Fixed Income Securities

Fixed income securities are financial instruments that provide fixed (in both time and amount) payments. They include treasury bills, notes and bonds, municipal bonds, corporate bonds, securitized bank lending such as asset-backed securities (ABS) etc.

Zero Coupon Bonds

- **Zero coupon bonds**
  A zero coupon bond is a financial instrument that makes only one payment at the maturity. It is usually sold with discount and therefore is also known as a pure discount bond. Coupon bonds, on the other hand, make payments to holders at specific time points before maturity. It will be shown that a coupon bond can be viewed theoretically as a collection of zero coupon bonds.

- **Face value / par value**
  The face value (par value) of a zero coupon bond is the amount of payment the bond holder receives at the maturity.

Zero Coupon Bonds

- **Present value / market price**
  Pricing of a zero coupon bond is related to the risk free interest rate. For a 10 year zero coupon bond with a face value at \$100 and interest rate at r =4%, the (fair) market price (present value) is

  $$\mathrm{PV} = \frac{\$100}{1.04^{10}} = \$67.56.$$

  Here the interest rate is compounded yearly. If the interest rate is r =2%, then

  $$\mathrm{PV} = \frac{\$100}{1.02^{10}} = \$82.03.$$

  For r =6%,

  $$\mathrm{PV} = \frac{\$100}{1.06^{10}} = \$55.84.$$

  In general, a higher interest rate entails a deeper discount. For a zero coupon with T years maturity and rate r, compounded annually,

  $$\mathrm{PV} = \frac{\mathrm{PAR}}{(1+r)^T}.$$

Zero Coupon Bonds

- **Periodic compounding**
  If the interest is r, which is compounded semi-annually, then

$$\mathrm{PV} = \frac{\mathrm{PAR}}{(1 + r/2)^{2T}}.$$

  In general, if it is compounded n times a year, then

$$\mathrm{PV} = \frac{\mathrm{PAR}}{(1 + r/n)^{nT}}.$$

Zero Coupon Bonds

- **Continuous compounding**
  If the interest is compounded continuously ($n \to \infty$), then

$$\mathrm{PV} = \frac{\mathrm{PAR}}{e^{rT}} = \mathrm{PAR}e^{-rT}. \tag{2}$$

Again, for PAR=\$100 and r =2%, 4% and 6%, we have

$$\mathrm{PV} = \$100e^{-0.02\times 10} = 81.87,$$

$$\mathrm{PV} = \$100e^{-0.04\times 10} = 67.03,$$

$$\mathrm{PV} = \$100e^{-0.06\times 10} = 54.88.$$

Coupon Bonds

- **Coupon bonds**
  A coupon bond makes, in additional to the principal at the maturity, periodic coupon payments to the holder. For a T-year coupon bond that makes coupon payment C every year, if interest rate is r and compounded yearly, then

$$\mathrm{PV} = \sum_{t=1}^{T} \frac{C}{(1+r)^t} + \frac{\mathrm{PAR}}{(1+r)^T};$$

  if it is compounded continuously, then

$$\mathrm{PV} = \sum_{t=1}^{T} \frac{C}{e^{rt}} + \frac{\mathrm{PAR}}{e^{rT}}.$$

Coupon Bonds

More generally, if coupon payments $C_1, \ldots, C_m$ are made at $t_1 < \ldots, < t_m$ with continuous compounding rate r, then

$$\mathrm{PV} = \sum_{i=1}^{m} \frac{C_i}{e^{rt_i}} + \frac{\mathrm{PAR}}{e^{rT}}. \tag{3}$$

Equation (3) shows that a coupon bond is equivalent to a collection of zero coupon bonds.

- **Default time and probability**
  Price of a bond depends not only on the risk-free rate, but also on credit worthiness of the bond issuer. The credit worthiness may be quantified in terms of the probability distribution of the default time. Let X denote the default time, which is positive random variable. For simplicity, suppose that X follows the exponential distribution with hazard β. Then $P(X > t) = e^{-\beta t}$. With default and no recovery, equation (2) needs to be modified to

$$\mathrm{PV} = E\left[I(X > T)\frac{\mathrm{PAR}}{e^{rT}}\right] = \frac{\mathrm{PAR}}{e^{(\beta+r)T}},$$

which is as if the interest rate were $\beta + r$. Thus, β represents its difference with the risk free rate and is known as the spread.

Default

For coupon bond and with possibility of default, the present value calculation that corresponds to equation (3) is

$$\mathrm{PV} = E\left[\sum_{i=1}^{m} I(X > t_i)\frac{C_i}{e^{rt_i}} + I(X > T)\frac{\mathrm{PAR}}{e^{rT}}\right] = \sum_{i=1}^{m} \frac{C_i}{e^{(\beta+r)t_i}} + \frac{\mathrm{PAR}}{e^{(\beta+r)T}}.$$

We can also calculate the one-year default probability $P(X \leq 1) = 1 - e^{-\beta}$, which is approximately equal to β when β is small.

- **Yield to Maturity (YTM)**
  The yield to maturity is essentially the internal rate of return of the bond at the current price. For a T-year zero coupon bond with interest compounded n times a year, the yield is r that satisfies

$$\text{Price} = \frac{\text{PAR}}{(1 + r/n)^{nT}}.$$

  Given Price and PAR, you can solve the above equation to get YTM

$$r = n\left[\left(\frac{\text{PAR}}{\text{Price}}\right)^{\frac{1}{nT}} - 1\right].$$

Note that Price must be smaller than PAR, i.e., there is a discount. Thus, suppose you just purchased a 3-year zero coupon bond with face value PAR= 1000 dollars for Price= 850 dollars. Then the yearly interest rate is effectively

$$r = \left(\frac{1000}{850}\right)^{\frac{1}{3}} - 1 \approx 0.05567.$$

If coupons are paid n times a year with amount C per coupon, then its yield r satisfies

$$\text{Price} = \frac{\text{PAR}}{(1 + r/n)^{nT}} + \sum_{i=1}^{nT} \frac{C}{(1 + r/n)^i}.$$

Although in general there is no explicit solution, r is unique and can be solved numerically.

The corresponding formula for YTM under continuous compounding is mathematically simpler:

$$\text{Price} = \frac{\text{PAR}}{e^{rT}} \qquad \text{for } T - \text{year, zero} - \text{coupon}$$

or

$$r = \frac{1}{T} \log\left(\frac{\text{PAR}}{\text{Price}}\right).$$

For the example, we have

$$r = \frac{1}{3} \log\left(\frac{1000}{850}\right) \approx 0.05417.$$

For a coupon bond, the formula becomes (r may be solved implicitly)

$$\text{Price} = \sum_{i=1}^{m} \frac{C_i}{e^{rt_i}} + \frac{\text{PAR}}{e^{rT}}.$$

- **Spot rate**
  T-year spot rate is the YTM of a T-year zero coupon bond.

- **Price-yield curves**
  The Price is clearly a monotone decreasing function of Yield. The relationship as represented by the corresponding curve is called the price-yield curve.

- **Duration (DUR)**
  The duration, DUR, is defined by

$$\mathrm{DUR} = -\frac{1}{\mathrm{PRICE}} \times \frac{\Delta\mathrm{PRICE}}{\Delta\mathrm{YIELD}}.$$

  Thus, for a T-year zero coupon bond, $\mathrm{DUR} = T$; for a coupon bond,

$$\mathrm{DUR} = \frac{1}{\mathrm{PRICE}}\left(\sum_{i=1}^{m} \frac{C_i}{e^{rt_i}} t_i + \frac{\mathrm{PAR}}{e^{rT}} T\right).$$

- **Forward rates**
  The forward rates, $r_1, r_2, \ldots,$ compounded yearly, are defined recursively as follow. Let $P(n)$ be the price for the n-year zero coupon bond with face value PAR. Then

$$P(1) = \frac{\mathrm{PAR}}{1 + r_1}$$

$$\cdots$$

$$P(n) = \frac{\mathrm{PAR}}{(1 + r_1)(1 + r_2)\cdots(1 + r_n)}$$

$$\cdots$$

  The continuously compounded rates are similarly defined

$$P(n) = \frac{\mathrm{PAR}}{\exp(r_1 + r_2 + \cdots + r_n)}.$$

- **Continuous forward rates**
  Continuous forward rate function, $r(t)$, is defined by

$$P(t) = \mathrm{PAR} \times e^{-\int_0^t r(s)ds},$$

where $P(t)$ is the price of a t-year zero coupon bond with face value PAR. Note that this is an integral equation with solution

$$r(t) = -\frac{d}{dt} \log P(t).$$

Let $y(t)$ be the yield for t-year zero coupon bond under continuous compounding. Then we have

$$\mathrm{PAR} \times e^{-ty(t)} = P(t) = \mathrm{PAR} \times e^{-\int_0^t r(s)ds}$$

or

$$y(t) = \frac{1}{t}\int_0^t r(s)ds.$$

In other words, $y(t)$ is the average forward rate over the period $[0,\ t]$.

Price-yield Curves

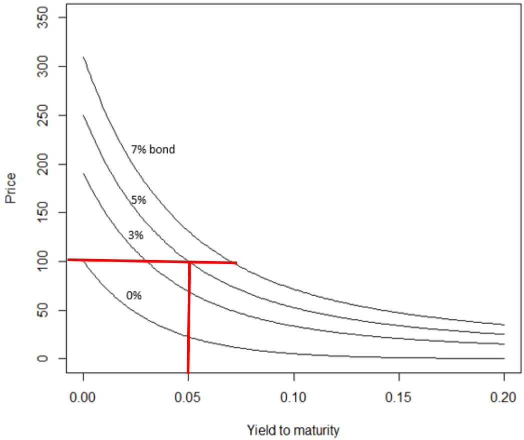

Figure: Price-yield curves and coupon rate. All bonds shown have a maturity of 30 years and are paid semi-annually. The **coupon rates** are indicated on the respective curves and prices are expresses as a percentage of par.

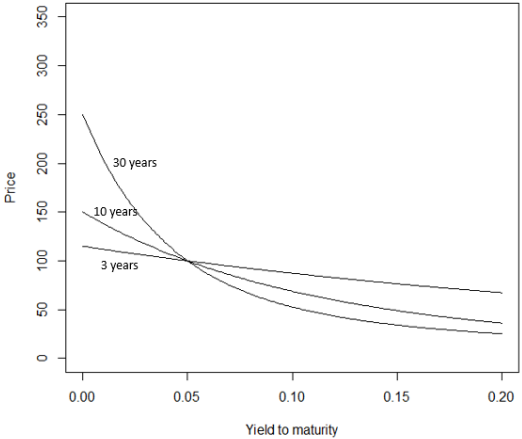

Figure: Price-yield curves and maturities. The price-yield curves is shown for three maturities. All bonds have a **5**% coupon paid annually.