Solution to Practice Questions

1 Interest Rate

1. Let r be the yearly compounded rate. Expected present value of all the payments is

$$p \times 0 + (1-p)p\frac{C}{1+r} + (1-p)^2 p \sum_{t=1}^{2} \frac{C}{(1+r)^t} + (1-p)^3 p \sum_{t=1}^{3} \frac{C}{(1+r)^t}$$
$$+(1-p)^4 p \sum_{t=1}^{4} \frac{C}{(1+r)^t} + \mathbb{P}(X \geq 6)\left(\sum_{t=1}^{5} \frac{C}{(1+r)^t} + \frac{\text{PAR}}{(1+r)^5}\right),$$

which equals 785.979.

2. (a)

$$\text{return} = \frac{\text{Payment} - 800}{800} = \frac{1000 \cdot I(\text{no deafult}) - 800}{800},$$

where $I(\cdot)$ is the indicator function. Therefore,

$$\mathbb{E}(\text{return}) = \frac{1000 \cdot 0.9 - 800}{800} = \frac{1}{8}$$

and

$$\text{SD}(\text{return}) = \sqrt{\left(\frac{1000}{800}\right)^2 \cdot 0.1(0.9)} = \frac{3}{8}.$$

(I am using the formulae for expected value and variance of a Bernoulli random variable)

(b)

$$\text{return} = \frac{1000(I(\text{A no default}) + I(\text{B no default})) - 1600}{1600}.$$

As in (a),

$$\mathbb{E}(\text{return}) = \frac{1}{8}.$$

Since the two default events are mutually exclusive (note that this does not mean they are independent), $I(\text{A default})I(\text{B default}) = 0$ and thus

$$\begin{aligned} & \text{Cov}(I(\text{A no default}), I(\text{B no default})) \\ = \; & \mathbb{E}(I(\text{A no default})I(\text{B no default})) - \mathbb{E}(I(\text{A no default}))\mathbb{E}(I(\text{B no default})) \\ = \; & \mathbb{E}(1 - I(\text{A default})) - I(\text{B default})) + I(\text{A default})I(\text{B default})) - 0.9(0.9) \\ = \; & 1 - 0.1 - 0.1 - 0.9(0.9) = -0.01. \end{aligned}$$

Hence,

$$\begin{aligned} & \text{Var}(I(\text{A no default}) + I(\text{B no default})) \\ = \; & \text{Var}(I(\text{A no default})) + \text{Var}(I(\text{B no default})) + 2\text{Cov}(I(\text{A no default}), I(\text{B no default})) \\ = \; & 0.1(0.9) + 0.1(0.9) + 2(-0.01) = 0.16 \end{aligned}$$

and

$$\text{SD}(\text{return}) = \sqrt{\left(\frac{1000}{1600}\right)^2 \cdot 0.16} = 0.25.$$

3. (a) Let s be the 5-year spot rate.

$$\begin{aligned} s &= \frac{1}{5}\int_0^5 r(t)dt = \frac{1}{5}\int_0^5 0.03 + 0.001t + 0.0002t^2 dt \\ &= \frac{1}{5}\left[0.03t + \frac{0.001}{2}t^2 + \frac{0.0002}{3}t^3\right]_0^5 \\ &= 0.03417. \end{aligned}$$

(b) Price $= \text{PAR}e^{-5s} = 0.843\text{PAR}$.

4. WLOG, assume PAR = 1.

$$\begin{aligned} P_0 &= e^{-\int_0^8 0.04+0.001t\,dt} = 0.703 \\ P_{0.5} &= e^{-\int_0^{7.5} 0.03+0.013t\,dt} = 0.5539809 \\ \text{Return} &= \frac{P_{0.5} - P_0}{P_0} = -0.212. \end{aligned}$$

2 Portfolio Theory

1. Suppose we assign weight w to asset 1 and $1 - w$ to asset 2.

$$\begin{aligned} Var(wR_1 + (1-w)R_2) =& w^2 Var(R_1) + 2w(1-w)Cov(R_1, R_2) + (1-w)^2 Var(R_2) \\ =& w^2 20^2 + 2w(1-w)20 \times 10 \times \rho + (1-w)^2 10^2. \end{aligned}$$

Taking derivative w.r.t. w, we get

$$w = \frac{1-2\rho}{5-4\rho}.$$

Plugging in $\rho = 0, 0.3, -0.3$ we get the answer.

2. Use equation (16.5) of the book (page 471)

$$w_T = \frac{V_1\sigma_2^2 - V_2\rho_{12}\sigma_1\sigma_2}{V_1\sigma_2^2 + V_2\sigma_1^2 - (V_1+V_2)\rho_{12}\sigma_1\sigma 2},$$

where $V_1 = \mu_1 - \mu_f$, $V_2 = \mu_2 - \mu_f$.

3. (a) $\mu_R = \mu_f + (\mu_M - \mu_f) \times \frac{\sigma_R}{\sigma_M} = 0.023 + (0.1 - 0.023) \times 0.05/0.12 = 0.05508$.

(b) $Cov(R_A, R_M) = \beta Var(R_M) = \beta \times (0.12)^2 = 0.004$ so $\beta = 0.004/0.12^2 = 0.27$.

(c) β of the portfolio is $(1.5 + 1.8)/2 = 1.65$, so the expected return is

$$\mu_f + \beta(\mu_M - \mu_f) = 0.023 + 1.65 \times (0.1 - 0.023) = 0.15.$$

The σ_ϵ for the portfolio is $\sigma_\epsilon = \sqrt{\frac{1}{2^2}(0.08^2 + 0.10^2)} = 0.064$, therefore the standard deviation of the return of the portfolio is

$$\sqrt{\beta^2\sigma_M^2 + \sigma_\epsilon^2} = \sqrt{1.65^2 \times 0.12^2 + 0.064^2} = 0.208.$$

3 Copula

1. Denote $W = U + V$. Using definition of Kendall's τ.

$$
\begin{aligned}
\tau =& E\left[\operatorname{sign}\left((U_t - U_s)(W_t - W_s)\right)\right] \\
=& 1 - 2P((U_t - U_s)(W_t - W_s) < 0) \\
=& 1 - 2P(U_t > U_s, W_t < W_s) - 2P(U_t < U_s, W_t > W_s) \\
=& 1 - 4P(U_t > U_s, W_t < W_s) \\
(*)\ =& 1 - 4P(U_t > U_s, V_t = 0, V_s = 1) \\
(**)\ =& 1 - 4P(U_t > U_s)P(V_t = 0)P(V_s = 1) \\
=& 1 - 4 \times 0.5 \times (1-p)p = 2p^2 - 2p + 1
\end{aligned}
$$

$(*)$ holds because that $U_t > U_s, W_t < W_s$ can only happen when $V_t = 0, V_s = 1$ (note V_t and V_s are binary). $(**)$ holds because U_t, U_s, V_t, V_s are all independent of each other. Note that this holds as long as U is a continuous variable with support in $(0, 1)$.

For Spearman's ρ, from definition we have

$$
\rho = \operatorname{Corr}\left(F_U(U), F_W(W)\right) = \frac{E(F_U(U)F_W(W)) - E(F_U(U)) \times E(F_W(W))}{SD(F_U(U)) \times SD(F_W(W))},
$$

where F_U and F_W are the CDF of U and W. Note that $F_U(U)$ and $F_V(V)$ both follows uniform distribution (this is the property of CDF), so we have

$$
\rho = \frac{E(F_U(U)F_W(W)) - \frac{1}{2} \times \frac{1}{2}}{\frac{1}{12}}.
$$

Now we only need to compute $E(F_U(U)F_W(W))$. We have $F_U(U) = U$ (U follows uniform distribution). Computing F_W is slightly trickier, we have, for $x \in [0, 1]$

$$
F_W(x) = P(W = U + V < x) = P(U < x, V = 0) = P(V = 0)P(U < x) = (1-p)x.
$$

For $x \in [1, 2]$

$$
F_W(x) = P(W = U + V < x) = P(V = 0) + P(U < x - 1, V = 1) = (1-p) + p \times (x - 1).
$$

Therefore we have

$$
\begin{aligned}
E(F_U(U)F_W(W)) =& E(UF_W(U + V)) \\
=& P(V = 0)E(UF_W(U)|V = 0) + P(V = 1)E(UF_W(U + 1)) \\
=& (1-p)E((1-p)U^2) + pE(U[(1-p) + pU]) \\
=& \frac{(1-p)^2}{3} + \frac{p(1-p)}{2} + \frac{p^2}{3}.
\end{aligned}
$$

Therefore we have

$$
\rho = \frac{\frac{(1-p)^2}{3} + \frac{p(1-p)}{2} + \frac{p^2}{3} - \frac{1}{2} \times \frac{1}{2}}{\frac{1}{12}} = 2p^2 - 2p + 1.
$$

Now Pearson's correlation is much more straightforward, Here we have

$$
\begin{aligned}
Corr(U,W) =& \frac{Cov(U,U+V)}{SD(U)\times SD(W)} \\
=& \frac{Cov(U,U)+Cov(U,V)}{\sqrt{var(U)}\sqrt{var(U)+var(V)}} \\
=& \frac{\frac{1}{12}}{\sqrt{\frac{1}{12}}\sqrt{\frac{1}{12}+p(1-p)}} \\
=& \frac{1}{\sqrt{1+12p(1-p)}}.
\end{aligned}
$$

2. (a) $F_X(x) = F_\theta(x,\infty) = 1-e^{-x}$. X has an exponential distribution with parameter 1. The same holds for Y.

   (b) By definition,

$$
\begin{aligned}
C_\theta(x,y) &= \mathbb{P}(F_X(X)\le x, F_Y(Y)\le y) \\
&= \mathbb{P}(X\le F_X^{-1}(x), Y\le F_Y^{-1}(y)) \\
&= \mathbb{P}(X\le -\log(1-x), Y\le -\log(1-y)) \\
&= F_\theta(-\log(1-x), -\log(1-y)).
\end{aligned}
$$

3. Same as Problem 6 in Homework 5, just plug in different numbers.

4 Risk Management

1. First we compute the normalizing constant

$$
Z = \int_{-\infty}^{+\infty} \exp(-\lambda|x|)dx = 2\int_0^{+\infty} \exp(-\lambda|x|)dx = \frac{2}{\lambda}.
$$

For $l > 0$, we have

$$
P(\mathcal{L} > l) = \frac{1}{Z}\int_l^{+\infty} \exp(-\lambda|x|)dx = \frac{\lambda}{2}\times\frac{\exp(-\lambda l)}{\lambda} = \frac{\exp(-\lambda l)}{2}.
$$

Solving $P(\mathcal{L} > \mathrm{VaR}(0.05)) = \frac{\exp(-\lambda \mathrm{VaR}(0.05))}{2} = 0.05$, we get

$$
\mathrm{VaR}(0.05) = -\frac{\log(0.1)}{\lambda} = \frac{\log(10)}{\lambda}.
$$

We also have, for $l > 0$

$$
\mathrm{ES}(0.05) = E(\mathcal{L}|\mathcal{L} > l) = \frac{\frac{1}{Z}\int_l^{+\infty} x\exp(-\lambda|x|)dx}{P(\mathcal{L} > l)},
$$

where

$$
\int_l^{+\infty} x\exp(-\lambda|x|)dx = \frac{\exp(-\lambda l)(1+\lambda l)}{\lambda^2}.
$$

Plut in $l = \text{VaR}(0.05) = \frac{\log(10)}{\lambda}$, we have

$$\text{ES}(0.05) = \frac{\lambda}{2} \times \frac{0.1(1+\log(10))}{\lambda^2}\frac{1}{0.05} = \frac{\log(10)}{\lambda} + \frac{1}{\lambda}.$$

Note (ignore this if you are not interested): the interesting pattern that $\text{ES}(0.05) = \text{VaR}(0.05) + \frac{1}{\lambda}$. For the double exponential distribution, this would actually hold in general. i.e.

$$\text{ES}(\alpha) = \text{VaR}(\alpha) + \frac{1}{\lambda}$$

for all $\alpha < 0.5$. This is related to the memoryless property of exponential distribution.

2. (a) We first compute the mean and variance of the portfolio return. We have $\mu_1 = 1/7$, $\mu_2 = 2/7$ and $\mu_3 = 4/7$, so $\mu_P = \sum_{i=1}^3 w_i\mu_i = 0.044286$ and

$$\sigma_P^2 = \sum_{i=1}^{3} w_i^2\sigma_i^2 + \sum_{i\neq j} w_i w_j \sigma_{ij} = \sum_{i=1}^{3} w_i^2\sigma_i^2 + 2\sum_{i<j} w_i w_j \sigma_{ij} = 0.195876,$$

where σ_{ij} denotes the covariance between the returns of stocks i and j. Using the formula of normal VaR, we have $\text{VaR}(0.05) = -350000[\mu_P + \sigma_P(-z_{0.05})] = 97265$.

(b) $\text{ES}(0.05) = 350000(-\mu_P + \frac{\sigma_P}{0.05}\phi(z_{0.05})) = 125913$.

3. Recall that $\text{VaR}(\alpha)$ is defined such that $\mathbb{P}(L > \text{VaR}(\alpha)) = \alpha$. Hence, $\text{VaR}(s) = F_L^{-1}(1-s)$, where F_L is the cdf of L. Note that

$$\text{ES}(\alpha) = \frac{\int_0^\alpha \text{VaR}(s)ds}{\alpha} = \frac{\int_\infty^{\text{VaR}(\alpha)} -lf_L(l)dl}{\alpha} = \frac{\int_{\text{VaR}(\alpha)}^\infty lf_L(l)dl}{P(L > \text{VaR}(\alpha))} = \mathbb{E}(L|L > \text{VaR}(\alpha)) = \text{CTE}(\alpha).$$

The second equality follows by using the change of variable $l = \text{VaR}(s) = F_L^{-1}(1-s)$, which gives $F_L(l) = 1 - s$ and so $ds = -f_L(l)$.