Stat Review

HUDM 6122: Multivariate Analysis

- Expected value
- Linear Transformations
- Covariance

Mean, Variance, Covariance, Correlation

$X_1, X_2 \equiv$ Random Variables $\quad a,b,c \equiv$ constants

$$E\{X_1\} = \mu_1 \quad V\{X_1\} = E\left\{(X_1-\mu_1)^2\right\} = \sigma_1^2 = \sigma_{11} \quad E\{X_2\} = \mu_2 \quad V\{X_2\} = \sigma_2^2 = \sigma_{22}$$

$$\text{COV}\{X_1, X_2\} = E\{(X_1-\mu_1)(X_2-\mu_2)\} = \sigma_{12}$$

$$E\{cX_1\} = cE\{X_1\} = c\mu_1 \qquad V\{cX_1\} = E\left\{(cX_1 - c\mu_1)^2\right\} = c^2 E\left\{(X_1-\mu_1)^2\right\} = c^2\sigma_{11}$$

$$\text{COV}\{aX_1, bX_2\} = E\{(aX_1 - a\mu_1)(bX_2 - b\mu_2)\} = abE\{(X_1-\mu_1)(X_2-\mu_2)\} = ab\sigma_{12}$$

$$E\{aX_1 + bX_2\} = aE\{X_1\} + bE\{X_2\} = a\mu_1 + b\mu_2$$

$$V\{aX_1 + bX_2\} = E\left\{\left[(aX_1 + bX_2) - (a\mu_1 + b\mu_2)\right]^2\right\} = E\left\{\left[a(X_1-\mu_1) + b(X_2-\mu_2)\right]^2\right\}$$

$$= E\left\{\left[a^2(X_1-\mu_1)^2 + b^2(X_2-\mu_2)^2 + 2ab(X_1-\mu_1)(X_2-\mu_2)\right]\right\} = a^2\sigma_{11} + b^2\sigma_{22} + 2ab\sigma_{12}$$

$$\text{CORR}\{aX_1, bX_2\} = \frac{\text{COV}\{aX_1, bX_2\}}{\sqrt{V\{aX_1\}}\sqrt{V\{bX_2\}}} = \frac{ab\sigma_{12}}{\sqrt{a^2\sigma_{11}}\sqrt{b^2\sigma_{22}}} = \frac{\sigma_{12}}{\sqrt{\sigma_{11}}\sqrt{\sigma_{22}}} = \text{CORR}\{X_1, X_2\} = \rho_{12}$$

Random Vectors and Matrices

$$\mathbf{X} = \begin{bmatrix} X_{11} & X_{12} & \cdots & X_{1p} \\ X_{21} & X_{22} & \cdots & X_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ X_{n1} & X_{n2} & \cdots & X_{np} \end{bmatrix} \qquad E\{\mathbf{X}\} = \begin{bmatrix} E\{X_{11}\} & E\{X_{12}\} & \cdots & E\{X_{1p}\} \\ E\{X_{21}\} & E\{X_{22}\} & \cdots & E\{X_{2p}\} \\ \vdots & \vdots & \ddots & \vdots \\ E\{X_{n1}\} & E\{X_{n2}\} & \cdots & E\{X_{np}\} \end{bmatrix} \qquad \underset{r_A \times n}{\mathbf{A}}, \underset{p \times c_B}{\mathbf{B}} \text{ constants} \Rightarrow E\{\mathbf{AXB}\} = \mathbf{A}E\{\mathbf{X}\}\mathbf{B}$$

Random Vectors: Shown for case of n=3, generalizes to any n:

$$\text{Random variables: } X_1, X_2, X_3 \quad \Rightarrow \quad \mathbf{X} = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} \qquad \text{Expectation: } \mathbf{E}\{\mathbf{X}\} = \begin{bmatrix} E\{X_1\} \\ E\{X_2\} \\ E\{X_3\} \end{bmatrix} = \boldsymbol{\mu}_{\mathbf{X}}$$

Variance-Covariance Matrix for a Random Vector:

$$\sigma^2\{\mathbf{X}\} = E\left\{\left[\mathbf{X} - \mathbf{E}\{\mathbf{X}\}\right]\left[\mathbf{X} - \mathbf{E}\{\mathbf{X}\}\right]'\right\} = \mathbf{E}\left\{\begin{bmatrix} X_1 - E\{X_1\} \\ X_2 - E\{X_2\} \\ X_3 - E\{X_3\} \end{bmatrix}\begin{bmatrix} X_1 - E\{X_1\} & X_2 - E\{X_2\} & X_3 - E\{X_3\} \end{bmatrix}\right\} =$$

$$= \mathbf{E}\left\{\begin{bmatrix} \left(X_1 - E\{X_1\}\right)^2 & \left(X_1 - E\{X_1\}\right)\left(X_2 - E\{X_2\}\right) & \left(X_1 - E\{X_1\}\right)\left(X_3 - E\{X_3\}\right) \\ \left(X_2 - E\{X_2\}\right)\left(X_1 - E\{X_1\}\right) & \left(X_2 - E\{X_2\}\right)^2 & \left(X_2 - E\{X_2\}\right)\left(X_3 - E\{X_3\}\right) \\ \left(X_3 - E\{X_3\}\right)\left(X_1 - E\{X_1\}\right) & \left(X_3 - E\{X_3\}\right)\left(X_2 - E\{X_2\}\right) & \left(X_3 - E\{X_3\}\right)^2 \end{bmatrix}\right\} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} = \boldsymbol{\Sigma}_{\mathbf{X}}$$

Independence and covariance

The p random variables $X_1, X_2, \ldots, X_p$ are (statistically) *independent* if their joint density can be factored as

$$f(x_1, x_2, \ldots, x_p) = f_1(x_1)f_2(x_2) \ldots f_p(x_p)$$

Theorem: If X_i and X_k are independent, then

$$\text{cov}(X_i, X_k) = 0$$

Note: The converse is not true in general!

Mean and Variance of Linear Functions of X

$$\underset{k\times p}{\mathbf{A}} \equiv \text{matrix of fixed constants} \qquad \underset{p\times 1}{\mathbf{X}} \equiv \text{random vector}$$

$$\underset{k\times 1}{\mathbf{W}} = \mathbf{AX} = \begin{bmatrix} a_{11} & \cdots & a_{1p} \\ \vdots & & \vdots \\ a_{k1} & \cdots & a_{kp} \end{bmatrix} \begin{bmatrix} X_1 \\ \vdots \\ X_p \end{bmatrix} \equiv \text{ random vector: } \mathbf{W} = \begin{bmatrix} W_1 \\ \vdots \\ W_k \end{bmatrix} = \begin{bmatrix} a_{11}X_1 + ... + a_{1p}X_p \\ \vdots \\ a_{k1}X_1 + ... + a_{kp}X_p \end{bmatrix}$$

$$\mathbf{E}\left\{\mathbf{W}\right\} = \begin{bmatrix} E\left\{W_1\right\} \\ \vdots \\ E\left\{W_k\right\} \end{bmatrix} = \begin{bmatrix} E\left\{a_{11}X_1 + ... + a_{1p}X_p\right\} \\ \vdots \\ E\left\{a_{k1}X_1 + ... + a_{kp}X_p\right\} \end{bmatrix} = \begin{bmatrix} a_{11}E\left\{X_1\right\} + ... + a_{1p}E\left\{X_p\right\} \\ \vdots \\ a_{k1}E\left\{X_1\right\} + ... + a_{kp}E\left\{X_p\right\} \end{bmatrix} =$$

$$= \begin{bmatrix} a_{11} & \cdots & a_{1p} \\ \vdots & & \vdots \\ a_{k1} & \cdots & a_{kp} \end{bmatrix} \begin{bmatrix} E\left\{X_1\right\} \\ \vdots \\ E\left\{X_p\right\} \end{bmatrix} = \mathbf{AE}\left\{\mathbf{X}\right\} = \mathbf{A}\boldsymbol{\mu}_{\mathbf{X}}$$

$$\boldsymbol{\sigma}^2\left\{\mathbf{W}\right\} = \mathbf{E}\left\{\left[\mathbf{AX} - \mathbf{AE}\left\{\mathbf{X}\right\}\right]\left[\mathbf{AX} - \mathbf{AE}\left\{\mathbf{X}\right\}\right]'\right\} = \mathbf{E}\left\{\left[\mathbf{A}\left(\mathbf{X} - \mathbf{E}\left\{\mathbf{X}\right\}\right)\right]\left[\mathbf{A}\left(\mathbf{X} - \mathbf{E}\left\{\mathbf{X}\right\}\right)\right]'\right\} =$$

$$= \mathbf{E}\left\{\left[\mathbf{A}\left(\mathbf{X} - \mathbf{E}\left\{\mathbf{X}\right\}\right)\right]\left[\left(\mathbf{X} - \mathbf{E}\left\{\mathbf{X}\right\}\right)'\mathbf{A}'\right]\right\} = \mathbf{AE}\left\{\left(\mathbf{X} - \mathbf{E}\left\{\mathbf{X}\right\}\right)\left(\mathbf{X} - \mathbf{E}\left\{\mathbf{X}\right\}\right)'\right\}\mathbf{A}' = \mathbf{A}\boldsymbol{\Sigma}_{\mathbf{X}}\mathbf{A}'$$

Standard Deviation and Correlation Matrices

Variance-Covariance Matrix: $V\left\{\mathbf{X}\right\} = \mathbf{\Sigma_X} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1p} \\ \sigma_{12} & \sigma_{22} & \cdots & \sigma_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{1p} & \sigma_{2p} & \cdots & \sigma_{pp} \end{bmatrix}$

Standard Deviation Matrix: $\mathbf{V}^{1/2} = \text{diag}\left\{\sqrt{\sigma_{ii}}\right\} = \begin{bmatrix} \sqrt{\sigma_{11}} & 0 & \cdots & 0 \\ 0 & \sqrt{\sigma_{22}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sqrt{\sigma_{pp}} \end{bmatrix}$ $\quad \mathbf{V}^{-1/2} = \text{diag}\left\{1\big/\sqrt{\sigma_{ii}}\right\} = \begin{bmatrix} 1\big/\sqrt{\sigma_{11}} & 0 & \cdots & 0 \\ 0 & 1\big/\sqrt{\sigma_{22}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1\big/\sqrt{\sigma_{pp}} \end{bmatrix}$

Correlation: $\rho_{ij} = \dfrac{\sigma_{ij}}{\sqrt{\sigma_{ii}}\sqrt{\sigma_{jj}}}$ $\quad$ Correlation Matrix: $\boldsymbol{\rho} = \begin{bmatrix} 1 & \rho_{12} & \cdots & \rho_{1p} \\ \rho_{12} & 1 & \cdots & \rho_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{1p} & \rho_{2p} & \cdots & 1 \end{bmatrix} = \mathbf{V}^{-1/2}\mathbf{\Sigma_X}\mathbf{V}^{-1/2} \quad \Rightarrow \quad \mathbf{V}^{1/2}\boldsymbol{\rho}\mathbf{V}^{1/2} = \mathbf{V}^{1/2}\mathbf{V}^{-1/2}\mathbf{\Sigma_X}\mathbf{V}^{-1/2}\mathbf{V}^{1/2} = \mathbf{\Sigma_X}$

Partitioned Covariance Matrix

Suppose the p variables can be split into 2 groups: Group 1: $X_1, \ldots, X_q$ Group 2: $X_{q+1}, \ldots, X_p$

$$\underset{p\times 1}{\mathbf{X}} = \begin{bmatrix} X_1 \\ \vdots \\ X_q \\ -- \\ X_{q+1} \\ \vdots \\ X_p \end{bmatrix} = \begin{bmatrix} \mathbf{X}^{(1)} \\ -- \\ \mathbf{X}^{(2)} \end{bmatrix} \qquad \boldsymbol{\mu}_{\mathbf{X}} = E\{\mathbf{X}\} = \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_q \\ -- \\ \mu_{q+1} \\ \vdots \\ \mu_p \end{bmatrix} = \begin{bmatrix} \boldsymbol{\mu}^{(1)} \\ -- \\ \boldsymbol{\mu}^{(2)} \end{bmatrix}$$

$$E\left\{(\mathbf{X}-\boldsymbol{\mu}_{\mathbf{X}})(\mathbf{X}-\boldsymbol{\mu}_{\mathbf{X}})'\right\} = E\left\{\begin{bmatrix} \mathbf{X}^{(1)}-\boldsymbol{\mu}^{(1)} \\ -- \\ \mathbf{X}^{(2)}-\boldsymbol{\mu}^{(2)} \end{bmatrix}\begin{bmatrix} \left(\mathbf{X}^{(1)}-\boldsymbol{\mu}^{(1)}\right)' & | & \left(\mathbf{X}^{(2)}-\boldsymbol{\mu}^{(2)}\right)' \end{bmatrix}\right\} = E\left\{\begin{bmatrix} \left(\mathbf{X}^{(1)}-\boldsymbol{\mu}^{(1)}\right)\left(\mathbf{X}^{(1)}-\boldsymbol{\mu}^{(1)}\right)' & \left(\mathbf{X}^{(1)}-\boldsymbol{\mu}^{(1)}\right)\left(\mathbf{X}^{(2)}-\boldsymbol{\mu}^{(2)}\right)' \\ \left(\mathbf{X}^{(2)}-\boldsymbol{\mu}^{(2)}\right)\left(\mathbf{X}^{(1)}-\boldsymbol{\mu}^{(1)}\right)' & \left(\mathbf{X}^{(2)}-\boldsymbol{\mu}^{(2)}\right)\left(\mathbf{X}^{(2)}-\boldsymbol{\mu}^{(2)}\right)' \end{bmatrix}\right\}$$

$$= \begin{bmatrix} \boldsymbol{\Sigma}_{\mathbf{X}}^{(11)} & \boldsymbol{\Sigma}_{\mathbf{X}}^{(12)} \\ \boldsymbol{\Sigma}_{\mathbf{X}}^{(21)} & \boldsymbol{\Sigma}_{\mathbf{X}}^{(22)} \end{bmatrix} = \boldsymbol{\Sigma}_{\mathbf{X}} = \begin{bmatrix} \sigma_{11} & \cdots & \sigma_{1q} & \sigma_{1,q+1} & \cdots & \sigma_{1p} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \sigma_{1q} & \cdots & \sigma_{qq} & \sigma_{q,q+1} & \cdots & \sigma_{qp} \\ \sigma_{1,q+1} & \cdots & \sigma_{q,q+1} & \sigma_{q+1,q+1} & \cdots & \sigma_{q+1,p} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \sigma_{1p} & \cdots & \sigma_{qp} & \sigma_{q+1,p} & \cdots & \sigma_{pp} \end{bmatrix}$$

Note: $\boldsymbol{\Sigma}_{\mathbf{X}}^{(21)} = \boldsymbol{\Sigma}_{\mathbf{X}}^{(12)}\,'$

Similar Partitioning for sample mean vector and Variance-Covariance Matrix: $\overline{\mathbf{x}} = \begin{bmatrix} \overline{x}_1 \\ \vdots \\ \overline{x}_q \\ -- \\ \overline{x}_{q+1} \\ \vdots \\ \overline{x}_p \end{bmatrix} = \begin{bmatrix} \overline{\mathbf{x}}^{(1)} \\ -- \\ \overline{\mathbf{x}}^{(2)} \end{bmatrix}$ $\qquad \mathbf{S}_n = \begin{bmatrix} \mathbf{S}_n^{(11)} & \mathbf{S}_n^{(12)} \\ \mathbf{S}_n^{(21)} & \mathbf{S}_n^{(22)} \end{bmatrix}$

Matrix Inequalities and Maximization

Cauchy-Schwarz Inequality $\underset{p\times 1}{\mathbf{b}},\ \underset{p\times 1}{\mathbf{d}}$

$(\mathbf{b}'\mathbf{d})^2 \le (\mathbf{b}'\mathbf{b})(\mathbf{d}'\mathbf{d})$ with equality only if $\mathbf{b} = c\mathbf{d}$ for some constant c

Extended Cauchy-Schwarz Inequality $\underset{p\times 1}{\mathbf{b}},\ \underset{p\times 1}{\mathbf{d}},\ \underset{p\times p}{\mathbf{B}}$ $\mathbf{B} \equiv$ positive definite

$(\mathbf{b}'\mathbf{d})^2 \le (\mathbf{b}'\mathbf{B}\mathbf{b})(\mathbf{d}'\mathbf{B}^{-1}\mathbf{d})$ with equality only if $\mathbf{b} = c\mathbf{B}^{-1}\mathbf{d}$ or $\mathbf{d} = c\mathbf{B}\mathbf{b}$ for some constant c

Maximization Lemma $\underset{p\times p}{\mathbf{B}}$ $\mathbf{B} \equiv$ positive definite $\underset{p\times 1}{\mathbf{d}} \equiv$ given vector $\underset{p\times 1}{\mathbf{x}} \equiv$ arbitrary non-zero vector

$$\max_{\mathbf{x}\neq\mathbf{0}} \frac{(\mathbf{x}'\mathbf{d})^2}{\mathbf{x}'\mathbf{B}\mathbf{x}} = \mathbf{d}'\mathbf{B}^{-1}\mathbf{d} \quad @ \quad \mathbf{x} = c\mathbf{B}^{-1}\mathbf{d} \quad \text{for any constant } c \neq 0$$

Maximization of Quadratic Forms for Points on Unit Sphere

$\underset{p\times p}{\mathbf{B}}$ $\mathbf{B} \equiv$ positive definite with eigenvalues and eigenvectors: $(\lambda_1, \mathbf{e}_1), \ldots, (\lambda_p, \mathbf{e}_p)$ $\lambda_1 \ge \ldots \ge \lambda_p$

$$\max_{\mathbf{x}\neq\mathbf{0}} \frac{\mathbf{x}'\mathbf{B}\mathbf{x}}{\mathbf{x}'\mathbf{x}} = \lambda_1 \quad @ \quad \mathbf{x} = \mathbf{e}_1 \qquad \min_{\mathbf{x}\neq\mathbf{0}} \frac{\mathbf{x}'\mathbf{B}\mathbf{x}}{\mathbf{x}'\mathbf{x}} = \lambda_p \quad @ \quad \mathbf{x} = \mathbf{e}_p$$

$$\max_{\mathbf{x}\neq\mathbf{0}\perp\mathbf{e}_1,\ldots,\mathbf{e}_k} \frac{\mathbf{x}'\mathbf{B}\mathbf{x}}{\mathbf{x}'\mathbf{x}} = \lambda_{k+1} \quad @ \quad \mathbf{x} = \mathbf{e}_{k+1} \qquad k = 1, \ldots, p-1$$

Matrix Inequalities and Maximization

Example:
Let $\boldsymbol{b}'$ = [2, -1, 4, 0] and $\boldsymbol{d}'$ = [-1, 3, -2, 1]. Verify the Cauchy-Scwartz inequality $(\boldsymbol{b}'\boldsymbol{d})^2 \leq (\boldsymbol{b}'\boldsymbol{b})(\boldsymbol{d}'\boldsymbol{d})$

Multivariate Statistics

Chapter 3

$\boldsymbol{X}$-Matrix, Mean Vector, Deviation Vectors

Data: p Variables/Characteristics Observed on n Experimental/Sampling Units:

$$\mathbf{X}=\begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1p} \\ x_{21} & x_{22} & \cdots & x_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{np} \end{bmatrix}=\begin{bmatrix} \mathbf{x}_1' \\ \mathbf{x}_2' \\ \vdots \\ \mathbf{x}_n' \end{bmatrix}=\begin{bmatrix} \mathbf{y}_1 & \mathbf{y}_2 & \cdots & \mathbf{y}_p \end{bmatrix} \qquad \mathbf{x}_j'=\begin{bmatrix} x_{j1} & x_{j2} & \cdots & x_{jp} \end{bmatrix} \quad j=1,...,n \qquad \mathbf{y}_i=\begin{bmatrix} x_{1i} \\ x_{2i} \\ \vdots \\ x_{ni} \end{bmatrix} \quad i=1,...,p$$

Mean Vector and Vector of a Single Mean:

$$\overline{\mathbf{x}}=\begin{bmatrix} \overline{x}_1 \\ \overline{x}_2 \\ \vdots \\ \overline{x}_p \end{bmatrix} \qquad \overline{x}_i=\frac{\sum_{j=1}^{n} x_{ji}}{n} \quad i=1,...,p \qquad \overline{\mathbf{x}}=\frac{1}{n}\mathbf{X}'\mathbf{1}_n \qquad \overline{x}_i\mathbf{1}_n=\begin{bmatrix} \overline{x}_i \\ \overline{x}_i \\ \vdots \\ \overline{x}_i \end{bmatrix}=\left[\mathbf{y}_i'\left(\frac{1}{\sqrt{n}}\mathbf{1}_n\right)\right]\left(\frac{1}{\sqrt{n}}\mathbf{1}_n\right) \quad i=1,...,p$$

Projection of $\mathbf{y}_i$ on the vector $\mathbf{1}_n$ is $\overline{x}_i\mathbf{1}_n$ which has Length: $\sqrt{\overline{x}_i\mathbf{1}_n'\overline{x}_i\mathbf{1}_n}=\left|\overline{x}_i\right|\sqrt{n}$

Vector of Deviations for a Single Variable, Sums of Squares and Cross-Products, and Correlation:

$$\mathbf{d}_i=\begin{bmatrix} x_{1i}-\overline{x}_i \\ x_{2i}-\overline{x}_i \\ \vdots \\ x_{ni}-\overline{x}_i \end{bmatrix}=\mathbf{y}_i-\overline{x}_i\mathbf{1}_n \quad i=1,...,p \qquad \mathbf{d}_i'\mathbf{d}_i=\sum_{j=1}^{n}\left(x_{ji}-\overline{x}_i\right)^2 \qquad \mathbf{d}_i'\mathbf{d}_k=\sum_{j=1}^{n}\left(x_{ji}-\overline{x}_i\right)\left(x_{jk}-\overline{x}_k\right)=L_{\mathbf{d}_i}L_{\mathbf{d}_k}\cos\left(\theta_{ik}\right) \;\Rightarrow\; r_{ik}=\frac{\sum_{j=1}^{n}\left(x_{ji}-\overline{x}_i\right)\left(x_{jk}-\overline{x}_k\right)}{\sqrt{\sum_{j=1}^{n}\left(x_{ji}-\overline{x}_i\right)^2}\sqrt{\sum_{j=1}^{n}\left(x_{jk}-\overline{x}_k\right)^2}}=\cos\left(\theta_{ik}\right) \;\Rightarrow\; \theta_{ik}=\cos^{-1}\left(r_{ik}\right)$$

$$L_{\mathbf{d}_i}=\sqrt{\mathbf{d}_i'\mathbf{d}_i}=\sqrt{\sum_{j=1}^{n}\left(x_{ji}-\overline{x}_i\right)^2}=\sqrt{ns_{ii}}$$

$\boldsymbol{X}$-Matrix, Mean Vector, Deviation Vectors

Example: Given the data matrix

$$\boldsymbol{X} = \begin{bmatrix} 9 & 1 \\ 5 & 3 \\ 1 & 2 \end{bmatrix}$$

a) Graph the data, calculate $\bar{\boldsymbol{x}}$ and locate it on the graph.

b) Calculate the deviation vectors $\boldsymbol{d}_1$ and $\boldsymbol{d}_2$

c) Calculate the lengths and angle between the two deviation vectors from part b). Relate to r, s_{11} and s_{22}.

Deviation, SSCP, and Variance-Covariance Matrices

Matrix of Deviations, Sum of Squares Cross-Products (SSCP) Matrix, and Sample Variance-Covariance Matrix ($\mathbf{S}_n$):

$$\mathbf{E} = \begin{bmatrix} \mathbf{d}_1 & \mathbf{d}_2 & \cdots & \mathbf{d}_p \end{bmatrix} = \begin{bmatrix} x_{11}-\bar{x}_1 & x_{12}-\bar{x}_2 & \cdots & x_{1p}-\bar{x}_p \\ x_{21}-\bar{x}_1 & x_{22}-\bar{x}_2 & \cdots & x_{2p}-\bar{x}_p \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1}-\bar{x}_1 & x_{n2}-\bar{x}_2 & \cdots & x_{np}-\bar{x}_p \end{bmatrix}$$

$$SSCP = \mathbf{E}'\mathbf{E} = \begin{bmatrix} \sum_{j=1}^{n}\left(x_{j1}-\bar{x}_1\right)^2 & \sum_{j=1}^{n}\left(x_{j1}-\bar{x}_1\right)\left(x_{j2}-\bar{x}_2\right) & \cdots & \sum_{j=1}^{n}\left(x_{j1}-\bar{x}_1\right)\left(x_{jp}-\bar{x}_p\right) \\ \sum_{j=1}^{n}\left(x_{j1}-\bar{x}_1\right)\left(x_{j2}-\bar{x}_2\right) & \sum_{j=1}^{n}\left(x_{j2}-\bar{x}_2\right)^2 & \cdots & \sum_{j=1}^{n}\left(x_{j2}-\bar{x}_2\right)\left(x_{jp}-\bar{x}_p\right) \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{j=1}^{n}\left(x_{j1}-\bar{x}_1\right)\left(x_{jp}-\bar{x}_p\right) & \sum_{j=1}^{n}\left(x_{j2}-\bar{x}_2\right)\left(x_{jp}-\bar{x}_p\right) & \cdots & \sum_{j=1}^{n}\left(x_{jp}-\bar{x}_p\right)^2 \end{bmatrix}$$

$$\mathbf{S}_n = \frac{1}{n}\mathbf{E}'\mathbf{E} = \frac{1}{n}\begin{bmatrix} \sum_{j=1}^{n}\left(x_{j1}-\bar{x}_1\right)^2 & \sum_{j=1}^{n}\left(x_{j1}-\bar{x}_1\right)\left(x_{j2}-\bar{x}_2\right) & \cdots & \sum_{j=1}^{n}\left(x_{j1}-\bar{x}_1\right)\left(x_{jp}-\bar{x}_p\right) \\ \sum_{j=1}^{n}\left(x_{j1}-\bar{x}_1\right)\left(x_{j2}-\bar{x}_2\right) & \sum_{j=1}^{n}\left(x_{j2}-\bar{x}_2\right)^2 & \cdots & \sum_{j=1}^{n}\left(x_{j2}-\bar{x}_2\right)\left(x_{jp}-\bar{x}_p\right) \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{j=1}^{n}\left(x_{j1}-\bar{x}_1\right)\left(x_{jp}-\bar{x}_p\right) & \sum_{j=1}^{n}\left(x_{j2}-\bar{x}_2\right)\left(x_{jp}-\bar{x}_p\right) & \cdots & \sum_{j=1}^{n}\left(x_{jp}-\bar{x}_p\right)^2 \end{bmatrix}$$

Random Samples of Vectors of Units

$$\mathbf{X} = \begin{bmatrix} X_{11} & X_{12} & \cdots & X_{1p} \\ X_{21} & X_{22} & \cdots & X_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ X_{n1} & X_{n2} & \cdots & X_{np} \end{bmatrix} = \begin{bmatrix} \mathbf{X}_1' \\ \mathbf{X}_2' \\ \vdots \\ \mathbf{X}_n' \end{bmatrix} \qquad \mathbf{X}_j' = \begin{bmatrix} X_{j1} & X_{j2} & \cdots & X_{jp} \end{bmatrix} \quad j = 1, \ldots, n$$

Row Vectors $\equiv$ independent observations from common joint probability distribution: $f(\mathbf{x}) = f(x_1, \ldots, x_p)$

$\Rightarrow \quad \mathbf{X}_1, \ldots, \mathbf{X}_n$ can be treated as a Random Sample from $f(\mathbf{x})$

$\Rightarrow$ joint density is $f(\mathbf{x}_1) \cdots f(\mathbf{x}_n)$ where $f(\mathbf{x}_j) = f(x_{j1}, \ldots, x_{jp})$

Treating Units (Rows) as a Random Sample from an Underlying Population of Units on the p Variables:

$$E\{X_{ji}\} = \mu_i \qquad V\{X_{ji}\} = E\left\{\left(X_{ji} - \mu_i\right)^2\right\} = \sigma_{ii} \qquad \mathrm{COV}\{X_{ji}, X_{jk}\} = E\left\{\left(X_{ji} - \mu_i\right)\left(X_{jk} - \mu_k\right)\right\} = \sigma_{ik}$$

For Random Samples, Rows are Independent (Note: Not a Time-Series where rows are the same units being observed over Time):

$$\mathrm{COV}\{X_{ji}, X_{lk}\} = 0 \quad j \neq l, \forall\, i, k$$

$$E\{\mathbf{X}_j\} = \boldsymbol{\mu}_{\mathbf{X}} = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_p \end{bmatrix} \qquad V\{\mathbf{X}_j\} = E\left\{\left(\mathbf{X}_j - \boldsymbol{\mu}_{\mathbf{x}}\right)\left(\mathbf{X}_j - \boldsymbol{\mu}_{\mathbf{x}}\right)'\right\} = \boldsymbol{\Sigma}_{\mathbf{X}} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1p} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{p1} & \sigma_{p2} & \cdots & \sigma_{pp} \end{bmatrix} \qquad \mathrm{COV}\{\mathbf{X}_j, \mathbf{X}_l\} = E\left\{\left(\mathbf{X}_j - \boldsymbol{\mu}_{\mathbf{x}}\right)\left(\mathbf{X}_l - \boldsymbol{\mu}_{\mathbf{x}}\right)'\right\} = \mathbf{0}_{p \times p} \quad j \neq l$$

Expectation and Covariance Matrix of Sample Mean Vector

$$E\left\{\mathbf{X}_j\right\} = \boldsymbol{\mu}_{\mathbf{X}} = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_p \end{bmatrix} \qquad V\left\{\mathbf{X}_j\right\} = E\left\{\left(\mathbf{X}_j - \boldsymbol{\mu}_{\mathbf{x}}\right)\left(\mathbf{X}_j - \boldsymbol{\mu}_{\mathbf{x}}\right)'\right\} = \boldsymbol{\Sigma}_{\mathbf{X}} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1p} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{p1} & \sigma_{p2} & \cdots & \sigma_{pp} \end{bmatrix}$$

$$\mathrm{COV}\left\{\mathbf{X}_j, \mathbf{X}_l\right\} = E\left\{\left(\mathbf{X}_j - \boldsymbol{\mu}_{\mathbf{x}}\right)\left(\mathbf{X}_l - \boldsymbol{\mu}_{\mathbf{x}}\right)'\right\} = \mathbf{0}_{p\times p}$$

$$\text{Let } \overline{\mathbf{X}} = \frac{1}{n}\left(\mathbf{X}_1 + \ldots + \mathbf{X}_n\right) = \frac{1}{n}\left(\begin{bmatrix} X_{11} \\ X_{12} \\ \vdots \\ X_{1p} \end{bmatrix} + \ldots + \begin{bmatrix} X_{n1} \\ X_{n2} \\ \vdots \\ X_{np} \end{bmatrix}\right) = \begin{bmatrix} \overline{X}_1 \\ \overline{X}_2 \\ \vdots \\ \overline{X}_p \end{bmatrix}$$

$$E\left\{\overline{\mathbf{X}}\right\} = E\left\{\frac{1}{n}\left(\mathbf{X}_1 + \ldots + \mathbf{X}_n\right)\right\} = \frac{1}{n} nE\left\{\mathbf{X}_j\right\} = \boldsymbol{\mu}_{\mathbf{X}}$$

$$V\left\{\overline{\mathbf{X}}\right\} = E\left\{\left(\overline{\mathbf{X}} - \boldsymbol{\mu}_{\mathbf{x}}\right)\left(\overline{\mathbf{X}} - \boldsymbol{\mu}_{\mathbf{x}}\right)'\right\} = E\left\{\left[\frac{1}{n}\sum_{j=1}^{n}\left(\mathbf{X}_j - \boldsymbol{\mu}_{\mathbf{X}}\right)\right]\left[\frac{1}{n}\sum_{l=1}^{n}\left(\mathbf{X}_l - \boldsymbol{\mu}_{\mathbf{X}}\right)\right]'\right\} = E\left\{\frac{1}{n^2}\sum_{j=1}^{n}\sum_{l=1}^{n}\left(\mathbf{X}_j - \boldsymbol{\mu}_{\mathbf{X}}\right)\left(\mathbf{X}_l - \boldsymbol{\mu}_{\mathbf{X}}\right)'\right\} =$$

$$E\left\{\frac{1}{n^2}\sum_{j=1}^{n}\left(\mathbf{X}_j - \boldsymbol{\mu}_{\mathbf{X}}\right)\left(\mathbf{X}_j - \boldsymbol{\mu}_{\mathbf{X}}\right)'\right\} = \frac{1}{n^2} n\boldsymbol{\Sigma}_{\mathbf{X}} = \frac{1}{n}\boldsymbol{\Sigma}_{\mathbf{X}}$$

Expectation of Sample Variance-Covariance Matrix

$$\mathbf{S}_n = \frac{1}{n}\mathbf{E}'\mathbf{E} = \frac{1}{n}\begin{bmatrix} \sum_{j=1}^{n}\left(x_{j1}-\bar{x}_1\right)^2 & \sum_{j=1}^{n}\left(x_{j1}-\bar{x}_1\right)\left(x_{j2}-\bar{x}_2\right) & \cdots & \sum_{j=1}^{n}\left(x_{j1}-\bar{x}_1\right)\left(x_{jp}-\bar{x}_p\right) \\ \sum_{j=1}^{n}\left(x_{j1}-\bar{x}_1\right)\left(x_{j2}-\bar{x}_2\right) & \sum_{j=1}^{n}\left(x_{j2}-\bar{x}_2\right)^2 & \cdots & \sum_{j=1}^{n}\left(x_{j2}-\bar{x}_2\right)\left(x_{jp}-\bar{x}_p\right) \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{j=1}^{n}\left(x_{j1}-\bar{x}_1\right)\left(x_{jp}-\bar{x}_p\right) & \sum_{j=1}^{n}\left(x_{j2}-\bar{x}_2\right)\left(x_{jp}-\bar{x}_p\right) & \cdots & \sum_{j=1}^{n}\left(x_{jp}-\bar{x}_p\right)^2 \end{bmatrix} = \frac{1}{n}\sum_{j=1}^{n}\left[\left(\mathbf{X}_j-\overline{\mathbf{X}}\right)\left(\mathbf{X}_j-\overline{\mathbf{X}}\right)'\right]$$

$$= \frac{1}{n}\sum_{j=1}^{n}\left[\mathbf{X}_j\mathbf{X}_j' - \mathbf{X}_j\overline{\mathbf{X}}' - \overline{\mathbf{X}}\mathbf{X}_j' + \overline{\mathbf{X}}\overline{\mathbf{X}}'\right] = \frac{1}{n}\sum_{j=1}^{n}\mathbf{X}_j\mathbf{X}_j' - \overline{\mathbf{X}}\overline{\mathbf{X}}' - \overline{\mathbf{X}}\overline{\mathbf{X}}' + \overline{\mathbf{X}}\overline{\mathbf{X}}' = \frac{1}{n}\sum_{j=1}^{n}\mathbf{X}_j\mathbf{X}_j' - \overline{\mathbf{X}}\overline{\mathbf{X}}'$$

Aside: Let $\mathbf{V}$ be a random vector with mean vector and Variance-Covariance matrix: $\boldsymbol{\mu}_{\mathrm{V}}$ and $\boldsymbol{\Sigma}_{\mathrm{V}}$:

$$\boldsymbol{\Sigma}_{\mathrm{V}} = E\left\{\left(\mathbf{V}-\boldsymbol{\mu}_{\mathrm{V}}\right)\left(\mathbf{V}-\boldsymbol{\mu}_{\mathrm{V}}\right)'\right\} = E\left\{\mathbf{V}\mathbf{V}' - \mathbf{V}\boldsymbol{\mu}_{\mathrm{V}}' - \boldsymbol{\mu}_{\mathrm{V}}\mathbf{V}' + \boldsymbol{\mu}_{\mathrm{V}}\boldsymbol{\mu}_{\mathrm{V}}'\right\} = E\left\{\mathbf{V}\mathbf{V}'\right\} - \boldsymbol{\mu}_{\mathrm{V}}\boldsymbol{\mu}_{\mathrm{V}}' - \boldsymbol{\mu}_{\mathrm{V}}\boldsymbol{\mu}_{\mathrm{V}}' + \boldsymbol{\mu}_{\mathrm{V}}\boldsymbol{\mu}_{\mathrm{V}}' = E\left\{\mathbf{V}\mathbf{V}'\right\} - \boldsymbol{\mu}_{\mathrm{V}}\boldsymbol{\mu}_{\mathrm{V}}'$$

$$\Rightarrow \quad E\left\{\mathbf{V}\mathbf{V}'\right\} = \boldsymbol{\Sigma}_{\mathrm{V}} + \boldsymbol{\mu}_{\mathrm{V}}\boldsymbol{\mu}_{\mathrm{V}}'$$

$$\Rightarrow \quad E\left\{\mathbf{S}_n\right\} = E\left\{\frac{1}{n}\sum_{j=1}^{n}\mathbf{X}_j\mathbf{X}_j' - \overline{\mathbf{X}}\overline{\mathbf{X}}'\right\} = \left[\frac{1}{n}n\left(\boldsymbol{\Sigma}_{\mathrm{X}} + \boldsymbol{\mu}_{\mathrm{X}}\boldsymbol{\mu}_{\mathrm{X}}'\right)\right] - \left[\frac{1}{n}\boldsymbol{\Sigma}_{\mathrm{X}} + \boldsymbol{\mu}_{\mathrm{X}}\boldsymbol{\mu}_{\mathrm{X}}'\right] = \frac{n-1}{n}\boldsymbol{\Sigma}_{\mathrm{X}}$$

Defining: $\mathbf{S} = \frac{n}{n-1}\mathbf{S}_n = \frac{1}{n-1}\sum_{j=1}^{n}\left[\left(\mathbf{X}_j-\overline{\mathbf{X}}\right)\left(\mathbf{X}_j-\overline{\mathbf{X}}\right)'\right] \quad \Rightarrow \quad E\left\{\mathbf{S}\right\} = \boldsymbol{\Sigma}_{\mathrm{X}}$

Generalized Variance

$$\mathbf{S} = \begin{bmatrix} s_{11} & s_{12} & \cdots & s_{1p} \\ s_{12} & s_{22} & \cdots & s_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ s_{1p} & s_{2p} & \cdots & s_{pp} \end{bmatrix} \qquad s_{ik} = \frac{1}{n-1}\sum_{j=1}^{n}\left(x_{ji}-\bar{x}_i\right)\left(x_{jk}-\bar{x}_k\right) \quad i,k=1,\ldots,p \quad \text{Generalized Sample Variance} \;=\; \det(\mathbf{S}) \;=\; |\mathbf{S}|$$

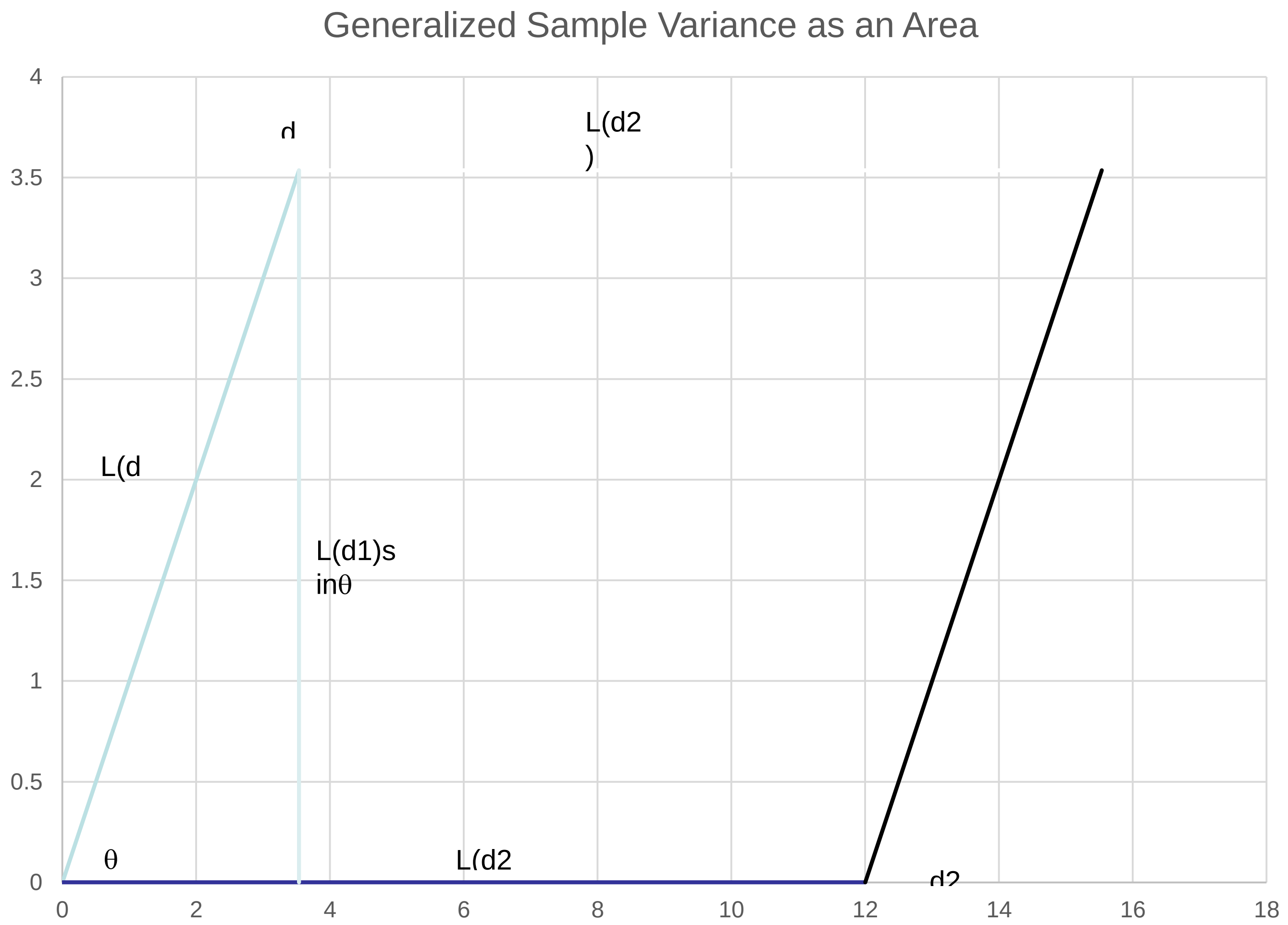
$$p=2: \quad \mathbf{S} = \begin{bmatrix} s_{11} & s_{12} \\ s_{12} & s_{22} \end{bmatrix} = \begin{bmatrix} s_{11} & r_{12}\sqrt{s_{11}}\sqrt{s_{22}} \\ r_{12}\sqrt{s_{11}}\sqrt{s_{22}} & s_{22} \end{bmatrix} \;\Rightarrow\; |\mathbf{S}| = s_{11}s_{22} - r_{12}^2 s_{11}s_{22} = s_{11}s_{22}\left(1-r_{12}^2\right) = s_{11}s_{22}\left(1-\cos^2\theta_{12}\right) = s_{11}s_{22}\sin^2\theta_{12}$$

Area of Trapezoid created from vectors $\mathbf{d}_1, \mathbf{d}_2$ (plot on next slide): Area $= L_{\mathbf{d}_1} L_{\mathbf{d}_2} \left|\sin\theta_{12}\right| \qquad \theta_{12} = \cos^{-1} r_{12}$

$$L_{\mathbf{d}_1} = \sqrt{\sum_{j=1}^{n}\left(x_{j1}-\bar{x}_1\right)^2} = \sqrt{(n-1)s_{11}} \qquad L_{\mathbf{d}_2} = \sqrt{\sum_{j=1}^{n}\left(x_{j2}-\bar{x}_2\right)^2} = \sqrt{(n-1)s_{22}}$$

$$\Rightarrow \quad \text{Area} \;=\; (n-1)\sqrt{s_{11}s_{22}}\left|\sin\theta_{12}\right| \quad \Rightarrow \quad |\mathbf{S}| = \frac{\text{Area}^2}{(n-1)^2} \qquad \text{For general } p: \quad |\mathbf{S}| = \frac{\text{Volume}^2}{(n-1)^p}$$

Volume of a hyperellipsoid centered at $\bar{\mathbf{x}}$: $\left\{\mathbf{x} : \left(\mathbf{x}-\bar{\mathbf{x}}\right)'\mathbf{S}^{-1}\left(\mathbf{x}-\bar{\mathbf{x}}\right) \le c^2\right\} = k_p |\mathbf{S}|^{1/2} c^p \qquad k_p = \dfrac{2\pi^{p/2}}{p\Gamma\left(p/2\right)}$



Generalized Sample Variance as an Area
4
3.5
3
2.5
2
1.5
1
0.5
0
0
2
4
6
8
10
12
14
16
18
d
L(d2
)
L(d
L(d1)s
inθ
θ
L(d2
d2

Matrix Form of Sample Mean, Covariance, Correlation

$$\bar{\mathbf{x}} = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \vdots \\ \bar{x}_p \end{bmatrix} = \frac{1}{n}\mathbf{X}'\mathbf{1}_n \qquad n \times p \text{ matrix of Column means: } \begin{bmatrix} \bar{x}_1 \mathbf{1}_n & \bar{x}_2 \mathbf{1}_n & \cdots & \bar{x}_p \mathbf{1}_n \end{bmatrix} = \mathbf{1}_n \bar{\mathbf{x}}' = \frac{1}{n}\mathbf{1}_n\mathbf{1}_n{}'\mathbf{X} = \frac{1}{n}\mathbf{J}_n\mathbf{X}$$

Note: $\frac{1}{n}\mathbf{J}_n \frac{1}{n}\mathbf{J}_n = \frac{1}{n^2}\mathbf{J}_n\mathbf{J}_n = \frac{1}{n^2}n\mathbf{J}_n = \frac{1}{n}\mathbf{J}_n$

$n \times p$ matrix of Deviations from Column means: $\mathbf{E} = \mathbf{X} - \frac{1}{n}\mathbf{J}_n\mathbf{X} = \left(\mathbf{I}_n - \frac{1}{n}\mathbf{J}_n\right)\mathbf{X}$

Note: $\left(\mathbf{I}_n - \frac{1}{n}\mathbf{J}_n\right)\left(\mathbf{I}_n - \frac{1}{n}\mathbf{J}_n\right) = \mathbf{I}_n - \frac{1}{n}\mathbf{J}_n - \frac{1}{n}\mathbf{J}_n + \frac{1}{n^2}\mathbf{J}_n\mathbf{J}_n = \mathbf{I}_n - \frac{1}{n}\mathbf{J}_n$

$p \times p$ matrix of Sums of Squares and Cross-Products and Estimated Variance-Covariance Matrix:

$$\mathbf{E}'\mathbf{E} = \mathbf{X}'\left(\mathbf{I}_n - \frac{1}{n}\mathbf{J}_n\right)'\left(\mathbf{I}_n - \frac{1}{n}\mathbf{J}_n\right)\mathbf{X} = \mathbf{X}'\left(\mathbf{I}_n - \frac{1}{n}\mathbf{J}_n\right)\mathbf{X} \quad \Rightarrow \quad \mathbf{S} = \frac{1}{n-1}\mathbf{E}'\mathbf{E} = \frac{1}{n-1}\mathbf{X}'\left(\mathbf{I}_n - \frac{1}{n}\mathbf{J}_n\right)\mathbf{X}$$

$p \times p$ Standard Deviation Matrix and its Inverse:

$$\mathbf{D}^{1/2} = \begin{bmatrix} \sqrt{s_{11}} & 0 & \cdots & 0 \\ 0 & \sqrt{s_{22}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sqrt{s_{pp}} \end{bmatrix} \qquad \mathbf{D}^{-1/2} = \begin{bmatrix} 1/\sqrt{s_{11}} & 0 & \cdots & 0 \\ 0 & 1/\sqrt{s_{22}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1/\sqrt{s_{pp}} \end{bmatrix} \qquad \mathbf{R} = \mathbf{D}^{-1/2}\mathbf{S}\mathbf{D}^{-1/2} \qquad \mathbf{S} = \mathbf{D}^{1/2}\mathbf{R}\mathbf{D}^{1/2}$$

Linear Combinations of Sample Values

$$\mathbf{X} = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_p \end{bmatrix} \qquad \mathbf{c}'\mathbf{X} = c_1 X_1 + \ldots + c_p X_p \quad \text{observed value for Unit } j: \quad \mathbf{c}'\mathbf{x}_j = c_1 x_{j1} + \ldots + c_p x_{jp} \qquad j = 1,\ldots,n$$

Sample Mean: $\dfrac{\mathbf{c}'\mathbf{x}_1 + \ldots + \mathbf{c}'\mathbf{x}_n}{n} = \dfrac{1}{n}\mathbf{c}'\left(\mathbf{x}_1 + \ldots + \mathbf{x}_n\right) = \mathbf{c}'\bar{\mathbf{x}}$

Sample Variance: $\dfrac{1}{n-1}\sum_{j=1}^{n}\left(\mathbf{c}'\mathbf{x}_j - \mathbf{c}'\bar{\mathbf{x}}\right)^2 = \dfrac{1}{n-1}\sum_{j=1}^{n}\left[\mathbf{c}'\left(\mathbf{x}_j - \bar{\mathbf{x}}\right)\right]^2 = \dfrac{1}{n-1}\sum_{j=1}^{n}\left[\mathbf{c}'\left(\mathbf{x}_j - \bar{\mathbf{x}}\right)\left(\mathbf{x}_j - \bar{\mathbf{x}}\right)'\mathbf{c}\right] = \mathbf{c}'\dfrac{\sum_{j=1}^{n}\left(\mathbf{x}_j - \bar{\mathbf{x}}\right)\left(\mathbf{x}_j - \bar{\mathbf{x}}\right)'}{n-1}\mathbf{c} = \mathbf{c}'\mathbf{S}\mathbf{c}$

Second Linear Combination: $\mathbf{b}'\mathbf{X}$:

Sample Mean: $\mathbf{b}'\bar{\mathbf{x}}$ Sample Variance: $\mathbf{b}'\mathbf{S}\mathbf{b}$

Sample Variance of $\mathbf{b}'\mathbf{X}$ and $\mathbf{c}'\mathbf{X}$: $\dfrac{1}{n-1}\sum_{j=1}^{n}\left[\mathbf{b}'\left(\mathbf{x}_j - \bar{\mathbf{x}}\right)\mathbf{c}'\left(\mathbf{x}_j - \bar{\mathbf{x}}\right)\right] = \dfrac{1}{n-1}\sum_{j=1}^{n}\left[\mathbf{b}'\left(\mathbf{x}_j - \bar{\mathbf{x}}\right)\left(\mathbf{x}_j - \bar{\mathbf{x}}\right)'\mathbf{c}\right] = \mathbf{b}'\dfrac{\sum_{j=1}^{n}\left(\mathbf{x}_j - \bar{\mathbf{x}}\right)\left(\mathbf{x}_j - \bar{\mathbf{x}}\right)'}{n-1}\mathbf{c} = \mathbf{b}'\mathbf{S}\mathbf{c}$

Generalized to q Linear Combinations: $a_{i1}X_1 + \ldots + a_{ip}X_p \quad i = 1,\ldots q \qquad \mathbf{AX}$ where $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1p} \\ a_{21} & a_{22} & \cdots & a_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ a_{q1} & a_{q2} & \cdots & a_{qp} \end{bmatrix}$

Sample Mean: $\mathbf{A}\bar{\mathbf{x}}$ Sample Variance: $\mathbf{ASA}'$