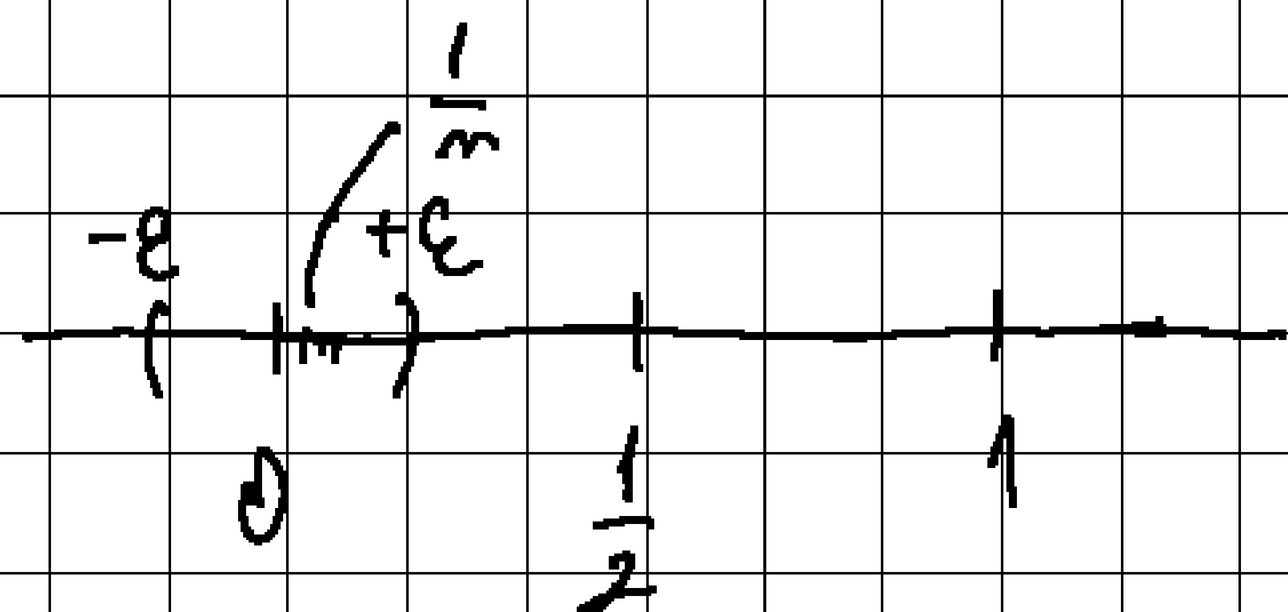
1. Find the accumulation points

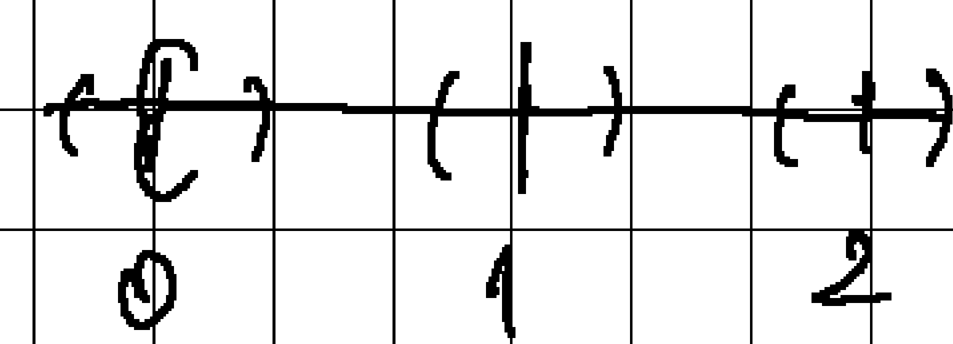
$A = [0,1] \cup \{2\}$

$x \in \overline{\mathbb{R}}$ is an accumulation point of A $\Leftrightarrow$

$\Leftrightarrow$ $\forall\, V \in \mathcal{V}(x)$ $\quad V \cap (A \setminus \{x\}) \neq \emptyset$

$\frac{1}{n} > \infty$





$A' = [0, 1]$

$B = \mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$

$B' = \{\pm \infty\}$

$C = \{0.1,\ 0.11,\ 0.111,\ \dots\}$

$C' = \{0.1111\dots\} = \{\frac{1}{9}\}$

2. Find a function that is discontinous everywhere but $|f|$ is cont. everywhere.

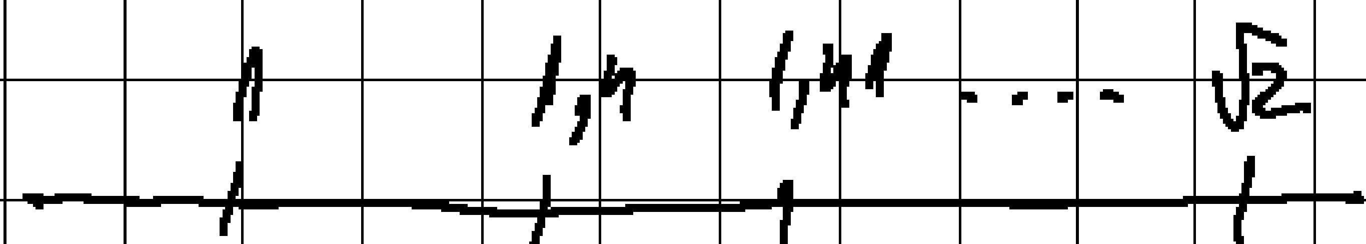
$f : \mathbb{R} \to \mathbb{R}$

f is cont. at $x_0 \Leftrightarrow \lim\limits_{x \to x_0} f(x) = f(x_0)$

$$f(x) = \begin{cases} 1 & , x \in \mathbb{R} \setminus \mathbb{Q} \\ 0 & , x \in \mathbb{Q} \end{cases}$$

Dirichlet function $\Rightarrow$

$\Rightarrow$ discontinous everywhere

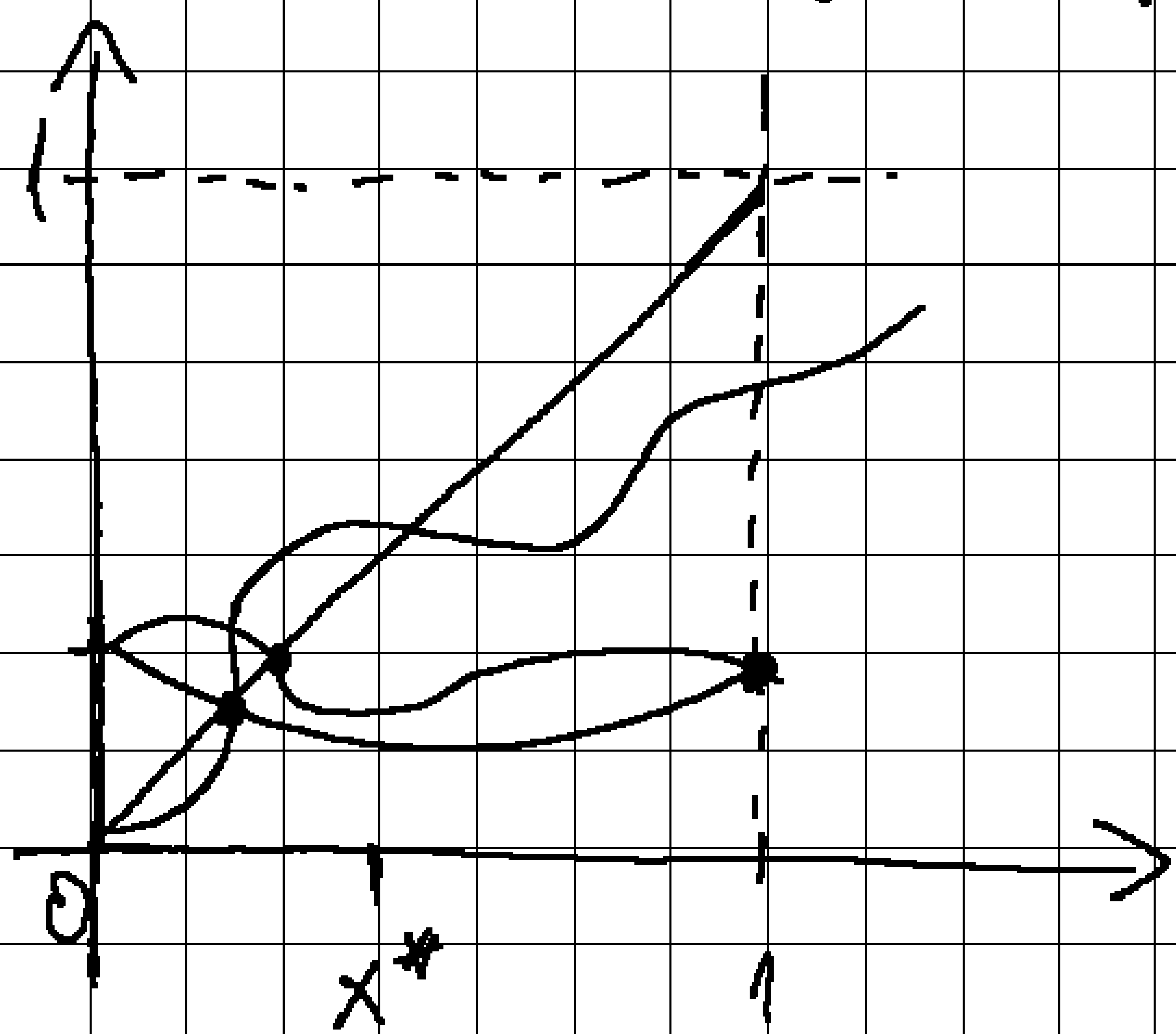


$f(1) = f(1{,}4) = f(1{,}41) = 0 \qquad f(\sqrt{2}) = 1$

$$f(x) = \begin{cases} 1 & , x \in \mathbb{R} \setminus \mathbb{Q} \\ -1 & , x \in \mathbb{Q} \end{cases}$$

$\Rightarrow f$ disc. everywhere but $|f| = 1$ cont. everywhere

3. $f:[a,b]\to[a,b]$ continuous. Prove that f has at least one fixed point $x^* \in [a,b]: f(x^*)=x^*$



$(x^*, f(x^*)) = (x^*, x^*)$

$f(a) \geq a$
$f(b) \leq b$

Let $g:[a,b]\to\mathbb{R}$
$g(x) = f(x) - x$

$$\left.\begin{array}{l} g(a) = f(a) - a \geq 0 \\ g(b) = f(b) - b \leq 0 \\ \quad g \text{ continuous} \end{array}\right\} \Rightarrow \exists\, c \in [a,b]: g(c) = 0 \Rightarrow$$

$\Rightarrow f(c) = c \Rightarrow c$ is a fixed point of f.

4. Study continuity and differentiability of f and f'

$f : \mathbb{R} \to \mathbb{R}$ $\qquad f(x) = \begin{cases} x^2 \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

f is differentiable at $x_0 \Leftrightarrow \lim\limits_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \in \mathbb{R}$

$\lim\limits_{x \to 0} f(x) = \lim\limits_{x \to 0} x^2 \sin \frac{1}{x}$

$$\left.\begin{array}{l} \sin \frac{1}{x} \in [-1, 1] \Rightarrow |f(x)| \leq x^2 \\ \lim\limits_{x \to 0} x^2 = 0 \\ |f(x)| \geq 0 \end{array}\right\} \Rightarrow \lim\limits_{x \to 0} f(x) = 0 = f(0)$$

$$\left.\begin{array}{l} \Rightarrow f \text{ is cont. in } 0 \\ f \text{ is cont on } \mathbb{R}^* \end{array}\right\} \Rightarrow$$

$\Rightarrow f.$ is continous everywhere

I $x \neq 0 \Rightarrow f'(x) = 2x \cdot \sin \frac{1}{x} + x^2 \cdot \cos \frac{1}{x} \cdot \left(-\frac{1}{x^2}\right) =$

$= 2x \sin \frac{1}{x} - \cos \frac{1}{x}$

II $x = 0 \Rightarrow f'(0) = \lim\limits_{x \to 0} \frac{f(x) - f(0)}{x - x_0} = \lim\limits_{x \to 0} \frac{\cancel{x^2} \cdot \sin \frac{1}{x} - 0}{\cancel{x}} =$

$= \lim\limits_{x \to 0} \overset{\nearrow 0}{x} \cdot \underbrace{\sin \frac{1}{x}}_{\in [-1, 1]} = 0 \in \mathbb{R} \Rightarrow f.$ is diff. on $\mathbb{R}$

$$f'(x) = \begin{cases} 2x \sin \frac{1}{x} - \cos \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

$$\lim_{x \to 0} f'(x) = \lim_{x \to 0} \left(2x \cdot \sin \frac{1}{x} - \cos \frac{1}{x} \right) = \lim_{x \to 0} \underbrace{2x}_{\to 0} \cdot \underbrace{\sin \frac{1}{x}}_{\in [-1,1]} -$$

$$- \lim_{x \to 0} \cos \frac{1}{x} = \underbrace{- \lim_{x \to 0} \cos \frac{1}{x}}_{\text{limit does not exist}}$$

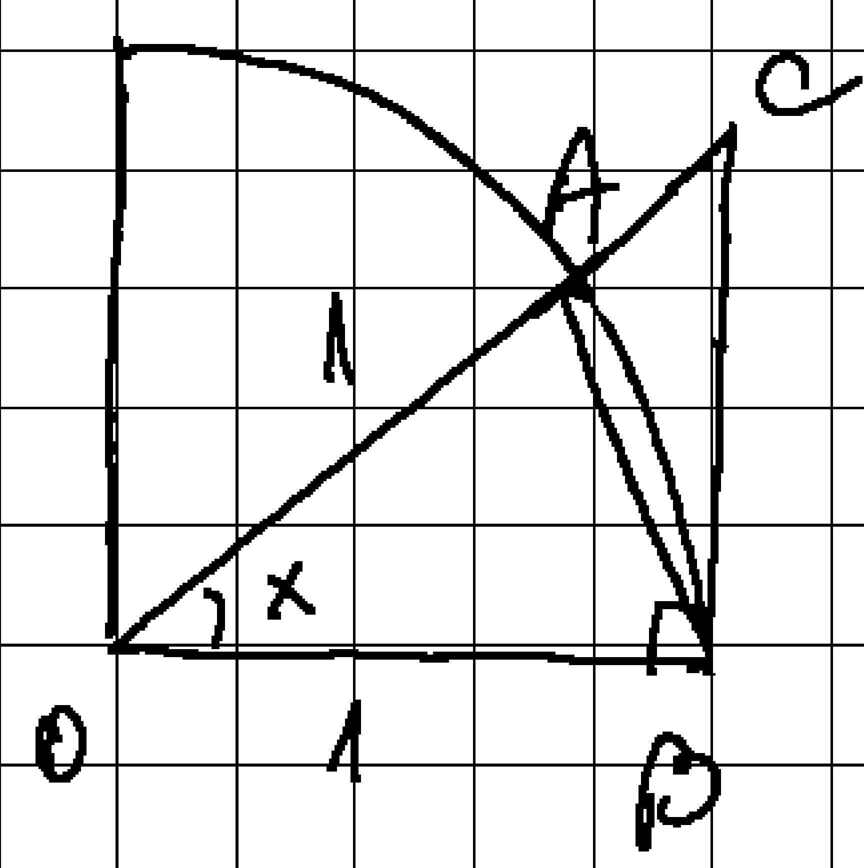
$\Rightarrow f'$ is not cont at 0

but f' is cont on $\mathbb{R}^*$

$\Rightarrow f'$ is not diff at 0

f' is diff on $\mathbb{R}^*$

5. Prove that $\lim\limits_{x\to 0} \frac{\sin x}{x} = 1$ $\quad (\sin x)' = \cos x$



$\Delta OAB < A\ \widehat{OAB} < \Delta OBC$

$\frac{\sin x}{2} < \frac{x}{2} < \frac{\tan x}{2}$

$\sin x < x < \tan x$

$\frac{1}{\sin x} > \frac{1}{x} > \frac{1}{\tan x} \quad | \sin x$

$1 > \frac{\sin x}{x} > \cos x \quad | \lim\limits_{x\to 0}$

$1 > \lim\limits_{x\to 0} \frac{\sin x}{x} > 1 \Rightarrow \lim\limits_{x\to 0} \frac{\sin x}{x} = 1$

$\lim\limits_{h\to 0} \frac{\sin(x+h) - \sin x}{h} = \lim\limits_{h\to 0} \frac{\sin x \cdot \cos h + \sin h \cdot \cos x - \sin x}{h} =$

$= \lim\limits_{h\to 0} \frac{\sin x(\cos h - 1) + \sin h \cdot \cos x}{h} =$

$= \lim\limits_{h\to 0} \frac{\sin x(\cos h - 1)}{h} + \lim\limits_{h\to 0} \underbrace{\frac{\sin h}{h}}_{1} \cdot \cos x =$

$= \lim\limits_{h\to 0} \frac{\sin x(\cos h - 1)}{h}$

$$\lim_{x\to x_0} \frac{\sin x - \sin(x_0)}{x - x_0} = \lim_{x\to x_0} \frac{\cancel{2}\cdot \sin\left(\frac{x-x_0}{2}\right)\cos\frac{x+x_0}{2}}{\frac{x-x_0}{2}} \cdot \frac{1}{\cancel{2}} =$$

$$= \lim_{x\to x_0} \cos\left(\frac{x+x_0}{2}\right) = \cos(x_0) = \sin'(x_0)$$

6. Find the limits

a) $\lim_{x\to\infty} \frac{[x]}{x} =$ $\qquad x = [x] + \{x\}$

$$= \lim_{x\to\infty} \frac{x - \{x\}}{x} = \lim_{x\to\infty} \frac{x}{x} - \lim_{x\to\infty} \frac{\{x\}}{x} = 1 - 0 = 1$$

b) $\lim_{x\to\infty} x\left(\ln(x+2) - \ln(x+1)\right) =$

$$= \lim_{x\to\infty} x \cdot \ln \frac{x+2}{x+1} = \lim_{x\to\infty} \ln\left(\frac{x+2}{x+1}\right)^x =$$

$$= \ln \lim_{x\to\infty} \left(1 + \frac{x+2}{x+1} - 1\right)^x = \ln \lim_{x\to\infty} \left[\left(1 + \frac{1}{x+1}\right)^{x+1}\right]^{\frac{1}{x+1}} =$$

$$= \ln e^{\lim_{x\to\infty} \frac{x}{x+1}} = \ln e^1 = \ln e = 1$$

c) $\lim\limits_{x\to 0}(\cos x)^{\frac{1}{x^2}} = \lim\limits_{x\to 0}(1+\cos x-1)^{\frac{1}{x^2}} =$

$= \lim\limits_{x\to 0}\left[(1+\cos x-1)^{\frac{1}{\cos x-1}}\right]^{\frac{\cos x-1}{x^2}} =$

$= \lim\limits_{x\to 0} e^{\frac{\cos x-1}{x^2}} = e^{\lim\limits_{x\to 0}\frac{\cos x-1}{x^2}} \overset{l'H}{=} e^{\lim\limits_{x\to 0}\frac{-\sin x}{2x}} =$

$\overset{l'H}{=} e^{\lim\limits_{x\to 0}\frac{-\cos x}{2}} = e^{-\frac{1}{2}} = \frac{1}{\sqrt{e}}$

d) $\lim\limits_{x\to 0} x^x = \lim\limits_{x\to 0} e^{\ln x^x} = \lim\limits_{x\to 0} e^{x\ln x} =$

$= e^{\lim\limits_{x\to 0}\frac{x}{\frac{1}{x}}} = e^{\lim\limits_{x\to 0}\frac{1}{x}\cdot(-x^2)} = e^0 = 1$

7. Find the n-th derivative

a) $f:(-1,\infty)\to R \qquad f(x)=\ln(1+x)$

$(\ln(1+x))' = \frac{1}{1+x} = f'(x)$

$f''(x) = \left(\frac{1}{1+x}\right)' = \frac{-(1+x)'}{(1+x)^2} = -\frac{1}{(1+x)^2} = -(1+x)^{-2}$

$f'''(x) = \left(-(1+x)^{-2}\right)' = -1\cdot(-2)(1+x)^{-3} = 2(1+x)^{-3}$

$f^{(n)}(x) \overset{?}{=} \frac{(-1)^{n-1}(n-1)!}{(1+x)^n}$

$$\left(f^{(n)}(x)\right)' = \left(\frac{(-1)^{n-1}(n-1)!}{(1+x)^n}\right)' = \left(\left((-1)^{n-1}(n-1)!\right)\cdot(1+x)^{-n}\right)' =$$

$$= \left((-1)^{n-1}\cdot(n-1)!\right)\cdot(-n)(1+x)^{-n-1} =$$

$$= (-1)^n \cdot n! \cdot (1+x)^{-(n+1)} = \frac{(-1)^n \cdot n!}{(1+x)^{n+1}} = f^{(n+1)}(x)$$

b) $f : \mathbb{R} \Rightarrow \mathbb{R}$ $\quad f(x) = \sin x$

$f'(x) = \cos x$

$f''(x) = -\sin x$

$f'''(x) = -\cos x$

$f^{(4)}(x) = \sin x = f(x)$

$f^{(4k)}(x) = \sin x$

$f^{(4k+1)}(x) = \cos x$

$f^{(4k+2)}(x) = -\sin x$

$f^{(4k+3)}(x) = -\cos x$

for c), d) use formula $(f\cdot g)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} \cdot f^{(k)} \cdot g^{(n-k)}$