Find the lower and upper bounds, sup, inf, max, min

Let x be a real number and A a set

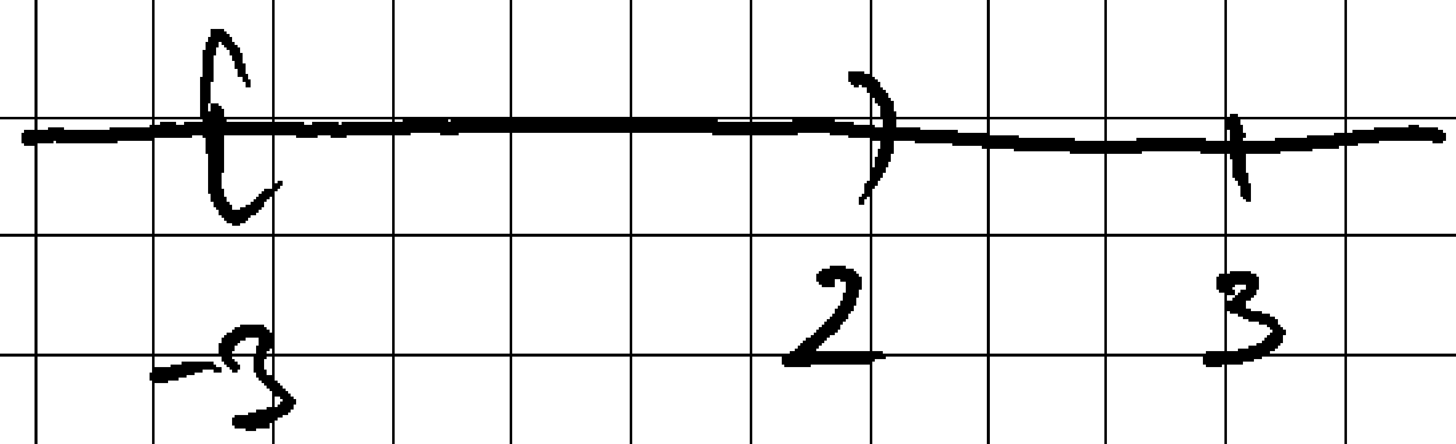
x is a lower bound of A ($x \in lb(A)$) if $x \leq a \ \forall a \in A$

x is a upper bound of A ($x \in ub(A)$) if $x \geq a \ \forall a \in A$

Sup A is the lowest upper bound of A
inf A is the greatest lower bound of A

max A is the sup A if it's included in A
min A is the inf A if it's included in A

1) a) $[-3,2) \cup \{3\} =: A$



$lb(A) = (-\infty, -3]$

$ub(A) = [3, \infty)$

$\inf(A) = -3 \quad , \quad \sup(A) = 3$

$-3 \in A \Rightarrow \min A = -3$

$3 \in A \Rightarrow \max A = 3$

b) $(-1,1] \cup (2,\infty) = B$

$lb(B) = (-\infty, -1)$

$ub(B) = \emptyset$

$\inf(B) = -1 \notin B \Rightarrow \nexists \min(B)$

$\sup(B) = \infty \qquad \nexists \max(B)$

c) $(-5,5) \cap \mathbb{Z} =: C = \{-4,-3,-2 \dots 1,2,3,4\}$

$lb(C) = (-\infty, -4]$

$ub(C) = [4, \infty)$

$\sup(C) = +4$, $\inf(C) = -4$

$-4 \in C \Rightarrow \min(C) = -4$

$4 \in C \Rightarrow \max(C) = 4$

d) $lb(\emptyset) = ub(\emptyset) = \mathbb{R}$

$\inf(\emptyset) = +\infty$ $\quad \nexists \min(\emptyset)$

$\sup(\emptyset) = -\infty$ $\quad \nexists \max(\emptyset)$

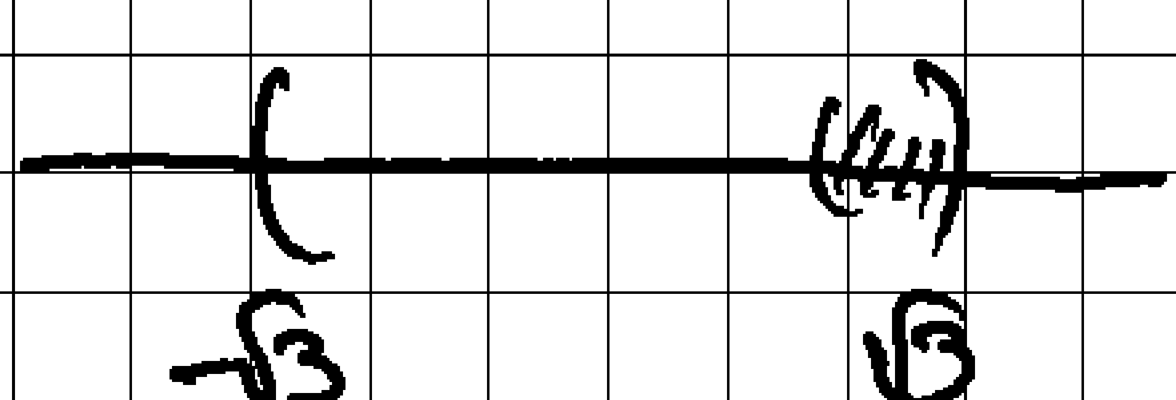
2) Find the sup, inf, max, min

a) $A = \{x \in \mathbb{Q} \mid x^2 < 3\} = (-\sqrt{3}, \sqrt{3}) \cap \mathbb{Q}$

$\sup A = \sqrt{3}$, $\inf A = -\sqrt{3}$

$\sqrt{3} \notin A \Rightarrow \nexists \max A$,

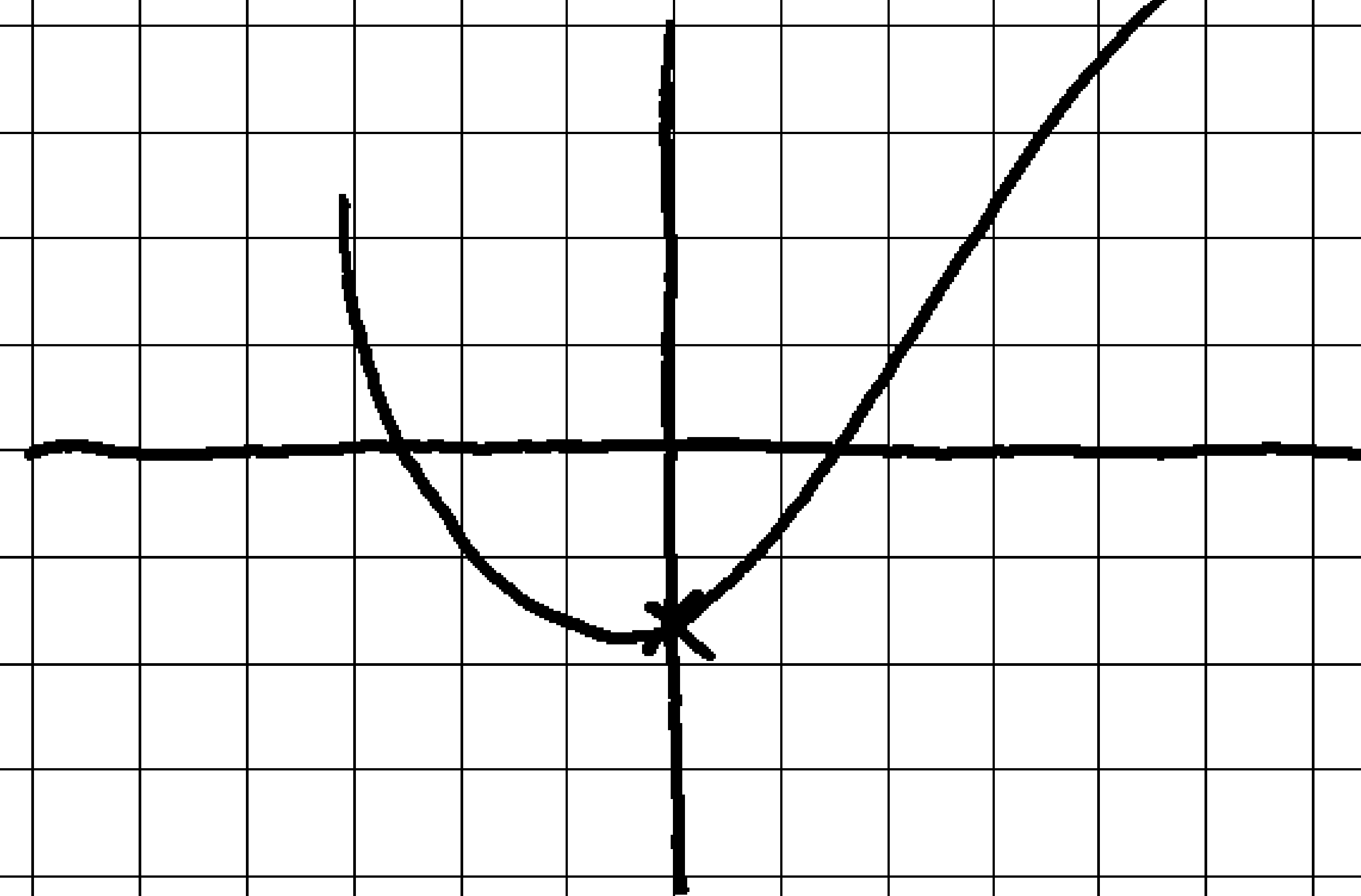
$-\sqrt{3} \notin A \Rightarrow \nexists \min A$



b) $\{x^2-4x+3 \mid x \in \mathbb{R}\} =: A$

$\sup A = +\infty$

$$\frac{-\Delta}{4a} = -\frac{16-12}{4} = -1$$



$\inf A = -1$

$\min A = -1$, $\nexists \max A$

c) $\{\frac{n}{n+1} \mid n \in \mathbb{N}^*\} = \{\underset{0{,}5}{\frac{1}{2}}, \underset{0{,}66}{\frac{2}{3}}, \underset{0{,}75}{\frac{3}{4}}, \dots\} = C$

$\inf C = \frac{1}{2} \Rightarrow \min C = \frac{1}{2}$

$\sup C = 1 \Rightarrow \nexists \max C$

d) $\{2^{-k} + 3^{-m} \mid k, m \in \mathbb{N}^*\} = \{\frac{1}{2}+\frac{1}{3}, \frac{1}{2}+\frac{1}{9}, \dots$

$\frac{1}{4}+\frac{1}{3}, \frac{1}{8}+\frac{1}{3}, \dots\} = D$

$\inf D = 0 \Rightarrow \nexists \min D$

$\sup D = \frac{5}{6} \Rightarrow \max D = \frac{5}{6}$

3) Suppose that S is a nonempty set and bounded above. Show that the set:
$-S = \{-x \mid x \in S\}$ is bounded below and
$\inf(-S) = -\sup S$

S is bounded above $\Rightarrow \exists\, a \in \mathbb{R},\ a \geq x \quad \forall x \in S$

$-a \leq -x \quad \forall x \in S \Leftrightarrow -a \leq -y \quad \forall y \in -S \Rightarrow$
$\Rightarrow -a$ is from $lb(-S)$, $-a \in lb(-S)$

$\inf(-S) = -\sup S$

S is bounded from above $\Rightarrow$
$\Rightarrow \exists\, \alpha \in \mathbb{R} : \alpha = \sup S$
$-S$ is bounded from below $\Rightarrow$
$\Rightarrow \exists\, \beta \in \mathbb{R} : \beta = \inf(-S)$
We need to prove that: $\beta = -\alpha$

$$\left.\begin{matrix} x \leq y \\ y \leq x \end{matrix}\right\} \Rightarrow x = y$$

$$\left.\begin{matrix} -\alpha \in lb(-S) \\ \beta = \inf(-S) \\ \text{(largest lower bound)} \end{matrix}\right\} \Rightarrow -\alpha \leq \beta \quad (1)$$

$\beta = \inf(-S) \Rightarrow \forall y \in -S$

$\Downarrow$

$\beta \in lb(-S)$

$\beta \le y \;|\cdot(-1) \Rightarrow -\beta \ge -y \in S$

$\Rightarrow -\beta \in ub(S)$ $\Rightarrow \alpha \le -\beta \;|\cdot(-1)$

$\Downarrow$

$-\alpha \ge \beta \; (2)$

$\alpha = \sup(S)$
(smallest upper bound)

$(1), (2) \Rightarrow -\alpha = \beta$

4. Let $f: D \to \mathbb{R}$, $g: D \to \mathbb{R}$ be two functions on a nonempty set D. Prove that $\inf_{x\in D}(f(x)+g(x)) \ge \inf_{x\in D}(f(x)) + \inf_{x\in D}(g(x))$

$\inf_{x\in D}(f(x)) \le f(x), \forall x \in D$

$\inf_{x\in D}(g(x)) \le g(x), \forall x \in D$

$\oplus \; \inf_{x\in D} f(x) + \inf_{x\in D} g(x) \le$

$\le f(x) + g(x)$

$\Rightarrow \inf\limits_{x\in D} f(x) + \inf\limits_{x\in D} g(x)$ is a lower bound

of $\{ f(x) + g(x) \mid x \in D \}$

$\inf (f(x) + g(x))$ is the largest lower bound

of $\{ f(x) + g(x) \mid x \in D \}$

$\Rightarrow$

$$\Rightarrow \inf_{x\in D} f(x) + \inf_{x\in D} g(x) \leq \inf_{x\in D} (f(x) + g(x))$$

Example: for strict inequality

$f(x) = x$

$g(x) = x$

$D = [-1, 1]$

$\inf\limits_{x\in D} f(x) = -1$

$\inf\limits_{x\in D} g(x) = -1$

$\oplus$ $f(x) + g(x) = 0$

$$-2 < 0$$

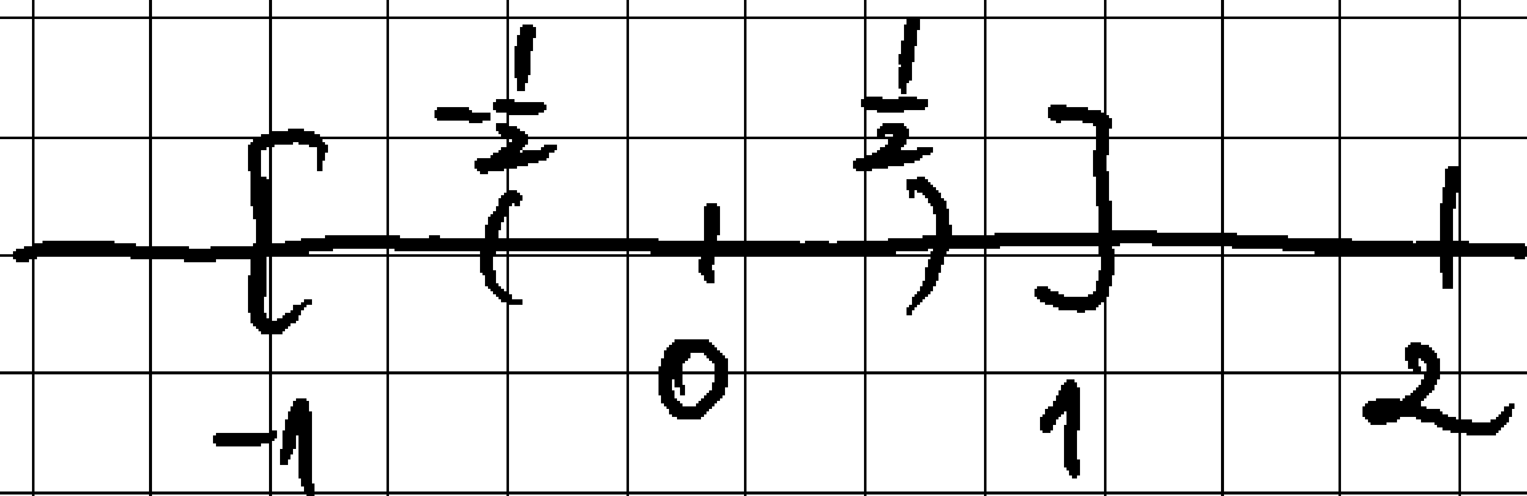
6. For which of the following are neighborhoods of 0?

U is a neighborhood of $x \iff$

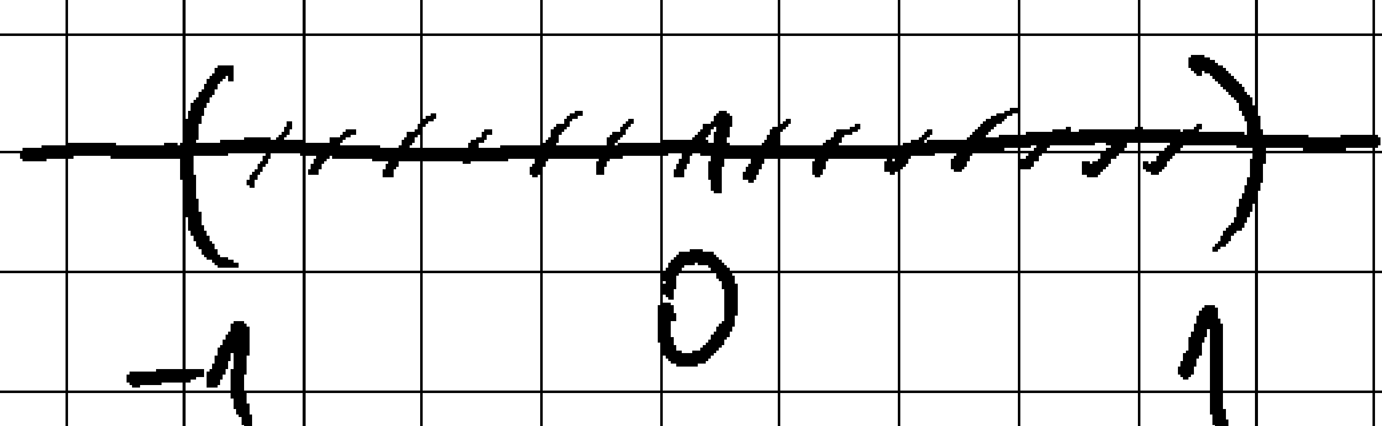
$\iff \exists\, \varepsilon > 0,\ (x-\varepsilon, x+\varepsilon) \subseteq U$

a) $[-1, 1] \cup \{2\}$

$0 \in \left(-\frac{1}{2}, \frac{1}{2}\right) \subseteq A \Rightarrow A \in V(0)$

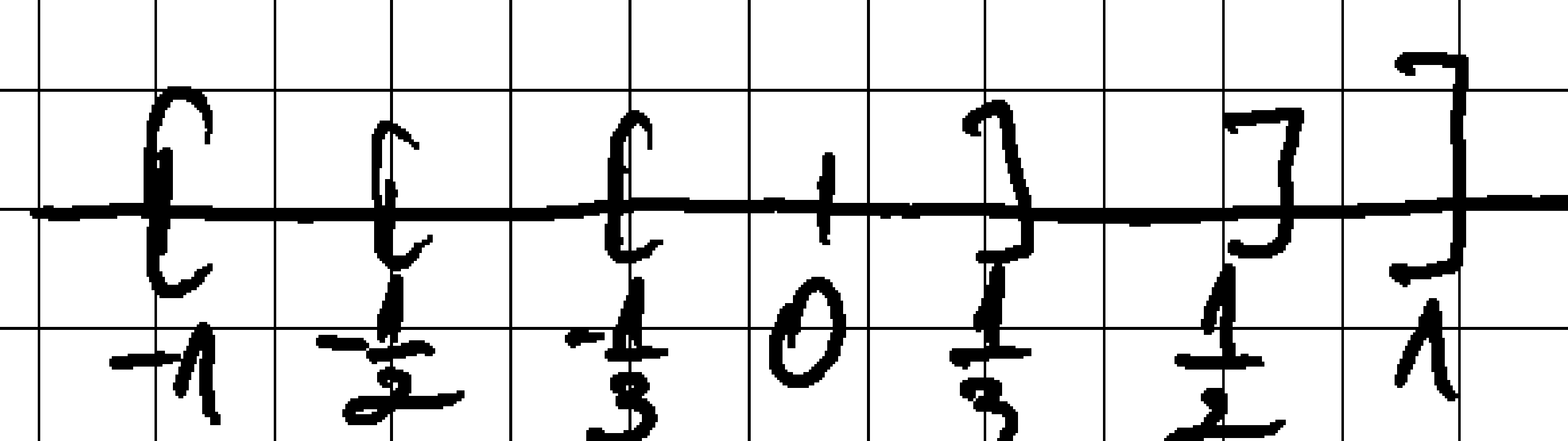


b) $(-1, 1) \cap \mathbb{Q} = B$

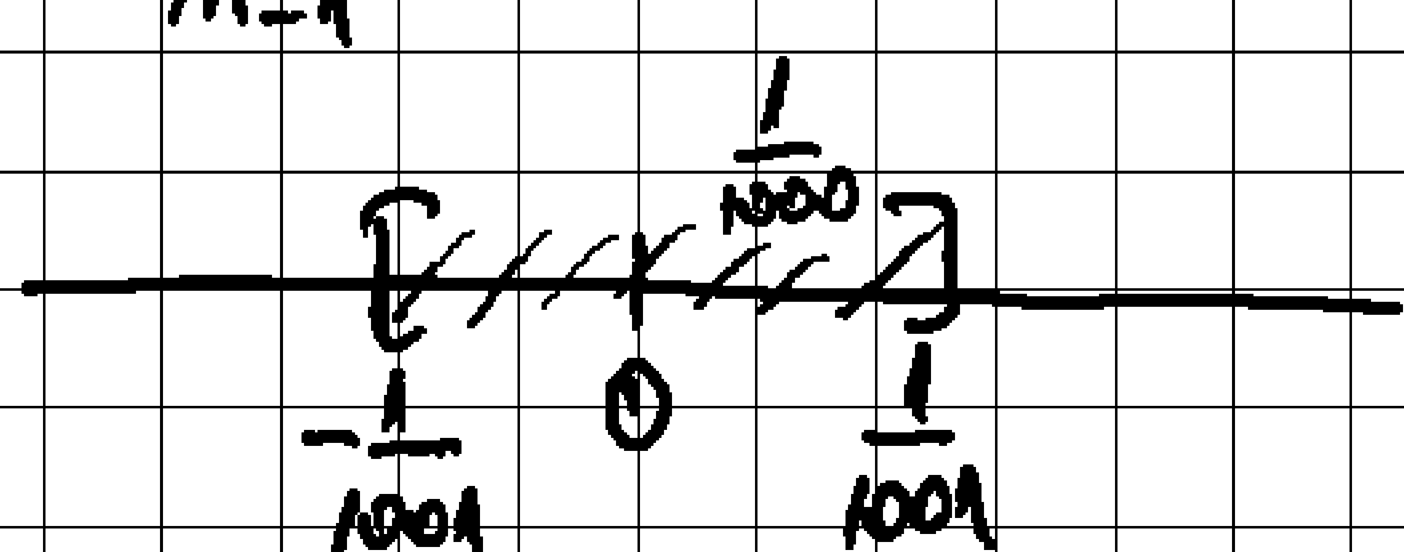


Any open interval $(-\varepsilon, \varepsilon)$ for $\varepsilon > 0$ contain real numbers so it cannot be included in $B \Rightarrow B \notin V(0)$

c) $\bigcap_{n=1}^{\infty} \left[-\frac{1}{n}, \frac{1}{n}\right] = \{0\}$

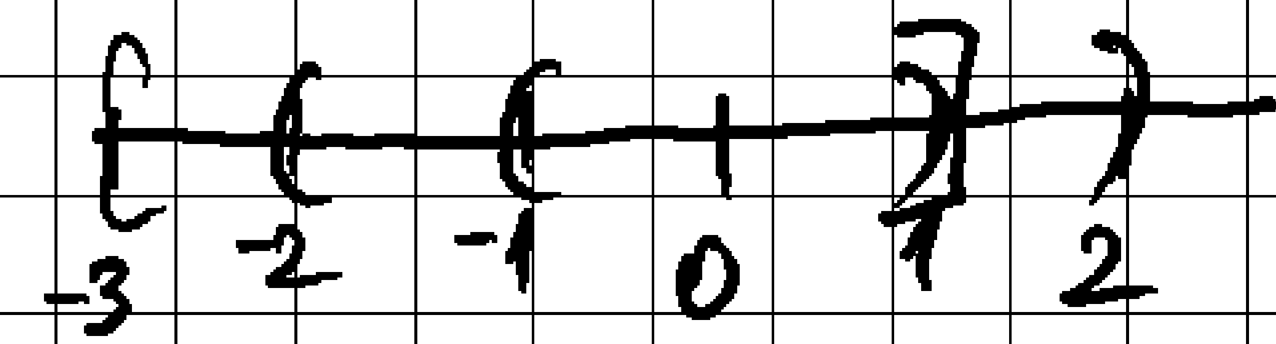


$C \in V(0)$



7. Let $x \in \mathbb{R}$ and $u, v \in \mathcal{V}(x)$.

Prove that $u \cap V \in \mathcal{V}(x)$

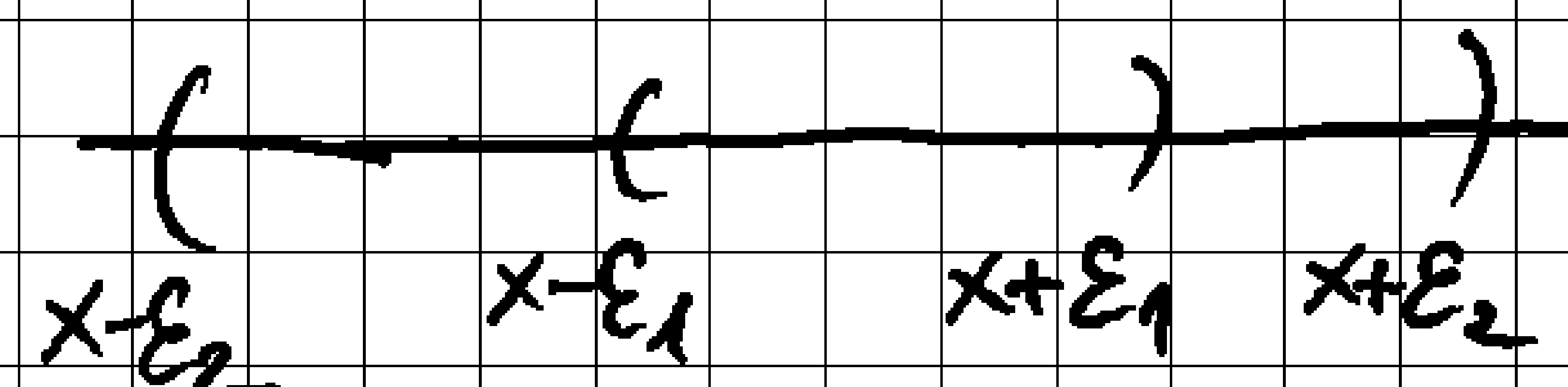


$u \in \mathcal{V}(x) \Rightarrow$

$(x-\varepsilon_1, x+\varepsilon_1) \subseteq u$

$V \in \mathcal{V}(x) \Rightarrow \exists\, \varepsilon_2 > 0 : \ (x-\varepsilon_2, x+\varepsilon_2) \subseteq V$

Let $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$



Example: $\varepsilon = \varepsilon_1 \leq \varepsilon_2$

$\Rightarrow (x-\varepsilon, x+\varepsilon) = (x-\varepsilon_1, x+\varepsilon_1) \subseteq (x-\varepsilon_2, x+\varepsilon_2)$

$x+\varepsilon_1 \leq x+\varepsilon_2$

$x-\varepsilon_1 \geq x-\varepsilon_2$

$\Rightarrow (x-\varepsilon, x+\varepsilon) \subseteq U \cap V$

9. Find the interior and closure

a) $(1,2] = A$

$\operatorname{int} A = \mathring{A} = \{x \in \mathbb{R} \mid \exists\, \varepsilon > 0 : (x-\varepsilon, x+\varepsilon) \subseteq A\}$

$\operatorname{int}(1,2] = (1,2)$

$\operatorname{cl} A = \overline{A} = \{x \in \mathbb{R} \mid \forall\, V \in \mathcal{V}(x),\ V \cap A \neq \emptyset\}$

$\operatorname{cl}(1,2] = [1,2]$