Chapter 6: 1, ~~3~~, ~~4~~, ~~5~~, ~~6~~, ~~7~~, ~~8~~, ~~9~~, ~~10~~

~~2~~, ~~4~~

Quadratic curves in $\mathbb{E}^2$: (ellipses)

Q quadratic

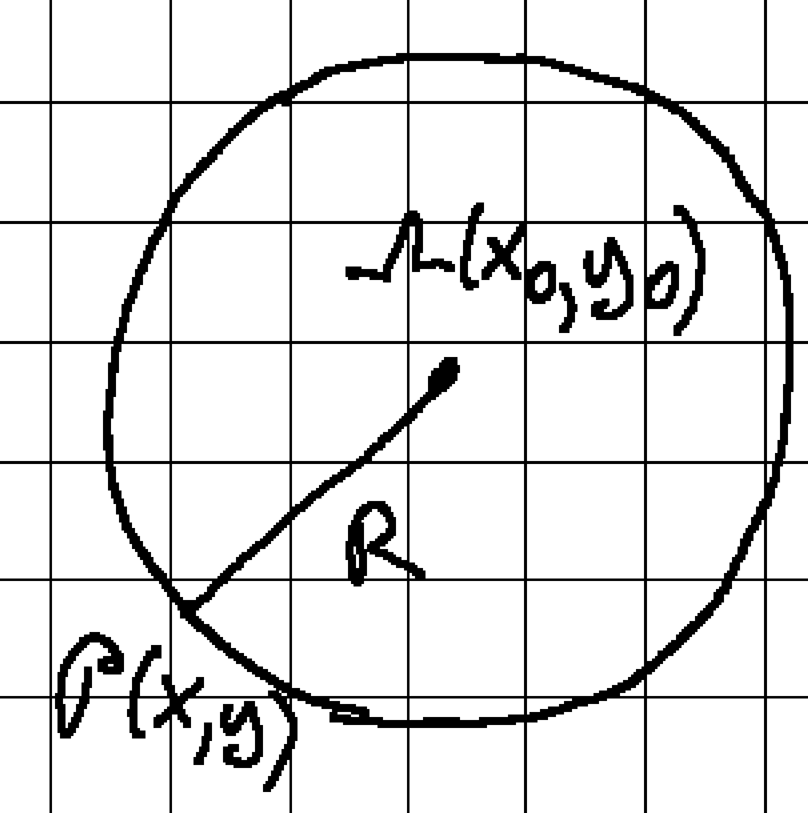
$$Q: \; a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_{10}x + 2a_{01}y + a_{00} = 0$$

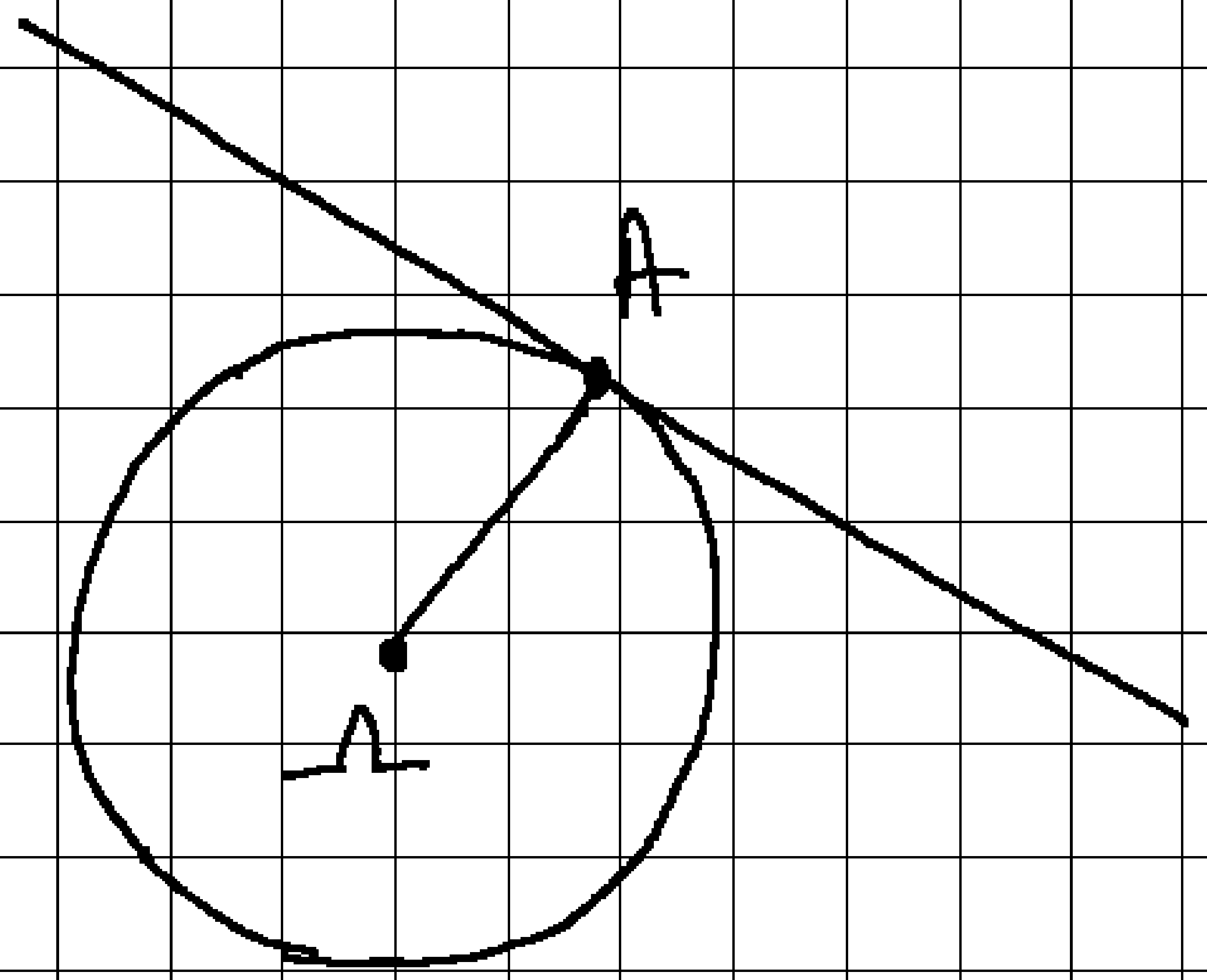
<u>Circle:</u> locus of points in the plane whose distance to a fixed point $\Omega(x_0, y_0)$ is a constant $R > 0$

$$\mathcal{C}(\Omega, R): \; (x-x_0)^2 + (y-y_0)^2 = R^2$$ (eq of a circle) implicit

parametric:

$$\mathcal{E}(\Omega, R): \begin{cases} x = x_0 + R\cos t \\ y = y_0 + R\sin t \end{cases}$$





$$T_{\mathcal{C},A} \perp \Omega A$$

6.1. Find the equation of the circle

a) of diameter [AB]

$A(1,2)$, $B(-3,-1)$

b) passing through $A(3,1)$ and $B(-1,3)$ and having the center on the line:

$$l: 3x - y - 2 = 0$$

a) $AB = \sqrt{4^2+3^2} = 5$

$$x_0 = \frac{x_A + x_B}{2} = -1$$

$$y_0 = \frac{y_A + y_B}{2} = \frac{1}{2}$$

$$(x+1)^2 + \left(y - \frac{1}{2}\right)^2 = \frac{5^2}{4}$$

b) $$\begin{cases} 3x_0 - y_0 - 2 = 0 \\ (3-x_0)^2 + (1-y_0)^2 = R^2 \\ (-1-x_0)^2 + (3-y_0)^2 = R^2 \end{cases} \Leftrightarrow \begin{cases} 3x_0 - y_0 - 2 = 0 \\ 9 - 6x_0 + x_0^2 + 1 - 2y_0 + y_0^2 = R^2 \quad (2) \\ 1 + 2x_0 + x_0^2 + 9 - 6x_0 + y_0^2 = R^2 \quad (3) \end{cases}$$

$(2)-(3) \Rightarrow \cancel{8} - 8x_0 - \cancel{8} + 4y_0 = 0$

$$\begin{cases} 2x_0 - y_0 = 0 \\ 3x_0 - y_0 - 2 = 0 \end{cases} \Bigg| \; -$$

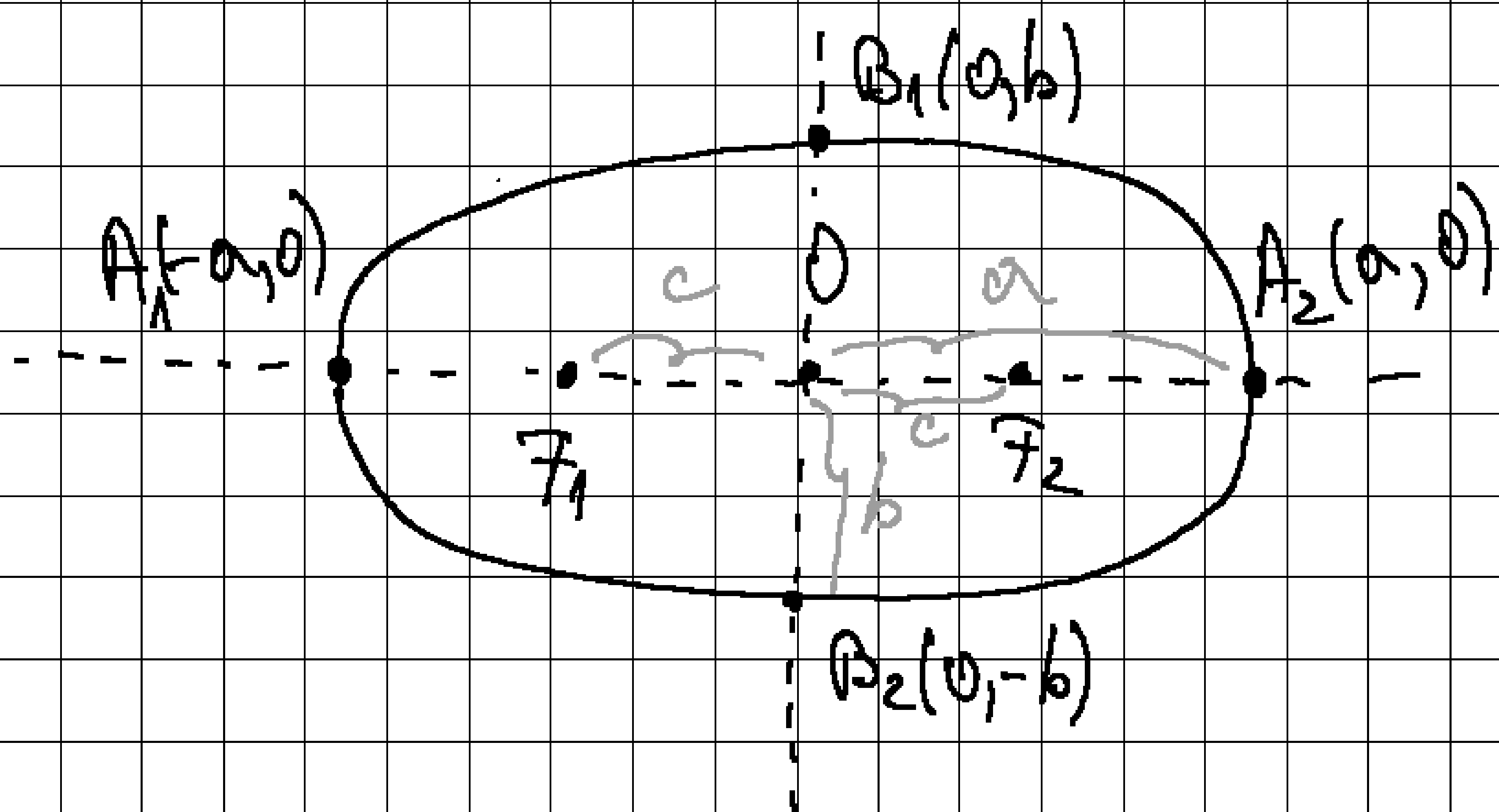
$$-x_0+2=0 \Rightarrow x_0=2$$

$$4-y_0=0 \Rightarrow y_0=4$$

$$(3-2)^2+(1-4)^2=R^2 \Rightarrow R=\sqrt{10}$$

$$\Rightarrow C: (x-2)^2+(y-4)^2=10$$

Ellipse: locus of points in the plane whose sum of distances to two distinct points (the focal points, the foci) is a constant $2a$



if we fix a reference system $K(O,\vec{i},\vec{j})$ where O midpoint of F_1F_2 and $\vec{i} \in D(F_1,F_2)$ then we have the canonical equation:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \quad \text{where } a,b>0,\ a>b$$

$$c=\sqrt{a^2-b^2}$$

Proof: $B_1F_1 + B_2F_2 = 2a$

$$B_1F_1 = B_2F_2 = \sqrt{c^2+b^2}$$

$$\Rightarrow \sqrt{c^2+b^2} = a \Rightarrow c^2 = a^2 - b$$

$$e = \frac{c}{a}$$ (excentricity)

6.3. Determine the foci of the ellipse

$9x^2 + 25y^2 = 225 \quad | :225$

$$\frac{x^2}{5^2} + \frac{y^2}{3^2} = 1$$

$$c = \sqrt{5^2 - 3^2} = 4 \Rightarrow F_1 = (-4, 0)$$

$$F_2 = (4, 0)$$

6.4. Determine the intersection of the line $\ell: x + 2y - 7 = 0$ and the ellipse $\mathcal{E}: x^2 + 3y^2 - 25 = 0$

$$\begin{cases} x + 2y - 7 = 0 \\ x^2 + 3y^2 - 25 = 0 \end{cases} \Leftrightarrow \begin{cases} x = 7 - 2y \\ (7-2y)^2 + 3y^2 - 25 = 0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow 49 - 28y + 4y^2 + 3y^2 - 25 = 0$$

$$\Leftrightarrow 7y^2 - 28y + 24 = 0$$

$$\Delta = 284 \qquad y_{1,2} = \frac{28 \pm 4\sqrt{7}}{14} = \frac{14 \pm 2\sqrt{7}}{7}$$

$$y_1 = \frac{14+2\sqrt{7}}{7} \Rightarrow x_1 = 7 - \frac{28+4\sqrt{7}}{7}$$

$$y_2 = \frac{14-2\sqrt{7}}{7} \Rightarrow x_2 = 7 - \frac{28-4\sqrt{7}}{7}$$

$$l: \ y = kx + m$$

$$C: \ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$l \cap C: \begin{cases} \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \\ y = kx + m \end{cases} \Leftrightarrow \begin{cases} \frac{x^2}{a^2} + \frac{(kx+m)^2}{b^2} = 1 \\ y = kx+m \end{cases}$$

$$\begin{cases} x^2\left(\frac{1}{a^2} + \frac{k^2}{b^2}\right) + \frac{2km}{b^2} \cdot x + \frac{m^2}{b^2} - 1 = 0 \\ y = kx+m \end{cases}$$

$$\Delta = \frac{4k^2m^2}{b^4} - 4 \cdot \frac{b^2+a^2k^2}{a^2b^2} \cdot \frac{m^2-b^2}{b^2} =$$

$$= \frac{4}{a^2b^4}\left(\cancel{a^2k^2m^2} - b^2m^2 + b^4 - \cancel{a^2k^2m^2} + b^2a^2k^2\right) =$$

$$= \frac{4}{a^2b^2}\left(b^2 - m^2 + a^2k^2\right)$$

$b^2+a^2k^2-m^2$	intersection
< 0	none
$= 0$	one intersection (tangent)
> 0	two intersections point (secant

If l tangent to $\mathcal{E}$:

$$m^2 = b^2 + a^2k^2$$

$$\Rightarrow m = \pm\sqrt{b^2 + a^2k^2}$$

So the eqn. of the tangent line with slope k is

$$y = kx \pm \sqrt{b^2 + a^2k^2}$$

$$T_{(x_0, y_0)}\mathcal{E}_{a,b} : \frac{xx_0}{a^2} + \frac{yy_0}{b^2} = 1$$

$$\Leftrightarrow \frac{x_0}{a^2}(x - x_0) + \frac{y_0}{b^2}(y - y_0) = 0$$

The tangent to $\mathcal{E}_{a,b}$ in the point $(x_0, y_0) \in \mathcal{E}_{a,b}$

6.6. Determine the equation of a line that is orthogonal to $l: 2x - 2y - 13 = 0$ and tangent to the ellipse $\mathcal{E}: x^2 + 4y^2 - 20 = 0$

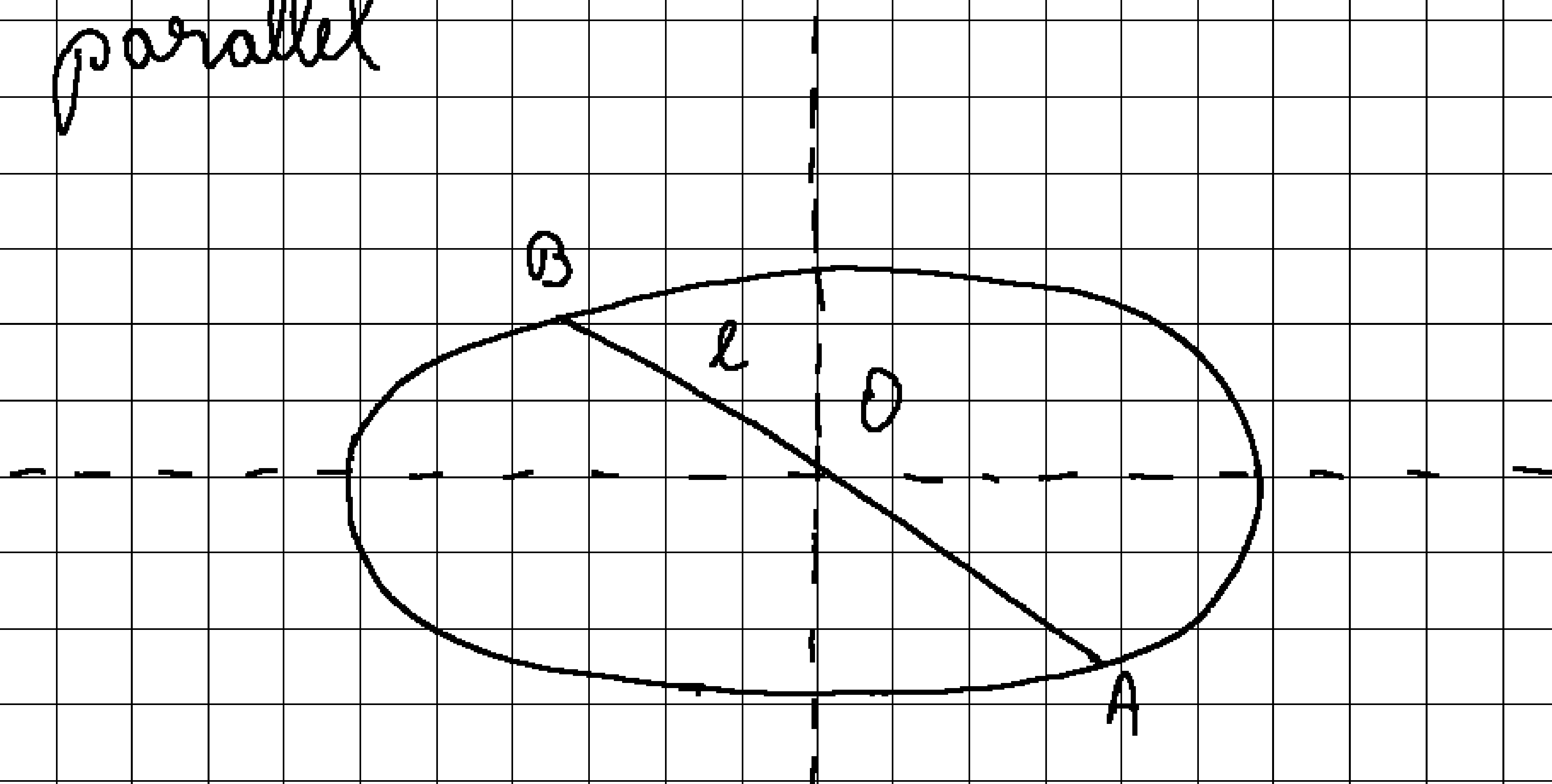
$$k_l = 1 \Rightarrow k_t = -1$$

$$\mathcal{E}: \frac{x^2}{20} + \frac{y^2}{5} = 1$$

$$m = \pm\sqrt{5 + 20} = \pm 5$$

$$t: y = -x \pm 5$$

6.7. A diameter of an ellipse is the line segment determined by the intersection points of the ellipse with a line passing throught the center of the ellipse. Show that the tangent lines to an ellipse at the endpoints of a diameter are parallel



Show that $T_A\mathcal{E} \parallel T_B\mathcal{E}$

$$\mathcal{E}: \ \frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

$$l: \ y = kx$$

$$AB: \begin{cases} \frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \\ y=kx \end{cases} \Leftrightarrow \frac{x^2}{a^2}+\frac{x^2k^2}{b^2}=1 \Leftrightarrow$$

$$\Leftrightarrow x^2\left(\frac{1}{a^2}+\frac{k^2}{b^2}\right)=1 \Rightarrow x_0=\pm\sqrt{\left(\frac{1}{a^2}+\frac{k^2}{b^2}\right)^{-1}}$$

$$T_{A,\mathcal{E}}: \quad \frac{x_A x}{a^2} + \frac{y_A y}{b^2} = 1$$

$$x_A = \left(\sqrt{\frac{1}{a^2} + \frac{k^2}{b^2}}\right)^{-1}$$

$$x_B = -\left(\sqrt{\frac{1}{a^2} + \frac{k^2}{b^2}}\right)^{-1}$$

$$y_0 = \pm k \sqrt{\left(\frac{1}{a^2} + \frac{k^2}{b^2}\right)^{-1}}$$

$$T_{A,\mathcal{E}}: \quad y = \frac{b^2\left(1 - \frac{x_A x}{a^2}\right)}{y_A}$$

$$\Rightarrow m_{T_A} = \frac{b^2 x_A}{y_A a^2}$$

$$m_{T_B} = \frac{b^2 x_B}{y_B a^2}$$

$$x_B = -x_A \Rightarrow y_B = -y_A$$

$$\Rightarrow m_{T_A} = m_{T_B} \Rightarrow T_A \parallel T_B$$

6.* Consider the ellipse $\mathcal{E}: x^2+4y^2=1$. Find the lines that contain $P(2,3)$ and are tangents to $\mathcal{E}$.

$$\ell_P : y-3 = m(x-2)$$

or

$$\ell_P : x=2$$

if $\ell_P : x=2$

$$\ell_P \cap \mathcal{E} : \begin{cases} x=2 \\ 4+4y^2=1 \end{cases} \Rightarrow \text{there is no intersection}$$

if $\ell_P : y-3 = m(x-2)$

$$\ell_P \cap \mathcal{E} : \begin{cases} y=3+m(x-2) \\ x^2+4y^2=1 \end{cases}$$

$$\Rightarrow \begin{cases} y=3+m(x-2) \\ x^2+4\left(9+m^2(x-2)^2+6m(x-2)\right)=1 \end{cases}$$

$$x^2(1+4m^2)+x(-16m^2+24m)+36-4m^2-48m=0$$

$$\Delta = 64\cdot(2m^2-3m)^2-16(1+4m^2)(-m^2-12m+9)$$

Next we solve $\Delta=0$ and we get m