affine varieties

$a + V$

$a \in \mathbb{E}^n$, $V \subseteq V^n$

dim 1: line

dim 2: plane

Vector equation:

line: $A(x_A, y_A)$

$\vec{v}(x_{\vec{v}}, y_{\vec{v}})$

line l in $\mathbb{E}^2$

(Consider a reference system $K = (O, B)$)

$\forall M \in \mathbb{E}^2 : \exists \lambda \in \mathbb{R} : \underbrace{\vec{OM}}_{=\vec{r}_M} = \underbrace{\vec{OA}}_{=\vec{r}_A} + \lambda \vec{v}$ (vector equation)

$\Leftrightarrow \begin{cases} x = x_A + \lambda x_{\vec{v}} \\ y = y_A + \lambda y_{\vec{v}} \end{cases}$ (param. eqn. of l)

symmetric form:

$\Leftrightarrow \dfrac{x - x_A}{x_{\vec{v}}} = \dfrac{y - y_A}{y_{\vec{v}}}$ if $x_{\vec{v}}, y_{\vec{v}} \neq 0$

if $x_{\vec{v}} = 0$: l : $x = x_A$

implicit form:

$$y_{\vec{v}}(x-x_A)-x_{\vec{v}}(y-y_A)=0$$

Explicit form: $y=\underbrace{m}_{\text{slope}}x+n$

2.1, 2.2

Determine parametric and Cartesian equations for the line $l \subseteq \mathbb{E}^2$ and describe its direction vectors.

a) $l \ni A(1,2)$ and
$l \parallel \vec{a}(3,-1)$

b) $l \ni O$, $l \parallel \vec{b}(4,5)$

c) $l \ni M(1,7)$, $l \parallel Oy$

d) $l \ni M(2,4)$, $N(2,-5)$

a) Parametric equation:

$$\begin{cases} x = 1 + 3\lambda \\ y = 2 - \lambda \end{cases}$$

direction vector

$D(\ell) = \langle (3, -1) \rangle$

Symmetric equation:

$$\frac{x-1}{3} = \frac{y-2}{-1}$$

implicit form:

$$-1(x-1) = 3(y-2)$$

$$-x + 1 - 3y - 6 = 0$$

Explicit form: $3y = 7 - x$

$$y = \frac{7}{3} - \frac{1}{3}x$$

b) Parametric eq:

$$\begin{cases} x = 0 + 4\lambda \\ y = 0 + 5\lambda \end{cases}$$

$D(\ell) = \langle (4, 5) \rangle$

Sym. eq:

$$\frac{x}{4} = \frac{y}{5}$$

imp. form:

$$5x = 4y \Leftrightarrow 5x - 4y = 0$$

Exp. form: $y = \frac{5}{4}x$

c) param. eq

$\vec{v}(0,1)$

$$\begin{cases} x = 1 + 0\lambda \\ y = 7 + \lambda \end{cases}$$

$D(\ell) = \langle (0,1) \rangle$

Sym. form.
imp. form.
exp. form:

$\Bigg\} \Rightarrow \quad x = 1$

d) param. eq.

$D(\ell) = \langle (0,-9) \rangle$

basically all the vectors with the same direction so all the multiples of $(0,-9)$

$$\vec{MN} = \vec{r}_N - \vec{r}_M = (0,-9)$$

$$\begin{cases} x = 2 + 0\lambda \\ y = 4 - 9\lambda \end{cases}$$

Cart eq: $x - 2 = 0$

2.5. Determine the equation of the line parallel to $\vec{v}$ and passing through $S \cap T$ if:

1. $\vec{v} = (2,4)$, $S: 3x - 2y - 7 = 0$

$T: 2x + 3y = 0$

$\underline{S \cap T}$:

$$\begin{cases} 3x - 2y - 7 = 0 & / \cdot 2 \\ 2x + 3y = 0 & / \cdot 3 \end{cases} \quad -$$

$$-13y - 14 = 0$$

$$-13y = 14 \Rightarrow y = -\frac{14}{13}$$

$$3x + \frac{28}{13} - 7 = 0 \Leftrightarrow$$

$$\Leftrightarrow 39x + 28 - 91 = 0$$

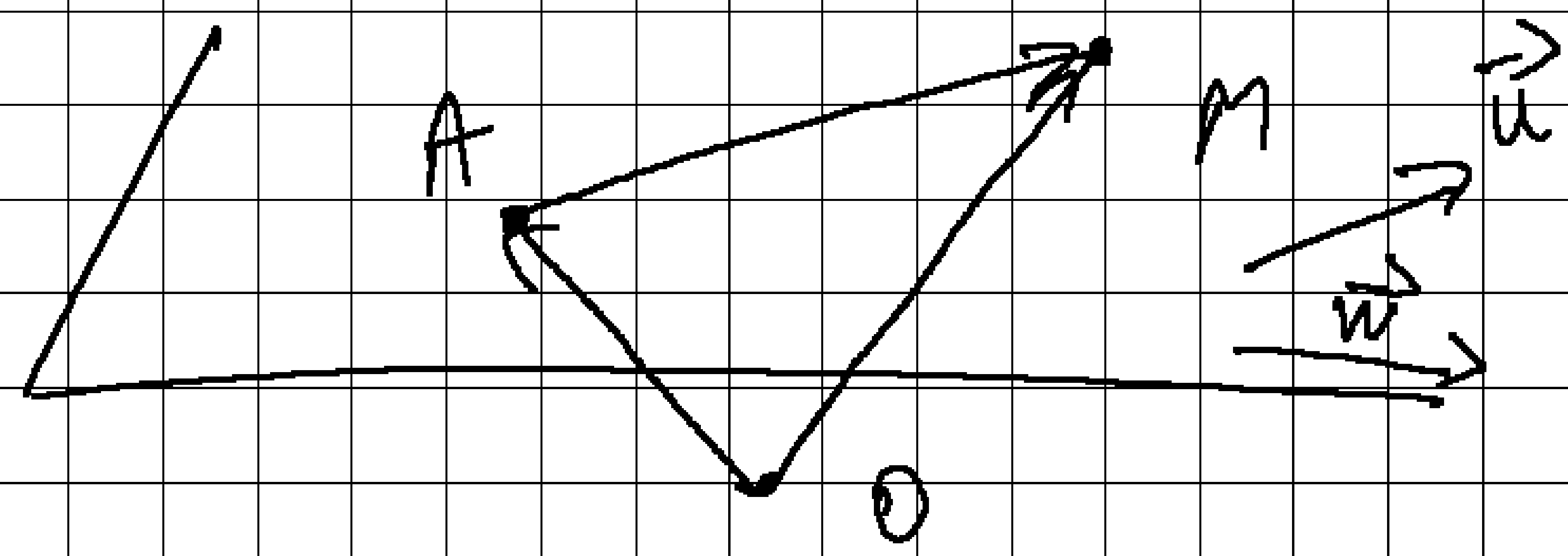
$$\Leftrightarrow 39x = 63$$

$$\Leftrightarrow x = \frac{63}{39} = \frac{21}{13}$$

$$S \cap T = U\left(\frac{21}{13}, \frac{-14}{13}\right)$$

Param eq: $l: \begin{cases} x = \frac{21}{13} + 2\lambda \\ y = -\frac{14}{13} + 4\lambda \end{cases}$

Π plane ($\dim_{\mathbb{R}} D(\Pi) = 2$)



$A \in \Pi$, $\vec{v}, \vec{w} \parallel \Pi$ $(\vec{v}, \vec{w} \in D(\Pi))$

$\vec{r}_M = \vec{r}_A + \vec{AM}$

$\vec{AM} \in D(\Pi)$

$\vec{v}, \vec{w}$ lin. indep., so they are a basis of $D(\Pi)$

$\Rightarrow \exists!\ \lambda, \mu \in \mathbb{R}$:

$$\vec{r}_M = \vec{r}_A + \lambda \vec{v} + \mu \vec{w}$$

(vector equation)

Param. eq:

$$(S)\ \begin{cases} x = x_A + \lambda x_{\vec{v}} + \mu x_{\vec{w}} \\ y = y_A + \lambda y_{\vec{v}} + \mu y_{\vec{w}} \\ z = z_A + \lambda z_{\vec{v}} + \mu z_{\vec{w}} \end{cases} \qquad \lambda, \mu \in \mathbb{R}$$

(S) is compatible (with respect to λ, μ) iff

Cartesian eqn.

$$\begin{vmatrix} x-x_A & y-y_A & z-z_A \\ x_{\vec{v}} & y_{\vec{v}} & z_{\vec{v}} \\ x_{\vec{w}} & y_{\vec{w}} & z_{\vec{w}} \end{vmatrix} = 0$$

S, P, Q, R coplanar (in the same plan) iff

$$\begin{vmatrix} x_S & y_S & z_S & 1 \\ x_P & y_P & z_P & 1 \\ x_Q & y_Q & z_Q & 1 \\ x_R & y_R & z_R & 1 \end{vmatrix}$$

implicit form: $Ax + By + Cz + D = 0$

↳ normal vectors of π are the ones in $\langle (A, B, C) \rangle$

2.10 Determine Cartesian eqn. for the plane π

a) $\pi : \begin{cases} x = 2+3u-4v \\ y = 4-v \\ z = 2+3u \end{cases}$

b) $\pi : \begin{cases} x = u+v \\ y = u-v \\ z = 5+6u-4v \end{cases}$

a) $\begin{vmatrix} x-2 & y-4 & z-2 \\ 3 & 0 & 3 \\ -4 & -1 & 0 \end{vmatrix} = 0$

$-3z+6-12y+48+3x-6=0$

$3x-12y-3z+48=0$

$x-4y-z+16=0$

b) $\begin{vmatrix} x & y & z-5 \\ 1 & 1 & 6 \\ 1 & -1 & -4 \end{vmatrix} = 0 \iff -4x+6y-(z-5)-(z-5)-4y+$

$+6x=0$

$\iff 2x+10y-2z+10=0 \iff x+5y-z+5=0$

2.11. Determine param. eq. for Π:

a) $\Pi: \quad 3x-6y+z=0$

b) $\Pi: \quad 2x-y-z-3=0$

Find $D(\Pi)$.

a) $x=\lambda$

$y=\mu$

$3\lambda-6\mu+z=0$

$$\begin{cases} x=\lambda \\ y=\mu \\ z=-3\lambda+6\mu \end{cases}$$

$D(\Pi) = \langle (1,0,-3),(0,1,6) \rangle$

b) $x=\lambda$

$y=\mu$

$2\lambda-\mu-z-3=0$

$$\begin{cases} x=\lambda \\ y=\mu \\ z=2\lambda-\mu-3 \end{cases}$$

$D(\Pi) = \langle (1,0,2),(0,1,-1) \rangle$

2.14. Determine the relative positions of the planes.

Skew ($\nexists$ intersect)

incident ($\exists$ intersect)

a) $\pi_1: \ x+2y+2z-1=0$

$\pi_2: \ x+2y-3z-1=0$

b) $\pi_1: \ x+2y+3z-1=0$

$\pi_2: \ 2x+y+3z-2=0$

$\pi_3: \ x+2y+3z+2=0$

a) $\pi_1 \cap \pi_2 = \begin{cases} x+2y+2z-1=0 & (1) \\ x+2y-3z-1=0 & (2) \end{cases}$

$(1)-(2) \Rightarrow \ 6z=0 \Rightarrow z=0$

$x+2y-1=0 \Rightarrow x=1-2y$

$\pi_1 \cap \pi_2 = \{(1-2a, a, 0) \mid a \in \mathbb{R}\}$

Exercises that we were supposed to cover:

- 18
- 19
- 26
- 27
- 30

from set 2