$\vec{v}, \vec{w}$ vectors

$$\vec{v} \bullet \vec{w} = \begin{cases} 0, & \text{if } \vec{v} \circ \vec{w} \text{ is } 0 \\ |\vec{v}| \cdot |\vec{w}| \cdot \cos(\vec{v}, \vec{w}), & \text{otherwise} \end{cases}$$

the dot product (the scalar product)

Alternate notation:

$$\vec{v} \cdot \vec{w} = \langle \vec{v}, \vec{w} \rangle$$ ⇐ scalar product not a subspace

Properties:

- biliniarity: $\forall \alpha, \beta \in R$, $\forall \vec{v}_1, \vec{v}_2, \vec{w} \in V$

$$(\alpha v_1 + \beta v_2) \bullet w = \alpha \vec{v}_1 \bullet w + \beta v_2 \bullet w$$

- symmetry: $\forall \vec{v}, \vec{w}$:

$$\vec{v} \bullet \vec{w} = \vec{w} \bullet \vec{v}$$

- positive definiteness: $\forall \vec{v} \in V$

$$\vec{v} \cdot \vec{v} \in R_{\geq 0}$$

if $\vec{v} \neq \vec{0} \Rightarrow \vec{v} \cdot \vec{v} > 0$

Consequences:

- $\vec{v} \bullet \vec{v} = |\vec{v}|^2$
- $\vec{v}^2 - \vec{w}^2 = (\vec{v} - \vec{w}) \bullet (\vec{v} + \vec{w})$

If we have K an orthonormal reference system and $[\vec{v}]_K = \begin{pmatrix} x_v \\ y_v \end{pmatrix}$

$$[\vec{w}]_K = \begin{pmatrix} x_w \\ y_w \end{pmatrix}$$

$$\Rightarrow \vec{v} \bullet \vec{w} = x_v x_w + y_v y_w$$

$K = (O, \vec{O}_1, \dots, \vec{O}_n)$ orthonormal if

$\forall\, i, j,\ i \neq j : \ \vec{v}_i \bullet O_j = 0$

$\forall\, i : \ \vec{v}_i \bullet \vec{v}_i = |\vec{v}_i|^2 = 1$

3.2. $\vec{m}$ and $\vec{n}$ unit vectors s.t. $\sphericalangle(\vec{m},\vec{n}) = 120^0$

Determine $\sphericalangle(\vec{a},\vec{b})$ where:

$$\vec{a} = 2\vec{m} + 4\vec{n}$$
$$\vec{b} = \vec{m} - \vec{n}$$

$$\vec{a} \bullet \vec{b} = (2\vec{m} + 4\vec{n}) \bullet (\vec{m} - \vec{n}) =$$

$$= 2\underbrace{|\vec{m}|^2}_{=1} + 2\vec{m}\cdot\vec{n} - 4\underbrace{|\vec{n}|^2}_{=1} =$$

$$= -2 + 2\cdot\vec{m}\cdot\vec{n}$$

$$\vec{m}\bullet\vec{n} = |m|\cdot n \cdot \cos 120^0 = -\frac{1}{2}$$

$$\vec{a}\bullet\vec{b} = -2-1 = -3$$

$$|\vec{a}|^2 = (2\vec{m} + 4\vec{n})(2\vec{m} + 4\vec{n}) =$$

$$= 4\underbrace{|\vec{m}|^2}_{=1} + 16\vec{m}\cdot\vec{n} + 16\underbrace{|\vec{n}|^2}_{=1} = 20 - 8 = 12$$

$$\Rightarrow |\vec{a}| = 2\sqrt{3}$$

$$|\vec{b}|^2 = (\vec{m}-\vec{n})(\vec{m}-\vec{n}) = |\vec{m}|^2 - 2\vec{m}\vec{n} + |\vec{n}|^2 =$$

$$= 2 + 1 = 3 \quad \Rightarrow \quad |\vec{b}| = \sqrt{3}$$

$$\cos(\vec{a},\vec{b}) = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\cdot|\vec{b}|} = \frac{-3}{2\sqrt{3}\sqrt{3}} = -\frac{3}{6} = -\frac{1}{2} \Rightarrow$$

$$\Rightarrow m(\vec{a},\vec{b}) = 120^0$$

3.3. Fix an orthonormal basis:

$\vec{a}(2,1,0)$

$\vec{b}(0,-2,1)$

Find the angle between the diagonals of the parallelogram spanned by $\vec{a}$ and $\vec{b}$

$\vec{d_1} = \vec{a} + \vec{b}$

$\vec{d_2} = \vec{a} - \vec{b}$

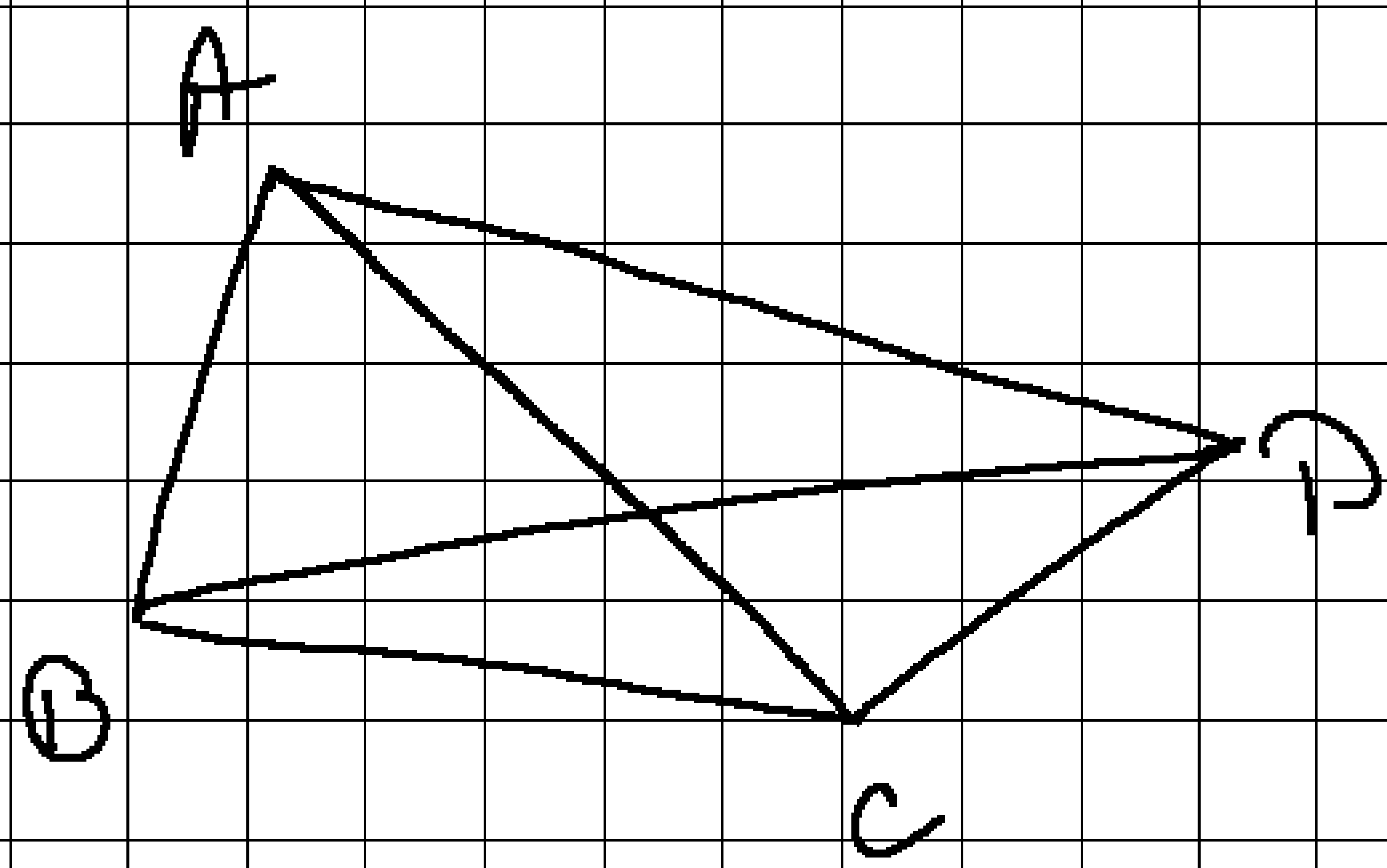
$\vec{d_1} \bullet \vec{d_2} = (\vec{a} + \vec{b}) \bullet (\vec{a} - \vec{b}) = \vec{a}^2 - \vec{b}^2 =$

$= |\vec{a}|^2 - |\vec{b}|^2 = 2^2 + 1^2 + 0^2 - 0^2 - 2^2 - 1^2 = 0 \Rightarrow$

$\Rightarrow \vec{d_1} \perp \vec{d_2}$

3.7. ABCD tetrahedron. Show that:

$$\cos(\vec{AB}, \vec{CD}) = \frac{AD^2 + BC^2 - AC^2 - BD^2}{2 \cdot AB \cdot CD}$$ (3D law of cosine)



exercise 3.6

Law of cosines in 2D:

ΔABC

$$BC^2 = AB^2 + AC^2 - 2 AB\, AC \cdot \cos(\widehat{BAC})$$

$$BC^2 = AB^2 + AC^2 - 2 \cdot \vec{AB} \cdot AC$$

Proof:

$$AB^2 + AC^2 - BC^2 = (AB^2 - BC^2) + AC^2 =$$

$$= (\vec{AB} - \vec{BC})(\vec{AB} + \vec{BC}) + AC^2 = (\vec{AB} - \vec{BC})\, AC^2 + AC^2 =$$

$$= (\vec{AB} - \vec{BC} + \vec{AC}) \cdot \vec{AC} = (\vec{r}_B - \vec{r}_A - \vec{r}_C + \vec{r}_B + \vec{r}_C - \vec{r}_A) \cdot \vec{AC} =$$

$$= (2\vec{r}_B - 2\vec{r}_A)\, \vec{AC} = 2\vec{AB} \cdot \overline{AC}$$

$$AD^2 + BC^2 - AC^2 - BD^2 = (AD^2 - AC^2) + (BC^2 - BD^2) =$$

$$= (\vec{AD} - \vec{AC})(\vec{AD} + \vec{AC}) + (\vec{BC} - \vec{BD})(\vec{BC} + \vec{BD}) =$$

$$= (\vec{CA} + \vec{AD})(\vec{AD} + \vec{AC}) + (\vec{DB} + \vec{BC})(\vec{BC} + \vec{BD}) =$$

$$= \vec{CD}(\vec{AD} + \vec{AC}) + \vec{DC}(\vec{BC} + \vec{BD}) =$$

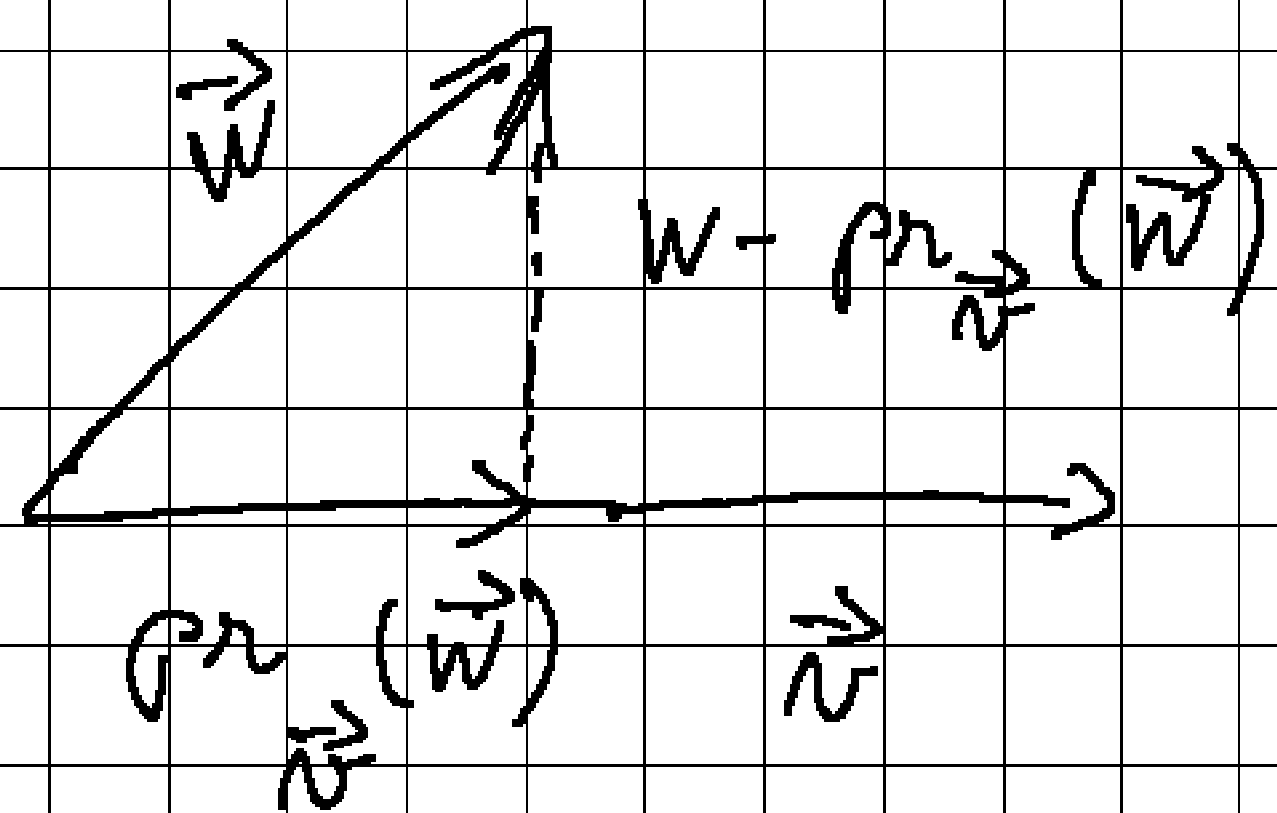
$$= \vec{CD}(\vec{AD} + \vec{AC} - \vec{BC} - \vec{BD}) =$$

$$= \vec{CD}(\vec{AB} + \vec{AB})$$

$$= \vec{CD} \cdot 2\vec{AB} \quad \Rightarrow$$

$$\Rightarrow \frac{AD^2 + BC^2 - AC^2 - BD^2}{2\, AB \cdot CD} = \cos(\vec{AB}, \vec{CD})$$

To get an orthonormal basis from a general one: → Gram-Schmidt



$$\vec{w}\cdot\vec{v} = \underbrace{|\vec{w}|\cdot\cos(\vec{v},\vec{w})}\cdot|\vec{v}| =$$

$$= \left|pr_{\vec{v}}(\vec{w})\right|\cdot|\vec{v}|$$

$$\Rightarrow pr_{\vec{v}}|\vec{w}| = \frac{\vec{w}\cdot\vec{v}}{|\vec{v}|^2}\cdot\vec{v} = \frac{\vec{w}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\cdot\vec{v}$$

$\mathcal{B} = (\vec{v}_1, \dots, \vec{v}_n)$ basis of V^n to get an orthonormal basis:

1. $$\vec{v}_1^{\,1} = \vec{v}_1$$

$$\vec{v}_2^{\,1} = \vec{v}_2 - \overbrace{pr_{v_1^1}(\vec{v}_2)}^{=\frac{\vec{v}_2\cdot\vec{v}_1^{\,1}}{\vec{v}_1^{\,1}\cdot\vec{v}_1^{\,1}}\cdot\vec{v}_1^{\,1}}$$

$$\vec{v}_3^{\,1} = \vec{v}_3 - pr_{\langle\vec{v}_1^{\,1},\vec{v}_2^{\,1}\rangle}(\vec{v}_3) = \vec{v}_3 - \frac{\vec{v}_3\cdot\vec{v}_1^{\,1}}{\vec{v}_1^{\,1}\cdot\vec{v}_1^{\,1}}\cdot\vec{v}_1^{\,1} -$$

$$- \frac{\vec{v}_3\cdot\vec{v}_2^{\,1}}{\vec{v}_2^{\,1}\cdot\vec{v}_2^{\,1}}\cdot\vec{v}_2^{\,1}$$

Repeat until we have $\vec{v}_n^{\,1}$

2. Normalize:

$$\vec{v}_i'' = \frac{\vec{v}_i'}{|\vec{v}_i'|}$$

$\Rightarrow B'' = (\vec{v}_1'', \ldots, \vec{v}_n'')$ orthonormal basis

3.10. $\vec{v}_1\ (0,1,10)$

$\vec{v}_2\ (2,1,0)$

$\vec{v}_3\ (-1,0,1)$

Use Gram-Schmidt to get an orthonormal basis containing v_1

1. $\vec{v}_1' = \vec{v}_1 \Rightarrow \vec{v}_1' = (0,1,10)$

2. $$\vec{v}_2' = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} - \frac{2\cdot 0 + 1\cdot 1 + 0\cdot 0}{1} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$

3. $v_3' = v_3 - pr_{\vec{v}_1'}(v_3) - pr_{\vec{v}_2'}(v_3) =$

$$= \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - \frac{-1\cdot 0 + 0\cdot 1 + 1\cdot 0}{1} \cdot \vec{v}_1' - \frac{2\cdot(-1) + 0\cdot 0 + 0\cdot 1}{4} \cdot v_2' =$$

$$= \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + \frac{1}{2} \cdot \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Normalize:

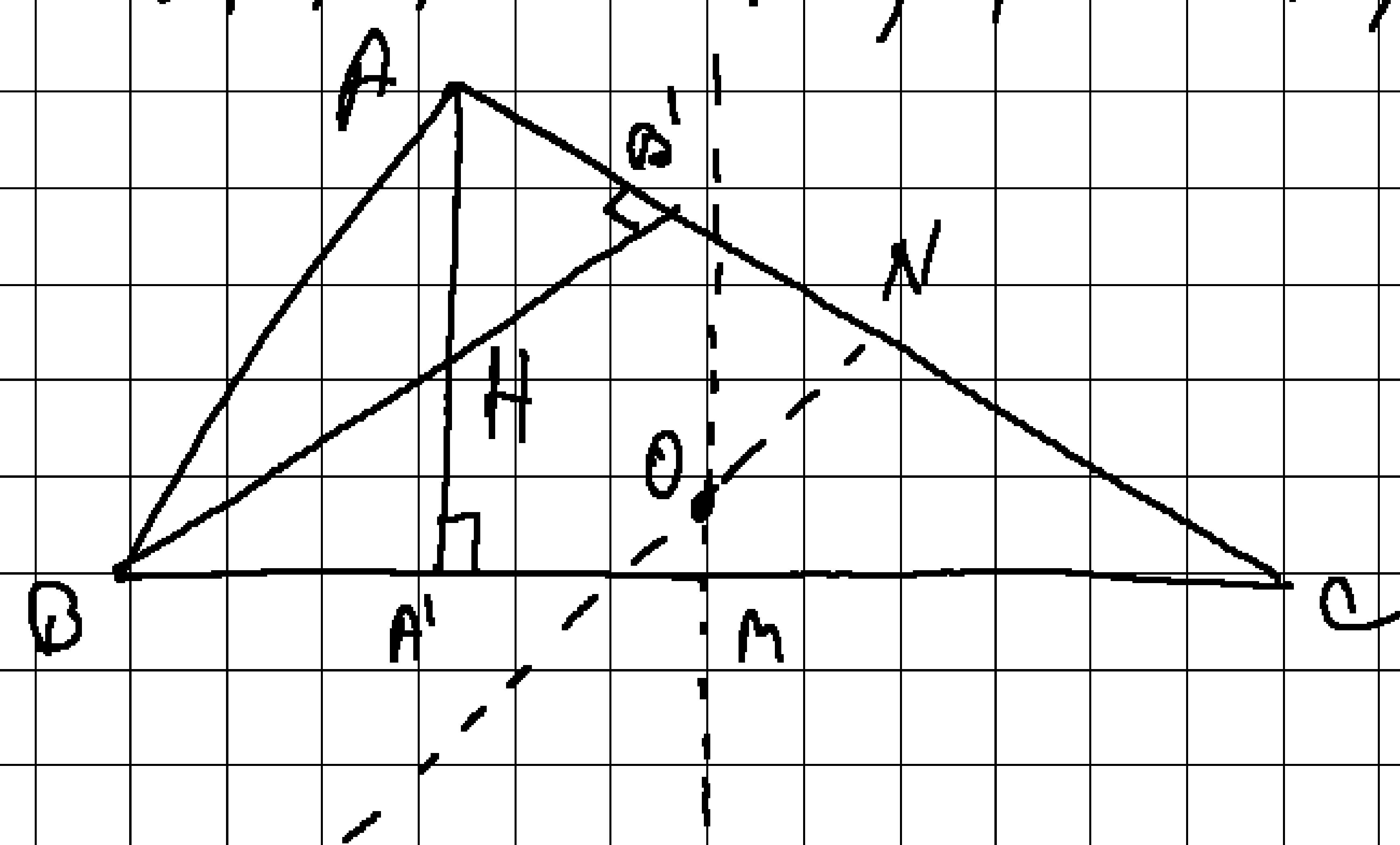
$$\vec{v}_1'' = \frac{\vec{v}_1'}{|\vec{v}_1'|} = v_1'$$

$$\vec{v}_3'' = \frac{\vec{v}_3'}{|\vec{v}_3'|} = v_3'$$

$$\vec{v}_2'' = \frac{\vec{v}_2'}{|\vec{v}_2'|} = \frac{1}{2}\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

3.18. Determine the circumcenter and the orthocenter of ΔABC with:

$A(1,2)$, $B(3,-2)$, $C(5,6)$



$$m_{AC} = \frac{y_C - y_A}{x_C - x_A} = \frac{6-2}{5-1} = \frac{4}{4} = 1$$

$$NO \perp AC \Rightarrow m_{NO} \cdot m_{AC} = -1$$

$$m_{NO} = -1$$

N mid of $AC \Rightarrow N(3,4)$

$NO: \ (y - y_N) = (x - x_N)\, m_{NO}$

$NO: \ y - 4 = -x + 3$

$NO: \ y = 7 - x$

M mid of $BC \Rightarrow M(4,2)$

$m_{BC} = \frac{6+2}{5-3} = 4$

$m_{MO} = -\frac{1}{4}$

$\Rightarrow MO: \ y - 2 = (x-4)\left(-\frac{1}{4}\right)$

$MO: \ y - 2 = -\frac{1}{4}x + 1$

$MO: \ y = -\frac{1}{4}x + 3$

$\{O\} \in MO \cap NO \Rightarrow \ 7 - x_O = -\frac{1}{4}x_O + 3$

$\Rightarrow \frac{3}{4}x_O = 4 \Rightarrow x_O = \frac{16}{3}$

$\Rightarrow y_O = 7 - \frac{16}{3} = \frac{5}{3}$

orthocenter:

1, 4, 9, 12, 20

$m_{AA'} = m_{MO}$

$\Rightarrow \quad AA' : y - y_A = m_{AA'} \cdot (x - x_A)$

$y - 2 = \left(-\frac{1}{4}\right)(x-1)$

$m_{BB'} = m_{ON}$

$BB' : \quad y + 2 = (-1)(x-3)$

$y + 2 = -x + 3$

$H: \begin{cases} y - 2 = -\frac{1}{4}x + \frac{1}{4} \\ y + 2 = -x + 3 \end{cases}$

$y_H = -\frac{1}{4} x_H + \frac{9}{4}$

$\Rightarrow \quad -\frac{1}{4}x_H + \frac{9}{4} + 2 = -x_H + 3$

$\frac{3}{4} x_H = -\frac{5}{4}$

$\Rightarrow x_H = -\frac{5}{3}$

$y_H = \frac{5}{12} + \frac{9}{4} = \frac{32}{12}$