LDECC:

$$x'' + x' + x = t \quad , \quad x = x(t)$$

1.4.1. Find the general solution

a) $x' + 6x = 0$

b) $x'' + 4x' + x = 0$

g) $x'' + x' + x = 0$

h) $x^{(4)} - x = 0$

Step I: the characteristic equation $\Rightarrow$

$\Rightarrow r_1, \dots, r_n$ roots $\in \mathbb{R} \vee \mathbb{C}$

Step III: $\{x_1, \dots, x_n\}$ - fund sys. sol $\Rightarrow$

$\Rightarrow x = C_1 x_1 + C_2 x_2 + \dots + C_n x_n \quad , \quad C_1, \dots, C_n \in \mathbb{R}$

Step II:

a) $r_1 \in \mathbb{R}$ simple root $\Rightarrow x_1 = e^{r_1 t}$

b) $r_1 = r_2 = \dots = r_k \in \mathbb{R}$ multiple roots $\Rightarrow \begin{cases} x_1 = e^{r_1 t} \\ x_2 = t e^{r_2 t} \\ \vdots \\ x_k = t^{k-1} e^{r_k t} \end{cases}$

c) $r_{1,2} = \alpha \pm i\beta \in \mathbb{C} \setminus \mathbb{R} \Rightarrow \begin{cases} x_1 = e^{\alpha t} \cos \beta t \\ x_2 = e^{\alpha t} \sin \beta t \end{cases}$

d) $r_{1,2,3,4} = \alpha \pm i\beta$ double root $\Rightarrow \begin{cases} x_1 = e^{\alpha -} \cos \beta t \\ x_2 = e^{\alpha t} \sin \beta t \\ x_3 = t e^{\alpha t} \cos \beta t \\ x_4 = t e^{\alpha t} \sin \beta t \end{cases}$

a) $x' + 6x = 0$

Step I: $r^1 + 6r^0 = 0$ (the char. eq)

$r + 6 = 0$

$r_1 = -6 \Rightarrow x_1 = e^{-6t}$

$\Rightarrow x = C \cdot e^{-6t}, \quad C \in \mathbb{R}$

b) $x'' + 4x' + 4x = 0$

$r^2 + 4r + 4 = 0$

$(r+2)^2 = 0 \Rightarrow r_1 = r_2 = -2 \quad \begin{cases} x_1 = e^{-2t} \\ x_2 = t \cdot e^{-2t} \end{cases}$

$\Rightarrow x = C_1 \cdot e^{-2t} + C_2 t \cdot e^{-2t}, \quad C_1, C_2 \in \mathbb{R}$

f) $x'' + x' + x = 0$

$r^2 + r + 1 = 0$

$\Delta = -3 \Rightarrow r_1 = \dfrac{-1 + i\sqrt{3}}{2} = -\dfrac{1}{2} + i\dfrac{\sqrt{3}}{2}$

$r_2 = \dfrac{-1 - i\sqrt{3}}{2} = -\dfrac{1}{2} - i\dfrac{\sqrt{3}}{2}$

$\Rightarrow x_1 = e^{-\frac{t}{2}} \cos\dfrac{\sqrt{3}}{2}t$

$x_2 = e^{-\frac{t}{2}} \sin\dfrac{\sqrt{3}}{2}t$

$\Rightarrow x = c_1 \cdot e^{-\frac{t}{2}} \cos\dfrac{\sqrt{3}}{2}t + c_2 e^{-\frac{t}{2}} \sin\dfrac{\sqrt{3}}{2}t$

h) $x^{(4)} - x = 0$

$r^4 - r^0 = 0$

$r^4 - 1 = 0$

$(r^2 - 1)(r^2 + 1) = 0$

$\rightarrow r^2 - 1 = 0 \Rightarrow r^2 = 1 \begin{cases} r_1 = 1 \\ r_2 = -1 \end{cases}$

$\rightarrow r^2 + 1 = 0 \Rightarrow r^2 = -1 \begin{cases} r_3 = i \\ r_4 = -i \end{cases}$

$x_1 = e^{r_1 t} = e^{t}$

$x_2 = t e^{r_2 t} = t e^{-t}$

$x_3 = \cos t$

$x_4 = \sin t$

$\Rightarrow x = c_1 \cdot e^{t} + c_2 e^{-t} + c_3 \cos t + c_4 \sin t$

$c_{1,2,3,4} \in \mathbb{R}$

1.4.2.

a) $e^{-3t}$ and $e^{5t}$

b) $5e^{-3t}$ and $-3e^{5t}$

d) $5te^{-3t}$ and $-3e^{5t}$

f) $(5-3t)e^{-3t}$

e) $(t-1)^2$

g) $-t\sin 3t$

a) $\left.\begin{aligned} x_1 = e^{-3t} &\Rightarrow r_1 = -3 \\ x_2 = e^{5t} &\Rightarrow r_2 = 5 \end{aligned}\right\} \Rightarrow (r+3)(r-5) = 0$

$\Rightarrow r^2 - 2r - 15 = 0$

$x'' - 2x' - 15x = 0$

$x = C_1 e^{-3t} + C_2 e^{5t}$ general sol

b) $\left.\begin{aligned} x_1 = 5e^{-3t} &\Rightarrow r_1 = -3 \\ x_2 = -3e^{5t} &\Rightarrow r_2 = 5 \end{aligned}\right\} \Rightarrow (r+3)(r-5)$

$\Rightarrow r^2 - 2r - 15 = 0$

$x'' - 2x' - 15x = 0$

$x = C_1 5e^{-3t} + C_2 (-3e^{5t})$ general sol

$x = C_3 e^{-3t} + C_4 e^{5t}$ general sol

d) $x_1 = 5te^{-3t} \Rightarrow r_{1,2} = -3 \Rightarrow$

$\Rightarrow x_2 = e^{-3t} \Rightarrow r_2 = -3$

$x_3 = -3e^{5t} \Rightarrow r_3 = 5$

$\Rightarrow$

$\Rightarrow (r+3)^2(r-5) = 0$

$\Rightarrow (r^2+6r+9)(r-5) = 0 \Rightarrow$

$\Rightarrow r^3 + r^2 - 21r - 45 = 0 \Rightarrow$

$\Rightarrow x''' + x'' - 21x - 45 = 0 \Rightarrow$

$\Rightarrow x = C_1 t e^{-3t} + C_2 e^{-3t} + C_3 e^{5t} \quad , C_{1,2,3} \in \mathbb{R}$

f) $(5-3t)e^{-3t}$

$5e^{-3t} - 3te^{-3t}$

$5e^{-3t}$ and $-3te^{-3t}$

$x_1 = 5e^{-3t} \Rightarrow r_1 = -3$

$x_2 = -3te^{-3t} \Rightarrow r_2 = -3$

$(r+3)^2 = 0 \Leftrightarrow r^2 + 6r + 9 = 0$

$x'' + 6x' + 9x = 0$

$x = C_1 \cdot e^{-3t} + C_2 \cdot t \cdot e^{-3t} \quad , C_1, C_2 \in \mathbb{R}$

σ) $(t-1)^2 = t^2 - 2t + 1$

$x_1 = t^2 \cdot e^{0t} \Rightarrow r_1 = 0$

$x_2 = t \cdot e^{0t} \Rightarrow r_2 = 0$

$x_3 = 1 \cdot e^{0 \cdot t} \Rightarrow r_3 = 0$

$r^3 = 0 \Rightarrow x''' = 0$

$x = c_1 \cdot t^2 + c_2 \cdot t + c_3 + 1$

g) $-t \sin 3t$

$x_1 = -t \sin 3t \cdot e^{0t} \Rightarrow x_2 = t \cos 3t \cdot e^{0t}$

$x_3 = \sin 3t \cdot e^{0t}$

$x_4 = \cos 4t \cdot e^{0t}$

$\Rightarrow r_{1,3} = 0 + 3i$

$r_{2,4} = 0 - 3i$

$(r - 3i)^2 (r + 3i)^2 = 0$

$(r^2 + 9)^2 = 0 \Leftrightarrow$

$\Leftrightarrow r^4 + 18r^2 + 81 = 0$

$x^{(4)} + 18x'' + 81x = 0$

$\Rightarrow x = c_1 t \sin 3t + c_2 t \cos 3t + c_3 \sin 3t + c_4 \cos 3t$

$c_1, c_2, c_3, c_4 \in \mathbb{R}$

1.4.4.

$$\text{IVP}\begin{cases} x'' + \pi^2 x = 0 \\ x(0) = 0 \\ x'(0) = \eta,\ \eta \in \mathbb{R} \end{cases}$$

1.4.5.

$$\text{BVP}\begin{cases} x'' + x = 0 \\ x(0) = 0 \\ x(\pi) = 0 \end{cases}$$

1.4.4. $x'' + \pi^2 x = 0$

$r^2 + \pi^2 = 0$

$r_{1,2} = \pm i\pi \Rightarrow x_1 = e^{0t}\cos\pi t$

$x_2 = e^{0t}\sin\pi t$

gen sol. of eq. $x = C_1\cos\pi t + C_2\sin\pi t$

$x(0) = 0 \Rightarrow \quad C_1\cos\pi\cdot 0 + C_2\sin\pi\cdot 0 = 0$

$C_1 + 0 = 0$

$C_1 = 0$

$x' = -\pi C_1\sin(\pi t) + \pi C_2\cos(\pi t)$ $C_1 = 0$

$x'(0) = \eta \Rightarrow -\pi C_1\sin(\pi\cdot 0) + \pi C_2\cos(\pi\cdot 0) = \eta$

$\Leftrightarrow \quad 0 + \pi C_2 = \eta \Rightarrow C_2 = \frac{\eta}{\pi}$

gen sol IVP $\quad x = \frac{\eta}{t}\sin(\pi t)$

$x'' + x = 0$

$r^2 + r^0 = 0$

$r^2 + 1 = 0 \Rightarrow r^2 = -1 \nearrow r_1 = i$
$\searrow r_2 = -i$

$x_1 = e^{0t} \cdot \cos t$

$x_2 = e^{0t} \cdot \sin t$

$x = c_1 \cos t + c_2 \sin t \quad c_1, c_2 \in \mathbb{R}$

general sol. of the equation

$x(0) = 0 \Rightarrow c_1 \cos 0 + c_2 \sin 0 = 0 \Rightarrow c_1 = 0$

$x(\pi) = 0 \Rightarrow c_1 \cos \pi + c_2 \sin \pi = 0 \Rightarrow c_1 = 0$

$x = c_2 \cdot \sin t \quad , c_2 \in \mathbb{R}$

gen. sol. of BVP

1.4.6. $\lambda = ?$ s.t. $\exists\, x \neq 0$

$x = 2\pi$ periodic

$x =$ sol of $x'' + \lambda x = 0$

$x'' + \lambda x = 0$

$r^2 + \lambda r^0 = 0$

$r^2 + \lambda = 0$

I $\lambda < 0 \Rightarrow r^2 = \underbrace{-\lambda}_{>0}$

$\Rightarrow r_1, r_2 \in \mathbb{R}$

$r_1 = \sqrt{-\lambda} \Rightarrow x_1 = e^{\sqrt{-\lambda}t}$

$r_2 = -\sqrt{-\lambda} \Rightarrow x_2 = e^{-\sqrt{-\lambda}t}$

$x = C_1 e^{\sqrt{-\lambda}t} + C_2 e^{-\sqrt{-\lambda}t}$

$x \neq 0 \Rightarrow C_1 = C_2 \neq 0$

this function can't be periodic

II $\lambda > 0 \Rightarrow r^2 = -\lambda$

$\Rightarrow r_1, r_2 \in \mathbb{C} \setminus \mathbb{R}$

$r_1 = \sqrt{\lambda}\, i \Rightarrow x_1 = \sin\sqrt{\lambda}\, t$

$r_2 = -\sqrt{\lambda}\, i \Rightarrow x_2 = \cos\sqrt{\lambda}\, t$

$\Rightarrow x = c_1 \sin\sqrt{\lambda}\, t + c_2 \cos\sqrt{\lambda}\, t$

main period $= 2\pi$

$\Rightarrow \sqrt{\lambda}\, t = 2\pi \Rightarrow t = \frac{2\pi}{\sqrt{\lambda}}$

any period is $n \cdot t \Rightarrow n \cdot \frac{2\pi}{\sqrt{\lambda}}$

$\Rightarrow n \cdot \frac{2\pi}{\sqrt{\lambda}} = 2\pi \Rightarrow \lambda = n^2, n \in \mathbb{Z}$

$\boxed{III}$ $\lambda = 0$

$r^2 = 0$

$\Rightarrow r_{1,2} = 0$

$\Rightarrow x_1 = e^{0t}$

$x_2 = t \cdot e^{0t}$

$x = c_1 \cdot e^{0t} + c_2 t e^{0t} = c_1 + c_2 t$

$\Rightarrow$ not a periodic function