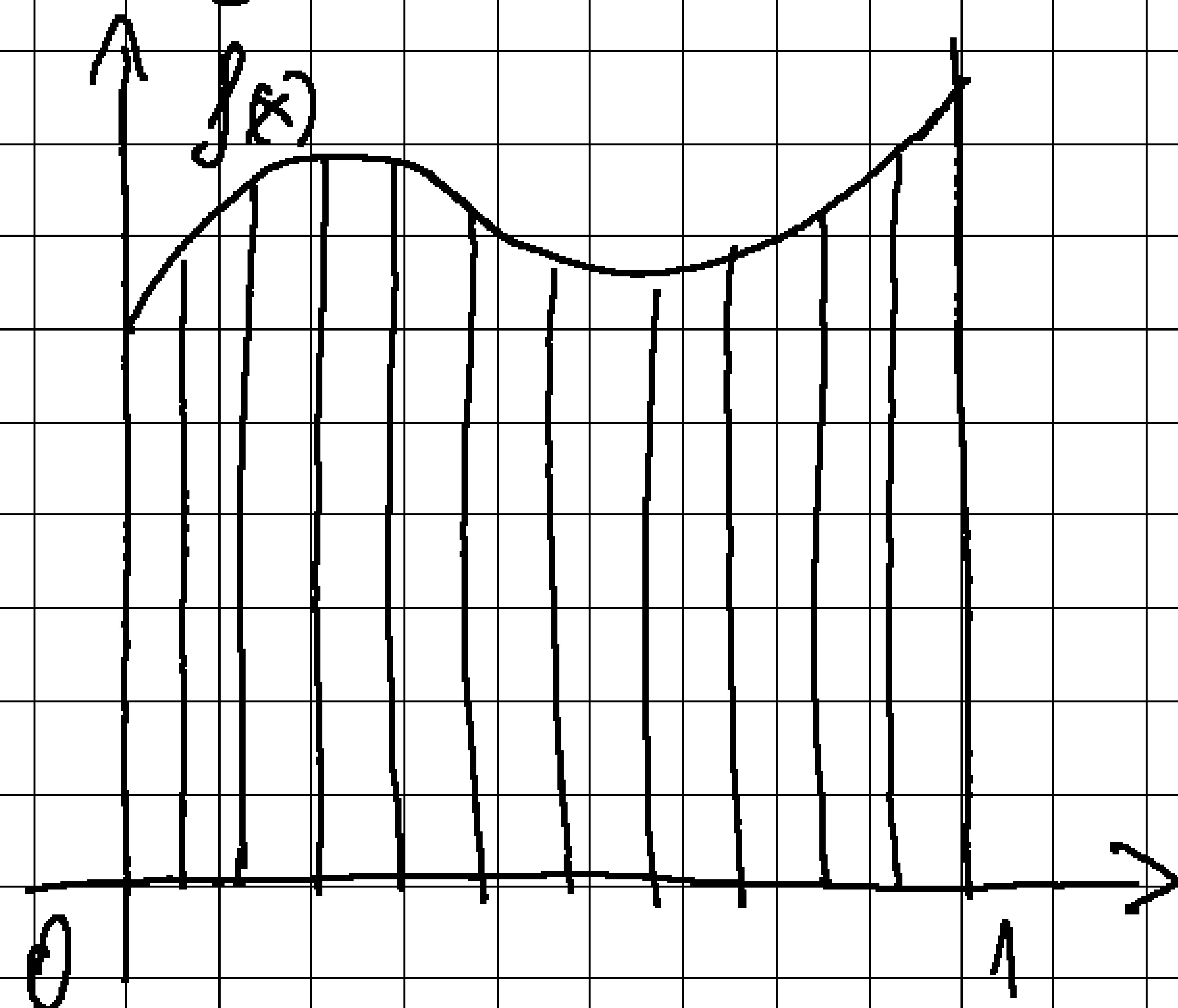
1. Compute the limits using Riemann integrals



Riemann sum: $\frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right) \longrightarrow \int_0^1 f(x)\,dx$

a) $\lim_{n\to\infty}\left(\frac{1}{n+1}+\frac{1}{n+2}+\ldots+\frac{1}{2n}\right)=$

$=\lim_{n\to\infty}\left(\frac{1}{n}\cdot\frac{1}{1+\frac{1}{n}}+\frac{1}{n}\cdot\frac{1}{1+\frac{2}{n}}+\ldots+\frac{1}{n}\cdot\frac{1}{1+\frac{n}{n}}\right)=$

$=\lim_{n\to\infty}\frac{1}{n}\left(\frac{1}{1+\frac{1}{n}}+\frac{1}{1+\frac{2}{n}}+\ldots+\frac{1}{1+\frac{n}{n}}\right)=$

$=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{1}{1+\frac{k}{n}}=\int_0^1\frac{1}{1+x}\,dx=\ln(1+x)\Big|_0^1=$

$=\ln 2$

c) $\lim\limits_{n\to\infty} \frac{\sqrt[n]{n!}}{n} = \lim\limits_{n\to\infty} \frac{\sqrt[n]{1\cdot 2\cdot 3\cdot \ldots \cdot n}}{n} = \lim\limits_{n\to\infty} \left(\prod_{k=1}^{n} k\right)^{\frac{1}{n}} \cdot \frac{1}{n} =$

$$= \lim_{n\to\infty} \underbrace{\left(\prod_{k=1}^{n} \frac{k}{n}\right)^{\frac{1}{n}}}_{x_n}$$

$$\ln x_n = \ln\left(\prod_{k=1}^{n} \frac{k}{n}\right)^{\frac{1}{n}} = \frac{1}{n} \ln \prod_{k=1}^{n} \frac{k}{n} =$$

$$= \frac{1}{n} \cdot \sum_{k=1}^{n} \ln\left(\frac{k}{n}\right) \longrightarrow \int_0^1 1\cdot \ln x \, dx =$$

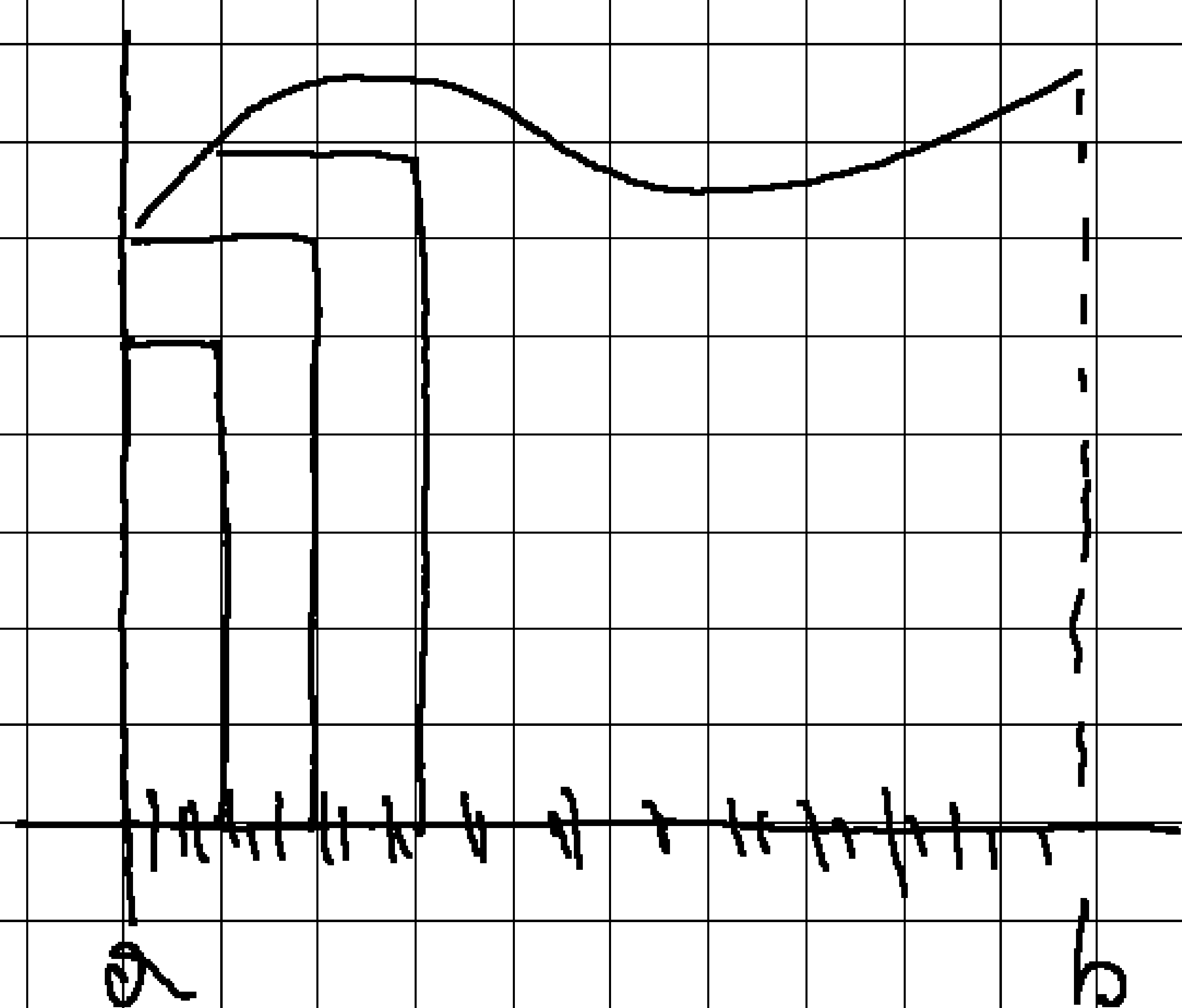
$$= x \ln x \Big|_0^1 - 1 = -\lim_{t\to 0} t \ln t - 1 =$$

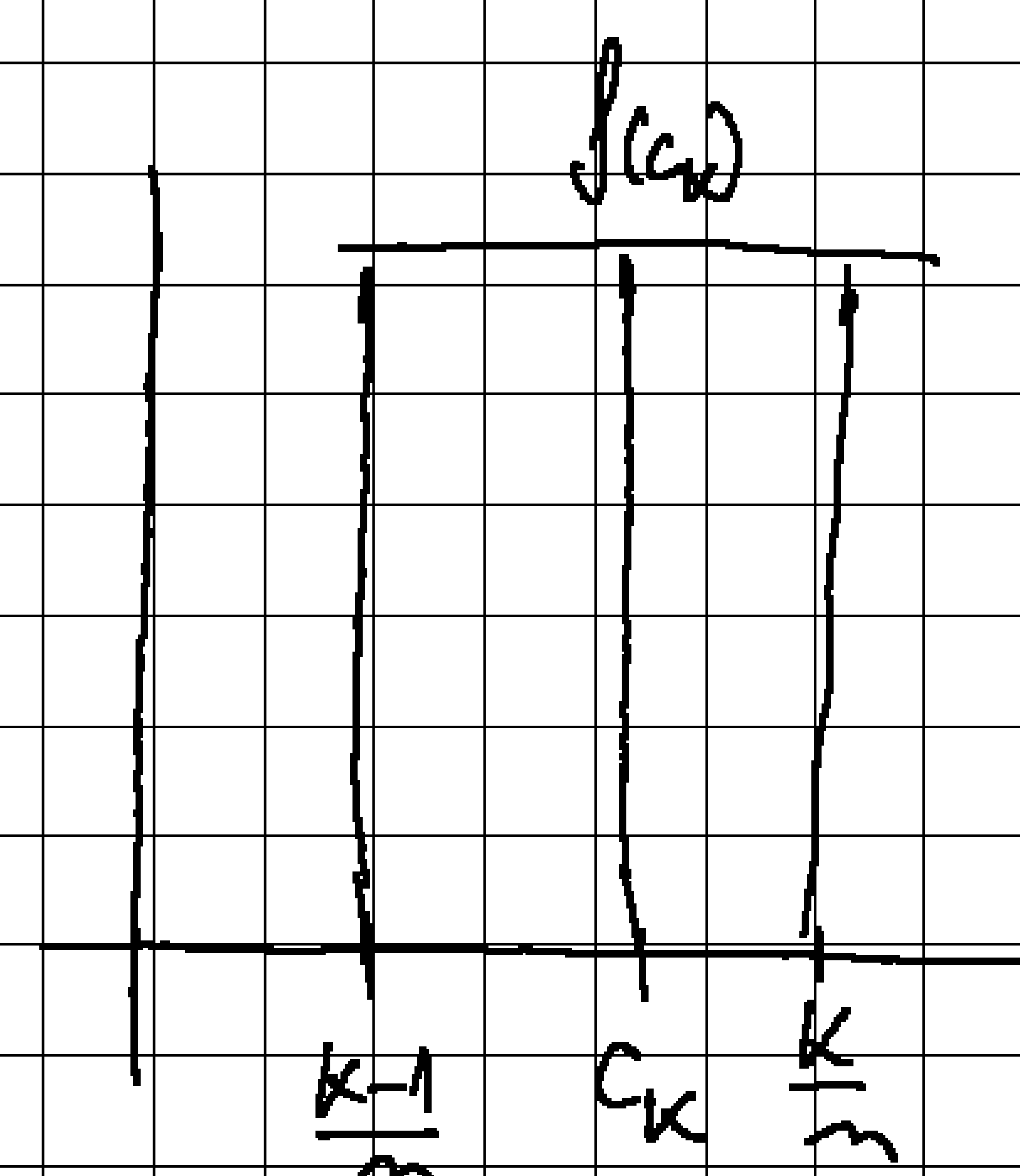
$$= -\lim_{t\searrow 0} \frac{\ln t}{\frac{1}{t}} - 1 = -\lim_{t\searrow 0} \frac{\frac{1}{\cancel{t}}}{-\frac{1}{t^{\cancel{2}}}} - 1 = \lim_{t\searrow 0} t - 1 =$$

$$= -1 \Rightarrow \ln x_n \to -1 \Rightarrow x_n \to \frac{1}{e}$$

2) Study the Riemann integrability of $f:[0,1]\to\mathbb{R}$

for $\begin{cases} 1, & x\in\mathbb{Q} \\ 0, & x\in\mathbb{R}\setminus\mathbb{Q} \end{cases}$





Consider the partition $P = \left\{ \left[\frac{k-1}{n}, \frac{k}{n}\right) \,\middle|\, k=\overline{1,n} \right\}$

$$\begin{cases} \text{for } c_k \in \left[\frac{k-1}{n}, \frac{k}{n}\right) \cap \mathbb{Q} : \sigma(f,P); \quad \lim\limits_{n\to\infty} \frac{1}{n} \sum\limits_{k=1}^{n} f(c_k) = 1 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \| \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \sigma(f,P) \\ \text{for } c_k \in \left[\frac{k-1}{n}, \frac{k}{n}\right) \cap (\mathbb{R}\setminus\mathbb{Q}) : \sigma(f,P) = 0 \end{cases}$$

$\Rightarrow \nexists \lim\limits_{n\to\infty} \sigma(f,P) \Rightarrow f$ is not integrable

3) Compute the following improper integrals:

a) $$\int_1^2 \frac{1}{x(x-2)}\,dx = \lim_{t \nearrow 2} \int_1^t \frac{dx}{x(x-2)} =$$

$$= \frac{1}{2} \lim_{t \nearrow 2} \int_1^t \frac{x-(x-2)}{x(x-2)}\,dx = \frac{1}{2} \lim_{t \nearrow 2} \int_1^t \frac{x}{x(x-2)} - \frac{x-2}{x(x-2)}\,dx =$$

$$= \frac{1}{2} \lim_{t \nearrow 2} \left(\int_1^t \frac{\cancel{x}}{\cancel{x}(x-2)}\,dx - \int_1^t \frac{\cancel{x-2}}{x\cancel{(x-2)}}\,dx \right) =$$

$$= \frac{1}{2} \left(\lim_{t \nearrow 2} \ln|(x-2)| \Big|_1^t - \ln x \Big|_1^2 \right) = \frac{1}{2} \Big(\lim_{t \nearrow 2} \underbrace{\ln|t-2|}_{=-\infty} - \ln 2 \Big) =$$

$$= -\infty$$

b) $$\int_0^\infty x e^{-x^2} = \lim_{t \to \infty} \int_0^t x e^{-x^2}\,dx = \lim_{t \to \infty} -\frac{1}{2} \int_0^t -2x \cdot e^{-x^2}\,dx =$$

$$= \lim_{t \to \infty} -\frac{1}{2} e^{-x^2} \Big|_0^t = -\frac{1}{2} \cdot \lim_{t \to \infty} \left(e^{-t^2} - e^0 \right) =$$

$$= -\frac{1}{2} \lim_{t \to \infty} \left(\frac{1}{e^{t^2}} - 1 \right) = \frac{1}{2}$$

c) $\int_0^1 \frac{\ln x}{\sqrt{x}} dx = \lim_{t \searrow 0} \int_t^1 \frac{\ln x}{\sqrt{x}} dx =$

$= \lim_{t \searrow 0} 2 \int_t^1 \frac{\ln x}{2\sqrt{x}} dx = 2 \lim_{t \searrow 0} \int_t^1 (\sqrt{x})' \cdot \ln x \, dx =$

$= 2 \cdot \lim_{t \searrow 0} \sqrt{x} \cdot \ln x \Big|_t^1 - \int_t^1 \sqrt{x} \cdot \frac{1}{x} =$

$(f \cdot g)' = f' \cdot g + f \cdot g' \ / \int$

$\Rightarrow f \cdot g - \int f \cdot g' = \int f' g$

$= 2 \cdot \lim_{t \searrow 0} \left(-\sqrt{t} \ln t - \int_t^1 \frac{x^{\frac{1}{2}}}{x} dx \right) =$

$= 2 \cdot \lim_{t \searrow 0} \left(-\frac{\ln t}{\frac{1}{\sqrt{t}}} - 2\sqrt{x} \Big|_t^1 \right) \overset{L'H}{=}$

$= 2 \cdot \lim_{t \searrow 0} \left(\frac{-\frac{1}{t}}{-\frac{1}{2} t^{-\frac{3}{2}}} - 2 + 2\sqrt{t} \right) =$

$= 2 \cdot (0 - 2 + 0) = -4$

4.) Study the convergence of the improper integrals

a) $\int_1^\infty \frac{1}{x\sqrt{1+x^2}}\,dx$

$\int_1^\infty \frac{1}{x^p}\,dx$ conv $\Leftrightarrow p > 1$

$0 \le f(x) \le g(x)$

$\lim_{x \to b} \frac{f(x)}{g(x)}$ $b \in (0,\infty)$ $\Rightarrow$

$\Rightarrow \int f(x)\,dx$ and $\int g(x)\,dx$ have same nature

Let $g(x) = \frac{1}{x^2}$

$\frac{1}{x\sqrt{1+x^2}} \le \frac{1}{x^2}$

$\int_1^\infty \frac{1}{x^2}\,dx$ is convergent $(2 > 1)$

$\Rightarrow \int_1^\infty \frac{1}{x\sqrt{1+x^2}}\,dx$ is conv

b) $\int_0^{\frac{\pi}{2}} \frac{1}{\cos x}\, dx$

$\int_a^b \frac{dx}{(b-x)^p}$ and $\int_a^b \frac{dx}{(x-a)^p}$ are conv $\Leftrightarrow p > 1$

$\cos x = \sin\left(\frac{\pi}{2} - x\right)$ $\quad \frac{\sin t}{t} \quad t = \frac{\pi}{2} - x$

$$\int_0^{\frac{\pi}{2}} \frac{1}{\cos x}\, dx = \int_0^{\frac{\pi}{2}} \frac{1}{\sin\left(\frac{\pi}{2}-x\right)}\, dx$$

Let $g(x) = \frac{1}{\frac{\pi}{2} - x}$

$$\left.\begin{aligned} &\lim_{x \to \frac{\pi}{2}} \frac{\frac{1}{\frac{\pi}{2}-x}}{\frac{1}{\sin\left(\frac{\pi}{2}-x\right)}} = 1 \\ &\int_0^{\frac{\pi}{2}} \frac{1}{\frac{\pi}{2}-x}\, dx \text{ - is divergent} \end{aligned}\right\} \Rightarrow \int_0^{\frac{\pi}{2}} \frac{1}{\cos x}\, dx \text{ has the same nature as } \int_0^{\frac{\pi}{2}} \frac{1}{\frac{\pi}{2}-x}\, dx$$

$\Rightarrow \int_0^{\frac{\pi}{2}} \frac{1}{\cos x}\, dx$ is divergent

5. Using the integral test study the convergence.

a) $\sum\limits_{n\geq 1} \frac{1}{n^p}, p>0$

integral test: $\int_1^\infty f(x)\,dx$ same nature as $\sum\limits_{n\geq 1}^{\infty} f(n)$

Same nature as $\int_1^\infty \frac{1}{x^p}\,dx = \lim\limits_{t\to\infty} \int_1^\infty \frac{dx}{x^p} =$

$$= \lim_{t\to\infty}\left(\frac{t^{-p+1}}{-p+1} - \frac{1}{-p+1}\right)$$

conv: $-p+1 < 0 \Rightarrow p > 1$

div: $-p+1 > 0 \Rightarrow p < 1$

$p = 1 \Rightarrow$ check separately