1. c) x^2-y^2 subject to $x^2+y^2=1$

$f(x,y)=x^2-y^2$

$g(x,y)=x^2+y^2$

$c=1$

$$L(x,y,\lambda)=f(x,y)+\lambda(g(x,y)-c)=$$

$$=x^2-y^2+\lambda(x^2+y^2-1)$$

$$\frac{\partial L}{\partial x}=2x+2x\lambda$$

$$\frac{\partial L}{\partial y}=-2y+2y\lambda$$

$$\frac{\partial L}{\partial \lambda}=x^2+y^2-1$$

$$\begin{cases} 2x+2x\lambda=0 \Rightarrow 2x(1+\lambda)=0 \Rightarrow \mathrm{I}_1\ x=0 \quad \mathrm{I}_2\ \lambda=-1 \\ -2y+2y\lambda=0 \Rightarrow 2y(-1+\lambda)=0 \Rightarrow \mathrm{II}_1\ y=0 \quad \mathrm{II}_2\ \lambda=1 \\ x^2+y^2-1=0 \end{cases}$$

I_1 $x=0 \Rightarrow y^2-1=0 \Rightarrow y=\pm 1$

I_2 $\lambda=-1 \Rightarrow -2y+2y\cdot(-1)=0 \Rightarrow -4y=0 \Rightarrow y=0 \Rightarrow x=\pm 1$

II_1 $y=0 \Rightarrow x^2-1=0 \Rightarrow x=\pm 1$

II_2 $\lambda=1 \Rightarrow 2x+2x\cdot 1=0 \Rightarrow 4x=0 \Rightarrow x=0 \Rightarrow y=\pm 1$

$I_1 = II_2$ and $I_2 = II_1$

$f(x,y) = f(0,1) = f(0,-1) = 1$
$f(x,y) = f(1,0) = f(-1,0) = 1$ } local extremum of f

f) $x^3+y^3+z^3$ subject to $x^2+y^2+z^2=1$

$f(x,y,z) = x^3+y^3+z^3$

$g(x,y,z) = x^2+y^2+z^2$

$c=1$

$$L(x,y,z,\lambda) = f(x,y,z) + \lambda(g(x,y,z)-c) =$$
$$= x^3+y^3+z^3 + \lambda(x^2+y^2+z^2-1)$$

$$\frac{\partial L}{\partial x} = 3x^2 + 2x\lambda$$

$$\frac{\partial L}{\partial y} = 3y^2 + 2y\lambda$$

$$\frac{\partial L}{\partial z} = 3z^2 + 2z\lambda$$

$$\frac{\partial L}{\partial \lambda} = x^2+y^2+z^2-1$$

$$\begin{cases} 3x^2+2x\lambda=0 \Rightarrow 2x\lambda=-3x^2 \Rightarrow \lambda=-\dfrac{3x^2}{2x}=-\dfrac{3x}{2} \Rightarrow \\ 3y^2+2y\lambda=0 \qquad \Rightarrow -3x=2\lambda \Rightarrow x=-\dfrac{2\lambda}{3} \\ 3z^2+2z\lambda=0 \qquad \Rightarrow y=-\dfrac{2\lambda}{3},\ z=-\dfrac{2\lambda}{3} \\ x^2+y^2+z^2-1=0 \Rightarrow \left(-\dfrac{2\lambda}{3}\right)^2+\left(-\dfrac{2\lambda}{3}\right)^2+\left(-\dfrac{2\lambda}{3}\right)^2-1=0 \Rightarrow \end{cases}$$

$$\Rightarrow \frac{4\lambda^2+4\lambda^2+4\lambda^2}{9}-1=0 \Rightarrow 12\lambda^2=9 \quad |:3 \Rightarrow$$

$$\Rightarrow 4\lambda^2=3 \Rightarrow \lambda^2=\frac{3}{4} \Rightarrow \lambda=\sqrt{\frac{3}{4}}=\frac{\sqrt{3}}{2}$$

$$3x^2+\cancel{2}x\cdot\frac{\sqrt{3}}{\cancel{2}}=0 \Rightarrow 3x^2+\sqrt{3}x=0 \Rightarrow 3x^2=-\sqrt{3}x \Rightarrow$$

$$\Rightarrow 3x=-\sqrt{3} \Rightarrow x=-\frac{\sqrt{3}}{3}$$

$$3y^2+2y\lambda=0 \Rightarrow 3y^2+\sqrt{3}y=0 \Rightarrow 3y^2=-\sqrt{3}y \Rightarrow$$

$$\Rightarrow y=-\frac{\sqrt{3}}{3}$$

$$3z^2+2z\lambda=0 \Rightarrow z=-\frac{\sqrt{3}}{3}$$

$$f(x,y,z)=f\left(-\frac{\sqrt{3}}{3},-\frac{\sqrt{3}}{3},-\frac{\sqrt{3}}{3}\right)=\left(-\frac{\sqrt{3}}{3}\right)^3+\left(-\frac{\sqrt{3}}{3}\right)^3+\left(-\frac{\sqrt{3}}{3}\right)^3=$$

$$=-\frac{\sqrt{3}}{9}-\frac{\sqrt{3}}{9}-\frac{\sqrt{3}}{9}=-\frac{\cancel{3}\sqrt{3}}{\cancel{9}}=-\frac{\sqrt{3}}{3} \Rightarrow \text{local extremum of } f$$