(1) $\begin{cases} C^-(u,v) = \max(u+v-1, 0) & \text{, when } u=v \\ C^+(u,v) = \min(u,v) & \text{, when } u=1-v \\ C(u,v) = P(U\le u, V\le v) \end{cases}$

① Let (u,v) be an ~~an~~ arbitrary point:

$\begin{cases} C(u,v) \le C(u,1) = u \\ C(u,v) \le C(1,v) = v \end{cases} \Rightarrow C(u,v) \le \min(u,v) = C^+(u,v)$

② Like $P(A+B) = P(A) + P(B) - P(AB)$

$P(U\le u \text{ or } V\le v) = P(U\le u) + P(V\le v) - P(V\le v, U\le u)$

$= C(u,1) + C(1,v) - C(u,v)$

$\because P(U\le u \text{ or } V\le v) \le 1 \quad \therefore C(u,1) + C(1,v) - C(u,v) \le 1$

$\therefore C(u,v) \ge C(u,1) + C(1,v) - 1 = u+v-1$

$\because C(u,v) \ge 0$

$\therefore C(u,v) \ge \max(0, u+v-1)$

(2) $\rho_\tau = E[\mathrm{sign}\{(Y_1-Y_1^*)(Y_2-Y_2^*)\}]$

Here: $\rho_\tau = E[\mathrm{sign}\{(Y-Y^*)(X-X^*)\}]$

① $\rho_\tau = E[\mathrm{sign}\{(X-X^*)(\frac{1}{Y}-\frac{1}{Y^*})\}]$

$= E[\mathrm{sign}\{(X-X^*)(Y-Y^*)\frac{1}{YY^*}\}]$. since YY^* must greater than 0

$= -0.15$

② $\rho_\tau = E[\mathrm{sign}\{(\frac{1}{Y}-\frac{1}{Y^*})(\frac{1}{X}-\frac{1}{X^*})\}]$

$= E[\mathrm{sign}\{(Y^*-Y)(X^*-X)\frac{1}{YY^*}\frac{1}{XX^*}\}]$, since $X, X^*, Y, Y^* > 0$

$= 0.15$

(3) Spearman Rank correlation and kendall's tau all calculate the correlation between X and Y. when X and Y increase or decrease at the same time. Then these two value converge to 1. When X increase lead to the decrease of Y, then these two value converge to -1.

△ However, Person's correlation calculate the Linear relation between X and Y. when X and Y are Linear correlated, then the absolute value of Person's correlation converge to 1. when these two X and

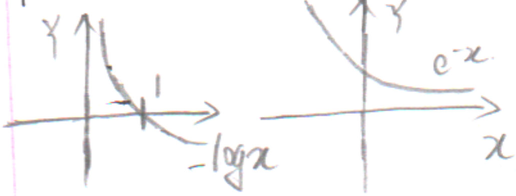
$\tau(X,Y) = E[\text{sign}(X-X^*)(Y-Y^*)]$

Y are not linearly correlated, then the value is 0.

4 Here $Y=X^2$, when $X \sim U[0,1]$. clearly, we can see that x increase lead to Y increase at the same time. We can image that the rank must march at two data sets. Thus, Spearman rank and Kendall's tau are both equal to 1. But, we also know. $Y=X^2$ is not a linear relation. thus. Person's correlation must be smaller than 1.

(4). $g_\alpha(u) = (-\log u)^\alpha \Rightarrow C_\alpha(u,v) = e^{-[(-\log u)^\alpha + (-\log v)^\alpha]^{\frac{1}{\alpha}}}$, prove $C \le C^+$

first, we know that both u and v are $[0,1]$, and $\alpha \to +\infty$



As we can see, u and v in the $-\log x$ function, the one that is smaller will have higher value. when $\alpha \to \infty$, then the one with higher value would ~~den~~ dominate the whole value and after we have exponential $\frac{1}{\alpha}$, the value ~~the~~ turn back to the original scale. Thus, the one with smaller value would dominate the result.

Thus: $C_\alpha(u,v) = \min(u,v)$

(5). Here $F_1(T) = F_2(T) = 3.46\%$, $T=1$, $r=4\%$, $S=1000000$, $\rho=0.5$

(a) $FTD = Se^{-rT}[F_1(T) + F_2(T) - C(F_1(T), F_2(T))]$

$= 1000000 \times e^{-4\% \times 1} \times [3.46\% + 3.46\% - C(3.46\%, 3.46\%)]$

$\because$ Their default time satisfied Gaussian copula:

$C(F_1(T), F_2(T)) \sim N\left(\begin{pmatrix} 3.46\% \\ 3.46\% \end{pmatrix}, \begin{pmatrix} 1 & 0.5 \\ 0.5 & 1 \end{pmatrix}\right)$

USE the R, we know $C(3.46\%, 3.46\%) =$ ~~0.00417924~~ 0.0072767897

$\therefore FTD =$ ~~85780~~ 59495.1664

(b) If we consider both default within one year

$FTD = S \times e^{-rT} P(\tau_1 \le T, \tau_2 \le T) = 1000000 \times e^{-4\% \times 1} \times C(0.0346, 0.0346)$

$=$ ~~4734.207~~ 6991.46278