Ex: $$\sum_{n\geq 0} \frac{n!}{a(a+1)\dots(a+n)}, \quad a>0$$

$$\frac{x_{n+1}}{x_n} = \frac{\cancel{(n+1)!}}{\cancel{a(a+1)\dots(a+n)}(a+n+1)} \cdot \frac{\cancel{a(a+1)\dots(a+n)}}{\cancel{n!}} =$$

$$= \frac{n+1}{a+n+1} = \frac{n+1}{n+1+a} \longrightarrow 1$$

Ratio test $\Rightarrow$ inconclusive

Raabe-Duhamel test:

$$n\left(\frac{x_n}{x_{n+1}} - 1\right) = n\left(\frac{n+1+a}{n+1} - 1\right) =$$

$$= n\left(\cancel{1} + \frac{a}{n+1} - \cancel{1}\right) = \frac{n \cdot a}{n+1} = \frac{a \cdot n}{n+1} \longrightarrow a$$

R.D. test: if $a>1 \Rightarrow$ convergent series

if $a<1 \Rightarrow$ divergent series

if $a=1 \Rightarrow \sum_{n\geq 0} \frac{n!}{(n+1)!} = \sum_{n\geq 0} \frac{1}{n+1} = \infty$

Alternating series:

Ex: $\sum_{n\geq 1} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots = \ln 2$

- convergence:

Let $S_n = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots + \frac{(-1)^{n+1}}{n}$

$$S_{2n+2} - S_{2n} = 1 - \frac{1}{2} + \dots + \frac{(-1)^{2n+1}}{2n} + \frac{(-1)^{2n+2}}{2n+1} + \frac{(-1)^{2n+3}}{2n+2}$$

$$-\left(\text{———} \ // \ \text{———} \right) = \frac{1}{2n+1} - \frac{1}{2n+2} > 0$$

$\Rightarrow S_{2n+2} > S_{2n} \Rightarrow (S_{2n})$ increasing

- $S_{2n+3} - S_{2n+2} = \frac{(-1)^{2n+3}}{2n+2} + \frac{(-1)^{2n+4}}{2n+3} = -\frac{1}{2n+2} + \frac{1}{2n+3} < 0$,)

$\frac{1}{2n+3} < \frac{1}{2n+2}$

$\Rightarrow S_{2n+3} < S_{2n+1} \Rightarrow (S_{2n+1})$ is decreasing

- $S_{2n+1} - S_{2n} = \frac{(-1)^{2n+2}}{2n+1} = \frac{1}{2n+1} \Rightarrow$

$\Rightarrow S_{2n+1} = S_{2n} + \frac{1}{2n+1} > S_{2n} \Rightarrow S_{2n+1} > S_{2n}$

$\Downarrow$

$\Downarrow \lim_{n\to\infty} S_{2n+1} = \lim_{n\to\infty} S_{2n} \quad (1)$ | $(S_{2n}), (S_{2n+1})$ bounded

• (S_{2n}), (S_{2n+1}) are convergent (monotone, bounded) (2)

(1), (2) $\Rightarrow$ (S_n) convergent and $\lim_{n\to\infty} S_n = \lim_{n\to\infty} S_{n+1} =$

$= \lim_{n\to\infty} S_{2n}$

• $S_{2n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots + \frac{1}{2n-1} - \frac{1}{2n}$

$= 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{2n} - 2\left(\frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{2n}\right) =$

$= 1 + \frac{1}{2} + \dots + \frac{1}{2n} - \left(1 + \frac{1}{2} + \dots + \frac{1}{n}\right)$

Recall (Seminar): $\underbrace{1 + \frac{1}{2} + \dots + \frac{1}{n} - \ln n}_{a_n} \longrightarrow \gamma \in (0,1)$

$\Rightarrow = \underbrace{1 + \frac{1}{2} + \dots + \frac{1}{2n} - \ln 2n}_{} + \ln 2n - \left(\underbrace{1 + \frac{1}{2} + \frac{1}{n} - \ln n}_{} + \ln n\right)$

$= a_{2n} - a_n + \ln 2n - \ln n =$

$= a_{2n} - a_n + \ln 2$

$\Rightarrow \gamma - \gamma + \ln 2 = \ln 2$

$$\boxed{\lim_{n\to\infty} S_{2n} = \ln 2 = \lim_{n\to\infty} S_n}$$

Theorem 5 (Leibniz test)

$$x_n \searrow 0 \Rightarrow \sum_{n\geq 1} (-1)^n x_n \text{ convergent}$$

Proof:

Let $S_n = \sum_{k=1}^{n} (-1)^k \cdot x_k$

(S_n) convergent $\Leftrightarrow$ (S_n) Cauchy $\left(S_{n+p} - S_n \right) \to 0$ as $n \to \infty$, $p \in \mathbb{N}$

- $$|S_{n+p} - S_n| = \left| (-1)^{n+1} x_{n+1} + (-1)^{n+2} x_{n+2} + \dots + (-1)^{n+p} x_{n+p} \right|$$
$$= \left| x_{n+1} - x_{n+2} + x_{n+3} - x_{n+4} + \dots + (-1)^{n+p} x_{n+p} \right|$$

$$x_{n+1} \underbrace{- x_{n+2} + x_{n+3}}_{\leq 0} \underbrace{- x_{n+4} + \dots}_{\leq 0} \underbrace{- x_{n+p-1} + x_{n+p}}_{\leq 0}$$

or

$$x_{n+1} \underbrace{- x_{n+2} + x_{n+3}}_{\leq 0} \underbrace{- x_{n+4} + \dots}_{\leq 0} \underbrace{+ (- x_{n+p})}_{\leq 0} \leq x_{n+2}$$

$x_{n+3} \leq x_{n+2}$

$-x_{n+2} + x_{n+3} \leq 0$

$\Rightarrow |S_{n+p} - S_n| \leq x_{n+1} \longrightarrow 0 \Rightarrow (S_n)$ Cauchy $\Rightarrow$

$\Rightarrow (S_n)$ convergent

Absolutely convergent Series:

Ex: $\sum \frac{(-1)^{n+1}}{n}$ convergent to $\ln 2$

$\sum \frac{1}{n}$ divergent

$\Rightarrow \sum \frac{(-1)^{n+1}}{n}$ is <u>not</u> absolutely convergent

$\Rightarrow$ Conditionally convergent

Prop: $\sum_{n\geq 1} |x_n|$ convergent $\Rightarrow \sum_{n\geq 1} x_n$ convergent

Proof: Let $S_n = \sum_{k=1}^{n} |x_k|$, $s_n = \sum_{k=1}^{n} x_k$

$\sum_{n\geq 1} |x_n|$ convergent $\Rightarrow S_n$ Cauchy sequence $\Rightarrow$
$\Rightarrow |S_{n+p} - S_n| \to 0$

$$|s_{n+p} - s_n| = |x_{n+1} + \dots x_{n+p}| \leq |x_{n+1}| + \dots + |x_{n+p}| =$$
$$= S_{n+p} - S_n$$

$|s_{n+p} - s_n| \leq S_{n+p} - S_n \Rightarrow s_{n+p} - s_n \to 0$ as $n \to \infty$
$\Rightarrow s_n$ is Cauchy

$\Rightarrow (s_n)$ convergent $\Rightarrow \sum_{n\geq 1} x_n$ convergent

Theorem 8 (Cauchy)

$$x_{\gamma(1)} + x_{\gamma(2)} + x_{\gamma(3)} + \dots + x_{\gamma(n)} + \dots =$$

$$= x_1 + x_2 + \dots + x_n + \dots$$

$\uparrow$

if $\sum |x_n|$ is convergent

(absolutelly convergent)

Theorem 10 (Riemann)

$$x_{\gamma(1)} + x_{\gamma(2)} + \dots + x_{\gamma(n)} + \dots \neq x_1 + x_2 + \dots + x_n$$

$\downarrow$

if $\sum (x_n)$ is divergent

Ex: $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots = \ln 2$

$$= \underbrace{1 - \frac{1}{2}}_{\frac{1}{2}} - \frac{1}{4} + \underbrace{\frac{1}{3} - \frac{1}{6}}_{\frac{1}{6}} - \frac{1}{8} + \dots =$$

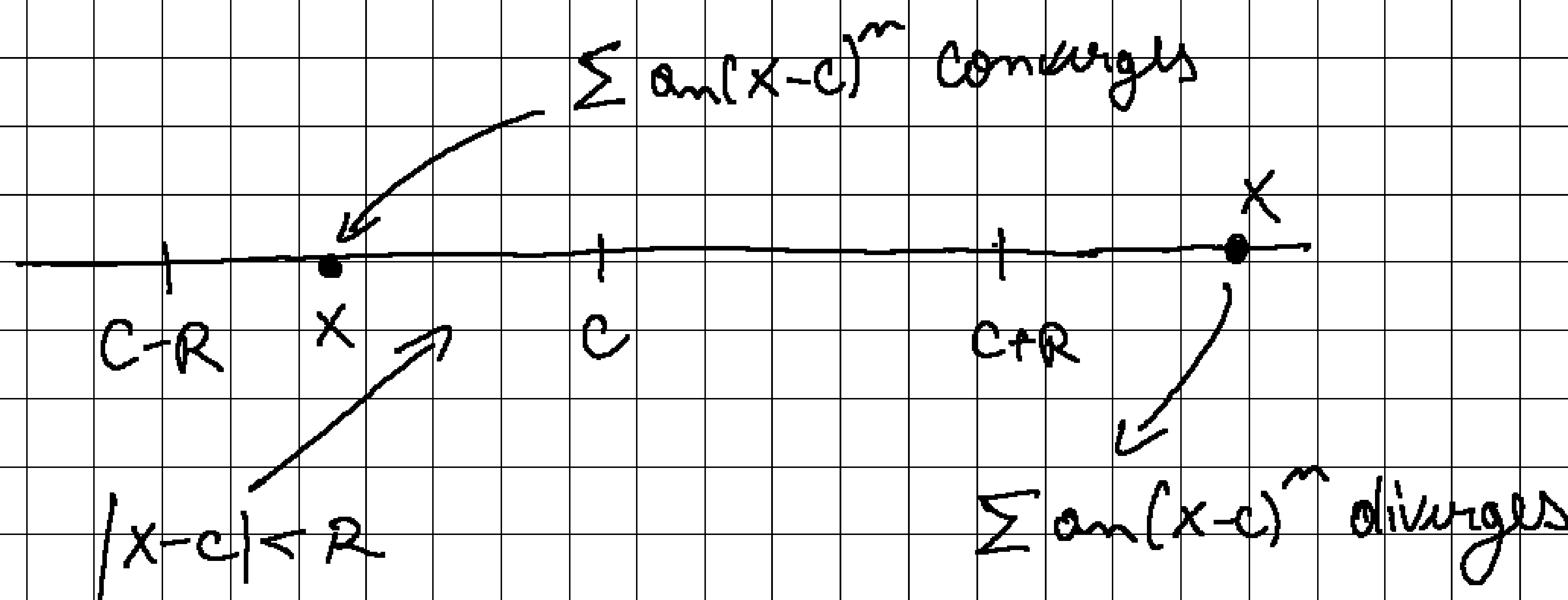
$$= \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \dots = \frac{1}{2}\Big(\underbrace{1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots}_{\ln 2}\Big) =$$

$$= \frac{1}{2} \cdot \ln 2 = \frac{\ln 2}{2}$$

Power series:

$$\sum_{n=0}^{\infty} a_n (x-c)^n$$

Theorem 13:



Theorem 14:

$$\sum a_n (x-c)^n$$

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = L \Rightarrow R = \frac{1}{L}$$

Proof:

We study $\sum_{n\geq 1} \underbrace{|a_n|\,|x-c|^n}_{x_n}$

Root test: $\lim_{n\to\infty} \sqrt[n]{x_n} = \lim_{n\to\infty} \sqrt[n]{|a_n|} \cdot |x-c| = L\cdot|x-c|$

if $L\,|x-c| < 1 \Rightarrow \sum |a_n|\,|x-c|^n$ convergent

if $|x-c| < \underbrace{\frac{1}{L}}_{=R} \Rightarrow \sum |a_n|\,|x-c|^n$ convergent

if $|x-c| > \frac{1}{L} \Rightarrow \sum |a_n|\,|x-c|^n$ divergent

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L \Rightarrow \lim_{n\to\infty}\sqrt[n]{|a_n|} = L \Rightarrow \text{conclusion}$$

Ex: $\sum_{n\geq 0} x^n$, $a_n = 1$, $c = 0$

$L = 1 \Rightarrow$ if $|x| < 1 \Rightarrow$ the series converges absolutely