Lecture 2: Local Independence & Unidimensionality Assumptions / Normal Ogive & Logistic Models

Assumptions – Local Independence & Unidimensionality

Local independence (LI) implies that the probability of observing an n-item response pattern $U = (U_1, U_2, U_3, \ldots U_n)$ given ability θ, written $P(U|\theta)$, can be expressed as

$$P(U|\theta) = P(U_1|\theta)P(U_2|\theta)\ldots P(U_n|\theta).$$

Example: If the response pattern for an examinee on three items is (1, 1, 0), then the assumption of LI implies that

Unidimensionality is a requirement in most item response theory (IRT) models. A test is unidimensional provides it measures only one trait or ability. If a test is unidimensional, then a single trait exists such that all item pairs are locally independent. We can think of the dimensionality of a test as the number of dimensions that must be specified in order to achieve LI.

Example 1: Lazarsfeld & Henry (1968)

Suppose that a group of 1,000 persons is asked whether they have read the last issue of magazines A and B. Their responses are:

	Read A	Did not read A	Total
Read B	260	240	500
Did not read B	140	360	500
Total	400	600	1000

Now, suppose that we have information on the respondents' educational levels, dichotomized as high and low. When the 1,000 people are divided into these two groups, readership is observed to be:

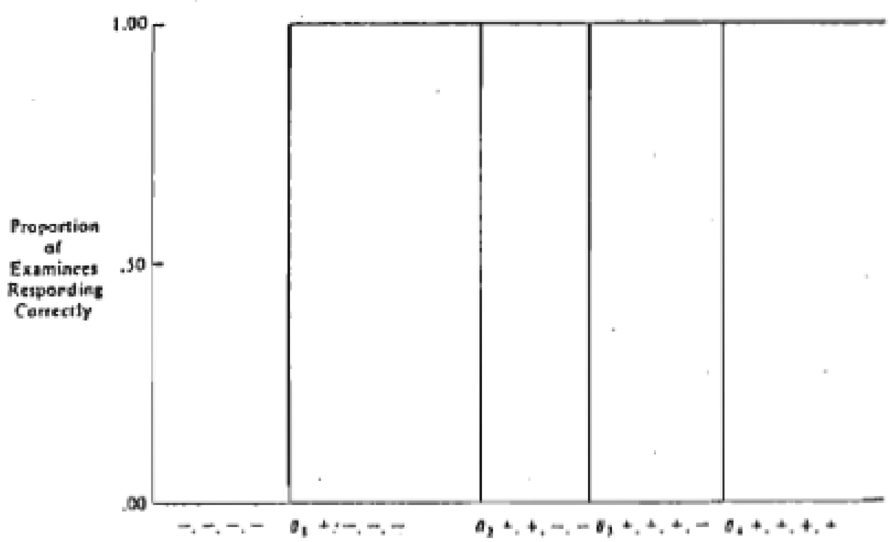
- *High Education*

	Read A	Did not read A	Total
Read B	240	60	300
Did not read B	160	40	200
Total	400	100	500

- *Low Education*

	Read A	Did not read A	Total
Read B	20	80	100
Did not read B	80	320	400
Total	100	400	500

<u>Example 2</u>: Suppose we have a four-item test and for a sample of 200 examinees we only observe the following response patterns: 0000, 1000, 1100, 1110, and 1111. We can then represent both items and examinees with respect to a Guttman scale:



- Joint distribution of responses for the entire population:

		Item 4 +	Item 4 −	
Item 3	+	.20	.20	.4
	−	.00	.60	.6
		.2	.8	

- Joint distribution of response for the subpopulation having $\theta = \theta_3$:

		Item 4 +	Item 4 −	
Item 3	+	.00	.00	.00
	−	.00	1.00	1.00
		.00	1.00	

Since this same type of 2×2 matrix will be observed for all item pairs, we obviously obtain local independence through specification of one dimension, so the test is unidimensional.

In application of IRT models, the potential for local dependence can also be checked. One index of local dependence is Yen's Q_3 statistic. Essentially Q_3 is a correlation between the residuals of item scores based on the item response model: $d_{ik} = u_{ik} - \hat{P}_i(\hat{\theta}_k)$. Then for a pair of items i and j, $Q_{3ij} = r_{d_i, d_j}$, where r represents the correlation between variables. Yen (1984) suggests that under an assumption of local independence, the distribution of Q_3 should be approximately normally distributed. Thus, to check local independence, one could construct a normal QQ-plot or apply some other test to check for normality.

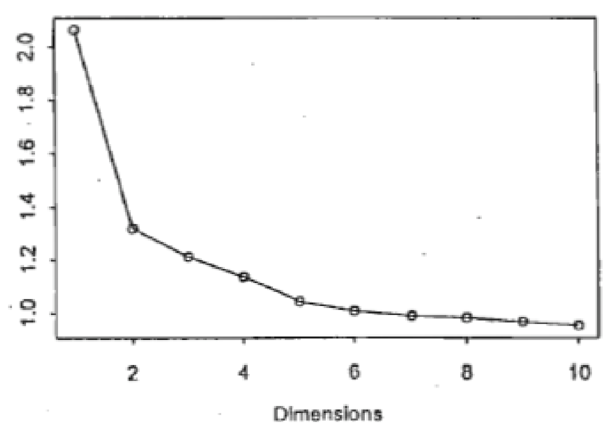
Yen. W. M. (1984). Effect of Local Item Dependence on the Fit and Equating Performance of the Three-Parameter Logistic Model. *Applied Psychological Measurement*, *8*, 125 – 145.

<u>Some tools for checking unidimensionality</u>

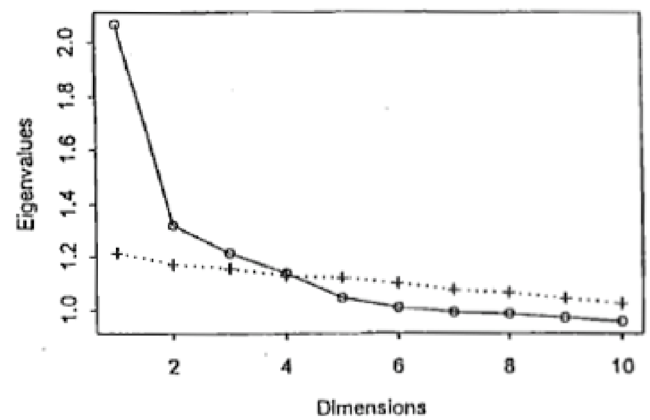
- Eigenvalue plots
- Parallel analysis; modified parallel analysis
- Full information item factor analysis (TESTFACT), limited information nonlinear factor analysis (NOHARM)
- Some nonparametric procedures: Stout's essential dimensionality procedure (DIMTEST)

<u>Example</u>: Law School Admissions Test Data – the Analytical Reasoning section (500 examinees, 24 items scored dichotomously, 1 = correct, 0 = incorrect).

- Eigenvalue plots: Examine the eigenvalues from a linear factor analysis, and determine how large the first eigenvalue is compared to the second.



- Parallel analysis: Compare the eigenvalue plots above with one generated from a random normal dataset with the same sample size and the same number of variables, but in this case with the variables assumed independent.



- TESTFACT: Compare the fit of a one-factor model to a two-factor model – the difference in G^2 can be regarded as a χ^2 variable with degrees of freedom equal to the difference across models in the number of parameters.
  - Fit of a one-factor model: $\chi^2 = 8038.45$, $df = 441$
  - Fit of a two-factor model: $\chi^2 = 7836.18$, $df = 418$
  - Difference: $\chi^2 = 102.27$, $df = 23$

- Stout's essential dimensionality procedure (DIMTEST): The basic idea is partition examinees into groups that obtained the same test score, and determine whether there exists a subset of items whose inter-item covariance are greater than would be expected if the test were unidimensional.
  - Step 1: Identify a subset of items in the dataset that appear to measure a second dimension. Call this subtest Assessment subtest 1 (AT1).
  - Step 2: Among the remaining test items, select a subtest containing an equal number of items as AT1 and whose items are of approximately the same difficulty as the items in AT1. Call this subtest Assessment subtest 2 (AT2). All of the remaining items comprise a Partitioning subtest (PT). Each examinee can be assigned a PT score based on the sum of item scores on the partitioning test.
  - Step 3: For examinees having the same PT score, compute the sum of inter-item covariances on AT1, and then sum these inter-item covariances across all possible PT scores. Repeat the process for AT2. The difference between the sums computed for the AT1 and AT2 subtests can be used to derive a t-statistic. A positive value suggests the inter-item covariances in the AT1 subtest are greater than would be expected by chance, and that a null hypothesis of unidimensionality can be rejected.

Monotonicity implies that as θ increases, the probability of correct response to the item should also increase, or $P(\theta_j) \geq P(\theta_k)$ whenever $\theta_j \geq \theta_k$.

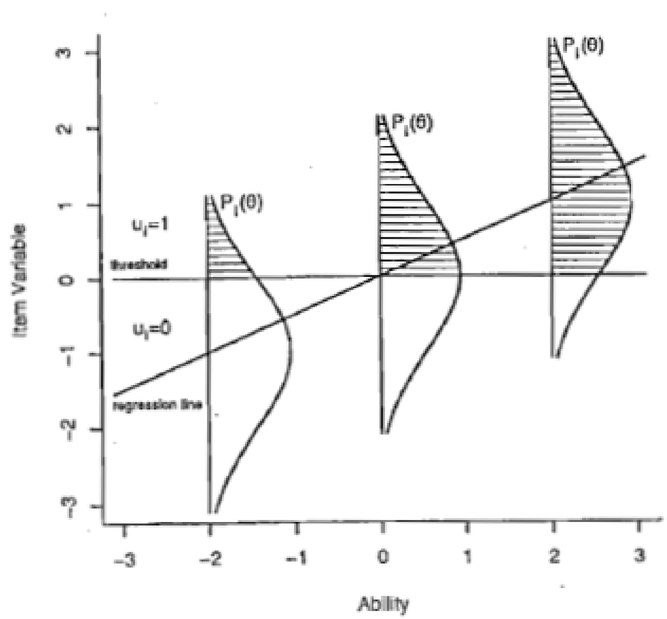
IRT Models – Normal Ogive vs. Logistic Models

IRT models assume a particular functional relationship between the latent trait or ability measured by the test, denoted θ, and the probability of correct response to the item, $P_i(\theta)$. The latent ability scale for θ ranges from $-\infty$ to $+\infty$.

- Normal Ogive models

The normal ogive shape is typically justified as follows:

- For any item, an item variable Γ_i, representing a propensity to answer the item correctly, is a linear function of θ.
- At each θ_j, there exists a conditional distribution of Γ_i that is normal with mean $\mu_j|\theta_j$ and variance $\sigma_j^2|\theta_j$.
- The variance $\sigma_j^2|\theta_j$ is assumed constant over all θ.
- Each item is also associated with a critical value γ_i, such that if $\Gamma_i > \gamma_i$, the item is answered correctly, and if not the item is answered incorrectly. The probability of correct response at a given θ is determined by the area under the normal curve above γ_i at that θ.



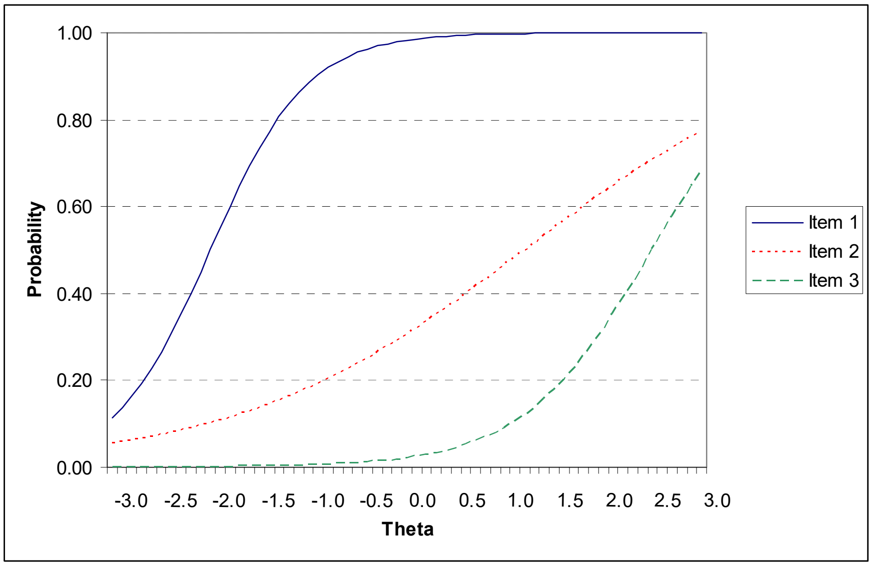
In a normal ogive model in IRT, parameters analogous to the slope of the regression line and threshold γ_i control the slope and location of the item characteristic curve (ICC). The cumulative normal evaluated at a point θ_j is then computed as:

$$\Phi(Z) = \int_{-Z_{ij}=-(\theta_j-\mu_i)/\sigma_i}^{\infty} \frac{1}{\sqrt{2\pi}} e^{\frac{-z^2}{2}} dz.$$

The normal ogive model uses $\beta_i = \mu_i$ and $\alpha_i = \frac{1}{\sigma_i}$ as parameters defining the form of the normal ogive ICC. Then the probability of correct response to an item i for a person having ability level θ_j is written as: $P_i(\theta_j) = \int_{-Z_{ij}=-\alpha_i(\theta_j-\beta_i)}^{\infty} \frac{1}{\sqrt{2\pi}} e^{\frac{-z^2}{2}} dz.$

Example:

Item	1	2	3
α_i	1.2	0.4	0.9
β_i	-2.0	1.2	2.5



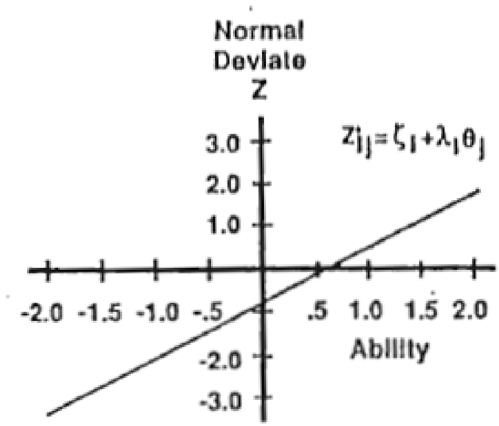
Notice that the β_i parameter, which controls the location of the ICC, also corresponds to the θ value at which exactly 50% of the examinees answer the item correctly. The α_i parameter controls the slope of the ICC. In fact, when θ is close to being equal to β_i, the ICC is nearly linear, with slope equal to $\alpha_i / \sqrt{2\pi}$ (Lord & Novick, 1968, p. 368).

With respect to the normal distribution, $\alpha_i = \frac{1}{\sigma_i}$ and $\beta_i = \mu_i$, so that

$$Z_{ij} = \frac{\theta_j - \mu_i}{\sigma_i} = \alpha_i(\theta_j - \beta_i).$$

Example: Item characteristic curve based on Normal Ogive Model, $\alpha_i = 1.2$ and $\beta_i = 0.6$ (Baker & Kim, 2004, p.10)

θ_j	$\theta_j - \beta_i$	Z_{ij}	$P_i(\theta_j)$
-2.0	-2.6	-3.12	.0003
-1.5	-2.1	-2.52	.0059
-1.0	-1.6	-1.92	.0274
-.5	-1.1	-1.32	.0934
0	-.6	-.72	.2358
.5	-.1	-.12	.4522
1.0	+.4	+.48	.6844
1.5	+.9	+1.08	.8599
2.0	+1.4	+1.68	.9535



This can be advantageous in thinking about estimation as we all have familiarity with how linear regression functions can be estimated.

- Logistic Models

A logistic model has the following general form:

$$P_i(\theta_j) = \frac{\exp[Z_{ij}]}{1+\exp[Z_{ij}]},$$

where $Z_{ij} = \alpha_i^*(\theta_j - \beta_i)$. In logistic models, Z_{ij} is referred to as the logit, β_i is a location (difficulty) parameter, and α_i^* is the discrimination or scale parameter equal to the reciprocal of the standard deviation of the logistic function.

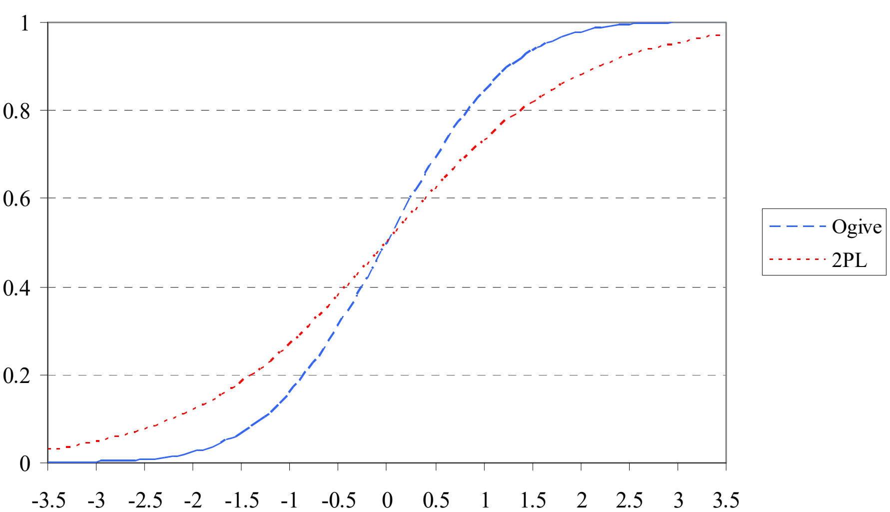
<u>Example</u>: Item characteristic curve based on Logistic Model, $\alpha_i^* = 1.2$ and $\beta_i = 0.6$ (p.15)

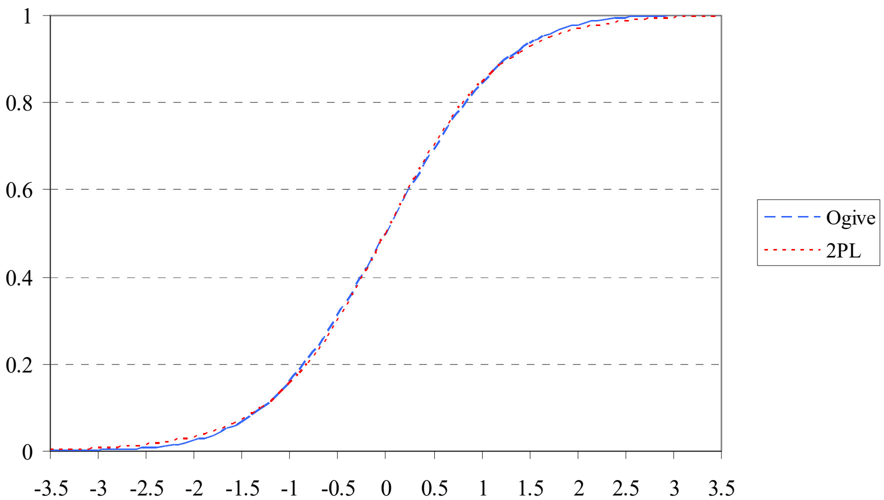
θ_j	$\theta_j - \beta_i$	Z_{ij}	$P_i(\theta_j)$
-2.0	-2.6	-3.12	.0423
-1.5	-2.1	-2.52	.0745
-1.0	-1.6	-1.92	.1279
-.5	-1.1	-1.32	.2108
0	-.6	-.72	.3274
.5	-.1	-.12	.4700
1.0	+.4	+.48	.6178
1.5	+.9	+1.08	.7465
2.0	+1.4	+1.68	.8429

The normal ogive and logistic parameterizations are very similar when the logit in the logistic model is multiplied by $D = 1.702$. In fact, the following relationship can be established:

$$\left|\Phi(Z_{ij}) - \Psi(1.702 Z_{ij})\right| < .01, \text{ for } -\infty \le \theta_j \le \infty,$$

where Ψ denotes the logistic function and Φ the normal ogive function. Equivalently, the discrimination parameter in the logistic model can be multiplied by 1.702 to result in a close match to the normal ogive model ($\alpha = 1.702\alpha^*$).





An attractive feature of the logistic model is that it is in closed form, and thus does not require evaluating an integral. The logistic function is also associated with the log-odds of getting the item correct:

$$\log\left[\frac{P_i(\theta_j)}{Q_i(\theta_j)}\right] = \alpha_i(\theta_j - \beta_i)$$

As with the normal deviate in the normal ogive model, we refer to $\alpha_i(\theta_j - \beta_i)$ as the logistic deviate when considering the logistic model. Unlike the normal deviate, the logistic deviate has a non-linear association with θ.

The model above is more specifically called the two parameter logistic (2PL) model, and is attributed to Birnbaum (1968):

$$P_i(\theta_j) = \frac{\exp[\alpha_i(\theta_j - \beta_i)]}{1 + \exp[\alpha_i(\theta_j - \beta_i)]}.$$

Rasch or one parameter logistic (1PL) model

$$P_i(\theta_j) = \frac{\exp[(\theta_j - \beta_i)]}{1 + \exp[(\theta_j - \beta_i)]}$$

Rasch model has nice statistical properties which make it easier to estimate than the 2PL. The main problem in IRT estimation is to estimate the item (structural) parameters in the presence of unknown examinee ability (incidental) parameters. The Rasch model is attractive because it has a simple sufficient statistic for examinee ability – the *total test score*. As a result, test scores can substitute for the latent abilities when estimating the item parameters. Another attractive feature of the Rasch model is that it possesses a property called ***specific objectivity*** (i.e., item-free & examinee-free measure).

Specific objectivity implies that "1) comparisons between persons are invariant over the specific items used to measure them, and 2) comparisons between items are invariant over the specific persons used to calibrate them."

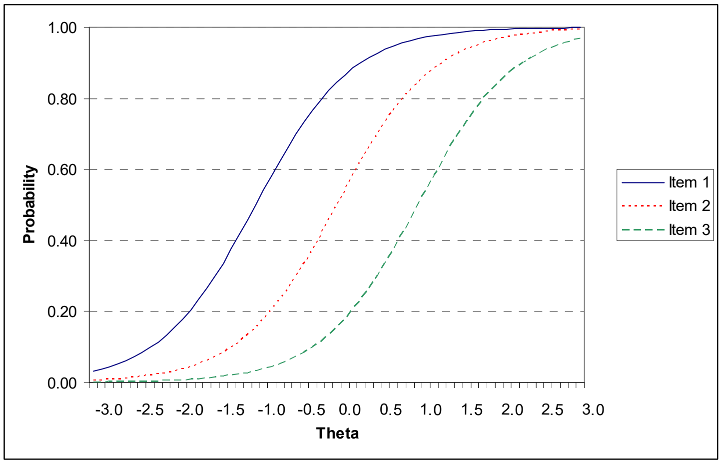
$$1)\ \ln\left[\frac{P_i(\theta_1)}{Q_i(\theta_1)}\right] - \ln\left[\frac{P_i(\theta_2)}{Q_i(\theta_2)}\right] = \theta_1 - \theta_2.$$

$$2)\ \ln\left[\frac{P_1(\theta_j)}{Q_1(\theta_j)}\right] - \ln\left[\frac{P_2(\theta_j)}{Q_2(\theta_j)}\right] = b_2 - b_1.$$

The primary disadvantage of the Rasch model relative to the 2PL is that it is not able to account for items that may vary with respect to their discriminating power.

Example:

Item	1	2	3
β_i	-1.0	0.0	1.0



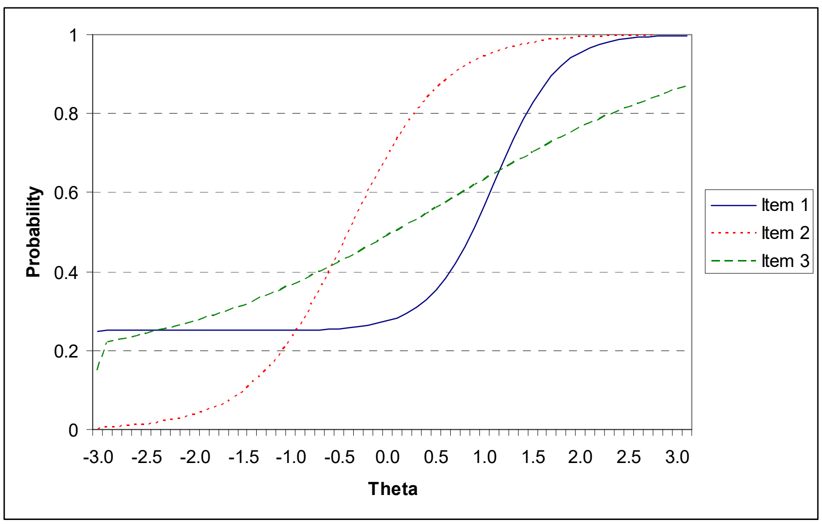
Three parameter logistic model (3PL) – Birnbaum's Three parameter model

$$P_i(\theta_j) = c_i + (1 - c_i)\frac{\exp[\alpha_i(\theta_j - \beta_i)]}{1 + \exp[\alpha_i(\theta_j - \beta_i)]}$$

The parameter c_i is sometimes called a "guessing" parameter, but more generally defines the lower asymptote of the ICC for the 3PL. Although attractive from a practical standpoint for multiple choice items, the guessing parameter is difficult to estimate with any accuracy. Large standard errors will be observed whenever the difficulty and discrimination parameters assume extreme values.

Example:

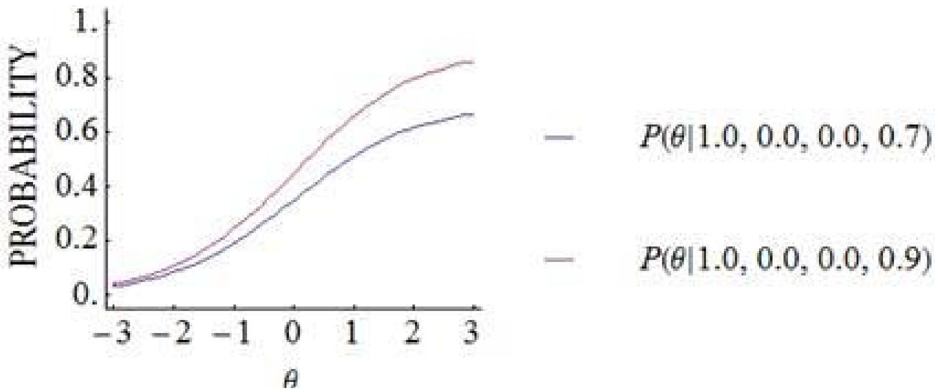
	Item Parameter		
Test Item	β_i	α_i	c_i
1	1.00	1.80	0.25
2	-0.50	1.20	0.00
3	0.50	0.40	0.15

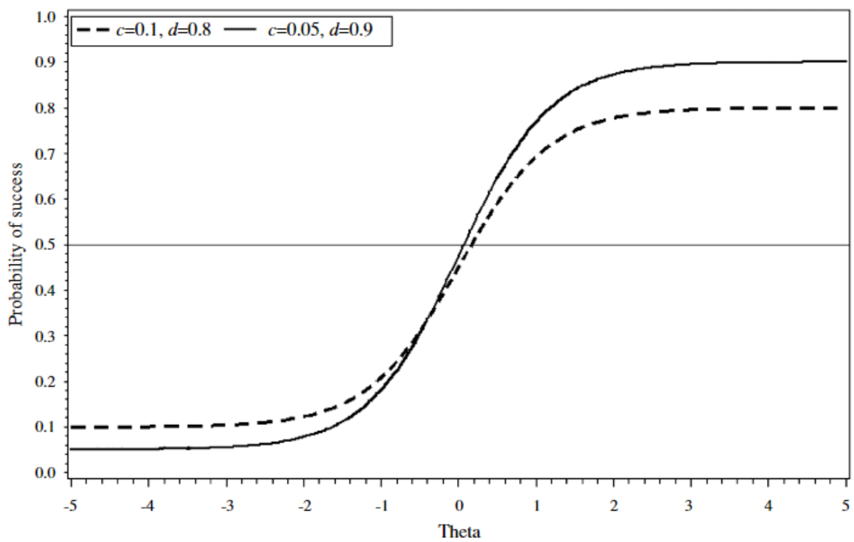


Four parameter logistic (4PL) model

Barton and Lord (1981) introduced an upper asymptote parameter, expressed by the lowercase *d*, into the 3PL model, resulting in the 4PL model:

$$P_i(\theta_j) = c_i + \frac{(d - c_i)}{1 + \exp[-\alpha_i(\theta_j - \beta_i)]} \qquad (0 \le c \le d \le 1)$$



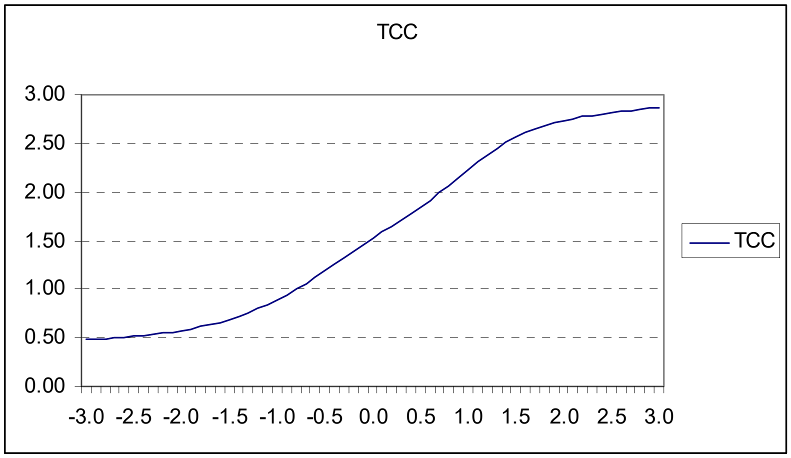


Test characteristic curves (TCCs)

Even though IRT models performances at the item level, we can look at the relationship between test scores and θ:

$$T(\theta_j) = \sum_{i}^{N} P_i(\theta_j).$$

The form of TCC is dependent on the parameters of the items that comprise the test. The curve represents the expected test scores conditional on θ_j. The shape of the TCC will generally not be normal ogive or logistic in form like the ICCs.



Exercise: Plot ICCs and TCC of the following items.

Test item	Item parameter		
	β_i	α_i	c_i
1	1.00	1.80	0.00
2	1.00	0.80	0.00
3	1.00	1.80	0.25
4	-1.50	1.80	0.00
5	-0.50	1.20	0.10
6	0.50	0.40	0.15