Relations

$$r = (\underset{\text{sets}}{\underset{\downarrow}{A}}, \underset{\text{sets}}{\underset{\downarrow}{B}}, \underset{\text{graph}}{\underset{\downarrow}{R}})$$

$R \subseteq A \times B$

$A = B \Rightarrow$ homogeneous relation

$r = (A, A, R)$

- reflexivity : $\forall x \in A : x \, r \, x$
- simetry : $\forall x, y \in A : [x \, r \, y \Rightarrow y \, r \, x]$
- transitivity : $\forall x, y, z \in A : [x \, r \, y, \; y \, r \, z \Rightarrow x \, r \, z]$

all 3 $\Rightarrow$ equivalence relation

Ex 1:

r, s, t, v hom. relations defined on $A = \{2,3,4,5,6\}$

by $\quad x\, r\, y \Leftrightarrow x < y$

$\quad x\, s\, y \Leftrightarrow x \mid y$

$\quad x\, t\, y \Leftrightarrow \gcd(x,y) = 1$

$\quad x\, v\, y \Leftrightarrow x \equiv y \pmod 3$

Write the graphs R, S, T, V of the relations

$R = \{(2,3), (2,4), (2,5), (2,6), (3,4), (3,5), (3,6)$
$(4,5), (4,6), (5,6)\}$

$S = \{(2,2), (2,4), (2,6), (3,3), (3,6), (4,4), (5,5),$
$(6,6)\}$

$T = \{(2,3), (2,5), (3,4), (3,5), (5,6), (4,5), (5,4),$
$(6,5), (3,2), (5,2), (4,3), (5,3)\}$

$V = \{(2,5), (3,6), (5,2), (6,3), (2,2), (3,3), (4,4),$
$(5,5), (6,6)\}$

Ex 2: Give examples of relations having each one of the properties of reflexivity, symmetry and transitivity but not the others:

$t, \cancel{r}, \cancel{s}$:

$A = \mathbb{R}$, $x \, r \, y \Leftrightarrow x < y$

another example:

$A = \{1,2,3\}$ $R = \{(1,2),(2,3),(1,3)\}$

$s, \cancel{r}, \cancel{t}$

$B = \{1,2,3,4\}$, $x \, r \, y \Leftrightarrow x+y=5$

another example:

$A = \{1,2,3\}$ $R = \{(1,2),(2,1)\}$

$r, \cancel{s}, \cancel{t}$:

$C = \{1,2,3\}$

$R = \{(1,1),(2,2),(3,3),(1,2),(2,3)\}$

Th: M set

$$\mathcal{E}(M) \xleftrightarrow[\text{bijection}]{\sim} \mathcal{P}(M)$$

$\mathcal{E}(M)$ ⇓ set of equivalence relations on M

$\mathcal{P}(M)$ ↘ set of partitions of M

$$r \longmapsto M/r$$

M/r " quotient set

$$r_{\mathcal{P}} \longleftarrow\!\!\!\mid \mathcal{P}$$

$$M/r = \{\, r\langle x\rangle \mid x \in M \,\}$$

$$r\langle x\rangle = \{\, y \in M \mid x \, r \, y \,\}$$

$x \, r_{\mathcal{P}} \, y \Leftrightarrow \exists\, A \in \mathcal{P}$ so that $x, y \in A$

Ex 5:

$M = \{1,2,3,4\}$, r_1, r_2 homogeneous relations on M and $\Pi_1, \Pi_2 \subseteq \mathcal{P}(M)$

$R_1 = \Delta_M \cup \{(1,2),(2,1),(1,3),(3,1),(2,3),(3,2)\}$

$\Delta_M = \{(1,1),(2,2),(3,3),(4,4)\}$

$R_2 = \Delta_M \cup \{(1,2),(2,3)\}$

$\Pi_1 = \{\{1\},\{2\},\{3,4\}\}$

$\Pi_2 = \{\{1\},\{1,2\},\{3,4\}\}$

(i) Are r_1, r_2 equivalences on M?
if so, write their associated partition

(ii) Are Π_1, Π_2 partitions of M?
if so, write the graph(s) of their associated equivalence.

(i) r_1 - equivalence relation (we checked)

$M/r_1 = \{\{1,2,3\}, \{4\}\}$

Midterm exercise (i), (ii)

$r_1\langle 1\rangle = \{1,2,3\}$

$r_1\langle 4\rangle = \{4\}$

r_2 - is not a equivalence relation because:

$(1,2) \in R_2$ | $\Rightarrow$ r_2 is not symmetrical
but $(2,1) \notin R_2$ | $\Rightarrow$ not an equivalence

(ii) Π_1 does not have overlapping elements and every element of M can be found in an element of Π_1

$R_{\Pi_1} = \{(1,1), (2,2), (3,3), (4,4), (3,4), (4,3)\}$

$\{1\} \cap \{1,2\} = \{1\} \neq \emptyset \Rightarrow \Pi_2$ is not a partition

$r = (A, B, R)$ relation

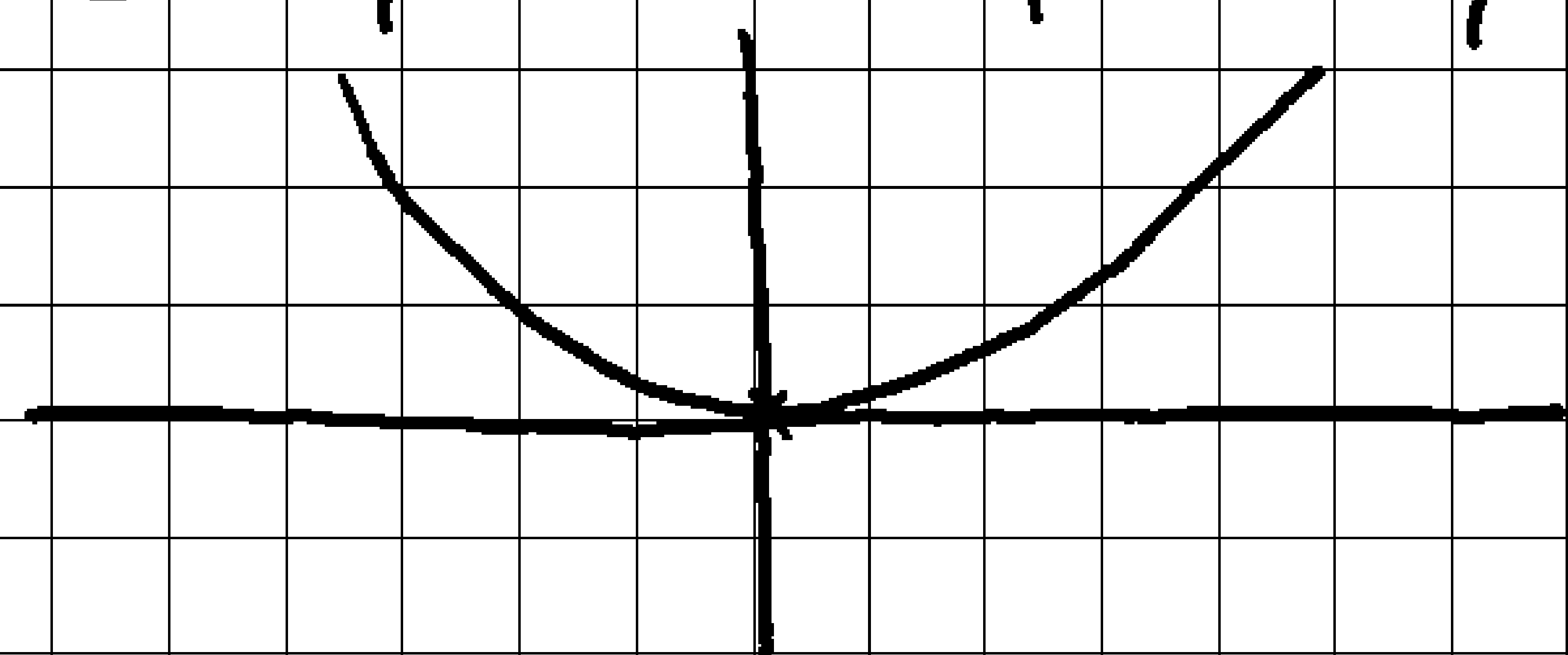
r function $\Leftrightarrow \forall x \in A, |r\langle x\rangle| = 1$

if r is a function: say $r\langle x\rangle = \{(x, f(x))\}$

$R = \{(x, f(x)) \mid x \in A\}$

Ex: $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$

$Grf = \{(x, x^2) \mid x \in \mathbb{R}\}$



Ex 9: $M = \{0, 1, 2, 3\}$, $h = (\mathbb{Z}, M, H)$

$H = \{(x,y) \in \mathbb{Z} \times M \mid \exists z \in \mathbb{Z} : x = 4z + y\}$

is h a function?

take $x \in \mathbb{Z} \Rightarrow \exists\ y \in \{0, 1, 2, 3\}$

$x \equiv y \pmod 4$

But $x \equiv y \pmod 4 \Leftrightarrow \exists\ z \in \mathbb{Z}$

$x = 4z + y$

$\Rightarrow h$ is indeed a function

Ex6: Define on $\mathbb{C}$ the relations

$$z_1 \, r \, z_2 \Leftrightarrow |z_1| = |z_2|$$

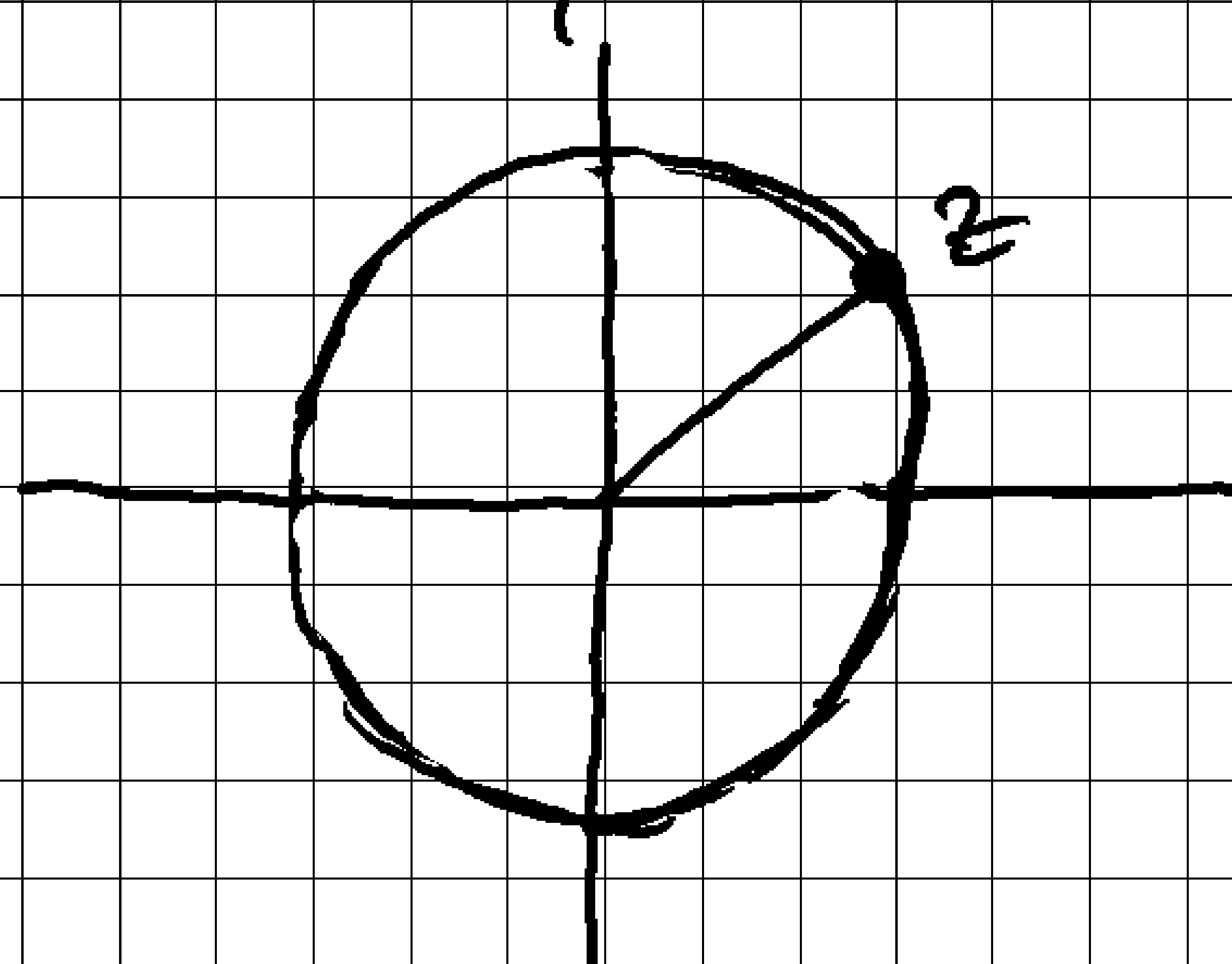
$$z_1 \, s \, z_2 \Leftrightarrow \arg(z_1) = \arg(z_2)$$

$$z_1 = z_2 = 0$$

Show that r and s are equivalence relations and find $\mathbb{C}/r$ and $\mathbb{C}/s$ (geometrical interpretation)

$$\mathbb{C}/r = \{ r\langle z \rangle \mid z \in \mathbb{C} \}$$

$$r\langle z \rangle = \{ w \in \mathbb{C} \mid |z| = |w| \}$$



$$\mathbb{C}/s = \{ s\langle z \rangle \mid z \in \mathbb{C} \}$$

$z \neq 0$ $\quad s\langle z \rangle = \{ w \in \mathbb{C} \mid \arg z = \arg w \} = 0$

$$s\langle 0 \rangle = 0$$