Seminar 10

3. $f \in Hom_{\mathbb{R}}(\mathbb{R}^3, \mathbb{R}^4)$

$f(e_1) = (1,2,3,4)$

$f(e_2) = (4,3,2,1)$

$f(e_3) = (-2,1,4,1)$

$$[f]_{EE'} = \left([f(e_1)]_{E'}, [f(e_2)]_{E'}, [f(e_3)]_{E'} \right) =$$

$$= \begin{pmatrix} 1 & 4 & -2 \\ 2 & 3 & 1 \\ 3 & 2 & 4 \\ 4 & 1 & 1 \end{pmatrix}$$

$\operatorname{Im} f = \{ f(v) \mid v \in \mathbb{R}^3 \} = \{ w \in \mathbb{R}^4 \mid \exists\, v = (x,y,z) :$
$f(v) = w \} = \{ w = (a,b,c,d) \mid \exists\, v = (x,y,z) : [f]_{EE'} \cdot$
$\cdot [v]_{E'} = [w]_{E'} \} =$

$$= \left\{ w = (a,b,c,d) \mid \exists\, v = (x,y,z) : \begin{pmatrix} 1 & 4 & -2 \\ 2 & 3 & 1 \\ 3 & 2 & 4 \\ 4 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \right.$$

$$\left(\begin{array}{ccc|c} 1 & 4 & -2 & a \\ 2 & 3 & 1 & b \\ 3 & 2 & 4 & c \\ 4 & 1 & 1 & d \end{array}\right) \underset{\substack{L_3 \leftarrow L_3 - 3L_1 \\ L_4 \leftarrow L_4 - 4L_1}}{\overset{L_2 \leftarrow L_2 - 2L_1}{\sim}} \left(\begin{array}{ccc|c} 1 & 4 & -2 & a \\ 0 & -5 & 5 & b-2a \\ 0 & -10 & 10 & c-3a \\ 0 & -15 & 9 & d-4a \end{array}\right)$$

$$\underset{L_4 \leftarrow L_4 - 3L_2}{\overset{L_3 \leftarrow L_3 - 2L_1}{\sim}} \left(\begin{array}{ccc|c} 1 & 4 & -2 & a \\ 0 & -5 & 5 & b-2a \\ 0 & 0 & 0 & a-2b+c \\ 0 & 0 & -6 & 2a-3b+d \end{array}\right) \overset{L_3 \leftarrow L_4}{\sim} \left(\begin{array}{ccc|c} 1 & 4 & -2 & a \\ 0 & -5 & 5 & b-2a \\ 0 & 0 & -6 & 2a-3b+d \\ 0 & 0 & 0 & a-2b+c \end{array}\right)$$

the system is compatible iff $a-2b+c=0$

$$\operatorname{Im} f = \{(a,b,c,d) \mid a-2b+c=0\} =$$

$$= \{(2b-c, b, c, d) \mid b,c,d \in \mathbb{R}\} =$$

$$= \langle (2,1,0,0), (-1,0,1,0), (0,0,0,1) \rangle$$

We now extract the basis

$$\begin{pmatrix} 2 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \overset{L_1 \leftrightarrow L_2}{\sim} \begin{pmatrix} -1 & 0 & 1 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \overset{L_2 \leftarrow L_2 + 2L_2}{\sim}$$

$$\begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \Rightarrow \text{a basis of the Image is:}$$

$$((-1,0,1,0), (0,1,2,0), (0,0,0,1))$$

$$\Rightarrow \operatorname{Dim} \operatorname{Im} f = 3$$

Seminar 11

3. $\mathbb{R}_2[X] = \{ f \in \mathbb{R}[X] \mid degree(f) \leq 2 \}$

$E = (1, X, X^2)$, $B = (1, X-a, (X-a)^2)$

$B' = (1, X-b, (X-b)^2)$

T_{EB} , T_{BE} , $T_{BB'}$

$T_{EB} = [id]_{BE}$

$1 = 1$

$X-a = 1 \cdot X + (-a) \cdot 1$

$(X-a)^2 = X^2 - 2aX + a^2 = X^2 + (-2a)X + a^2 \cdot 1$

$[1]_E = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$[X-a]_E = \begin{pmatrix} -a \\ 1 \\ 0 \end{pmatrix}$ $\Rightarrow T_{EB} = \begin{pmatrix} 1 & -a & a^2 \\ 0 & 1 & -2a \\ 0 & 0 & 1 \end{pmatrix}$

$[(X-a)^2]_E = \begin{pmatrix} a^2 \\ -2a \\ 1 \end{pmatrix}$ $T_{BE} = T_{EB}^{-1} = \begin{pmatrix} 1 & -a & a^2 \\ 0 & 1 & -2a \\ 0 & 0 & 1 \end{pmatrix}^{-1}$

$$\left(\begin{array}{ccc|ccc} 1 & -a & a^2 & 1 & 0 & 0 \\ 0 & 1 & -2a & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \underset{L_1 \leftarrow L_1 - a^2 L_3}{\overset{L_2 \leftarrow L_2 + 2aL_3}{\sim}}$$

$$\left(\begin{array}{ccc|ccc} 1 & -a & 0 & 1 & 0 & -a \\ 0 & 1 & 0 & 0 & 1 & 2a \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \underset{\sim}{L_1 \leftarrow L_1 + aL_2}$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & a & a^2 \\ 0 & 1 & 0 & 0 & 1 & 2a \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \Rightarrow T_{BE} = \begin{pmatrix} 1 & a & a^2 \\ 0 & 1 & 2a \\ 0 & 0 & 1 \end{pmatrix}$$

I first method

$$T_{BB'} = [id]_{B',B} = [id]_{EB} [id]_{B'E}$$

$$[id]_{B'E} = \left([1]_E \quad [x-b]_E \quad [(x-b)^2]_E \right) = \begin{pmatrix} 1 & -b & b^2 \\ 0 & 1 & -2b \\ 0 & 0 & 1 \end{pmatrix}$$

$$T_{BB'} = [id]_{B',B} = \begin{pmatrix} 1 & a & a^2 \\ 0 & 1 & 2a \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -b & b^2 \\ 0 & 1 & -2b \\ 0 & 0 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & -a-b & b^2+a^2-2ab \\ 0 & 1 & 2a-2b \\ 0 & 0 & 1 \end{pmatrix}$$

II second method

$$1 = 1$$

$$x-b = 1(x-a) + a-b$$

$$[x-b]_E = \begin{pmatrix} a-b \\ 1 \\ 0 \end{pmatrix}$$

$$(x-b)^2 = x^2 - 2xb + b^2 =$$

$$= (x-a)^2 + 2ax - a^2 - 2xb + b^2 =$$

$$= (x-a)^2 + 2ax - 2xb - a^2 + b^2 =$$

$$= (x-a)^2 + 2a(x-a) - 2b(x-a) + 2a^2 - 2ab$$

$$-a^2 + b^2 =$$

$= (x-a)^2 + (2a-ab)(x-a) + (a^2 - 2ab + b^2)$

$f: V \to V'$

$b_1, b_2 \quad b_1', b_2'$

$$[f]_{b_1, b_1'} = [id]_{b_2', b_1'} [f]_{b_2, b_2'} [id]_{b_1, b_2} =$$

$$= T_{b_1', b_2'} \cdot [f]_{b_2, b_2'} T_{b_2, b_1}$$

$A, B \in M_{m,n}(K)$

$A \sim B \Leftrightarrow \exists T \in GL_m(K)$
$\exists S \in GL_n(K)$

$\Leftrightarrow B = T \cdot A \cdot S$

$\Leftrightarrow \exists f \in Hom(K, K)$
s.t. $B = [f]_{b_1, b_1'}$
$A = [f]_{b_2, b_2'}$
with b_1, b_2 bases of K^n
b_1', b_2' bases of K^m

$A, B \in M_n(K)$

$\underset{\text{similar}}{A \simeq B} \Leftrightarrow \exists S \in GL(K)$
$B = S^{-1} A S$

$A \in M_n(K)$ diagonalizable
$\Leftrightarrow \exists b$ basis of K^n s.t.
$[A]_b = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \lambda_n \end{pmatrix}$

Seminar 10.

5. $\mathbb{R}_2[X]$

$E=(1,X,X^2)$, $B=(1,X-1,X^2+1)$

$\varphi \in End_{\mathbb{R}}(\mathbb{R}_2[X])$

$$\varphi(a_0+a_1X+a_2X^2)=(a_0+a_1)+(a_1+a_2)X+(a_0+a_2)X^2$$

$[\varphi]_E=?$ $[\varphi]_B=?$

$$\begin{aligned}\varphi(1)&=1+X^2\\ \varphi(X)&=1+X\\ \varphi(X^2)&=X+X^2\end{aligned} \Rightarrow [\varphi]_E=\begin{pmatrix}1&1&0\\0&1&1\\1&0&1\end{pmatrix}$$

$$[\varphi]_B=[id]_{E,B}[\varphi]_E\cdot[id]_{B,E}=$$

$$=[id]_{B,E}^{-1}\cdot[\varphi]_E\cdot[id]_{B,E}$$

$$[id]_{B,E}=\left([1]_E,[X-1]_E,[X^2+1]_E\right)=\begin{pmatrix}1&-1&1\\0&1&0\\0&0&1\end{pmatrix}$$

$$A^T=\begin{pmatrix}1&0&0\\-1&1&0\\1&0&1\end{pmatrix}$$

$$A^*=\begin{pmatrix}1&1&-1\\0&1&0\\0&0&1\end{pmatrix}=A^{-1}$$

$$[\varphi]_B=\begin{pmatrix}1&1&-1\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&1&0\\0&1&1\\1&0&1\end{pmatrix}\begin{pmatrix}1&-1&1\\0&1&0\\0&0&1\end{pmatrix}$$

$$=\begin{pmatrix}0&2&0\\0&1&1\\1&-1&2\end{pmatrix}$$