$$\mathbb{R}^n = \{ (x_1, x_2, \dots, x_n) \mid x_i \in \mathbb{R} \ , i \in \{1, 2, \dots, n\} \}$$

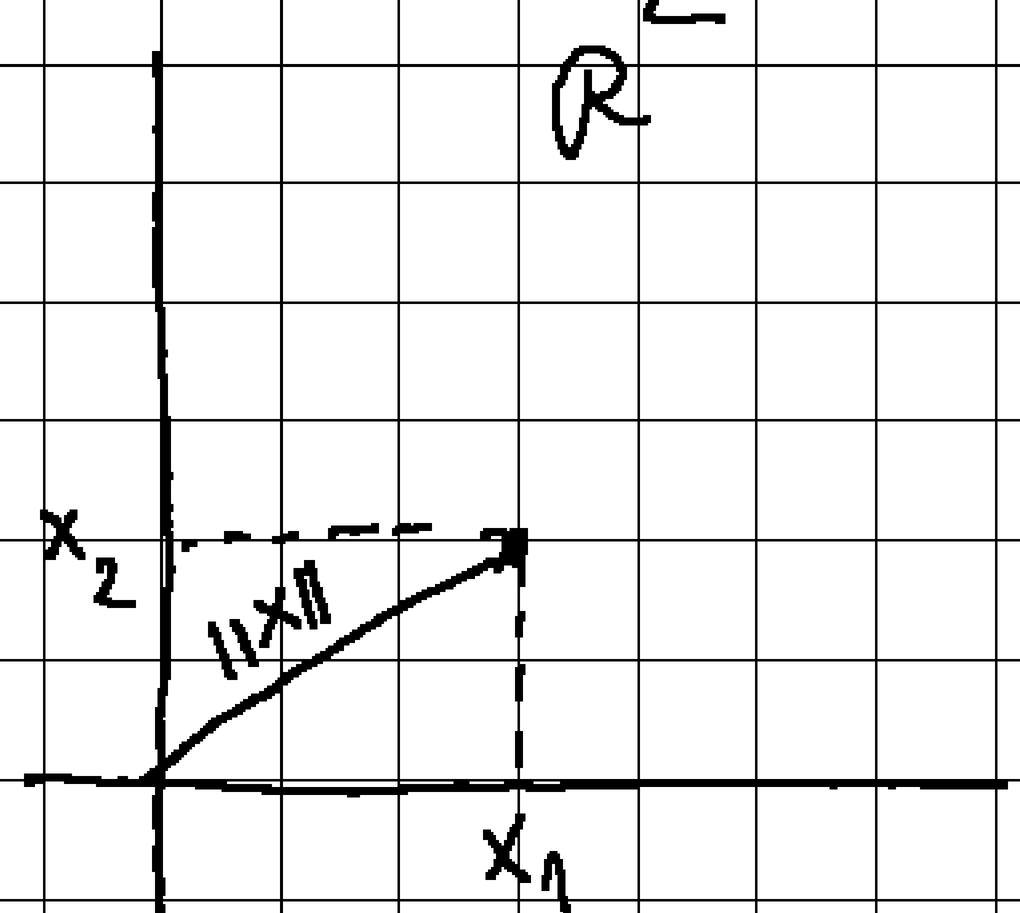
$$x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$$

$$y = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n$$

$$\langle x, y \rangle = x \cdot y = x_1 \cdot y_1 + x_2 \cdot y_2 + \dots + x_n \cdot y_n =$$

$$= \sum_{i=1}^{n} x_i \cdot y_i$$ - dot product (scalar product)

$\mathbb{R}^2$



$$\|x\| = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2} = \sqrt{\langle x, x \rangle}$$ - norm of x (Euclidean)

1. Prove that for any $x, y \in \mathbb{R}^n$ the following identities hold:

a) $\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$

$\langle x+y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$

$\langle \alpha \cdot x, y\rangle = \alpha \langle x, y\rangle$

$\|x+y\|^2 = \langle x+y, x+y\rangle = \langle x, x+y\rangle + \langle y, x+y\rangle =$

$= \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle =$

$= \|x\|^2 + \langle x, y\rangle + \langle x, y\rangle + \|y\|^2 =$

$= \|x\|^2 + 2\langle x, y\rangle + \|y\|^2 \quad (1)$

$\|x-y\|^2 = \langle x-y, x-y\rangle = \langle x, x-y\rangle + \langle -y, x-y\rangle =$

$= \langle x, x\rangle + \langle x, -y\rangle + \langle -y, x\rangle + \langle -y, -y\rangle =$

$= \|x\|^2 - \langle x, y\rangle - \langle x, y\rangle + \|y\|^2 =$

$= \|x\|^2 - 2\langle x, y\rangle + \|y\|^2 \quad (2)$

$(1), (2) \Rightarrow \|x\|^2 + 2\langle x, y\rangle + \|y\|^2 + \|x\|^2 - 2\langle x, y\rangle + \|y\|^2 =$

$= 2(\|x\|^2 + \|y\|^2)$

b) $\langle x, y \rangle = \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2 \right)$

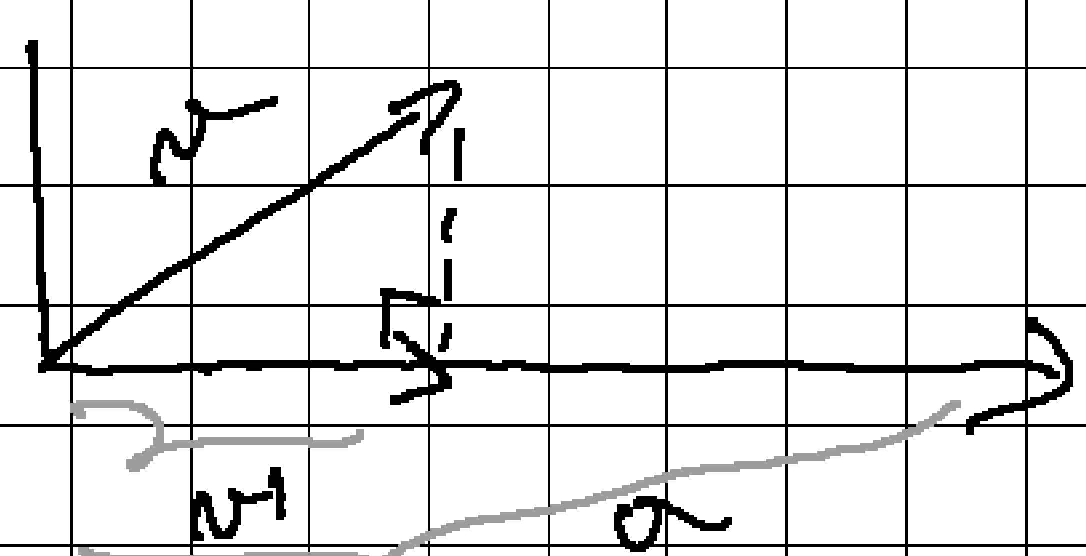
$$(1),(2) \Rightarrow \|x\|^2 + 2\langle x,y\rangle + \|y\|^2 - \left(\|x\|^2 - 2\langle x,y\rangle + \|y\|^2 \right) =$$

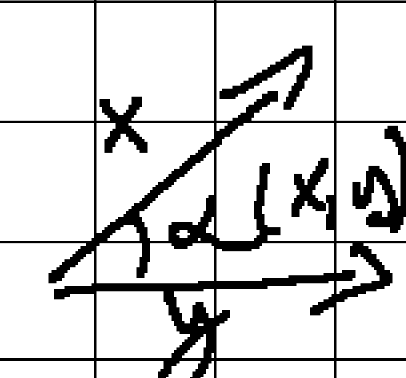
$$= \|x+y\|^2 - \|x-y\|^2$$

$$= 4\langle x,y\rangle \Rightarrow$$

$$\Rightarrow \langle x,y\rangle = \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2 \right)$$

3. Find the orthogonal projection of a vector $v \in \mathbb{R}^2$ onto a vector $a \in \mathbb{R}^2$





$$\langle x,y\rangle = \|x\| \cdot \|y\| \cdot \cos(\alpha(x,y))$$

if $x \perp y \Rightarrow \langle x,y\rangle = 0$ $\quad \left(\alpha(x,y) = \frac{\pi}{2},\ \cos\frac{\pi}{2} = 0\right)$

$$v' = a \cdot \frac{\|v'\|}{\|a\|} \quad (1)$$

$$\left.\begin{aligned} \langle v,a\rangle &= \|v\| \cdot \|a\| \cdot \cos\alpha \\ \cos\alpha &= \frac{\|v'\|}{\|v\|} \end{aligned}\right\} \Rightarrow \langle v,a\rangle = \cancel{\|v\|} \cdot \|a\| \cdot \frac{\|v'\|}{\cancel{\|v\|}} =$$

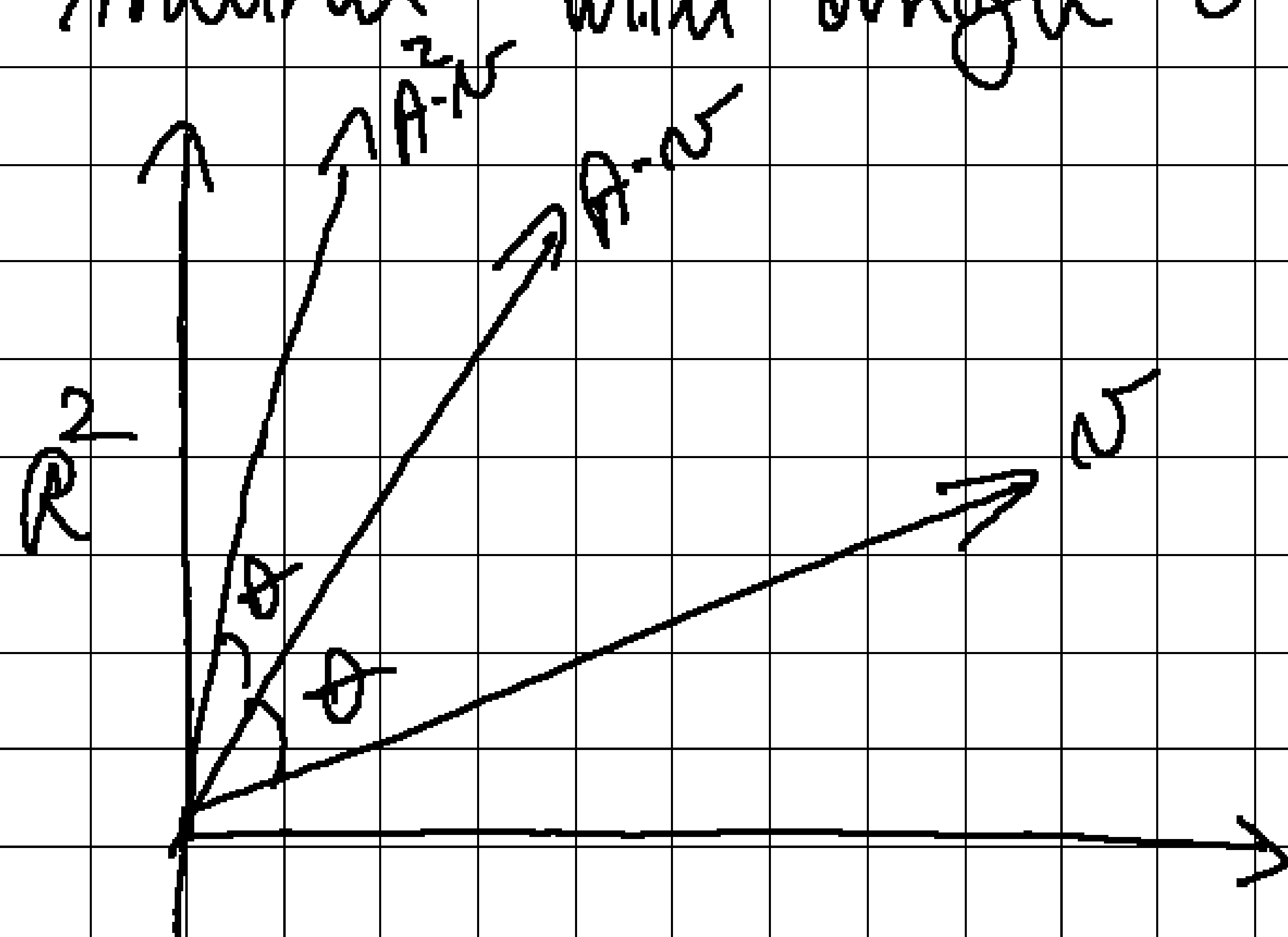
$$= \|a\| \cdot \|v'\|$$

$$\Rightarrow \|v'\| = \frac{\langle v, a\rangle}{\|a\|} \quad (2)$$

$$(1),(2) \Rightarrow v' = a \cdot \frac{\langle v, a\rangle}{\|a\|} \cdot \frac{1}{\|a\|} = \frac{a \cdot \langle v, a\rangle}{\|a\|^2} =$$

$$= \frac{\langle v, a\rangle}{\langle a, a\rangle} \cdot a$$

4. Show that $\overbrace{\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}}^{=A}$ is a rotation matrix with angle θ in $\mathbb{R}^2$



$v = (v_1, v_2) \in \mathbb{R}^2$

$$A \cdot v = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} =$$

$$= \begin{pmatrix} \cos\theta \cdot v_1 - \sin\theta\, v_2 \\ \sin\theta \cdot v_1 + \cos\theta\, v_2 \end{pmatrix}$$

^{2} ^{2} $+$

$$\|A\cdot v\|^2 = (\cos\theta\, v_1)^2 - \cancel{2\cos\theta\, v_1 \cdot \sin\theta\, v_2} + (\sin\theta\, v_2)^2 +$$

$$+ (\sin\theta\, v_1)^2 + \cancel{2\sin\theta\, v_1 \cos\theta\, v_2} + (\cos\theta\, v_2)^2 =$$

$$= v_1^2(\cos^2\theta + \sin^2\theta) + v_2^2(\sin^2\theta + \cos^2\theta) =$$

$$= v_1^2 + v_2^2 = \|v\|^2$$

$$\langle v, A\cdot v\rangle = v_1^2 \cdot \cos\theta - \cancel{v_1\cdot v_2 \cdot \sin\theta} + \cancel{v_1\cdot v_2 \sin\theta}$$

$$+ v_2^2 \cos\theta = \cos\theta\,(v_1^2 + v_2^2) = \cos\theta \cdot \|A\cdot v\|^2 =$$

$$= \cos\theta \cdot \|v\| \cdot \|Av\|$$

$$\Rightarrow \alpha(v, Av) = \theta$$

9. $\|x\|_p = \left(|x_1|^p + \dots + |x_n|^p\right)^{\frac{1}{p}}, \quad p \geq 1$

$\|x\|_\infty = \max\{|x_1|, \dots, |x_n|\}$

Draw the unit ball in $\mathbb{R}^2$ for the p-norm with $p \in \{1, 2, \infty\}$

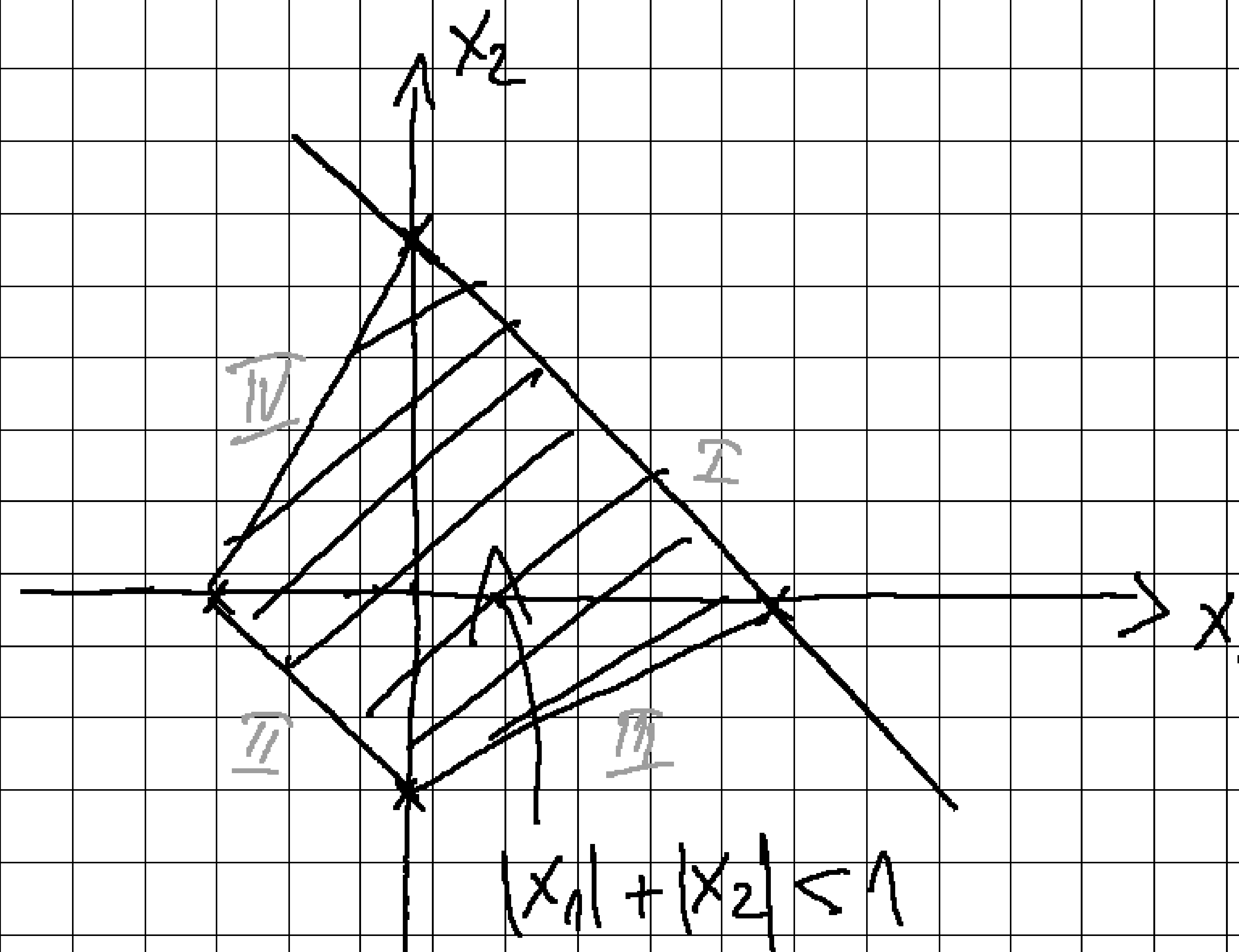
$$p = 1 \Rightarrow \|x\|_1 = (|x_1| + |x_2|)$$

$|x_1| + |x_2| = 1$

I $x_1, x_2 > 0 \Rightarrow$

$\Rightarrow x_1 + x_2 = 1$

$x_2 = 1 - x_1$ (take values $x_1 = 0,\ x_2 = 1$; $x_2 = 0,\ x_1 = 1$)



(>1 would be outside the square)

II $x_1, x_2 < 0$

$-x_1 - x_2 = 1$ ($x_1 = -1,\ x_2 = 0$; $x_1 = 0,\ x_2 = -1$)

$-x_1 = 1 + x_2$

III $x_1 > 0,\ x_2 < 0$

$x_1 - x_2 = 1$

$x_1 = 1 + x_2$

IV $x_1 < 0,\ x_2 > 0$

$-x_1 + x_2 = 1$

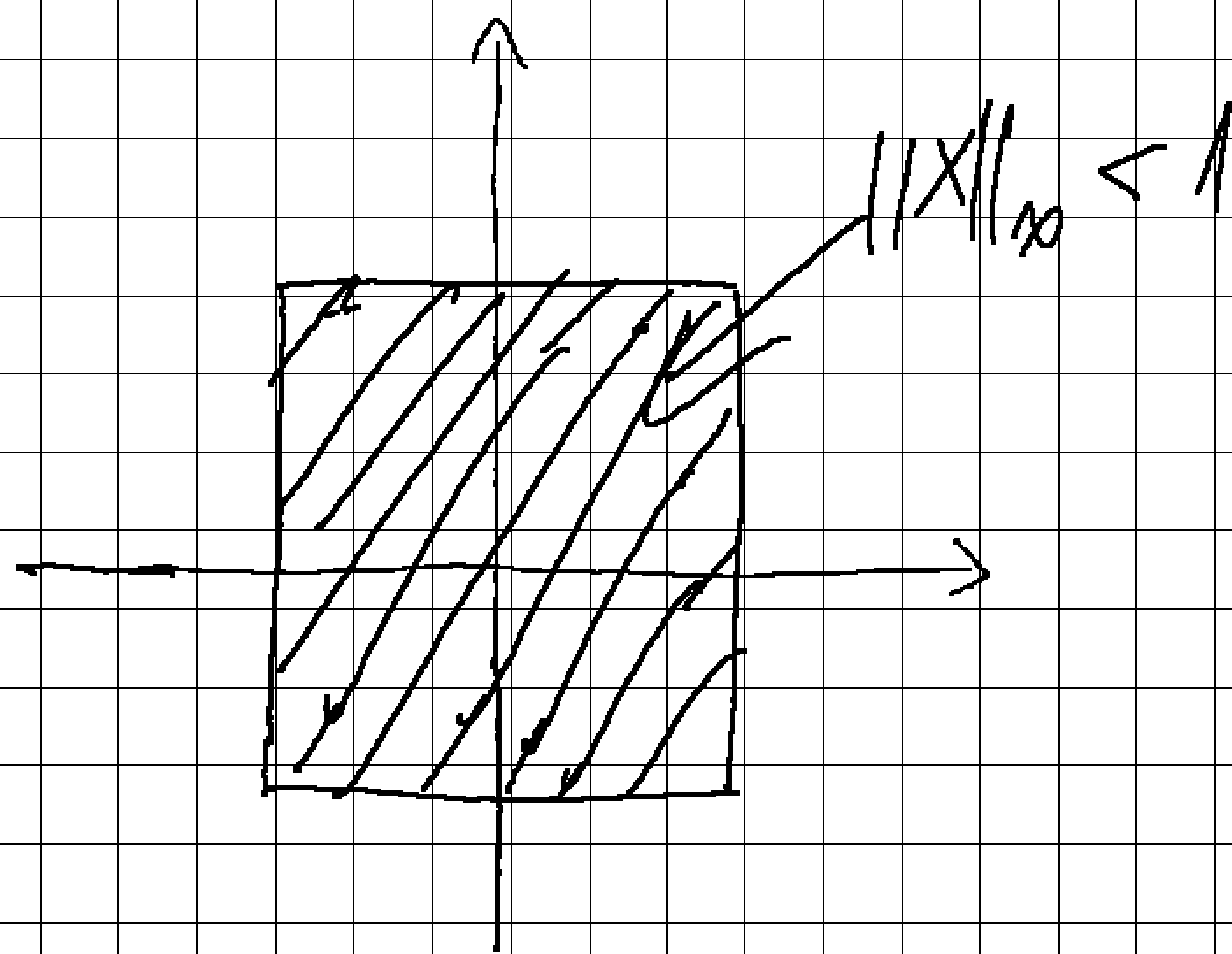
$-x_1 = 1 - x_2$

$B(x, r) = \{ y \in \mathbb{R}^n \mid \|x - y\| < r \}$ - ball centered at x with radius r

$\overline{B}(x, r) = \{ y \in \mathbb{R}^n \mid \|x - y\| \leq r \}$ - closed unit ball

$p = \infty \Rightarrow \|x\|_\infty = \max\{ |x_1|, |x_2| \}$

$\|x\|_\infty < 1$



if $\|X\| = 1 \Rightarrow$ I $X_1 = 1$ or $X_1 = -1$

and

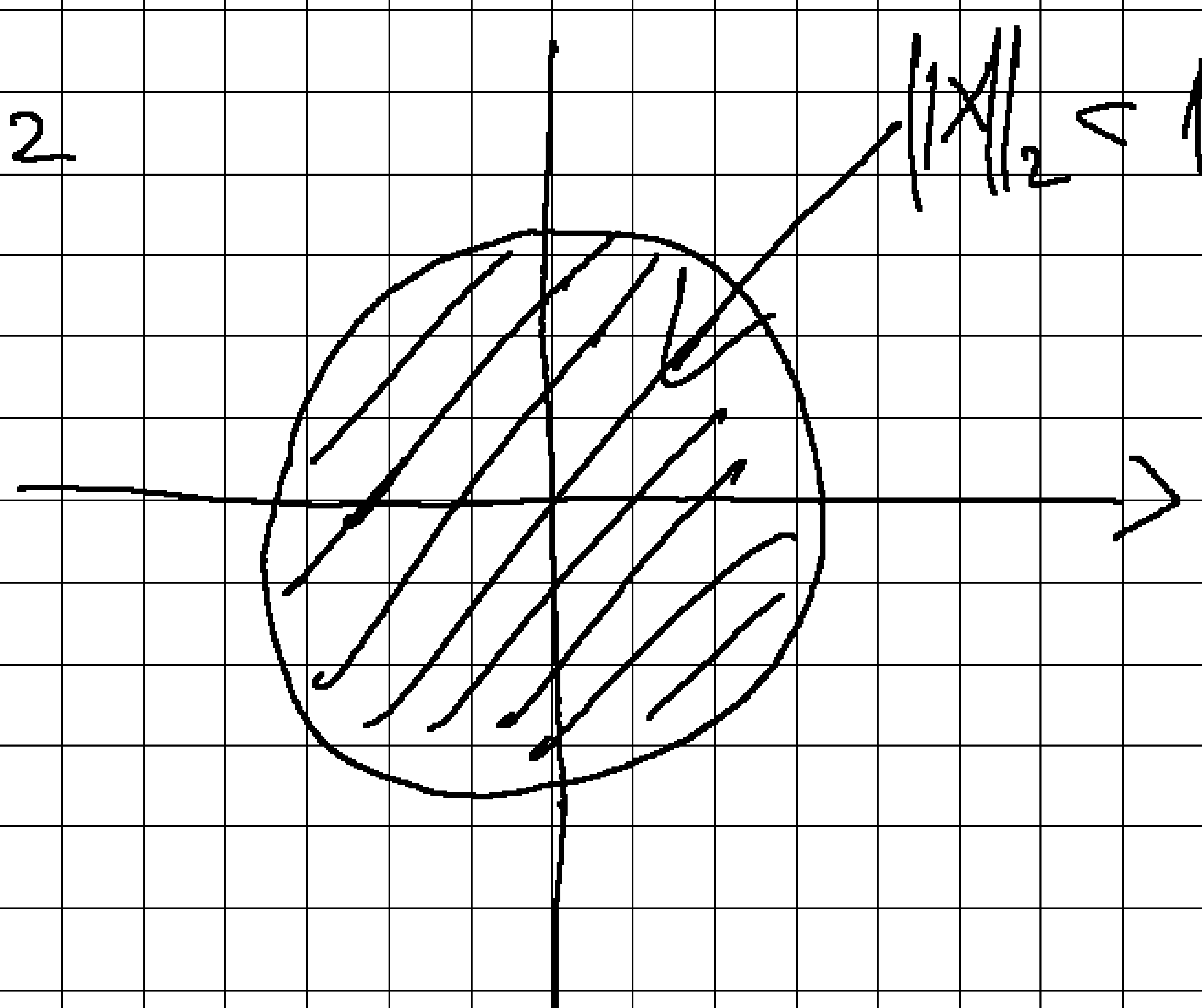
$x_2 \in [-1, 1]$

II $X_2 = 1$ or $X_2 = -1$

and

$x_1 \in [-1, 1]$

if $p = 2$

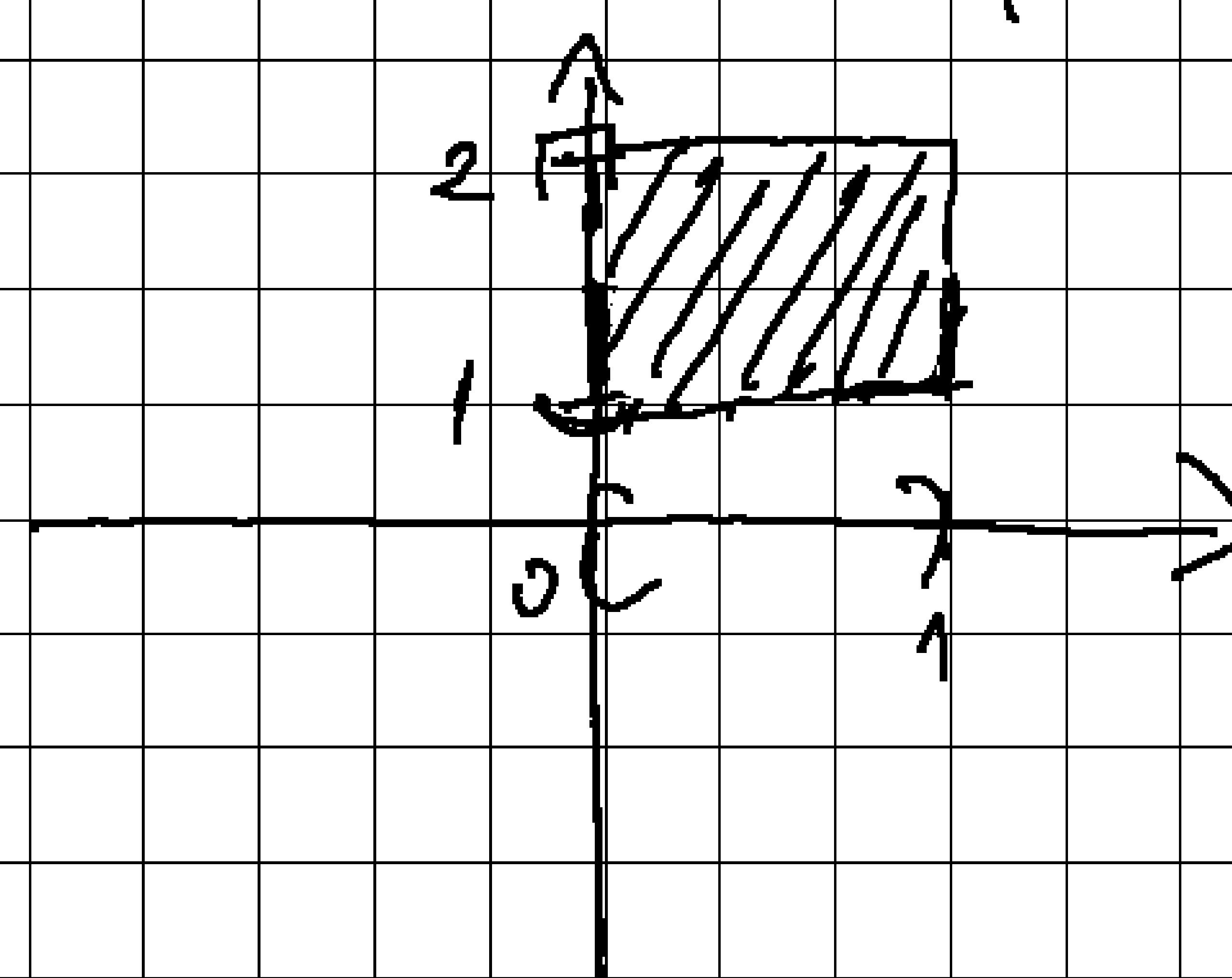


7. Find the interior, closure and boundary

a) $A=[0,1)\times(1,2]$

$\operatorname{int} A = \{y \in \mathbb{R}^2 \mid \exists r>0,\ B(y,r) \subseteq A\} =$

$= (0,1)\times(1,2)$



$\operatorname{cl} A = \{y \in \mathbb{R}^2 \mid \forall r>0,\ B(y,r) \cap A \neq \emptyset\} =$

$= [0,1]\times[1,2]$

$\operatorname{bd}(A) = \operatorname{cl} A \setminus \operatorname{int} A = \{0\}\times[1,2] \cup [0,1]\times\{2\} \cup$

$\cup \{1\}\times[1,2] \cup [0,1]\times\{1\}$