PHY254 Homework Problems #4, 2020

1. **Small oscillations about a circular orbit:** Consider a satellite of mass m in a circular orbit of radius $r = R$ around a planet of mass M. The orbit of the satellite is at the minimum of an effective potential given by

$$V_{\text{eff}}(r) = \frac{L^2}{2mr^2} - \frac{GMm}{r} \ , \tag{1}$$

where $L = m\omega R^2$ is the angular momentum of the satellite and ω is its orbital angular frequency. Suppose that the rocket engines on the satellite are fired away from the planet very briefly, giving the satellite a small nudge in the radial direction.

   (a) Make an argument that the subsequent motion of the satellite has the same angular momentum L after the rockets fire as it had before.

   (b) Make an argument that, considering only the radial motion, the satellite executes simple harmonic motion in r about its former circular orbit at radius R.

   (c) Show that the frequency of the simple harmonic motion ω_{SHM} is the same as the original circular orbital frequency ω.

   What is happening here is that the satellite enters a new, slightly elliptical orbit. The new orbit is closed because the radial motion has the same frequency as the circular motion.

2. **Space elevator:** Morin exercise 5.65 page 187.

   This is a conceptually hard question. Here are some hints: In part (a), you can treat the satellite as a point mass. In part (b), the elevator cable is not a point mass, so you must integrate the gravitational force on each little radial segment of the cable of length dr. When you have an expression for the net force on the cable F_{net}, you can remember that $F_{\text{net}} = ma_{\text{cm}}$, where a_{cm} is the acceleration of the center of mass, whose location and acceleration you know.

Space elevators are the perennial subjects of science fiction. Unfortunately, there is probably no material strong enough to make the cable needed to build one. Show that this is so by calculating the tension T in the cable for an Earth scale elevator. Suppose that the cable has cross-sectional area A and density ρ_{cable}. The quantity $T/(A\rho_{\text{cable}}) = T/\sigma$ is called the *specific strength* if T is just at the breaking strength of the cable material. Consult the table on the Wikipedia page for specific strength to see if there exists any material strong enough.

3. **Orbiting the Pluto - Charon system:** Pluto and its moon Charon form a close binary planet system. The *New Horizons* spacecraft passed through this system in July 2015 without getting into orbit. Let's see what orbits around this binary object might look like.

   Pluto has a mass of $M_{\text{p}} = 1.3 \times 10^{22}$ Kg and radius R_{p} of 1200 km. Charon has a mass of $M_{\text{c}} = 1.6 \times 10^{21}$ Kg and a radius R_{c} of 600 km. The distance between their centers D_{pc} is about 20 000 km.

   (a) Show the the radial position of the center of mass R_{cm} of this binary system, measured from the centre of Pluto, is actually well outside the radius R_{p} of Pluto. Thus, Pluto and Charon both orbit an empty point in space that is somewhere between them. See http://en.wikipedia.org/wiki/File:Pluto-Charon_System.gif.

   (b) Find the period in Earth days of the orbits of Pluto and Charon about their center of mass, assuming circular orbits.

   Now we want to plot the orbit of a spacecraft of negligible mass m around this binary system. To keep things simple, we will fix Pluto and Charon to lie on the x axis, with the centre of mass at the origin. Suppose that Pluto is located at $x_{\text{p}} = -R_{\text{cm}}$ and Charon is located at $x_{\text{c}} = D_{\text{pc}} - R_{\text{cm}}$. We will consider satellite orbits with initial conditions of the form $\vec{r}_0 = (0, y_0)$ and $\vec{v}_0 = (v_0, 0)$.

   This is a version of the "restricted three body problem", in which the two gravitating bodies are nailed down and only the satellite moves.

(c) First, make a "spherical cow" approximation to get the overall scale of the orbit. Assume the whole Pluto - Charon system is approximately a point of mass $M = M_{\rm p} + M_{\rm c}$ at the origin and use this to estimate the speed v and period T for a circular orbit with a radius of 25 000 km (which is somewhat outside the orbit of Charon).

(d) Modify the sample code `pluto_charon_orbit_template.py` to plot some realistic orbits for initial conditions $\vec{r}_0 = (0, y_0) = (0, 25\,000\,\text{km})$ and $\vec{v}_0 = (v_0, 0)$, with v_0 the orbit speed you estimated above. Use a time step that is about $1/100$ of the estimate orbit period T, or less. Plot about 10 - 100 orbits.

(e) Try a few nearby values of y_0, v_0 and plot some of the interesting orbits you find. Do you think the orbital dynamics are chaotic? How would you check if they are, computationally?

Often the satellite will eventually make a close encounter with one of the point-like planets, causing it to be violently ejected from the system. This is tricky to avoid.

Notes:

This Euler-Cromer time stepping code is not ideal for orbit problems because it does not conserve angular momentum exactly. Realistic numbers will require a very small time step. The python function `odeint` would do a better job, as discussed on the computational assignment.