Stat GR5204   **Name (Print):** _____

Fall 2017   **Student UNI:** _____

Time Limit: 1 hour 30 minutes   **Signature:** _____

This exam contains 5 problems. Answer all of them. Point values are in parentheses. You **must show your work** to get credit for your solutions - correct answers without work will not be awarded points.

No calculator will be allowed in the exam. You are allowed to bring one sheet of A4 paper with notes written on both sides by yourself. Normal tables, if needed, will be provided during the exam.

1	25 pts	
2	15 pts	
3	15 pts	
4	15 pts	
5	30 pts	
TOTAL	100 pts	

1. (25 points) (7 + 8 + 10) Suppose that $X_1, \ldots, X_n$ are i.i.d Exponential(θ), (i.e., $f(x, \theta) = \theta e^{-\theta x} \mathbf{1}_{(0,\infty)}(x)$).

   (a) Find the MLE $\hat{\theta}_n$ of θ.

   (b) Find the exact sampling distribution of $\hat{\theta}_n$.

(c) Find the limiting distribution of $\log \bar{X}_n$, properly standardized.

2. (15 points) (8+7) Let $X_1, \ldots, X_n$ be i.i.d. Exponential(λ). Find the distribution of $2\lambda n \bar{X}$ (where $\bar{X} = \sum_{i=1}^{n} X_i/n$), and use this result to find a $(1 - \alpha)$ confidence interval for λ.

3. (15 points) (9 + 6) A company has manufactured certain objects and has printed a serial number on each manufactured object. The serial numbers start at 1 and end at N, where N is the number of objects manufactured. The problem is to estimate N (the parameter). A simple random sample of size n is drawn with replacement from the lot; let $X_1, X_2, \ldots, X_n$ denote the numbers on the object that come up. Thus, $X_1, \ldots, X_n$ are i.i.d random variables with common mass function $p(x, N) := \mathbb{P}(X_1 = x)$.

(a) Find $p(x, N)$ and compute $\mu := \mathbb{E}(X_1)$.

(b) Obtain a method of moments estimator of N.

4. (15 points) Suppose that $X_1, \ldots, X_n$ form a random sample from a normal distribution with unknown mean θ and variance σ^2. Assuming that $\theta \neq 0$, determine the asymptotic distribution of $\bar{X}_n^3$, where $\bar{X}_n := \sum_{i=1}^n X_i/n$.

5. (30 points) $(10 + 5 + 10 + 5)$ Consider the hierarchical model

$$Y|\Lambda \sim \text{Poisson}(\Lambda) \quad \text{and} \quad \Lambda \sim \text{Gamma}(\alpha, \beta), \quad \text{where } \alpha, \beta \text{ are known}, .$$

Note that the density of Λ is given by

$$f(y|\alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} e^{-y\beta} y^{\alpha-1}, \quad \text{for } y \geq 0, \quad (\text{and } 0 \text{ otherwise}).$$

(a) Find the marginal distribution of Y.

(b) Find the mean and variance of Y.

(c) Find the posterior distribution of Λ given the observation Y, and identify the distribution with its parameters.

(d) Find the mean of the posterior distribution.