1. Find the general solution of each of the following equations looking first for some solutions of the form $x = t^r$, $r \in \mathbb{R}$

a) $t^2x'' - 8tx' + 20x = 0$, $t \in 0,\infty$

$x' = rt^{r-1}$ $\quad$ $x'' = r(r-1)t^{r-2}$

$t^2 \cdot r(r-1)t^{r-2} - 8t \cdot rt^{r-1} + 20t^r = 0$

$t^r(r(r-1) - 8r + 20) = 0 \iff$

$\iff t^r(r^2 - 9r + 20) = 0$

$t^r \neq 0 \Rightarrow r^2 - 9r + 20 = 0$

$\Delta = 80 - 81 = 1 \Rightarrow r_{1,2} = \frac{9 \pm 1}{2} \begin{cases} r_1 = 5 \\ r_2 = 4 \end{cases}$

general solution: $x(t) = c_1t^5 + c_2t^4$

3. $\begin{cases} x' = -y(x^2+y^2) \\ y' = x(x^2+y^2) \end{cases}$

a) Does this system have other equilibria besides $(0,0)$? Justify.

$x, y \neq 0$

Let $\alpha(x,y)$ be an eq. point

$$\Rightarrow \begin{cases} x' = 0 \\ y' = 0 \end{cases} \Rightarrow \begin{cases} -y(x^2+y^2) = 0 \\ x(x^2+y^2) = 0 \\ x, y \neq 0 \end{cases}$$

$$\Rightarrow \quad x^2 + y^2 = 0$$

$$x^2 = -y^2 \Rightarrow \quad (x,y) = (0,0)$$

$\Rightarrow$ $(0,0)$ is the only eq. point.

b) $\varphi(t,1,0) = (\cos t, \sin t)$

$\varphi(t,2,0) = (2\cos t, 2\sin t)$ } verify

$\varphi(t,1,0) \Rightarrow$ } sol of (1)
$x(0) = 1$
$y(0) = 0$

$$\begin{cases} (\cos t)' = -\sin t\,(\cos^2 t + \sin^2 t) \\ (\sin t)' = \cos t\,(\cos^2 t + \sin^2 t) \end{cases} \Leftrightarrow$$

$\Leftrightarrow \begin{cases} -\sin t = -\sin t \cdot 1 \quad \text{True} \\ \cos t = \cos t \cdot 1 \quad \text{True} \end{cases}$

$\cos 0 = 1$

$\sin 0 = 0$

$\Rightarrow \varphi(t,1,0)$ is a flow

$\varphi(t,2,0)$

$$\begin{cases} (2\cos 4t)' = -2\sin 4t\left((2\cos 4t)^2 + (2\sin 4t)^2\right) \\ (2\sin 4t)' = -2\cos 4t\left((2\cos 4t)^2 + (2\sin 4t)^2\right) \\ 2\cos(4\cdot 0) = 2 \\ 2\sin(4\cdot 0) = 0 \end{cases}$$

c) Decide the validity of the statement

"Any sol is periodic"

$$\varphi(t,\eta,0) = (\eta\cos(\eta^2 t),\ \eta\sin(\eta^2 t)) \quad \eta > 0$$

$\Rightarrow x, y$ periodic