STAT GU4261/GR5261 Statistical Methods in Finance

Spring 2018

Homework 6 Suggested Solution

Due date: 11 Mar 2017 (Sun)

Note: In this suggested solution, R is used to compute the quantiles of normal and t-distributions if required. If you have used the rounded-off figures provided in the normal table, you will have slightly different results, but it is fine.

1. Denote μ and σ as the mean and standard deviation of the loss L, then $\mu = -0.002$ and $\sigma = 0.016$. The required probability is

$$P(L > \mu + \sigma) = P(\tfrac{L-\mu}{\sigma} \sim N(0,1) > 1) = 0.159.$$

2. (a) Recall $-\text{VaR}(\alpha)$ is the α-th quantile of R, $-z_\alpha$ is the α-th quantile of $N(0,1)$, hence $\text{VaR}(0.1) = -1000000[0.002 + 0.016(-z_{0.1})] = 18500$.

   (b) $\text{VaR}(0.05) = -1000[0.1 + 0.2(-z_{0.05})] = 229$.

3. (a) If $-t_\alpha(\nu)$ denotes the α-th quantile of a t-distribution with degree of freedom ν, then $\text{VaR}(0.1) = -1000[0.002 + 0.016(-t_{0.1}(2))] = 28.2$.

   (b) $\text{VaR}(0.1) = -1000[0.002 + 0.016(-t_{0.1}(5))] = 21.6$.

4. (a) Note that $w_A = 1/3$, $w_B = 2/3$, and $R_P = w_A R_A + w_B R_B \sim N(\mu_P, \sigma_P^2)$, where $\mu_P = w_A\mu_A + w_B\mu_B$ and $\sigma_P^2 = w_A^2\sigma_A^2 + w_B^2\sigma_B^2 + 2\rho w_A w_B \sigma_A \sigma_B$. If the returns are independent, then $\rho = 0$ and $\text{VaR}(0.05) = -1500[\mu_P + \sigma_P(-z_{0.05})] = 34.3$.

   (b) If $\rho = 0.3$, then $\text{VaR}(0.05) = -1500[\mu_P + \sigma_P(-z_{0.05})] = 38.7$.

   (c) If $\rho = -0.3$, then $\text{VaR}(0.05) = -1500[\mu_P + \sigma_P(-z_{0.05})] = 29.4$.

5. (a) We first compute the cdf, as it is also required in (b). When $a \le -1$, we have

$$\begin{aligned}
ZF(a) &= -\int_{-\infty}^{a} \frac{x+1}{(x^2+1)^2}dx = -\int_{-\infty}^{a} \frac{1/2}{(x^2+1)^2}d(x^2) - \int_{-\pi/2}^{\tan^{-1}a} \frac{\sec^2 x}{(\tan^2 x+1)^2}dx \\
&= \frac{1}{2(x^2+1)}\bigg|_{-\infty}^{a} - \int_{-\pi/2}^{\tan^{-1}a} \cos^2 x dx = \frac{1}{2(a^2+1)} - \int_{-\pi/2}^{\tan^{-1}a} \frac{1+\cos 2x}{2}dx \\
&= \frac{1}{2(a^2+1)} - \left[\frac{x}{2} + \frac{\sin 2x}{4}\right]_{-\pi/2}^{\tan^{-1}a} = \frac{1}{2(a^2+1)} - \frac{\tan^{-1}a}{2} - \frac{\pi}{4} - \frac{\sin 2(\tan^{-1}a)}{4} \\
&= \frac{1}{2(a^2+1)} - \frac{\tan^{-1}a}{2} - \frac{\pi}{4} - \frac{\sin(\tan^{-1}a)\cos(\tan^{-1}a)}{2} \\
&= \frac{1}{2(a^2+1)} - \frac{\tan^{-1}a}{2} - \frac{\pi}{4} - \frac{a\cdot 1}{2\sqrt{a^2+1}\sqrt{a^2+1}} = -\frac{a-1}{2(a^2+1)} - \frac{\tan^{-1}a}{2} - \frac{\pi}{4}.
\end{aligned}$$

This implies $ZF(-1) = \frac{1}{2} - \frac{-\pi/4}{2} - \frac{\pi}{4} = \frac{1}{2} - \frac{\pi}{8}$. When $a > -1$, we have

$$\begin{aligned}
ZF(a) = ZF(-1) + \int_{-1}^{a} \frac{x+1}{(x^2+1)^2}dx &= \frac{1}{2} - \frac{\pi}{8} + \int_{-1}^{a} \frac{x+1}{(x^2+1)^2}dx \\
&= \frac{1}{2} - \frac{\pi}{8} - \frac{1}{2(x^2+1)}\Big|_{-1}^{a} + \left[\frac{x}{2} + \frac{\sin 2x}{4}\right]_{-\pi/4}^{\tan^{-1} a} \\
&= \frac{1}{2} - \frac{\pi}{8} - \frac{1}{2(a^2+1)} + \frac{1}{4} + \frac{\tan^{-1} a}{2} + \frac{\pi}{8} + \frac{\sin 2(\tan^{-1} a)}{4} + \frac{1}{4} \\
&= 1 - \frac{1}{2(a^2+1)} + \frac{\tan^{-1} a}{2} + \frac{\sin 2(\tan^{-1} a)}{4} = 1 + \frac{a-1}{2(a^2+1)} + \frac{\tan^{-1} a}{2}.
\end{aligned}$$

It follows from $F(\infty) = 1$ that $Z = 1 - 0 + \frac{\pi/2}{2} = 1 + \pi/4$.

(b) VaR(0.05) is the value of a such that $F(a) = 0.95$. But $F(-1) = \frac{1/2-\pi/8}{1+\pi/4} < 0.95$, so we have $a > -1$. To this end, we equate the formula of $F(a)$ for $a > -1$ obtained above to 0.95 and solve for a (using equation solver, eg. **fzero** in MATLAB or **uniroot** in R) to get $a = 2.4637$.

6. (a) Recall VaR(α) is the $(1-\alpha)$-th quantile of L, and $z_\alpha = -z_{1-\alpha}$ is the $(1-\alpha)$-th quantile of $N(0,1)$. If $L \sim N(\mu, \sigma^2)$, then $\mathrm{VaR}(\alpha) = \mu + \sigma z_\alpha$, therefore

$$\begin{aligned}
\mathrm{ES}(\alpha) = E(L|L \geq \mathrm{VaR}(\alpha)) &= \frac{1}{\alpha}\int_{\mathrm{VaR}(\alpha)}^{\infty} \frac{x}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx \\
&= \frac{1}{\alpha}\int_{\frac{\mathrm{VaR}(\alpha)-\mu}{\sigma}}^{\infty} \frac{\mu+\sigma y}{\sqrt{2\pi\sigma^2}} e^{-\frac{y^2}{2}} \sigma dy = \frac{1}{\alpha}\left[\mu \int_{z_\alpha}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy + \sigma \int_{z_\alpha}^{\infty} \frac{y}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy\right] \\
&= \mu + \frac{\sigma}{\alpha}\left[-\frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}\right]_{z_\alpha}^{\infty} = \mu + \frac{\sigma}{\alpha}\frac{1}{\sqrt{2\pi}} e^{-\frac{z_\alpha^2}{2}} = \mu + \frac{\sigma}{\alpha}\phi(z_\alpha).
\end{aligned}$$

(b) Using the above formula, we have $\mathrm{ES}(0.05) = 100000(-0.04 + \frac{0.18}{0.05}\phi(z_\alpha)) = 33129$.

(c) We calculate $\mu_P = w_A\mu_A + w_B\mu_B$ and $\sigma_P^2 = w_A^2\sigma_A^2 + w_B^2\sigma_B^2 + 2\rho w_A w_B \sigma_A \sigma_B$, where $w_A = w_B = 1/2$, then $\mathrm{ES}(0.05) = 100000(-\mu_P + \frac{\sigma_P}{0.05}\phi(z_\alpha)) = 24760$, which is smaller than that in (b) because of diversification.