1. Study the convergence of the following series:

a) $\sum_{n\geq 1} \underbrace{\frac{1\cdot 3\cdot \ldots \cdot (2n-1)}{2\cdot 4\cdot \ldots \cdot 2n}}_{x_n}$

Ratio test $\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \frac{1\cdot 3\cdot \ldots \cdot (2n-1)(2n+1)}{2\cdot 4\cdot \ldots \cdot 2n(2n+2)} \cdot \frac{2\cdot 4\cdot \ldots \cdot 2n}{1\cdot 3\cdot \ldots \cdot (2n-1)}$

R-D. $R = \lim_{n\to\infty} n\left(\frac{x_n}{x_{n+1}} - 1\right) = \lim_{n\to\infty} n\left(\frac{2n+2}{2n+1} - \frac{2n+1}{1}\right) =$

$= \lim_{n\to\infty} n \cdot \frac{\cancel{2n}+2-\cancel{2n}-1}{2n+1} = \frac{1}{2} < 1 \Rightarrow$ The series is divergent

Raabe-Duhamel

$n\left(\frac{x_n}{x_{n+1}} - 1\right) \to l$

$l < 1 \Rightarrow$ diverg.

$l > 1 \Rightarrow$ conv.

$l = 1 \Rightarrow$ inconclusive

c) $\sum_{n\geq 1} \underbrace{a^{\ln n}}_{x_n}$, $a > 0$

Ratio test: $\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \frac{a^{\ln n+1}}{a^{\ln n}} = \lim_{n\to\infty} a^{\ln(n+1)-\ln n} =$

$= \lim_{n\to\infty} a^{\ln \frac{n+1}{n}} = a^{\lim_{n\to\infty} \ln \frac{n+1}{n}} = a^{\ln \lim_{n\to\infty} \frac{n+1}{n}} = a^{\ln 1} = 1$

$\Rightarrow$ inconclusive

R.D. $\lim_{n\to\infty} n\left(\frac{x_n}{x_{n+1}}-1\right)=\lim_{n\to\infty} n\left(a^{\ln\frac{n}{n+1}}-1\right)=$

$$=\lim_{n\to\infty} n\cdot\frac{a^{\ln\frac{n}{n+1}}-1}{\ln\frac{n}{n+1}}\cdot\ln\frac{n}{n+1}=$$ $\lim_{x\to 0}\frac{a^x-1}{x}=\ln a$

$$=\lim_{n\to\infty}\ln a\cdot n\cdot\ln\frac{n}{n+1}=\ln a\cdot\lim_{n\to\infty}\ln\left(\frac{n}{n+1}\right)^n=$$

$$=\ln a\lim_{n\to\infty}\ln\underbrace{\left(1-\frac{1}{n+1}\right)^n}_{\to\frac{1}{e}}=\ln a\cdot\ln\frac{1}{e}=-\ln a=$$

$$=\ln\frac{1}{a}$$

If $\ln\frac{1}{a}<1\Rightarrow a>\frac{1}{e}\Rightarrow\sum x_n$ is divergent

If $\ln\frac{1}{a}>1\Rightarrow a<\frac{1}{e}\Rightarrow\sum x_n$ is convergent

If $\underbrace{\ln\frac{1}{a}=1}_{\Rightarrow a=\frac{1}{e}}\Rightarrow$ inconclusive $\Rightarrow$ we have to check it

$$\sum_{n\ge 1} a^{\ln n}=\sum_{n\ge 1}\frac{1}{e^{\ln n}}=\sum_{n\ge 1}\frac{1}{n}\Rightarrow \text{div.}$$

d) $\sum\limits_{n\geq 1} \frac{a^n \cdot n!}{n^n}$, $a>0$

R.D. test: $\lim\limits_{n\to\infty} n\left(\frac{x_n}{x_{n+1}} - 1\right) =$

$$= \lim_{n\to\infty} n\cdot\left(\frac{\frac{\cancel{a^n}\cdot\cancel{n!}}{n^n}}{\frac{\cancel{a^{n+1}}\cdot\cancel{(n+1)!}}{(n+1)^{n+1}}} - 1\right) =$$

$$= \lim_{n\to\infty} n\cdot\left(\frac{(n+1)^{\cancel{n+1}}}{n^n}\cdot\frac{1}{a\cancel{(n+1)}} - 1\right) =$$

$$= \lim_{n\to\infty} n\cdot\left(\frac{1}{a}\left(\frac{n+1}{n}\right)^n - 1\right) = \lim_{n\to\infty} n\cdot\left(\frac{1}{a}\left(1+\frac{1}{n}\right)^n - 1\right) =$$

$$= \lim_{n\to\infty} n\cdot\left(\frac{e}{a} - 1\right)$$

Ratio test

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \frac{a}{e}$$

I.1 $\frac{a}{e} > 1 \Rightarrow e < a \Rightarrow \sum\limits_{n\geq 1} x_n$ is divergent

I.2 $\frac{a}{e} < 1 \Rightarrow e > a \Rightarrow \sum\limits_{n\geq 1} x_n$ is convergent

II $\frac{a}{e} - 1 = 0 \Rightarrow a = e$

$\text{II} \quad \frac{a}{e} - 1 = 0 \Rightarrow a = e$

Meth I: $\left(1+\frac{1}{n}\right)^n < e$

$$\frac{x_{n+1}}{x_n} = \frac{a}{\left(1+\frac{1}{n}\right)^n} > \frac{a}{e}$$

$$\frac{x_{n+1}}{x_n} = \frac{\overset{\shortparallel}{e}}{\left(1+\frac{1}{n}\right)^n} > 1$$

$$\Rightarrow \boxed{x_{n+1} > x_n} > 0$$

$$\Downarrow$$

$$\lim_{n\to\infty} x_n > 0 \Rightarrow \sum x_n \text{ div.}$$

Method II: Use R.D.

$$\lim_{n\to\infty} n\left(\frac{1}{e}\overbrace{\left(1+\frac{1}{n}\right)^n}^{e^{\ln n\left(1+\frac{1}{n}\right)}} - 1\right) = \lim_{n\to\infty} n\cdot\left(\frac{1}{e}\cdot e^{n\ln\left(1+\frac{1}{n}\right)}\right) =$$

$$= \lim_{n\to\infty} n\left(e^{\underbrace{n\ln\left(1+\frac{1}{n}\right)-1}_{\to 0}} - 1\right) = \lim_{n\to\infty} n\cdot\left(n\cdot\ln\left(1+\frac{1}{n}\right)-1\right) =$$

$$= \lim_{n\to\infty} \frac{n\cdot\ln\left(1+\frac{1}{n}\right)-1}{\frac{1}{n}} \quad \text{l'Hopital} \ldots \quad \text{(longer version)}$$

2) Study convergence and absolute convergence

a) $\displaystyle\sum_{n\geq 1} \frac{(-1)^{n+1}}{\sqrt{n(n+1)}}$

$\sum x_n$ abs. conv. $\Leftrightarrow$ $\sum |x_n|$ conv.

abs. conv. $\Rightarrow$ conv.

$$\left|\frac{(-1)^{n+1}}{\sqrt{n(n+1)}}\right| = \frac{1}{\sqrt{n(n+1)}}$$

$$\sum_{n\geq 1} \frac{1}{\sqrt{n(n+1)}}$$

Comparasion test 2: $\displaystyle\frac{\frac{1}{\sqrt{n(n+1)}}}{\frac{1}{n}} = \frac{n}{\sqrt{n(n+1)}} \to 1 \in (0,\infty) \Rightarrow$

$\displaystyle\Rightarrow \sum_{n\geq 1} \frac{1}{\sqrt{n(n+1)}}$ div. $\displaystyle\Rightarrow \sum_{n\geq 1} \frac{(-1)^{n+1}}{\sqrt{n(n+1)}}$ is not abs. conv.

$$\left.\begin{array}{l} + - + - \ldots \\ |x_n| \searrow 0 \end{array}\right\} \xRightarrow{\text{Leibniz Test}} \sum x_n \text{ is convergent}$$

$\displaystyle\sum_{n\geq 1} \frac{(-1)^{n+1}}{\sqrt{n(n+1)}}$ is alternating

$$\left|\frac{(-1)^{n+1}}{\sqrt{n(n+1)}}\right| = \frac{1}{\sqrt{n(n+1)}} \quad \text{decreasing and} \quad \frac{1}{\sqrt{n(n+1)}} \to 0$$

$\sum\limits_{n\geq 1} \frac{(-1)^{n+1}}{\sqrt{n(n+1)}}$ conv. but not abs. conv. $\Rightarrow$ semi-conv.

b) $\sum\limits_{n\geq 1} (-1)^n \cdot \sin\frac{1}{n}$

$n \geq 1 \Rightarrow \frac{1}{n} > 0$; $0 < \frac{1}{n} \leq 1$

alt. series and decreasing $\overset{\text{Leibnitz}}{\underset{\text{Test}}{\Longrightarrow}}$ conv. series. (1)

$\lim\limits_{n\to\infty} \sin\frac{1}{n} = 0$

$$\sum_{n\geq 1}\left|(-1)^n \cdot \sin\frac{1}{n}\right| = \sum_{n\geq 1} \sin\frac{1}{n}$$

$$\lim_{n\to\infty} \frac{\sin\frac{1}{n}}{\frac{1}{n}} = 1 \in (0,\infty)$$ $\frac{x_n}{y_n} \to l$ (comparison test 2)

$\sum \frac{1}{n} = \infty \Rightarrow$ div. $\Rightarrow$ not abs. conv. (2)

From (1) and (2) $\Rightarrow$ semi-conv.

c) $\sum_{n\geq 1} \frac{\sin n}{n^2}$ $\qquad \frac{\sin n}{n} \cdot \frac{1}{n}$

$$\sum_{n\geq 1} \left| \frac{\sin n}{n^2} \right| = \sum_{n\geq 1} \frac{|\sin n|}{n^2}$$

$$\left.\begin{array}{l} \frac{|\sin n|}{n^2} \leq \frac{1}{n^2} \\ \sum_{n\geq 1} \frac{1}{n^2} \text{ is conv.} \end{array}\right\} \Rightarrow \sum_{n\geq 1} \left|\frac{\sin}{n^2}\right| \text{ is conv} \Rightarrow$$

$$\Rightarrow \sum_{n\geq 1} \frac{\sin}{n^2} \text{ is abs. conv.}$$

3) Prove by differentiating the geometric series that, for $|x| < 1$

$$\sum_{n=1}^{\infty} n x^n = \frac{x}{(1-x)^2} \quad , \quad \sum_{n=2}^{\infty} n(n-1)x^n = \frac{2x^2}{(1-x)^3}$$

$$\sum_{n=1}^{\infty} x^n = 1 + x + x^2 + \ldots = \frac{1}{1-x} \text{ abs conv for } |x| < 1$$

Differentiating: $\sum_{n=1}^{\infty} n x^{n-1} = \frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \ldots \quad \Big| \cdot x$

$$\sum_{n\geq 1} n x^n = \frac{x}{(1-x)^2}$$

$$\Rightarrow \text{Diff. } \sum_{n\geq 2} n\cdot(n-1)x^{n-2} = 2 \cdot \frac{1}{(1-x)^3} \quad \Big| \cdot x^2 \Rightarrow$$

$$\Rightarrow \sum_{n\geq 2} n(n-1)x^n = \frac{2x^2}{(1-x)^3}$$

$$\left(\frac{1}{1-x}\right)' = \left[(1-x)^{-1}\right]' = -1\cdot(1-x)'\cdot(1-x)^{-2} = (1-x)^{-2} =$$

$$= -1\cdot(-1)\cdot\frac{1}{(1-x)^2}$$

$$\left[\frac{1}{(1-x)^2}\right]' = \left[(1-x)^{-2}\right]' = 2(1-x)^{-3}$$

4) Prove by integrating the geometric series

$$\sum_{n=1}^{\infty} \frac{x^n}{n} = -\ln(1-x) \qquad \sum_{n=1}^{\infty} (-1)^{n+1}\frac{x^n}{n} = \ln(1+x)$$

$$\sum_{n=0}^{\infty} t^n = 1+t+t^2+\dots = \frac{1}{1-t} \quad \Big| \int_0^x dt$$

$$\sum_{n=0}^{\infty} \int_0^x t^n\,dt = \sum_{n\geq 0} \frac{t^{n+1}}{n+1}\Big|_0^x = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = \sum_{n=1}^{\infty} \frac{x^n}{n} =$$

$$= -\ln(1-t)\Big|_1^x = -\ln(1-x)$$

we replace x with $-x$

we can do this bcs the series is abs. conv.

second one:

$$\sum_{n=1}^{\infty} \frac{(-x)^n}{n} = -\ln(1+x) \quad \Big| \cdot(-1) \Leftrightarrow$$

$$\Leftrightarrow \sum_{n\geq 1} \frac{(-1)^{n+1}\cdot x^n}{n} = \ln(1+x)$$

5) Use the formula from ex. 4)

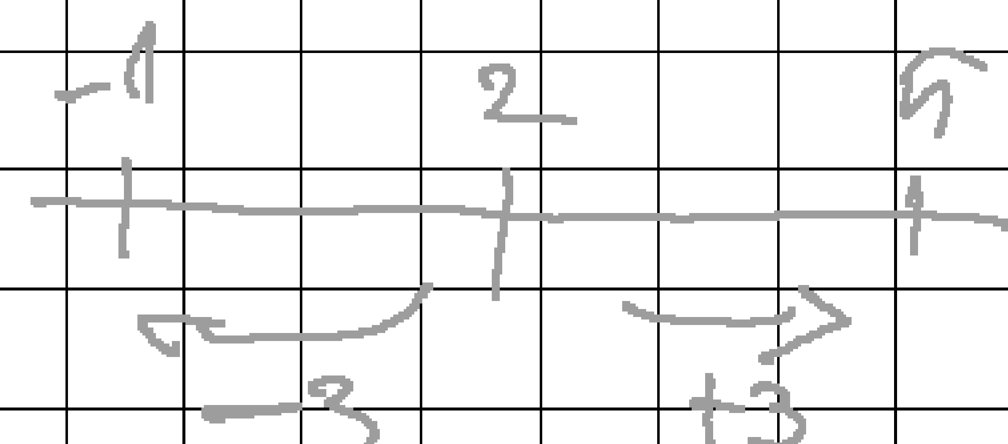
6) Find the radius of convergence and the conv. set.

a) $\sum\limits_{n\geq 1} \frac{(x-2)^n}{(n+1)\cdot 3^n}$ $\qquad$ $\sum\limits_{n=0}^{\infty} a_n\cdot(x-c)^n$ power series centered at c

$$\left|\frac{x_{n+1}}{x_n}\right| = \left|\frac{\cancel{(x-2)^{n+1}}}{(n+2)\cdot \cancel{3^{n+1}}}\cdot\frac{(n+1)\cdot\cancel{3^n}}{\cancel{(x-2)^n}}\right| = \left|\frac{x-2}{3}\cdot\frac{n+1}{n+2}\right| \Rightarrow$$

$\Rightarrow \frac{|x-2|}{3} < 1 \quad \Rightarrow \quad |x-2| < 3$ then the series is conv.

$\Rightarrow R = 3$ - radius of. conv.



$\exists\, R\in(0,\infty)$, the series is conv. for $|x-c|<R$
the series is diverg. for $|x-c|>R$

check: $|x-2|=3 \Rightarrow x\in\{-1,5\}$

$x=-1 \Rightarrow \sum \frac{(-1)^n}{n+1}$ is. conv. (Leibnitz test)

$x=5 \Rightarrow \sum \frac{1}{n+1}$ is. diverg.

The conv. set is: $[-1,5)$