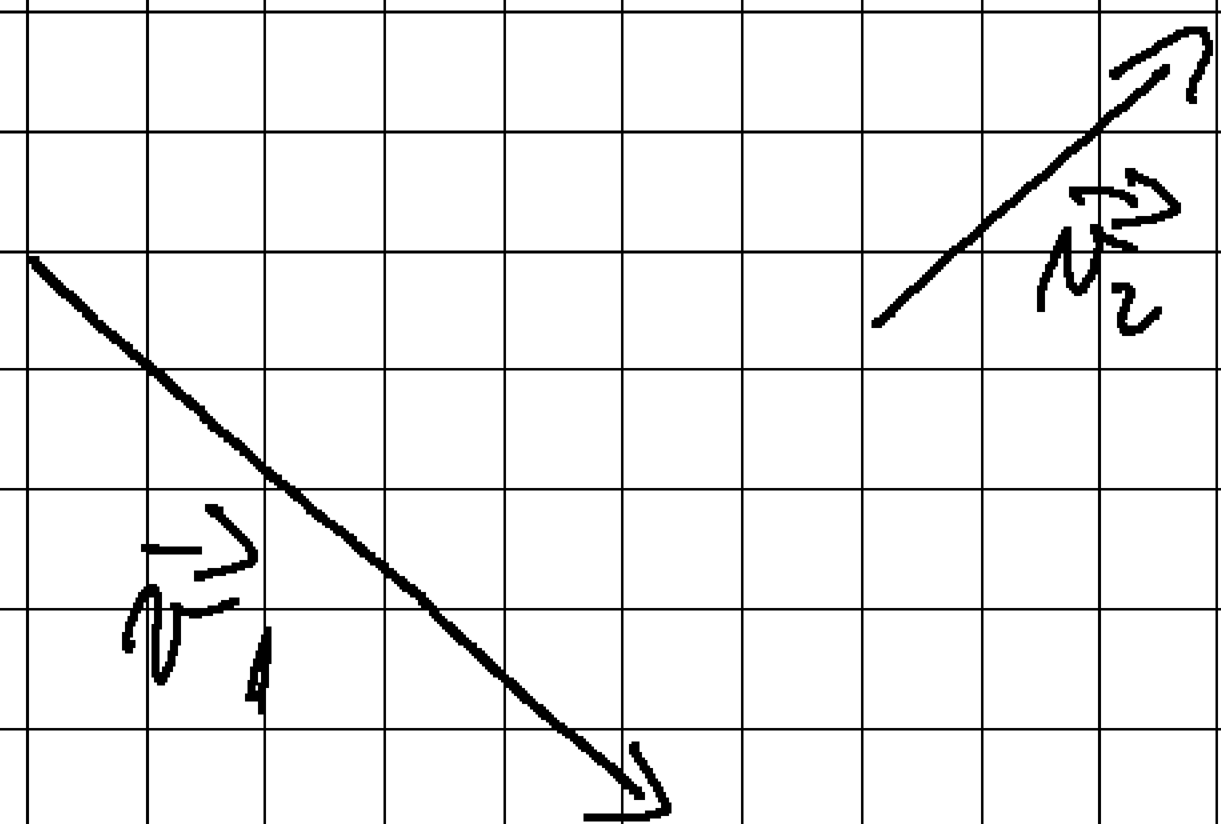
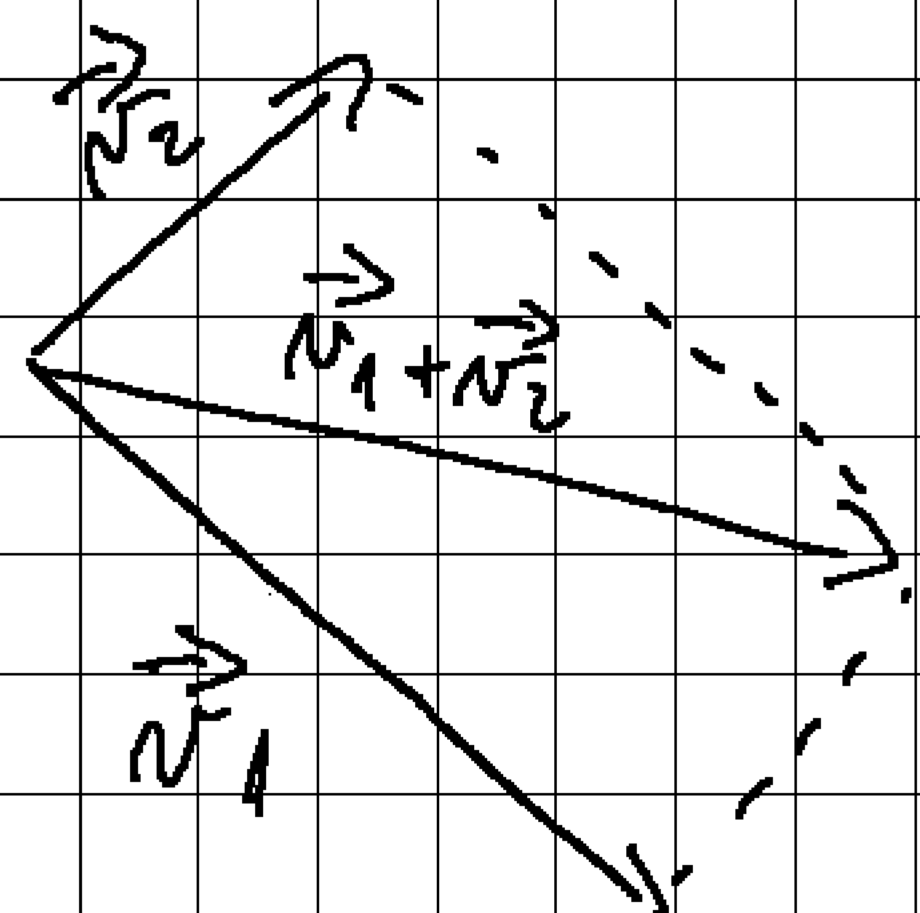
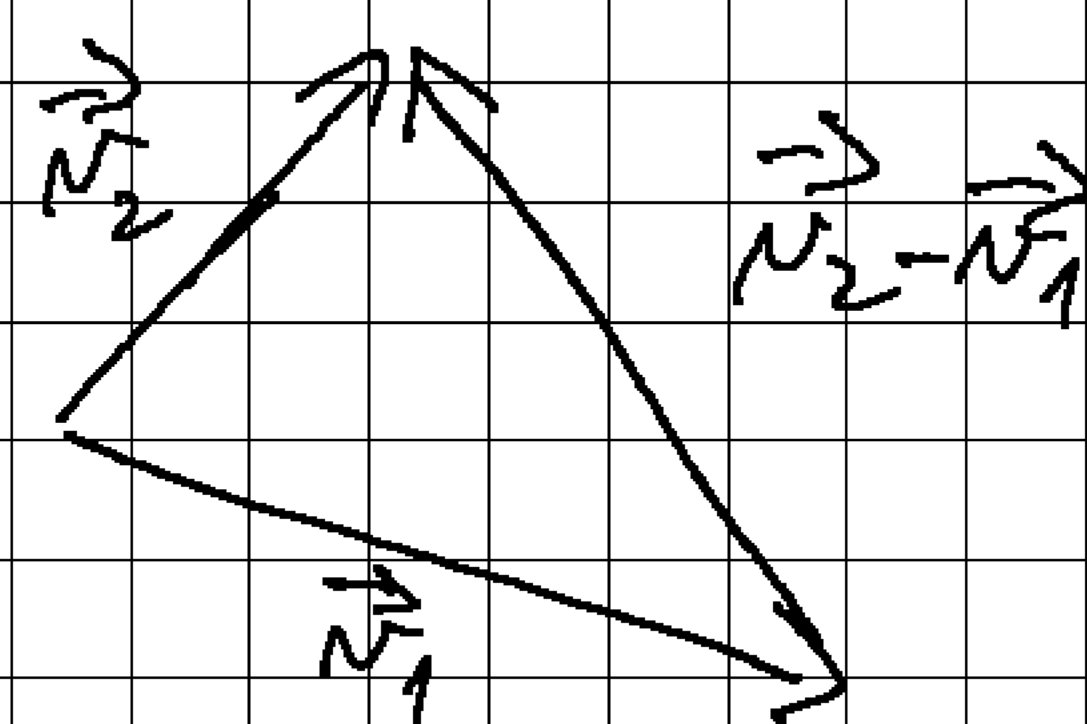
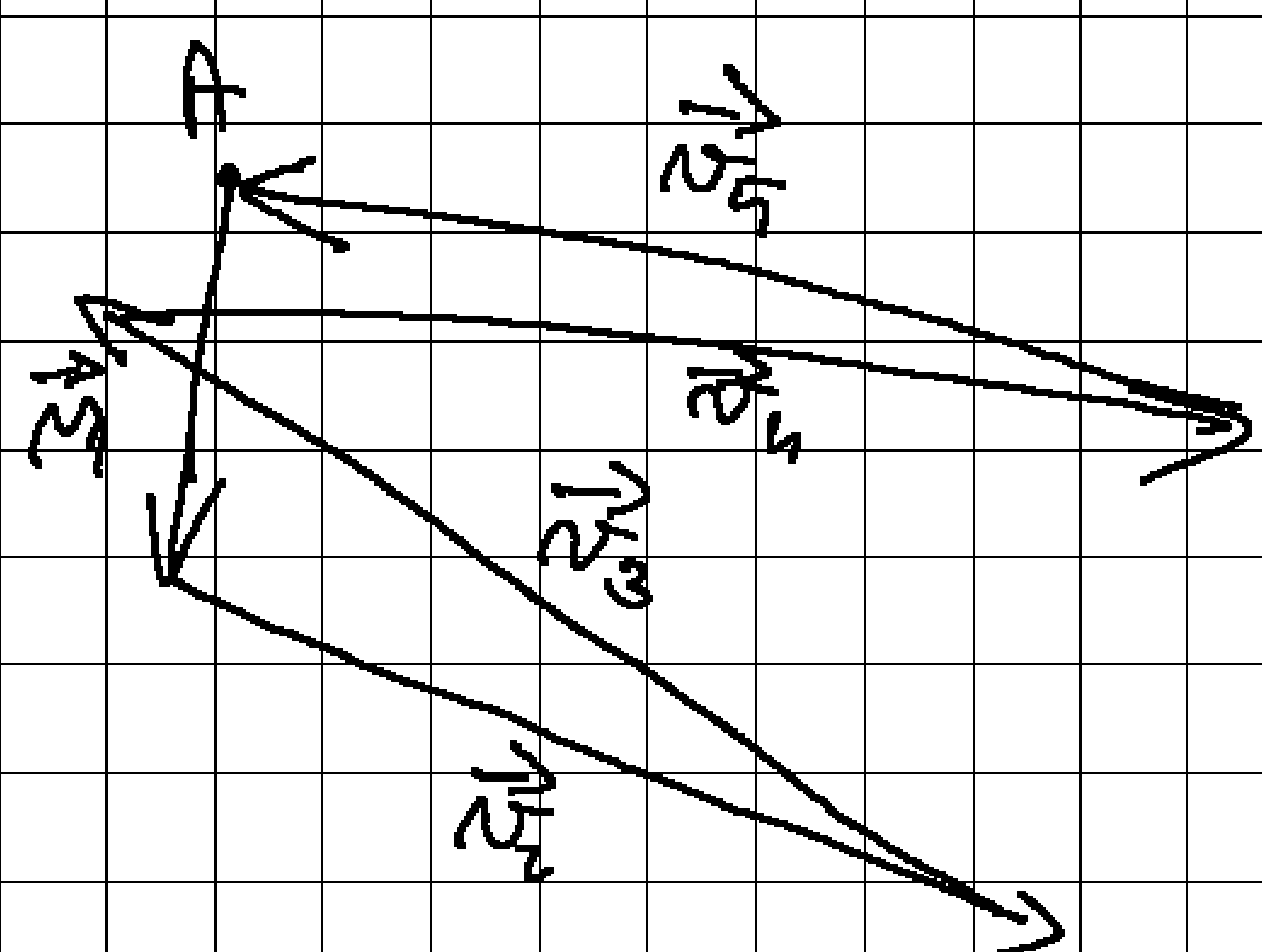
E^n Euclidean Space usually $n=2$ or $n=3$

V^n the vector space of vectors in E^n









$\vec{v}_1+\vec{v}_2+\vec{v}_3+\vec{v}_4+\vec{v}_5=\vec{0}$ (we start and end at A)

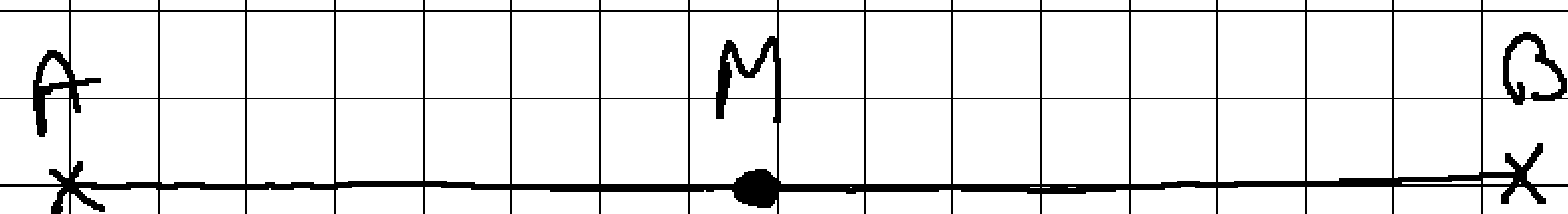
Fix $O \in E^n$

We denote $\forall A \in E^n$:

$$\vec{OA} = \vec{r}_A$$

the position vector of A with respect to O

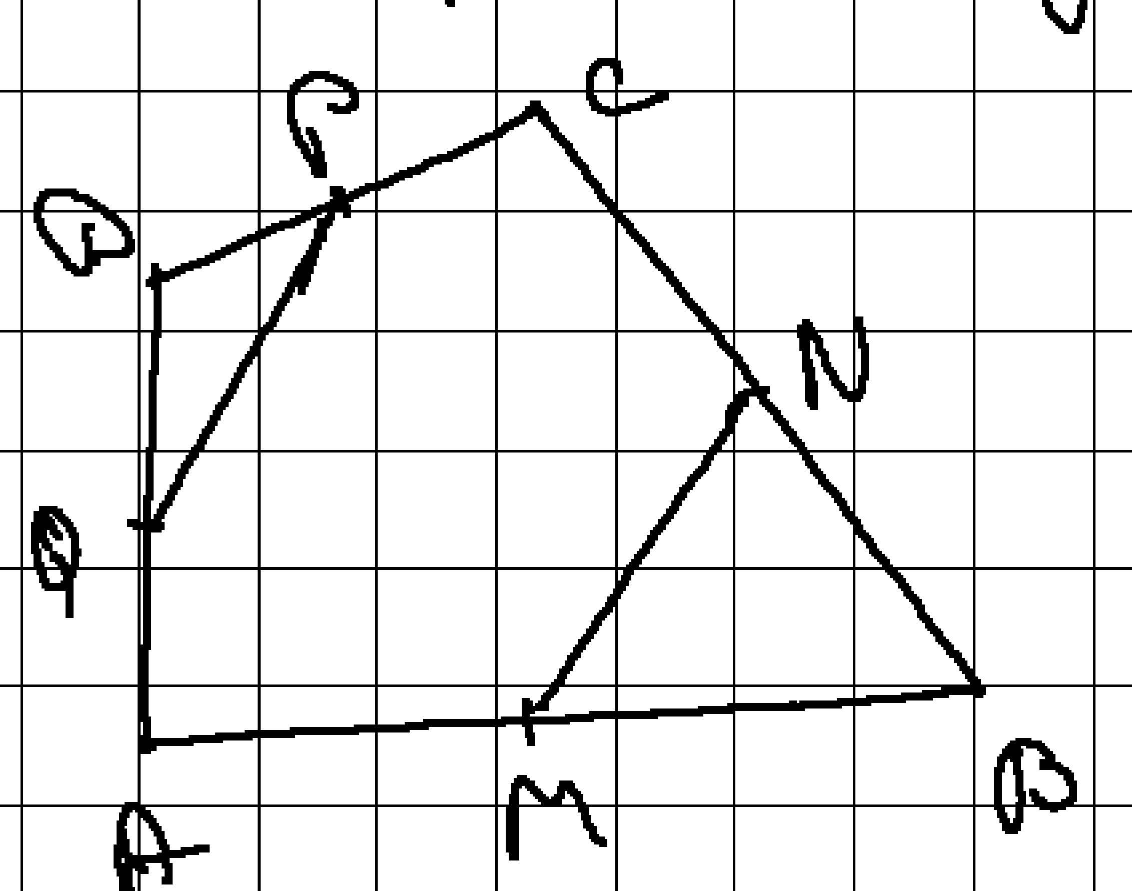
$A, B \in E^n$; $\vec{AB} = \vec{r}_B - \vec{r}_A$



M midpoint of $[AB]$

$$\vec{r}_M = \frac{\vec{r}_A + \vec{r}_B}{2}$$

Ex1: $ABCD$ quadrilateral, M, N, P, Q midpoints of $[AB]$, $[BC]$, $[CD]$, $[DA]$. Show that $\vec{MN} + \vec{PQ} = \vec{O}$. Deduce that the midpoints of the sides of $ABCD$ form a parallelogram.



$$\vec{MN} = \vec{r}_N - \vec{r}_M$$

$$\vec{PQ} = \vec{r}_Q - \vec{r}_P$$

$$\vec{r}_M = \frac{\vec{r}_A + \vec{r}_B}{2} \qquad \vec{r}_P = \frac{\vec{r}_D + \vec{r}_C}{2}$$

$$\vec{r}_N = \frac{\vec{r}_C + \vec{r}_B}{2} \qquad \vec{r}_Q = \frac{\vec{r}_A + \vec{r}_D}{2}$$

$$\Rightarrow \vec{MN} + \vec{PQ} = \frac{\vec{r}_B + \vec{r}_C}{2} - \frac{\vec{r}_A + \vec{r}_B}{2} + \frac{\vec{r}_A + \vec{r}_D}{2} - \frac{\vec{r}_B + \vec{r}_C}{2} =$$

$$= \frac{\vec{r}_B + \vec{r}_C - \vec{r}_A - \vec{r}_B + \vec{r}_A + \vec{r}_D - \vec{r}_B - \vec{r}_C}{2} = 0$$

b) $$\left.\begin{array}{l} \vec{MN} = -\vec{PQ} = \vec{QP} \\ \vec{MN} = \vec{QP} \Rightarrow \text{so the vectors have the same direction} \\ MN \parallel QP \end{array}\right\} \Rightarrow MNQP \text{ parallelogram}$$

$$MNPQ \text{ - parallelogram} \Leftrightarrow \vec{r}_N + \vec{r}_Q = \vec{r}_M + \vec{r}_P$$

$$\Leftrightarrow \vec{r}_M - \vec{r}_N = \vec{r}_Q - \vec{r}_P$$

$$\Leftrightarrow \vec{NM} = \vec{PQ}$$

$$\vec{v} \parallel \vec{w} \Leftrightarrow \exists\, \alpha \in \mathbb{R} : \vec{v} = \alpha \vec{w}$$

1.8. (Ex. a reference system) Decide if the given points are collinear:

a) $P(3,-5)$, $Q(-1,2)$, $R(-5,9)$

c) $A(1,0,-1)$, $Q(0,-1,2)$, $R(-1,-2,5)$

b) $A(11,2)$, $B(1,-3)$, $C(31,13)$

d) $A(-1,-1,-4)$, $B(1,1,0)$, $C(2,2,2)$

A,B,C collinear $\Leftrightarrow$ rank $(\vec{AB},\vec{BC},\vec{CA})=1$ $\Leftrightarrow$

$\Leftrightarrow \exists \alpha \in \mathbb{R}: \vec{AB} = \alpha \vec{BC}$

a) $\vec{PQ} = \vec{r}_Q - \vec{r}_P = (-1,2)-(3,-5) = (-4,7)$

$\vec{QR} = \vec{r}_R - \vec{r}_Q = (-5,9)-(-1,2) = (-4,7)$

$\Rightarrow \vec{PQ} = \vec{QR} \Rightarrow P,Q,R$ coll.

b) $\vec{AB} = \vec{r}_B - \vec{r}_A = (1,-3)-(11,2) = (-10,-5)$

$\vec{BC} = \vec{r}_C - \vec{r}_B = (31,13)-(1,-3) = (30,16)$

$\frac{30}{-10} \neq \frac{16}{-5} \Rightarrow \vec{AB},\vec{BC}$ are not parallel

$\Rightarrow A,B,C$ not collinear

c) $\vec{PQ} = \vec{r}_Q - \vec{r}_P = (0,-1,2) - (1,0,-1) = (-1,-1,3)$

$\vec{QR} = \vec{r}_R - \vec{r}_Q = (-1,-2,5) - (0,-1,2) = (-1,-1,3)$

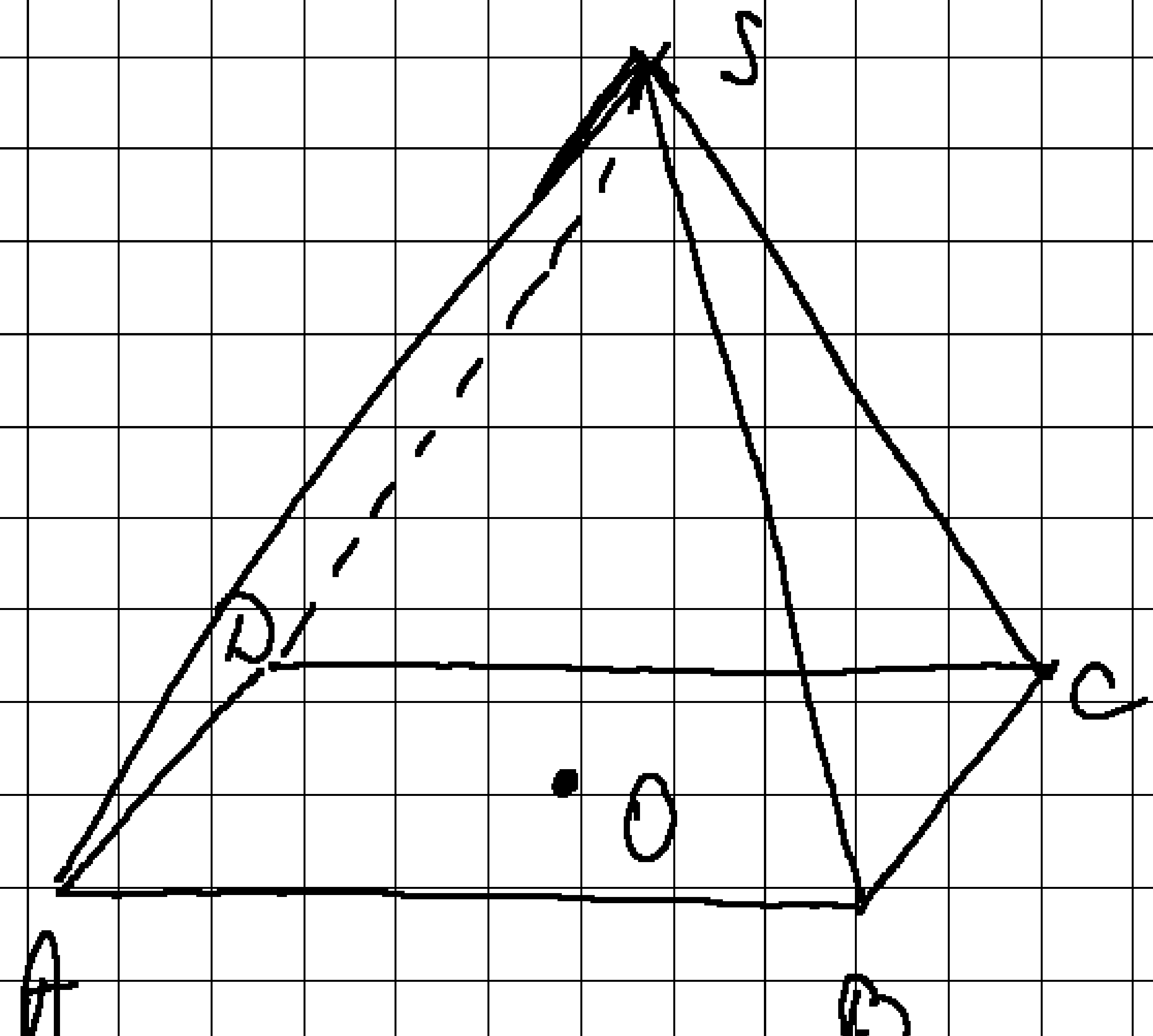
$\Rightarrow \vec{PQ} = \vec{QR} \Rightarrow$ P, Q, R are collinear

d) $\vec{AB} = \vec{r}_B - \vec{r}_A = (1,1,0) - (-1,-1,-4) = (2,2,4)$

$\vec{BC} = \vec{r}_C - \vec{r}_B = (2,2,2) - (1,1,0) = (1,1,2)$

$\frac{2}{1} = \frac{2}{1} = \frac{4}{2} = 2 \Rightarrow \vec{AB} \parallel \vec{BC} \Rightarrow$ A, B, C collinear

1.11. SABCD is a pyramid with apex S and base the parallelogram ABCD with centre O.
Show that $\vec{SA} + \vec{SB} + \vec{SC} + \vec{SD} = 4\vec{SO}$



$\vec{SA} + \vec{SB} + \vec{SC} + \vec{SD} =$

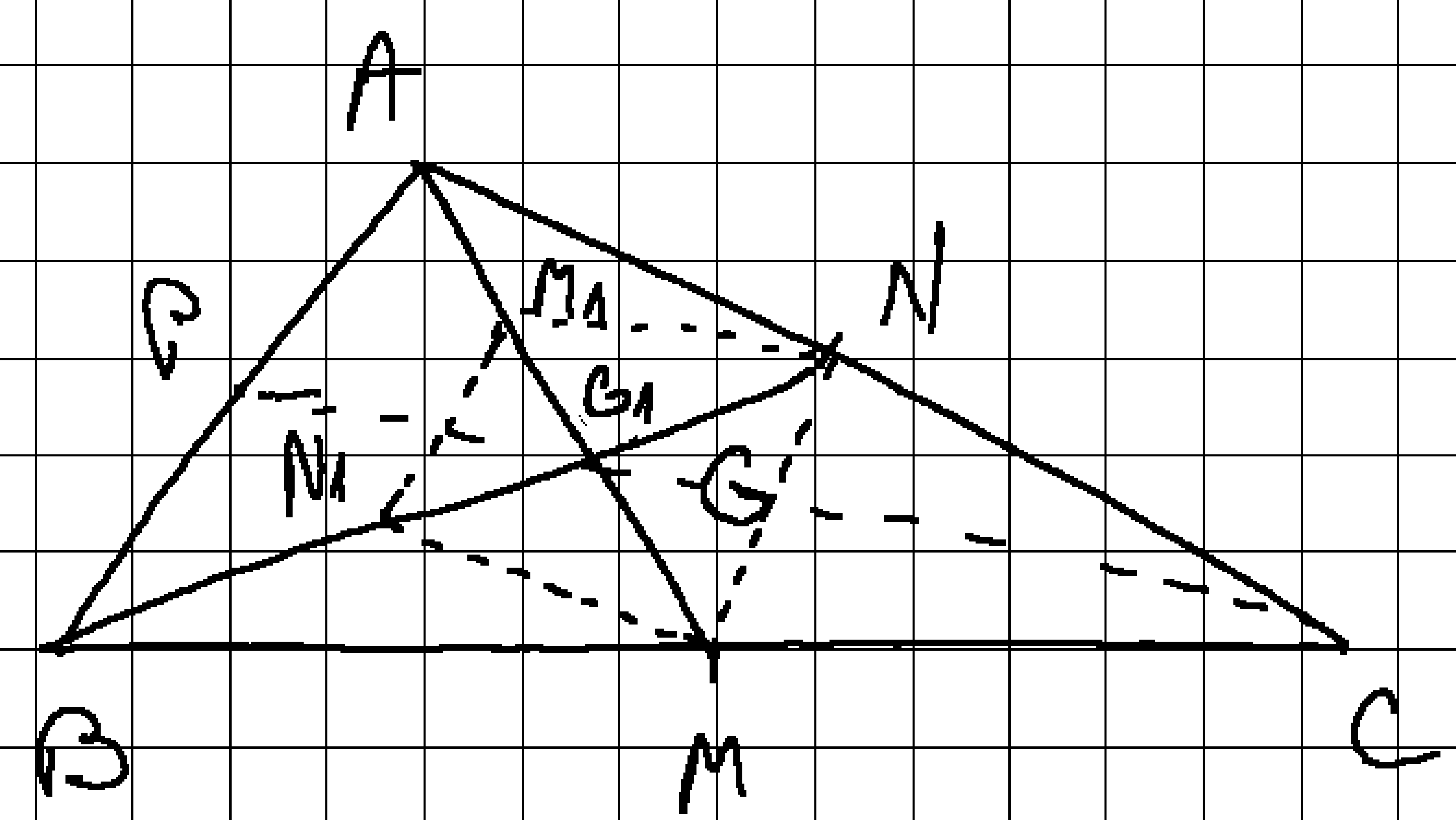
$= \vec{SO} + \vec{OA} + \vec{SO} + \vec{OB} + \vec{SO} + \vec{OC}$
$+ \vec{SO} + \vec{OD} = 4\vec{SO} + \vec{OA} + \vec{OB}$
$+ \vec{OC} + \vec{OD}$

ABCD - parallelogram $\Rightarrow \vec{OA} = \vec{CO} \Rightarrow \vec{OA} + \vec{OC} = \vec{0}$

$\vec{OB} = \vec{DO} \Rightarrow \vec{OB} + \vec{OD} = \vec{0}$

$\Rightarrow \vec{SA} + \vec{SB} + \vec{SC} + \vec{SD} = 4\vec{SO}$

1.7. Show that the medians of a triangle intersect in one point and deduce the ratio which the common intersection point devides the medians.



Let $\{G\} = AM \cap BN$

We must show that C, G, P are collinear.

$A, B \in E^n$

$T \in AB \Leftrightarrow \vec{r}_T = \alpha \vec{r}_A + (1-\alpha)\vec{r}_B$ for $\alpha \in \mathbb{R}$

$T \in [AB] \Leftrightarrow \vec{r}_T = \alpha \vec{r}_A + (1-\alpha)\vec{r}_B$, $\alpha \in [0,1]$

Let M_1 be the midpoint of AG and N_1 the midpoint of BG

MN midline in $\Delta ABC \Rightarrow MN \parallel AB$

$MN = \frac{AB}{2}$

M_1N_1 midline in $AGB \Rightarrow M_1N_1 \parallel AB$

$M_1N_1 = \frac{1}{2} AB$

parallel & equal

$\Rightarrow M_1N_1 \overset{\parallel}{=} MN$

$\Rightarrow MNM_1N_1$ parallelogram $\Rightarrow G$ midpoint of MM_1 and NN_1

Say $CP \cap AM = \{G_1\}$

by the same argument G_1 is the midpoint of $[MM_1]$

$\Rightarrow G = G_1$

$\Rightarrow AM \cap BN \cap CP = \{G\}$

G midpoint of MM_1

M_1 midpoint of $AG \Rightarrow \frac{GM}{AG} = \frac{1}{2}$

II method:

$AM \cap BN = \{G\}$

$G \in AM \Rightarrow \vec{r_G} = \alpha \vec{r_A} + (1-\alpha)\vec{r_M} =$

$= \alpha \vec{r_A} + \frac{1-\alpha}{2} \vec{r_B} + \frac{1-\alpha}{2} \vec{r_C}$

$G \in BN \Rightarrow \vec{r_G} = \beta \vec{r_B} + (1-\beta)\vec{r_N} =$

$= \beta \vec{r_B} + \frac{1-\beta}{2} \vec{r_A} + \frac{1-\beta}{2} \vec{r_C}$

$$\Rightarrow \left(\alpha - \frac{1-\beta}{2}\right)\vec{r}_A + \left(\frac{1-\beta}{2} - \beta\right)\vec{r}_B + \left(\frac{1-\alpha}{2} - \frac{1-\beta}{2}\right)\vec{r}_C = \vec{0}$$

$\vec{AB} = \vec{v}$, $\vec{AC} = \vec{w}$ are lin. indep.

$$\vec{r}_B = \vec{v} + \vec{r}_A$$
$$\vec{r}_C = \vec{w} + \vec{r}_A$$

$$\Rightarrow \left(\alpha - \frac{1-\beta}{2}\right)\vec{r}_A + \left(\frac{1-\alpha}{2} - \beta\right)(\vec{v} + \vec{r}_A) + \left(\frac{\beta-\alpha}{2}\right)(\vec{w} + \vec{r}_A) = \vec{0}$$

$$\Rightarrow \underbrace{\left(\alpha - \frac{1-\beta}{2} + \frac{1-\alpha}{2} - \beta + \frac{\beta-\alpha}{2}\right)}_{=0}\vec{r}_A +$$

$$+ \left(\frac{1-\alpha}{2} - \beta\right)\vec{v} + \left(\frac{\beta-\alpha}{2}\right)\cdot\vec{w} = \vec{0}$$

$$\Rightarrow \left(\frac{1-\alpha}{2} - \beta\right)\vec{v} + \frac{\beta-\alpha}{2}\vec{w} = \vec{0}$$

$\vec{v}, \vec{w}$ lin. indep. $\Rightarrow \begin{cases} \beta = \dfrac{1-\alpha}{2} \\ \dfrac{\beta-\alpha}{2} = 0 \end{cases}$

$$\Rightarrow \alpha = \beta$$

$$\alpha = \frac{1-\alpha}{2} \Rightarrow 1-\alpha = 2\alpha \Rightarrow \alpha = \frac{1}{3}$$

$$\Rightarrow \vec{r_G} = \frac{1}{3}\cdot\vec{r_A} + \frac{1-\frac{1}{3}}{2}\vec{r_B} + \frac{1-\frac{1}{3}}{2}\vec{r_C} =$$

$$= \frac{1}{3}\left(\vec{r_A}+\vec{r_B}+\vec{r_C}\right)$$

$$\vec{GP} = \vec{r_P} - \vec{r_G} = \frac{1}{2}\vec{r_A} + \frac{1}{2}\vec{r_B} - \frac{1}{3}\vec{r_A} - \frac{1}{3}\vec{r_B} - \frac{1}{3}\vec{r_C} =$$

$$= \frac{1}{6}\vec{r_A} + \frac{1}{6}\vec{r_B} - \frac{1}{3}\vec{r_C}$$

$$\vec{CP} = \vec{r_P} - \vec{r_C} = \frac{1}{2}\vec{r_A} + \frac{1}{2}r_B - \vec{r_C} = \frac{1}{3}\cdot\vec{GP}$$

$\Rightarrow C, G, P$ - coll.