LNDECC : $\underbrace{x'' + a_1 x' + a_2 x}_{L[x]=0} = \underbrace{f(t)}$, $a_{1,2} \in \mathbb{R}$

$\Downarrow$

$x = x_h + x_p$

x_h - general sol of LHDECC

x_p - a particular sol of LNDECC

1) $f(t) = P_m(t) \Rightarrow x_p = t^q \cdot Q_m(t)$

q = multiple order of "0" as root of characteristic eq

$r_1 \neq 0 \neq r_2 \Rightarrow q = 0$

$r_1 = 0 \Rightarrow q = 1$, $r_1 = r_2 = 0 \Rightarrow q = 2$

2) $f(t) = P_m(t) e^{\alpha t} \Rightarrow x_p = t^q Q_m(t) e^{\alpha t}$

q = multiple order of "α" as root of char eq

3) $\left.\begin{array}{l} f(t) = P_m(t) e^{\alpha t} \cos\beta t \\ f(t) = P_m(t) e^{\alpha t} \sin\beta t \end{array}\right\} \Rightarrow$

$\Rightarrow x_p = t^q e^{\alpha t} \left[Q_m(x) \sin\beta t + R_m(x) \cos\beta t \right]$

q = multiple order of "$\alpha + i\beta$" as root of char eq

1.5.1. T / F ?

a) All of the solutions of $x'' + 3x' + x = 1$ satisfy
$\lim_{t\to\infty} x(t) = 1$

b) The solution of the IVP $\begin{cases} x'' + 4x = 1 \\ x(0) = \frac{5}{4} \\ x'(0) = 0 \end{cases}$

Satisfies $x(\pi) = \frac{5}{4}$

c) The equation $x' = 3x + t^3$ admits a polynomial sol (of degree 3)

a) Step 1: Solve the associated homogeneous equation

$x'' + 3x' + x = 0$

$r^2 + 3r + 1 = 0$

$\Delta = 5$

$r_{1,2} = \frac{-3 \pm \sqrt{5}}{2}$

$r_{1,2}$ simple real solutions $\Rightarrow$

$\Rightarrow e^{\frac{-3-\sqrt{5}}{2}t}, e^{\frac{-3+\sqrt{5}}{2}t}$ solutions

$x_h = C_1 e^{\frac{-3-\sqrt{5}}{2}t} + C_2 e^{\frac{-3+\sqrt{5}}{2}t}, \quad C_1, C_2 \in \mathbb{R}$

Step 2: $x_p = ?$

$$x'' + 3x' + x = 1$$

a constant

$$\underset{P_0(t)}{\underbrace{f(t) = 1}} \Rightarrow x_p = Q_0(t) = K$$

$$K'' + 3K' + K = 1 \Rightarrow K = 1$$

Step 3: $$x = x_h + x_p = C_1 e^{\frac{-3-\sqrt{5}}{2}t} + C_2 e^{\frac{-3+\sqrt{5}}{2}t} + 1$$

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty}\Big(\underbrace{C_1 e^{\frac{-3-\sqrt{5}}{2}t}}_{=0} + \underbrace{C_2 e^{\frac{-3+\sqrt{5}}{2}t}}_{=0} + 1\Big) =$$

$$= 1$$

b) $$\begin{cases} x'' + 4x = 1 \\ x(0) = \frac{5}{4} \\ x'(0) = 0 \end{cases}$$ Sol.is. $x(\pi) = \frac{5}{4}$

Step 1: $x'' + 4x = 0$

$r^2 + 4 = 0$

$(r-2i)(r+2i) = 0 \Rightarrow r_{1,2} = \pm 2i \in \mathbb{C}\setminus\mathbb{R} \Rightarrow$

$\Rightarrow x_1 = e^{0t}\cdot \sin 2t$

$x_2 = \cos 2t$

$\Rightarrow x_h = C_1 \cdot \sin 2t + C_2 \cdot \cos 2t$

Step2: $x_p = ?$ sol of $x'' + 4x = 1$

$f(t) = 1 = P_0(t) \Rightarrow x_p = Q_0(t) = k$

$k'' + 4k = 1 \Rightarrow 4k = 1 \Rightarrow k = \frac{1}{4} \Rightarrow x_p = \frac{1}{4}$

Step3: $x = x_h + x_p = c_1 \cdot \sin 2t + c_2 \cdot \cos 2t + \frac{1}{4}$ - gen. sol. of the nonhomog eq

$x(0) = \frac{5}{4}$

$x(0) = c_1 \sin 0 + c_2 \cos 0 + \frac{1}{4} = \frac{5}{4} \Rightarrow c_2 + \frac{1}{4} = \frac{5}{4}$

$\Rightarrow c_2 = 1$

$x'(t) = 2t\, c_1 \cos 2t - 2t\, c_2 \sin 2t$

$x'(0) = 0 \Leftrightarrow 2c_1 = 0 \Rightarrow c_1 = 0$

$\Rightarrow x = \cos 2t + \frac{1}{4}$ - general sol. of IVP

$x(\pi) = \cos 2\pi + \frac{1}{4} = 1 + \frac{1}{4} = \frac{5}{4}$

c) $x' = 3x + t^3$

$x' - 3x = t^3$

Step 1: $x' - 3x = 0$

$r - 3 = 0$

$r = 3$

$\Rightarrow x_1 = e^{3t}$

$\Rightarrow x_h = C_1 \cdot e^{3t}$

Step 2: $x_p = ?$ - sol of $x' - 3x = t^3$

$x_p = at^3 + bt^2 + ct + d$ (stated in the prob, sol of degree 3)

$x_p' = 3at^2 + 2bt + c$

$3at^2 + 2bt + c - 3at^3 - 3bt^2 - 3ct - 3d = t^3$

$(-3a-1)t^3 + (3a-3b)t^2 + (2b-3c)t + c - 3d = 0 \quad , \forall t$

$$\Rightarrow \begin{cases} -3a-1=0 & \Rightarrow a = -\frac{1}{3} \\ 3a-3b=0 & \Rightarrow b = -\frac{1}{3} \\ 2b-3c=0 & \Rightarrow 3c = -\frac{2}{3} \Rightarrow c = -\frac{2}{9} \\ c-3d=0 & \Rightarrow 3d = -\frac{2}{9} \Rightarrow d = -\frac{2}{27} \end{cases}$$

$$\Rightarrow x_p = -\frac{1}{3}t^3 - \frac{1}{3}t^2 - \frac{2}{9}t - \frac{2}{27}$$

Step 3: $X = x_h + x_p = c_1 e^{3t} - \frac{1}{3}t^3 - \frac{1}{3}t^2 - \frac{2}{9}t - \frac{2}{27}$

1.5.2. $\lambda \in \mathbb{R}$. Find the general solution of $x'' - x = e^{\lambda t}$ knowing that, depending on λ, it has a particular sol either of the form $a e^{\lambda t}$ or of the form $a t e^{\lambda t}$

Step 1: $x'' - x = 0$ $\Rightarrow x_1 = e^{t}, \; x_2 = e^{-t}$

$r^2 - 1 = 0$

$r^2 = 1 \Rightarrow r = \pm 1$

$$x_h = c_1 \cdot e^{t} + c_2 \cdot e^{-t}$$

Step 2: $\left.\begin{array}{l} x'' - x = e^{\lambda t} \\ x_p = a \cdot e^{\lambda t} \end{array}\right\} \Rightarrow (a \cdot e^{\lambda t})'' - a e^{\lambda t} = e^{\lambda t} \Rightarrow$

$$\Rightarrow (a\lambda e^{\lambda t})' - a e^{\lambda t} = e^{\lambda t} \Rightarrow a\lambda^2 e^{\lambda t} - a e^{\lambda t} = e^{\lambda t}$$

$$\Rightarrow e^{\lambda t}(a\lambda^2 - a) = e^{\lambda t} \; / : e^{\lambda t} \Rightarrow a\lambda^2 - a = 1$$

$$\Rightarrow a(\lambda^2 - 1) = 1 \Rightarrow a = \frac{1}{\lambda^2 - 1}, \quad \lambda^2 - 1 \neq 0$$

Case 1: $\lambda \neq \pm 1 \Rightarrow x_p = \frac{1}{\lambda^2 - 1} \cdot e^{\lambda t}$

Case 2: $\lambda = 1 \Rightarrow x_p = a \cdot t e^t$ - sol of the eq $x'' - x = e^t$

$$x_p = a \cdot t \cdot e^t \Rightarrow x_p' = ae^t + ate^t$$

$$x_p'' = ae^t + ae^t + ate^t$$

$$x'' - x = e^t \Rightarrow ae^t + ae^t + ate^t - ate^t = e^t$$

$$\Rightarrow 2ae^t = e^t \Rightarrow 2a = 1 \Rightarrow a = \frac{1}{2}$$

$$x_p = \frac{1}{2} t e^t$$

Case 3: $\lambda = -1 \Rightarrow x_p = a t e^{-t}$ sol of the eq $x'' - x = e^{-t}$

$$x_p' = ae^{-t} - ate^{-t}$$

$$x_p'' = -ae^{-t} - ae^{-t} + ate^{-t}$$

$$-ae^{-t} - ae^{-t} + \cancel{ate^{-t}} - \cancel{ate^{-t}} = e^{-t}$$

$$-2ae^{-t} = e^{-t} \Rightarrow -2a = 1 \Rightarrow a = -\frac{1}{2}$$

$$x_p = -\frac{1}{2} t e^{-t}$$

Step 3: Case 1: $\lambda \neq \pm 1 \Rightarrow x = x_h + x_p$

$$\Rightarrow x = c_1 e^{t} + c_2 e^{-t} + \frac{1}{\lambda^2 - 1} e^{\lambda t}$$

Case 2: $\lambda = 1$

$$\Rightarrow x = c_1 \cdot e^{t} + c_2 \cdot e^{-t} + \frac{1}{2} t e^{t}$$

Case 3: $\lambda = -1$

$$\Rightarrow x = c_1 \cdot e^{t} + c_2 \cdot e^{-t} - \frac{1}{2} t e^{-t} \quad , c_1, c_2 \in \mathbb{R}$$

1.5.3. $w > 0$, $\varphi(\cdot, w)$ = sol of the

IVP $\begin{cases} x'' + x = \cos wt \\ x(0) = x'(0) = 0 \end{cases}$

i) $w \neq 0$ find a sol $x_p = a \cos wt + b \sin wt$ of the eq

ii) $x_p = ?$ s.t. $x_p = t(a \cos t + b \sin t)$ sol of $x'' + x = \cos t$ $(w = 1)$

iii) $\varphi(\cdot, w) = ?$ $w > 0$

iv) Prove $\lim_{w \to 1} \varphi(t, w) = \varphi(t, 1)$, $\forall t \in \mathbb{R}$

i), ii) - we only deal with step 2.

i) $(a\cos wt + b\sin wt)'' + a\cos wt + b\sin wt = \cos wt$

$-aw^2\cos wt - bw^2\sin wt + a\cos wt + b\sin wt = \cos wt$

$\cos wt(-aw^2 + a - 1) + \sin wt(-bw^2 + b) = 0$

$$\begin{cases} -aw^2 + a - 1 = 0 \Rightarrow a = \dfrac{1}{1-w^2}, \quad w \neq 1, w > 0 \\ -bw^2 + b = 0 \Rightarrow b = 0 \end{cases}$$

$x_p = \dfrac{1}{1-w^2} \cdot \cos wt$

ii) Homework: $a = 0, \ b = \frac{1}{2}$ $\qquad$! $w = 1$

$x_p = \frac{1}{2} \cdot \sin t$

iii) $r^2 + 1 = 0 \Rightarrow r = \pm i$

$x_h = c_1 \cdot \cos t + c_2 \cdot \sin t$

$\varphi(t, w) = c_1 \cos t + c_2 \cdot \sin t + \dfrac{1}{1-w^2}\cos wt$

$$\left.\begin{array}{l} x(0) = 0 \\ x'(0) = 0 \end{array}\right\} \Rightarrow \varphi(t, w) = \dots\dots$$

From (ii) $\Rightarrow$ $\left.\begin{array}{l} x(t,1) = C_1 \cdot \cos t + c_2 \cdot \sin t + \frac{1}{2} t \sin t \\ x(0) = 0 \\ x'(0) = 0 \end{array}\right\} \Rightarrow$

$\Rightarrow \varphi(t,1) = \ldots$

iv) $\lim_{w \to 1} \varphi(t,w) = \ldots \quad = \varphi(t,1)$