Real numbers

Let $A \subseteq \mathbb{R}$. We define $x \in \mathbb{R}$ to be:

a lower bound for A if $x \leq a$, $\forall a \in A$

an upper bound for A if $x \geq a$, $\forall a \in A$

$lb(A) := \{x \in \mathbb{R} \mid x \leq A, \forall a \in A\}$ the set of lower bounds of A

$ub(A) := \{x \in \mathbb{R} \mid x \geq A, \forall a \in A\}$ the set of upper bounds of A

$$\min(A) \in lb(A) \cap A$$

$$\max(A) \in ub(A) \cap A$$

- $\min(A) \leq a$, $\forall a \in A$

  $\min(A) \in A$

- $\max(A) \geq a$, $\forall a \in A$

  $\max(A) \in A$

Ex: $A = [0,1)$

- $lb(A) = (-\infty, 0]$

  $ub(A) = [1, \infty)$, $ub(A) \cap A = \emptyset$

  $\min(A) = 0$, $\nexists \max(A)$

Definition:

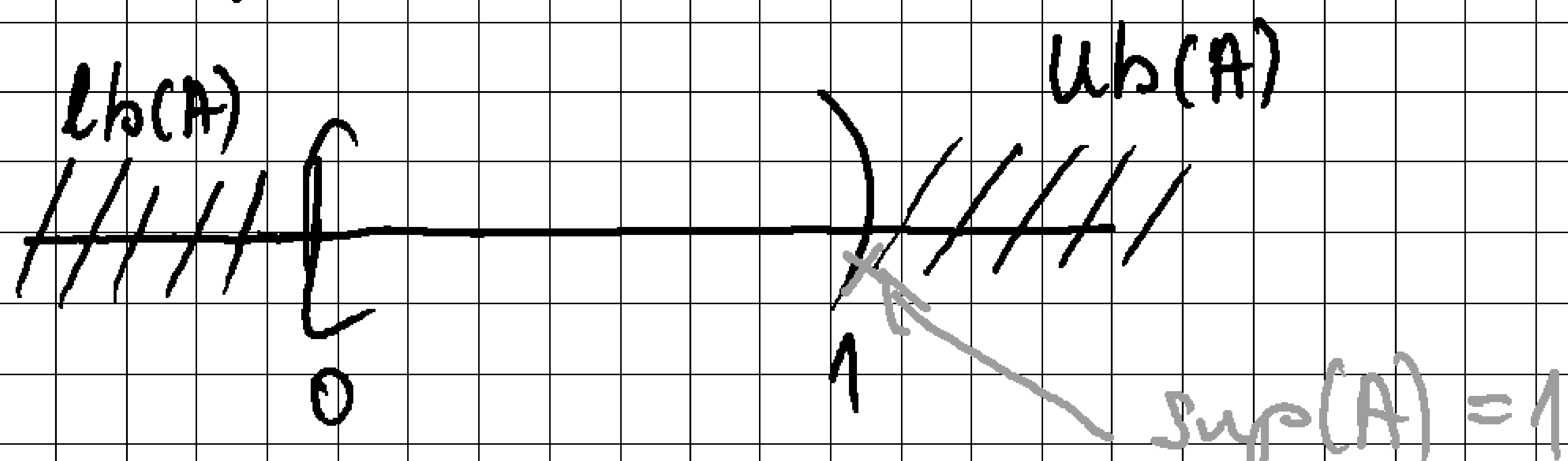
We say that $x \in \mathbb{R}$ is the supremum of $A \subseteq \mathbb{R}$, $x := \sup(A)$, if an only if:

1. $x \geq a, \forall a \in A$, that is $x \in ub(A)$
2. if u is an upper bound for A, then $x \leq u$

The supremum is the least upper bound, i.e. $\sup(A) := \min(ub(A))$

Ex: $A = [0,1]$

- $\sup(A) = 1$

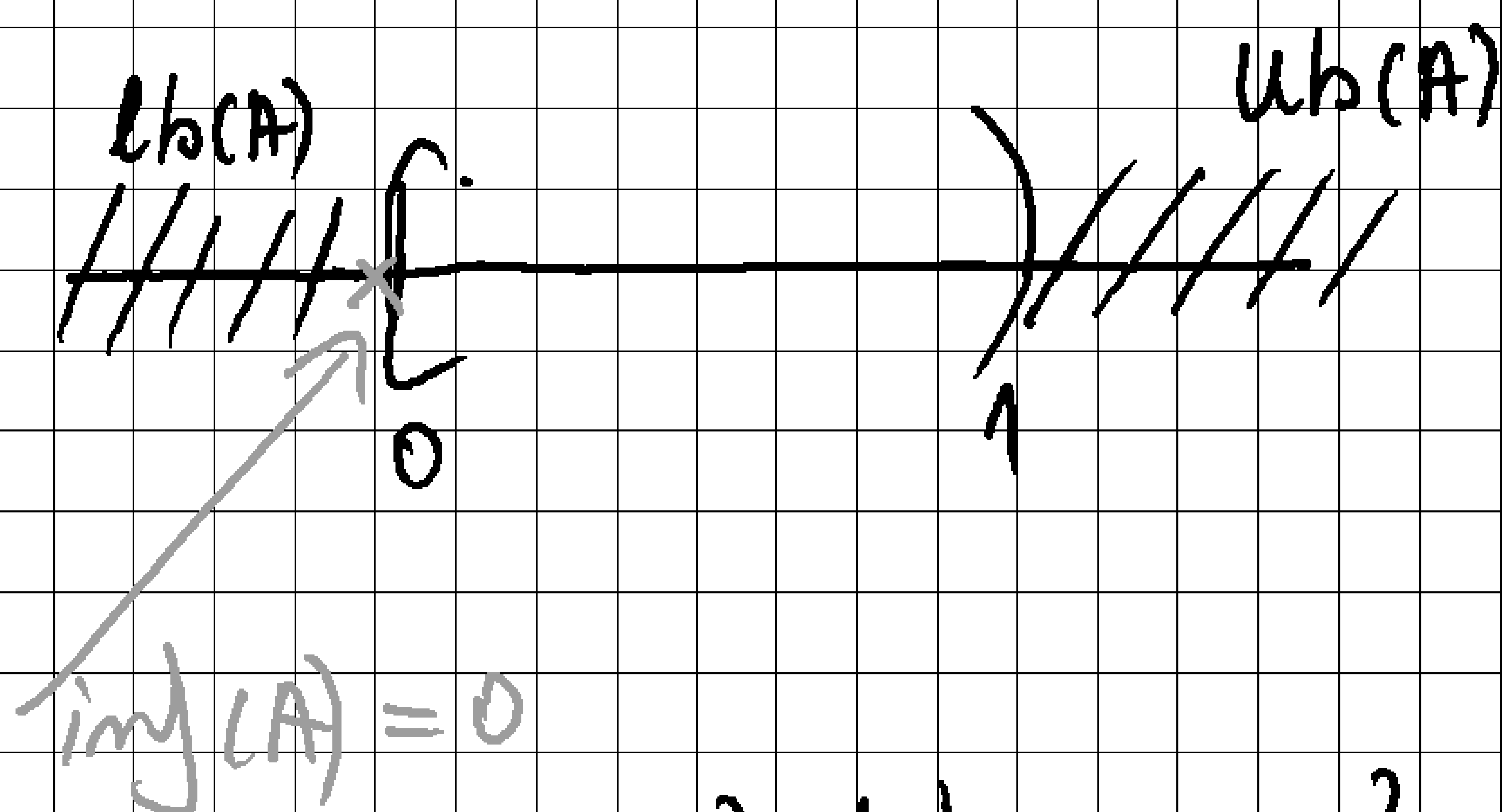


Definition :

We say that $x \in \mathbb{R}$ is the infimum of $A \subseteq \mathbb{R}$, $x := \inf(A)$, if an only if :

1. $x \leq a$, $\forall a \in A$, that is $x \in lb(A)$
2. If u is a lower bound for A, then $x \geq u$

The infimum is the greatest lower bound, i.e. $\inf(A) := \max(lb(A))$



Ex: 1) $A = \{ \frac{1}{n} \mid n \in \mathbb{N}^* \} = \{ 1, \frac{1}{2}, \frac{1}{3} \ldots \}$

$\sup(A) = 1 = \max(A)$,

$\inf(A) = 0$, $\nexists \min(A)$

2) $A = \{x \in \mathbb{Q} \mid x^2 \leq 2\} = [-\sqrt{2}, \sqrt{2}] \cap \mathbb{Q}$

$\sup(A) = \sqrt{2}$, $\nexists \max(A)$

$\inf(A) = -\sqrt{2}$, $\nexists \min(A)$

Definition: (completeness Axiom)

Every set $A \subseteq \mathbb{R}$ that is bounded above has a supremum. Similarly, every set $A \subseteq \mathbb{R}$ that is bounded below has an infimum

if A is bounded, then $\exists \sup(A), \inf(A)$

Theorem:

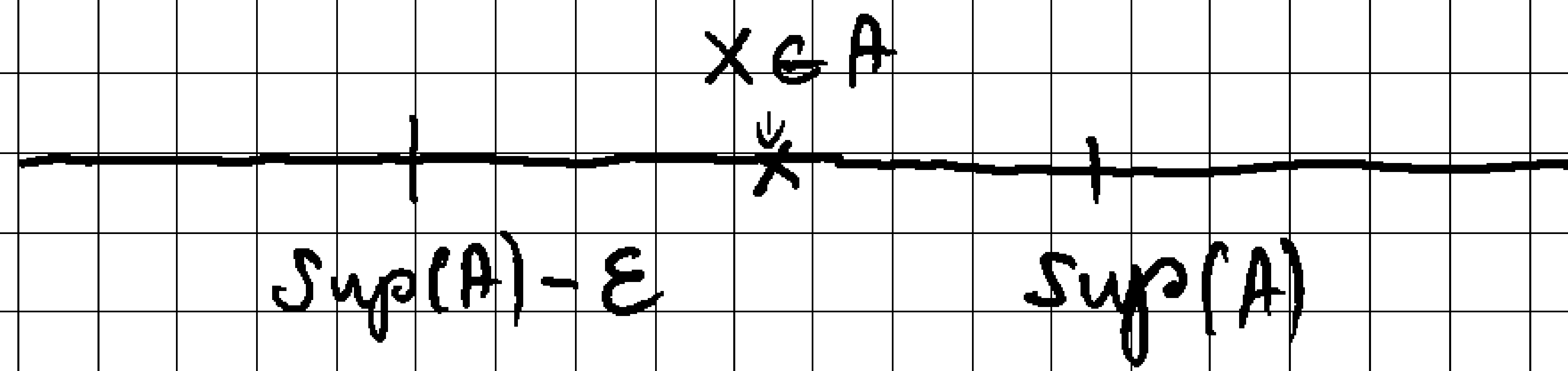
Let $A \subseteq \mathbb{R}$ be a bounded set. For $\sup(A)$ and $\inf(A)$ the following are true:

$\forall \varepsilon > 0$, $\exists x \in A$ such that

$$\sup(A) - \varepsilon < x$$

$\forall \varepsilon > 0$, $\exists x \in A$ such that

$$x < \inf(A) + \varepsilon$$



Proof:

- $\sup(A) \geq a, \forall a \in A$
- $\forall u \in ub(A), u \geq a, \forall a \in A$
  $u \geq \sup(A)$

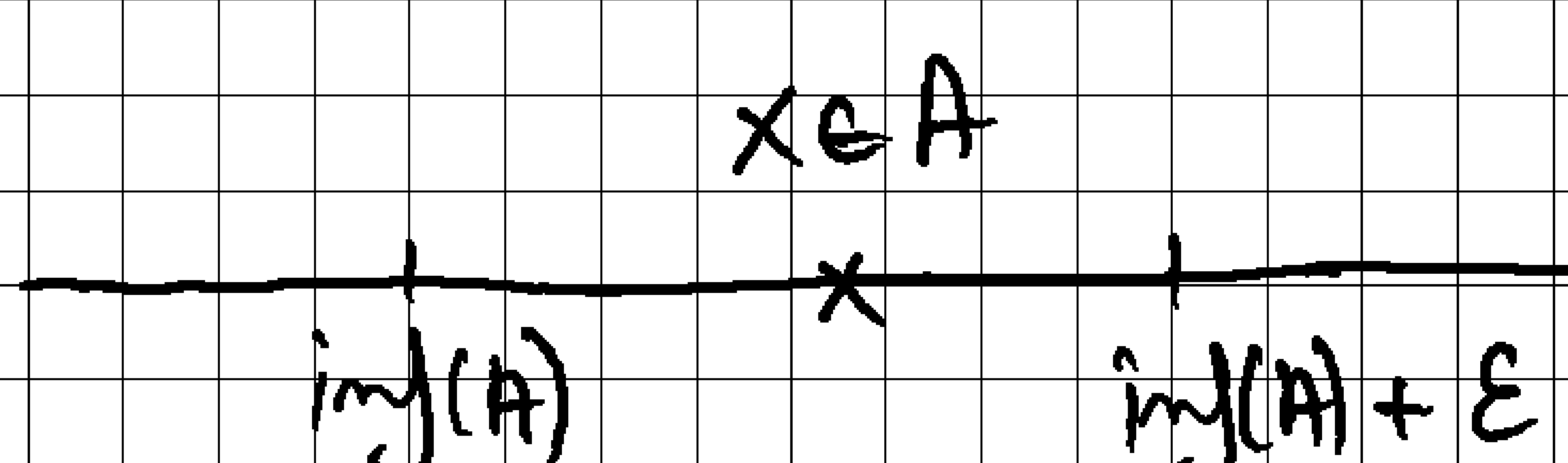
Let $\varepsilon > 0$, let $y = \sup(A) - \varepsilon < \sup(A)$

Since $y < \sup(A) \Rightarrow y \notin ub(A)$

$u \in ub(A)$ if $u \geq a, \forall a \in A$

$y \notin ub(A) \Rightarrow \exists x \in A$ s.t. $y < x$
such that

$$y = \sup(A) - \varepsilon < x \leq \sup(A)$$



Proposition:

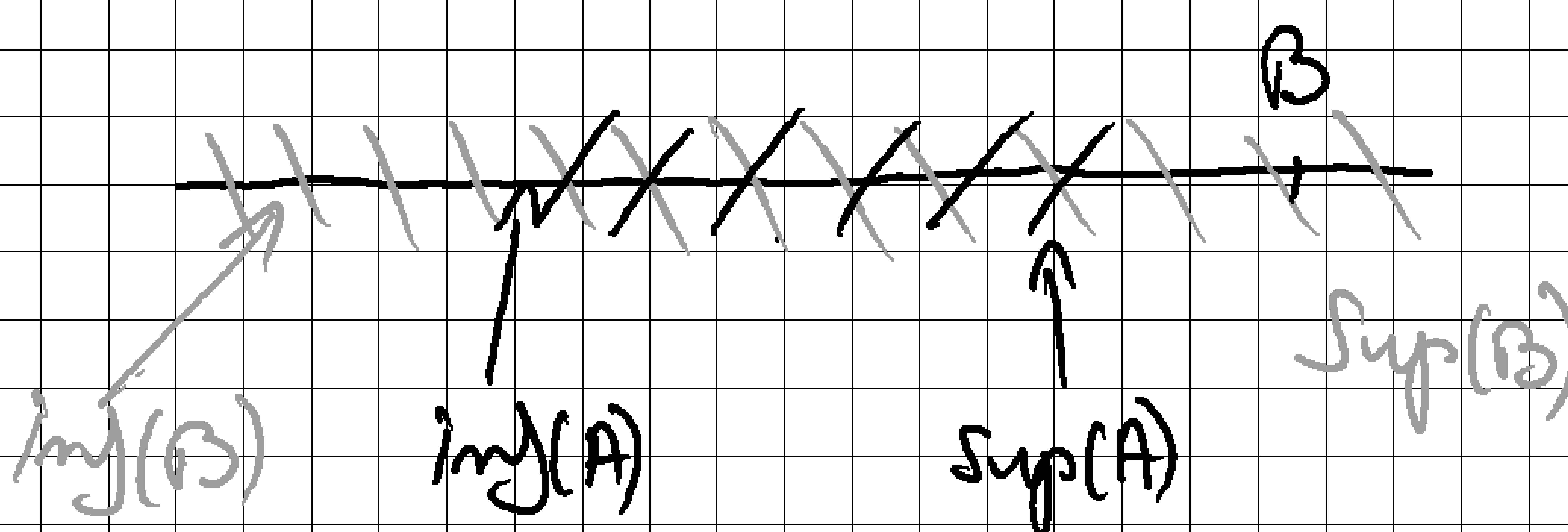
Let $A \subseteq B \subseteq \mathbb{R}$ be (nonempty) bounded sets

Then:

$$\inf(B) \le \inf(A) \le \sup(A) \le \sup(B)$$

$$\sup(A \cup B) = \max\{\sup(A), \sup(B)\}$$

$$\inf(A \cup B) = \min\{\inf(A), \inf(B)\}$$



$\inf(B) \in lb(A)$, $\inf(A) \ge \inf(B)$

$\sup(B) \in ub(A)$, $\sup(A) \le \sup(B)$

Definition:

Define the extended real line $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, \infty\}$ where ∞ and $-\infty$ are such that

$$\forall x \in \mathbb{R}, \ -\infty < x < \infty$$

if a set A is not bounded above, we define $\sup(A) = \infty$

if a set A is not bounded below, we define $\inf(A) = -\infty$

Definition:

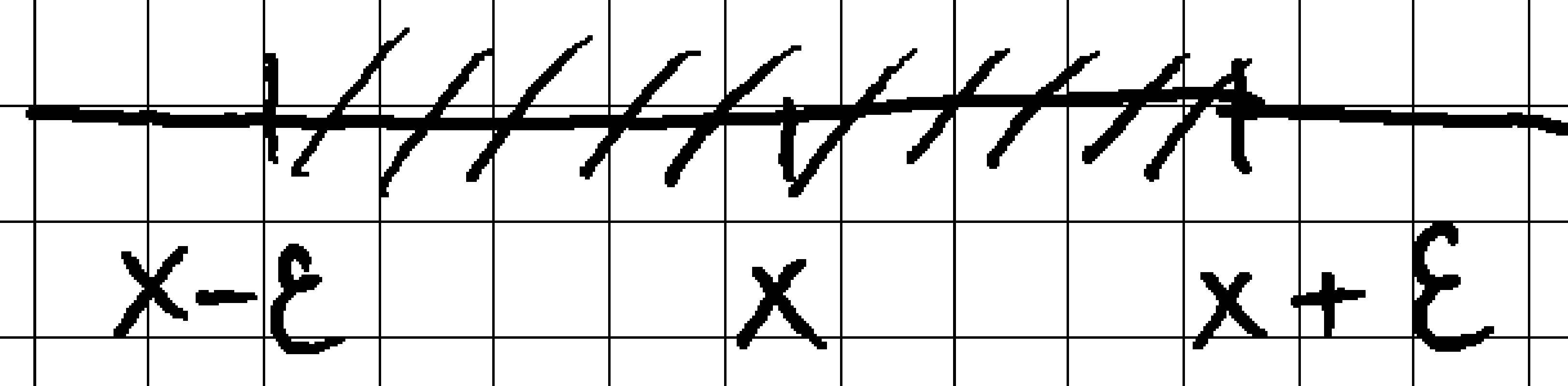
A set $V \subseteq \mathbb{R}$ is a neighborhood (vecinity) of $x \in \mathbb{R}$ if

$$\exists \, \varepsilon > 0 \text{ such that } (x-\varepsilon, x+\varepsilon) \subseteq V$$

A set $V \subseteq \mathbb{R}$ is a neighborhood of ∞ if $\exists a \in \mathbb{R}$ such that $(a, \infty) \subseteq V$

A set $V \subseteq \mathbb{R}$ is a neighborhood of $-\infty$ if $\exists a \in \mathbb{R}$ such that $(-\infty, a) \subseteq V$

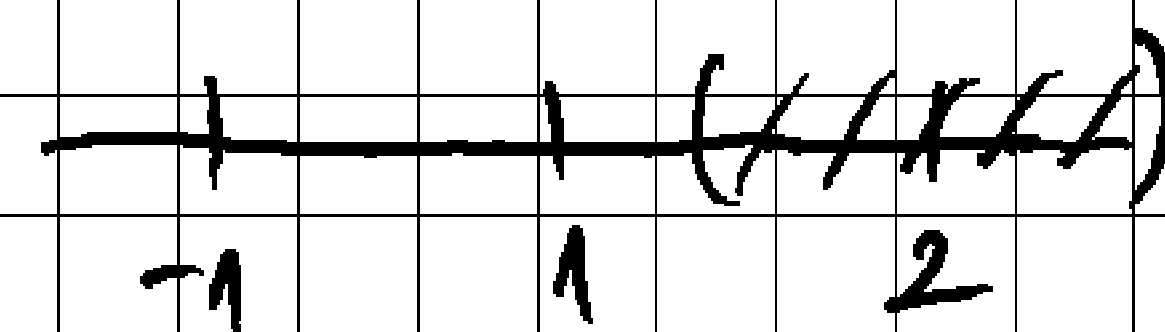
$$V(x) = V \subseteq \mathbb{R} \mid V \text{ is a neighborhood of } x\}$$



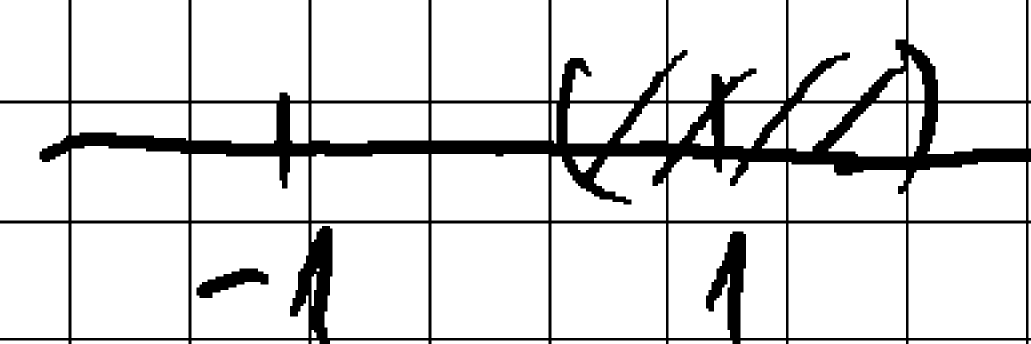
$V \in \mathcal{V}(x) \rightarrow V$ is a neighborhood of x

Ex: $(-1,1) \in \mathcal{V}(0)$

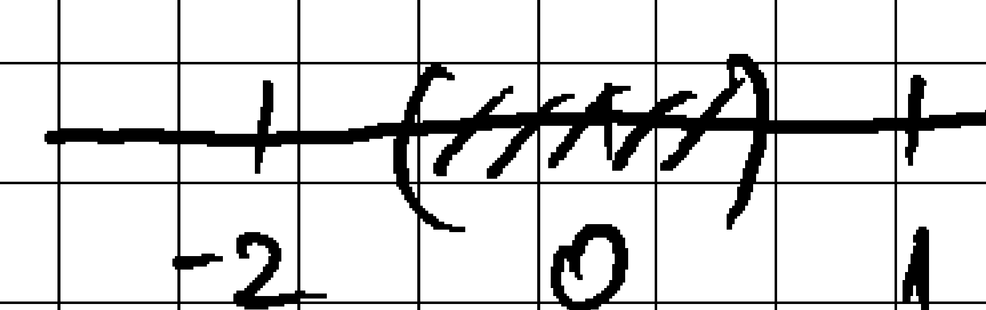
$(-1,1) \notin \mathcal{V}(2)$



$(-1,1) \in \mathcal{V}(1)$



$(-2,1) \in \mathcal{V}(0)$



Definition:

Let $A \subseteq \mathbb{R}$. The following set is called the interior of A

$$\operatorname{int}(A) = \{ x \in \mathbb{R} \mid \exists\, V \in \mathcal{V}(x) \text{ s.t. } V \subseteq A \}$$

and the following set is called the closure of A

$$\operatorname{cl}(A) = \{ x \in \mathbb{R} \mid \forall\, V \in \mathcal{V}(x),\ V \cap A \neq \emptyset \}$$

Proposition:

For any $A \subseteq \mathbb{R}$, it holds that

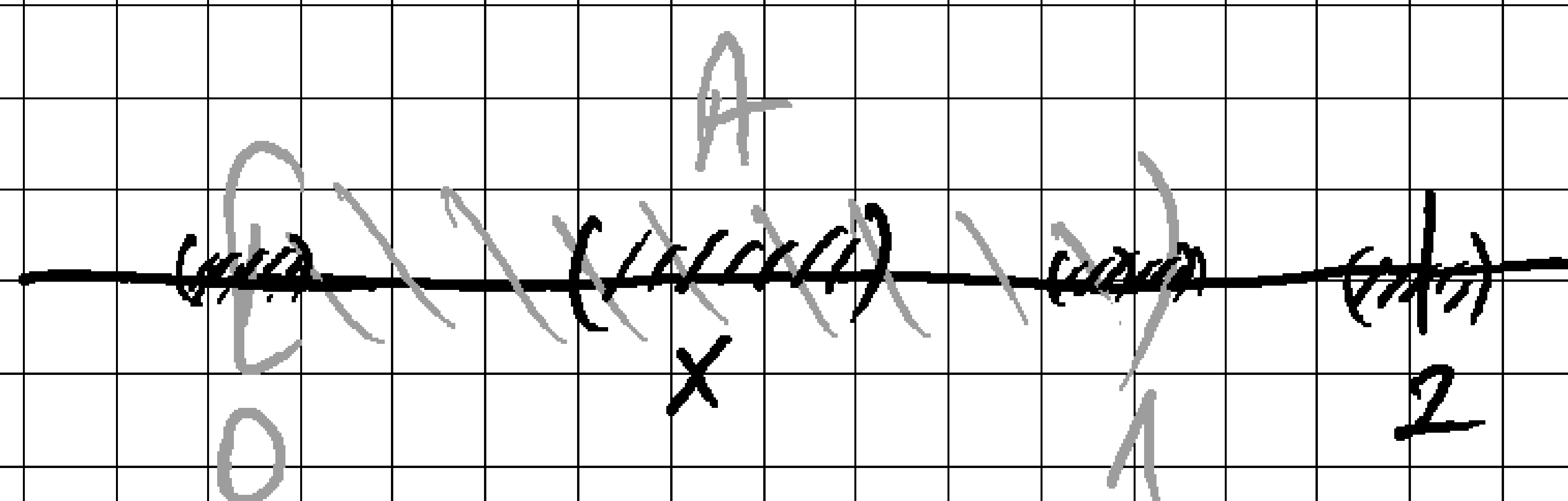
$int(A) \subseteq A \subseteq cl(A)$

Definition:

if $A = int(A)$, then A is called open.

if $A = cl(A)$, then A is called closed.

Ex: $A = [0,1)$



- $int(A) = (0,1)$

  $0 \notin int(A)$, $1 \notin int(A)$

- $cl(A) = [0,1]$

  $2 \notin cl(A)$