1) $f: \mathbb{R}^2 \to \mathbb{R}$

$f(x,y) = x^2 + xy$

a) Find the gradient of f and the direction of steepest descent at $(1,0)$

$\nabla f(x,y) = \left(\frac{\partial f}{\partial x}(x,y), \frac{\partial f}{\partial y}(x,y) \right)$ - gradient

$-\nabla f(x,y)$ - dir of steepest descent

$\nabla f(x,y) = (2x+y, x)$

$\nabla f(0,1) = (2,1)$ - gradient at $(1,0)$

$-\nabla f(1,0) = (-2,-1)$ - dir. of steepest descent

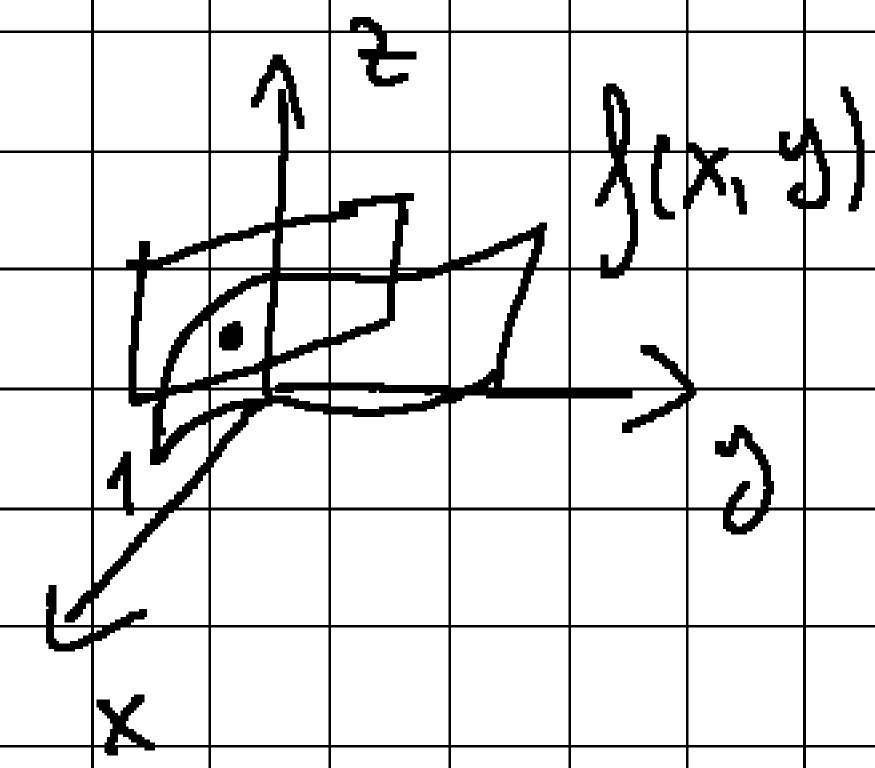
b) Find the directional derivative at point $(1,0)$ in the direction of $\vec{i} + \vec{j}$

$\vec{i} + \vec{j} = (1,0) + (0,1) = (1,1)$

$\langle \nabla f(1,0), (1,1) \rangle = \langle (2,1), (1,1) \rangle = 2 \cdot 1 + 1 \cdot 1 = 3$

c) Find the equation of the tangent plane to the surface $z = f(x,y)$ at the point $(1,0,1)$

Example 9.21. from lecture



tangent plane:

$\nabla f(x_0,y_0,z_0) \cdot (x-x_0, y-y_0, z-z_0) = 0$

$L = \{(x,y,z) \in \mathbb{R}^3 \mid f(x,y,z) = 0\}$

Let $g(x,y,z) = f(x,y) - z = 0$ level curve

$\nabla g(x,y,z) = (2x+y, x, -1)$

$g(x,y,z) = x^2 + xy - z$

$\nabla g(1,0,1) = (2,1,-1)$

tangent plane: $\nabla g(1,0,1) \cdot (x-1, y, z-1) = 0$

$\Leftrightarrow (2,1,-1) \cdot (x-1, y, z-1) = 0$

$\Leftrightarrow (2x-2, y, -z+1) = 0$

$\Leftrightarrow (2x-2+y-z+1) = 0$

2. Find the equation of the tangent line to the elipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at an arbitary point (x_0, y_0)

$$\nabla f(x_0, y_0)\bullet(x-x_0, y-y_0)=0$$

$f(x,y)=\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ level curve

$$\nabla f(x,y)=\left(\frac{2x}{a^2}, \frac{2y}{b^2}\right)$$

$$\nabla f(x_0,y_0)=\left(\frac{2x_0}{a^2}, \frac{2y_0}{b^2}\right)$$

$$\nabla f(x_0,y_0)\cdot(x-x_0, y-y_0)=0 \Leftrightarrow$$

$$\Leftrightarrow \left(\frac{2x_0}{a^2}, \frac{2y_0}{b^2}\right)\cdot(x-x_0, y-y_0)=0 \Leftrightarrow$$

$$\Leftrightarrow \left(\frac{2xx_0-2x_0^2}{a^2}, \frac{2yy_0-2y_0^2}{b^2}\right)=0 \quad \Big| \cdot \frac{1}{2}$$

$$\Leftrightarrow \left(\frac{xx_0-x_0^2}{a^2}, \frac{yy_0-y_0^2}{b^2}\right)=0$$

$$\frac{xx_0}{a^2}+\frac{yy_0}{b^2}-\frac{x_0^2}{a^2}-\frac{y_0^2}{b^2}=0$$

from pb. statement

$$y=\left(\overbrace{\frac{x_0^2}{a^2}+\frac{y_0^2}{b^2}}^{=1}-\frac{xx_0}{a^2}\right) \quad \Big| \cdot \frac{b^2}{y_0} \Leftrightarrow$$

$$\Leftrightarrow y=\frac{b^2}{y_0}-\frac{xx_0b^2}{a^2y_0}$$

3) $f : \mathbb{R}^n \to \mathbb{R}$

$f(x) = \frac{1}{2} \|x\|^2$

Find the gradient of f and the directional derivative $D_v f(x)$ in two ways: using the gradient and the definition.

$$f(x_1, x_2, \dots, x_n) = \frac{1}{2}(x_1^2 + x_2^2 + \dots + x_n^2)$$

$$\nabla f(x_1, x_2, \dots, x_n) = \left(\frac{df}{dx_1}(x), \dots, \frac{df}{dx_n}(x) \right)$$

$$\frac{df}{dx_i}(x) = \frac{1}{2} \cdot 2x_i = x_i$$

$$\nabla f(x_1, x_2, \dots, x_n) = (x_1, x_2, \dots x_n) = x$$

$$D_v f(x) = \nabla f(x_1, x_2, \dots, x_n) \cdot v = x \cdot v = \langle x, v \rangle$$

$$D_v f(x) = \lim_{h \to \infty} \frac{f(x + h \cdot v) - f(x)}{h}$$

$$\frac{f(x + h \cdot v) \cdot f(x)}{h} = \frac{1}{2} \cdot \frac{\|x + h \cdot v\|^2 - \|x\|^2}{h} =$$

$$\|a + b\|^2 = \|a\|^2 + \|b\|^2 + 2\langle a, b \rangle$$

$$= \frac{\frac{1}{2}(\|x\|^2 + \|h \cdot v\|^2 + 2\langle x, h \cdot v \rangle + \|x\|^2)}{h} =$$

$$= \frac{\frac{1}{2}\left(\|h\cdot v\|^2 + 2 \langle x, h\cdot v\rangle \right.}{h}$$

$$D_v f(x) = \lim_{h\to 0} \frac{\frac{1}{2}\left(h^2 \cdot \|v\|^2 + 2\langle x, h\cdot v\rangle\right)}{h} =$$

$$= \lim_{h\to 0}\left(\underbrace{\frac{1}{2}\cdot h\, \|v\|^2}_{0} + \frac{\langle x, h\, v\rangle}{h} \right) =$$

$$= \lim_{h\to 0} \frac{\cancel{h}\cdot\langle x, v\rangle}{\cancel{h}} = \langle x, v\rangle$$

4) Let $D = \mathrm{diag}(d_1, \dots, d_n)$ and $f: \mathbb{R}^n \to \mathbb{R}$ $f(x) = \frac{1}{2} x^t D \cdot X$. Prove that $\nabla f(x) = DX$ and $H(x) = D$.

$$H(x) = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1}(x) & \frac{\partial^2 f}{\partial x_1 \partial x_2}(x) & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n}(x) \\ \vdots & & & \\ \frac{\partial^2 f}{\partial x_n \partial x_1}(x) & \cdots & \cdots & \frac{\partial^2 f}{\partial x_n^2}(x) \end{pmatrix}$$

$\nabla f(x) =$

$$f(x) = \frac{1}{2}(x_1, x_2, \dots, x_n) \begin{pmatrix} d_1 & & & 0 \\ 0 & d_2 & & \\ \vdots & & \ddots & \\ 0 & 0 & \cdots & d_n \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} =$$

$$= \frac{1}{2}(d_1 x_1, d_2 x_2, \dots d_n x_n) \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} =$$

$$= \frac{1}{2}(d_1 x_1^2 + d_2 x_2^2 + \dots + d_n x_n^2)$$

$$\nabla f(x) = (d_1 x_1, d_2 x_2, \dots, d_n x_n) = D \cdot X$$

$$\frac{\partial f}{\partial x_i} = x_i d_i$$

$$\frac{\partial^2 f}{\partial x_j \partial x_i} = \begin{cases} 0, & j \neq i \\ d_i, & j = i \end{cases}$$

$$H(x) = \begin{pmatrix} d_1 & & & 0 \\ & d_2 & & \\ & & \ddots & \\ 0 & & & d_n \end{pmatrix} = D$$

6) $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$

$x = g_1(u, v)$

$y = g_2(u, v)$

$f(x, y) = (f \circ g)(u, v)$,

$g = (g_1, g_2)$

Prove that:

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial u}$$

$$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial v}$$

chain rule: $D(f \circ g)(\cdot) = Df(g(\cdot)) \cdot Dg(\cdot)$

$(f \circ g)(u, v) = f(g_1(u, v), g_2(u, v))$

$g(u, v) = (g_1(u, v), g_2(u, v))$

$$Dg(u, v) = \begin{pmatrix} \nabla g_1(u, v) \\ \nabla g_2(u, v) \end{pmatrix} = \begin{pmatrix} \frac{\partial g_1}{\partial u}, & \frac{\partial g_1}{\partial v} \\ \frac{\partial g_2}{\partial u}, & \frac{\partial g_2}{\partial v} \end{pmatrix} = Dg(\cdot)$$

$$Df(x,y) = \nabla f(x,y) = \left(\frac{\partial f}{\partial x}(x,y), \frac{\partial f}{\partial y}(x,y) \right)$$

$$Df(g_1, g_2) = \left(\frac{\partial f}{\partial x}(g_1,g_2), \frac{\partial f}{\partial y}(g_1,g_2) \right) = \underset{f}{Df}(g(\cdot))$$

from chain rule

$$D(f \circ g)(u,v) = \nabla f(u,v) = \left(\frac{\partial f}{\partial u}(u,v), \frac{\partial f}{\partial v}(u,v) \right)$$

$$= D(f \circ g)(\cdot)$$