V K - vector space

$v_1, v_2, \dots, v_n \in V$ are linearly independent if

$\forall\, \alpha_1, \alpha_2, \dots, \alpha_n \in K$

$\alpha_1 \cdot v_1 + \alpha_2 v_2 + \dots + \alpha_n \cdot v_n = 0$

We have $\alpha_1 = \alpha_2 = \dots = \alpha_n = 0$

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if $v_1, \dots, v_n \in K^m$

the number of linear independent vectors in $\{v_1, \dots, v_n\} =$

$$= \operatorname{rank}\begin{pmatrix} v_1 \\ \hline v_2 \\ \hline \vdots \\ \hline v_n \end{pmatrix} = \operatorname{rank}\left(v_1^t \,|\, v_2^t \,|\, \dots \,|\, v_n^t \right)$$

So:

$v_1, v_2, \dots, v_n$ lin. indep. $\Leftrightarrow \operatorname{rank}\begin{pmatrix} v_1 \\ \hline v_2 \\ \hline \vdots \\ \hline v_n \end{pmatrix} = n$

Ex1: $v_1=(1,-1,0)$, $v_2=(2,1,1)$, $v_3=(1,5,2)\in\mathbb{R}^3$

Prove that:

(i) v_1, v_2, v_3 are linearly dependent and determine a dependence relationship

(ii) v_1, v_2 are linearly independent

(i) Let $\alpha_1,\alpha_2,\alpha_3\in\mathbb{R}$ s.t. $\alpha_1\cdot v_1+\alpha_2\cdot v_2+\alpha_3\cdot v_3=0$

$$\alpha_1(1,-1,0)+\alpha_2(2,1,1)+\alpha_3(1,5,2)=(0,0,0)$$

$$\begin{cases}\alpha_1+2\alpha_2+\alpha_3=0\\ -\alpha_1+\alpha_2+5\alpha_3=0\\ \alpha_2+2\alpha_3=0\Rightarrow\alpha_2=-2\alpha_3\end{cases}$$

$$\Rightarrow\begin{cases}\alpha_1-4\alpha_3+\alpha_3=0\\ -\alpha_1-2\alpha_3+5\alpha_3=0\\ \alpha_2=-2\alpha_3\end{cases}\Rightarrow\begin{cases}\alpha_1-3\alpha_3=0\\ -\alpha_1+3\alpha_3=0\\ \alpha_2=-2\alpha_3\end{cases}\overset{(+)}{\Rightarrow}\begin{matrix}\alpha_2=-2\alpha_3\\ \alpha_1=3\alpha_3\end{matrix}$$

$$\Rightarrow 3v_1-2v_2+v_3=0$$

(ii) $\alpha_1v_1+\alpha_2v_2=0$

$$\alpha_1(1,-1,0)+\alpha_2(2,1,1)=0$$

$$\begin{cases}\alpha_1+2\alpha_2=0\\ -\alpha_1+\alpha_2=0\\ \alpha_2=0\end{cases}\Rightarrow\alpha_1=0$$

$\Rightarrow v_1, v_2$ are lin. indep.

3. $v_1 = (1, a, 0)$, $v_2 = (a, 1, 1)$, $v_3 = (1, 0, a)$

Determine $a \in \mathbb{R}$ s.t. v_1, v_2, v_3 are lin. indep.

midterm exam problem

Let $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$

$\alpha_1 \cdot v_1 + \alpha_2 \cdot v_2 + \alpha_3 \cdot v_3 = 0 \Leftrightarrow$

$\Leftrightarrow \alpha_1 \cdot (1, a, 0) + \alpha_2 (a, 1, 1) + \alpha_3 (1, 0, a) = (0, 0, 0)$

$$\Leftrightarrow \begin{cases} \alpha_1 + a\alpha_2 + \alpha_3 = 0 \\ a\alpha_1 + \alpha_2 = 0 \\ \alpha_2 + a\alpha_3 = 0 \end{cases} \Leftrightarrow \begin{pmatrix} 1 & a & 1 \\ a & 1 & 0 \\ 0 & 1 & a \end{pmatrix} \cdot \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\Rightarrow$ System is homogenous thus it has the triveal sol. if $\det A \neq 0$, the sol is unique, which is what we need.

$\Rightarrow \det A = 0 \Leftrightarrow a + a - a^3 = 0 \Leftrightarrow a(2 - a^2) = 0 \Leftrightarrow$

$\Leftrightarrow a \in \{0, \pm\sqrt{2}\}$

$\Rightarrow v_1, v_2, v_3$ are linearly indep. iff $a \in \mathbb{R} \setminus \{0, \pm\sqrt{2}\}$

V K - vector space

$(v_1, v_2, \dots, v_n)$ basis of V

$\Leftrightarrow (v_1, \dots, v_n)$ system of generators for V

$v_1, \dots, v_n$ lin. indep.

shorter version

OR : $\Leftrightarrow$ $\forall\, v \in V: \exists!\ \alpha_1, \alpha_2, \dots, \alpha_n : \alpha_1 v_1 + \dots + \alpha_n v$

if $B = (v_1, \dots, v_n)$ basis, then the coordinate vector of v in the basis B is:

$$[v]_B = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}$$

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8. $\mathbb{R}_2[X] = \{ f \in \mathbb{R}[X] \mid \deg f \leq 2 \}$

Similar to last year midterm

show that the lists

$E = (1, X, X^2)$ and $B = (1, X-a, (X-a)^2)$ for some $a \in \mathbb{R}$ are bases of $\mathbb{R}_2[X]$ and $\forall\, f = a_0 + a_1 X + a_2 X^2$ find $[f]_E$ and $[f]_B$.

for $E = (1, x, x^2)$

1. lin. indep.

Let $a_0, a_1, a_2 \in \mathbb{R}$ s.t. $a_0 \cdot 1 + a_1 \cdot x + a_2 \cdot x^2 = 0$

$\Rightarrow a_0 = a_1 = a_2 = 0 \Rightarrow E$ lin. ind.

2. System. gen.

Let $f = a_0 + a_1 x + a_2 x^2$

$f = \alpha_1 \cdot v_1 + \alpha_2 \cdot v_2 + \alpha_3 \cdot v_3 =$

$= \alpha_1 \cdot 1 + \alpha_2 \cdot x + \alpha_3 x^2$

$\Rightarrow \begin{cases} \alpha_1 = a_0 \\ \alpha_2 = a_1 \\ \alpha_3 = a_3 \end{cases} \Rightarrow E$ - sys. of generators

1, 2 $\Rightarrow E$ is a basis.

$[f]_e = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix}$

$B = (1, x-a, x-a^2)$ (Same type of ex. but diff approach)

Let $f = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3$

$f = \alpha_1 \cdot 1 + \alpha_2 \cdot (x-a) + \alpha_3 \cdot (x-a)^2$

$f = \alpha_1 \cdot 1 + \alpha_2 x - \alpha_2 a + \alpha_3 x^2 - 2ax \cdot \alpha_3 + \alpha_3 \cdot a^2$

$f = \alpha_3 a^2 + \alpha_1 - \alpha_2 a + x(\alpha_2 - 2a\alpha_3) + \alpha_3 x^2$

$$\begin{cases} \alpha_1 - \alpha_2 a + \alpha_3 a^2 = a_0 \\ \alpha_2 - 2a\alpha_3 = a_1 \\ \alpha_3 = a_2 \end{cases}$$

$$\begin{cases} \alpha_1 - a \cdot a_1 - 2a^2 a_2 + a_2 a^2 = a_0 \\ \alpha_2 = a_1 + 2a a_2 \\ \alpha_3 = a_2 \end{cases}$$

$$\begin{cases} \alpha_1 = a_0 + a \cdot a_1 + a^2 \cdot a_2 \\ \alpha_2 = a_1 + 2a a_2 \\ \alpha_3 = a_2 \end{cases}$$

We have a unique solution $\Rightarrow$ $\mathcal{B}$ is a basis

$$[f]_{\mathcal{B}} = \begin{pmatrix} a_0 + a \cdot a_1 + a^2 \cdot a_2 \\ a_1 + 2a a_2 \\ a_2 \end{pmatrix}$$

7. $A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $A_2 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, $A_3 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$,

$A_4 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \in \mathcal{M}_2(\mathbb{R})$

Prove that the list $\mathcal{B} = (A_1, A_2, A_3, A_4)$ is a basis of $\mathcal{M}_2(\mathbb{R})$ and find $\left[\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}\right]_{\mathcal{B}}$

Let $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{R}$

Not. $M = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$

$M = \alpha_1 A_1 + \alpha_2 A_2 + \alpha_3 A_3 + \alpha_4 A_4 =$

$$= \begin{pmatrix} \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 & \alpha_2 + \alpha_3 + \alpha_4 \\ \alpha_3 + \alpha_4 & \alpha_1 + \alpha_4 \end{pmatrix}$$

$$\begin{cases} \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 2 \\ \alpha_2 + \alpha_3 + \alpha_4 = 1 \quad \Rightarrow \alpha_1 = 1 \\ \alpha_3 + \alpha_4 = 1 \\ \alpha_1 + \alpha_4 = 0 \quad \Rightarrow \alpha_4 = -1 \end{cases}$$

$\alpha_3 = 2$

$\alpha_2 = 0$

$$[\,]_{\mathcal{B}} = \begin{pmatrix} 1 \\ 0 \\ 2 \\ -1 \end{pmatrix}$$

$M = \begin{pmatrix} x & y \\ z & t \end{pmatrix}$

$$\begin{cases} \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = x \\ \alpha_2 + \alpha_3 + \alpha_4 = y \quad \Rightarrow \alpha_1 = x - y \\ \alpha_3 + \alpha_4 = z \\ \alpha_1 + \alpha_4 = t \quad \Rightarrow \alpha_4 = t - x + y \end{cases}$$

$$\alpha_3 = z - t + x$$

$$\alpha_2 = y - z + \cancel{t} - \cancel{x} + \cancel{y} - \cancel{t} + \cancel{x} - \cancel{y} \Rightarrow \alpha_2 = y - z$$

$\alpha_1, \alpha_2, \alpha_3, \alpha_4$ - unique sol $\Rightarrow \beta$ is a basis

4. $v_1 = (1, -2, 0, -1)$, $v_2 = (2, 1, 1, 0)$, $v_3 = (0, a, 1, 2) \in \mathbb{R}^4$

Find $a \in \mathbb{R}$ s.t. v_1, v_2, v_3 are linearly dependent

$$A = \begin{pmatrix} 1 & -2 & 0 & -1 \\ 2 & 1 & 1 & 0 \\ 0 & a & 1 & 2 \end{pmatrix}$$

$$\begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix} = 5 \neq 0 \Rightarrow \text{Rank } A \geq 2$$

v_1, v_2, v_3 linearly dependent $\Leftrightarrow$ rank $A < 3 \Leftrightarrow$

$\Leftrightarrow$ every 3×3 minor in A is 0

$$A = \begin{vmatrix} 1 & -2 & 0 \\ 2 & 1 & 1 \\ 0 & a & 1 \end{vmatrix} = 0 \Leftrightarrow 5 - a = 0 \Rightarrow a = 5$$

We compute the other minors ....