Risk Management

Statistical Methods in Finance

Risk Management (See Chapter 19)

- Types of risk
  -Market risk
  -Credit risk
  -Liquidity risk
  -Operational risk
- It is important to identify, measure and control risks.
- One important concept to measure market risk is value at risk.

Value at Risk (VaR)

- **Definition of Value at Risk (VaR)**
  Consider a single asset or portfolio. Let R be its rate of return and $L = -R$ the loss. The (unit currency) value at risk (VaR) of level α or confidence $1 - \alpha$ is denoted by $VaR(\alpha)$ that satisfies the following

$$P(L > VaR(\alpha)) = \alpha.$$

  In the case of a portfolio of size S dollars, the VaR becomes $S \times VaR(\alpha)$ dollars.

- **Quantile**
  VaR is closely related to quantile or quantile function. For a random variable X with distribution F, its τth quantile is $F^{-1}(\tau)$. In particular, 0.5th quantile is known as the median. Thus, $VaR(\alpha)$ is the $(1 - \alpha)$th quantile of L or $-VaR(\alpha)$ is the αth quantile of R.

Parametric Modeling

- **Parametrization**
  A convenient approach is to parametrize the underlying distribution for R and specify the relevant parameters. Simple parametric models include the normal (Gaussian) model and the t model.

- **Normal family**
  Suppose R follows $N(\mu, \sigma^2)$ and let z_α denote the $1-\alpha$th quantile of $N(0,1)$ ($-z_\alpha$ is the αth quantile of $N(0,1)$). Obviously $(R-\mu)/\sigma$ is standard normal. Thus,
  $\alpha = P((R-\mu)/\sigma \leq -z_\alpha) = P(L \geq \sigma z_\alpha - \mu)$.
  So $VaR(\alpha) = \sigma z_\alpha - \mu$.

- **t family**
  Suppose instead $(R-\mu)/\sigma$ follows the central t-distribution with k degrees of freedom. Then $VaR(\alpha) = \sigma t_\alpha(k) - \mu$, where $t_\alpha(k)$ is the αth quantile of central t with k degrees of freedom.

- **Example 1**
  Suppose we have a portfolio of size $1,000,000$ dollars.
- **Normal**
  Suppose that the rate of return follows normal distribution with mean $\mu = 5\%$ and standard deviation $\sigma = 15\%$. Then the value at risk of the portfolio at $\alpha = 0.05$ is

$$1,000,000 \times (0.15 \times 1.64 - 0.05) = \$196,000.$$

- **t distributions**
  If we use t with $df = 5$, a similar calculation gives VaR=\$252,000; with $df = 3$, it gives VaR=\$303,000.

Example 2 (Diversification)

Example 2 Suppose we have two assets. Their rates of return have the same mean $\mu = 5\%$ and standard deviation 15% and they are jointly normal. Consider the portfolio of equal weights, i.e., the weights are $(50\%, 50\%)$. The size of the portfolio is again 1,000,000 dollars.

Case 1: The two assets are independent. Then the value at risk of the portfolio at $\alpha = 0.05$ is \$124,463.

Case 2: The correlation between the returns of two assets is 0.5. Then the value at risk of the portfolio at $\alpha = 0.05$ is \$163,673.

Case 3: The correlation between the returns of two assets is -0.5. Then the value at risk of the portfolio at $\alpha = 0.05$ is \$73,364.

Expected Shortfall

The value at risk only gives the cutoff point that the loss will exceed with certain probability (α). It does not quantify the amount of loss if such an event occurs. The expected shortall, $ES(\alpha)$, fills in the gap. It is defined as

$$ES(\alpha) = \frac{\int_0^{\alpha} VaR(s)ds}{\alpha}.$$

For Example 1,

under normal, $ES(\alpha)$=\$259,000;

under $t(5)$, $ES(\alpha)$=\$384,000;

under $t(3)$, $ES(\alpha)$=\$531,000.

Estimation of VaR

- **Parametric Approach** This approach first fits a parametric model (f_θ) using historical return data and then uses the fitted CDF ($F_{\hat{\theta}}$) to find $\widehat{VaR}(\alpha) = -F_{\hat{\theta}}^{-1}(\alpha)$, where $F_{\hat{\theta}}^{-1}(\alpha)$ is the αth quantile. If the size of the current position is S, then its estimated VaR at level α becomes $\widehat{VaR}(\alpha) = -S \times F_{\hat{\theta}}^{-1}(\alpha)$.
- **Nonparametric Approach** In this case, we use the empirical distribution from the historical data

$$\hat{F}(x) = \frac{1}{T} \sum_{t=1}^{T} I(R_t \leq x)$$

to estimate VaR: $\widehat{VaR}(\alpha) = -S \times \hat{F}^{-1}(\alpha)$.

Semiparametric model and Hill estimator

- There is an interesting compromise between the nonparametric approach and the parametric approach.

- The nonparametric estimator is feasible for large α, but not for small α. For example, if the sample had 1000 returns, then reasonably accurate estimation of the 0.05-quantile is feasible, but not estimation of the 0.0005- quantile. Parametric estimation can estimate VaR for any value of α but is sensitive to misspecification of the tail when α is small. Therefore, a methodology intermediary between totally nonparametric and parametric estimation is attractive.

- **Estimation of VaR assuming polynomial tails**

  The approach assumes that the return density has a polynomial left tail, or equivalently that the loss density has a polynomial right tail. Under this assumption, it is possible to use a nonparametric estimate of $VaR(\alpha_0)$ for a large value of α_0 to obtain estimates of $VaR(\alpha_1)$ for small values of α_1.

Semiparametric model and Hill estimator

It is assumed that the return density has a polynomial left tail:

$$f(y) \sim Ay^{-(a+1)}, \quad \text{as } y \to -\infty,$$

where $A > 0$ is a constant and $a > 0$ is the tail index. Therefore,

$$P(R \leq y) \sim \int_{-\infty}^{y} f(u)du = \frac{A}{a} y^{-a}, \quad \text{as} \quad y \to -\infty.$$

If $y_1 > 0$ and $y_2 > 0$, then

$$\frac{P(R < -y_1)}{P(R < -y_2)} \approx \left(\frac{y_1}{y_2}\right)^{-a}$$

Semiparametric model and Hill estimator

Now suppose that $y_1 = VaR(\alpha_1)$ and $y_2 = VaR(\alpha_0)$, where $0 < \alpha_1 < \alpha_0$. Then we have

$$\frac{\alpha_1}{\alpha_0} = \frac{P\left(R < -VaR(\alpha_1)\right)}{P\left(R < -VaR(\alpha_0)\right)} \approx \left(\frac{VaR(\alpha_1)}{VaR(\alpha_0)}\right)^{-a},$$

or

$$\frac{VaR(a_1)}{VaR(a_0)} \approx \left(\frac{\alpha_0}{\alpha_1}\right)^{1/a}.$$

Now dropping the subscript "1" of α_1,

$$VaR(\alpha) \approx VaR(\alpha_0)\left(\frac{\alpha_0}{\alpha}\right)^{1/a}.$$

It can be used to estimate $VaR(\alpha)$ when we have an estimate of $VaR(\alpha_0)$ and an estimate of a. The value of α_0 must be large enough that $VaR(\alpha_0)$ can be accurately estimated, but α can be any value less than α_0.

Semiparametric approach

A model combining parametric and nonparametric components is called semiparametric, so the previous estimator is semiparametric because the tail index is specified by a parameter, but otherwise the distribution is unspecified.

- **Estimation of ES**

To find a formula for ES, we will assume further that for some $c < 0$, the returns density satisfies

$$f(y) = A|y|^{-(a+1)}, y \leq c.$$

Then, for any $d \leq c$,

$$P(R \leq d) = \int_{-\infty}^{d} A|y|^{-(a+1)} dy = \frac{A}{a}|d|^{-a},$$

and the conditional density of R given that $R \leq d$ is

$$f(y|R \leq d) = \frac{Ay^{-(a+1)}}{P(R \leq d)} = a|d|^{a}|y|^{-(a+1)}.$$

Semiparametric model and Hill estimator

It follows that for $a > 1$

$$E(|R||R \le d) = a|d|^a \int_{-\infty}^{d} |y|^{-a} dy = \frac{a}{a-1}|d|.$$

(For $a \le 1$, this expection is $+\infty$.) If we let $d = -VaR(\alpha)$, then we see that

$$ES(\alpha) = \frac{a}{a-1} VaR(\alpha) = \frac{1}{1-a^{-1}} VaR(\alpha), \text{ if } a > 1.$$

This formula enables one to estimate $ES(\alpha)$ using an estimate of $VaR(\alpha)$ and an estimate of a.

- **Estimating the tail index a**

Regression estimator of the tail index

It follows from

$$P(R \leq y) \sim \int_{-\infty}^{y} f(u)du = \frac{A}{a} y^{-a}, \quad \text{as} \quad y \to -\infty,$$

that

$$\log\{P(R \leq -y)\} = \log(L) - a\log(y),$$

where $L = A/a$.

Semiparametric model and Hill estimator

If $R_{(1)}$, ..., $R_{(n)}$ are the order statistics of the returns, then the number of observed returns less than or equal to $R_{(k)}$ is k, so we estimate $\log\{P(R \leq R_{(k)})\}$ to be $\log(k/n)$. Then, we have

$$\log(k/n) \approx \log(L) - a\log(-R_{(k)}).$$

In other words,

$$\log(-R_{(k)}) \approx (1/a)\log(L) - (1/a)\log(k/n)$$

The approximation is expected to be accurate only if R(k) is large, which means k is small, perhaps only 5%, 10%, or 20% of the sample size n.

Semiparametric model and Hill estimator

If we plot the points $[\{\log(k/n), \log(-R_{(k)})\}]_{k=1}^{m}$ for m equal to a small percentage of n, say 10%, then we should see these points fall on roughly a straight line. Moreover, if we fit the straight-line model to these points by least squares, then the estimated slope, call it $\hat{\beta}_1$, estimates $-1/a$. Therefore, we will call $-1/\hat{\beta}_1$ the regression estimator of the tail index.

Hill estimator

The Hill estimator of the left tail index a of the return density f uses all data less than a constant $c < 0$, where c is sufficiently small that

$$f(y) \leq A|y|^{-(a+1)}$$

is assumed to be true for $y < c$. The choice of c is crucial and will be discussed later.

Semiparametric model and Hill estimator

Let $y_{(1)}$, ..., $y_{(n)}$ be order statistics of the returns and $n(c)$ be the number of y_i less than or equal to c. The conditional density of Y_i given that $Y_i \leq c$ is

$$a|c|^a|y|^{-(a+1)}.$$

Therefore, the conditional likelihood is

$$L(a) = \left(\frac{a|c|^a}{|y_{(1)}|^{a+1}}\right)\left(\frac{a|c|^a}{|y_{(2)}|^{a+1}}\right)\cdots\left(\frac{a|c|^a}{|y_{(n(c))}|^{a+1}}\right),$$

and the log-likelihood is

$$\log\{L(a)\} = \sum_{i=1}^{n(c)}\{\log(a) + a\log(|c|) - (a+1)\log(|y_{(i)}|)\}.$$

Semiparametric model and Hill estimator

The score equation is

$$\frac{n(c)}{a} = \sum_{i=1}^{n(c)} \log(y_{(i)}/c).$$

By solving the equation, the Hill estimator is

$$\hat{a}^{Hill}(c) = \frac{n(c)}{\sum_{i=1}^{n(c)} \log(y_{(i)}/c)}.$$

Note that $y_{(i)} \leq c < 0$, so that $y_{(i)}/c$ is positive.

The choice of c

Usually c is equal to one of y_1, ..., Y_n, so that $c = y_{(n(c))}$, and therefore choosing c means choosing $n(c)$. The choice involves a bias-variance tradeoff. If $n(c)$ is too large, then $f(y) = A|y|^{(a+1)}$ will not hold for all values of $y < c$, causing bias. If $n(c)$ is too small, then there will be too few y_i below c and $\hat{a}^{Hill}(c)$ will be highly variable and unstable because it uses too few data. However, we can hope that there is a range of values of $n(c)$ where $\hat{a}^{Hill}(c)$ is reasonably constant because it is neither too biased nor too variable.

A Hill plot is a plot of $\hat{a}^{Hill}(c)$ versus $n(c)$ and is used to find this range of values of $n(c)$. In a Hill plot, one looks for a range of $n(c)$ where the estimator is nearly constant and then chooses $n(c)$ in this range.

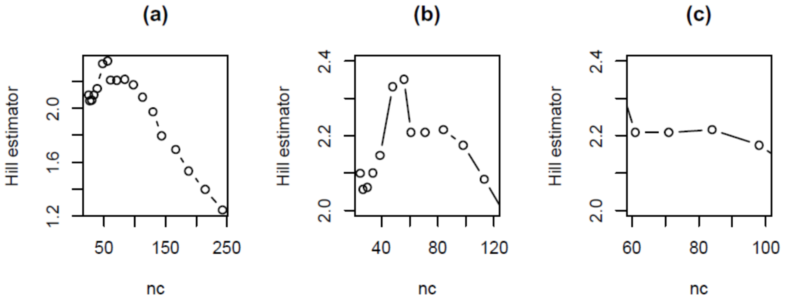

Figure: Estimation of tail index by applying a Hill plot to the daily returns on the S&P 500 for 1000 consecutive trading days ending on March 4, 2003. (a) Full range of $n(c)$. (b) Zoom in to $n(c)$ between 25 and 120. (c) Zoom in further to $n(c)$ between 60 and 100. Source: Ruppert (2011).