Damping of Primordial Gravitational Waves by Free Streaming Neutrinos in the Matter Dominated Era

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1 Introduction

In this project, we consider tensor perturbations in the FLRW metric, and their evolution throughout RD, MD, and ΛD. After deriving an inhomogeneous differential equation for the propagation of GWs, we specifically consider the effect of free-streaming neutrinos. The anisotropic stress generated by these neutrinos effects the decay of modes entering the horizon during RD and early MD. This effect is best seen through numerical integration, comparing the decaying solution of the homogeneous equation with the neutrino-dampened solution.

2 Setup

Consider a small perturbation in the metric, $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, and decompose it into scalar, vector and tensor parts:

$$h_{00} = -2\psi$$

$$h_{0i} = \beta_i + \partial_i\gamma$$

$$h_{ij} = -2\phi\delta_{ij} + (\partial_i\partial_j - \frac{1}{3}\delta_{ij}\nabla^2)\lambda + \frac{1}{2}(\partial_i\epsilon_j + \partial_j\epsilon_i) + h_{ij}^{TT}$$

where $\psi, \phi, \gamma, \lambda$ are scalar fields, β_i, ϵ_i are transverse vector fields, and h_{ij}^{TT} is a traceless, transverse tensor field. Working in the FLRW metric,

$$ds^2 = a^2(\eta)(\eta_{\mu\nu} + h_{\mu\nu})dx^\mu dx^\nu$$

we form the following scalar combinations (Bardeen variables):

$$\Phi = -\phi - \frac{1}{6}\nabla^2\lambda + \mathcal{H}\left(\lambda - \frac{1}{2}\frac{d\lambda}{d\eta}\right)$$

$$\Psi = -\psi + \frac{1}{a}\frac{d}{d\eta}\left[a\left(\gamma - \frac{1}{2}\frac{d}{d\eta}\right)\right]$$

In the Newtonian Gauge ($\lambda = \gamma = \beta_i = 0$), the perturbed metric is

$$ds^2 = -a^2(1+2\Psi)d\eta^2 + a^2\left[(1+2\Phi)\delta_{ij} + \frac{1}{2}(\partial_i\epsilon_j + \partial_j\epsilon_i) + h_{ij}^{TT}\right]dx^i dx^j.$$

Considering only the tensor perturbations, the metric becomes:

$$ds^2 = a^2\left[-d\eta^2 + (\delta_{ij} + h_{ij}^{TT})dx^i dx^j\right]$$

from which we can form the Einstein Tensor (computing Christoffels, etc)

$$\delta G^i_j = \frac{1}{2a^2}\left[(h_{ij}^{TT})'' + 2\mathcal{H}(h_{ij}^{TT})' - \nabla^2 h_{ij}^{TT}\right].$$

Applying Einstein's Field Equation, we obtain

$$(h_{ij}^{TT})'' + 2\mathcal{H}(h_{ij}^{TT})' - \nabla^2 h_{ij}^{TT} = 16\pi G a^2 \sigma_{ij}^{TT},$$

where SVT decomposition of $T_{\mu\nu}$ gives the traceless, transverse tensor field σ_{ij}^{TT} of the anisotropic stress (similar to above with h_{ij}^{TT}). Going into momentum space, and expanding h_{ij}^{TT} in the basis of the polarization tensors (e_{ij}^A), we have

$$\tilde{h}_{ij}^{TT}(\eta, \mathbf{k}) = \sum_{A=+,\times} e_{ij}^A(\hat{\mathbf{k}})\tilde{h}_A(\eta, \mathbf{k})$$

$$\tilde{\sigma}_{ij}^{TT}(\eta, \mathbf{k}) = \sum_{A=+,\times} e_{ij}^A(\hat{\mathbf{k}})\tilde{\sigma}_A(\eta, \mathbf{k})$$

which yields two independent equations for $\tilde{h}_A(\eta, k)$:

$$\tilde{h}_A'' + 2\mathcal{H}\tilde{h}_A' + k^2\tilde{h}_A = 16\pi G a^2 \tilde{\sigma}_A.$$

3 Neutrinos

After neutrinos decoupled from other types of matter in the early universe (at T ≈ 1 MeV), they became free-streaming; essentially traveling across space at near light speed in uniformly random directions without interacting with anything.

Generally, relativistic free-streaming particles generate an anisotropic stress-energy tensor of the form

$$\tilde{\sigma}_A(\eta, \mathbf{k}) = -4\rho_n u(\eta)\int_{\eta_{\nu dec}}^{\eta} d\eta' \frac{j_2[k(\eta - \eta')]}{k^2(\eta - \eta')^2}\tilde{h}_A'(\eta', k)$$

where j_2 is the spherical Bessel function, defined by

$$j_2(x) = -\frac{\sin x}{x} - \frac{3\cos x}{x^2} + \frac{3\sin x}{x^3}$$

We further simplify the differential equation for $\tilde{h}_A$ by using the following relation

$$16\pi G a^2 \rho_\nu(\eta) = 6\Omega_\nu(\eta)\mathcal{H}^2$$

where Ω_ν is the neutrino density parameter $\Omega_\nu(\eta) = \frac{\rho_\nu(\eta)}{\rho_c(\eta)}$. Now our differential equation becomes the following integro-differential equation

$$\tilde{h}_A'' + 2\mathcal{H}\tilde{h}_A' + k^2\tilde{h}_A = -24\Omega_\nu(\eta)\mathcal{H}^2(\eta)\int_{\eta_{\nu dec}}^{\eta} d\eta' \frac{j_2[k(\eta - \eta')]}{k^2(\eta - \eta')^2}\tilde{h}_A'(\eta', k)$$

We still need to know what $\Omega_\nu(\eta)$ exactly is before we proceed. In the radiation-dominated era, we can say that

$$\begin{aligned}\Omega_\nu(\eta) &= \frac{\Omega_\nu a^{-4}}{\Omega_R a^{-4} + \Omega_M a^{-3} + (1-\Omega)a^{-2} + \Omega_\Lambda} \\ &\approx \frac{\Omega_\nu}{\Omega_R(1 + a(\eta)/a_{eq})} \\ &= f_\nu(0)\frac{1}{1 + a(\eta)/a_{eq}} \\ &\equiv f_\nu(\eta)\end{aligned} \tag{1}$$

We now make some change of variables and substitutions, to more closely match Weinberg's derivation. Firstly, we let $u \equiv k\eta$, and we replace $\mathcal{H}(\eta)$ with its definition from the Friedman equation: $\frac{a'(\eta)}{a(\eta)}$. Now the integro-differential equation looks like this:

$$\tilde{h}_A''(u) + \frac{2a'(u)}{a(u)}\tilde{h}_A'(u) + k^2\tilde{h}_A(u) = -24f_\nu(u)\left(\frac{a'(u)}{a(u)}\right)^2\int_0^u dU\frac{j_2(u-U)}{(u-U)^2}\tilde{h}_A'(U)$$

We substitute $\tilde{h}_A(u) = \tilde{h}_A(0)\chi(u)$, and find that the constant $\tilde{h}_A(0) \neq 0$ factors out of the equation to give the equivalent equation in $\chi(u)$

$$\chi''(u) + \frac{2a'(u)}{a(u)}\chi'(u) + k^2\chi(u) = -24f_\nu(u)\left(\frac{a'(u)}{a(u)}\right)^2\int_0^u dU\frac{j_2(u-U)}{(u-U)^2}\chi'(U)$$

We have the initial conditions $\chi(0) = 1$ and $\chi'(0) = 0$

4 Results

We want to determine the damping of the neutrinos on the gravitational waves formed during crucial eras of the early universe, such as during inflation, from the first order electroweak phase transition, and from perturbations in Quantum Field Theory. In particular, we want to investigate the damping on the gravitational waves formed from those eras that reenter the cosmological horizon well into the matter-dominated era of the universe.

We first verify Weinberg's and Maggiore's proposed solution for $u >> 1$, where Weinberg suggests the solution approaches the homogeneous solution

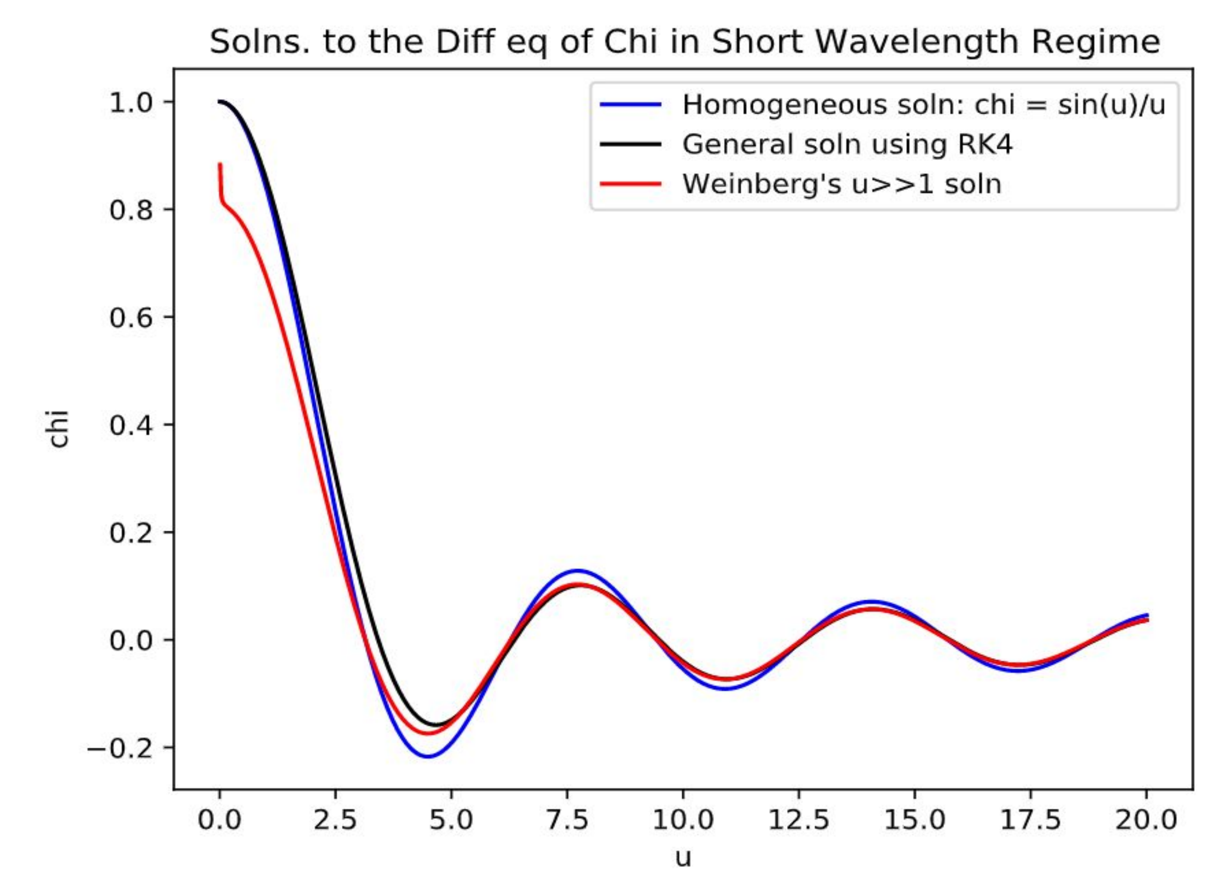
$$\chi(u) \to A\frac{\sin(u+\delta)}{u}$$

and Maggiore suggests the solution approaches

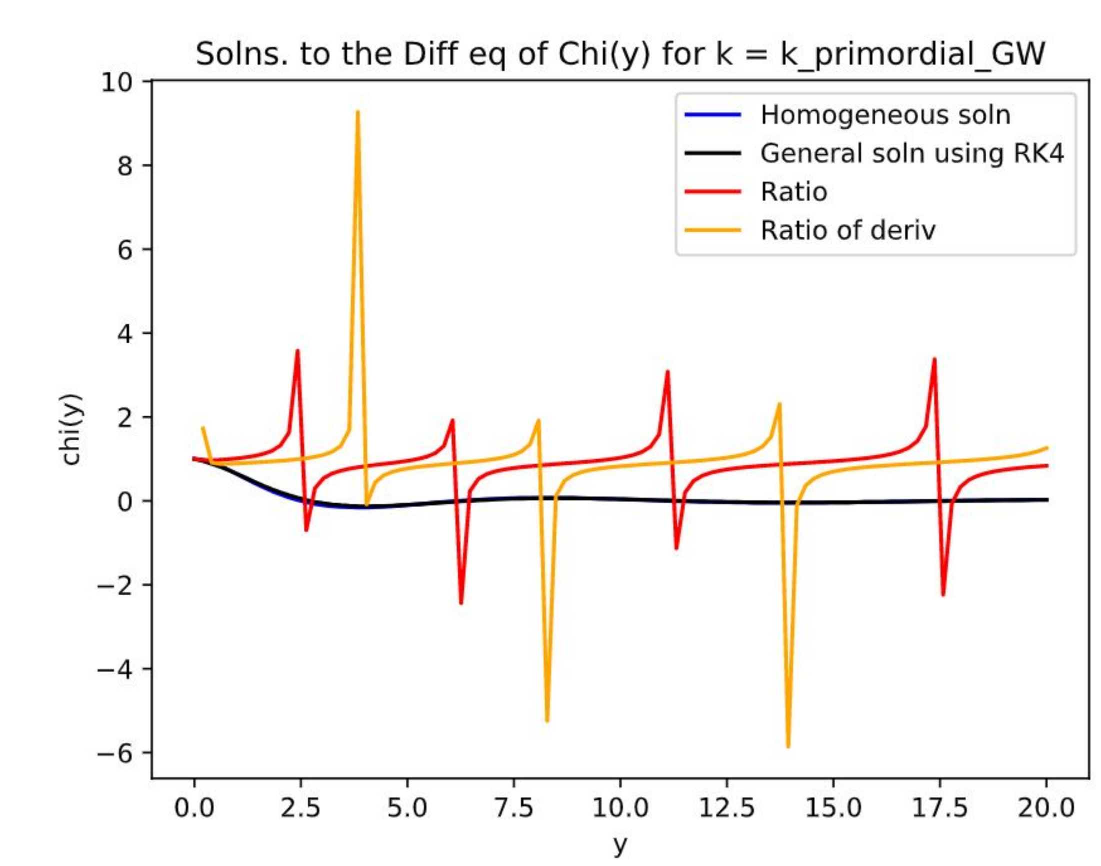
$$\chi(u) \to A\frac{\sin(u)}{u}$$

with both of them saying that $A \approx 0.80$.

We numerically calculated the solution for u >> 1, shown below in black.



We see that it does agree with Weinberg's solution (and Maggiore's for small enough **δ**). We then numerically calculated the general integro-differential equation, and looked at the ratio between the general solution and the homogeneous, and likewise for their derivatives to see the effect of the damping.



We see the same finite spikes and relatively flat regions that Weinberg mentions.

Conclusion

We successfully reproduced the results of Weinberg and Maggiore for the neutrino damping of gravitational waves by numerically solving the integro-differential equation of the tensor perturbation. We want to apply the results for the wave vectors in the range of those gravitational waves produced in the previously mentioned eras after they reenter the horizon. We want to see if there really is a substantial damping of 0.644, the value of squared ratio of the derivatives of the general and homogeneous solutions.

Literature Cited

- Weinberg, S. (2004). Damping of tensor modes in cosmology. Physical Review D, 69(2), 023503. https://arxiv.org/abs/astro-ph/0306304
- Maggiore, M. (2018) Gravitational Waves, Volume 2, Astrophysics and Cosmology. Oxford University Press, Oxford. https://doi.org/10.1093/oso/9780198570899.001.0001