Chapter 7: $\cancel{1}, 2, 4, \cancel{6}, 7, 8$

$$Q: \; a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_{10}x + 2a_{01}y + a_{00} = 0$$

isometries: (rotation + translation)

$$\lambda_1 x^2 + \lambda_2 y^2 = h$$

or

$$\lambda_1 x^2 + \lambda_2 y = h$$

$$\lambda_1, \lambda_2 \in \{-1, 1\}$$

$r = \text{rank } Q$	$(p, r) - p$	equations	name
2	$(0,2)$ or $(2,0)$	$x^2+y^2+1=0$	imaginary ellipse
2	$(1,1)$	$x^2-y^2-1=0$	hyperbola
2	$(0,2)$ or $(2,0)$	$x^2+y^2-1=0$	ellipse
2	$(0,2)$ or $(2,0)$	$x^2+y^2=0$	two complex lines
2	$(1,1)$	$x^2-y^2=0$	two real lines
1	$(0,1)$ or $(1,0)$	$x^2+1=0$	two complex lines
1	$(1,0)$	$x^2-1=0$	two real lines
1	$(1,0)$	$x^2=0$	a real double-line
1	$(0,1)$ or $(1,0)$	$x^2-y=0$	parabola

7.2. Write down a quadratic eq. with associated matrix A and find the matrix $M \in SO(2)$ which diagonalizes A

a) $\begin{pmatrix} 6 & 2 \\ 2 & 9 \end{pmatrix}$

$$P_A(x) = \det(A - x I_n) = \begin{vmatrix} 6-x & 2 \\ 2 & 9-x \end{vmatrix} =$$

$$= (6-x)(9-x) - 4 = x^2 - 15x + 50$$

$$\lambda_{1,2} = \frac{15 \pm \sqrt{25}}{2} \quad \begin{cases} \lambda_1 = 10 \\ \lambda_2 = 5 \end{cases}$$

$$S(\lambda_1) = \left\{ (x,y) \;\middle|\; (A - \lambda_1 I_2) \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$$

$$\begin{pmatrix} -4 & 2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} -4x + 2y = 0 \\ 2x - y = 0 \end{cases} \Rightarrow y = 2x \qquad S(\lambda_1) = \langle (1,2) \rangle$$

→ the norm of the vector so it's normalized

We chose $v_1 = \frac{1}{\sqrt{5}} (1,2)$

$$S(\lambda_2) = \left\{ (x,y) \;\middle|\; \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\} = \left\{ (x,y) \;\middle|\; \begin{cases} x + 2y = 0 \\ 2x + 4y = 0 \end{cases} \right\} =$$

$$= \{ (-2y, y) \mid y \in \mathbb{R} \} = \langle (-2,1) \rangle$$

We chose $v_2 = \frac{1}{\sqrt{5}} (2,-1)$

This gives us

$$M_{\mathcal{B}',\mathcal{B}} = \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \end{pmatrix}$$

$$\det M_{\mathcal{B}',\mathcal{B}} = -1$$

We want $\det M = 1$. So instead of chosing $v_2 = \frac{1}{\sqrt{5}}(-2,1)$ we chose $v_2 = \frac{1}{\sqrt{5}}(-2,1)$

$$\Rightarrow M_{\mathcal{B}',\mathcal{B}} = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$$

Regarding the quadratic eq., let us take:

can be anything (random)

$$Q: \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 6 & 2 \\ 2 & 9 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 2 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + 5 = 0$$

$$Q: \begin{pmatrix} 6x+2y & 2x+9y \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} + 2x + 3y + 5 = 0$$

$$Q: 6x^2 + 4xy + 9y^2 + 2x + 3y + 5 = 0$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = M \begin{pmatrix} x' \\ y' \end{pmatrix} = M_{\mathcal{B}',\mathcal{B}} \begin{pmatrix} x' \\ y' \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} =$$

$$= \frac{1}{\sqrt{5}} \begin{pmatrix} x' - 2y' \\ 2x' + y' \end{pmatrix}$$

$$Q: \overbrace{\begin{pmatrix} x' & y' \end{pmatrix} \cdot M^T}^{(x,y)} \cdot \begin{pmatrix} 6 & 2 \\ 2 & 9 \end{pmatrix} \cdot \underbrace{M \cdot \begin{pmatrix} x' \\ y' \end{pmatrix}}_{\begin{pmatrix} x \\ y \end{pmatrix}} +$$

$$+ \begin{pmatrix} 2 & 3 \end{pmatrix} \cdot M \begin{pmatrix} x' \\ y' \end{pmatrix} + 5 = 0$$

$$Q: \begin{pmatrix} x' & y' \end{pmatrix} \cdot \begin{pmatrix} 10 & 0 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} + \frac{1}{\sqrt{5}} \cdot (2,3) \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} \cdot \begin{pmatrix} x' \\ y' \end{pmatrix} + 5 = 0$$

$$Q: 10x'^2 + 5y'^2 + \frac{8}{\sqrt{5}} x' - \frac{1}{\sqrt{5}} y' + 5 = 0$$

$$Q: \left(10x'^2 + \frac{8}{\sqrt{5}} x'\right) + \left(5y'^2 - \frac{1}{\sqrt{5}} y'\right) + 5 = 0$$

$$Q: 10\left(x'^2 + 2 \cdot \frac{2}{5\sqrt{5}} x' + \frac{4}{125}\right) + 5\left(y'^2 - 2 \cdot \frac{1}{10\sqrt{5}} y' + \frac{1}{500}\right)$$

$$+ 5 - \frac{8}{25} - \frac{1}{100} = 0$$

$$Q: 10\left(x' + \frac{2}{5\sqrt{5}}\right)^2 + 5 \cdot \left(y' - \frac{1}{10\sqrt{5}}\right)^2 + \frac{467}{100} = 0$$

$$x'' = x' + \frac{2}{5\sqrt{5}}$$

$$y'' = y' - \frac{1}{10\sqrt{5}}$$

$$\Rightarrow Q: 10x''^2 + 5y''^2 + \frac{467}{100} = 0$$

$\Rightarrow$ imaginary ellipse

So far the transformations have been isometries but if we want we can do

$$\begin{cases} x''' = \sqrt{\dfrac{1000}{467}}\, x'' \\ y''' = \sqrt{\dfrac{700}{467}}\, y'' \end{cases} \Rightarrow Q:\ x'''^2 + y'''^2 + 1 = 0$$

7.4. Write down the associated matrix and bring the equation to the canonical form.

a) $Q:\ -x^2 + xy + y^2 = 0$

coef of x^2 coef of xy over 2

$$M_Q = \begin{pmatrix} -1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}$$

coef of y^2

$$P_{M_Q}(x) = \begin{vmatrix} -1-x & \frac{1}{2} \\ \frac{1}{2} & -1-x \end{vmatrix} = (1+x)(1+x) - \frac{1}{4} =$$

$$= x^2 + 2x + \frac{3}{4}$$

$$\lambda_{1,2} = \frac{-2 \pm \sqrt{4-3}}{2} = \frac{-2\pm 1}{2} \begin{cases} -\frac{1}{2} \\ -\frac{3}{2} \end{cases}$$

$$M_\varphi\begin{pmatrix}x\\y\end{pmatrix} = \lambda_1\begin{pmatrix}x\\y\end{pmatrix}$$

$$\begin{pmatrix}-1 & \frac{1}{2}\\ \frac{1}{2} & -1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = -\frac{1}{2}\begin{pmatrix}x\\y\end{pmatrix}$$

$$\begin{cases}-x+\frac{1}{2}y = -\frac{1}{2}x\\ \frac{1}{2}x - y = -\frac{1}{2}y\end{cases} \Leftrightarrow \begin{cases}-\frac{1}{2}x+\frac{1}{2}y = 0\\ \frac{1}{2}x-\frac{1}{2}y=0\end{cases} \Rightarrow S(\lambda_1) = \langle(1,1)\rangle$$

$$\Rightarrow v_1 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

$$M_\varphi\begin{pmatrix}x\\y\end{pmatrix} = \lambda_2\begin{pmatrix}x\\y\end{pmatrix}$$

$$\begin{cases}-x+\frac{1}{2}y = -\frac{3}{2}x\\ \frac{1}{2}x - y = -\frac{3}{2}x\end{cases} \Leftrightarrow \begin{cases}\frac{1}{2}x+\frac{1}{2}y = 0\\ \frac{1}{2}x+\frac{1}{2}y=0\end{cases}$$

$$\Rightarrow S(\lambda_2) = \langle(1,-1)\rangle$$

$$\Rightarrow v_2 = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$$

$$M = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$$

$$|M| = \frac{1}{2}(-1 \quad -1) \Rightarrow v_2 = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

$$\Rightarrow M = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \cdot (x \quad y)\ M_Q \begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$$\Rightarrow (x' \quad y')\ \underbrace{M^T \cdot M_Q \cdot M}_{= \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & -\frac{3}{2} \end{pmatrix}} \begin{pmatrix} x' \\ y' \end{pmatrix} = 0$$

$\Rightarrow$ skip the multiplications because those are the eigenvalues

$$\Rightarrow \left(-\frac{1}{2}x' \quad -\frac{3}{2}y'\right)\begin{pmatrix} x' \\ y' \end{pmatrix} = 0$$

$$\Rightarrow -\frac{1}{2}x'^2 + \frac{3}{2}y'^2 = 0 \qquad / \cdot (-2)$$

$$\Rightarrow x'^2 + 3y'^2 = 0$$

$$\begin{cases} x'' = x' \\ y'' = \sqrt{3}\,y' \end{cases} \qquad \Rightarrow \quad x''^2 + y''^2 = 0$$

Lagrange method: (Method II)

$$-x^2+xy-y^2=0$$

$$-(x^2-xy)-y^2=0$$

$$-\left(x^2-2x\cdot\frac{y}{2}+\frac{y^2}{4}\right)+\frac{y^2}{4}-y^2=0$$

$$-\left(x-\frac{y}{2}\right)^2-\frac{3y^2}{4}=0$$

$$\left(x-\frac{y}{2}\right)^2+\frac{3}{4}y^2=0$$

$$\begin{cases} x'=x-\frac{y}{2} \\ y'=\frac{\sqrt{3}}{2}y \end{cases} \Rightarrow x'^2+y'^2=0$$

7.7. Find the canonical eq.

a) $Q:\ 5x^2+4xy+8y^2-32x-56y+80=0$

$$Q:\ (5x^2+4xy)+8y^2-32x-56y+80=0$$

$$Q:\ \left(5x^2+2\sqrt{5}x\cdot\frac{2}{\sqrt{5}}y+\frac{4}{5}y^2\right)+8y^2-\frac{4}{5}y^2-32x-56y+80=0$$

$$Q:\ \left(\sqrt{5}x+\frac{2}{\sqrt{5}}y\right)^2+\frac{36}{5}y^2-32x-56y+80=0$$

$$x'=\sqrt{5}x+\frac{2}{\sqrt{5}}y$$

$$\sqrt{5}x=x'-\frac{2}{\sqrt{5}}y \Rightarrow x=\frac{1}{\sqrt{5}}x'-\frac{2}{5}y$$

$$Q: \; x'^2 + \frac{36}{5}y^2 - \frac{32}{\sqrt{5}}x' + \frac{32\cdot 2}{5}y - 56y + 80 = 0$$

$$Q: \; x'^2 - \frac{32}{\sqrt{5}}x' + \frac{36}{5}y^2 + \frac{64}{5}y - 56y + 80 = 0$$

$$Q: \; \left(x'^2 - \frac{32}{\sqrt{5}}x'\right) + \left(\frac{36}{5}y^2 - \frac{216}{5}y\right) + 80 = 0$$

$$Q: \left(x'^2 - 2x' \cdot \frac{16}{\sqrt{5}} + \frac{256}{5}\right) + \left(\frac{36}{5}y - 2\cdot\frac{6}{\sqrt{5}}\cdot y\cdot\frac{18}{\sqrt{5}} + \frac{18^2}{5}\right) -$$

$$- \frac{256}{5} - \frac{18^2}{5} + 80 = 0$$

$$x'' = x' - \frac{16}{\sqrt{5}}$$

$$y'' = \frac{6}{\sqrt{5}}y - \frac{18}{\sqrt{5}}$$

$$Q: \; x''^2 + y''^2 = \frac{180}{5}$$

$$\left.\begin{aligned} x''' &= \sqrt{\frac{5}{180}}\; x'' \\ y''' &= \sqrt{\frac{5}{180}}\cdot y'' \end{aligned}\right\} \Rightarrow Q: \; x'''^2 + y'''^2 = 1$$