13.2. Using the parity check matrix

$$H = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}$$ and the syndroms and coset leaders

Syndrome	000	001	010	011
coset leader	000000	001000	010000	000010

Syndrome	100	101	110	111
coset leader	100000	000110	000100	000011

Decode: 101110, 011000, 001011, 111111, 110011

$\sim$ equiv. relation on a set $A \Rightarrow A/\sim \{[\underbrace{<x>}_{\hookrightarrow \text{coset}}] \mid x \in A\}$

For us: $A = \mathbb{R}_2^n$

$v \sim w \Leftrightarrow v - w \in C$

To decode:

- Multiply the vector by H to get the syndrome
- Look up the syndrome in the table, find it's associated coset leader l
- Use l to correct v

$$v \rightsquigarrow v + l$$

- Extract the message from the corrected vector $v + l$

$$H \cdot v_1 = \begin{pmatrix} 1&0&0&1&0&1 \\ 0&1&0&1&1&1 \\ 0&0&1&0&1&1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\Rightarrow \quad l = 000000 \quad \Rightarrow \quad l + v_1 = 101110$

message $= 110$

$$H \cdot v_2 = \begin{pmatrix} 1&0&0&1&0&1 \\ 0&1&0&1&1&1 \\ 0&0&1&0&1&1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$\Rightarrow l = 000010 \Rightarrow l + v_2 = 011010$

$m_2 = 010$

$$H \cdot v_3 = \begin{pmatrix} 1&0&0&1&0&1 \\ 0&1&0&1&1&1 \\ 0&0&1&0&1&1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$\Rightarrow l = 000110 \Rightarrow v_3 + l = 001101 \Rightarrow$ message $= 101$

$$H \cdot v_4 = \begin{pmatrix} 1&0&0&1&0&1 \\ 0&1&0&1&1&1 \\ 0&0&1&0&1&1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$\Rightarrow l = 000110 \Rightarrow v_4 + l = 111001 \Rightarrow$ message $= 001$

$$H \cdot v_5 = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$\Rightarrow l = 010000 \Rightarrow v_5 + l = 100011 \Rightarrow$ message $= 011$

$$H \cdot v_6 = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\Rightarrow l = 000000 \Rightarrow v_6 + l = 001101 \Rightarrow$ message $= 101$

Syndrome	000	001	010	011
coset leader	000000	001000	010000	000010

for 101 we add together the column 4 and 5 $\Rightarrow$ coset leader is 000010 (with mds with mds)

Syndrome	100	101	110	111
coset leader	100000	000110	000100	000011

$$H \cdot \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad H \cdot \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad H \cdot \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$H \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

You can also look at H at the columns, column 4 is 110 so the coset leader is 000100 (1 on 4th position bcs it's the 4th column)

5. Construct a table of coset leaders and syndromes for the (7,4) - code with

$$H = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 \end{pmatrix}$$

S	000	001	010	011	100	101	110	111
C.l.	0000000	0010000	0100000	0110000	1000000	1010000	1100000	1110000

$$H \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \Rightarrow l = 1000000 \Rightarrow l+v = 1100100 \Rightarrow m = 100$$

8. Construct a table of coset leaders and syndromes for the (7,3)-code generated by $p = 1+x^2+x^3+x^4 \in \mathbb{Z}_2[x]$

$m_1 = 100$

$P_1 = 1$

$P_1 \cdot x^{n-k} = x^4$

x^4	$1+x^2+x^3+x^4$
$1+x^2+x^3+x^4$	1
$1+x^2+x^3$	

$Q_1 = 1+x^2+x^3+x^4 \Rightarrow 1011100$

$m_2 = 010$

$P_2 = X$

$P_2 \cdot X^{m-\alpha} = X^5$

$X^5 \mid 1+X^2+X^3+X^4$

$X+X^3+X^4+X^5 \mid X+1$

$1+X^2+X^3+X^4$

$1+X^2+X$

$Q_2 = 1+X^2+X+X^5$

111001_0

$m_3 = 001 \Rightarrow P_3 = X$

$P_3 \cdot X^4 = X^6$

$X^6 \mid 1+X^2+X^3+X^4$

$X^2+X^4+X^5+X^6 \mid X^2+X$

$X^2+X^4+X^5$

$X+X^3+X^4+X^5$

$X+X^2+X^3$

$Q_3 = X+X^2+X^3+X^6 \Rightarrow 0111001$

$$G = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$H = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \end{pmatrix}$$

S	Ct
0000	00000000
0001	00010000
0010	00100000
0011	00110000
0100	01000000
0101	00001100
0110	01100000
0111	00000001
1000	10000000
1001	00000011
1010	00011000
1011	00001000
1100	11000000
1101	11010000
1110	00000010
1111	00010100

6. Determine the parity check matrix and all syndromes and coset leaders of the $(7,3)$-code with generator matrix $G = \left(\frac{P}{I_3}\right) \in M_{7,3}(\mathbb{Z}_2)$ where

$$P = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$H = \begin{pmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{pmatrix}$$

$$G = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

S	00	01	10	11
CL	00000	01000	10000	00010

Decode:
10110

$$\begin{pmatrix} 10111 \\ 01010 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow \quad l = 00010 \quad \Rightarrow l + v = 10100$$