if $A, B \in M_{m,n}(K)$ and $A \sim B$ then
$\operatorname{rank} A = \operatorname{rank} B$

So if E is the row echilon form of the matrix A,
then $\operatorname{rank} A =$ number of nonzero rows in E

9.2. $\begin{pmatrix} 1 & -1 & 3 & 2 \\ -2 & 0 & 3 & -1 \\ -1 & 2 & 0 & -1 \end{pmatrix} \underset{L_3 \leftarrow L_3 + L_1}{\overset{L_2 \leftarrow L_2 + 2L_1}{\sim}} \begin{pmatrix} 1 & -1 & 3 & 2 \\ 0 & -2 & 9 & 3 \\ 0 & 1 & 3 & 1 \end{pmatrix} \sim$

$\overset{L_3 \leftrightarrow L_2}{\sim} \begin{pmatrix} 1 & -1 & 3 & 2 \\ 0 & 1 & 3 & 1 \\ 0 & -2 & 9 & 3 \end{pmatrix} \overset{L_3 \leftarrow L_3 + 2L_2}{\sim} \begin{pmatrix} 1 & -1 & 3 & 2 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 15 & 5 \end{pmatrix}$

$\Rightarrow \operatorname{rank} A = 3$

9.3. $\begin{pmatrix} \beta & 1 & 3 & 4 \\ 1 & \alpha & 3 & 3 \\ 2 & 3\alpha & 4 & 7 \end{pmatrix}, (\alpha, \beta \in \mathbb{R}) \overset{L_1 \leftrightarrow L_2}{\sim}$

$\sim \begin{pmatrix} 1 & \alpha & 3 & 3 \\ \beta & 1 & 3 & 4 \\ 2 & 3\alpha & 4 & 7 \end{pmatrix} \underset{L_3 \leftarrow L_3 - 2L_1}{\overset{L_2 \leftarrow L_2 - \beta L_1}{\sim}} \begin{pmatrix} 1 & \alpha & 3 & 3 \\ 0 & 1-\beta\alpha & 3-3\beta & 4-3\beta \\ 0 & \alpha & -2 & 1 \end{pmatrix}$

$$L_2 \leftrightarrow L_3 \sim \begin{pmatrix} 1 & \alpha & 3 & 3 \\ 0 & \alpha & -2 & 1 \\ 0 & 1-\beta\alpha & 3-3\beta & 4-3\beta \end{pmatrix}$$

$$\text{I } \alpha = 0 \Rightarrow \begin{pmatrix} 1 & 0 & 3 & 3 \\ 0 & 0 & -2 & 1 \\ 0 & 1 & 3-3\beta & 4-3\beta \end{pmatrix} \overset{L_2 \leftrightarrow L_3}{\sim}$$

$$\sim \begin{pmatrix} 1 & 0 & 3 & 3 \\ 0 & 1 & 3-3\beta & 4-3\beta \\ 0 & 0 & -2 & 1 \end{pmatrix} \Rightarrow \text{rank } A = 3$$

$\text{II } \alpha \neq 0$

$$A \sim \begin{pmatrix} 1 & \alpha & 3 & 3 \\ 0 & 1 & -\frac{2}{\alpha} & \frac{1}{\alpha} \\ 0 & 1-\beta\alpha & 3-3\beta & 4-3\beta \end{pmatrix} \overset{L_3 \leftarrow L_3 - (1-\beta\alpha)L_2}{\sim}$$

$$\sim \begin{pmatrix} 1 & \alpha & 3 & 3 \\ 0 & 1 & -\frac{2}{\alpha} & \frac{1}{\alpha} \\ 0 & 0 & x & y \end{pmatrix}$$

$$x = 3-3\beta + \frac{2}{\alpha}(1-\beta\alpha) = 3-3\beta+\frac{2}{\alpha}-2\beta =$$
$$= 3-5\beta+\frac{2}{\alpha}$$
$$y = 4-3\beta-\frac{1}{\alpha}(1-\beta\alpha) = 4-3\beta-\frac{1}{\alpha}+\beta = 4-2\beta-\frac{1}{\alpha}$$

$$\text{rank } A = \begin{cases} 2 & \text{if } x=y=0 \\ 3 & \text{otherwise} \end{cases}$$

$$x=y=0 \Leftrightarrow \begin{cases} 3-5\beta+\frac{2}{\alpha}=0 \\ 4-2\beta-\frac{1}{\alpha}=0 \end{cases} \Leftrightarrow \begin{cases} 3-5\beta+2\gamma=0 \\ 4-2\beta-\gamma=0 \end{cases}$$

$$\frac{1}{\alpha}=\gamma$$

$$\Leftrightarrow -5\beta = -2\gamma-3 \Rightarrow \beta = \frac{2\gamma+3}{5}$$

$$4-2\cdot\frac{2\gamma+3}{5}-3=0 \Leftrightarrow 20-4\gamma-6-5\gamma=0 \Leftrightarrow$$

$$\Leftrightarrow 14-9\gamma=0$$
$$\gamma=\frac{14}{9} \Rightarrow \alpha=\frac{9}{14}$$
$$\beta=\frac{11}{9}$$

$$\operatorname{rank} A = \begin{cases} 2, & \text{if } \alpha = \frac{3}{14},\ \beta = \frac{11}{9} \text{ or } \alpha = 0 \\ 3 & \text{otherwise} \end{cases}$$

$$\forall A \in \mathcal{M}_n(K): \left(A \mid I_n\right) \xrightarrow{\text{Gauss-Jordan elimination}} \cdots \sim \left(I_n \mid A^{-1}\right)$$

9.4. Compute the inverse of the matrix:

$$A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 2 & 1 & -2 & 0 & 1 & 0 \\ 2 & -2 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow[L_3 \leftarrow L_3 - 2L_1]{L_2 \leftarrow L_2 - 2L_1} \left(\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & -3 & -6 & -2 & 1 & 0 \\ 0 & -6 & -3 & -2 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{L_2 \leftarrow \frac{1}{3}L_2} \left(\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & \frac{2}{3} & -\frac{1}{3} & 0 \\ 0 & -6 & -3 & -2 & 0 & 1 \end{array}\right) \xrightarrow{L_3 \leftarrow L_3 + 6L_2} \left(\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & \frac{2}{3} & -\frac{1}{3} & 0 \\ 0 & 0 & 9 & 2 & -2 & 1 \end{array}\right)$$

$$\xrightarrow{L_3 \leftarrow \frac{1}{9}L_3} \left(\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & \frac{2}{3} & -\frac{1}{3} & 0 \\ 0 & 0 & 1 & \frac{2}{9} & -\frac{2}{9} & \frac{1}{9} \end{array}\right) \xrightarrow[L_1 \leftarrow L_1 - 2L_3]{L_2 \leftarrow L_2 - 2L_3}$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & \frac{5}{9} & \frac{4}{9} & -\frac{2}{9} \\ 0 & 1 & 0 & \frac{2}{9} & \frac{1}{9} & -\frac{2}{9} \\ 0 & 0 & 1 & \frac{2}{9} & -\frac{2}{9} & \frac{1}{9} \end{array}\right) \underset{L_1 \leftarrow L_1 - 2L_2}{\sim}$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{9} & \frac{2}{9} & \frac{2}{9} \\ 0 & 1 & 0 & \frac{2}{9} & \frac{1}{9} & -\frac{2}{9} \\ 0 & 0 & 1 & \frac{2}{9} & -\frac{2}{9} & \frac{1}{9} \end{array}\right)$$

$$\underbrace{\quad\quad}_{I_3} \quad \underbrace{\quad\quad}_{A^{-1}}$$

$$A^{-1} = \begin{pmatrix} \frac{1}{9} & \frac{2}{9} & \frac{2}{9} \\ \frac{2}{9} & \frac{1}{9} & -\frac{2}{9} \\ \frac{2}{9} & -\frac{2}{9} & \frac{1}{9} \end{pmatrix}$$

$$A \cdot A^{-1} = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{9} & \frac{2}{9} & \frac{2}{9} \\ \frac{2}{9} & \frac{1}{9} & -\frac{2}{9} \\ \frac{2}{9} & -\frac{2}{9} & \frac{1}{9} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Verify if it's good

$$= I_3$$

We can use Gauss elimination to extract a basis from a system of generators.

9.9. Determine the dimensions of the subspaces $S, T, S+T, S\cap T$ and find bases for $S, T, S+T$

$S = \langle (1,0,4), (2,1,0), (1,1,-4) \rangle$

$T = \langle (-3,-2,4), (5,2,4), (-2,0,-8) \rangle$

We find a basis for S

$$S: \begin{pmatrix} 1 & 0 & 4 \\ 2 & 1 & 0 \\ 1 & 1 & -4 \end{pmatrix} \underset{L_2 \leftarrow L_2 - 2L_1}{\overset{L_3 \leftarrow L_3 - L_1}{\sim}} \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & -8 \\ 0 & 1 & -8 \end{pmatrix} \overset{L_3 \leftarrow L_3 - L_2}{\sim}$$

$$\sim \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & -8 \\ 0 & 0 & 0 \end{pmatrix} \underset{\text{form}}{\overset{\text{Echelon}}{\Longrightarrow}} \big((1,0,4), (0,1,-8)\big)$$

$\hookrightarrow$ basis for S

$\Rightarrow \dim S = 2$

$$T: \begin{pmatrix} -3 & -2 & 4 \\ 5 & 2 & 4 \\ -2 & 0 & -8 \end{pmatrix} \xrightarrow[L_2 \leftarrow 3L_2 + 5L_1]{L_3 \leftarrow -3L_3 + 2L_1} \begin{pmatrix} -3 & -2 & 4 \\ 0 & -4 & 32 \\ 0 & -4 & 32 \end{pmatrix}$$

$$\xrightarrow{L_3 \leftarrow L_3 - L_2} \begin{pmatrix} -3 & -2 & 4 \\ 0 & -4 & 32 \\ 0 & 0 & 0 \end{pmatrix} \xRightarrow[\text{form}]{\text{Echelon}} \big((-3,-2,4), (0,-4,32) \big)$$

$\hookrightarrow$ basis for T

$\dim T = 2$

$$S+T: \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & -8 \\ -3 & -2 & 4 \\ 0 & -4 & 32 \end{pmatrix} \xrightarrow{L_3 \leftarrow L_3 + 3L_1}$$

$$\sim \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & -8 \\ 0 & -2 & 16 \\ 0 & -4 & 32 \end{pmatrix} \xrightarrow[L_4 \leftarrow L_4 + 4L_4]{L_3 \leftarrow L_3 + 2L_2} \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & -8 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow$$

$$\xRightarrow[\text{form}]{\text{Ec.}} \big((1,0,4), (0,1,-8) \big)$$

$\hookrightarrow$ bases for $S+T$

$\dim S+T = 2$

$$\dim S+T = \dim S + \dim T - \dim S\cap T$$

$$\Rightarrow \dim S\cap T = 2+2-2 = 2$$