1.a)

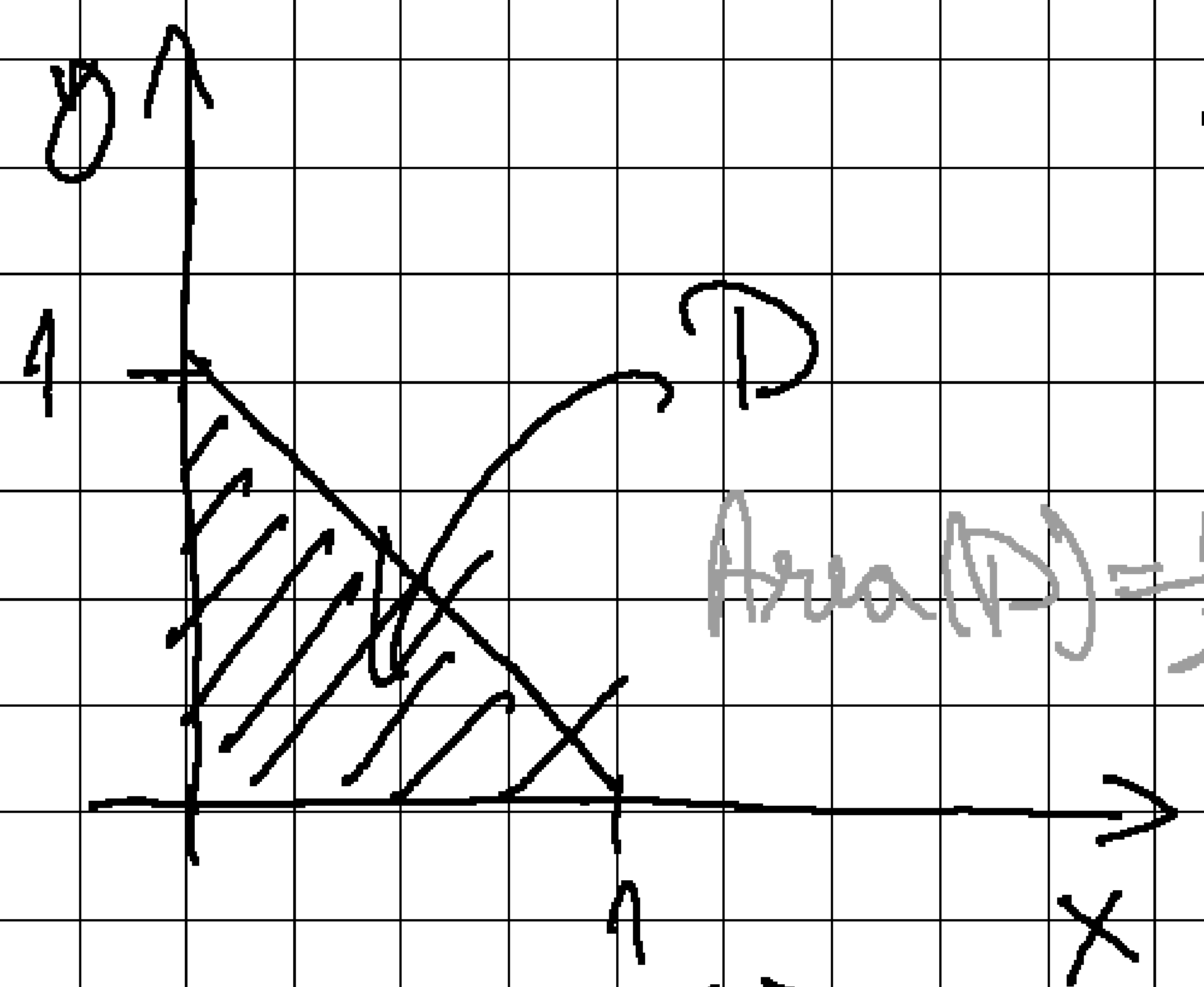
$$I = \int_0^1 \int_0^x \sin x^2 \, dy \, dx = \int_0^1 \sin x^2 \cdot y \Big|_{y=0}^{y=x} dx =$$

$$= \int_0^1 x \sin x^2 \, dx = -\frac{1}{2} \cos x^2 \Big|_{x=0}^{x=1} = -\frac{1}{2}(\cos 1 - 1) =$$

$$= \frac{1}{2}(1 - \cos 1)$$

2. Compute the following integrals by doing a change of variables.

a) $\iint_D e^{\frac{x-y}{x+y}} \, dx \, dy$, $D = \{(x,y) \in \mathbb{R}^2 \mid x \geq 0, y \geq 0, x+y \leq 1\}$



Let $u = x - y$, $v = x + y$ | $+$

$$\boxed{x = \frac{u+v}{2}}$$

$$y = \frac{v-u}{2}$$

$v \in [0,1]$

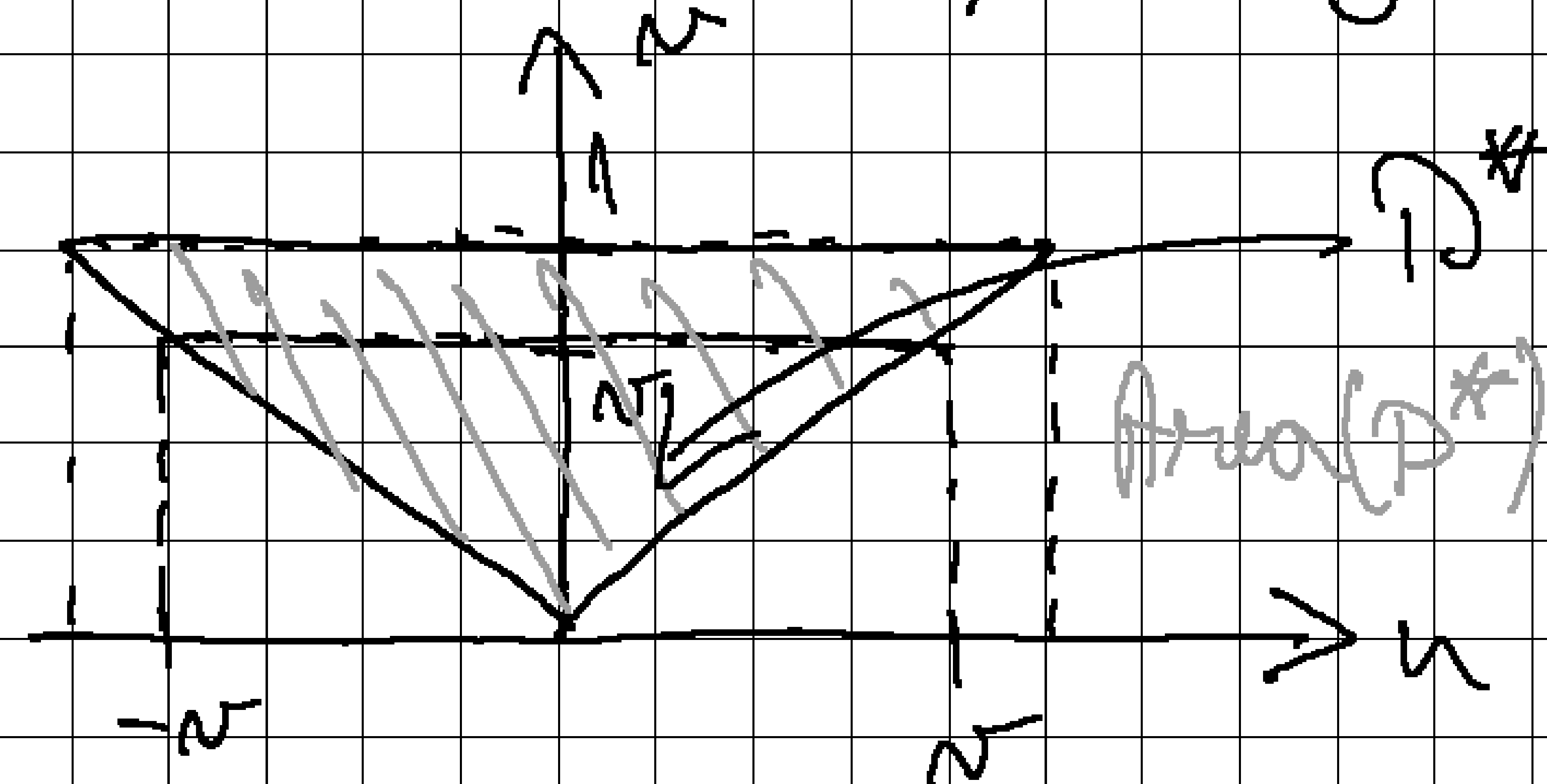
$0 \leq x \Rightarrow \frac{u+v}{2} \geq 0 \ |\cdot 2$

$\Rightarrow u \geq -v$

$0 \leq y \Rightarrow \frac{v-u}{2} \geq 0 \ |\cdot 2$

$\Rightarrow u \leq v$

$u \in [-v, v]$



Theorem 11.13

$$\iint_D f(x,y)\,dx\,dy =$$

we take the abs. value

$$= \iint_{D^*} f(x(u,v), y(u,v)) \cdot \left|\det(J)\right| du\,dv$$

$$J = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} \qquad x = \frac{u+v}{2} \quad y = \frac{v-u}{2}$$

$$J = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$\det J = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

$$I = \iint_{D^*} e^{\frac{u}{v}} \cdot \frac{1}{2}\,du\,dv = \frac{1}{2}\int_0^1 \int_{-v}^{v} e^{\frac{u}{v}}\,du\,dv =$$

$$\frac{\partial}{\partial u}\left(e^{\frac{u}{v}}\right) = \frac{1}{v} e^{\frac{u}{v}}$$

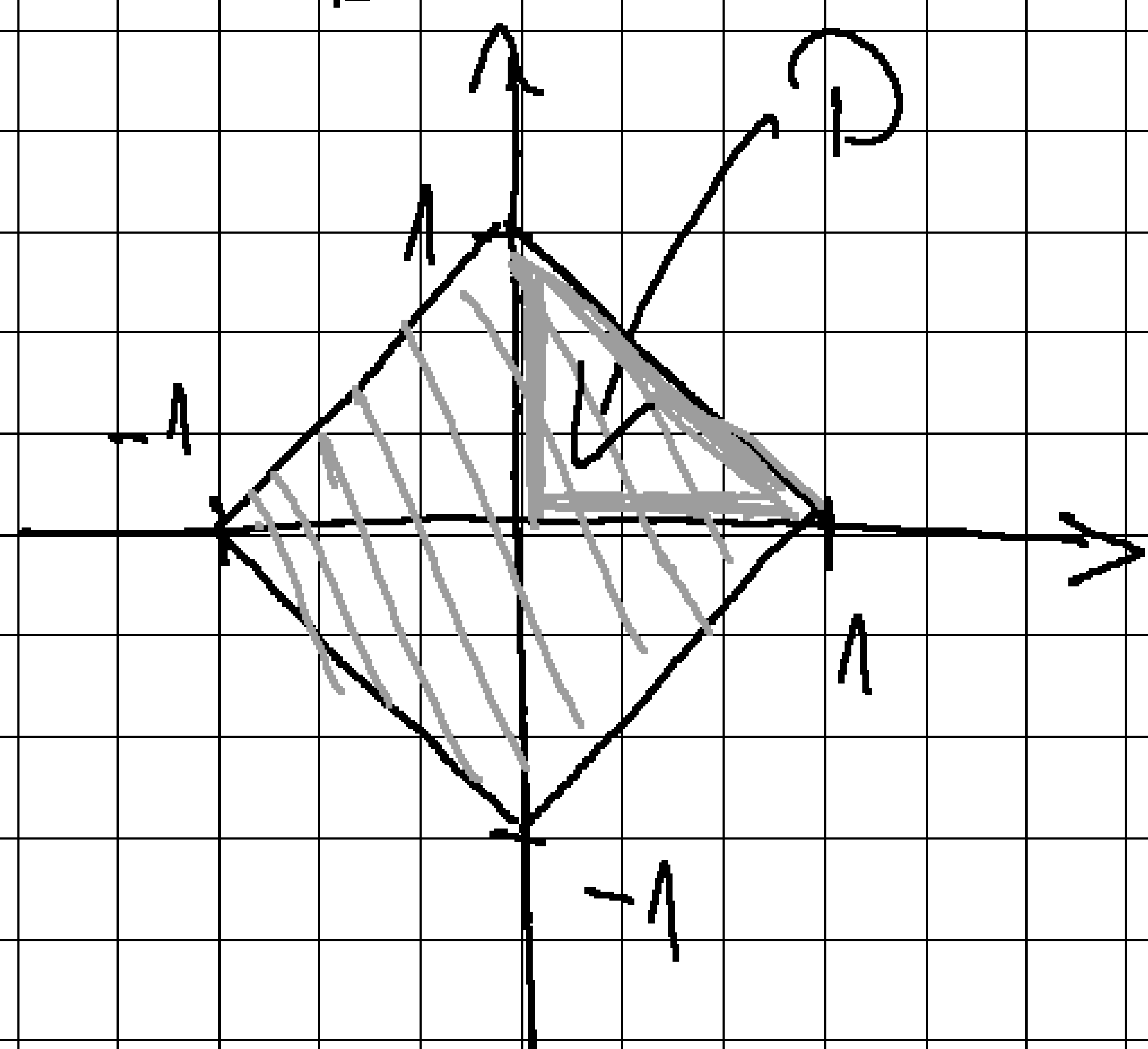
$$= \frac{1}{2}\int_0^1 v \cdot e^{\frac{u}{v}} \Big|_{u=-v}^{u=v} dv =$$

$$\left(e^{\frac{u}{v}}\right) \qquad \left(e^{-\frac{u}{v}}\right) = e^{-1}$$

$$= \frac{1}{2}\int_0^1 v \cdot e - v \cdot \frac{1}{e}\,dv = \frac{1}{2}\int_0^1 v\left(e - \frac{1}{e}\right) dv =$$

$$= \frac{1}{2}\left(e - \frac{1}{e}\right) \cdot \frac{v^2}{2}\Big|_{v=0}^{v=1} = \frac{1}{4}\left(e - \frac{1}{e}\right)$$

b) $\iint\limits_D \left(\frac{x-y}{x+y+2}\right)^2 dx\,dy$, $D=\{(x,y)\in\mathbb{R}^2 \mid |x|+|y|\le 1\}$



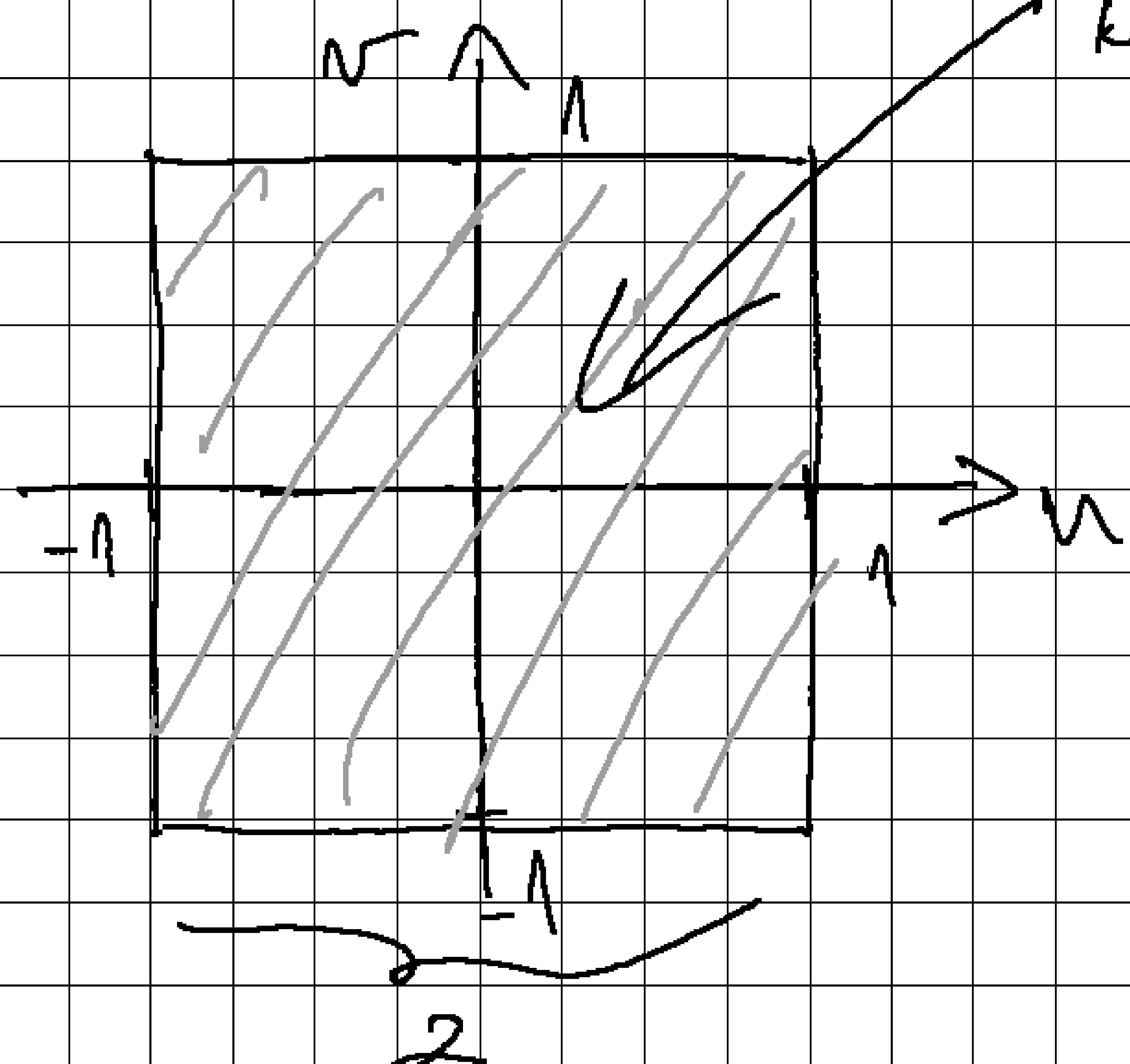
$$\begin{cases} u = x-y \\ v = x+y \end{cases}$$

$v\in[-1,1]$

$u\in[-1,1]$

$x=\frac{u+v}{2}$

$y=\frac{v-u}{2}$



$\text{Area}(D)=2$

$\text{Area}(D^*)=4$

$|\det J| = 2$

$J=\begin{pmatrix} \frac12 & \frac12 \\ -\frac12 & \frac12 \end{pmatrix}$

$$\iint\limits_D \left(\frac{x-y}{x+y+2}\right)^2 dx\,dy = \iint\limits_{D^*} \left(\frac{u}{v+2}\right)^2 \cdot \frac12\, du\,dv =$$

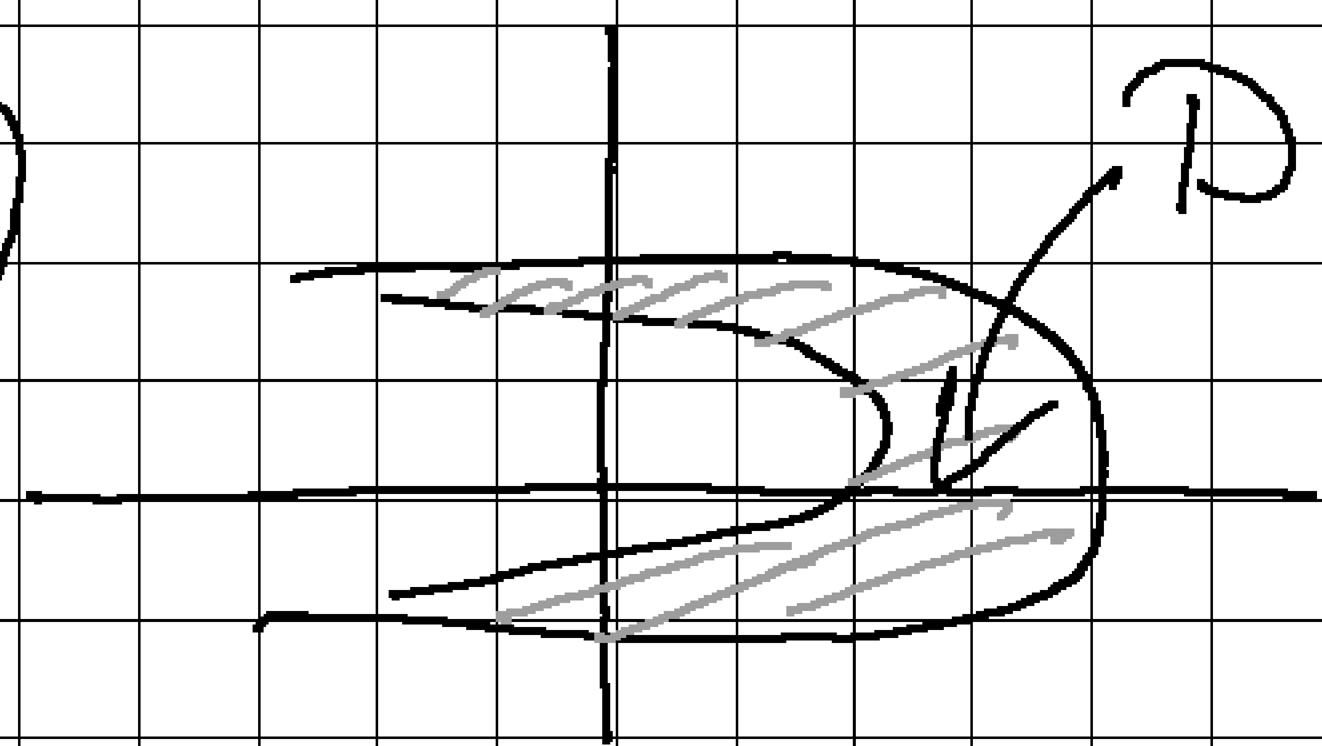
$$\left(\frac{u}{v+2}\right)^2 = u^2\cdot\left(\frac{1}{v+2}\right)^2$$

$$=\frac12 \int_{-1}^{1}\int_{-1}^{1} \frac{u^2}{(v+2)^2}\,du\,dv =$$

$$=\frac12 \int_{-1}^{1} u^2\,du \cdot \int_{-1}^{1} \frac{1}{(v+2)^2}\,dv = \frac12\cdot \frac{u^3}{3}\Bigg|_{u=-1}^{u=1} \cdot \left(-\frac{1}{v+2}\right)\Bigg|_{v=-1}^{v=1} =$$

$$= \frac{1}{2} \cdot \frac{2}{3} \cdot \left(-\frac{1}{3} + \frac{1}{1}\right) = \frac{1}{3} \cdot \frac{2}{3} = \frac{2}{9}$$

3 c)



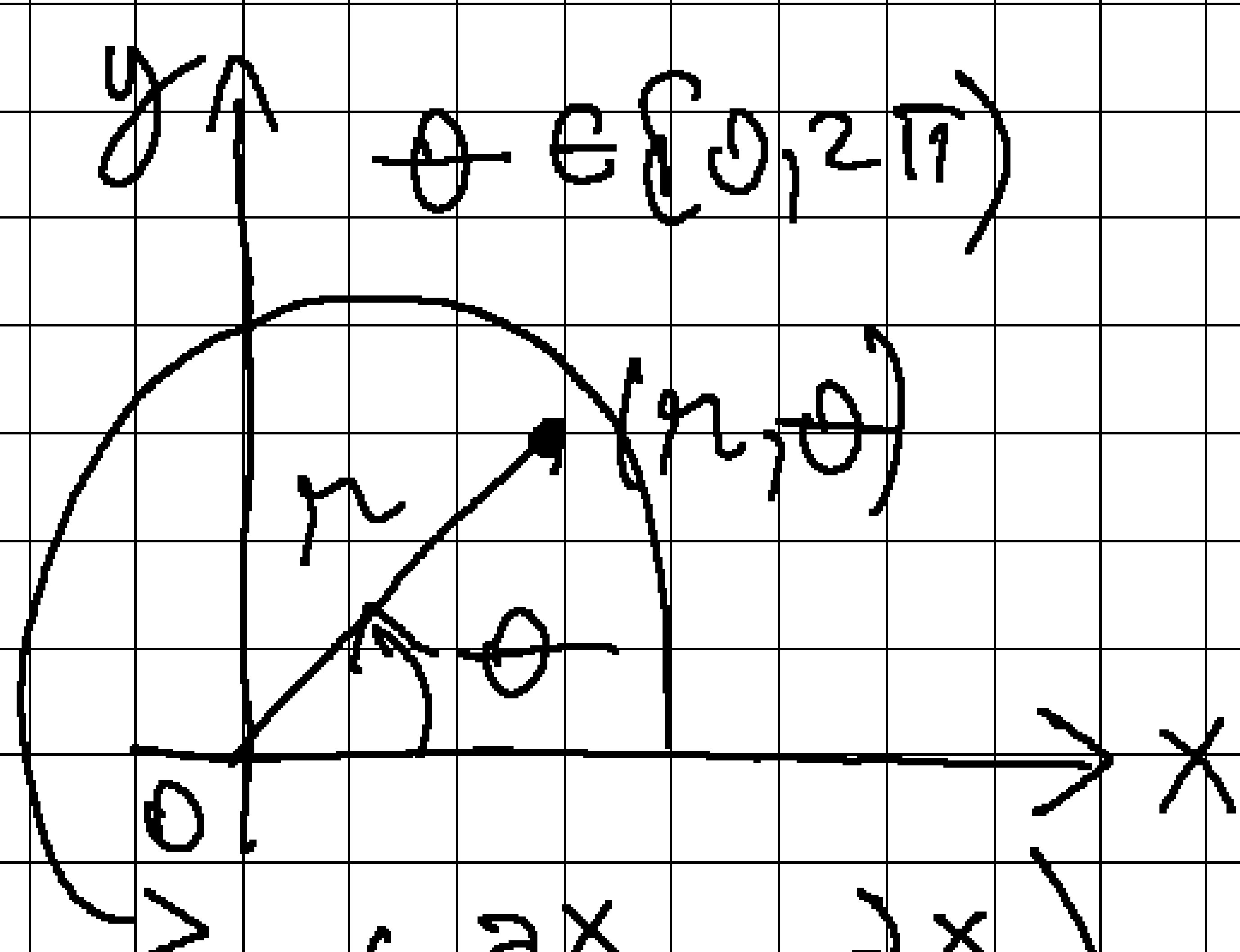
a) Use polar coordinates

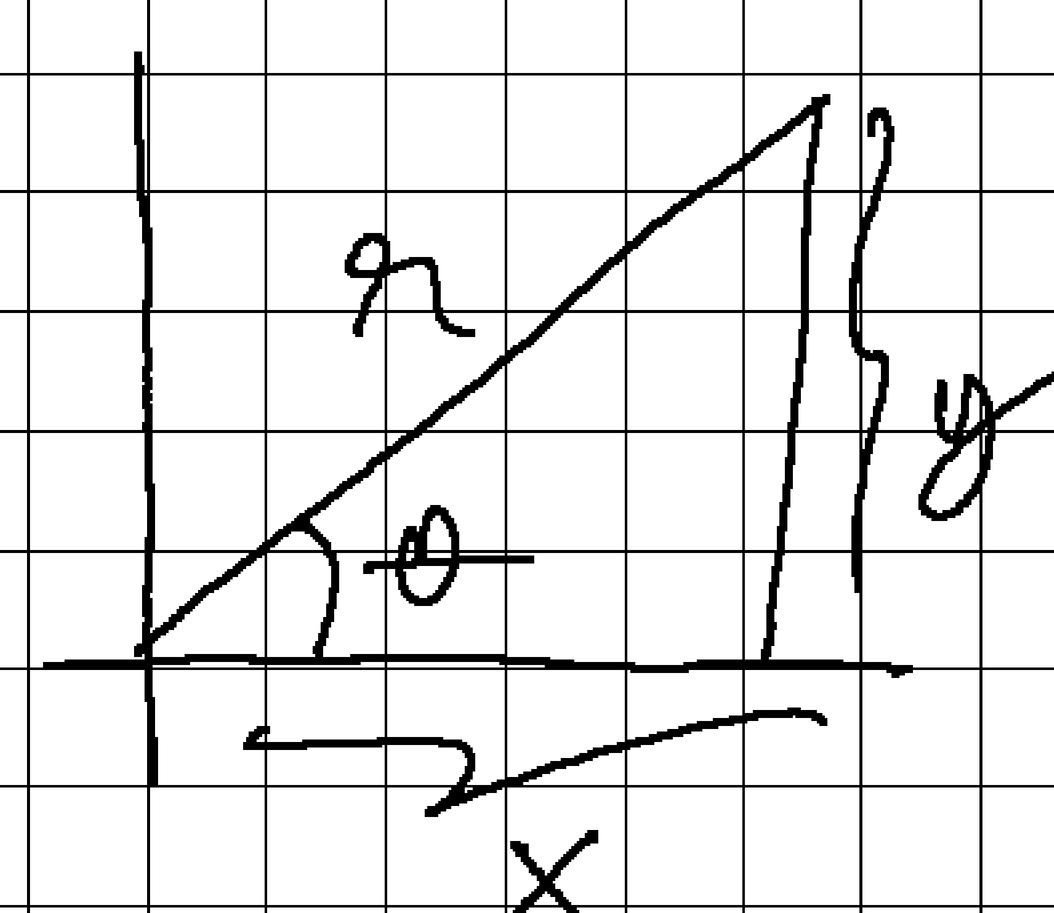
$$\iint_D \sqrt{x^2+y^2}\,dx\,dy, \quad D = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \le 4\}$$

$\theta \in [0, 2\pi)$

$x = r \cdot \cos\theta$

$y = r \cdot \sin\theta$





$$J = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$

$$\cos\theta = \frac{x}{r}$$

$$\det J = r\cos^2\theta + r\sin^2\theta = r \ge 0$$

$$r^2 = x^2+y^2 \le 4 \overset{r\ge 0}{\Longrightarrow} r \in [0,2]$$

$$\iint_D \sqrt{x^2+y^2}\,dx\,dy = \int_0^{2\pi}\int_0^2 \sqrt{r^2}\cdot r\,dr\,d\theta =$$

$$= \int_0^{2\pi}\int_0^2 r^2\,dr\,d\theta = \int_0^{2\pi} 1\,d\theta \int_0^2 r^2\,dr = 2\pi\cdot \frac{r^3}{3}\Big|_0^2 =$$

$$= \frac{16\pi}{3}$$

4. Find the area of the region bounded by the curve defined by the equation.

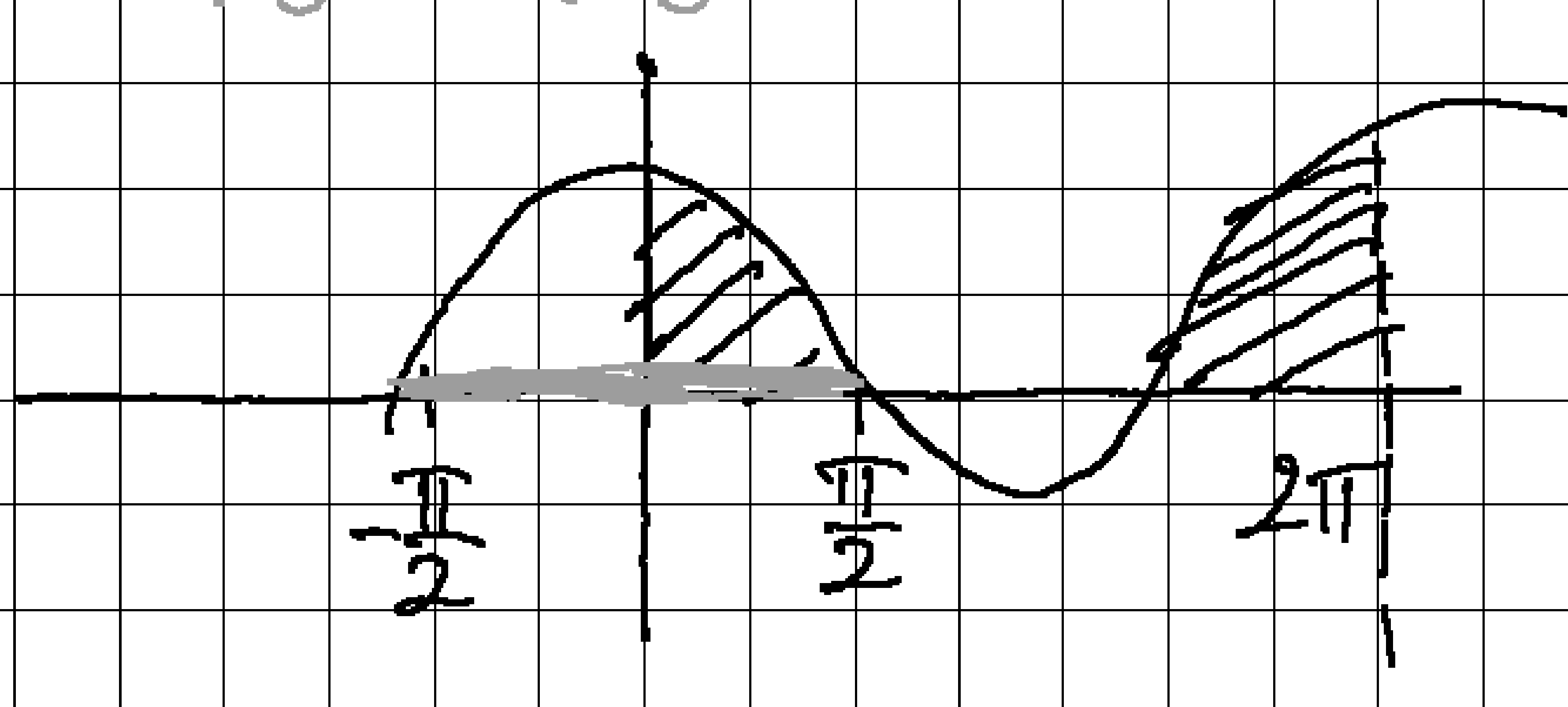
a) $(x^2+y^2)^2 = a^2(x^2-y^2)$, $x>0$, $a>0$ (1)

$x = r\cos\theta$, $y = r\sin\theta$

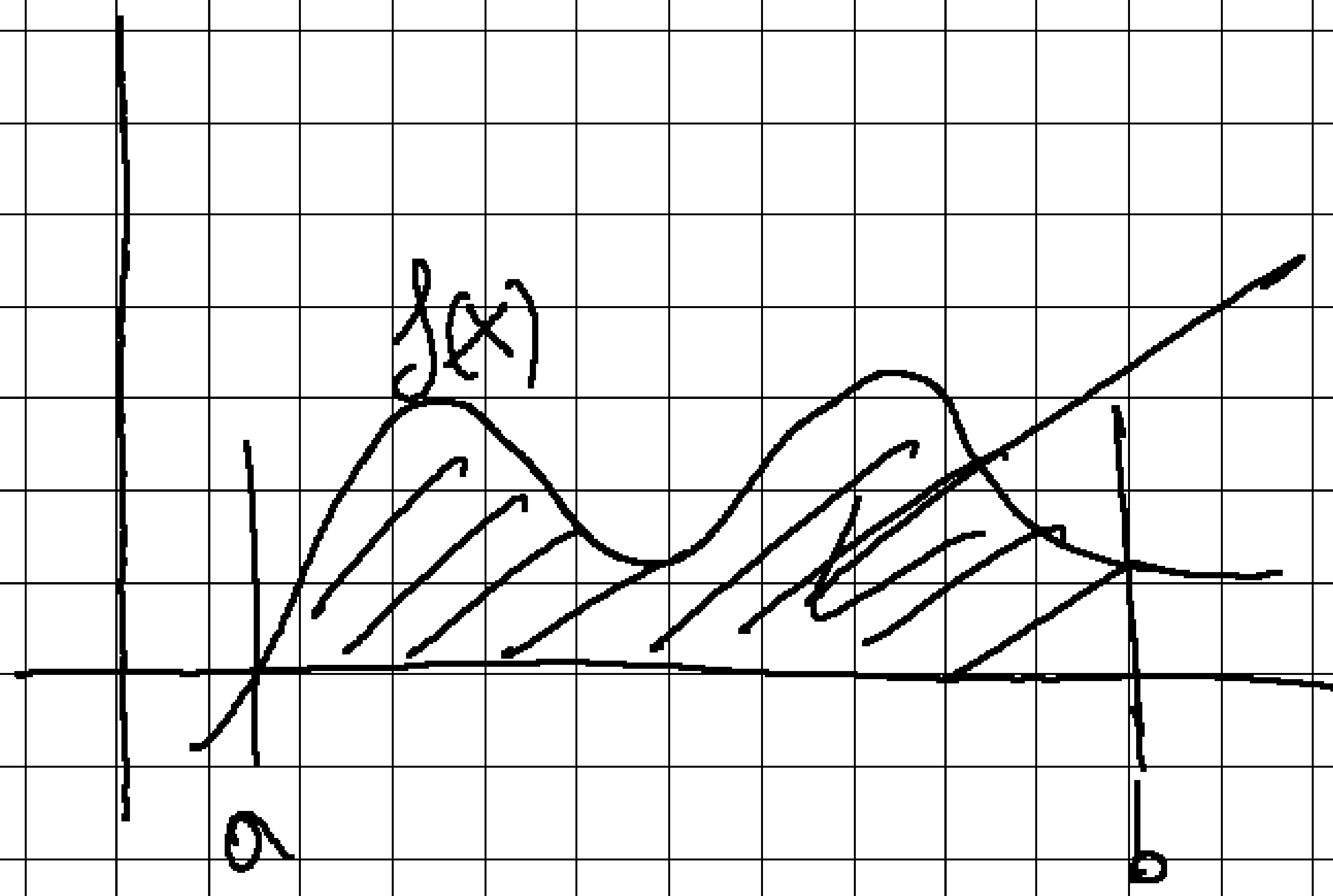
$$(1) \Leftrightarrow \underbrace{(r^2)^2}_{(a,b)} = a^2(r^2\cos^2\theta - r^2\sin^2\theta) =$$

$$= a^2 r^2 (\underbrace{\cos^2\theta - \sin^2\theta}_{\cos 2\theta}) = a^2 r^2$$

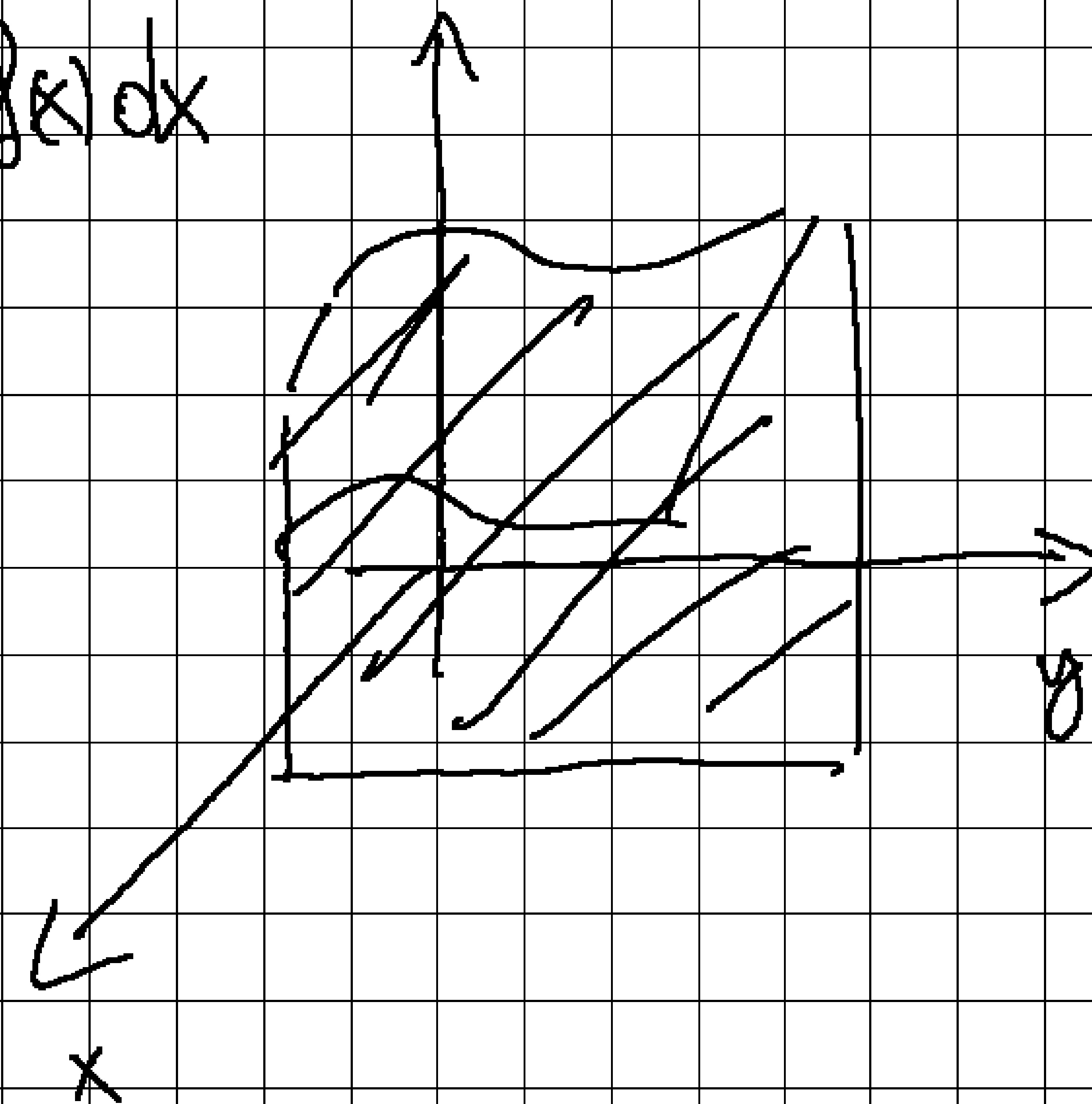
$$\overset{r>0}{\Longrightarrow} \underbrace{r^2}_{\ge 0} = \underbrace{a^2\cos 2\theta}_{\ge 0} \Rightarrow 2\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \Rightarrow \theta \in \left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$$

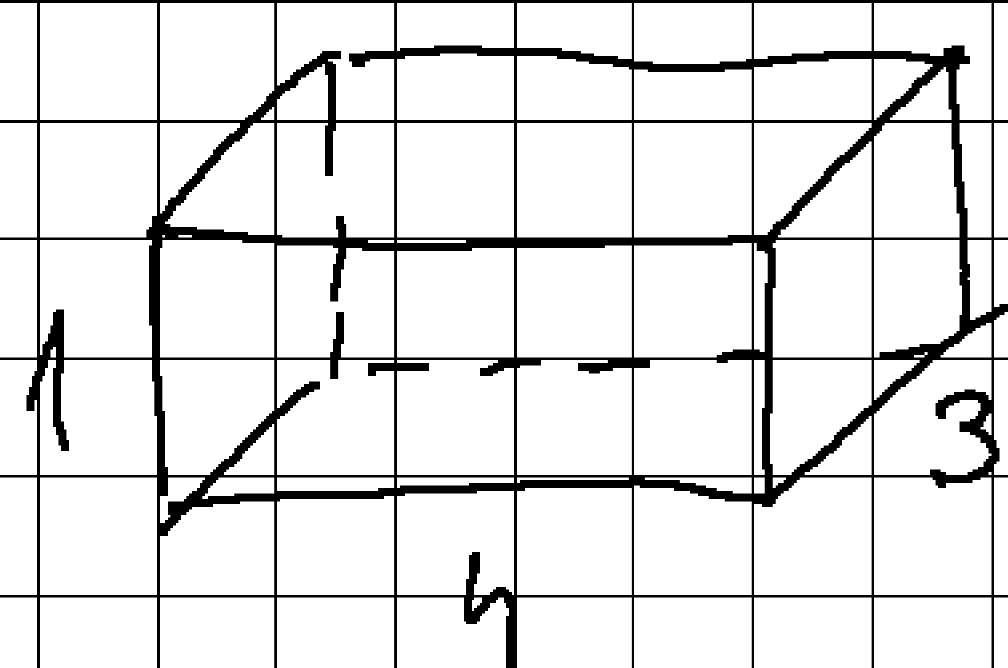


$r = a \cdot \sqrt{\cos 2\theta}$



$\int_0^b f(x)\,dx$





volume

$V = 12$

$$\text{Area} = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \int_0^{a\sqrt{\cos 2\theta}} 1 \cdot r \, dr \, d\theta =$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{r^2}{2} \Big|_{r=0}^{r=a\sqrt{\cos 2\theta}} d\theta =$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{a^2 \cos 2\theta}{2} \, d\theta = \frac{a^2}{2} \sin 2\theta \cdot \frac{1}{2} \Big|_{\theta = -\frac{\pi}{4}}^{\theta = \frac{\pi}{4}}$$

$$= \frac{a^2}{2}$$

5. $\iiint_D \sqrt{x^2+y^2+z^2}\,dx\,dy\,dz$,

$D = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 \le az\}$ $\quad a \ge 0$

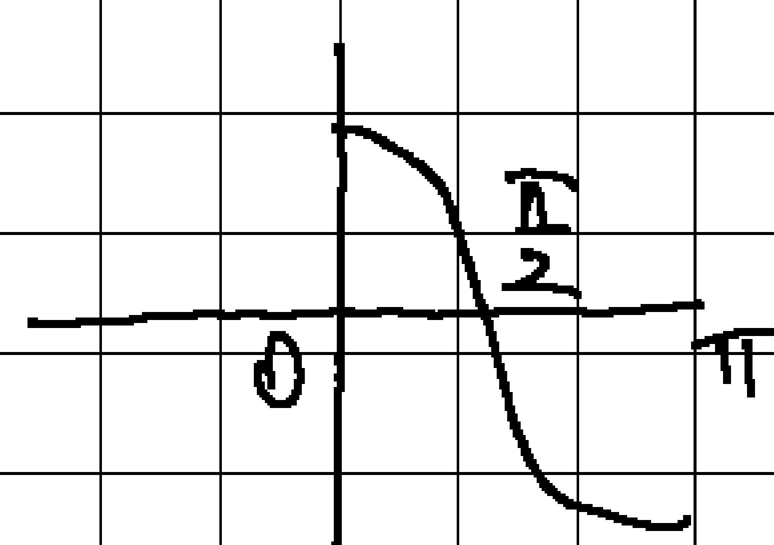
Spherical: $\begin{cases} x = (r\sin\varphi)\cos\theta & \theta \in [0,2\pi) \\ y = (r\sin\varphi)\sin\theta & \varphi \in [0,\pi) \\ z = r\cos\varphi \end{cases}$

$|\det J| = r^2 \sin\varphi$

$\underbrace{x^2+y^2+z^2}\le az$

$r^2 \le a r\cos\varphi \overset{r\ne 0}{\Rightarrow} r \le a\cos\varphi \Rightarrow$

$\Rightarrow \varphi \in [0,\frac{\pi}{2})$



$r \in [0, a\cos\varphi]$

$\theta \in [0,2\pi)$

$$I = \int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^{a\cos\varphi} r \cdot r^2 \sin\varphi \, dr\, d\varphi\, d\theta$$

6. $u = \dfrac{x}{a}$

$v = \dfrac{y}{b}$

$w = \dfrac{z}{c}$

$$J = \begin{pmatrix} a & & 0 \\ & b & \\ 0 & & c \end{pmatrix}$$

$\det J = abc$