Portfolio/Asset Allocation

Statistical Methods in Finance

Review of basic terminologies

- Price

$$P_t - \text{price of the asset at time t}$$

- Return (single-period net return)

$$R_t = \frac{P_t - P_{t-1}}{P_{t-1}} = \frac{P_t}{P_{t-1}} - 1$$

- k-period (net) return

$$R_t(k) = \frac{P_t - P_{t-k}}{P_{t-k}} = \frac{P_t}{P_{t-k}} - 1$$

Price, returns, and other terminologies

- Gross return (single-period)

$$\frac{P_t}{P_{t-1}} = 1 + R_t$$

- k-period (gross) return

$$\frac{P_t}{P_{t-k}} = 1 + R_t(k)$$

- Multiplicative relationship of gross returns

$$\begin{aligned}\frac{P_t}{P_{t-k}} &= \frac{P_t}{P_{t-1}} \cdot \frac{P_t - 1}{P_{t-2}} \cdots \frac{P_{t-k+1}}{P_{t-k}} \\ &= \prod_{j=0}^{k-1} (1 + R_{t-j})\end{aligned}$$

Statistical models

- Single asset models (longitudinal)
  1. R_t, $t = 1, 2, \ldots$ are uncorrelated with same mean and variance μ and σ^2) (Yes).
  2. R_t, $t = 1, 2, \ldots$ are iid $N(\mu, \sigma^2)$ (Later).
  3. $\{R_t\}$ is a time series, e.g. AR, MA, ARCH, GARCH (Later).
- Log returns

  It is sometimes mathematically convenient to consider the logarithm of (gross) return
$$\tilde{R}_t = log(1 + R_t).$$
  It follows that the k-period gross return has the following expression
$$1 + R_t(k) = \exp\{\tilde{R}_t + \tilde{R}_{t-1} + \cdots + \tilde{R}_{t-k+1}\}.$$

Statistical models

- Joint modeling of n asset returns
  For a market with n assets, denote their returns by $R_{i,t}, i = 1, 2, \cdots, n, t = 1, 2, \cdots$. We need both cross sectional and longitudinal modeling of the returns to capture the dynamics of the market behavior over time. For the time being, we assume that $\mathbf{R}_t = (R_{1,t}, R_{2,t} ..., R_{n,t}), t = 1, ..., m$ are uncorrelated random vectors with same mean vector $\boldsymbol{\mu}$ and covariance matrix Σ.

Portfolio construction - single period

- A market consists of n assets with their returns given by $(R_1, ..., R_n)$ for a single period.
- We consider the mean-variance portfolio theory of Markowitz.
- A portfolio is specified by a set of weights, $\{w_i, i = 1, \cdots, n\}$, such that $\sum w_i = 1$.
- Notation:

$$\mu_i = ER_i, \quad \sigma_{ij} = Cov(R_i, R_j)$$

$$\mu = \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_n \end{pmatrix}, \Sigma = \begin{pmatrix} \sigma_{11} & \cdots & \sigma_{1n} \\ \vdots & \ddots & \vdots \\ \sigma_{n1} & \cdots & \sigma_{nn} \end{pmatrix}.$$

Clearly, $\sigma_{ii} = \sigma_i^2$.

Portfolio construction

- Properties:
  1. For a portfolio specified by weights $\{w_i, i = 1, \cdots, n\}$, the mean and variance of the portfolio return can be expressed by

$$\mu = \sum_{i=1}^{n} w_i \mu_i$$

$$\sigma^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j \sigma_{ij}.$$

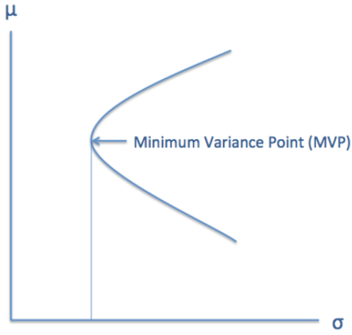
  2. Given two portfolios, $R^{(1)} = \sum_i w_i^{(1)} R_i$ and $R^{(2)} = \sum w_i^{(2)} R_i$, we may form a new portfolio as a weighted average of the two

$$R = \alpha R^{(1)} + (1 - \alpha) R^{(2)}.$$

     The weights for this new portfolio are thus $w_i = \alpha w_i^{(1)} + (1 - \alpha) w_i^{(2)}$.

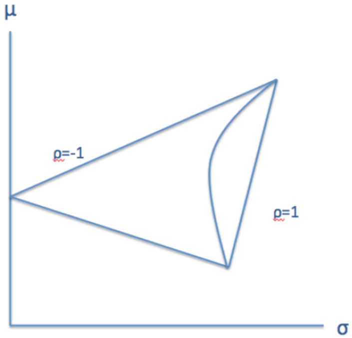
Portfolio construction

- Properties:
  3. Feasible Region: A set of all points in the $\sigma - \mu$ diagram attained by portfolios.
  4. Feasible region is convex to the left (proved by Cauchy-Schwarz):
  $$\sigma \leq \alpha\sigma^{(1)} + (1-\alpha)\sigma^{(2)}.$$
  5. Efficient frontier and minimum variance point



Portfolio construction

- Properties:
  6. A market with two assets:



- Betting wheel (Luenberger, 98)

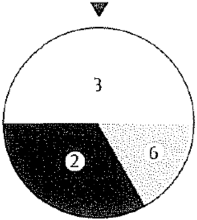


Figure: **Betting wheel** It is possible to bet on any segment of the wheel. If that segment is chosen by the spin, the better receives the amount indicated times the bet

Markowitz problem

- The Markowitz problem is described as finding weights so that, for a given level of return, the variance (standard deviation) of the corresponding portfolio is minimized.

- Example: Suppose that $\mu_1 = \mu_2 = \cdots = \mu_n = \mu^*$ with $\sigma_{ij} = 0, i \neq j$. Then, the mean return of all portfolios must be the same as μ^*, i.e. the feasible set is a horizontal line segment starting from the MVP as the left end point. The efficient frontier coincides with MVP $= (\sigma_{\min}, \mu^*)$, where

$$\sigma_{\min} = \frac{1}{\sqrt{\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2} + \cdots + \frac{1}{\sigma_n^2}}}$$

Markowitz problem

- Example: Let $n = 2$ and $\mu_1 > \mu_2$. The efficient frontier is identical to the upper half of the feasible set. It is characterized by the relationship

$$\sigma = \sqrt{\left(\frac{\mu - \mu_2}{\mu_1 - \mu_2}\right)^2 \sigma_1^2 + \left(\frac{\mu_1 - \mu}{\mu_1 - mu_2}\right) \sigma_2^2 + 2\frac{(\mu - \mu_2)(\mu_1 - \mu)}{(\mu_1 - \mu_2)^2}\sigma_{12}}.$$

The segment, $\mu_2 \leq \mu \leq \mu_1$, corresponds to the feasible set without short selling.

Markowitz problem

- Markowitz optimal portfolio problem:

$$\min \sum_{i,j} \sigma_{ij} w_i w_j$$

subject to

$$\sum_i w_i u_i = \mu$$
$$\sum_i w_i = 1$$

(Note: under no short selling, $w_i \geq 0$.)

Markowitz problem

- Short selling allowed: When short selling is permitted, we may use the Lagrange multiplier method to get the following $n+2$ linear equations with $n+2$ variables $(w_1, \cdots, w_n, \lambda_1, \lambda_2)$.

$$\sum_{j=1}^{n} \sigma_{ij} w_j - \lambda_1 \mu_i - \lambda_2 = 0, i = 1, \cdots, n$$

$$\sum_{j=1}^{n} w_j \mu_j = \mu,$$

$$\sum_{j=1}^{n} w_j = 1.$$

- Example (Luenberger, 98)
  Let $n = 3$, $\sigma_{ij} = 0$, $i \neq j$, $\sigma_1 = \sigma_2 = \sigma_3 = 1$ and $\mu_1 = 1$, $\mu_2 = 2$, $\mu_3 = 3$. For example, the previous linear equations lead to

$$w_1 = \frac{4}{3} - \frac{\mu}{2}, \quad w_2 = \frac{1}{3}, \quad w_3 = \frac{\mu}{2} - \frac{2}{3}$$

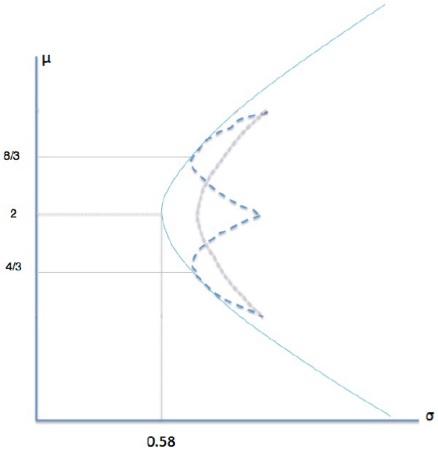
  The standard deviation of this portfolio is

$$\sigma = \sqrt{\frac{7}{3} - 2\mu + \frac{\mu^2}{2}} = \sqrt{\frac{1}{3} + \frac{1}{2}(\mu - 2)^2}$$

  Thus, MVP: $\mu^* = 2, \sigma^* = \frac{1}{\sqrt{3}} = 0.58$. In addition, if short selling is not permitted, then $\frac{4}{3} \leq \mu \leq \frac{8}{3}$.

Markowitz problem

- Example 6 (continued, Luenberger, 98)



Two-Fund Theorem

- Two-Fund Theorem:
  From two minimum variance portfolios (funds), one can construct any minimum variance portfolio as a linear combination of these two funds. In addition, any linear combination of minimum variance portfolios is again a minimum variance portfolio.
  (Notes: Two fund theorem can by derived by the $n+2$ linear equations in the Lagrange multiplier approach.)

Inclusion of a risk free asset

Consider a risk-free asset with a constant rate of return μ_f.

- μ_f is unique (no arbitrage).
- $\sigma_f = 0$.
- The 0-variance entails that such a portfolio must be the MVP and thus on the minimum variance set. By the two-fund theorem, we only need to find a second portfolio on the minimum variance frontier.
- Suppose now that R_* is the second portfolio. For any α, its combination with the risk-free asset, $R(\alpha) = \alpha R_* + (1 - \alpha)\mu_f$ must also be on the minimum variance frontier by the two-fund theorem. But $\mu(\alpha) = ER(\alpha) = \alpha\mu_* + (1 - \alpha)\mu_f$ and $\sigma(\alpha) = \alpha\sigma_* + (1 - \alpha)\sigma_f$, which means that the minimum variance frontier is a straight line. Since there cannot be any feasible point to the left of this line, it must be the tangent line.

One-Fund Theorem

- One-Fund Theorem:
  There is a fund (portfolio), denoted by R_M, in the market of all risky assets such that $\{R(\alpha) = (1-\alpha)\mu_f + \alpha R_M, \alpha \geq 0\}$ is the efficient frontier.

- We shall called R_M in the one-fund theorem the tangent portfolio or market portfolio (in an efficient market).

- A simple way to find it is by solving the following linear equations

$$\sum_{j=1}^{n} \sigma_{ij}\tilde{w}_j = \mu_i - \mu_f, i = 1, \cdots, n$$

  and then define

$$w_i = \frac{\tilde{w}_i}{\sum_{j=1}^{n} \tilde{w}_j}$$