1. b) $\lim_{n\to\infty} \frac{\sqrt[n]{e} + 2\sqrt[n]{e^2} + \ldots + n\sqrt[n]{e^n}}{n^2} =$

$$= \lim_{n\to\infty} \frac{1}{n^2} \cdot \left(e^{\frac{1}{n}} + 2 \cdot e^{\frac{2}{n}} + \ldots + n \cdot e^{\frac{n}{n}}\right) =$$

$$= \lim_{n\to\infty} \frac{1}{n^2} \cdot \sum_{k=1}^{n} k \cdot e^{\frac{k}{n}} = \int_0^1 x e^x \, dx = xe^x - e^x \Big|_0^1 = e-1$$

d) $\lim_{n\to\infty} \sqrt[n]{\sin\frac{\pi}{2n} \cdot \sin\frac{2\pi}{2n} \cdot \ldots \cdot \sin\frac{(n-1)\pi}{2n}}$

$$S_n = \sqrt[n]{\sin\frac{\pi}{2n} \cdot \sin\frac{2\pi}{2n} \cdot \ldots \cdot \sin\frac{(n-1)\pi}{2n}} \quad / \ln$$

$$\ln(S_n) = \frac{1}{n}\left(\ln \sin\frac{\pi}{2n} + \ln \sin\frac{2\pi}{2n} + \ldots + \ln \sin\frac{(n-1)\pi}{2n}\right)$$

$$\ln(S_n) = \frac{1}{n} \sum_{k=1}^{n-1} \ln\left(\sin\frac{k\pi}{2n}\right) \quad / \lim_{n\to\infty}$$

$$\lim_{n\to\infty} \ln(S_n) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n-1} \ln\left(\sin\frac{k\pi}{2n}\right)$$

$$= \int_0^1 \ln\left(\sin\frac{\pi x}{2}\right) dx$$

$u = \frac{\pi x}{2} \quad du = \frac{\pi}{2}$

$$= \frac{2}{\pi} \cdot \int_0^{\frac{\pi}{2}} \ln(\sin u) \, du$$

$v = \ln(\sin u)$

$dV = 1\, du$

$$\frac{2}{\pi}\int_0^{\frac{\pi}{2}} \ln(\sin u)\,du = \frac{2}{\pi}\left(\overset{0}{\overset{\parallel}{u\cdot\ln(\sin(u))\Big|_0^{\frac{\pi}{2}}}} - \right.$$

$$\left. - \int_0^{\frac{\pi}{2}} \frac{u}{\operatorname{tg} u}\,du\right) = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{u}{\operatorname{tg} u}\,du =$$

$$= \frac{2}{\pi}\cdot\pi\cdot\left(0 - \ln(\sin 0)\right) = -2$$

$$\Rightarrow \lim_{n\to\infty} S_n = e^{-2}$$

3. $\int_0^{\infty} e^{-x}\sin x\,dx$

$u = e^{-x} \qquad dv = \sin x\,dx$

$du = -e^{-x} \qquad v = -\cos x$

$$\int e^{-x}\sin x = -e^{-x}\cos x + \int e^{-x}\cos x\,dx$$

$u = e^{-x} \qquad dv = \cos x\,dx$

$du = -e^{-x} \qquad v = \sin x$

$$\int e^{-x}\cos x = e^{-x}\sin x + \int e^{-x}\sin x\,dx$$

$$\int e^{-x}\sin x\,dx = -e^{-x}\cos x + e^{-x}\sin x + \int e^{-x}\sin x\,dx$$

$$2\int e^{-x}\sin x\,dx = -e^{-x}\cos x + e^{-x}\sin x \quad /:2$$

$$\int e^{-x}\sin x\,dx = -\frac{1}{2}e^{-x}\cos x + \frac{1}{2}e^{-x}\sin x$$

$$\int_0^{\infty} e^{-x}\sin x\,dx = \lim_{a\to\infty}\int_0^{a} e^{-x}\sin x\,dx$$

$$\lim_{a\to\infty}\left(-\frac{1}{2}e^{-a}\cos a + \frac{1}{2}e^{-a}\sin a\right) = \frac{1}{2}$$

$$\Rightarrow \int_0^{\infty} e^{-x}\sin x\,dx = \frac{1}{2}$$