$(G, \cdot)$ - group
- „$\cdot$" operation
- „$\cdot$" associative
- „$\cdot$" has a natural element
- $\forall x \in G$ invertible with respect to „$\cdot$"

$(G, \cdot)$ group, $H \subseteq G$
H subgroup of G if $(H, \cdot)$ group

Characterisation theorem:
$(G, \cdot)$-group, $H \subseteq G$

$$\underset{\text{(subgroup)}}{H \leq G} \Leftrightarrow \begin{array}{l} \text{(i) } H \neq \emptyset \\ \text{(ii) } \forall x, y \in H,\ x \cdot y \in H \\ \text{(iii) } \forall x \in H,\ x^{-1} \in H \end{array} \left\} \Leftrightarrow \begin{array}{l} \forall x, y \in H \\ x \cdot y^{-1} \in H \end{array} \right.$$

3.) Prove that $H = \{ z \in \mathbb{C} \mid |z| = 1 \}$ is a subgroup of $(\mathbb{C}^*, \cdot)$, but it is not a subgroup of $(\mathbb{C}, +)$.

i) $H \neq \emptyset$ because $i \in H$

ii) Let $z_1 z_2 \in H$ and $z_1 = a+bi$, $z_2 = c+di$, $a,b,c,d \in \mathbb{R}$

$$\sqrt{a^2+b^2} = \sqrt{c^2+d^2} = 1$$

$$z_1 \cdot z_2 = (a+bi)(c+di) = ac + adi + bci + bdi^2 =$$

$$= (ac-bd) + (ad+bc)i$$

$$|z_1 z_2| = \sqrt{(ac-bd)^2 + (ad+bc)^2} =$$

$$= \sqrt{a^2c^2 + b^2d^2 - 2abcd + a^2d^2 + b^2c^2 + 2abcd} =$$

$$= \sqrt{a^2c^2 + b^2d^2 + a^2d^2 + b^2c^2} = \sqrt{(a^2+b^2)(c^2+d^2)} = 1,$$

so $z_1 z_2 \in H$

iii) Let $z \in H$, $z = a+bi$

$$z^{-1} = \frac{1}{a+bi} = \frac{a-bi}{a^2-b^2i^2} = \frac{a-bi}{a^2+b^2} = \frac{a}{a^2+b^2} + \frac{-b}{a^2+b^2} i$$

$$|z^{-1}| = \sqrt{\left(\frac{a}{a^2+b^2}\right)^2 + \left(\frac{-b}{a^2+b^2}\right)^2} = \sqrt{\frac{a^2+b^2}{a^2+b^2}} = 1$$

so $z^{-1} \in H$

From all of these $H \cong (\mathbb{C}^*, \cdot)$

Let $z_1 = i$, $z_2 = i \in H \Rightarrow z_1 + z_2 = 2i \notin H$

5) $n \in \mathbb{N}$, $n \geq 2$. Prove that

i) $GL_n(\mathbb{C}) = \{A \in M_n(\mathbb{C}) \mid \det(A) \neq 0\}$ is a stable subset of the monoid $(M_n(\mathbb{C}), \cdot)$

ii) $(GL_n(\mathbb{C}), \cdot)$ is a group, called the general linear group of rank n

iii) $SL_n(\mathbb{C}) = \{A \in M_n(\mathbb{C}) \mid \det(A) = 1\}$ is a subgroup of the group $(GL_n(\mathbb{C}), \cdot)$.

(i) Let $x, y \in GL_n$, $x \cdot y \overset{?}{\in} GL_n$ (1)

$\det(xy) = \det x \cdot \det y$ (2)

(1),(2) $\Rightarrow$ concl.

(ii) We accept that $(A \cdot B) \cdot C = A \cdot (B \cdot C)$

$\det(I_n) = 1 \neq 0 \Rightarrow I_n \in GL_n(\mathbb{C})$

Let $A \in GL_n(\mathbb{C}) \Rightarrow \det A \neq 0$, we can say that

$A^{-1} = \frac{1}{\det A} \cdot A^* \Rightarrow GL_n(\mathbb{C})$ group

iii) $\det(I_n) = 1 \Rightarrow I_n \in SL_n(\mathbb{C}) \neq \emptyset$

• $\det A_1, A_2 \in SL_n(\mathbb{C}) \Rightarrow \det(A_1) = \det(A_2) = 1$

$\det(A_1 ; A_2) = 1 \Rightarrow A_1 A_2 \in SL_n$

$$\left.\begin{array}{l} \det A = 1 \neq \emptyset \Rightarrow \exists A^{-1} \text{ s.t. } A^{-1} \cdot A = I_n \\ \det I_n = \det A = 1 \end{array}\right\} \Rightarrow$$

$\Rightarrow \det A^{-1} = 1 \Rightarrow A^{-1} \in SL_n(\mathbb{C})$

$\Rightarrow SL_n(\mathbb{C}) \subseteq GL_n(\mathbb{C})$

$(R, +, \cdot)$ – ring if:

- $(R, +)$ – abilian group
- $(R, \cdot)$ – semigroup
- distributivity

$(R, +, \cdot)$ ring, $S \subseteq R$

S subring of R if $(S, +, \cdot)$ ring

charm. theorem. for subrings

$(R, +, \cdot)$ ring, $S \subseteq R$

$S \subseteq R$ (subring) $\Leftrightarrow$ (i) $S \neq \emptyset$

(ii) $(S,+) \subseteq (R,+)$

(iii) $(S,\cdot) \subseteq (R,\cdot)$

(ii) $\Rightarrow \forall\, x,y \in S$, $x-y \in S$ $\Leftrightarrow$ $\forall\, x,y \in S$ $x+y \in S$, $\forall x \in S$, $-x \in S$

(iii) $\Leftrightarrow \forall\, x,y \in S$, $x \cdot y \in S$

6. Show that the following sets are subrings of the corresponding rings:

i) $\mathbb{Z}[i] = \{a+bi \mid a,b \in \mathbb{Z}\}$ in $(\mathbb{C},+,\cdot)$

$1,2 \in \mathbb{Z} \Rightarrow 1+2i \in \mathbb{Z}[i] \Rightarrow \mathbb{Z}[i] \neq \emptyset$ (1)

Let $x,y \in \mathbb{Z}[i] \Rightarrow \begin{cases} x = a+bi, & a,b \in \mathbb{Z} \\ y = c+di, & c,d \in \mathbb{Z} \end{cases}$

$x-y = a+bi-c-di = a-c+(b-d)i$

$$\left.\begin{matrix} a,c \in \mathbb{Z} \Rightarrow a-c \in \mathbb{Z} \\ b,d \in \mathbb{Z} \Rightarrow b-d \in \mathbb{Z} \end{matrix}\right\} \Rightarrow x-y \in \mathbb{Z}[i], \forall x,y \in \mathbb{Z}[i] \quad (2)$$

$x \cdot y = (a+bi)(c+di) = ac + adi + bci + bdi^2 =$

$= ac + (ad+bc) \cdot i - bd$

$a, b, c, d \in \mathbb{Z} \Rightarrow \begin{cases} ac - bd \in \mathbb{Z} \\ ad + bc \in \mathbb{Z} \end{cases}$

$\Rightarrow xy \in \mathbb{Z}[i], \forall x, y \in \mathbb{Z}[i]$ (3)

(1), (2), (3) $\Rightarrow \mathbb{Z}[i] \subseteq (\mathbb{C}, +, \cdot)$

$(G_1, \cdot), (G_2, *)$ group

$f : G_1 \Rightarrow G_2$ group homomorphism

if $\forall x, y \in G_1 \quad f(x * y) = f(x) * f(y)$

$(R_1, +, \cdot), (R_2, \oplus, \odot)$ rings:

$f : R_1 \Rightarrow R_2$ ring homomorphism if:

$\forall x, y \in R_1 \quad f(x+y) = f(x) \oplus f(y)$

$f(x \cdot y) = f(x) \odot f(y)$

If R_1, R_2 unital rings then f is a unital ring homo if $f(1_A) = 1_B$

7. (ii) Let $g : \mathbb{C}^* \to GL_2(\mathbb{R})$ be defined by $g(a+bi) = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$. Show that g is a group homomorphism between $(\mathbb{C}^*, \cdot)$ and $(GL_2(\mathbb{R}, \cdot)$.

$$g((a+bi)(c+di)) = g(a+bi) \cdot g(c+di)$$

$$g((a+bi)(c+di)) = g(ac+adi+bci+bdi^2) =$$

$$= g(ac-bd+(ad+bc)i) = \begin{pmatrix} ac-bd & ad+bc \\ -ad-bc & ac-bd \end{pmatrix} (1)$$

$$g(a+bi) \cdot g(c+di) = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \begin{pmatrix} c & d \\ -d & c \end{pmatrix} =$$

$$= \begin{pmatrix} ac-bd & ad+bc \\ -ad-bc & ac-bd \end{pmatrix} = \begin{pmatrix} ac-bd & ad+bc \\ -(ac+bc) & ac-bd \end{pmatrix} (2)$$

$(1),(2) \Rightarrow g((a+bi)(c+di)) = g(a+bi) \cdot g(c+di) \Rightarrow$

$\Rightarrow g$ is a group homomorphism

10. Let $M = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \middle| \, a, b \in \mathbb{R} \right\} \subseteq M_2(\mathbb{R})$.

Show that $(M, +, \cdot)$ is a field isomorphic to $(\mathbb{C}, +, \cdot)$

M field $\begin{cases} (M, +) \text{ abelian group} \\ (M^*, \cdot) \text{ abelian group} \\ \text{distributivity} \end{cases}$

$x, y \in M \Rightarrow x + y \in M$

$x = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ $\qquad x + y = \begin{pmatrix} a+c & b+d \\ -b-d & a+c \end{pmatrix} \in M$

$y = \begin{pmatrix} c & d \\ -d & c \end{pmatrix}$

$(M_2(\mathbb{R}), +)$ is associative and com.

$(M, +)$ is associative and com.

$O_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in M$

$\forall x \in M \overset{?}{\Rightarrow} \exists -x \in M$

$x = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$

$-x = \begin{pmatrix} -a & -b \\ b & -a \end{pmatrix} \in M$

$x, y \in M \overset{?}{\Rightarrow} x \cdot y \in M$

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix}\begin{pmatrix} c & d \\ -d & c \end{pmatrix} = \begin{pmatrix} ac-bd & ad+bc \\ -(bc+ad) & ac-bd \end{pmatrix} \in M$$

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in M$$

$$X \in M \Rightarrow X = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

$$\det X = a^2+b^2$$

$$X^* = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

$$X^{-1} = \frac{1}{\det X} \cdot X^* = \frac{1}{a^2+b^2} \cdot \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \in M$$

distributivity is inherited from $M_2(\mathbb{R})$

$\Rightarrow$ concl.

We had to show that $M \cong \mathbb{C}$

we want to find a homomorphism

$f: M \to \mathbb{C}$

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix} \to a+bi$$