$$S \leq_K V \Leftrightarrow \begin{cases} S \neq \emptyset \quad (0 \in S) \\ \forall\ v_1, v_2 \in S,\ v_1 + v_2 \in S \\ \forall\ k \in K,\ \forall v \in S,\ k \cdot v \in S \end{cases} \Leftrightarrow$$

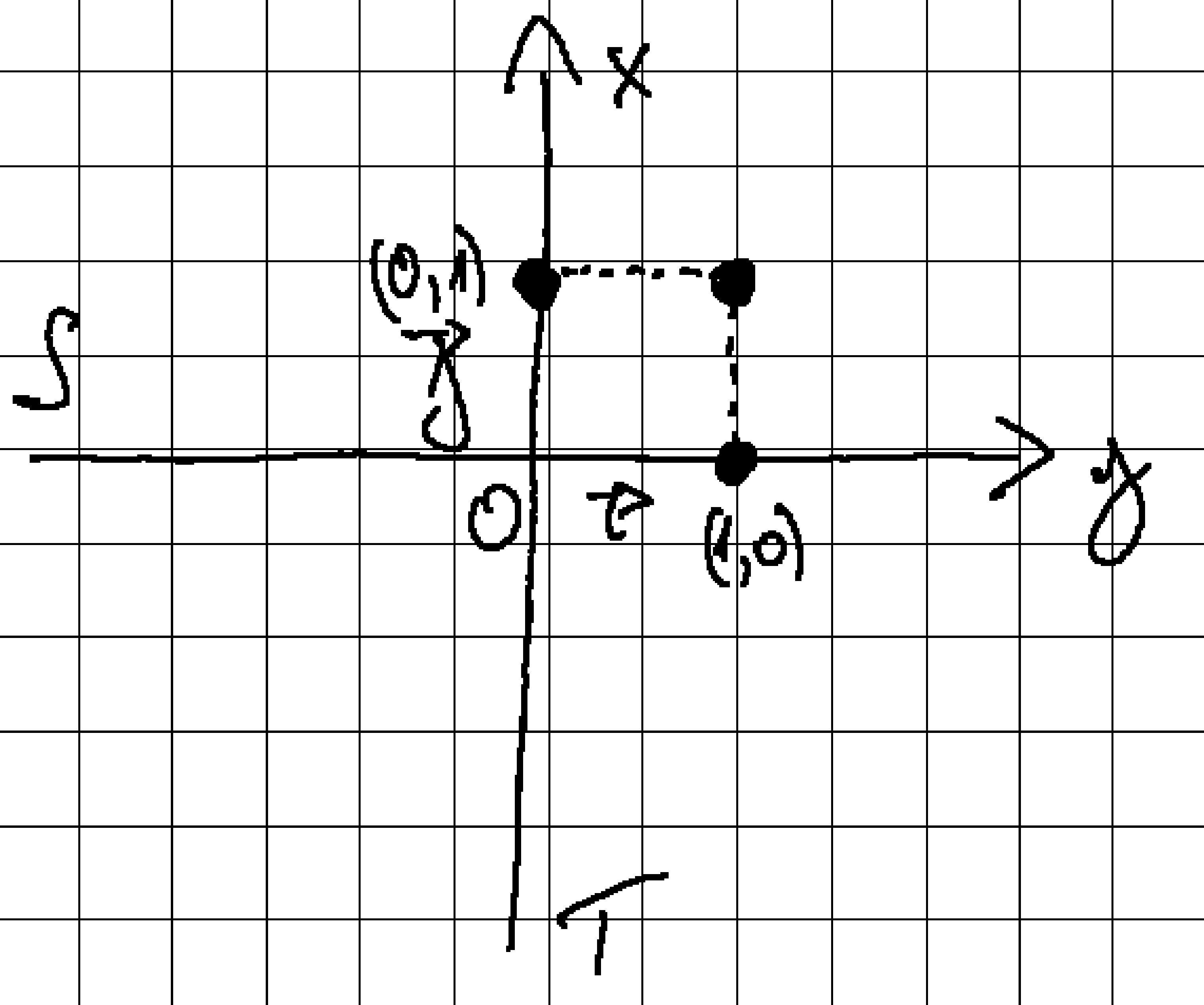
$$\Leftrightarrow \begin{cases} S \neq \emptyset \quad (0 \in S) \\ \forall k_1, k_2 \in K,\ \forall\ v_1, v_2 \in S,\ k_1 \cdot v_1 + k_2 \cdot v_2 \in S \end{cases}$$

<u>Proof</u>: $0 \in \bigcap_{i \in J} S_i \neq \emptyset$, because $0 \in S_i,\ \forall\ i \in J\ (S_i \leq_K V)$

- let $k_1, k_2 \in K$ and $v_1, v_2 \in \bigcap_{i \in J} S_i \Rightarrow$

$$\left.\begin{array}{l} \Rightarrow v_1, v_2 \in S_i,\ \forall\ i \in J \\ \quad S_i \leq_K V,\ \forall\ i \in J \end{array}\right\} \Rightarrow k_1 \cdot v_1 + k_2 \cdot v_2 \in S_i,\ \forall\ i \in J \Rightarrow$$

$\Rightarrow k_1 \cdot v_1 + k_2 \cdot v_2 \in \bigcap_{i \in J} S_i$ hence $\bigcap_{i \in J} S_i \leq_K V$



Proof:

(i) $\underline{L \leq_K V}$:

- $0 = 0 \cdot x \in X \neq \emptyset \Rightarrow$ finite linear combination of x

$\Rightarrow 0 \in L$

- let $k_1, k_2 \in K$ and $v_1, v_2 \in L$

$\Downarrow$

v_1, v_2 are finite linear combinations of vectors from X

$\Rightarrow k_1 \cdot v_1 + k_2 \cdot v_2$ is also a finite linear combinati of vectors from X

$\Rightarrow k_1 \cdot v_1 + k_2 \cdot v_2 \in L$ hence $L \leq_K V$

(ii) $\underline{X \subseteq L}$

$\forall x \in X, \quad x = 1 \cdot x \in L$

(iii) Let $S \leq_K V$ with $X \subseteq S$ we show that $L \subseteq S$

$$\left.\begin{array}{l} \text{Let } v = \underbrace{k_1 \cdot x_1}_{\in X \subseteq S} + \dots + \underbrace{k_n \cdot x_n}_{\in X \subseteq S} \in L \\ \\ S \leq_K V \end{array}\right\} \Rightarrow v \in S$$

$\underline{\text{Proof}}$: $S + T = \langle S \cup T \rangle$

$\boxed{\subseteq}$ Let $v \in S+T \Rightarrow v = s + t$ wit $s \in S, t \in T \Rightarrow$

$\Rightarrow v = k \cdot s + k \cdot t \in \langle S \cup T \rangle$

$\boxed{\supseteq}$ Let $v \in \langle S \cup T \rangle \Rightarrow v = k_1 \cdot v_1 + k_2 \cdot v_2 + \dots + k_n \cdot v_n$

with $k_1, k_2 \dots k_n \in K$, $v_1, v_2 \dots v_n \in S + T$

$i = \{ i \in \{1, \dots, n\} \mid v_i \in S \}$

$j = \{1, \dots, n\} \setminus i$

$$v = \sum_{i \in J} k_i \cdot \underset{\substack{\in \\ S}}{v_i} + \sum_{j \in J} k_j \cdot \underset{\substack{\in \\ T}}{v_j} \in S + T$$

$\boxed{\Rightarrow}$ Suppose $V = S \oplus T \Rightarrow V = S+T$ & $S \cap T = \{0\}$

$\Rightarrow \forall v \in V, v = s+t$ with $s \in S, t \in T$

For uniqueness, suppose that:

$v = s'+t'$ with $s' \in S, t' \in T$

$\Rightarrow s+t = s'+t' \Rightarrow \underbrace{s-s'}_{\in S} = \underbrace{t'-t}_{\in T} \in S \cap T = \{0\} \Rightarrow$

$\Rightarrow s = s'$ & $t = t'$

$\boxed{\Leftarrow}$ Suppose $\forall v \in V, \exists! s \in S, t \in T : v = s+t$

Let $v \in V \Rightarrow v = s+t$, with $s \in S, t \in T \Rightarrow$

$\Rightarrow V \subseteq S+T \Rightarrow V = S+T$

Suppose $S \cap T = \{u\} \Rightarrow u = \underset{\in S}{u} + \underset{\in T}{0} = \underset{\in S}{0} + \underset{\in T}{u} \xRightarrow[\text{uniqueness of writing}]{}$

$\Rightarrow u = 0 \Rightarrow S \cap T = \{0\}$

$\Rightarrow V = S \oplus T$

$\boxed{\Rightarrow}$ Suppose f is a K-linear map

$\forall\ k_1, k_2 \in K,\ \forall\ v_1, v_2 \in V$, we have:

$$f(k_1 v_1 + k_2 \cdot v_2) = f(k_1 \cdot v_1) + f(k_2 v_2) =$$

$$= k_1 f(v_1) + k_2 f(v_2)$$

$\boxed{\Leftarrow}$ Apply hypothesis for $k_1 = 1,\ k_2 = 1$

$$\Rightarrow f(v_1 + v_2) = f(v_1) + f(v_2)$$

& for $k_2 = 0$

$$\Rightarrow f(k_1 \cdot v_1) = k_1 \cdot f(v_1)$$

I $\underbrace{\mathrm{Ker}}_{\substack{\text{kernel} \\ \text{(nucleu)}}} f \leq_K V$

- $f(0) = 0' \Rightarrow 0 \in \mathrm{Ker}\, f$
- Let $k_1, k_2 \in K$ and $v_1, v_2 \in \mathrm{Ker}\, f$. We show

that $k_1 v_1 + k_2 v_2 \in \mathrm{Ker}\, f$

we have $f(k_1 \cdot v_1 + k_2 \cdot v_2) = k_1 \cdot \underbrace{f(v_1)}_{\substack{\| \\ 0' \\ (v_1 \in \mathrm{Ker}\, f)}} + k_2 \cdot \underbrace{f(v_2)}_{\substack{\| \\ 0' \\ (v_2 \in \mathrm{Ker}\, f)}} = 0' \Rightarrow$

$\Rightarrow k_1 v_1 + k_2 v_2 \in \mathrm{Ker}\, f$

$\underline{\text{II}}$ $\operatorname{Im} f \leq_K V$

- $0' = f(0) \in \operatorname{Im} f$
- Let $k_1, k_2 \in K$ and $v_1', v_2' \in \operatorname{Im} f$. We show

that: $k_1 \cdot v_1' + k_2 \cdot v_2' \in \operatorname{Im} f$

$$\left.\begin{array}{l} v_1' = f(v_1) \text{ for some } v_1 \in V \\ v_2' = f(v_2) \text{ for some } v_2 \in V \end{array}\right\} \Rightarrow$$

$$\Rightarrow k_1 \cdot v_1' + k_2 \cdot v_2' = k_1 \cdot f(v_1) + k_2 \cdot f(v_2) = f(k_1 v_1 + k_2 v_2) \in \operatorname{Im} f$$