# Multivariate Normal Distribution

With a brief introduction to other probability distributions

# Univariate Normal (Gaussian) Distribution

- Bell-shaped distribution with tendency for individuals to clump around the group median/mean
- Used to model many biological phenomena
- Many *estimators* have approximate normal sampling distributions (see Central Limit Theorem)
- Notation: $X \sim N(\mu, \sigma^2)$ where $\mu$ is mean and $\sigma^2$ is the variance

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}}, -\infty < x < \infty, -\infty < \mu < \infty, \sigma > 0$$

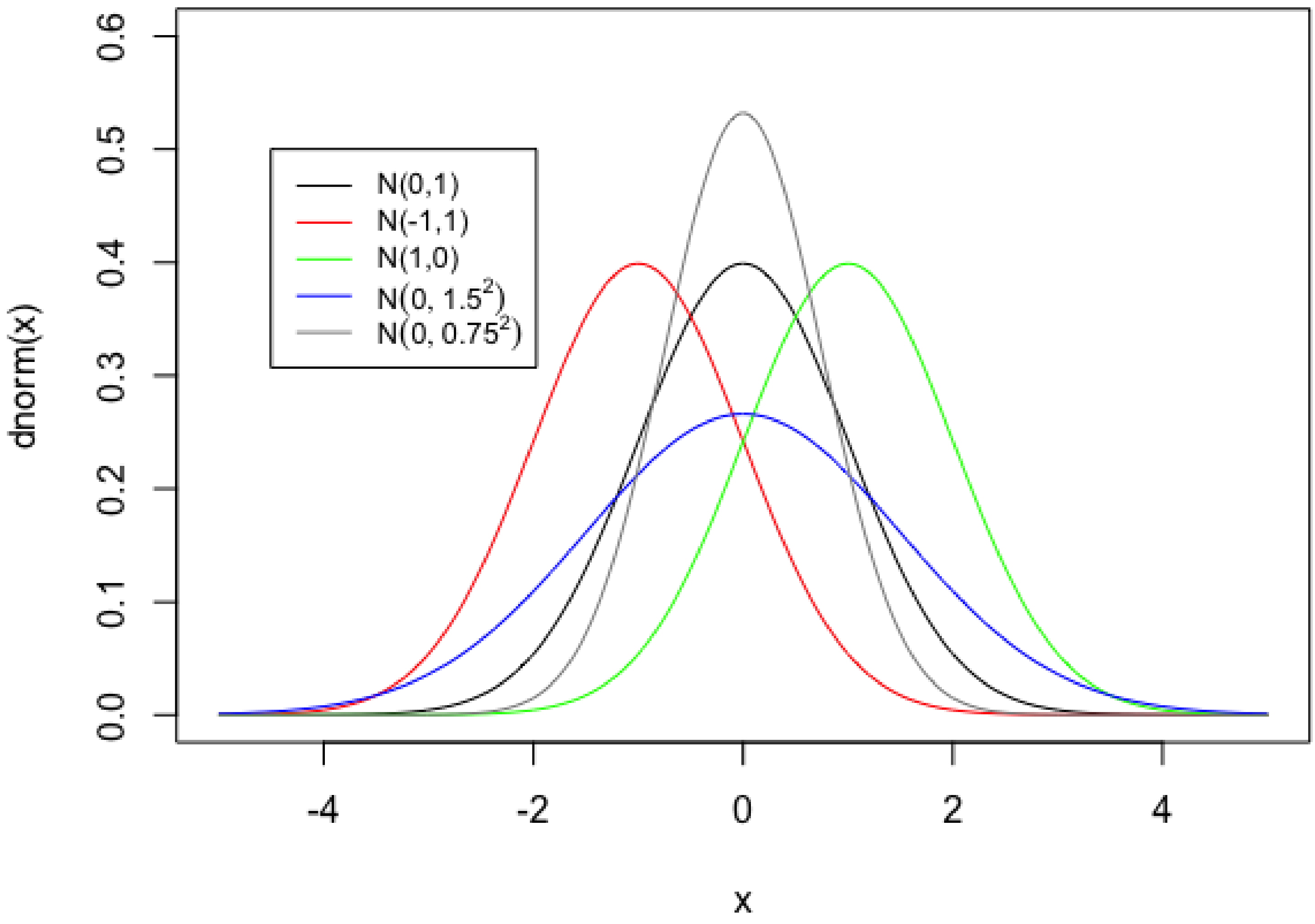
**Obtaining Probabilities and Quantiles in R:**

**To obtain: $F(x) = P(X \le x)$ Use Function: pnorm(x, $\mu$, $\sigma$)**

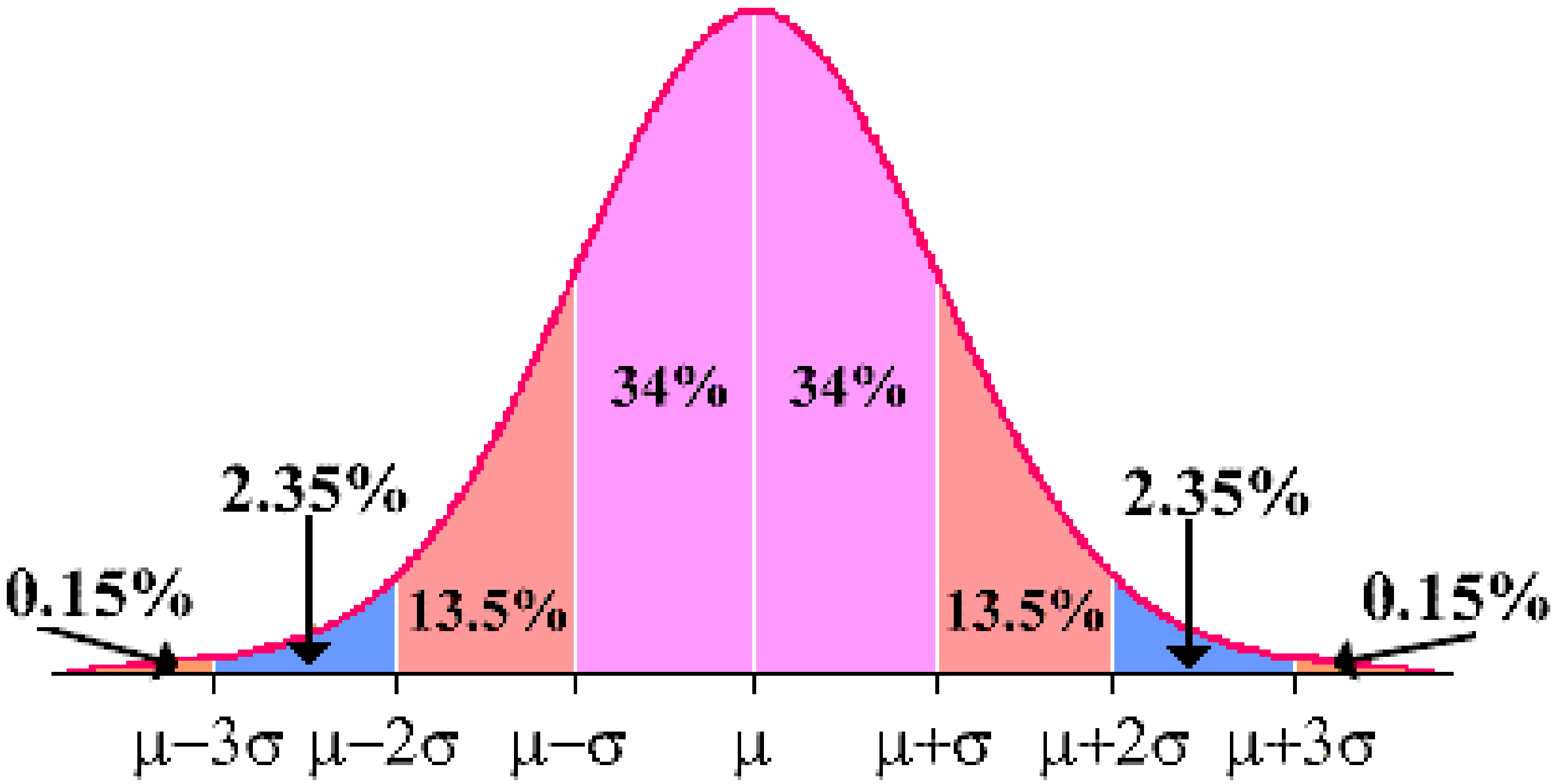
**To obtain the $p^{th}$ quantile: $P(X \le x_p) = p$ Use Function: qnorm(p, $\mu$, $\sigma$)**

**Virtually all statistics textbooks give the cdf (or upper tail probabilities) for standardized normal random variables: $z=(x-\mu)/\sigma \sim N(0,1)$**

# Normal Distribution – Density Functions (pdf)



# Empirical Rule

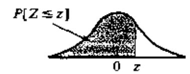


Second Decimal Place of z

Integer part and first decimal place of z



TABLE 1 STANDARD NORMAL PROBABILITIES




| z | .00 | .01 | .02 | .03 | .04 | .05 | .06 | .07 | .08 | .09 |
|---|---|---|---|---|---|---|---|---|---|---|
| .0 | .5000 | .5040 | .5080 | .5120 | .5160 | .5199 | .5239 | .5279 | .5319 | .5359 |
| .1 | .5398 | .5438 | .5478 | .5517 | .5557 | .5596 | .5636 | .5675 | .5714 | .5753 |
| .2 | .5793 | .5832 | .5871 | .5910 | .5948 | .5987 | .6026 | .6064 | .6103 | .6141 |
| .3 | .6179 | .6217 | .6255 | .6293 | .6331 | .6368 | .6406 | .6443 | .6480 | .6517 |
| .4 | .6554 | .6591 | .6628 | .6664 | .6700 | .6736 | .6772 | .6808 | .6844 | .6879 |
| .5 | .6915 | .6950 | .6985 | .7019 | .7054 | .7088 | .7123 | .7157 | .7190 | .7224 |
| .6 | .7257 | .7291 | .7324 | .7357 | .7389 | .7422 | .7454 | .7486 | .7517 | .7549 |
| .7 | .7580 | .7611 | .7642 | .7673 | .7703 | .7734 | .7764 | .7794 | .7823 | .7852 |
| .8 | .7881 | .7910 | .7939 | .7967 | .7995 | .8023 | .8051 | .8078 | .8106 | .8133 |
| .9 | .8159 | .8186 | .8212 | .8238 | .8264 | .8289 | .8315 | .8340 | .8365 | .8389 |
| 1.0 | .8413 | .8438 | .8461 | .8485 | .8508 | .8531 | .8554 | .8577 | .8599 | .8621 |
| 1.1 | .8643 | .8665 | .8686 | .8708 | .8729 | .8749 | .8770 | .8790 | .8810 | .8830 |
| 1.2 | .8849 | .8869 | .8888 | .8907 | .8925 | .8944 | .8962 | .8980 | .8997 | .9015 |
| 1.3 | .9032 | .9049 | .9066 | .9082 | .9099 | .9115 | .9131 | .9147 | .9162 | .9177 |
| 1.4 | .9192 | .9207 | .9222 | .9236 | .9251 | .9265 | .9279 | .9292 | .9306 | .9319 |
| 1.5 | .9332 | .9345 | .9357 | .9370 | .9382 | .9394 | .9406 | .9418 | .9429 | .9441 |
| 1.6 | .9452 | .9463 | .9474 | .9484 | .9495 | .9505 | .9515 | .9525 | .9535 | .9545 |
| 1.7 | .9554 | .9564 | .9573 | .9582 | .9591 | .9599 | .9608 | .9616 | .9625 | .9633 |
| 1.8 | .9641 | .9649 | .9656 | .9664 | .9671 | .9678 | .9686 | .9693 | .9699 | .9706 |
| 1.9 | .9713 | .9719 | .9726 | .9732 | .9738 | .9744 | .9750 | .9756 | .9761 | .9767 |
| 2.0 | .9772 | .9778 | .9783 | .9788 | .9793 | .9798 | .9803 | .9808 | .9812 | .9817 |
| 2.1 | .9821 | .9826 | .9830 | .9834 | .9838 | .9842 | .9846 | .9850 | .9854 | .9857 |
| 2.2 | .9861 | .9864 | .9868 | .9871 | .9875 | .9878 | .9881 | .9884 | .9887 | .9890 |
| 2.3 | .9893 | .9896 | .9898 | .9901 | .9904 | .9906 | .9909 | .9911 | .9913 | .9916 |
| 2.4 | .9918 | .9920 | .9922 | .9925 | .9927 | .9929 | .9931 | .9932 | .9934 | .9936 |
| 2.5 | .9938 | .9940 | .9941 | .9943 | .9945 | .9946 | .9948 | .9949 | .9951 | .9952 |
| 2.6 | .9953 | .9955 | .9956 | .9957 | .9959 | .9960 | .9961 | .9962 | .9963 | .9964 |
| 2.7 | .9965 | .9966 | .9967 | .9968 | .9969 | .9970 | .9971 | .9972 | .9973 | .9974 |
| 2.8 | .9974 | .9975 | .9976 | .9977 | .9977 | .9978 | .9979 | .9979 | .9980 | .9981 |
| 2.9 | .9981 | .9982 | .9982 | .9983 | .9984 | .9984 | .9985 | .9985 | .9986 | .9986 |
| 3.0 | .9987 | .9987 | .9987 | .9988 | .9988 | .9989 | .9989 | .9989 | .9990 | .9990 |
| 3.1 | .9990 | .9991 | .9991 | .9991 | .9992 | .9992 | .9992 | .9992 | .9993 | .9993 |
| 3.2 | .9993 | .9993 | .9994 | .9994 | .9994 | .9994 | .9994 | .9995 | .9995 | .9995 |
| 3.3 | .9995 | .9995 | .9995 | .9996 | .9996 | .9996 | .9996 | .9996 | .9996 | .9997 |
| 3.4 | .9997 | .9997 | .9997 | .9997 | .9997 | .9997 | .9997 | .9997 | .9997 | .9998 |
| 3.5 | .9998 | .9998 | .9998 | .9998 | .9998 | .9998 | .9998 | .9998 | .9998 | .9998 |

# Chi-Square Distribution

- Indexed by "degrees of freedom ($\nu$)" $X \sim \chi_{\nu}^{2}$
- $Z \sim N(0,1) \Rightarrow Z^2 \sim \chi_1^{2}$
- Assuming Independence:

$$X_1, \ldots, X_n \sim \chi^2_{\nu_i} \quad i = 1, \ldots, n \quad \Rightarrow \quad \sum_{i=1}^{n} X_i \sim \chi^2_{\sum \nu_i}$$

Density Function:

$$f(x) = \frac{1}{\Gamma\left(\frac{\nu}{2}\right) 2^{\nu/2}} x^{(\nu/2)-1} e^{-x/2} \quad x > 0, \nu > 0 \qquad E\{X\} = \nu \quad V\{X\} = 2\nu$$

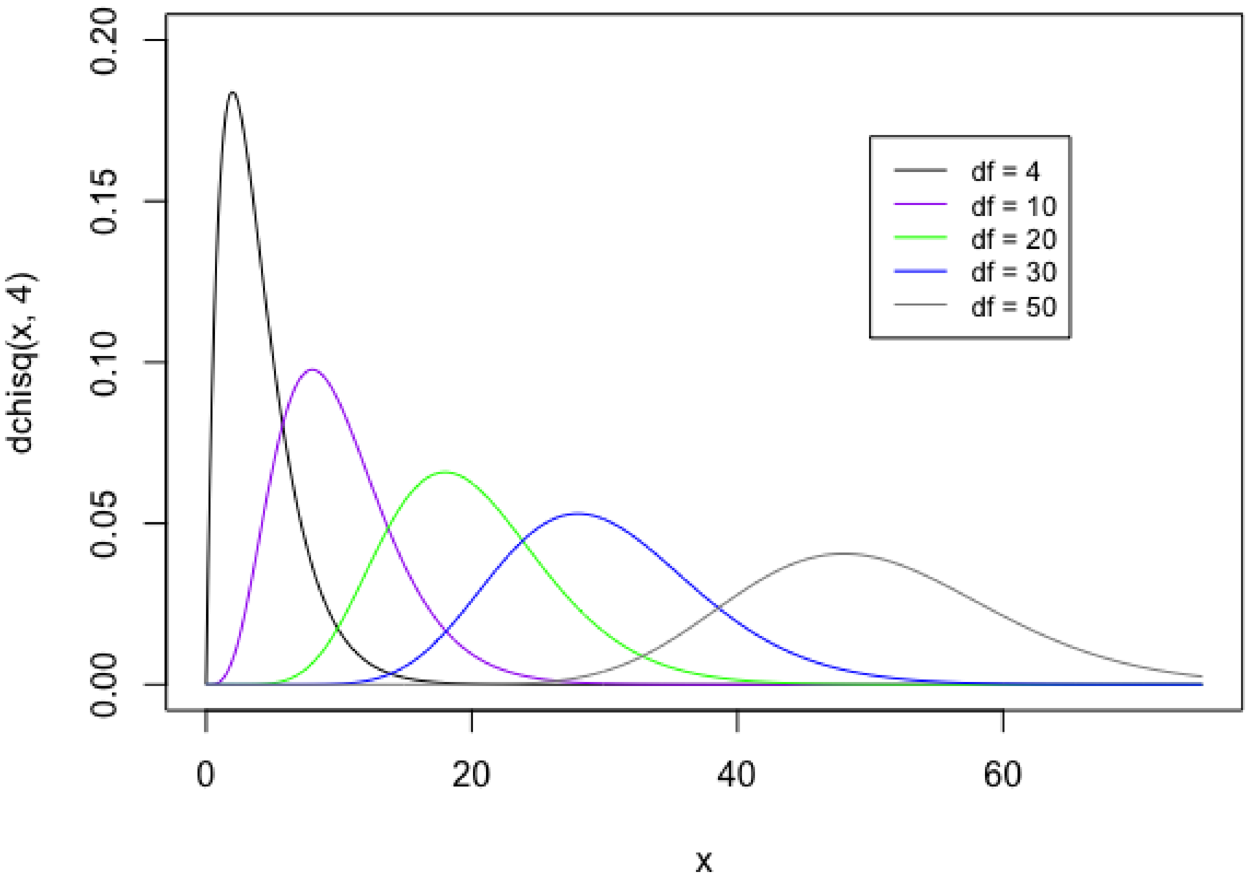
**Obtaining Probabilities in R:**

**To obtain: 1-F(x) = P(X≥x)** **Use Function:** **pchisq(x, ν)**

**To obtain quantiles: P(X≤$x_p$) = p** **Use Function:** **qchisq(x, ν)**

**Virtually all statistics textbooks give upper tail cut-off values for commonly used upper (and sometimes lower) tail probabilities**

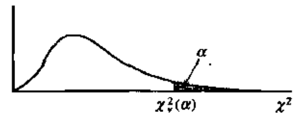
# Chi-Square Distributions



# Critical Values for Chi-Square Distributions (Mean=ν, Variance=2ν)



**BLE 3** $\chi^2$ DISTRIBUTION PERCENTAGE POINTS




| d.f. | α | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| ν | .990 | .950 | .900 | .500 | .100 | .050 | .025 | .010 | .005 |
| 1 | .0002 | .004 | .02 | .45 | 2.71 | 3.84 | 5.02 | 6.63 | 7.88 |
| 2 | .02 | .10 | .21 | 1.39 | 4.61 | 5.99 | 7.38 | 9.21 | 10.60 |
| [illegible] | .11 | .35 | .58 | 2.37 | 6.25 | 7.81 | 9.35 | 11.34 | 12.84 |
| [illegible] | .30 | .71 | 1.06 | 3.36 | 7.78 | 9.49 | 11.14 | 13.28 | 14.86 |
| 5 | .55 | 1.15 | 1.61 | 4.35 | 9.24 | 11.07 | 12.83 | 15.09 | 16.75 |
| [illegible] | .87 | 1.64 | 2.20 | 5.35 | 10.64 | 12.59 | 14.45 | 16.81 | 18.55 |
| [illegible] | 1.24 | 2.17 | 2.83 | 6.35 | 12.02 | 14.07 | 16.01 | 18.48 | 20.28 |
| 8 | 1.65 | 2.73 | 3.49 | 7.34 | 13.36 | 15.51 | 17.53 | 20.09 | 21.95 |
| 9 | 2.09 | 3.33 | 4.17 | 8.34 | 14.68 | 16.92 | 19.02 | 21.67 | 23.59 |
| [illegible] | 2.56 | 3.94 | 4.87 | 9.34 | 15.99 | 18.31 | 20.48 | 23.21 | 25.19 |
| [illegible] | 3.05 | 4.57 | 5.58 | 10.34 | 17.28 | 19.68 | 21.92 | 24.72 | 26.76 |
| [illegible] | 3.57 | 5.23 | 6.30 | 11.34 | 18.55 | 21.03 | 23.34 | 26.22 | 28.30 |
| 13 | 4.11 | 5.89 | 7.04 | 12.34 | 19.81 | 22.36 | 24.74 | 27.69 | 29.82 |
| [illegible] | 4.66 | 6.57 | 7.79 | 13.34 | 21.06 | 23.68 | 26.12 | 29.14 | 31.32 |
| [illegible] | 5.23 | 7.26 | 8.55 | 14.34 | 22.31 | 25.00 | 27.49 | 30.58 | 32.80 |
| 16 | 5.81 | 7.96 | 9.31 | 15.34 | 23.54 | 26.30 | 28.85 | 32.00 | 34.27 |
| 17 | 6.41 | 8.67 | 10.09 | 16.34 | 24.77 | 27.59 | 30.19 | 33.41 | 35.72 |
| [illegible] | 7.01 | 9.39 | 10.86 | 17.34 | 25.99 | 28.87 | 31.53 | 34.81 | 37.16 |
| [illegible] | 7.63 | 10.12 | 11.65 | 18.34 | 27.20 | 30.14 | 32.85 | 36.19 | 38.58 |
| 20 | 8.26 | 10.85 | 12.44 | 19.34 | 28.41 | 31.41 | 34.17 | 37.57 | 40.00 |
| 21 | 8.90 | 11.59 | 13.24 | 20.34 | 29.62 | 32.67 | 35.48 | 38.93 | 41.40 |
| [illegible] | 9.54 | 12.34 | 14.04 | 21.34 | 30.81 | 33.92 | 36.78 | 40.29 | 42.80 |
| [illegible] | 10.20 | 13.09 | 14.85 | 22.34 | 32.01 | 35.17 | 38.08 | 41.64 | 44.18 |
| 24 | 10.86 | 13.85 | 15.66 | 23.34 | 33.20 | 36.42 | 39.36 | 42.98 | 45.56 |
| 25 | 11.52 | 14.61 | 16.47 | 24.34 | 34.38 | 37.65 | 40.65 | 44.31 | 46.93 |
| [illegible] | 12.20 | 15.38 | 17.29 | 25.34 | 35.56 | 38.89 | 41.92 | 45.64 | 48.29 |
| [illegible] | 12.88 | 16.15 | 18.11 | 26.34 | 36.74 | 40.11 | 43.19 | 46.96 | 49.64 |
| 28 | 13.56 | 16.93 | 18.94 | 27.34 | 37.92 | 41.34 | 44.46 | 48.28 | 50.99 |
| [illegible] | 14.26 | 17.71 | 19.77 | 28.34 | 39.09 | 42.56 | 45.72 | 49.59 | 52.34 |
| [illegible] | 14.95 | 18.49 | 20.60 | 29.34 | 40.26 | 43.77 | 46.98 | 50.89 | 53.67 |
| [illegible] | 22.16 | 26.51 | 29.05 | 39.34 | 51.81 | 55.76 | 59.34 | 63.69 | 66.77 |
| 50 | 29.71 | 34.76 | 37.69 | 49.33 | 63.17 | 67.50 | 71.42 | 76.15 | 79.49 |
| [illegible] | 37.48 | 43.19 | 46.46 | 59.33 | 74.40 | 79.08 | 83.30 | 88.38 | 91.95 |
| [illegible] | 45.44 | 51.74 | 55.33 | 69.33 | 85.53 | 90.53 | 95.02 | 100.43 | 104.21 |
| [illegible] | 53.54 | 60.39 | 64.28 | 79.33 | 96.58 | 101.88 | 106.63 | 112.33 | 116.32 |
| [illegible] | 61.75 | 69.13 | 73.29 | 89.33 | 107.57 | 113.15 | 118.14 | 124.12 | 128.30 |
| [illegible] | 70.06 | 77.93 | 82.36 | 99.33 | 118.50 | 124.34 | 129.56 | 135.81 | 140.17 |

# Student's *t*-Distribution

- Indexed by "degrees of freedom ($\nu$)" $X \sim t_\nu$
- Z ~ N(0,1), X ~ $\chi_\nu^2$
- Assuming Independence of Z and X:

$$T = \frac{Z}{\sqrt{X/\nu}} \sim t_\nu$$

Density Function:

$$f(t) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\pi\nu}\,\Gamma\left(\frac{\nu}{2}\right)}\left(1+\frac{t^2}{\nu}\right)^{-\frac{\nu+1}{2}} \quad -\infty < t < \infty \quad \nu > 0 \qquad E\{T\} = 0 \ (\nu > 1) \quad V\{T\} = \frac{\nu}{\nu-2} \ (\nu > 2)$$

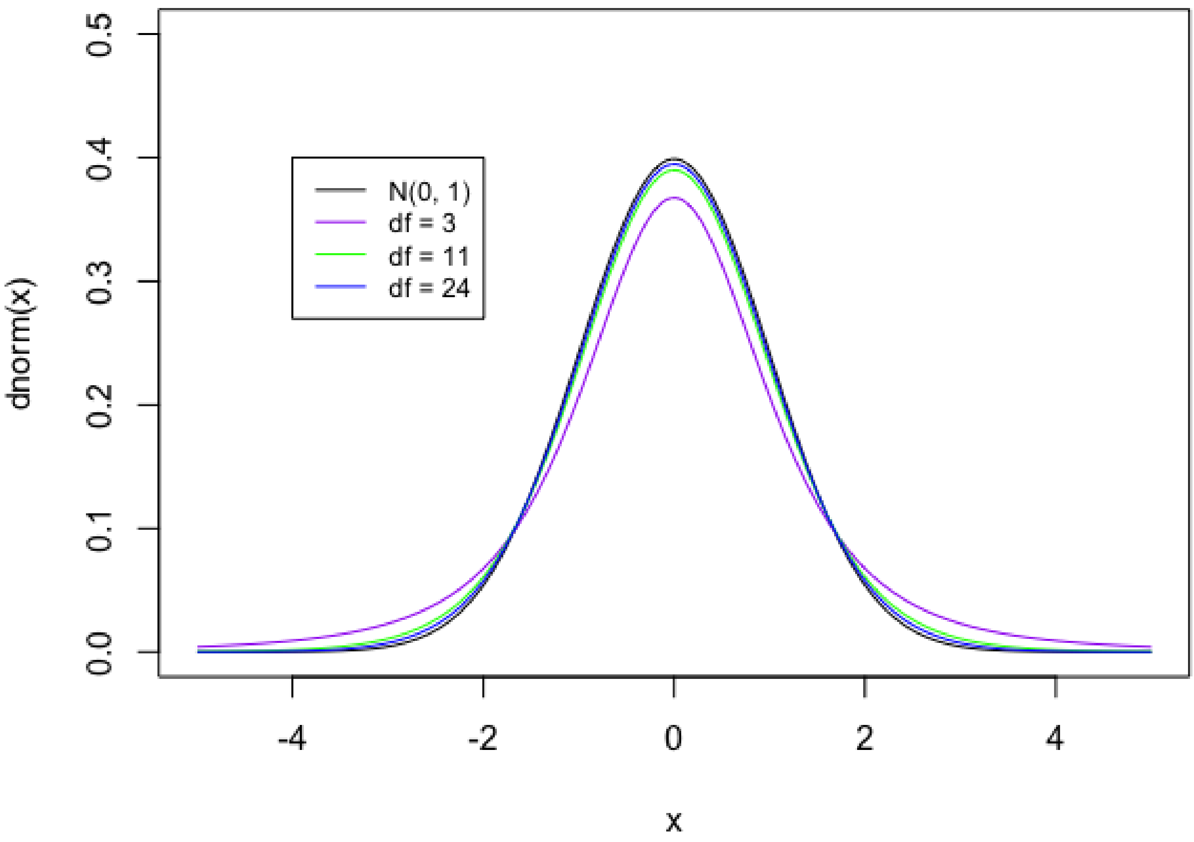
**Obtaining Probabilities /Quantiles in R:**

**To obtain: F(t) = P(T≤t) pt(t, ν)**

**To obtain: p^th quantile qt(p, ν)**

**Virtually all statistics textbooks give upper tail cut-off values for commonly used upper tail probabilities**

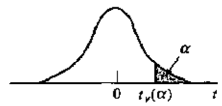
# *t* distributions



# Critical Values for Student’s t-Distributions



**TABLE 2** STUDENT'S $t$-DISTRIBUTION PERCENTAGE POINTS




| d.f. $\nu$ | $\alpha$ .250 | .100 | .050 | .025 | .010 | .00833 | .00625 | .005 | .0025 |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 1.000 | 3.078 | 6.314 | 12.706 | 31.821 | 38.190 | 50.923 | 63.657 | 127.321 |
| 2 | .816 | 1.886 | 2.920 | 4.303 | 6.965 | 7.649 | 8.860 | 9.925 | 14.089 |
| 3 | .765 | 1.638 | 2.353 | 3.182 | 4.541 | 4.857 | 5.392 | 5.841 | 7.453 |
| 4 | .741 | 1.533 | 2.132 | 2.776 | 3.747 | 3.961 | 4.315 | 4.604 | 5.598 |
| 5 | .727 | 1.476 | 2.015 | 2.571 | 3.365 | 3.534 | 3.810 | 4.032 | 4.773 |
| 6 | .718 | 1.440 | 1.943 | 2.447 | 3.143 | 3.287 | 3.521 | 3.707 | 4.317 |
| 7 | .711 | 1.415 | 1.895 | 2.365 | 2.998 | 3.128 | 3.335 | 3.499 | 4.029 |
| 8 | .706 | 1.397 | 1.860 | 2.306 | 2.896 | 3.016 | 3.206 | 3.355 | 3.833 |
| 9 | .703 | 1.383 | 1.833 | 2.262 | 2.821 | 2.933 | 3.111 | 3.250 | 3.690 |
| 10 | .700 | 1.372 | 1.812 | 2.228 | 2.764 | 2.870 | 3.038 | 3.169 | 3.581 |
| 11 | .697 | 1.363 | 1.796 | 2.201 | 2.718 | 2.820 | 2.981 | 3.106 | 3.497 |
| 12 | .695 | 1.356 | 1.782 | 2.179 | 2.681 | 2.779 | 2.934 | 3.055 | 3.428 |
| 13 | .694 | 1.350 | 1.771 | 2.160 | 2.650 | 2.746 | 2.896 | 3.012 | 3.372 |
| 14 | .692 | 1.345 | 1.761 | 2.145 | 2.624 | 2.718 | 2.864 | 2.977 | 3.326 |
| 15 | .691 | 1.341 | 1.753 | 2.131 | 2.602 | 2.694 | 2.837 | 2.947 | 3.286 |
| 16 | .690 | 1.337 | 1.746 | 2.120 | 2.583 | 2.673 | 2.813 | 2.921 | 3.252 |
| 17 | .689 | 1.333 | 1.740 | 2.110 | 2.567 | 2.655 | 2.793 | 2.898 | 3.222 |
| 18 | .688 | 1.330 | 1.734 | 2.101 | 2.552 | 2.639 | 2.775 | 2.878 | 3.197 |
| 19 | .688 | 1.328 | 1.729 | 2.093 | 2.539 | 2.625 | 2.759 | 2.861 | 3.174 |
| 20 | .687 | 1.325 | 1.725 | 2.086 | 2.528 | 2.613 | 2.744 | 2.845 | 3.153 |
| 21 | .686 | 1.323 | 1.721 | 2.080 | 2.518 | 2.601 | 2.732 | 2.831 | 3.135 |
| 22 | .686 | 1.321 | 1.717 | 2.074 | 2.508 | 2.591 | 2.720 | 2.819 | 3.119 |
| 23 | .685 | 1.319 | 1.714 | 2.069 | 2.500 | 2.582 | 2.710 | 2.807 | 3.104 |
| 24 | .685 | 1.318 | 1.711 | 2.064 | 2.492 | 2.574 | 2.700 | 2.797 | 3.091 |
| 25 | .684 | 1.316 | 1.708 | 2.060 | 2.485 | 2.566 | 2.692 | 2.787 | 3.078 |
| 26 | .684 | 1.315 | 1.706 | 2.056 | 2.479 | 2.559 | 2.684 | 2.779 | 3.067 |
| 27 | .684 | 1.314 | 1.703 | 2.052 | 2.473 | 2.552 | 2.676 | 2.771 | 3.057 |
| 28 | .683 | 1.313 | 1.701 | 2.048 | 2.467 | 2.546 | 2.669 | 2.763 | 3.047 |
| 29 | .683 | 1.311 | 1.699 | 2.045 | 2.462 | 2.541 | 2.663 | 2.756 | 3.038 |
| 30 | .683 | 1.310 | 1.697 | 2.042 | 2.457 | 2.536 | 2.657 | 2.750 | 3.030 |
| 40 | .681 | 1.303 | 1.684 | 2.021 | 2.423 | 2.499 | 2.616 | 2.704 | 2.971 |
| 60 | .679 | 1.296 | 1.671 | 2.000 | 2.390 | 2.463 | 2.575 | 2.660 | 2.915 |
| 120 | .677 | 1.289 | 1.658 | 1.980 | 2.358 | 2.428 | 2.536 | 2.617 | 2.860 |
| ∞ | .674 | 1.282 | 1.645 | 1.960 | 2.326 | 2.394 | 2.498 | 2.576 | 2.813 |

# *F*-Distribution

- Indexed by 2 "degrees of freedom ($\nu_1, \nu_2$)" $W \sim F_{\nu 1, \nu 2}$
- $X_1 \sim \chi_{\nu 1}^{\;2}$, $\quad X_2 \sim \chi_{\nu 2}^{\;2}$
- Assuming Independence of $X_1$ and $X_2$:

$$W = \frac{X_1/\nu_1}{X_2/\nu_2} \sim F_{\nu_1,\nu_2}$$

Density Function:

$$f(w) = \frac{\Gamma\left(\frac{\nu_1+\nu_2}{2}\right)}{\Gamma\left(\frac{\nu_1}{2}\right)\Gamma\left(\frac{\nu_2}{2}\right)} \left(\frac{\nu_1}{\nu_2}\right)^{\frac{\nu_1}{2}} w^{\frac{\nu_1}{2}-1} \left(\frac{1}{2^{\frac{\nu_1+\nu_2}{2}}}\right) \left(\frac{1}{2}\right)^{-\frac{\nu_1+\nu_2}{2}} \left[1+\frac{w\nu_1}{\nu_2}\right]^{-\frac{\nu_1+\nu_2}{2}} \qquad w > 0 \quad \nu_1, \nu_2 > 0$$

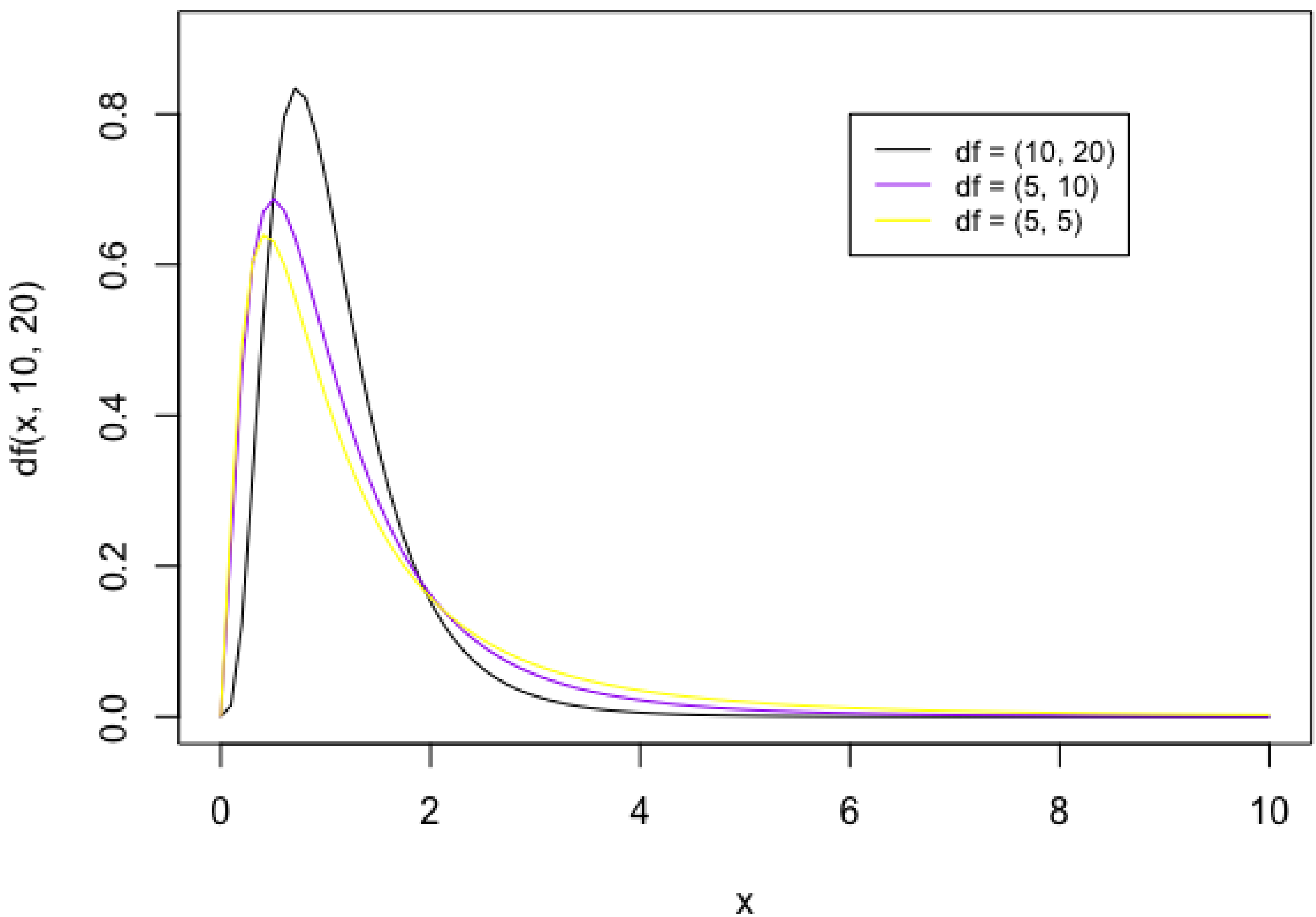
$$E\{W\} = \frac{\nu_2}{\nu_2 - 2} \quad (\nu_2 > 2) \qquad V\{W\} = \left(\frac{\nu_2}{\nu_2-2}\right)^2 \frac{\nu_1+\nu_2-2}{\nu_1(\nu_2-4)} \quad (\nu_2 > 4)$$

**Obtaining Probabilities/Quantiles in R:**

**To obtain: F(w) = P(W≤w): pf(w, $\nu_1$, $\nu_2$)**

**p^th quantile: qf(p, $\nu_1$, $\nu_2$)**

**Virtually all statistics textbooks give upper tail cut-off values for some upper probabilities**

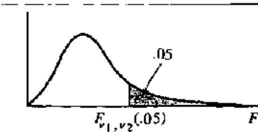


df = (10, 20)
df = (5, 10)
df = (5, 5)
df(x, 10, 20)
0.0
0.2
0.4
0.6
0.8
0
2
4
6
8
10
x

# Critical Values for F-distributions   P(F ≤ Table Value) = 0.95



TABLE 5 *F*-DISTRIBUTION PERCENTAGE POINTS ($\alpha$ = .05)




| $\nu_2 \backslash \nu_1$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 12 | 15 | 20 | 25 | 30 | 40 | 60 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 161.5 | 199.5 | 215.7 | 224.6 | 230.2 | 234.0 | 236.8 | 238.9 | 240.5 | 241.9 | 243.9 | 246.0 | 248.0 | 249.3 | 250.1 | 251.1 | 252.2 |
| 2 | 18.51 | 19.00 | 19.16 | 19.25 | 19.30 | 19.33 | 19.35 | 19.37 | 19.38 | 19.40 | 19.41 | 19.43 | 19.45 | 19.46 | 19.46 | 19.47 | 19.48 |
| 3 | 10.13 | 9.55 | 9.28 | 9.12 | 9.01 | 8.94 | 8.89 | 8.85 | 8.81 | 8.79 | 8.74 | 8.70 | 8.66 | 8.63 | 8.62 | 8.59 | 8.57 |
| 4 | 7.71 | 6.94 | 6.59 | 6.39 | 6.26 | 6.16 | 6.09 | 6.04 | 6.00 | 5.96 | 5.91 | 5.86 | 5.80 | 5.77 | 5.75 | 5.72 | 5.69 |
| 5 | 6.61 | 5.79 | 5.41 | 5.19 | 5.05 | 4.95 | 4.88 | 4.82 | 4.77 | 4.74 | 4.68 | 4.62 | 4.56 | 4.52 | 4.50 | 4.46 | 4.43 |
| 6 | 5.99 | 5.14 | 4.76 | 4.53 | 4.39 | 4.28 | 4.21 | 4.15 | 4.10 | 4.06 | 4.00 | 3.94 | 3.87 | 3.83 | 3.81 | 3.77 | 3.74 |
| 7 | 5.59 | 4.74 | 4.35 | 4.12 | 3.97 | 3.87 | 3.79 | 3.73 | 3.68 | 3.64 | 3.57 | 3.51 | 3.44 | 3.40 | 3.38 | 3.34 | 3.30 |
| 8 | 5.32 | 4.46 | 4.07 | 3.84 | 3.69 | 3.58 | 3.50 | 3.44 | 3.39 | 3.35 | 3.28 | 3.22 | 3.15 | 3.11 | 3.08 | 3.04 | 3.01 |
| 9 | 5.12 | 4.26 | 3.86 | 3.63 | 3.48 | 3.37 | 3.29 | 3.23 | 3.18 | 3.14 | 3.07 | 3.01 | 2.94 | 2.89 | 2.86 | 2.83 | 2.79 |
| 10 | 4.96 | 4.10 | 3.71 | 3.48 | 3.33 | 3.22 | 3.14 | 3.07 | 3.02 | 2.98 | 2.91 | 2.85 | 2.77 | 2.73 | 2.70 | 2.66 | 2.62 |
| 11 | 4.84 | 3.98 | 3.59 | 3.36 | 3.20 | 3.09 | 3.01 | 2.95 | 2.90 | 2.85 | 2.79 | 2.72 | 2.65 | 2.60 | 2.57 | 2.53 | 2.49 |
| 12 | 4.75 | 3.89 | 3.49 | 3.26 | 3.11 | 3.00 | 2.91 | 2.85 | 2.80 | 2.75 | 2.69 | 2.62 | 2.54 | 2.50 | 2.47 | 2.43 | 2.38 |
| 13 | 4.67 | 3.81 | 3.41 | 3.18 | 3.03 | 2.92 | 2.83 | 2.77 | 2.71 | 2.67 | 2.60 | 2.53 | 2.46 | 2.41 | 2.38 | 2.34 | 2.30 |
| 14 | 4.60 | 3.74 | 3.34 | 3.11 | 2.96 | 2.85 | 2.76 | 2.70 | 2.65 | 2.60 | 2.53 | 2.46 | 2.39 | 2.34 | 2.31 | 2.27 | 2.22 |
| 15 | 4.54 | 3.68 | 3.29 | 3.06 | 2.90 | 2.79 | 2.71 | 2.64 | 2.59 | 2.54 | 2.48 | 2.40 | 2.33 | 2.28 | 2.25 | 2.20 | 2.16 |
| 16 | 4.49 | 3.63 | 3.24 | 3.01 | 2.85 | 2.74 | 2.66 | 2.59 | 2.54 | 2.49 | 2.42 | 2.35 | 2.28 | 2.23 | 2.19 | 2.15 | 2.11 |
| 17 | 4.45 | 3.59 | 3.20 | 2.96 | 2.81 | 2.70 | 2.61 | 2.55 | 2.49 | 2.45 | 2.38 | 2.31 | 2.23 | 2.18 | 2.15 | 2.10 | 2.06 |
| 18 | 4.41 | 3.55 | 3.16 | 2.93 | 2.77 | 2.66 | 2.58 | 2.51 | 2.46 | 2.41 | 2.34 | 2.27 | 2.19 | 2.14 | 2.11 | 2.06 | 2.02 |
| 19 | 4.38 | 3.52 | 3.13 | 2.90 | 2.74 | 2.63 | 2.54 | 2.48 | 2.42 | 2.38 | 2.31 | 2.23 | 2.16 | 2.11 | 2.07 | 2.03 | 1.98 |
| 20 | 4.35 | 3.49 | 3.10 | 2.87 | 2.71 | 2.60 | 2.51 | 2.45 | 2.39 | 2.35 | 2.28 | 2.20 | 2.12 | 2.07 | 2.04 | 1.99 | 1.95 |
| 21 | 4.32 | 3.47 | 3.07 | 2.84 | 2.68 | 2.57 | 2.49 | 2.42 | 2.37 | 2.32 | 2.25 | 2.18 | 2.10 | 2.05 | 2.01 | 1.96 | 1.92 |
| 22 | 4.30 | 3.44 | 3.05 | 2.82 | 2.66 | 2.55 | 2.46 | 2.40 | 2.34 | 2.30 | 2.23 | 2.15 | 2.07 | 2.02 | 1.98 | 1.94 | 1.89 |
| 23 | 4.28 | 3.42 | 3.03 | 2.80 | 2.64 | 2.53 | 2.44 | 2.37 | 2.32 | 2.27 | 2.20 | 2.13 | 2.05 | 2.00 | 1.96 | 1.91 | 1.86 |
| 24 | 4.26 | 3.40 | 3.01 | 2.78 | 2.62 | 2.51 | 2.42 | 2.36 | 2.30 | 2.25 | 2.18 | 2.11 | 2.03 | 1.97 | 1.94 | 1.89 | 1.84 |
| 25 | 4.24 | 3.39 | 2.99 | 2.76 | 2.60 | 2.49 | 2.40 | 2.34 | 2.28 | 2.24 | 2.16 | 2.09 | 2.01 | 1.96 | 1.92 | 1.87 | 1.82 |
| 26 | 4.23 | 3.37 | 2.98 | 2.74 | 2.59 | 2.47 | 2.39 | 2.32 | 2.27 | 2.22 | 2.15 | 2.07 | 1.99 | 1.94 | 1.90 | 1.85 | 1.80 |
| 27 | 4.21 | 3.35 | 2.96 | 2.73 | 2.57 | 2.46 | 2.37 | 2.31 | 2.25 | 2.20 | 2.13 | 2.06 | 1.97 | 1.92 | 1.88 | 1.84 | 1.79 |
| 28 | 4.20 | 3.34 | 2.95 | 2.71 | 2.56 | 2.45 | 2.36 | 2.29 | 2.24 | 2.19 | 2.12 | 2.04 | 1.96 | 1.91 | 1.87 | 1.82 | 1.77 |
| 29 | 4.18 | 3.33 | 2.93 | 2.70 | 2.55 | 2.43 | 2.35 | 2.28 | 2.22 | 2.18 | 2.10 | 2.03 | 1.94 | 1.89 | 1.85 | 1.81 | 1.75 |
| 30 | 4.17 | 3.32 | 2.92 | 2.69 | 2.53 | 2.42 | 2.33 | 2.27 | 2.21 | 2.16 | 2.09 | 2.01 | 1.93 | 1.88 | 1.84 | 1.79 | 1.74 |
| 40 | 4.08 | 3.23 | 2.84 | 2.61 | 2.45 | 2.34 | 2.25 | 2.18 | 2.12 | 2.08 | 2.00 | 1.92 | 1.84 | 1.78 | 1.74 | 1.69 | 1.64 |
| 60 | 4.00 | 3.15 | 2.76 | 2.53 | 2.37 | 2.25 | 2.17 | 2.10 | 2.04 | 1.99 | 1.92 | 1.84 | 1.75 | 1.69 | 1.65 | 1.59 | 1.53 |
| 120 | 3.92 | 3.07 | 2.68 | 2.45 | 2.29 | 2.18 | 2.09 | 2.02 | 1.96 | 1.91 | 1.83 | 1.75 | 1.66 | 1.60 | 1.55 | 1.50 | 1.43 |
| ∞ | 3.84 | 3.00 | 2.61 | 2.37 | 2.21 | 2.10 | 2.01 | 1.94 | 1.88 | 1.83 | 1.75 | 1.67 | 1.57 | 1.51 | 1.46 | 1.39 | 1.32 |

# Multivariate Normal Distribution

$$X = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_p \end{bmatrix}, \mu_x = E(X) = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_p \end{bmatrix}, \Sigma_X = \sigma^2(X) = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1p} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{1p} & \sigma_{2p} & \cdots & \sigma_{pp} \end{bmatrix}$$

Multivariate normal density function:

$$f(x) = (2\pi)^{-p/2} \left| \Sigma_X^{-1/2} \right| e^{-\frac{1}{2}(X-\mu_X)' \Sigma_X^{-1} (X-\mu_X)}$$

Notation: $X \sim \mathrm{N}_p(\mu_X, \Sigma_X)$

Results:

$$X_i \sim N(\mu_i, \sigma_{ii}), \qquad i = 1, \dots, p$$
$$\mathrm{cov}\left(X_i, X_j\right) = \sigma_{ij}, \qquad i \neq j$$

Note: If *A* is a full rank matrix of constants, then:

$$W = AX \sim \mathrm{N}\,(A_{\mu x}, A\Sigma_x A')$$

# Results Involving Multivariate Normal - I

If $\mathbf{\Sigma}_{\mathbf{X}}$ is positive definite with eigenvalue/eigenvector pairs $\left(\lambda_i, \mathbf{e}_i\right) \quad i = 1, \ldots, p$

Then $\mathbf{\Sigma}_{\mathbf{X}}^{-1}$ is positive definite with eigenvalue/eigenvector pairs $\left(1/\lambda_i, \mathbf{e}_i\right) \quad i = 1, \ldots, p$

$$\Rightarrow \quad \mathbf{\Sigma}_{\mathbf{X}} = \sum_{i=1}^{n} \lambda_i \mathbf{e}_i \mathbf{e}_i' = \mathbf{P \Lambda P}' \qquad \mathbf{\Sigma}_{\mathbf{X}}^{-1} = \sum_{i=1}^{n} \left(\frac{1}{\lambda_i}\right) \mathbf{e}_i \mathbf{e}_i' = \mathbf{P \Lambda}^{-1} \mathbf{P}'$$

where $\mathbf{P} = \left[\mathbf{e}_1 \quad \mathrm{L} \quad \mathbf{e}_p\right] \quad \mathbf{\Lambda} = \mathrm{diag}\left\{\lambda_i\right\}$

Contours of constant density for $p$-dim MVN are ellipsoids wrt $\mathbf{x}$ s.t. $\left(\mathbf{x} - \boldsymbol{\mu}_{\mathbf{X}}\right)' \mathbf{\Sigma}_{\mathbf{X}}^{-1} \left(\mathbf{x} - \boldsymbol{\mu}_{\mathbf{X}}\right) = c^2$

Ellipsoids cenetered at $\boldsymbol{\mu}_{\mathbf{X}}$ with axes $\pm c\sqrt{\lambda_i}\mathbf{e}_i \quad i = 1, \ldots, p$

Setting $c^2 = \chi_p^2(\alpha)$ s.t. $P\left(\chi_p^2 \geq \chi_p^2(\alpha)\right) = \alpha$

$\Rightarrow$ the solid ellipsoid $\left(\mathbf{x} - \boldsymbol{\mu}_{\mathbf{X}}\right)' \mathbf{\Sigma}_{\mathbf{X}}^{-1} \left(\mathbf{x} - \boldsymbol{\mu}_{\mathbf{X}}\right) \leq \chi_p^2(\alpha)$ has probability $1 - \alpha$

$\mathrm{COV}\left\{X_i, X_j\right\} = 0 \quad \Rightarrow \quad X_i, X_j$ independent

# Results Involving Multivariate Normal - II

Conditional Distributions of components are MVN: $\left(\mathbf{X}_1 \equiv q \times 1, \quad \mathbf{X}_2 \equiv (p-q) \times 1\right)$

$$\mathbf{X} = \begin{bmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{bmatrix} \sim N_p\left(\boldsymbol{\mu}_{\mathbf{X}}, \boldsymbol{\Sigma}_{\mathbf{X}}\right) \qquad \boldsymbol{\mu}_{\mathbf{X}} = \begin{bmatrix} \boldsymbol{\mu}_1 \\ \boldsymbol{\mu}_2 \end{bmatrix} \qquad \boldsymbol{\Sigma}_{\mathbf{X}} = \begin{bmatrix} \boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12} \\ \boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_{22} \end{bmatrix} \qquad \text{s.t. } \left|\boldsymbol{\Sigma}_{22}\right| > 0$$

$\Rightarrow \quad \mathbf{X}_1 \mid \mathbf{X}_2 = \mathbf{x}_2 \sim N_q \quad$ with

Mean: $\boldsymbol{\mu}_1 + \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}\left(\mathbf{x}_2 - \boldsymbol{\mu}_2\right) = \boldsymbol{\mu}_{1|\mathbf{x}_2}$ and Variance-Covariance Matrix: $\boldsymbol{\Sigma}_{11} - \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}\boldsymbol{\Sigma}_{21} = \boldsymbol{\Sigma}_{11|\mathbf{x}_2}$

$$\Rightarrow \quad f\left(\mathbf{x}_1 \mid \mathbf{x}_2\right) = \left(2\pi\right)^{-q/2}\left|\boldsymbol{\Sigma}_{11|\mathbf{x}_2}\right|^{-1/2} \exp\left\{-\frac{1}{2}\left(\mathbf{x}_1 - \boldsymbol{\mu}_{1|\mathbf{x}_2}\right)' \boldsymbol{\Sigma}_{11|\mathbf{x}_2}^{-1}\left(\mathbf{x}_1 - \boldsymbol{\mu}_{1|\mathbf{x}_2}\right)\right\} \quad -\infty < x_{1i} < \infty \; i = 1, \ldots, q$$

Note that the conditional mean depends on the specific level(s) $\mathbf{x}_2$, the conditional variance does not.

Special Case: $p = 2, q = 1 \qquad X_1 \mid X_2 = x_2 \sim N\left(\mu_{1|x_2}, \sigma_{11|x_2}\right)$

$$\mu_{1|x_2} = \mu_1 + \frac{\sigma_{12}}{\sigma_{22}}\left(x_2 - \mu_2\right) \qquad \sigma_{11|x_2} = \sigma_{11} - \frac{\sigma_{12}^2}{\sigma_{22}} = \sigma_{11} - \left(\frac{\sigma_{11}}{\sigma_{11}}\right)\frac{\sigma_{12}^2}{\sigma_{22}} = \sigma_{11}\left(1 - \rho_{12}^2\right)$$

$$\Rightarrow \quad f\left(x_1, x_2\right) = \frac{1}{\sqrt{2\pi\sigma_{11}\left(1 - \rho_{12}^2\right)}} \exp\left\{-\frac{1}{2\sigma_{11}\left(1 - \rho_{12}^2\right)}\left(x_1 - \left(\mu_1 + \frac{\sigma_{12}}{\sigma_{22}}\left(x_2 - \mu_2\right)\right)\right)^2\right\} \quad -\infty < x_1 < \infty$$

# Example with $p$ = 2

Joint Distribution:

$$f(x_1,x_2)=\frac{1}{2\pi\sqrt{\sigma_1^2\sigma_2^2\left(1-\rho^2\right)}}\exp\left\{-\left(\frac{1}{2\left(1-\rho^2\right)}\right)\left[\frac{\left(x_1-\mu_1\right)^2}{\sigma_1^2}-\frac{2\rho\left(x_1-\mu_1\right)\left(x_2-\mu_2\right)}{\sigma_1\sigma_2}+\frac{\left(x_2-\mu_2\right)^2}{\sigma_2^2}\right]\right\}\quad -\infty<x_1,x_2<\infty$$

Marginal (aka Unconditional) Distributions:

$$f_1\left(x_1\right)=\frac{1}{\sqrt{2\pi\sigma_1^2}}\exp\left\{-\frac{\left(x_1-\mu_1\right)^2}{2\sigma_1^2}\right\}\quad -\infty<x_1<\infty$$

$$f_2\left(x_2\right)=\frac{1}{\sqrt{2\pi\sigma_2^2}}\exp\left\{-\frac{\left(x_2-\mu_2\right)^2}{2\sigma_2^2}\right\}\quad -\infty<x_2<\infty$$

$$X_1\sim N\left(\mu_1,\sigma_1^2\right)\qquad X_2\sim N\left(\mu_2,\sigma_2^2\right)$$

Conditional Distributions:

$$f\left(x_2\,|\,x_1\right)=\frac{1}{\sqrt{2\pi\sigma_2^2\left(1-\rho^2\right)}}\exp\left\{-\left(\frac{1}{2\left(1-\rho^2\right)\sigma_2^2}\right)\left[x_2-\left(\mu_2+\frac{\left(x_1-\mu_1\right)\rho\sigma_2}{\sigma_1}\right)\right]^2\right\}\quad -\infty<x_2<\infty$$

$$f\left(x_1\,|\,x_2\right)=\frac{1}{\sqrt{2\pi\sigma_1^2\left(1-\rho^2\right)}}\exp\left\{-\left(\frac{1}{2\left(1-\rho^2\right)\sigma_1^2}\right)\left[x_1-\left(\mu_1+\frac{\left(x_2-\mu_2\right)\rho\sigma_1}{\sigma_2}\right)\right]^2\right\}\quad -\infty<x_1<\infty$$

$$X_2\,|\,X_1=x_1\sim N\left[\mu_2+\frac{\left(x_1-\mu_1\right)\rho\sigma_2}{\sigma_1},\sigma_2^2\left(1-\rho^2\right)\right]\qquad X_1\,|\,X_2=x_2\sim N\left[\mu_1+\frac{\left(x_2-\mu_2\right)\rho\sigma_1}{\sigma_2},\sigma_1^2\left(1-\rho^2\right)\right]$$

# Results Involving Multivariate Normal - III

$\mathbf{X}_1,\ldots,\mathbf{X}_n$ independendent with common variance matrices: $\mathbf{X}_j \sim N_p\left(\boldsymbol{\mu}_j,\boldsymbol{\Sigma}\right)$

$$\mathbf{W}_1 = a_1\mathbf{X}_1 + \ldots + a_n\mathbf{X}_n \sim N_p\left(\sum_{j=1}^{n} a_j\boldsymbol{\mu}_j, \left(\sum_{j=1}^{n} a_j^2\right)\boldsymbol{\Sigma}\right)$$

$$\mathbf{W}_2 = b_1\mathbf{X}_1 + \ldots + b_n\mathbf{X}_n \sim N_p\left(\sum_{j=1}^{n} b_j\boldsymbol{\mu}_j, \left(\sum_{j=1}^{n} b_j^2\right)\boldsymbol{\Sigma}\right)$$

$\mathbf{W}_1, \mathbf{W}_2$ are jointly distributed normal with

$$\boldsymbol{\mu}_{\mathbf{W}_1\mathbf{W}_2} = \begin{bmatrix} \sum_{j=1}^{n} a_j\boldsymbol{\mu}_j \\ \sum_{j=1}^{n} b_j\boldsymbol{\mu}_j \end{bmatrix} \quad \text{and} \quad \boldsymbol{\Sigma}_{\mathbf{W}_1\mathbf{W}_2} = \begin{bmatrix} \left(\sum_{j=1}^{n} a_j^2\right)\boldsymbol{\Sigma} & \left(\sum_{j=1}^{n} a_j b_j\right)\boldsymbol{\Sigma} \\ \left(\sum_{j=1}^{n} a_j b_j\right)\boldsymbol{\Sigma} & \left(\sum_{j=1}^{n} b_j^2\right)\boldsymbol{\Sigma} \end{bmatrix}$$

# Results Involving Multivariate Normal - IV

**Theorem (p. 163):** Let $\boldsymbol{X} \sim N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$. Then:

(a) $(\boldsymbol{X} - \boldsymbol{\mu})' \boldsymbol{\Sigma}^{-1}(\boldsymbol{X} - \boldsymbol{\mu}) \sim \chi^2_p$

(b) The probability that $\boldsymbol{X}$ is inside the solid ellipsoid

$$\{\boldsymbol{X}: (\boldsymbol{X} - \boldsymbol{\mu})' \boldsymbol{\Sigma}^{-1}(\boldsymbol{X} - \boldsymbol{\mu}) \leq \chi^2_p(\alpha)\}$$

is $1 - \alpha$, where $\chi^2_p(\alpha)$ denotes the upper $\alpha$ percentile of the $\chi^2_p$ distribution.

# Exercise

Let $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_{60}$ be a random sample of size 60 from a four-variate normal distribution with mean $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$. Specify the distribution of:

(a) $\bar{\boldsymbol{X}}$

(b) $(\boldsymbol{X}_1 - \boldsymbol{\mu})' \boldsymbol{\Sigma}^{-1}(\boldsymbol{X}_1 - \boldsymbol{\mu})$

(c) $n(\bar{\boldsymbol{X}} - \boldsymbol{\mu})' \boldsymbol{\Sigma}^{-1}(\bar{\boldsymbol{X}} - \boldsymbol{\mu})$

# Multivariate Normal Likelihood Function

$\mathbf{X}_1,\ldots,\mathbf{X}_n \equiv$ random sample from $N_p(\boldsymbol{\mu},\boldsymbol{\Sigma})$ with joint density:

$$\prod_{j=1}^{n}\left[(2\pi)^{-p/2}|\boldsymbol{\Sigma}|^{-1/2}\exp\left\{-\frac{1}{2}(\mathbf{x}_j-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{x}_j-\boldsymbol{\mu})\right\}\right]=(2\pi)^{-np/2}|\boldsymbol{\Sigma}|^{-n/2}\exp\left\{-\frac{1}{2}\sum_{j=1}^{n}(\mathbf{x}_j-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{x}_j-\boldsymbol{\mu})\right\}$$

Exponential term multiplied by $-1/2$:

$$\sum_{j=1}^{n}(\mathbf{x}_j-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{x}_j-\boldsymbol{\mu})=\mathrm{tr}\left\{\sum_{j=1}^{n}(\mathbf{x}_j-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{x}_j-\boldsymbol{\mu})\right\}=\mathrm{tr}\left\{\sum_{j=1}^{n}\boldsymbol{\Sigma}^{-1}(\mathbf{x}_j-\boldsymbol{\mu})(\mathbf{x}_j-\boldsymbol{\mu})'\right\}=\mathrm{tr}\left\{\boldsymbol{\Sigma}^{-1}\sum_{j=1}^{n}(\mathbf{x}_j-\boldsymbol{\mu})(\mathbf{x}_j-\boldsymbol{\mu})'\right\}$$

$$\sum_{j=1}^{n}(\mathbf{x}_j-\boldsymbol{\mu})(\mathbf{x}_j-\boldsymbol{\mu})'=\sum_{j=1}^{n}(\mathbf{x}_j-\bar{\mathbf{x}}+\bar{\mathbf{x}}-\boldsymbol{\mu})(\mathbf{x}_j-\bar{\mathbf{x}}+\bar{\mathbf{x}}-\boldsymbol{\mu})'=\sum_{j=1}^{n}(\mathbf{x}_j-\bar{\mathbf{x}})(\mathbf{x}_j-\bar{\mathbf{x}})'+\sum_{j=1}^{n}(\bar{\mathbf{x}}-\boldsymbol{\mu})(\bar{\mathbf{x}}-\boldsymbol{\mu})'+2\sum_{j=1}^{n}(\mathbf{x}_j-\bar{\mathbf{x}})(\bar{\mathbf{x}}-\boldsymbol{\mu})'$$

$$=\sum_{j=1}^{n}(\mathbf{x}_j-\bar{\mathbf{x}})(\mathbf{x}_j-\bar{\mathbf{x}})'+\sum_{j=1}^{n}(\bar{\mathbf{x}}-\boldsymbol{\mu})(\bar{\mathbf{x}}-\boldsymbol{\mu})'+2\left(\sum_{j=1}^{n}(\mathbf{x}_j-\bar{\mathbf{x}})\right)(\bar{\mathbf{x}}-\boldsymbol{\mu})'=\sum_{j=1}^{n}(\mathbf{x}_j-\bar{\mathbf{x}})(\mathbf{x}_j-\bar{\mathbf{x}})'+n(\bar{\mathbf{x}}-\boldsymbol{\mu})(\bar{\mathbf{x}}-\boldsymbol{\mu})'+0$$

Joint Density / Likelihood Function:

$$(2\pi)^{-np/2}|\boldsymbol{\Sigma}|^{-n/2}\exp\left\{-\frac{1}{2}\mathrm{tr}\left\{\boldsymbol{\Sigma}^{-1}\left[\sum_{j=1}^{n}(\mathbf{x}_j-\bar{\mathbf{x}})(\mathbf{x}_j-\bar{\mathbf{x}})'+n(\bar{\mathbf{x}}-\boldsymbol{\mu})(\bar{\mathbf{x}}-\boldsymbol{\mu})'\right]\right\}\right\}$$

# Maximum Likelihood Estimator of $\boldsymbol{\mu}$

Likelihood Function:

$$(2\pi)^{-np/2}|\boldsymbol{\Sigma}|^{-n/2}\exp\left\{-\frac{1}{2}\mathrm{tr}\left\{\boldsymbol{\Sigma}^{-1}\left[\sum_{j=1}^{n}(\mathbf{x}_j-\bar{\mathbf{x}})(\mathbf{x}_j-\bar{\mathbf{x}})'+n(\bar{\mathbf{x}}-\boldsymbol{\mu})(\bar{\mathbf{x}}-\boldsymbol{\mu})'\right]\right\}\right\}=$$

$$(2\pi)^{-np/2}|\boldsymbol{\Sigma}|^{-n/2}\exp\left\{-\frac{1}{2}\left[\mathrm{tr}\left\{\boldsymbol{\Sigma}^{-1}\left[\sum_{j=1}^{n}(\mathbf{x}_j-\bar{\mathbf{x}})(\mathbf{x}_j-\bar{\mathbf{x}})'\right]\right\}+\mathrm{tr}\left\{\boldsymbol{\Sigma}^{-1}n(\bar{\mathbf{x}}-\boldsymbol{\mu})(\bar{\mathbf{x}}-\boldsymbol{\mu})'\right\}\right]\right\}$$

$$(2\pi)^{-np/2}|\boldsymbol{\Sigma}|^{-n/2}\exp\left\{-\frac{1}{2}\left[\mathrm{tr}\left\{\boldsymbol{\Sigma}^{-1}\left[\sum_{j=1}^{n}(\mathbf{x}_j-\bar{\mathbf{x}})(\mathbf{x}_j-\bar{\mathbf{x}})'\right]\right\}+n(\bar{\mathbf{x}}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\bar{\mathbf{x}}-\boldsymbol{\mu})\right]\right\}$$

Maximum Likelihood Estimator for $\boldsymbol{\mu}$ :

$$L(\boldsymbol{\mu},\boldsymbol{\Sigma})=(2\pi)^{-np/2}|\boldsymbol{\Sigma}|^{-n/2}\exp\left\{-\frac{1}{2}\left[\mathrm{tr}\left\{\boldsymbol{\Sigma}^{-1}\left[\sum_{j=1}^{n}(\mathbf{x}_j-\bar{\mathbf{x}})(\mathbf{x}_j-\bar{\mathbf{x}})'\right]\right\}\right]\right\}\exp\left\{-\frac{1}{2}\left[n(\bar{\mathbf{x}}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\bar{\mathbf{x}}-\boldsymbol{\mu})\right]\right\}$$

maximized when $\hat{\boldsymbol{\mu}}=\bar{\mathbf{X}}$ $\Rightarrow$

$\exp\left\{-\frac{1}{2}\left[n(\bar{\mathbf{x}}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\bar{\mathbf{x}}-\boldsymbol{\mu})\right]\right\}=1$ is at its maximum since $\boldsymbol{\Sigma}^{-1}$ is positive definite

# Maximum Likelihood Estimator of Σ

Result: $\mathbf{B} \equiv p \times p$ positive definite, scalar $b > 0$, $\boldsymbol{\Sigma} \equiv$ positive definite:

$$\frac{1}{|\boldsymbol{\Sigma}|^{b}} \exp\left\{-\frac{1}{2}\operatorname{tr}\left\{\boldsymbol{\Sigma}^{-1}\mathbf{B}\right\}\right\} \leq \frac{1}{|\mathbf{B}|^{b}}(2b)^{bp} e^{-bp} \quad \text{with equality holding at } \boldsymbol{\Sigma} = \left(\frac{1}{2b}\right)\mathbf{B}$$

Maximum Likelihood Estimator for $\boldsymbol{\Sigma}$ evaluated at $\hat{\boldsymbol{\mu}}$:

$$L(\boldsymbol{\mu}, \boldsymbol{\Sigma}) = (2\pi)^{-np/2}|\boldsymbol{\Sigma}|^{-n/2} \exp\left\{-\frac{1}{2}\left[\operatorname{tr}\left\{\boldsymbol{\Sigma}^{-1}\left[\sum_{j=1}^{n}\left(\mathbf{x}_j - \bar{\mathbf{x}}\right)\left(\mathbf{x}_j - \bar{\mathbf{x}}\right)'\right]\right\}\right]\right\} \quad \text{setting} \quad b = \frac{n}{2} \quad \mathbf{B} = \sum_{j=1}^{n}\left(\mathbf{x}_j - \bar{\mathbf{x}}\right)\left(\mathbf{x}_j - \bar{\mathbf{x}}\right)'$$

$$\Rightarrow \quad \hat{\boldsymbol{\Sigma}} = \left(\frac{1}{2(n/2)}\right)\sum_{j=1}^{n}\left(\mathbf{x}_j - \bar{\mathbf{x}}\right)\left(\mathbf{x}_j - \bar{\mathbf{x}}\right)' = \frac{1}{n}\sum_{j=1}^{n}\left(\mathbf{x}_j - \bar{\mathbf{x}}\right)\left(\mathbf{x}_j - \bar{\mathbf{x}}\right)' = \frac{n-1}{n}\mathbf{S}$$

Likelihood Function evaluated at the observed ML estimates:

$$L\left(\hat{\boldsymbol{\mu}}, \hat{\boldsymbol{\Sigma}}\right) = (2\pi)^{-np/2}\left|\hat{\boldsymbol{\Sigma}}\right|^{-n/2} \exp\left\{-\frac{1}{2}\left[\operatorname{tr}\left\{\left(\hat{\boldsymbol{\Sigma}}\right)^{-1}\left[\sum_{j=1}^{n}\left(\mathbf{x}_j - \bar{\mathbf{x}}\right)\left(\mathbf{x}_j - \bar{\mathbf{x}}\right)'\right]\right\}\right]\right\} = (2\pi)^{-np/2}\left|\hat{\boldsymbol{\Sigma}}\right|^{-n/2} e^{-np/2}$$

Note: $\left|\hat{\boldsymbol{\Sigma}}\right| = \left|\frac{n-1}{n}\mathbf{S}\right| = \left(\frac{n-1}{n}\right)^{p}|\mathbf{S}| \Rightarrow$

$$L\left(\hat{\boldsymbol{\mu}}, \hat{\boldsymbol{\Sigma}}\right) = (2\pi)^{-np/2} e^{-np/2}\left(\frac{n-1}{n}\right)^{p}|\mathbf{S}| \quad = \text{constant} \times \text{generalized inverse}$$

# Results for ML Estimators and Large-Sample Properties

$\boldsymbol{\theta} \equiv$ Parameter vector $\quad h(\boldsymbol{\theta}) \equiv$ function of $\boldsymbol{\theta}$

ML Estimate of $h(\boldsymbol{\theta}) \equiv h\left(\hat{\boldsymbol{\theta}}\right)$

Sufficient Statistics: Joint density of $\mathbf{x}_1,\dots,\mathbf{x}_n$ depends only on the observed data through:

$\bar{\mathbf{x}}$ and $(n-1)\mathbf{S} = \sum_{j=1}^{n}\left(\mathbf{x}_j - \bar{\mathbf{x}}\right)'\left(\mathbf{x}_j - \bar{\mathbf{x}}\right) \quad \Rightarrow \quad \bar{\mathbf{X}}$ and $\mathbf{S}$ are Sufficient Statistics

Sampling Distributions of $\bar{\mathbf{X}}$ and $\mathbf{S}$ under Normality and Independence:

$\bar{\mathbf{X}} \sim N_p\left(\boldsymbol{\mu}, \frac{1}{n}\boldsymbol{\Sigma}\right) \qquad (n-1)\mathbf{S} \sim$ Wishart w/ df $= n-1 \qquad \bar{\mathbf{X}}$ and $\mathbf{S}$ are independent

Wishart distribution with $m$ d.f. : Distribution of $\sum_{j=1}^{m}\mathbf{Z}_j\mathbf{Z}_j'$ where $\mathbf{Z}_j \sim NID(\mathbf{0},\boldsymbol{\Sigma})$

$\mathbf{X}_1,\dots,\mathbf{X}_n$ independent with mean $\boldsymbol{\mu}$ and finite variance-covariance $\boldsymbol{\Sigma}$ with Large $n$ and $n-p$:

$\sqrt{n}\left(\bar{\mathbf{X}} - \boldsymbol{\mu}\right)$ is approximately distributed $N_p(\mathbf{0},\boldsymbol{\Sigma})$

$n\left(\bar{\mathbf{X}} - \boldsymbol{\mu}\right)'\mathbf{S}^{-1}\left(\bar{\mathbf{X}} - \boldsymbol{\mu}\right)$ is approximately distributed $\chi_p^2$

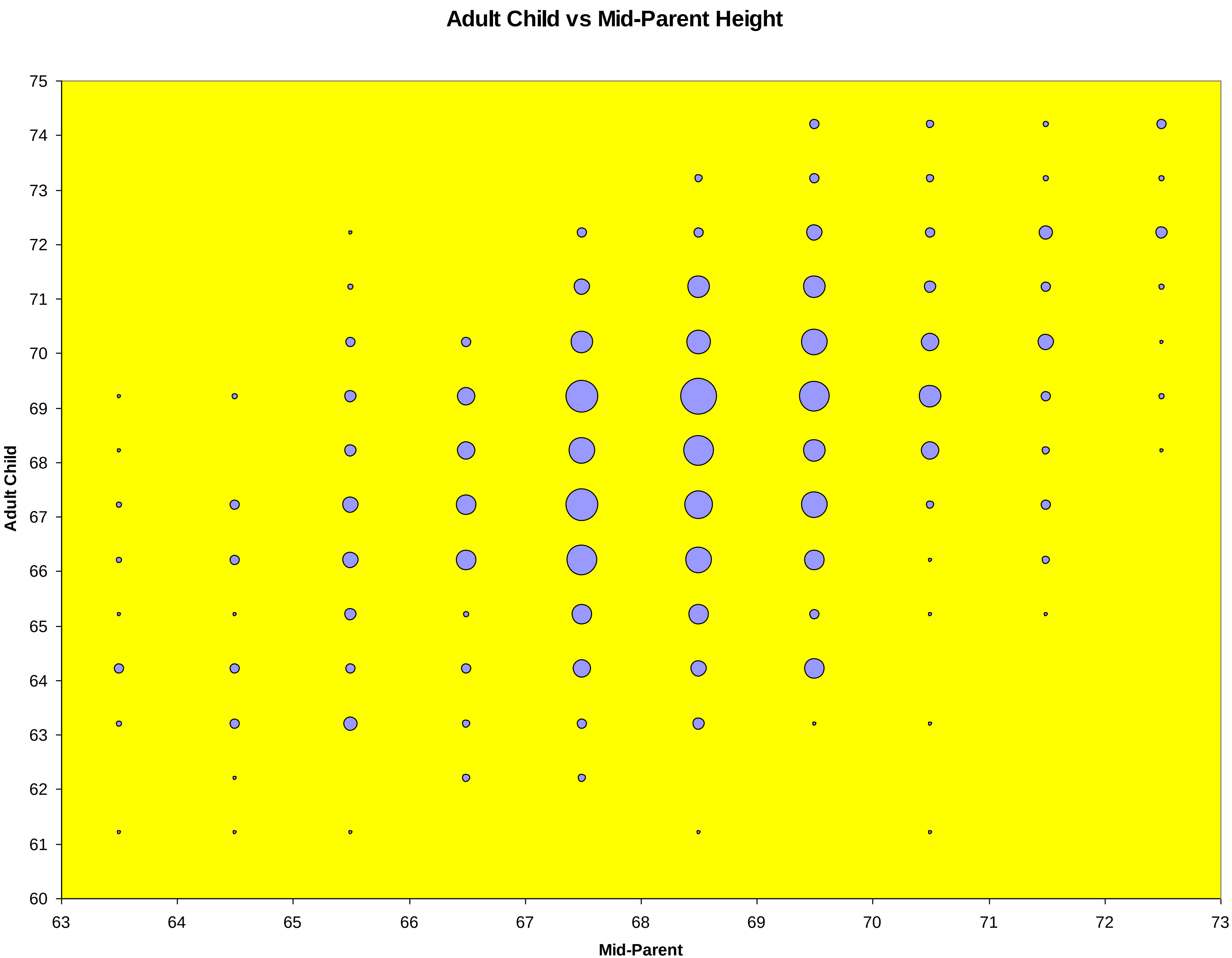
# Assessing the Assumption of Normality

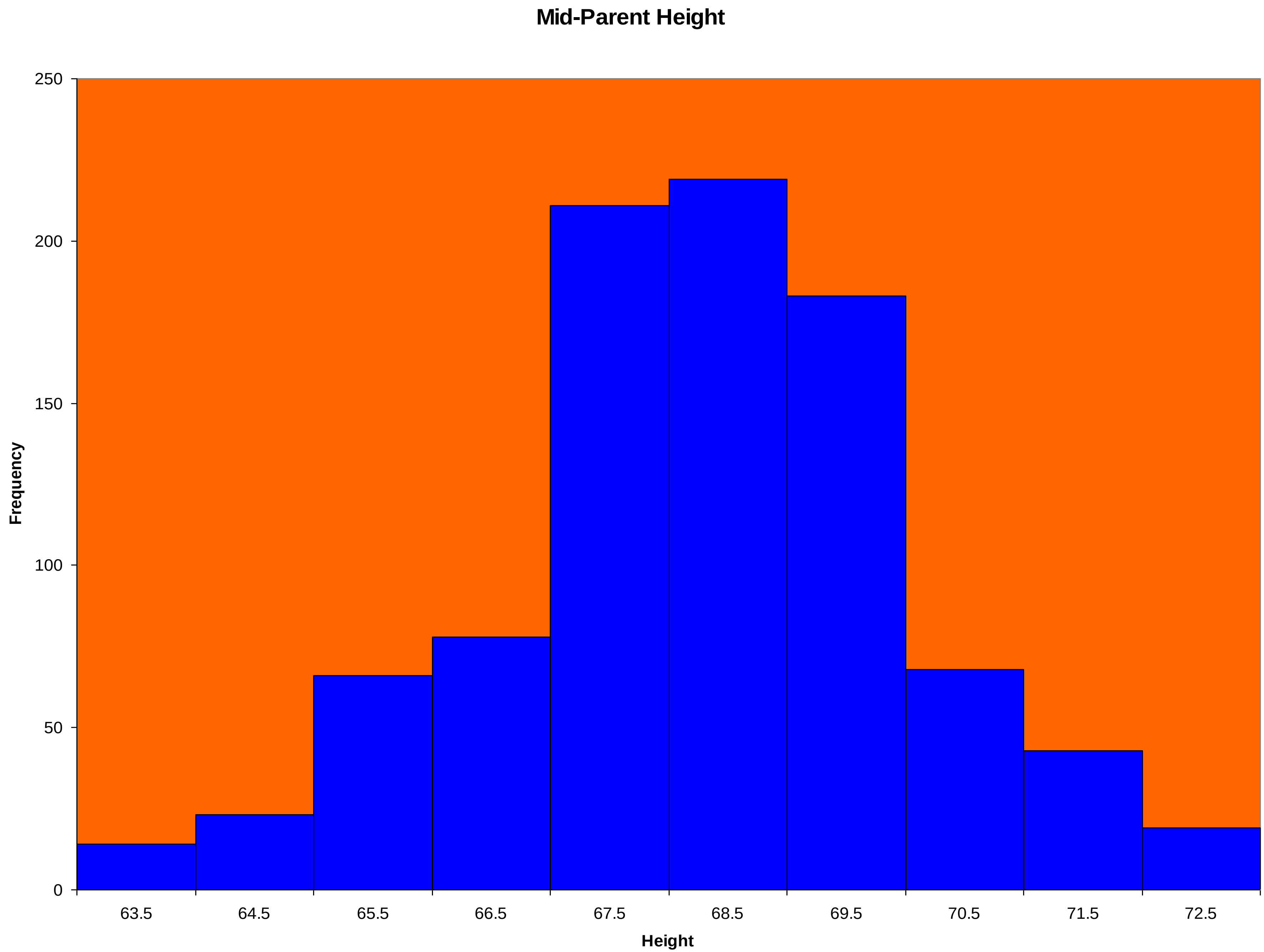
Evaluating normality of the univariate distributions:

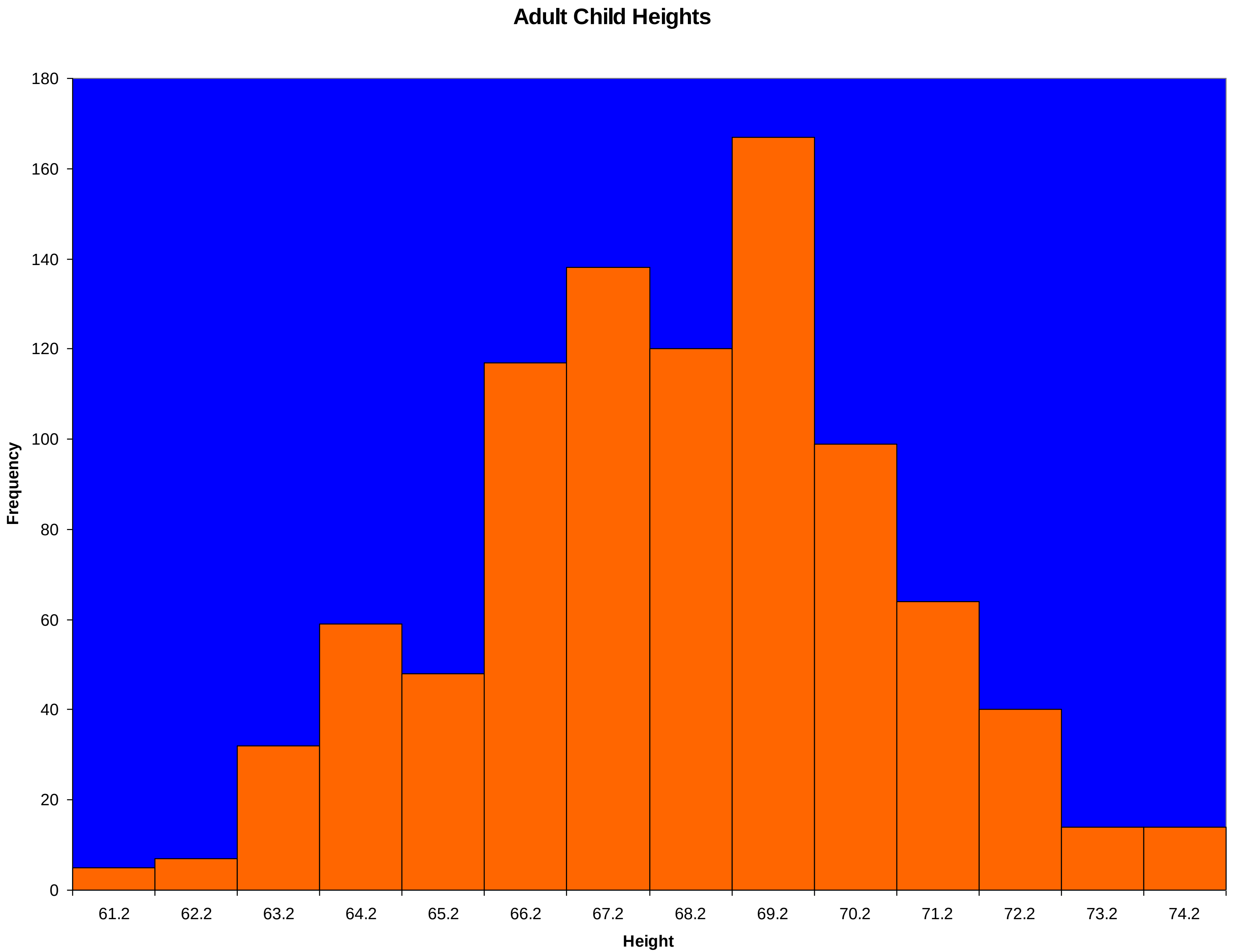
- Histogram: compare bin proportions with empirical rule;
- Q-Q Plot: plot ordered sample observation $x_{(j)}$ vs. the $(j - 3/8)/(n + ¼)$ quantile of the normal distribution (should be a 45-degree line);
- Hypotheses tests (Shapiro-Wilks, Anderson-Darling, Kolmogorov-Smirnov, …): note small *p*-value rejects normality!

# Data – Heights of Adult Children and Parents

- Adult Children Heights are reported by inch, in a manner so that the median of the grouped values is used for each (62.2”,…,73.2” are reported by Galton).
  - He adjusts female heights by a multiple of 1.08
  - We use 61.2” for his “Below”
  - We use 74.2” for his “Above”
- Mid-Parents Heights are the average of the two parents’ heights (after female adjusted). Grouped values at median (64.5”,…,72.5” by Galton)
  - We use 63.5” for “Below”
  - We use 73.5” for “Above”


Adult Child vs Mid-Parent Height
Adult Child
Mid-Parent
75
74
73
72
71
70
69
68
67
66
65
64
63
62
61
60
63
64
65
66
67
68
69
70
71
72
73


Mid-Parent Height
250
200
150
100
50
0
Frequency
63.5
64.5
65.5
66.5
67.5
68.5
69.5
70.5
71.5
72.5
Height


Adult Child Heights
Frequency
180
160
140
120
100
80
60
40
20
0
61.2
62.2
63.2
64.2
65.2
66.2
67.2
68.2
69.2
70.2
71.2
72.2
73.2
74.2
Height

# Joint Density Function

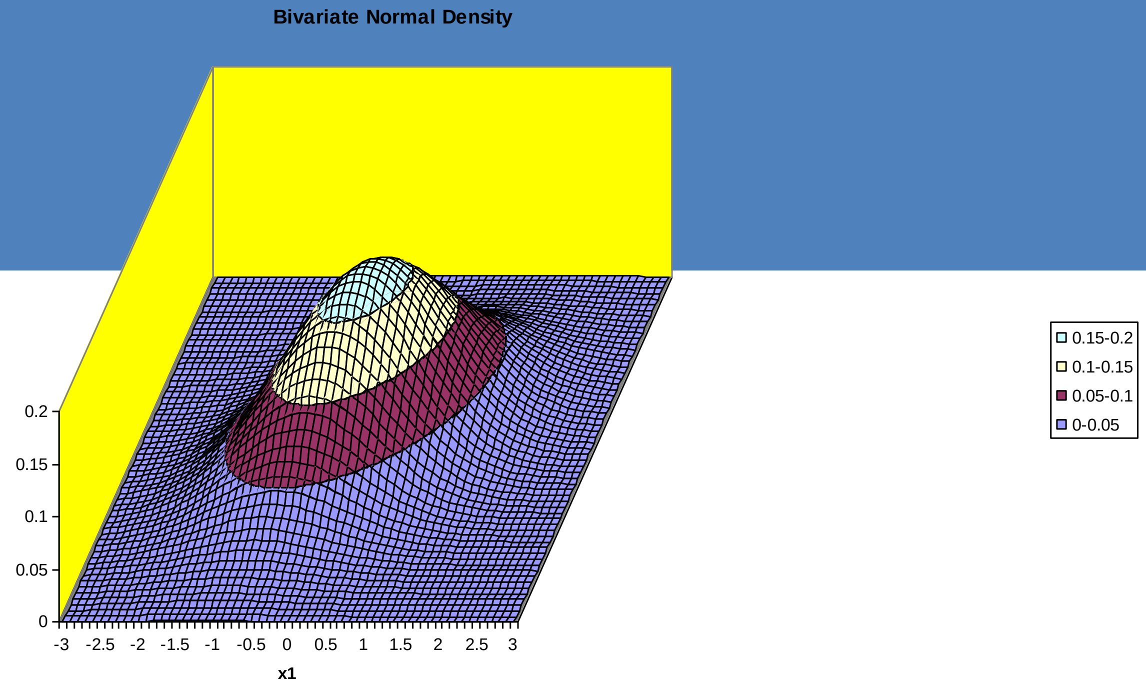
$$f(x_1,x_2)=\frac{1}{2\pi\sqrt{\sigma_1^2\sigma_2^2\left(1-\rho^2\right)}}\exp\left\{-\left(\frac{1}{2\left(1-\rho^2\right)}\right)\left[\frac{\left(x_1-\mu_1\right)^2}{\sigma_1^2}-\frac{2\rho\left(x_1-\mu_1\right)\left(x_2-\mu_2\right)}{\sigma_1\sigma_2}+\frac{\left(x_2-\mu_2\right)^2}{\sigma_2^2}\right]\right\}\qquad -\infty<y_1,y_2<\infty$$

where:

$$\mu_1=E\left\{X_1\right\}\qquad \sigma_1^2=V\left\{X_1\right\}$$

$$\mu_2=E\left\{X_2\right\}\qquad \sigma_2^2=V\left\{X_2\right\}$$

$$\rho=\frac{E\left\{\left(X_1-\mu_1\right)\left(X_2-\mu_2\right)\right\}}{\sigma_1\sigma_2}$$




$\mu_1=\mu_2=0 \quad \sigma_1=\sigma_2=1 \quad \rho=0.4$