Stat Gr5204: Practice Final	**Name (Print):** ________________
Fall 2017	**Student UNI:** ________________
Time Limit: 2 hours 40 minutes	**Signature:** ________________

This exam contains 8 problems. Answer all of them. Point values are in parentheses. You **must show your work** to get credit for your solutions - correct answers without work will not be awarded points.

No calculator will be allowed in the exam. You are allowed to bring 2 sheets of A4 paper with notes written on both sides by yourself. Normal tables, if needed, will be provided during the exam.

1	10 pts	
2	15 pts	
3	15 pts	
4	10 pts	
5	15 pts	
6	15 pts	
7	10 pts	
8	10 pts	
TOTAL	100 pts	

1. (10 points) (5 + 5) Suppose that we have a random sample $X_1, \ldots, X_n$ from a $N(0, \sigma^2)$ population.

   (i) Find the form of the most powerful test of $H_0 : \sigma^2 = \sigma_0^2$ versus $H_1 : \sigma^2 = \sigma_1^2$, where $0 < \sigma_0 < \sigma_1$.

   (ii) For a given value of α, the size of the Type I error, find the explicit form of the critical value of the above test.

2. (15 points) (5 + 10) Let $X_1, \ldots, X_n$ be i.i.d Uniform$(\theta, \theta + 1)$, where $\theta \in \mathbb{R}$ is unknown. To test $H_0 : \theta = 0$ versus $H_1 : \theta > 0$ we can use the procedure:

$$\text{reject } H_0 \text{ if } Y_n \geq 1 \text{ or } Y_1 \geq k,$$

where k is a constant, $Y_1 = \min\{X_1, \ldots, X_n\}$, and $Y_n = \max\{X_1, \ldots, X_n\}$.

(i) Determine k so that the test will have size α.

(ii) Find an expression for the power function of the above test (for all θ).

3. (15 points) (3 + 3 + 4 + 5) We obtain observations $Y_1, \ldots, Y_n$ which can be described by the relationship

$$Y_i = \theta x_i^2 + \epsilon_i,$$

where $x_1, \ldots, x_n$ are fixed constants and $\epsilon_1, \ldots, \epsilon_n$ are i.i.d $N(0, \sigma^2)$.

(i) Find the least squares estimator $\hat{\theta}$ of θ.

(ii) Is $\hat{\theta}$ unbiased?

(iii) Find the distribution of $\hat{\theta}$.

(iv) How would you test the hypothesis $H_0 : \theta = 0$ versus $H_1 : \theta \neq 0$? Describe the test and the critical value.

4. (10 points) (3 + 4 + 3) Suppose that $X_1, \ldots, X_n$ are i.i.d Exp$(1/\mu)$, where $\mathbb{E}(X_1) = \mu > 0$.

   (i) Find the mean and variance of $\bar{X}_n = \sum_{i=1}^n X_i/n$. Hence, find the asymptotic distribution of $\bar{X}_n$ (properly standardized).

   (ii) Let $T = \log \bar{X}_n$. Find the corresponding asymptotic distribution of T (properly standardized).

   (iii) How can the asymptotic distribution of T be used to construct an approximate $(1 - \alpha)$ confidence interval (CI) for μ? Explain your answer and give the desired CI.

5. (20 points) (3 + 4 + 3 + 5) Suppose that $X_1, X_2, \ldots, X_n$ are i.i.d $N(\theta, 1)$, where $\theta \in \mathbb{R}$ is unknown. Let $\psi = \mathbb{P}_\theta(X_1 > 0)$.

   (a) Find the maximum likelihood estimator $\hat{\psi}$ of ψ.

   (b) Find an approximate 95% confidence interval for ψ.

(c) Let $Y_i = \mathbf{1}\{X_i > 0\}$, for $i = 1, \ldots, n$. Define $\tilde{\psi} = (1/n)\sum_{i=1}^{n} Y_i$. Show that $\tilde{\psi}$ is a consistent estimator of ψ.

(d) Which estimator of ψ, $\hat{\psi}$ or $\tilde{\psi}$, is more preferable in this model.

6. (20 points) (3 + 4 + 8) Suppose that

$$Y_{ij} \sim N(\theta_i, \sigma^2), \quad i = 1, \ldots, k; \quad j = 1, \ldots, n_i,$$

where θ_i, for $i = 1, \ldots, k$, and σ^2 are unknown.

(a) Find the distribution of $\bar{Y}_{i\cdot} = \frac{1}{n_i} \sum_{i=1}^{n_i} Y_{ij}$.

(b) Suppose that the goal is to estimate $\tau = \sum_{i=1}^{k} a_i \theta_i$, where a_i, for $i = 1, \ldots, k$, are known constants. Note that $\hat{\tau} = \sum_{i=1}^{k} a_i \bar{Y}_{i\cdot}$ is a natural estimator of τ. Find the distribution of $\hat{\tau}$.

(c) Let us consider testing for $H_0 : \tau = 0$ versus $H_1 : \tau \neq 0$. Describe a test procedure — the test statistic, its exact distribution (under H_0) and the critical value.

7. (10 points) (5 + 5) Suppose that we observe $X_1, \ldots, X_n$, where

$$\begin{aligned} X_i|\theta_i &\sim N(\theta_i, \sigma^2), \quad i = 1, \ldots, n, \quad \text{independent,} \\ \theta_i &\sim N(\mu, \tau^2), \quad i = 1, \ldots, n, \quad \text{independent.} \end{aligned}$$

   (a) Find that the marginal distribution of X_i.

   (b) Also, show that, marginally, $X_1, \ldots, X_n$ are i.i.d.

8. (10 points) (6 + 4) Let $W_1, W_2, \ldots, W_k$ be unbiased estimators of a parameter θ with $\text{Var}(W_i) = \sigma_i^2$ and $\text{Cov}(W_i, W_j) = 0$ if $i \neq j$.

   (a) Show that among all estimators of the form $\sum_{i=1}^k a_i W_i$, where a_i's are constants and $\mathbb{E}_\theta(\sum_i a_i W_i) = \theta$, the estimator $W^* = \frac{\sum_i W_i/\sigma_i^2}{\sum_i 1/\sigma_i^2}$ has minimum variance.

   (b) Show that $\text{Var}(W^*) = \frac{1}{\sum_i 1/\sigma_i^2}$.