Sequence : $\{x_1, x_2, x_3, \dots\} = (x_n)$

$x_n \longrightarrow \infty$ iff $\forall a \in \mathbb{R}$, $x_n \in (a, \infty), \forall n \geq N_a$

if and only if

$\exists N_a \in \mathbb{N}$ s.t.

$x_n > a$

Q1: $(x_n) = \{-1, 1, -1, 1, \dots\}$

$x_1 = -1$, $x_{n+1} = -x_n$

$x_n = (-1)^n$

is bounded but not convergent

Q4: $x_n = \frac{(-1)^n}{n+1}$, $|x_n| < \frac{1}{n+1}$

Every convergent sequence is monotone : FALSE

Proposition 3: Any convergent sequence is bounded

Proof: Let $\varepsilon = 7$ (be fixed)

(x_n) convergent $\Rightarrow \exists N_7 \in N$, $\forall n \geq N_7$

$|x_n - l| < 7$, $\forall n \geq N_7$

(*) $x_n \in (l-7, l+7)$

The set $\{x_1, x_2, \dots, x_{N_7}\}$ is finite

$\Rightarrow$ it is bounded (**)

(*) + (**) $\Longrightarrow$ (x_n) is bounded

Theoreme 4 (Weierstrass)

Any monotone and bounded sequence is convergent

Proof: Let (x_n) be increasing: $x_{n+1} \geq x_n$, $\forall n \in \mathbb{N}$

- x_n is bounded $\Rightarrow S = \{x_n \mid n \in \mathbb{N}\}$ is bounded $\Rightarrow$

$\Rightarrow \sup(S) \in \mathbb{R}$

Let $\varepsilon > 0$. Then $\exists\, x_{N_\varepsilon} \in S$ s.t.

$$\sup(S) - \varepsilon < x_{N_\varepsilon} \leq \sup(S)$$

- (x_n) is increasing $\Rightarrow \sup(S) - \varepsilon < x_{N_\varepsilon} \leq x_N \leq \sup(S)$,

$$\forall\, n \geq N_\varepsilon$$

$$\sup(S) - x_N < \varepsilon,\ \forall\, n \geq N_\varepsilon \Rightarrow x_n \to \sup(S)$$

Proposition 5

Any monotone sequence has a limit in $\overline{\mathbb{R}}$

Theorem 6: (Squeeze / Sandwich theorem)

Proof: Let $l = \lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} z_n \in \mathbb{R}$

(*) • $x_n \to l \Rightarrow \exists\, N_1 \in \mathbb{N} : |x_n - l| < \varepsilon, \forall\, n \geq N_1$

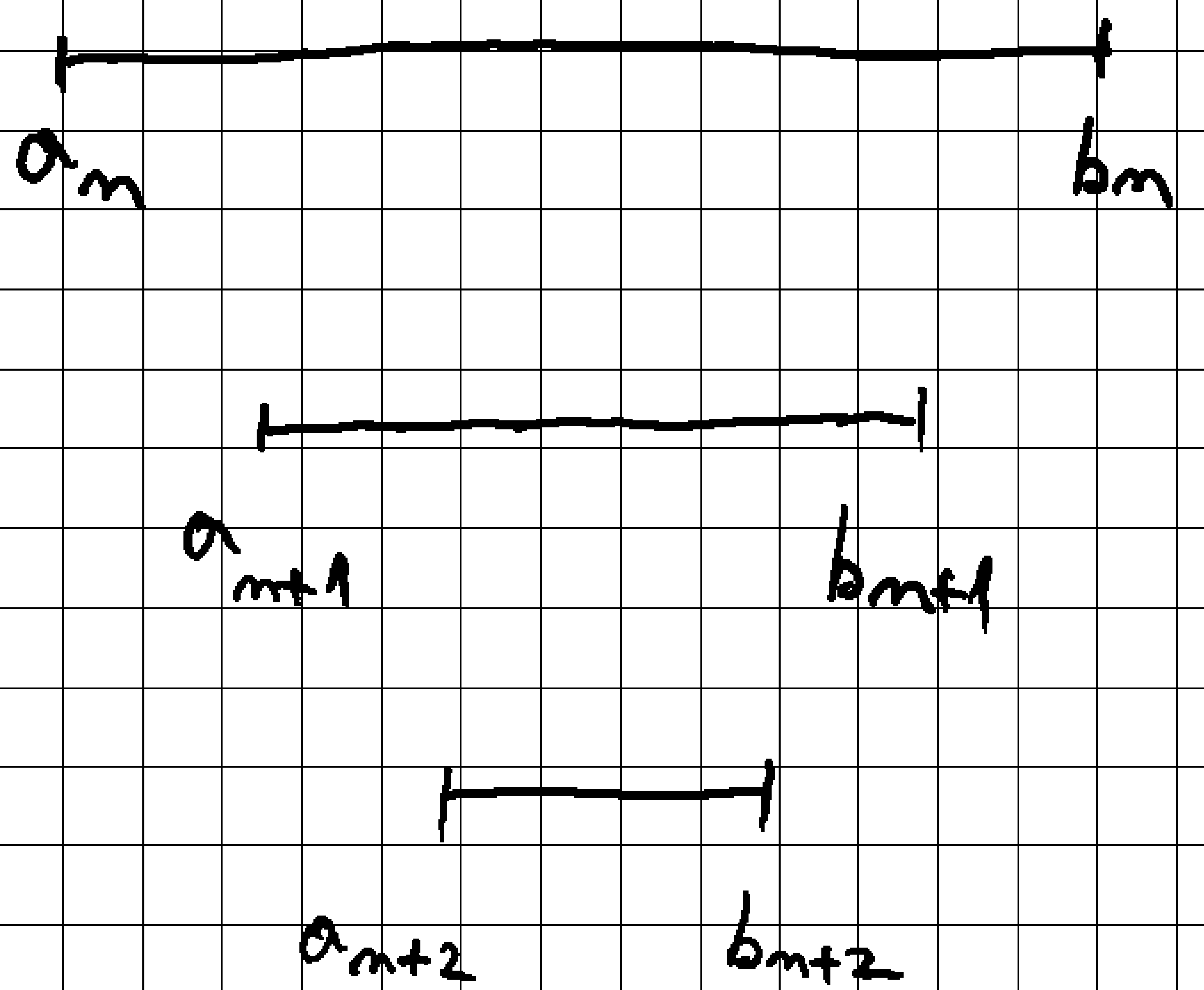
(**) • $z_n \to l \Rightarrow \exists\, N_2 \in \mathbb{N} : |z_n - l| < \varepsilon, \forall\, n \geq N_2$

(***) • $x_n \leq y_n \leq z_n,\ \forall\, n \geq n_0$

$$|y_n - l| < \varepsilon,\ \forall\, n \geq \max\{N_1, N_2, n_0\}$$

((*), (**), (***))

Theorem 7 (Cantor's nested intervals)



$I_n = [a_n, b_n]$

Proof: Let $A = (a_n) = \{a_n \mid n \in \mathbb{N}\}$

$$B = (b_n) = \{b_n \mid n \in \mathbb{N}\}$$

Let $k \in \mathbb{N}$

$$\left.\begin{array}{l} a_k \le \sup(A) \le b_k \\ b_k \ge \inf(B) \ge a_k \end{array}\right|$$

Pass to the limit $k \to \infty$

$$\lim_{n\to\infty} (a_n - b_n) = 0$$

$$\Downarrow$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = X$$

$$\left.\begin{array}{l} X \le \sup(A) \le X \\ X \ge \inf(B) \ge X \end{array}\right\} \Rightarrow \sup(A) = \inf(B) = X$$

$$X \in \bigcap_{n=1}^{\infty} I_n$$

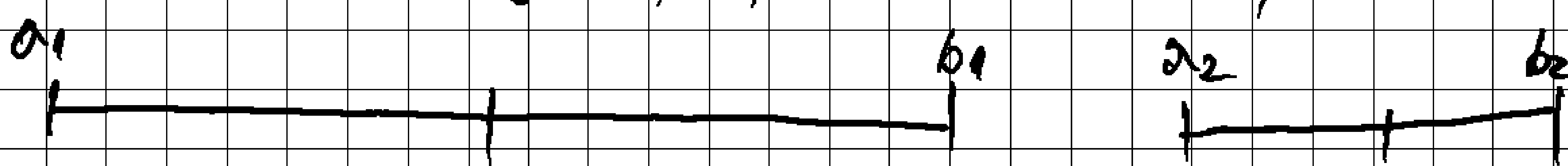
Theorem 8 (Bolzano - Weierstrass)

Any bounded sequence has a convergent subsequence

Subsequence: $\{x_1, x_2, x_3, \dots\} = (x_{n_k})$

Proof of Bolzano - Weierstrass:

- Let $a_1 = \inf(x_n)$, $b_1 = \sup(x_n)$, $I_1 = [a_1, b_1]$



- Split I_1 in two halves

- Chose the one that contains infinetely many terms of (x_n) and call it $I_2 = [a_2, b_2]$

e.g. $a_2 = a_1$, $b_2 = \frac{a_1 + b_1}{2}$

- iterate this process.

$$I_{n+1} \subseteq I_n \quad , \quad I_n = [a_n, b_n]$$

$$b_n - a_n = \frac{b_1 - a_1}{2^{n-1}} \Rightarrow \lim_{n \to \infty} (b_n - a_n) = 0$$

- Use Cantor's Nested intervals theorem:

$$\bigcap_{n=1}^{\infty} I_n = \{x\}$$

- Take $x_{n_k} \in [a_k, b_k]$ (out of infinitely many others)

$$\lim_{k \to \infty} x_{n_k} = x$$

Proposition 12:

Any Cauchy sequence is bounded

Proof: Set $\varepsilon = 1$ (fixed) Then $\exists N_1 \in \mathbb{N}$ s.t.

$$|x_m - x_n| < 1, \forall m, n \geq N_1$$

$$\Rightarrow |x_n - x_{N_1}| < 1, \forall n \geq N_1 \Rightarrow$$

$\Rightarrow x_n$ is bounded after N_1

- $\{x_1, x_2, \ldots, x_n\}$ finite $\Rightarrow$ bounded $\rightarrow (x_n)$ bounded

Ex:

$x_n = 1 + \frac{1}{2} + \dots + \frac{1}{n}$ is not Cauchy, hence not convergent

$x_{n+1} = x_n + \frac{1}{n+1} > x_n \Rightarrow (x_n)$ is increasing $\Rightarrow$

$\Rightarrow \exists \lim\limits_{n\to\infty} x_n = \infty$

- $x_{2n} - x_n = \underbrace{\frac{1}{n+1}}_{>\frac{1}{2n}} + \underbrace{\frac{1}{n+2}}_{>\frac{1}{2n}} + \dots + \frac{1}{2n} > n \cdot \frac{1}{2n} = \frac{1}{2}$

So $x_{2n} - x_n > \frac{1}{2}, \forall n \in \mathbb{N} \Rightarrow$

$\Rightarrow (x_n)$ is not Cauchy $\Rightarrow$ not convergent

Theorem 11:

Ex: $f(x) = x^3$

$f'(0) = f''(0) = 0$

$f'''(0) = 6$