Factor Analysis

Part II

Recall:

The Factor Analysis Model

$$x_1 = \mu_1 + \lambda_{11}F_1 + \lambda_{12}F_2 + \ldots + \lambda_{1m} F_m + \varepsilon_1$$
$$x_2 = \mu_2 + \lambda_{21}F_1 + \lambda_{22}F_2 + \ldots + \lambda_{2m} F_m + \varepsilon_2$$
$$\ldots$$
$$x_p = \mu_p + \lambda_{p1}F_1 + \lambda_{p2}F_2 + \ldots + \lambda_{pm} F_m + \varepsilon_p$$

or

$$\boldsymbol{X} - \boldsymbol{\mu} = \boldsymbol{LF} + \boldsymbol{\varepsilon}$$

which implies

$$\boldsymbol{\Sigma} = \mathrm{cov}(\boldsymbol{X}) = \boldsymbol{LL}' + \boldsymbol{\Psi}$$

where

$F_1, F_2, \ldots, F_m$ are called the **common factors**

Hypothesis Test for the Number of Common Factors

Testing the adequacy of the the m common factor model is equivalent to testing

$$H_0: \boldsymbol{\Sigma} = \boldsymbol{L}\boldsymbol{L}' + \boldsymbol{\Psi}$$

where

$\boldsymbol{\Sigma}$ is $p \times p$, $\boldsymbol{L}$ is $p \times m$, and $\boldsymbol{\Psi}$ is $p \times p$

vs.

H_1: $\boldsymbol{\Sigma}$ is any other positive definite matrix

Test Statistic

Bartlett's correction to the likelihood ration statistic:

Reject H_0 at level of significance α if

$$\left(n - 1 - \frac{2p + 4m + 5}{6}\right) \ln \frac{\left|\hat{\boldsymbol{L}}\hat{\boldsymbol{L}}' + \hat{\boldsymbol{\Psi}}\right|}{|\boldsymbol{S}_n|} > \chi^2(\alpha)$$

where the df of the chi-square are

$$\mathrm{df} = \frac{(p - m)^2 - p - m}{2}$$

Assumptions: n and $n - p$ are "large".

Notes:

- We must have $m < \frac{1}{2}\left(2p + 1 - \sqrt{8p + 1}\right)$ or the df will be negative.

For example, if $p = 4$, then $\frac{1}{2}\left(2p + 1 - \sqrt{8p + 1}\right) \approx 1.6$, so we can't have 2 factors!

- Retaining the null hypothesis is a good news! Otherwise we must increase the number of factors.
- However, adding more factors should be done carefully using some judgement even if they are ”significant” as often happens if m is small relative to p.

Factor Rotation

If $\hat{\boldsymbol{L}}$ is the $p \times m$ matrix of estimated factor loadings (obtained by any method), then let

$$\hat{\boldsymbol{L}}^* = \hat{\boldsymbol{L}}\boldsymbol{T}$$

where $\boldsymbol{T}$ is an orthogonal matrix ($\boldsymbol{T}'\boldsymbol{T} = \boldsymbol{T}\boldsymbol{T}' = \boldsymbol{I}$).

Then $\hat{\boldsymbol{L}}^*$ is a $p \times m$ matrix of *rotated* loadings. Moreover,

$$\hat{\boldsymbol{L}}\hat{\boldsymbol{L}}' + \hat{\boldsymbol{\Psi}} = \hat{\boldsymbol{L}}\boldsymbol{T}\boldsymbol{T}'\hat{\boldsymbol{L}}' + \hat{\boldsymbol{\Psi}} = \hat{\boldsymbol{L}}^*\hat{\boldsymbol{L}}^{*\prime} + \hat{\boldsymbol{\Psi}}$$

That is, the estimated covariance matrix remains unchanged! Thus from a mathematical point of view it is whether $\hat{\boldsymbol{L}}$ or $\hat{\boldsymbol{L}}^*$ is used, because the factor model is overparametrized and has many solutions.

Notes:

- Since the original loadings may not be easily interpretable, it is a common practice to rotate them until a “simpler structure” is achieved.
- Ideally, we want to see a pattern of loadings such that each variable loads heavily on a single factor and has small loadings on the remaining factors.
- It is not always possible to obtain such simple structure ☹
- There are graphical and analytical methods for choosing the optimal rotation.

Example 9.10: *Stock price data*

Recall the data were collected for $n = 103$ weekly rates of return on $p = 5$ stocks (as in Example 8.5). We will use $m = 2$ factors as before we used 2 PCs

Without rotation:

Factor 1: general economic conditions (market)

Factor 2: differentiate the industries

With rotation:

Factor 1: economic forces that cause bank stocks to move together

Factor 2: economic conditions for oil stocks

Without rotation

Principal component method:

	PC1	PC2	h2	u2	com
JPM	0.73	-0.44	0.73	0.27	1.6
C	0.83	-0.28	0.77	0.23	1.2
WFC	0.73	-0.37	0.67	0.33	1.5
RDS	0.60	0.69	0.85	0.15	2.0
XOM	0.56	0.72	0.83	0.17	1.9

MLE method:

	Factor1	Factor2
JPM	0.121	0.754
C	0.328	0.786
WFC	0.188	0.650
RDS	0.997	
XOM	0.685	

	Factor1	Factor2
JPM	**0.763**	
C	**0.819**	0.232
WFC	0.668	0.108
RDS	0.113	**0.991**
XOM	0.108	**0.677**

After rotation