Stat GR5204
Fall 2017
Time Limit: 1 hours 20 minutes

Name (Print): ____________________
Student UNI: ____________________
Signature: ____________________

This exam contains 5 problems. Answer all of them. Point values are in parentheses. You **must show your work** to get credit for your solutions - correct answers without work will not be awarded points.

No calculator will be allowed in the exam. You are allowed to bring one sheet of A4 paper with notes written on both sides by yourself. Normal tables, if needed, will be provided during the exam.

1	25 pts	
2	15 pts	
3	15 pts	
4	15 pts	
5	30 pts	
TOTAL	100 pts	

1. (25 points) (7 + 8 + 10) Suppose that $X_1, \ldots, X_n$ are i.i.d Exponential(θ), (i.e., $f(x,\theta) = \theta e^{-\theta x}\mathbf{1}_{(0,\infty)}(x)$).

(a) Find the MLE $\hat{\theta}_n$ of θ.

$$f(x_1, \cdots, x_n, \theta) = \theta^n e^{-\theta \sum_{i=1}^{n} x_i}$$

$$\text{Log } f(x_1, \cdots, x_n, \theta) = n \log\theta - \theta \sum_{i=1}^{n} x_i$$

$$\text{Let } 0 = \frac{\partial \text{Log} f}{\partial \theta} = \frac{n}{\theta} - \sum_{i=1}^{n} x_i \Rightarrow \hat{\theta}_n = \frac{n}{\sum_{i=1}^{n} x_i}$$

(b) Find the exact sampling distribution of $\hat{\theta}_n$.

$$\sum_{i=1}^{n} X_i \sim \Gamma(n, \theta),$$

$$f_{\sum_{i=1}^{n} X_i}(x) = \frac{\theta^n x^{n-1}}{\Gamma(n)} e^{-\theta x} 1_{(0,\infty)}(x)$$

$$F(\hat{\theta}_n, x) = P(\hat{\theta}_n \leq x)$$

$$= P\left(\sum_{i=1}^{n} X_i \geq \frac{n}{x}\right) = \int_{\frac{n}{x}}^{\infty} \frac{\theta^n t^{n-1}}{\Gamma(n)} e^{-\theta t} \, dt$$

$$f_{\hat{\theta}_n}(x) = -\frac{\theta^n \left(\frac{n}{x}\right)^{n-1}}{\Gamma(n)} e^{-\frac{\theta n}{x}} \cdot \left(\frac{n}{x}\right)'$$

$$= \frac{\theta^n n^n}{x^{n+1} \Gamma(n)} e^{-\frac{\theta n}{x}} \quad 1(x>0)$$

(c) Find the limiting distribution of $\log \bar{X}_n$, properly standardized.

$$EX_1 = \frac{1}{\theta} \qquad \bar{X}_n \xrightarrow{a.s} \frac{1}{\theta}$$

$$\sqrt{n}\left(\bar{X}_n - \frac{1}{\theta}\right) \xrightarrow{d} N(0, \theta^{-2})$$

By delta method, $\dfrac{\sqrt{n}\left(\ln \bar{X}_n - \ln \frac{1}{\theta}\right)}{\dfrac{d(\ln \frac{1}{\theta})}{d(\frac{1}{\theta})}}$

$\cancel{\sqrt{n}\left(\bar{X}_n - \frac{1}{\theta}\right)}$

$$\xrightarrow{d} N(0, \theta^{-2})$$

$$\Rightarrow \sqrt{n}\left(\ln \bar{X}_n + \ln \theta\right) \xrightarrow{d} N(0, 1)$$

2. (15 points) (8+7) Let $X_1, \ldots, X_n$ be i.i.d. Exponential(λ). Find the distribution of $2\lambda n\bar{X}$ (where $\bar{X} = \sum_{i=1}^n X_i/n$), and use this result to find a $(1-\alpha)$ confidence interval for λ.

Solution: Denote $Y_i = 2\lambda X_i$, then Y_i has pdf

$$f_{Y_i}(y_i) = \frac{1}{2}e^{-\frac{1}{2}y_i}, \quad y_i > 0.$$

From this we can find that $Y_1, \ldots, Y_n$ are iid $\chi^2(2)$ and $2\lambda n\overline{X} = \sum_{i=1}^n Y_i$ follows $\chi^2(2n)$ distribution.

$$P(\chi^2_{\alpha/2,2n} \le 2\lambda n\overline{X} \le \chi^2_{1-\alpha/2,2n}) = 1 - \alpha$$

$$P\left(\frac{\chi^2_{\alpha/2,2n}}{2n\overline{X}} \le \lambda \le \frac{\chi^2_{1-\alpha/2,2n}}{2n\overline{X}}\right) = 1 - \alpha$$

Thus the $(1-\alpha)$ confidence interval for λ is

$$\left(\frac{\chi^2_{\alpha/2,2n}}{2n\overline{X}}, \frac{\chi^2_{1-\alpha/2,2n}}{2n\overline{X}}\right)$$

3. (15 points) (9 + 6) A company has manufactured certain objects and has printed a serial number on each manufactured object. The serial numbers start at 1 and end at N, where N is the number of objects manufactured. The problem is to estimate N (the parameter). A simple random sample of size n is drawn with replacement from the lot; let $X_1, X_2, \ldots, X_n$ denote the numbers on the object that come up. Thus, $X_1, \ldots, X_n$ are i.i.d random variables with common mass function $p(x, N) := \mathbb{P}(X_1 = x)$.

   (a) Find $p(x, N)$ and compute $\mu := \mathbb{E}(X_1)$.

$$p(x, N) = \frac{1}{N}$$

$$\mu := E(X_1) = \frac{1+N}{2}$$

   (b) Obtain a method of moments estimator of N.

$$E\bar{X} = \frac{1+N}{2}$$

So the moment estimator is $2\bar{X} - 1$

4. (15 points) Suppose that $X_1, \ldots, X_n$ form a random sample from a normal distribution with unknown mean θ and variance σ^2. Assuming that $\theta \neq 0$, determine the asymptotic distribution of $\bar{X}_n^3$, where $\bar{X}_n := \sum_{i=1}^n X_i/n$.

By CLT: $$\frac{\sum_{i=1}^{n} X_i - n\theta}{\sigma\sqrt{n}} \xrightarrow{d} N(0,1)$$

$$\Rightarrow \sqrt{n}(\bar{X}_n - \theta) \xrightarrow{d} N(0, \sigma^2)$$

By Delta Method

$$\frac{\sqrt{n}(\bar{X}_n^3 - \theta^3)}{3\theta^2} \xrightarrow{d} N(0, \sigma^2)$$

$$\Rightarrow \sqrt{n}(\bar{X}_n^3 - \theta^3) \xrightarrow{d} N(0, 9\theta^4\sigma^2)$$

5. (30 points) (10 + 5 + 10 + 5) Consider the hierarchical model

$$Y|\Lambda \sim \text{ Poisson}(\Lambda) \text{ and } \Lambda \sim \text{ Gamma}(\alpha, \beta), \text{ where } \alpha, \beta \text{ are known},.$$

Note that the density of Λ is given by

$$f(y|\alpha,\beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} e^{-y\beta} y^{\alpha-1}, \quad \text{for } y \geq 0, \quad \text{(and 0 otherwise)}.$$

(a) Find the marginal distribution of Y.

$$\cancel{f_Y(y)} \quad P(Y=y) = E\,P(Y=y|\Lambda)$$

$$= E\,\frac{\Lambda^y}{y!} e^{-\Lambda} = \frac{1}{y!} E\,\Lambda^y e^{-\Lambda}$$

$$= \frac{1}{y!} \frac{\beta^\alpha}{\Gamma(\alpha)} \int_{\mathbb{R}} \Lambda^y e^{-\Lambda} e^{-\Lambda\beta} \Lambda^{\alpha-1} d\Lambda$$

$$= \frac{\beta^\alpha}{y!\,\Gamma(\alpha)} \int_{\mathbb{R}} e^{-(1+\beta)\Lambda} \Lambda^{\alpha+y-1} d\Lambda$$

$$= \frac{\beta^\alpha}{y!\,\Gamma(\alpha)} \frac{\Gamma(\alpha+y)}{(1+\beta)^{\alpha+y}}$$

(b) Find the mean and variance of Y.

$$E(Y|\Lambda) = \Lambda$$

$$EY = E\Lambda = \frac{\alpha}{\beta}$$

$$E(Y^2|\Lambda) = \Lambda + \Lambda^2 \Rightarrow EY^2 = E\Lambda + E\Lambda^2 = \frac{\alpha}{\beta} + E\Lambda^2$$

$$\Rightarrow Var(Y) = EY^2 - (EY)^2 = \frac{\alpha}{\beta} + Var(\Lambda) = \frac{\alpha}{\beta} + \frac{\alpha}{\beta^2}$$

(c) Find the posterior distribution of Λ given the observation Y, and identity the distribution with its parameters.

Solution: The posterior distribution of λ given $X = x$ is

$$\begin{aligned}\pi(\lambda|x) \propto \pi(\lambda) \times f_\lambda(x) &= \frac{\beta^\alpha}{\Gamma(\alpha)} e^{-\lambda\beta} \lambda^{\alpha-1} \times \frac{\lambda^x}{x!} e^{-\lambda} \\ &\propto e^{-\lambda(1+\beta)} \lambda^{x+\alpha-1}.\end{aligned}$$

Thus, the posterior distribution of λ given $X = x$ is Gamma$(x + \alpha, \beta + 1)$.

(d) Find the mean of the posterior distribution.

Solution: Thus the mean of the posterior distribution is $(X + \alpha)/(\beta + 1)$.