Sample test 1:

1\. a) Find a particular solution of the form $x_p(t) = at^2 + bt$ (where the real coeficient a, b have to be determined) for: $x'' + 3x' = t$

$$x'_p(t) = 2at + b$$

$$x''_p(t) = 2a$$

$$x'' + 3x' = t \Leftrightarrow \underline{2a} + \underline{\underline{6at}} + \underline{3b} = \underline{\underline{t}}$$

$$\begin{cases} \Rightarrow \text{I} \quad 6at = t \Rightarrow a = \frac{1}{6} \\ \Rightarrow \text{II} \quad 2a + 3b = 0 \Rightarrow \frac{1}{3} + 3b = 0 \\ \Rightarrow b = -\frac{1}{9} \end{cases}$$

$$\Rightarrow x_p(t) = \frac{1}{6}t^2 - \frac{1}{9}t$$

b) Find a first degree polynomial solution for

$x'' + 3x' = 3$ $\Rightarrow 0 = 3$ „F”

because we have a const there

I $x_p(t) = A$

$x'_p(t) = 0$

$x''_p(t) = 0$

(we assume a particular sol of degree 0)

the result wouldn't work so we try with next degree

II $x_p(t) = At$

$x'_p(t) = A \quad x''_p(t) = 0$

$\Rightarrow 0 + 3A = 3 \Rightarrow A = 1$

$\Rightarrow x_p(t) = t$

c) Find the general solution of the differential equation: $x'' + 3x' = t + 3$

$x'' + 3x' = 0$

$r^2 + 3r = 0 \Leftrightarrow r(r+3) = 0 \Rightarrow r = 0$ and $r = -3$

$\Rightarrow x_h(t) = C_1 + C_2 e^{-3t}$

$x'' + 3x' = t + 3$

$x_p(t) = At^2 + Bt + C$

$x_p'(t) = 2At + B$

$x_p''(t) = 2A$

$\underline{2A} + \underline{6At} + \underline{3B} = \underline{t} + \underline{3}$

$6At = t \Rightarrow A = \frac{1}{6}$

$2A + 3B = 3$

$\Rightarrow \frac{1}{3} + 3B = 3 \Rightarrow B = \frac{8}{9}$

$\Rightarrow x_p(t) = \frac{1}{6}t^2 + \frac{8}{9}t + C$

we just take $C = 0$ for simplicity

$\Rightarrow x_p(t) = \frac{1}{6}t^2 + \frac{8}{9}t$

(If we take a polynomial with one degree higher then the degree of the differential eq as the particular sol we get the correct result in most cases i've seen, this applies for polynomials only as we can have exponential functions: ex: $x'' + 3x' = e^{2t}$

$\Rightarrow x_p(t) = Ae^{2t}$

sinusoidal functions:

$x'' + 3x' = \sin(2t) \Rightarrow$

$\Rightarrow x_p(t) = A\sin(2t) + B\cos(2t)$

$$x(t) = x_h(t) + x_p(t) = C_1 + C_2 e^{-3t} + \frac{1}{6}t^2 + \frac{8}{9}t$$

d) Find the solution of the ivp $\begin{cases} x'' + 3x' = t \\ x(0) = 5 \\ x'(0) = 0 \end{cases}$

$$x'' + 3x' = 0$$

$$r^2 + 3r = 0 \quad \Rightarrow \quad r(r+3) = 0 \quad \rightarrow \quad r = 0, \; r = -3$$

$$\Rightarrow x_h(t) = C_1 + C_2 e^{-3t}$$

$$x'' + 3x' = t$$

$$x_p(t) = At^2 + Bt + C$$

$$x_p(t)' = 2At + B$$

$$x_p(t)'' = 2A$$

$$\Rightarrow \underline{\underline{2A}} + \underline{6At} + \underline{\underline{3B}} = \underline{t}$$

$$\Rightarrow 6A = 1 \Rightarrow A = \frac{1}{6} \qquad \text{set } C = 0$$

$$2A + 3B = 0 \Rightarrow B = -\frac{1}{9}$$

$$\Rightarrow x_p(t) = \frac{1}{6}t^2 - \frac{1}{9}t$$

$$\Rightarrow x(t) = x_h(t) + x_p(t) = C_1 + C_2 e^{-3t} + \frac{1}{6}t^2 - \frac{1}{9}t$$

$$x(0) = 5 \iff c_1 + c_2 e^0 + \frac{1}{6} \cdot 0^2 - \frac{1}{9} \cdot 0 = 5$$

$$\Rightarrow c_1 + c_2 = 5 \quad (1)$$

$$x'(t) = -3c_2 e^{-3t} + \frac{1}{3} t - \frac{1}{9}$$

$$x'(0) = 0 \Rightarrow -3c_2 + \frac{1}{3} \cdot 0 - \frac{1}{9} = 0$$

$$\Rightarrow -3c_2 = \frac{1}{9} \Rightarrow c_2 = -\frac{1}{27}$$

$$(1) \Rightarrow c_1 - \frac{1}{27} = 5 \Rightarrow c_1 = \overset{27}{5} + \frac{1}{27} = \frac{136}{27}$$

$\Rightarrow$ sol of ivp is: $x(t) = \dfrac{136}{27} - \dfrac{1}{27} e^{-3t} + \dfrac{1}{6} t^2 - \dfrac{1}{9} t$

2. a) Find the linear homogeneous differential equation with constant coefficients of minimal order that has the general solution:

$C_1 + C_2 e^{-t} + C_3 \cos t + C_4 \sin t$, where C_1, C_2, C_3, C_4 - real const

Step I: C_1 - constant term

$\Rightarrow$ root $r=0$ in characteristic eq

$C_2 e^{-t}$ - exponential term

$\Rightarrow$ root $r=-1$

$C_3 \cos t$ and $C_4 \sin t$ $\Rightarrow$ roots $r=i$ and $r=-i$

Step II: $(r-0)(r+1)(r-i)(r+i)=0$

$r(r+1)(r^2+1)=0$

$r(r^3+r^2+r+1)=0$

$r^4+r^3+r^2+r=0$

$\Rightarrow x'''' + x''' + x'' + x' = 0$

b) Let the equation: $x^{IV} + x''' + x'' + x' = 0$

Find the general solution of this equations. Find the constants such that the general solution is periodic. Find the main period.

$$r^4 + r^3 + r^2 + r = 0$$

$$r(r^3 + r^2 + r + 1) = 0$$

I $r = 0$

II $r^3 + r^2 + r + 1 = 0$

$$r^2(r+1) + (r+1) = 0$$

$$(r^2+1)(r+1) = 0$$

$\Rightarrow r+1 = 0 \Rightarrow r = 1$

$r^2+1 = 0 \Rightarrow r = \pm i$

General sol: $x(t) = C_1 + C_2 e^{-t} + C_3 \cos t + C_4 \sin t$

The sol is periodic if $C_1 = C_2 = 0$ (non periodic terms)

$\Rightarrow x(t) = C_3 \cos t + C_4 \sin t$

The period for sin and cos is 2π

$\Rightarrow$ the main period is $T = 2\pi$

3. Consider the following linear planar system

$\dot{x} = x - 4y \quad , \quad \dot{y} = x + y$

a) Find its general solution and its flow

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 1 & -4 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\det(A - \lambda I_2) = 0$$

$$\begin{vmatrix} 1-\lambda & -4 \\ 1 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - (-4)\cdot 1 = \lambda^2 - 2\lambda + 5$$

$$\lambda^2 - 2\lambda + 5 = 0$$

$$\Delta = 4 - 20 = -16$$

$$\lambda_{1,2} = \frac{2 \pm \sqrt{-16}}{2} \quad \begin{cases} \lambda_1 = 1+2i \\ \lambda_2 = 1-2i \end{cases}$$

I $\lambda_1 = 1+2i$

$$\begin{pmatrix} 1-(1+2i) & -4 \\ 1 & 1-(1+2i) \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = 0$$

$$\begin{cases} -2i w_1 - 4 w_2 = 0 \Rightarrow -4i^2 w_2 - 4 w_2 = 0 \Leftrightarrow 0 = 0 \Rightarrow \text{we can chose } w_2 = 1 \\ w_1 - 2i w_2 = 0 \Rightarrow w_1 = 2i w_2 = 2i \end{cases}$$

$$\Rightarrow v_1 = \begin{pmatrix} 2i \\ 1 \end{pmatrix}$$

II $\lambda = 1-2i$

$$\begin{pmatrix} 2i & -4 \\ 1 & 2i \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} 2iw_1 - 4w_2 = 0 \Rightarrow 4w_2 - 4w_2 = 0 \Leftrightarrow 0=0 \Rightarrow \text{we take } w_2 = 1 \\ w_1 + 2iw_2 = 0 \Rightarrow w_1 = -2iw_2 = -2i \end{cases}$$

$$v_2 = \begin{pmatrix} -2i \\ 1 \end{pmatrix}$$

$$X(t) = e^{\lambda t} \cdot v$$

I $\lambda_1 = 1+2i$ $\quad v_1 = \begin{pmatrix} 2i \\ 1 \end{pmatrix}$

$$\Rightarrow X(t) = e^{(1+2i)t}\begin{pmatrix} 2i \\ t \end{pmatrix}$$

II $\lambda_2 = 1-2i$ $\quad v_2 = \begin{pmatrix} -2i \\ 1 \end{pmatrix}$

$$\Rightarrow X(t) = e^{(1-2i)t}\begin{pmatrix} -2i \\ t \end{pmatrix}$$

Use Euler's formula: $e^{i\theta} = \cos\theta + i\sin\theta$

$$e^{(1+2i)t} = e^t e^{2it} = e^t\left(\cos(2t) + i\sin(2t)\right)$$

$$e^{(1-2i)t} = e^t e^{-2it} = e^t\left(\cos(2t) - i\sin(2t)\right)$$

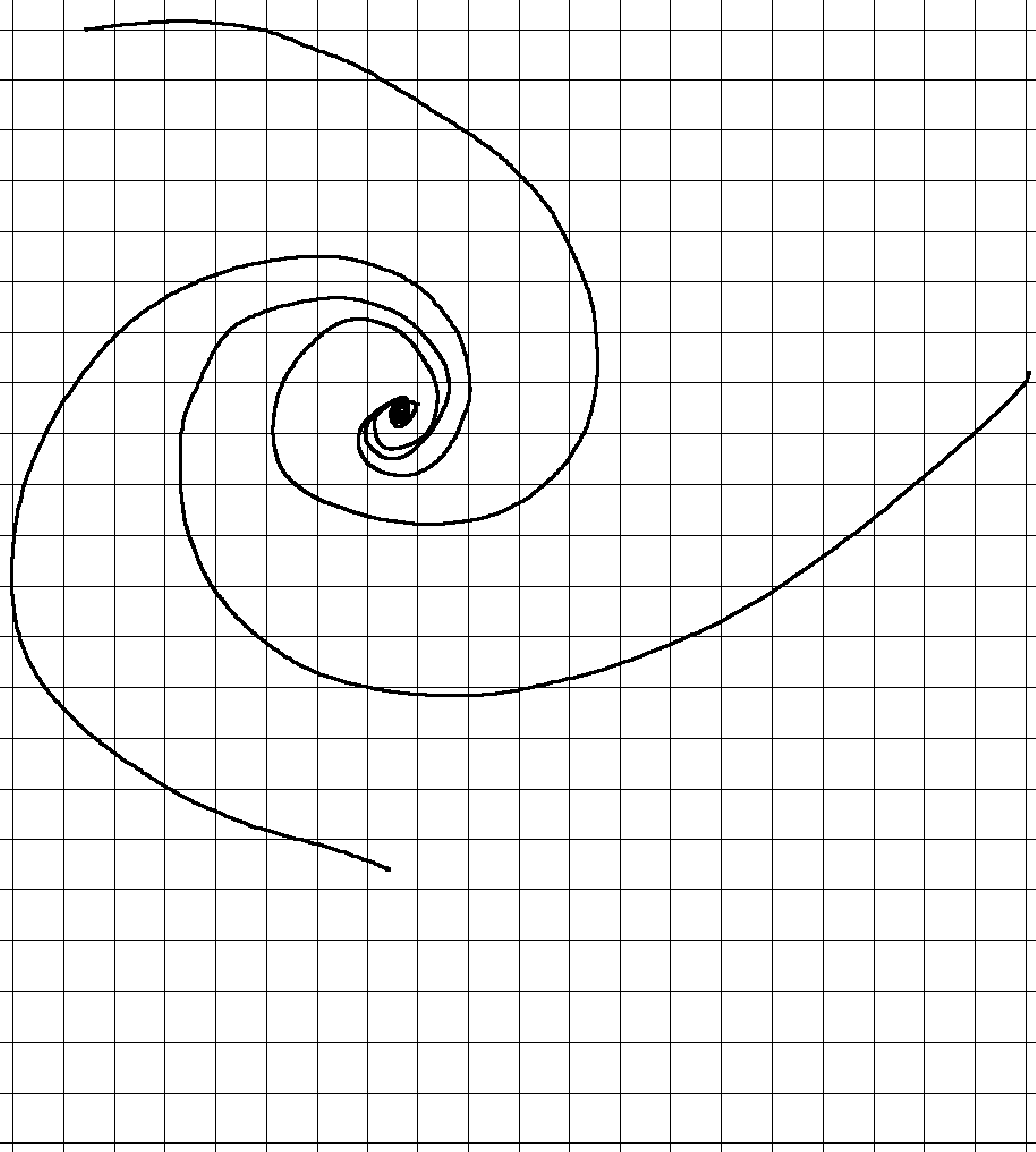
$$\Rightarrow \begin{cases} e^t\left(\cos(2t) + i\sin(2t)\right)\begin{pmatrix} 2i \\ t \end{pmatrix} = e^t\left(\begin{pmatrix} -2\sin 2t \\ \cos 2t \end{pmatrix} + i\begin{pmatrix} 2\cos 2t \\ \sin 2t \end{pmatrix}\right) \quad (1) \\ e^t\left(\cos(2t) - i\sin(2t)\right)\begin{pmatrix} -2i \\ t \end{pmatrix} = e^t\left(\begin{pmatrix} 2\sin 2t \\ \cos 2t \end{pmatrix} - i\begin{pmatrix} 2\cos 2t \\ \sin 2t \end{pmatrix}\right) \quad (2) \end{cases}$$

$$(1)+(2) = e^t\left(\begin{pmatrix}-2\sin 2t\\ \cos 2t\end{pmatrix} + \begin{pmatrix}2\sin 2t\\ \cos 2t\end{pmatrix}\right)$$

General sol:

$$\begin{pmatrix}x(t)\\ y(t)\end{pmatrix} = e^t\left(A\begin{pmatrix}-2\sin 2t\\ \cos 2t\end{pmatrix} + B\begin{pmatrix}2\sin 2t\\ \cos 2t\end{pmatrix}\right)$$

The flow:



b) Study the type and stability of it's equilibrium point (0,0)

Analyze eigenvalues

I check real part:

1. both real parts are negative
   $\Rightarrow$ trajectories are attracted to the equilibrium point, indicating stability
2. both real parts are positive
   $\Rightarrow$ the trajectories are repelled from the equilibrium point, indicating instability
3. real parts with diff. signs.
   $\Rightarrow$ equilibrium point is a saddle point which is unstable.

II Check Imaginary part.

- purely real eigenvalues $\Rightarrow$ No oscillations
- complex eigenvalues $\Rightarrow$ presence of oscillations (the system exhibits spiraling trajectories)

For our system the eigenvalues are:

$$\lambda_1 = 1+2i$$

$$\lambda_2 = 1-2i$$

$\left.\begin{array}{l}\text{Real part: both positive} \Rightarrow \text{repeller, unstable} \\ \text{imaginary part} \Rightarrow \text{oscillatory behavior}\end{array}\right\} \Rightarrow$

$\Rightarrow$ the type is a spiral.

Sample test 2:

1. Represent the phase portrait of the scalar dynamical system

$$\dot{x} = ax - 1 \quad \text{where } a \in \mathbb{R}^{*} \text{ is a parameter}$$

Determine the stability character of the equilibrium point using the linearization method. Find $\varphi(t, 1/a)$ and determine the property of $\varphi(t, 1)$. Take into account that you may need to consider distinct cases depending on a.

Step I: Equilibrium points occur when $\dot{x} = 0$

$$ax - 1 = 0 \Rightarrow x = \frac{1}{a}$$

Step II: Stability analysis using linearization

$$\dot{x} = ax - 1 \Rightarrow f(x) = ax - 1$$

$f'(x) = a$

$f'\left(\frac{1}{a}\right) = a$

if $a < 0 \Rightarrow$ equilibrium point is stable

if $a > 0 \Rightarrow$ equilibrium point is unstable

Step III: Find the solution $\varphi(t, \frac{1}{a})$

Solve $\dot{x} = ax - 1$ with initial cond. $x(0) = \frac{1}{a}$ (from step I)

general sol: $x(t) = Ce^{at} + \frac{1}{a}$

$x(0) = \frac{1}{a} \Rightarrow \frac{1}{a} = C + \frac{1}{a} \Rightarrow C = 0$

$\Rightarrow x(t) = \frac{1}{a}$

Step IV: Determine the property of $\varphi(t, 1)$

Solve $\dot{x} = ax - 1$ with initial cond. $x(0) = 1$

general sol: $x(t) = Ce^{at} + \frac{1}{a}$

$x(0) = 1 \Rightarrow 1 = C + \frac{1}{a} \Rightarrow C = 1 - \frac{1}{a}$

$\Rightarrow x(t) = \left(1 - \frac{1}{a}\right) e^{at} + \frac{1}{a}$

Step 5: Phase portrait

Case 1: $a > 0 \Rightarrow \quad \rightarrow x = \frac{1}{a} \leftarrow$ ↙ arrows pointing towards $\frac{1}{a}$

$\rightarrow x < \frac{1}{a} \rightarrow \frac{1}{a} \leftarrow x > \frac{1}{a} \leftarrow$

Case 2: $a < 0 \Rightarrow \quad \leftarrow x = \frac{1}{a} \rightarrow$ « arrows pointing away from $\frac{1}{a}$

$\leftarrow x < \frac{1}{a} \leftarrow \frac{1}{a} \rightarrow x > \frac{1}{a} \rightarrow$

Case 3: $a = 0 \Rightarrow x$ decreases linearly over time

2. Consider the following linear planar system.

$$\dot{x} = 4y \qquad \dot{y} = -x$$

a) Find its general solution and its flow.

$$\begin{cases} \dot{x} = 4y \\ \dot{y} = -x \end{cases}$$

$$\ddot{x} = 4\dot{y} = 4\cdot(-x) = -4x \Rightarrow \ddot{x} + 4x = 0$$

$$r^2 + 4 = 0$$

$$\Rightarrow r = \pm 2i$$

$$x(t) = C_1 \cos 2t + C_2 \sin 2t$$

$$\dot{x} = 4y \Leftrightarrow$$

$$\Leftrightarrow 4y = \frac{d}{dt}(C_1 \cos 2t + C_2 \sin 2t)$$

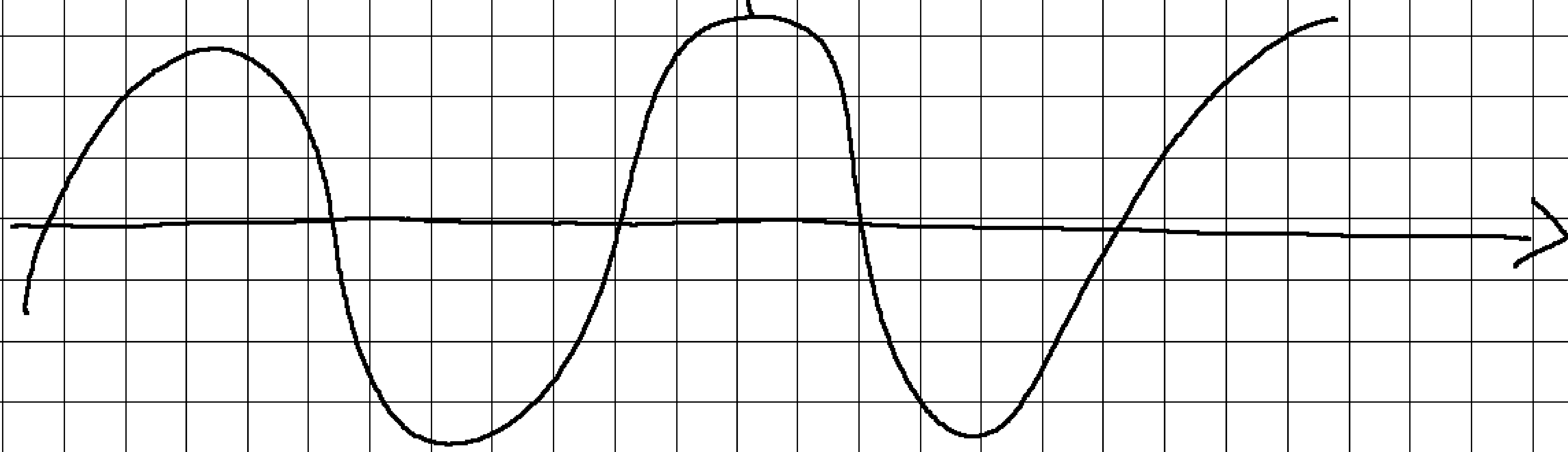
$$4y = -2C_1 \sin 2t + 2C_2 \cos 2t$$

$$y = \frac{1}{2}(-C_1 \sin 2t + C_2 \cos 2t)$$

general sol: $\begin{cases} x(t) = C_1 \cos 2t + C_2 \sin 2t \\ y(t) = \frac{1}{2}(-C_1 \sin 2t + C_2 \cos 2t) \end{cases}$

$$\begin{cases} x(0) = C_1 \cos 0 + C_2 \sin 0 = C_1 \\ y(0) = \frac{1}{2}(-C_1 \sin 0 + C_2 \cos 0) = \frac{1}{2} C_2 \end{cases} \Rightarrow \begin{array}{l} C_1 = x_0 \\ C_2 = 2y_0 \end{array}$$

The flow $\varphi_t(x_0, y_0) = \left(x_0 \cos 2t + 2y_0 \sin 2t, -\frac{1}{2}x_0 \sin 2t + y_0 \cos 2t\right)$



b) Study the type and stability of its equilibrium point $(0,0)$

$$A = \begin{pmatrix} 0 & 4 \\ -1 & 0 \end{pmatrix}$$

$$\det(A - \lambda I_2) = 0$$

$$\begin{vmatrix} -\lambda & 4 \\ -1 & -\lambda \end{vmatrix} = 0 \Leftrightarrow \lambda^2 + 4 = 0 \Rightarrow \lambda = \pm 2i$$

$\Rightarrow$ eigenvalues are purely imaginary $\Rightarrow$

$\Rightarrow$ oscilations $\Rightarrow$ type is a center

Stability: Neutrally stable (because it's a center trajectories remain bounded in closed orbits)

c) Find the shape of the orbits:

Square and add up the general sol:

$$x^2(t) = (C_1 \cos 2t + C_2 \sin 2t)^2$$
$$4y^2(t) = (-C_1 \sin 2t + C_2 \cos 2t)^2 \quad \Bigg| \; (+)$$

$$x^2(t) + 4y^2(t) = (C_1 \cos 2t + C_2 \sin 2t)^2 + (-C_1 \sin 2t + C_2 \cos 2t)^2$$

$$\Leftrightarrow x^2(t) + 4y^2(t) = C_1^2 \cos^2 2t + \cancel{2C_1C_2 \cos 2t \sin 2t} + C_2^2 \sin^2 2t +$$
$$+ C_1^2 \sin^2 2t - \cancel{2C_1C_2 \sin 2t \cos 2t} + C_2^2 \cos^2 2t$$

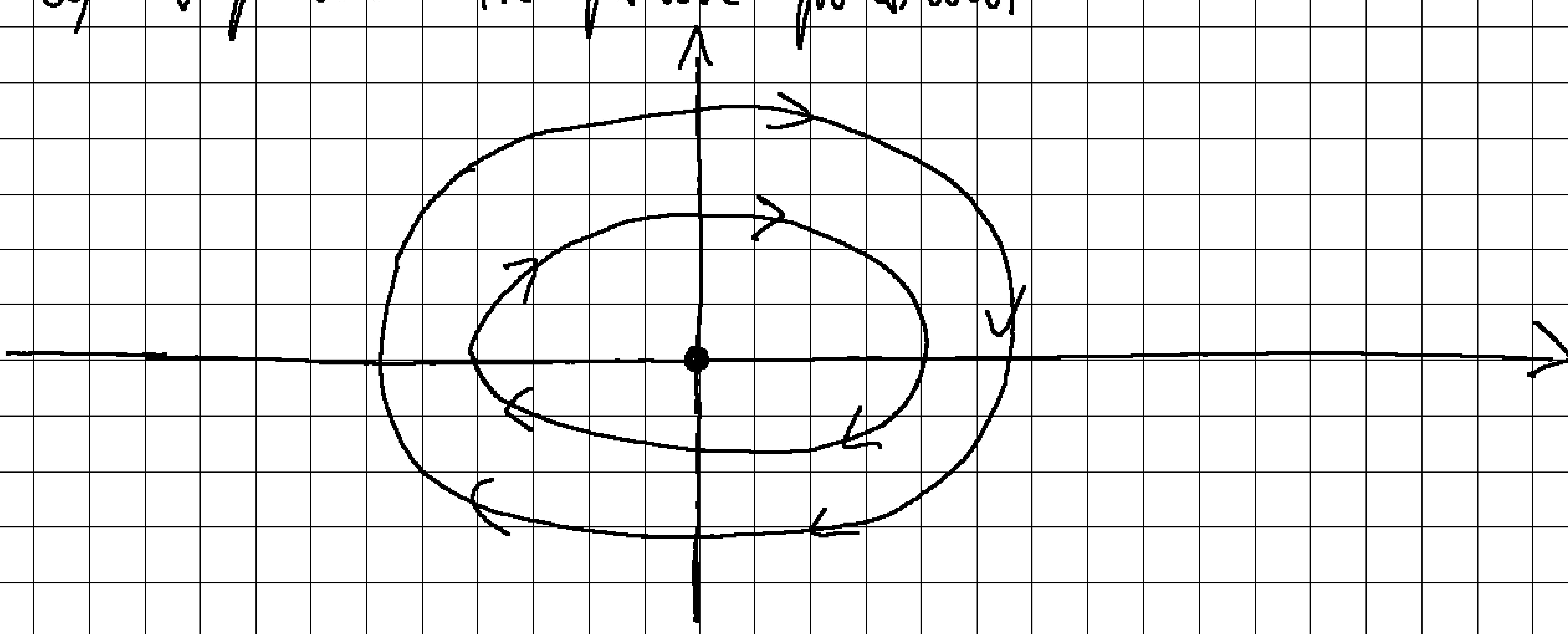
$$x^2(t) + 4y^2(t) = C_1^2(\underbrace{\cos^2 2t + \sin^2 2t}_{=1}) + C_2^2(\underbrace{\sin^2 2t + \cos^2 2t}_{=1}) \Leftrightarrow$$

$$\Leftrightarrow x^2(t) + 4y^2(t) = C_1^2 + C_2^2$$

Let $C = C_1^2 + C_2^2$

$\Rightarrow x^2 + 4y^2 = C \Rightarrow$ Shape is an ellipse

d) Represent its phase portrait



3. a) Find a particular solution of the form $x_p(t) = a t^2 e^t$ (where the real coefficient a has to be determined for $x'' - 2x' + x = e^t$

$$x_p(t) = a t^2 e^t$$

$$x_p'(t) = a(2te^t + t^2e^t) = 2ate^t + at^2e^t$$

$$x_p''(t) = 2ae^t + 2ate^t + 2ate^t + at^2e^t$$

$$x'' - 2x' + x = e^t \Rightarrow 2ae^t + \cancel{2ate^t} + \cancel{2ate^t} + \cancel{at^2e^t} -$$
$$- \cancel{4ate^t} - \cancel{2at^2e^t} + \cancel{at^2e^t} = e^t$$

$$\Rightarrow 2ae^t = e^t \Rightarrow 2a = 1 \Rightarrow a = \frac{1}{2}$$

$$\Rightarrow x_p(t) = \frac{1}{2} t^2 e^t$$

b) Find the solution of the ivp $\begin{cases} x'' - 2x' + x = 0 \\ x(0) = 5 \\ x'(0) = 0 \end{cases}$

$$r^2 - 2r + 1 = 0$$
$$(r-1)^2 = 0 \Rightarrow r = 1$$

$$x_h(t) = (C_1 + C_2 t)e^t$$

$$x(t) = x_h(t) + x_p(t) = (C_1 + C_2 t)e^t + \frac{1}{2}t^2e^t$$

$x(0)=5 \Rightarrow \quad C_1=5$

$x'(t) = (5+C_2t)'e^t + (5+C_2t)e^t + \left(\frac{1}{2}t^2\right)' + \left(\frac{1}{2}t^2\right)e^t$

$x'(t) = (C_2+5+C_2t+t)e^t$

$x'(0)=0 \Rightarrow \quad 5+C_2=0 \Rightarrow C_2=-5$

$\Rightarrow$ sol of IVP is $\quad x(t) = (5-5t)e^t + \frac{1}{2}t^2e^t$

Exercises:

1) Find the flow of $\dot{x} = -3(x-21)$

$$\dot{x} = -3(x-21)$$

$$\frac{dx}{dt} = -3(x-21) \Leftrightarrow \frac{dx}{x-21} = -3\,dt \quad / \int \Leftrightarrow$$

$$\Leftrightarrow \int \frac{1}{x-21}\,dx = -\int 3\,dt$$

$$\Leftrightarrow \ln|x-21| + c_1 = -3t + c_2$$

Let $c_2 - c_1 = c$

$$\ln|x-21| = -3t + c$$

$$|x-21| = e^{-3t+c}$$

$$|x-21| = e^{-3t} \cdot e^{c}$$

Let $e^{c} = k$ - a const

$$|x-21| = e^{-3t} \cdot k$$

$$\Rightarrow \text{I} \quad x-21 = e^{-3t} \cdot k$$

$$\text{II} \quad x-21 = -k \cdot e^{-3t}$$

So k can be any const

$$\Rightarrow \quad x-21 = c \cdot e^{-3t}$$

$$\Rightarrow x(t) = c \cdot e^{-3t} + 21$$

2. a) Find a solution of the form $x_p = a\cos t + b\sin t$ of $x'' + x' + x = 2\cos t$

$$x_p' = -a\sin t + b\cos t$$

$$x_p'' = -a\cos t - b\sin t$$

$$\cancel{-a\cos t} \cancel{- b\sin t} - a\sin t + b\cos t + \cancel{a\cos t} + \cancel{b\sin t} = 2\cos t$$

$$-a\sin t + b\cos t = 2\cos t$$

$$\Rightarrow \quad a = 0$$
$$b = 2$$

$$\Rightarrow \quad x_p = 2\sin t$$

b) Find the unique solution of the ivp

$$\begin{cases} x'' + x' + x = 2\cos 2t \\ x(0) = 0 \\ x'(0) = 0 \end{cases}$$

$$x'' + x' + x = 0$$

$$r^2 + r + 1 = 0$$

$$\Delta = -3 \quad \Rightarrow r_{1,2} = \frac{-1 \pm \sqrt{-3}}{2} \begin{cases} r_1 = \frac{-1 + \sqrt{3}i}{2} \\ r_2 = \frac{-1 - \sqrt{3}i}{2} \end{cases}$$

$$\Rightarrow x_h(t) = C_1 e^{-\frac{1}{2}t} \cos \frac{\sqrt{3}}{2}t + C_2 e^{-\frac{1}{2}t} \sin \frac{\sqrt{3}}{2}t$$

$$x'' + x' + x = 2\cos 2t$$

$$x_p = A\cos 2t + B\sin 2t$$

$$x_p' = -2A\sin 2t + 2B\cos 2t$$

$$x_p'' = -4A\cos 2t - 4B\sin 2t$$

$$-4A\cos 2t - 4B\sin 2t - 2A\sin 2t + 2B\cos 2t + A\cos 2t + B\sin 2t$$
$$= 2\cos 2t$$

$$\Leftrightarrow -3A\cos 2t + 2B\cos 2t - 3B\sin 2t - 2A\sin 2t = 2\cos 2t$$

$$\Rightarrow \begin{cases} -3A + 2B = 2 \Rightarrow {}^{3/}-3A + \frac{4A}{-3} = 2 \Rightarrow -\frac{13A}{3} = 2 \Rightarrow \\ -3B - 2A = 0 \Rightarrow -3B = 2A \Rightarrow B = \frac{2A}{-3} = \frac{12}{39} \end{cases}$$

$$\Rightarrow A = -\frac{6}{13}$$

$$\Rightarrow x_p(t) = -\frac{6}{13}\cos 2t + \frac{12}{39}\sin 2t$$

$$\Rightarrow x(t) = C_1 e^{-\frac{1}{2}t}\cos\frac{\sqrt{3}}{2}t + C_2 e^{-\frac{1}{2}t}\sin\frac{\sqrt{3}}{2}t - \frac{6}{13}\cos 2t +$$
$$+ \frac{12}{39}\sin 2t$$

$$x(0) = 0 \Leftrightarrow C_1 - \frac{6}{13} = 0 \Rightarrow C_1 = \frac{6}{13}$$

$$x'(t) = C_1 \cdot \left(-\frac{1}{2} e^{-\frac{1}{2}t} \cdot \cos \frac{\sqrt{3}}{2}t - e^{-\frac{1}{2}t} \cdot \sin \frac{\sqrt{3}}{2}t \cdot \frac{\sqrt{3}}{2} \right) +$$

$$+ C_2 \left(-\frac{1}{2} e^{-\frac{1}{2}t} \cdot \sin \frac{\sqrt{3}}{2}t + e^{-\frac{1}{2}t} \cdot \cos \frac{\sqrt{3}}{2}t \cdot \frac{\sqrt{3}}{2} \right) + \frac{6}{13} \sin 2t$$

$$+ \frac{12}{39} \cos 2t$$

$$x'(0) = 0 \iff -\frac{C_1}{2} + C_2 \cdot \frac{\sqrt{3}}{2} + \frac{12}{39} = 0$$

$$\iff -\frac{\cancel{6}^{3}}{\cancel{26}_{13}} + C_2 \cdot \frac{\sqrt{3}}{2} + \frac{\cancel{12}^{4}}{\cancel{39}_{13}} = 0$$

$$\iff C_2 \cdot \frac{\sqrt{3}}{2} = \frac{3}{13} - \frac{4}{13}$$

$$\iff C_2 = -\frac{1}{13} \cdot \frac{2}{\sqrt{3}} = -\frac{2}{13\sqrt{3}}$$

$$x(t) = \frac{6}{13} e^{-\frac{1}{2}t} \cos \frac{\sqrt{3}}{2}t - \frac{2}{13\sqrt{3}} e^{-\frac{1}{2}t} \sin \frac{\sqrt{3}}{2}t - \frac{6}{13} \cos 2t +$$

$$+ \frac{12}{39} \sin 2t$$

3. Find a polynomial solution of $x' = -2x + 7t^3$

$x_p(t) = at^3 + bt^2 + ct + d$

$x_p'(t) = 3at^2 + 2bt + c$

$\Rightarrow \quad 3at^2 + 2bt + c = -2at^3 - 2bt^2 - 2ct - 2d + 7t^3$

$\Leftrightarrow \quad 3at^2 + 2bt + c + 2at^3 + 2bt^2 + 2ct + 2d - 7t^3 = 0$

$\Leftrightarrow \quad (2a - 7)t^3 + (3a + 2b)t^2 + (2b + 2c)t + (c + 2d) = 0$

$$\Rightarrow \begin{cases} 2a - 7 = 0 & \Rightarrow a = \frac{7}{2} \\ 3a + 2b = 0 & \Rightarrow 2b = -\frac{21}{2} \Rightarrow b = -\frac{21}{4} \\ 2b + 2c = 0 & \Rightarrow 2c = \frac{21}{2} \Rightarrow c = \frac{21}{4} \\ c + 2d = 0 & \end{cases}$$

$\Rightarrow \quad 2d = -\frac{21}{4} \Rightarrow d = \frac{21}{8}$

$\Rightarrow \quad x_p(t) = \frac{7}{2}t^3 - \frac{21}{4}t^2 + \frac{21}{4}t + \frac{21}{8}$

4. Find the general solution of

a) $x' + \frac{1}{t}x = e^{-3t}$

$x' + \frac{1}{t}x = 0$

$x' = -\frac{1}{t}x$

$\frac{dx}{dt} = -\frac{1}{t} \cdot x \iff \frac{1}{x}dx = -\frac{1}{t}dt \ / \int \iff$

$\iff \ln|x| + c_1 = -\ln|t| + c_2$ Let $c_2 - c_1 = c$

$\iff \ln|x| + \ln|t| = c$

$\iff \ln|xt| = c$

$\iff xt = c \Rightarrow x_h = \frac{c}{t}$

$x_p(t) = A \cdot e^{-3t} \Rightarrow x_p(t) = \frac{1}{3t+1} e^{-3t}$

$x_p' = -3A \cdot e^{-3t}$

$\Rightarrow x(t) = \frac{c}{t} + \frac{1}{3t+1} \cdot e^{-3t}$

$-3A \cdot e^{-3t} + \frac{1}{t} \cdot A \cdot e^{-3t} = e^{-3t}$

$\iff \left(3A + \frac{A}{t}\right) \cdot e^{-3t} = e^{-3t}$

$-3A + \frac{A}{t} = 1 \Rightarrow 3At + A = 1 \Rightarrow A(3t+1) = 1$

$\Rightarrow A = \frac{1}{3t+1}$

b) $x' + 3t^2 x = -1$

$x' + 3t^2 x = 0$

$x' = -3t^2 x \Leftrightarrow \frac{dx}{dt} = -3t^2 x \Leftrightarrow \frac{1}{x} dx = -3t^2 dt \quad / \int$

$\Rightarrow \ln|x| + c_1 = -t^3 + c_2$ Let $c_2 - c_1 = c$

$\ln|x| = -t^3 + c$

$|x| = e^{-t^3+c}$

$|x| = e^{-t^3} \cdot e^c$ Let $e^c = k$

$\Rightarrow$ I $x = k e^{-t^3}$

II $x = -k e^{-t^3}$ Let $k = c$ for any const

$\Rightarrow x_h(t) = c \cdot e^{-t^3}$

$x_p = A \Rightarrow x_p = -\frac{1}{3t^2}$

$x_p' = 0$

$3t^2 \cdot A = -1 \Rightarrow A = -\frac{1}{3t^2}$

$\Rightarrow x(t) = c \cdot e^{-t^3} - \frac{1}{3t^2}$

5. We consider the scalar lin syst. $\dot{x} = x - 2x^3$
Find its equilibria and study their stability using the linearization method. Represent the phase portrait. Find $\varphi(t,0)$. Describe the properties of $\varphi(t,0.2)$ and $\varphi(t,20)$.

Step I: Find equilibria

$\dot{x} = 0 \Rightarrow \quad x - 2x^3 = 0 \quad \Rightarrow x(1-2x^2) = 0$

I $x = 0$

II $1 - 2x^2 = 0 \Rightarrow \quad 2x^2 = 1 \Rightarrow x = \pm\sqrt{\frac{1}{2}}$

equilibria are $x = 0$, $x = \sqrt{\frac{1}{2}}$, $x = -\sqrt{\frac{1}{2}}$

Step 2: Study stability using linearization method

$f(x) = x - 2x^3$

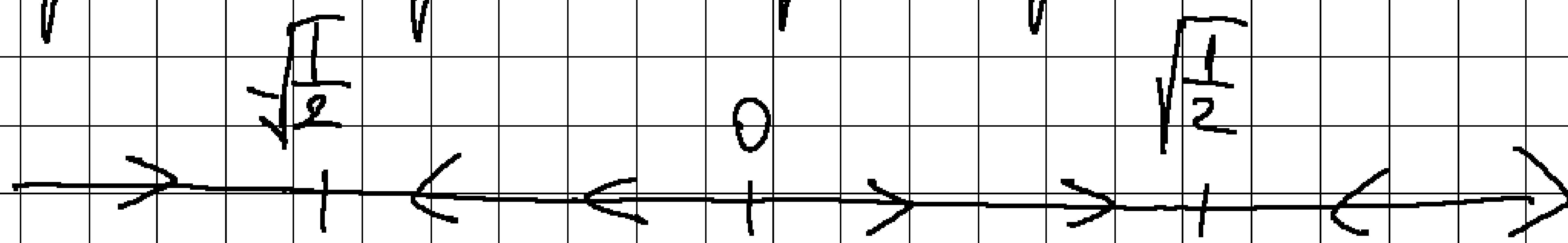
$f'(x) = 1 - 6x^2$

I $x = 0 \Rightarrow f'(0) = 1 > 0 \quad \Rightarrow$ unstable equilibrium

II $x = \sqrt{\frac{1}{2}} \Rightarrow f'\left(\sqrt{\frac{1}{2}}\right) = 1 - 3 = -2 < 0 \Rightarrow$ stable equilibrium

III $x = -\sqrt{\frac{1}{2}} \Rightarrow f'\left(-\sqrt{\frac{1}{2}}\right) = 1 - 3 = -2 < 0 \Rightarrow$ stable equilibrium

Step III: Represent phase portrait



Step IV: Find $\varphi(t,0{,}2)$, $\varphi(t,20)$

The solutions $\varphi(t,x_0)$ represent the trajectories of the system starting from the initial conditions $x(0)=0{,}2$ and $x(0)=20$

I $x(0)=0{,}2$

- $0{,}2$ is closer to $x=0$ (is unstable) $\Rightarrow$ it will move away and approach $x=\sqrt{\frac{1}{2}}$

II $x(0)=20$ $\Rightarrow$ will move towards $x=\sqrt{\frac{1}{2}}$

6. Specify the type and stability of the linear system $\dot{x} = x + y \quad , \dot{y} = -2x + 4y$

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\det(A - \lambda I_2) = 0$$

$$\begin{vmatrix} 1-\lambda & 1 \\ -2 & 4-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 - 5\lambda + 6 = 0$$

$$\Delta = 25 - 24 = 1$$

$$\Rightarrow \lambda_{1,2} = \frac{5 \pm 1}{2} \quad \begin{cases} \lambda_1 = 3 \\ \lambda_2 = 2 \end{cases}$$

Both eigenvalues are positive $\Rightarrow$ unstable and the type is a repeller

7. Find a global first integral of $\dot{x} = -7y$, $\dot{y} = 9x$. Represent its phase portrait

8. Find the equilibria and study their stability for

$\dot{x} = -x + xy \qquad \dot{y} = -2y + 3y^2$

$\dot{x} = 0 \Rightarrow -x + xy = 0 \Rightarrow x(-1+y) = 0$

$\Rightarrow x = 0 \qquad y = 1$

$\dot{y} = 0 \Rightarrow -2y + 3y^2 = 0 \qquad y(-2+3y) = 0$

$\Rightarrow y = 0 \qquad y = \frac{2}{3}$

eq. points: $(x,y) = \left\{ (0,0), \left(0,\frac{2}{3}\right) \right\}$

$$J = \begin{pmatrix} \frac{d\dot{x}}{dx} & \frac{d\dot{x}}{dy} \\ \frac{d\dot{y}}{dx} & \frac{d\dot{y}}{dy} \end{pmatrix} = \begin{pmatrix} -1+y & x \\ 0 & -2+6y \end{pmatrix}$$

I $(x,y) = (0,0)$

$$J(0,0) = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix} \Rightarrow \lambda_1 = -1 \quad \lambda_2 = -2$$

both negative $\Rightarrow$ attractor (stable)

$\boxed{\mathrm{II}}$ $(x,y) = (0, \frac{2}{3})$

$$J(0,\tfrac{2}{3}) = \begin{pmatrix} -1+\frac{2}{3} & 0 \\ 0 & -2+6\cdot\frac{2}{3} \end{pmatrix} = \begin{pmatrix} -\frac{1}{3} & 0 \\ 0 & 2 \end{pmatrix}$$

$\Rightarrow \lambda_1 = -\frac{1}{3}$, $\lambda_2 = 2$ (oposite signs) $\Rightarrow$

$\Rightarrow$ saddle point (unstable)