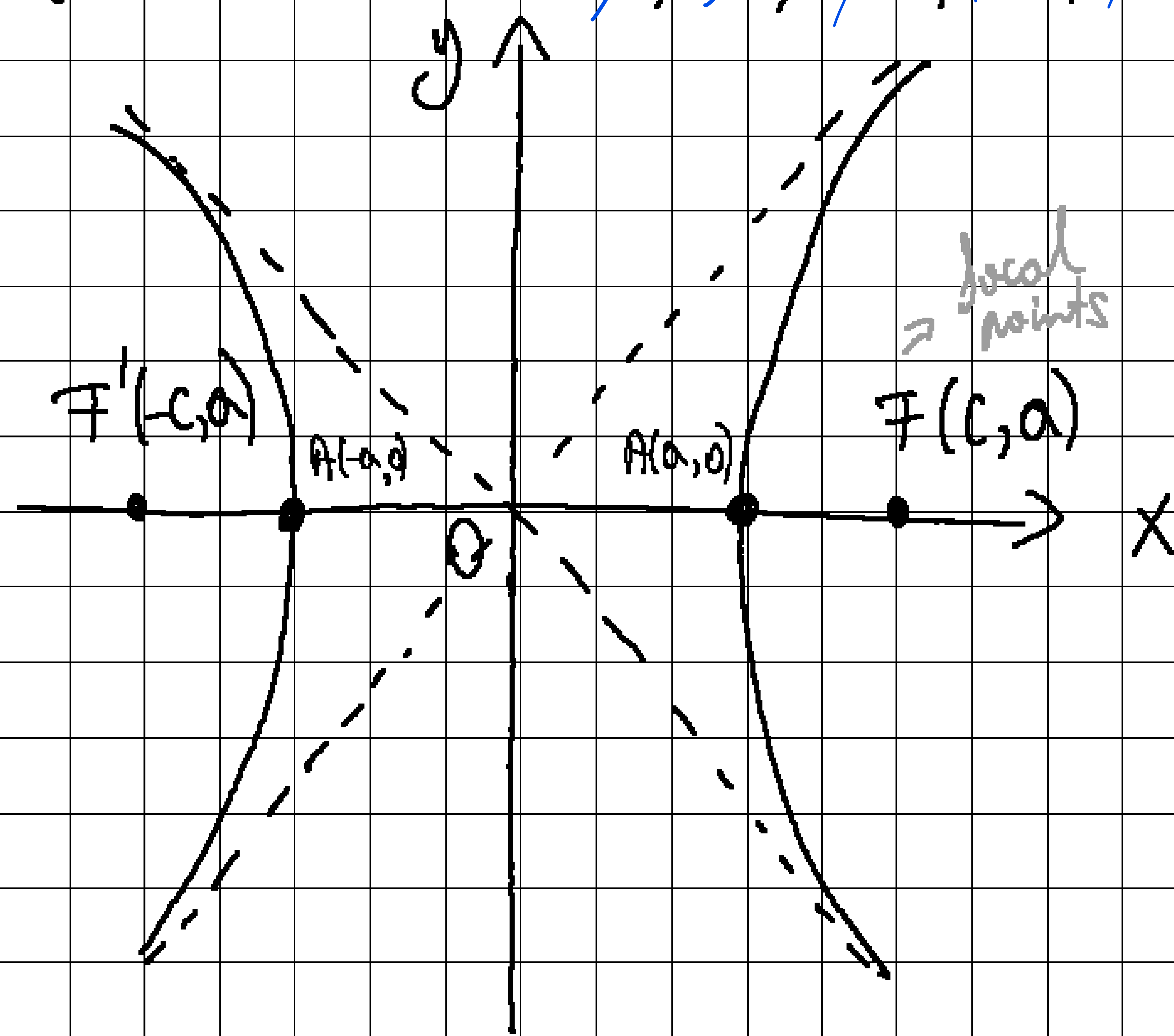
Problems: $2, \cancel{8}, \cancel{9}, \cancel{10}, \cancel{18}, \cancel{20}, \cancel{26}, \cancel{27}, \cancel{28}$ $\cancel{23}$



$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Assymptotes: $y = \frac{b}{a}x$

$y = -\frac{b}{a}x$

$c^2 = a^2 + b^2$

18. Determine the tangents to the hyperbola $H: x^2 - y^2 = 16$ which contain the point $M(-1, 7)$

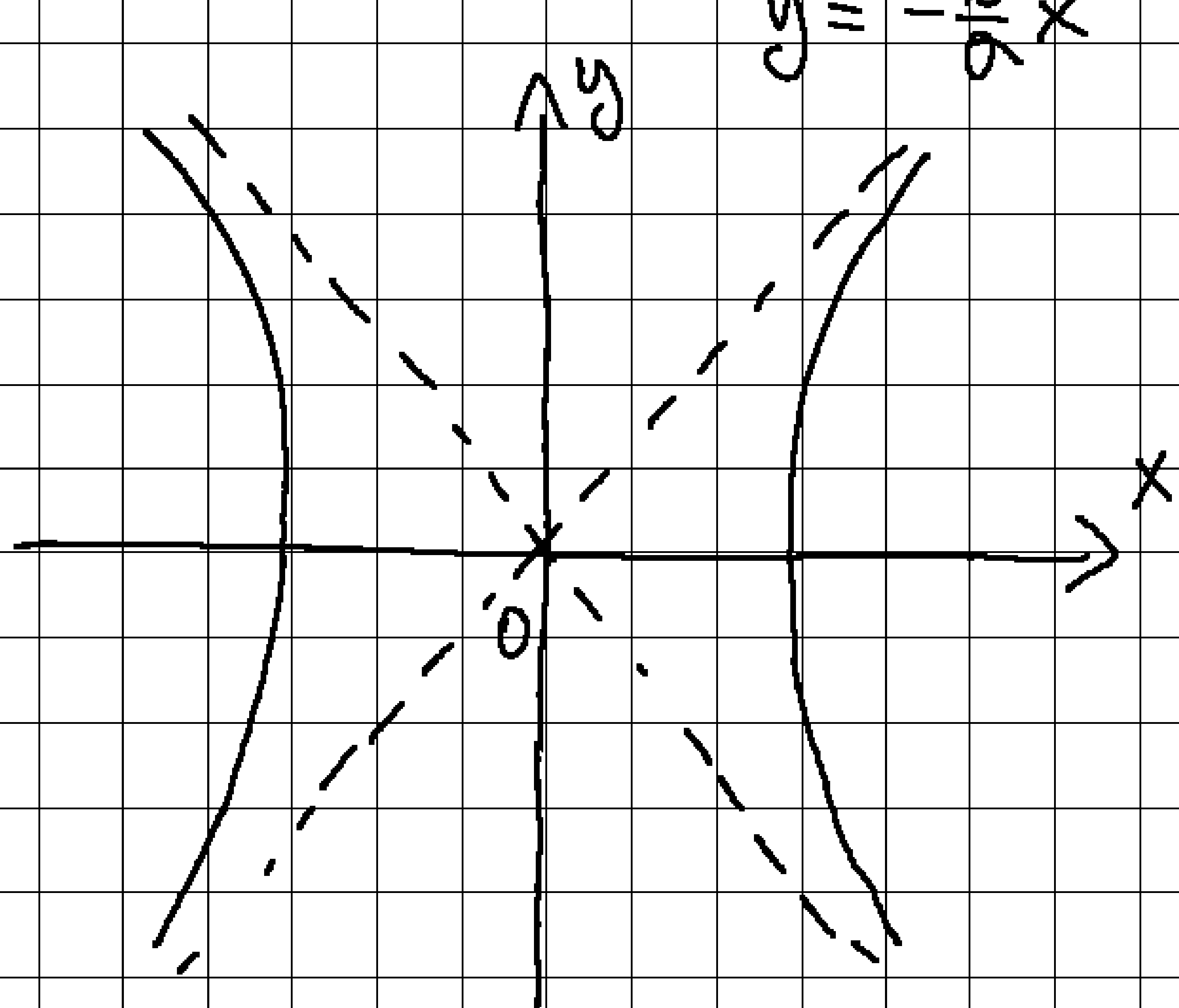
(Find the tangent through M)

$x^2 - y^2 = 16 \quad | : 16$

$\frac{x^2}{16} - \frac{y^2}{16} = 1 \Rightarrow a = 4, \ b = 4$

Assymptotes: $y = \frac{b}{a}x \Rightarrow y = x$

$y = -\frac{b}{a}x \Rightarrow y = -x$



$$\boxed{\frac{x x_0}{a^2} - \frac{y y_0}{b^2} = 1}$$

tangent through a point

Check if M is on the hyperbola

$(-1)^2-(7)^2=16 \Leftrightarrow 1-49=16 \quad \notin H$

then:

Let $P(x_0, y_0)$ be a point on H

$$\frac{x x_0}{16}-\frac{y y_0}{16}=1$$

$$\frac{-1 x_0}{16}-\frac{7 y_0}{16}=1$$

$$-x_0-7y_0=16$$

$$\begin{cases} -x_0-7y_0=16 \Rightarrow x_0=-7y_0-16 \\ x_0^2-y_0^2=16 \end{cases}$$

$$(-7y_0-16)^2-y_0^2=16$$

$$(-7y_0-16)(-7y_0+16)-y_0^2=16$$

$$49y_0^2-\cancel{112}y_0+\cancel{112}y_0+256-y_0^2=16$$

$$48y_0^2+240=0 \Leftrightarrow y_0^2=5 \Leftrightarrow y_0=\pm 5$$

$$x_0=-7\cdot y_0-16 \Rightarrow x_{0_1}=-51 \Rightarrow P(-51,5)$$

$$x_{0_2}=19 \qquad P(19,-5)$$

then we just use the formula from page 1

Method 2: $\begin{cases} y-7 = m(x+1) \\ x^2-y^2=16 \end{cases}$ (eq of line passing through M)

impose the condition $\Delta = 0$

find m, replace m in the first eq $\Rightarrow$ tangents

20. Find the area of the triangle determined by the asymptotes of the hyperbola $H: \frac{x^2}{4}-\frac{y^2}{9}-1=0$ and the line $l: 9x+2y-24=0 \Rightarrow y=\frac{-9x}{2}+12$

$l_1: \quad y=\frac{3}{2}x$

$l_2: \quad y=-\frac{3}{2}x$

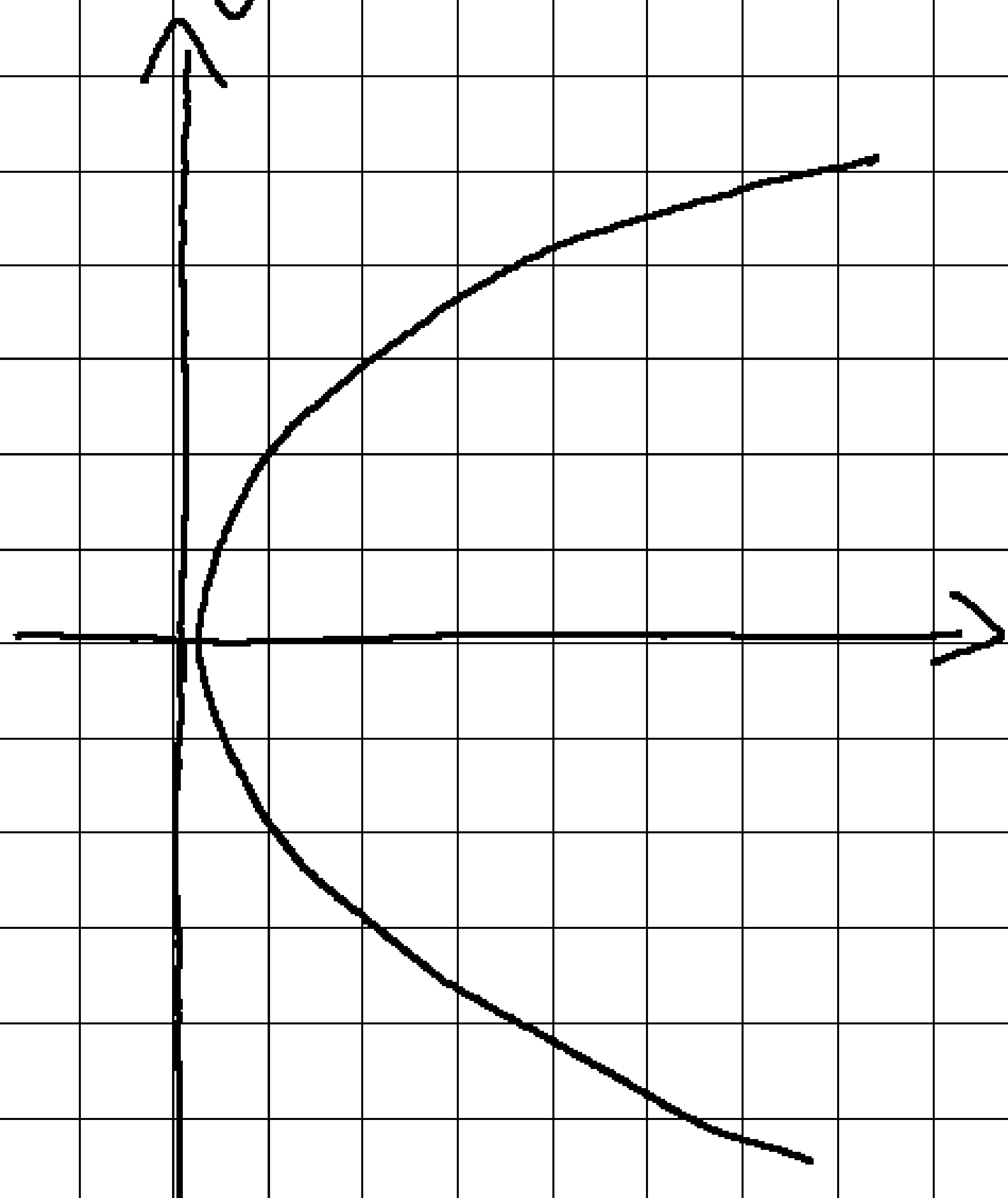
$l_1 \cap l: \begin{cases} y=\frac{3}{2}x \\ y=\frac{-9x}{2}+12 \end{cases} \Rightarrow 3x=-9x+24 \Rightarrow x=2, y=3$

$l_2 \cap l: \begin{cases} y=-\frac{3}{2}x \\ y=-\frac{9x}{2}+12 \end{cases} \Rightarrow -3x=-9x+24 \Rightarrow x=4, y=-6$

$l_1 \cap l_2: \begin{cases} y=\frac{3}{2}x \\ y=-\frac{3}{2}x \end{cases} \Rightarrow x=0, y=0$

$$A = \left| \frac{1}{2} \begin{vmatrix} 0 & 0 & 1 \\ 2 & 3 & 1 \\ 4 & -6 & 1 \end{vmatrix} \right| = \left| \frac{1}{2} \cdot \begin{vmatrix} 2 & 3 \\ 4 & -6 \end{vmatrix} \right| = \left| \frac{1}{2} \cdot -24 \right|$$
$$= 12$$

26. For which value k is the line $y = kx+2$ tangent to the parabola $P: y^2 = 4x$?



$$\begin{cases} y = kx+2 \\ y^2 = 4x \end{cases}$$

$$(kx+2)^2 = 4x \iff$$

$$\iff k^2x^2 + 4kx + 4 = 4x$$

$$\iff k^2x^2 + 4(k-1)x + 4 = 0 \qquad , k \neq 0$$

$$\Delta = 16(k^2-2k+1) - 4k^2 \cdot 4 = 16k^2 - 32k + 16 - 16k^2 =$$

$$= -32k + 16$$

The system has 1 sol $\Rightarrow \Delta = 0$

$$\Delta = 0 \Rightarrow k = \frac{1}{2} \neq 0$$

ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \Rightarrow T_P: \frac{x x_0}{a^2} + \frac{y y_0}{b^2} = 1$

Hiperbola $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \Rightarrow T_P: \frac{x x_0}{a^2} - \frac{y y_0}{b^2} = 1$

Parabola: $y^2 = 2px \Rightarrow T_P: y y_0 = p(x + x_0)$

Work iff the point is on the figure

27. Consider the parabola $P: y^2 = 16x$. Determine the tangents to P which are

a) parallel to the line $\ell: 3x - 2y + 30 = 0$

$\ell' \parallel \ell$

$m = \frac{3}{2}$

$\ell': y = mx + k$
$= \frac{3}{2}x + k$

$\begin{cases} y = \frac{3}{2}x + k \\ y^2 = 16x \end{cases} \Leftrightarrow \left(\frac{3}{2}x + k\right)^2 = 16x \Leftrightarrow$

$\Leftrightarrow \frac{9}{4}x^2 + 3k + k^2 = 16x \Leftrightarrow \frac{9}{4}x^2 + (3k - 16)x + k^2 = 0$

$\Delta = (3k - 16)^2 - 9k^2 = -96k + 256$

$\Delta = 0 \Rightarrow k = \frac{8}{3} \Rightarrow y = \frac{3}{2}x + \frac{8}{3}$

b) perpendicular to the line $l; 4x+2y+7=0$

$l' \perp l$

$m_l = -2 \quad , \quad m_{l'} = \frac{1}{2}$

$l': \quad y = \frac{1}{2}x + k$

$$\begin{cases} y = \frac{1}{2}x + k \\ y^2 = 16x \end{cases}$$

$\frac{1}{4}x^2 + kx + k^2 = 16x$

$\frac{1}{4}x^2 + (k-16)x + k^2 = 0$

$\Delta = (k-16)^2 - k^2 = -32k + 256$

$\Delta = 0 \Rightarrow k = 8$

$\Rightarrow y = \frac{1}{2}x + 8$

28. Determine the tangents to the parabola $P: y^2 = 16x$ which containt the point $P(-2,2)$

$p = 8$

$$\begin{cases} 2y_0 = 8(x_0 - 2) \\ y_0^2 = 16x_0 \end{cases} \Leftrightarrow \begin{cases} y_0 = 4(x_0 - 2) \\ 16(x_0 - 2)^2 = 16 \end{cases}$$

$\Leftrightarrow x_0^2 - 4x_0 + 4 = x_0 \Leftrightarrow$

$\Leftrightarrow x_0^2 - 5x_0 + 4 = 0 \Leftrightarrow$

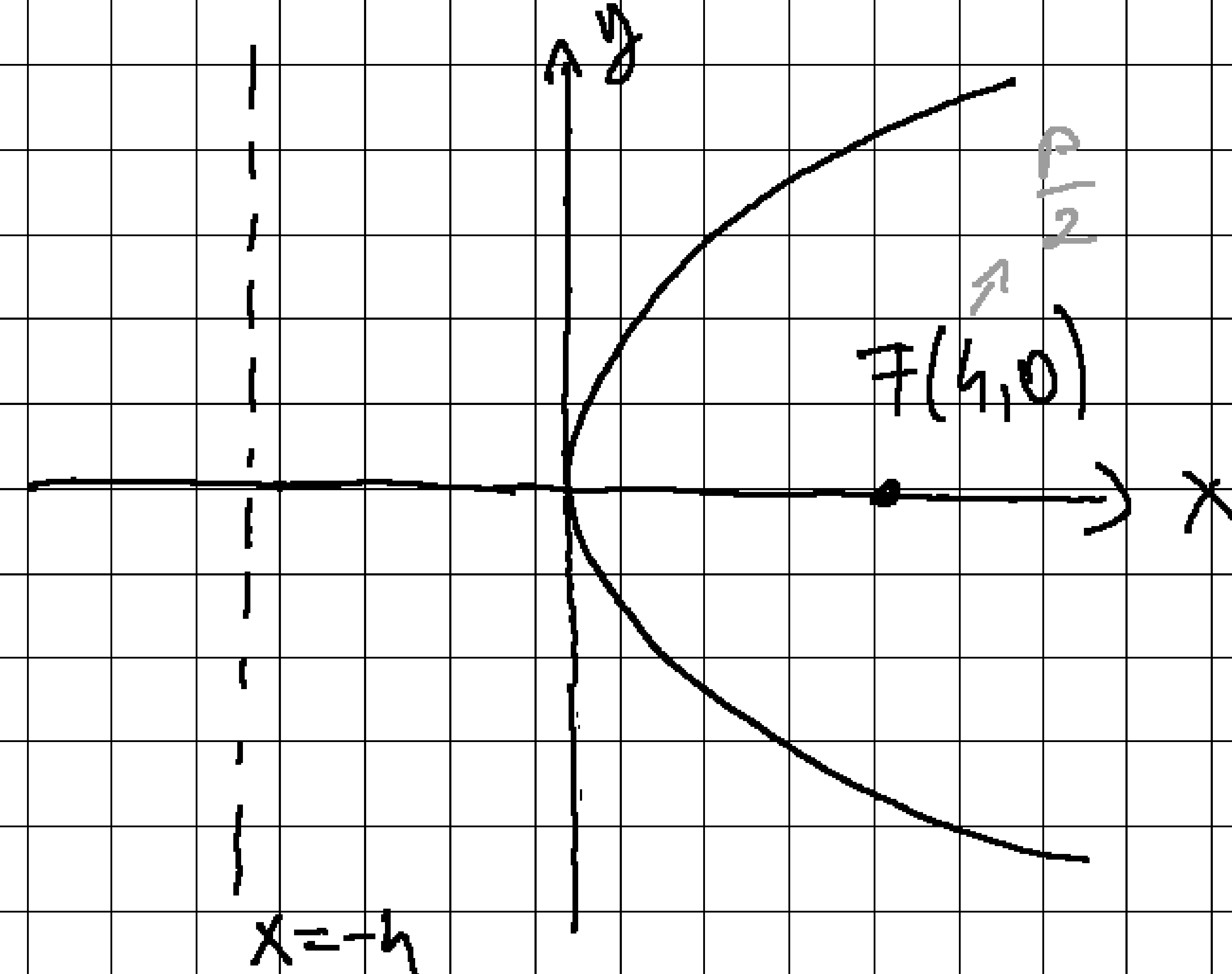
$\Leftrightarrow (x_0 - 1)(x_0 - 4) = 0 \Rightarrow x_0 = 1 \Rightarrow y_0 = -4$

$x_0 = 4 \Rightarrow y_0 = 8$

$T_1: -4y = 8(x+1)$

$T_2: 8y = 8(x+4)$

$y = x + 4$



10. Determine the common tangents to the ellipses:

$$\frac{x^2}{45}+\frac{y^2}{9}=1 \quad \text{and} \quad \frac{x^2}{9}+\frac{y^2}{18}=1$$

$$\left.\begin{array}{l} y=mx\pm\sqrt{45m^2+9} \\ y=mx\pm\sqrt{9m^2+18} \end{array}\right\} \Rightarrow 45m^2+9=9m^2+18$$

$$36m^2=9$$

$$m=\pm\frac{1}{2}$$

$$y=\pm\frac{1}{2}x+\sqrt{45\cdot\frac{1}{4}+9}=\pm\frac{1}{2}x+\frac{9}{2}$$

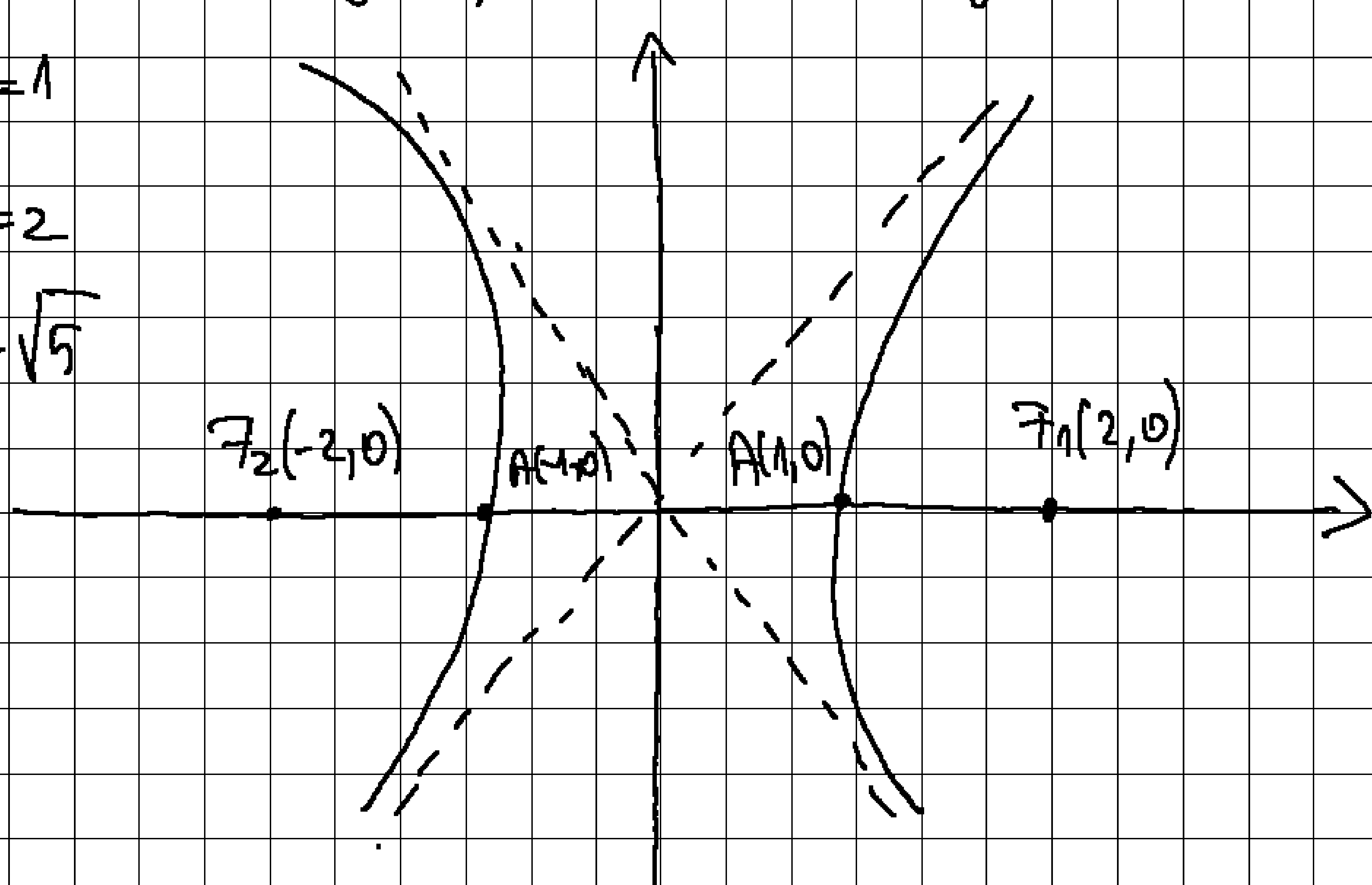
23. Consider the hyperbola $H: x^2-\frac{y^2}{4}-1=0$ with focal points F_1 and F_2. Find the points M situated on the hyperbola such that:

a) The angle $\sphericalangle F_1MF_2$ is right.

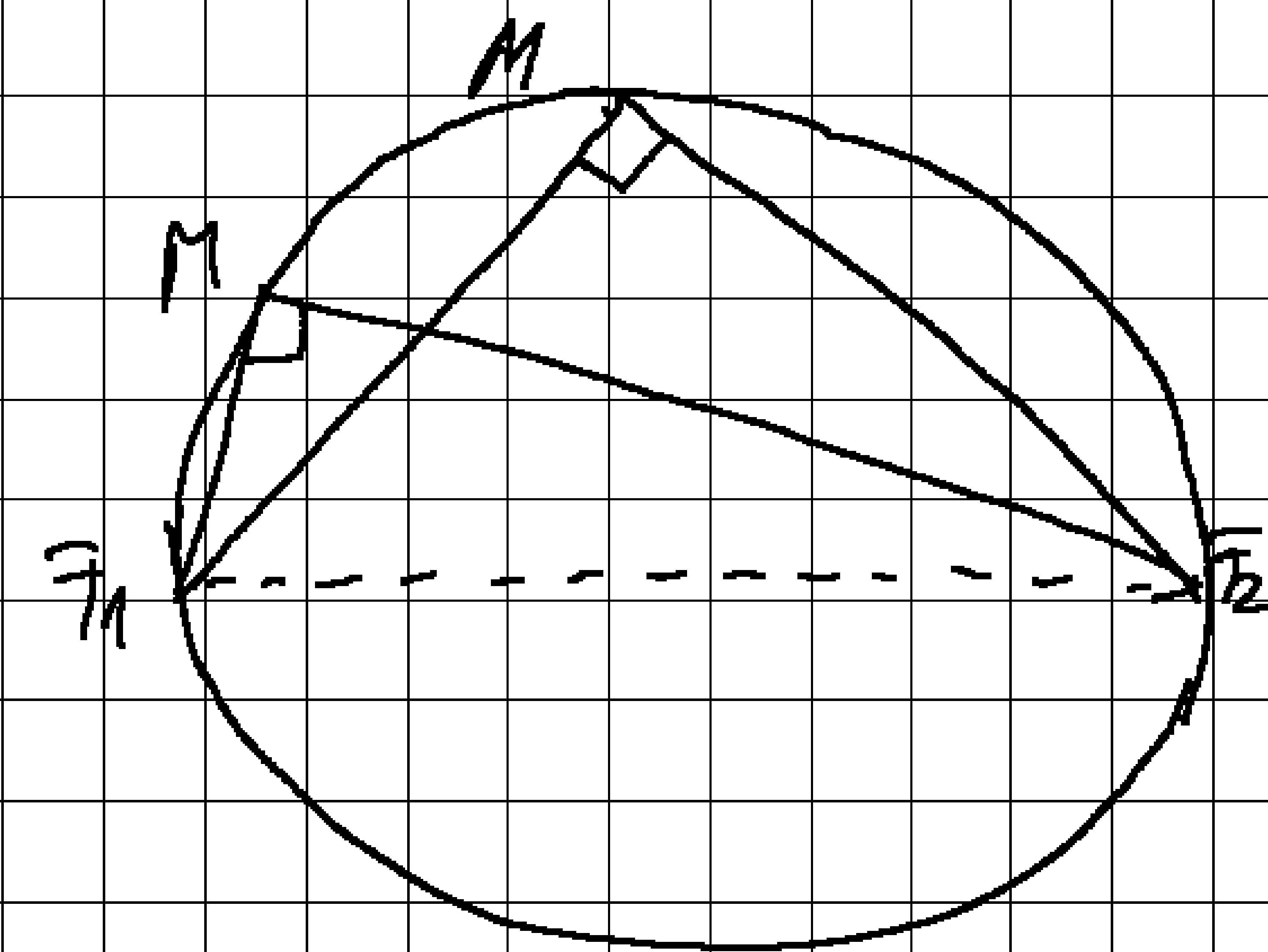
$a=1$

$b=2$

$c=\sqrt{5}$



$H: \ |MF_2 - MF_1| = a$



$C: \quad x^2 + y^2 = 5$

$$\begin{cases} x^2+y^2=5 \Rightarrow x^2 = 5-y^2 \\ x^2 - \frac{y^2}{4} = 1 \end{cases} \Rightarrow y_{1,2} = \pm \frac{4}{\sqrt{5}}$$

$5 - y^2 - \frac{y^2}{4} = 1$

$5 - \frac{5y^2}{4} = 1$

$20 - 4y^2 = 4$

I $\ y = \frac{4}{\sqrt{5}} \Rightarrow x_{1,2} = \pm \frac{3}{\sqrt{5}}$

I $\ y = -\frac{4}{\sqrt{5}} \Rightarrow x_{1,2} = \pm \frac{3}{\sqrt{5}}$