Solving Mass-Spring-Damper System

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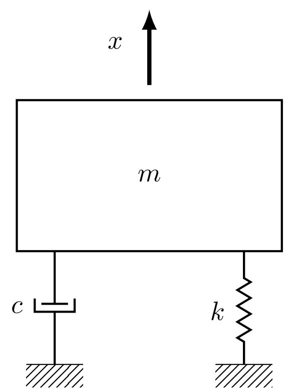
1 Problem Statement

Consider the following Mass-Spring-Damper (MSD) system in one dimension:



depicted at equilibrium point, $x = 0$, where a block of mass, m, is placed on the top of a spring with stiffness coefficient, k, natural length l and shrinkage, L. Additionally there is a damper attached with damping coefficient c. The arrow indicates the positive direction.

2 Deriving the ODE describing the system

We need to develop a differential equation that will describe the displacement of the object at any time t. In order to do that we need to recall Newton's Second Law of Motion: $F = ma = m\ddot{x}$ and we need to consider the forces acting on the block. There are three forces acting on the block:

Gravity, F_g The force of gravity which acts on the object, accounting for the sign of the negative direction, is given by: $F_g = -mg$.

Spring, F_s The force exerted on the object by the spring is described by the Hooke's Law and is given by: $F_s = -k(x - L)$.

Damping, F_d The force exerted by the damper opposes any movement and is given by: $F_d = -c\dot{x}$.

External Forces, $F(t)$ There are no external forces, i.e. $F(t) = 0$.

Accounting for all the above gives us the following expression for the Newton's Second Law:

$$m\ddot{x} = -mg - k(x - L) - c\dot{x} + F(t) \tag{1}$$

after expanding the second term on the right hand side (r.h.s) and setting $F(t)$ to zero we get:

$$m\ddot{x} = -mg - kx + kL - c\dot{x} \tag{2}$$

When the block is at rest in its equilibrium there are only two forces acting on it: F_g and F_s. Since the object is at rest these two forces must be canceling each other out, i.e.:

$$mg = kL \tag{3}$$

Finally, after dividing by m, the ODE describing the system is:

$$\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = 0 \tag{4}$$

3 Reducing the order of the equation

The state space representation of a system replaces the n^{th} order ODE currently describing the system with n first order ODEs, by introducing the following substitutions:

$$S(t) = \begin{cases} S_1 = x \\ S_2 = \dot{x} \end{cases} \tag{5}$$

then:

$$\frac{dS(t)}{dt} = \begin{cases} S_1' = \dot{x} \\ S_2' = \ddot{x} \overset{(4)}{=} -\frac{k}{m}x - \frac{c}{m}\dot{x} \end{cases} \tag{6}$$

after expressing (6) in terms of (5), we get:

$$\frac{dS(t)}{dt} = \begin{cases} S_1' = S_2 \\ S_2' = -\frac{k}{m}S_1 - \frac{c}{m}S_2 \end{cases} \tag{7}$$

which written in matrix form is:

$$\frac{dS(t)}{dt} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix} S(t) \tag{8}$$

where:

$$\mathcal{F}(t, S(t)) = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix} \tag{9}$$

4 Prepare for numerical solution

In order to apply a numerical sheme to solve our system we need to write $\mathcal{F}(t, S(t))$ from (9) as:

```
function [dS] = myMSD(t, S0, m, c, k)
% Input:
%   t - 1x1 scalar - time.
%   S0 - 2x1 column vector - initial displacement and velocity.
%   m - 1x1 scalar - mass of object.
%   c - 1x1 scalar - damping coefficient.
%   k - 1x1 scalar - spring stiffness coefficient.
% Output:
%   dS - 2x1 column vector - displacement and velocity after time t
    .

  dS = [0, 1; -k/m, -c/m] * S0;

end
```

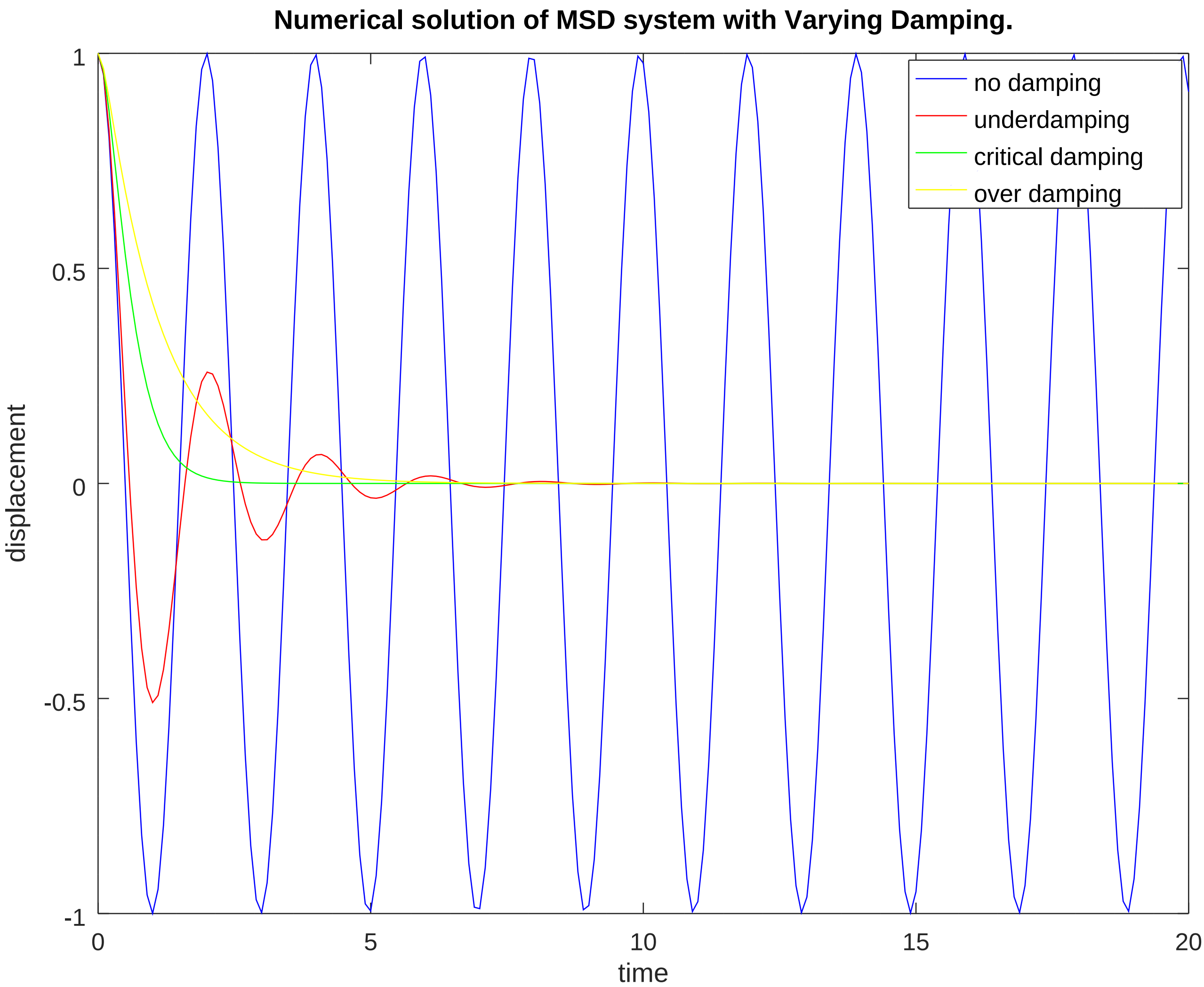
Listing 1: Change of State of the MSD system.

Now let's solve the equation in the cases of: no damping, under damping, critical damping and over damping. To achieve this we use the following code:

```
% Script solving and a plotting the solutions to the Mass-Spring-
    Damper system.

% Define system parameters.
m = 1; k = 10;
t = 0 : 0.1 : 20;
x0 = 1; u0 = 0; S0 = [x0; u0];

% Solve the ODE.
c = 0;                                  % no damping.
[T1, S1] = ode45(@myMSD, t, S0, m, c, k);
c = 2 * sqrt(k * m) - 5;           % under damping.
[T2, S2] = ode45(@myMSD, t, S0, m, c, k);
c = 2 * sqrt(k * m);                % critical damping.
[T3, S3] = ode45(@myMSD, t, S0, m, c, k);
c = 2 * sqrt(k * m) + 5;           % over damping.
[T4, S4] = ode45(@myMSD, t, S0, m, c, k);

% Draw displacement vs time of the MSD system.
plot(T1, S1(:, 1), 'b', T2, S2(:, 1), 'r', T3, S3(:, 1), 'g', T4,
    S4(:, 1), 'y' )
title('Numerical solution of MSD system with Varying Damping.')
xlabel('time')
ylabel('displacement')
legend('no damping', 'under damping', 'critical damping', 'over
    damping')
```

Listing 2: Solving the MSD ODE for different damping cases.

the above code produces the following 2D plot of the displacement versus time for initial conditions: $x_0 = 0$ and $u_0 = 1$:


Numerical solution of MSD system with Varying Damping.
no damping
underdamping
critical damping
over damping
displacement
time
1
0.5
0
-0.5
-1
0
5
10
15
20

Listings