2. Find the sum for each of the following series

a) $$\sum_{n\geq 2} \ln\left(1-\frac{1}{n^2}\right) = \sum_{n\geq 2} \ln\left(\left(1-\frac{1}{n}\right)\left(1+\frac{1}{n}\right)\right) =$$

$$= \sum_{n\geq 2} \ln\left(1-\frac{1}{n}\right) + \ln\left(1+\frac{1}{n}\right) = \sum_{n\geq 2} \ln\left(1-\frac{1}{n}\right) + \sum_{n\geq 2} \ln\left(1+\frac{1}{n}\right) =$$

$$\sum_{n\geq 2} \ln\left(1-\frac{1}{n}\right) = \ln\left(1-\frac{1}{2}\right) + \ln\left(1-\frac{1}{3}\right) + \ln\left(1-\frac{1}{4}\right) + \ldots + \ln\left(1-\frac{1}{n}\right) =$$

$$= \ln\left(\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right)\cdot\ldots\cdot\left(1-\frac{1}{n}\right)\right) =$$

$$= \ln\left(\frac{1}{\cancel{2}}\cdot\frac{\cancel{2}}{\cancel{3}}\cdot\frac{\cancel{3}}{\cancel{4}}\cdot\ldots\cdot\frac{\cancel{n-1}}{n}\right) = \ln\frac{1}{n}$$

$$\sum_{n\geq 2} \ln\left(1+\frac{1}{n}\right) = \ln\left(\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\left(1+\frac{1}{4}\right)\cdot\ldots\cdot\left(1+\frac{1}{n}\right)\right) =$$

$$= \ln\left(\frac{\cancel{3}}{2}\cdot\frac{\cancel{4}}{\cancel{3}}\cdot\frac{\cancel{5}}{\cancel{4}}\cdot\ldots\cdot\frac{n+1}{\cancel{n}}\right) = \ln\left(\frac{1}{2}\cdot(n+1)\right)$$

$$\Rightarrow \sum_{n\geq 2} \ln\left(1-\frac{1}{n^2}\right) = \sum_{n\geq 2} \ln\left(1-\frac{1}{n}\right) + \sum_{n\geq 2} \ln\left(1+\frac{1}{n}\right) =$$

$$= \ln\frac{1}{n} + \ln\frac{n+1}{2} = \ln\left(\frac{n+1}{2n}\right) =$$

$$= \lim_{n\to\infty} \ln\left(\frac{n+1}{2n}\right) = \left(\frac{\infty}{\infty}\right) = \ln \lim_{n\to\infty} \frac{\cancel{n}\left(1+\frac{1}{n}\right)}{2\cancel{n}} = \ln\frac{1}{2}$$

b) $\sum_{n\geq 1} \frac{n+1}{3^n}$

We can use the formula for the sum of an infinite geometric series

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$$

$$\sum_{n\geq 1} \frac{n+1}{3^n} = \sum_{n\geq 1} \frac{n}{3^n} + \sum_{n\geq 1} \frac{1}{3^n}$$

$$\sum_{n\geq 1} \frac{n}{3^n} = \frac{1}{3} + \frac{2}{3^2} + \frac{3}{3^3} + \ldots = \frac{\frac{1}{3}}{1-\frac{1}{3}} = \frac{\frac{1}{3}}{\frac{2}{3}} = \frac{1}{2} \quad (1)$$

$a = \frac{1}{3}$ start term $\qquad r = \frac{1}{3}$

$$\sum_{n\geq 1} \frac{1}{3^n} = \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \ldots = \frac{\frac{1}{3}}{1-\frac{1}{3}} = \frac{\frac{1}{3}}{\frac{2}{3}} = \frac{1}{2} \quad (2)$$

$a = \frac{1}{3} \qquad r = \frac{1}{3}$

$$(1),(2) \Rightarrow \sum_{n\geq 2} \frac{n+1}{3^n} = \frac{1}{2} + \frac{1}{2} = \frac{2}{2} = 1$$

c) $\sum_{n\geq 1} \frac{n}{n^4+n^2+1}$

$n^4+n^2+1 = (n^2)^2+2n^2+1-n^2 = (n^2+1)^2-n^2 =$

$= (n^2+1-n)(n^2+1+n)$

We see that:

$(n^2+1+n)-(n^2+1-n) = 2n$

$$\sum_{n\geq 1} \frac{n}{n^4+n^2+1} = \sum_{n\geq 1} \frac{1}{2}\cdot\frac{2n}{(n^2+1-n)(n^2+1+n)} =$$

$$= \sum_{n\geq 1} \frac{1}{2}\cdot\frac{(n^2+1+n)-(n^2+1-n)}{(n^2+1-n)(n^2+1+n)} =$$

$$= \sum_{n\geq 1} \frac{1}{2}\left(\frac{\cancel{(n^2+1+n)}}{(n^2+1-n)\cancel{(n^2+1+n)}} - \frac{\cancel{(n^2+1-n)}}{\cancel{(n^2+1-n)}(n^2+1+n)}\right) =$$

$$= \sum_{n\geq 1} \frac{1}{2}\cdot\left(\frac{1}{n^2+1-n} - \frac{1}{n^2+1+n}\right) = 0$$

$$\sum_{n\geq 1} \frac{1}{n^2+1-n} = \lim_{n\to\infty} \frac{1}{n^2+1-n} = 0$$

$$\sum_{n\geq 0} \frac{1}{n^2+1+n} = \lim_{n\to\infty} \frac{1}{n^2+1+n} = 0$$

4. Study if the following series are convergent or divergent

a) $\sum_{n\geq 1} \frac{x^n}{n^p}$, $x>0$, $p\in\mathbb{N}$

Ratio test:

$$L=\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{\frac{x^{n+1}}{(n+1)^p}}{\frac{x^n}{n^p}}=\lim_{n\to\infty}\frac{\cancel{x^{n+1}}^{\,x}}{(n+1)^p}\cdot\frac{n^p}{\cancel{x^n}}=$$

$$=\lim_{n\to\infty}x\cdot\left(\frac{n}{n+1}\right)^p=x\cdot\lim_{n\to\infty}\left(\frac{\cancel{n}}{\cancel{n}(1+\frac{1}{n})}\right)^p=x\cdot 1^p=x$$

I $\quad 0\leq x<1 \Rightarrow$ the series converges

II $\quad$ if $x>1 \Rightarrow$ the series diverges

b) $\sum_{n\geq 2}\frac{1}{(\ln n)^{\ln n}}=$

$$\left.\begin{array}{l} a^{\ln b}=b^{\ln a} \\ a=\ln n,\ b=n \end{array}\right\}\Rightarrow(\ln n)^{\ln n}=n^{\ln(\ln n)}$$

$$=\sum_{n\geq 2}\frac{1}{n^{\ln(\ln n)}},\quad \begin{array}{l} n\geq e^{e^2},\ \ln(\ln(e^{e^2}))\geq 2 \\ \text{bcs. } \ln(\ln(e^{e^2}))=2 \end{array}$$

$\sum_{n\geq 1}\frac{1}{n^p}$ convergent $\Leftrightarrow p>1$

$\sum_{n\geq 2}\frac{1}{n^{\ln(\ln n)}}$ convergent $\Rightarrow \sum_{n\geq 2}\frac{1}{(\ln n)^{\ln n}}$ convergent

c) $\sum_{n\geq 1}\left(\sqrt[n]{n}-1\right)$

Root test:

$a_n = \sqrt[n]{n} - 1$

$$\lim_{n\to\infty}\sqrt[n]{a_n} = \lim_{n\to\infty}\sqrt[n]{\sqrt[n]{n}-1} = \lim_{n\to\infty}\sqrt[n]{1-1} = 0$$

As n approaches infinity
the term $\sqrt[n]{n}$ approaches 1

$\lim_{n\to\infty}\sqrt[n]{a_n} = 0 < 1 \Rightarrow$ the series converges

$\Rightarrow \sum_{n\geq 1}\sqrt[n]{n}-1$ is convergent

5. Number of sides at iteration n:

it 0: 3 sides

it 1: $3 \cdot 4^1 = 12$ sides

it 2: $3 \cdot 4^2 = 48$ sides

it 3: $3 \cdot 4^3 = 192$ sides

$\vdots$

it n: $3 \cdot 4^n$ sides

$$P = 3 \cdot 4^n \cdot \left(\frac{1}{3}\right)^n = \frac{3 \cdot 4^n}{3^n} = \frac{4^n}{3^{n-1}}$$

$$\lim_{n \to \infty} P = \lim_{n \to \infty} \frac{4^n}{3^{n-1}} = \left(\frac{\infty}{\infty}\right) \overset{l'H}{=} \lim_{n \to \infty} \frac{4^n \cdot \ln 4}{3^{n-1} \cdot \ln 3} =$$

$$= \frac{\ln 4}{\ln 3} \lim_{n \to \infty} \frac{4^n}{3^{n-1}} = \left(\frac{\infty}{\infty}\right) \overset{l'H}{=} \frac{\ln 4}{\ln 3} \lim_{n \to \infty} \frac{4^n \ln 4}{3^{n-1} \ln 3} \longrightarrow \infty$$

As we can see, the limit does not converge as n goes to infinity, the limit approaches ∞.