# EEE 3342C: Digital Systems

## Chapter 3: Boolean Algebra (continued)

This chapter in the book includes:

3.1 Multiplying Out and Factoring Expressions
3.2 Exclusive-OR and Equivalence Operations
3.3 The Consensus Theorem
3.4 Algebraic Simplification of Switching Expressions

# Multiplying Out and Factoring Using Theorems

- We saw before how to multiply out
  - Keep multiplying until there are no parentheses left
- We saw before factoring
  - Keep factoring until everything is in the parentheses
- Given an expression in product-of-sums form, the corresponding sum-of-products expression can be obtained by multiplying out, using the two distributive laws:

$$X(Y + Z) = XY + XZ \qquad (3\text{-}1)$$

$$(X + Y)(X + Z) = X + YZ \qquad (3\text{-}2)$$

- In addition, the following theorem is very useful for factoring and multiplying out:

$$(X + Y).(X' + Z) = XZ + X'Y \qquad (3\text{-}3)$$

- Proof:

If X=0, we have: $(0+Y).(1+Z) = (0.Z) + (1.Y) \rightarrow Y.1 = 1.Y \rightarrow Y = Y$

If X=1, we have: $(1+Y).(0+Z) = 1.Z + 0.Y \rightarrow 1.Z = 1.Z \rightarrow Z = Z$

- The opposite is:

$AB + A'C = (A + C).(A' + B)$

$(A+B).(A'+C) = AC + A'B$

# Example

- Multiply out this expression

  $(Q + AB').(C'D + Q')$

- You can multiply out all the terms
- But if you use the theorem in the box above, you end up with a simplified form

  It is equal to: $QC'D + Q'AB'$

# Example 2

*Rule #1* $(X+Y)(X+Z) = X + YZ$

*Rule #2* $(X+Y)(X'+Z) = XZ + X'Y$

- **Multiply out** this expression
- We are going to use the theorems in the boxes

$$(A+B+C').(A+B+D).(A+B+E).(A+D'+E).(A'+C)$$

Apply rule #1
X=A+B, Y=C', Z=D

$$= (A+B+C'D).(A+B+E).(A+D'+E).(A'+C)$$

Apply rule #1
X=A+B, Y=C'D, Z=E

$$= (A+B+C'DE).(A+D'+E).(A'+C)$$

Apply rule #2
X=A, Y=D'+E, Z=C

$$= (A+B+C'DE).[AC + A'(D'+E)]$$

$$= (A+B+C'DE).(AC+A'D'+A'E)$$

$$= AC + ABC + A'BD' + A'BE + A'C'DE$$

ABC disappears
$AC+ABC = AC.(1+B) = AC$

$$=AC + A'BD' + A'BE + A'C'DE$$

Note : if we were to multiply out by brute force, we would generate 162 terms, and 158 of these terms would then have to be eliminated to simplify the expression

# Example 2

*Rule #1* $(X+Y)(X'+Z) = XZ + X'Y$

*Rule #2* $X + YZ = (X+Y)(X+Z)$

- **Factor out** this expression

$AC + A'BD' + A'BE + A'C'DE$

- We are going to use the theorem in the box

$= \underline{A}\underline{C} + \underline{A}'(\underline{BD' + BE + C'DE})$

Apply rule #1
$X=A$, $Y=BD'+BE+C'DE$, $Z=C$

$= (A + \underline{BD' + BE} + C'DE)\ (A' + C)$

Apply rule #2
$X=A+C'DE$, $Y=B$, $Z=D'+E$

$= [\underline{A + C'DE} + \underline{B}(\underline{D'+E})]\ (A'+C)$

C'DE disappears
$X+XY=X$

$= (A+C'DE+B)(A+\underline{C'DE}+D'+\underline{E})(A'+C)$

Apply rule #2'
$X=A+B$, $Y=C'$, $Z=D$, $W=E$

$= (A+C'DE+B)(A+D'+E)(A'+C)$

$= (A+B+C')(A+B+D)(A+B+E)(A+D'+E)(A'+C)$

*This is similar to rule#2 … Rule #2'* $X + YZW = (X+Y)(X+Z)(X+W)$

# Exclusive OR (XOR) and Equivalence

Also written,
A XOR B

Also called, the equivalence operation,
$A \equiv B$

| A B | A+B | $A \oplus B$ | A xnor B |
|---|---|---|---|
| 0 0 | 0 | 0 | 1 |
| 0 1 | 1 | 1 | 0 |
| 1 0 | 1 | 1 | 0 |
| 1 1 | 1 | 0 | 1 |

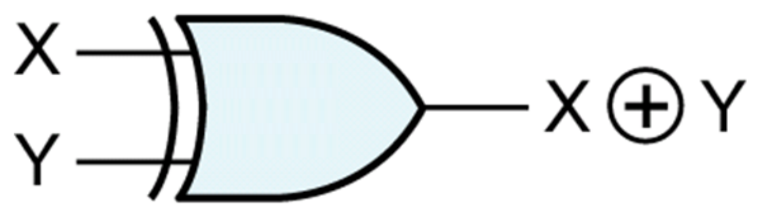
It is called equivalence operation because it is equal to 1 when A=B

$A \oplus B = A'B + AB'$ it's equal to 1 at terms 01 (A'B) and 10 (AB')

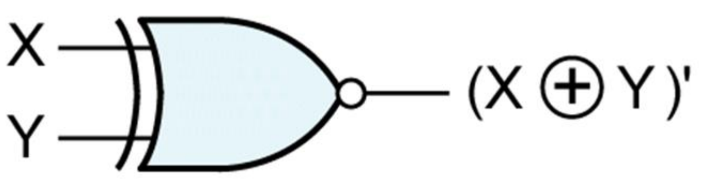
$A \equiv B = AB + A'B'$ it's equal to 1 at terms 00 (A'B') and 11 (AB)

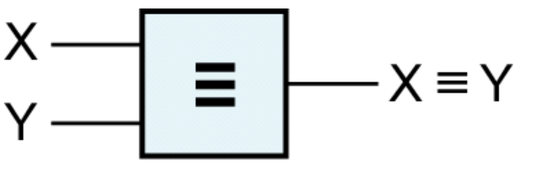
# Exclusive OR (XOR)

- This is the symbol for XOR




- Exclusive NOR and equivalence are the same thing



# XOR Theorems

$X \oplus 0 = X$

$X \oplus 1 = X'$

$X \oplus X = 0$

$X \oplus X' = 1$

$X \oplus Y = Y \oplus X$

$(X \oplus Y) \oplus Z = X \oplus (Y \oplus Z) = X \oplus Y \oplus Z$

$X(Y \oplus Z) = XY \oplus XZ$

| $X$ | $Y$ | $X \oplus Y$ |
|---|---|---|
| 0 | 0 | 0 |
| 0 | 1 | 1 |
| 1 | 0 | 1 |
| 1 | 1 | 0 |

$(X \oplus Y)' = X \oplus Y' = X' \oplus Y = \underbrace{XY + X'Y'}_{\text{This is XNOR or equivalence}}$

# Equivalence Theorems

$(0 \equiv 0) = 1$

$(0 \equiv 1) = 0$

$(1 \equiv 0) = 0$

$(1 \equiv 1) = 1$

| $X$ | $Y$ | $X \equiv Y$ |
|---|---|---|
| 0 | 0 | 1 |
| 0 | 1 | 0 |
| 1 | 0 | 0 |
| 1 | 1 | 1 |

$(X \equiv Y) = X'Y' + XY$

# Example

Show by Boolean algebra that XOR and equivalence are the complement of each other

$(X \oplus Y)' = (X'Y + XY')'$

$= (X+Y').(X'+Y)$

$= XY + X'Y'$

$= (X \equiv Y)$

Apply DeMorgan's
$(A+B)' = A'B'$, $(AB)' = A'+B'$

Apply $(A+B)(A'+C) = AC + A'B$
Here, $A=X$, $B=Y'$, $C=Y$

# Example 2

- Simplify this expression so that it is written using AND, OR, invert

$X \equiv Y = X'Y' + XY$

$X \oplus Y = X'Y + XY'$

$F = (A'B \equiv C) + (B \oplus AC')$

$F = [(A'B)C + (A'B)'C'] + [B'(AC') + B(AC')']$

$= A'BC + (A+B')C' + AB'C' + B(A'+C)$

$= A'BC + AC'+B'C' + AB'C' + BA'+BC$

Apply DeMorgan's

$(X+Y)' = X'Y'$, $(XY)' = X'+Y'$

$= B(A'C+A'+C) + C'(A+B'+AB')$

$= B(A'+C) + C'(A+B')$

$A'C+C = C(A'+1) = C.1 = C$

$A+AB' = A(1+B') = A.1 = A$

# Example 3

Simplify this expression

$F = A' \oplus B \oplus C$

$\boxed{X \oplus Y = X'Y + XY'}$

$F = [(A')'B + A'B'] \oplus C$

$= (AB + A'B') \oplus C$

$= (AB + A'B')'C + (AB + A'B')C'$

$= (A'B + AB')C + (AB + A'B')C'$

$= A'BC + AB'C + ABC' + A'B'C'$

AB + A'B' is the complement of A'B + AB'

The first one is 'equivalence' or XNOR, the second one is XOR

# Consensus Theorem

$$XY + X'Z + YZ = XY + X'Z \qquad (3\text{-}20)$$

Proof:

$$\begin{aligned} XY + X'Z + YZ &= XY + X'Z + (X+X')YZ \\ &= XY + X'Z + XYZ + X'YZ \\ &= XY(1+Z) + X'Z(1+Y) \\ &= XY + X'Z \end{aligned}$$

**Dual Form:**

$$(X + Y)(X' + Z)(Y + Z) = (X + Y)(X' + Z) \qquad (3\text{-}21)$$

# Example

$XY + X'Z + YZ = XY + X'Z$

Simplify this expression using the consensus theorem

$F = a'b' + ac + bc' + b'c + ab$

*consider:* $a'b' + ac + b'c = a'b' + ac$

$F = a'b' + ac + bc' + ab$

*consider:* $ac + bc' + ab = ac + bc'$

$F = a'b' + ac + bc'$

# Last Part from Exercises

- Using theorems, factor the following expression

$$\boxed{X + YZ = (X + Y)(X + Z)}$$

$$G = \overbrace{A + BCD}^{X} + \overbrace{D}^{Y} \overbrace{(E + F)}^{Z}$$

$$= (A + BCD + D)\,(A + BCD + E + F)$$

$$= [A + D(BC + 1)]\,(A + BCD + E + F)$$

$$= \underbrace{(A + D)}_{W = A + D}\,(A + E + F + B)\,(A + E + F + C)\,\underbrace{(A + E + F + D)}_{V = E + F}$$

$$W\,(W + V) = W + VW = W(1 + V) = W$$

$$G = (A + D)\,(A + E + F + B)\,(A + E + F + C)$$

This is the last part of the exercises on Chapter 2 & 3.

# Simplifying Boolean Expressions

1. ***Combining terms.*** Use the theorem $XY + XY' = X$ to combine two terms.

   $abc'd' + abcd' = abd'$   $[X = abd', Y = c]$   (3-24)

2. ***Eliminating terms.*** Use the theorem $X + XY = X$ to eliminate redundant terms if possible; then try to apply the consensus theorem ($XY + X'Z + YZ = XY + X'Z$) to eliminate any consensus terms.

   $a'b + a'bc = a'b$   $[X = a'b]$

   $a'bc' + bcd + a'bd = a'bc' + bcd$   $[X = c, Y = bd, Z = a'b]$   (3-24)

***Section 3.4 (p. 68-69)***

# Simplifying Boolean Expressions

**3.** ***Eliminating literals.*** Use the theorem $X + X'Y = X + Y$ to eliminate redundant literals. Simple factoring may be necessary before the theorem is applied.

$$
\begin{aligned}
A'B + A'B'C'D' + ABCD' &= A'(B + B'C'D') + ABCD' \\
&= A'(B + C'D') + ABCD' \\
&= B(A' + ACD') + A'C'D' \\
&= B(A' + CD') + A'C'D' \\
&= A'B + BCD' + A'C'D' \qquad (3\text{-}26)
\end{aligned}
$$

# Simplifying Boolean Expressions

**4. *Adding redundant terms*** *. Chose to combine or eliminate other terms , ex. Add YZ to X'Y+XZ or XY to X*

$WX + XY + X'Z' + WY'Z'$ *(add WZ' by consensus theorem)*

$= WX + XY + X'Z' + WY'Z' + WZ'$ *(eliminate WY'Z')*

$= WX + XY + X'Z' + WZ'$ *(eliminate WZ')*

$= WX + XY + X'Z'$ *(3-27)*

$$XY + X'Z + YZ = XY + X'Z$$

# Example (consensus Theorem)

- Simplify the following logic function using the consensus theorem.

$$F = \cancel{ABCD} + B'CDE + A'B' + \underline{BCE'} + \underline{ACDE}$$

$$F = \cancel{B'CDE} + \underline{A'B'} + BCE' + \underline{ACDE}$$

$$F = A'B' + BCE' + ACDE$$

**- Comprehensive example illustrates use of all four methods**

| | |
|---|---|
| *Combining Terms* | $XY + XY' = X$ |
| *Eliminating Terms* | $X + XY = X$ |
| *Eliminating Literals* | $X + X'Y = X + Y$ |

$$A'B'C'D' + A'BC'D' + A'BD + \cancel{A'BC'D} + ABCD + ACD' + B'CD'$$

$$= A'C'D' + A'BD + ABCD + ACD' + B'CD'$$

$$= A'C'D' + BD(A' + AC) + ACD' + B'CD'$$

$$= A'C'D' + A'BD + BCD + ACD' + B'CD' + ABC$$ *add consensus(ABC)*

consensus ACD'

$$= A'C'D' + A'BD + \cancel{BCD} + \cancel{ACD'} + B'CD' + ABC$$

consensus BCD

$$= A'C'D' + A'BD + B'CD' + ABC$$