Let $A \in \mathbb{R}^{m \times n}$ be a full rank matrix, with $m<n$ and $b \in \mathbb{R}^m$. Consider the under-determined linear system $Ax = b$, $x \in \mathbb{R}^n$ (having more unknowns than equations)

1. Prove that the matrix AA^T is positive semi-definite (using the definition). Using that A has full rank (its rows are independent) prove that AA^t is positive definite, hence invertible.

$$\left.\begin{array}{l} \forall v \in \mathbb{R}^m, \text{ let } v^T A \cdot A^T \cdot v \\ \qquad \text{let } W = A^T \cdot v, \; W \in \mathbb{R}^n \end{array}\right\} \Rightarrow$$

$$\Rightarrow v^T A A^T v = v^T A (A^T \cdot v) = (Av)^T (Av) = W^T \cdot W$$

I squared norm of W (always non-negative)

II W - image of v

From I and II $\Rightarrow$

$\Rightarrow AA^T$ is positive semi-definite

A has full rank $\Rightarrow$ its null space contains only the zero vector.

Let $v \in \mathbb{R}^m$ such that $AA^t \cdot v = 0 \Rightarrow A(A^t v) = 0$

$\Rightarrow A^t v = 0 \Rightarrow v$ is in the null space of AA^T and A has full rank $\Rightarrow v$ is the zero vector $\Rightarrow$

$\Rightarrow AA^T$ is positive definite

AA^T is positive definite
$\forall\ v \neq 0,\ v^T A A^T v > 0 \Rightarrow AA^T v \neq 0$ $\Big\}\Rightarrow$

$\Rightarrow AA^T$ is invertible

2. We want to find the solution of $Ax = b$ that has the minimum norm by solving

$$\min_{x \in \mathbb{R}^n} \|x\|^2 \quad \text{subject to} \quad Ax = b$$

Using Lagrange multipliers, prove that the minimum norm solution x^* is given by

$$x^* = A^T (AA^T)^{-1} \cdot b$$

The matrix $A^T(AA^T)^{-1}$ is called the (right) generalized inverse (or pseudo-inverse) of A.

$$\mathcal{L}(x, \lambda) = \|x\|^2 + \lambda^T (Ax - b)$$

$$\frac{\partial L}{\partial x} = 2x + A^T \lambda = 0$$

$$\frac{\partial L}{\partial \lambda} = Ax - b = 0$$

$$\begin{cases} 2x + A^T\lambda = 0 \Rightarrow x = -\frac{1}{2} A^T \lambda \\ Ax - b = 0 \quad \Rightarrow -\frac{1}{2} AA^T \lambda = b \Rightarrow \lambda = -2(AA^T)^{-1} \cdot b \end{cases}$$

$$\Rightarrow x = -\frac{1}{2} A^T \lambda = A^T (AA^T)^{-1} \cdot b = x^*$$

3. Consider another solution X of $AX = b$. Prove that it satisfies $(X - X^*)^T X^* = 0$ and $\|X\| \geq \|X^*\|$

$(X - X^*)^T X^* = 0$?
$\|X\| \geq \|X\|^*$?

$$\left.\begin{aligned} &(X-X^*)^T X^* = 0 \\ &X^* = A^T (AA^T)^{-1} \cdot b \end{aligned}\right\} \Rightarrow X = X^* + (X - X^*) \Rightarrow$$

$$\Rightarrow (X-X^*)^T X^* = (X^* + (X - X^*))^T X^* = (X^*)^T X^* + (X - X^*)^T \cdot X^*$$

$$(X - X^*)^T X^* = (X - A^T (AA^T)^{-1} \cdot b)^T A^T (AA^T)^{-1} b =$$

$$= (X^T - b^T (AA^T)^{-T} A) X^* = (X^T - b^T (AA^T)^{-1} A) X^* =$$

$$= (X^T - b^T) X^* = 0$$

$$\|X\|^2 = \|(X^* + (X - X^*))\|^2 = \|X^*\|^2 + \|(X - X^*)\|^2$$

$$\|X - X^*\|^2 = \|X\|^2 - \|X^*\|^2 \Rightarrow \|X\|^2 - \|X^*\|^2 \geq 0 \Rightarrow$$

$$\Rightarrow \|X\|^2 \geq \|X^*\|^2 \Rightarrow \|X\| \geq \|X^*\|$$

4. Consider the following minimization problem (known as regularized least squares, Tikhonov regularization or ridge regression)

$$\|Ax - b\|^2 + \alpha \|x\|^2 \to \min,$$

where $\alpha > 0$ is called a regularization parameter. Note that a larger α will increase the penalization of the norm (regularization term). Prove that its solution x_α is given by $(A^TA+\alpha I)x_\alpha = A^Tb$

$$J(x) = \|Ax-b\|^2 + \alpha\|x\|^2$$

$$\frac{\partial J}{\partial x} = 2A^T(Ax-b) + 2\alpha x$$

$$2A^T(Ax-b) + 2\alpha x = 0 \Leftrightarrow A^TAx - A^Tb + \alpha x = 0 \Leftrightarrow$$

$$\Leftrightarrow (A^TA+\alpha I)x = A^Tb \Leftrightarrow$$

$$\Leftrightarrow x = (A^TA+\alpha I)^{-1}\cdot A^Tb = x_\alpha$$

7. Prove that $(A^TA+\alpha I)$ is positive definite (using the definition), hence invertible. Prove that $x_\alpha = (A^TA+\alpha I)^{-1}A^Tb \to x^*$ as $\alpha \to 0$

$(A^TA+\alpha I)$ is positive definite $\Leftrightarrow$

$$\Leftrightarrow \forall v \neq 0,\ v^T(A^TA+\alpha I)v > 0$$

$$v^T(A^TA+\alpha I)v = \underbrace{v^TA^TA\cdot v}_{\geq 0} + \underbrace{\alpha v^Tv}_{>0} > 0 \Rightarrow$$

$\Rightarrow (A^TA+\alpha I)$ is positive definite $\Rightarrow$ is invertible

Get $x_\alpha = (A^TA+\alpha I)^{-1}A^Tb$

As $\alpha \to 0 \Rightarrow (A^T A + \alpha I) \to A^T A$

$$\lim_{\alpha \to 0} x_\alpha = \lim_{\alpha \to 0} (A^T A + \alpha I)^{-1} A^T b = \lim_{\alpha \to 0} ((A^T A + \alpha I))^{-1} A^T b$$

$$= (A^T A)^{-1} A^T b = x^*$$