Capital Asset Pricing Model (CAPM)

Statistical Methods in Finance

Inclusion of a risk free asset

Consider a risk-free asset with a constant rate of return μ_f.

- μ_f is unique (no arbitrage).
- $\sigma_f = 0$.
- The 0-variance entails that such a portfolio must be the MVP and thus on the minimum variance set. By the two-fund theorem, we only need to find a second portfolio on the minimum variance frontier.
- Suppose now that R_* is the second portfolio. For any α, its combination with the risk-free asset, $R(\alpha) = \alpha R_* + (1-\alpha)\mu_f$ must also be on the minimum variance frontier by the two-fund theorem. But $\mu(\alpha) = ER(\alpha) = \alpha\mu_* + (1-\alpha)\mu_f$ and $\sigma(\alpha) = \alpha\sigma_* + (1-\alpha)\sigma_f$, which means that the minimum variance frontier is a straight line. Since there cannot be any feasible point to the left of this line, it must be the tangent line.

One-Fund Theorem

- One-Fund Theorem:
  There is a fund (portfolio), denoted by R_M, in the market of all risky assets such that $\{R(\alpha) = (1-\alpha)\mu_f + \alpha R_M, \alpha \geq 0\}$ is the efficient frontier.

- We shall called R_M in the one-fund theorem the tangent portfolio or market portfolio (in an efficient market).

- A simple way to find it is by solving the following linear equations

$$\sum_{j=1}^{n} \sigma_{ij}\tilde{w}_j = \mu_i - \mu_f, i = 1, \cdots, n$$

  and then define

$$w_i = \frac{\tilde{w}_i}{\sum_{j=1}^{n} \tilde{w}_j}$$

One-Fund Theorem

Example (Luenberger, 98) Let $n = 3$, $\sigma_{ij} = 0, i \neq j$, $\sigma_1 = \sigma_2 = \sigma_3 = 1$ and $\mu_1 = 1, \mu_2 = 2, \mu_3 = 3$. If in addition to the three risky assets, we also have a risk-free asset with rate of return $\mu_f = 0.1$, then we have $\tilde{w}_1 = 1 - 0.1 = 0.9$, $\tilde{w}_2 = 1.9$ and $\tilde{w}_3 = 2.9$. Therefore $w_1 = 9/57$, $w_2 = 19/57$ and $w_2 = 29/57$.

Capital Asset Pricing Model

- The presence of the risk-free asset entails that the efficient frontier is a straight line with intercept μ_f (risk-free rate) and is tangential to the feasible set (efficient frontier) of the market without the risk-free asset. The tangent portfolio, whose return is denoted by R_M with its risk-return on the tangent point, has a special role. Any rational investment portfolio is a linear combination of the risk-free asset and the tangent portfolio.

- In an efficient market, every investor is rational or mean-variance optimizer. Therefore, their investment strategies should converge to the tangent line. As a consequence, when the equilibrium is reached, weights of the tangent portfolio should be the market capitalization weights. That is, for stock i, its weight should be its total capitalization divided by the total market capitalization.

Capital Asset Pricing Model

- **Capital Market Line**. The line (on σ-μ plane) that forms the efficient frontier is known as the capital market line (CML). It satisfies the following equation:

$$\mu = \mu_f + \frac{\mu_M - \mu_f}{\sigma_M}\sigma.$$

- **Sharpe Ratio**. The slope of CML,

$$\frac{\mu_M - \mu_f}{\sigma_M},$$

is called the Sharpe ratio, which measures the excess return (in excess of the risk-free return) over risk (standard deviation).

- **Security Market Line**. The market portfolio has a special relationship with any individual asset or fund. Let R_i be an asset (or fund) with mean return μ_i. Let $\sigma_{i,M} = Cov(R_i, R_M)$, its covariance with the tangent portfolio. Define $\beta_i = \sigma_{i,M}/\sigma_M^2$. Then the following equation holds

$$\mu_i = \mu_f + \beta_i(\mu_M - \mu_f).$$

By viewing μ_i as a function of β_i, it is a straight line with intercept μ_f and slope $\mu_M - \mu_f$ and is known as the security market line (SML). In other words, every (β_i, μ_i) lines in the SML.

Capital Asset Pricing Model

- The SML can be derived by considering the set of portfolios, $R_t = tR_i + (1-t)R_M$, for all t. Calculating the corresponding mean and variance gives

$$\mu_t = E(R_t) = t\mu_i + (1-t)\mu_M$$

$$\sigma_t^2 = Var(R_t) = t^2\sigma_i^2 + (1-t)^2\sigma_M^2 + 2t(1-t)\sigma_{i,M}.$$

By noting that the efficient frontier must also be tangent to the curve (σ_t, μ_t) at $t = 0$, we can show that the SML equation must be satisfied.

- Note that for the market (tangent) portfolio, its beta is 1 and for the money market (risk-free asset), its beta is 0. If an asset or fund is uncorrelated with the market portfolio, then its beta must be 0.

- The CML may be used to evaluate investment options.

Capital Asset Pricing Model

Example (Luenberger, 98) An oil drilling company is currently priced at \$875 a share. Suppose that the expected share price after one year is \$1000 and that the standard deviation of the return is $\sigma_{oil} = 40\%$. Furthermore, the risk-free rate is 10% and the market portfolio has an expected return 17% with a standard deviation 12%.

Example (continued)

- **Evaluation** From the above specification, we know that investing in the oil drilling company will result in an expected return

$$\mu_{oil} = \frac{1000 - 875}{875} = 14\%.$$

On the other hand, at σ =40%, the CML implies the expected return

$$\mu = 0.10 + \frac{0.17 - 0.10}{0.12} \times 0.40 = 33\%.$$

Thus, the expected return of the oil drilling venture is well below the CML so that the investment itself does not appear to be a good one.

Example (continued)

- **Pricing** The SML may be used as a pricing tool. Suppose that instead of \$875 the fair price P_0 for the oil drilling company is to be determined. Suppose we know that its beta is 0.6. The SML implies

$$\frac{1000 - P_0}{P_0} = 0.1 + 0.6 \times (0.17 - 0.10)$$

or

$$P_0 = \frac{1000}{1 + 0.1 + 0.6 \times (0.17 - 0.10)} = 876.$$

So the price is right, even though itself is not a good investment.

- **Security Characteristic Line**. The SML tells us that for asset i,

$$E(R_i) = \mu_f + \beta_i[E(R_M) - \mu_f].$$

If we remove the expectation in front of both R_i and R_M, then the above equation becomes a regression (security characteristic line)

$$R_i = \mu_f + \beta_i(R_M - \mu_f) + \epsilon_i,$$

where, by definition, $\epsilon_i = R_i - \mu_f + \beta_i(R_M - \mu_f)$. It is easy to see that
(i) ϵ_i is uncorrelated with R_M;
(ii) $\sigma_i^2 = \beta_i^2\sigma_M^2 + \sigma_{\epsilon_i}^2$.

From (ii), the risk σ_i^2 associated with asset i is decomposed into two parts, $\beta_i^2\sigma_M^2$ (systematic risk) due to market and $\sigma_{\epsilon_i}^2$ (idiosyncratic risk) that is diversifiable.

Capital Asset Pricing Model

- For a particular asset i, we may run regression analysis to test a statistical hypothesis concerning the security characteristic line. To do so, we need data from multiple periods, $t = 1, \ldots, T$. Note that the SCL is a linear regression of asset excess returns over market excess returns, without intercept. Let $Y_t = R_{it} - \mu_{ft}$ and $X_t = R_{Mt} - \mu_{ft}$. Thus the SCL is a linear regression model of Y on X through origin. In particular, we may test the hypothesis that the intercept, also known as α, is indeed 0.

- Because the SCL also applies to any fund, we may use it to evaluate a mutual fund. Fitting $Y = \alpha + \beta X + \epsilon$, we can find an estimate of α (Jensen index). A positive α means that the fund out performs the market.

- Being considered jointly over $i = 1, \ldots, n$, this is a multivariate linear regression model with a single factor R_M.