Stat GR5205 Lecture 3

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Least squares estimator for simple linear regression

- Slope

$$\hat{\beta}_1 = \rho_{x,y} \frac{s_y}{s_x}.$$

- The fitted regression line

$$(x - \bar{x}) = \rho_{x,y} \frac{s_y}{s_x} (y - \bar{y})$$

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The sampling distribution

- $E(\hat{\beta}_0) = \beta_0 \quad E(\hat{\beta}_1) = \beta_1$
- The variances are

$$Var(\hat{\beta}_1) = \frac{\sigma^2}{\sum_{i=1}^n (x_i - \bar{x})^2}, \quad Var(\hat{\beta}_0) = \sigma^2 \left[\frac{1}{n} + \frac{\bar{x}^2}{\sum_{i=1}^n (x_i - \bar{x})^2}\right]$$

- $Var(\hat{\beta}_0) = Var(\bar{Y}) + Var(\bar{x}\hat{\beta}_1)$. Derive $Cov(\bar{Y}, \hat{\beta}_1) = 0$
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Confidence interval

- About confidence interval
- A confidence interval of β_0 and β_1.

$$[\beta_i - 1.96 \times SD(\beta_i), \quad \beta_i + 1.96 \times SD(\beta_i)]$$

Confidence interval

- What if σ^2 unknown?

Inference with unknown σ^2

- The noise level

$$\sigma^2 = Var(\varepsilon_i) = E(y_i - \beta_0 - \beta_1 x_i)^2$$

- Point estimate

$$\hat{\sigma}^2 = \frac{1}{n-2} \sum_{i=1}^{n} (y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i)^2$$

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About normal (Gaussian) distribution

- $Z_1, ..., Z_n$ are independent and identically distributed $N(0, 1)$
- The χ^2 distribution with degrees of freedom n

$$Z_1^2 + ... + Z_n^2$$

Inference with unknown σ^2

- Distribution of $\hat{\sigma}^2$

$$\frac{n-2}{\sigma^2}\hat{\sigma}^2 \sim \chi^2_{n-2}$$

Derive sample $(n-1)s_x^2/\sigma^2 \sim \chi^2_{n-1}$ and $\bar{x}$ is independent of s_x^2.

- About χ^2-distribution
- $\hat{\sigma}^2$ is independent of $(\hat{\beta}_0, \hat{\beta}_1)$.
- Confidence interval of $\hat{\beta}_i$ when σ^2 is unknown.

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On the Student's t-distribution

- $Z \sim N(0,1)$ and $\gamma^2 \sim \frac{\chi_v^2}{v}$
- $T = \frac{Z}{\gamma} \sim t_v$
- About t-distribution

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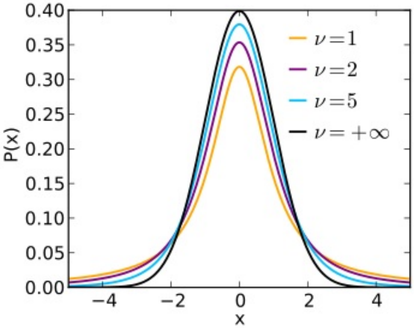

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- Normal approximation of t-distribution
- Rationale

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Prediction of a new observation

- Prediction of a new observation when parameters are known
- When σ^2 is known
- When σ^2 is unknown

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Simultaneous confidence band

- Simultaneous confidence band versus usual confidence interval
- The confidence band

$$\hat{\mu}_x \pm \lambda SD(\hat{\mu}_x)$$

where

$$\lambda^2 = 2F(1-\alpha; 2, n-2)$$

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Comparing the intervals

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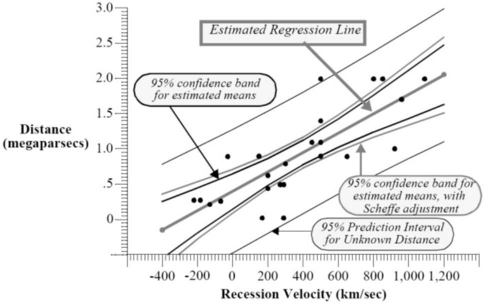
The 95% confidence band on the population regression line, the 95% confidence interval band for single mean estimates, and a 95% prediction interval band for the Big Bang example



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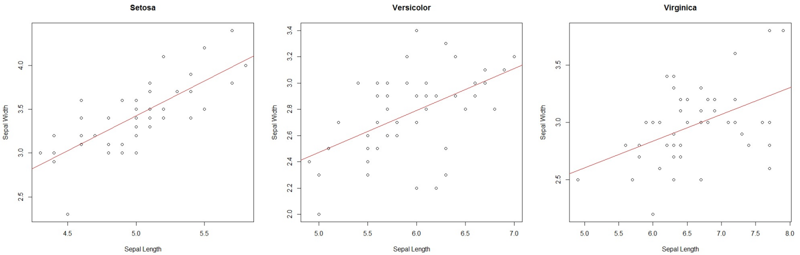

Some descriptive statistics

- The coefficient of determination (a.k.a. the R^2)

$$R^2 = 1 - \frac{\sum_{i=1}^{n}(y_i - \hat{y}_i)^2}{\sum_{i=1}^{n}(y_i - \bar{y})^2}$$

- Another representation for the simple linear regression

$$R^2 = \rho^2$$

- Derive analysis of variance of simple linear regression

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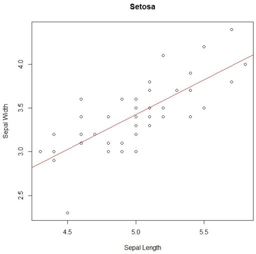

- The fitted regression

$$y = -0.57 + 0.80x$$

and $\hat{\sigma} = 0.26$.

- Question: whether plants with longer sepal tend to have wider sepal?

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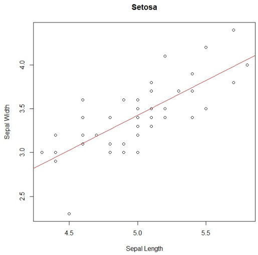

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- Testing of hypotheses
- Basic setting
  1. Two families of models without overlap: the null hypothesis (H_0) and the alternative hypothesis (H_1)
  2. Which family were the data sampled from?

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  - Type I error
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- Formulation

$$H_0 : \beta_1 = 0 \quad versus \quad H_1 : \beta_1 \neq 0$$

- $\hat{\beta}_1 = 0.80$ and $SD(\hat{\beta}_1) = 0.10$.
- The sampling distribution of β_1

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