STAT GU4261/GR5261 Statistical Methods in Finance

Spring 2018

Homeowork 5 Suggested Solution

Due date: 1 Mar 2017 (Thu)

1. Since C is a multivariate distribution function,

$$C(u,v) \leq C(u,1) = u, \; C(u,v) \leq C(1,v) = v.$$

Therefore $C(u,v) \leq \min\{u,v\}$. On the other hand,

$$C(u,v) = \mathbb{P}(U \leq u, V \leq v) = \mathbb{P}(U \leq u) + \mathbb{P}(V \leq v) - \mathbb{P}(\{U \leq u\} \cup \{V \leq v\}) \geq u + v - 1.$$

Also, as C is a distribution function, $C(u,v) \geq 0$, so $C(u,v) \geq \max\{u+v-1, 0\}$.

2. It is straightforward to see that

$$\text{sign}\left[(X_t - X_s)\left(\frac{1}{Y_t} - \frac{1}{Y_s}\right)\right] = -\text{sign}\left[(X_t - X_s)(Y_t - Y_s)\right]$$

and

$$\text{sign}\left[\left(\frac{1}{X_t} - \frac{1}{X_s}\right)\left(\frac{1}{Y_t} - \frac{1}{Y_s}\right)\right] = \text{sign}\left[(X_t - X_s)(Y_t - Y_s)\right].$$

Thus we have

$$\begin{aligned} \tau(X, 1/Y) &= -\tau(X,Y) = -0.55 \\ \tau(1/X, 1/Y) &= \tau(X,Y) = 0.55. \end{aligned}$$

3. Since Y is a monotone transformation of X (note that X is positive), any rank correlation will be 1. More rigorously, we can prove using the definitions of Kendall's τ and Spearman's ρ.

   - For Kendall's τ, note that $(X_t - X_s)(Y_t - Y_s) = (X_t - X_s)(X_t^2 - X_s^2) > 0$ almost surely for $X_t, X_s \overset{iid}{\sim}$ Uniform$(0,1)$, so $\text{sign}\left[(X_t - X_s)(Y_t - Y_s)\right] = 1$ almost surely. Therefore,

   $$\tau = E\left\{\text{sign}\left[(X_t - X_s)(Y_t - Y_s)\right]\right\} = E(1) = 1.$$

   - For Spearman's ρ. We can actually prove that $F(X) = G(Y)$. This is because $G(y) = P(Y \leq y) = P(X \leq \sqrt{y}) = F(\sqrt{y})$ for $0 \leq y \leq 1$. Therefore $G(Y) = F(\sqrt{Y}) = F(\sqrt{X^2}) = F(X)$. As a result, we have $\rho(X,Y) = \text{Corr}[F(X), G(Y)] = 1$.
   - For Pearson's correlation, $\text{Corr}(X,Y) = 1$ if and only if $Y = aX + b$ for some $a > 0$ and $b \in \mathbb{R}$ almost surely. But obviously Y is not a linear function of X so the Pearson correlation should be less than 1. (You can also just compute the Pearson's correlation and find that it is less than 1).

4. <u>Solution 1:</u> Observe that

$$\exp\{-[(-\log\min(u,v))^\alpha + (-\log\min(u,v))^\alpha]^{1/\alpha}\} \le \exp\{-[(-\log u)^\alpha + (-\log v)^\alpha]^{1/\alpha}\} \le \min(u,v).$$

The first term can be simplified to $\exp\{2^{1/\alpha}\log\min(u,v)\} = \exp\{2^{1/\alpha}\}\min(u,v) \to \min(u,v)$ as $\alpha \to \infty$. By sandwich theorem, the result follows.

<u>Alternative solution:</u> If $u = v$,

$$C_\alpha(u,v) = \exp\{-2^{1/\alpha}(-\log u)\} \to u \quad \text{as } \alpha \to \infty.$$

If $u < v$, define

$$f(\alpha) := -\log C_\alpha(u,v) = [(-\log u)^\alpha + (-\log v)^\alpha]^{1/\alpha}.$$

Note that

$$\begin{aligned}
\lim_{\alpha\to\infty} \log f(\alpha) &= \lim_{\alpha\to\infty} \frac{1}{\alpha}\log[(-\log u)^\alpha + (-\log v)^\alpha] \\
&= \lim_{\alpha\to\infty} \frac{(-\log u)(-\log u)^\alpha + (-\log v)(-\log v)^\alpha}{[(-\log u)^\alpha + (-\log v)^\alpha]} \\
&= \lim_{\alpha\to\infty} \frac{-\log u + (-\log v)\left(\frac{-\log v}{-\log u}\right)^\alpha}{1 + \left(\frac{-\log v}{-\log u}\right)^\alpha} \\
&= -\log u.
\end{aligned}$$

The last equality follows as $-\log v < -\log u$ when $u < v$, which gives $\left(\frac{-\log v}{-\log u}\right)^\alpha \to 0$ as $\alpha \to \infty$. Hence, $C_\alpha(u,v) \to u$ as $\alpha \to \infty$. By symmetry, the case $v < u$ is the same and the required statement is proved.

5. Remark: $C_{\text{Gaussian}}(u,v) = \Phi_2(\Phi^{-1}(u), \Phi^{-1}(v))$.

   (a)

$$\text{Price} = 1000000 \times e^{-0.04}(0.0346 + 0.0346 - C_{\text{Gaussian}}(0.0346, 0.0346)) = 59495.17.$$

   (b)

$$\text{Price} = 1000000 \times e^{-0.04}(C_{\text{Gaussian}}(0.0346, 0.0346)) = 6991.463.$$

   R Code:

```
library(mvtnorm)

C = pmvnorm(lower = c(-Inf,-Inf), upper = c(qnorm(0.0346),qnorm(0.0346)),
sigma = matrix( c(1,0.5,0.5,1),2,2))

# (a)
price = 1000000*exp(-0.04)*(0.0346 + 0.0346 - C)

# (b)
price = 1000000*exp(-0.04)*C
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6. Denote F_A and F_B as the CDF of T_A and T_B. From the problem we know that the joint CDF of $(F_A(T_A), F_B(T_B))$ is the Gumbel copula with $\alpha = 2$.

$$\begin{aligned}
F_A(1) = P(T_A \le 1) =& 1 - e^{-0.01} = 0.00995 \\
F_B(1) = P(T_B \le 1) =& 1 - e^{-0.02} = 0.0198 \\
P(T_A \le 1, T_B \le 1) =& P(F_A(T_A) \le F_A(1), F_B(T_B) \le F_B(1)) \\
=& \exp\left[-\left\{(-\log F_A(1))^2 + (-\log F_B(1))^2\right\}^{1/2}\right] = 0.00235 \\
P(T_A \le 1 \text{ or } T_B \le 1) =& P(T_A \le 1) + P(T_B \le 1) - P(T_A \le 1, T_B \le 1) = 0.0274
\end{aligned}$$

7. Fair value = expected payoff = $10^6 \times 0.00235 = 2350$.

   For $\alpha = 1$, $P(T_A \le 1, T_B \le 1) = 0.000197$, so Fair value = 197.