1. Use Lagrange multipliers to find the local extremum points.

a) $\underbrace{x^2+y^2}_{f}$ subject to $x-y+1=0$

$y=x+1$

min / max of $f(x)$ subject to $g(x)=c$

$L(x,\lambda) = f(x) + \lambda(g(x)-c)$ → Lagrange function

Lagrange multiplier

$\nabla L = 0$ , $\frac{\partial L}{\partial \lambda} = 0$

$f(x,y) = x^2+y^2$

$g(x,y) = x-y+1$

$c=0$

$L(x,y,\lambda) = f(x,y) + \lambda \cdot g(x,y)$

$= x^2+y^2+\lambda(x-y+1)$

$= x^2+y^2+\lambda x-\lambda y+\lambda$

$\frac{\partial L}{\partial x} = 2x+\lambda = 0$

$\frac{\partial L}{\partial y} = 2y-\lambda = 0$

$\frac{\partial L}{\partial \lambda} = x-y+1 = 0$

$$\begin{cases} 2x+\lambda=0 \\ 2y-\lambda=0 \\ x-y+1=0 \end{cases} \quad \stackrel{(+)}{\Rightarrow} \quad 2x+2y=0 \Rightarrow x+y=0 \Rightarrow x=-y$$

$$x-y+1=0 \Rightarrow -2y+1=0 \Rightarrow y=\frac{1}{2} \qquad x=-\frac{1}{2}$$

$$2y-\lambda=0 \Leftrightarrow 1-\lambda=0 \Rightarrow \lambda=1$$

$$f(x,y)=f\left(-\frac{1}{2},\frac{1}{2}\right)=\left(-\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2=\frac{1}{4}+\frac{1}{4}=\frac{1}{2}$$

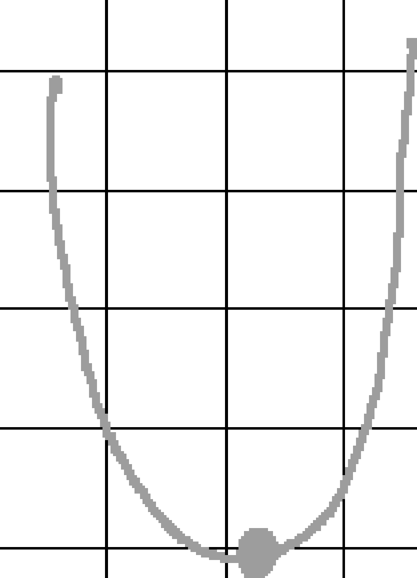
local extremum of $f$

Method II

$x-y+1=0$ - constraint $\Rightarrow y=x+1$

$x^2+y^2=0$

$x^2+y^2=x^2+(x+1)^2=2x^2+2x+1$



$ax^2+bx+c$

$-\frac{b}{2a}=-\frac{1}{2}$

minimum

b) $(x+y)^2$ subject to $x^2+y^2=1$

$f(x,y) = (x+y)^2$

$g(x,y) = x^2+y^2$

$c = 1$

$$L(x, y, \lambda) = f(x,y) + \lambda(g(x,y) - c) =$$

$$= (x+y)^2 + \lambda(x^2+y^2-1)$$

$$\frac{\partial L}{\partial x} = 2x + 2y + 2x\lambda = 0$$

$$\frac{\partial L}{\partial y} = 2x + 2y + 2y\lambda = 0$$

$$\frac{\partial L}{\partial \lambda} = x^2+y^2-1 = 0$$

$$\begin{cases} 2x+2y+2x\lambda = 0 \\ 2x+2y+2y\lambda = 0 \\ x^2+y^2-1 = 0 \end{cases} \Rightarrow \quad \left\{ \ominus \Rightarrow 2x\lambda - 2y\lambda = 0 \Rightarrow \right.$$

$$\Rightarrow \lambda(2x-2y) = 0 \quad \begin{array}{l} \rightarrow \mathrm{I}\ \lambda = 0 \\ \rightarrow \mathrm{II}\ x = y \end{array}$$

I $\lambda = 0$

$$\begin{cases} 2x+2y=0 \Rightarrow x=-y \\ 2x+2y=0 \\ x^2+y^2-1=0 \Rightarrow 2y^2=1 \Rightarrow y=\pm\frac{1}{\sqrt{2}} \Rightarrow x=\mp\frac{1}{\sqrt{2}} \end{cases}$$

II $x=y$

$$\begin{cases} 4x+2x\lambda \Rightarrow 2x(2+\lambda)=0 \xrightarrow{x\neq 0} 2+\lambda=0 \Rightarrow \lambda=-2 \\ 2x^2-1=0 \Rightarrow x^2=\frac{1}{2} \Rightarrow x=\pm\frac{1}{\sqrt{2}} \Rightarrow y=\pm\frac{1}{\sqrt{2}} \end{cases}$$

$$f(x,y)=f\left(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)=0=f\left(\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right) - \text{min}$$

$$f(x,y)=f\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)=\left(\frac{2}{\sqrt{2}}\right)^2=2=f\left(-\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right)$$

max

2. Find the minimum value of

$\underbrace{\frac{1}{2}\left(x_1^2+x_2^2+x_3^2\right)}_{=f}$ subject to

b) $\underbrace{x_1+x_2+x_3}_{=g}=3$ and $\underbrace{x_1+2x_2+3x_3}_{=h}=12$

$$L=(x_1,x_2,x_3,\lambda,\mu)=f(x_1,x_2,x_3)+$$

$$+\lambda\left(g(x_1,x_2,x_3)-3\right)+\mu\left(h(x_1,x_2,x_3)-12\right)$$

$$L(x_1, x_2, x_3, \lambda, \mu) = \frac{1}{2}(x_1^2 + x_2^2 + x_3^2) + \lambda(x_1 + x_2 + x_3 - 3) +$$

$$+ \mu(x_1 + 2x_2 + 2x_3 - 12)$$

$$\left.\begin{array}{ll} (1) & \frac{\partial L}{\partial x_1} = x_1 + \lambda + \mu = 0 \\ (2) & \frac{\partial L}{x_2} = x_2 + \lambda + 2\mu = 0 \\ (3) & \frac{\partial L}{x_3} = x_3 + \lambda + 3\mu = 0 \end{array}\right\} \oplus \quad \underbrace{3}_{= x_1 + x_2 + x_3} + 3\lambda + 6\mu = 0$$

$$\Rightarrow \lambda = -1 - 2\mu$$

$$(4) \quad \frac{\partial L}{\partial \lambda} = x_1 + x_2 + x_3 - 3 = 0$$

$$(5) \quad \frac{\partial L}{\partial \mu} = x_1 + 2x_2 + 3x_3 - 12 = 0$$

$$1 \cdot (1) + 2 \cdot (2) + 3 \cdot (3) = x_1 + 2x_2 + 3x_3 +$$

$$+ \lambda + \mu + 2\lambda + 4\mu$$

$$+ 3\lambda + 9\mu = 12 + 6\lambda + 14\mu$$

$$\Rightarrow 12 - 6 - 12\mu + 14\mu = 0 \Rightarrow \mu = -3$$

$$\Rightarrow \lambda = 5$$

$$f(-2, 1, 4) = \frac{1}{2}((-2)^2 + 1^2 + 4^2) =$$

$$= \frac{21}{2} \text{ - minimum}$$

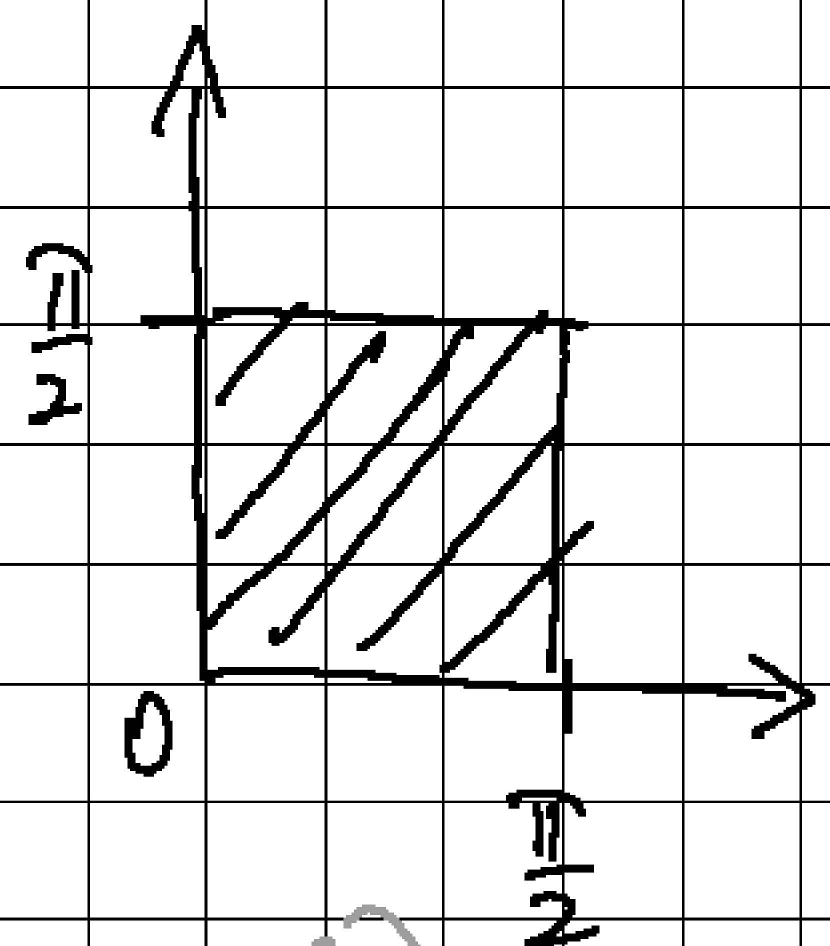
$$(1) \Rightarrow x_1 = -2$$

$$(2) \Rightarrow x_2 = 1$$

$$(3) \Rightarrow x_3 = 4$$

3. Compute the integrals:

a) $$\iint_R \cos x \sin y \, dx\, dy, \quad R = \underbrace{\left[0, \tfrac{\pi}{2}\right]}_{x} \times \underbrace{\left[0, \tfrac{\pi}{2}\right]}_{y}$$




$$= \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \cos x \sin y \, dx\, dy = \int_0^{\frac{\pi}{2}} \sin y \underbrace{\int_0^{\frac{\pi}{2}} \cos x \, dx}_{\text{constant}} \, dy =$$

$$= \int_0^{\frac{\pi}{2}} \sin y \, dy \cdot \int_0^{\frac{\pi}{2}} \cos x \, dx = -\cos y \Big|_0^{\frac{\pi}{2}} \cdot \sin x \Big|_0^{\frac{\pi}{2}} =$$

$$= \Big(-\underbrace{\cos \tfrac{\pi}{2}}_{0} + \underbrace{\cos 0}_{1}\Big)\Big(\underbrace{\sin \tfrac{\pi}{2}}_{1} - \underbrace{\sin 0}_{0}\Big) = 1$$

b) $\iint\limits_R \frac{1}{(x+y)^2}\,dx\,dy$, $R = \underbrace{[1,2]}_{x} \times \underbrace{[0,1]}_{y}$

$$\int_1^2\int_0^1 \frac{1}{(x+y)^2}\,dy\,dx$$

$$\int_0^1 \frac{1}{(x+y)^2}\,dy \overset{\substack{y-\text{var}\\ x-\text{const}}}{=\!=} -\frac{1}{x+y}\Big|_{y=0}^{y=1} = -\frac{1}{x+1}+\frac{1}{x}$$

$$\int_1^2\int_0^1 \frac{1}{(x+y)^2}\,dy\,dx = \int_1^2\left(-\frac{1}{x+1}+\frac{1}{x}\right)dx = \left(-\ln(x+1)+\ln x\right)\Big|_1^2$$

$$= -\ln 3+\ln 2+\ln 2-\underbrace{\ln 1}_{=0} = \ln\frac{4}{3}$$

II If we change the order of the end points we get the same result

$$\int_0^1\int_1^2 \frac{1}{(x+y)^2}\,dx\,dy$$

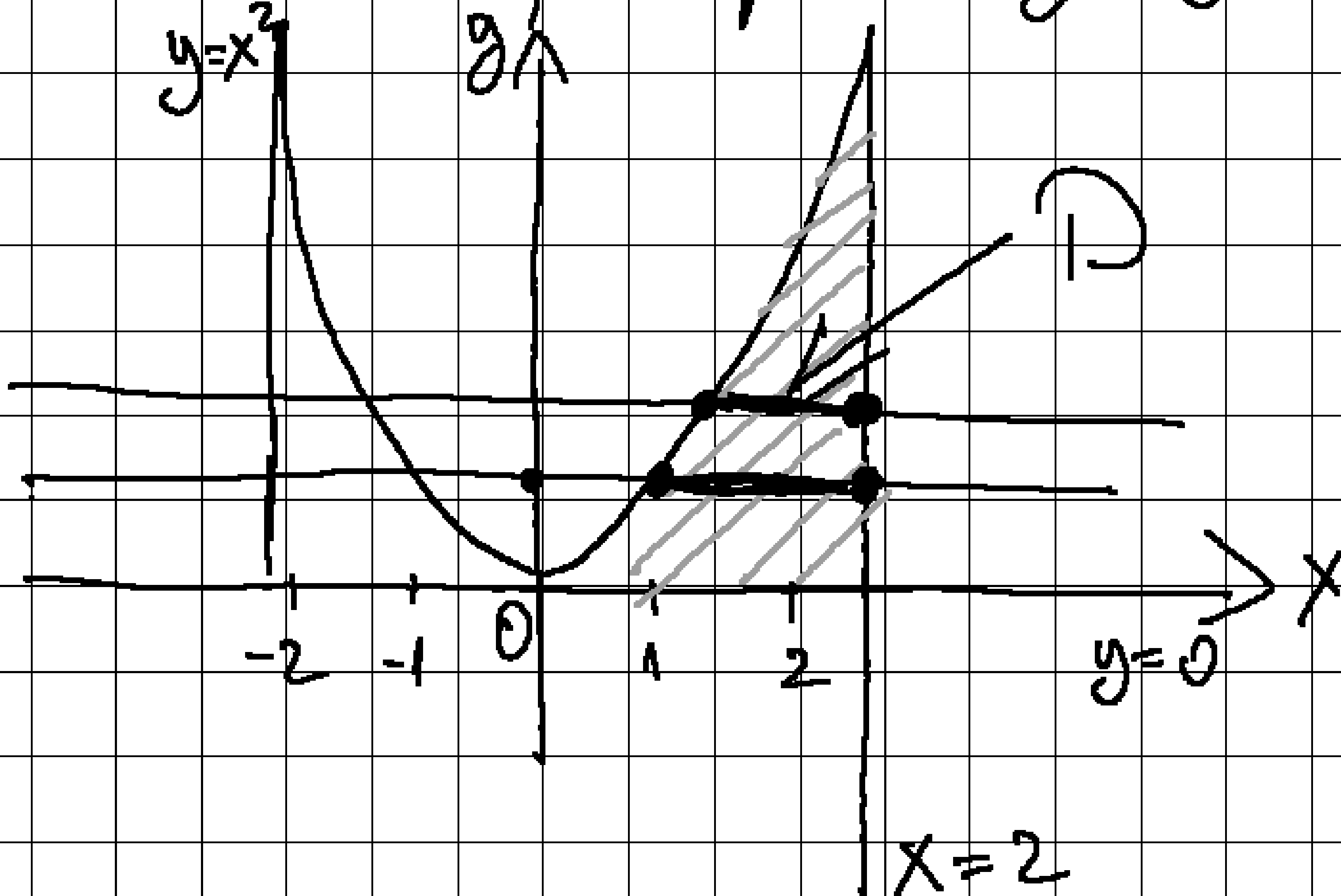
$$\int_1^2 \frac{1}{(x+y)^2}\,dx = -\frac{1}{x+y}\Big|_{x=1}^{x=2} = -\frac{1}{2+y}+\frac{1}{1+y}$$

$$\int_0^1\left(-\frac{1}{2+y}+\frac{1}{1+y}\right)dy = -\ln(2+y)+\ln(1+y)\Big|_0^1 =$$

$$= -\ln 3+\ln 2+\ln 2-\ln 1 = \ln\frac{4}{3}$$

4. Let $D \subseteq \mathbb{R}^2$ be a subset bounded by the parabola $y = x^2$ and the lines $x = 2$ and $y = 0$

a) Express $D$ as a simple set with respect to the x-axis, respectively y-axis

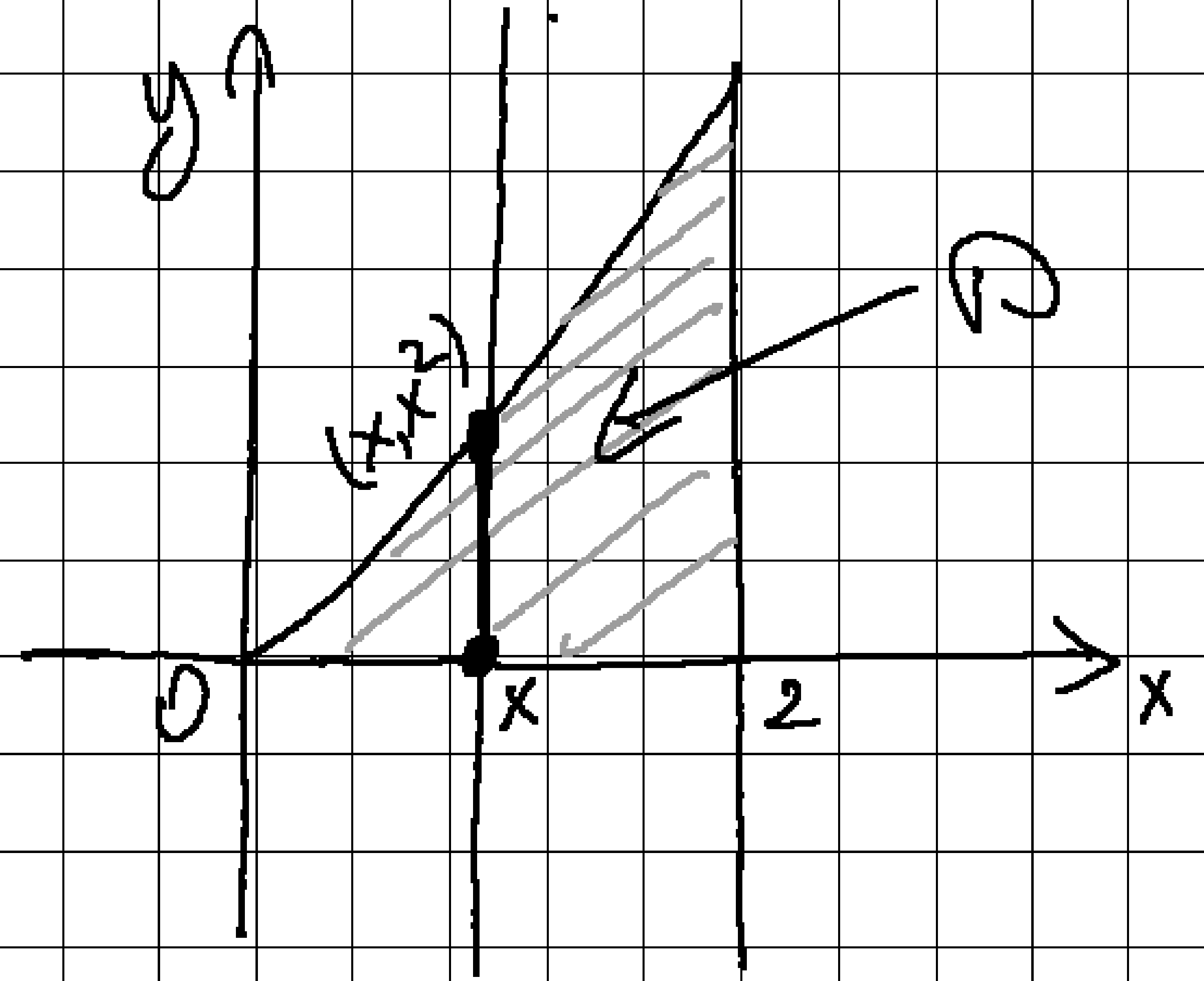



it only intersects with one line

Simple w.r.t. x-axis

$$D = \{(x,y) \in \mathbb{R}^2 \mid a \le y \le b,\ f_1(y) \le x \le f_2(y)\}$$

$$= \{(x,y) \in \mathbb{R}^2 \mid 0 \le y \le 4,\ \sqrt{y} \le x \le 2\}$$




w.r.t y-axis

$$D = \{(x,y) \in \mathbb{R}^2 \mid 0 \le x \le 2,\ 0 \le y \le x^2\}$$

b) Compute $\iint_D xy\,dx\,dy$ in two ways

$$\iint_D xy\,dx\,dy = \int_0^4 \int_{\sqrt{y}}^2 xy\,dx\,dy =$$

$$= \int_0^4 y\left(\frac{x^2}{2}\right)\Big|_{x=\sqrt{y}}^{x=2} dy = \int_0^4 y\left(2 - \frac{y}{2}\right) dy =$$

$$= \int_0^4 2y - \frac{y^2}{2}\,dy = 2\int_0^4 y\,dy - \frac{1}{2}\int_0^4 y^2\,dy =$$

$$= 2 \cdot \frac{y^2}{2}\Big|_0^4 - \frac{1}{2}\,\frac{y^3}{3}\Big|_0^4 = 16 - \frac{64}{6} = \frac{16}{3}$$

$$\int_0^2 \int_0^{x^2} xy \, dy \, dx = \int_0^2 x \cdot \frac{y^2}{2} \Big|_0^{x^2} dx = \int_0^2 \left( x \cdot \frac{x^4}{2} - 0 \right) dx$$

$$= \int_0^2 \frac{x^5}{2} \, dx = \frac{x^6}{12} \Big|_0^2 = \frac{\cancel{64}}{\cancel{12}} = \frac{16}{3}$$