1) Find the sum of the following series

$X_1, X_2, X_3, \dots, X_i \in \mathbb{R}$

$$\sum_{i=1}^{\infty} x_i = \sum_{i\geq 1} x_i = \lim_{n\to\infty} \underbrace{\sum_{\varepsilon=1}^{n} x_\varepsilon}_{S_n} \quad , \quad S_n = X_1+X_2+\dots+X_n$$

a) $$\sum_{n\geq 1} \frac{2}{3^n} = \frac{2}{3} + \frac{2}{3^2} + \frac{2}{3^3} + \dots\dots =$$

$$1+r+r^2+\dots+r^n = \frac{1-r^{n+1}}{1-r} \qquad 1+r+r^2+\dots+r^n+\dots = \frac{1}{1-r}$$

$$= \frac{2}{3}\left(1+\frac{1}{3}+\frac{1}{3^2}+\dots+\frac{1}{3^n}\right) = \frac{2}{3}\cdot \lim_{n\to\infty} \frac{\left(\frac{1}{3}\right)^n - 1}{\frac{1}{3}-1} =$$

$$\lim_{n\to\infty} \sum_{i\geq 1} X_i = \lim_{n\to\infty} \frac{2}{3}\cdot\frac{\left(\frac{1}{3}\right)^n-1}{-\frac{2}{3}} = -\frac{\cancel{2}}{\cancel{3}}\cdot\frac{\cancel{3}}{-\cancel{2}} = 1$$

b) $$\sum_{n\geq 1} \frac{2n+1}{n!} = \quad , \quad e = \sum_{n\geq 0} \frac{1}{n!}$$

$$= \frac{3}{1!} + \frac{5}{2!} + \dots + \frac{2n+1}{n!} + \dots = \sum_{n\geq 1}\left(\frac{2n}{n!} + \frac{1}{n!}\right) =$$

$$= 2\sum_{n\geq 1}\frac{1}{(n-1)!} + \sum_{n\geq 1}\frac{1}{n!} = 2\cdot\sum_{n\geq 0}\frac{1}{n!} + \sum_{n\geq 1}\frac{1}{n!} =$$

$$= 2\cdot e + \sum_{n\geq 0}\frac{1}{n!} - 1 = 2\cdot e + e - 1 = 3\cdot e - 1$$

c) $\sum_{n\geq 1} \frac{1}{4n^2-1} = \sum_{n\geq 1} \frac{1}{(2n-1)(2n+1)} =$

$= \sum_{n\geq 1} \frac{1}{2}\left(\frac{1}{2n-1} - \frac{1}{2n+1}\right) = \lim_{n\to\infty} \frac{1}{2}\Big(\underbrace{\frac{1}{1} - \cancel{\frac{1}{3}}}_{n=1} + \underbrace{\cancel{\frac{1}{3}} - \cancel{\frac{1}{5}}}_{n=2} + \dots$

$+ \cancel{\frac{1}{2n-3}} - \cancel{\frac{1}{2n-1}} + \cancel{\frac{1}{2n-1}} - \frac{1}{2n+1}\Big) = \lim_{n\to\infty} \frac{1}{2}\left(1 - \frac{1}{2n+1}\right) = \frac{1}{2}$

d) $\sum_{n\geq 1} \frac{1}{n(n+1)(n+2)} = \frac{1}{2} \sum_{n\geq 1} \frac{n+2-n}{n(n+1)(n+2)}$

$= \frac{1}{2}\sum_{n\geq 1}\left(\frac{1}{n(n+1)} - \frac{1}{(n+1)(n+2)}\right) = \frac{1}{2}\sum_{n\geq 1} \frac{n+1-n}{n(n+1)} - \frac{(n+2)-(n+1)}{(n+1)(n+2)} =$

$= \frac{1}{2}\sum_{n\geq 1}\left(\frac{1}{n} - \frac{1}{n+1} - \frac{1}{n+1} + \frac{1}{n+2}\right) =$

$= \frac{1}{2}\sum_{n\geq 1}\left(\frac{1}{n} - \frac{1}{n+1}\right) - \frac{1}{2}\sum_{n\geq 1}\left(\frac{1}{n+1} - \frac{1}{n+2}\right) =$

$= \frac{1}{2}\lim_{n\to\infty}\Big(1 - \underbrace{\frac{1}{n+1}}_{\to 0}\Big) - \frac{1}{2}\lim_{n\to\infty}\Big(\frac{1}{2} - \underbrace{\frac{1}{n+2}}_{\to 0}\Big) = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$

Method 2: $\frac{1}{2}\sum_{n\geq 1}\Big(\underset{\substack{1\cdot 2\\ 2\cdot 3\\ 3\cdot 4}}{\frac{1}{n(n+1)}} - \underset{\substack{2\cdot 3\\ 3\cdot 4}}{\frac{1}{(n+1)(n+2)}}\Big) =$

$= \frac{1}{2}\sum_{n\geq 0}\frac{1}{(n+1)(n+2)} - \sum_{n\geq 1}\frac{1}{(n+1)(n+2)} =$

$= \frac{1}{2}\left(\frac{1}{1\cdot 2} + \sum_{n\geq 1} \cancel{\frac{1}{(n+1)(n+2)}} - \sum_{n\geq 1} \cancel{\frac{1}{(n+1)(n+2)}}\right) = \frac{1}{4}$

k) $S = \sum_{n\geq 1} \frac{n}{2^n} = \frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \dots =$

$= \frac{1}{2} + \frac{1+1}{2^2} + \frac{2+1}{2^3} + \dots + \frac{n-1+1}{2^n} + \dots =$

$= \sum_{n\geq 1} \frac{1}{2^n} + \left(\frac{1}{2^2} + \frac{2}{2^3} + \dots \frac{n-1}{2^n} + \dots \right) =$

$= \sum_{n\geq 1} \frac{1}{2^n} + \frac{1}{2} \underbrace{\sum_{n\geq 1} \frac{n}{2^n}}_{= S} = 1 + \frac{1}{2} \cdot S$

$S = 1 + \frac{1}{2} S \Rightarrow \frac{1}{2} \cdot S = 1 \Rightarrow S = 2$

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method 2:

$\sum_{n\geq 1} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots = \frac{1}{2}\left(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots\right) =$

$= \frac{1}{2} \cdot \frac{1}{1 - \frac{1}{2}} = 1$

$S = \sum_{n\geq 1} \frac{n}{2^n} = \sum_{n\geq 1} \frac{n+1-1}{2^n} = 2 \underbrace{\sum_{n\geq 1} \frac{n+1}{2^{n+1}}}_{S - \frac{1}{2}} - \underbrace{\sum_{n\geq 1} \frac{1}{2^n}}_{=1}$

$\Rightarrow S = 2\left(S - \frac{1}{2}\right) - 1$

f) $\sum_{n\geq 1} \frac{1}{\sqrt[3]{n}} = \sum_{n\geq 1} \frac{1}{n^{\frac{1}{3}}} = \sum_{n\geq 1} \frac{1}{n} \to \infty \Rightarrow$

$\Rightarrow \sum_{n\geq 1} \frac{1}{\sqrt[3]{n}} \to \infty$

$$\sum_{n\geq 1} \frac{1}{n} = \infty = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\dots$$

$$\vee$$

$$1+\frac{1}{2}+\underbrace{\frac{1}{4}+\frac{1}{4}}_{\frac{1}{2}}+\underbrace{\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}}_{\frac{1}{2}}+\dots$$

3) Study if the following series are convergent or divergent

a) $\sum_{n \geq 2} \frac{1}{\ln n}$

$e^x \geq x, \forall x \geq 2$

$x \geq \ln x, \forall x \geq 2$

$\frac{1}{x} \leq \frac{1}{\ln x}$

$$\left.\begin{array}{l} \underbrace{\sum_{n \geq 2} \frac{1}{n}}_{\infty} \leq \sum_{n \geq 2} \frac{1}{\ln n} \end{array}\right\} \Rightarrow \sum_{n \geq 2} \frac{1}{\ln n} = \infty$$

$\Rightarrow$ divergent

Comparison test 1:

if $x_n \leq y_n$

if $\sum x_n$ div. $\Rightarrow \sum y_n$ div.

$\sum y_n$ conv. $\Rightarrow \sum x_n$ conv.

b) $\sum_{n\geq 1} \frac{1}{n\sqrt{n+1}} < \sum_{n\geq 1} \frac{1}{n\sqrt{n}} = \sum_{n\geq 1} \frac{1}{n\cdot n^{\frac{1}{2}}} =$

$= \sum_{n\geq 1} \frac{1}{n^{\frac{3}{2}}}$ convergent $\left(\frac{3}{2} > 1\right) \Rightarrow \sum_{n\geq 1} \frac{1}{n\sqrt{n+1}}$ convergent

$\sum_{n\geq 1} \frac{1}{n^p}$ convergent $\Leftrightarrow p > 1$

c) $\sum_{n\geq 1} \frac{\ln(1+\frac{1}{n})}{n}$

$\sum x_n$ conv $\Rightarrow \lim_{n\to\infty} x_n = 0$

Comparison test 2:

$\lim_{n\to\infty} \frac{x_n}{y_n} = l$

$l \in (0,\infty) \Rightarrow \sum x_n$ and $\sum y_n$ have the same nature

$l = \infty$ $\sum y_n$ div. $\Rightarrow \sum x_n$ div

$l = 0$ $\sum y_n$ conv $\Rightarrow \sum x_n$ conv.

Let $y_n = \frac{1}{n^2}$

$\lim_{n\to\infty} \frac{x_n}{y_n} = \lim_{n\to\infty} \frac{\frac{\ln(1+\frac{1}{n})}{n}}{\frac{1}{n^2}} = \lim_{n\to\infty} n\cdot \ln\left(1+\frac{1}{n}\right) =$

$\left.\begin{array}{l} = \lim_{n\to\infty} \ln\left(1+\frac{1}{n}\right)^n = \ln e = 1 \\ \sum_{n\geq 1} y_n = \sum_{n\geq 1} \frac{1}{n^2} \text{ convergent} \end{array}\right\} \Rightarrow \sum_{n\geq 1} x_n$ convergent

d) $\sum_{n\geq 1} \frac{n!}{n^n}$

d'Alembert

Ratio test

$\frac{x_{n+1}}{x_n} \to l$

$l < 1 \Rightarrow \sum x_n$ convergent

$l > 1 \Rightarrow \sum x_n$ divergent

$l = 1 \Rightarrow$ inconclusive

$$\frac{\frac{(n+1)!}{(n+1)^{n+1}}}{\frac{n!}{n^n}} = \frac{\cancel{(n+1)!}}{(n+1)^{\cancel{n+1}}} \cdot \frac{n^n}{\cancel{n!}} = \frac{n^n}{(n+1)^n} = \left(\frac{n}{n+1}\right)^n =$$

$$= \left(1 - \frac{1}{n+1}\right)^n = \frac{1}{e} < 1 \Rightarrow \sum_{n\geq 1} x_n \text{ is convergent}$$

e) $\sum_{n\geq 1} \left(\frac{n}{n+1}\right)^{n^2}$

Root test

$\sqrt[n]{x_n} \to l$

$l < 1 \Rightarrow \sum x_n$ convergent

$l > 1 \Rightarrow \sum x_n$ divergent

$l = 1 \Rightarrow$ inconclusive

$$\sqrt[n]{x_n} = \sqrt[n]{\left(\frac{n}{n+1}\right)^{n^2}} = \left[\left(\frac{n}{n+1}\right)^{n^{\cancel{2}}}\right]^{\frac{1}{\cancel{n}}} = \left(1 - \frac{1}{n+1}\right)^n \xrightarrow{n\to\infty}$$

ratio test

$$\to \frac{1}{e} < 1 \Rightarrow \sum_{n\geq 1} \left(\frac{n}{n+1}\right)^{n^2} \text{ convergent}$$

f) $\sum_{n\geq 1} \frac{1}{n \ln n}$

Condensation test: (x_n) decreasing $x_n > 0$

$\sum_{n\geq 1} x_n$ conv $\Leftrightarrow$ $\sum_{n\geq 0} 2^n \cdot x_{2^n}$ conv

$$\frac{1}{\cancel{2^n} \cdot \ln 2^n} \cdot \cancel{2^n} = \frac{1}{\ln 2^n} = \frac{1}{n \ln 2}$$

$$\sum_{n\geq 1} \frac{1}{n\ln 2} = \frac{1}{\ln 2} \cdot \sum_{n\geq 1} \underbrace{\frac{1}{n}}_{\infty} = \infty \Rightarrow \sum_{n\geq 1} \frac{1}{n \ln n} \text{ divergent}$$

Ex: $\sum \frac{n}{2^n}$

$$\sum_{k\geq 1}^{n} k \cdot x^k = x \cdot \sum k \cdot x^{k-1} =$$

$$1 + r + r^2 + \dots + r^n = \frac{1 - r^{n+1}}{1-r} \Big/ ()'$$

$$1 + 2r + \dots + n r^{n-1}, \quad r = \frac{1}{2}$$

$$= x \sum (x^k)' = x \cdot \Big(\underbrace{\sum_{k=1}^{n} x^k}_{S_n}\Big)'$$