Theorem: Any two bases of a vector space have the same number of elements.

Proof:

Take $\mathcal{B} = (v_1, \dots, v_m)$

and $\mathcal{B}' = (v_1', \dots, v_n')$ be bases of a vector space V over K

$$\left.\begin{array}{l} \mathcal{B} \text{ is lin. indep.} \\ \mathcal{B}' \text{ is a system of gens} \end{array}\right\} \xRightarrow{\text{Steinitz}} m \leq n$$

$$\left.\begin{array}{l} \mathcal{B}' \text{ is lin indep.} \\ \mathcal{B} \text{ is a system of gens} \end{array}\right\} \xRightarrow{\text{Steinitz}} n \leq m$$

Theorem: Let V be a vector space over K with $\dim V = n$ and $X = (u_1, \dots, u_n)$ a list of vectors in V. Then X is linearly independent in V $\Leftrightarrow$ X is a system of generators for V

$\boxed{\Rightarrow}$ Assume that $X = (u_1, \dots, u_n)$ is lin. indep.

$\dim V = n \Rightarrow \exists$ basis

$\mathcal{B} = (v_1, \dots, v_n)$ of V

$\Rightarrow \mathcal{B}$ is a system of generators for V

Steinitz (ii) $\Rightarrow$ n vectors from $\mathcal{B}$ can be replaced by the vectors of X obtaining again a system of gens. for V. So all vectors of $\mathcal{B}$ can be replaced by those from X and we get a system of gens for V $\Rightarrow$ X is a system of gens for V.

$\boxed{\Leftarrow}$ Assume that X is a system of gens for V. Suppose that X is linearly independent.

$\Rightarrow \exists\, j \in \{1, \dots, n\} : \quad u_j = \sum_{\substack{i=1 \\ i \neq j}}^{n} k_i u_i \quad , k_i \in K$
$i \in \{1, \dots, n\} \setminus \{j\}$

$$V = \langle X \rangle = \langle u_1, \dots, u_{j-1}, u_j, u_{j+1}, \dots, u_n \rangle =$$

$$= \langle u_1, \dots, u_{j-1}, \sum_{\substack{i=1 \\ i \neq j}}^{n} k_i u_i, u_{j+1}, \dots, u_n \rangle =$$

$$= \langle u_1, \dots, u_{j-1}, u_{j+1}, \dots, u_n \rangle$$

$\Rightarrow V$ has a system of gens with $n-1$ vectors

But $\dim V = n$ and the minimum no. of vectors in a system of gens is n $\Rightarrow$ contrad.

$\Rightarrow$ So X is lin indep

Theorem: Any linearly independent list of vectors in a vector space can be completed to a basis of the vector space.

Proof: Let $X = (u_1, \dots, u_m)$ be lin. indep.

Let $B = (v_1, \dots, v_n)$ be a basis of V

$\Rightarrow B$ is a system of gens for V

$\xRightarrow{\text{Steinitz}}$ $m \le n$ and m vectors from B can be replaced by those from X obtaining again a system of gens for V

Say (!) the first m are replaced

$(u_1, \dots, u_m, v_{m+1}, \dots, v_n)$ is a system of gens for V $\Big\}$ $\xRightarrow{Th}$

$\dim V = n$

$\Rightarrow (u_1, \dots, u_m, v_{m+1}, \dots, v_n)$ are lin. indep.

and so it is a basis of V

Ex: (e_1, e_2, e_3) is a basis $\Leftrightarrow$

$$\Leftrightarrow \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{vmatrix} \neq 0$$

Theorem: Let V be a vector space over K and let $S \leq V$. Then there exists $\bar{S} \leq V$ such that $V = S \oplus \bar{S}$. In particular, $\dim V = \dim S + \dim \bar{S}$

Proof: Let $(u_1, \dots, u_m)$ be a basis of S. Then it can be completed to a basis $B = (u_1, \dots, u_m, v_{m+1}, \dots, v_n)$ of V. We consider

$$\bar{S} = (v_{m+1}, \dots, v_n)$$

and we prove that $V = S \oplus \bar{S}$

$V = S \oplus \bar{S}$

$$\forall\, v \in V,\ v = \underbrace{\sum_{i=1}^{m} k_i u_i}_{\in S} + \underbrace{\sum_{j=m+1}^{n} k_j \underset{\in \bar{S}}{v_j}}_{\in \bar{S}} \in S + \bar{S} \Rightarrow$$

(uniquely)

(because $S \leq V, \bar{S} \leq V$)

$\Rightarrow V = S + \bar{S}$

• We show that $S \cap \bar{S} = \{0\}$. Let $v \in S \cap \bar{S} \Rightarrow v \in S$ and $v \in \bar{S} \Rightarrow v = \sum_{i=1}^{m} \underbrace{k_i}_{\in K} u_i$ and $v = \sum_{j=m+1}^{n} \underbrace{k'_j}_{\in K} v_j \Rightarrow$

$$\Rightarrow \boxed{\sum_{i=1}^{m} k_i u_i - \sum_{j=m+1}^{n} k'_j v_j = 0}$$

But $(u_1, \ldots, u_m, v_{m+1}, \ldots, v_n)$ is a basis of V

$\Rightarrow$ lin. indep.

$\Rightarrow$

$$\Rightarrow \begin{cases} k_i = 0, & i = \overline{1,m} \\ k'_j = 0, & j = \overline{m+1,n} \end{cases}$$

$$\Rightarrow v = 0 \Rightarrow S \cap \bar{S} = \{0\}$$