Portfolio construction - single period

- A market consists of n assets with their returns given by $(R_1, ..., R_n)$ for a single period.
- We consider the mean-variance portfolio theory of Markowitz.
- A portfolio is specified by a set of weights, $\{w_i, i = 1, \cdots, n\}$, such that $\sum w_i = 1$.
- If $w_i \geq 0$, then short selling is not allowed.
- Notation:

$$\mu_i = ER_i, \quad \sigma_{ij} = Cov(R_i, R_j)$$

$$\mu = \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_n \end{pmatrix}, \Sigma = \begin{pmatrix} \sigma_{11} & \cdots & \sigma_{1n} \\ \vdots & \ddots & \vdots \\ \sigma_{n1} & \cdots & \sigma_{nn} \end{pmatrix}.$$

Portfolio construction

- Facts:
  1. For a portfolio specified by weights $\{w_i, i = 1, \cdots, n\}$, the mean and variance of the portfolio return can be expressed by

$$\mu = \sum_{i=1}^{n} w_i \mu_i$$

$$\sigma^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j \sigma_{ij}.$$

  2. Given two portfolios, $R^{(1)} = \sum_i w_i^{(1)} R_i$ and $R^{(2)} = \sum w_i^{(2)} R_i$, we may form a new portfolio as a weighted average of the two

$$R = \alpha R^{(1)} + (1 - \alpha) R^{(2)}.$$

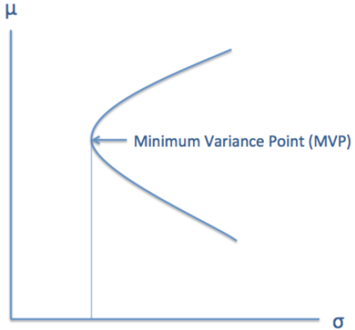
     The weights for this new portfolio are thus $w_i = \alpha w_i^{(1)} + (1 - \alpha) w_i^{(2)}$.

Portfolio construction

- 3. Feasible Region: A set of all points in the $\sigma - \mu$ diagram attained by portfolios.
  4. Feasible region is convex to the left (proved by Cauchy-Schwarz):

$$\sigma \leq \alpha\sigma^{(1)} + (1-\alpha)\sigma^{(2)}.$$

  5. Efficient frontier and minimum variance point



Markowitz problem

- The Markowitz problem is described as finding weights so that, for a given level of return, the variance (standard deviation) of the corresponding portfolio is minimized.
- Example: Suppose that $\mu_1 = \mu_2 = \cdots = \mu_n = \mu^*$ with $\sigma_{ij} = 0, i \neq j$. Then, the mean return of all portfolios must be the same as μ^*, i.e. the feasible set is a horizontal line segment starting from the MVP as the left end point. The efficient frontier coincides with MVP $= (\sigma_{\min}, \mu^*)$, where

$$\sigma_{\min} = \frac{1}{\sqrt{\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2} + \cdots + \frac{1}{\sigma_n^2}}}$$

Markowitz problem

- Markowitz optimal portfolio problem:

$$\min \sum_{i,j} \sigma_{ij} w_i w_j$$

subject to

$$\sum_i w_i u_i = \mu$$

$$\sum_i w_i = 1$$

(Note: under no short selling, $w_i \geq 0$.)

Markowitz problem

- Short selling allowed: When short selling is permitted, we may use the Lagrange multiplier method to get the following $n+2$ linear equations with $n+2$ variables $(w_1, \cdots, w_n, \lambda_1, \lambda_2)$.

$$\sum_{j=1}^{n} \sigma_{ij} w_j - \lambda_1 \mu_i - \lambda_2 = 0, i = 1, \cdots, n$$

$$\sum_{j=1}^{n} w_j \mu_j = \mu,$$

$$\sum_{j=1}^{n} w_j = 1.$$

- Example (Luenberger, 98)
  Let $n = 3$, $\sigma_{ij} = 0$, $i \neq j$, $\sigma_1 = \sigma_2 = \sigma_3 = 1$ and $\mu_1 = 1$, $\mu_2 = 2$, $\mu_3 = 3$. Then, the previous linear equations lead to

  $$w_1 = \frac{4}{3} - \frac{\mu}{2}, \quad w_2 = \frac{1}{3}, \quad w_3 = \frac{\mu}{2} - \frac{2}{3}$$

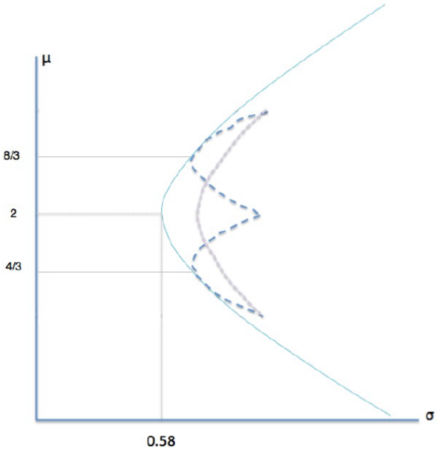
  The standard deviation of this portfolio is

  $$\sigma = \sqrt{\frac{7}{3} - 2\mu + \frac{\mu^2}{2}} = \sqrt{\frac{1}{3} + \frac{1}{2}(\mu - 2)^2}$$

  Thus, MVP: $\mu^* = 2, \sigma^* = \frac{1}{\sqrt{3}} = 0.58$. In addition, if short selling is not permitted, then $\frac{4}{3} \leq \mu \leq \frac{8}{3}$.

Markowitz problem

- Example 6 (continued, Luenberger, 98)



Two-Fund Theorem

- Two-Fund Theorem:
  From two minimum variance portfolios (funds), one can construct any minimum variance portfolio as a linear combination of these two funds. In addition, any linear combination of minimum variance portfolios is again a minimum variance portfolio.
  (Notes: Two fund theorem can by derived by the $n+2$ linear equations in the Lagrange multiplier approach.)