Department of Electrical Engineering
and Computer Science

University of Central Florida

EEE 3342: Digital Systems

Chapter 2: Boolean Algebra

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Boolean Algebra

- It is algebra on binary numbers
- Invented by mathematician and philosopher George Boole in 1847
- Claude Shannon first applied Boolean algebra to circuits in 1939

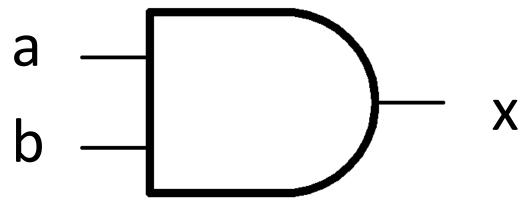
- The variables (like X, Y) can be either 0 or 1

Operation: AND

- x = a AND b

  x is equal to 1 if a =1 and b = 1

  X is equal to 0 otherwise
- Also written as: x = a.b or x=ab



This is an AND gate

Below is the truth table for x = a.b

The truth table lists all the possible input and the corresponding output

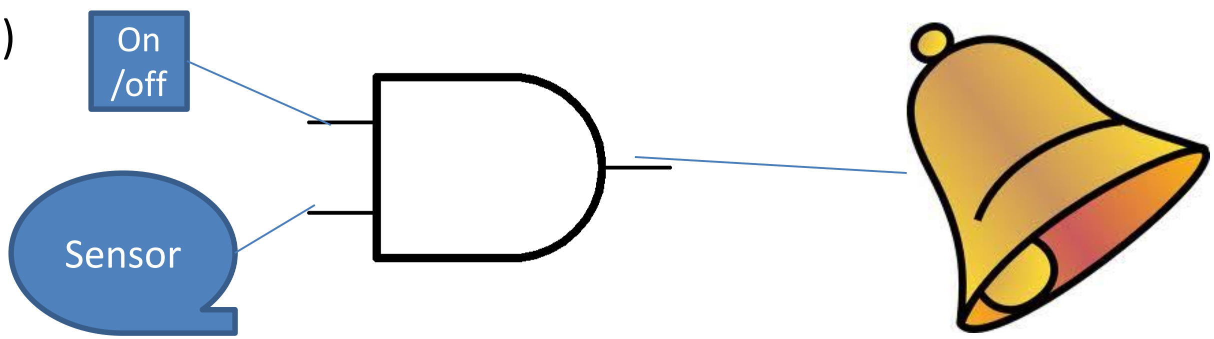
a	b	x
0	0	0
0	1	0
1	0	0
1	1	1

Example of Using AND Gate

- Home alarm system

Alarm is armed (ready mode)

There is someone in the home



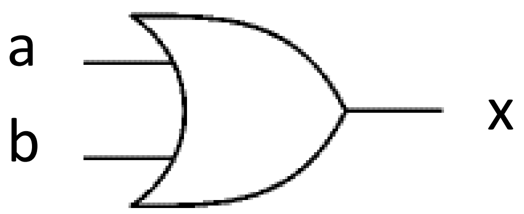
0 0	0	Alarm is off, no one in the home → Bell silent
0 1	0	Alarm is off, intruder in the home → Bell silent
1 0	0	Alarm is on, no one in the home → Bell silent
1 1	1	Alarm is on, intruder in the home → Bell rings

Basic Operation: OR

- x = a OR b

  x is equal to 1 if a =1 or b = 1

  X is equal to 0 otherwise
- Also written as: x = a + b



This is an OR gate

Below is the truth table for
x = a + b

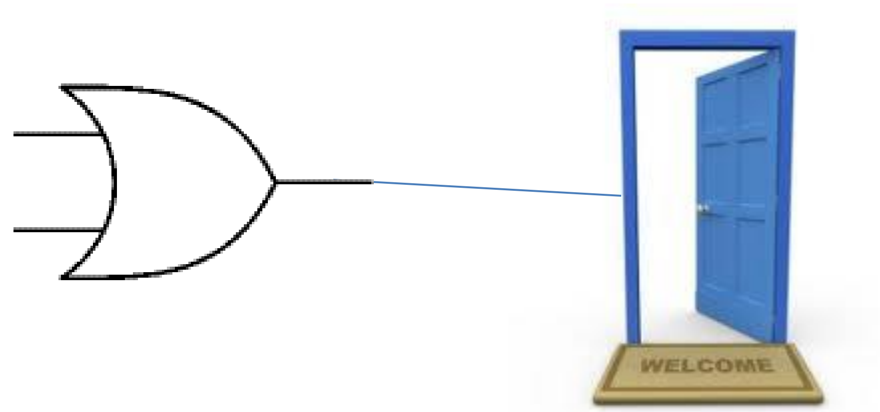
a	b	x
0	0	0
0	1	1
1	0	1
1	1	1

Example of Using OR Gate

- Building door

Apartment 1 opens door

Apartment 2 opens door



#1	#2	
0	0	0
0	1	1
1	0	1
1	1	1

Apt1 doesn’t open door, Apt 2 doesn’t open door → Door is closed

Apt 1 doesn’t open door, Apt 2 opens door → Door is open

Apt 1 opens door, Apt 2 doesn’t open door → Door is open

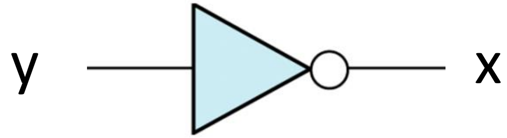
Apt 1 and Apt 2 open door → Door is open

Basic Operation: Inverse

- Also called complement
- $x = y'$
- Also written as $x = \overline{y}$

  x is equal to 1 if y = 0

  X is equal to 0 if y = 1
- x is the complement or inverse of y



This is an inverter

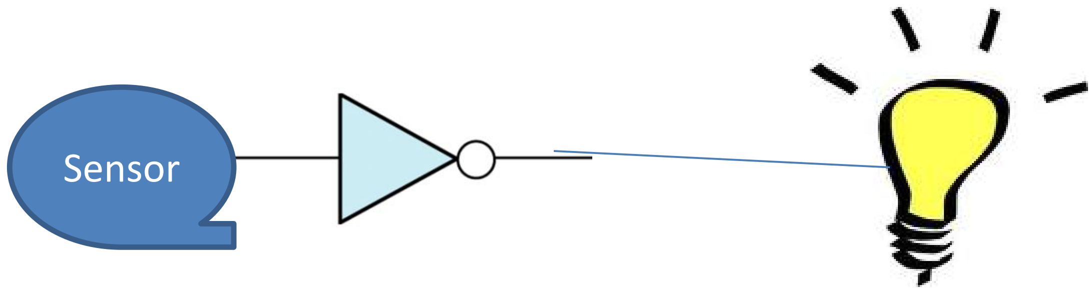
Below is the truth table for $x = y'$

y	x
0	1
1	0

Example of Using an Inverter

- Automatic light switch

Light sensor
Gives 0 when it's dark
Gives 1 when there's light



y	x
0	1
1	0

It's dark → The light goes on

There is light (it's daytime) → The light is off

Let’s Recap

AND Operation

0 . 0 = 0
0 . 1 = 0
1 . 0 = 0
1 . 1 = 1

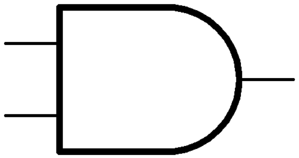
OR Operation

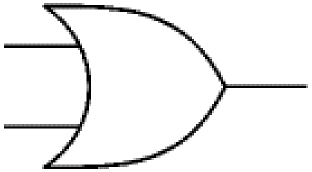
0 + 0 = 0
0 + 1 = 1
1 + 0 = 1
1 + 1 = 1

Inverse Operation

0’ = 1
1’ = 0

Pronounced:
(zero not equals 1)
(inverse of zero is 1)
(complement of zero is 1)





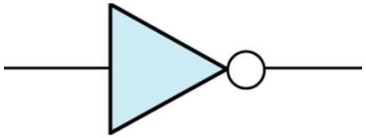
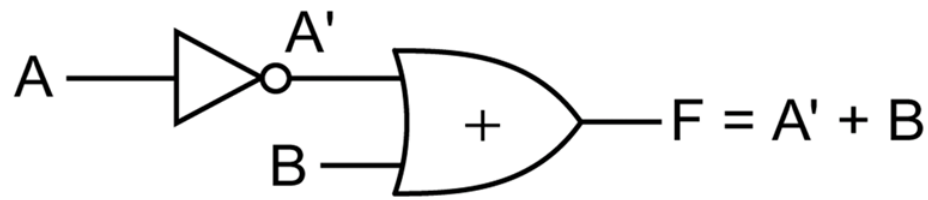


Diagram figures

Boolean Expression

- It is made of Boolean operations (and, or, inverse)
- F = A’ + B



- Write the truth table for the function F

  (All possible inputs and corresponding outputs)

Remember:
2 inputs (n=2 bits)

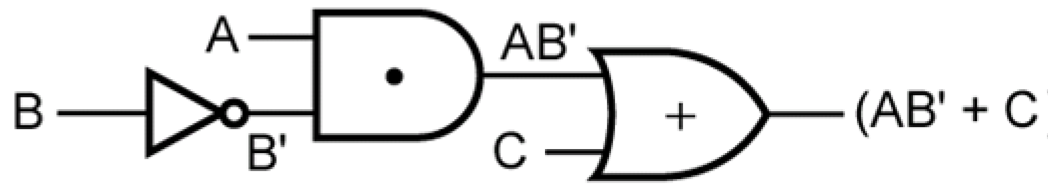
2^n possible input combination

From 0 to 2^n-1
From 0 to 3 in this case

A	B	A'	$F = A' + B$
0	0	1	1
0	1	1	1
1	0	0	0
1	1	0	1

Boolean Expressions and Truth Tables

- F = AB’ + C
- Draw the diagram



- Write the truth table of F

A B C	B’	AB’	AB’+C
0 0 0	1	0	0
0 0 1	1	0	1
0 1 0	0	0	0
0 1 1	0	0	1
1 0 0	1	1	1
1 0 1	1	1	1
1 1 0	0	0	0
1 1 1	0	0	1

Two Functions can be Equivalent

- We have two functions

  F = AB’ + C

  G = (A + C) . (B’ + C)

- Are these two functions equivalent?
  - Write the truth table of these 2 functions
  - Look at the output
  - They are equivalent if the output is the same for all input combinations

A B C	B’	AB’	AB’+C	A+C	B’+C	(A+C)(B’+C)
0 0 0	1	0	0	0	1	0
0 0 1	1	0	1	1	1	1
0 1 0	0	0	0	0	0	0
0 1 1	0	0	1	1	1	1
1 0 0	1	1	1	1	1	1
1 0 1	1	1	1	1	1	1
1 1 0	0	0	0	1	0	0
1 1 1	0	0	1	1	1	1

Yes, F and G are equivalent . They have the same output for any input.

Theorems

$A + 0 = A$ $\quad$ $A \cdot 1 = A$

$A + 1 = 1$ $\quad$ $A \cdot 0 = 0$

Idempotent laws:

$A + A = A$ $\quad$ $A \cdot A = A$

Involution law:

$(A')' = A$

Laws of complementarity

$A + A' = 1$ $\quad$ $A \cdot A' = 0$

You can prove these theorems by doing a truth table.
Let's show that: $A + A' = 1$
If $A = 0$, then we have: $0 + 0' = 0 + 1 = 1$ correct
If $A = 1$, then we have: $1 + 1' = 1 + 0 = 1$ correct
Then the theorem is correct

A can be substituted by an expression:

A: XYZ
We have: $A \cdot 0 = 0$
So, $XYZ \cdot 0 = 0$
(from $A \cdot 0 = 0$)

And

$XYZ + (XYZ)' = 1$
(from $A + A' = 1$)

Commutative, Associative and Distributive Laws

Commutative Law

- X.Y = Y.X   X + Y = Y + X

Associative Law

- (X + Y) + Z = X + (Y + Z) = X + Y + Z
- (XY).Z = X.(YZ) = XYZ
  - Prove this one
  - Do a truth table

X	*Y*	*Z*	*XY*	*YZ*	*(XY)Z*	*X(YZ)*
0	0	0	0	0	0	0
0	0	1	0	0	0	0
0	1	0	0	0	0	0
0	1	1	0	1	0	0
1	0	0	0	0	0	0
1	0	1	0	0	0	0
1	1	0	1	0	0	0
1	1	1	1	1	1	1

Distributive law

- $X(Y + Z) = XY + XZ$

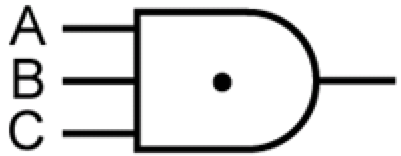
- $X + YZ = (X + Y)(X + Z)$
  - Let's prove this
  - We can do a truth table, but we can also do algebra

$$
\begin{aligned}
(X + Y)(X + Z) &= XX + XZ + XY + YZ \\
&= X + XZ + XY + YZ \\
&= X(1 + Z + Y) + YZ \\
&= X(1) + YZ \\
&= X + YZ
\end{aligned}
$$

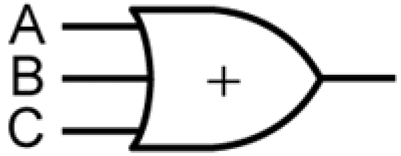
Correct

More than Two Inputs

- AND
  - The result is 1 iff (if and only if), A=1, B=1, C=1
  - Otherwise it’s 0



- OR
  - The results is 1 if A=1, or B=1, or C=1



Simplification Theorems

- To simplify an expression means to make it smaller

$$XY + XY' = X \qquad (X + Y)(X + Y') = X$$

$$X + XY = X \qquad X(X + Y) = X$$

$$(X + Y')Y = XY \qquad XY' + Y = X + Y$$

- We can prove them by truth table or by algebra

$XY + XY' = X (Y + Y') = X (1) = X$

$X + XY = X (1 + Y) = X (1) = X$

$(X + Y')Y = XY + YY' = XY + 0 = XY$

$$\underbrace{XY' + Y = (Y + X) (Y + Y')}_{\text{By distributive law (a previous slide)}} = (X + Y) . 1 = X + Y$$

Example 1

- Simplify

  $Z = A'BC + A'$

  $Z = A'(BC + 1)$

  $= A'(1)$

  $= A'$

Example 2

- Simplify this expression

$$Z = [\underbrace{A + B'C} + \underbrace{D + EF}] \,.\, [\underbrace{A + B'C} + \underbrace{(D + EF)'}]$$

Observe the repetition of terms

You can write this out in a long expression, but if you observe the repetition of terms, you can do substitution

Let $X = A + B'C$ and $Y = D + EF$

Then, $Z = (X + Y) \,.\, (X + Y')$

A theorem in a previous slide, $Z = X$

If you don't remember the theorem

$$(X+Y)(X+Y') = XX + XY' + XY + YY'$$
$$= X\,(1 + Y' + Y)$$
$$= X$$

Then, $Z = A + B'C$

Example 3

- Simplify this expression

$$Z = \underbrace{(AB + C)} \,.\, (B'D + C'E') + \underbrace{(AB + C)}'$$

Observe the repetition of terms

You can write this out in a long expression, but if you observe the repetition of terms, you can do substitution

Let $X = AB + C$ and $Y = B'D + C'E'$

Then, $Z = XY + X'$

A theorem in a previous slide, $Z = X' + Y$

If you don't remember the theorem

$$\begin{aligned} XY + X' &= (X + X').(Y+X') \\ &= 1 \,.\, (Y + X') \\ &= X' + Y \end{aligned}$$

Then, $Z = (AB + C)' + B'D + C'E'$

Multiplying Out a Boolean Expression

- To fully multiply out an expression, keep multiplying until there are no parentheses
- It's ok to simplify
- Multiply out: (A + BC) . (A + D + E)

AA + AD + AE + ABC + BCD + BCE

= A + AD + AE + ABC + BCD + BCE

= A (1 + D + E + BC) + BCD + BCE

= A + BCD + BCE

Sum-of-Products

- If you multiply out an expression, it becomes a sum-of-products
- Sum-of-products look like this:

  AB’ + CD’E + AC’E’

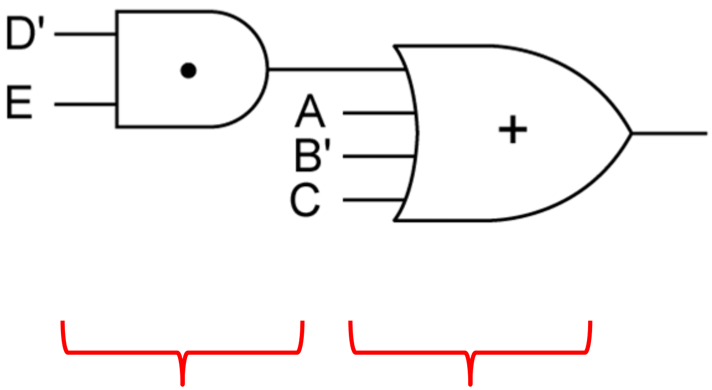
  or ABC’ + DEFG + H

  or A + B’ + C + D’E , but not (A+B) CD +EF

Look at these, you can’t multiply them anymore!

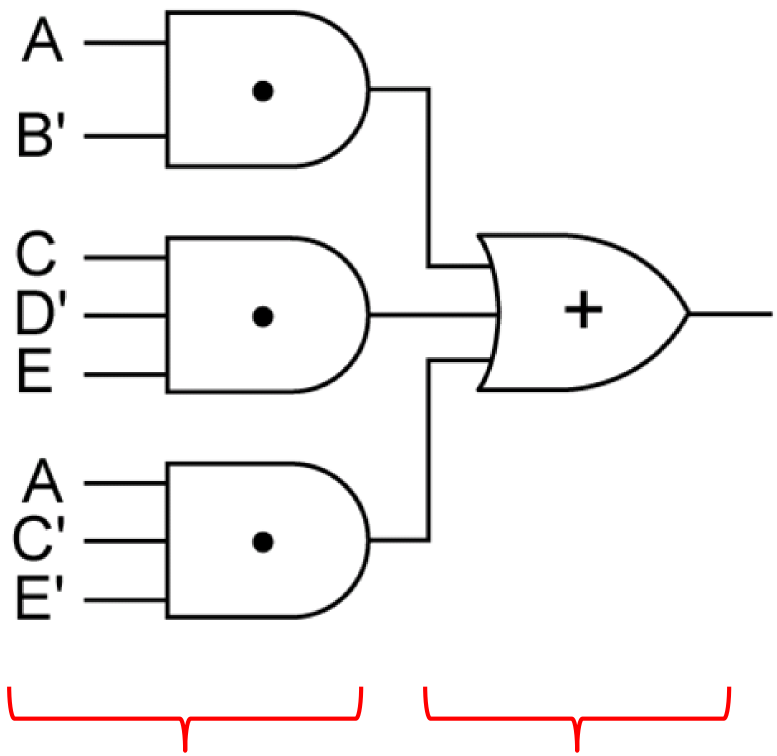
The digital circuit diagram of sum-of-products is 2-level

$A + B' + C + D'E$



1 level   1 level

$AB' + CD'E + AC'E'$



1 level   1 level

Factoring a Boolean Expression

- You can keep factoring a Boolean expression until everything is inside the parentheses
- Multiply out: AB’ + C’D

You need to use this rule:
X + YZ = (X + Y) . (X + Z)

AB’ + C’D

= (AB’ + C’).(AB’ + D)

Step-by-step is shown in the box below

= (C’ + A).(C’ + B’).(D + A).(D + B’)

Let X = AB’
We have: X + C’D
Apply the rule, we have: (X + C’).(X + D)
Substitute back in X, we have: (AB’ + C’) . (AB’ + D)

Product of Sums

- If you keep factoring out an expression, it becomes a product of sums
- The expression in the previous slide became a product of sums
- Product of sums look like this

  $(A + B').(C' + D' + E).(A + C' + E')$

  $(A + B).(C + D + E).F \quad \leftrightarrow \quad (A + B).(C + D + E).(F)$

  $AB'C(D' + E) \quad \leftrightarrow \quad (A).(B').(C).(D' + E)$

  But not $(A+B)(C+D)+EF$

Example

Factor out: $C'D + C'E' + G'H$

Start by factoring out regularly,
Then use the rule in the box below

$= C'(D + E') + \underbrace{G'H}$

Apply the rule in the box below
Consider this equal to X
And think $(X + C').(X + D + E')$

$= (G'H + C').(G'H + \underbrace{D + E'})$

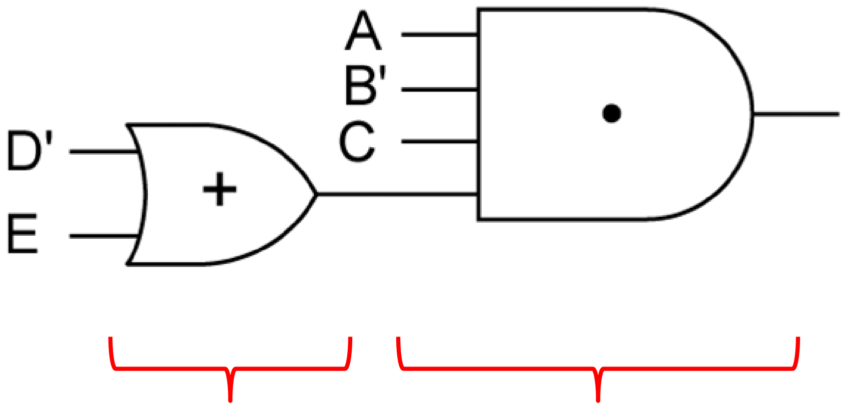
Consider this equal to X
And think $(X + G').(X + H)$

$= (C' + G').(C' + H).(D + E' + G').(D + E' + H)$

You need to use this rule:
$X + YZ = (X + Y) . (X + Z)$

The digital circuit diagram of product-of-sums is 2-level

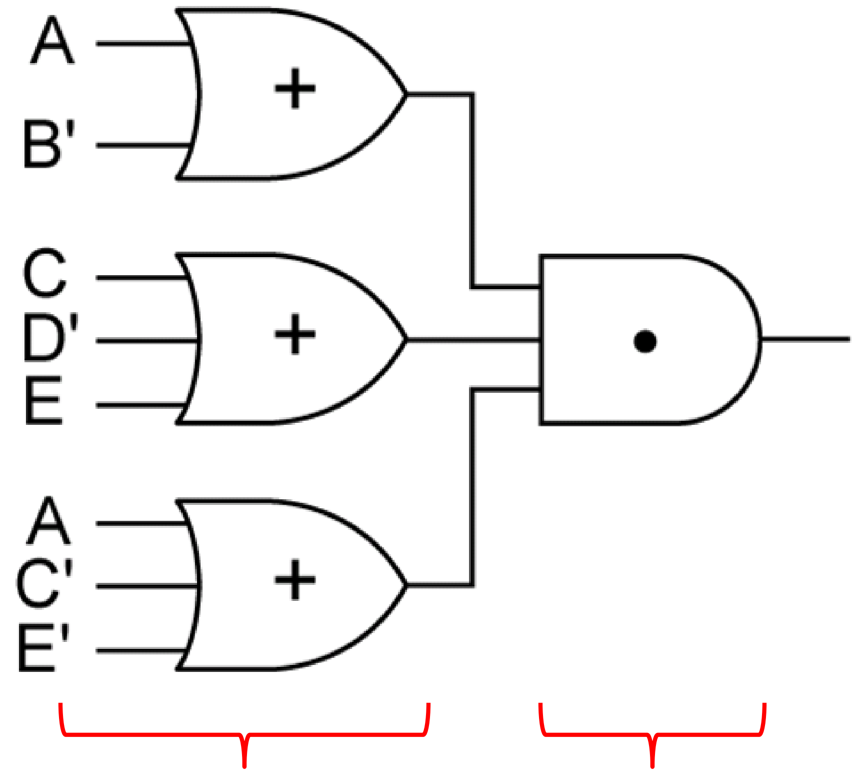
$AB'C(D' + E)$



1 level

1 level

$(A + B')(C + D' + E)(A + C' + E')$



1 level

1 level

DeMorgan’s Laws

<u>First law:</u> (A + B)’ = A’B’

OR becomes AND

<u>Second law:</u> (AB)’ = A’ + B’

AND becomes OR

Prove DeMorgan’s Laws. Do a truth table

A	B	A’	B’	A+B	(A+B)’	A’B’	AB	(AB)’	A’+B’
0	0	1	1	0	1	1	0	1	1
0	1	1	0	1	0	0	0	1	1
1	0	0	1	1	0	0	0	1	1
1	1	0	0	1	0	0	1	0	0

First law

Second law

DeMorgan's Laws General Form

$(A+B+C+D+E+F+\ldots)' = A'B'C'D'E'F'\ldots$

$(ABCDEF\ldots)' = A'+B'+C'+D'+E'+F'+\ldots$

DeMorgan's Laws short form (prev. slide)

$(A + B)' = A'B'$

$(AB)' = A' + B'$

How is DeMorgan's laws useful?

They allow us to find the complement of an expression

~Remember, the complement is the inverse~

Example

DeMorgan's Laws

$(X + Y)' = X'Y'$

$(XY)' = X' + Y'$

Find the complement of this expression: $(A' + B).C'$

It is: $[(A' + B).C']'$

Apply: $(XY)' = X' + Y'$

$= (A' + B)' + (C')'$

$= (A' + B)' + C$

Apply: $(X + Y)' = X'Y'$

$= (A')'.B' + C$

$=AB' + C$

Example 2

Find the complement of: (AB’ + C).D’ + E

(X + Y)’ = X’Y’

(XY)’ = X’ + Y’

[(AB’ + C).D’ + E]’

= [(AB’ + C).D’]’ . E’

= [(AB’ + C)’ + D] . E’

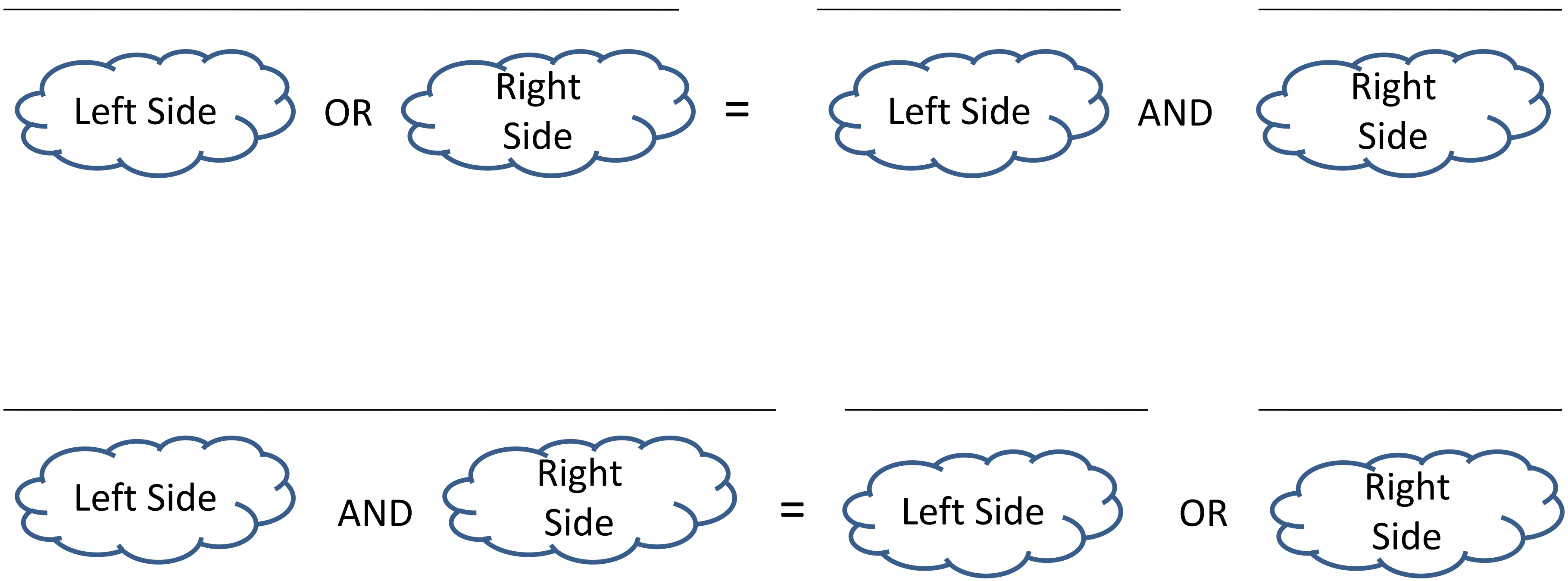
= [(A’+B).C’ + D] . E’

= (A’C’ + BC’ + D) . E’

= A’C’E’ + BC’E’ + DE’

Remember, with DeMorgan’s,
the OR becomes AND
the AND becomes OR

Visualizing DeMorgan's Laws



The bar on top means invert

$\overline{A}$ is the same as A'

We have, $F = A'B + AB'$. Find the complement of F.

$$
\begin{aligned}
F' &= (A'B + AB')' \\
&= (A'B)' \,.\, (AB')' \\
&= (A + B') \,.\, (A' + B) \\
&= AA' + AB + A'B' + BB' \\
&= 0 + AB + A'B' + 0 \\
&= AB + A'B'
\end{aligned}
$$

Verify this result with a truth table

Duality

- We have a Boolean expression
- To obtain its dual, replace AND with OR and OR with AND
- Replace 1 with 0 and 0 with 1
- The variables stay the same
- The invert signs stay the same

$(XYZ)^D = X + Y + Z$

$(X + Y + Z)^D = XYZ$

Duality

- When you have a big expression, start from the outwards and go inwards

$(AB' + C)^D$

$= (\ldots) \cdot C$

$= (A + B') \cdot C$

Find the dual of: $(AB' + C).D' + E$

Start from outwards and go inwards

$[(AB' + C).D' + E]^D$
$= (\ldots\ldots\ldots\ldots\ldots\ldots) . E$
$= [(\ldots\ldots\ldots\ldots) + D'] . E$
$= [(\ldots\ldots) . C + D'] . E$
$= [(A + B').C + D'] . E$

This is the dual. We can multiply it out in the two lines below

$= (AC + B'C + D').E$
$= ACE + B'CE + D'E$

You may also write all at once

Another Way to Find the Dual

- Complement the whole expression (using DeMorgan's)
- Then, complement every variable

We want to find the dual of: [(AB' + C).D' + E]'

In a previous slide, we found its complement by using DeMorgan. We got: A'C'E' + BC'E' + DE'

Now let's complement every variable, we get:

ACE + B'CE + D'E

Correct answer. In the previous slide, we got this answer.

Let’s Summarize Duality

- Find the dual of: AB’ + C using the two methods

- <u>Method 1:</u> Replace AND with OR. Also, replace OR with AND. Start from outward to inward.

$$(AB' + C)^D$$
$$= (\ldots\ldots) \,.\, C$$
$$= \boxed{(A + B') \,.\, C}$$

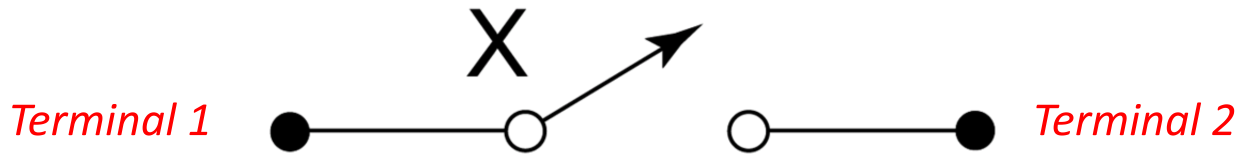
- <u>Method 2:</u> Find the complement. Then, invert every variable

Complement: $(AB' + C)' = (AB')' \,.\, C' = (A'+B) \,.\, C'$

Invert every variable: $\boxed{(A + B').\, C}$

Switches

- This is a switch
- If X=0, the switch is open
  - The switch in the figure is open; the two terminals are not connected



- If X=1, the switch is closed
  - It means the two terminals are connected

- What can we do with the switches?
  - We can write the Boolean expression with switches

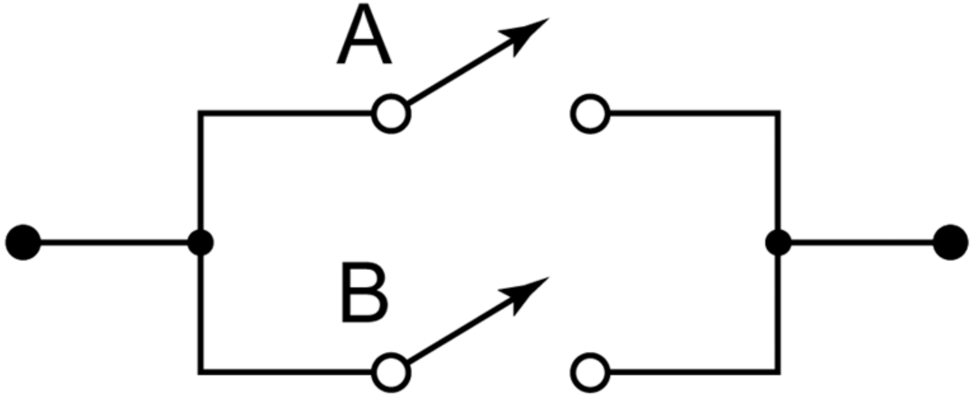
Example

- F = A.B
- Draw a circuit with switches that implements F

- Remember, when A=1, B=1, the switches will close
- The two terminals are connected

Example 2

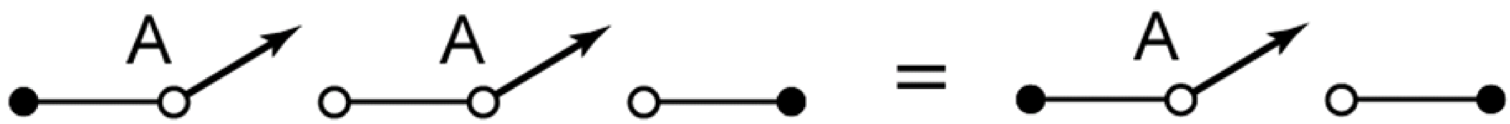
- Draw the switch circuit for F = A + B



- If A=1 or B=1 or A=B=1, the two terminals will be connected

Simplification

- We learned that: A.A = A
- The same applied in the switch circuit

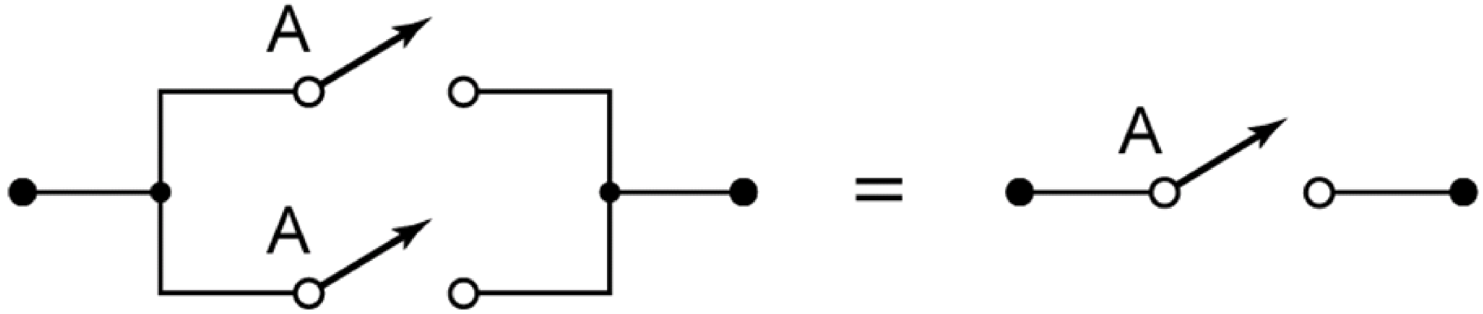


- On the left side:
  - If A=1, the two terminals will connect
  - If A=0, the two terminals will not be connected

- So, we can replace the left side with one switch controlled by A

Simplification

- We learned that A + A = A
- The same applied when we draw the switch circuit

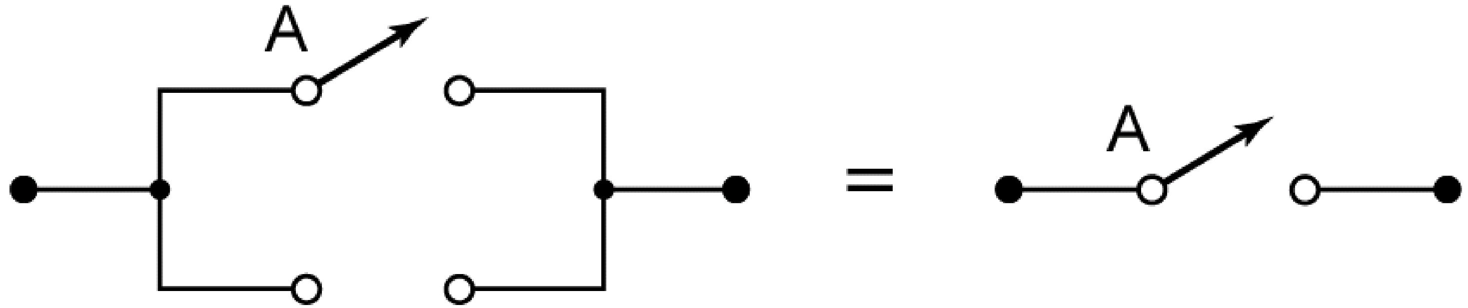


- On the left side:
  - The terminals will connect if A=1
  - They will disconnect if A=0

- So, it is equivalent to one switch that's controlled by A

We learned that: $A + 0 = A$

This is shown in the switch circuit

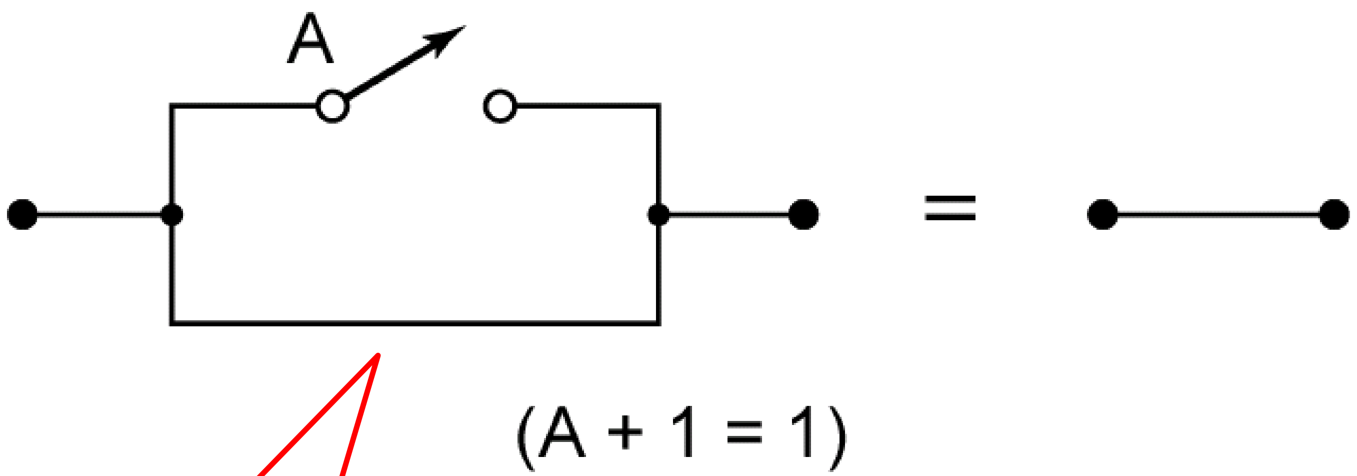


$(A + 0 = A)$

This switch is always open.
So it is equivalent to zero.

We learned that: A + 1 = 1

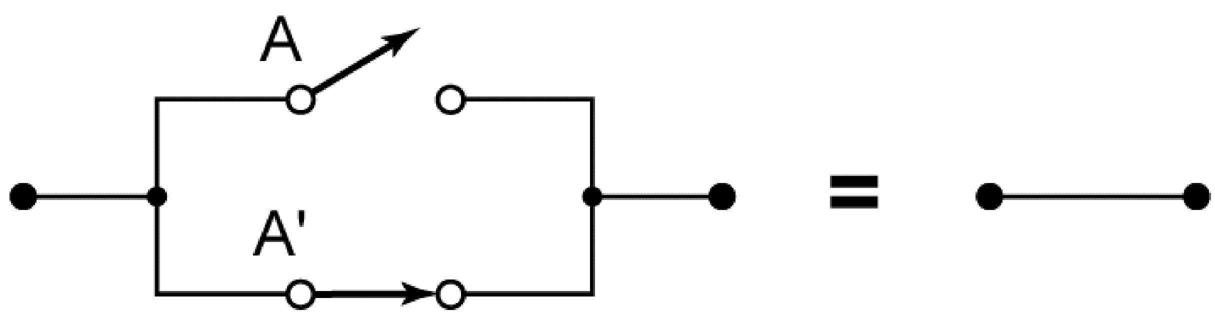
This is shown in the switch circuit



The bottom part is equivalent to 1 since it’s always closed.

$A + A' = 1$

This is represented in the switch circuit below.



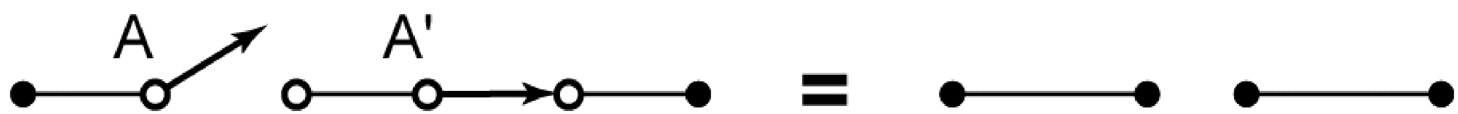
$(A + A' = 1)$

$A \cdot A' = 0$

This is shown in the circuit below.

If one switch closes, the other opens.

This circuit is always open.



(A · A' = 0)

We learned:

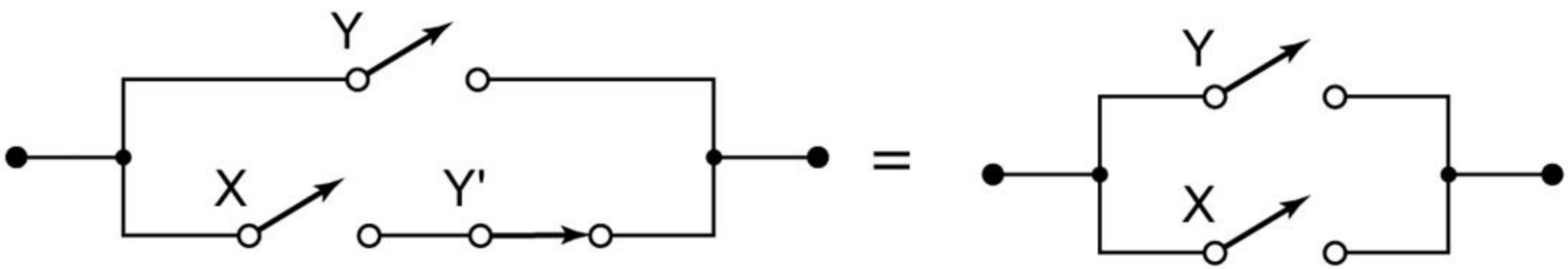
XY’ + Y = X + Y

XY’ + Y

Apply 2[nd] distributive law, we get:

(Y+Y’).(Y+X) =

1.(Y+X) = X+Y



XY’ + Y

X + Y

The circuit on the left side:

If Y=1, the circuit closes

If X=1, there are two cases of Y

  If Y=1, it closes on the top side

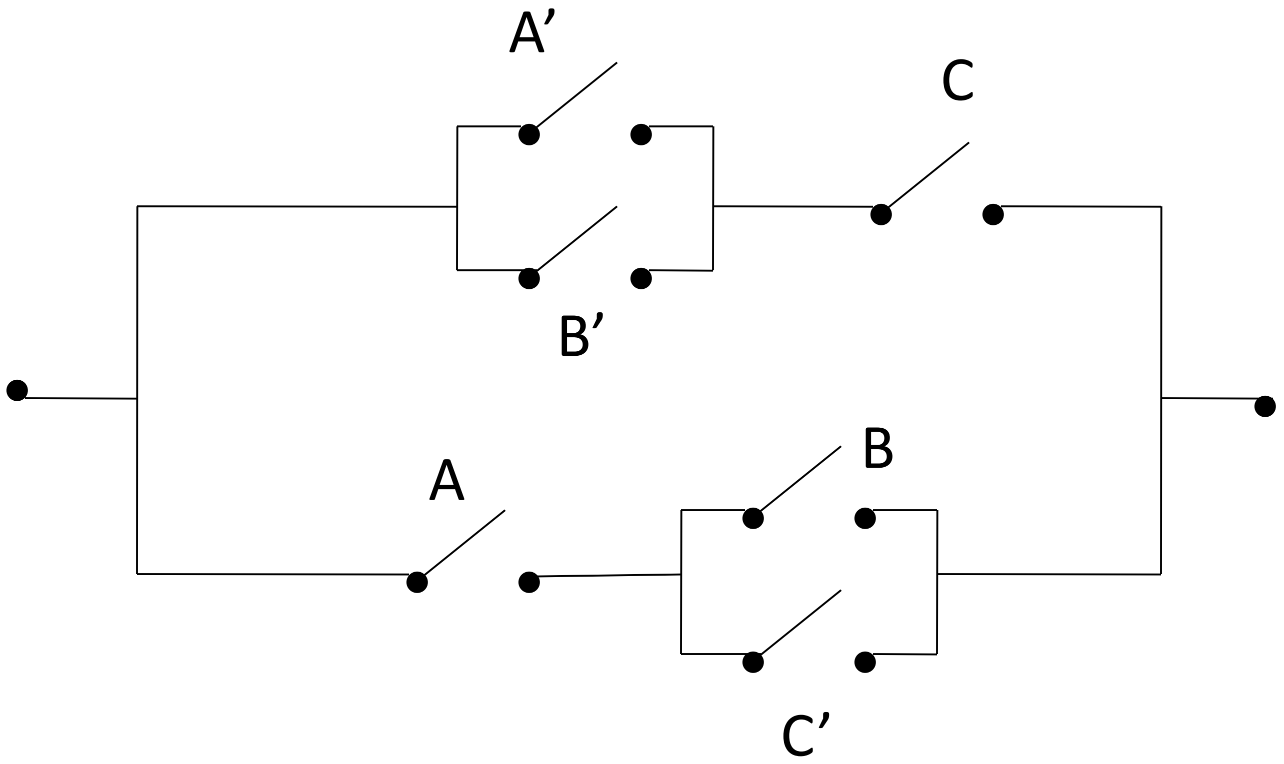
  If Y=0, the switch on the low side closes (since it’s controlled by Y’)

So, If X=1, the circuit closes

Example

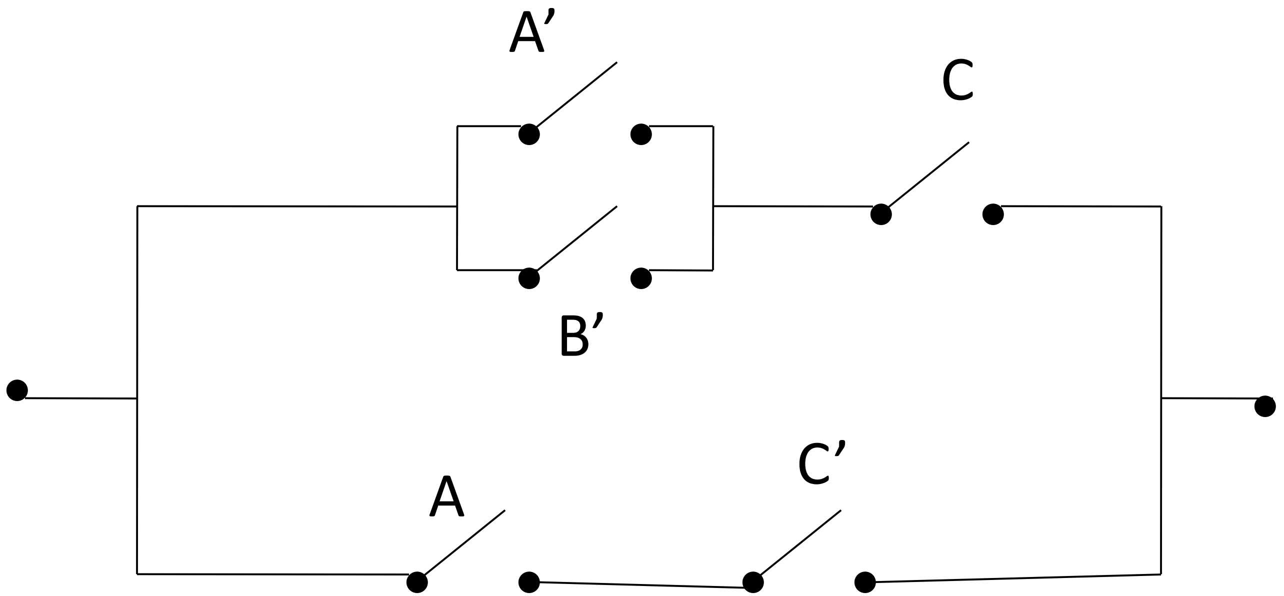
- Draw the switch circuit for:

  $F = (A'+B').C + A.(B + C')$



Example 2

Write the Boolean expression that corresponds to this switch circuit



F = (A’ + B’).C + (A.C’)