Ch 5: ~~16~~, ~~18~~, ~~19~~, 20.3, ~~22~~, ~~23~~, 24

$f: \mathbb{E}^n \to \mathbb{E}^n \qquad f(P) = A \cdot P + b$

$(\text{lin} f)(P) = A \cdot P \qquad$ affine morphism

f isometry $\Leftrightarrow \forall A, B \in \mathbb{E}^n$

$\text{dist}(A, B) = \text{dist}(f(A), f(B))$

$\Leftrightarrow f(P) = A \cdot P + b$ and $A \in O(n) = \{ X \in M_n(\mathbb{R}) : X^{-1} = X^T \}$

$\Leftrightarrow f(P) = A \cdot P + b$ and $AA^T = I_n$

Classification:

f isometry
- direct: $\det f = 1$
- indirect: $\det f = -1$

$n = 2$

direct isometry
- translate: $\text{Fix}(f) = \emptyset$
- rotation: $\text{Fix}(f) = \{ * \}$
- indentity: $\text{Fix}(f) = \mathbb{E}^2$

indirect isometry
- reflection w.r. to a line: $\text{Fix}(f) = \ell$
- glide reflection

$f : \mathbb{E}^n \to \mathbb{E}^n$

$\text{Fix}(f) = \{ P \in \mathbb{E}^n \mid f(P) = P \}$

5.16. Let F be the isometry obtained by applying $Rot_{-\frac{\pi}{3}}$ after a translation $T_{(-2,5)}$.

Determine the inverse transformation T^{-1}

$F = Rot_{-\frac{\pi}{3}} \circ T_{(-2,5)} : \mathbb{E}^2 \to \mathbb{E}^2$

$$\left(\begin{array}{cc|c} \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ \hline 0 & 0 & 1 \end{array}\right) \left(\begin{array}{cc|c} 1 & 0 & -2 \\ 0 & 1 & 5 \\ \hline 0 & 0 & 1 \end{array}\right) \cdot \begin{pmatrix} X_1 \\ X_2 \\ 1 \end{pmatrix} =$$

$$= \underbrace{\begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} & -1 + \frac{5\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} & \sqrt{3} + \frac{5}{2} \\ 0 & 0 & 1 \end{pmatrix}}_{= m} \begin{pmatrix} X_1 \\ X_2 \\ 1 \end{pmatrix}$$

$$m = \frac{1}{2} \begin{pmatrix} 1 & \sqrt{3} & -2 + 5\sqrt{3} \\ -\sqrt{3} & 1 & 2\sqrt{3} + 5 \\ 0 & 0 & 2 \end{pmatrix}$$

$$\left[\widehat{Rot_{-\frac{\pi}{3}}}\right]^{-1} = \left[\widehat{Rot_{\frac{\pi}{3}}}\right]$$

$$\left[T_{(2,-5)}\right]^{-1} = \left[T_{(2,-5)}\right]$$

$$F^{-1} = \left(Rot_{-\frac{\pi}{3}} \circ T_{(-2,5)}\right)^{-1}$$

$$F^{-1} = T_{(2,-5)} \circ Rot_{\frac{\pi}{3}}$$

$$\left[\widehat{F^{-1}}\right] = \left(\begin{array}{cc|c} \frac{1}{\sqrt{2}} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ \hline 0 & 0 & 1 \end{array}\right)\left(\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -5 \\ \hline 0 & 0 & 1 \end{array}\right) \quad 17$$

$$= \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 2 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & -5 \\ 0 & 0 & 1 \end{pmatrix}$$

5.18. $f : \mathbb{E}^2 \to \mathbb{E}^2$

$$f(P) = \frac{1}{\sqrt{13}} \begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix} \cdot P + \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

Show that f is a rotation, find it's centre, it's angle of rotation and the inverse f^{-1}.

$f(P) = A \cdot P + b$

If f rotation $\Rightarrow Tr(A) = 2\cos\theta$

$$M = [\operatorname{lin} f] = \frac{1}{\sqrt{13}} \begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix}$$

$$M \cdot M^T = \frac{1}{\sqrt{13}} \begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ -3 & 2 \end{pmatrix} \cdot \frac{1}{\sqrt{13}} = \frac{1}{13} \begin{pmatrix} 13 & 0 \\ 0 & 13 \end{pmatrix} = I_2 \Rightarrow$$

$\Rightarrow M \in O(2) \Rightarrow$ isometry (1)

$$\det M = \begin{vmatrix} 2 & -3 \\ 3 & 2 \end{vmatrix} \cdot \left(\frac{1}{\sqrt{3}}\right)^2 = 4 + 9 \cdot \frac{1}{13} = 1$$

$\Rightarrow M \in SO(2) \Rightarrow$ direct isometry (2)

$\overset{(1)(2)}{\Rightarrow} f$ is a rotation

$P(x,y)$

$f(P) = P$

$$\frac{1}{\sqrt{13}}\begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \Leftrightarrow$$

$$\Leftrightarrow \frac{1}{\sqrt{13}}\begin{pmatrix} 2x-3y+1 \\ 3x+2y-2 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$\sqrt{13}$

$-2\sqrt{13}$

the calculation was wrong

$$\begin{cases} \sqrt{13}\,x = 2x-3y+1 \\ \sqrt{13}\,y = 3x+2y-2 \end{cases} \Rightarrow \begin{cases} y = \dfrac{x(2-\sqrt{13})+1}{3} \\ \sqrt{13}\,\dfrac{x(2-\sqrt{13})+1}{3} = 3x + 2\,\dfrac{x(2-\sqrt{13})+1}{3} - 2 \quad \Big| \cdot 3 \end{cases}$$

$$\Leftrightarrow x\left[\sqrt{13}(2-\sqrt{13}) - 9 - 2(2-\sqrt{13})\right] = -\sqrt{13}+2-6$$

$$\Leftrightarrow x\left[\cancel{2\sqrt{13}} - 13 - 9 - 4 + \cancel{2\sqrt{13}}\right] = -\sqrt{13}-4$$

$$\Leftrightarrow x\left[4\sqrt{13}-26\right] = -\sqrt{13}-4$$

$$\Rightarrow x = \frac{-\sqrt{13}-4}{4\sqrt{13}-26} \Rightarrow y = \ldots$$

(we get the centre)

$$Tr_M = \frac{4}{\sqrt{13}} = 2\cos\theta \Rightarrow \cos\theta = \frac{2}{\sqrt{13}}$$

The inverse is found just like the ex. before.

For $n=3$

direct
- rotation arround an axis $Rot_{l,\theta}$
  $Fix(f) = l$
- indentity, $Fix(l) = \mathbb{E}^2$
  $Fix(l) = \emptyset$
- translation
- glide rotation

$T_{\vec{v}} \circ Rot_{l,\theta}$, where $\vec{v} \in D(l)$

indirect
- reflection w.r. to a plane π
  $Fix(f) = \pi$
- glide reflection
  $T_{\vec{v}} \circ Ref_{\pi}$ where $\vec{v} \in D(\pi)$
- rotation-reflection
  $Rot_{l,\theta} \circ Ref_{\pi}$ where $l \perp \pi$

if $f : \mathbb{E}^3 \to \mathbb{E}^3$ rotation then $Tr([\text{lin} f]) = 2\cos\theta + 1$

19. Verify that $A=\frac{1}{3}\begin{pmatrix}-1 & 2 & -2\\ -2 & -2 & -1\\ -2 & 1 & 2\end{pmatrix}\in SO(3)$. Determine the axis and the angle of rotation.

Possible ex. at the exam

$$A\cdot A^T=\frac{1}{3}\begin{pmatrix}-1 & 2 & -2\\ -2 & -2 & -1\\ -2 & 1 & 2\end{pmatrix}\begin{pmatrix}-1 & -2 & -2\\ 2 & -2 & 1\\ -2 & -1 & 2\end{pmatrix}\cdot\frac{1}{3}=$$

$$=\frac{1}{9}\begin{pmatrix}9 & 0 & 0\\ 0 & 9 & 0\\ 0 & 0 & 9\end{pmatrix}=I_3\Rightarrow A\in O(3)$$

$$\det A=\left(\frac{1}{3}\right)^3\begin{vmatrix}-1 & 2 & -2\\ -2 & -2 & -1\\ -2 & 1 & 2\end{vmatrix}=\frac{1}{27}\cdot 27=1\Rightarrow A\in SO(3)$$

$$\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\frac{1}{3}\begin{pmatrix}-1 & 2 & -2\\ -2 & -2 & -1\\ -2 & 1 & 2\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}\Leftrightarrow\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\frac{1}{3}\begin{pmatrix}-x+2y-2z\\ -2x-2y-z\\ -2x+y+2z\end{pmatrix}$$

$$\Leftrightarrow\begin{cases}3x=-x+2y-2z\\ 3y=-2x-2y-z\\ 3z=-2x+y+2z\end{cases}\Leftrightarrow\begin{cases}2x=y-z\\ 5y=-2x-z\Rightarrow 5y=-y+z-z\Rightarrow y=0\\ z=-2x+y\end{cases}$$

$$\Rightarrow\begin{cases}2x=-z\Rightarrow x=-\frac{1}{2}z\\ z=-2x\Leftrightarrow z=z\\ y=0\end{cases}$$

$$\Rightarrow \mathrm{Fix}(A) = \{(\alpha, 0, -2\alpha)\} \quad \alpha \in \mathbb{R}$$

$$\Rightarrow l : \begin{cases} x = \alpha \\ y = 0 \\ z = -2\alpha \end{cases}$$ is the axis of rotation

$$Tr(A) = 2\cos\theta + 1 \Rightarrow -1 = 2\cos\theta + 1$$

$$\Rightarrow -2 = 2\cos\theta$$

$$\Rightarrow \cos\theta = -1$$

$$\Rightarrow \theta = -\pi$$

Euler - Rodrigues:

$$Rot_{l,\theta}(P) = \cos\theta \cdot P + \sin\theta(\vec{v} \times P) + (1-\cos\theta)\langle v, P\rangle v$$

where $v \in D(l)$

$|\vec{v}| = 1$

22. Using the Euler-Rodrigues formula write the matrix form of a rotation arround the axis $\mathbb{R}\vec{v}$ where $\vec{v}(1,1,0)$. Use this matrix form to give a parametrization of a cylinder with axis $\mathbb{R}\vec{v}$

$$\vec{w} = \frac{1}{|\vec{v}|} \cdot \vec{v} = \frac{(1,1,0)}{\sqrt{2}}$$ (normalize the vector)

$$l : \begin{cases} x = \frac{1}{\sqrt{2}} \cdot \lambda \\ y = \frac{1}{\sqrt{2}} \cdot \lambda \\ z = 0 \end{cases}$$

$$Rot_{l,\theta}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \cos\theta \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \sin\theta \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ x & y & z \end{vmatrix} +$$

$$+ (1-\cos\theta)\left(\frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}y\right)\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}$$

$$= \cos\theta \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \sin\theta \left(\frac{1}{\sqrt{2}} z - \frac{1}{\sqrt{2}} z + \frac{1}{\sqrt{2}} y - \frac{1}{\sqrt{2}} x \right) +$$

$$+ (1 - \cos\theta) \cdot \frac{1}{2} \begin{pmatrix} x+y \\ x+y \\ 0 \end{pmatrix} =$$

$$= \begin{pmatrix} \cos\theta & 0 & 0 \\ 0 & \cos\theta & 0 \\ 0 & 0 & \cos\theta \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} -\frac{1}{\sqrt{2}} \sin\theta & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} \sin\theta & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} +$$

$$+ \begin{pmatrix} \frac{1-\cos\theta}{2} & \frac{1-\cos\theta}{2} & 0 \\ \frac{1-\cos\theta}{2} & \frac{1-\cos\theta}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} =$$