1. Check the following properties for ↓ ('nor'), ↑ ('nand') and ⊕ ('xor') connectives using the truth table method.

5. distribution of '↓' connective over '↑' connective

$$\underbrace{p \overset{\vee}{\downarrow} (q \overset{\wedge}{\uparrow} r)}_{u} \equiv \underbrace{(p \overset{\vee}{\downarrow} q) \overset{\wedge}{\uparrow} (p \overset{\vee}{\downarrow} r)}_{v}$$

$p \downarrow q :\overset{def}{=} \neg(p \vee q)$ nor

$p \uparrow q :\overset{def}{=} \neg(p \wedge q)$ nand

Step 1 $u \equiv v$ (u and v are logically equivalent)

iff $\forall\ i,\ i(u) = i(v)$

iff u and v have identical truth tables

Step 2

	p	q	r	$q\uparrow r$	u $p\downarrow(q\uparrow r)$	$p\downarrow q$	$p\downarrow r$	v $(p\downarrow q)\uparrow(p\downarrow r)$
i_1	T	T	T	F	F	F	F	T
i_2	T	T	F	T	F	F	F	T
i_3	T	F	T	T	F	F	F	T
i_4	T	F	F	T	F	F	F	T
i_5	F	T	T	F	T	F	F	T
i_6	F	T	F	T	F	F	T	T
i_7	F	F	T	T	F	T	F	T
i_8	F	F	F	T	F	T	T	F

Step 3 u and v don't have identical truth tables $\Rightarrow u \not\equiv v$

„$\downarrow$" cannot be distributive over „$\uparrow$"

3. Using the truth table method, check whether the following logical consequences hold:

5 $\underbrace{p \to q}_{u} \models \underbrace{(\neg p \to q) \to q}_{v}$

Step 1 $u \models v$ (v is a logical consequence of u)

if $\forall i$, $i(u) = T$ then $i(v) = T$

all the models of u are also models of v

	p	q	r	$p \to q$	$p \to r$	$q \land r$	$p \to q \land r$	v
i_1	T	T	T	T	T	T	T	T
i_2	T	T	F	T	F	F	F	T
i_3	T	F	T	F	T	F	F	F
i_4	T	F	F	F	F	F	F	T
i_5	F	T	T	T	T	T	T	T
i_6	F	T	F	T	T	F	T	T
i_7	F	F	T	T	T	F	T	T
i_8	F	F	F	T	T	F	T	T

Step 3 All the models of of V ($i_1, i_2, i_5, i_6, i_7, i_8$) are also models of v or write

$i_1, i_2, i_5, i_6, i_7, i_8\ (u) = T$

— " — $(v) = T$

2. Using the truth table method decide what kind of formula (consistent, inconsistent, tautology, contingent) is U_j, $j \in \{1, \dots 8\}$

Write all the models and anti-models of U_j, $j \in \{1, 2, \dots 8\}$

5. $U_5 = \neg p \vee q \wedge r \to q \wedge \neg p$

Step 1

1. U is consistent if $\exists\, i,\ i(U) = T$, i - model of U
2. U is inconsistent if $\forall\, i,\ i(U) = F$, has no models only antimodels
3. U is valid (tautology) if $\forall\, i,\ i(U) = T$, has only models $\models U$
4. U is contingent if $\exists\, i_1,\ i_1(U) = T$, i_1 - model

   $\exists\, i_2,\ i_2(U) = F$, i_2 - anti-model

Step 2	p	q	r	$\neg p$	$q \wedge r$	$\overbrace{\neg p \vee q \wedge r}^{u}$	$\overbrace{q \wedge \neg p}^{v}$	U_5
i_1	T	T	T	F	T	T	F	F
i_2	T	T	F	F	F	F	F	T
i_3	T	F	T	F	F	F	F	T
i_4	T	F	F	F	F	F	F	T
i_5	F	T	T	T	T	T	T	T
i_6	F	T	F	T	F	T	T	T
i_7	F	F	T	T	F	T	F	F
i_8	F	F	F	T	F	T	F	F

U_5 is contigent having 5 models $(i_2, i_3, i_4, i_5, i_6)$, $i_2, i_3, i_4, i_5, i_6(U) = T$

and 3 anti-models (i_1, i_7, i_8), $i_1, i_7, i_8(U) = F$

$i_2 : \{p, q, r\} \Rightarrow \{T, F\}$

$i_2(p) = T$

$i_2(q) = T$

$i_2(r) = F$

$i_2(U_5) = T$

MODEL

$i_1 : \{p, q, r\} \Rightarrow \{T, F\}$

$i_1(p) = T$

$i_1(q) = T$

$i_1(r) = T$

$i_1(U_5) = F$

ANTI-MODEL

5.5 Transform the formulas U_j, $j \in \{1,2,\dots 8\}$ into their equivalent conjunctive and disjunctive normal forms. Using one of these forms prove that U_j, $j \in \{1,2,\dots 8\}$ are valid formulas in propositional logic

$$U_5 = (p \overset{1}{\to} q) \wedge (p \wedge q \overset{2}{\to} r) \overset{3}{\to} (p \overset{4}{\to} r) \equiv$$

We apply the normalization algorithm

$A \to B \equiv \neg A \vee B$

replace 1,2,4

$$\equiv (\neg p \vee q) \wedge (\neg(p \vee q) \vee r) \Rightarrow (\neg p \vee r) \equiv$$

replace 3

$$\equiv \neg((\neg p \vee q) \wedge (\neg(p \vee q) \vee r)) \vee (\neg p \vee r) \equiv$$

d'Morgan's law

$$\equiv (p \wedge \neg q) \vee (p \wedge q \wedge \neg r) \vee \neg p \vee r$$

(DNF with 4 cubes)

apply distribution laws

$$\equiv \underbrace{(p \vee p \vee \neg p \vee r)}_{=T} \wedge \underbrace{(\neg q \vee p \vee \neg p \vee r)}_{=T} \wedge \underbrace{(p \vee q \vee \neg p \vee r)}_{=T} \wedge$$

$$\wedge \underbrace{(\neg q \vee q \vee \neg p \vee r)}_{=T} \wedge \underbrace{(p \vee \neg r \vee \neg p \vee r)}_{=T} \wedge \underbrace{(q \vee \neg r \vee \neg p \vee r)}_{=T}$$

CNF with 6 clauses $\Rightarrow$ U_5 is a tautology

1. U is valid if all the clauses of CNF(U) are valid

A clause is valid if it contains a pair of opposite literals

2. U is inconsistent if all the cubes of DNF(U) are inconsistent. A cube is inconsistent if it contains a pair of opposite literals.

$$p \vee \neg p \vee q \equiv T$$

$$p \wedge \neg p \wedge r$$

1. – theoretical part for ex 5.
2. – theoretical part for ex 7.

7. Using the appropriate normal form, prove that the following formulas are inconsistent.

$$U_5 = p \wedge (q \to r) \wedge ((p \wedge q) \wedge \neg(p \wedge r))$$

$$U_5 \underset{1}{\overset{\text{replace}}{\equiv}} p \wedge (\neg q \vee r) \wedge p \wedge q \wedge (\neg p \vee \neg r) \equiv$$

$$\equiv p \wedge (\neg q \vee r) \wedge q \wedge (\neg p \vee \neg r)$$ CNF with 4 clauses

distrib.

$$\equiv (p \wedge \neg q \wedge q \wedge \neg p) \vee (p \wedge \neg q \wedge q \wedge \neg r) \vee$$

$$\vee (p \wedge r \wedge q \wedge \neg p) \vee (p \wedge r \wedge q \wedge \neg p)$$ DNF with 4 cubes

$$\equiv F \vee F \vee F \vee F = F$$

$\Rightarrow U_5$ – is inconsistent.

6. Using the appropriate normal form write all the models of the following.

$$U_5 = p \lor \neg(q \land \neg r) \to p \land q \land \neg r$$

apply normalization algorithm

$$\equiv (\neg p \land q \land \neg r) \lor (p \land q \land \neg r)$$ DNF with 2 cubes.

Step 1 1. The cubes of DNF(U) provide the models of U
2. The clauses of CNF(U) provide the anti-models of U

1 - theory for ex 6.
2 - theory for ex 8.

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cube $\neg p \land q \land \neg r \equiv T$ provides 1 model

$i_1 : \{p, q, r\} \to \{T, F\}$

$i_1(p) = F$, $i_1(q) = T$, $i_1(r) = F$, $i_1(U_5) = T$

cube $p \land q \land \neg r \equiv T$ provides 1 model

$i_2 : \{p, q, r\} \to \{T, F\}$

$i_2(p) = T$, $i_2(q) = T$, $i_2(r) = F$

8. Find the anti-models of U

$CNF(U) = (p \vee \neg q \vee r) \wedge (\neg p \vee \neg q) \wedge \neg r$

clause $p \vee \neg q \vee r \equiv F$ provides 1 anti-model

$i_1 : \{p, q, r\} \to \{T, F\}$

$i_1(p) = F \quad i_1(q) = T \quad i_1(r) = F \quad i_1(U) = F$

clause $\neg p \vee \neg q \equiv F$ provides 2 anti-models

$i_2, i_3 : \{p, q, r\} \to \{T, F\}$

$i_2, i_3(p) = T \; , \; i_2, i_3(q) = T \; , \; i_2(r) = T \quad i_3(r) = F \quad i_2, i_3(U) = F$

clause $\neg r \equiv F$ provides 4 anti-models

$i_4, i_5, i_6, i_7 : \{p, q, r\} \to \{T, F\}$

$i_4(p) = T \quad i_4(q) = T$

$i_5(p) = T \quad i_5(q) = F$

$i_6(p) = F \quad i_6(q) = T$

$i_7(p) = F \quad i_7(q) = F$

$i_4, i_5, i_6, i_7(r) = T$

$i_2 = i_4$

U has 6 antimodels : $i_1, i_2, i_3, i_5, i_6, i_7(U) = F$