V, V' K - vector sp., B_1, B_2 - bases of V

B'_1, B'_2 bases of V', $f \in Hom_K(V, V')$

$B_1 = (w_1, \dots, w_n)$

$$[f]_{B_1, B'_1} = \left([f(v_1)]_{B'} \dots [f(v_n)]_{B'} \right)$$

$\forall f_1, f_2 \in Hom(V, V')$:

$$[f_1 + f_2]_{B,B'} = [f_1]_{B,B'} + [f_2]_{B,B'}$$

$\forall \alpha \in K$:

$$[\alpha f]_{B,B'} = \alpha [f]_{B,B'}$$

$f \in Hom_K(V, V')$, $g \in Hom_K(V', V'')$

B, B', B'' bases for V, V', V''

$g \circ f \in Hom_K(V, V'')$

$$\boxed{[g \circ f]_{B,B''} = [g]_{B',B''} \cdot [f]_{B,B'}}$$

Base-change:

$T_{B_1,B_1'}$ - base change matrix from B_1 to B_1'

$$[f]_{B_1',B_2'} = [id_{V'}]_{B_2,B_2'} \cdot [f]_{B_1,B_2} \cdot [id_V]_{B_1',B_1}$$

For base change matrices we have:

$$[id]_{B,B'} = [id]^{-1}_{B',B}$$

If we want to convert the coordinates of a vector from a basis to another then we have:

$$[v]_{B'} = \underbrace{[id]_{B,B'}}_{T_{B',B}} \cdot [v]_B$$

2. in $\mathbb{R}^2$ we consider the bases

$\mathcal{B} = (v_1, v_2) = ((1,2),(1,3))$

$\mathcal{B}' = (v_1', v_2') = ((1,0),(2,1))$

$f, g \in End_{\mathbb{R}}(\mathbb{R}^2)$ with $[f]_{\mathcal{B}} = \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}$

$[g]_{\mathcal{B}'} = \begin{pmatrix} -7 & -13 \\ 5 & 7 \end{pmatrix}$. Determine $[2f]_{\mathcal{B}}$,

$[f+g]_{\mathcal{B}}$ and $[f \circ g]_{\mathcal{B}'}$

$$[2f]_{\mathcal{B}} = 2[f]_{\mathcal{B}} = 2\begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ -2 & -2 \end{pmatrix}$$

$$[f+g]_{\mathcal{B}} = [f]_{\mathcal{B}} + [g]_{\mathcal{B}}$$

$$[g]_{\mathcal{B}} = [id]_{\mathcal{B}',\mathcal{B}} \cdot [g]_{\mathcal{B}'} \cdot [id]_{\mathcal{B},\mathcal{B}'}$$

$$[id]_{\mathcal{B},\mathcal{B}'} = \left([v_1]_{\mathcal{B}'} \; [v_2]_{\mathcal{B}'} \right)$$

$v_1 = \alpha v_1' + \beta v_2' \iff$

$$\Leftrightarrow (1,2) = \alpha(1,0) + \beta(2,1) \Leftrightarrow$$

$$\Leftrightarrow (1,2) = (\alpha,0) + (2\beta,\beta) \Leftrightarrow$$

$$\Leftrightarrow (1,2) = (\alpha+2\beta, \beta)$$

$$\Rightarrow \begin{cases} \alpha + 2\beta = 1 \Leftrightarrow \alpha = -3 \\ \beta = 2 \end{cases} \Rightarrow [v_1]_{\beta'} = \begin{pmatrix} -3 \\ 2 \end{pmatrix}$$

$$v_2 = \alpha v_1' + \beta v_2' \Leftrightarrow (1,3) = \alpha(1,0) + \beta(2,1) \Leftrightarrow$$

$$\Leftrightarrow (1,3) = (\alpha,0) + (2\beta,\beta) \Leftrightarrow$$

$$\Leftrightarrow (1,3) = (\alpha+2\beta, \beta) \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \alpha + 2\beta = 1 \Rightarrow \alpha = -5 \\ \beta = 3 \end{cases} \Rightarrow [v_2]_{\beta'} = \begin{pmatrix} -5 \\ 3 \end{pmatrix}$$

$$[id]_{\beta,\beta'} = \begin{pmatrix} -3 & -5 \\ 2 & 3 \end{pmatrix} \qquad \det([id]_{\beta,\beta'}) = 1 \neq 0$$

$$[id]_{\beta',\beta} = \left([id]_{\beta,\beta'}\right)^{-1} = \begin{pmatrix} 3 & 5 \\ -2 & -3 \end{pmatrix}$$

$$[id]^{t}_{\beta',\beta} = \begin{pmatrix} -3 & 2 \\ -5 & 3 \end{pmatrix}$$

$$[g]_B = \begin{pmatrix} 3 & 5 \\ -2 & -3 \end{pmatrix}\begin{pmatrix} -7 & -13 \\ 5 & 7 \end{pmatrix}\begin{pmatrix} -3 & -5 \\ 2 & 3 \end{pmatrix} =$$

$$= \begin{pmatrix} 4 & -4 \\ -1 & 5 \end{pmatrix}\begin{pmatrix} -3 & -5 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} -20 & -32 \\ 13 & 20 \end{pmatrix}$$

$$[f+g]_B = [f]_B + [g]_B = \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix} + \begin{pmatrix} -20 & -32 \\ 13 & 20 \end{pmatrix}$$

$$= \begin{pmatrix} -19 & -30 \\ 12 & 19 \end{pmatrix}$$

$$[f \circ g]_{B'} = [f]_{B'} \cdot [g]_{B'}$$

$$[f]_{B'} = [id]_{B,B'} \cdot [f]_B \cdot [id]_{B',B}$$

$$[f]_{B'} = \begin{pmatrix} -3 & -5 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix} \cdot \begin{pmatrix} 3 & 5 \\ -2 & -3 \end{pmatrix}$$

$$[f]_{B'} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 3 & 5 \\ -2 & -3 \end{pmatrix} = \begin{pmatrix} 8 & 13 \\ -5 & 8 \end{pmatrix}$$

$$[f \circ g]_{\beta'} = \begin{pmatrix} 8 & 13 \\ -5 & -8 \end{pmatrix} \cdot \begin{pmatrix} -7 & -13 \\ 5 & 7 \end{pmatrix} =$$

$$= \begin{pmatrix} 9 & -13 \\ -5 & 9 \end{pmatrix}$$

Say we have $f: V \to V$ linear map.
How do we find eigenvalues and eigenvectors?

1. $f \in End_{\mathbb{R}}(\mathbb{R}^2)$, $f(x,y) = (3x+3y, 2x+4y)$

(i) Determine the eigenvalues and the eigenvectors of f

(ii) Write a basis of $\mathbb{R}^2$ consisting of eigenvectors of f.

Step 1: Write the matrix of f in a basis (if convenient, chose the canonical basis)

$$[f]_E = \left([f(e_1)]_E \ [f(e_2)]_E \right)$$

$$\left.\begin{array}{l} f(e_1) = f(1,0) = (3,2) \\ f(e_2) = f(0,1) = (3,4) \end{array}\right\} \Rightarrow [f]_E = \begin{pmatrix} 3 & 3 \\ 2 & 4 \end{pmatrix} = A$$

Step 2: Write the characteristic poly. of A

$$P_A(X) = \det(A - X I_n)$$

$$P_A(X) = \begin{vmatrix} 3-X & 3 \\ 2 & 4-X \end{vmatrix} = (3-X)(4-X) - 6 =$$

$$= 12 - 4X - 3X + X^2 - 6 = X^2 - 7X + 6$$

Step 3: the eigenvalues of f are the roots of $P_A(X)$

$$\lambda_{1,2} = \frac{7 \pm \sqrt{49-24}}{2} = \frac{7 \pm \sqrt{25}}{2} = \frac{7 \pm 5}{2}$$

$$\Rightarrow \lambda_1 = 6, \ \lambda_2 = 1$$

Step 4: For each eigenvalue λ we determine the eigenspace (the space of eigenvectors along with 0):

$$S(\lambda) = \{ v \in V \mid [f]_E \cdot [v]_E = \lambda \cdot [v]_E \}$$

If $\lambda = \lambda_1 = 6$

$$\begin{pmatrix} 3 & 3 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 6 \cdot \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{cases} 3x + 3y = 6x \\ 2x + 4y = 6y \end{cases}$$

$$\Rightarrow \begin{cases} 3y = 3x \\ 2x = 2y \end{cases} \Leftrightarrow y = x$$

$$\Rightarrow S(\lambda_1) = \{ (x, x) \mid x \in \mathbb{R} \} = \langle (1,1) \rangle$$

If $\lambda = \lambda_2 = 1$

$$\begin{pmatrix} 3 & 3 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{cases} 3x + 3y = x \\ 2x + 4y = y \end{cases} \Leftrightarrow \begin{cases} 2x = -3y \\ 2x = -3y \end{cases} \Rightarrow y = -\frac{2x}{3}$$

$$\Rightarrow S(\lambda_2) = \left\{ \left(x, -\frac{2x}{3}\right) \,\middle|\, x \in \mathbb{R} \right\} = \left\langle \left(1, -\frac{2}{3}\right) \right\rangle$$
$$= \langle (3,-2) \rangle$$

(ii) To get a basis of eigenvectors we put together the basis of the eigenspaces.

$$\Rightarrow \mathcal{B} = \big((1,1), (3,-2) \big)$$

5. Compute the eigenvalues and the eigenvectors for:

$$A = \begin{pmatrix} 3 & 1 & 0 \\ -4 & -1 & 0 \\ -4 & -8 & -2 \end{pmatrix}$$

$$P_A(x) = \begin{vmatrix} 3-x & 1 & 0 \\ -4 & -1-x & 0 \\ -4 & -8 & -2-x \end{vmatrix} = 0$$

$$\Leftrightarrow (3-x)(-1-x)(-2-x) + 4(-2-x) = 0$$

$$\Leftrightarrow (-2-x)\left[(3-x)(-1-x)+4\right] = 0$$

$$\Rightarrow \lambda_1 = -2$$

$(3-x)(-1-x)+4=0$

$-3-3x+x+x^2+4=0$

$x^2-2x+1=0 \Rightarrow \lambda_2=\lambda_3=1$

If $\lambda=\lambda_1=-2$

$$\begin{pmatrix} 3 & 1 & 0 \\ -4 & -1 & 0 \\ -4 & -8 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = (-2)\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{cases} 3x+y=-2x \\ -4x-y=-2y \\ -4x-8y-2z=-2z \end{cases} \Leftrightarrow \begin{cases} y=-5x \\ -4x+5x=10x \\ -4x+40x=0 \end{cases} \Leftrightarrow$$

$\Leftrightarrow \quad y=-5x$

$x=10x \Rightarrow x=0 \Rightarrow y=0$

$\Rightarrow S(\lambda_1)=\{(0,0,z) \mid z\in\mathbb{R}\} = \langle(0,0,1)\rangle$

If $\lambda=\lambda_2=1$

$$\begin{pmatrix} 3 & 1 & 0 \\ -4 & -1 & 0 \\ -4 & -8 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 1\cdot\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{cases} 3x+y=x \\ -4x-y=y \\ -4x-8y-2z=z \end{cases} \Leftrightarrow \begin{cases} 2x+y=0 \\ -4x-2y=0 \\ -4x-8y-3z=0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} y=-2x \\ -4x+4x=0 \\ -4x+16x-3z=0 \end{cases} \Leftrightarrow \begin{cases} y=-2x \\ 0=0 \\ 12x-3z=0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} y=-2x \\ z=4x \end{cases}$$

$$S(\lambda_2)=\{(x,-2x,4x) \mid x\in\mathbb{R}\}=\langle(1,-2,4)\rangle$$