Seminar 8 exercises:

1. $A = \begin{pmatrix} 1 & 4 & 2 \\ 2 & 3 & 1 \\ 3 & 0 & -1 \end{pmatrix} \qquad X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \qquad B = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$

$\det A = -3 + 12 + 0 - 18 + 8 + 0 = -1 \neq 0 \Rightarrow$

$\Rightarrow A$ is invertible

$$A = \left(\begin{array}{ccc|ccc} 1 & 4 & 2 & 1 & 0 & 0 \\ 2 & 3 & 1 & 0 & 1 & 0 \\ 3 & 0 & -1 & 0 & 0 & 1 \end{array}\right) \quad \begin{array}{l} L_2 \leftarrow L_2 - 2L_1 \\ L_3 \leftarrow L_3 - 3L_1 \end{array}$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 4 & 2 & 1 & 0 & 0 \\ 0 & -5 & -3 & -2 & 1 & 0 \\ 0 & -12 & -7 & -3 & 0 & 1 \end{array}\right) \quad L_2 \leftarrow -\frac{1}{5} L_2$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 4 & 2 & 1 & 0 & 0 \\ 0 & 1 & \frac{3}{5} & \frac{2}{5} & -\frac{1}{5} & 0 \\ 0 & -12 & -7 & -3 & 0 & 1 \end{array}\right) \quad L_3 \leftarrow L_3 + 12L_2$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 4 & 2 & 1 & 0 & 0 \\ 0 & 1 & \frac{3}{5} & \frac{2}{5} & -\frac{1}{5} & 0 \\ 0 & 0 & \frac{1}{5} & \frac{9}{5} & -\frac{12}{5} & 1 \end{array}\right) \quad L_3 \leftarrow L_3 \cdot 5$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 4 & 2 & 1 & 0 & 0 \\ 0 & 1 & \frac{3}{5} & \frac{2}{5} & -\frac{1}{5} & 0 \\ 0 & 0 & 1 & 9 & -12 & 5 \end{array}\right) \begin{array}{l} L_1 \leftarrow L_1 - 2L_3 \\ L_2 \leftarrow L_2 - \frac{3}{5}L_3 \end{array}$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 4 & 0 & -17 & 24 & -10 \\ 0 & 1 & 0 & -5 & 7 & -3 \\ 0 & 0 & 1 & 9 & -12 & 5 \end{array}\right) \quad L_1 \leftarrow L_1 - 4L_2$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -4 & 2 \\ 0 & 1 & 0 & -5 & 7 & -3 \\ 0 & 0 & 1 & 9 & -12 & 5 \end{array}\right)$$

$$A \cdot A^{-1} = \begin{pmatrix} 1 & 4 & 2 \\ 2 & 3 & 1 \\ 3 & 0 & -1 \end{pmatrix} \begin{pmatrix} 3 & -4 & 2 \\ -5 & 7 & -3 \\ 9 & -12 & 5 \end{pmatrix} =$$

$$= \begin{pmatrix} 1\cdot 3 + 4\cdot(-5) + 2\cdot 9 & 1\cdot(-4) + 4\cdot 7 + 2\cdot(-12) & 1\cdot 2 + 4\cdot(-3) + 2\cdot 5 \\ 2\cdot 3 + 3\cdot(-5) + 1\cdot 9 & 2\cdot(-4) + 3\cdot 7 + 1\cdot(-12) & 2\cdot 2 + 3\cdot(-3) + 1\cdot 5 \\ 3\cdot 3 + 0\cdot(-5) + (-1)\cdot 9 & 3\cdot(-4) + 0\cdot 7 - 1\cdot(-12) & 3\cdot 2 + 0\cdot(-3) - 1\cdot 5 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I_3$$

$$AX = B$$

$$AX = \begin{pmatrix} x_1 + 4x_2 + 2x_3 \\ 2x_1 + 3x_2 + x_3 \\ 3x_1 \quad\quad - x_3 \end{pmatrix} = B$$

$$\begin{cases} x_1 + 4x_2 + 2x_3 = 1 \\ 2x_1 + 3x_2 + x_3 = 0 \\ 3x_1 - x_3 = 2 \end{cases}$$

$$\left(\begin{array}{ccc|c} 1 & 4 & 2 & 1 \\ 2 & 3 & 1 & 0 \\ 3 & 0 & -1 & 2 \end{array}\right) \underset{L_3 \leftarrow L_3 - 3L_1}{\overset{L_2 \leftarrow L_2 - 2L_1}{\sim}} \left(\begin{array}{ccc|c} 1 & 4 & 2 & 1 \\ 0 & -5 & -3 & -2 \\ 0 & -12 & -7 & -1 \end{array}\right)$$

$$\underset{L_2 \leftarrow -\frac{1}{5}L_2}{\sim} \left(\begin{array}{ccc|c} 1 & 4 & 2 & 1 \\ 0 & 1 & \frac{3}{5} & \frac{2}{5} \\ 0 & -12 & -7 & -1 \end{array}\right) \overset{L_3 \leftarrow L_3 + 12L_2}{\sim} \left(\begin{array}{ccc|c} 1 & 4 & 2 & 1 \\ 0 & 1 & \frac{3}{5} & \frac{2}{5} \\ 0 & 0 & -\frac{1}{5} & \frac{19}{5} \end{array}\right)$$

$$\begin{cases} x_1+4x_2+2x_3=1 \Leftrightarrow \\ x_2+\frac{3}{5}x_3=\frac{2}{5} \Rightarrow x_2+\frac{3\cdot 19}{5}=\frac{2}{5} \Leftrightarrow x_2=\frac{2}{5}-\frac{57}{5} \Leftrightarrow \\ -\frac{1}{5}x_3=\frac{19}{5} \quad /:5 \Leftrightarrow -x_3=19 \Rightarrow x_3=19 \end{cases}$$

$$\Leftrightarrow x_2=-\frac{55}{5}=-11$$

$$\Leftrightarrow x_1+4\cdot(-11)+2\cdot 19=1 \Leftrightarrow x_1-44+38=1 \Leftrightarrow$$

$$\Leftrightarrow x_1-6=1 \Rightarrow x_1=7$$

5. (ii) $\begin{cases} 2x+5y+z=7 \\ x+2y-z=3 \\ x+y-4z=2 \end{cases}$

$$\left(\begin{array}{ccc|c} 2 & 5 & 1 & 7 \\ 1 & 2 & -1 & 3 \\ 1 & 1 & -4 & 2 \end{array}\right) \overset{L_1 \leftrightarrow L_2}{\sim} \left(\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 2 & 5 & 1 & 7 \\ 1 & 1 & -4 & 2 \end{array}\right)$$

$$\underset{L_3 \leftarrow L_3 - L_1}{\overset{L_2 \leftarrow L_2 - 2L_1}{\sim}} \left(\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 1 & 3 & 1 \\ 0 & -1 & -3 & -1 \end{array}\right) \overset{L_3 \leftarrow L_3 + L_2}{\sim} \left(\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\begin{cases} x+2y-z=3 \Leftrightarrow x+2-6z-z=3 \Leftrightarrow x=1+7z \\ y+3z=1 \Rightarrow y=1-3z \Rightarrow y=1-3\alpha \\ z=\alpha \qquad\qquad x=1+7\alpha \end{cases}$$

2. (i) $\begin{cases} x_1 + x_2 + x_3 - 2x_4 = 5 \\ 2x_1 + x_2 - 2x_3 + x_4 = 1 \\ 2x_1 - 3x_2 + x_3 + 2x_4 = 3 \end{cases}$

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & -2 & 5 \\ 2 & 1 & -2 & 1 & 1 \\ 2 & -3 & 1 & 2 & 3 \end{array}\right) \underset{L_3 \leftarrow L_3 - 2L_1}{\overset{L_2 \leftarrow L_2 - 2L_1}{\sim}}$$

$$\sim \left(\begin{array}{cccc|c} 1 & 1 & 1 & -2 & 5 \\ 0 & -1 & -4 & 5 & -9 \\ 0 & -5 & -1 & 6 & -7 \end{array}\right) \overset{L_3 \leftarrow L_3 - 5L_2}{\sim}$$

$$\sim \left(\begin{array}{cccc|c} 1 & 1 & 1 & -2 & 5 \\ 0 & -1 & -4 & 5 & -9 \\ 0 & 0 & 19 & -19 & 38 \end{array}\right)$$

$\begin{cases} x_1 + x_2 + x_3 - 2x_4 = 5 \\ -x_2 - 4x_3 + 5x_4 = -9 \Leftrightarrow -x_2 - 8 - 4\alpha + 5\alpha = -9 \\ 19x_3 - 19x_4 = 38 \quad /:19 \Leftrightarrow x_3 - x_4 = 2 \end{cases}$

$x_3 = 2 + \alpha$

$x_4 = \alpha$

$$\Leftrightarrow \quad x_1 + 1 + \cancel{\alpha} + 2 + \cancel{\alpha} - \cancel{2\alpha} = 5 \Rightarrow x_1 = 2$$

$$-x_2 + \alpha = -1 \Rightarrow x_2 = 1 + \alpha$$

(ii) $$\begin{cases} x_1 - 2x_2 + x_3 + x_4 = 1 \\ x_1 - 2x_2 + x_3 - x_4 = -1 \\ x_1 - 2x_2 + x_3 + 5x_4 = 5 \end{cases}$$

$$\left(\begin{array}{cccc|c} 1 & -2 & 1 & 1 & 1 \\ 1 & -2 & 1 & -1 & -1 \\ 1 & -2 & 1 & 5 & 5 \end{array}\right) \begin{array}{l} L_2 \leftarrow L_2 - L_1 \\ L_3 \leftarrow L_3 - L_1 \end{array}$$

$$\left(\begin{array}{cccc|c} 1 & -2 & 1 & 1 & 1 \\ 0 & 0 & 0 & -2 & -2 \\ 0 & 0 & 0 & 4 & 4 \end{array}\right)$$

$$\begin{cases} x_1 - 2x_2 + x_3 + x_4 = 1 \Rightarrow x_1 = 2\alpha - \beta - 1 \\ -2x_4 = -2 \Rightarrow x_4 = 1 \\ 4x_4 = 4 \Rightarrow x_4 = 1 \end{cases}$$

$x_2 = \alpha$

$x_3 = \beta$

Seminar 9:

1. Rank $A = ?$

$$A = \begin{pmatrix} 0 & 2 & 3 \\ 2 & 4 & 3 \\ 1 & 1 & 1 \\ 2 & 2 & 4 \end{pmatrix} \underset{L_1 \leftrightarrow L_3}{\sim} \begin{pmatrix} 1 & 1 & 1 \\ 2 & 4 & 3 \\ 0 & 2 & 3 \\ 2 & 2 & 4 \end{pmatrix} \underset{L_3 \leftarrow L_3 - 2L_1}{\overset{L_2 \leftarrow L_2 - 2L_1}{\sim}}$$

$$\sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & 2 \end{pmatrix} \underset{L_3 \leftarrow L_3 - L_2}{\sim} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 2 \end{pmatrix} \underset{L_4 \leftarrow L_4 - L_3}{\sim}$$

$$\sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \text{Rank } A = 3$$

5.

$$A = \begin{pmatrix} 1 & 4 & 2 \\ 2 & 3 & 1 \\ 3 & 0 & -1 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & 4 & 2 & 1 & 0 & 0 \\ 2 & 3 & 1 & 0 & 1 & 0 \\ 3 & 0 & -1 & 0 & 0 & 1 \end{array}\right) \quad \begin{array}{l} L_2 \leftarrow L_2 - 2L_1 \\ \sim \\ L_3 \leftarrow L_3 - 3L_1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 4 & 2 & 1 & 0 & 0 \\ 0 & -5 & -3 & -2 & 1 & 0 \\ 0 & -12 & -7 & -3 & 0 & 1 \end{array}\right) \quad \begin{array}{l} L_2 \leftarrow -\frac{1}{5}L_2 \\ \sim \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 4 & 2 & 1 & 0 & 0 \\ 0 & 1 & \frac{3}{5} & \frac{2}{5} & -\frac{1}{5} & 0 \\ 0 & -12 & -7 & -3 & 0 & 1 \end{array}\right) \quad \begin{array}{l} L_3 \leftarrow L_3 + 12L_2 \\ \sim \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 4 & 2 & 1 & 0 & 0 \\ 0 & 1 & \frac{3}{5} & \frac{2}{5} & -\frac{1}{5} & 0 \\ 0 & 0 & \frac{1}{5} & 9 & 0 & 1 \end{array}\right) \quad \begin{array}{l} L_3 \leftarrow L_3 \cdot 5 \\ \sim \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 4 & 2 & 1 & 0 & 0 \\ 0 & 1 & \frac{3}{5} & \frac{2}{5} & -\frac{1}{5} & 0 \\ 0 & 0 & 1 & \frac{9}{5} & 0 & \frac{1}{5} \end{array}\right)$$

$L_2 \leftarrow L_2 - \frac{3}{5} L_3$

$L_1 \leftarrow L_1 - 2 L_3$

$$\left(\begin{array}{ccc|ccc} 1 & 4 & 0 & -\frac{13}{5} & 0 & -\frac{2}{5} \\ 0 & 1 & 0 & -\frac{17}{25} & -\frac{1}{5} & -\frac{3}{25} \\ 0 & 0 & 1 & \frac{9}{5} & 0 & \frac{1}{5} \end{array}\right)$$

$L_1 \leftarrow L_1 - 4 L_2$

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{I_3} \Bigg| \underbrace{\begin{pmatrix} \frac{3}{25} & \frac{4}{5} & \frac{2}{25} \\ -\frac{17}{25} & -\frac{1}{5} & -\frac{3}{25} \\ \frac{9}{5} & 0 & \frac{1}{5} \end{pmatrix}}_{A^{-1}}$$

7. $X = (v_1, v_2, v_3, v_4)$

$v_1 = (1,0,4)$, $v_2 = (2,1,0)$, $v_3 = (1,5,-36)$, $v_4 = (2,10,-72)$

$\dim \langle X \rangle = ?$ basis of $\langle X \rangle = ?$

$$X = \begin{pmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & 5 & 10 \\ 4 & 0 & -36 & -72 \end{pmatrix} \underset{L_3 \leftarrow L_3 - 4L_1}{\sim} \begin{pmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & 5 & 10 \\ 0 & -8 & -40 & -80 \end{pmatrix}$$

$$\underset{L_3 \leftarrow L_3 + 8L_2}{\sim} \begin{pmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & 5 & 10 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

basis of $\langle X \rangle = \big((1,2,1,2) , (0,1,5,10) \big)$

$\dim \langle X \rangle = 2$

$$X = \begin{pmatrix} 1 & 0 & 4 \\ 2 & 1 & 0 \\ 1 & 5 & -36 \\ 2 & 10 & -72 \end{pmatrix} \underset{\substack{L_3 \leftarrow L_3 - L_1 \\ L_4 \leftarrow L_4 - 2L_1}}{\overset{L_2 \leftarrow L_2 - 2L_1}{\sim}} \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & -8 \\ 0 & 5 & -40 \\ 0 & 10 & -80 \end{pmatrix}$$

$$\underset{L_4 \leftarrow L_4 - 10L_2}{\overset{L_3 \leftarrow L_3 - 5L_2}{\sim}} \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & -8 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

basis of $\langle X \rangle = \big((1,0,4), (0,1,-8) \big)$

$\dim \langle X \rangle = 2$

8. $X = (v_1, v_2, v_3)$

$v_1 = (1, 0, 4, 3)$, $v_2 = (0, 2, 3, 1)$, $v_3 = (0, 4, 6, 2)$

$\dim \langle X \rangle = ?$, basis of $\langle X \rangle = ?$

$$X = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 4 \\ 4 & 3 & 6 \\ 3 & 1 & 2 \end{pmatrix} \underset{L_4 \leftarrow L_4 - 3L_1}{\overset{L_3 \leftarrow L_3 - 4L_1}{\sim}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 4 \\ 0 & 3 & 6 \\ 0 & 1 & 2 \end{pmatrix}$$

$$\overset{L_2 \leftarrow \frac{1}{2} L_2}{\sim} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 3 & 6 \\ 0 & 1 & 2 \end{pmatrix} \underset{L_4 \leftarrow L_4 - L_2}{\overset{L_3 \leftarrow L_3 - 3L_2}{\sim}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

basis of $\langle X \rangle = \{ \underbrace{(1, 0, 4, 3)}_{v_1}, \underbrace{(0, 2, 3, 1)}_{v_2} \}$

$\dim \langle X \rangle = 2$

$$X = \begin{pmatrix} 1 & 0 & 4 & 3 \\ 0 & 2 & 3 & 1 \\ 0 & 4 & 6 & 2 \end{pmatrix} \underset{\sim}{L_3 \leftarrow L_3 - 2L_2} \begin{pmatrix} 1 & 0 & 4 & 3 \\ 0 & 2 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

basis of $\langle X \rangle = \big((1,0,4,3), (0,2,3,1) \big)$

$\dim \langle X \rangle = 2$

10. Determine the dimension of the subspaces $S, T, S+T$ and $S \cap T$ of the real vector space $\mathbb{R}^4$ and a basis for the first three of them, where:

$$S = \langle (1,2,-1,-2), (3,1,1,1), (-1,0,1,-1) \rangle$$

$$T = \langle (2,5,-6,-5), (-1,2,-7,-3) \rangle$$

$$S = \begin{pmatrix} 1 & 2 & -1 & -2 \\ 3 & 1 & 1 & 1 \\ -1 & 0 & 1 & -1 \end{pmatrix} \begin{matrix} L_2 \leftarrow L_2 - 3L_1 \\ L_3 \leftarrow L_3 + L_1 \end{matrix}$$

$$\begin{pmatrix} 1 & 2 & -1 & -2 \\ 0 & -5 & 4 & 7 \\ 0 & 2 & 0 & -3 \end{pmatrix} \xrightarrow{L_2 \leftarrow \frac{1}{5} L_2} \begin{pmatrix} 1 & 2 & -1 & -2 \\ 0 & -1 & \frac{4}{5} & \frac{7}{5} \\ 0 & 2 & 0 & -3 \end{pmatrix} \xrightarrow{L_3 \leftarrow L_3 + 2L_2}$$

$$\begin{pmatrix} 1 & 2 & -1 & -2 \\ 0 & -1 & \frac{4}{5} & \frac{7}{5} \\ 0 & 0 & \frac{8}{5} & -\frac{1}{5} \end{pmatrix} \Rightarrow$$

$\Rightarrow$ basis of $S = \left((1,2,-1,-2), \left(0,-1,\frac{4}{5},\frac{7}{5}\right), \left(0,0,\frac{8}{5},-\frac{1}{5}\right) \right)$

$\dim S = 3$

$$T = \begin{pmatrix} 2 & 5 & -6 & -5 \\ -1 & 2 & -7 & -3 \end{pmatrix} \xrightarrow{L_1 \leftrightarrow L_2} \begin{pmatrix} -1 & 2 & -7 & -3 \\ 2 & 5 & -6 & -5 \end{pmatrix}$$

$$\xrightarrow{L_2 \leftarrow L_2 + 2L_1} \begin{pmatrix} -1 & 2 & -7 & -3 \\ 0 & 9 & -20 & -11 \end{pmatrix}$$

$\Rightarrow$ basis of $T = \big((-1,2,-7,-3),(0,9,-20,-11)\big)$

$\dim T = 2$

$$S+T = \begin{pmatrix} 1 & 2 & -1 & -2 \\ 0 & -1 & \frac{4}{5} & \frac{7}{5} \\ 0 & 0 & \frac{8}{5} & -\frac{1}{5} \\ -1 & 2 & -7 & -3 \\ 0 & 9 & -20 & -11 \end{pmatrix} \xrightarrow{L_4 \leftarrow L_4 + L_1}$$

$$\begin{pmatrix} 1 & 2 & -1 & -2 \\ 0 & -1 & \frac{4}{5} & \frac{7}{5} \\ 0 & 0 & \frac{8}{5} & -\frac{1}{5} \\ 0 & 4 & -8 & -5 \end{pmatrix} \xrightarrow{L_4 \leftarrow L_4 + 4L_2} \begin{pmatrix} 1 & 2 & -1 & -2 \\ 0 & -1 & \frac{4}{5} & \frac{7}{5} \\ 0 & 0 & \frac{8}{5} & -\frac{1}{5} \\ 0 & 0 & -\frac{24}{5} & \frac{3}{5} \end{pmatrix}$$

$$\xrightarrow[L_4 \leftarrow L_4 \cdot \frac{1}{5}]{L_3 \leftarrow L_3 \cdot \frac{1}{5}} \begin{pmatrix} 1 & 2 & -1 & -2 \\ 0 & -1 & \frac{4}{5} & \frac{7}{5} \\ 0 & 0 & 8 & -1 \\ 0 & 0 & -24 & 3 \end{pmatrix} \xrightarrow{L_4 \leftarrow L_4 + 3L_3} \begin{pmatrix} 1 & 2 & -1 & -2 \\ 0 & -1 & \frac{4}{5} & \frac{7}{5} \\ 0 & 0 & 8 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

basis of $S+T = \left((1,2,-1,-2), (0,-1,\frac{4}{5},\frac{7}{5}), (0,0,8,-1) \right)$

$\dim S+T = 3$

$$\overbrace{\dim S+T}^{=3} = \overbrace{\dim S}^{=3} + \overbrace{\dim T}^{=2} - \dim S \cap T$$

$$\Rightarrow \quad \dim S \cap T = 2$$

Seminar 10:

1. $f \in \mathrm{End}_{\mathbb{R}}(\mathbb{R}^3)$, $f(x,y,z) = (x+y, y-z, 2x+y+z)$

$[f]_E = ?$ $\quad E = (e_1, e_2, e_3)$ – canonical basis for $\mathbb{R}^3$

$f(e_1) = f(1,0,0) = (1,0,2)$
$f(e_2) = f(0,1,0) = (1,1,1)$
$f(e_3) = f(0,0,1) = (0,-1,1)$

$$[f]_E = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 2 & 1 & 1 \end{pmatrix}$$

3. $f \in Hom_{\mathbb{R}}(\mathbb{R}^3, \mathbb{R}^4)$ defined by

$f(e_1) = (1,2,3,4)$, $f(e_2) = (4,3,2,1)$, $f(e_3) = (-2,1,4,1)$

on the elements of the canonical basis of $\mathbb{R}^3$

Determine :

i) $f(v)$ for every $v \in \mathbb{R}^3$, $v = (x_1, x_2, x_3)$

$$f(v) = x_1 f(e_1) + x_2 \cdot f(e_2) + x_3 \cdot f(e_3) =$$

$$= x_1(1,2,3,4) + x_2(4,3,2,1) + x_3(-2,1,4,1) =$$

$$= (x_1 + 4x_2 - 2x_3, 2x_1 + 3x_2 + x_3, 3x_1 + 2x_2 + 4x_3, 4x_1 + x_2 + x_3)$$

ii) the matrix of f in the canonical bases

$E' = (e_1', e_2', e_3') = ((1,0,0), (0,1,0), (0,0,1))$

$f(e_1') = (1,2,3,4)$

$f(e_2') = (4,3,2,1)$

$f(e_3') = (-2,1,4,1)$

$$[f]_{E,E'} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \\ -2 & 1 & 4 & 1 \end{pmatrix}$$

iii) a basis and the dimension of Ker f and Im f

Let $u \in \mathbb{R}$, $u = (x, y, z, t)$

$u \in \operatorname{Ker} f \Leftrightarrow [f]_{EE'} \cdot [u]_E = 0$

$$\left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 0 \\ 4 & 3 & 2 & 1 & 0 \\ -2 & 1 & 4 & 1 & 0 \end{array}\right) \underset{L_3 \leftarrow L_3 + 2L_1}{\overset{L_2 \leftarrow L_2 - 4L_1}{\sim}} \left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 0 \\ 0 & -5 & -10 & -15 & 0 \\ 0 & 5 & 10 & 9 & 0 \end{array}\right)$$

$$\overset{L_3 \leftarrow L_3 + L_2}{\sim} \left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 0 \\ 0 & -5 & -10 & -15 & 0 \\ 0 & 0 & 0 & -6 & 0 \end{array}\right)$$

$$\begin{cases} x + 2y + 3z + 4t = 0 \Leftrightarrow x - 4\alpha + 3\alpha = 0 \Rightarrow x = \alpha \\ -5y - 10z - 15t = 0 \Leftrightarrow -5y - 10\alpha = 0 \Leftrightarrow -5y = 10\alpha \\ -6t = 0 \Rightarrow t = 0 \end{cases}$$

$5y = -10\alpha \mid :5$

$y = -2\alpha$

Let $z = \alpha$

$\Rightarrow \operatorname{Ker} f = \{(\alpha, -2\alpha, \alpha, 0) \mid \alpha \in \mathbb{R}\}$

$(\alpha, -2\alpha, \alpha, 0) = \alpha(1, -2, 1, 0)$

$\operatorname{Ker} f = \langle (1, -2, 1, 0) \rangle$

$\dim \operatorname{Ker} f = 1$

$$\operatorname{Im} f = \{ w = (a,b,c) \mid \exists\, u = (x,y,z,t),\ f(u) = w \} =$$

$$= \{ w \mid \exists\, u \in \mathbb{R}^4 : [f]_{EE'} \cdot [u]_E = [w]_E \} =$$

$$= \left\{ w = (a,b,c) \mid \exists\, u = (x,y,z,t) : \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \\ -2 & 1 & 4 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \right.$$

$$\left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & a \\ 4 & 3 & 2 & 1 & b \\ -2 & 1 & 4 & 1 & c \end{array}\right) \underset{L_3 \leftarrow L_3 + 2L_1}{\overset{L_2 \leftarrow L_2 - 4L_1}{\sim}} \left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & a \\ 0 & -5 & -10 & -15 & b-4a \\ 0 & 5 & 10 & 9 & c+2a \end{array}\right)$$

$$\underset{L_3 \leftarrow L_3 + L_2}{\sim} \left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & a \\ 0 & -5 & -10 & -15 & b-4a \\ 0 & 0 & 0 & -6 & c-2a+b \end{array}\right)$$

???

5. $\mathbb{R}_2[X] = \{ f \in \mathbb{R}[X] \mid \text{degree}(f) \leq 2 \}$ and its bases $E = (1, X, X^2)$ and $B = (1, X-1, X^2+1)$.

Consider $\varphi \in \text{End}_{\mathbb{R}}(\mathbb{R}_2[X])$ defined by:

$$\varphi(a_0 + a_1X + a_2X^2) = (a_0+a_1) + (a_1+a_2)X + (a_0+a_2)X^2$$

Determine the matrices $[\varphi]_E$ and $[\varphi]_B$

$$\left.\begin{aligned} \varphi(1) &= 1 + X^2 \\ \varphi(X) &= 1 + X \\ \varphi(X^2) &= X + X^2 \end{aligned}\right\} \Rightarrow [\varphi]_E = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

$$\left.\begin{aligned} \varphi(1) &= 1 + X^2 \\ \varphi(X-1) &= X - X^2 \\ \varphi(X^2+1) &= 1 + X + 2X^2 \end{aligned}\right\} \Rightarrow [\varphi]_B = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & 1 & 2 \end{pmatrix}$$

Seminar 11:

3. In the real vector space $\mathbb{R}_2[X] = \{f \in \mathbb{R}[X] \mid \text{degree}(f) \le 2\}$ consider the bases $E = (1, X, X^2)$, $B = (1, X-a, (X-a)^2)$ $(a \in \mathbb{R})$ and $B' = (1, X-b, (X-b)^2)$ $(b \in \mathbb{R})$.

Determine the matrices of change of bases $T_{EB}, T_{BE}, T_{BB'}$

$$[1]_E = (1,0,0)$$

$$[X-a]_E = (-a,1,0) \Rightarrow T_{EB} = \begin{pmatrix} 1 & 0 & 0 \\ -a & 1 & 0 \\ a^2 & -2a & 1 \end{pmatrix}$$

$$[(X-a)^2]_E = (a^2, -2a, 1)$$

$$\left.\begin{array}{l} [1]_B = (1,0,0) \\ [X]_B = X-a = (-a,1,0) \\ [X^2]_B = (X-a)^2 = X^2 - 2aX + a^2 = (a^2, -2a, 1) \end{array}\right\} \Rightarrow$$

$$\Rightarrow T_{BE} = \begin{pmatrix} 1 & 0 & 0 \\ -a & 1 & 0 \\ a^2 & -2a & 1 \end{pmatrix}$$

$$T_{BB'} = \begin{pmatrix} 1 & 0 & 0 \\ -b & 1 & 0 \\ b^2 & -2b & 1 \end{pmatrix}$$

$$[1]_B = (1,0,0) \quad [X-b]_B = (-b,1,0) \quad [(X-b)^2]_B = (b^2,-2b,1)$$

6. Compute the eigenvalues and the eigenvectors

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$P_A(X) = \begin{vmatrix} -\lambda & 0 & 0 & 1 \\ 0 & -\lambda & 1 & 0 \\ 0 & 1 & -\lambda & 0 \\ 1 & 0 & 0 & -\lambda \end{vmatrix} = 0$$

$$= -\lambda \cdot \begin{vmatrix} -\lambda & 1 & 0 \\ 1 & -\lambda & 0 \\ 0 & 0 & -\lambda \end{vmatrix} - 1 \cdot \begin{vmatrix} 0 & -\lambda & 1 \\ 0 & 1 & -\lambda \\ 1 & 0 & 0 \end{vmatrix} =$$

$$= -\lambda(-\lambda^3 + \lambda) - (\lambda^2 - 1) =$$

$$= \lambda^2(\lambda^2 - 1) - (\lambda^2 - 1) = (\lambda^2 - 1)(\lambda^2 - 1) = (\lambda^2 - 1)^2$$

$$(\lambda^2 - 1)^2 = 0 \Leftrightarrow \lambda^2 - 1 = 0 \Leftrightarrow \lambda^2 = 1 \Leftrightarrow \begin{array}{l} \lambda_1 = 1 \\ \lambda_2 = -1 \end{array}$$

if $\lambda = \lambda_1 = 1$

$$\begin{pmatrix} -1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = 1 \cdot \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix}$$

$$\begin{cases} -x+t = x \Rightarrow -2t+t = 2t \Rightarrow t=0 \\ -y+z = y \Rightarrow -2z+z = 2z \Rightarrow z=0 \\ y-z = z \Rightarrow y = 2z = 0 \\ x-t = t \Rightarrow x = 2t = 0 \end{cases}$$

$$S(\lambda_1) = \langle (0,0,0,0) \rangle$$

if $\lambda = \lambda_2 = -1$

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = -1 \cdot \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix}$$

$x+t = -x \Rightarrow -2t+t = 2t \Rightarrow t=0$

$y+z = -y \Rightarrow -2z+z = 2z \Rightarrow z=0$

$y+z = -z \Rightarrow y = -2z = 0$

$x+t = -t \Rightarrow x = -2t \Rightarrow x=0$

$$S(\lambda_2) = \langle (0,0,0,0) \rangle$$

7.
$$A = \begin{pmatrix} x & 0 & y \\ 0 & x & 0 \\ y & 0 & x \end{pmatrix}$$

$$P_A(x) = \begin{vmatrix} x-\lambda & 0 & y \\ 0 & x-\lambda & 0 \\ y & 0 & x-\lambda \end{vmatrix} = (x-\lambda)^3 - (x-\lambda)\cdot y^2 =$$

$$= (x-\lambda)\left((x-\lambda)^2 - y^2\right)$$

$P_A(x) = 0 \Rightarrow$ I $x - \lambda = 0 \Rightarrow \lambda = x$

II $(x-\lambda)^2 - y^2 = 0 \Leftrightarrow (x-\lambda)^2 = y^2 \mid \sqrt{\ } \Leftrightarrow$

$\Leftrightarrow x - \lambda = y \Rightarrow -\lambda = y - x \Rightarrow \lambda = x - y$

if $\lambda = x$

$$\begin{pmatrix} 0 & 0 & y \\ 0 & 0 & 0 \\ y & 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = x \cdot \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

$$\begin{cases} y \cdot v_3 = x \cdot v_1 \\ y \cdot v_1 = x \cdot v_3 \end{cases}$$

$y \cdot v_3 = x \cdot v_1 \Leftrightarrow y \cdot v_3 = x \cdot \frac{x \cdot v_3}{y} \Leftrightarrow y^2 \cdot v_3 = x^2 \cdot v^3$

$y \cdot v_1 = x \cdot v_3 \Rightarrow v_1 = \frac{x \cdot v_3}{y}$

$\Rightarrow v_1 = v_3$

$\Leftrightarrow y^2 = x^2$

$\Leftrightarrow y = x$

$$S(\lambda_1) = \{ (v_1, v_2, v_1) \mid v_1, v_2 \in \mathbb{R} \} = \langle (1,0,1), (0,1,0) \rangle$$

if $\lambda = x - y$

$$\begin{pmatrix} y & 0 & y \\ 0 & y & 0 \\ y & 0 & y \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = (x-y) \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

$$\begin{cases} y \cdot v_1 + y \cdot v_3 = (x-y) \cdot v_1 \Leftrightarrow y \cdot v_3 = 0 \Rightarrow v_3 = 0 \\ y \cdot v_2 = (x-y) \cdot v_2 \quad | : v_2 \Rightarrow x - y = y \Rightarrow x = 2y \\ y \cdot v_1 + y \cdot v_3 = (x-y) \cdot v_3 \Leftrightarrow y(v_1 + v_3) = y \cdot v_3 \quad | : y \end{cases}$$

$$\Leftrightarrow v_1 + v_3 = v_3$$

$$\Leftrightarrow v_1 = 0$$

$$S(\lambda_2) = \{ (0, v_2, 0) \mid v_2 \in \mathbb{R} \} = \langle (0,1,0) \rangle$$

Seminar 12:

7. Find G and H

The (4,1) - code generated by $p = 1+X+X^2+X^3 \in \mathbb{Z}_2[X]$

$m = \overline{1} = 1$

$g_m = 1 \cdot X^3 = X^3$

$$\begin{array}{l|l} X^3 & 1+X+X^2+X^3 \\ \hline X^3+X^2+X+1 & 1 \\ \hline X^2+X+1 & \end{array}$$

$\underbrace{X^2+X+1}_{= R_m}$

$R_m + g_m = X^3+X^2+X+1$

$v = 1111$

$$G = \left(\begin{array}{c} P \\ \hline I_k \end{array}\right) = \left(\begin{array}{c} 1 \\ 1 \\ 1 \\ \hline 1 \end{array}\right) \qquad P = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$H = \left(I_{n-k} \,\middle|\, P \right) = \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

4. A code is defined by the generator matrix $G = \left(\frac{P}{i_3}\right) \in \mathcal{M}_{5,3}(\mathbb{Z}_2)$, where $P = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

Write down the parity check matrix and all the code words.

$$G = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad H = \left(\begin{array}{cc|ccc} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 \end{array}\right)$$

$\Rightarrow 2^3 = 8$ codewords

[0 0 0]
[0 0 1]
[0 1 0]
[0 1 1]
[1 0 0]
[1 0 1]
[1 1 0]
[1 1 1]

$$G \cdot \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$G \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

$$G \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$G \cdot \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

$$G \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$G \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

$$G \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

$$G \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

3. Write down all the words in the (6,3)-code generated by $p = 1 + X^2 + X^3 \in \mathbb{Z}_2[X]$

i think the correct solution is to find G then multiply it by all the codewords

I $\quad 0 \cdot p = 0 \Rightarrow 000000$

II $\quad 1 \cdot p = 1 + X^2 + X^3 \Rightarrow 101100$

III $\quad X \cdot p = X + X^3 + X^4 \Rightarrow 010110$

IV $\quad (1+X) \cdot p = 1 + X^2 + X^3 + X + X^3 + X^4 = 1 + X + X^2 + 2X^3 + X^4$

$\Rightarrow 111010$

V $\quad X^2 \cdot p = X^2 + X^4 + X^5 \Rightarrow 001011$

VI $\quad (X^2+1) \cdot p = X^2 + X^4 + X^5 + 1 + X^2 + X^3 = 1 + 2X^2 + X^3 + X^4 + X^5$

$\Rightarrow 100111$

VII $\quad (X^2+X) \cdot p = X^2 + X^4 + X^5 + X + X^3 + X^4 = X + X^2 + X^3 + 2X^4 + X^5$

$\Rightarrow 011101$

VIII $\quad (X^2+X+1) \cdot p = X^2 + X^4 + X^5 + X + X^3 + X^4 + 1 + X^2 + X^3 =$

$= 1 + X + 2X^2 + 2X^3 + 2X^4 + X^5$

$\Rightarrow 110001$

2. Are $1+x^3+x^4+x^6+x^7$ and $x+x^2+x^3+x^6$ code words in the polynomial (8,4)-code generated by $p = 1+x^2+x^3+x^4 \in \mathbb{Z}_2[X]$

$$
\begin{array}{l|l}
1+x^3+x^4+x^6+x^7 & 1+x^2+x^3+x^4 \\
\hline
x^3+x^5+x^6+x^7 & x^3+x \\
\hline
1+x^4+x^5 & \\
\hline
x+x^3+x^4+x^5 & \\
\hline
1+x+x^3 &
\end{array}
$$

$\Rightarrow$ the remainder is not 0 so this is not a codeword

$$
\begin{array}{l|l}
x+x^2+x^3+x^6 & 1+x^2+x^3+x^4 \\
\hline
x^2+x^4+x^5+x^6 & x^2+x \\
\hline
x+x^3+x^4+x^5 & \\
\hline
x+x^3+x^4+x^5 & \\
\hline
0 &
\end{array}
$$

$\Rightarrow$ is a codeword

Seminar 13:

7. Construct a table of coset leaders and syndromes for the (3,1) code generated by $p = 1+X+X^2 \in \mathbb{Z}_2[X]$

$m_1 = 1$

$p_1 = 1$

$g_m = 1 \cdot X^2 = X^2$

$$\begin{array}{l|l} X^2 & 1+X+X^2 \\ \hline 1+X+X^2 & 1 \\ \hline 1+X & \end{array}$$

$R_m + g_m = 1+X+X^2$

$v = 111$

$$G = \left(\frac{P}{i_k}\right) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad H = (i_{n-k} \mid P) = \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \end{array}\right)$$

S	00	01	10	11
CL	000	010	100	001

3. A $(7,4)$-code is defined by the equations
$u_1 = u_4 + u_5 + u_7$, $u_2 = u_4 + u_6 + u_7$, $u_3 = u_4 + u_5 + u_6$,
where u_4, u_5, u_6, u_7 are the message digits and u_1, u_2, u_3 are the check digits. Write its generator matrix and parity check matrix. Decode the received words 0000111 and 0001111

$$u_1 = u_4 + u_5 + u_7$$
$$u_2 = u_4 + u_6 + u_7$$
$$u_3 = u_4 + u_5 + u_6$$

$$P = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

$$\Rightarrow G = \left(\frac{P}{I_k}\right) = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad H = \left(I_{n-k} \mid P\right) = \left(\begin{array}{ccc:cccc} 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{array}\right)$$

$$H \cdot \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \Rightarrow e = 0000000 \quad \Rightarrow e + v_1 = 0000111$$
$$\Rightarrow m = 0111$$

$$H \cdot \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad \Rightarrow e = 0001000 \Rightarrow e + v_2 = 0000111$$
$$\Rightarrow m = 0111$$

4. Find the syndromes of all the received words in the (3,2)-parity check code and in the (3,1)-repeating code.

(3,2)-parity check code

data	Parity bit	codeword	syndrome
0 0	0	000	0
0 1	1	011	1
1 0	1	101	1
1 1	0	110	0

(3,1)-repeating code

data	codeword	syndrome
0	000	0
1	111	0

$$\begin{pmatrix} 1 & a & 1 & \vdots & a \\ 0 & 1 & -1 & \vdots & -a \\ 0 & 1-a^2 & 1-a & \vdots & (1-a)^2 \end{pmatrix} \; L_3 \leftarrow L_3 - (1-a^2)L_2$$

$$\sim \begin{pmatrix} 1 & a & 1 & \vdots & a \\ 0 & 1 & -1 & \vdots & -a \\ 0 & 0 & 2-a-a^2 & \vdots & -a^3+a^2-a+1 \end{pmatrix}$$

if $2-a-a^2=0 \Leftrightarrow -a^2-a+2=0$

$\Delta = b^2-4ac = 1-4\cdot(-1)\cdot 2 = 9$

$a_{1,2} = \frac{1\pm 3}{-2}$ $\to a_1 = \frac{4}{-2} = -2$

$\to a_2 = \frac{-2}{-2} = 1$

if $a=1 \Rightarrow \begin{pmatrix} 1 & 1 & 1 & \vdots & 1 \\ 0 & 1 & -1 & \vdots & -1 \\ 0 & 0 & 0 & \vdots & 0 \end{pmatrix}$

$x+y+z=1 \Rightarrow x = 1+1-\alpha-\alpha = 2-2\alpha$

$y-z=-1 \Rightarrow y=-1+\alpha$

$z=\alpha$

if $a=-2 \Rightarrow \begin{pmatrix} 1 & -2 & 1 & \vdots & -2 \\ 0 & 1 & -1 & \vdots & 2 \\ 0 & 0 & 0 & \vdots & 11 \end{pmatrix} \Rightarrow$ the system is incompatible

ii) $2-a-a^2 \neq 0$

$$\Rightarrow \begin{cases} x + ay + z = a \\ y - z = -a \Rightarrow y = -a + \dfrac{-a^3+a^2-a+1}{2-a-a^2} \\ (2-a-a^2)z = -a^3+a^2-a+1 \\ \Rightarrow z = \dfrac{-a^3+a^2-a+1}{2-a-a^2} \end{cases}$$

$$x = -ay - z + a = a^2 + \frac{a^4-a^3+a^2-a}{2-a-a^2} - \frac{-a^3+a^2-a+1}{2-a-a^2} + a$$

$$\left(\begin{array}{cc|cc} -3 & -5 & 1 & 0 \\ 2 & 3 & 0 & 1 \end{array}\right) \overset{L_1 \leftrightarrow L_2}{\sim} \left(\begin{array}{cc|cc} 2 & 3 & 0 & 1 \\ -3 & -5 & 1 & 0 \end{array}\right)$$

$$\overset{L_2 \leftarrow \frac{1}{2}L_2}{\sim} \left(\begin{array}{cc|cc} 1 & \frac{3}{2} & 0 & \frac{1}{2} \\ -3 & -5 & 1 & 0 \end{array}\right) \overset{L_2 \leftarrow L_2+3L_1}{\sim} \left(\begin{array}{cc|cc} 1 & \frac{3}{2} & 0 & \frac{1}{2} \\ 0 & -\frac{1}{2} & 1 & \frac{3}{2} \end{array}\right)$$

$$\overset{L_1 \leftarrow L_1+3L_2}{\sim} \left(\begin{array}{cc|cc} 1 & 0 & 3 & 5 \\ 0 & -\frac{1}{2} & 1 & \frac{3}{2} \end{array}\right) \overset{L_2 \leftarrow -2L_2}{\sim} \left(\begin{array}{cc|cc} 1 & 0 & 3 & 5 \\ 0 & 1 & -2 & -3 \end{array}\right)$$

$r = x^2 + x \quad (5,3) \quad G, H = ? \quad P = ?$

$m = 100$

$f_m = 1$

$g_m = 1 \cdot x^2 = x^2$

$$\begin{array}{l|l} x^2 & x^2+x \\ \hline x^2+x & 1 \\ \hline x = r_m & \end{array}$$

$g_m + r_m = x + x^2 \Rightarrow 01100 \Rightarrow m = 100$

$H = (I_{n-k} \mid P)$

$H = \left(\begin{array}{cc|ccc} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 \end{array}\right)$

$m = 010$

$f_m = x$

$g_m = x \cdot x^2 = x^3$

$$\begin{array}{l|l} x^3 & x^2+x \\ \hline x^3+x^2 & x+1 \\ \hline x^2 & \\ x^2+x & \\ \hline x & \\ \shortparallel & \\ r_m & \end{array}$$

$g_m + r_m = x^3 + x$

$v = 01010 \Rightarrow m = 010$

$m = 001$

$f_m = x^2$

$g_m = x^2 \cdot x^2 = x^4$

$$\begin{array}{l|l} x^4 & x^2+x \\ \hline x^4+x^3 & x^2+x+1 \\ \hline x^3 & \\ x^3+x^2 & \\ \hline x^2 & \\ x^2+x & \\ \hline x = r_m & \end{array}$$

$g_m + r_m = x^4 + x$

$v = 01001 \Rightarrow m = 001$

$G, H = ?$ $(7,3)$ code

$p = 1 + x^2 + x^3 + x^4$

100

$l_m = 1$

$g_m = l_m \cdot x^{n-k} = 1 \cdot x^4 = x^4$

$$
\begin{array}{l|l}
x^4 & 1+x^2+x^3+x^4 \\
\hline
1+x^2+x^3+x^4 & 1 \\
\hline
1+x^2+x^3 = r_m &
\end{array}
$$

$r_m + g_m = 1 + x^2 + x^3 + x^4$

$v = 1011100$

$m = 100$

$H = (I_{n-k} \mid P)$

$$
= \left(\begin{array}{cccc|ccc}
1 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 0 & 1
\end{array}\right)
$$

$$
G = \left(\frac{P}{I_k}\right) = \left(\begin{array}{ccc}
1 & 1 & 0 \\
0 & 1 & 1 \\
1 & 1 & 1 \\
1 & 0 & 1 \\
\hline
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array}\right)
$$

001

$l_m = x^2$

$g_m = l_m \cdot x^{n-k} = x^6$

$$
\begin{array}{l|l}
x^6 & 1+x^2+x^3+x^4 \\
\hline
x^2+x^4+x^5+x^6 & x^2+x \\
x^2+x^4+x^5 & \\
\hline
x+x^3+\cancel{x^4}+\cancel{x^5} & \\
\hline
x+x^3+x^2 = r_m &
\end{array}
$$

$r_m + g_m = x + x^2 + x^3 + x^6$

$v = 0111001$

010

$l_m = x$

$g_m = l_m \cdot x^{n-k} = x \cdot x^4 = x^5$

$$
\begin{array}{l|l}
x^5 & 1+x^2+x^3+x^4 \\
\hline
x+x^3+x^4+x^5 & x+1 \\
x+x^3+x^4 & \\
\hline
1+x^2+\cancel{x^3}+\cancel{x^4} &
\end{array}
$$

$r_m + g_m = 1 + \cancel{x} + x^2 + x^5$

$v = 1110010$

error detecting & correcting

$$H = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \end{pmatrix}$$

$d(C) \geq 1$ there are no zero columns

$d(C) > 2$ there are no identical columns

$d(C) > 3$ there are no 3 colums that sum up to a zero column

$$C_1 + C_3 + C_4 + C_5 = 0$$

$$C_2 + C_4 + C_5 + C_6 = 0$$

error detecting $= 4 - 1 = 3$ errors

error correcting $= \frac{4-1}{2} = \frac{3}{2} = 1$ error

13.2

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{array}\right) \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$m = 101110$

$cl = 000000$

$m + cl = 101\underline{110} \Rightarrow v = 110$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{array}\right) \cdot \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$m = 011000$

$cl = 000010$

$m + cl = 011\underline{010} \Rightarrow v = 010$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{array}\right) \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$m = 111111$

$cl = 000110$

$m + cl = 111001$

$v = 001$

syndromes	000	001	010	011	100	101	110	111
cosetleader	0000000	0010000	0100000	0000001	1000000	0000010	000010	0001000

$2^3 = 8$

$\frac{8}{2} = 4/2 = 2/2 = 1$

0 0 0

0 0 1

0 1 0

0 1 1

1 0 0

1 0 1

1 1 0

1 1 1

4. $\begin{cases} x + 4y + 3z = 1 \\ 2x + 2y + z = 0 \\ 3x \quad\;\; - z = \lambda \end{cases}$

a)

$$\left(\begin{array}{ccc|c} 1 & 4 & 3 & 1 \\ 2 & 2 & 1 & 0 \\ 3 & 0 & -1 & \lambda \end{array}\right) \underset{L_3 \leftarrow L_3 - 3L_1}{\overset{L_2 \leftarrow L_2 - 2L_1}{\sim}} \left(\begin{array}{ccc|c} 1 & 4 & 3 & 1 \\ 0 & -6 & -5 & -2 \\ 0 & -12 & -10 & \lambda - 3 \end{array}\right)$$

$$\overset{L_3 \leftarrow L_3 - 2L_2}{\sim} \left(\begin{array}{ccc|c} 1 & 4 & 3 & 1 \\ 0 & -6 & -5 & -2 \\ 0 & 0 & 0 & \lambda + 1 \end{array}\right)$$

I $\lambda + 1 \neq 0 \Rightarrow$ the system is incompatible

II $\lambda + 1 = 0 \Rightarrow \lambda = -1 \Rightarrow$ the system is comp

$\begin{cases} x + 4y + 3z = 1 \Rightarrow x + \frac{4 - 10\alpha}{3} + {}^{3)}3\alpha = 1 \Rightarrow \\ -6y - 5z = -2 \Rightarrow -6y = -2 + 5\alpha \Rightarrow 6y = 2 - 5\alpha \end{cases}$

$\Rightarrow y = \frac{2 - 5\alpha}{6}$

$z = \alpha$

$\Rightarrow x + \frac{4 - \alpha}{3} = 1 \Rightarrow x = {}^{3)}1 - \frac{4 - \alpha}{3} \Rightarrow x = \frac{-1 - \alpha}{3}$

$\Rightarrow x = \frac{-1 - \alpha}{3}, \; y = \frac{2 - 5\alpha}{6}, \; z = \alpha$

b) $f \in End_R(R_3)$, $f(x,y,z) = (x-y, x+y-2z, -5y+z)$

$f(1,0,0) = (1,1,0)$

$f(0,1,0) = (-1,1,-5) \quad \Rightarrow [f]_B = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & -2 \\ 0 & -5 & 1 \end{pmatrix} = A$

$f(0,0,1) = (0,-2,1)$

$$P_A(X) = |\det A - \lambda I_n| = \begin{vmatrix} 1-\lambda & -1 & 0 \\ 1 & 1-\lambda & -2 \\ 0 & -5 & 1-\lambda \end{vmatrix} =$$

$$= (1-\lambda) \cdot \begin{vmatrix} 1-\lambda & -2 \\ -5 & 1-\lambda \end{vmatrix} = (1-\lambda)\left((1-\lambda)^2 - 10\right) =$$

$$= (1-\lambda)(1-2\lambda+\lambda^2-10) = (1-\lambda)(\lambda^2-2\lambda-9)$$

I $1-\lambda = 0 \Rightarrow \lambda = 1$

II $\lambda^2 - 2\lambda - 9 = 0$

$$\lambda_{1,2} = \frac{-b \pm \sqrt{\Delta}}{2a} = \frac{2 \pm \sqrt{40}}{2} = \frac{2 \pm 2\sqrt{10}}{2} = 1 \pm \sqrt{10}$$

$$\Delta = b^2 - 4ac = 4 + 36 = 40$$

$$S(\lambda) = \{ v \in V \mid [f]_E \cdot [v]_E = \lambda \cdot [v]_E \}$$

if $\lambda = 1$

$$\begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & -2 \\ 0 & -5 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 1 \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{cases} x - y = x \Rightarrow y = 0 \\ x + y - 2z = y \Rightarrow x - 2z = 0 \Rightarrow x = 2z \\ -5y + z = z \Rightarrow y = 0 \end{cases}$$

$$S(\lambda_1) = \{ (2z, 0, z) \mid z \in \mathbb{R} \} = \langle (2,0,1) \rangle$$

if $\lambda = 1 + \sqrt{10}$

$$\begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & -2 \\ 0 & -5 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = (1+\sqrt{10}) \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{cases} x - y = (1+\sqrt{10})x \Rightarrow \cancel{x} - y = \cancel{x} + x\sqrt{10} \Rightarrow y = -x\sqrt{10} \Rightarrow y = 0 \\ x + y - 2z = (1+\sqrt{10})y \Rightarrow x - \cancel{x\sqrt{10}} - \cancel{10}x = -\cancel{x\sqrt{10}} - \cancel{10}x \Rightarrow \\ -5y + z = (1+\sqrt{10})z \Rightarrow 5x\sqrt{10} + \cancel{z} = \cancel{z} + z\sqrt{10} \Rightarrow \end{cases}$$

$$\Rightarrow 5x = z \Rightarrow z = 0$$

$$\Rightarrow x = 0$$

$S(\lambda_2) = \langle (0,0,0) \rangle$

if $\lambda = 1-\sqrt{10}$

$$\begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & -2 \\ 0 & -5 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = (1-\sqrt{10})\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{cases} x-y = (1-\sqrt{10})x \Rightarrow \cancel{x}-y = \cancel{x}-x\sqrt{10} \Rightarrow y = x\sqrt{10} \Rightarrow y=0 \\ x+y-2z = (1-\sqrt{10})y \Rightarrow x+\cancel{x\sqrt{10}}-1\cancel{0}x = \cancel{x\sqrt{10}}-1\cancel{0}x \Rightarrow x=0 \\ -5y+z = (1-\sqrt{10})z \Rightarrow -5x\sqrt{10}+\cancel{z} = \cancel{z}-z\sqrt{10} \Rightarrow \\ \Rightarrow -5x = -z \Rightarrow 5x = z \Rightarrow z=0 \end{cases}$$

$S(\lambda_3) = \langle 0,0,0 \rangle$

5. $(5,2)$-code $\quad H = \left(\begin{array}{ccc|cc} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{array}\right)$

S	000	001	010	011	100	101	110	111
CL	00000	00100	01000	01100	10000	00001	00010	01001

$$H \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow e = 00000 \Rightarrow e + v = 00000 + 10101$$

$$= 10101$$

$$\Rightarrow m = 101$$

I 3 Determine the rank of the matrix

$$A = \begin{pmatrix} 2 & 0 & 1 & -1 & 3 \\ 4 & 1 & 9 & -2 & 10 \\ 0 & 1 & 7 & 5 & -1 \\ 6 & 0 & 3 & -3 & \alpha \end{pmatrix}$$ by using elementary operations

Discussion on $\alpha \in \mathbb{R}$

$$\begin{pmatrix} 2 & 0 & 1 & -1 & 3 \\ 4 & 1 & 9 & -2 & 10 \\ 0 & 1 & 7 & 5 & -1 \\ 6 & 0 & 3 & -3 & \alpha \end{pmatrix} \underset{L_4 \leftarrow L_4 - 3L_1}{\overset{L_2 \leftarrow L_2 - 2L_1}{\sim}} \begin{pmatrix} 2 & 0 & 1 & -1 & 3 \\ 0 & 1 & 7 & 0 & 4 \\ 0 & 1 & 5 & 7 & -7 \\ 0 & 0 & 0 & 0 & \alpha - 9 \end{pmatrix} \sim$$

$$\overset{L_3 \leftarrow L_3 - L_2}{\sim} \begin{pmatrix} 2 & 0 & 1 & -1 & 3 \\ 0 & 1 & 7 & 0 & 4 \\ 0 & 0 & -2 & 7 & -11 \\ 0 & 0 & 0 & 0 & \alpha - 9 \end{pmatrix}$$

I $\alpha - 9 \neq 0 \Rightarrow$ the system is incompatible $\Rightarrow \operatorname{rank} A = 4$

II $\alpha - 9 = 0 \Rightarrow \alpha = 9 \Rightarrow$ the system is compatible $\Rightarrow \operatorname{rank} A = 3$

4. b) $f \in \mathrm{End}_{\mathbb{R}}(\mathbb{R}^3)$, $f(x,y,z) = (-2y-3z, x+3y+3z, z)$

Write the matrix of f in the canonical basis $e = (e_1, e_2, e_3)$ of $\mathbb{R}^3$

$$f(e_1) = f(1,0,0) = (0,1,0)$$

$$f(e_2) = f(0,1,0) = (-2,3,0)$$

$$f(e_3) = f(0,0,1) = (-3,3,1)$$

$$[f]_E = \begin{pmatrix} 0 & -2 & -3 \\ 1 & 3 & 3 \\ 0 & 0 & 1 \end{pmatrix}$$

II 3. Solve the system $\begin{cases} x+4y+2z=1 \\ 2x+3y+z=0 \\ 3x+\quad -z=\lambda \end{cases}$ by the Gauss method. Discussion on λ

$$\left(\begin{array}{ccc|c} 1 & 4 & 2 & 1 \\ 2 & 3 & 1 & 0 \\ 3 & 0 & -1 & \lambda \end{array}\right) \underset{L_3 \leftarrow L_3 - 3L_1}{\overset{L_2 \leftarrow L_2 - 2L_1}{\sim}} \left(\begin{array}{ccc|c} 1 & 4 & 2 & 1 \\ 0 & -5 & -3 & -2 \\ 0 & -12 & -7 & \lambda-3 \end{array}\right)$$

$$\overset{L_2 \leftarrow -\frac{1}{5}L_2}{\sim} \left(\begin{array}{ccc|c} 1 & 4 & 2 & 1 \\ 0 & 1 & \frac{3}{5} & \frac{2}{5} \\ 0 & -12 & -7 & \lambda-3 \end{array}\right) \overset{L_3 \leftarrow L_3 + 12L_2}{\sim}$$

$$\left(\begin{array}{ccc|c} 1 & 4 & 2 & 1 \\ 0 & 1 & \frac{3}{5} & \frac{2}{5} \\ 0 & 0 & \frac{1}{5} & \lambda+\frac{9}{5} \end{array}\right) \overset{L_3 \leftarrow 5 \cdot L_3}{\sim} \left(\begin{array}{ccc|c} 1 & 4 & 2 & 1 \\ 0 & 1 & \frac{3}{5} & \frac{2}{5} \\ 0 & 0 & 1 & 5\lambda+9 \end{array}\right)$$

$$\begin{cases} x+4y+2z=1 \Rightarrow x-12\alpha-20+10\alpha+18=1 \Leftrightarrow \\ y+\frac{3}{5}z=\frac{2}{5} \Rightarrow y+3\alpha+\frac{27}{5}=\frac{2}{5} \Rightarrow y=-3\alpha-5 \\ z=5\alpha+9 \end{cases}$$

$$\Leftrightarrow x-2\alpha-2=1 \Rightarrow x=2\alpha+3$$

4. b) $f \in \mathrm{End}_{\mathbb{R}}(\mathbb{R}^3)$ $f(x,y,z) = (x-y,\ x+y-2z,\ -5y+z)$

$f(e_1) = f(1,0,0) = (1,1,0)$

$f(e_2) = f(0,1,0) = (-1,1,-5)$

$f(e_3) = f(0,0,1) = (0,-2,1)$

$$[f]_E = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & -2 \\ 0 & -5 & 1 \end{pmatrix}$$

III 3. $A^{-1} = ?$, $A = \begin{pmatrix} 1 & 4 & 2 \\ 2 & 3 & 1 \\ 3 & 0 & -1 \end{pmatrix}$

$$\left(\begin{array}{ccc|ccc} 1 & 4 & 2 & 1 & 0 & 0 \\ 2 & 3 & 1 & 0 & 1 & 0 \\ 3 & 0 & -1 & 0 & 0 & 1 \end{array}\right) \underset{L_3 \leftarrow L_3 - 3L_1}{\overset{L_2 \leftarrow L_2 - 2L_1}{\sim}} \left(\begin{array}{ccc|ccc} 1 & 4 & 2 & 1 & 0 & 0 \\ 0 & -5 & -3 & -2 & 1 & 0 \\ 0 & -12 & -7 & -3 & 0 & 1 \end{array}\right)$$

$$\underset{L_2 \leftarrow -\frac{1}{5}L_2}{\sim} \left(\begin{array}{ccc|ccc} 1 & 4 & 2 & 1 & 0 & 0 \\ 0 & 1 & \frac{3}{5} & \frac{2}{5} & -\frac{1}{5} & 0 \\ 0 & -12 & -7 & -3 & 0 & 1 \end{array}\right) \underset{L_3 \leftarrow L_3 + 12L_2}{\sim}$$

$$\left(\begin{array}{ccc|ccc} 1 & 4 & 2 & 1 & 0 & 0 \\ 0 & 1 & \frac{3}{5} & \frac{2}{5} & -\frac{1}{5} & 0 \\ 0 & 0 & \frac{1}{5} & \frac{9}{5} & -\frac{12}{5} & 1 \end{array}\right) \underset{L_1 \leftarrow L_1 - 10L_3}{\overset{L_2 \leftarrow L_2 - 3L_3}{\sim}} \left(\begin{array}{ccc|ccc} 1 & 4 & 0 & -17 & 24 & -10 \\ 0 & 1 & 0 & -5 & 7 & -3 \\ 0 & 0 & \frac{1}{5} & \frac{9}{5} & -\frac{12}{5} & 1 \end{array}\right)$$

$$\underset{L_3 \leftarrow L_3 \cdot 5}{\sim} \left(\begin{array}{ccc|ccc} 1 & 4 & 0 & -17 & 24 & -10 \\ 0 & 1 & 0 & -5 & 7 & -3 \\ 0 & 0 & 1 & 9 & -12 & 5 \end{array}\right) \underset{L_1 \leftarrow L_1 - 4L_2}{\sim}$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -4 & 2 \\ 0 & 1 & 0 & -5 & 7 & -3 \\ 0 & 0 & 1 & 9 & -12 & \frac{1}{7} \end{array}\right)$$

$$\underbrace{}_{I_3} \qquad \underbrace{}_{A^{-1}}$$

4. b) $f \in End_{\mathbb{R}}(\mathbb{R}^3)$ $\quad f(x,y,z) = (2x, y+2z, -y+4z)$

$$f(1,0,0) = (2,0,0)$$
$$f(0,1,0) = (0,1,-1) \quad \Rightarrow [f]_B = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & -1 & 4 \end{pmatrix}$$
$$f(0,0,1) = (0,2,4)$$

$$P_A(\lambda) = \begin{vmatrix} 2-\lambda & 0 & 0 \\ 0 & 1-\lambda & 2 \\ 0 & -1 & 4-\lambda \end{vmatrix} = (2-\lambda)(\lambda^2 - 5\lambda + 6)$$

$$(2-\lambda)(\lambda^2-5\lambda+6) = 0$$

I $\quad 2-\lambda = 0 \Rightarrow \lambda_1 = 2$

II $\quad \lambda^2 - 5\lambda + 6 = 0$ $\qquad \lambda_{2,3} = \frac{5 \pm 1}{2} \begin{cases} \lambda_2 = \frac{6}{2} = 3 \\ \lambda_3 = \frac{4}{2} = 2 \end{cases}$

$\Delta = b^2 - 4ac = 1$

$$\lambda_1 = \lambda_3 = 2$$

in conclusion we only have 2 distinct eigenvalues so f is not diagonizable