Ch 8: ~~11~~, ~~14~~, 17, ~~18~~, ~~21~~, ~~24~~

8.14. Determine the tangent plane of the hyperboloid $H^1_{2,3,1}$: $\frac{x^2}{4}+\frac{y^2}{9}-\frac{z^2}{1}=1$ in the point $M(2,3,1)$

Show that the tangent plane in M intersects the surface in two lines.

$$\nabla f = \left(\frac{2x_0}{4}, \frac{2y_0}{9}, -2z_0\right)$$

$$T_{(x_0,y_0,z_0)} H_{2,3,1}: \nabla f \cdot (x-x_0, y-y_0, z-z_0) = 0$$

$$\frac{x_0}{2}(x-x_0)+\frac{2y_0}{9}(y-y_0)-2z_0(z-z_0)=0$$

$$x-2+\frac{2}{3}(y-3)-2(z-1)=0$$

$$3x-\cancel{6}+2y-6-6z+\cancel{6}=0$$

$$T_M H^1_{2,3,1} \quad 3x+2y-6z-6=0$$

$$T_M H \cap H: \begin{cases} 3x+2y-6z-6=0 \Rightarrow z = \frac{3x+2y-6}{6} \\ \frac{x^2}{4}+\frac{y^2}{9}-z^2=1 \end{cases}$$

$$\frac{x^2}{4}+\frac{y^2}{9}-\left(\frac{3x+2y-6}{6}\right)^2=1$$

$$\cancel{\frac{x^2}{4}} + \cancel{\frac{y^2}{9}} - \frac{\cancel{9x^2}+\cancel{4y^2}+36+12xy-36x+24y}{36} = 1 \iff$$

$$-36-12xy+36x+24y=36$$

$$-xy+3x+2y=6$$

$$x(-y+3)+(3-y)(-2)=0$$

$$(3-y)(x-2)=0$$

$$x=2 \Rightarrow 2y-6z=0 \Rightarrow y-3z=0$$

$$\Rightarrow T_H \cap H: \begin{cases} x=2 \\ y-3z=0 \end{cases}$$

$$y=3 \Rightarrow 3x-6z=0 \Rightarrow x-2z=0$$

$$\Rightarrow T_H \cap H: \begin{cases} x-2z=0 \\ y=3 \end{cases}$$

24 Which of the following is a hyperboloid?

a) $\varphi: 2xz + 2xy + 2yz = 1$

b) $\varphi: 5x^2 + 3y^2 + xz = 1$

c) $\varphi: 2xy + 2yz + y + z = 2$

b) $5x^2 + 3y^2 + xz = 1$

$$(\sqrt{5}x)^2 + \underbrace{xz}_{2\sqrt{5}x \cdot \frac{1}{2\sqrt{5}} z} + \left(\frac{1}{2\sqrt{5}} z\right)^2 + 3y^2 - \left(\frac{1}{2\sqrt{5}}\right) z^2 = 1$$

$$\left(\sqrt{5}x + \frac{z}{2\sqrt{5}}\right)^2 + (\sqrt{3}y)^2 - \left(\frac{1}{2\sqrt{5}} z\right)^2 = 1$$

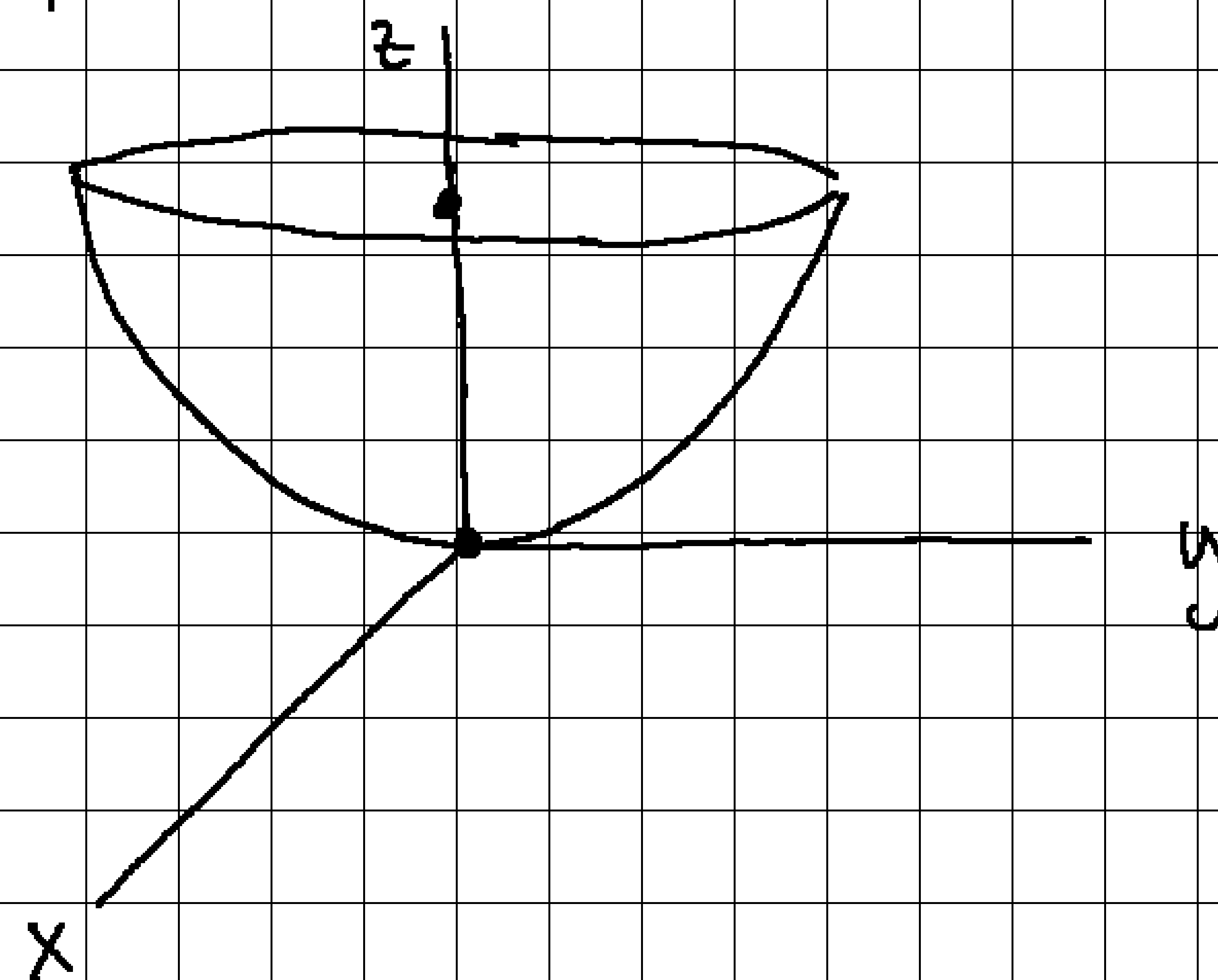
$$x' = \sqrt{5}x + \frac{z}{2\sqrt{5}}$$

$$y = \sqrt{3}y$$

$$z' = \frac{1}{2\sqrt{5}} z$$

$$x'^2 + y'^2 - z'^2 = 1 \quad \Rightarrow \text{hyperboloid of one sheet}$$

21. Use a parametrization of a parabola and a rotation matrix to deduce a parametrization of an elliptic paraboloid of revolution



The parabola has the equation: $P\begin{cases} x=0 \\ y^2=2pz \end{cases}$

$$P:\begin{cases} x=0 \\ y=t \\ z=\frac{t^2}{2p} \end{cases}, t\in\mathbb{R}$$

$$Rot_{Oz,\theta} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \theta\in[0,2\pi)$$

$$\Rightarrow P^{\ell}: \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ t \\ \frac{t^2}{2p} \end{pmatrix} =$$

$$= \begin{pmatrix} -t\sin\theta \\ t\cos\theta \\ \frac{t^2}{2p} \end{pmatrix}$$

We see that the surface has an implicit equation

$$x'^2 + y'^2 - 2pz = 0$$

$$\Rightarrow \frac{x'^2}{p} + \frac{y'^2}{p} = 2z$$

$$\Rightarrow P^\ell = P^\ell_{p,p}$$

18. Determine the tangent plane of the elliptic paraboloid $P: \frac{x^2}{5} + \frac{y^2}{3} = z$ which are parallel to the plane $\Pi: x - 3y + 2z - 1 = 0$

$$\vec{n}_\Pi = (1, -3, 2)$$

$$f = \frac{x^2}{5} + \frac{y^2}{3} - z = 0$$

$$\nabla f = \left(\frac{2x_0}{5}, \frac{2y_0}{3}, -1\right)$$

$$\vec{n}_{T_P} = \nabla f = \left(\frac{2x_0}{5}, \frac{2y_0}{3}, -1\right)$$

tangent plane

$$\frac{\frac{2x_0}{5}}{1} = \frac{\frac{2y_0}{3}}{-3} = \frac{-1}{2}$$

$$\frac{2x_0}{5} = -\frac{2y_0}{9} = -\frac{1}{2}$$

$$x_0 = -\frac{5}{4} \quad y_0 = \frac{9}{4} \quad z_0 = \frac{x_0^2}{5} + \frac{y_0^2}{3} = \frac{32}{16} = 2$$

$$\frac{2x_0}{5}(x-x_0)+\frac{2y_0}{3}(y-y_0)-(z-z_0)=0 \Leftrightarrow$$

$$\Leftrightarrow -\frac{2}{5}\cdot\frac{5}{4}\left(x+\frac{5}{4}\right)+\frac{2}{3}\cdot\frac{9}{4}\left(\cancel{y}-\frac{9}{4}\right)-(z-2)=0$$

$$\Leftrightarrow -\frac{1}{2}x-\frac{5}{8}+\frac{3}{2}y-\frac{27}{8}-z+2=0 \quad |\cdot 8$$

$$-4x-5+12y-27-8z+16=0$$

$$T_P: -4x+12y-8z-16=0$$

$$T_P: -x+3y-4z-4=0$$

11. For the surface S with parametrization

$$\varphi:\begin{cases} x=4\cos(s)\cos(t) \\ y=4\sin(s)\cos(t) \\ z=2\sin(t) \end{cases} \qquad s\in[0,2\pi) \qquad t\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right)$$

a) Give an equation of S

b) Find the parameters of the point $P(3,\sqrt{3},1)$

c) Calculate a parametrization of the tangent plane T_PS using partial derivatives

d) Give an equation of T_PS

a) $x^2+y^2 = 16\cos^2 t\,(\cos^2 s + \sin^2 s) = 16\cos^2 t$

$\Rightarrow \frac{1}{16}x^2 + \frac{1}{16}y^2 + \frac{1}{4}z^2 = 1$

b) $\begin{cases} 4\cos(s)\cos(t) = 3 \\ 4\sin(s)\cos(t) = \sqrt{3} \\ 2\sin t = 1 \end{cases}$

$\sin t = \frac{1}{2} \Rightarrow t = \frac{\pi}{6}$

$\begin{cases} \frac{4\sqrt{3}}{2}\cdot\cos(s) = 3 \\ \frac{4\cdot\sqrt{3}}{2}\sin(s) = \sqrt{3} \end{cases}$

$\begin{cases} \cos(s) = \frac{\sqrt{3}}{2} \\ \sin(s) = \frac{1}{2} \end{cases}$

$\Rightarrow s = \frac{\pi}{6}$

c) $D\left(T_{(\frac{\pi}{6},\frac{\pi}{6})} S\right) = \left\langle \frac{\partial F}{\partial s}\left(\frac{\pi}{6},\frac{\pi}{6}\right), \frac{\partial F}{\partial s}\left(\frac{\pi}{6},\frac{\pi}{6}\right)\right\rangle$

$\frac{\partial F}{\partial s}\left(\frac{\pi}{6},\frac{\pi}{6}\right) = \langle -4\sin(s)\cos(t),\ 4\cos s\cdot\cos t,\ 0\rangle =$

$= \langle -4\cdot\frac{1}{2}\cdot\frac{\sqrt{3}}{2},\ 4\cdot\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{2},\ 0\rangle = \langle -\sqrt{3}, 3, 0\rangle$

$$\frac{\partial F}{\partial t} = \left(-4\cos s \sin t\ ,\ -4\sin s \sin t\ ,\ 2\cos t\right) =$$

$$\frac{\partial F}{\partial t}\left(\frac{\pi}{6},\frac{\pi}{6}\right) = \left(-4\cdot\frac{\sqrt{3}}{2}\cdot\frac{1}{2},\ -4\frac{1}{2}\cdot\frac{1}{2},\ 2\frac{\sqrt{3}}{2}\right) = \left(-\sqrt{3},-1,\sqrt{3}\right)$$

$$T_{\left(\frac{\pi}{6},\frac{\pi}{6}\right)}S : \begin{vmatrix} x-3 & y-\sqrt{3} & z-1 \\ -\sqrt{3} & 3 & 0 \\ -\sqrt{3} & -1 & \sqrt{3} \end{vmatrix} = 0$$

d) $T_{(3,\sqrt{3},0)}S : \frac{1}{16}\cdot x\cdot 3 + \frac{1}{16}\cdot y\cdot\sqrt{3} + \frac{1}{4}z\cdot 0 = 1$