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Programming Languages
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For all tasks, $\mathcal{L}$ is the set of all possible lists of numbers where $\forall x \in \mathcal{L}$, then for all elements e of x, $e \in \mathbb{N}$
We let $P(x)$ be the property $max(x) \leq sum(x)$ and choose $max(\emptyset) = -\infty,\ sum(\emptyset) = 0$ where $\emptyset$ is the empty list.

1 Task 1

Must have 2 examples that are not the empty list (the trivial example).

Proof. Let $x = [1; 2; 3]$

$$\begin{aligned} max(x) &= 3 \\ sum(x) &= 6 \\ max(x) &\leq sum(x) \\ P(x) &\text{ holds} \end{aligned}$$

$\square$

2 Task 2

Proof. Let $x \in \mathcal{L}$
Base Case: $x = \emptyset$

$$\begin{aligned} max(x) &= -\infty \\ sum(x) &= 0 \\ max(x) &\leq sum(x) \\ P(x) &\text{ holds} \end{aligned}$$

Inductive Step: Let $x \in \mathcal{L}, e \in \mathbb{N}$ where e is a new element to be added to x and assume $P(x)$ holds. $len(x)$ is the length of a list x. We want to prove $P(e :: x)$

$$\begin{aligned} len(e :: x) &= len(x) + 1 \\ max(e :: x) &= max(max(x), e) \\ sum(e :: x) &= sum(x) + e \\ P(e :: x) &= \\ max(e :: x) &\leq sum(e :: x) \\ len(e :: x) \cdot max(e :: x) &\leq len(e :: x) \cdot sum(e :: x) \\ max(e :: x) \cdot len(e :: x) &= max(e :: x) \cdot len(x) + max(e :: x) \\ &= (len(x) + 1) \cdot (max(max(x), e)) \end{aligned}$$

$$\begin{aligned} sum(e :: x) \cdot len(e :: x) &= (len(x) + 1) \cdot (sum(x) + e) \\ &= len(x)sum(x) + sum(x) + len(x)e + e \\ (len(x) + 1) \cdot (max(max(x), e)) &\leq len(x)sum(x) + sum(x) + len(x)e + e \end{aligned}$$

$max(max(x), e))$ Will either be $max(x)$ or e, so we prove both.

$$\begin{aligned}
&max(x)\\
len(x) \cdot max(x) + max(x) &\leq len(x) \cdot sum(x) + sum(x) + len(x) \cdot e + e\\
(len(x) + 1)max(x) &\leq (len(x) + 1)sum(x) + len(x) \cdot e + e\\
max(x) &\leq sum(x) + \frac{len(x) \cdot e + e}{len(x) + 1}
\end{aligned}$$

$$\begin{aligned}
&e\\
len(x) \cdot e + e &\leq len(x) \cdot sum(x) + sum(x) + len(x) \cdot e + e\\
0 &\leq len(x) \cdot sum(x) + sum(x)
\end{aligned}$$

□

3 Task 3

Proof. Let $x \in \mathcal{L}$
Base Case: $x = \emptyset$

$$\begin{aligned}
max(x) &= -\infty \\
sum(x) &= 0 \\
max(x) &\leq sum(x) \\
P(x) &\text{ holds}
\end{aligned}$$

Inductive Step: Let $x \in \mathcal{L}, e \in \mathbb{N}$ where e is a new element to be added to x and assume $P(x)$ holds for all lists shorter than $e :: x$. $len(x)$ is the length of a list x. We want to prove $P(e :: x)$

$$\begin{aligned}
len(e :: x) &= len(x) + 1 \\
max(e :: x) &= max(max(x), e) \\
sum(e :: x) &= sum(x) + e \\
P(e :: x)& \\
max(e :: x) &\leq sum(e :: x) \\
len(e :: x) \cdot max(e :: x) &\leq len(e :: x) \cdot sum(e :: x) \\
max(e :: x) \cdot len(e :: x) &= max(e :: x) \cdot len(x) + max(e :: x) \\
&= (len(x) + 1) \cdot (max(max(x), e))
\end{aligned}$$

$$\begin{aligned}
sum(e :: x) \cdot len(e :: x) &= (len(x) + 1) \cdot (sum(x) + e) \\
&= len(x)sum(x) + sum(x) + len(x)e + e \\
(len(x) + 1) \cdot (max(max(x), e)) &\leq len(x)sum(x) + sum(x) + len(x)e + e
\end{aligned}$$

$max(max(x), e))$ Will either be $max(x)$ or e, so we prove both.

$$
\begin{aligned}
&max(x) \\
len(x) \cdot max(x) + max(x) &\leq len(x) \cdot sum(x) + sum(x) + len(x) \cdot e + e \\
(len(x) + 1)max(x) &\leq (len(x) + 1)sum(x) + len(x) \cdot e + e \\
max(x) &\leq sum(x) + \frac{len(x) \cdot e + e}{len(x) + 1}
\end{aligned}
$$

$$
\begin{aligned}
&e \\
len(x) \cdot e + e &\leq len(x) \cdot sum(x) + sum(x) + len(x) \cdot e + e \\
0 &\leq len(x) \cdot sum(x) + sum(x)
\end{aligned}
$$

$\square$

4 Task 4

Proof. Let $x \in \mathcal{L}$
Base Case: $x = \emptyset$

$$max(x) = -\infty$$
$$sum(x) = 0$$
$$max(x) \leq sum(x)$$
$$P(x) \text{ holds}$$

Inductive Step: Let $x \in \mathcal{L}, e \in \mathbb{N}$ where e is a new element to be added to x and assume $P(x)$ holds. We want to prove $P(e :: x)$

$$P(e :: x) =$$
$$max(e :: x) \leq sum(e :: x)$$
$$max(max(x), e) \leq sum(x) + e$$

$max(max(x), e))$ Will either be $max(x)$ or e, so we prove both.

$$max(x)$$
$$max(x) \leq sum(x) + e$$

$$e$$
$$e \leq sum(x) + e$$
$$0 \leq sum(x)$$

□