PHY407: Lab 8

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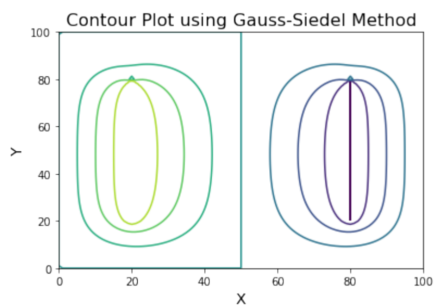
Work Distribution: We worked on this assignment together in-person, so it was split quite evenly. We brainstormed pseudocodes, and then the base code for each question was collaborated upon. We reused the code we wrote as necessary to answer all of the questions. We alternated between parts for these questions, and then switched and checked each other's work at the end.

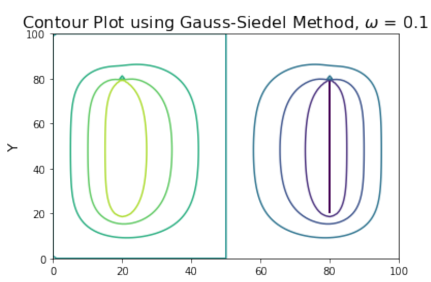
*All Python code and outputs are included in the Quercus submission as .py files.

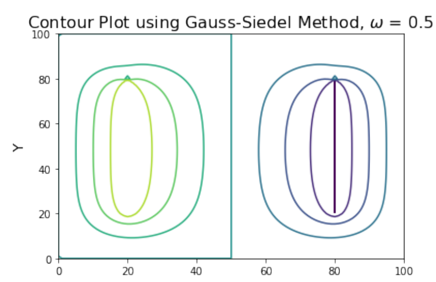
Question 1: Electrostatics and Laplace's equation

a) Code and plot



b) Code, plot, and written answer





Notice that the plots for ω values of 0.1 and 0.5 are virtually the same as the plot produced using the Gauss-Siedel method without overrelaxation. The main difference is in the runtime of the Python code. We notice that $\omega = 0.5$ has the fastest runtime.

Question 2: Simulating the shallow water system

a) Written answers

Equation 6 is given to us as:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -g\frac{\partial \eta}{\partial x}$$
$$\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x}(uh) = 0$$

We can rearrange the first part as follows:

$$\frac{\partial u}{\partial t} = -g\frac{\partial \eta}{\partial x} - u\frac{\partial u}{\partial x}$$
$$= -g\frac{\partial \eta}{\partial x} - \frac{1}{2}\frac{\partial u^2}{\partial x}$$
$$= -\frac{\partial}{\partial x}(\frac{1}{2}u^2 + g\eta)$$

If we apply the second part of Equation 6, we can get:

$$\frac{\partial \eta}{\partial t} = -\frac{\partial}{\partial x}(uh)$$

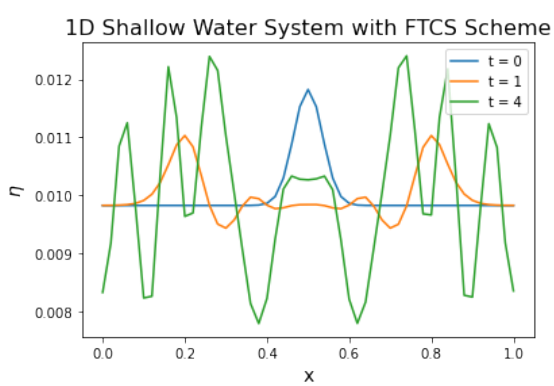
Using these equations, we can find that:

$$\frac{\partial}{\partial t}\vec{u} = -\begin{bmatrix} \frac{\partial}{\partial x}(\frac{1}{2}u^2 + g\eta) \\ \frac{\partial}{\partial x}[u(\eta - \eta_b] \end{bmatrix}$$

If we apply the FTCS scheme, our equation will look something like this:

$$\frac{u_{i+1} - u_i}{\Delta t} = -\frac{(\frac{1}{2}(u_{i+1})^2 - \frac{1}{2}(u_{i-1})^2 + g\eta_{i+1} - g\eta_{i-1})}{2\Delta x}$$
$$\frac{\eta_{i+1} - \eta_i}{\Delta t} = \frac{u_{i+1}(\eta_{i+1} - \eta_{b,i+1}) - u_{i-1}(\eta_{i-1} - \eta_{b,i-1})}{2\Delta x}$$

b) Code and plots



c) Derivation

For the first linear equation, we immediately notice that u = 0, and so the $u\frac{\partial u}{\partial x} = 0$, therefore we are left with:

$$\frac{\partial u}{\partial t} = -g\frac{\partial \eta}{\partial x} \tag{1}$$

For the second linear equation, we can use the chain rule:

$$\frac{\partial \eta}{\partial t} + (H - \eta_b)\frac{\partial u}{\partial x} + u\frac{\partial (H - \eta_b)}{\partial x} = 0$$

Since u = 0 and $\eta_b = constant$, we can simplify to:

$$\frac{\partial \eta}{\partial t} = -H\frac{\partial u}{\partial x} \tag{2}$$

Applying the FTCS scheme to these equations (1) and (2) gives:

$$\frac{u_{i+1} - u_i}{\Delta t} = -g\frac{\eta_{i+1} - \eta_{i-1}}{2\Delta x} \tag{3}$$

$$\frac{\eta_{i+1} - \eta_i}{\Delta t} = -H\frac{u_{i+1} - u_{i-1}}{2\Delta x} \tag{4}$$

Applying the Fourier transform and noticing the identity $e^{ik\Delta x} - e^{-ik\Delta x} = 2\sin k\Delta x$, we get the following matrix equation:

$$\begin{bmatrix} u_{i+1} \\ \eta_{i+1} \end{bmatrix} = \begin{bmatrix} 1 & -g\frac{\Delta t}{\Delta x}\sin k\Delta x \\ -H\frac{\Delta t}{\Delta x}\sin k\Delta x & 1 \end{bmatrix} \begin{bmatrix} u_i \\ \eta_i \end{bmatrix} \tag{5}$$

This Matrix equation gives us the following eigenvalues:

$$\lambda = 1 \pm i\frac{\Delta t}{\Delta x}\sqrt{gH}\sin k\Delta x$$

Therefore:

$$|\lambda| = \sqrt{1 + \frac{\Delta t}{\Delta x}^2 \sqrt{gH}\sin^2 k\Delta x} \tag{6}$$

Notice that everything to the right-hand side of the + sign will always be positive, therefore $|\lambda|$ will always be greater than 1 and will thus be UNSTABLE.