STAT GU4261/GR5261 Statistical Methods in Finance

Spring 2018

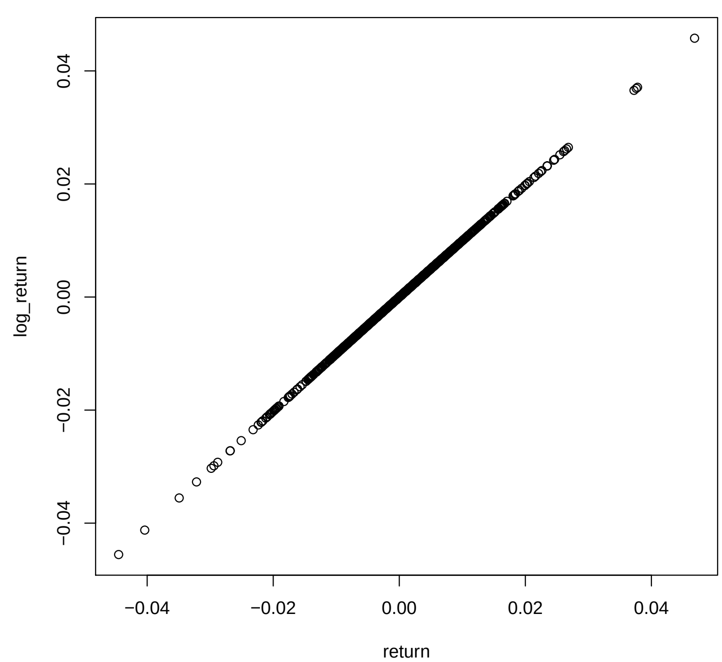
Homeowork 1 Suggested Solution

Due date: 6 Feb 2017 (Tue)

P.15, Problem 9 in textbook:
Mean = 0.05
Standard deviation = 0.2.
Note: since the question is not very clear, the answers computed from the simulation are also accepted. But I think the question is trying to see if you understand which parts in the code represent the mean and standard deviation of the log-returns (for 1 year).

P.15, Problem 11 in textbook:
The code gives a vector of daily prices $(P_0, P_1, \ldots, P_n)$ since $P_k = 120e^{r_1 + \cdots r_k}$, where r_k is the log-return at day k.
Note that the `cumsum` function is to compute the cumulative sum of a vector. Suppose that `x = c(1,2,3)`, them `cumsum(x) = c(1,3,6)`.

P.15, Problem 12 in textbook:



P.16-17 Exercise 1 in textbook:

(a)

$$\mathbb{P}(P_1 < 990) = \mathbb{P}\bigg(\log \frac{P_1}{1000} < \log \frac{990}{1000} \bigg) = \mathbb{P}\bigg(\mathcal{N}(0.001, 0.015^2) < \log \frac{990}{1000} \bigg) = 0.231.$$

In R: `pnorm(log(990/1000),0.001,0.015)`

(b)

$$\mathbb{P}(P_5 < 990) = \mathbb{P}\bigg(\log \frac{P_5}{1000} < \log \frac{990}{1000} \bigg) = \mathbb{P}\bigg(\mathcal{N}(0.005, 5 \cdot 0.015^2) < \log \frac{990}{1000} \bigg) = 0.327.$$

In R: `pnorm(log(990/1000),0.005,sqrt(5)*0.015)`

P.16-17 Exercise 3 in textbook:

$$\mathbb{P}(P_2 \geq 90) = \mathbb{P}\bigg(\log \frac{P_2}{80} \geq \log \frac{90}{80} \bigg) = \mathbb{P}\bigg(\mathcal{N}(0.16, 2 \cdot 0.15^2) \geq \log \frac{90}{80} \bigg) = 0.579.$$

In R: `1-pnorm(log(90/80),0.16,sqrt(2)*0.15)`

P.16-17 Exercise 10 in textbook:

$$\mathbb{P}(P_{20} > 100) = \mathbb{P}\bigg(\log \frac{P_{20}}{97} > \log \frac{100}{97} \bigg) = \mathbb{P}\bigg(\mathcal{N}(20 \cdot 0.0002, 20 \cdot 0.03^2) > \log \frac{100}{97} \bigg) = 0.422.$$

In R: `1-pnorm(log(100/97),20*0.0002,sqrt(20)*0.03)`

P.40-43 Exercise 1 in textbook:

(a) $y_{20} = \frac{1}{20} \int_0^{20} (0.028 + 0.00042t)dt = 0.0322.$

(b) $P = 1000e^{-\int_0^{15} (0.028+0.00042t)dt} = 626.7.$

P.40-43 Exercise 3 in textbook:

(a) Coupon rate > current yield if and only if price > par. Therefore, it is selling above par.

(b) Price > par if and only if copun rate > current yield > yield to maturity. Hence, yield to maturity is below 2.8%.

Remark: Let y be the yield. The bond price is

$$P = \sum_{t=1}^{T} \frac{C}{(1+y)^t} + \frac{\text{PAR}}{(1+y)^T}.$$

Divided both sides by P and moving terms, we obtain

$$\frac{C}{P} = \frac{1 - \frac{\text{PAR}}{P} \frac{1}{(1+y)^T}}{\sum_{t=1}^{T} \frac{1}{(1+y)^t}}.$$

Since

$$\sum_{t=1}^{T} \frac{1}{(1+y)^t} = \frac{\frac{1}{(1+y)^{T+1}} - \frac{1}{1+y}}{\frac{1}{1+y} - 1} = \frac{1 - \frac{1}{(1+y)^T}}{y},$$

we have

$$\frac{C}{P} = y \times \frac{1 - \frac{\text{PAR}}{P}\frac{1}{(1+y)^T}}{1 - \frac{1}{(1+y)^T}}.$$

As $\frac{C}{P}$ is the current yield, we see that current yield > yield to maturity if and only if price > PAR.

<u>P.40-43 Exercise 8 in textbook:</u>

(a) $828 = 1000e^{-5y_5}$ implies $y_5 = 0.0377$. When the forward rate is constant, it is equal to the yield to maturity. Hence, $r = 0.0377$.

(b) $P = 1000e^{-4(0.042)} = 845$.

(c) Return $= \frac{845-828}{828} = 0.0205$.

<u>P.40-43 Exercise 11 in textbook:</u>

$$P = 100e^{-\int_0^{15}(0.033+0.0012t)dt} = 53.259.$$

<u>P.40-43 Exercise 12 in textbook:</u>

$$\begin{aligned}
P_0 &= \text{Par} \times e^{-\int_0^8 0.04+0.001t dt} = 0.703 \\
P_{0.5} &= \text{Par} \times e^{-\int_0^{7.5} 0.03+0.0013t dt} = 0.770 \\
\text{Return} &= \frac{P_{0.5} - P_0}{P_0} = 0.09465.
\end{aligned}$$

<u>P.40-43 Exercise 16 in textbook:</u>

(a)

$$P_0 = 1000e^{-(0.04+0.001(10))10} = 606.5037.$$

(b)

$$P_1 = 1000e^{-(0.042+0.001(9))9} = 631.9152$$

$$\text{Return} = \frac{P_1 - P_0}{P_0} = 0.04185.$$

<u>P.40-43 Exercise 22 in textbook:</u>

(a)

$$P = \sum_{i=1}^{8} C_i \times e^{-\int_0^{i/2} r(t)dt} = 1100.87,$$

where $C_i = 21$ for $i = 1, \ldots, 7$ and $C_8 = 21 + 1000$.

(b)

$$\text{Duration} = \sum_{i=1}^{8} \frac{C_i \times e^{-\int_0^{i/2} r(t)dt}}{P} \times \frac{i}{2} = 3.741.$$

Note: the formula of duration is given on page Page 42 in textbook.

Other questions:

(1) $\mathbb{P}(U \le u) = \mathbb{P}(F(X) \le u) = \mathbb{P}(X \le F^{-1}(u)) = F(F^{-1}(u)) = u$. Hence, $U \sim \text{Unif}(0,1)$.

(2) (a) By chain rule and fundamental theorem of calculus,

$$f_Y(y) = \frac{d}{dy}\mathbb{P}(Y \le y) = \frac{d}{dy}\mathbb{P}(X \le \log Y) = \frac{d\log y}{dy}\frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{(\log y-\mu)^2}{2\sigma^2}} = \frac{1}{y\sqrt{2\pi\sigma^2}}e^{-\frac{(\log y-\mu)^2}{2\sigma^2}}.$$

(b) We first prove the claim in the hint. Note that

$$\begin{aligned}
\mathrm{E}(e^{Xt}) &= \int_{-\infty}^{\infty} e^{xt}\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}dx \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{x^2-2x\mu+\mu^2-2x\sigma^2 t}{2\sigma^2}\right\}dx \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{[x-(\mu+\sigma^2 t)]^2-2\mu\sigma^2 t-\sigma^4 t^2}{2\sigma^2}\right\}dx \\
&= \exp\left\{\mu t+\frac{1}{2}\sigma^2 t^2\right\}\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{[x-(\mu+\sigma^2 t)]^2}{2\sigma^2}\right\}dx \\
&= \exp\left\{\mu t+\frac{1}{2}\sigma^2 t^2\right\}.
\end{aligned}$$

Hence, the mean is $\mathbb{E}(e^X) = e^{\mu+\frac{1}{2}\sigma^2}$ and

$$\begin{aligned}
\mathrm{Var}(Y) &= \mathrm{Var}(e^X) \\
&= \mathrm{E}(e^X)^2 - [\mathrm{E}(e^X)]^2 \\
&= E(e^{2X}) - [\mathrm{E}(e^X)]^2 \\
&= e^{2\mu+2\sigma^2} - [e^{\mu+\sigma^2/2}]^2 \\
&= e^{2\mu+\sigma^2}(e^{\sigma^2}-1).
\end{aligned}$$

(3) Denote A to be the corresponding matrix in the question.

(i) $\det(A-\lambda I) = 0$ gives $(1-\lambda)^2 - \rho^2 = 0$. Hence, the eigenvalues are $1 \pm \rho$. Largest eigenvalue is $\max\{1-\rho, 1+\rho\} = 1+|\rho|$.

(ii) $\det(A-\lambda I) = 0$ gives $(1-\lambda)^3 - (1-\lambda)\rho^2 = 0$. Hence, the eigenvalues are 1 and $1 \pm \rho$. Largest eigenvalue is $\max\{1, 1-\rho, 1+\rho\} = 1+|\rho|$.

(4) Since X and Y are iid, $\mathrm{Var}(X) = \mathrm{Var}(Y)$ and $\mathrm{Cov}(X,Y) = 0$. Hence, $\mathrm{Cov}(X+Y, X-Y) = \mathrm{Var}(X) - \mathrm{Var}(Y) - 2\,\mathrm{Cov}(X,Y) = 0$. That is, they are uncorrelated.

Two ways to show that they are not independent:

(i) Recall that two random variables U and V are said to be independent if

$$\mathbb{P}(U \in A, V \in B) = \mathbb{P}(U \in A)\mathbb{P}(V \in B), \quad \text{for any } A, B \subset \mathbb{R}. \tag{1}$$

To show that $X+Y$ and $X-Y$ are not independent, note that

$$\mathbb{P}(X+Y \le 1, X-Y > 1) = 0,$$

as the event $\{X+Y \le 1, X-Y > 1\} = \emptyset$. However, it is clear that $\mathbb{P}(X+Y \le 1) > 0$ and $\mathbb{P}(X-Y > 1) > 0$. Therefore, (1) does not hold and they are not independent.

(ii) Let $U := X + Y$ and $V := X - Y$. Then $X = \frac{U+V}{2}$ and $Y = \frac{U-V}{2}$. The Jacobian is

$$J = \left| \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} \right| = \left| \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \right| = -\frac{1}{2}.$$

Hence,

$$f_{U,V}(u,v) = f_{X,Y}\left(\frac{u+v}{2}, \frac{u-v}{2}\right)|J| = \frac{e^{-u}}{2} 1(-u < v < u).$$

Clearly, they cannot be independent.

Remark:

- Always remember that the density function should also include the support. It is wrong to write something like $f_{U,V}(u,v) = \frac{e^{-u}}{2}$ without specifying the domain of definition of the function. For example, the equality $f_{U,V}(u,v) = \frac{e^{-u}}{2}$ does not hold for any $u, v \in \mathbb{R}$.
- The joint density $f_{X,Y}(x,y) = e^{-x}e^{-y}1(x > 0)1(y > 0)$. That's why $f_{X,Y}(\frac{u+v}{2}, \frac{u-v}{2}) = e^{-u}1(\frac{u+v}{2} > 0)1(\frac{u-v}{2} > 0) = e^{-u}1(-u < v < u)$.

Remark: if X and Y were iid normal r.v.'s, then $X + Y$ and $X - Y$ are independent. So it is wrong to say something like $X + Y$ annd $X - Y$ are independent because both terms involve X and Y.