Chapter 7: 9, 10 a b

Chapter 8: 3, 4, 5, 10, 14

$$Q: \quad a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_{10}x + 2a_{01}y + a_{00} = 0$$

$$M_Q = \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix}$$

$$\hat{M}_Q = \left(\begin{array}{cc|c} a_{11} & a_{12} & a_{10} \\ a_{12} & a_{22} & a_{01} \\ \hline a_{10} & a_{01} & a_{00} \end{array}\right)$$

$$\hat{\Delta} = \det \hat{M}_Q$$

$$D = \det M_Q$$

$$T = \operatorname{Tr} M_Q \quad \text{(Trace)}$$

$\hat{\Delta}$	D	T	Curve Q
$\hat{\Delta}=0$	$D>0$		a point
	$D=0$		two lines or the empty set
	$D<0$		two lines
$\hat{\Delta}\neq 0$	$D>0$	$T<0$	an ellipse
	$D>0$	$T>0$	the empty set
	$D=0$		a parabola
	$D<0$		a hiperbola

7.9. Discuss the type of the curve

$x^2 + \lambda xy + y^2 - 6x - 16 = 0$ in terms of $\lambda \in \mathbb{R}$

(Homework: Solve this using the techniques from last time, diagonalization & Lagrange)

$$M_q = \begin{pmatrix} 1 & \frac{\lambda}{2} \\ \frac{\lambda}{2} & 1 \end{pmatrix} \qquad \hat{M}_q = \begin{pmatrix} 1 & \frac{\lambda}{2} & -3 \\ \frac{\lambda}{2} & 1 & 0 \\ -3 & 0 & -16 \end{pmatrix}$$

$$\det M_q = 1 - \frac{\lambda^2}{4}$$

$$\det \hat{M}_q = 4\lambda^2 - 25$$

I $\lambda = \frac{5}{2} \Rightarrow \left.\begin{array}{l} \hat{D} = 0 \\ D < 0 \end{array}\right\} \Rightarrow$ two lines

II $\lambda = -\frac{5}{2} \Rightarrow \left.\begin{array}{l} \hat{D} = 0 \\ D < 0 \end{array}\right\} \Rightarrow$ two lines

III $|\lambda| > 2$ & $\lambda \neq \pm\frac{5}{2}$

$\Rightarrow \left.\begin{array}{l} \hat{D} \neq 0 \\ D < 0 \end{array}\right\} \Rightarrow$ hyperbola

IV $\lambda = \pm 2$

$\Rightarrow \left.\begin{array}{l} \hat{D} \neq 0 \\ D = 0 \end{array}\right\} \Rightarrow$ parabola

V $\lambda \in (-2, 2) \Rightarrow \left.\begin{array}{l} \hat{D} \neq 0 \\ D > 0 \\ T > 0 \end{array}\right\} \Rightarrow$ empty set

7.10. Decide (using the Lagrange method) what surfaces are discended by the equations:

a) $x^2+2y^2+z^2+xy+yz+zx=1$

b) $xy+yz+zx=1$

a) $x^2+2y^2+z^2+xy+yz+zx=1 \Leftrightarrow$

$\Leftrightarrow \frac{x^2}{2}+\frac{y^2}{4}+\frac{z^2}{4}+2xy\cdot\frac{1}{2}+2yz\cdot\frac{1}{4}+2zx\cdot\frac{1}{2}+\frac{7y^2}{4}+\frac{3z^2}{4}+\frac{1}{2}yz=1$

$\Leftrightarrow \left(x+\frac{y}{2}+\frac{z}{2}\right)^2+\frac{7}{4}y^2+\frac{3z^2}{4}+\frac{1}{2}yz=1$

$x' = x+\frac{y}{2}+\frac{z}{2}$

$\Leftrightarrow (x')^2+\frac{7}{4}y^2+2\cdot\frac{\sqrt{7}}{2}yz\cdot\frac{1}{2\sqrt{7}}+\frac{1}{28}z^2+\frac{20}{28}z^2=1$

$\Leftrightarrow (x')^2+\left(\frac{\sqrt{7}}{2}y+\frac{1}{2\sqrt{7}}z\right)^2+\frac{5}{7}z^2=1$

$y' = \frac{\sqrt{7}}{2}y+\frac{1}{2\sqrt{7}}z$

$\Leftrightarrow (x')^2+(y')^2+\frac{5}{7}z^2=1$

$z' = \frac{5}{7}z$

$\Rightarrow (x')^2+(y')^2+(z')^2 \Rightarrow$ ellipsoid

b) We don't have squares, so we make them.

$X = x' + y$

$Q: \quad (x'+y)y + yz + z(x'+y) = 1 \Leftrightarrow$

$\Leftrightarrow \quad x'y + y^2 + 2yz + zx' = 1$

$\Leftrightarrow \quad \left(y^2 + 2\cdot\frac{1}{2}yx' + 2yz + \frac{x'^2}{4} + z^2 + x'z\right) - z^2 - \frac{x'^2}{4}$

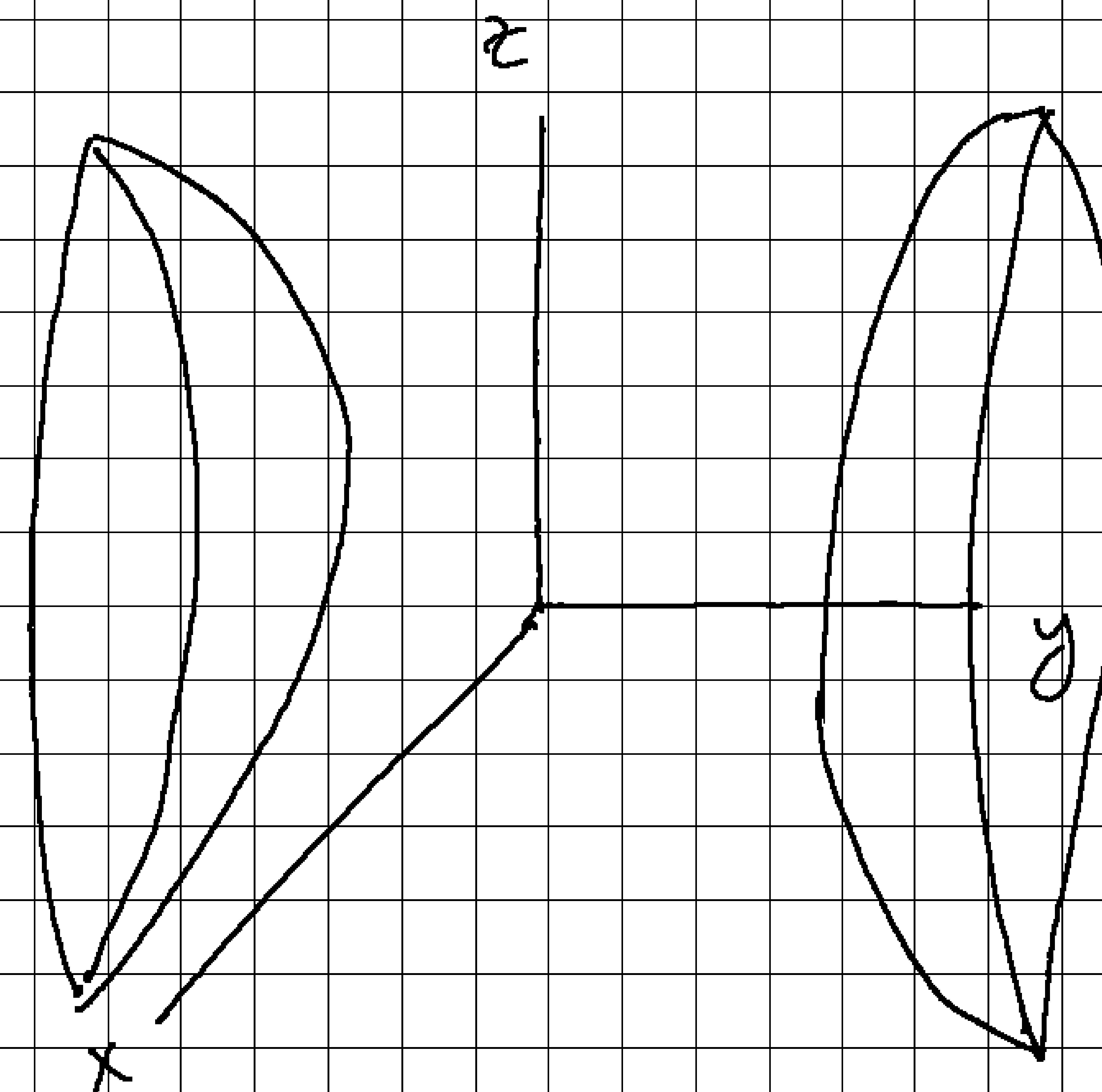
$\Leftrightarrow \quad \left(y + \frac{x'}{2} + z\right)^2 - \frac{x'^2}{4} - z^2 = 1$

$y' = y + \frac{x'}{2} + z$

$y'^2 - \frac{x'^2}{4} - z^2 = 1$

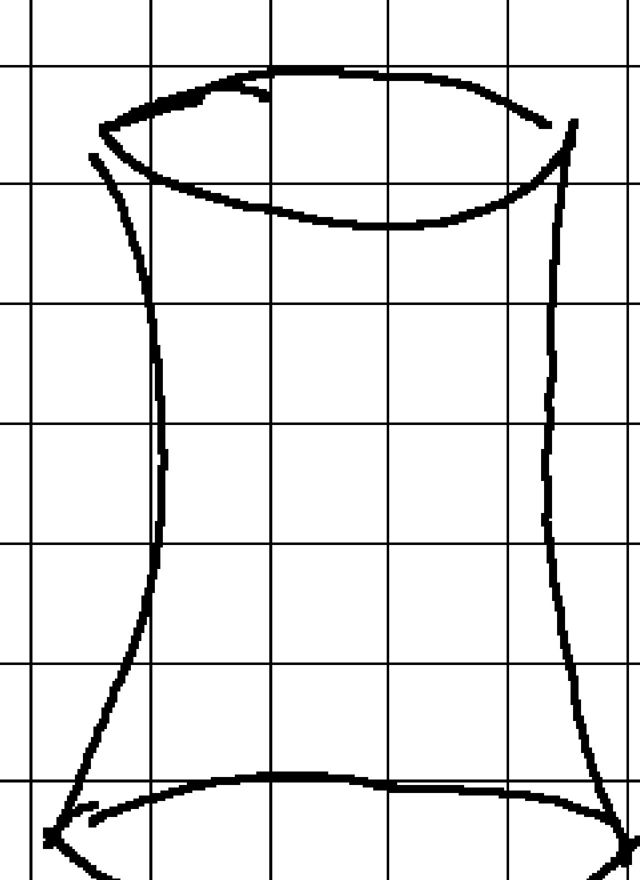
$\Rightarrow \frac{x'^2}{4} - y'^2 + z^2 = -1$

$\Rightarrow$ hyperboloid of two sheets.



$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \Rightarrow$ ellipsoid

$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \Rightarrow$ hyperboloid of one sheet (one negative)



$\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = -1 \Rightarrow$ hyperboloid of two sheets

$\frac{x^2}{p} + \frac{y^2}{q} = 2z \Rightarrow$ elliptic paraboloid

$\frac{x^2}{p} - \frac{y^2}{2} = 2z \Rightarrow$ hyperbolic paraboloid

Chapter 8:

$S : f(x,y,z) = 0$

surface

$T_{(x_0,y_0,z_0)} S : \nabla f \cdot (x \cdot x_0, y \cdot y_0, z x_0) = 0$

$$\frac{\partial f}{\partial x}(x_0,y_0,z_0) \cdot (x + x_0) + \dots = 0$$

8.4. Determine the tangent planes to the ellipsoid

$$\mathcal{E}_{2,3,2\sqrt{2}} : \frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{8} = 1$$

Which are parallel to the plane $\pi : 3x - 2y + 5z + 1 = 0$

$$T_{(x_0,y_0,z_0)} \mathcal{E} : \frac{x_0 x}{4} + \frac{y_0 y}{9} + \frac{z_0 z}{8} = 1$$

$$\left.\begin{array}{l} \vec{n}_{\pi} = (3,-2,5) \\ \vec{n}_{T_p \mathcal{E}} = \left(\frac{x_0}{4}, \frac{y_0}{9}, \frac{z_0}{8}\right) \\ \pi \parallel T_p \mathcal{E} \end{array}\right\} \Rightarrow \frac{\frac{x_0}{4}}{3} = \frac{\frac{y_0}{9}}{-2} = \frac{\frac{z_0}{8}}{5}$$

$$\Rightarrow \quad \frac{x_0}{12} = \frac{y_0}{-18} = \frac{z_0}{40}$$

$$\Rightarrow \begin{cases} y_0 = \frac{-18x_0}{12} = \frac{-3x_0}{2} \\ z_0 = \frac{40x_0}{12} = \frac{10x_0}{3} \end{cases}$$

$$^{18/}\frac{x_0^2}{4} + {}^{2/}\frac{9x_0^2}{36} + \frac{100x_0^2}{72} = 1$$

$$18x_0^2 + 18x_0^2 + 100x_0^2 = 72$$

$$136x_0^2 = 72$$

$$x_0^2 = \frac{72}{136}$$

$$x_0^2 = \frac{9}{17} \quad \Rightarrow \quad x_0 = \pm\sqrt{\frac{9}{17}}$$

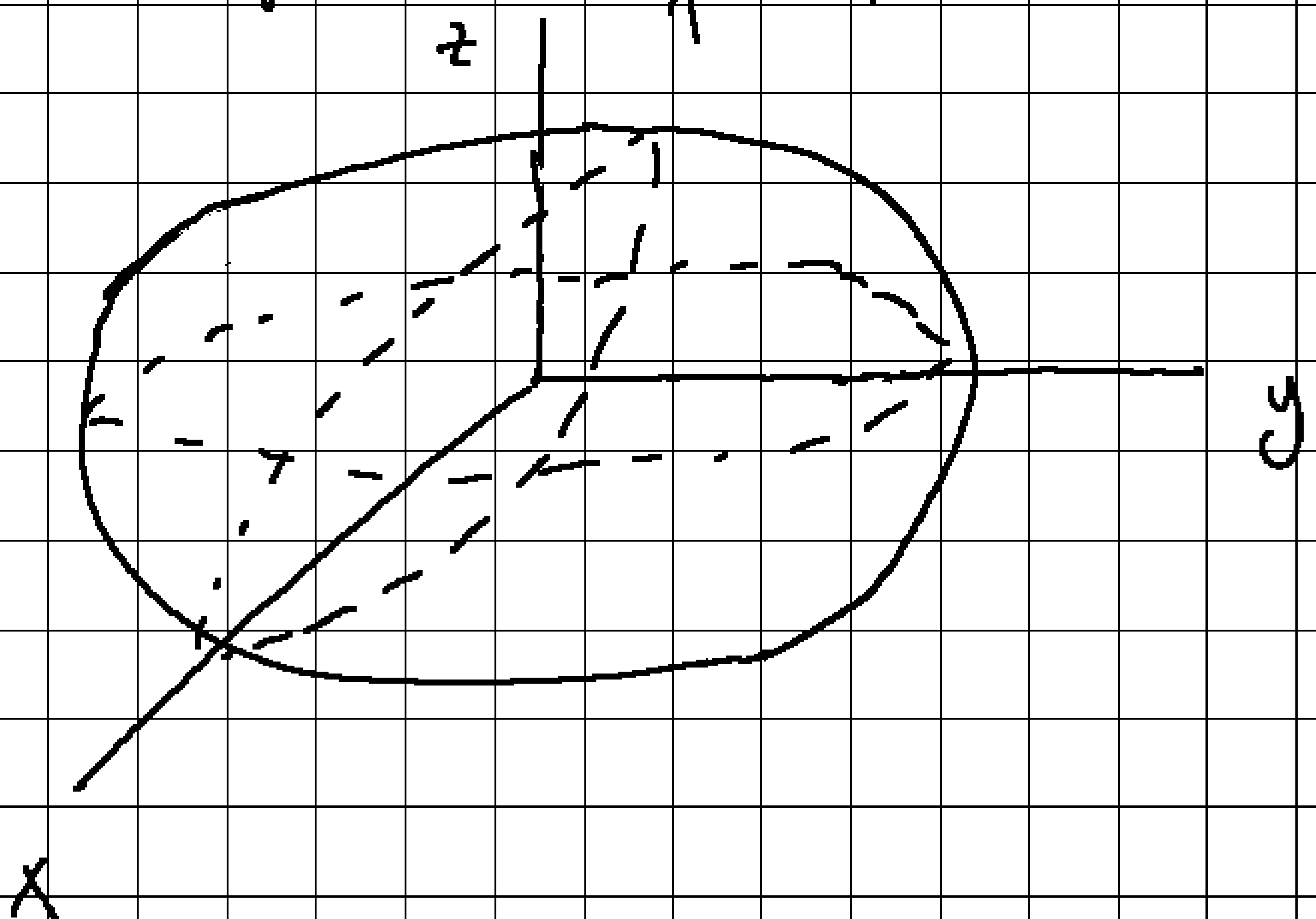
$$\Rightarrow \quad y_0 = \mp \frac{3}{2}\sqrt{\frac{9}{17}}$$

$$\Rightarrow \quad z_0 = \pm \frac{10}{3}\sqrt{\frac{9}{17}}$$

$$T_{P_1}\mathcal{E}: \sqrt{\frac{9}{17}} \cdot \frac{x}{4} - \frac{3\sqrt{9}}{2\sqrt{17}} \cdot \frac{y}{9} + \frac{10\sqrt{9}}{3\sqrt{17}} \cdot \frac{z}{8} = 1$$

$$T_{P_2}\mathcal{E}: -\sqrt{\frac{9}{17}} \cdot \frac{x}{4} + \frac{3\sqrt{9}}{2\sqrt{17}} \cdot \frac{y}{9} + \frac{10\sqrt{9}}{3\sqrt{7}} \cdot \frac{z}{8} = 1$$

8.10. Use a parametrization of an ellipse and a rotation matrix to deduce a parametrization of an ellipsoid of revolution



The eq of an ellipse in the xoy plane is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$ which we can parametrize

$$\begin{cases} x = a\cos\theta \\ y = b\sin\theta \end{cases}$$

$\theta \in [0, 2\pi)$

in 3D this is:

$$\begin{cases} x = a\cos\theta \\ y = b\sin\theta \\ z = 0 \end{cases}$$

$$[Rot_{Oy}] = \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix}$$

$$[Rot\, O_z] = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$\Rightarrow$ so the surface of revolution has the eq:

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos t & 0 & -\sin t \\ 0 & 1 & 0 \\ \sin t & 0 & \cos t \end{pmatrix} \begin{pmatrix} a\cos\theta \\ b\sin\theta \\ 0 \end{pmatrix} =$$

$$= \begin{pmatrix} a\cos t\cos\theta \\ b\sin\theta \\ a\sin t\cos\theta \end{pmatrix}$$

$$\Rightarrow S: \frac{x'^2}{a^2} + \frac{y'^2}{b^2} + \frac{z'^2}{a^2} = 1$$ (not required)