plane which contains the line $\begin{bmatrix}-1\\0\\0\end{bmatrix} + \langle \begin{bmatrix}2\\-1\\0\end{bmatrix} \rangle$ and is tangent to the quadric

$x^2+2y^2-z^2+1=0$

$$\begin{bmatrix}-1\\0\\0\end{bmatrix} + t\begin{bmatrix}2\\-1\\0\end{bmatrix} \Rightarrow \begin{cases} x=-1+2t \\ y=-t \\ z=0 \end{cases}$$

eq of a plane $ax+by+cz+d=0$

$$a(-1+2t)+b\cdot(-t)+c\cdot 0+d=0$$

$$-a+2at-bt+d=0$$

$$t(2a-b)-a+d=0$$

$$\Rightarrow \begin{cases} 2a-b=0 \Rightarrow b=2a \\ -a+d=0 \Rightarrow d=a \end{cases}$$

$$\Rightarrow ax+2ay+cz+a=0$$

$$\Rightarrow a\left(x+2y+\frac{c}{a}z+1\right)=0$$

$$\Rightarrow x+2y+\frac{c}{a}z+1=0$$

$$\left.\begin{aligned} &n=\left(1,2,\frac{c}{a}\right) \\ &\nabla f=(2x_0,4y_0,-2z_0) \end{aligned}\right\} \Rightarrow$$ At the point of tangency these are proportional

$$\Rightarrow \begin{cases} 1=2x_0 \Rightarrow x_0=\frac{1}{2} \\ 2=4y_0 \Rightarrow y_0=\frac{1}{2} \\ \frac{c}{a}=-2z_0 \end{cases}$$

$x^2+2y^2-z^2+1=0$

$\Leftrightarrow \left(\frac{1}{2}\right)^2+2\cdot\left(\frac{1}{2}\right)^2-z_0^2+1=0$

$\Leftrightarrow \frac{1}{4}+\frac{1}{2}-z_0^2+1=0$

$\Leftrightarrow \frac{7}{4}-z_0^2=0 \Rightarrow z_0=\frac{\sqrt{7}}{2}$

$\Rightarrow \frac{c}{a}=-2\frac{\sqrt{7}}{2}=-\sqrt{7}$

$\Rightarrow x+2y-\sqrt{7}z+1=0$

23. Determine the generators of the paraboloid $\frac{x^2}{16}-\frac{y^2}{4}=z$ which are parallel to the plane $3x+2y-4z=0$

$n=(3,2,-4)$

Generator: $r(t)=r_0+td$, $r_0=(x_0,y_0,z_0)$

$d=(a,b,c)$

direction vector

Parallel cond: $d\cdot n=0$

$(a,b,c)\cdot(3,2,-4)=0 \Rightarrow 3a+2b-4c=0$

Let $a=2$, $b=1$ $\Rightarrow 6+2-4c=0 \Rightarrow c=2$

$\Rightarrow d=(2,1,2)$

$\Rightarrow r(t)=r_0\cdot t(2,1,2) \Rightarrow r(t)=(x_0+2t, y_0+t, z_0+2t)$

Where (x_0,y_0,z_0) is any point on the paraboloid.