Algorithms for Data Science

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Network flows

Outline

1. Flow networks
   - Applications
2. The residual graph and augmenting paths
3. The Ford-Fulkerson algorithm for max flow
4. Correctness of the Ford-Fulkerson algorithm
5. Application: max bipartite matching

Today

1. Flow networks
   - Applications
2. The residual graph and augmenting paths
3. The Ford-Fulkerson algorithm for max flow
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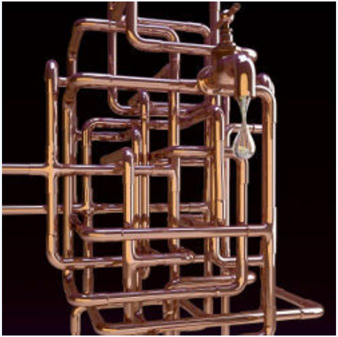
Modeling transportation networks



Source: Communications of the ACM, Vol. 57, No. 8

Can model a fluid network or a highway system by a **graph**:
edges carry *traffic*, nodes are *switches* where traffic gets diverted.

Flow networks

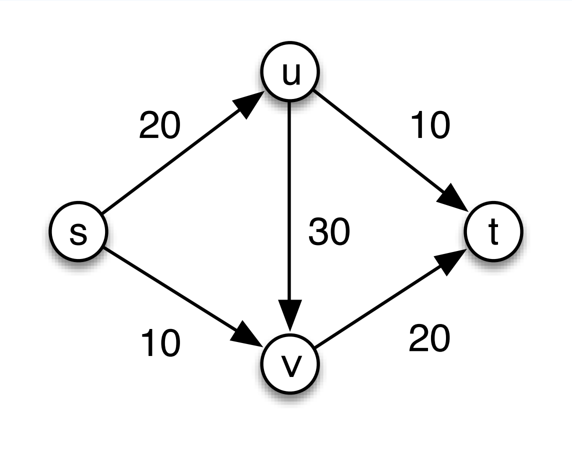
A flow network $G = (V, E)$ is a directed graph such that

1. Every edge has a capacity $c_e \geq 0$. *A1: integer capacities*
2. There is a single source $s \in V$. *A2: no edge enters s*
3. There is a single sink $t \in V$. *A3: no edge leaves t*

Two more assumptions for the purposes of the analysis

- *A4: If $(u, v) \in E$ then $(v, u) \notin E$.*
- *A5: Every $v \in V - \{s, t\}$ is on some s-t path.*
  Hence G has $m \geq n - 1$ edges.

An example flow network



Flows

Given a flow network G, an s-t flow f in G is a function

$$f : E \rightarrow R^{+}.$$

Intuitively, the flow $f(e)$ on edge e is the amount of *traffic* that edge e carries.

Two kinds of constraints that every flow must satisfy

1. Capacity constraints: for all $e \in E$, $0 \leq f(e) \leq c_e$.

2. Flow conservation: for all $v \in V - \{s, t\}$,

$$\sum_{(u,v)\in E} f(u,v) = \sum_{(v,w)\in E} f(v,w) \tag{1}$$

In words, the flow into node v equals the flow out of v, or

$$\sum_{e \text{ into } v} f(e) = \sum_{e \text{ out of } v} f(e)$$

A cleaner equation for flow conservation constraints

Define

1. $f^{\text{out}}(v) = \sum_{e \text{ out of } v} f(e)$
2. $f^{\text{in}}(v) = \sum_{e \text{ into } v} f(e)$

So we can rewrite equation (1) as: for all $v \in V - \{s, t\}$

$$f^{\text{in}}(v) = f^{\text{out}}(v) \tag{2}$$

The value of a flow

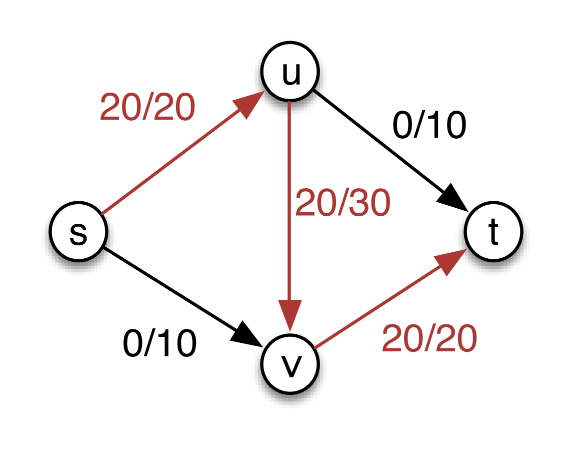
Definition 1.

The value of a flow f, denoted by $|f|$, is

$$|f| = \sum_{e \text{ out of } s} f(e) = f^{\text{out}}(s)$$

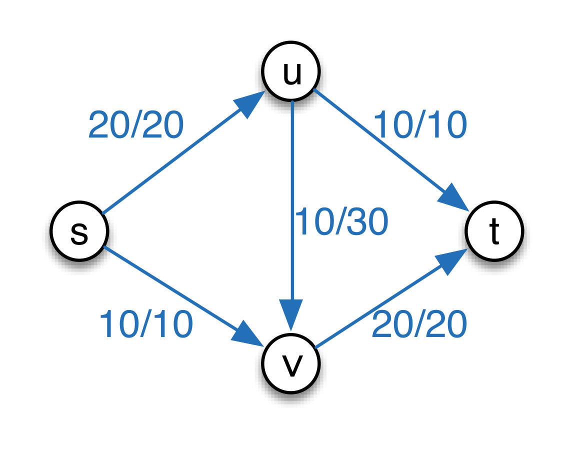
Exercise: show that $|f| = f^{\text{in}}(t)$.

An example flow of value 20



A flow f of value 20.

A flow of value 30



A (max) flow of value 30.

Max flow problem

Input: (G, s, t, c) such that

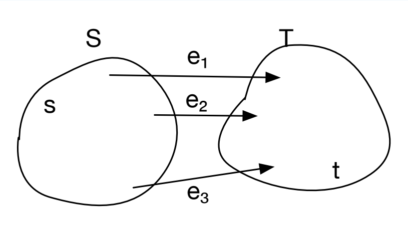
- $G = (V, E)$ is a flow network;
- $s, t \in V$ are the source and sink respectively;
- c is the (integer-valued) capacity function.

Output: a flow of maximum possible value

s-t cuts in flow networks

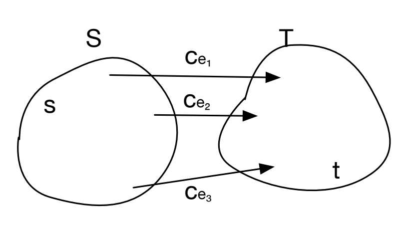
Definition 2.

An s-t cut (S, T) in G is a bipartition of the vertices into two sets S and T, such that $s \in S$ and $t \in T$.



A natural upper bound for the max value of a flow

- Flow f must cross (S, T) to go from source s to sink t.
- So it uses some (at most all) of the capacity of the edges crossing from S to T.



- So, intuitively, the value of the flow cannot exceed

$$\sum_{\text{e out of } S} c_e$$

Max flow and min cut

Definition 3.

The capacity $c(S,T)$ of an s-t cut (S,T) is defined as

$$c(S,T) = \sum_{\text{e out of } S} c_e.$$

$\triangle$ Note asymmetry in the definition of $c(S,T)$!

So, *intuitively,* the value of the max flow is upper bounded by the capacity of *every* cut in the flow network, that is,

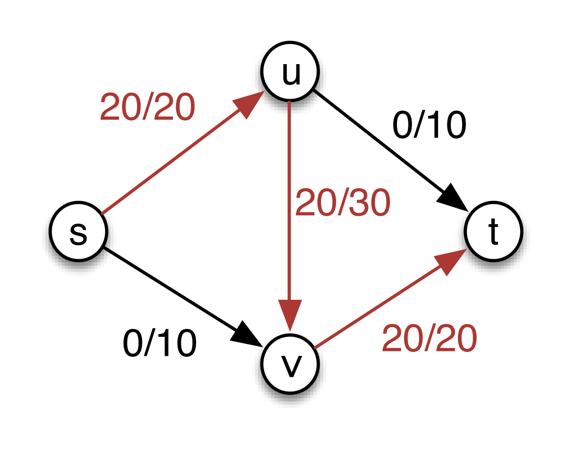
$$\max_f |f| \leq \min_{(S,T) \text{ cut in } G} c(S,T) \tag{3}$$

Applications of max-flow and min-cut

- Find a set of edges of smallest capacity whose deletion disconnects the network (min cut)
- Bipartite matching (max flow) —*coming up*
- Airline scheduling (max flow)
- Baseball elimination (max flow)
- Distribution of goods to cities (max flow)
- Image segmentation (min cut)
- Survey design (max flow)
- ...

Today

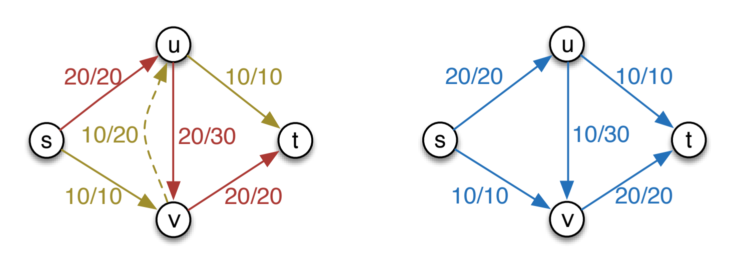
“Undoing” flow



A flow f of value 20.

Goal: *undo* 10 units of flow along (u, v), divert it along (u, t).

Pushing flow back



- "Push back" 10 units of flow along (v, u).
- Send 10 more units from s to t along the green path edges $(s, v), (v, u), (u, t)$.
- New flow f' (on the right) with value 30.

Pushing flow forward and backward

By pushing flow back on (v, u), we created an s-t path on which we are pushing flow

- forward, on edges with leftover capacity (e.g, (s, v));
- backward, on edges that are already carrying flow so as to divert it to a different direction (e.g., (u, v)).

The residual graph G_f

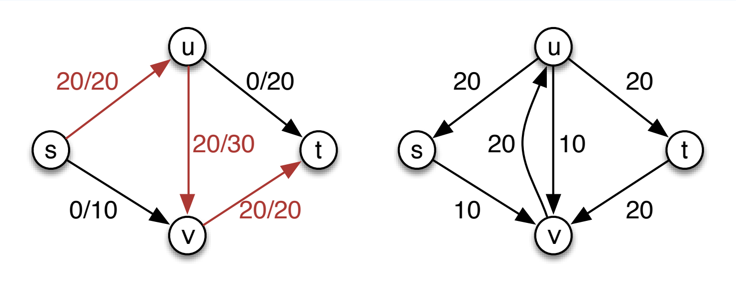
Definition 4.

Given flow network G and flow f, the residual graph G_f has

- the same vertices as G;
- for every edge $e = (u, v) \in E$ with $f(e) < c_e$, an edge $e = (u, v)$ with residual capacity $c_f(e) = c_e - f(e)$ (forward edge);
- for every edge $e = (u, v) \in E$ such that $f(e) > 0$, an edge $e^r = (v, u)$ in G_f with residual capacity $c_f(e^r) = f(e)$ (backward edge).

So G_f has $\leq 2m$ edges and every $e \in G_f$ has $c_f(e) > 0$.

Example residual graph



Left: a graph G and a flow f of value 20.

Right: the residual graph G_f for flow network G and flow f.

The residual graph G_f as a roadmap for augmenting f

1. Let P be a **simple** s-t path in G_f.
2. Augment f by pushing extra flow on P.

Question: How much flow can we push on P without violating capacity constraints in G_f?

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Definition 5.

The **bottleneck** capacity $c(P)$ of a simple path P is the minimum residual capacity of **any** edge of P. In symbols

$$c(P) = \min_{e \in P} c_f(e).$$

The residual graph G_f as a roadmap for augmenting f

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Definition 5.

The **bottleneck** capacity $c(P)$ of a simple path P is the minimum residual capacity of **any** edge of P. In symbols

$$c(P) = \min_{e \in P} c_f(e).$$

Answer: The max amount of flow we can safely push on **every** edge of P is $c(P)$.

The augmented flow f'

Let P be an augmenting path in the residual graph G_f.

We obtain an **augmented flow** f' as follows:

1. For a **forward** edge $e \in P$, set $f'(e) = f(e) + c(P)$
2. For a **backward** edge $e^r = (u, v) \in P$, let $e = (v, u) \in G$; set $f'(e) = f(e) - c(P)$
3. For $e \in E$ but not in P, $f'(e) = f(e)$.

Claim 1.

f' is a flow.

Pseudocode for subroutine `Augment`

Input: a flow f, and an augmenting path P in G_f
Output: the augmented flow f'

```
Augment(f, P)
  for each edge (u, v) ∈ P do
    if e = (u, v) is a forward edge then
      f(e) = f(e) + c(P)
    else
      f(v, u) = f(v, u) − c(P)
    end if
  end for
  return f
```

Today

The Ford-Fulkerson algorithm

```
Ford-Fulkerson(G = (V, E, c), s, t)
    for all e ∈ E do f(e) = 0
    end for
    while there is an s-t path in G_f do
        Let P be a simple s-t path in G_f
        f' = Augment(f, P)
        Set f = f'
        Set G_f = G_{f'}
    end while
    Return f
```

Running time analysis

The algorithm terminates if the following facts are both true.

Fact 6.

Every iteration of the while loop returns a flow increased by an integer amount.

Fact 7.

There is a finite upper bound to the flow.

Running time analysis

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Proof of Fact 7.

Let U be the largest edge capacity, that is, $U = \max_e c_e$. Then

$$|f| \leq \sum_{e \text{ out of } s} c_e \leq nU.$$

$\square$

f increases by an integer amount after Augment(f, P)

Proof of Fact 6.

It follows from the following claims.

Claim 2.

During execution of the Ford-Fulkerson algorithm, the flow values $\{f(e)\}$ and the residual capacities in G_f are all integers.

Claim 3.

Let f be a flow in G and P a simple s-t path in G_f with residual capacity $c(P) > 0$. Then after Augment(f, P)

$$|f'| = |f| + c(P) \geq |f| + 1.$$

$\square$

f increases by an integer amount after Augment(f, P)

Proof of Claim 3.

Recall that $|f| = f^{\text{out}}(s)$.

1. Since P is an s-t path, it contains some edge out of s, say (s, u).
2. Since P is simple, it does not contain any edge entering s (P is in G_f, where there could be edges entering s!).
3. Since no edge enters s in G, (s, u) is a forward edge in G_f, thus its augmented flow is $f(s, u) + c(P) \geq f(s, u) + 1$.
4. Since no other edge going out of s is updated, it follows that the value of f' is

$$|f'| = |f| + c(P) \geq |f| + 1.$$

□

Running time of Ford-Fulkerson

1. Claim 3 guarantees at most nU iterations.
2. The running time of each iteration is bounded as follows:
   - $O(m+n)$ to create G_f using adjacency list representation.
   - $O(m+n)$ to run BFS or DFS to find the augmenting path.
   - $O(n)$ for Augment(f, P) since P has at most $n-1$ edges.

   $\Rightarrow$ Hence one iteration requires $O(m)$ time.

The running time of Ford-Fulkerson is $O(mnU)$.

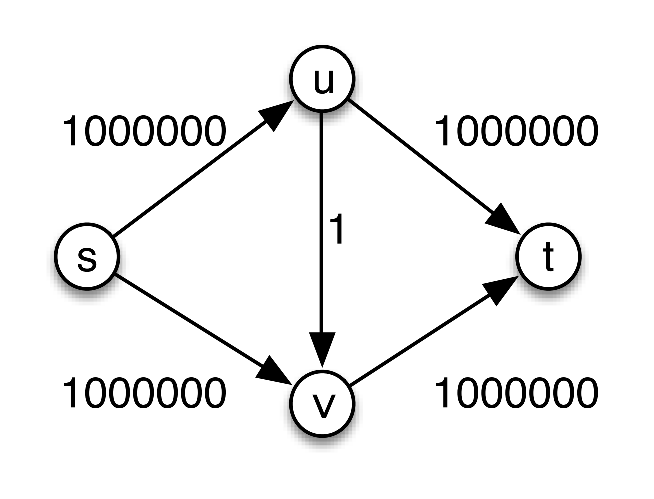
Definition 8 (Pseudo-polynomial algorithms).

An algorithm is pseudo-polynomial if it is polynomial in the size of the input when the **numeric** part of the input is encoded in **unary**.

Remark 1.

Ford-Fulkerson is a pseudo-polynomial time algorithm.

Problems with pseudo-polynomial running times



Improved algorithms

- FF can be made polynomial: use BFS instead of DFS
  - Edmonds-Karp: $O(nm^2)$, Dinitz: $O(n^2m)$, other improvements: $O(nm\log n)$, $O(n^3)$
- **Unit** capacities: $O(\min\{m^{3/2}, mn^{2/3}\})$ [EvenTarjan1975]
  - Improved for sparse graphs: $\tilde{O}(m^{10/7})$ [Madry2013]
- **Integral** capacities: $O(\min\{m^{3/2}, mn^{2/3})\log(n^2/m)\log U)$ [GoldbergRao1998]
  - Improved: $\tilde{O}(m\sqrt{n}\log^2 U)$ [LeeSidfort2014]; *also improves for dense graphs with unit capacities*
- **Real** capacities: $O(nm\log(n^2/m))$
  - Improved: $O(nm)$ [Orlin2013]

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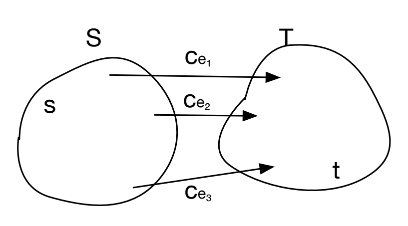
A natural upper bound for the max value of a flow

- An s-t cut (S, T) in G is a bipartition of the vertices into two sets S and T, such that $s \in S$ and $t \in T$.



- The capacity $c(S, T)$ of s-t cut (S, T) is $\sum_{\text{e out of } S} c_e$.
- Then intuitively

$$\max_f |f| \leq \min_{(S,T) \text{ cut in } G} c(S, T) \tag{4}$$

Roadmap for proving optimality of Ford-Fulkerson

Let f be the flow *upon termination* of the Ford-Fulkerson algorithm. Recall that $|f| = f^{\text{out}}(s)$.

1. Exhibit a specific s-t cut (S^*, T^*) in G such that

$$|f| = \text{capacity of the cut } (S^*, T^*)$$

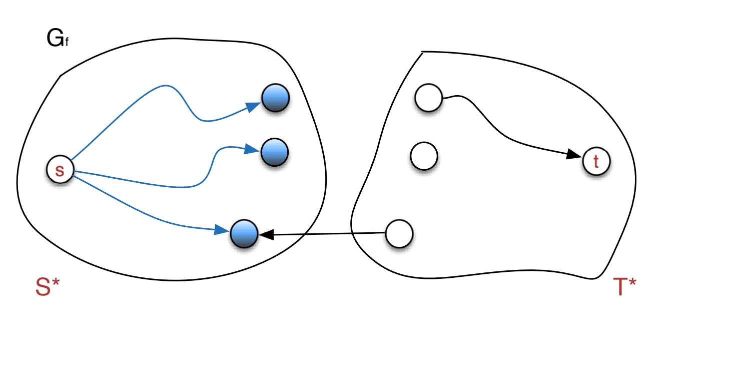
2. Show that $|f|$ cannot exceed the capacity of **any** cut in G.

3. Conclude that f is a maximum flow.
   - And (S^*, T^*) is a cut of minimum capacity.

Ford-Fulkerson terminates when $\nexists$ s-t path in G_f

Consider the residual graph G_f *upon termination* of the algorithm. Let (S^*, T^*) be the cut in G_f where

- S^* is the set of nodes reachable from the source s;
- T^* contains every other node.



Is (S^*, T^*) *also a cut in* G*?*

On the cut (S^*, T^*)

1. (S^*, T^*) is an s-t cut: that is, $s \in S^*$, $t \in T^*$. *Why?*

2. In G_f, no edge crosses from S^* to T^*. *Why?*

3. Hence, if $e = (x, y) \in E$ with $x \in S^*$ and $y \in T^*$, then $f(e) = c_e$ (thus $e \notin E_f$).

4. Similarly, if $e' = (u, v) \in E$ with $u \in T^*$ and $v \in S^*$, then $f(e') = 0$. *Why?*

On the cut (S^*, T^*)

1. (S^*, T^*) is an s-t cut: that is, $s \in S^*$, $t \in T^*$. *Why?*

   Because there is no s-t path in G_f.

2. In G_f, no edge crosses from S^* to T^*. *Why?*

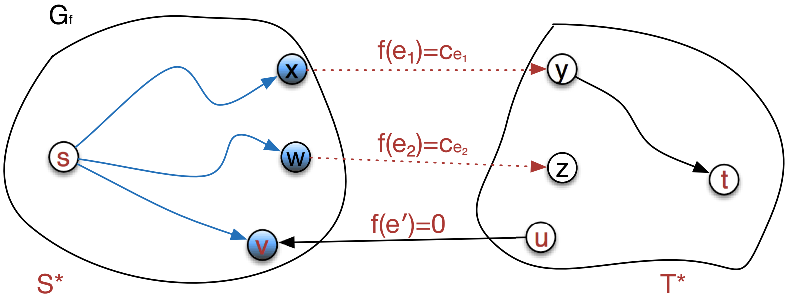
   If (u, v) crosses from S^* to T^*, thus $u \in S^*, v \in T^*$, then $\exists$ s-v path in G_f. Hence $v \in S^*$; contradiction.

3. Hence, if $e = (x, y) \in E$ with $x \in S^*$ and $y \in T^*$, then $f(e) = c_e$ (thus $e \notin E_f$).

4. Similarly, if $e' = (u, v) \in E$ with $u \in T^*$ and $v \in S^*$, then $f(e') = 0$. *Why?*

   If $f(e') > 0$, then $(v, u) \in E_f$, with $c_f(v, u) = f(e') > 0$. Contradicts our second observation.

Our observations on (S^*, T^*) in a diagram



In G, every edge e crossing from S^* to T^* satisfies $f(e)=c_e$ (of course, such e does not appear in G_f).
Every edge e' in G crossing from T^* to S^* satisfies $f(e')=0$.

Net flow across a cut

Definition 9.

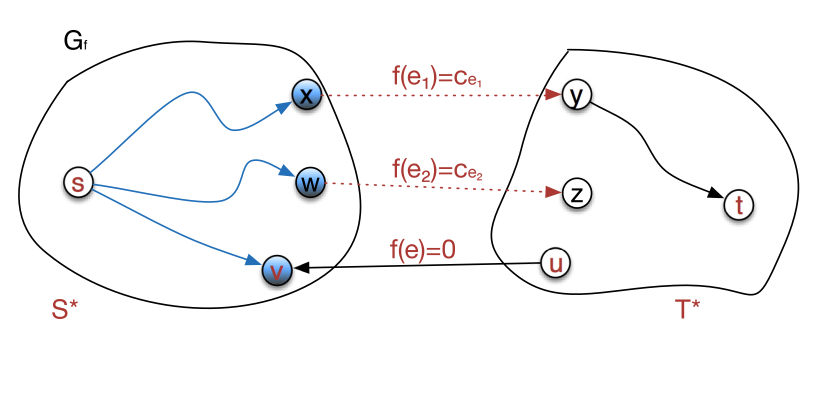
The net flow across an s-t cut (S, T) is the amount of flow leaving the cut minus the amount of flow entering the cut

$$f^{\text{out}}(S) - f^{\text{in}}(S), \tag{5}$$

where

1. $f^{\text{out}}(S) = \sum_{e \text{ out of } S} f(e)$
2. $f^{\text{in}}(S) = \sum_{e \text{ into } S} f(e)$

Net flow across (S^*, T^*) equals capacity of (S^*, T^*)



$$
\begin{aligned}
f^{\text{out}}(S^*) - f^{\text{in}}(S^*) &= \sum_{e \text{ out of } S^*} f(e) - \sum_{e \text{ into } S^*} f(e) \\
&= \sum_{e \text{ out of } S^*} c_e - 0 \\
&= c(S^*, T^*) \qquad (6)
\end{aligned}
$$

Roadmap revisited

Let f be the flow upon termination of the Ford-Fulkerson algorithm.

1. Exhibit a specific s-t cut (S^*, T^*) in G such that the

$$|f| = c(S^*, T^*).$$

   Not quite there yet!
   - *We exhibited (S^*, T^*) with net flow equal to its capacity.*
   - *We need to relate the net flow across (S^*, T^*) to $|f|$ (that is, the flow out of s).*
   - *In particular,* **if we showed them equal**, *then we'd have $|f| = c(S^*, T^*)$.*

2. Show that $|f|$ cannot exceed the capacity of **any** cut in G.
3. Conclude that f is a maximum flow.

net flow across any s-t cut $= |f|$

Recall that

- $f^{\text{out}}(S) = \sum_{e \text{ out of } S} f(e)$
- $f^{\text{in}}(S) = \sum_{e \text{ into } S} f(e)$
- net flow across $(S, T) \triangleq f^{\text{out}}(S) - f^{\text{in}}(S)$

Lemma 10.

Let f be any s-t flow, and (S, T) any s-t cut. Then

$$|f| = f^{out}(S) - f^{in}(S).$$

Proof of Lemma 10

First, rewrite the flow out of s in terms of the flow on the vertices on S:

$$|f| \quad = \quad f^{\text{out}}(s) = \sum_{v \in S} \Big(f^{\text{out}}(v) - f^{\text{in}}(v) \Big) \tag{7}$$

since

- $f^{\text{in}}(s) = 0$;
- for every $v \in S - \{s\}$, the terms in the right-hand side of (7) cancel out because of flow conservation constraints.

Next, rewrite the right-hand side of equation 7 in terms of the *edges* that participate in these sums.
There are three types of edges.

Proof of Lemma 10 (cont'd)

1. Edges with both endpoints in S: such edges appear once in the first sum in equation 7 and once in the second, hence their flows cancel out.
2. Edges with the tail in S and head in T (out of S): such edges contribute to the first sum, $\sum_{v \in S} f^{\text{out}}(v)$, in equation 7 so they appear with a $+$.
3. Edges with the tail in T and head in S (into S): such edges contribute to the second sum, $\sum_{v \in S} f^{\text{in}}(v)$, in equation 7 so they appear with a $-$.

In effect, the right-hand side of equation 7 becomes

$$\sum_{e \text{ out of } S} f(e) - \sum_{e \text{ into } S} f(e).$$

The lemma follows.

The value of a flow cannot exceed capacity of any cut

Corollary 11.

Let f be any s-t flow and (S, T) any s-t cut. Then

$$|f| \leq c(S, T).$$

Proof.

$$|f| = f^{\text{out}}(S) - f^{\text{in}}(S) \leq f^{\text{out}}(S) \leq c(S, T).$$

$\square$

Putting everything together

- By Corollary 11, the value of a flow cannot exceed the capacity of any cut; in particular,

$$|f| \leq c(S^*, T^*).$$

- By Lemma 10, $|f|$ equals the net flow across any s-t cut; in particular,

$$|f| = f^{\text{out}}(S^*) - f^{\text{in}}(S^*).$$

- From (6), the net flow across (S^*, T^*) equals $c(S^*, T^*)$. Hence the above becomes

$$|f| = f^{\text{out}}(S^*) - f^{\text{in}}(S^*) = c(S^*, T^*).$$

$\Rightarrow$ Thus the flow computed by Ford-Fulkerson is a maximum flow because it cannot be increased anymore.

The max-flow min-cut theorem

Theorem 12.

If f is an s-t flow such that there is no s-t path in G_f, then there is an s-t cut (S^*, T^*) in G such that $|f| = c(S^*, T^*)$. Therefore, f is a max flow and (S^*, T^*) is a cut of min capacity.

Theorem 13 (Max-flow Min-cut).

In every flow network, the maximum value of an s-t flow equals the minimum capacity of an s-t cut.

Integrality theorem

Recall the following claim.

Claim 4.

During execution of the Ford-Fulkerson algorithm, the flow values $\{f(e)\}$ and the residual capacities in G_f are all integers.

Combine with Theorem 12 to conclude:

Theorem 14 (Integrality theorem).

If all capacities in a flow network are integers, then there is a maximum flow for which every flow value $f(e)$ is an integer.

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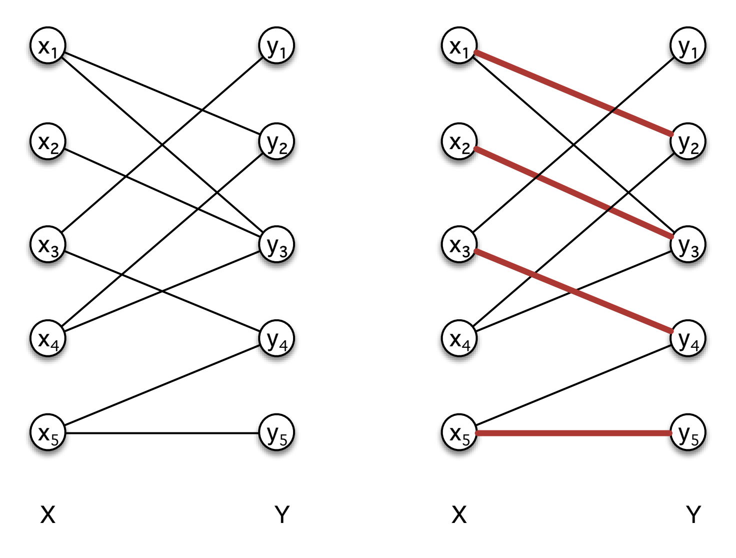
Bipartite Matching



Matchings

Definition 15.

A matching M is a subset of edges where every vertex in $X \cup Y$ appears at most once.

Example: $\{(x_1, y_2), (x_2, y_3), (x_3, y_4), (x_5, y_5)\}$ is a matching.

Perfect matching: every vertex in $X \cup Y$ appears exactly once.

- Not always possible: e.g., $|X| \neq |Y|$.

Maximum matching still desirable in applications.

- If we had an algorithm to find maximum matching then we could also find a perfect matching, if one exists (*why?*).

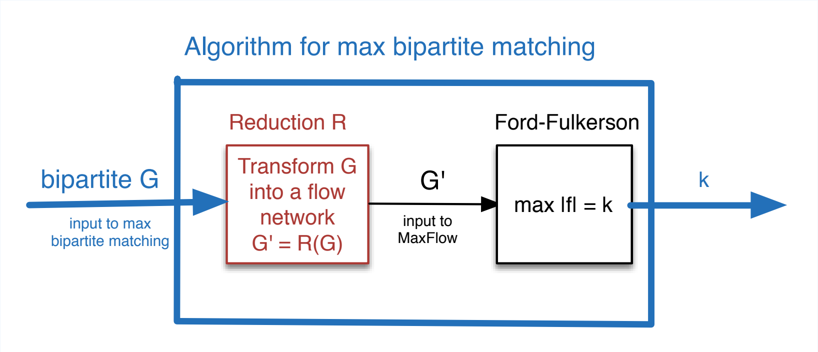
Finding maximum matchings in bipartite graphs

Idea: Use the Ford-Fulkerson algorithm to find maximum (or perfect) matchings in bipartite graphs.

Strategy: reformulate the problem as a max flow problem which we know how to solve (reduction).

To this end, we need to transform our input bipartite graph into a flow network.

A diagram of the algorithm for max bipartite matching



Remark 2.

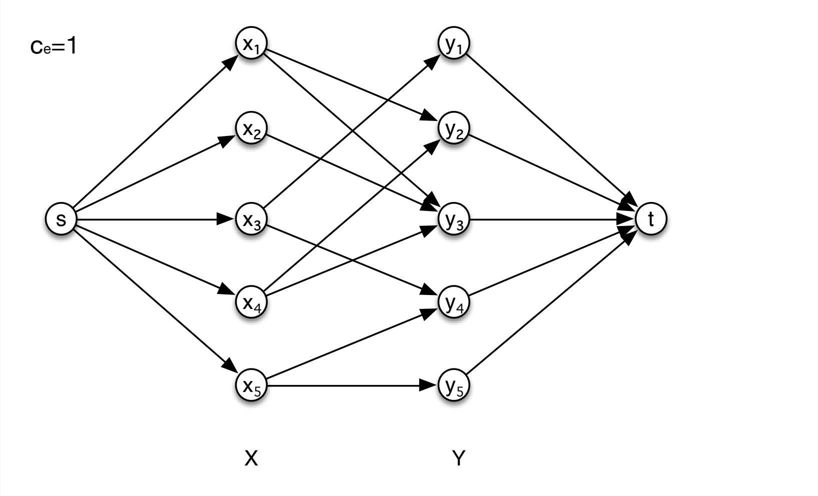
1. *The reduction R must be efficient (polynomial in the size of G).*
2. *G and G' should be equivalent, in the sense that G has a max matching of size k if and only if the max flow in G' has value k.*

Deriving a flow network given a bipartite graph

Given a bipartite graph $G = (X \cup Y, E)$, we construct a flow network G' as follows.

- Add a source s.
- Add a sink t.
- Add (s, x) edges for all $x \in X$.
- Add (y, t) edges for all $y \in Y$.
- Direct all $e \in E$ from X to Y.
- Assign to every edge capacity of 1.

The flow network for the example bipartite graph



Computing matchings in G from flows in G'

- $G = (X \cup Y, E)$ is the bipartite graph
- G' is the derived flow network

Claim 5.

The size of the maximum matching in G equals the value of the maximum flow in G'. The edges of the matching are the edges that carry flow from X to Y in G'.

Proof of Claim 5

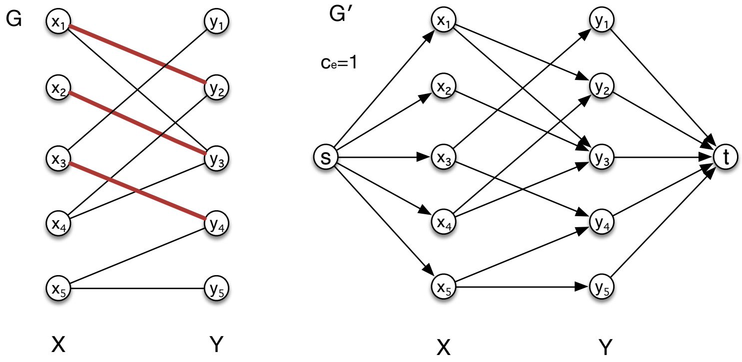
The claim follows if we show the following two statements (*why?*).

1. (⇒ Forward direction) For any matching M in G, we can construct a flow f in G' with value equal to the size of M, that is, $|f| = |M|$.

2. (⇐ Reverse direction) Given a max flow f' in G', we can construct a matching M' in G, with size equal to the value of the max flow, that is, $|M'| = |f'|$.

(1. $\Rightarrow$) From a matching M to a flow f with $|f| = |M|$

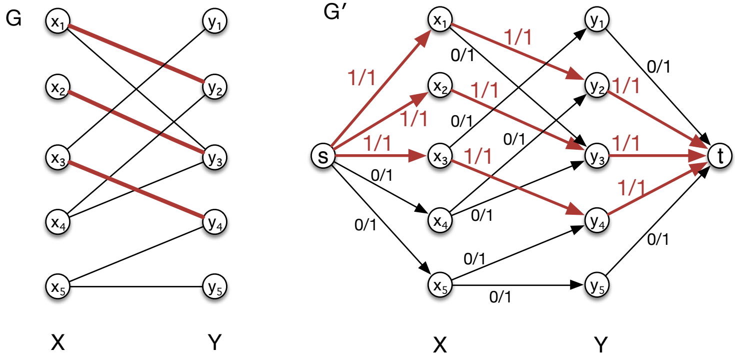
Let $|M| = k$.



Given matching M (the red edges in G), construct an integral flow f in G′, such that the value of f equals the number of edges in M.

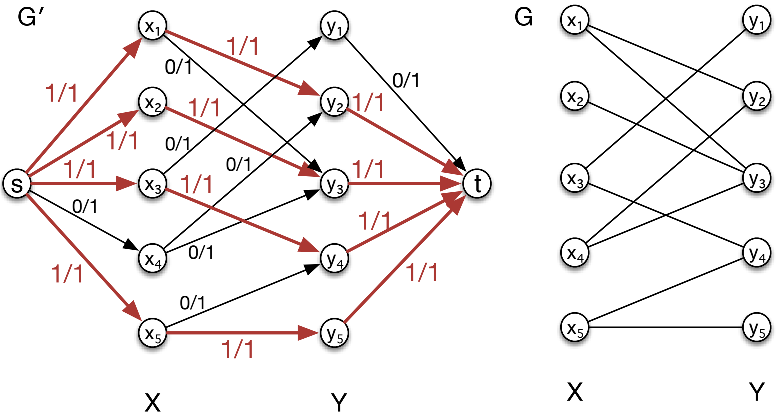
(1. $\Rightarrow$) From a matching M to a flow f with $|f| = |M|$

Let $|M| = k$. Send one unit of flow along each of the k edge-disjoint s-t paths that use the edges in M; then $|f| = k$.



Given matching M (the red edges in G), construct the integral flow f in G′.
Then the value of f equals the number of edges in M.

(2. $\Leftarrow$) From max flow f' to M' with $|M'| = |f'|$



Given integral max flow f′ in G′, construct a matching M′ in G so that the number of edges in M′ equals the value of f′.

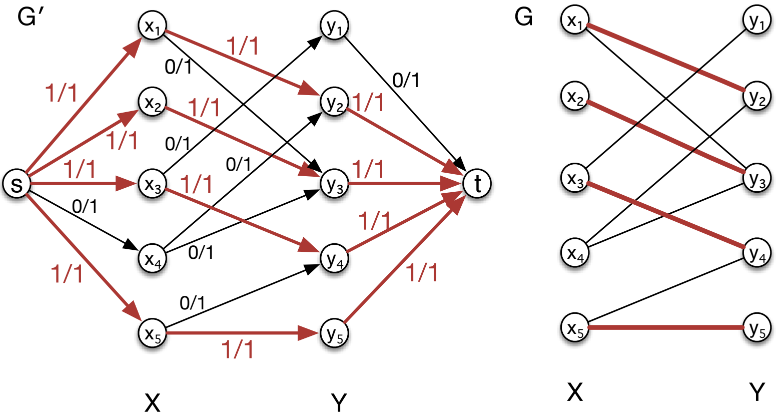
From max flow f' to matching M' with $|M'| = |f'|$

Given a max flow f' in G' with $|f'| = k$, we want to select a set of edges M' in G so that M' is a matching of size k.

- By the integrality theorem, there is an integer-valued flow f of value k.
- Then for every edge e, $f(e) = 0$ or $f(e) = 1$ (*why?*).
- Define the following matching M':

$$M' = \Big\{ e = (x, y) : x \in X, y \in Y \text{ and } f(e) = 1 \Big\}.$$

Obtaining a matching M' from an integral flow f



Given integral flow f in G′, construct matching M′ (the red edges in G), so that the number of edges in M′ equals the value of f.

M' is a matching of size k

We need to show the following two facts.

1. **Fact 1**: M' is a matching.
2. **Fact 2**: M' has size k.

Proof of Fact 1.

Must show that every node in G' appears at most once in M'.

- Each node in X is the tail of at most one edge in M' (*flow conservation constraints*).
- Each node in Y is the head of at most one edge in M' (*flow conservation constraints*).

$\square$

M' has size k

Proof of Fact 2.

- Consider the cut (S, T) where $S = \{s\} \cup X$, $T = Y \cup \{t\}$.
- We will compute its net flow.

  1. By definition, the net flow of (S, T) is

     $$f^{\text{out}}(S) - f^{\text{in}}(S) = |M'|$$

     since
     - the only edges that carry flow out of S are the edges in M';
     - the flow into S is 0 (no edges enter S).
  2. By Lemma 10, the net flow across (S, T) equals $|f|$; hence net flow across $(S, T) = k$.

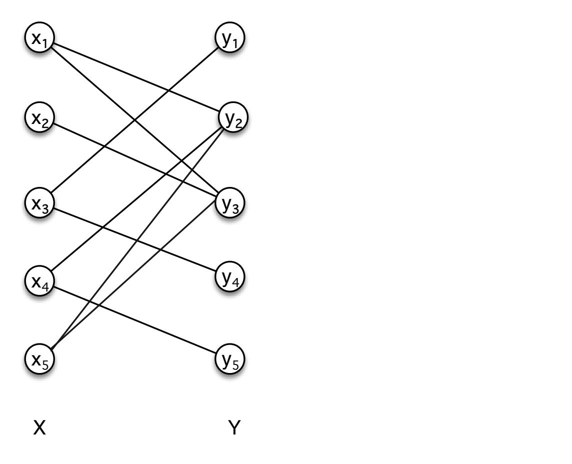
$\Rightarrow$ Thus $|M'| = k$.

$\square$

Time for finding max matching in bipartite graphs

1. Ford-Fulkerson: $O(mnU) = O(mn)$
2. Improved: $O(m\sqrt{n})$ [HopcroftKarp, Karzanov 1973]
3. Improved further for sparse ($m = O(n)$) graphs: $\tilde{O}(m^{10/7})$ [Madry2013]

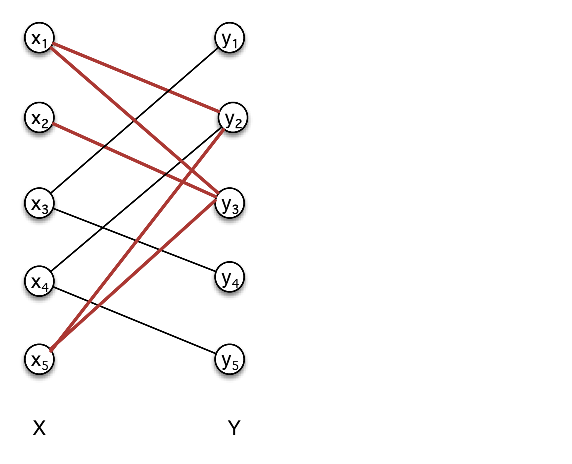
Perfect matchings in bipartite graphs with $|X| = |Y| = n$ (Hall's theorem)



Is there a matching of size n, that is, a perfect matching in G?

A necessary condition for a perfect matching to exist

For every subset A of nodes in X, there are at least as many neighbors of A in Y. In symbols, $\forall A \subseteq X, |N(A)| \geq |A|$.

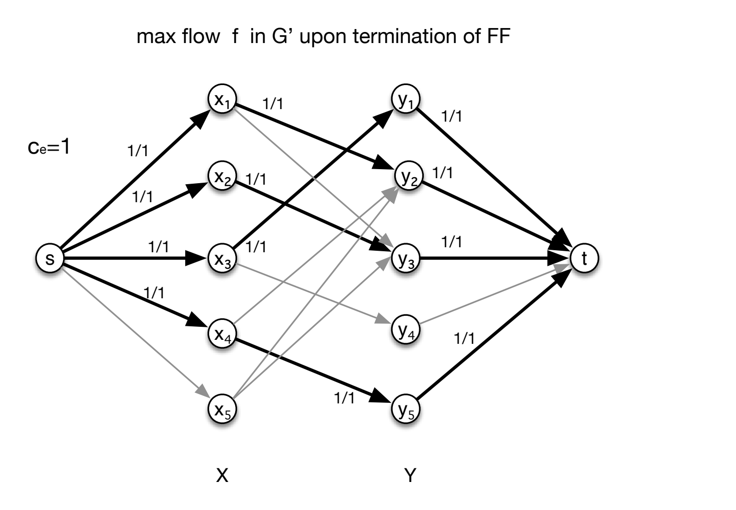


Is this a **sufficient** *condition as well?*

That is, if G does not have a perfect matching, is there always a subset $A \subseteq X$ such that $|N(A)| < |A|$?

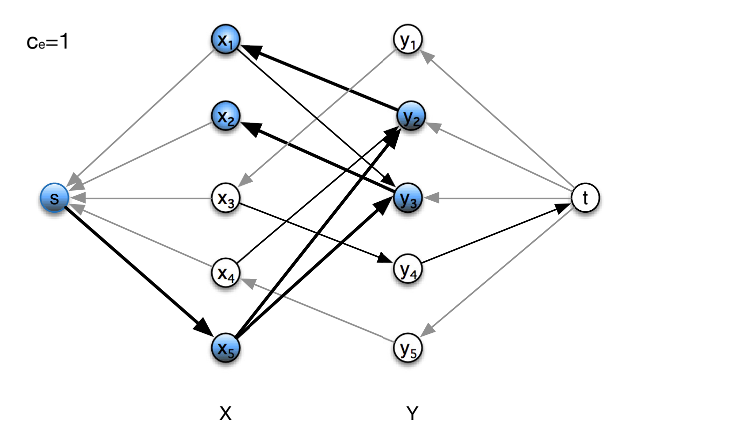
- Given bipartite G, transform it into a flow network G'.
- Run Ford-Fulkerson on G'.
- Assume $\max |f| < n$. We want to exhibit a set A as above.
- Since $\max |f| < n$, we know that $\min_{(S,T)} c(S,T) = \max |f| < n$.
- Consider the residual graph G_f upon termination of FF.
- Define the cut (S^*, T^*) as before, that is, S^* consists of all vertices reachable from s and T^* of everything else.
- We claim that the set $A = S^* \cap X$ satisfies $|A| > |N(A)|$.

A max flow in G'



The residual graph upon termination of FF

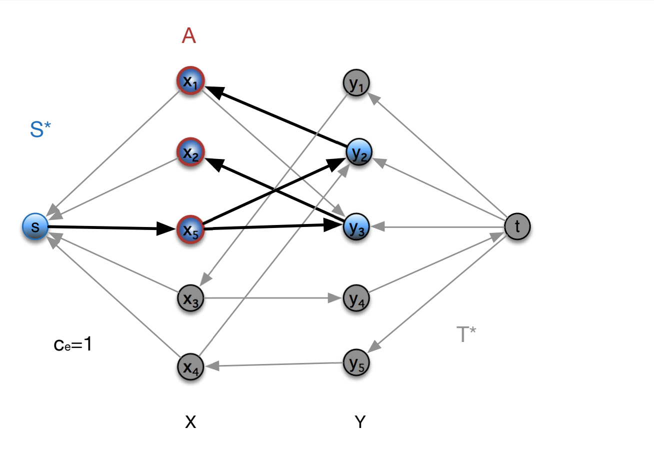
Define the cut (S^*, T^*) where S^* consists of all vertices reachable from s in G_f and T^* of everything else.



Here S^* consists of the blue vertices.

The set A that satisfies $|A| > |N(A)|$

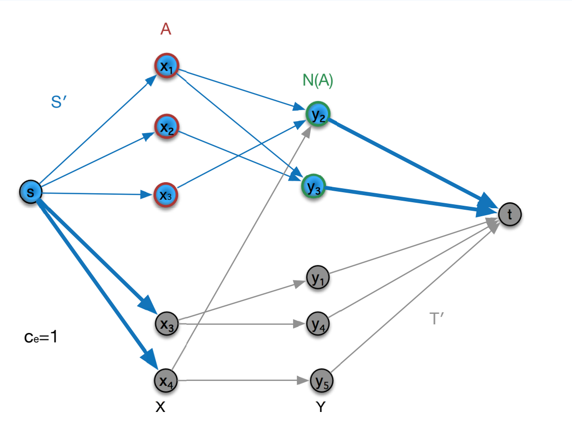
Clearly S^* consists of s, some vertices in X *(why?)* and possibly some vertices in Y. We set $A = S^* \cap X$.



The residual graph slightly rearranged to group the vertices in A together.

Augmenting S^* to include all neighbors of A

Note that S^* might contain some neighbors of A.
Augment S^* by moving *all* neighbors of A into S^*.
Let S' be the resulting set, $T' = V - S'$.



Blue nodes belong to S': among them, nodes with red contour are in A and nodes with green contour in $N(A)$. Silver nodes belong to T'.

How does $c(S', T')$ compare to $c(S^*, T^*)$?

For every node u we move from T^* to S'

- we add 1 to $c(S^*, T^*)$ because of the edge (u, t) that now crosses from S' to T';
- we subtract at least 1 from $c(S^*, T^*)$ because u is a neighbor of at least one node $v \in A$, hence the edge (v, u) no longer crosses from S^* to T^*.

Hence $c(S', T') < c(S^*, T^*)$.

Since (S^*, T^*) is a min cut, and $\max |f| < n$, we have

$$c(S', T') < n.$$

The capacity of (S', T')

What exactly is the capacity of (S', T')*?*

- $n - |A|$ edges cross from $s \in S'$ to T'
- $|N(A)|$ edges cross from S' to $t \in T'$

Hence $c(S', T') = n - |A| + |N(A)|$.

Since $c(S', T') < n$, we have

$$n - |A| + |N(A)| < n.$$

Hence

$$|A| > |N(A)|.$$