1. a) $\begin{cases} \dot{x} = -y \\ \dot{y} = 5x \end{cases}$

eq. point is at $(0,0)$, we study it's stability

$X = \begin{pmatrix} x \\ y \end{pmatrix}$ $\quad X' = AX \Rightarrow A = \begin{pmatrix} 0 & -1 \\ 5 & 0 \end{pmatrix}$

$\det(A-\lambda) = 0 \Leftrightarrow \begin{vmatrix} -\lambda & -1 \\ 5 & -\lambda \end{vmatrix} = 0 \Leftrightarrow$

$\Leftrightarrow \lambda^2 + 5 = 0 \Rightarrow \lambda_{1,2} = \pm i\sqrt{5} \in \mathbb{C} \setminus \mathbb{R}$

$\Rightarrow$ center, stable

$\dfrac{dy}{dx} = \dfrac{5x}{-y} \Leftrightarrow -y\,dy = 5x\,dx \overset{\text{integrate}}{\Longrightarrow}$

$\Rightarrow -\int y\,dy = 5\int x\,dx \Leftrightarrow -\frac{1}{2}y^2 + C_1 = \frac{5}{2}x^2 + C_2 \quad C_1, C_2 \in \mathbb{R}$

$2(C_1 - C_2) = C \in \mathbb{R} \quad \Rightarrow \quad 5x^2 + y^2 = C$

$H: \mathbb{R}^2 \Rightarrow \mathbb{R} \quad H(x,y) = 5x^2 + y^2$

$f_2 = 5x$
$f_1 = -y$

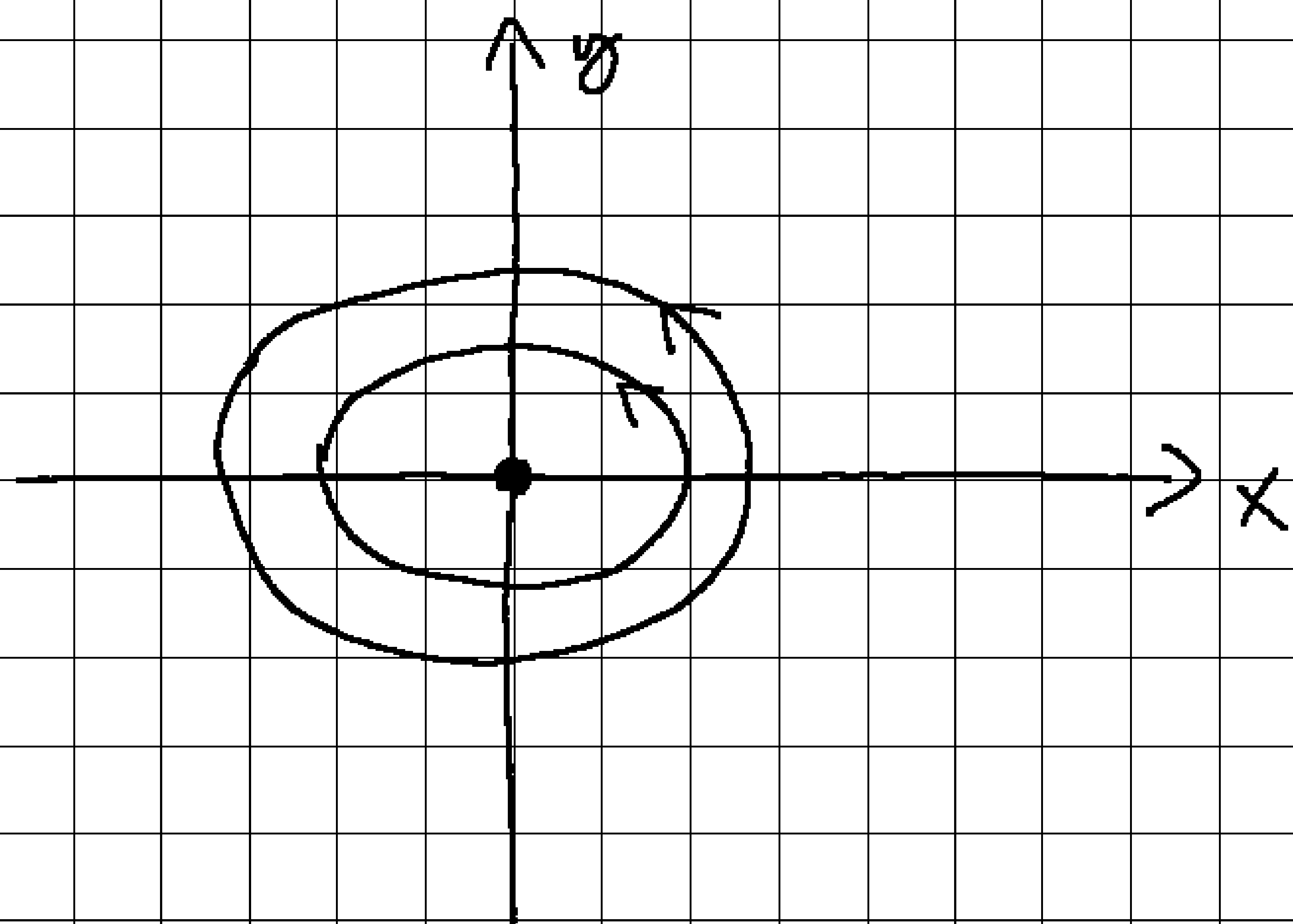
$\dfrac{dH}{dx}(x,y) \cdot f_1(x,y) + \dfrac{dH}{dy}(x,y) \cdot f_2(x,y) = 0 \Leftrightarrow$

$\Leftrightarrow 10x \cdot (-y) + 2y \cdot 5x = 0$

$-10xy + 10xy = 0$ "T" $\forall t \in \mathbb{R} \Rightarrow$ H is a global first integral

$H(x,y) = c \in \mathbb{R}$ fixed and arbitrary

$\Leftrightarrow \quad 5x^2 + y^2 = c$



$x < 0 \Rightarrow y' > 0$

$x > 0 \Rightarrow y' < 0$

$y < 0 \Rightarrow x' < 0$

$y > 0 \Rightarrow x' > 0$

b) $\begin{cases} x' = -x \\ y' = 5y \end{cases} \Rightarrow A = \begin{pmatrix} -1 & 0 \\ 0 & 5 \end{pmatrix}$ (diag m.) $\Rightarrow$

$\Rightarrow \lambda_{1,2} = -1, 5 \Rightarrow$

(i) eq point $= (0,0)$

$\Rightarrow \lambda_1 < 0 < \lambda_2 \Rightarrow$

$\Rightarrow$ saddle point

(ii) (iii) $\quad \dfrac{dH}{dt} = \dfrac{dH}{dx} \cdot x' + \dfrac{dH}{dy} \cdot y' = 0 \quad (1)$

$\begin{cases} \dfrac{dx}{dt} = -x \\ \dfrac{dy}{dt} = 5y \end{cases} \Leftrightarrow \dfrac{dx}{dy} = \dfrac{-x}{5y} \Leftrightarrow \dfrac{dy}{5y} = \dfrac{dx}{-x} \Rightarrow$

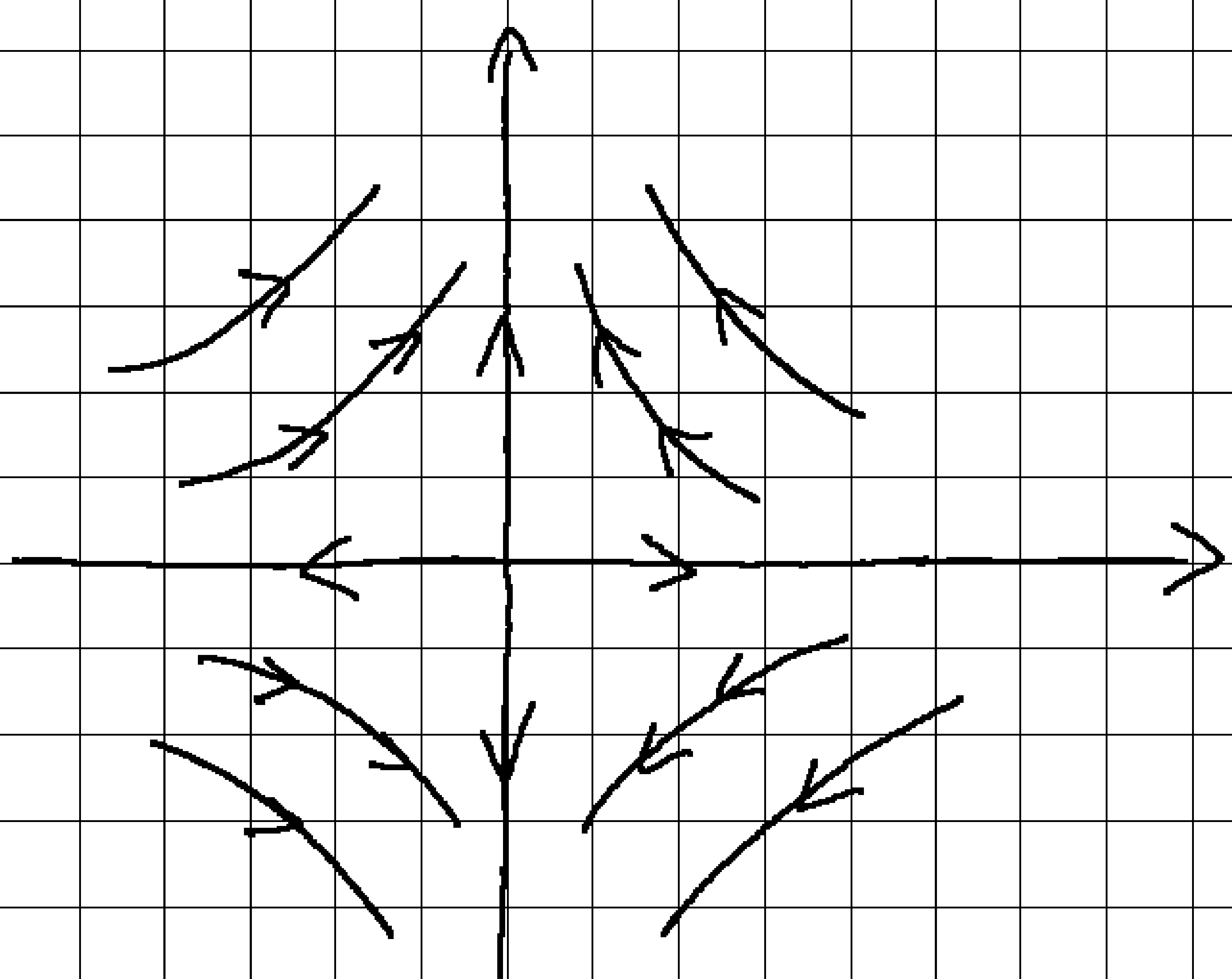
$\Rightarrow \displaystyle\int \frac{dy}{y} = -5 \int \frac{dx}{x} \Leftrightarrow$

$\Leftrightarrow \quad \ln|y| + 5\ln|x| = \ln|c| \quad c \in \mathbb{R}$

$\Leftrightarrow \quad yx^5 = C \quad , \quad H(x,y) = yx^5$

Replacing in (1) $\Leftrightarrow$

$\Leftrightarrow \quad \cancel{5x^4y} \cdot (-x) + \cancel{x^5 \cdot 5y} = 0 \quad \Rightarrow$ H is a global first integral



$y=0 \Rightarrow \begin{cases} x>0 \ , \ x'<0 \\ x<0 \ , \ x'>0 \end{cases}$

$x=0 \Rightarrow \begin{cases} y>0 \ , \ y'>0 \\ y<0 \ , \ y'<0 \end{cases}$

c) $\begin{cases} x' = -3x \\ y' = -2y \end{cases}$ $\quad \begin{pmatrix} -3 & 0 \\ 0 & -2 \end{pmatrix}$ (diag matrix) $\Rightarrow$

eq point $= (0,0)$ $\quad \Rightarrow \begin{array}{l} \lambda_1 = -3 \\ \lambda_2 = -2 \end{array} \quad \lambda_1, \lambda_2 < 0$

$\Rightarrow$ node (global retractor)

$\Rightarrow$ there is no global first integral

$\mathcal{U} = \{ (x,y) \mid x, y \in \mathbb{R}^* \} = \mathcal{U}_1 \cup \mathcal{U}_2$

$\frac{dx}{dy} = \frac{-2y}{-3x} \Rightarrow 3\frac{dy}{y} = 2\frac{dx}{x} \Big| \int \Rightarrow$

$\Rightarrow \ln|y^3| = \ln|x^2| + \ln c \Rightarrow \ln\left|\frac{y^3}{x^2}\right| = \ln|c|, c \in \mathbb{R}$

$\Rightarrow \frac{y^3}{x^2} = c$

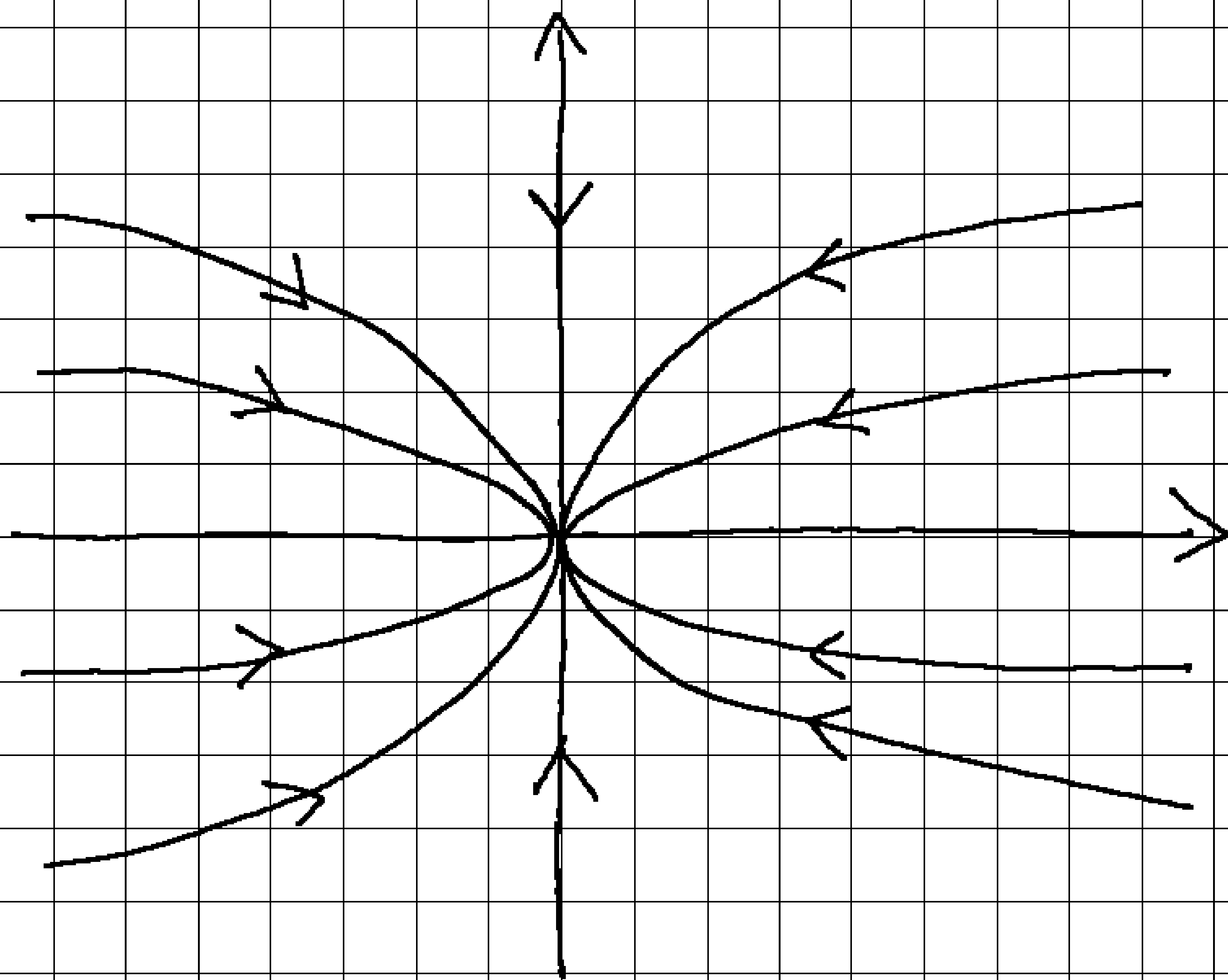
$H(x,y) = c = \frac{y^3}{x^2}$

$\frac{dH}{dx} \cdot f_1 + \frac{dH}{dy} f_2 = 0 \Leftrightarrow y^3(-2x^{-3}) \cdot (-3x) + x^2(-2y) \cdot 3y^2 = 0$

$y^3 \cdot 6x^{-2} - x^{-2} \cdot 6y^3 = 0$ „T" $\Rightarrow$

$\Rightarrow$ H is a first integral

(not global)



d) $\begin{cases} x' = x - y \\ y' = x + y \end{cases}$ $\qquad A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

eq point $= (0,0)$

i) $\det(A - \lambda I_2) = 0$

$\begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)^2 + 1 = 0$

$\Leftrightarrow \lambda^2 - 2\lambda + 2 = 0 \Rightarrow \lambda_{1,2} = 1 \pm i \in \mathbb{C} / \mathbb{R}$

$\Rightarrow \begin{cases} \mathrm{Re}(\lambda_{1,2}) \neq 0 \Rightarrow \text{focus} \\ \mathrm{Re}(\lambda_{1,2}) > 0 \Rightarrow \text{global repeller} \end{cases}$

$\Rightarrow$ no global integral

$$\frac{dy}{dx} = \frac{x+y}{x-y} \Rightarrow \text{not separable}$$

$\Rightarrow$ polar coordinates

$$x = \rho\cos\theta \qquad y = \rho\sin\theta$$

$$x^2+y^2 = \rho^2\cos^2\theta + \rho^2\sin^2\theta$$

$$x^2+y^2 = \rho^2(\underbrace{\cos^2\theta + \sin^2\theta}_{=1})$$

$$x^2+y^2 = \rho^2$$

$$\tan\theta = \frac{y}{x}$$

$$\tan\theta = \frac{y}{x}$$

$$\frac{\theta'}{\cos^2\theta} = \frac{xy' - x'y}{x^2} \Rightarrow \frac{\theta'}{\cos^2\theta} = \frac{x(x+y)-(x-y)y}{x^2} \Leftrightarrow$$

$$\Leftrightarrow \frac{\theta'}{\cos^2\theta} = \frac{x^2+\cancel{xy}-\cancel{xy}+y^2}{x^2} \Leftrightarrow \frac{\theta'}{\cos^2\theta} = \frac{x^2+y^2}{x^2} \Leftrightarrow$$

$$\Leftrightarrow \frac{\theta'}{\cos^2\theta} = \frac{\rho^2}{x^2} \Rightarrow \theta' = \cos^2\theta\cdot\left(\frac{\rho}{x}\right)^2 = \cos^2\theta\,\frac{\rho^2}{\rho^2\cos^2\theta}$$

$$\Rightarrow \theta' = 1 > 0 \Rightarrow \text{trigonometric sense}$$

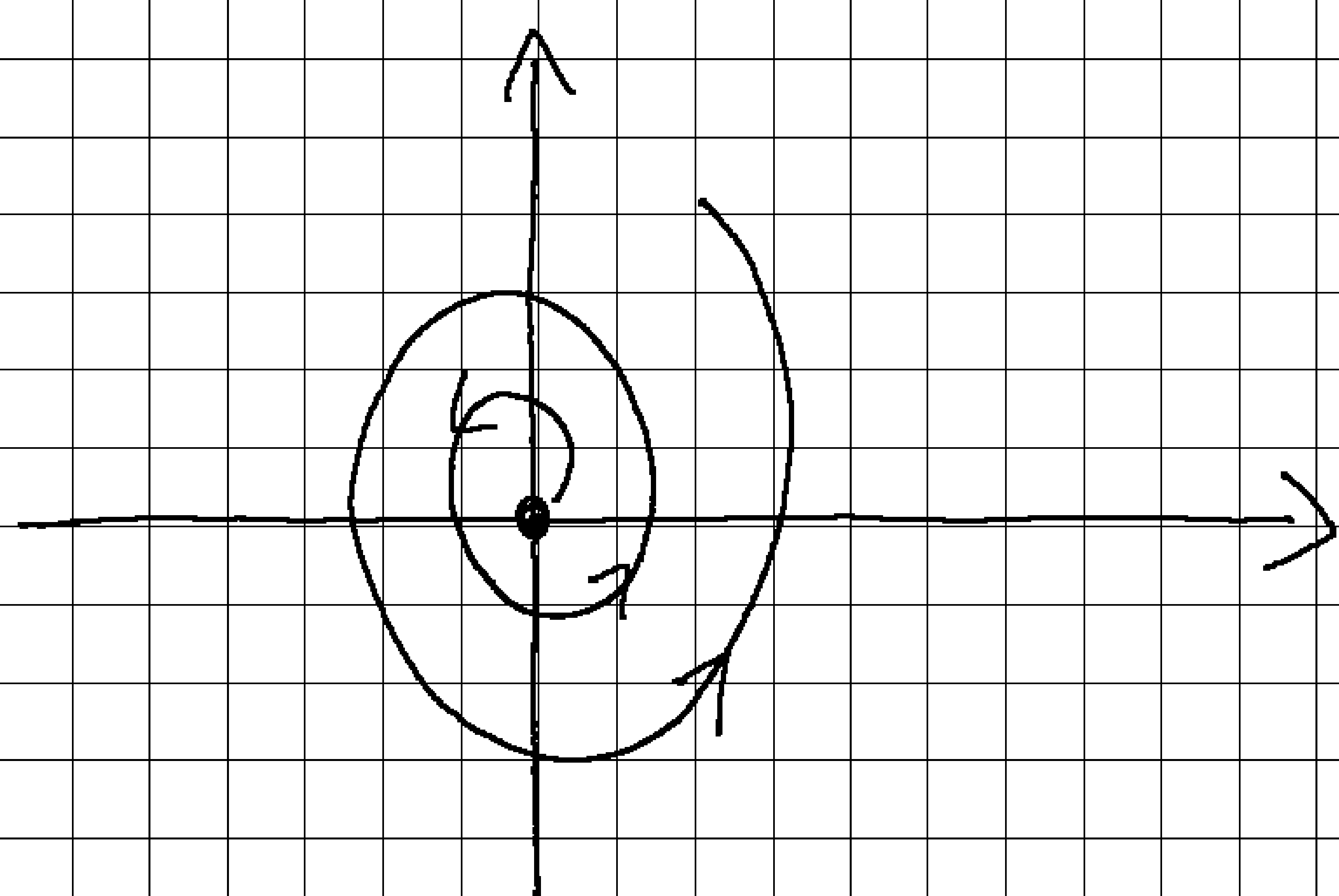
$x^2+y^2=\varphi^2$

$x\cdot x' + y\cdot y' = \varphi\cdot\varphi' \Leftrightarrow x(x\cdot y)+(x+y)\cdot y = \varphi\cdot\varphi'$

$\Leftrightarrow x^2 - xy + xy + y^2 = \varphi\cdot\varphi'$

$\Leftrightarrow x^2+y^2 = \varphi\cdot\varphi' \Leftrightarrow \varphi^2 = \varphi\varphi'$

$\Rightarrow \boxed{\varphi' = \varphi}$



2. $\begin{cases} \dot{x} = x(1-x) \\ \dot{y} = y(3-y) \end{cases}$

$\dot{x} = 0 \Leftrightarrow x(1-x) = 0 \Rightarrow x \in \{0, 1\}$

$\dot{y} = 0 \Rightarrow y(3-y) = 0 \Rightarrow y \in \{0, 3\}$

$\Rightarrow eq \in \{(0,0), (0,3), (1,0), (1,3)\}$

$$J = \begin{pmatrix} \frac{d\dot{x}}{dx} & \frac{d\dot{x}}{dy} \\ \frac{d\dot{y}}{dx} & \frac{d\dot{y}}{dy} \end{pmatrix} = \begin{pmatrix} 1-2x & 0 \\ 0 & 3-2y \end{pmatrix}$$

I $(0,0) \Rightarrow J(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \Rightarrow \lambda_1 = 1, \lambda_2 = 3$

$\Rightarrow$ node repeller

II $(0,3) \Rightarrow J(0,3) = \begin{pmatrix} 1 & 0 \\ 0 & -3 \end{pmatrix} \Rightarrow \lambda_1 = 1, \lambda = -3$

$\Rightarrow$ saddle point

III $(1,0) \Rightarrow J(1,0) = \begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix} \Rightarrow \lambda_1 = -1 \quad \lambda_2 = 3$

$\Rightarrow$ Saddle point

IV $(1,3) \Rightarrow J(1,3) = \begin{pmatrix} -1 & 0 \\ 0 & -3 \end{pmatrix} \Rightarrow \lambda_1 = -1 \quad \lambda_2 = -3$

$\Rightarrow$ stable node attractor

3. $\begin{cases} x = ax - 5y \\ \dot{y} = x - 2y \end{cases}$ has a center

i) $a = ?$

$$A = \begin{pmatrix} a & -5 \\ 1 & -2 \end{pmatrix}$$

$$\det(A - \lambda I_2) = 0 \Rightarrow \begin{vmatrix} a-\lambda & -5 \\ 1 & -2-\lambda \end{vmatrix} = 0 \Rightarrow$$

$\Rightarrow (a-\lambda)(-2-\lambda) + 5 = 0$

$\Rightarrow -2a - a\lambda + 2\lambda + \lambda^2 + 5 = 0$

$\Rightarrow \lambda^2 + (2-a)\lambda + 5 - 2a = 0$

$\Delta = (2-a)^2 - 4(5-2a)$

$\Delta = a^2 - 4a + 4 - 20 + 8a = a^2 + 4a - 16$

$$\lambda_{1,2} = \frac{a-2}{2} \pm \frac{\sqrt{a^2+4a-16}}{2}$$

$\rightarrow$ real part

$$Re(\lambda) = 0 \Rightarrow \frac{a-2}{2} = 0 \Rightarrow \boxed{a=2}$$

For $a=2 \Rightarrow \Delta = -4 \Rightarrow \lambda_{1,2} = \pm\frac{2i}{2} = \pm i$

ii) $\det(A) = 0 \Rightarrow \begin{vmatrix} a & -5 \\ 1 & -2 \end{vmatrix} = 0$

$$\Rightarrow -2a+5=0$$

$$\Rightarrow a = \frac{5}{2}$$