Exam 2023 June 27: First Sample

1. Find the general solution of

a) the difference equation

$$X_{k+2} - \sqrt{3}\, X_{k+1} + X_k = 0$$

$$r^2 - \sqrt{3}\, r + 1 = 0$$

$$\Delta = 3 - 4 = -1$$

$$r_{1,2} = \frac{\sqrt{3} \pm \sqrt{-1}}{2} = \frac{\sqrt{3} \pm i}{2}$$

$$z = \rho(\cos\theta + i\sin\theta)$$

$$\rho = \sqrt{\left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \frac{4}{4} = 1$$

$$\left.\begin{aligned} \cos\theta &= \frac{\sqrt{3}}{2} \\ \sin\theta &= \frac{1}{2} \end{aligned}\right\} \Rightarrow \theta = \frac{\pi}{6}$$

$$z^k = \left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)^k = \cos\frac{\pi}{6}\cdot k + i\sin\frac{\pi}{6} k$$

$$X_k = C_1 \cdot \cos\frac{\pi}{6} k + C_2 \cdot i\sin\frac{\pi}{6} k$$

b) the differential equation $x'' - \sqrt{3}x' + x = 0$

$r^2 - \sqrt{3}r + r = 0$

$r_{1,2} = \frac{\sqrt{3} \pm i}{2}$

$x_1 = e^{\frac{\sqrt{3}}{2}t} \cdot \cos \frac{i}{2} t$

$x_2 = e^{\frac{\sqrt{3}}{2}t} \cdot \sin \frac{i}{2} t$

$x_h = C_1 e^{\frac{\sqrt{3}}{2} \cdot t} \cdot \cos \frac{i}{2} t + C_2 \cdot e^{\frac{\sqrt{3}}{2}t} \cdot \sin \frac{1}{2} t$

2. We consider the ideal pendulum system

$\dot{x} = y \qquad \dot{y} = -g \sin x$

a) Find the expression of a global first integral. Check it using the corresponding first order partial differential equation.

$H(x,y) = ? \quad \Rightarrow \quad \frac{dH}{dx} = 0$

Let $H(x,y) = \frac{1}{2}y^2 + V(x)$

$V(x) = ?$

$F = -\frac{dV}{dx}$

where $F = -g \sin x$

$\Rightarrow \quad \frac{dV}{dx} = g \sin x \quad / \int$

$V(x) = \int g \sin x \, dx$

$V(x) = -g \cos x + C$
Let $C = 0$
$\Big\} = V(x) = -g \cos x$

$$H(x,y) = \frac{1}{2}y^2 - g\cos x$$

$$\frac{dH}{dt} = \frac{\partial H}{\partial x}\cdot\dot{x} + \frac{\partial H}{\partial y}\cdot\dot{y} = g\sin x\cdot y - g\sin x\cdot y = 0$$

$$\frac{\partial H}{\partial x} = g\sin x$$

$$\frac{\partial H}{\partial y} = y$$

b) if $(2\pi, 0)$ is an equilibrium point, is it hyperbolic? is it stable?

$\dot{x} = 0 \quad \dot{y} = 0 \Rightarrow$ is an eq.

for $(2\pi, 0)$

$$\dot{x} = y = 0$$

$$\dot{y} = -g\sin x = -g\sin 2\pi = 0$$

$\Rightarrow$ True, it's an eq

$$J = \begin{pmatrix} \frac{\partial \dot{x}}{\partial x} & \frac{\partial \dot{x}}{\partial y} \\ \frac{\partial \dot{y}}{\partial x} & \frac{\partial \dot{y}}{\partial y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -g\cos x & 0 \end{pmatrix}$$

at $(2\pi, 0)$

$$\Rightarrow J = \begin{pmatrix} 0 & 1 \\ -g\cos 2\pi & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -g & 0 \end{pmatrix}$$

$$\det(J - \lambda I) = \det\begin{pmatrix} -\lambda & 1 \\ -g & -\lambda \end{pmatrix}$$

$$\begin{vmatrix} -\lambda & 1 \\ -g & -\lambda \end{vmatrix} = \lambda^2 + g$$

$$\lambda^2 + 9 = 0 \Rightarrow \lambda^2 = -9 \Rightarrow \lambda = \pm 3i$$

$\Rightarrow$ no real parts

$\Rightarrow$ eigenvalues are purely imaginary $\Rightarrow$ oscilations $\Rightarrow$ is a center $\Rightarrow$

$\Rightarrow$ neutrally stable

$\Rightarrow$ is not hyperbolic (real part of eigenvalues are non-zero)

3. Find the values of $h>0$ such that the attractor equilibrium point of $x' = x^2+5x+6$ is also an attractor fixed point of the discrete dynamical system associated to the Euler's numerical formula with stepsize $h>0$ for the given differential equation

Step 1: identify eq points

$x^2+5x+6=0$ $\qquad x_{1,2} = \frac{-5\pm1}{2}$ $\quad \begin{cases} x_1 = -2 \\ x_2 = -3 \end{cases}$

$\Delta = 25-24 = 1$

$\Rightarrow$ The eq points are $x=-2$ and $x=-3$

Step 2: Stability of eq. points

$f(x) = x^2+5x+6$

$f'(x) = 2x+5$

$f'(-2) = 1 > 0 \Rightarrow$ repeller (unstable)

$f'(-3) = -1 < 0 \Rightarrow$ attractor (stable)

Step 3: Euler's method

Euler's method for solving $x' = f(x)$

$$x_{n+1} = x_n + h\cdot f(x_n)$$

$$x_{n+1} = x_n + h\cdot(x_n^2+5x_n+6)$$

Step 4: Fixed points of the discrete system

A fixed point x^* statisfies:

$$x^* = x^* + h(x^{*2} + 5x^* + 6)$$

$$0 = h(x^{*2} + 5x^* + 6) \quad , \quad h > 0$$

$$\Rightarrow \quad x^{*2} + 5x^* + 6 = 0$$

$\Rightarrow$ the discrete points are $x = -2$ and $x = -3$

Step 5: Stability of the fixed points

$$g(x) = x + h(x^2 + 5x + 6)$$

$$g'(x) = 1 + h(2x + 5)$$

$$\left.\begin{array}{l} g'(-2) = 1 + h \\ \text{needs to be an attractor} \end{array}\right\} \Rightarrow |1+h| < 1 \quad \Rightarrow$$

$$\Rightarrow \quad -1 < 1+h < 1 \quad \Rightarrow \quad -2 < h < 0$$

We are interested in $h > 0$

$$\left.\begin{array}{l} g'(-3) = 1 - h \\ \text{needs to be an attractor} \end{array}\right\} \Rightarrow |1-h| < 1 \Rightarrow -1 < 1-h < 1 \Rightarrow$$

$$\Rightarrow 0 < h < 2$$

Exam 2023 June 2027: Sample 2

1. Find the values of $h>0$ such that the attractor eq. point of $x'=x^2-x-6$ is also attractor fixed point of the discrete dynamical system associated to the Euler's numerical formula with stepsize $h>0$ for the given diff. eq.

Step 1: identify eq points

$x^2-x-6=0$ $\quad x_{1,2}=\frac{1\pm 5}{2}$ $\begin{cases} x_1=3 \\ x_2=-2 \end{cases}$

$\Delta=25$

The eq. points are $x=3$ and $x=-2$

Step 2: Stability of eq points

$f(x)=x^2-x-6$

$f'(x)=2x-1$

$f'(3)=5 \quad >0 \Rightarrow$ repeller (unstable)

$f'(-2)=-5<0 \Rightarrow$ attractor (stable)

Step 3: Euler's Method

$$x_{n+1}=x_n+h\,f(x_n)$$

$$x_{n+1}=x_n+h\cdot(x_n^2-x_n-6)$$

Step 4: Fixed points of the discrete system

A fixed point x^* statisfies:

$$x^* = x^* + h(x^{*2} + 5x^* + 6)$$

$$0 = h(x^{*2} + 5x^* + 6) \quad , h > 0$$

$$x^{*2} + 5x^* + 6 = 0$$

$\Rightarrow$ the discrete points are $x = 3$ and $x = -2$

Step 5: Stability of fixed points

$$g(x) = x + h(x^2 - x - 6)$$

$$g'(x) = 1 + h(2x - 1)$$

I $g'(3) = 1 + 5h$
needs to be an attractor $\Big\} \Rightarrow |1+5h| < 1 \Rightarrow$

$\Rightarrow \quad -1 < 1+5h < 1 \quad \Rightarrow \quad -\frac{2}{5} < h < 0 \qquad$ (we need $h > 0$)

II $g'(-2) = 1 - 5h$
needs to be an attractor $\Big\} \Rightarrow |1-5h| < 1 \Rightarrow$

$\Rightarrow -1 < 1 - 5h < 1 \Rightarrow -2 < -5h < 0 \Rightarrow 0 < h < \frac{2}{5}$

2. Find the general solution of:

a) the difference equation $x_{k+2} + \sqrt{2}\, x_{k+1} + x_k = 0$

$r^2 + \sqrt{2}\, r + 1 = 0$

$\Delta = -2 \Rightarrow \quad r_{1,2} = \frac{-\sqrt{2} \pm i\sqrt{2}}{2}$

$z = \rho(\cos\theta + i\sin\theta)$

$\rho = \sqrt{\left(-\frac{\sqrt{2}}{2}\right)^2 + \left(\frac{\sqrt{2}}{2}\right)^2} = 1$

$\left.\begin{array}{l} \cos\theta = -\frac{\sqrt{2}}{2} \\ \sin\theta = \frac{\sqrt{2}}{2} \end{array}\right\} \Rightarrow \theta = \frac{\pi}{4}$

$z^k = \left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)^k = \cos\frac{\pi}{4}\cdot k + i\sin\frac{\pi}{4}\cdot k$

$x_k = C_1 \cdot \cos\frac{\pi}{4}\cdot k + C_2 \cdot i\sin\frac{\pi}{4}\cdot k$

b) the differential equation $x'' + \sqrt{2}\, x' + x = 0$

$r^2 + \sqrt{2}\, r + 1 = 0$

$\Delta = -2 \Rightarrow r_{1,2} = \frac{-\sqrt{2} \pm i\sqrt{2}}{2}$

$\left.\begin{array}{l} x_1 = e^{-\frac{\sqrt{2}}{2}t} \cdot \cos\frac{\sqrt{2}}{2}t \\ x_2 = e^{-\frac{\sqrt{2}}{2}t} \cdot \sin\frac{\sqrt{2}}{2}t \end{array}\right\} \Rightarrow \quad x_k = C_1 \cdot e^{-\frac{\sqrt{2}}{2}t} \cdot \cos\frac{\sqrt{2}}{2}t + C_2 \cdot e^{-\frac{\sqrt{2}}{2}t} \cdot \sin\frac{\sqrt{2}}{2}t$

3. We consider the prey-predator system

$\dot{x} = x(1-y) \quad \dot{y} = -y(2-x)$

a) Find the expression of a first integral in $(0,\infty)\times(0,\infty)$

Check it using the corresponding first order partial diff. eq

$H(x,y) = ?$ such that $\frac{dH}{dt} = \frac{\partial H}{\partial x}\dot{x} + \frac{\partial H}{\partial y}\cdot\dot{y} = 0$

Let $H(x,y) = f(x) + g(y)$

$$\left.\begin{aligned} &\frac{\partial H}{\partial x}\cdot\dot{x} + \frac{\partial H}{\partial y}\cdot\dot{y} = 0 \\ &\frac{\partial H}{\partial x} = f'(x) \\ &\frac{\partial H}{\partial y} = g'(y) \end{aligned}\right\} \begin{aligned} &\Rightarrow f'(x)\cdot x(1-y) - g'(y)\,y(2-x) = 0 \\ &\Rightarrow f'(x)\cdot x(1-y) = g'(y)\cdot y(2-x) \\ &\Rightarrow f'(x)\cdot x - f'(x)xy = 2g'(y)\cdot y - g'(y)\cdot xy \end{aligned}$$

$\Rightarrow -f'(x)\cdot xy = -g'(y)\cdot xy \Rightarrow f'(x) = g'(y)$

and for the remaining terms $\quad f'(x)\cdot x = 2g'(y)\cdot y \quad (2)$

Let $f'(x) = k$ and $g'(y) = 2k$

$f'(x) = k \;/\int \Rightarrow f(x) = kx + C_1 \quad \Rightarrow f(x) = x$

$g'(y) = 2k \;/\int \Rightarrow g(y) = 2ky + C_2 \quad \Rightarrow g(y) = 2y$

$(2) \Rightarrow kx = 2ky \Rightarrow x = 2y \Rightarrow$ Let $k = 1$

b) if $(2,1)$ is an equilibrium point, is it hyperbolic? is it stable?

$\dot{x} = 0 \quad \dot{y} = 0$

for $(2,1)$

$$\left.\begin{aligned} \dot{x} &= 2(1-1) = 0 \\ \dot{y} &= -1(2-2) = 0 \end{aligned}\right\} \Rightarrow \text{it's an eq}$$

$$J = \begin{pmatrix} \frac{\partial \dot{x}}{\partial x} & \frac{\partial \dot{x}}{\partial y} \\ \frac{\partial \dot{y}}{\partial x} & \frac{\partial \dot{y}}{\partial y} \end{pmatrix} = \begin{pmatrix} 1-y & -x \\ y & -(2-x) \end{pmatrix}$$

at $(2,1) \Rightarrow \begin{pmatrix} 0 & -2 \\ 1 & 0 \end{pmatrix}$

$$\det(J - \lambda I) = \begin{vmatrix} -\lambda & -2 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 + 2$$

$\lambda^2 = -2 \Rightarrow \lambda = i\sqrt{2}$

$\Rightarrow$ no real parts

$\Rightarrow$ eigenvalues are purely imaginary $\Rightarrow$ oscilations $\Rightarrow$ is a center $\Rightarrow$

$\Rightarrow$ neutrally stable

$\Rightarrow$ is not hyperbolic (real part of eigenvalues are non-zero)

Exam 2023 June 28:

1. For each $K>0$ we consider the differential equation $\dot{x} = -k(x-21)$ which is the model of Newton for cooling processes, here $x(t)$ being the temperature of a cup of tea at time t. The time is measured in minutes, and the temperature in celsius degrees.

a) Find it's flow

b) An experiment revealed the following fact: a cup of tea with initial temperature of $49^{\circ}C$ has, after 10 minutes, $37^{\circ}C$. Find the initial temperature of a cup of tea such that after 20 minutes the tea has $37^{\circ}C$.

Solved in Seminar_4

2. Find the solution of the IVP:

$$\begin{cases} \theta'' + 9\theta = 0 \\ \theta(0) = \frac{\pi}{2} \\ \theta'(0) = 0 \end{cases}$$

which models the motion of a simple gravity pendulum. After how much time the pendulum will return to the initial state? Here $\theta(t)$ is the angle at time t between the rod and the vertical. The time is measured in minutes, and the angle in radians.

Step I:

$r^2 + 9 = 0$

$r = \pm 3i$

$\theta(t) = C_1 \cdot \cos 3t + C_2 \sin 3t$

$\theta(0) = \frac{\pi}{2} \Rightarrow C_1 \cdot \cos 0 + C_2 \sin 0 = \frac{\pi}{2} \Rightarrow C_1 = \frac{\pi}{2}$

$\theta'(t) = -3C_1 \sin 3t + 3C_2 \cos 3t$

$\theta'(0) = -3C_1 \sin 0 + 3C_2 \cos 0 = 3C_2$

$\Rightarrow 3C_2 = 0 \Rightarrow C_2 = 0$

$\theta(t) = \frac{\pi}{2} \cos 3t$

Step 2: Determine the period of the Pendulum's Motion

$\theta(t) = \frac{\pi}{2}\cos 3t$ has angular freq $w = 3$

The period $T = \frac{2\pi}{w} = \frac{2\pi}{3}$ $\Rightarrow$ the time it takes for the pendulum to complete one full oscillation and return to its initial state.

3. Let $g: I \to \mathbb{R}$ be a C^1 map such that $g'(x) \neq 0$ for all x in the interval I. Assume that there exists $r \in I$ such that $g(r) = 0$. Prove that for $\eta \in I$ sufficiently close to r the unique solution $(x_k)_{k \geq 0}$ of the IVP

$$\begin{cases} x_{k+1} = x_k - \dfrac{g(x_k)}{g'(x_k)} \\ x_0 = \eta \end{cases}$$

satisfies $\lim\limits_{k \to \infty} x_k = r$

4. We consider the IVP $y' = 1 + xy^2$, $y(0) = 0$. Write the Euler numerical formula on the interval $[0,1]$ with step-size $h = 0,02$. Specify the initial values and the number of steps necessary to find the approximate value of $\varphi(0,5)$ and, respectively, of $\varphi(1)$. Compute the approximate value of $\varphi(0,04)$. Here with φ is denoted the exact solution of the given IVP.

Euler's Method formula:

$$y_{n+1} = y_n + h f(x_n, y_n)$$

$$x_{n+1} = x_n + h$$

$f(x,y) = 1 + xy^2$

$x_0 = 0$, $y_0 = 0$, $h = 0,02$

$x_1 = x_0 + h = 0,02$

$y_1 = y_0 + h(1 + x_0 y_0^2) = 0,02$

$x_2 = x_1 + h = 0,04$

$y_2 = y_1 + h(1 + x_1 y_1^2) = 0,02 + 0,02 \cdot 1,000008 = 0,04$

$\Rightarrow \varphi(0,04) \simeq 0,04$

$\varphi(0,5)$

$$N = \frac{0,5}{0,02} = 25$$

$$x_3 = x_2 + h = 0,06$$

$$y_3 = h_2 + h(1 + x_2 y_2^2) = 0,06$$

we do for 25 steps $\Rightarrow \varphi(0,5) \simeq 0,5$

$\varphi(1) \Rightarrow$ we iterate 50 times $\Rightarrow \varphi(1) \simeq 1$

$$N = \frac{1}{0,02} = 50$$

Exam 2023 June 14:

1. a) Let $A \in M_n(\mathbb{R})$ and $\eta \in \mathbb{R}^n$. Write a representation formula for the solution of the IVP

$$x' = AX \qquad X(0) = \eta$$

$$X(t) = e^{At} X(0) \Rightarrow X(t) = e^{At} \cdot \eta$$

$$e^{At} = \sum_{K=0}^{\infty} \frac{(At)^K}{K!}$$

$$x'(t) = \frac{d}{dt} e^{At} \cdot \eta = A \cdot e^{At} \cdot \eta = A \cdot X(t)$$

b) Let $t \in \mathbb{R}$. Using the definition of the matrix exponential, compute

$$e^{\begin{pmatrix}0&0\\0&0\end{pmatrix}}, \quad e^{t\begin{pmatrix}0&1\\0&0\end{pmatrix}}, \quad e^{t\begin{pmatrix}1&0\\0&1\end{pmatrix}}, \quad e^{t\begin{pmatrix}0&-1\\1&0\end{pmatrix}}$$

$$e^{At} = \sum_{k=0}^{\infty} \frac{(At)^k}{k!} \qquad e^{A} = \sum_{k=0}^{\infty} \frac{A^k}{k!}$$

$$e^{\begin{pmatrix}0&0\\0&0\end{pmatrix}} = \sum_{k=0}^{\infty} \frac{A^k}{k!} = I + \sum_{k=1}^{\infty} \frac{(0)^k}{k!} = I = \begin{pmatrix}1&0\\0&1\end{pmatrix}$$

$$A^1 = \begin{pmatrix}0&0\\0&0\end{pmatrix} \dots \quad A^k = \begin{pmatrix}0&0\\0&0\end{pmatrix}$$

$$e^{t\begin{pmatrix}0&1\\0&0\end{pmatrix}} = I + A\cdot t + \sum_{k=0}^{\infty} \frac{(At)^k}{k!} = I + At = \begin{pmatrix}1&t\\0&1\end{pmatrix}$$

$$A^1 = \begin{pmatrix}0&1\\0&0\end{pmatrix} \quad A^2 = \begin{pmatrix}0&0\\0&0\end{pmatrix} \dots\dots \quad A^k = \begin{pmatrix}0&0\\0&0\end{pmatrix}$$

$$e^{t\begin{pmatrix}1&0\\0&1\end{pmatrix}} = I + tA + \frac{t^2 A}{2!} + \frac{t^3 A}{3!} + \dots = \left(\sum_{k=0}^{\infty} \frac{t^k}{k!}\right)\cdot A = e^t \cdot A = \begin{pmatrix}e^t&0\\0&e^t\end{pmatrix}$$

$$A^1 = \begin{pmatrix}1&0\\0&1\end{pmatrix} \quad A^2 = \begin{pmatrix}1&0\\0&1\end{pmatrix} \Rightarrow A^k = A$$

$$e^{t\begin{pmatrix}0&-1\\1&0\end{pmatrix}} = I + A\cdot t + \frac{(At)^2}{2!} + \frac{(At)^3}{3!} + \frac{(At)^4}{4!} + \dots =$$

$$= \left(I + \frac{t^2}{2!} I + \frac{t^4}{4!} I + \dots\right) + \left(At + \frac{t^3}{3!}\cdot A + \frac{t^5}{5!} A + \dots\right) = \cos t \cdot I + \sin t \cdot A =$$

$$A^1 = \begin{pmatrix}0&-1\\1&0\end{pmatrix} \quad A^2 = \begin{pmatrix}1&0\\0&1\end{pmatrix} = I \quad A^3 = \begin{pmatrix}0&-1\\1&0\end{pmatrix} = A \quad A^4 = I \quad = \begin{pmatrix}\cos t & -\sin t\\ \sin t & \cos t\end{pmatrix}$$

c) Let $A, J, P \in \mathcal{M}_n(\mathbb{R})$ and assume that P is invertible and $A = PJP^{-1}$. Prove that $e^A = Pe^JP^{-1}$

$$\left.\begin{aligned} e^A &= \sum_{k=0}^{\infty} \frac{A^k}{k!} \\ A &= PJP^{-1} \end{aligned}\right\} \Rightarrow e^{PJP^{-1}} = \sum_{k=0}^{\infty} \frac{(PJP^{-1})^k}{k!} = \sum_{k=0}^{\infty} \frac{PJ^kP^{-1}}{k!}$$

$$\Rightarrow e^{PJP^{-1}} = P \cdot \left(\sum_{k=0}^{\infty} \frac{J^k}{k!} \right) \cdot P^{-1} = Pe^JP^{-1}$$

$(PJP^{-1})^k = PJ^kP^{-1}$ proved by induction

$$(PJP^{-1})^{k+1} = (PJP^{-1})^k (PJP^{-1}) = (PJ^kP^{-1})(PJP^{-1}) = PJ^k(P^{-1}P) \cdot J \cdot P^{-1}$$

$$= PJ^k \cdot I \cdot J \cdot P^{-1} = PJ^{k+1} \cdot P^{-1}$$

2. We consider the planar Lotka-Volterra system

$\dot{x} = x(1-y) \qquad \dot{y} = y(2-x)$

a) Find the expression of a first integral in $(0,\infty)\times(0,\infty)$. Check it using the corresponding first order partial differential equation

$H(x,y) = ?$ such that $\frac{\partial H}{\partial x}\dot{x} + \frac{\partial H}{\partial y}\cdot\dot{y} = 0$

$\frac{\partial H}{\partial x}\cdot x(1-y) + \frac{\partial H}{\partial y}\cdot y(2-x) = 0$

Let $H(x,y) = x^a y^b$

$\frac{\partial H}{\partial x} = a x^{a-1} y^b$

$\frac{\partial H}{\partial y} = b x^a y^{b-1}$

$\Rightarrow a x^a y^b (1-y) + b x^a y^b (2-x) = 0$

$\Rightarrow x^a y^b \left(a(1-y) + b(2-x) \right) = 0$

$\Rightarrow a(1-y) + b(2-x) = 0$

$a + 2b = 0 \Rightarrow a = -2b$

$a\cdot(-y) + b\cdot(-x) = 0$

$-b(2y+x) = 0$

Let $b = 1 \Rightarrow a = -2$

$\Rightarrow H(x,y) = \frac{y}{x^2}$

$\frac{\partial H}{\partial x}\cdot\dot{x} + \frac{\partial H}{\partial y}\cdot\dot{y} = \frac{-2y}{x^3}\cdot x(1-y) + \frac{1}{x^2}\cdot y(2-x) = 0$

b) if (2,1) is an eq point, is it hyperbolic? is it stable

$\dot{x} = 0 \qquad \dot{y} = 0$

$$\left.\begin{aligned} \dot{x} &= 2(1-1) = 0 \\ \dot{y} &= 2(2-2) = 0 \end{aligned}\right\} \Rightarrow \text{it's an eq.}$$

$$J = \begin{pmatrix} \frac{\partial \dot{x}}{\partial x} & \frac{\partial \dot{x}}{\partial y} \\ \frac{\partial \dot{y}}{\partial x} & \frac{\partial \dot{y}}{\partial y} \end{pmatrix} = \begin{pmatrix} 1-y & -x \\ -y & 2-x \end{pmatrix}$$

at $(2,1) \Rightarrow \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

$$\det(J - \lambda I) = \begin{vmatrix} -\lambda & -2 \\ -1 & -\lambda \end{vmatrix} = \lambda^2 + 2$$

$\lambda^2 - 2 = 0 \Rightarrow \lambda = \pm\sqrt{2}$

- eigen values with diff signs ⇒ saddle point ⇒ unstable
- no imaginary part ⇒ no oscillations

⇒ is hyperbolic (real part of eigenvalues are non-zero)

3. Find the solution of each of the following ivp's

a) $x' + 3x = 2 \qquad x(0) = 0$

$$x' + 3x = 0$$

$$r + 3 = 0 \Rightarrow r = -3 \Rightarrow x_h = C_1 \cdot e^{-3t}$$

$$\left.\begin{array}{l} x' + 3x = 2 \\ x_p = k \end{array}\right\} \Rightarrow k' + 3k = 2 \Rightarrow k = \frac{2}{3}$$

$$\Rightarrow x(t) = x_h + x_p = C_1 e^{-3t} + \frac{2}{3}$$

$$x(0) = 0 \Rightarrow C_1 + \frac{2}{3} = 0 \Rightarrow C_1 = -\frac{2}{3}$$

$$\Rightarrow x(t) = -\frac{2}{3} \cdot e^{-3t} + \frac{2}{3}$$

b) $x_{k+1} + 3x_k = 2 \qquad x_0 = 0$

$$r + 3 = 2 \Rightarrow r = -3 \Rightarrow x_k^h = C_1 \cdot (-3)^k$$

$$x_k^p = A \Rightarrow A + 3A = 2 \Rightarrow A = \frac{1}{2}$$

$$\Rightarrow x_k^p = \frac{1}{2}$$

$$\left.\begin{array}{l} \Rightarrow x_k = C_1 \cdot (-3)^k + \frac{1}{2} \\ x_0 = 0 \Rightarrow C_1 + \frac{1}{2} = 0 \Rightarrow C_1 = \frac{1}{2} \end{array}\right\} \Rightarrow x_k = \frac{1}{2} \cdot (-3)^k + \frac{1}{2}$$

Exam June 14 - 2023:

1. We consider the scalar difference equation:

$$X_{k+1} = X_k + \lambda X_k (2 - X_k) \quad , \quad k \in \mathbb{N}$$

whose unknown is the sequence of real numbers $(X_k)_{k \geq 0}$, and where $\lambda \in (0,1)$ is a parameter

a) Find its constant solutions (fixed points) and study their stability.

$X_{k+1} = X_k = X^*$

$X^* = X^* + \lambda X^* (2 - X^*)$

$0 = \lambda X^* (2 - X^*)$

$$\left.\begin{array}{l} \text{I} \;\; X^* = 0 \\ \text{II} \;\; 2 - X^* = 0 \Rightarrow X^* = 2 \end{array}\right\} \Rightarrow \text{the fixed points are } \begin{array}{l} X^* = 0 \\ X^* = 2 \end{array}$$

$f(x) = x + \lambda x (2 - x)$

$f'(x) = 1 + \lambda (2 - x) - \lambda x = 1 + 2\lambda - 2\lambda x$

I $f'(0) = 1 + 2\lambda$

$$\left.\begin{array}{l} |1 + 2\lambda| < 1 \;\; \text{to be stable} \\ \lambda \in (0,1) \end{array}\right\} \Rightarrow 1 + 2\lambda > 1 \Rightarrow \text{is not stable}$$

I $f'(2) = 1+2\lambda-4\lambda = 1-2\lambda$

$|1-2\lambda| < 1$ to be stable $\Rightarrow -1 < 1-2\lambda < 1 \Rightarrow$

$\Rightarrow -2 < -2\lambda < 0$

$0 > \lambda > 0$

True for $\lambda \in (0,1)$

$\Rightarrow x^* = 2$ is stable

b) Fix $\lambda = 0{,}5$. if there is an attractor, estimate its basin of attraction using the stair-step (cobweb) diagram

$$X_{k+1} = X_k + 0{,}5\, X_k(2 - X_k)$$

$$X_{k+1} = 2X_k - 0{,}5\, X_k^2$$

$$f(x) = 2x - 0{,}5\, x^2$$

Let $X_0 = 1$

$x^* = 2 \Rightarrow$ stable fixed point $\Rightarrow$ attractor

$X_1 = f(X_0) = 1{,}5$

$X_2 = f(X_1) = 1{,}8$

$X_3 = f(X_2) = 1{,}99$

$X_4 = f(X_3) > 2$

$\Rightarrow 0 \leq X_k < 4$ to move closer to 2

2. We consider the planar systems

$$\begin{cases} x' = -2x \\ y' = x - \sqrt{5}\,y \end{cases} \quad \text{and} \quad \begin{cases} x' = -2x \\ y' = x + 3x^2 - \sqrt{5}\,(y + y^3) \end{cases}$$

a) Find the flow of the linear one, denoted $\varphi(t, \eta)$

$$\begin{cases} x' = -2x \\ x(0) = \eta, \ \eta > 0 \end{cases}$$

$$\frac{dx}{dt} = -2x \Rightarrow \frac{dx}{x} = -2\,dt \ \Big/ \int \Rightarrow \int \frac{1}{x}\,dx = \int -2\,dt \Rightarrow$$

$$\Rightarrow \ln|x| = -2t + \ln|c| \Rightarrow \ln|x| = \ln e^{-2t} + \ln|c| \Rightarrow$$

$$\Rightarrow \ln|x| = \ln|c \cdot e^{-2t}| \Rightarrow x = c \cdot e^{-2t}$$

$$x(0) = \eta \Rightarrow c = \eta \Rightarrow x = \eta e^{-2t}$$

$$\begin{cases} y' = x - \sqrt{5}\,y \\ y(0) = \eta_2, \ \eta_2 > 0 \end{cases}$$

$$\frac{dy}{dt} = \eta e^{-2t} - \sqrt{5}\,y$$

. . . . .