Introduction to Vectors and Matrices

Matrices

- Definition: A matrix is a rectangular array of numbers
- In many applications, the rows of a matrix will represent individual cases (people, items, plants, animals, ...) and columns will represent attributes or characteristics
- The dimension of a matrix is its number of rows and columns, often denoted as *r* x *c* (*r* rows by *c* columns)
- Can be represented in full form or abbreviated form:

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1c} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2c} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{ic} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{r1} & a_{r2} & \cdots & a_{rj} & \cdots & a_{rc} \end{bmatrix} = \left[a_{ij} \right] \quad i = 1, \ldots, r; \; j = 1, \ldots, c$$

Special Types of Matrices

Square Matrix: Number of rows = # of Columns $(r = c)$

$$\mathbf{A} = \begin{bmatrix} 20 & 32 & 50 \\ 12 & 28 & 42 \\ 28 & 46 & 60 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

Vector: Matrix with one column (column vector) or one row (row vector)

$$\mathbf{C} = \begin{bmatrix} 57 \\ 24 \\ 18 \end{bmatrix} \qquad \mathbf{D} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{bmatrix} \qquad \mathbf{E}' = \begin{bmatrix} 17 & 31 \end{bmatrix} \qquad \mathbf{F}' = \begin{bmatrix} f_1 & f_2 & f_3 \end{bmatrix}$$

Transpose: Matrix formed by interchanging rows and columns of a matrix (use "prime" to denote transpose)

$$\underset{2\times 3}{\mathbf{G}} = \begin{bmatrix} 6 & 15 & 22 \\ 8 & 13 & 25 \end{bmatrix} \qquad \underset{3\times 2}{\mathbf{G}'} = \begin{bmatrix} 6 & 8 \\ 15 & 13 \\ 22 & 25 \end{bmatrix}$$

$$\underset{r\times c}{\mathbf{H}} = \begin{bmatrix} h_{11} & \cdots & h_{1c} \\ \vdots & & \vdots \\ h_{r1} & \cdots & h_{rc} \end{bmatrix} = \begin{bmatrix} h_{ij} \end{bmatrix} \quad i = 1,\ldots,r;\ j = 1,\ldots,c \quad \Rightarrow \quad \underset{c\times r}{\mathbf{H}'} = \begin{bmatrix} h_{11} & \cdots & h_{r1} \\ \vdots & & \vdots \\ h_{1c} & \cdots & h_{rc} \end{bmatrix} = \begin{bmatrix} h_{ji} \end{bmatrix} \quad j = 1,\ldots,c;\ i = 1,\ldots,r$$

Matrix Equality: Matrices of the same dimension, and corresponding elements in same cells are all equal:

$$\mathbf{A} = \begin{bmatrix} 4 & 6 \\ 12 & 10 \end{bmatrix} = \mathbf{B} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \quad \Rightarrow \quad b_{11} = 4,\ b_{12} = 6,\ b_{21} = 12,\ b_{22} = 10$$

Matrix Addition and Subtraction

Two matrices ***A*** and ***B*** of the same dimensions can be added, where the sum ***A*** + ***B*** has $(i, j)^{th}$ entry equal to $a_{ij} + b_{ij}$

Example (2.4 on p. 55): If $\boldsymbol{A} = \begin{bmatrix} 0 & 3 & 1 \\ 1 & -1 & 1 \end{bmatrix}$ and $\boldsymbol{B} = \begin{bmatrix} 1 & -2 & -3 \\ 2 & 5 & 1 \end{bmatrix}$, then

$$\boldsymbol{A} + \boldsymbol{B} = \begin{bmatrix} 0+1 & 3-2 & 1-3 \\ 1+2 & -1+5 & 1+1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -2 \\ 3 & 4 & 2 \end{bmatrix}$$

Note: Subtraction is defined in a similar way.

Exercise: Find ***A*** – ***B*** in the above example.

Matrix Multiplication

Multiplication of a Matrix by a Scalar (single number):

$$k = 3 \quad \mathbf{A} = \begin{bmatrix} 2 & 1 \\ -2 & 7 \end{bmatrix} \quad \Rightarrow \quad k\mathbf{A} = \begin{bmatrix} 3(2) & 3(1) \\ 3(-2) & 3(7) \end{bmatrix} = \begin{bmatrix} 6 & 3 \\ -6 & 21 \end{bmatrix}$$

Multiplication of a Matrix by a Matrix (#cols($\mathbf{A}$) = #rows($\mathbf{B}$)):

$$\text{If } c_A = r_B : \underset{r_A \times c_A}{\mathbf{A}} \ \underset{r_B \times c_B}{\mathbf{B}} = \underset{r_A \times c_B}{\mathbf{AB}} = \left[ab_{ij} \right] \ i = 1, \ldots, r_A; \ j = 1, \ldots, c_B$$

$ab_{ij} \equiv$ sum of the products of the $c_A = r_B$ elements of i$^{\text{th}}$ row of $\mathbf{A}$ and j$^{\text{th}}$ column of $\mathbf{B}$:

$$\underset{3\times 2}{\mathbf{A}} = \begin{bmatrix} 2 & 5 \\ 3 & -1 \\ 0 & 7 \end{bmatrix} \quad \underset{2\times 2}{\mathbf{B}} = \begin{bmatrix} 3 & -1 \\ 2 & 4 \end{bmatrix}$$

$$\underset{3\times 2}{\mathbf{A}} \ \underset{2\times 2}{\mathbf{B}} = \underset{3\times 2}{\mathbf{AB}} = \begin{bmatrix} 2(3)+5(2) & 2(-1)+5(4) \\ 3(3)+(-1)(2) & 3(-1)+(-1)(4) \\ 0(3)+7(2) & 0(-1)+7(4) \end{bmatrix} = \begin{bmatrix} 16 & 18 \\ 7 & -7 \\ 14 & 8 \end{bmatrix}$$

$$\text{If } c_A = r_B = c : \underset{r_A \times c_A}{\mathbf{A}} \ \underset{r_B \times c_B}{\mathbf{B}} = \underset{r_A \times c_B}{\mathbf{AB}} = \left[ab_{ij} \right] = \left[\sum_{k=1}^{c} a_{ik} b_{kj} \right] \ i = 1, \ldots, r_A; \ j = 1, \ldots, c_B$$

Special Matrix Types

Symmetric Matrix: Square matrix with a transpose equal to itself: **A = A'**:

$$\mathbf{A} = \begin{bmatrix} 6 & 19 & -8 \\ 19 & 14 & 3 \\ -8 & 3 & 1 \end{bmatrix} \qquad \mathbf{A'} = \begin{bmatrix} 6 & 19 & -8 \\ 19 & 14 & 3 \\ -8 & 3 & 1 \end{bmatrix} = \mathbf{A}$$

Diagonal Matrix: Square matrix with all off-diagonal elements equal to 0:

$$\mathbf{A} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} b_1 & 0 & 0 \\ 0 & b_2 & 0 \\ 0 & 0 & b_3 \end{bmatrix}$$

Note:Diagonal matrices are symmetric (not vice versa)

Identity Matrix: Diagonal matrix with all diagonal elements equal to 1 (acts like multiplying a scalar by 1):

$$\underset{3\times 3}{\mathbf{I}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \underset{3\times 3}{\mathbf{A}} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \Rightarrow \mathbf{IA} = \mathbf{AI} = \mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Scalar Matrix: Diagonal matrix with all diagonal elements equal to a single number"

$$\begin{bmatrix} k & 0 & 0 & 0 \\ 0 & k & 0 & 0 \\ 0 & 0 & k & 0 \\ 0 & 0 & 0 & k \end{bmatrix} = k \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = k \underset{4\times 4}{\mathbf{I}}$$

1-Vector and matrix and zero-vector:

$$\underset{r\times 1}{\mathbf{1}} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \qquad \underset{r\times r}{\mathbf{J}} = \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{bmatrix} \qquad \underset{r\times 1}{\mathbf{0}} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Note: $\underset{1\times r}{\mathbf{1}'}\underset{r\times 1}{\mathbf{1}} = \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = r \qquad \underset{r\times 1}{\mathbf{1}}\underset{1\times r}{\mathbf{1}'} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix} = \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{bmatrix} = \underset{r\times r}{\mathbf{J}}$

Linear Dependence

- **Definition:** A set of vectors $\boldsymbol{x}_1$, $\boldsymbol{x}_2$, … , $\boldsymbol{x}_k$ is said to be *linearly dependent* if there exist k numbers a_1, a_2, … , a_k, not all zero, such that

$$a_1\boldsymbol{x}_1 + a_2\boldsymbol{x}_2 + \ldots + a_k\boldsymbol{x}_k = 0$$

- Otherwise the set of vectors is said to be *linearly independent*.

- **Example**: Let $\boldsymbol{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\boldsymbol{x}_2 = \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix}$, $\boldsymbol{x}_3 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

Then $2\boldsymbol{x}_1 - \boldsymbol{x}_2 + 3\boldsymbol{x}_3 = 0$

Thus, $\boldsymbol{x}_1$, $\boldsymbol{x}_2$, $\boldsymbol{x}_3$ are a linearly dependent set of vectors.

Linear Dependence and Rank of a Matrix

- Linear Dependence: When a linear function of the columns (rows) of a matrix produces a zero vector (one or more columns (rows) can be written as linear function of the other columns (rows))
- Rank of a matrix: Number of linearly independent columns (rows) of the matrix. Rank cannot exceed the minimum of the number of rows or columns of the matrix. rank(**A**) ≤ $\min(r_A, c_A)$
- A matrix if full rank if rank(**A**) = $\min(r_A, c_A)$

$$\underset{2\times 2}{\mathbf{A}} = \begin{bmatrix} 1 & -3 \\ -4 & 12 \end{bmatrix} = \begin{bmatrix} \underset{2\times 1}{\mathbf{A}_1} & \underset{2\times 1}{\mathbf{A}_2} \end{bmatrix} \qquad 3\mathbf{A}_1 + \mathbf{A}_2 = \mathbf{0} \quad \text{Columns of } \mathbf{A} \text{ are linearly dependent} \;\; \text{rank}(\mathbf{A}) = 1$$

$$\underset{2\times 2}{\mathbf{B}} = \begin{bmatrix} 4 & -3 \\ 4 & 12 \end{bmatrix} = \begin{bmatrix} \underset{2\times 1}{\mathbf{B}_1} & \underset{2\times 1}{\mathbf{B}_2} \end{bmatrix} \qquad 0\mathbf{B}_1 + 0\mathbf{B}_2 = \mathbf{0} \quad \text{Columns of } \mathbf{B} \text{ are linearly independent} \;\; \text{rank}(\mathbf{B}) = 2$$

Geometry of Vectors

- A vector of order *n* is a point in *n*-dimensional space
- The line running through the origin and the point represented by the vector defines a 1-dimensional subspace of the n-dim space
- Any p linearly independent vectors of order n, $p < n$ define a p-dimensional subspace of the n-dim space
- Two vectors $\mathbf{x}$ and $\mathbf{y}$ are orthogonal if $\mathbf{x}'\mathbf{y} = \mathbf{y}'\mathbf{x} = 0$ and form a 90° angle at the origin
- Two vectors $\mathbf{x}$ and $\mathbf{y}$ are linearly dependent if they form a 0° or 180° angle at the origin

Geometry of Vectors - II

Length of a vector: $\text{length}(\mathbf{x}) = L_{\mathbf{x}} = \sqrt{\mathbf{x}'\mathbf{x}}$

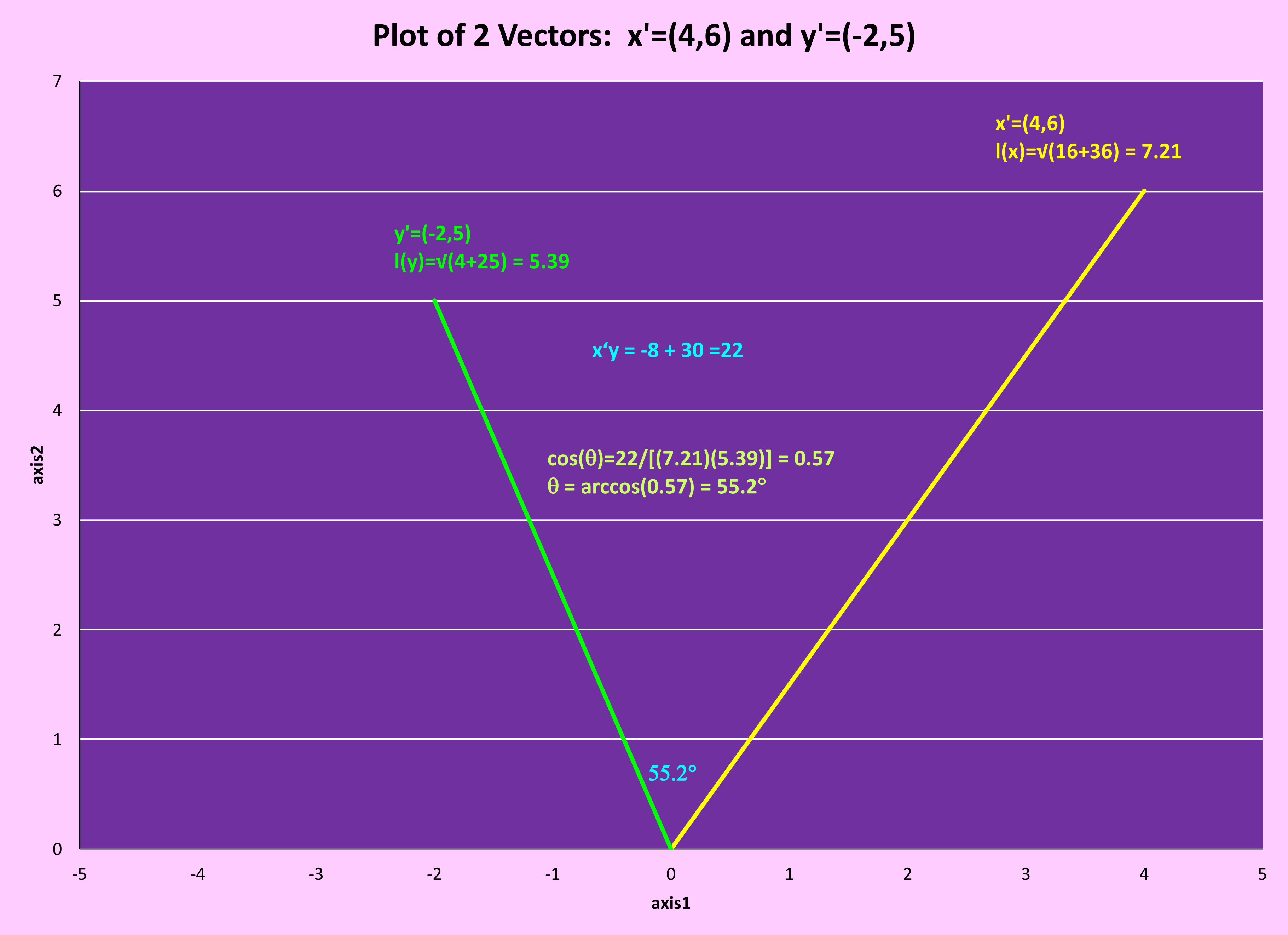
Cosine of Angle between 2 vectors: $\cos(\theta) = \dfrac{\mathbf{x}'\mathbf{y}}{\sqrt{\mathbf{x}'\mathbf{x}}\sqrt{\mathbf{y}'\mathbf{y}}}$

$$\Rightarrow \quad \theta = \arccos\left(\frac{\mathbf{x}'\mathbf{y}}{\sqrt{\mathbf{x}'\mathbf{x}}\sqrt{\mathbf{y}'\mathbf{y}}}\right) \quad \text{in degrees}$$

Projection of $\mathbf{x}$ on $\mathbf{y}$: $\dfrac{\mathbf{x}'\mathbf{y}}{\mathbf{y}'\mathbf{y}}\mathbf{y} = \dfrac{\mathbf{x}'\mathbf{y}}{L_{\mathbf{y}}}\left(\dfrac{1}{L_{\mathbf{y}}}\right)\mathbf{y}$

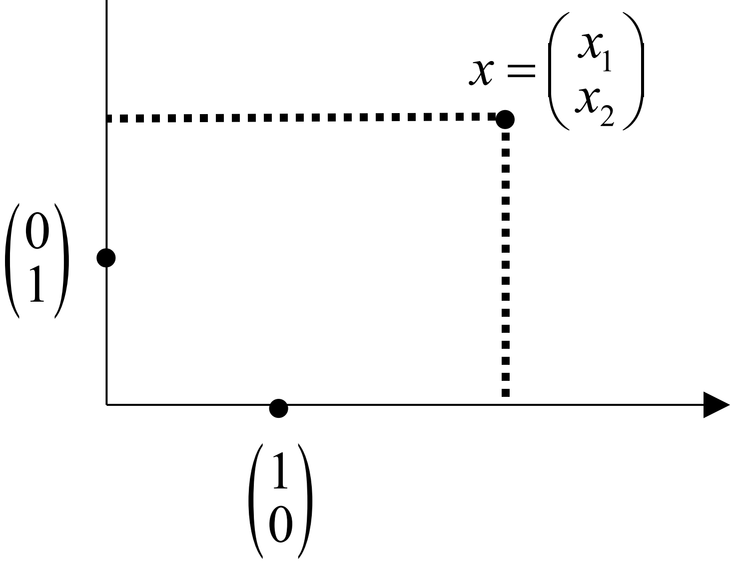
with length $\dfrac{|\mathbf{x}'\mathbf{y}|}{L_{\mathbf{y}}} = L_{\mathbf{x}}\left|\dfrac{\mathbf{x}'\mathbf{y}}{L_{\mathbf{x}}L_{\mathbf{y}}}\right| = L_{\mathbf{x}}|\cos(\theta)|$

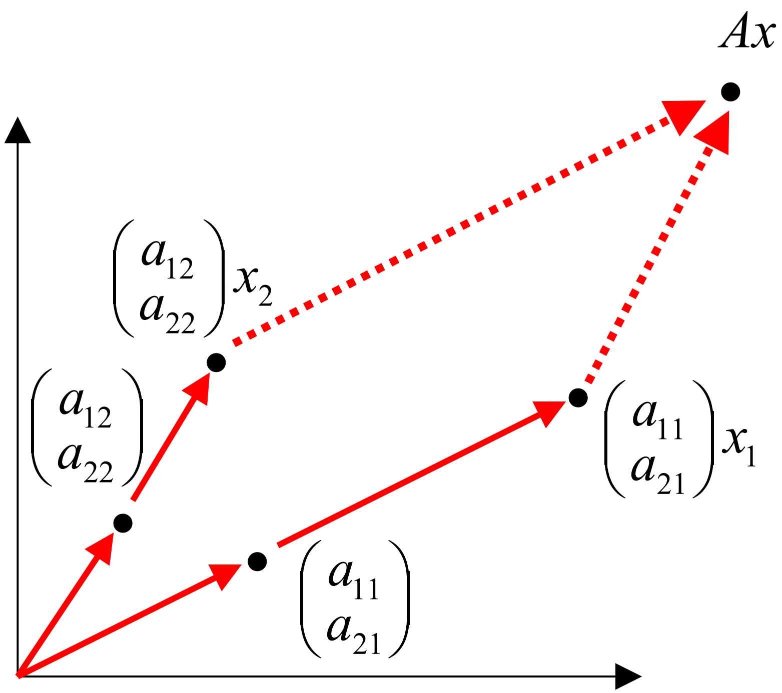
If two vectors each have mean 0 among their elements then θ is the product moment correlation between the two vectors



Plot of 2 Vectors: x'=(4,6) and y'=(-2,5)
x'=(4,6)
l(x)=√(16+36) = 7.21
y'=(-2,5)
l(y)=√(4+25) = 5.39
x'y = -8 + 30 =22
cos(θ)=22/[(7.21)(5.39)] = 0.57
θ = arccos(0.57) = 55.2°
55.2°
axis2
axis1
0
1
2
3
4
5
6
7
-5
-4
-3
-2
-1
0
1
2
3
4
5

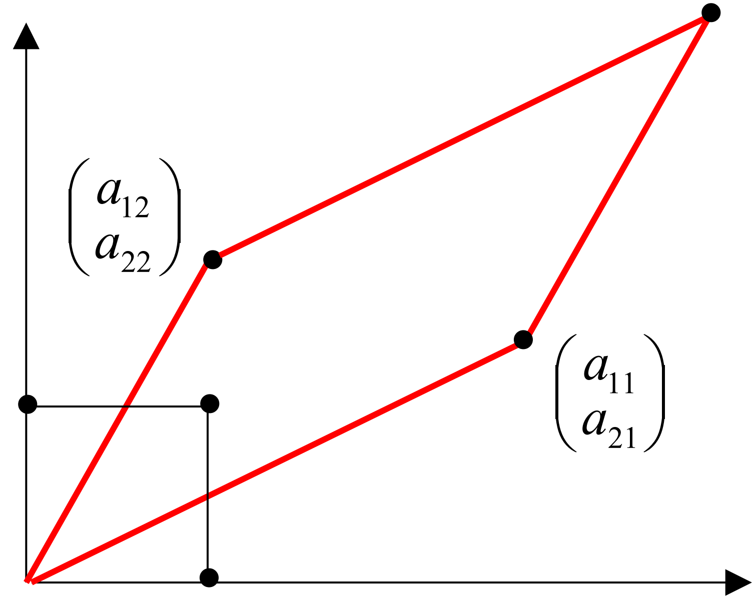
- **Graphical Depiction of Matrix multiplication:**





DETERMINANTS OF SQUARE MATRICES

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$



$$\Rightarrow \quad \det(A)$$

$$= \; a_{11}a_{22} - a_{21}a_{21}$$

$$\Rightarrow \quad |\det(A)|$$

$=$ ***Area of the image of the unit square under A***

Matrix Inverse

- Note: For scalars (except 0), when we multiply a number, by its reciprocal, we get 1:
  $2(1/2)=1$ $\qquad x(1/x)=x(x^{-1})=1$
- In matrix form if **A** is a square matrix and full rank (all rows and columns are linearly independent), then **A** has an inverse: $\mathbf{A}^{-1}$ such that: $\mathbf{A}^{-1}\,\mathbf{A} = \mathbf{A}\,\mathbf{A}^{-1} = \mathbf{I}$

$$\mathbf{A}=\begin{bmatrix}2 & 8\\ 4 & -2\end{bmatrix} \qquad \mathbf{A}^{-1}=\begin{bmatrix}\frac{2}{36} & \frac{8}{36}\\ \frac{4}{36} & \frac{-2}{36}\end{bmatrix} \qquad \mathbf{A}^{-1}\mathbf{A}=\begin{bmatrix}\frac{2}{36} & \frac{8}{36}\\ \frac{4}{36} & \frac{-2}{36}\end{bmatrix}\begin{bmatrix}2 & 8\\ 4 & -2\end{bmatrix}=\begin{bmatrix}\frac{4}{36}+\frac{32}{36} & \frac{16}{36}-\frac{16}{36}\\ \frac{8}{36}-\frac{8}{36} & \frac{32}{36}+\frac{4}{36}\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=\mathbf{I}$$

$$\mathbf{B}=\begin{bmatrix}4 & 0 & 0\\ 0 & -2 & 0\\ 0 & 0 & 6\end{bmatrix} \quad \mathbf{B}^{-1}=\begin{bmatrix}1/4 & 0 & 0\\ 0 & -1/2 & 0\\ 0 & 0 & 1/6\end{bmatrix} \quad \mathbf{B}\mathbf{B}^{-1}=\begin{bmatrix}4(1/4)+0+0 & 0+0+0 & 0+0+0\\ 0+0+0 & 0+(-2)(-1/2)+0 & 0+0+0\\ 0+0+0 & 0+0+0 & 0+0+6(1/6)\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}=\mathbf{I}$$

Use of Inverse Matrix – Solving Simultaneous Equations

$\mathbf{AY} = \mathbf{C}$ where $\mathbf{A}$ and $\mathbf{C}$ are matrices of of constants, $\mathbf{Y}$ is matrix of unknowns

$\Rightarrow \mathbf{A}^{-1}\mathbf{AY} = \mathbf{A}^{-1}\mathbf{C} \quad \Rightarrow \quad \mathbf{Y} = \mathbf{A}^{-1}\mathbf{C}$ (assuming $\mathbf{A}$ is square and full rank)

Equation 1: $12y_1 + 6y_2 = 48$ Equation 2: $10y_1 - 2y_2 = 12$

$$\mathbf{A} = \begin{bmatrix} 12 & 6 \\ 10 & -2 \end{bmatrix} \qquad \mathbf{Y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \qquad \mathbf{C} = \begin{bmatrix} 48 \\ 12 \end{bmatrix} \qquad \mathbf{Y} = \mathbf{A}^{-1}\mathbf{C}$$

$$\Rightarrow \quad \mathbf{A}^{-1} = \frac{1}{12(-2) - 6(10)} \begin{bmatrix} -2 & -6 \\ -10 & 12 \end{bmatrix} = \frac{1}{84} \begin{bmatrix} 2 & 6 \\ 10 & -12 \end{bmatrix}$$

$$\mathbf{Y} = \mathbf{A}^{-1}\mathbf{C} = \frac{1}{84} \begin{bmatrix} 2 & 6 \\ 10 & -12 \end{bmatrix} \begin{bmatrix} 48 \\ 12 \end{bmatrix} = \frac{1}{84} \begin{bmatrix} 96 + 72 \\ 480 - 144 \end{bmatrix} = \frac{1}{84} \begin{bmatrix} 168 \\ 336 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

Note the wisdom of waiting to divide by $|\mathbf{A}|$ at end of calculation!

Useful Matrix Results

All rules assume that the matrices are conformable to operations:

Addition Rules:

$\mathbf{A}+\mathbf{B}=\mathbf{B}+\mathbf{A} \qquad (\mathbf{A}+\mathbf{B})+\mathbf{C}=\mathbf{A}+(\mathbf{B}+\mathbf{C})$

Multiplication Rules:

$\mathbf{(AB)C}=\mathbf{A(BC)} \qquad \mathbf{C}(\mathbf{A}+\mathbf{B})=\mathbf{CA}+\mathbf{CB} \qquad k(\mathbf{A}+\mathbf{B})=k\mathbf{A}+k\mathbf{B} \quad k \equiv \text{ scalar}$

Transpose Rules:

$(\mathbf{A}')'=\mathbf{A} \qquad (\mathbf{A}+\mathbf{B})'=\mathbf{A}'+\mathbf{B}' \qquad (\mathbf{AB})'=\mathbf{B}'\mathbf{A}' \qquad (\mathbf{ABC})'=\mathbf{C}'\mathbf{B}'\mathbf{A}'$

Inverse Rules (Full Rank, Square Matrices):

$(\mathbf{AB})^{-1}=\mathbf{B}^{-1}\mathbf{A}^{-1} \qquad (\mathbf{ABC})^{-1}=\mathbf{C}^{-1}\mathbf{B}^{-1}\mathbf{A}^{-1} \qquad (\mathbf{A}^{-1})^{-1}=\mathbf{A} \qquad (\mathbf{A}')^{-1}=(\mathbf{A}^{-1})'$

NONSINGULAR SQUARE MATRICES

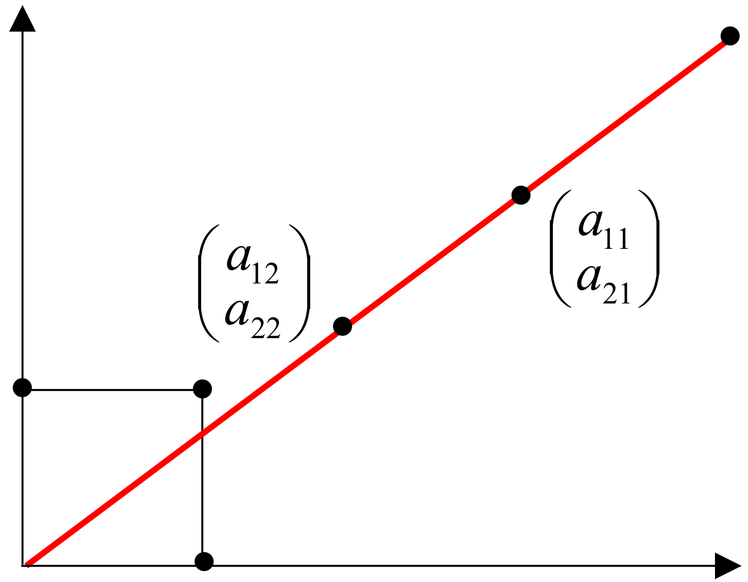
A^{-1} ***exists***

$$\Leftrightarrow \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} \textit{ and } \begin{pmatrix} a_{21} \\ a_{22} \end{pmatrix}$$

are ***not colinear***

$$\Leftrightarrow \boxed{\det\left(A\right) \neq 0}$$

$$\Leftrightarrow A \textit{ is } \textbf{\textit{nonsingular}}$$



PROPERTIES OF DETERMINANTS

- Another notation for det(A) is $|A|$
- $|A'| = |A|$
- $|AB| = |A||B|$
- $|A^{-1}| = |A|^{-1}$
- Partitioned matrices:

$$\begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix} = |A_{11}|\left|A_{22} - A_{21}A_{11}^{-1}A_{12}\right|$$
$$= |A_{22}|\left|A_{11} - A_{12}A_{22}^{-1}A_{21}\right|$$

Orthogonal Matrices

$$\mathbf{Q} = \begin{bmatrix} \mathbf{q}_1{}' \\ \mathbf{q}_2{}' \\ \vdots \\ \mathbf{q}_k{}' \end{bmatrix} \qquad \mathbf{Q}' = \begin{bmatrix} \mathbf{q}_1 & \mathbf{q}_2 & \cdots & \mathbf{q}_k \end{bmatrix} \qquad \mathbf{q}_i = \begin{bmatrix} q_{1i} \\ q_{2i} \\ \vdots \\ q_{ki} \end{bmatrix} \quad i = 1,\ldots,k \quad \text{Orthogonal} \Rightarrow \mathbf{QQ}' = \mathbf{Q}'\mathbf{Q} = \mathbf{I}$$

$$\mathbf{QQ}' = \mathbf{I} \quad \Rightarrow \quad \mathbf{Q}' = \mathbf{Q}^{-1} \qquad \mathbf{q}_i{}'\mathbf{q}_i = \sum_{l=1}^{k} q_{li}^2 = 1 \quad i = 1,\ldots,k \qquad \mathbf{q}_i{}'\mathbf{q}_j = \sum_{l=1}^{k} q_{li} q_{lj} = 0 \quad i \neq j$$

$\mathbf{QQ}' = \mathbf{I} \quad \Rightarrow$ Rows of $\mathbf{Q}$ are of length 1 and mutually perpendicular

$\mathbf{Q}'\mathbf{Q} = \mathbf{I} \quad \Rightarrow$ Columns of $\mathbf{Q}$ are of length 1 and mutually perpendicular

Example: Rotation Matrix - Rotate 2-dimensional Axes by $\theta = 22.5^\circ$ counterclockwise

$$\mathbf{T} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} 0.9239 & 0.3827 \\ -0.3827 & 0.9239 \end{bmatrix} \qquad \mathbf{T}' = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} 0.9239 & -0.3827 \\ 0.3827 & 0.9239 \end{bmatrix}$$

$$\mathbf{TT}' = \begin{bmatrix} \cos^2\theta + \sin^2\theta & -\cos\theta\sin\theta + \sin\theta\cos\theta \\ -\sin\theta\cos\theta + \cos\theta\sin\theta & \sin^2\theta + \cos^2\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{I}$$

Trace of a matrix

Def: If A is a square matrix, then $\mathrm{tr}(A) = \sum_{i=1}^{n} a_{ii}$

Properties:

- $\mathrm{tr}(A + B) = \mathrm{tr}(A) + \mathrm{tr}(B)$
- $\mathrm{tr}(AB) = \mathrm{tr}(BA)$
- $\mathrm{tr}(A'A) = \mathrm{tr}\,(AA') = \sum_{i=1}^{n} \sum_{j=1}^{p} a_{ij}^2$

Eigenvalues and Eigenvectors

Def: If $\boldsymbol{A}$ is a square matrix and λ is a scalar and x is a nonzero vector such that

$$\boldsymbol{A}x = \lambda x$$

then we say that λ is an eigenvalue of $\boldsymbol{A}$ and x is its corresponding eigenvector.

Note: To find the eigenvalues we solve $|\boldsymbol{A} - \lambda \boldsymbol{I}| = 0$

Note: If $\boldsymbol{A}$ is $n \times n$ then $\boldsymbol{A}$ has n eigenvalues $\lambda_1, \ldots, \lambda_n$.
The λ's are not necessarily all distinct, or nonzero or real numbers.

Properties:

- $\mathrm{tr}(A) = \sum_{i=1}^{n} \lambda_i$
- $|A| = \prod_{i=1}^{n} \lambda_i$

Spectral Decomposition

Def: If $\boldsymbol{A}$ is a symmetric matrix then

$$\boldsymbol{A} = \boldsymbol{CDC}'$$

where $\boldsymbol{C} = (\boldsymbol{e}_1, \dots , \boldsymbol{e}_n)$ contains the eigenvectors of $\boldsymbol{A}$ and $\boldsymbol{D} = \text{diag}(\lambda_1, \dots , \lambda_n)$ is a diagonal matrix with the eigenvalues of $\boldsymbol{A}$

Equivalently, this means

$$\boldsymbol{A} = \lambda_1 \boldsymbol{e}_1 \boldsymbol{e}_1' + \lambda_2 \boldsymbol{e}_2 \boldsymbol{e}_2' + \cdots + \lambda_n \boldsymbol{e}_n \boldsymbol{e}_n'$$

If $\lambda_i > 0$ for all $i = 1, \dots , n$ then $\boldsymbol{A}$ is called positive definite and we can define square root matrix

$$\boldsymbol{A}^{1/2} = \boldsymbol{CD}^{1/2}\boldsymbol{C}'$$

where $D^{1/2} = \text{diag}(\sqrt{\lambda_1}, \dots, \sqrt{\lambda_n})$

Properties:

- $\boldsymbol{A}^{1/2}\,\boldsymbol{A}^{1/2} = \boldsymbol{A}$
- $\boldsymbol{A}^{-1} = \boldsymbol{CD}^{-1}\boldsymbol{C}'$