PS Written Ex1   Exam 2:

1. A software firm produces accounting programs. The probability that one of their programs is defective is 0.2. A chain store purchases 20 accounting programs from that firm.

a) Find the probability that exactly 3 programs are defective.

The probability of exactly k successes in a binomial distribution is given by

$$P(X=k) = C_k^n \cdot p^k \cdot (1-p)^{n-k}$$

$n=20$, $k=3$, $p=0.2$

$$C_k^n = \frac{n!}{k!(n-k)!}$$

$$P(X=3) = C_3^{20} \cdot (0.2)^3 \cdot (0.8)^{17}$$

$$C_3^{20} = \frac{20 \cdot 19 \cdot 18}{3 \cdot 2 \cdot 1} = 1140$$

$$P(X=3) = 1140 \cdot (0.2)^3 \cdot (0.8)^{17} = 0.354$$

b) Find the probability that at least 80% of the programs are working properly.

80% work $\Rightarrow$ 20% don't work

$$\Rightarrow \quad X \leq 20\% \cdot 20 \Rightarrow X \leq \frac{20}{100} \cdot 20 \Rightarrow X \leq 4$$

$$P(X \leq 4) = P(X=0) + P(X=1) + P(X=2) + P(X=3) + P(X=4)$$

$$= 0{,}01 + 0{,}05 + 0{,}13 + 0{,}35 + 0{,}21 = 0{,}78$$

c) Let X denote the number of programs that are working properly. Find the probability distribution function of X. What type of distribution is it?

$X = 20 - Y$, where Y - num. of defective prog.

When a situation involves a fixed number of independent trials (n) with a constant probability (p) of success or failure, it follows a binomial distribution

Y follows a binomial distribution

$$Y \sim \text{Binomial}(n=20, p=0.2)$$

X also follows a binomial distribution

$$X \sim \text{Binomial}(n=20, p=0.8)$$

the probability distribution function is:

$$P(X=K) = C_K^{20} \cdot (0.8)^K \cdot (0.2)^{20-K} \quad , K = 0,1\dots,20$$

d) What is the expected number of programs that are working properly

For a binomial distribution, the expected value is given by:

$$E(X) = n \cdot p$$

$n = 20$, $p = 0.8$

$$E(X) = 20 \cdot 0.8 = 16$$

2. Let $X_1, X_2 \dots X_n$ be a random sample drawn from a Poisson distribution with parameter λ, unknown

$$P(X;\lambda) = \frac{\lambda^x e^{-\lambda}}{x!} \qquad x = 0, 1, 2, \dots$$

$E(X) = \lambda$ $\quad$ $V(X) = \lambda$

Mean $\quad$ variance

a) Find the maximum likelihood estimator, $\hat{\lambda}$, for λ

The likelihood function is the product of the PMFs for all n independent observations.

$$L(\lambda) = \prod_{i=1}^{n} P(X_i;\lambda)$$

$$L(\lambda) = \prod_{i=1}^{n} \frac{\lambda^{x_i} e^{-\lambda}}{x_i!}$$ $\quad$ $x_i!$ are const. we ignore them

$$L(\lambda) = \lambda^{\sum_{i=1}^{n} x_i} \cdot e^{-n\lambda}$$

$$l(\lambda) = \ln(L(\lambda)) = \left(\sum_{i=1}^{n} x_i\right) \cdot \ln(\lambda) - n\lambda$$

$$\frac{\partial l(\lambda)}{\partial \lambda} = \frac{\sum_{i=1}^{n} x_i}{\lambda} - n$$

$$\frac{\sum_{i=1}^{n} x_i}{\lambda} - n = 0 \Rightarrow \lambda = \frac{\sum_{i=1}^{n} x_i}{n} \qquad \hat{\lambda} = \bar{X} = \frac{\sum_{i=1}^{n} x_i}{n}$$

b) is it an absolutely correct estimator? Explain

I Check unbiasedness

estimator is unbiased if $E(\bar{\lambda}) = \lambda$

$$\left.\begin{aligned} E(\bar{\lambda}) &= E\left(\frac{\sum_{i=1}^{n} X_i}{n}\right) = \frac{1}{n}\sum_{i=1}^{n} E(X_i) \\ E(X) &= \lambda \end{aligned}\right\} \Rightarrow E(\bar{X}) = \frac{1}{\cancel{n}}\cdot\cancel{n}\cdot\lambda \Rightarrow$$

$\Rightarrow \bar{\lambda}$ is unbiased

II Check efficiency

$$\left.\begin{aligned} V(\bar{\lambda}) &= V(\bar{X}) = \frac{V(X_i)}{n} \\ V(X_i) &= \lambda \end{aligned}\right\} \Rightarrow V(\bar{\lambda}) = \frac{\lambda}{n}$$

The Cramér-Rao bound for the variance of an unbiased estimator of λ is

$$\left.\begin{aligned} V(\bar{\lambda}) &\geq \frac{1}{i(\lambda)} = \frac{\lambda}{n} \\ i(\lambda) &= \frac{n}{\lambda} \\ V(\bar{\lambda}) &= \frac{\lambda}{n} \end{aligned}\right\} \Rightarrow \text{is efficient}$$

I, II $\Rightarrow$ $\bar{\lambda}$ is an absolutely correct estimator

c) Find the efficiency of $\bar{\lambda}$, $e(\bar{\lambda})$

$$e(\bar{\lambda}) = \frac{\text{Cramér-Rao lower bound}}{\text{Variance of the estimator}}$$

Cramér-Rao lower bound $= \dfrac{1}{i(\lambda)} = \dfrac{\lambda}{n}$

$$e(\lambda) = \frac{\frac{1}{i(\lambda)}}{V(\bar{\lambda})} = \frac{\frac{\lambda}{n}}{\frac{\lambda}{n}} = 1$$

d) At the significance level $\alpha \in (0,1)$, find a most powerful test for testing $H_0: \lambda = 1$ against $H_1: \lambda = 2$

The Neyman-Pearson lemma states that the most powerful test for a given signifance level α is

$$\Lambda = \frac{L(\lambda=1)}{L(\lambda=2)}$$

$$\Lambda = \frac{1 \cdot \sum_{i=1}^{n} x_i \cdot e^{-n \cdot 1}}{2 \cdot \sum_{i=1}^{n} x_i \cdot e^{-n \cdot 2}}$$

???

PS Written Ex$_2$:

1) The battery of a particular car brand starts with probability 0.95. Find the probability of the following events:

a) A: the battery only starts on the 5th attempt.

$p = 0.95$

$q = 1 - p = 0.05$

$$P(X=k) = q^{k-1} \cdot p$$

$$P(A) = P(X=5) = q^{5-1} \cdot p = q^4 \cdot p = (0.05)^4 \cdot 0.95 \simeq 0.00000594$$

b) B: The battery starts on the first at least 20 consecutive attempts.

$$P(B) = p^{20} = 0.95^{20} \simeq 0.35$$

2. Let $X \in Exp(\mu)$. Find the pdf of $Y = \sqrt{X}$.

$$f_X(x) = \begin{cases} \frac{1}{\mu} \cdot e^{-\frac{x}{\mu}} & , x \geq 0 \\ 0 & , x < 0 \end{cases}$$

To find the PDF of $Y = g(X)$:

$$f_Y(y) = f_X(g^{-1}(y)) \cdot \left| \frac{d}{dy} g^{-1}(y) \right|$$

$$\left.\begin{array}{l} Y=\sqrt{X} \Rightarrow X=Y^2 \\ g^{-1}(y)=g(X) \end{array}\right\} \Rightarrow g^{-1}(y)=y^2$$

$$g^{-1}(y)=y^2 \quad \Big/ \frac{d}{dy} \Rightarrow \frac{d}{dy}g^{-1}(y)=\frac{d}{dy}(y^2)=2y$$

$$f_Y(y)=f_X\left(g^{-1}(y)\right)\cdot\left|\frac{d}{dy}g^{-1}(y)\right|$$

$$\left.\begin{array}{l} f_Y(y)=f_X(y^2)\cdot|2y| \\ f_X(y^2)=\frac{1}{\mu}e^{-\frac{y^2}{\mu}} \end{array}\right\} \Rightarrow f_Y(y)=\frac{1}{\mu}e^{-\frac{y^2}{\mu}}\cdot 2y$$

3. Let $X_1, X_2 \dots X_n$ be a random sample drawn from a distribution with pdf $(x;\theta)=\frac{2}{\theta^2}x$, for $0<x<\theta$, with $\theta>0$ unknown.

a) Find the method of moments estimator, $\bar{\theta}$, for θ

$$\boxed{E(X)=\int_0^\theta x\cdot f(x;\theta)\,dx}$$

$$E(X)=\int_0^\theta x\cdot\frac{2}{\theta^2}x\,dx=\frac{2}{\theta^2}\int_0^\theta x^2\,dx=\frac{2}{\theta^2}\cdot\frac{x^3}{3}\Big|_0^\theta=$$

$$=\frac{2}{\theta^2}\cdot\frac{\theta^3}{3}=\frac{2\theta}{3}$$

The sample mean
$\overline{X} = \frac{1}{n} \sum_{i=1}^{n} X_i$

$E(X) = \overline{X} \Rightarrow \frac{2\theta}{3} = \overline{X} \Rightarrow \overline{\theta} = \frac{3}{2}\overline{X}$

b) is $\overline{\theta}$ an absolutely correct estimator? Explain

Step I: Check for unbiasedness

unbiased if $E(\overline{\theta}) = \theta$

$$\left.\begin{aligned} E(\overline{\theta}) &= E\left(\frac{3}{2}\overline{X}\right) = \frac{3}{2} E(\overline{X}) \\ E(\overline{X}) &= E(X) = \frac{2\theta}{3} \end{aligned}\right\} \Rightarrow E(\overline{\theta}) = \frac{3}{2} \cdot \frac{2\theta}{3} = \theta$$

$\Rightarrow \overline{\theta}$ is unbiased

Step II: check for efficiency:

$$V(\overline{\theta}) = V\left(\frac{3}{2}\overline{X}\right) = \left(\frac{3}{2}\right)^2 V(\overline{X})$$

$V(aX) = a^2 \cdot V(X)$

$$V(\overline{X}) = \frac{V(X)}{n}$$

$V(X) = E(X^2) - (E(X))^2$

$$E(X^2) = \int_0^\theta x^2 \cdot f(x;\theta)\,dx = \int_0^\theta x^2 \cdot \frac{2}{\theta^2} x \, dx = \frac{2}{\theta^2} \int_0^\theta x^3 \, dx =$$

$$= \frac{2}{\theta^2} \cdot \frac{\theta^4}{4} = \frac{\theta^2}{2}$$

$$V(X) = E(X^2) - \left(E(X)\right)^2 = \frac{\theta^2}{2} - \left(\frac{2\theta}{3}\right)^2 = \frac{\theta^2}{2} - \frac{4\theta^2}{9} = \frac{9\theta^2 - 8\theta^2}{18} =$$

$$= \frac{\theta^2}{18}$$

$$V(\overline{X}) = \frac{V(X)}{n} = \frac{\theta^2}{18n}$$

$$V(\overline{\theta}) = \left(\frac{3}{2}\right)^2 \cdot V(\overline{X}) = \left(\frac{3}{2}\right)^2 \cdot \frac{\theta^2}{18n} = \frac{9}{4} \cdot \frac{\theta^2}{18n} = \frac{\theta^2}{8n}$$

(you have to calculate the lower bound and fisher info)

PS Written ex3:

1. A contestant participates in a game show where three important prizes are offered. His chances of winning the three prizes are $\frac{1}{6}$, $\frac{1}{3}$ and $\frac{1}{2}$, respectively.

a) Find the probability that the contestant win exactly one prize

Case 1 = Win first, lose the rest

Case 2 = Win the second, lose the rest

Case 3 = win the third, lose the rest

$$P(\text{Case }1) = \frac{1}{6}\cdot\left(1-\frac{1}{3}\right)\cdot\left(1-\frac{1}{2}\right) = \frac{1}{18}$$

$$P(\text{Case }2) = \left(1-\frac{1}{6}\right)\cdot\frac{1}{3}\cdot\left(1-\frac{1}{2}\right) = \frac{5}{36}$$

$$P(\text{case }3) = \left(1-\frac{1}{6}\right)\cdot\left(1-\frac{1}{3}\right)\cdot\frac{1}{2} = \frac{5}{18}$$

$$P(\text{Exactly one prize}) = P(\text{Case }1) + P(\text{Case }2) + P(\text{Case }3) =$$

$$= \frac{1}{18} + \frac{5}{36} + \frac{5}{18} = \frac{17}{36}$$

b) Find the probability that the contestant loses at least two prizes.

$P(\text{lose all}) = \left(1-\frac{1}{6}\right)\cdot\left(1-\frac{1}{3}\right)\cdot\left(1-\frac{1}{2}\right) = \frac{5}{18}$

$P(\text{win exactly one prize}) = \frac{17}{36}$ (calculated at point a)

$P(\text{lose at least two prizes}) = P(\text{lose all}) + P(\text{win exactly one prize})$

$= \frac{5}{18} + \frac{17}{36} = \frac{27}{36} = \frac{3}{4}$

c) Let X denote the number of prizes won by the contestant. Find the probability distribution function of X.

X - number of prizes won $\Rightarrow X = \{0, 1, 2, 3\}$

$P(X=0) = \frac{5}{18}$ (calculated at point b)

$P(X=1) = \frac{17}{36}$ (calculated at point a)

$P(X=2) = P(\text{case } 1) + P(\text{case } 2) + P(\text{case } 3)$

Case 1 $\Rightarrow$ win first, second, lose the third

Case 2 $\Rightarrow$ win first, third, lose the second

Case 3 $\Rightarrow$ win second, third, lose the first

$P(\text{case } 1) = \frac{1}{6} \cdot \frac{1}{3} \cdot \left(1 - \frac{1}{2}\right) = \frac{1}{36}$

$P(\text{case } 2) = \frac{1}{6} \cdot \left(1 - \frac{1}{3}\right) \cdot \frac{1}{2} = \frac{2}{36}$

$P(\text{case } 3) = \left(1 - \frac{1}{6}\right) \cdot \frac{1}{3} \cdot \frac{1}{2} = \frac{5}{36}$

$P(X=2) = \frac{1}{36} + \frac{2}{36} + \frac{5}{36} = \frac{8}{36} = \frac{2}{9}$

$P(X=3) = \frac{1}{6} \cdot \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{36}$

$$P(X) = \begin{cases} \frac{5}{18}, & \text{if } X=0 \\ \frac{17}{36}, & \text{if } X=1 \\ \frac{2}{9}, & \text{if } X=2 \\ \frac{1}{36}, & \text{if } X=3 \end{cases}$$

d) How many prizes can the contestant expect to win?

$$E(X) = \sum_{x=0}^{3} x \cdot P(X=x) =$$

$$= 0 \cdot \frac{5}{18} + 1 \cdot \frac{17}{36} + 2 \cdot \frac{2}{9} + 3 \cdot \frac{1}{36} = \frac{36}{36} = 1$$

2. Let $X_1, X_2 \dots X_n$ be a random sample drawn from a Gamma$(2, 3\theta)$ distribution, with $\theta > 0$ unknown. (for $X \sim$ Gamma(a, b), the pdf is $f(x; a, b) = \frac{1}{b^a \Gamma(a)} x^{a-1} e^{-x/b}$, $x > 0$, $E(x) = ab$, $V(x) = ab^2$)

a) Find the maximum likelyhood estimator, $\bar{\theta}$, for θ

$$\boxed{\Gamma(n) = (n-1)!}$$

$$a = 2, \; b = 3\theta$$

$$f(x; a, b) = \frac{1}{(3\theta)^2 \cdot \underbrace{\Gamma(2)}_{1}} x^{2-1} \cdot e^{-x/3\theta} =$$

$$= \frac{1}{9\theta^2} x \cdot e^{-x/3\theta} = f(x, \theta)$$

$$E(x) = a \cdot b = 6\theta$$

$$V(x) = ab^2 = 2 \cdot (3\theta)^2 = 18\theta^2$$

$$L(\theta) = \prod_{i=1}^{n} f(x_i; \theta) = \prod_{i=1}^{n} \frac{1}{9\theta^2} x_i e^{-x_i/3\theta} =$$

$$= \left(\frac{1}{9\theta^2}\right)^n \prod_{i=1}^{n} x_i \cdot e^{-x_i/3\theta}$$

$$\ln L(\theta) = n \cdot \ln\left(\frac{1}{9\theta^2}\right) + \ln\left(\prod_{i=1}^{n} X_i\right) - \frac{\sum_{i=1}^{n} X_i}{3\theta}$$

$$n \cdot \ln\left(\frac{1}{9\theta^2}\right) = n \cdot (\ln 1 - \ln(9\theta^2)) = -n \ln(9\theta^2) = -n \ln 9 - 2n \ln\theta$$

$$\ln\left(\prod_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} \ln X_i$$

$$\Rightarrow \ln L(\theta) = -n\ln 9 - 2n\ln\theta + \sum_{i=1}^{n}\ln x_i - \frac{\sum_{i=1}^{n} x_i}{3\theta} =$$

$$\frac{\partial}{\partial\theta}\ln L(\theta) = -\frac{2n}{\theta} + \frac{\sum_{i=1}^{n} x_i}{3\theta^2} \qquad \left(\frac{1}{\theta}\right)' = -\left(\frac{1}{\theta^2}\right)$$

$$-\frac{2n}{\theta} + \frac{\sum_{i=1}^{n} x_i}{3\theta^2} = 0 \Rightarrow \frac{\sum_{i=1}^{n} x_i}{3\theta^2} = \frac{2n}{\theta} \Rightarrow \sum_{i=1}^{n} x_i = 6n\theta \Rightarrow$$

$$\Rightarrow \bar{\theta} = \frac{\sum_{i=1}^{n} x_i}{6n} = \frac{\bar{X}}{6}$$

b) is it an absolutely correct estimator? Explain

Step I: check for unbiasedness

$$E(\bar{\theta}) = \theta$$

$$\left.\begin{array}{l} E(\bar{\theta}) = E\left(\frac{\bar{X}}{6}\right) = \frac{1}{6}E(\bar{X}) \\ \qquad\qquad E(X) = 6\theta \end{array}\right\} \Rightarrow E(\bar{\theta}) = \frac{1}{6}\cdot 6\theta = \theta$$

Step II: check for efficiency (solved at c)

c) Find the efficiency of $\bar{\theta}$, $e(\bar{\theta})$

$$e(\bar{\theta}) = \frac{CRLB}{V(\bar{\theta})}$$

$$CRLB = \frac{1}{i(\theta)}$$

$$\boxed{i(\theta) = -E\left[\frac{\partial^2}{\partial\theta^2}\ln L(\theta)\right]}$$

$$\frac{\partial}{\partial\theta}\ln L(\theta) = -\frac{2n}{\theta} + \frac{\sum_{i=1}^{n} X_i}{3\theta^2}$$

$$\frac{\partial^2}{\partial\theta^2}\ln L(\theta) = \frac{2n}{\theta^2} - \frac{2\sum_{i=1}^{n} X_i}{3\theta^3}$$

$$i(\theta) = -E\left(\frac{2n}{\theta^2} - \frac{2\sum_{i=1}^{n} X_i}{3\theta^3}\right)$$

$E\left(\frac{2n}{\theta^2}\right) = \frac{2n}{\theta^2}$ because it's constants values

$$E\left(\sum_{i=1}^{n} X_i\right) = n \cdot E(X_i) = n \cdot 6\theta$$

$$E\left(\frac{2\sum_{i=1}^{n} X_i}{3\theta^3}\right) = \frac{2 \cdot n \cdot 6\theta}{3\theta^3} = \frac{4n}{\theta^2}$$

$$\Rightarrow i(\theta) = -\left(\frac{2n}{\theta^2} - \frac{4n}{\theta^2}\right) = \frac{2n}{\theta^2}$$

$$\Rightarrow CRLB = \frac{1}{i(\theta)} = \frac{\theta^2}{2n}$$

$$\left.\begin{aligned} V(\bar{\theta}) &= \left(\frac{1}{6}\right)^2 \cdot V(\bar{X}) \\ V(\bar{X}) &= \frac{V(X)}{n} = \frac{18\theta^2}{n} \end{aligned}\right\} \Rightarrow V(\bar{\theta}) = \frac{1}{36} \cdot \frac{18\theta^2}{n} = \frac{\theta^2}{2n}$$

$$e(\bar{\theta}) = \frac{\frac{\theta^2}{n}}{\frac{\theta^2}{n}} = 1 \quad \Rightarrow \text{is efficient}$$

PS Written Ex 4:

1. A basketball player makes a free throw with probability 0.7. Find the probability of the following events:

a) A: the player makes his first free throw only on the 4^{th} shot.

$P(A) = P(\text{miss}, \text{miss}, \text{miss}, \text{make}) = q^3 \cdot p$

$q = 1 - p = 1 - 0.7 = 0.3$

$\Rightarrow P(A) = (0.3)^3 \cdot 0.7 = 0.0189$

b) B: the player makes the first at least 10 consecutive free throws:

$P(B) = p^{10} = 0.7^{10} = 0.02$

2. Let $X \in N(0,1)$. Find the pdf of $Y = X^2$. What type of distribution is it.

$Y = X^2 \Rightarrow X = \pm\sqrt{Y}$

$f_Y(y) = f_X(x) \cdot \left|\frac{dx}{dy}\right|$

$x = \sqrt{y}$ or $x = -\sqrt{y}$

Symmetry from $X \in N(0,1)$

$\Downarrow$

we double the result

$\Rightarrow f_Y(y) = 2 \cdot f_X(x) \left|\frac{dx}{dy}\right|$

The PDF of X: (Standard normal distribution)

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

$$\frac{dx}{dy} = \frac{1}{2\sqrt{y}} \quad \text{from } x = \sqrt{y}$$

$$f_X(\sqrt{y}) = \frac{1}{\sqrt{2\pi}} \cdot e^{-(\sqrt{y})^2/2} = \frac{1}{\sqrt{2\pi}} \cdot e^{-y/2}$$

$$f_Y(y) = 2 \cdot f_X(\sqrt{y}) \cdot \frac{1}{2\sqrt{y}} = \cancel{2} \cdot \frac{1}{\sqrt{2\pi}} \cdot e^{-y/2} \cdot \frac{1}{\cancel{2}\sqrt{y}} =$$

$$= \frac{1}{\sqrt{2\pi y}} \cdot e^{-y/2}$$

3. Let $X_1, X_2 \dots X_n$ be a random sample drawn from a distribution with pdf $f(x;\theta) = \frac{1}{\theta}$, for $0 < x < \theta$, with $\theta > 0$ unknown

a) Find the method of moments estimator, $\bar{\theta}$, for θ

$$E(X) = \int_0^\theta x \cdot f(x,\theta)dx = \int_0^\theta x \cdot \frac{1}{\theta}\, dx = \frac{1}{\theta} \cdot \frac{x^2}{2} \Big|_0^\theta = \frac{\theta}{2}$$

$$\bar{X} = \frac{1}{n} \sum_{i=1}^{n} x_i$$

$$\Rightarrow \bar{X} = \frac{\theta}{2} \Rightarrow \bar{\theta} = 2\bar{X}$$

b) is θ an absolutely correct estimator? Explain.

Step I: is θ unbiased?

$$\left.\begin{aligned} E(\bar{\theta}) = E(2\bar{X}) = 2 \cdot E(\bar{X}) \\ E(X) = \frac{\theta}{2} \end{aligned}\right\} \Rightarrow E(\bar{\theta}) = \theta \Rightarrow \text{is unbiased}$$

Step II: $\bar{\theta}$ is not absolutely correct because it depends on the random sample mean $\bar{X}$, which introduces variability

PS Written Ex5:

1. The probability that the battery of a particular car brand does not start is 0.03. Find the probability of the following events:

a) A: the battery only starts on the 3rd attempt

$q = 0.03$

$p = 1 - q = 0.97$

$P(A) = P(\text{fail}, \text{fail}, \text{success}) = q \cdot q \cdot p = (0.03)^2 \cdot 0.97 =$

$= 0.0008$

b) B: the battery starts on the first at least 25 consecutive attempts.

$P(B) = (0.97)^{25} \simeq 0.47$

2. Let $X \in \chi^2(2,1)$. Find the PDF of $Y = \sqrt{X}$

$$f_X(x) = \frac{1}{2^{k/2}\,\Gamma(k/2)} x^{(k/2)-1} \cdot e^{-x/2}$$

$k=2$ from $\chi^2(2,1)$ $\qquad \Gamma(1) = 1$

$$f_X(x) = \frac{1}{2} \cdot e^{-x/2}$$

$$Y = \sqrt{X} \Rightarrow X = Y^2$$

$$\frac{dX}{dY} = 2Y$$

$$f_Y(y) = f_X(x) \cdot \left|\frac{dx}{dy}\right| = f_X(y^2) \cdot \left|\frac{d}{dy} y^2\right| =$$

$$= f_X(y^2) \cdot 2y = \frac{1}{2} \cdot e^{-y/2} \cdot 2y = e^{-y/2} \cdot y$$

3. Let $X_1, X_2 \dots X_n$ be a random Sample drawn from a distribution with pdf $f(x;\theta) = (1+\theta)x^{\theta}$, for $0 < x < 1$ with $\theta > -1$, unknown

a) Find the method of moments estimator, $\bar{\theta}$, for θ

$$E(X) = \int_0^1 x\, f(x;\theta)\,dx = \int_0^1 x(1+\theta)x^{\theta}\,dx =$$

$$= (1+\theta)\int_0^1 x^{\theta+1}\,dx = (1+\theta)\cdot\frac{1}{\theta+2} = \frac{1+\theta}{\theta+2}$$

$$\int x^a\,dx = \frac{x^{a+1}}{a+1}$$

$$\bar{X} = \frac{1+\theta}{\theta+2} \Rightarrow \bar{X}(\theta+2) = 1+\theta \Leftrightarrow \bar{X}\theta + 2\bar{X} = 1+\theta \Leftrightarrow$$

$$\Leftrightarrow \bar{X}\theta - \theta = 1 - 2\bar{X} \Rightarrow \theta(\bar{X}-1) = 1-2\bar{X}$$

$$\Rightarrow \bar{\theta} = \frac{1-2\bar{X}}{\bar{X}-1}$$

b) Find the maximum likelihood estimator, $\bar{\theta}$, for θ

$$L(\theta) = \prod_{i=1}^{n} f(x_i;\theta) = \prod_{i=1}^{n}(1+\theta)x_i^{\theta} =$$

$$= (1+\theta)^n \prod_{i=1}^{n} x_i^{\theta}$$

$$\ln L(\theta) = n\ln(1+\theta) + \theta\sum_{i=1}^{n}\ln x_i$$

$$\frac{d}{d\theta}\ln L(\theta) = \frac{n}{1+\theta} + \sum_{i=1}^{n} \ln x_i$$

$$\frac{n}{1+\theta} + \sum_{i=1}^{n} \ln x_i = 0 \Rightarrow \frac{n}{1+\theta} = -\sum_{i=1}^{n} \ln x_i \Leftrightarrow$$

$$\Leftrightarrow 1+\theta = -\frac{n}{\sum_{i=1}^{n} \ln x_i} \Rightarrow \theta = \frac{n}{-\sum_{i=1}^{n} \ln x_i} - 1 = \frac{n}{-n\overline{\ln x}} - 1 =$$

$$= \frac{1}{-\overline{\ln x}} - 1$$

PS Written EX 6:

1. The probability of a certain basketball player making a free throw is known to be 0.6. Find the probability of the following events

a) A: the player makes his first free throw only on the 5th shot.

$$q = 1 - p = 1 - 0.6 = 0.4$$

$$P(A) = (0.4)^4 \cdot 0.6 = 0.01$$

b) B: the player makes the first at least 7 consecutive free throws

$$P(B) = (0.6)^7 = 0.27$$

2. Let $X \in \chi^2(2, 1/2)$. Find the pdf of $Y = \sqrt{X}$.

$$f_X(x) = \frac{1}{b^{k/2}\Gamma(k/2)} x^{(k/2)-1} \cdot e^{-x/b}$$

$k = 2$, $b = 1/2$

$\Gamma(2/2) = \Gamma(1) = 1$

$$f_X(x) = \frac{1}{\frac{1}{2} \cdot 1} \cdot x^{1-1} \cdot e^{-x/(1/2)} =$$

$$= 2e^{-2x}$$

$Y = \sqrt{X} \Rightarrow X = Y^2$

$$f_Y(y) = f_X(x) \cdot \left|\frac{dx}{dy}\right| = f_X(y^2) \cdot 2y =$$

$$\frac{dx}{dy} = \frac{d}{dy} Y^2 = 2Y \qquad = 2e^{-2y^2} \cdot 2y = 4ye^{-2y^2}$$

$$f_Y(y) = \begin{cases} 4ye^{-2y^2}, & y > 0 \\ 0, & y \le 0 \end{cases}$$

3. Let $X_1, X_2 \dots X_n$ be a random sample drawn from a distribution with pdf $f(x;\theta) = \theta x^{\theta-1}$, for $0 < x < 1$, with $\theta > 0$, unknown

a) Find the method of moments estimator, $\bar{\theta}$, for θ

$$E(X) = \int_0^1 x\, f(x;\theta)\, dx = \int_0^1 x \cdot \theta x^{\theta-1}\, dx =$$

$$= \theta \int_0^1 x^{\theta}\, dx = \frac{\theta}{\theta+1}$$

$$\bar{X} = \frac{\theta}{\theta+1} \Rightarrow \bar{X}(\theta+1) = \theta \Leftrightarrow \bar{X}\theta + \bar{X} = \theta \Leftrightarrow$$

$$\Leftrightarrow \bar{X}\theta - \theta = -\bar{X} \Leftrightarrow \theta(\bar{X}-1) = -\bar{X}$$

$$\Rightarrow \bar{\theta} = \frac{\bar{X}}{1-\bar{X}}$$

b) Find the maximum likelihood estimator, $\bar{\theta}$, for θ

$$L(\theta) = \prod_{i=1}^{n} f(x_i;\theta) = \prod_{i=1}^{n} \theta x_i^{\theta-1} = \theta^n \prod_{i=1}^{n} x_i^{\theta-1}$$

$$\ln L(\theta) = n \ln\theta + (\theta-1)\sum_{i=1}^{n} \ln x_i$$

$$\frac{d}{d\theta} \ln L(\theta) = \frac{n}{\theta} + \sum_{i=1}^{n} \ln x_i$$

$$\frac{n}{\theta} + \sum_{i=1}^{n} \ln x_i = 0 \Rightarrow \theta = -\frac{n}{\sum_{i=1}^{n} \ln x_i}$$

$$\bar{\theta} = -\frac{1}{\ln \bar{X}}$$

PS Written 2024 22:

1. A student has to take two exams. His chances of passing the two tests are $\frac{4}{5}$ and $\frac{3}{4}$, respectively

a) Find the probability that the student passes exactly one exam.

$P(P_1) = \frac{4}{5}$

$P(F_1) = 1 - \frac{4}{5} = \frac{1}{5}$

$P(P_2) = \frac{3}{4}$

$P(F_2) = 1 - \frac{3}{4} = \frac{1}{4}$

$$P(A) = P(P_1) \cdot P(F_2) + P(F_1) \cdot P(P_2) =$$
$$= \frac{4}{5} \cdot \frac{1}{4} + \frac{1}{5} \cdot \frac{3}{4} = \frac{4}{20} + \frac{3}{20} = \frac{7}{20}$$

b) Let X denote the number of exams the student passes. Find the probability distribution function of X

$P(X=0) = P(F_1) \cdot P(F_2) = \frac{1}{5} \cdot \frac{1}{4} = \frac{1}{20}$

$P(X=1) = \frac{7}{20}$ (solved at point a)

$P(X=2) = P(P_1) \cdot P(P_2) = \frac{4}{5} \cdot \frac{3}{4} = \frac{12}{20}$

$$P(X) = \begin{cases} \frac{1}{20}, & X = 0 \\ \frac{7}{20}, & X = 1 \\ \frac{12}{20}, & X = 2 \end{cases}$$

c) How many tests is the student expected to pass

$$\boxed{E(X) = \sum x \cdot P(X)}$$

$$E(X) = 0 \cdot \frac{1}{20} + 1 \cdot \frac{7}{20} + 2 \cdot \frac{12}{20} = \frac{31}{20} = 1.55$$

2. The pdf of the random variable X is given by $f_X(x) = 2e^{-2x}$, $x > 0$. Find the pdf of $Y = \frac{1}{2}X - 1$

$$Y = \frac{1}{2}X - 1 \Rightarrow X = 2(Y+1)$$

$$X > 0 \Rightarrow 2(Y+1) > 0$$
$$Y > -1$$

$$\frac{dX}{dY} = \frac{d}{dY} \cdot 2(Y+1) = 2$$

$$f_Y(y) = f_X(x) \cdot \left|\frac{dx}{dy}\right| = f_X(2Y+2) \cdot 2 =$$

$$= 2e^{-4Y+4}$$

$$f_Y(y) = \begin{cases} 2e^{-4Y+4} & , y > -1 \\ 0 & , y \le -1 \end{cases}$$

3. Let $X_1, X_2 \dots X_n$ be a random sample drawn from a Gamma$(2, \theta)$ distribution, with $\theta > 0$ unknown.

For $X \in$ Gamma(a, b), the pdf is

$$f(x; a, b) = \frac{1}{b^a \Gamma(a)} \cdot x^{a-1} e^{-x/b}, \ x > 0$$

$E(X) = ab$, $V(X) = ab^2$ $\qquad a = 2 \quad b = \theta$

a) Find the method of moments estimator, $\bar{\theta}$, for θ

$$\bar{X} = E(X) \Leftrightarrow \bar{X} = 2\theta \Rightarrow \bar{\theta} = \frac{\bar{X}}{2}$$

b) Find the maximum likelihood estimator, $\bar{\theta}$, for θ

$$L(\theta) = \prod_{i=1}^{n} f(x_i; 2, \theta) =$$

$$= \prod_{i=1}^{n} \frac{1}{\theta^2 \underbrace{\Gamma(2)}_{=1}} x_i^{2-1} \cdot e^{-x_i/\theta} = \prod_{i=1}^{n} \frac{1}{\theta^2} x_i \cdot e^{-x_i/\theta} =$$

$$= \frac{1}{\theta^{2n}} \prod_{i=1}^{n} x_i \cdot e^{-x_i/\theta} = \frac{1}{\theta^{2n}} \cdot \prod_{i=1}^{n} x_i \cdot e^{-\sum x_i/\theta}$$

$$\ln L(\theta) = -2n \ln\theta + \sum_{i=1}^{n} \ln x_i - \frac{\sum x_i}{\theta}$$

$$\frac{d}{d\theta} \ln L(\theta) = -\frac{2n}{\theta} + \frac{\sum x_i}{\theta^2}$$

$$-\frac{2n}{\theta} + \frac{\sum x_i}{\theta^2} = 0 \Leftrightarrow \frac{\sum x_i}{\theta^2} = \frac{2n}{\theta}$$

$\sum X_i = 2n\theta \Rightarrow \overline{\theta} = \frac{\sum X_i}{2n} = \frac{\cancel{n}\overline{X}}{2\cancel{n}} = \frac{\overline{X}}{2}$

c) is $\overline{\theta}$ an absolute correct estimator? Explain

$E(\overline{\theta}) = E\left(\frac{\overline{X}}{2}\right) = \frac{E(\overline{X})}{2} = \frac{2\theta}{2} = \theta \Rightarrow$ is unbiased

$\overline{\theta}$ is not absolutely correct because it changes with each sample, $\overline{\theta} = \frac{\overline{X}}{2}$ it depends on the sample mean

PS Written 2024 15

1) Recent studies show that air traffic controllers correctly identify 95 out of 100 signals. 25 signals arrive in a 30 minute period.

a) Find the probability that exactly 24 signals are identified correctly.

$$p = \frac{95}{100} = 0.95$$

$$q = 1 - p = 0.05$$

$$n = 25$$

$$X \in \text{Binomial}(25, 0.95)$$

$$\boxed{P(X=k) = C_k^n \cdot p^k \cdot q^{n-k}} \qquad \boxed{C_k^n = \frac{n!}{k!(n-k)!}}$$

$$P(X=24) = C_{24}^{25} \cdot (0.95)^{24} \cdot (0.05)^1 = 25 \cdot 0.27 \cdot 0.05 =$$

$$= 0.34$$

$$C_{24}^{25} = \frac{25!}{24!(25-24)} = 25$$

b) Find the probability that at least 20 signals are identified correctly

$$P(X \geq 20) = 1 - P(X \leq 19)$$

c) Let X denote the number of correctly identified signals. Find the probability distribution function of X. What type of distribution is it?

Since we follow a fixed number of independent trials ($n=25$) each with a success probability of 0.95, the number of correctly identified signals follows a Binomial distribution

$$X \in \text{Binomial}(25, 0.95)$$

$$P(X=k) = C_k^{25} \cdot (0.95)^k \cdot (0.05)^{25-k} \quad , k = 0, 1, \dots, 25$$

d) Consider an hour and a half in which 25 signals arrive every half hour. Find the probability that at least 20 signals are identified correctly in every half hour.

for half an hour:

$$P(X \geq 20) = 1 - P(X \leq 19)$$

for an hour and a half:

$$P(X \geq 20 \text{ in each 3 half-hour periods}) = \left(P(X \geq 20)\right)^3$$

$$= \left(1 - P(X \leq 19)\right)^3$$

1. Let X be a random variable with

pdf $\begin{pmatrix} -2 & -1 & 0 & 2 \\ \frac{1}{8} & \frac{1}{4} & \frac{1}{2} & \frac{1}{8} \end{pmatrix}$

Find the pdf of $Y = 2X^2$

$X: \quad -2, -1, 0, 2$

$P(X=x): \quad \frac{1}{8}, \frac{1}{4}, \frac{1}{2}, \frac{1}{8}$

$Y = 2X^2$

I $X = -2 \Rightarrow Y = 2\cdot(-2)^2 = 8$

II $X = -1 \Rightarrow Y = 2\cdot(-1)^2 = 2$

III $X = 0 \Rightarrow Y = 2\cdot 0^2 = 0$

IV $X = 2 \Rightarrow Y = 2\cdot 2^2 = 8$

$Y: \quad 0, 2, 8$

$P(Y=8) = P(X=-2) + P(X=2) = \frac{1}{8} + \frac{1}{8} = \frac{1}{4}$

$P(Y=2) = P(X=2) = \frac{1}{8}$

$P(Y=0) = P(X=0) = \frac{1}{2}$

PDF of Y is $= \begin{pmatrix} 0 & 2 & 8 \\ \frac{1}{2} & \frac{1}{8} & \frac{1}{4} \end{pmatrix}$

3. Let $X_1, X_2, \dots, X_n$ be a random sample drawn from distribution with pdf $f(X;\theta) = \frac{\theta^X}{X!} e^{-\theta}$, $X = 0, 1, \dots$, $E(X) = V(X) = \theta$, with $\theta > 0$, unknown

a) Find the maximum likelihood estimator, $\bar\theta$, for θ

$$L(\theta) = \prod_{i=1}^{n} f(x_i;\theta) = \prod_{i=1}^{n} \frac{\theta^{x_i}}{x_i!} \cdot e^{-\theta}$$

$$\ln L(\theta) = \sum_{i=1}^{n} \Big(x_i \ln\theta - \theta - \underbrace{\ln x_i!}_{=0}\Big)$$

$$\frac{d}{d\theta} \ln L(\theta) = \sum_{i=1}^{n} \left(\frac{x_i}{\theta} - 1\right) = \sum_{i=1}^{n} \frac{x_i}{\theta} - n$$

$$\sum_{i=1}^{n} \frac{x_i}{\theta} - n = 0 \Rightarrow \sum_{i=1}^{n} \frac{x_i}{\theta} = n$$

$$\Leftrightarrow \frac{1}{\theta} \sum_{i=1}^{n} x_i = n \Leftrightarrow \bar\theta = \frac{1}{n} \sum_{i=1}^{n} x_i \Leftrightarrow \bar\theta = \frac{n \cdot \bar X}{n} \Leftrightarrow$$

$$\Leftrightarrow \bar\theta = \bar X$$

b) is it an absolutely correct estimator? Explain.

$$E(\bar\theta) = E(\bar X) = \frac{1}{n} \cdot \sum_{i=1}^{n} E(X_i) = \frac{n\theta}{n} = \theta \Rightarrow \text{is unbiased}$$

$$\bar X = \frac{1}{n} \cdot \sum_{i=1}^{n} X_i$$

$$E(X_i) = \theta$$

Step II: Check Variance:

$$V(\bar{\theta}) = V(\bar{X}) = V\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n^2} V\left(\sum_{i=1}^{n} X_i\right) = \frac{1}{n^2}\cdot n\theta$$

$$= \frac{\theta}{n}$$

$$\boxed{V(aX) = a^2 V(X)}$$

$$V\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} V(X_i) = \sum_{i=1}^{n} \theta = n\theta$$

$$V(X_i) = \theta$$

$\Rightarrow$ $\bar{\theta}$ is absolutely correct bcs it has variance $\frac{\theta}{n}$ = CRLB (calculated at c)

c) Find the efficiency of $\bar{\theta}$, $e(\bar{\theta})$

$$e(\bar{\theta}) = \frac{\text{Cramér-Rao Lower Bound (CRLB)}}{\text{Variance of the estimator } V(\bar{\theta})}$$

$$\text{CRLB} = \frac{1}{i_n(\theta)}, \quad i_n(\theta) = n\cdot i(\theta)$$

$$i(\theta) = E\left[\left(\frac{d}{d\theta}\ln f(X;\theta)\right)^2\right]$$

$$f(X;\theta) = \frac{\theta^X e^{-\theta}}{X!}$$

$$\ln f(x;\theta) = x\ln\theta - \theta - \ln x!$$

$$\frac{d}{d\theta}\ln f(X;\theta) = \frac{X}{\theta} - 1$$

$$i(\theta) = E\left[\left(\frac{X}{\theta}-1\right)^2\right] = E\left(\frac{X^2}{\theta^2} - 2\frac{X}{\theta} + 1\right) \Rightarrow$$

$$E(X) = \theta$$

$$\boxed{E(X^2) = V(X) + (E(X))^2} = \theta + \theta^2$$

$$\Rightarrow i(\theta) = \frac{1}{\theta^2}\cdot E(X^2) - \frac{2}{\theta}\cdot E(X) + E(1) = \frac{\theta+\theta^2}{\theta^2} - \frac{2\cancel{\theta}}{\cancel{\theta}} + 1 =$$

$$= \frac{\theta}{\theta^2} + \frac{\cancel{\theta^2}}{\cancel{\theta^2}} - 2 + 1 = \frac{1}{\theta}$$

$$i_n(\theta) = n \cdot i(\theta) = \frac{n}{\theta}$$

$$CRLB = \frac{1}{i_n(\theta)} = \frac{\theta}{n}$$

$$V(\bar{\theta}) = V(\bar{X}) = V\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n^2} V\left(\sum_{i=1}^{n} X_i\right) = \frac{1}{n^2}\sum_{i=1}^{n} V(X_i)$$

$\bar{\theta} = \bar{X}$ (from point a)

$$= \frac{1}{n^2}\cdot n\cdot\theta = \frac{\theta}{n}$$

$$\bar{X} = \frac{1}{n}\cdot\sum_{i=1}^{n} X_i \qquad V(X_i) = \theta$$

$$V(aX) = a^2 V(X)$$

$$e(\bar{\theta}) = \frac{CRLB}{V(\bar{\theta})} = \frac{\frac{\theta}{n}}{\frac{\theta}{n}} = 1$$

Note: its efficient if

$$V(\bar{\theta}) = \frac{1}{i_n(\theta)}$$

PS Written 2025:

1. Messages arrive at an electronic message center at random times, 1 message every 6 minutes

$X \in \text{Poisson}(\lambda t)$ $\qquad P(X=k) = \frac{\lambda^k \cdot e^{-\lambda}}{k!}$

$\lambda = \frac{1}{6}$

a) Find the probability of receiving at least 5 messages during one hour.

During one hour

$\lambda t = \frac{1}{6} \cdot 60 = 10$

$X \in \text{Poisson}(10) \Rightarrow \lambda = 10$

$P(X \geq 5) = 1 - P(X < 5) = 1 - (P(X=0) + P(X=1) + P(X=2) + P(X=3) + P(X=5))$

$P(X=0) = \frac{10^0 \cdot e^{-10}}{0!} = e^{-10}$

$P(X=1) = \frac{10^1 \cdot e^{-10}}{1!} = 10 \cdot e^{-10}$

$P(X=2) = \frac{100 \cdot e^{-10}}{2!} = 50 \cdot e^{-10}$

$P(X=3) = \frac{10^3 \cdot e^{-10}}{3!} = \frac{1000}{6} \cdot e^{-10}$

$P(X=4) = \frac{10^4 \cdot e^{-10}}{4!} = \frac{10000}{24} \cdot e^{-10}$

b) During one hour, if at least 6 messages were received what is the probability that 8 messages arrived?

$$P(X=8) = \frac{10^8 \cdot e^{-10}}{8!}$$

$$P(X=8 \mid X \geq 6) = \frac{P(X=8 \cap X \geq 6)}{P(X \geq 6)} = \frac{P(X=8)}{P(X \geq 6)} = \frac{P(X=8)}{1-P(X<6)}$$

$$P(X \geq 6) = 1 - P(X<6)$$

c) How many messages can be expected to arrive in 90 minutes

$$\boxed{E(X) = \lambda t}$$

$$E(X) = \frac{1}{6} \cdot 90 = 15$$

2. A die is rolled 2 times. Let X denote the number of 6's rolled. Find $E(X^2+1)$

$$E(X^2+1) = E(X^2) + 1$$

$$X \in \text{Binomial}\left(n=2, p=\frac{1}{6}\right)$$

$$E(X) = n \cdot p = \frac{2}{6} = \frac{1}{3}$$

$$V(X) = n \cdot p \cdot (1-p) = \frac{5}{18}$$

$$E(X^2) = V(X) \cdot (E(X))^2 = \frac{5}{18} \cdot \left(\frac{1}{3}\right)^2 = \frac{5}{18} \cdot \frac{1}{9} = \frac{7}{18}$$

$$E(X^2+1) = E(X^2) + 1 = \frac{7}{18} + 1 = \frac{25}{18}$$

3. Let X be a discrete random variable with pdf $P(X=3)=\theta$, $P(X=7)=1-\theta$, with $\theta \in (0,1)$ unknown. The sample $3,3,3,3,3,7,7,7$ is collected from this distribution. Find the method of moments estimator, $\bar{\theta}$, for θ

General formula

$$E(X)=\sum x_i\, P(X=x_i)$$

$$E(X)=\big(3\cdot P(X=3)\big)+\big(7\cdot P(X=7)\big)=3\theta+7-7\theta$$

$$=7-4\theta$$

$$\bar{X}=\frac{\sum x_i}{n}$$

$$\bar{X}=\frac{5\cdot 3+3\cdot 7}{9}=\frac{36}{9}=4$$

$$E(X)=\bar{X} \Leftrightarrow 7-4\theta=4 \Leftrightarrow -4\theta=-3 \Rightarrow \bar{\theta}=\frac{3}{4}$$

4. Let $X_1, X_2 \dots X_n$ be a random sample drawn from a distribution with pdf

$$f(x;p) = C_{x+1}^{x}\, p^2 (1-p)^x, \quad x = 0,1,\dots, \text{ with } p \in (0,1) \text{ unknown}$$

a) Find the maximum likelihood estimator, $\bar{p}$, for p

$$L(p) = \prod_{i=1}^{n} C_{x_i+1}^{x_i}\, p^2 (1-p)^{x_i} =$$

$$= p^{2n}(1-p)^{\sum_{i=1}^{n} x_i}$$

$$\ln L(p) = \ln\left(p^{2n}(1-p)^{\sum_{i=1}^{n} x_i}\right) = 2n \ln p + \sum_{i=1}^{n} x_i \cdot \ln(1-p)$$

$$\frac{d}{dp} L(p) = \frac{2n}{p} + \frac{\sum_{i=1}^{n} x_i}{1-p}$$

$$\frac{2n}{p} + \frac{\sum_{i=1}^{n} x_i}{1-p} = 0 \Leftrightarrow 2n(1-p) = p \cdot \sum_{i=1}^{n} x_i \Leftrightarrow$$

$$\Leftrightarrow 2n - 2np = p \cdot \sum_{i=1}^{n} x_i$$

$$2n = p \cdot \sum_{i=1}^{n} x_i + 2np$$

$$2n = p\left(\sum_{i=1}^{n} x_i + 2n\right) \Rightarrow \bar{p} = \frac{2n}{2n + \sum_{i=1}^{n} x_i}$$

b) At the 5% significance level, find a most powerful test for testing $H_0: p = 1/2$ against $H_1: p = 1/6$

$$\Lambda = \frac{L(p=1/2)}{L(p=1/6)}$$

$$L(p) = p^{2n} (1-p)^{\sum_{i=1}^{3} x_i}$$

$$L(p=1/2) = (1/2)^{2n} \cdot (1/2)^{\sum_{i=1}^{3} x_i} = (1/2)^{2n + \sum_{i=1}^{3} x_i}$$

$$L(p=1/6) = (1/6)^{2n} \cdot (5/6)^{\sum_{i=1}^{3} x_i}$$

$$\Lambda = \frac{(1/2)^{2n + \sum_{i=1}^{3} x_i}}{(1/6)^{2n} \cdot (5/6)^{\sum_{i=1}^{3} x_i}}$$

We reject H_0 in favor of H_1 if

$$\Lambda \leq K$$

where K is chosen to ensure 5% significance level

PS Written 2025 2;

1. Ten people donate blood at a clinic, 3 of which have blood type 0 ("universal donors"). Five people get in first.

$X \in \text{Hypergeometric}(N=10, K=3, n=5)$

$$P(X=k) = \frac{C_k^K \, C_{n-k}^{N-K}}{C_n^N}$$

a) Find the probability that at most 4 of them do not have blood type 0.

$$P(X \geq 1) = 1 - P(X=0) = 1 - \frac{21}{252}$$

$$P(X=0) = \frac{C_0^3 \, C_5^7}{C_5^{10}}$$

b) How many universal donors are expected to go in

$$E(X) = \frac{n \cdot K}{N} = 5 \cdot \frac{3}{10} = \frac{15}{10} = \frac{3}{2}$$

a) Let X be the number of universal donors that go in. Prove that $4P((X \le -2) \cup (X \ge 2)) \le 3$

$4P((X \le -2) \cup (X \ge 2)) \le 3$

$X \le -2$ is not possible (X can't be negative)

$4P(X \ge 2) \le 3$

$P(X \ge 2) = 1 - P(X < 2) = 1 - (P(X=0) + P(X=1)) =$

$= 1 - \frac{126}{252} = \frac{126}{252} = \frac{1}{2}$

$P(X=0) = \frac{C_0^3 C_5^7}{C_5^{10}} = \frac{21}{252}$

$P(X=1) = \frac{C_1^3 C_4^7}{C_5^{10}} = \frac{105}{252}$

$4P(X \ge 2) \le 3 \Leftrightarrow 4 \cdot \frac{1}{2} \le 3 \Leftrightarrow 2 \le 3$ True

2. Let $X \in N(0,1)$. Find the pdf of $Y = |X|+1$

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

$$|X| = Y-1$$

$$X = \pm(Y-1)$$

$$f_Y(y) = f_X(x)\left|\frac{dx}{dy}\right|$$

$$\frac{dx}{dy} = \frac{d}{dy}\pm(Y-1) = \pm 1$$

$$\left.\begin{array}{l} f_Y(y) = f_X(y-1)\cdot 1 + f_X(-(y-1))\cdot 1 \\ \quad f_X(x) \text{ is symmetric} \Rightarrow f_X(x) = f_X(-x) \end{array}\right\} \Rightarrow$$

$$\Rightarrow f_Y(y) = 2 f_X(y-1)$$

$$f_Y(y) = 2\cdot\frac{1}{\sqrt{2\pi}}\cdot e^{-(y-1)^2/2}$$

$$Y = |X|+1 \Rightarrow Y \geq 1$$

$$f_Y(y) = \begin{cases} 2\cdot\frac{1}{\sqrt{2\pi}}\cdot e^{-(y-1)^2/2}, & y \geq 1 \\ 0, & y < 1 \end{cases}$$

3. Every day, a company is offering a discount to $100\lambda\%$, $\lambda \in (0,1)$, of randomly selected customers. A person is observing that, over several days, out of 8 customers, the number of people who got the discount was 0, 1, 3, 1, 4, 0, 3, 4, 5 and 1. Based on these estimate λ

$$X \in \text{Binomial}(n=8, p=\lambda)$$

$$E(X) = np = 8\lambda$$

$$\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i = \frac{0+1+3+1+4+0+3+4+5+1}{10} = \frac{22}{10} = \frac{11}{5}$$

$$E(X) = \overline{X} \Rightarrow 8\lambda = \frac{11}{5} \Rightarrow \lambda = \frac{11}{40}$$

4. Let $X_1, X_2, \dots, X_n$ be a random sample drawn from a distribution with pdf

$$f(x;\mu) = \frac{1}{3\mu} e^{-\frac{x}{3\mu}} \quad , \text{ for } x > 0$$

$$E(X) = 3\mu \quad , \quad V(X) = 9\mu^2 \quad , \text{ with } \mu > 0 \text{ , unknown}$$

a) Find the maximum likelihood estimator, $\bar{\mu}$, for μ

$$L(\mu) = \prod_{i=1}^{n} f(x_i;\mu) = \prod_{i=1}^{n} \frac{1}{3\mu} e^{-\frac{x_i}{3\mu}} = \left(\frac{1}{3\mu}\right)^n \cdot e^{-\frac{\sum_{i=1}^{n} x_i}{3\mu}}$$

$$\ln L(\mu) = n \ln\left(\frac{1}{3\mu}\right) - \frac{1}{3\mu}\sum_{i=1}^{n} x_i = -n\ln(3\mu) - \frac{1}{3\mu}\sum_{i=1}^{n} x_i$$

$$\frac{d}{d\mu} \ln L(\mu) = -\frac{n}{3\mu}\cdot 3 + \frac{1}{9\mu^2}\cdot 3 \cdot \sum_{i=1}^{n} x_i =$$

$$= -\frac{n}{\mu} + \frac{1}{3\mu^2}\cdot\sum_{i=1}^{n} x_i$$

$$-\frac{n}{\mu} + \frac{1}{3\mu^2}\cdot\sum_{i=1}^{n} x_i = 0 \quad / \cdot 3\mu^2 \iff -n3\mu + \sum_{i=1}^{n} x_i = 0$$

$$\iff -n3\mu = -\sum_{i=1}^{n} x_i \iff \mu = \frac{\sum_{i=1}^{n} x_i}{3n} \iff \bar{\mu} = \frac{1}{3}\cdot\bar{X}$$

b) Show that $\bar{\mu}$ is absolutely correct and find its efficiency.

$$E(\bar{\mu}) = E\left(\frac{\overline{X}}{3}\right) = \frac{1}{3}\cdot E(\overline{X}) = \mu \Rightarrow \text{is unbiased}$$

$$E(X) = 3\mu$$

$$V(\bar{\mu}) = V\left(\frac{\overline{X}}{3}\right) = \frac{V(\overline{X})}{9} = \frac{9\mu^2}{9n} = \frac{\mu^2}{n}$$

$$V(\overline{X}) = \frac{V(X)}{n} = \frac{9\mu^2}{n}$$

$$e(\bar{\theta}) = \frac{\text{Cramér-Rao Lower Bound (CRLB)}}{\text{Variance of the estimator } V(\bar{\theta})}$$

$$CRLB = \frac{1}{i_n(\theta)} \quad , \quad i_n(\theta) = n\cdot i(\theta)$$

$$i(\theta) = E\left[\left(\frac{d}{d\theta}\ln f(X;\theta)\right)^2\right]$$

$$f(x;\mu) = \frac{1}{3\mu}e^{-\frac{x}{3\mu}}$$

$$\ln f(x;\mu) = -\ln(3\mu) - \frac{x}{3\mu}$$

$$\frac{d}{d\mu}\ln f(x;\mu) = -\frac{1}{\mu} + \frac{x}{3\mu^2}$$

$$i(\mu) = E\left(\left(-\frac{1}{\mu} + \frac{X}{3\mu^2}\right)^2\right) = E\left(\frac{1}{\mu^2} - \frac{2X}{3\mu^3} + \frac{X^2}{9\mu^4}\right)$$

$$i(\mu) = \frac{1}{\mu^2} - \frac{2}{3\mu^3} E(X) + \frac{1}{9\mu^4} \cdot E(X^2)$$

$E(X) = 3\mu \qquad V(X) = 9\mu^2$

$$E(X^2) = V(X) + (E(X))^2 = 9\mu^2 + 9\mu^2 = 18\mu^2$$

$$i(\mu) = \frac{1}{\mu^2} - \frac{2}{3\mu^3} \cdot 3\mu + \frac{1}{9\mu^4} \cdot 18\mu^2 = \frac{1}{\mu^2} - \frac{6\mu}{3\mu^3} + \frac{18\mu^2}{9\mu^4} =$$

$$= \frac{1}{\mu^2} - \frac{2}{\mu^2} + \frac{2}{\mu^2} = \frac{1}{\mu^2}$$

$$i_n(\mu) = \frac{n}{\mu^2}$$

$$CRLB = \frac{1}{i_n(\mu)} = \frac{\mu^2}{n}$$

$CRLB = V(\bar{\mu}) \Rightarrow$ its efficient

$$e(\bar{\mu}) = \frac{CRLB}{V(\bar{\mu})} = 1$$

PS Written 2025 2:

1. Let $X_1, X_2 \dots X_n$ be a random sample drawn from a distribution with pdf $f(x,\lambda) = \frac{1}{4\lambda} e^{-\frac{x}{4\lambda}}$, for $x > 0$

$E(X) = 4\lambda$, $V(X) = 16\lambda^2$, with $\lambda > 0$, unknown

a) Find the MLE, $\bar{\lambda}$, for λ

$$L(\lambda) = \prod_{i=1}^{n} f(x_i, \lambda) = \prod_{i=1}^{n} \frac{1}{4\lambda} \cdot e^{-\frac{x_i}{4\lambda}} =$$

$$= \left(\frac{1}{4\lambda}\right)^n \cdot e^{-\frac{\sum_{i=1}^{n} x_i}{4\lambda}}$$

$$\ln L(\lambda) = n \ln \frac{1}{4\lambda} - \frac{\sum_{i=1}^{n} x_i}{4\lambda} = -n \ln 4\lambda - \frac{\sum_{i=1}^{n} x_i}{4\lambda}$$

$$\frac{d}{d\lambda} \ln L(\lambda) = -\frac{n}{\lambda} + \frac{1}{4\lambda^2} \sum_{i=1}^{n} x_i$$

$$-\frac{n}{\lambda} + \frac{1}{4\lambda^2} \sum_{i=1}^{n} x_i = 0 \quad / \cdot 4\lambda^2 \iff -4\lambda \cdot n + \sum_{i=1}^{n} x_i = 0 \Rightarrow$$

$$\Rightarrow -4\lambda n = -\sum_{i=1}^{n} x_i \iff \lambda = \frac{\sum_{i=1}^{n} x_i}{4n} \Rightarrow \bar{\lambda} = \frac{1}{4} \cdot \bar{X}$$

b) Check that $\bar{\lambda}$ is absolutely correct and find its efficiency.

$$E(\bar{\lambda}) = E\left(\frac{1}{4}\bar{X}\right) = \frac{E(\bar{X})}{4} = \frac{4\lambda}{4} = \lambda \rightarrow \text{is unbiased}$$

$$V(\bar{\lambda}) = V\left(\frac{\bar{X}}{4}\right) = \frac{1}{16} V(\bar{X}) = \frac{\lambda^2}{n}$$

$$V(\bar{X}) = \frac{V(X)}{n} = \frac{16\lambda^2}{n}$$

$$i(\lambda) = E\left[\left(\frac{d}{d\lambda}\ln f(x,\lambda)\right)^2\right]$$

$$\frac{d}{d\lambda}\ln f(x,\lambda) = -\frac{1}{\lambda} + \frac{x}{4\lambda^2}$$

$$i(\lambda) = E\left(\left(-\frac{1}{\lambda} + \frac{x}{4\lambda^2}\right)^2\right) = E\left(\frac{1}{\lambda^2} - \frac{2x}{4\lambda^3} + \frac{x^2}{16\lambda^4}\right) =$$

$$= \frac{1}{\lambda^2} - \frac{2}{4\lambda^3}\cdot E(x) + \frac{1}{16\lambda^4}E(x^2) =$$

$$E(x^2) = V(x) + (E(x))^2 = 16\lambda^2 + 16\lambda^2 = 32\lambda^2$$

$$= \frac{1}{\lambda^2} - \frac{2}{4\lambda^3}\cdot 4\lambda + \frac{1}{16\lambda^4}\cdot 32\lambda^2 = \frac{1}{\lambda^2}$$

$$i_n(\lambda) = n\cdot i(\lambda) = \frac{n}{\lambda^2}$$

$$CRLB = \frac{1}{i_n(\lambda)} = \frac{\lambda^2}{n}$$

$CRLB = V(\bar{\lambda}) \Rightarrow$ it's efficient

$$e(\bar{\lambda}) = \frac{CRLB}{V(\bar{\lambda})} = 1$$

2. Let $X \sim N(0,1)$. Find the pdf of $Y = 2|X|$

$$f_X(x) = \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{x^2}{2}}$$

$$Y = 2|X| \Rightarrow |X| = \frac{Y}{2} \Rightarrow X = \pm \frac{Y}{2}$$

$$f_Y(y) = f_X(x) \cdot \left|\frac{dx}{dy}\right|$$

$$\frac{dx}{dy} = \frac{d}{dy} \cdot \left(\pm \frac{Y}{2}\right) = \pm \frac{1}{2}$$

$$f_Y(y) = f_X\left(\frac{Y}{2}\right) \cdot \frac{1}{2} + f_X\left(-\frac{Y}{2}\right) \cdot \frac{1}{2} = 2 \cdot \frac{1}{2} f_X\left(\frac{Y}{2}\right) =$$

$f_X(x)$ is symmetrical $\Rightarrow f_X(x) = f_X(-x)$

$$= f_X\left(\frac{Y}{2}\right) = \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{\left(\frac{Y}{2}\right)^2}{2}} = \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{Y^2}{8}}$$

$$f_Y(y) = \begin{cases} \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{Y^2}{8}}, & y > 0 \\ 0, & y \le 0 \end{cases}$$

3. A percentage of $100\theta\%$, $\theta \in (0,1)$, of parts that a manufacturing company is producing are defective. Over several weeks, out of 10 randomly selected items, the number of defective ones found each week was 0, 1, 2, 0, 3, 1, 2 and 1

$$X \in \text{Binomial}(n=10,\ p=\theta)$$

$$E(X) = n \cdot p = 10\theta$$

$$\overline{X} = \frac{0+1+2+0+3+1+2+1}{8} = \frac{\cancel{10}}{\cancel{8}} = \frac{5}{4}$$

$$E(X) = \overline{X} \Rightarrow 10\theta = \frac{5}{4} \Rightarrow \theta = \frac{5}{40} = \frac{1}{8}$$

4. There are 30 computers in a classroom, 4 of which are very slow. Twenty students come to class and are seated randomly each in front of a computer.

a) Find the probability that at most half of the slow computers are used.

$$X \in \text{Hypergeometric}\,(N=30,\ K=4,\ n=20)$$

$$P(X\leq 2) = P(X=0) + P(X=1) + P(X=2)$$

$$P(X=k) = \frac{C_k^K\ C_{n-k}^{N-K}}{C_n^N}$$

$$P(X=0) = \frac{C_0^4 \cdot C_{20}^{26}}{C_{20}^{30}}$$

$$P(X=1) = \frac{C_1^4 \cdot C_{19}^{26}}{C_{20}^{30}}$$

$$P(X=2) = \frac{C_2^4 \cdot C_{18}^{26}}{C_{20}^{30}}$$

$$P(X\leq 2) = \frac{C_0^4 \cdot C_{20}^{26}}{C_{20}^{30}} + \frac{C_1^4 \cdot C_{19}^{26}}{C_{20}^{30}} + \frac{C_2^4 \cdot C_{18}^{26}}{C_{20}^{30}}$$

b) How many slow computers are expected to be used

$$E(X) = \frac{n \cdot K}{N} = \frac{\cancel{20} \cdot 4}{\cancel{30}} = \frac{8}{3}$$

c) Let X be the number of students using a slow computer. Show that $9P(-3 < X < 3) \geq 1$

$$9P(-3 < X < 3) \geq 1$$

$$P(-3 < X < 3) = P(X < 3) = P(X=0) + P(X=1) + P(X=2)$$

$$\Rightarrow 9 \cdot \left(P(X=0) + P(X=1) + P(X=2)\right) \geq 1$$

$$\Rightarrow 9 \cdot 0.89 \geq 1 \Rightarrow 8.019 \geq 1 \quad \text{True}$$