PyLab 1: Mass-Spring Oscillations

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October 2, 2020

Abstract:

The purpose of this experiment was to collect data from a mass-spring system (damped and non-damped) and find the difference in accuracy between different methods of integration for calculating and plotting the displacement, velocity, and energy of the oscillating spring. The forward method and the symplectic method were both used for calculations. The symplectic method turned out to be the most accurate.

Introduction:

The experiment consisted of two parts. The first part included a regular mass-spring system where period, frequency, amplitude, and spring constant were all found/calculated by taking measurements and applying those measurements to Hooke's law and Newton's laws for springs. The second part was similar to the first, however, a damping disk was added to the mass for resistance. The damping constant was found along with the other values above.

Equipment:

In terms of equipment, the following was used throughout the experiment:

- Ruler - used to measure equilibrium point and diameter of damping disk
- Stopwatch - used to record time of oscillations
- Motion Sensor - used to record position vs time data for the spring
- Retort Stand - used to support the mass and spring
- Mass - the piece of equipment being analyzed
- Spring - used to oscillate the hanging mass
- Damping Disk - used to add resistance to the mass in the second part
- Scale - used to measure the mass

Procedure:

For the first part of the lab, a 0.2003kg mass was hung from a spring. The mass was found using a scale. Next, the equilibrium height relative to the motion sensor was recorded, with a ruler, to be 0.247m. This would be the reference point when analyzing the position vs time data. The mass then was then forced into oscillatory motion where ten oscillations were timed using a stopwatch. In this specific case, the mass made 10 oscillations in 7.33 seconds, and therefore, as shown below, the period of oscillation was 0.733s. A motion sensor was used to record the position vs time data of the mass-spring system.

In the second part of the experiment, all steps in the first part were repeated except the damping disk was attached to the spring and the mass was set into motion for 2 minutes instead of 10 oscillations. The period was found to be 0.733s for this part and the mass was also recorded to be 0.2003kg like the first part. The diameter of the disk was also measured to be 0.101m. The damping constant was found using the graphed data from the motion sensor and was 0.374Ns/m. The process of these calculations will be shown later in the report.

Results:

The results of the experiment are mainly covered in the python program and the answers to the questions, but a brief overview will be covered for more clarification.

Before plotting began, some values needed to be found. The amplitude of the oscillations was taken to be the distance between the point at which the mass was furthest from the sensor minus the equilibrium height. Therefore, the amplitude was given as:

$$A = 0.344 \pm 0.001m \; - \; 0.274 \pm 0.001m \; = 0.0968 \pm 0.001m$$

The period was found by analyzing the time difference between two crests on the position-time graph and dividing by the number of oscillations that took place in that time period:

$$T = \frac{6.70 \pm 0.01s - 0.10 \pm 0.01s}{9} = 0.733 \pm 0.01s$$

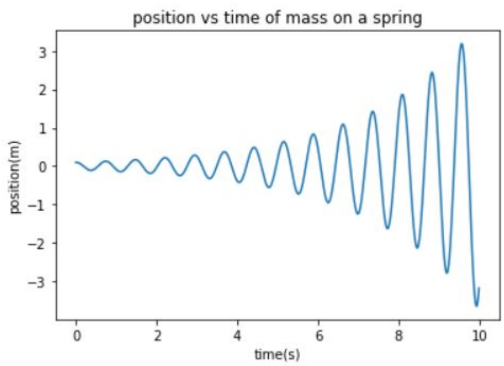
Next, the frequency was found by simply taking the inverse of the period, so:

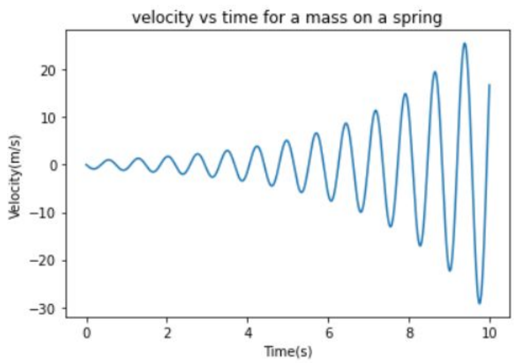
$$f = \frac{1}{T} = 1.36 \pm 0.01Hz$$

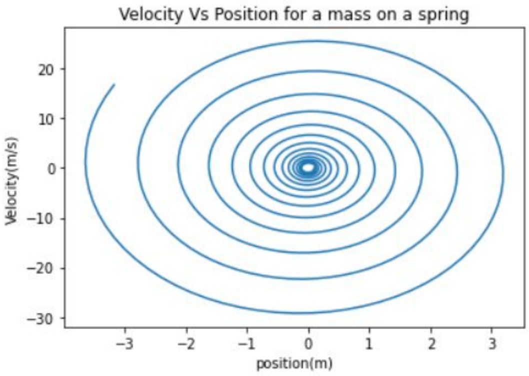
The spring constant was found using the formula for angular frequency:

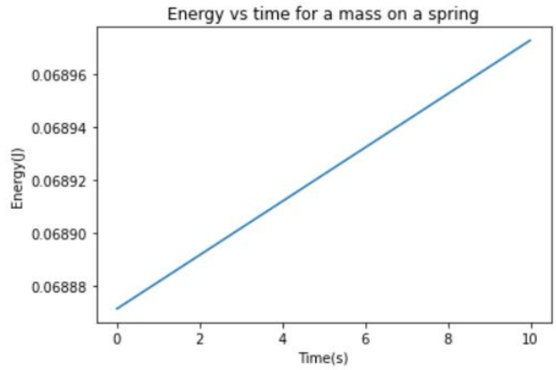
$$k = \frac{4\pi^2 m}{T^2} = \frac{4\pi^2(0.2003 \pm 0.0001)}{(0.733 \pm 0.01)^2} = 14.700 \pm 0.004N/m$$

After the data was calculated and recorded, the calculations for the position, velocity, and energy using the coupled equations were done (see python program). A position vs time, velocity vs time, energy vs time, and velocity vs position (phase) plot were all created and can be seen below:



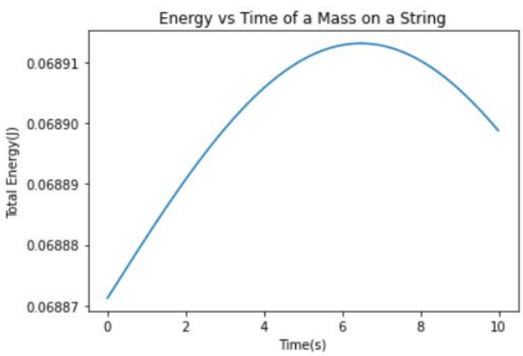


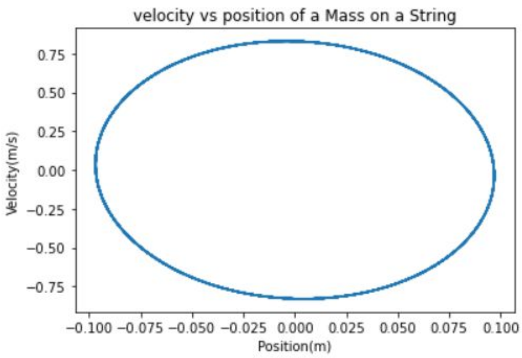




As evident from the plots above, they do not make sense. Position, velocity, and energy all increase over time, which defies the law of conservation of energy. A further analysis of this issue can be viewed in questions 4-6 below.

To fix this, the symplectic method was used to redo the calculations. This time only energy vs time and the phase plot were generated just to show that energy was now more accurately conserved:



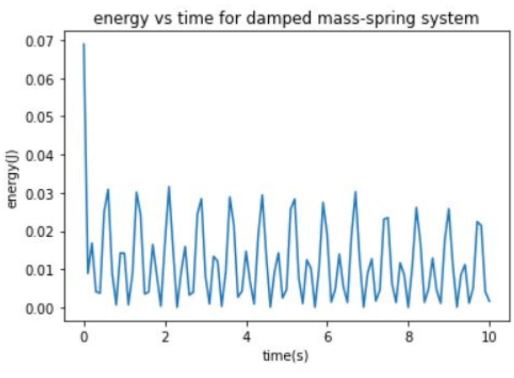


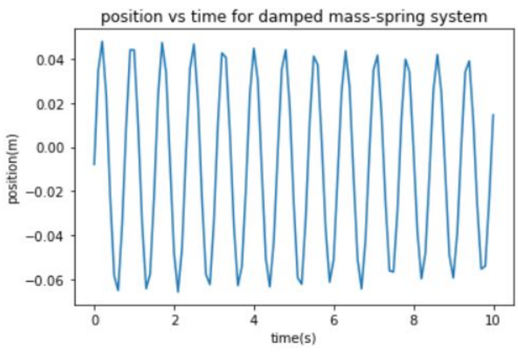
In the graphs above, it is clear that energy was almost completely conserved. The energy vs time plot shows a very minuscule variation in energy, but the oscillation shows that the total energy did not continuously increase like in the previous plot. Instead, it varied across an almost negligible amount of 0.00004J. The phase plot shows an ellipse. This means energy was conserved because across the semi-major axis, the velocity is 0 yet the position is at its minimum and maximum values. This is what was expected and since the spring is oscillating, these numbers points will be crossed multiple times, hence why it forms an ellipse. This also applies to the maximum and minimum velocity points where the position is 0m.

In the second part of the experiment, the decay term was added and some more values were calculated. The amplitude, period, frequency and spring constant were the same as part 1. The decay constant is given by $\gamma = 0.374 \pm 0.01 Ns/m$ and the decay rate is. The calculations behind the decay constant are in question 10.

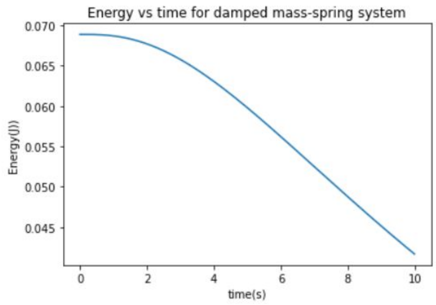
The position, velocity, and energy were calculated (see python program). After that, a position vs time plot and energy vs time plot was made and compared to plots made using the data collected from the motion sensor:

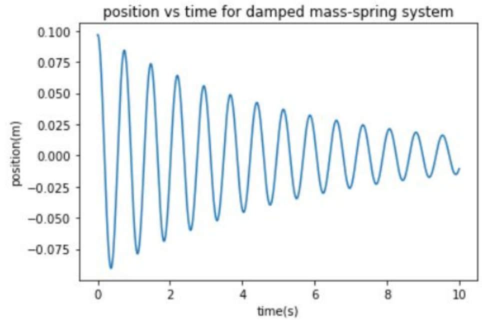
Experimental Data:





Theoretical Data:





It can be seen that the theoretical and experimental data look quite different. This will be discussed more in the error analysis as well as in question 12.

<u>Discussion</u>

As expected the symplectic integration method was more accurate than the forward Euler method. This is because the forward Euler method calculates each value using previously stored values. This means it does not take into account the change in the velocity or displacement between the present value and the previous values being used in the calculations. The symplectic method, however, uses present displacement values to calculate velocity as opposed to values calculated in the past and therefore it takes into account the change over that small time interval, making it more accurate.

In terms of the experimental vs theoretical data, they were off by quite a bit and this could be due to a few reasons. It is important to note that the theoretical calculations assume a perfect system. It assumes measurements with no error, completely vertical oscillation and an ideal spring. Of course, none of these are possible, and therefore, the experimental data was off as it takes these things into account.

<u>Conclusion:</u>

To conclude, the symplectic integration method was more accurate than the forward Euler method.

Questions

<u>Question 1:</u>

Starting from the fact that the net force in any given system is equal to the change in potential energy with respect to the position of the same system, Hooke's Law can be derived as follows:

$$m\vec{a} = -\frac{\mathrm{d}U}{\mathrm{d}\vec{y}}$$
$$m\frac{d^2\vec{y}}{dt^2} = -\frac{\mathrm{d}U}{\mathrm{d}\vec{y}}$$
$$m\frac{d^2\vec{y}}{dt^2} = -\frac{d}{dt}\left(\frac{1}{2}k\vec{y}^2\right)$$
$$m\frac{d^2\vec{y}}{dt^2} = -k\vec{y}$$
$$m\frac{d^2\vec{y}}{dt^2} + k\vec{y} = 0$$
$$\frac{d^2\vec{y}}{dt^2} + \frac{k}{m}\vec{y} = 0$$

In this derivation, the effects of gravity and friction are ignored and it is assumed the spring is oscillating perfectly vertical. This formula also works for small values of y. For large values of y, there are higher-order terms that must be accounted for.

Question 2:

Starting from the equation derived in Question 1, formula *b can be derived first as follows:

$$\frac{d^2\vec{y}}{dt^2} + \frac{k}{m}\vec{y} = 0$$
$$\frac{d\vec{v}}{dt} + \frac{k}{m}\vec{y} = 0$$
$$\frac{\vec{v}_{i+1} - \vec{v}_i}{\Delta t} + \Omega_0^2\vec{y}_i = 0$$
$$\vec{v}_{i+1} - \vec{v}_i + \Delta t\Omega_0^2\vec{y}_i = 0$$
$$\vec{v}_{i+1} = \vec{v}_i - \Delta t\Omega_0^2\vec{y}_i$$

Now converting the omega term back into the derivative of velocity with respect to time and multiplying through by the change in time, equation 8a can be derived:

$$\vec{v}_{i+1} = \vec{v}_i - \Delta t\Omega_0^2\vec{y}_i$$
$$\vec{v}_{i+1} = \vec{v}_i + \Delta t^2\frac{\vec{v}_i}{\Delta t}$$
$$\vec{v}_{i+1}\Delta t = \vec{v}_i\Delta t + \Delta t\vec{v}_i$$
$$\vec{y}_{i+1} = \vec{y}_i + \Delta t\vec{v}_i$$

Question 3:

Next, the spring constant, k, could be found using the angular frequency formula for a mass-spring system:

$$\omega = \frac{2\pi}{T} = \sqrt{\frac{k}{m}}$$
$$\frac{4\pi^2 m}{T^2} = k$$
$$k = \frac{4\pi^2(0.2003)}{(0.753)^2} = 14.7N/m$$

Question 4:

The graph is showing oscillations, however, the amplitude is increasing with time. This should not be the case since this would mean some energy is somehow being gained within the system as time goes on. In this system, energy is conserved and no external forces are acting on it, therefore the results shown in the graph are not physical relative to the experiment.

Question 5:

The energy plot suggests the system is gaining energy. This does explain the weird shape of the plots in the velocity vs time plot and the velocity vs position plot because given the formula for total energy, it is evident that the only variables changing are v and y, so in order for energy to increase either one of y and v has to increase or both have to. In the plots, it seems both are increasing with time.

Question 6:

The phase plot should be an ellipse because given that energy is conserved, the values for max and min position as well as max and min velocity should not change as there are no external forces adding energy to the system. This means that, since the spring is oscillating, the max and min values of v and y should be reached multiple times as the spring completes its oscillations and therefore an ellipse can be seen as the shape of the phase plot.

My phase plot is a spiral. This tells us that both the position and velocity values are increasing with time and therefore energy is not conserved. This is a non-physical result.

Question 7:

The leading error in the above numerical method was that higher-order terms were ignored and only linear terms on y were considered. Equation 5a should really be considered as:

$$\vec{p}(0) + \Delta t F(\vec{q}(t)) + \frac{\Delta t^2 F(\vec{q}(t))}{2!} \ldots$$

and 5b should be considered as:

$$\vec{q}(0) + \frac{\Delta t}{m}\vec{p}(t) + \frac{\Delta t^2(\vec{p}(t))}{2!m} \ldots$$

Therefore all terms with an order of 2 or higher were omitted and assumed to be very small causing some error.

Question 8:

When the symplectic method is used, the plot shapes turn out as expected and therefore it is evident that energy is now conserved within the calculations and plots. The energy plot is slightly curved, but only between the values 0.6887J and 0.6891J, therefore, the total energy is very close to conserved.

Question 9:

For this problem, the max velocity of the mass was used since the velocity of the spring is constantly changing. The density of air, the dynamic viscosity of air, and the diameter of the mass were also used. There, the Reynolds number for this system is:

$$R_e = \frac{\rho L v}{\eta} = \frac{(1.225)(0.101)(0.166)}{(1.81x10^{-5})} = 1134.7$$

Question 10:

Using the formulas for drag force and the average coefficient of drag for a circular face (1.17), the damping coefficient is:

$$F_d = -.5C\rho A|\vec{v}|\vec{v} = -\gamma|\vec{v}|$$
$$\Rightarrow \gamma = .5C\rho A|\vec{v}|$$
$$\Rightarrow \gamma = .5(1.17)\left(\pi(0.0505)^2\right)(0.166) = 5.69 \times 10^{-4} Ns/m$$

However, the experiment in section 1.4 was repeated (except with a damping plate attached to the mass) in order to find the actual decay rate. The mass and period were remeasured and found to be the same as in part 1.

After the steps in 1.4 were repeated, the graph generated by the motion sensor was analyzed to find the decay constant. The decay constant ($\frac{\gamma}{2}$) is the inverse time for which the amplitude falls to 1/e of the initial value. The initial amplitude, in this case, was 5.14cm and the time it took for the amplitude to fall to 5.14/e cm was 5.35s, therefore:

$$\frac{\gamma}{2} = \frac{1}{5.35 \pm 0.01}$$
$$\gamma = 0.374 \pm 0.01 Ns/m$$

Question 11:

The damped energy plot shows a slight decrease in energy as time goes on. This can be expected as the position vs time plot shows a decrease in amplitude over time suggesting that energy is being lost.

Question 12:

Comparing the theoretical and experimental graphs above, it is clear that the calculated data is more clear and has a higher decay rate than the experimental data. The experimental data was used with much larger time increments of 0.1s as opposed to the calculated data that had increments of 0.01s, making the calculated one more accurate. The calculated plots also show distance from equilibrium, whereas the experimental plot shows the distance from the sensor. It is important to note that the experimental data is collected in the real world, whereas the calculated data is calculated with equations that take into account some assumptions. The calculation will not accurately depict a non-perfect system with friction/tension in the spring, the possible distortion in the spring, and even the accurate measurements of time and mass as those were measured with equipment that has accuracy restrictions. The theoretical energy also shows a slight energy decay which is far more accurate than the oscillating and decaying energy in the experimental plots.