Theorem 1:

$\boxed{\Rightarrow}$ Suppose that $v_1, \dots, v_n$ are linearly dependent $\Rightarrow \exists\ k_1, \dots, k_n \in K$ <u>not</u> all zero s.t.

$$k_1 \cdot v_1 + \dots + k_n v_n = 0$$

Say $k_j \neq 0 \quad \Rightarrow k_1 \cdot v_1 + \dots + k_{j-1} v_{j-1} + \underline{k_j v_j} +$

$$+ k_{j+1} v_{j+1} + \dots + k_n \cdot v_n = 0$$

$\exists\ k_j^{-1} \cdot$

$\Uparrow$

$k_j \neq 0$

$$\Rightarrow \quad k_j v_j = -\sum_{\substack{i=1 \\ i \neq j}}^{n} k_i \cdot v_i \Rightarrow v_j = -\sum_{\substack{i=1 \\ i \neq j}}^{n} k_j^{-1} \cdot k_i \cdot v_i \Rightarrow$$

$$\Rightarrow v_j = \sum_{\substack{i=1 \\ i \neq j}}^{n} \alpha_i v_i \text{ , where } \alpha_i = k_j^{-1} \cdot k_i$$

$\boxed{\Leftarrow}$ Suppose that $\exists\ j \in \{1, \dots, n\}$ s.t.

$$v_j = \sum_{\substack{i=1 \\ i \neq j}}^{n} \alpha_i v_i \text{ for some } \alpha_i \in K, \forall\ i \in \{1, \dots, n\} \setminus \{j\}$$

$$\sum_{\substack{i=1 \\ i \neq j}}^{n} \alpha_i v_i - v_i = 0 \Rightarrow \alpha_1 v_1 + \dots + \alpha_{j-1} \cdot v_{j-1} + \underset{\neq 0}{(-1)} \cdot v_j +$$

$$+ \alpha_{j+1} \cdot v_{j+1} + \dots + \alpha_n v_n = 0$$

$\Rightarrow v_1, \dots, v_n$ are linearly dependant

$k_1 v_1 + k_2 \cdot v_2 = 0$ with $k_2 \neq 0 \Rightarrow v_2 = \underbrace{-k_2^{-1} k_1}_{=\alpha} \cdot v_1$

(v_1, v_2 linearly dependant)

Example II (Lecture)

Show that $e_1, \dots, e_n$ are linearly independent in the canonical vector space K^n over K:

Let $k_1, \dots, k_n \in K$ be such that $k_1 e_1 + \dots + k_n e_n = 0 \in K^n$

$\Rightarrow (k_1, 0, \dots, 0) + (0, k_2, 0, \dots, 0) + (0, \dots, 0, k_n) = (0, 0, \dots, 0) \in K^n$

$\Rightarrow (k_1, k_2, \dots, k_n) = (0, \dots, 0) \in K^n$

$\Rightarrow k_1 = k_2 = \dots = k_n = 0$

Hence $e_1, \dots, e_n$ are linearly independent

Theorem 2:

(ii) Let $v_1, \dots, v_n \in V$ that they are linearly dependant $\Leftrightarrow \exists\, k_1, \dots, k_n$ not all zero s.t.:

$k_1 v_1 + \dots + k_n v_n = 0$

$\Leftrightarrow \exists\, k_1, \dots, k_n$ not all zero s.t:

$k_1 \cdot (x_{11}, x_{12}, \dots, x_{1n}) + \dots + k_n (x_{n1}, x_{n2}, \dots, x_{nn}) = 0 \in K^n$

$\Leftrightarrow \exists\, k_1, \dots, k_n$ not all zero $\begin{cases} k_1 \cdot x_{11} + \dots + k_n x_{n1} = 0 \\ \dots \\ k_n \cdot x_{1n} + \dots + k_n \cdot x_{nn} \end{cases}$

$\Leftrightarrow$ (S) has a non-zero solution in the unknowns $k_1, \ldots, k_n$

$\Leftrightarrow$ the determinant of (S) is zero

Theorem 3:

$\boxed{\Rightarrow}$ Suppose that $\mathcal{B} = (v_1, \ldots, v_n)$ is a basis of V

$\Rightarrow \begin{cases} \mathcal{B} \text{ is linearly indep.} \\ \mathcal{B} \text{ is a system of generators, that is, } V = \langle \mathcal{B} \rangle \end{cases}$

$\exists\, k_1, \ldots, k_n \in K : v = k_1 v_1 + \ldots + k_n v_n$

Assume that we also have a writing:

$v = k_1' \cdot v_1 + \ldots + k_n' \cdot v_n$ for some $k_1', \ldots, k_n' \in K$

Substracting

$\Rightarrow \left.\begin{array}{l} 0 = (k_1 - k_1') \cdot v_1 + \ldots + (k_n - k_n') \cdot v_n \\ \text{But } v_1, \ldots, v_n \text{ are linearly independent} \end{array}\right\} \Rightarrow$

$\Rightarrow \begin{cases} k_1 - k_1' = 0 \\ \ldots \\ k_n - k_n' = 0 \end{cases} \Rightarrow \begin{cases} k_1 = k_1' \\ \ldots \\ k_n = k_n' \end{cases} \Rightarrow$ uniqueness

$\boxed{\Leftarrow}$ Suppose that $\forall v \in V$, $v = k_1 \cdot v_1 + \ldots + k_n v_n$ for some unique $k_1, \ldots, k_n \in K$

$\Rightarrow V = \langle B \rangle$

Let $k_1, \ldots, k_n \in K$ be such that $k_1 \cdot v_1 + \ldots + k_n v_n = 0$

But $0 \cdot v_1 + \ldots + 0 \cdot v_n = 0$

$\overset{\text{uniqueness of writing}}{\Rightarrow} \quad k_1 = 0, \ldots, k_n = 0$

So B is linearly independent

Example 4:

Let $g = a_0 + a_1 X + a_2 X^2 \in \mathbb{R}_2[X]$

We look for unique $k_1, k_2, k_3 \in \mathbb{R}$ such that

$$g = k_1 \cdot 1 + k_2 (X-1) + k_3 (X-1)^2$$

$$g = (k_1 - k_2 + k_3) \cdot 1 + (k_2 - 2k_3) X + k_3 \cdot X^2$$

$$\begin{cases} k_1 - k_2 + k_3 = a_0 & \Rightarrow k_1 = a_0 + a_1 + a_2 \\ k_2 - 2k_3 = a_1 & \Rightarrow k_2 = a_1 + 2a_2 \\ k_3 = a_2 & \end{cases}$$

unique!

Bases and linear maps:

Theorem 4:

Let $v \in V \Rightarrow \exists!\ k_1, \dots, k_n \in K : v = k_1 \cdot v_1 + \dots + k_n v_n \Rightarrow$

$\Rightarrow f(v) = k_1 \cdot f(v_1) + \dots + k_n \cdot f(v_n)$

Theorem 5:

(ii) $V = \langle X \rangle \Rightarrow f(V) = f(\langle X \rangle) \Rightarrow V' = \langle f(X) \rangle$