4.7 Using the given interpretation evaluate the follow formulas:

$$U = (\exists x) A(x) \wedge (\exists x) B(x) \rightarrow (\forall x)(A(x) \wedge B(x))$$

interpretation $i = \langle D, m \rangle$ where

$D = N$ (the set of natural numbers)

$m(A) : D \rightarrow \{T, F\}$, $m(A)(x)$: "x is a perfect square"

$m(B) : D \rightarrow \{T, F\}$, $m(B)(x)$: "x is divisible by 10"

$$V^I(U) = V^I((\exists x) A(x) \wedge (\exists x) Bx \rightarrow (\forall x)(A(x) \wedge B(x))) =$$

$$= V^I((\exists x) A(x) \wedge (\exists x) B(x)) \rightarrow V^I((\forall x)(A(x) \wedge B(x))) =$$

$$= V^I((\exists x) A(x)) \wedge V^I((\exists x) B(x)) \rightarrow V^I((\forall x)(A(x) \wedge B(x)))$$

$$= (\exists x)_{x \in N} \text{"x perfect sq"} \wedge (\exists x)_{x \in N} \text{"} x \vdots 10 \text{"} \rightarrow$$

$$\rightarrow (\forall x)_{x \in N} (\text{"x perfect sq"} \wedge \text{"} x \vdots 10 \text{"})$$

$$= T \wedge T \rightarrow F = T \rightarrow F = F$$

i is an anti model of U

5. Chose an arbritrary interpretation i and evaluate U and prove that it is a model

$$U = ((\exists x) A(x) \to (\forall x) B(x)) \to (\forall x)(A(x) \to B(x))$$

$i = \langle D, m \rangle$, $D = \{6, 4\}$,

$m(A(x)) =$ "$x \vdots 3$", $m(B(x)) =$ "x is an even number"

$(\forall x)\, U(x) \equiv U(a) \wedge U(b) \quad x \in \{a, b\}$

$(\exists x)\, U(x) \equiv U(a) \vee U(b) \quad x \in \{a, b\}$

$$v^I(U) = [v^I((\exists x) A(x)) \to v^I((\forall x) B(x))] \to v^I((\forall x)(A(x) \to B(x)))$$

$$= [(A(6) \vee A(4)) \to (B(6) \wedge B(4))] \to ((A(6) \to B(6)) \wedge$$

$$\wedge (A(4) \to B(4))) =$$

$$= [(\text{"}6 \vdots 3\text{"} \vee \text{"}4 \vdots 3\text{"}) \to (\text{"6 even"} \wedge \text{"4 even"})] \to$$

$$\to [(\text{"}6 \vdots 3\text{"} \to \text{"6 even"}) \wedge (\text{"}4 \vdots 3\text{"} \to \text{"4 even"})] =$$

$$= [(T \vee F) \to (T \wedge T)] \to [(T \to T) \wedge (F \to T)] =$$

$$= (T \to T) \to (T \wedge T) = T \to T = T$$

$\Rightarrow$ i is a model of U

6. Find an anti model for U

$$U_5 = (\forall x)(P(x) \vee Q(x)) \to (\forall x)P(x) \vee (\forall x)Q(x)$$

Let us consider the interpretation $i_2 = \langle D_2, m \rangle$ where $D_2 = \{5, 6\}$

$m(P) : \{5, 6\} \to \{T, F\}$, $m(P)(x) = $ " $x \vdots 5$ "

$m(Q)(x) = $ " $x \vdots 6$ "

$$v^{i_2}(U_5) = v^{i_2}((\forall x)(P(x) \vee Q(x)) \to v^{i_2}((\forall x)P(x)) \vee$$
$$\vee\, v^{i_2}((\forall x)Q(x)) =$$

$$= (5 \vdots 5 \vee 5 \vdots 6) \wedge (6 \vdots 5 \vee 6 \vdots 6) \to (5 \vdots 5 \wedge 6 \vdots 5) \vee (5 \vdots 6 \wedge$$
$$\wedge\, 6 \vdots 6) = (T \vee F) \wedge (F \vee T) \to (T \wedge F) \vee (F \wedge T) =$$

$$= T \wedge T \to F \vee F = T \to F = F$$

i is an anti model of U_5

1.5. Using the semantic tableaux method decide what kind (consistent, inconsistent, valid) of formula is U_j, $j \in \{1, 2, ..., 8\}$

if U_j, $j \in \{1, 2, ..., 8\}$ is consistent, find al it's models

$$U_5 = (r \vee q) \vee (p \Rightarrow \neg r) \to (p \leftrightarrow q)$$

Decomposition rules:

α rules

$A \wedge B$ | A | B

$\neg(A \vee B)$ | $\neg A$ | $\neg B$

$\neg(A \Rightarrow B)$ | A | $\neg B$

β rules:

$A \vee B$: A / B

$A \to B$: $\neg A$ / B

$A \Leftrightarrow B \equiv (A \Rightarrow B) \wedge (B \Rightarrow A)$

Theoretical rules

1. A branch is closed if it contains a pair of opposite literals ⊗

2. U is consistent if it has a complete and open sem. tableaux

at least one open branch

$U_5 = (r \lor q) \lor (p \to \neg r) \to (p \leftrightarrow q)$ (1) ✓

β for (1)

$\neg(r \lor q) \lor (p \to \neg r)$ (2) | $p \leftrightarrow q$ (3)

α for (2) ✓ | α for (3)

$\neg(r \lor q)$ (4) ✓ | $p \to q$ (6) ✓

$\neg(p \to \neg r)$ (5) | $q \to p$ (7)

α for (4) | β for (6)

$\neg r$ | $\neg p$ $\quad$ q

$\neg q$ | β for (7) $\quad$ β for (7)

α for (5) | $\neg q$ $\quad$ p $\quad$ $\neg q$ $\quad$ p

$\neg p$ | $\odot$ $\quad$ $\otimes$ $\quad$ $\otimes$ $\quad$ $\odot$

r $\otimes$

$$DNF(U_5) = (\neg r \land \neg q \land \neg p \land r) \lor (\neg p \land \neg q) \lor$$

$$\lor (\neg p \land p) \lor (q \land \neg q) \lor (q \land p) \equiv (\neg p \land \neg q) \lor (q \land p)$$

with 2 cubes

3. An open branch provides models of U

U_5 has an open sem. tableaux (2 open branches and 3 closed branches

so U_5 is consistent

Cube $\neg p \wedge \neg q$ provid the models $\overset{\equiv T}{}$

$i_1, i_2 : \{p, q, r\} \to \{T, F\}$

$i_{1,2}(p) = F$, $i_{1,2}(q) = F$ $\quad i_1(r) = T$

$i_2(r) = F$

Cube $q \wedge p$ provides the models $\overset{\equiv T}{}$

$i_3, i_4 : \{p, q, r\} \Rightarrow \{T, F\}$

$i_{3,4}(p) = T$, $i_{3,4}(q) = T$ $\quad i_3(r) = T$

$i_4(r) = F$

U_5 has 4 models $i_1(U_5) = i_2(U_5) = i_3(U_5) = i_4(U_5) = T$

2. Check the validity of the formula:

$U = (p \to (q \to r)) \to (p \wedge q \to r)$

4. U is inconsistent if it has a closed sem. tableaux when all the branches are closed.

5. U is valid if $\neg U$ has a closed semantic tableaux

$\neg U = \neg((p \to (q \to r)) \to (p \land q \to r))$ (1) ✓

| α for (1)

$p \to (q \to r)$ (2)

|

$\neg(p \land q \to r)$ (3) ✓

| α for (3)

$p \land q$ (4) ✓

|

$\neg r$

| α for (4)

p

|

q

β for (2)

$\neg p$ ⊗ $q \to r$ (5)

β for (5)

$\neg q$ ⊗ r ⊗

The sem.tab. of $\neg U$ is closed with 3 closed branches $(p, \neg p)$, $(q, \neg q)$, $(r, \neg r)$ so $\neg U$ is inconsistent and U is valid