Chapter 1:

1.1. $A_0, \dots, A_n$ - vertices of a polygon

Determine $\vec{A_0A_1} + \vec{A_1A_2} + \dots + \vec{A_{n-1}A_n} + \vec{A_nA_0}$

$$\left.\begin{aligned}
\vec{A_0A_1} + \vec{A_1A_2} &= \vec{A_0A_2} \\
\vec{A_2A_3} + \vec{A_3A_4} &= \vec{A_2A_4} \\
\vec{A_4A_5} + \vec{A_5A_6} &= \vec{A_4A_6} \\
&\vdots \\
\vec{A_{n-1}A_n} + \vec{A_nA_0} &= \vec{A_{n-1}A_0}
\end{aligned}\right\} \Rightarrow \vec{A_0A_1} + \vec{A_1A_2} + \dots + \vec{A_{n-1}A_n} + \vec{A_nA_0} = \vec{A_0}$$

1.2. Decide if the indicated vectors u, v, w can be represented with the vertices of a triangle

a) $u(7,3)$, $v(2,8)$, $w(-5,5)$

$u+v+w = (7,3)+(2,8)+(-5,5) = (4,16) \Rightarrow$

$\Rightarrow$ the triangle does not exist

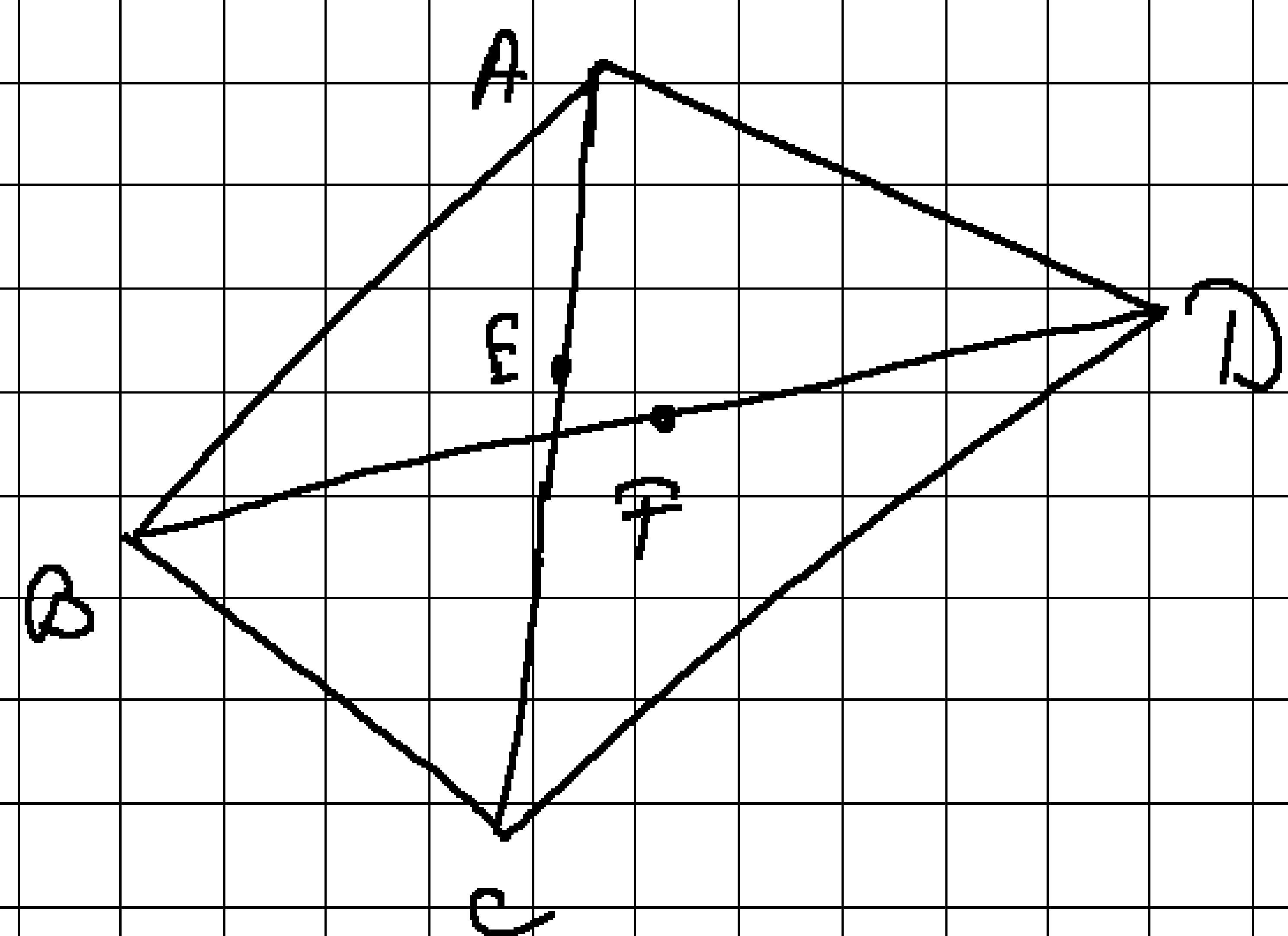
b) $u(1,2,-1)$ $v(2,-1,0)$, $w(1,-3,1)$

$u+v+w = (1,2,-1)+(2,-1,0)+(1,-3,1) = (2,-2,0)$

$\Rightarrow$ the triangle does not exist

1.4. Let $ABCD$ be a quadrilateral. Let E be the midpoint of $[AC]$ and let F be the midpoint of $[BD]$.
Show that

$$\vec{EF} = \frac{1}{2}(\vec{AB}+\vec{CD}) = \frac{1}{2}(\vec{AD}+\vec{CB})$$



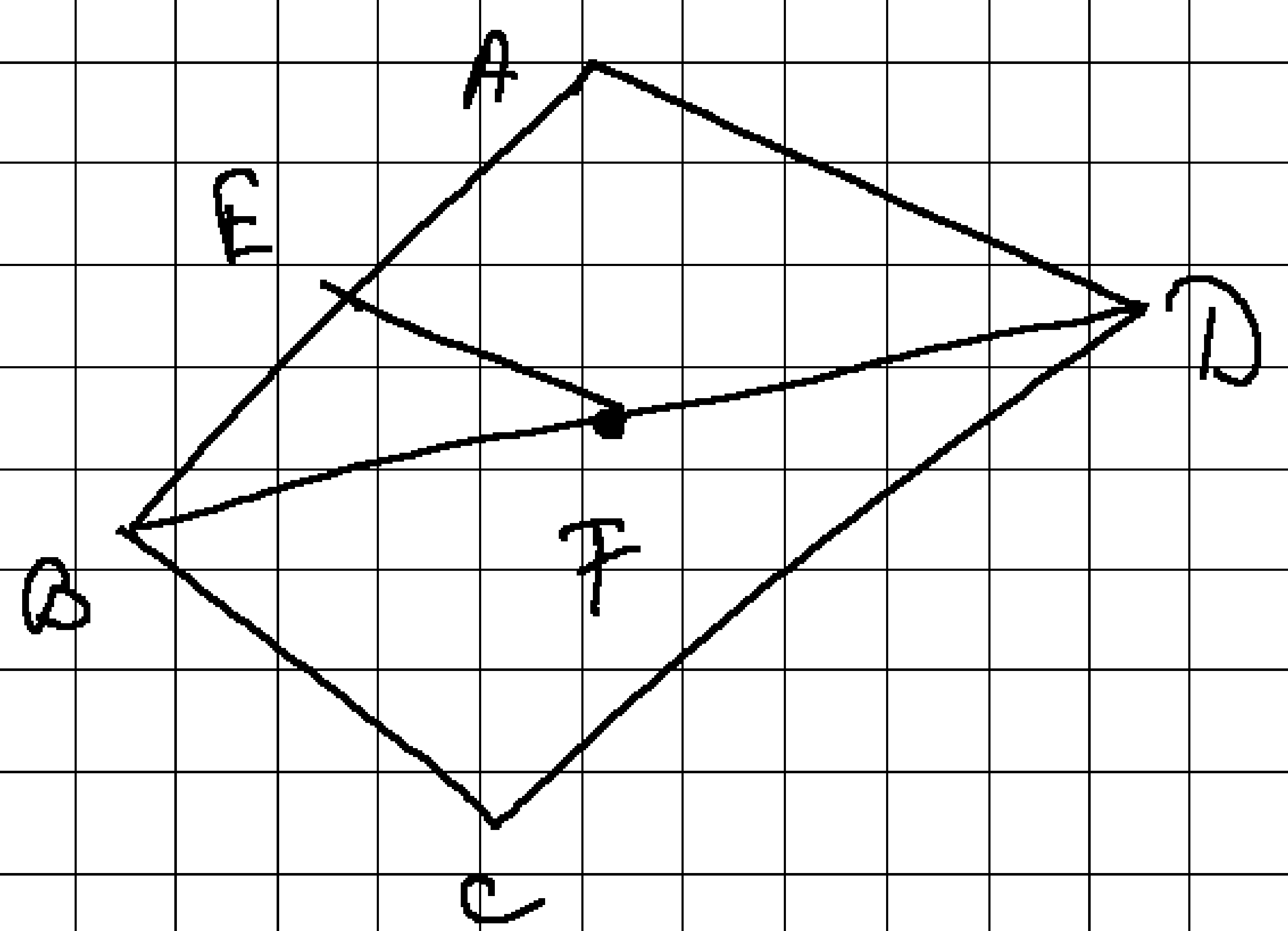
$$\vec{EF} = \vec{F} - \vec{E} = \left(\frac{\vec{B}+\vec{D}}{2}\right) - \left(\frac{\vec{A}+\vec{C}}{2}\right) = \frac{\vec{B}+\vec{D}-\vec{A}-\vec{C}}{2}$$

$$\left.\begin{array}{l} \vec{AB} = \vec{B}-\vec{A} \\ \vec{CD} = \vec{D}-\vec{C} \\ \vec{EF} = \frac{1}{2}(\vec{AB}+\vec{CD}) \end{array}\right\} \Rightarrow \vec{EF} = \frac{\vec{B}-\vec{A}+\vec{D}-\vec{C}}{2}$$

$$\left.\begin{array}{l} \vec{AD} = \vec{D}-\vec{A} \\ \vec{CB} = \vec{B}-\vec{C} \\ \vec{EF} = \frac{1}{2}(\vec{AD}+\vec{CB}) \end{array}\right\} \Rightarrow \vec{EF} = \frac{\vec{D}-\vec{A}+\vec{B}-\vec{C}}{2}$$

1.5. Let ABCD be a quadrilateral. Let E be the midpoint of [AB] and let F be the midpoint of [BD]. Show that:

$$\vec{EF} = \frac{1}{2}(\vec{AD}+\vec{BC})$$

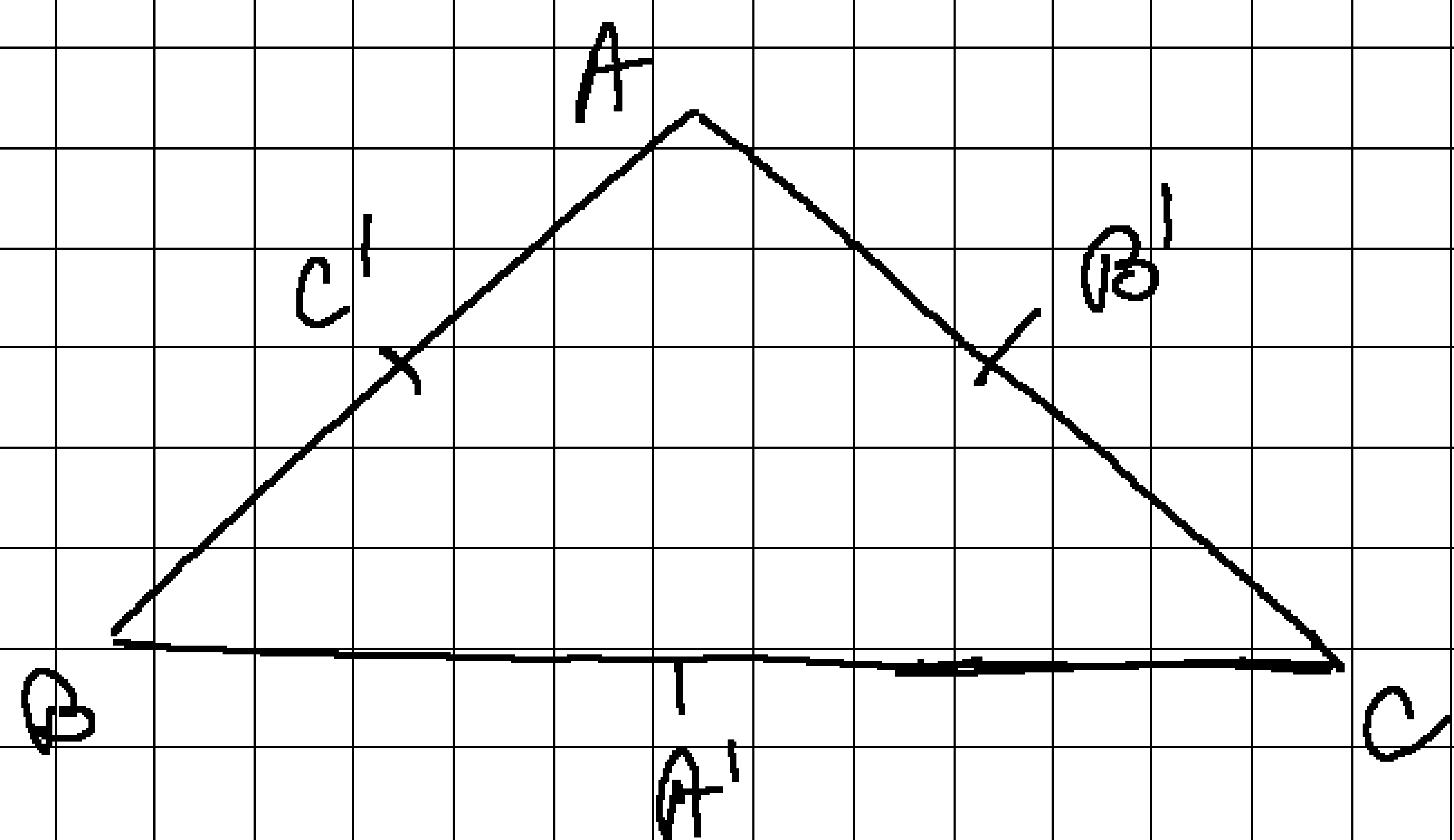


$$\vec{EF} = \vec{F} - \vec{E} = \frac{\vec{B}+\vec{D}}{2} - \frac{\vec{A}+\vec{B}}{2} = \frac{\vec{B}+\vec{D}-\vec{A}-\vec{B}}{2}$$

$$\left.\begin{array}{l} \vec{AD} = \vec{D}-\vec{A} \\ \vec{BC} = \vec{C}-\vec{B} \\ \vec{EF} = \frac{1}{2}(\vec{AD}+\vec{BC}) \end{array}\right\} \Rightarrow \vec{EF} = \frac{\vec{D}-\vec{A}+\vec{C}-\vec{B}}{2}$$

1.6. Let A', B' and C' be midpoints of the sides of a triangle ABC. Show that for any point O we have:

$$\vec{OA'}+\vec{OB'}+\vec{OC'}=\vec{OA}+\vec{OB}+\vec{OC}$$



$$\left.\begin{aligned}
\vec{A}' &= \frac{\vec{B}+\vec{C}}{2}\\
\vec{B}' &= \frac{\vec{A}+\vec{C}}{2}\\
\vec{C}' &= \frac{\vec{A}+\vec{B}}{2}\\
\vec{OA'} &= \vec{A}'-\vec{O}\\
\vec{OB'} &= \vec{B}'-\vec{O}\\
\vec{OC'} &= \vec{C}'-\vec{O}
\end{aligned}\right\} \Rightarrow \begin{aligned}
&\vec{OA'}+\vec{OB'}+\vec{OC'} = \vec{A}'-O+\vec{B}'-\vec{O}+\vec{C}'-\vec{O} =\\
&= \frac{\vec{B}+\vec{C}}{2}+\frac{\vec{A}+\vec{C}}{2}+\frac{\vec{A}+\vec{B}}{2}-3\vec{O}' =\\
&= \frac{2\vec{A}+2\vec{B}+2\vec{C}}{2}-3\vec{O}' = \vec{A}+\vec{B}+\vec{C}-3\vec{O}'
\end{aligned}$$

$$\left.\begin{aligned}
\vec{OA} &= \vec{A}-\vec{O}\\
\vec{OB} &= \vec{B}-\vec{O}\\
\vec{OC} &= \vec{C}-\vec{O}
\end{aligned}\right\} \Rightarrow \vec{OA}+\vec{OB}+\vec{OC} = \vec{A}+\vec{B}+c-3\vec{O}$$

$$\Rightarrow \vec{OA'}+\vec{OB'}+\vec{OC'}=\vec{OA}+\vec{OB}+\vec{OC}$$

1.9 Let ABCD be a tetrahedron. Determine the sums:

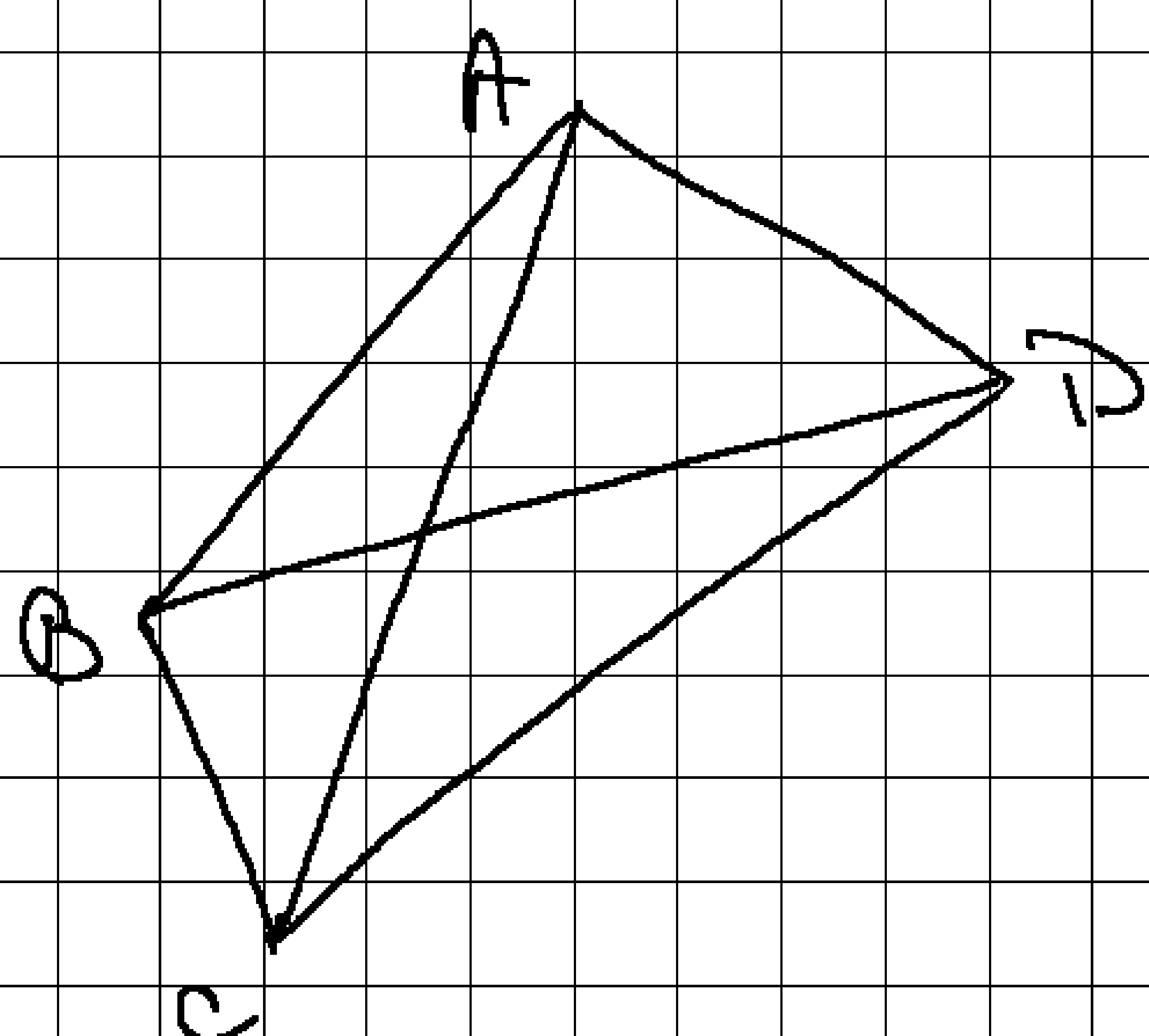
a) $\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CD} = \overrightarrow{AC}+\overrightarrow{CD} = \overrightarrow{AD}$

b) $\overrightarrow{AD}+\overrightarrow{BC}+\overrightarrow{DB} = \overrightarrow{AB}+\overrightarrow{BC} = \overrightarrow{AC}$

c) $\overrightarrow{AB}+\overrightarrow{CD}+\overrightarrow{BC}+\overrightarrow{DA} = \overrightarrow{AC}+\overrightarrow{CD}+\overrightarrow{DA} = \overrightarrow{AD}+\overrightarrow{DA} = \overrightarrow{A}$

1.10. Let ABCD be a tetrahedron. Show that:

$\overrightarrow{AD}+\overrightarrow{BC} = \overrightarrow{BD}+\overrightarrow{AC}$



$\overrightarrow{BD} = \overrightarrow{BC}+\overrightarrow{CD}$

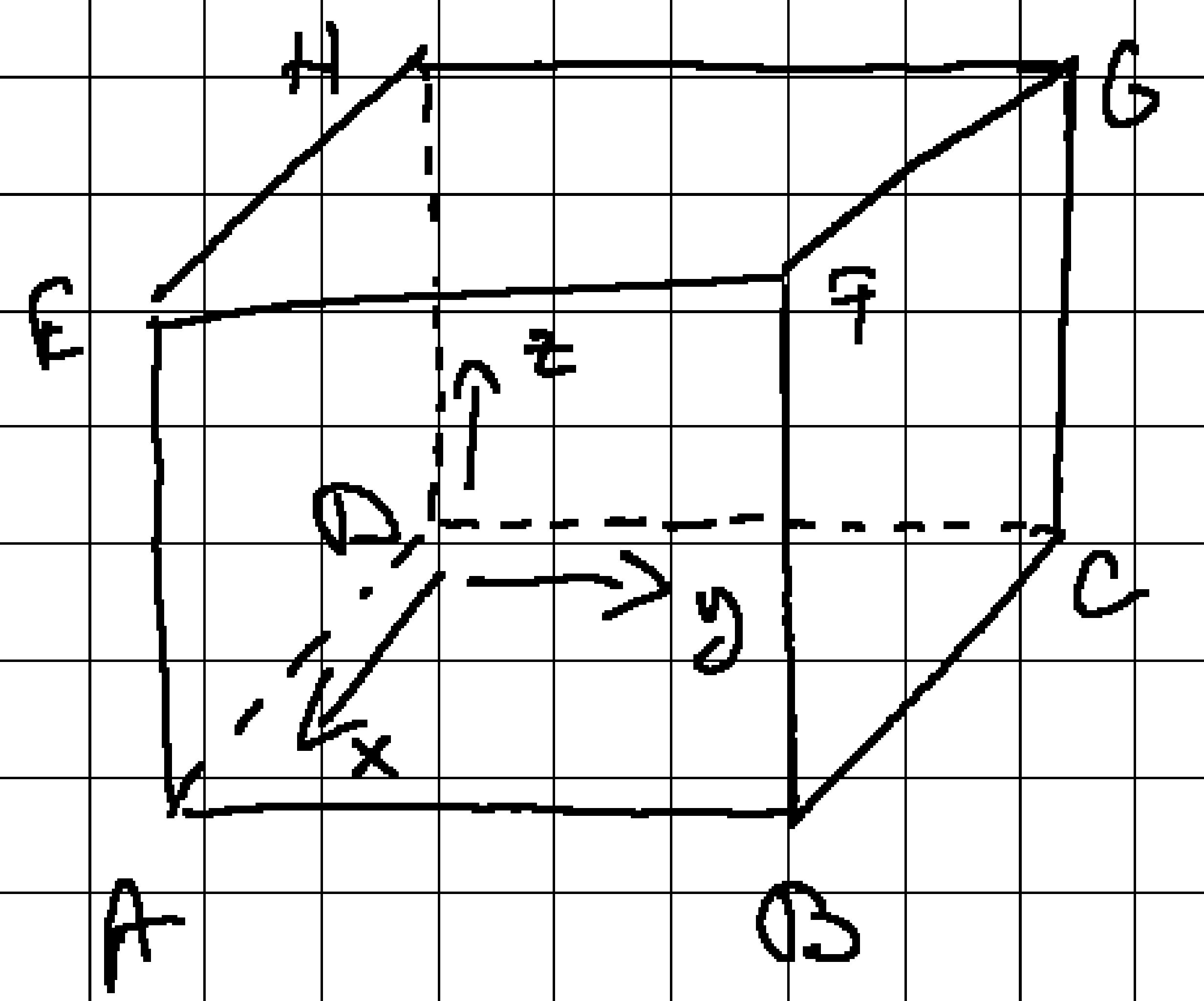
$\overrightarrow{AC} = \overrightarrow{AD}+\overrightarrow{DC}$

$\overrightarrow{BD}+\overrightarrow{AC} = \overrightarrow{BC}+\overrightarrow{CD}+\overrightarrow{AD}+\overrightarrow{DC} =$

$= \overrightarrow{BC}+\overrightarrow{AD}+\cancel{\overrightarrow{CD}}-\cancel{\overrightarrow{CD}} =$

$= \overrightarrow{BC}+\overrightarrow{AD}$

1.12. Give the coordinates of the vertices of the parallelepiped whose faces lie in the coordinate planes and in the planes $x=1$, $y=3$, $z=-2$



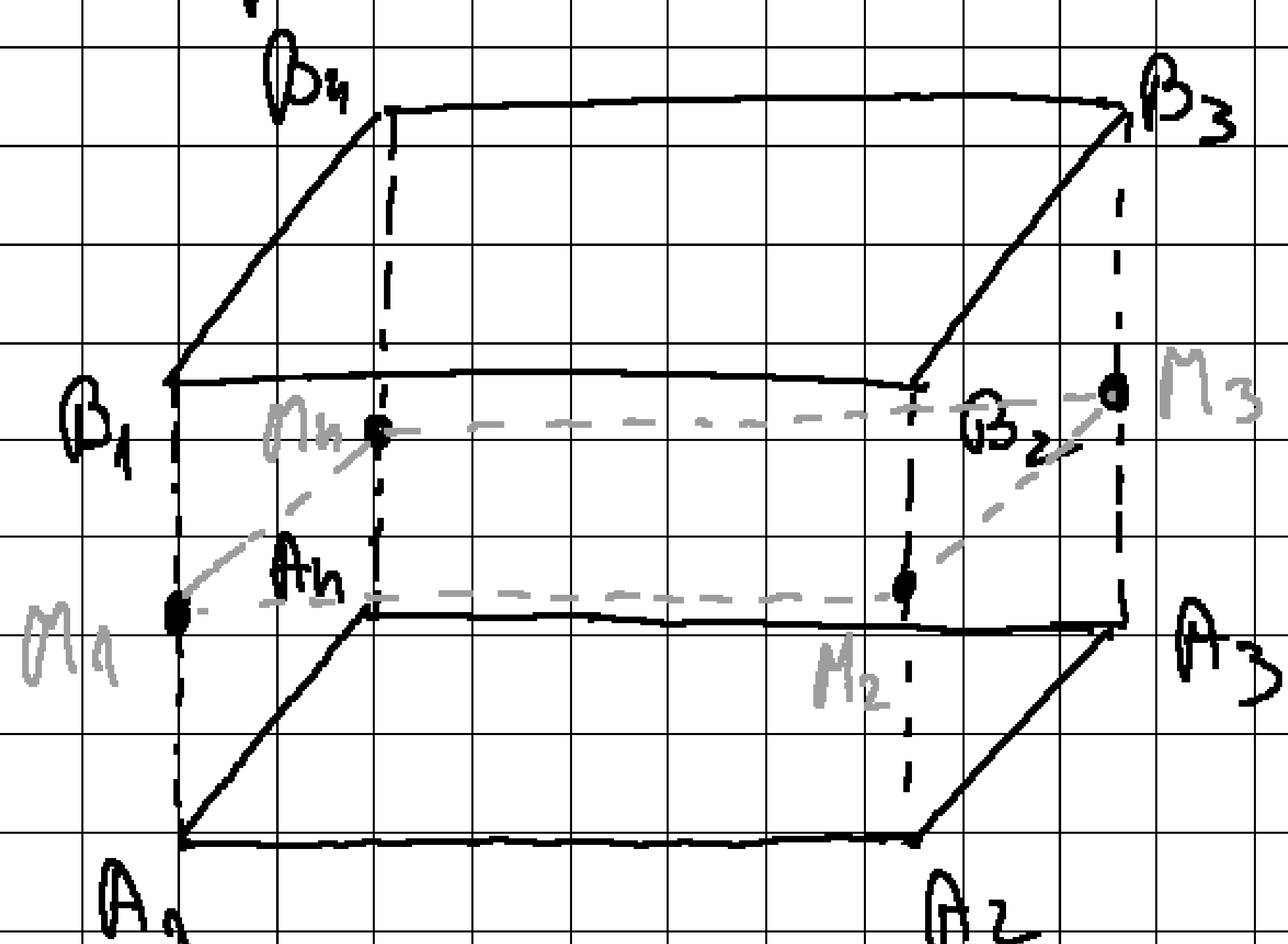
$D(0,0,0)$ $\quad$ $H(0,0,-2)$

$A(1,0,0)$ $\quad$ $E(1,0,-2)$

$B(1,3,0)$ $\quad$ $F(1,3,-2)$

$C(0,3,0)$ $\quad$ $G(0,3,-2)$

1.13 in E^3 consider the parallelograms $A_1A_2A_3A_4$ and $B_1B_2B_3B_4$. Show that the midpoints of the segments $[A_1B_1]$, $[A_2B_2]$, $[A_3B_3]$ and $[A_4B_4]$ are the vertices of a parallelogram



$\overrightarrow{M_1M_2} \parallel \overrightarrow{M_4M_3}$:

$$\overrightarrow{M_1M_2} = \vec{M_2} - \vec{M_1} = \frac{\vec{A_2}+\vec{B_2}}{2} - \frac{\vec{A_1}+\vec{B_1}}{2} = \frac{1}{2}\left(\vec{A_2}+\vec{B_2}-\vec{A_1}-\vec{B_1}\right)$$

$$\overrightarrow{M_4M_3} = \vec{M_3} - \vec{M_4} = \frac{\vec{A_3}+\vec{B_3}}{2} - \frac{\vec{A_4}+\vec{B_4}}{2} = \frac{1}{2}\left(\vec{A_3}+\vec{B_3}-\vec{A_4}-\vec{B_4}\right)$$

$$\overrightarrow{M_1M_2} \parallel \overrightarrow{M_4M_3} \Leftrightarrow \overrightarrow{M_1M_2} = k\cdot\overrightarrow{M_2M_4} \Leftrightarrow$$

$$\Leftrightarrow \quad \frac{1}{2}\left(\vec{A_2}+\vec{B_2}-\vec{A_1}-\vec{B_1}\right)=K\cdot\frac{1}{2}\left(\vec{A_3}+\vec{B_3}-\vec{A_4}-\vec{B_4}\right)$$

$$\Leftrightarrow \quad \frac{1}{2}\left(\vec{A_2}-\vec{A_1}+\vec{B_2}-\vec{B_1}\right)=K\cdot\frac{1}{2}\left(\vec{A_3}-\vec{A_4}+\vec{B_3}-\vec{B_4}\right)$$

$\vec{A_2}-\vec{A_1}=\vec{A_3}-\vec{A_4}$ because $A_1A_2A_3A_4$ - parallelogram

$\vec{B_2}-\vec{B_1}=\vec{B_3}-\vec{B_4}$ because $B_1B_2B_3B_4$ - parallelogram

$$\Rightarrow \quad \frac{1}{2}\left(\vec{A_3}-\vec{A_4}+\vec{B_3}-\vec{B_4}\right)=K\cdot\frac{1}{2}\left(\vec{A_3}-\vec{A_4}+\vec{B_3}-\vec{B_4}\right)$$

$\Rightarrow \overrightarrow{M_1M_2} \parallel \overrightarrow{M_3M_4} \Rightarrow$ the midpoints form a parallelogram

1.14. Which of the following sets of vectors form a basis:

a) $v(1,2)$, $w(3,4)$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c_1 \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \quad c_1, c_2 \in \mathbb{R}$$

$$\begin{array}{cc} c_1 & c_2 \end{array}$$
$$\left[\begin{array}{cc|c} 1 & 3 & x_1 \\ 2 & 4 & x_2 \end{array}\right]$$

$$\Rightarrow \quad c_1 + 3c_2 = x_1 \quad \Rightarrow c_1 = x_1 - 3c_2 = x_1 + \frac{3}{2}\cdot(x_2 - 2x_1)$$

$$2c_1 + 4c_2 = x_2 \quad \Rightarrow 2x_1 - 6c_2 + 4c_2 = x_2$$

$$\Rightarrow -2c_2 = x_2 - 2x_1$$

$$c_2 = -\frac{x_2 - 2x_1}{2}$$

when $x_1 = x_2 = 0$

$\Rightarrow \quad c_1 = c_2 = 0 \quad \Rightarrow$ vectors are linearly indep.

$\Rightarrow$ they form a basis

b) $u(-1,2,1)$, $v(2,1,1)$, $w(1,0,-1)$

$$\vec{X}=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=c_1\begin{bmatrix}-1\\2\\1\end{bmatrix}+c_2\begin{bmatrix}2\\1\\1\end{bmatrix}+c_3\begin{bmatrix}1\\0\\-1\end{bmatrix}$$

$$\left[\begin{array}{ccc|c}-1&2&1&x_1\\2&1&0&x_2\\1&1&-1&x_3\end{array}\right]\xrightarrow[l_3\leftarrow l_3+l_1]{l_2\leftarrow l_2+2l_1}\left[\begin{array}{ccc|c}-1&2&1&x_1\\0&5&2&x_2+2x_1\\0&3&0&x_3+x_1\end{array}\right]$$

$$\begin{cases}-c_1+2c_2+c_3=x_1 \Rightarrow c_1=2c_2+c_3-x_1=\frac{2x_3}{3}+\frac{x_2+x_1-\frac{x_3+x_1}{3}}{2}-x_1\\ 5c_2+2c_3=x_2+x_1\Rightarrow \frac{5x_3}{3}+2c_3=x_2\Rightarrow c_3=\frac{x_2+x_1-\frac{x_3+x_1}{3}}{2}\\ 3c_2=x_3+x_1\Rightarrow c_2=\frac{x_3+x_1}{3}\end{cases}$$

when $x_1=x_2=x_3=0$

$\Rightarrow c_1=c_2=c_3=0 \Rightarrow$ vectors are lin. indep.

$\Rightarrow$ form a basis

c) $u(-1,2,1)$, $v(2,1,1)$, $w(0,5,3)$

$$\vec{X} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = C_1 \cdot \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} + C_2 \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + C_3 \begin{bmatrix} 0 \\ 5 \\ 3 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} -1 & 2 & 0 & x_1 \\ 2 & 1 & 5 & x_2 \\ 1 & 1 & 3 & x_3 \end{array}\right] \xrightarrow[l_3 \leftarrow l_3 + l_1]{l_2 \leftarrow l_2 + 2l_1} \left[\begin{array}{ccc|c} -1 & 2 & 0 & x_1 \\ 0 & 5 & 5 & x_2 + 2x_1 \\ 0 & 3 & 3 & x_3 + x_1 \end{array}\right]$$

$$\xrightarrow[l_3 \leftarrow \frac{1}{3} l_3]{l_2 \leftarrow \frac{1}{5} l_2} \left[\begin{array}{ccc|c} -1 & 2 & 0 & x_1 \\ 0 & 1 & 1 & x_2 + 2x_1 \\ 0 & 1 & 1 & x_3 + x_1 \end{array}\right] \xrightarrow{l_3 \leftarrow l_3 - l_2} \left[\begin{array}{ccc|c} -1 & 2 & 0 & x_1 \\ 0 & 1 & 1 & x_2 + 2x_1 \\ 0 & 0 & 0 & x_3 - x_2 - x_1 \end{array}\right]$$

$$\begin{cases} -C_1 + 2C_2 = x_1 \Rightarrow -C_1 - C_3 = -x_1 - x_2 \Rightarrow C_1 + C_3 = x_1 + x_2 \\ C_2 + C_3 = x_2 + 2x_1 \Rightarrow C_2 = x_2 + 2x_1 - C_3 \end{cases}$$

$\Rightarrow$ if $x_1 = x_2 = x_3 = 0$

$\Rightarrow$ $C_1 + C_3 = 0$ $\Rightarrow C_1 - C_2 = 0$ $\Rightarrow$ vectors are lin. dependent

$C_3 = -C_2$

$\Rightarrow$ do not form a basis

1.19. With respect to the basis $\mathcal{B} = (i, j, k)$ consider the vectors $u = i + j$, $v = j + k$, $w = i + k$. Check that $\mathcal{B}' = (u, v, w)$ is a basis and give the base change matrix $M_{\mathcal{B}', \mathcal{B}}$.

$$u = 1 \cdot i + 1 \cdot j + 0 \cdot k$$
$$v = 0 \cdot i + 1 \cdot j + 1 \cdot k \quad \Rightarrow \quad M_{\mathcal{B}', \mathcal{B}} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$
$$w = 1 \cdot i + 0 \cdot j + 1 \cdot k$$

$$\vec{X} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = c_1 \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & x_1 \\ 1 & 1 & 0 & x_2 \\ 0 & 1 & 1 & x_3 \end{array}\right] \xrightarrow{l_2 \leftarrow l_2 - l_1} \left[\begin{array}{ccc|c} 1 & 0 & 1 & x_1 \\ 0 & 1 & -1 & x_2 - x_1 \\ 0 & 1 & 1 & x_3 \end{array}\right] \xrightarrow{l_3 \leftarrow l_3 - l_2}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & x_1 \\ 0 & 1 & -1 & x_2 - x_1 \\ 0 & 0 & 2 & x_3 - x_2 + x_1 \end{array}\right]$$

$$\Rightarrow \begin{cases} c_1 + c_3 = x_1 & \Rightarrow c_1 = x_1 - \dfrac{x_3 - x_2 + x_1}{2} \\ c_2 - c_3 = x_2 - x_1 & \Rightarrow c_2 = x_2 - x_1 + \dfrac{x_3 - x_2 + x_1}{2} \\ 2c_3 = x_3 - x_2 + x_1 & \Rightarrow c_3 = \dfrac{x_3 - x_2 + x_1}{2} \end{cases}$$

If $x_1=x_2=x_3=0 \Rightarrow c_1=c_2=c_3=0 \Rightarrow$ vectors are lin indep.

$\Rightarrow B'$ is a basis

1.18. Consider the two coordinate systems $K=(O,i,j,k)$ and $K'=(O',i',j',k')$. Determine the base change matrix from K to K' and the coordinates of the points

$$[A]_K=\begin{bmatrix}1\\4\\-1\end{bmatrix} \quad [B]_K=\begin{bmatrix}1\\5\\1\end{bmatrix} \quad [C]_K=\begin{bmatrix}-3\\7\\1\end{bmatrix} \quad [D]_K=\begin{bmatrix}0\\3\\1\end{bmatrix}$$

in the coordinates system K'. Determine the base change matrix from K' to K and use it with the prev. det. coordinates to calculate $[A]_K, [B]_K, [C]_K, [D]_K$.

$$[O']_K=\begin{bmatrix}4\\5\\-1\end{bmatrix} \qquad i'=-i-2j=\begin{bmatrix}-1\\-2\\0\end{bmatrix}$$

$$j'=-2i+j=\begin{bmatrix}-2\\1\\0\end{bmatrix}$$

$$k'=j+2k=\begin{bmatrix}0\\1\\2\end{bmatrix}$$

$$M_{K,K'}=[\mathrm{id}]_{K',K}=\begin{pmatrix}-1&-2&0\\-2&1&1\\0&0&2\end{pmatrix}$$

$$M_{K',K} = M^{-1}_{K,K'} = \begin{pmatrix} -\frac{1}{5} & -\frac{2}{5} & \frac{1}{5} \\ -\frac{2}{5} & \frac{1}{5} & -\frac{1}{10} \\ 0 & 0 & \frac{1}{2} \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} -1 & -2 & 0 & 1 & 0 & 0 \\ -2 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 \end{array}\right) \xrightarrow{l_2 \leftarrow l_2 - 2l_1} \left(\begin{array}{ccc|ccc} -1 & -2 & 0 & 1 & 0 & 0 \\ 0 & 5 & 1 & -2 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow[l_3 \leftarrow l_3 \cdot \frac{1}{2}]{l_1 \leftarrow l_1 \cdot (-1)} \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & -1 & 0 & 0 \\ 0 & 5 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{2} \end{array}\right) \xrightarrow{l_2 \leftarrow l_2 - l_3}$$

$$\longrightarrow \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & -1 & 0 & 0 \\ 0 & 5 & 0 & -2 & 1 & -\frac{1}{2} \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{2} \end{array}\right) \xrightarrow{l_2 \leftarrow l_2 \cdot \frac{1}{5}}$$

$$\longrightarrow \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & -\frac{2}{5} & \frac{1}{5} & -\frac{1}{10} \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{2} \end{array}\right) \xrightarrow{l_1 \leftarrow l_1 - 2l_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{5} & -\frac{2}{5} & \frac{1}{5} \\ 0 & 1 & 0 & -\frac{2}{5} & \frac{1}{5} & -\frac{1}{10} \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{2} \end{array}\right)$$

$$[A]_{K'} = M_{K',K} \cdot \left([A]_K - [O']_K\right) = \begin{pmatrix} -\frac{1}{5} & -\frac{2}{5} & \frac{1}{5} \\ -\frac{2}{5} & \frac{1}{5} & -\frac{1}{10} \\ 0 & 0 & \frac{1}{2} \end{pmatrix} \left(\begin{pmatrix} 1 \\ 4 \\ -1 \end{pmatrix} - \begin{pmatrix} 4 \\ 5 \\ -1 \end{pmatrix} \right)$$

$$= \begin{pmatrix} -\frac{1}{5} & -\frac{2}{5} & \frac{1}{5} \\ -\frac{2}{5} & \frac{1}{5} & -\frac{1}{10} \\ 0 & 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} -3 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{3}{5} + \frac{2}{5} \\ \frac{6}{5} - \frac{1}{5} \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$[B]_{K'} = M_{K',K} \cdot \left([B]_K - [O']_K\right) = \begin{pmatrix} -\frac{1}{5} & -\frac{2}{5} & \frac{1}{5} \\ -\frac{2}{5} & \frac{1}{5} & -\frac{1}{10} \\ 0 & 0 & \frac{1}{2} \end{pmatrix} \cdot \left(\begin{pmatrix} 1 \\ 5 \\ 1 \end{pmatrix} - \begin{pmatrix} 4 \\ 5 \\ -1 \end{pmatrix} \right)$$

$$= \begin{pmatrix} -\frac{1}{5} & -\frac{2}{5} & \frac{1}{5} \\ -\frac{2}{5} & \frac{1}{5} & -\frac{1}{10} \\ 0 & 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} -3 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} \frac{3}{5} + \frac{2}{5} \\ \frac{6}{5} - \frac{2}{10} \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$[C]_{K'} = M_{K',K} \cdot \left([C]_K - [O']_K\right) = \begin{pmatrix} -\frac{1}{5} & -\frac{2}{5} & \frac{1}{5} \\ -\frac{2}{5} & \frac{1}{5} & -\frac{1}{10} \\ 0 & 0 & \frac{1}{2} \end{pmatrix} \left(\begin{pmatrix} -3 \\ 7 \\ 1 \end{pmatrix} - \begin{pmatrix} 4 \\ 5 \\ -1 \end{pmatrix} \right)$$

$$= \begin{pmatrix} -\frac{1}{5} & -\frac{2}{5} & \frac{1}{5} \\ -\frac{2}{5} & \frac{1}{5} & -\frac{1}{10} \\ 0 & 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} -7 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} \frac{7}{5} - \frac{4}{5} + \frac{2}{5} \\ \frac{14}{5} + \frac{2}{5} - \frac{2}{10} \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{5} \\ 3 \\ 1 \end{pmatrix}$$

$$[D]_{K'} = M_{K',K} \cdot \left([D]_K - [O']_K\right) = \begin{pmatrix} -\frac{1}{5} & -\frac{2}{5} & \frac{1}{5} \\ -\frac{2}{5} & \frac{1}{5} & -\frac{1}{10} \\ 0 & 0 & \frac{1}{2} \end{pmatrix} \left(\begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} - \begin{pmatrix} 4 \\ 5 \\ -1 \end{pmatrix} \right)$$

$$= \begin{pmatrix} -\frac{1}{5} & -\frac{2}{5} & \frac{1}{5} \\ -\frac{2}{5} & \frac{1}{5} & -\frac{1}{10} \\ 0 & 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} -4 \\ -2 \\ 2 \end{pmatrix} = \begin{pmatrix} \frac{4}{5} + \frac{4}{5} + \frac{2}{5} \\ \frac{8}{5} - \frac{2}{5} - \frac{2}{10} \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$

$$[A]_K = [\vec{OA}]_K = [\vec{OO'} + \vec{O'A}]_K = [O'A]_K - [O'O]_K =$$

$$= M_{K,K'} \cdot \left([A]_{K'} - [O]_{K'}\right)$$

$$[O]_{K'} = [\vec{O'O}]_{K'} = -[\vec{OO'}]_{K'} = -\left(M_{K',K} \cdot [O']_K\right) =$$

$$= -\begin{pmatrix} -\frac{1}{5} & -\frac{2}{5} & \frac{1}{5} \\ -\frac{2}{5} & \frac{1}{5} & -\frac{1}{10} \\ 0 & 0 & \frac{1}{2} \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 5 \\ -1 \end{pmatrix} = -\begin{pmatrix} -\frac{4}{5} - \frac{10}{5} - \frac{1}{5} \\ -\frac{8}{5} + \frac{5}{5} + \frac{1}{10} \\ -\frac{1}{2} \end{pmatrix} = -\begin{pmatrix} -3 \\ -\frac{1}{2} \\ -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} 3 \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$$

$$[A]_K = \begin{pmatrix} -1 & -2 & 0 \\ -2 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix} \cdot \left(\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 3 \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \right) = \begin{pmatrix} -1 & -2 & 0 \\ -2 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} -2 \\ \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}$$

$$= \begin{pmatrix} 2 - \frac{2}{2} \\ 4 + \frac{1}{2} - \frac{1}{2} \\ 2 \cdot \left(-\frac{1}{2}\right) \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ -1 \end{pmatrix}$$

$$[B]_K = [\vec{OB}]_K = [\vec{OO'} + \vec{O'B}]_K = [O'B]_K - [\vec{O'O}]_K =$$

$$= M_{K,K'} \cdot \left([B]_{K'} - [O]_{K'}\right) = \begin{pmatrix} 1 \\ 5 \\ 1 \end{pmatrix}$$

$$[C]_K = [\vec{OC}]_K = [\vec{OO'} + \vec{O'C}]_K = [O'C] - [\vec{O'O}]_K = M_{K,K'} \cdot \left([C]_{K'} - [O]_{K'}\right) = \begin{pmatrix} -3 \\ 7 \\ 1 \end{pmatrix}$$

$$[D]_K = [\vec{OD}]_K = [\vec{OO'} + \vec{O'D}]_K = M_{K,K'} \cdot \left([D]_{K'} - [O]_{K'}\right) = \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}$$

Chapter 2:

2.18 Give Cartesian equations for the lines l

a) l contains the point $M_0(2,0,3)$ and is parallel to the vector $a(3,-2,-2)$

Symmetric form:

$$\frac{x-x_0}{a}=\frac{y-y_0}{b}=\frac{z-z_0}{c} \Leftrightarrow$$

$$\Leftrightarrow \frac{x-2}{3}=\frac{y}{-2}=\frac{z-3}{-2}$$

implicit form:

$$\frac{x-2}{3}=\frac{y}{-2}=\frac{z-3}{-2} \Rightarrow -2x+4+3y=\frac{z-3}{-2} \Rightarrow 4x-8-6y=z-3$$

$$\Leftrightarrow 4x-6y-z-5=0$$

explicit form:

b) l contains the point $A(1,2,3)$ and is parallel to the Oz-axis $\Rightarrow v(0,0,1)$

Symmetric form:

$x=1 \quad y=2 \quad z-3=0$

implicit form: $z-3=0$

Explicit form: $x=1 \quad y=2$

c) ℓ contains the points $M_1(1,2,3)$ and $M_2(4,4,4)$

$$\begin{cases} x = 1+3\lambda \\ y = 2+2\lambda \\ z = 3+\lambda \end{cases}$$

direction_vector $= M_2 - M_1 = (3,2,1)$

Symmetric form: $\frac{x-1}{3} = \frac{y-2}{2} = \frac{z-3}{1}$

implicit form:

$$\frac{x-1}{3} = \frac{y-2}{2} = \frac{z-3}{1} \Leftrightarrow 2x-2+3y-6 = \frac{z-3}{1}$$

$$\Leftrightarrow 2x+3y-z-5=0$$

Explicit form: $3y = 5-2x+z$

$$y = \frac{5-2x+z}{3}$$

2.19. Determine parametric equations for the line contained in the planes $x+y+2z-3=0$ and $x-y+z-1=0$

$$n_1 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \quad n_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

$$n_1 \times n_2 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \times \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{vmatrix} x & y & z \\ 1 & 1 & 2 \\ 1 & -1 & 1 \end{vmatrix} = x\begin{vmatrix} 1 & 2 \\ -1 & 1 \end{vmatrix} - y\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} + z\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}$$

$$= 3x + y - 2z$$

$$v_1 \times v_2 = \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}$$ (directional vector)

Find the point on the line:

$$\begin{cases} x+y+2z-3=0 \\ x-y+z-1=0 \end{cases} \Bigg| \oplus \quad \Rightarrow 2x+3z-4=0$$

$$\Rightarrow 2x+3z=4$$

$$\Rightarrow x = \frac{4-3z}{2}$$

$$\frac{4-3z}{2} + y + 2z - 3 = 0 \Leftrightarrow 4-3z+2y+4z-6=0 \Rightarrow$$

$$\Rightarrow 2y+z-2 \Rightarrow y = \frac{2-z}{2}$$

Note: $z = \lambda$

$$\Rightarrow \begin{cases} x = \dfrac{4-3\lambda}{2} \\ y = \dfrac{2-\lambda}{2} \\ z = \lambda \end{cases}$$

2.26. Determine the values a and d for which the line $\frac{x-2}{3}=\frac{y+1}{2}=\frac{z-3}{-2}$ is contained in the plane $ax+y-2z+d=0$.

$$\begin{cases} x=2+3\lambda \\ y=-1+2\lambda \\ z=3-2\lambda \end{cases}$$

$$ax+y-2z+d=0$$

$$a(2+3\lambda)+(-1+2\lambda)-2(3-2\lambda)+d=0$$

$$\Leftrightarrow a(2+3\lambda)-1+2\lambda-6+4\lambda+d=0$$

$$\Leftrightarrow a(2+3\lambda)-7+6\lambda+d=0$$

$$\Leftrightarrow 2a+\underline{3a\lambda}-7+\underline{6\lambda}+d=0$$

$$\Leftrightarrow \lambda(3a+6)+(2a-7+d)=0$$

equation must hold true for all values of λ

$$\Rightarrow \begin{cases} 3a+6=0 \Rightarrow 3a=-6 \Rightarrow a=-2 \\ 2a-7+d=0 \Rightarrow -4-7+d=0 \Rightarrow d=11 \end{cases}$$

2.27. In each of the following, find a Cartesian eq. of the plane in A^3 passing through Q and parallel to the lines l and l':

a) $Q(1,-1,-2)$ $\quad l: x-y=1,\ x+z=5$

$l': x=1,\ z=2$

Direction vectors:

$x-y=1 \Rightarrow x=1+y \Rightarrow x=1+\lambda$

$x+z=5 \Rightarrow z=5-x \Rightarrow z=4-\lambda$

$y=\lambda$

$\Rightarrow v_l=\begin{pmatrix}1\\1\\-1\end{pmatrix}$

$x=1,\ y=\lambda,\ z=2$

$\Rightarrow v_{l'}=\begin{pmatrix}0\\1\\0\end{pmatrix}$

$$v_l \times v_{l'} = \begin{pmatrix}1\\1\\-1\end{pmatrix} \times \begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{vmatrix}x & y & z\\1 & 1 & -1\\0 & 1 & 0\end{vmatrix} =$$

$$= x\cdot\begin{vmatrix}1 & -1\\1 & 0\end{vmatrix} - y\begin{vmatrix}1 & -1\\0 & 0\end{vmatrix} + z\begin{vmatrix}1 & 1\\0 & 1\end{vmatrix} =$$

$$= x + z = \begin{pmatrix}1\\0\\1\end{pmatrix} = n$$

$$n\cdot(r-Q)=0$$

$$\begin{pmatrix}1\\0\\1\end{pmatrix}\cdot\left(\begin{pmatrix}x\\y\\z\end{pmatrix}-\begin{pmatrix}1\\-1\\-2\end{pmatrix}\right)=0 \Leftrightarrow \begin{pmatrix}1\\0\\1\end{pmatrix}\begin{pmatrix}x-1\\y+1\\z+2\end{pmatrix}=0 \Leftrightarrow$$

$$\Leftrightarrow x-1+z+2=0 \Rightarrow x+z-1=0$$

b) $Q(0,1,3)$ $\quad l: x+y=-5 \ , \ x-y+2z=0$

$l': 2x-2y=1 \ , \ x-y+2z=1$

Direction vectors

$x+y=-5 \Rightarrow x=-5-y \Rightarrow x=-5-\lambda$

$x-y+2z=1 \Rightarrow -5-y-y+2z=1 \Rightarrow 2z=6+2y \Rightarrow z=3+y$

$y=\lambda \qquad \Rightarrow z=3+\lambda$

$$\Rightarrow v_l = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

$2x-2y=1 \Rightarrow x-y=\frac{1}{2} \Rightarrow x=y+\frac{1}{2} \Rightarrow x=\lambda+\frac{1}{2}$

$x-y+2z=1 \Rightarrow \cancel{y}+\frac{1}{2}-\cancel{y}+2z=1 \Rightarrow 2z=1-\frac{1}{2} \Rightarrow z=\frac{1}{4}$

$y=\lambda$

$$\Rightarrow v_{l'} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$v_l \times v_{l'} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{vmatrix} x & y & z \\ -1 & 1 & 1 \\ 1 & 1 & 0 \end{vmatrix} =$$

$$= x \cdot \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} - y \cdot \begin{vmatrix} -1 & 1 \\ 1 & 0 \end{vmatrix} + z \begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} =$$

$$= -x+y-2z = \begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix} = n$$

$n \cdot (r - Q) = 0 \Leftrightarrow$

$$\Leftrightarrow \begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix} \cdot \left(\begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} \right) = \begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} x \\ y-1 \\ z-3 \end{pmatrix} = -x + y - 1 - 2z + 6 =$$

$$\Rightarrow -x + y - 2z + 5 = 0$$

2.30. in each of the following, find Cartesian equations for the line ℓ in A^3 passing through Q, contained in the plane π and intersecting the line ℓ'.

a) $Q = (1,1,0)$, $\pi: 2x - y + z - 1 = 0$

$\ell': x = 2 - t,\ y = 2 + t,\ z = t$

$$2x - y + z - 1 = 0 \Leftrightarrow 4 - 2t - 2 - \cancel{t} + \cancel{t} - 1 = 0 \Leftrightarrow$$

$$\Leftrightarrow -2t + 1 = 0 \Leftrightarrow 2t = 1 \Rightarrow t = \frac{1}{2}$$

$$\Rightarrow x = {}^{2/}2 - \frac{1}{2} = \frac{3}{2} \qquad y = {}^{2/}2 + \frac{1}{2} = \frac{5}{2} \qquad z = \frac{1}{2}$$

$$\Rightarrow R\left(\frac{3}{2}, \frac{5}{2}, \frac{1}{2}\right)$$

$$\ell: R - Q = \left(\frac{3}{2}, \frac{5}{2}, \frac{1}{2}\right) - (1,1,0) = \left(\frac{1}{2}, \frac{3}{2}, \frac{1}{2}\right)$$

Parametric eq: $x = 1 + \frac{1}{2}\lambda$

$y = 1 + \frac{3}{2}\lambda$

$z = \frac{1}{2}\lambda$

Cartesian eq: $\dfrac{x-1}{\frac{1}{2}} = \dfrac{y-1}{\frac{3}{2}} = \dfrac{z}{\frac{1}{2}} \Leftrightarrow 2x-2 = \dfrac{2(y-1)}{3} = 2z$

b) $Q(-1,-1,-1)$, $\pi: x+y+z+3=0$

$l': x-2z+4=0,\ 2y-z=0$

$x-2z+4=0 \Rightarrow x = 2z-4 = 2t-4$

$2y-z=0 \Rightarrow y = \dfrac{z}{2} = \dfrac{t}{2}$

$z = t$

$2t-4+\dfrac{t}{2}+t+3=0 \Leftrightarrow {}^{2/}3t+\dfrac{t}{2}\,{}^{2/}-1=0 \Leftrightarrow 7t-2=0$

$\Leftrightarrow t = \dfrac{2}{7}$

$x = 2t-4 = \dfrac{4}{7} - {}^{7/}4 = \dfrac{4-28}{7} = -\dfrac{24}{7}$

$y = \dfrac{2}{14} = \dfrac{1}{7}$

$z = \dfrac{2}{7}$

$\Rightarrow R\left(-\dfrac{24}{7}, \dfrac{1}{7}, \dfrac{2}{7}\right)$

$l: R-Q = \left(-\dfrac{24}{7}, \dfrac{1}{7}, \dfrac{2}{7}\right) - (-1,-1,-1) =$

$= \left(-\dfrac{24}{7} + {}^{7/}1, \dfrac{1}{7} + {}^{7/}1, \dfrac{2}{7} + {}^{7/}1\right) = \left(-\dfrac{17}{7}, \dfrac{8}{7}, \dfrac{9}{7}\right)$

Parametric eq:

$$\begin{cases} x = -1 - \dfrac{17}{7}\lambda \\ y = -1 + \dfrac{8}{7}\lambda \\ z = -1 + \dfrac{9}{7}\lambda \end{cases}$$

Cartesian eq

$\dfrac{x+1}{\frac{17}{7}} = \dfrac{y+1}{\frac{8}{7}} = \dfrac{z+1}{\frac{9}{7}} \Leftrightarrow$

$\Leftrightarrow \dfrac{7}{17}(x+1) = \dfrac{7}{8}(y+1) = \dfrac{7}{9}(z+1)$

Chapter 3:

3.1. Let m and n be two unit vectors such that $\sphericalangle(m,n) = 60^\circ$. Determine the length of the diagonals in the parallelogram spanned by the vectors $a = 2m+n$ and $b = m-2n$.

$$d_1 = a+b = 2m+n+m-2n = 3m-n$$

$$d_2 = a-b = 2m+n-m+2n = m+3n$$

$$|d_1| = \sqrt{(3m-n)^2} = \sqrt{(3m-n)\bullet(3m-n)} =$$

$$= \sqrt{9\underbrace{|m|^2}_{=1} - 6\, m\bullet n + \underbrace{|n|^2}_{=1}} = \sqrt{9-3+1} = \sqrt{7}$$

$$m\bullet n = |m|\cdot|n|\cdot\cos 60^\circ = \frac{1}{2}$$

$$|d_2| = \sqrt{(m+3n)^2} = \sqrt{(m+3n)(m+3n)} =$$

$$= \sqrt{\underbrace{|m|^2}_{=1} + 6\cdot \underbrace{m\bullet n}_{=\frac{1}{2}} + 9\underbrace{|n|^2}_{=1}} = \sqrt{1+3+9} = \sqrt{13}$$

3.4. Let (i, j, k) be an orthonormal basis. Consider the vectors $q = 3i + j$ and $p = i + 2j + \lambda k$ with $\lambda \in \mathbb{R}$. Determine λ such that the cosine of the angle $\measuredangle(p, q) = \frac{5}{12}$

$$\cos(p, q) = \frac{p \cdot q}{|p| \cdot |q|} = \frac{5}{12}$$

$$p \cdot q = (i + 2j + \lambda k) \cdot (3i + j) = 3 + 2 + \lambda \cdot 0 = 5$$

$$|p| = \sqrt{(i + 2j + \lambda k) \cdot (i + 2j + \lambda k)} =$$

$$= \sqrt{1 + 4 + \lambda^2} = \sqrt{5 + \lambda^2}$$

$$|q| = \sqrt{(3i + j) \cdot (3i + j)} = \sqrt{9 + 1} = \sqrt{10}$$

$$\frac{5}{\sqrt{5 + \lambda^2} \cdot \sqrt{10}} = \frac{5}{12} \Leftrightarrow \sqrt{5 + \lambda^2} \cdot \sqrt{10} = 12 \Leftrightarrow$$

$$\Leftrightarrow \sqrt{5 + \lambda^2} = \frac{12}{\sqrt{10}} \quad /(\,)^2 \Leftrightarrow 5 + \lambda^2 = \frac{144}{10} \Leftrightarrow \lambda^2 = \frac{144}{10} - {}^{10/}5$$

$$\Leftrightarrow \lambda^2 = \frac{94}{10} \Leftrightarrow \lambda^2 = \frac{47}{5} \Leftrightarrow \lambda = \pm\sqrt{\frac{47}{5}}$$

3.9. Consider the vector v which is perpendicular on $a(4,-2,-3)$ and on $b(0,1,3)$. if v describes an acute angle with Ox and $|v|=26$ determine the components of v.

$$v = a \times b = \begin{vmatrix} x & y & z \\ 4 & -2 & -3 \\ 0 & 1 & 3 \end{vmatrix} =$$

$$= x \cdot \begin{vmatrix} -2 & -3 \\ 1 & 3 \end{vmatrix} - y \begin{vmatrix} 4 & -3 \\ 0 & 3 \end{vmatrix} + z \begin{vmatrix} 4 & -2 \\ 0 & 1 \end{vmatrix} =$$

$$= -3x - 12y + 4z = \begin{pmatrix} -3 \\ -12 \\ 4 \end{pmatrix}$$

$$|v| = \sqrt{(-3)^2 + (-12)^2 + 4^2} = \sqrt{9+144+16} = \sqrt{169} = 13$$

in order to satisfy $|v|=26$ we have to multiply v with 2

$\Rightarrow v = (-6, -24, 8)$

3.12. Determine a Cartesian equations for the line l in the following cases:

a) l contains the point $A(-2,3)$ and has an angle of 60^0 with the Ox-axis.

Method I:

$y = mx + c \Rightarrow 3 = \sqrt{3}\cdot(-2) + c \Rightarrow c = 3 + 2\sqrt{3}$

$m = tg(60^0) = \sqrt{3}$

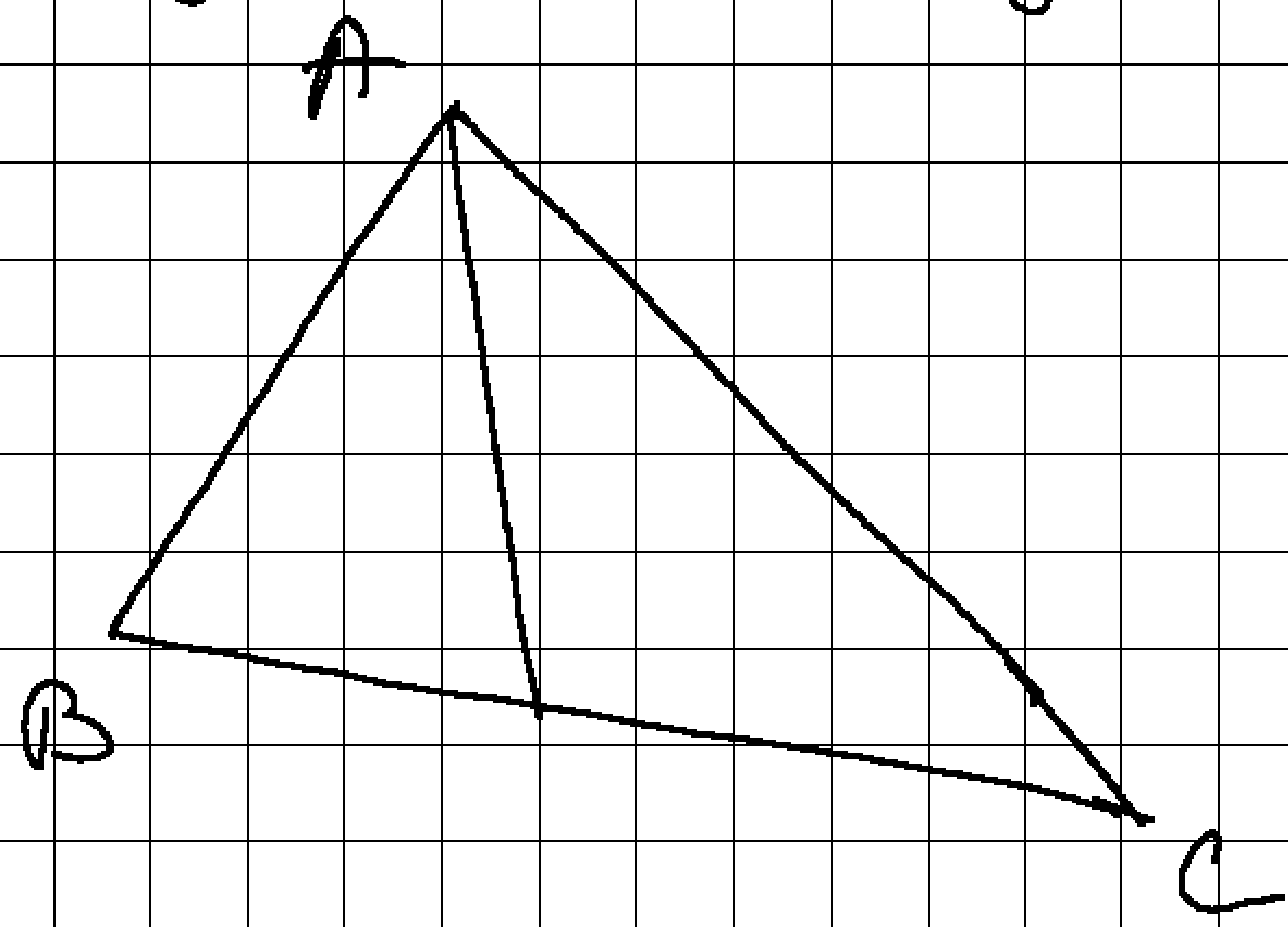
$\Rightarrow y = \sqrt{3}x + 3 + 2\sqrt{3} \Leftrightarrow y - 3 = \sqrt{3}(x+2)$

Method II:

$y - y_1 = m(x - x_1) \Rightarrow y - 3 = \sqrt{3}(x+2)$

$m = tg(60^0) = \sqrt{3}$

3.20. Let $A(1,-2)$, $B(5,4)$ and $C(-2,0)$ be the vertices of a triangle. Determine the equations of the angle bisectors for the angle $\measuredangle A$.



$$m_{AB} = \frac{y_B - y_A}{x_B - x_A} = \frac{4+2}{5-1} = \frac{6}{4} = \frac{3}{2}$$

$$m_{AC} = \frac{y_C - y_A}{x_C - x_A} = \frac{0+2}{-2-1} = -\frac{2}{3}$$

angle bisectors theorem:

$$y = x\,\frac{m_1 + m_2 \pm \sqrt{(1+m_1^2)(1+m_2^2)}}{1 + m_1 m_2} + c$$

where $m_1 = m_{AB}$, $m_2 = m_{BC}$, c – y intercept point

$$y = x\,\frac{\frac{3}{2} - \frac{2}{3} \pm \sqrt{\left(1+\left(\frac{3}{2}\right)^2\right)\left(1-\left(\frac{2}{3}\right)^2\right)}}{1-1} + c \Rightarrow y = c \Rightarrow c = -2$$

$$\Rightarrow y = -2 \Rightarrow y - 2 = 0$$

3.29. Determine an equation for each plane passing through $P(3,5,-7)$ and intersecting the coordinate axes in congruent segments.

Let $A(a,0,0)$, $B(0,b,0)$, $C(0,0,c)$

Congruent segments $\Rightarrow$ $a=b=c$

Equation of a plane passing through three points $P(x_1,y_1,z_1)$, $Q(x_2,y_2,z_2)$, $R(x_3,y_3,z_3)$

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix} = 0$$

For $P(3,5,-7)$, $A(a,0,0)$, $B(0,a,0)$

$$\Rightarrow \begin{vmatrix} x-3 & y-5 & z+7 \\ a-3 & -5 & 7 \\ -3 & a-5 & 7 \end{vmatrix} = 0$$

$$\Rightarrow (x-3)\begin{vmatrix} -5 & 7 \\ a-5 & 7 \end{vmatrix} - (y-5)\begin{vmatrix} a-3 & 7 \\ -3 & 7 \end{vmatrix} + (z+7)\begin{vmatrix} a-3 & -5 \\ -3 & a-5 \end{vmatrix} = 0$$

$$\Rightarrow (x-3)\cdot(-\cancel{35}-(7a-\cancel{35})) - (y-5)(7a-\cancel{21}+\cancel{21}) + (z+7)((a-3)(a-5)-15) = 0$$

$$\Rightarrow 7a(x-3) - 7a(y-5) + (a^2-8a)(z+7) = 0$$

3.31. Show that a parallelipiped with faces in the planes $2x+y-2z+6=0$ is rectangular.

$$2x-2y+z-8=0$$

$$x+2y+2z+1=0$$

$n_1=(2,1,-2)$

$n_2=(2,-2,1)$

$n_3=(1,2,2)$

A parallelipiped is a rectangular box iff the dot product of each pair of the normal vectors is zero.

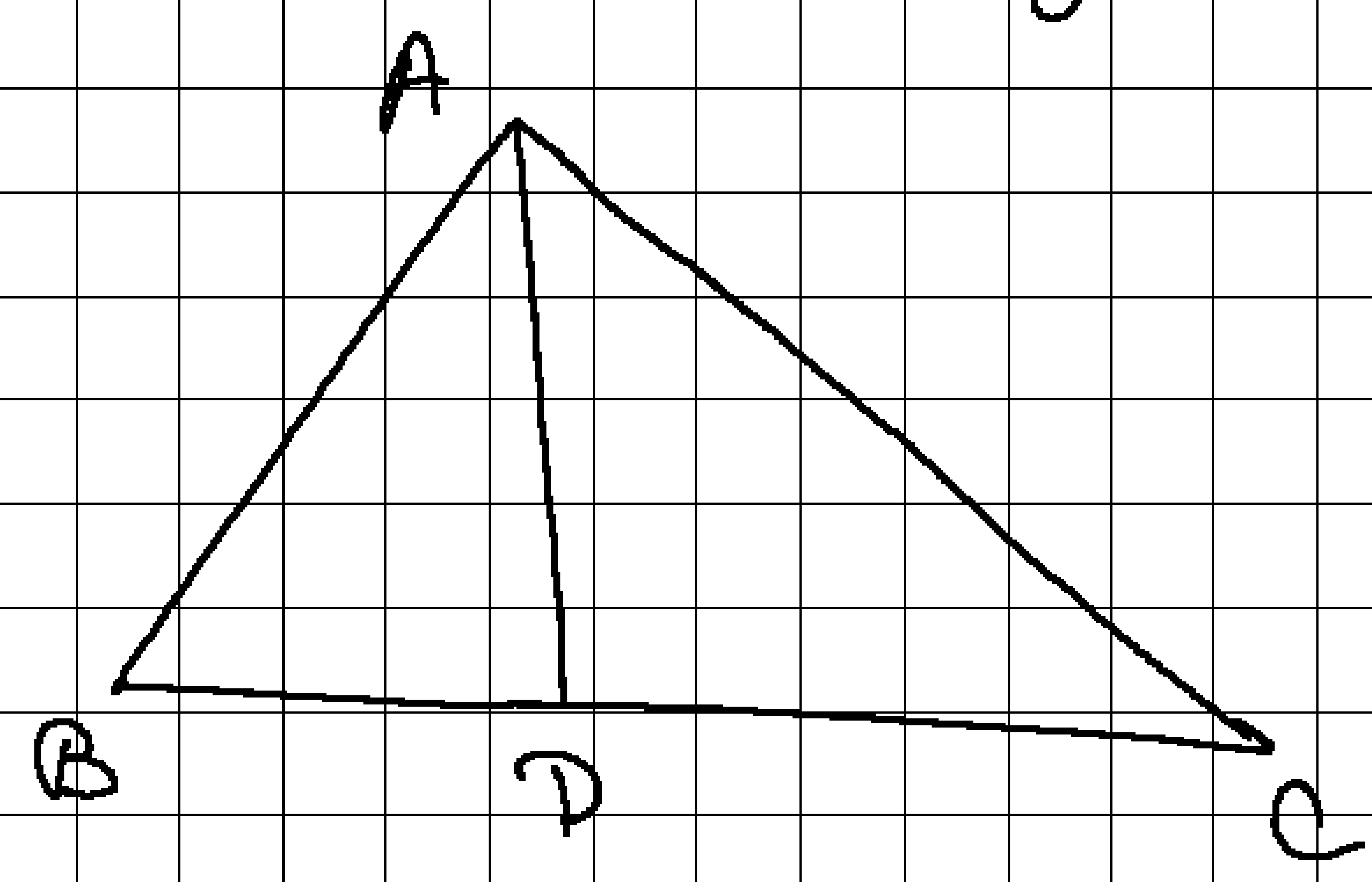
$n_1 \bullet n_2 = 2\cdot 2 + 1\cdot(-2) + (-2)\cdot 1 = 4-2-2=0$

$n_1 \bullet n_3 = 2\cdot 1 + 1\cdot 2 - 2\cdot 2 = 2+2-4=0$

$n_2 \bullet n_3 = 2\cdot 1 - 2\cdot 2 + 1\cdot 2 = 2-4+2=0$

$\Rightarrow$ the parallelipiped is a rectangular box.

3.35. Let $A(1,2,-7)$, $B(2,2,-7)$ and $C(3,4,-5)$ be vertices of a triangle. Determine the equation of the internal angle bisector of $\measuredangle A$



$$AB = \sqrt{(2-1)^2 + (2-2)^2 + (-7+7)^2} = 1$$

$$AC = \sqrt{(3-1)^2 + (4-2)^2 + (-5+7)^2} = \sqrt{4+4+4} = \sqrt{12} = 2\sqrt{3}$$

$$\frac{AB}{AC} = \frac{1}{2\sqrt{3}} = \frac{\sqrt{3}}{6}$$

AD - angle bisector of $\measuredangle A$

Angle bisector theorem $\Rightarrow$ $\frac{BD}{DC} = \frac{AB}{AC} = \frac{\sqrt{3}}{6}$

$$D = \frac{\frac{\sqrt{3}}{6}\cdot B + \left(1 - \frac{\sqrt{3}}{6}\right)\cdot C}{1 + \frac{\sqrt{3}}{6}} = \frac{\frac{\sqrt{3}}{6}\cdot(2,2,-7) + \left(1 - \frac{\sqrt{3}}{6}\right)\cdot(3,4,-5)}{1 + \frac{\sqrt{3}}{6}} =$$

$$= \frac{\left(\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, -\frac{7\sqrt{3}}{6}\right) + \left(3 - \frac{\sqrt{3}}{2}, 4 - \frac{2\sqrt{3}}{3}, -5 + \frac{5\sqrt{3}}{6}\right)}{1 + \frac{\sqrt{3}}{6}}$$

$$= \frac{\left(\frac{\sqrt{3}}{3} + 3 - \frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{3} + 4 - \frac{2\sqrt{3}}{3}, -\frac{7\sqrt{3}}{6} - 5 + \frac{5\sqrt{3}}{6}\right)}{1 + \frac{\sqrt{3}}{6}} =$$

$$= \frac{\left(3-\frac{\sqrt{3}}{6}, 4-\frac{\sqrt{3}}{3}, -5+\frac{2\sqrt{3}}{6}\right)}{1+\frac{\sqrt{3}}{6}} =$$

$$= \left(\frac{3-\frac{\sqrt{3}}{6}}{1+\frac{\sqrt{3}}{6}}, \frac{4-\frac{\sqrt{3}}{3}}{1+\frac{\sqrt{3}}{6}}, \frac{-9+\frac{2\sqrt{3}}{6}}{1+\frac{\sqrt{3}}{6}} \right)$$

$$y - y_1 = m(x - x_1)$$

$$m = \frac{y_D - y_A}{x_D - x_A} = \frac{\frac{4-\frac{\sqrt{3}}{3}}{1+\frac{\sqrt{3}}{6}} - 2}{\frac{3-\frac{\sqrt{3}}{6}}{1+\frac{\sqrt{3}}{6}} - 1} = \frac{\frac{4-\frac{\sqrt{3}}{3}-2-\frac{\sqrt{3}}{3}}{1+\frac{\sqrt{3}}{6}}}{\frac{3-\frac{\sqrt{3}}{6}-1-\frac{\sqrt{3}}{6}}{1+\frac{\sqrt{3}}{6}}} = \frac{\frac{2-\frac{2\sqrt{3}}{3}}{1+\frac{\sqrt{3}}{6}}}{\frac{2-\frac{2\sqrt{3}}{3}}{1+\frac{\sqrt{3}}{6}}} =$$

$$= \frac{2-\frac{2\sqrt{3}}{3}}{1+\frac{\sqrt{3}}{6}} \cdot \frac{1+\frac{\sqrt{3}}{6}}{2-\frac{2\sqrt{3}}{3}} = 1$$

$$AD: \quad y - y_A = m_{AD}(x - x_A)$$

$$y - 2 = 1(x-1)$$

$$y - 2 = x - 1$$

$$y = x + 1$$

3.38. Determine the orthogonal projection of the point $A(2,11,-5)$ on the plane $x+4y-3z+7=0$

Given a plane: $ax+by+cz+d=0$
and a point: $P(x_0,y_0,z_0)$

the orthogonal projection of the point onto the plane is given by $P'(x_0-aD,\ y_0-bD,\ z_0-cD)$

where $D=\dfrac{ax_0+by_0+cz_0+d}{\sqrt{a^2+b^2+c^2}}$ (distance from the point to the plane)

$n=(1,4,-3)$ $\qquad a=1,\ b=4,\ c=-3,\ d=7$

$A(2,11,-5)$ $\qquad x_0=2,\ y_0=11,\ z_0=-5$

$$D=\frac{2+44+15+7}{\sqrt{1+16+9}}=\frac{68}{\sqrt{26}}=\frac{34}{\sqrt{13}}$$

$$P'\left(2-\frac{34}{\sqrt{13}},\ 11-\frac{4\cdot 34}{\sqrt{13}},\ -5-\frac{3\cdot 34}{\sqrt{13}}\right)=$$

$$=P'\left(\frac{26-34\sqrt{13}}{13},\ \frac{143-136\sqrt{13}}{13},\ \frac{-65-102\sqrt{13}}{13}\right)$$

3.39. Determine the orthogonal reflection of the point $P(6,-5,5)$ in the plane $2x-3y+z-4=0$

Given a plane: $ax+by+cz+d=0$

and a point: $P(x_0, y_0, z_0)$

the orthogonal reflection of the point onto the plane is given by $P''(x_0-2aD,\ y_0-2bD,\ z_0-2cD)$

where $D=\dfrac{ax_0+by_0+cz_0+d}{\sqrt{a^2+b^2+c^2}}$ (distance from the point to the plane)

$n=(2,-3,1)$ $\qquad a=2,\ b=-3,\ c=1,\ d=-4$

$P(6,-5,5)$ $\qquad x_0=6,\ y_0=-5,\ z_0=5$

$$D=\frac{12+15+5-4}{\sqrt{4+9+1}}=\frac{28}{\sqrt{14}}=\frac{14}{\sqrt{7}}$$

$$P''\left(6-\frac{4\cdot 14}{\sqrt{7}},\ -5+\frac{6\cdot 14}{\sqrt{7}},\ 5-\frac{2\cdot 14}{\sqrt{7}}\right)=$$

$$=P''\left(6-\frac{56}{\sqrt{7}},\ -5+\frac{84}{\sqrt{7}},\ 5-\frac{28}{\sqrt{7}}\right)=$$

$$=P''\left(\frac{42-56\sqrt{7}}{7},\ \frac{-35+84\sqrt{7}}{7},\ \frac{35-28\sqrt{7}}{7}\right)$$

Chapter 4:

4.13. The points $A(1,2,-1)$, $B(0,1,5)$, $C(-1,2,1)$ and $D(2,1,3)$ are given with respect to an orthonormal coordinate system. Are the four points coplanar?

The points are coplanar if the volume of the parallelepiped formed by the vectors AB, AC, AD is zero, otherwise they form a three-dimensional figure in space

$V = |a \bullet (b \times c)|$

$$V = |AB \bullet (AC \times AD)|$$

$$AB = B - A = (0,1,5) - (1,2,-1) = (-1,-1,6)$$

$$AC = C - A = (-1,2,1) - (1,2,-1) = (-2,0,2)$$

$$AD = D - A = (2,1,3) - (1,2,-1) = (1,-1,4)$$

$$AC \times AD = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2 & 0 & 2 \\ 1 & -1 & 4 \end{vmatrix} = \vec{i} \cdot \begin{vmatrix} 0 & 2 \\ -1 & 4 \end{vmatrix} - \vec{j} \begin{vmatrix} -2 & 2 \\ 1 & 4 \end{vmatrix} + \vec{k} \begin{vmatrix} -2 & 0 \\ 1 & -1 \end{vmatrix} =$$

$$= 2i + 10j - 2k \Rightarrow (2,10,-2)$$

$$V = |AB \bullet (AC \times AD)| = |(-1,-1,6) \bullet (2,10,-2)| = |-2 - 10 - 12| =$$

$$= |-24| = 24 \neq 0$$

$\Rightarrow$ the points are not coplanar

4.17. With respect to an orthonormal system consider the vectors $a(8,4,1)$, $b(2,2,1)$ and $C(1,1,1)$. Determine a vector d satisfying the following properties.

a) the angles $\sphericalangle(d,a)$ and $\sphericalangle(d,b)$ are equal

b) d is orthogonal to c

c) (a,b,c) and (a,b,d) have the same orientation

$\boxed{a.}$ Equal angles $\sphericalangle(d,a)$ and $\sphericalangle(d,b)$

$$|\vec{a}| = \sqrt{8^2+4^2+1^2} = \sqrt{64+16+1} = \sqrt{81} = 9$$

$$|\vec{b}| = \sqrt{2^2+2^2+1^2} = \sqrt{9} = 3$$

$$\vec{a}' = \frac{\vec{a}}{|\vec{a}|} = \left(\frac{8}{9}, \frac{4}{9}, \frac{1}{9}\right)$$ (normalize vectors to unit vectors)

$$\vec{b}' = \frac{\vec{b}}{|\vec{b}|} = \left(\frac{2}{3}, \frac{2}{3}, \frac{1}{3}\right)$$

$\Rightarrow$ $d \bullet a' = d \bullet b'$ where $d = (x,y,z)$

$$x \cdot \frac{8}{9} + y \cdot \frac{4}{9} + z \cdot \frac{1}{9} = x \cdot \frac{2}{3}^{3/} + y \cdot \frac{2}{3}^{3/} + z \cdot \frac{1}{3}^{3/} \Leftrightarrow$$

$$\Leftrightarrow 8x + 4y + z = 6x + 6y + 3z \Leftrightarrow$$

$$\Leftrightarrow 2x - 2y - 2z = 0 \Leftrightarrow x - y - z = 0 \quad (1)$$

$\boxed{b.}$ $\vec{d}$ orthogonal to $\vec{c}$ $\Rightarrow$

$\Rightarrow$ $\vec{d} \bullet \vec{c} = 0$

$x \cdot 1 + y \cdot 1 + z \cdot 1 = 0 \Rightarrow x+y+z=0 \ (2)$

$(1) \Rightarrow z = x-y \quad (3) \quad \Rightarrow z = -y$

$(1),(2) \Rightarrow x+y+x-y=0$

$2x=0$

$x=0 \quad (4)$

$(1),(4) \Rightarrow 0-y-z=0$

$y=-z \quad (5)$

$\Rightarrow \vec{d}\,(0, y, -y)$

$\boxed{c.}$ (a,b,c) and (a,b,d) have the same orientation

$\Rightarrow$ $a \bullet (b \times c)$ and $a \bullet (b \times d)$ should have the same sign.

$a \bullet (b\times c) = (8,4,1) \bullet (1,-1,0) = 8-4=4 > 0$

$$b \times c = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{vmatrix} = i \cdot \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} - j \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} + k \begin{vmatrix} 2 & 2 \\ 1 & 1 \end{vmatrix} =$$

$= i - j = (1,-1,0)$

$a \cdot (b \times d) = (8,4,1) \cdot (-3y, 2y, 2y) = -24y + 8y + 2y = -14y$

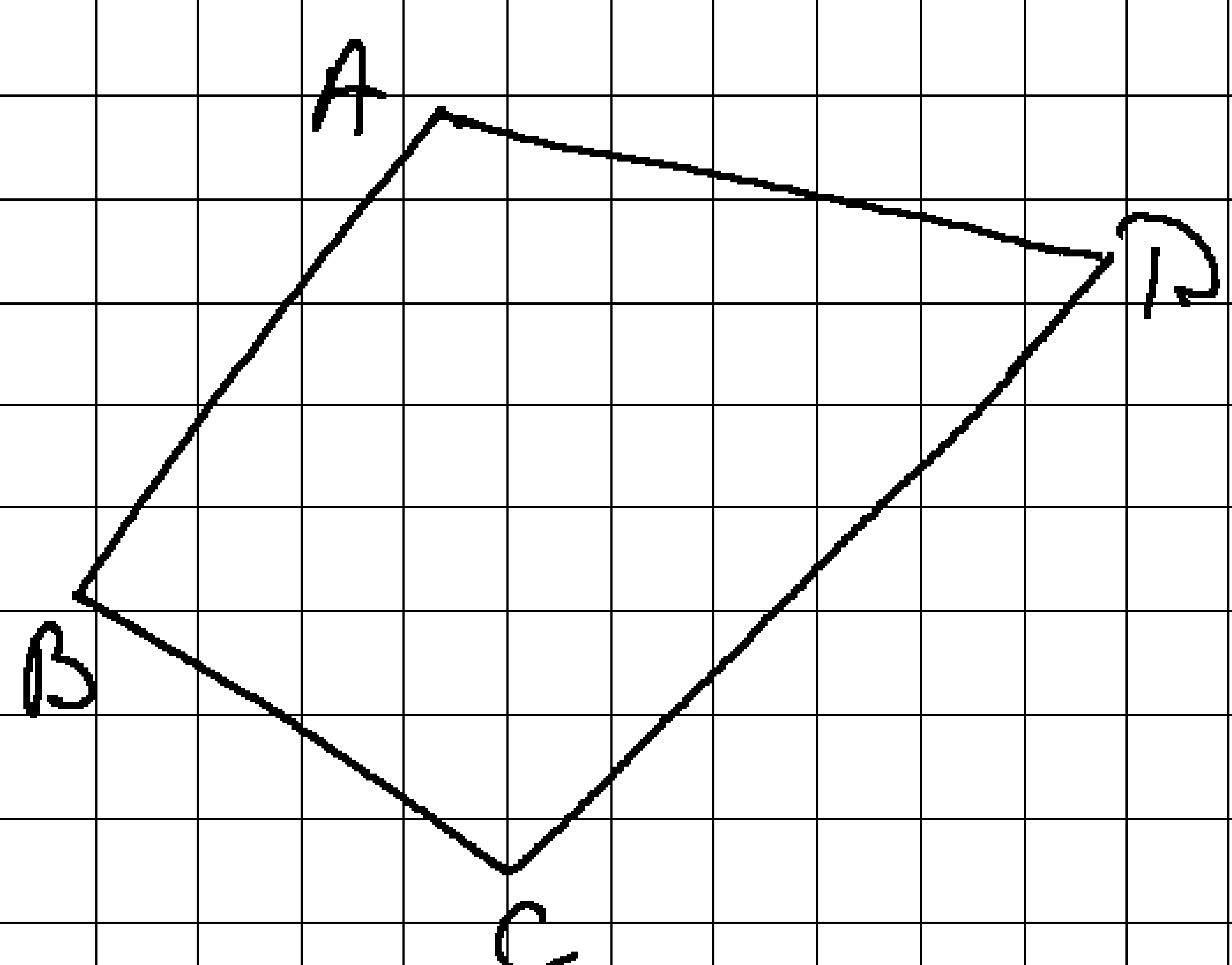
$$b \times d = \begin{vmatrix} i & j & k \\ 2 & 2 & 1 \\ 0 & y & -y \end{vmatrix} = i \begin{vmatrix} 2 & 1 \\ y & -y \end{vmatrix} - j \begin{vmatrix} 2 & 1 \\ 0 & -y \end{vmatrix} + k \begin{vmatrix} 2 & 2 \\ 0 & y \end{vmatrix} =$$

$$= -3y \cdot i + 2y \cdot j + 2y \cdot k = (-3y, 2y, 2y)$$

$\Rightarrow -14y > 0 \quad /:(-14) \Rightarrow y < 0$

$\Rightarrow \vec{d} = (0, y, -y), \quad y < 0$

4.18. For a tetrahedron with vertices $A(2,-1,1)$, $B(5,5,4)$, $C(3,2,-1)$ and $D(4,1,3)$ given with respect to an orthonormal system. Determine the common perpendicular of the sides AB and CD



$\vec{AB} = B - A = (5,5,4) - (2,-1,1)$
$= (3,6,3)$

$\vec{CD} = D - C = (4,1,3) - (3,2,-1) =$
$= (1,-1,4)$

$$\vec{AB} \times \vec{CD} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 6 & 3 \\ 1 & -1 & 4 \end{vmatrix} = \vec{i} \begin{vmatrix} 6 & 3 \\ -1 & 4 \end{vmatrix} - \vec{j} \begin{vmatrix} 3 & 3 \\ 1 & 4 \end{vmatrix} + \vec{k} \begin{vmatrix} 3 & 6 \\ 1 & -1 \end{vmatrix} =$$

$$= 27\vec{i} - 9\vec{j} - 9\vec{k} \Rightarrow (27, -9, -9)$$

$\Rightarrow$ the common perpendicular is the line parallel to $(27, -9, -9)$

4.12. Let (i, j, k) be a right oriented orthonormal basis. Consider the vectors $a = i + j$, $b = i - k$ and $c = k$. Determine if:

a) (a, b, c) is a basis of V^3

$a(1,1,0)$, $b(1,0,-1)$, $c(0,0,1)$

$$\vec{X} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = c_1 \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & x_1 \\ 1 & 0 & 0 & x_2 \\ 0 & -1 & 1 & x_3 \end{array}\right) \xrightarrow{l_2 \leftarrow l_2 - l_1} \left(\begin{array}{ccc|c} 1 & 1 & 0 & x_1 \\ 0 & -1 & 0 & x_2 - x_1 \\ 0 & -1 & 1 & x_3 \end{array}\right)$$

$$\xrightarrow{l_2 \leftarrow l_2 \cdot (-1)} \left(\begin{array}{ccc|c} 1 & 1 & 0 & x_1 \\ 0 & 1 & 0 & x_1 - x_2 \\ 0 & -1 & 1 & x_3 \end{array}\right) \xrightarrow[l_1 \leftarrow l_1 - l_2]{l_3 \leftarrow l_3 + l_2}$$

$$\longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & x_2 \\ 0 & 1 & 0 & x_1 - x_2 \\ 0 & 0 & 1 & x_1 + x_3 - x_2 \end{array}\right)$$

$$\begin{cases} c_1 = x_2 \\ c_2 = x_1 - x_2 \\ c_3 = x_1 + x_3 - x_2 \end{cases}$$

when $x_1 = x_2 = x_3 = 0 \Rightarrow c_1 = c_2 = c_3 = 0$

$\Rightarrow$ the vectors are lin. indep and form a basis

b) if it is a basis, decide if it is left or right oriented

$$a \bullet (b \times c) = (1,1,0) \bullet (0,-1,0) = 1 \cdot 0 + 1 \cdot (-1) = -1$$

$$b \times c = \begin{vmatrix} i & j & k \\ 1 & 0 & -1 \\ 0 & 0 & 1 \end{vmatrix} = i \cdot \begin{vmatrix} 0 & -1 \\ 0 & 1 \end{vmatrix} - j \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} + k \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} =$$

$$= -j \Rightarrow (0,-1,0)$$

$a \cdot (b \times c) = -1 < 0 \Rightarrow$ left oriented.

Midterm 1 2022 model:

Part 1: Consider the two coordinate systems $K=(O,i,j)$ and $K'(O',i',j')$ where

$$[O']_K=\begin{bmatrix}4\\0\end{bmatrix} \quad [i']_K=\begin{bmatrix}-1\\1\end{bmatrix} \quad [j']_K=\begin{bmatrix}0\\1\end{bmatrix}$$

and where K is a right oriented orthonormal system

Consider the triangle ABC where

$$[A]_K=\begin{bmatrix}2\\4\end{bmatrix} \quad [B]_K=\begin{bmatrix}4\\1\end{bmatrix} \quad [C]_K=\begin{bmatrix}4\\2\end{bmatrix}$$

Q1: Are the vectors $\vec{BC}$ and i orthogonal to eachother?

$$\vec{BC}=\vec{C}-\vec{B}=(4,2)-(4,1)=(0,1)$$

$\vec{BC}$ & i - orthogonal if their dot product is zero

$$\vec{BC}\bullet i=(0,1)\bullet(1,0)=0\cdot1+1\cdot0=0$$

$\Rightarrow$ they're orthogonal to eachother

P2: Determine the distance from B to the line AC

Distance from a point to a line in 2D

$$d = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}$$

where x_1, y_1 are the coordinates of the point

A, B, C are the coeficients of the equation of the line : $Ax + By + C = 0$

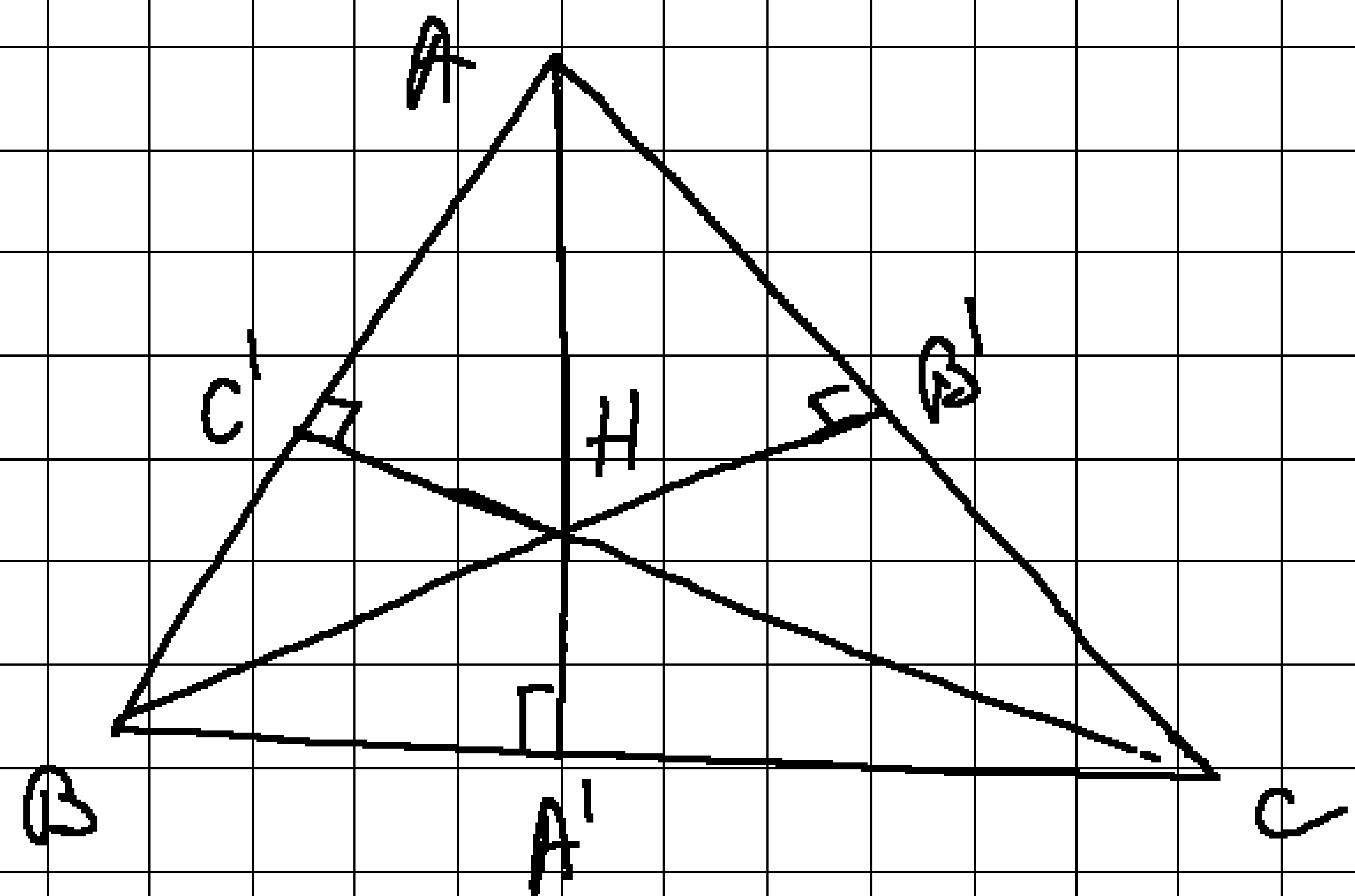
$$m_{AC} = \frac{y_C - y_A}{x_C - x_A} = \frac{2-4}{4-2} = -1$$

$$y - y_A = m(x - x_A) \Leftrightarrow y - 4 = -(x-2)$$

$$\Leftrightarrow y - 4 = -x + 2 \Leftrightarrow x + y - 6 = 0$$

$$d = \frac{|1 \cdot 4 + 1 \cdot 1 - 6|}{\sqrt{2}} = \frac{|4+1-6|}{\sqrt{2}} = \frac{1}{\sqrt{2}}$$

P3: With respect to K, determine the coordinates of the orthocenter of the triangle ABC



$$m_{AB} = \frac{y_B - y_A}{x_B - x_A} = \frac{1-4}{4-2} = -\frac{3}{2}$$

$$m_{BC} = \frac{y_C - y_B}{x_C - x_B} = \frac{2-1}{4-4} = \frac{1}{0} \Rightarrow \text{is undefined}$$

$$m_{AC} = \frac{y_C - y_A}{x_C - x_A} = \frac{2-4}{4-2} = -\frac{2}{2} = -1$$

$$m_{AA'} = -\frac{1}{m_{BC}}, \quad m_{BC} - \text{undefined} \Rightarrow m_{AA'} = 0$$

$$m_{BB'} = -\frac{1}{m_{AC}} = 1$$

$$m_{CC'} = -\frac{1}{m_{AB}} = \frac{2}{3}$$

Using $y - y_1 = m(x - x_1)$

AA': $y - 4 = 0 \Rightarrow y = 4$

BB': $y - 1 = 1(x - 4) \Rightarrow y - x + 3 = 0$

CC': $y - 2 = \frac{2}{3}(x - 4) \Leftrightarrow {}^{3/}\, y - 2 - \frac{2}{3}x + \frac{8}{3} = 0$

$\Leftrightarrow 3y - 2x + 2 = 0$

$$\begin{cases} y=4 \\ y-x+3=0 \Rightarrow 4-x+3=0 \Rightarrow -x=-7 \Rightarrow x=7 \\ 3y-2x+2=0 \end{cases}$$

$\Rightarrow H(7,4)$

P4: Determine an equation of the line AC with respect to the coordinate system K'.

$$M_{K,K'} = [id]_{K',K} = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}$$

$$M_{K',K} = M_{K,K'}^{-1}$$

$$\left(\begin{array}{cc|cc} -1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right) \xrightarrow{l_2 \leftarrow l_2 + l_1} \left(\begin{array}{cc|cc} -1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{array}\right) \xrightarrow{l_1 \leftarrow l_1 \cdot (-1)}$$

$$\longrightarrow \left(\begin{array}{cc|cc} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 1 \end{array}\right) \Rightarrow M_{K',K} = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}$$

$$[A]_{K'} = M_{K',K}\left([A]_K - [O']_K\right) = M_{K',K} \cdot \begin{bmatrix} 2-4 \\ 4-0 \end{bmatrix} =$$

$$= \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} -2 \\ 4 \end{pmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

$$[B]_{K'} = M_{K',K}\left([B]_K - [O']_K\right) = M_{K',K} \cdot \begin{bmatrix} 4-4 \\ 1-0 \end{bmatrix} =$$

$$= \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$[C]_{K'} = M_{K,K'} \cdot \left([C]_K - [O']_K \right) = M_{K,K'} \cdot \begin{pmatrix} 1-1 \\ 2-0 \end{pmatrix} =$$

$$= \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

$$m_{AC} = \frac{y_C - y_A}{x_C - x_A} = \frac{2-2}{0-2} = \frac{0}{-2} = 0$$

$$y - y_A = m(x - x_A) \Leftrightarrow y - 2 = 0 \Rightarrow y = 2$$

Part 2: Consider the right oriented orthonormal coordinate systems $K=(O,\vec{i},\vec{j},\vec{k})$ and the tetrahedron $ABCD$ where:

$$[A]_K=\begin{bmatrix}1\\2\\0\end{bmatrix}\quad [B]_K=\begin{bmatrix}1\\2\\1\end{bmatrix}\quad [C]_K=\begin{bmatrix}0\\3\\2\end{bmatrix}\quad [D]_K=\begin{bmatrix}1\\1\\1\end{bmatrix}$$

Moreover, denote by K' the coordinate system $(D,\vec{BA},\vec{DA},\vec{DC})$

P_5: Determine the distance from the point A to the line passing through B and D.

$$\boxed{d=\frac{|\vec{AB}\times\vec{AD}|}{|\vec{BD}|}}$$

Distance between two points:

2D: $d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$

3D: $d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$

$$\vec{AB}=B-A=\begin{bmatrix}1\\2\\1\end{bmatrix}-\begin{bmatrix}1\\2\\0\end{bmatrix}=\begin{bmatrix}0\\0\\1\end{bmatrix}$$

$$\vec{AD}=D-A=\begin{bmatrix}1\\1\\1\end{bmatrix}-\begin{bmatrix}1\\2\\0\end{bmatrix}=\begin{bmatrix}0\\-1\\1\end{bmatrix}$$

$$\vec{AB}\times\vec{AD}=\begin{pmatrix}\vec{i}&\vec{j}&\vec{k}\\0&0&1\\0&-1&1\end{pmatrix}=\vec{i}\cdot\begin{vmatrix}0&1\\-1&1\end{vmatrix}-\vec{j}\begin{vmatrix}0&1\\0&1\end{vmatrix}+\vec{k}\begin{vmatrix}0&0\\0&-1\end{vmatrix}$$

$$=\vec{i}\Rightarrow(1,0,0)$$

$$\vec{BD} = \vec{D} - \vec{B} = \begin{bmatrix}1\\1\\1\end{bmatrix} - \begin{bmatrix}1\\2\\1\end{bmatrix} = \begin{bmatrix}0\\-1\\0\end{bmatrix}$$

$$|\vec{BD}| = \sqrt{0^2+(-1)^2+0^2} = \sqrt{1} = 1$$

$$|\vec{AB}\times\vec{BD}| = \sqrt{1^2+0^2+0^2} = 1$$

$$d = \frac{1}{1} = 1$$

P_6: With respect to K, determine the orthogonal projection of the point A on the plane containing the triangle BCD.

Projection of a point A onto a plane defined by point B

$$P' = A - \big((A-B)\cdot n\big)n$$

where n – normal vector to the plane

$n = \vec{v}_1 \times \vec{v}_2$, v_1, v_2 – not parallel and originate from the same point

$$\vec{BC} = \vec{C} - \vec{B} = \begin{bmatrix}0\\3\\2\end{bmatrix} - \begin{bmatrix}1\\2\\1\end{bmatrix} = \begin{bmatrix}-1\\1\\1\end{bmatrix}$$

$$\vec{BD} = \vec{D} - \vec{B} = \begin{bmatrix}1\\1\\1\end{bmatrix} - \begin{bmatrix}1\\2\\1\end{bmatrix} = \begin{bmatrix}0\\-1\\0\end{bmatrix}$$

$$n = BC \times BD = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 1 & 1 \\ 0 & -1 & 0 \end{vmatrix} =$$

$$= \vec{i}\begin{vmatrix} 1 & 1 \\ -1 & 0 \end{vmatrix} - \vec{j}\begin{vmatrix} -1 & 1 \\ 0 & 0 \end{vmatrix} + \vec{k}\begin{vmatrix} -1 & 1 \\ 0 & -1 \end{vmatrix} = -\vec{i} + k \Rightarrow (-1, 0, 1)$$

$$P' = A - [(A - B) \bullet n]\, n =$$

$$= \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} - \left[\begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} \bullet \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right] n = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} - (-1)\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} =$$

$$= \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \Rightarrow P'(0, 2, 1)$$

P_7: With respect to k, calculate $(\vec{AB} \times \vec{AC}) \times k$

$$\vec{AB} = \vec{B} - \vec{A} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\vec{AC} = \vec{C} - \vec{A} = \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$$

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 0 & 1 \\ -1 & 1 & 2 \end{vmatrix} = \vec{i}\begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix} - \vec{j}\begin{vmatrix} 0 & 1 \\ -1 & 2 \end{vmatrix} + \vec{k}\begin{vmatrix} 0 & 0 \\ -1 & 1 \end{vmatrix}$$

$$= -\vec{i} - \vec{j} \Rightarrow (-1, -1, 0)$$

$$(\vec{AB} \times \vec{AC}) \times k = (-1, -1, 0) \times (0, 0, 1)$$

$$(-1,-1,0) \times (0,0,1) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & -1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \vec{i} \begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix} - \vec{j} \begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix} +$$

$$+ \vec{k} \begin{vmatrix} -1 & -1 \\ 0 & 0 \end{vmatrix} = -\vec{i} + \vec{j} = (-1,1,0)$$

P8: is K' left or right oriented?

$$\vec{BA} = \vec{A} - \vec{B} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

$$\vec{DA} = \vec{A} - \vec{D} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

$$\vec{DC} = \vec{C} - \vec{D} = \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$

$$\vec{BA} \bullet (\vec{DA} \times \vec{DC})$$

$$\vec{DA} \times \vec{DC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 1 & -1 \\ -1 & 2 & 1 \end{vmatrix} = \vec{i} \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} - \vec{j} \begin{vmatrix} 0 & -1 \\ -1 & 1 \end{vmatrix} + \vec{k} \begin{vmatrix} 0 & 1 \\ -1 & 2 \end{vmatrix} =$$

$$= 3\vec{i} + \vec{j} + \vec{k} = (3,1,1)$$

$$\vec{BA} \bullet (\vec{DA} \times \vec{DC}) = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} \bullet \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} = -1 < 0 \Rightarrow \text{left oriented}$$

Remark: If the box product is zero, the vectors are coplanar

P9. With respect to K', determine an equation of the plane which contains the triangle ABC

$$M_{K,K'} = id_{K',K} = \begin{pmatrix} \overrightarrow{DA} \\ \overrightarrow{DB} \\ \overrightarrow{DC} \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & -1 \\ -1 & 2 & 1 \end{pmatrix}$$

$$M_{K',K} = M_{K,K'}^{-1}$$

$$\left(\begin{array}{ccc|ccc} 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ -1 & 2 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{l_3 \leftarrow l_3 - 2l_2}$$

$$\left(\begin{array}{ccc|ccc} 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ -1 & 0 & 3 & 0 & -2 & 1 \end{array}\right) \xrightarrow{l_1 \leftarrow l_1 - l_3} \left(\begin{array}{ccc|ccc} 1 & 0 & -4 & 1 & 2 & -1 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ -1 & 0 & 3 & 0 & -2 & 1 \end{array}\right) \xrightarrow{l_3 \leftarrow l_3 + l_1}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -4 & 1 & 2 & -1 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \end{array}\right) \xrightarrow{l_3 \leftarrow l_3 \cdot (-1)} \left(\begin{array}{ccc|ccc} 1 & 0 & -4 & 1 & 2 & -1 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{array}\right) \xrightarrow[l_2 \leftarrow l_2 + l_3]{l_1 \leftarrow l_1 + 4l_3}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 5 & 2 & -1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{array}\right) \Rightarrow M_{K',K} = \begin{pmatrix} 5 & 2 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$[A]_{K'} = M_{K',K} \cdot \left([A]_K - [D]_K\right) = M_{K',K} \cdot \left(\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right) =$$

$$= M_{K',K} \cdot \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 5 & 2 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$$

$$[B]_{K'} = M_{K',K}\left([B]_K - [D]_K\right) = M_{K',K}\left(\begin{bmatrix}1\\2\\1\end{bmatrix} - \begin{bmatrix}1\\1\\1\end{bmatrix}\right) =$$

$$= \begin{pmatrix}5 & 2 & -1\\ 1 & 1 & 0\\ 1 & 0 & 0\end{pmatrix} \cdot \begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}2\\1\\0\end{pmatrix}$$

$$[C]_{K'} = M_{K',K} \cdot \left([C]_K - [D]_K\right) = M_{K',K} \cdot \left(\begin{bmatrix}0\\3\\2\end{bmatrix} - \begin{bmatrix}1\\1\\1\end{bmatrix}\right) =$$

$$= \begin{pmatrix}5 & 2 & -1\\ 1 & 1 & 0\\ 1 & 0 & 0\end{pmatrix} \begin{pmatrix}-1\\2\\1\end{pmatrix} = \begin{pmatrix}-2\\1\\-1\end{pmatrix}$$

$$[\vec{AB}]_{K'} = \vec{B} - \vec{A} = \begin{bmatrix}2\\1\\0\end{bmatrix} - \begin{bmatrix}3\\1\\0\end{bmatrix} = \begin{bmatrix}-1\\0\\0\end{bmatrix}$$

$$[\vec{AC}]_{K'} = \vec{C} - \vec{A} = \begin{bmatrix}-2\\1\\-1\end{bmatrix} - \begin{bmatrix}3\\1\\0\end{bmatrix} = \begin{bmatrix}-5\\0\\-1\end{bmatrix}$$

$$\vec{AB} \times \vec{AC} = \begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ -1 & 0 & 0\\ -5 & 0 & -1\end{vmatrix} = \vec{i}\begin{vmatrix}0 & 0\\ 0 & -1\end{vmatrix} - \vec{j}\begin{vmatrix}-1 & 0\\ -5 & -1\end{vmatrix} + \vec{k}\begin{vmatrix}-1 & 0\\ -5 & 0\end{vmatrix}$$

$= -\vec{j} \Rightarrow (0,-1,0) = n$ (the normal vector to the plane containing ABC)

Equation of a plane in 3D:

$$n \cdot (r - r_0) = 0$$, where $r = (x, y, z)$

r_0 - a point in the plane

$$n \bullet \left(r - [A]_{K'}\right) = 0 \quad , \quad r = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} \bullet \left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} - \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} \right) = 0$$

$$\begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} \bullet \begin{bmatrix} x-3 \\ y-1 \\ z \end{bmatrix} = 0 \Leftrightarrow -y+1=0 \Rightarrow y=1$$

Midterm model:

Part 1: Consider two coordinate systems $K=(O,i,j)$ and $K'=(O',i',j')$ where:

$$[O']_K=\begin{bmatrix}0\\1\end{bmatrix},\quad [i']_K=\begin{bmatrix}2\\1\end{bmatrix},\quad [j']_K=\begin{bmatrix}-1\\2\end{bmatrix}$$

and where K is a right oriented orthonormal system. Consider the triangle ABC where

$$[A]_K=\begin{bmatrix}1\\3\end{bmatrix}\quad [B]_K=\begin{bmatrix}-3\\1\end{bmatrix}\quad [C]_K=\begin{bmatrix}-1\\1\end{bmatrix}$$

P_1: Are the vectors $\vec{BA}$ and $\vec{CA}$ orthogonal to eachother

$$\vec{BA}=\vec{A}-\vec{B}=\begin{bmatrix}1\\3\end{bmatrix}-\begin{bmatrix}-3\\1\end{bmatrix}=\begin{bmatrix}4\\2\end{bmatrix}$$

$$\vec{CA}=\vec{A}-\vec{C}=\begin{bmatrix}1\\3\end{bmatrix}-\begin{bmatrix}-1\\1\end{bmatrix}=\begin{bmatrix}2\\2\end{bmatrix}$$

$$\vec{BA}\bullet\vec{CA}=\begin{bmatrix}4\\2\end{bmatrix}\bullet\begin{bmatrix}2\\2\end{bmatrix}=8+4=12\neq 0 \Rightarrow$$ the vectors are not orthogonal

P_2: With respect to K, determine a Cartesian equation for a line passing through C and having an angle of 45^0 with the line AB

$$\vec{AB} = \vec{B} - \vec{A} = \begin{bmatrix} -3 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -4 \\ -2 \end{bmatrix}$$

$$m_{AB} = \frac{y_B - y_A}{x_B - x_A} = \frac{1-3}{-3-1} = \frac{-2}{-4} = \frac{1}{2}$$

Let l: line passing through C

$$\tan\theta = \left| \frac{m_2 - m_1}{1 + m_1 \cdot m_2} \right|$$ – angle between two lines

m_1, m_2 – slopes

$$\tan 45 = \left| \frac{m_l - \frac{1}{2}}{1 + \frac{m_l}{2}} \right| \Leftrightarrow 1 = \left| \frac{m_l - \frac{1}{2}}{1 + \frac{m_l}{2}} \right| \Leftrightarrow$$

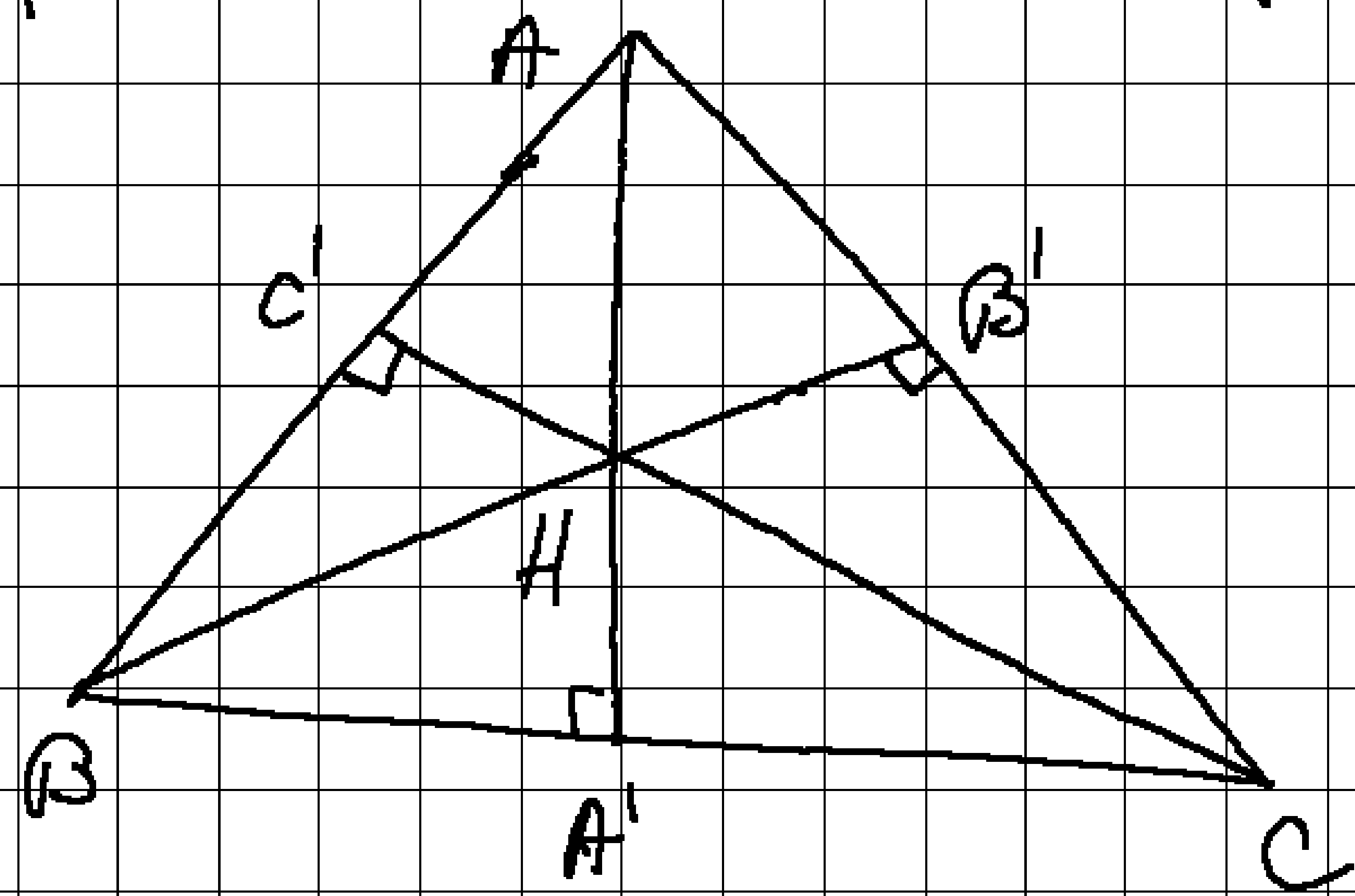
$$\Leftrightarrow 1 = \left| \frac{\frac{2m_l - 1}{2}}{\frac{2 + m_l}{2}} \right| \Leftrightarrow 1 = \left| \frac{2m_l - 1}{\cancel{2}} \cdot \frac{\cancel{2}}{2 + m_l} \right| \Leftrightarrow$$

$$\Leftrightarrow 1 = \frac{2m_l - 1}{2 + m_l} \Leftrightarrow 2 + m_l = 2m_l - 1 \Leftrightarrow m_l = 1$$

$$y - y_1 = m(x - x_1) \Leftrightarrow y - 1 = 1 \cdot (x + 1) \Leftrightarrow y - 1 = x + 1$$

$$\Leftrightarrow y = x + 2$$

P3: With respect to K, determine the coordinates of the circumcenter of the triangle ABC.



$$C' = \frac{A+B}{2} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \qquad B' = \frac{A+C}{2} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \qquad A' = \frac{B+C}{2} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$m_{AB} = \frac{y_B - y_A}{x_B - x_A} = \frac{1-3}{-3-1} = \frac{-2}{-4} = \frac{1}{2}$$

$$m_{AC} = \frac{y_C - y_A}{x_C - x_A} = \frac{-2}{-2} = 1$$

$$m_{BC} = \frac{y_C - y_B}{x_C - x_B} = \frac{0}{-4} = 0$$

$$\begin{cases} AB: \ y - 3 = \frac{1}{2}(x-1) = y - 3 = \frac{x-1}{2} \Leftrightarrow y = \frac{x-7}{2} \\ AC: \ y - 3 = 1(x+1) \Leftrightarrow y - 3 = x+1 \Leftrightarrow y = x+4 \end{cases}$$

$$x + 4 = \frac{x-7}{2} \Leftrightarrow 2x + 8 = x - 7 \Leftrightarrow x = -15$$

$$y = -15 + 4 = -11$$

$$\Rightarrow H(-15, -11)$$

P4: Determine an equation of the line AC with respect to the coordinate system K'.

$$M_{K,K'} = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}$$

$$M_{K',K} = M_{K,K'}^{-1}$$

$$\left(\begin{array}{cc|cc} 2 & -1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array}\right) \xrightarrow{l_1 \leftarrow l_1 \cdot \frac{1}{2}} \left(\begin{array}{cc|cc} 1 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 1 & 2 & 0 & 1 \end{array}\right) \xrightarrow{l_2 \leftarrow l_2 - l_1}$$

$$\left(\begin{array}{cc|cc} 1 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{5}{2} & -\frac{1}{2} & 1 \end{array}\right) \xrightarrow{l_3 \leftarrow l_3 \cdot \frac{2}{5}} \left(\begin{array}{cc|cc} 1 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & -\frac{1}{5} & \frac{2}{5} \end{array}\right) \longrightarrow$$

$$\xrightarrow{l_1 \leftarrow l_1 + \frac{1}{2} l_2} \left(\begin{array}{cc|cc} 1 & 0 & \frac{2}{5} & \frac{1}{5} \\ 0 & 1 & -\frac{1}{5} & \frac{2}{5} \end{array}\right)$$

$$[A]_{K'} = M_{K',K} \cdot \left([A]_K - [O']_K \right) = M_{K',K} \left(\begin{bmatrix} 1 \\ 3 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) =$$

$$= \begin{pmatrix} \frac{2}{5} & \frac{1}{5} \\ -\frac{1}{5} & \frac{2}{5} \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} \frac{3}{5} \\ \frac{3}{5} \end{pmatrix}$$

$$[C]_{K'} = M_{K',K} \cdot \left([C]_K - [O']_K \right) = M_{K',K} \cdot \left(\begin{bmatrix} -1 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$$

$$= \begin{pmatrix} \frac{2}{5} & \frac{1}{5} \\ -\frac{1}{5} & \frac{2}{5} \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{pmatrix} -\frac{2}{5} \\ \frac{1}{5} \end{pmatrix}$$

$$m_{AC} = \frac{y_C - y_A}{x_C - x_A} = \frac{\frac{1}{5} - \frac{3}{5}}{-\frac{2}{5} - \frac{3}{5}} = \frac{-\frac{2}{5}}{-1} = \frac{2}{5}$$

$$y - \frac{3}{5} = \frac{2}{5}\left(x + \frac{2}{5}\right) \Leftrightarrow y - \frac{3}{5} = \frac{2x+4}{5} \Leftrightarrow$$

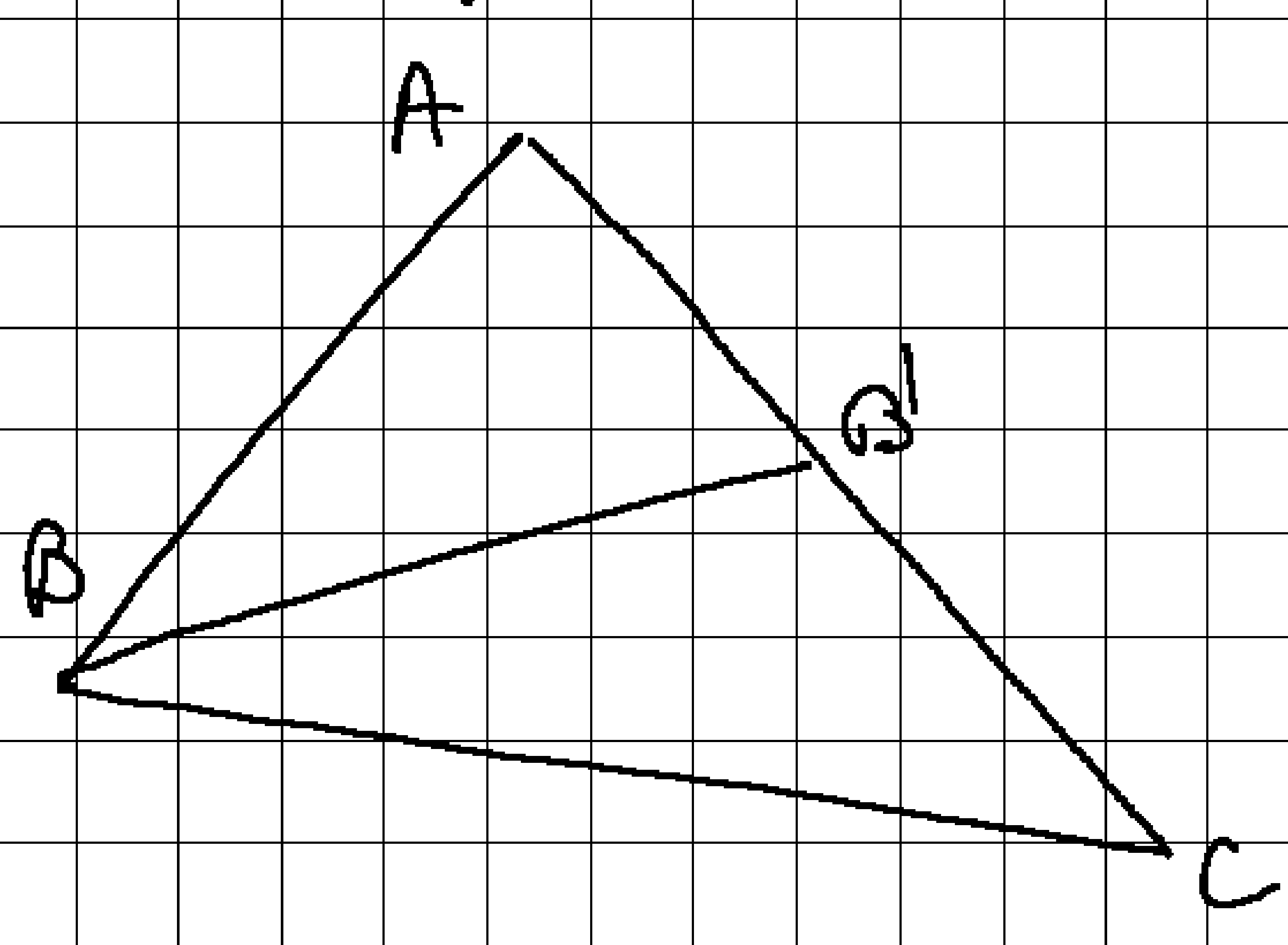
$$\Leftrightarrow y = \frac{2x+7}{5}$$

<u>Part 2:</u> Consider the right oriented orthonormal coordinate systems $K = (O, i, j, k)$ and the tetrahedron ABCD where:

$$[A]_K = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \quad [B]_K = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad [C]_K = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \quad [D]_K = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Moreover, denote by K' the coordinate system $(B, \vec{BA}, \vec{AC}, \vec{CD})$

P5: With respect to K, determine parametric equations for the interior angle bisector in the triangle ABC which passes through B.



$$\frac{|\vec{AB'}|}{|\vec{B'C}|} = \frac{|\vec{BA}|}{|\vec{BC}|}$$

$$\vec{BA} = \vec{A} - \vec{B} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

$$\vec{BC} = \vec{C} - \vec{B} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

$$|\vec{BA}| = \sqrt{(-1)^2} = 1$$

$$|\vec{BC}| = \sqrt{(-1)^2 + 1^2 + 1^2} = \sqrt{3}$$

$$\frac{|\vec{AB'}|}{|\vec{B'C}|} = \frac{|\vec{BA}|}{|\vec{BC}|} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

$$B' = \frac{\frac{\sqrt{3}}{3} \cdot A + \left(1 - \frac{\sqrt{3}}{3}\right) C}{1 + \frac{\sqrt{3}}{3}} = \frac{\begin{bmatrix} \frac{\sqrt{3}}{3} \\ \frac{\sqrt{3}}{3} \\ -\frac{\sqrt{3}}{3} \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ \frac{2\sqrt{3}}{3} \\ \frac{\sqrt{3}}{3} \end{bmatrix}}{1 + \frac{\sqrt{3}}{3}} =$$

$$= \frac{\left(\frac{\sqrt{3}}{3}, -\frac{\sqrt{3}}{3} + 2, -\frac{2\sqrt{3}}{3} + 1 \right)}{1 + \frac{\sqrt{3}}{3}} =$$

$$= \left(\frac{\frac{\sqrt{3}}{3}}{1 + \frac{\sqrt{3}}{3}}, \frac{-\frac{\sqrt{3}}{3} + 2}{1 + \frac{\sqrt{3}}{3}}, \frac{-\frac{2\sqrt{3}}{3} + 1}{1 + \frac{\sqrt{3}}{3}} \right)$$

$$m_{BB'} = \frac{y_{B'} - y_B}{x_{B'} - x_B} = \frac{\frac{-\frac{\sqrt{3}}{3} + 2}{1 + \frac{\sqrt{3}}{3}} - \overset{1+\frac{\sqrt{3}}{3}/}{1}}{\frac{\frac{\sqrt{3}}{3}}{1 + \frac{\sqrt{3}}{3}} - \overset{1+\frac{\sqrt{3}}{3}/}{1}} = \frac{\frac{-\frac{2\sqrt{3}}{3} - 1}{1 + \frac{\sqrt{3}}{3}}}{-\frac{1}{1 + \frac{\sqrt{3}}{3}}} =$$

$$= \frac{2\sqrt{3}}{3} + \overset{3/}{1} = \frac{2\sqrt{3} + 3}{3}$$

$$y - y_1 = m(x - x_1) \Leftrightarrow y - 1 = \frac{2\sqrt{3}+3}{3}(x-1) \Leftrightarrow$$

$$\Leftrightarrow y - 1 = \frac{2\sqrt{3}+3}{3}x + \frac{2\sqrt{3}+3}{3} \Leftrightarrow y = \frac{2\sqrt{3}+3}{3}x + \frac{2\sqrt{3}+6}{3}$$

$$\begin{cases} x = \lambda \\ y = \frac{2\sqrt{3}+3}{3}\lambda + \frac{2\sqrt{3}+6}{3} \end{cases}$$

P_6: With respect to K, determine the orthogonal reflection of the point A in the plane containing the triangle BCD

$P'' = 2P' - A$ where P' - projection of A onto the plane
A - the point we are projecting from

$$\vec{BC} = \vec{C} - \vec{B} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

$$\vec{BD} = \vec{D} - \vec{B} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$$

$$n = BC \times BD = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 1 & 1 \\ 0 & -1 & 0 \end{vmatrix} = \vec{i}\begin{vmatrix} 1 & 1 \\ -1 & 0 \end{vmatrix} - \vec{j}\begin{vmatrix} -1 & 1 \\ 0 & 0 \end{vmatrix} + \vec{k}\begin{vmatrix} -1 & 1 \\ 0 & -1 \end{vmatrix}$$

$$= \vec{i} + \vec{k} \Rightarrow n = (1, 0, 1)$$

$$P' = A - ((A-B)\bullet n)\, n$$

$$P' = \begin{bmatrix}1\\1\\-1\end{bmatrix} - \left(\left(\begin{bmatrix}1\\1\\-1\end{bmatrix} - \begin{bmatrix}1\\1\\0\end{bmatrix}\right)\bullet \begin{bmatrix}1\\0\\1\end{bmatrix}\right)\begin{bmatrix}1\\0\\1\end{bmatrix} =$$

$$= \begin{bmatrix}1\\1\\-1\end{bmatrix} - \left(\begin{bmatrix}0\\0\\-1\end{bmatrix}\bullet\begin{bmatrix}1\\0\\1\end{bmatrix}\right)\begin{bmatrix}1\\0\\1\end{bmatrix} =$$

$$= \begin{bmatrix}1\\1\\-1\end{bmatrix} + \begin{bmatrix}1\\0\\1\end{bmatrix} = \begin{bmatrix}2\\1\\0\end{bmatrix}$$

$$P'' = A - 2P' = \begin{bmatrix}1\\1\\-1\end{bmatrix} - 2\begin{bmatrix}2\\1\\0\end{bmatrix} = \begin{bmatrix}-3\\-2\\-1\end{bmatrix}$$

Q_7: With respect to K, calculate

$$(\vec{AB}\times\vec{AC})\times K + (AC\times K)\times\vec{AB}$$

$$\vec{AB} = \vec{B} - \vec{A} = \begin{bmatrix}1\\1\\0\end{bmatrix} - \begin{bmatrix}1\\1\\-1\end{bmatrix} = \begin{bmatrix}0\\0\\1\end{bmatrix}$$

$$\vec{AC} = \vec{C} - \vec{A} = \begin{bmatrix}0\\2\\1\end{bmatrix} - \begin{bmatrix}1\\1\\-1\end{bmatrix} = \begin{bmatrix}-1\\1\\2\end{bmatrix}$$

$$(\vec{AB}\times\vec{AC}) = \begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 0 & 0 & 1\\ -1 & 1 & 2\end{vmatrix} = \vec{i}\begin{vmatrix}0 & 1\\ 1 & 2\end{vmatrix} - \vec{j}\begin{vmatrix}0 & 1\\ -1 & 2\end{vmatrix} + \vec{k}\begin{vmatrix}0 & 0\\ -1 & 1\end{vmatrix}$$

$$= -\vec{i} - \vec{j} \Rightarrow (-1,-1,0)$$

$$(\vec{AB}\times\vec{AC})\times k = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & -1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \vec{i}\begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix} - \vec{j}\begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix}$$

$$+ \vec{k}\begin{vmatrix} -1 & -1 \\ 0 & 0 \end{vmatrix} = -\vec{i} + \vec{j} \Rightarrow (-1,0,1)$$

$$AC\times k = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 1 & 2 \\ 0 & 0 & 1 \end{vmatrix} = \vec{i}\begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} - \vec{j}\begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} + \vec{k}\begin{vmatrix} -1 & 1 \\ 0 & 0 \end{vmatrix}$$

$$= \vec{i} + \vec{j} \Rightarrow (1,1,0)$$

$$(AC\times k)\times AB = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \vec{i}\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} - \vec{j}\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + \vec{k}\begin{vmatrix} 1 & 1 \\ 0 & 0 \end{vmatrix}$$

$$= \vec{i} - \vec{j} \Rightarrow (1,-1,0)$$

$$(\vec{AB}\times\vec{AC})\times k + (AC\times k)\times\vec{AB} = (-1,-1,0) + (1,-1,0)$$

$$= (0,-2,0)$$

P8: is K' left or right oriented

$$\vec{BA} = \vec{A} - \vec{B} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

$$\vec{AC} = \vec{C} - \vec{A} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$$

$$\vec{CD} = \vec{D} - \vec{C} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}$$

$$\vec{BA} \bullet (\vec{AC} \times \vec{CD})$$

$$\vec{AC} \times \vec{CD} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 1 & 2 \\ 1 & -2 & -1 \end{vmatrix} = \vec{i} \begin{vmatrix} 1 & 2 \\ -2 & -1 \end{vmatrix} - \vec{j} \begin{vmatrix} -1 & 2 \\ 1 & -1 \end{vmatrix} + \vec{k} \begin{vmatrix} -1 & 1 \\ 1 & -2 \end{vmatrix}$$

$$= 3\vec{i} + \vec{j} + \vec{k} \Rightarrow \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$$

$$\vec{BA} \bullet (\vec{AC} \times \vec{CD}) = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} \bullet \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} = -1 < 0 \Rightarrow \text{left oriented}$$

P_9: With respect to K', determine an equation of the plane which contains triangle ACD

$$M_{K',K} = \begin{pmatrix} \vec{BA} \\ \vec{AC} \\ \vec{CD} \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1 \\ -1 & 1 & 2 \\ 1 & -2 & -1 \end{pmatrix}$$

$$M_{K,K'} = M_{K',K}^{-1}$$

$$\left(\begin{array}{ccc|ccc} 0 & 0 & -1 & 1 & 0 & 0 \\ -1 & 1 & 2 & 0 & 1 & 0 \\ 1 & -2 & -1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{l_3 \leftrightarrow l_1} \left(\begin{array}{ccc|ccc} 1 & -2 & -1 & 0 & 0 & 1 \\ -1 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \end{array}\right)$$

$$\xrightarrow[l_3 \leftarrow l_3 \cdot (-1)]{l_2 \leftarrow l_2 + l_1} \left(\begin{array}{ccc|ccc} 1 & -2 & -1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 & 0 & 0 \end{array}\right) \xrightarrow{l_2 \leftarrow l_2 - l_3} \left(\begin{array}{ccc|ccc} 1 & -2 & -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & 0 & 0 \end{array}\right)$$

$$\xrightarrow[l_1 \leftarrow l_1 + l_3]{l_2 \leftarrow l_2 \cdot (-1)} \left(\begin{array}{ccc|ccc} 1 & -2 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & 0 & 0 \end{array}\right) \xrightarrow{l_1 \leftarrow l_1 + 2l_2}$$

$$\longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 2 & 3 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & 0 & 0 \end{array}\right) \Rightarrow M_{K,K'} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ -1 & 0 & 0 \end{pmatrix}$$

$$[A]_{K'} = M_{K,K'}\left([A]_K - [B]_K\right) = M_{K,K'} \cdot \left(\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right) =$$

$$= \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} -3 \\ -1 \\ 0 \end{pmatrix}$$

$$[C]_{K'} = M_{K,K'} \cdot \left([C]_K - [B]_K\right) = M_{K,K'} \cdot \left(\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right) =$$

$$= \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix}$$

$$[D]_{K'} = M_{K,K'} \cdot \left([D]_K - [B]_K\right) = M_{K,K'} \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right)$$

$$= \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \\ 0 \end{pmatrix}$$

$$\vec{AC} = \vec{C} - \vec{A} = \begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} -3 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 7 \\ 2 \\ 1 \end{pmatrix}$$

$$\vec{AD} = \vec{D} - \vec{A} = \begin{pmatrix} -2 \\ -1 \\ 0 \end{pmatrix} - \begin{pmatrix} -3 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$n = \vec{AC} \times \vec{AD} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 7 & 2 & 1 \\ 1 & 0 & 0 \end{vmatrix} = \vec{i} \begin{vmatrix} 2 & 1 \\ 0 & 0 \end{vmatrix} - \vec{j} \begin{vmatrix} 7 & 1 \\ 1 & 0 \end{vmatrix} + \vec{k} \begin{vmatrix} 7 & 2 \\ 1 & 0 \end{vmatrix}$$

$$= \vec{j} - 2\vec{k} \Rightarrow n = (0, 1, -2)$$

$$n \cdot (r - r_0) = 0 \Leftrightarrow \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} x+3 \\ y+1 \\ z \end{pmatrix} = 0$$

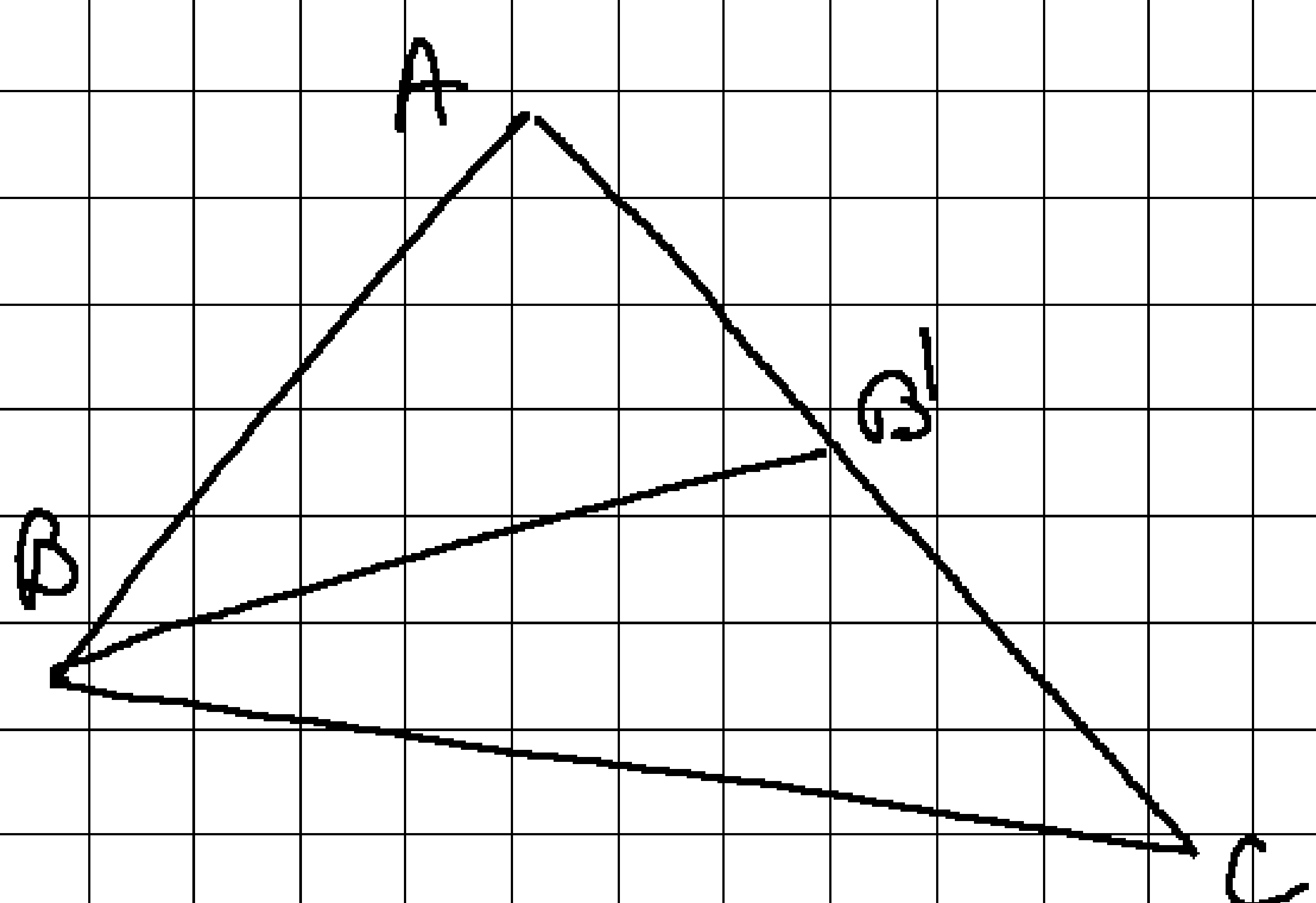
$$\Leftrightarrow y + 1 - 2z = 0 \Rightarrow y = 2z - 1$$

<u>Part 2:</u> Consider the right oriented orthonormal coordinate systems $K=(O,i,j,k)$ and the tetrahedron $ABCD$ where:

$$[A]_K=\begin{bmatrix}1\\1\\-1\end{bmatrix}\quad [B]_K=\begin{bmatrix}1\\1\\0\end{bmatrix}\quad [C]_K=\begin{bmatrix}0\\2\\1\end{bmatrix}\quad [D]_K=\begin{bmatrix}1\\0\\0\end{bmatrix}$$

Moreover, denote by K' the coordinate system $(B,\vec{BA},\vec{AC},\vec{CD})$

P5: With respect to K, determine parametric equations for the interior angle bisector in the triangle ABC which passes through B.



$$\frac{|\vec{AB'}|}{|\vec{B'C}|}=\frac{|\vec{BA}|}{|\vec{BC}|}$$

$$B'=(a,b,c)$$

$$\vec{AB'}=\vec{B'}-\vec{A}=\begin{bmatrix}a-1\\b-1\\c+1\end{bmatrix}\qquad \vec{B'C}=\vec{C}-\vec{B'}=\begin{bmatrix}-a\\2-b\\1-c\end{bmatrix}$$

$$\vec{BA}=\vec{A}-\vec{B}=\begin{bmatrix}0\\0\\-1\end{bmatrix}\qquad \vec{BC}=\vec{C}-\vec{B}=\begin{bmatrix}-1\\1\\1\end{bmatrix}$$

$$|\vec{AB'}|=\sqrt{(a-1)^2+(b-1)^2+(c+1)^2}\qquad |\vec{BA}|=\sqrt{(-1)^2}=1$$

$$|\vec{B'C}|=\sqrt{(-a)^2+(2-b)^2+(1-c)^2}\qquad |\vec{BC}|=\sqrt{3}$$

$$\frac{|\vec{AB}|}{|\vec{BC}|} = \frac{|\vec{BA}|}{|\vec{BC}|} \Leftrightarrow$$

$$\Leftrightarrow \frac{\sqrt{(a-1)^2+(b-1)^2+(c+1)^2}}{\sqrt{(-a)^2+(2-b)^2+(1-c)^2}} = \frac{1}{\sqrt{3}}$$

$$\Leftrightarrow \sqrt{3}\cdot\sqrt{(a-1)^2+(b-1)^2+(c+1)^2} = \sqrt{(-a)^2+(2-b)^2+(1-c)^2} \quad / ()^2$$

$$\Leftrightarrow 3\left[(a-1)^2+(b-1)^2+(c+1)^2\right] = (-a)^2+(2-b)^2+(1-c)^2$$