1. $k > 0$, we consider the diff eq:

$x' = -k(x-21)$ (the cooling process from model - Newton)

$x(t)$ = the temp of a cup of tea at the time t

a) flow = ?

b) experiment: a cup of tea with the initial temp of $49\,^{\circ}C$ has a temp of $37\,^{\circ}C$ in 10 minutes

Find the initial temp. of a cup of tea such that after 20 minutes the tea has $37\,^{\circ}C$

a) the flow is the solution of the IVP $\begin{cases} x' = -k(x-21) \\ x(0) = \eta, \ \eta > 0 \end{cases} \Rightarrow$

$\Rightarrow$ sol. $\varphi(t, \eta)$

$x' = -kx + 21k$

$x' + kx = 21k$ $\left(\begin{matrix}\text{non homog lin diff. eq} \\ \text{non homog const. coef. diff. eq}\end{matrix}\right)$

Step 1: Solve $x' + kx = 0$

$x' = -kx$ $\quad (= \underbrace{f(t)}_{-k} \cdot \underbrace{g(x)}_{x})$ - separable variable diff. eq

$\frac{dx}{dt} = -kx$

→ separate variables:

$$\frac{dx}{x} = -k\,dt$$

→ integrate with respect to each variable

$$\int \frac{dx}{x} = -k \int dt$$

$$\ln|x| = -kt + \underbrace{\ln|c|}_{\text{constant}}$$

$$\ln|x| = \ln e^{-kt} + \ln|c|$$

$$\ln|x| = \ln|c \cdot e^{-kt}|$$

$$x = c \cdot e^{-kt}$$

Step 2: $x' + kx = 21k$ (nonhomog)

$x_p = ?$

We will find x_p with the method of Lagrange (variation of constants method)

$$x_p = c(t) \cdot e^{-kt}$$

$$x'_p = c'(t) \cdot e^{-kt} + c(t) \cdot (-k) \cdot e^{-kt}$$

$$\underbrace{c'(t)\cdot e^{-kt} - \cancel{k\,c(t)\,e^{-kt}}}_{x_p'} + \underbrace{\cancel{k\,c(t)\,e^{-kt}}}_{x_p} = 21k$$

the terms with c(t) have to cancel out.

$$c'(t)\,e^{-kt} = 21k \Rightarrow c'(t) = 21k\,e^{+kt} \quad / \int dt$$

$$\Rightarrow c(t) = 21k \int e^{+kt}\,dt = 21k\cdot\frac{1}{k}\,e^{+kt}$$

$$\Rightarrow c(t) = 21e^{+kt} \Rightarrow x_p = \underbrace{21e^{+kt}}_{c(t)}\,e^{-kt}$$

$$\Rightarrow x_p = 21$$

Step 3: $x = x_h + x_p$

$$\boxed{x = C e^{-kt} + 21} \Rightarrow \text{sol of. the diff. eq}$$

$$x(0) = \eta$$

$$C\cdot e^{0} + 21 = \eta \Rightarrow C = \eta - 21$$

$\Rightarrow$ sol. of ivp: $\varphi(t,\eta) = (\eta - 21)\,e^{-kt} + 21$ (flow)

$$\varphi: \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$$

b) $X(0) = \eta$ = in. temp, $t_0 = 0$

$$\varphi(0,49) \xrightarrow{10'} \varphi(10,49) = 37 \quad (\eta = 49)$$

$$\varphi(0,49) = 49 \qquad \varphi(10,49) = (49-21)\,e^{-k\cdot 10} + 21 = 37$$

$$\tilde{\eta} = ? \qquad \varphi(0,\tilde{\eta}) = \tilde{\eta}$$

$$\xrightarrow{20'} \varphi(20,\tilde{\eta}) = 37$$

$$\varphi(20,\tilde{\eta}) = \ldots\ldots = 37$$

$$\varphi(10,49) = 28\cdot e^{-k\cdot 10} = 16 \Rightarrow e^{-k\cdot 10} = \frac{4}{7} \;/\ln(\,) \Rightarrow$$

$$\Rightarrow -k\cdot 10 = \ln\frac{4}{7} \Rightarrow k = -\frac{\ln\frac{4}{7}}{10}$$

$$\varphi(20,\tilde{\eta}) = \ldots = 37$$

$$(\tilde{\eta}-21)\,e^{-kt} + 21 = 37$$

$$e^{-kt} = \frac{16}{\tilde{\eta}-21} \Rightarrow e^{\frac{\ln\frac{4}{7}}{\cancel{10}}\cdot \cancel{20}^{2}} = \frac{16}{\tilde{\eta}-21} \Rightarrow$$

$$\Rightarrow \frac{16}{49} = \frac{16}{\tilde{\eta}-21} \Rightarrow \tilde{\eta} = 70$$

2. $0 < c < 1$; $x' = x(1-x) - cx$

a) Find the equilibria and study their stability using the linearization method.

b) Represent the phase portrait

c) $\xRightarrow[b)]{a)}$ if $x(t) > 0$ is the density of fish in a lake and $c \in (0,1)$ is the rate of fishing, try to predict the fate of the fish from the lake.

<u>lin. method:</u>

$f \in C^1(\mathbb{R})$, η^* s.t. $f(\eta^*) = 0$ → η^* = eq. point

a) if $f'(\eta^*) > 0 \Rightarrow \eta^*$ = repeller (unstable)

b) if $f'(\eta^*) < 0 \Rightarrow \eta^*$ = attractor (as. stable)

An equilibrium is a solution that is constant in time

Denote: equilibrium of diff. eq: x^* = solution const. in time

$\Rightarrow (x^*)' = 0$

$$0 = x(1-c) - cx$$

in order to find equilibrium point of diff. eq.

$x' = f(x) \Rightarrow f(x) = 0$

$$f(x) = 0 \Leftrightarrow x(1-x) - cx = 0 \Leftrightarrow$$

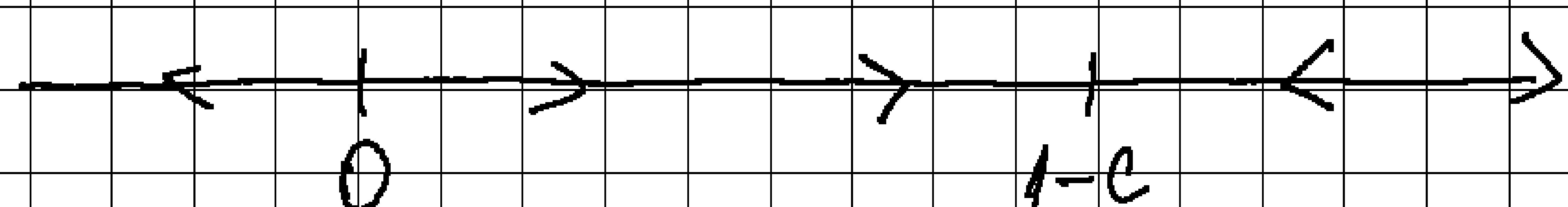
$$\Leftrightarrow x(1-x-c) = 0 \Rightarrow \begin{array}{l} x_1^* = 0 \\ x_2^* = 1-c \end{array} \quad \text{(eq. points)}$$

$$f(x) = x - x^2 - cx \Rightarrow f'(x) = 1 - 2x - c \Rightarrow$$

$$\Rightarrow f'(x_1^*) = 1 - c > 0 \Rightarrow x_1^* - \text{repeller}$$

$$c \in (0,1) \Rightarrow -c \in (-1,0) \;|+1 \Rightarrow 1 - c \in (0,1)$$

$$f'(x_2^*) = 1 - 2(1-c) = 1 - 2 + 2c - c = c - 1 < 0 \Rightarrow x_2^* - \text{attractor}$$



The orbits will be: $(-\infty, 0)$; $\{0\}$; $(0, 1-c)$; $\{1-c\}$;
$(1-c, \infty)$

if N is considered to be the optimal density $\Rightarrow$

$\Rightarrow N = 1 - c \Rightarrow c = 1 - N$

We have to chose any value between $(0, 1-N)$ because with a value from this interval the density of fish will increase in time.

3. Represent the phase portrait of $x' = x - x^3$ and from this representation find $\varphi(t,-1)$, $\varphi(t,0)$; then find the properties of $\varphi(t,-2)$, $\varphi(t,3)$, $\varphi(t,-\frac{1}{2})$

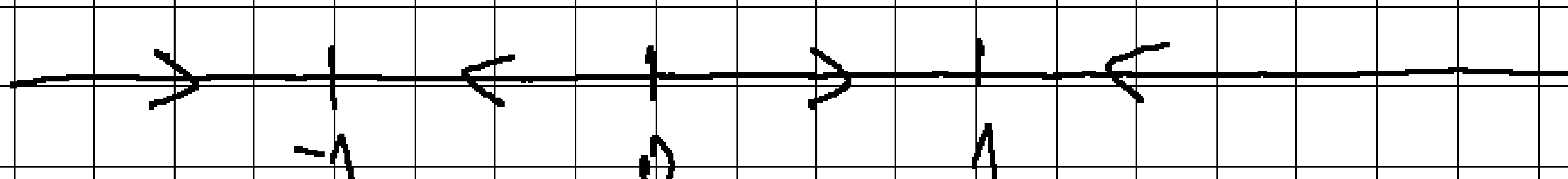
x^* eq point $\Leftrightarrow (x^*)' = 0 \Rightarrow x - x^3 = 0$

x^* - sol $\Rightarrow x(1-x^2) = 0 \Rightarrow$ $x_1^* = 0$, $x_2^* = 1$, $x_3^* = -1$

$1 - 3{x_1^*}^2 = 1 - 3 \cdot 0 = 1 > 0 \Rightarrow x_1^* = 0 \Rightarrow$ repeller

$1 - 3{x_2^*}^2 = 1 - 3 = -2 < 0 \Rightarrow x_2^* = 1 \Rightarrow$ attractor

$1 - 3{x_3^*}^2 = 1 - 3 = -2 < 0 \Rightarrow x_3^* = -1 \Rightarrow$ attractor



Orbits: $(-\infty,-1)$; $\{-1\}$; $(-1,0)$; $\{0\}$; $(0,1)$; $\{1\}$; $(1,\infty)$

$\varphi(t,-1) = -1$; $-1, 0$ - eq points

$\varphi(t,0) = 0$

ivp $\begin{cases} x' = x - x^3 \\ x(0) = \eta \end{cases}$

sol: $\varphi(t,\eta)$

$\varphi(t,\eta^*) = \eta^*$, η^* - eq point

$\varphi(t,-2) \in (-\infty,-1) \Rightarrow \varphi(t,-2)$ increasing

$\lim\limits_{t\to\infty} \varphi(t,-2) = -\infty$, $\lim\limits_{t\to -1} \varphi(t,-1) = -1$

$\varphi(t,3) \in (1,+\infty) \Rightarrow \varphi(t,3)$ is decreasing

$\lim\limits_{t\to\infty} \varphi(t,3) = 1$

$\lim\limits_{t\to 1} \varphi(t,3) = \infty$

$\varphi(t,-\frac{1}{2}) \in (-1,0) \Rightarrow$ is decreasing

$\lim\limits_{t\to 0} \varphi(t,-\frac{1}{2}) = -1$

$\lim\limits_{t\to -1} \varphi(t,-\frac{1}{2}) = +\infty$

(b) Represent the phase portrait of $x' = x - x^3 + 1$

eq. point. $x - x^3 + 1 = 0$

$x_1 \in \mathbb{R}$

$x_2, x_3 \in \mathbb{C} \setminus \mathbb{R}$

$x_1, x_2, x_3 \in \mathbb{R}$

There exists $x_1^* \in \mathbb{R}$ an eq. point

$f'(x) = 1 - 3x^2$

x		$-\frac{1}{\sqrt{3}}$		$\frac{1}{\sqrt{3}}$	1	x_1^*	2	
$f'(x)$	$- - - - -$	0	$+ + + + + +$	0		$-$		$- -$
$f(x)$	$\infty \searrow$	$+$	$\nearrow$	$+$		$\searrow 0$		$-\infty$
sign f	$+ + + + + +$		$+ + + + + +$			0		$- - -$

$f'(x) = 0 \Rightarrow x_{1,2} = \pm \frac{\sqrt{3}}{3}$

$f\left(-\frac{\sqrt{3}}{3}\right) = \frac{27 - 6\sqrt{3}}{27} > 0$

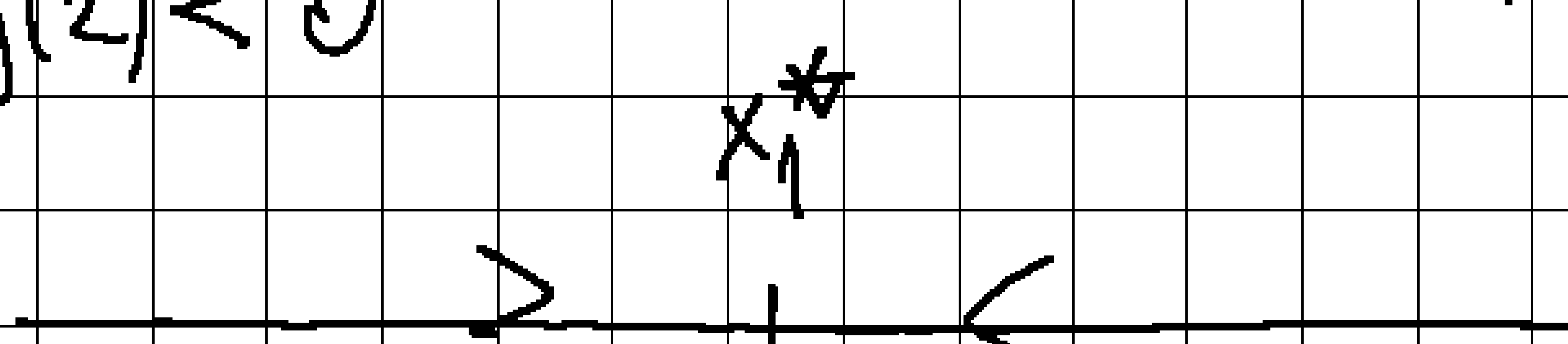
$f\left(\frac{\sqrt{3}}{3}\right) = \frac{6\sqrt{3} + 27}{27} > 0$

$\lim_{x \to \infty} f(x) = -\infty$

$\lim_{x \to -\infty} f(x) = \infty$

From the table $\Rightarrow$ we have only 1 real root for the eq. $\Rightarrow$ we have one eq point.

$f(1) > 0$
$f(2) < 0$ $\Rightarrow x_1^* \in (1, 2)$



$f'(x^*) < 0 \Rightarrow x_1^* =$ attractor

Orbits: $(-\infty, x_1^*)$; $\{x_1^*\}$; (x_1^*, ∞)

Basin of attraction:

$$A_{x_1^*} = \mathbb{R}$$ (whole set of orbits that go to the sol)