# ALGORITHMS AND DATA STRUCTURES

CSci 4041

FACULTY: NIKOS PAPANIKOLOPOULOS

# ALGORITHMS

* AN ALGORITHM IS ANY WELL-DEFINED COMPUTATIONAL PROCEDURE THAT TAKES SOME VALUE, OR SET OF VALUES, AS INPUT AND PRODUCES SOME VALUE, OR SET OF VALUES, AS OUTPUT.

* AN ALGORITHM CAN BE VIEWED AS A TOOL FOR SOLVING A WELL-SPECIFIED COMPUTATIONAL PROBLEM.

* AN ALGORITHM IS SAID TO BE CORRECT IF, FOR EVERY INPUT INSTANCE, IT PROVIDES THE CORRECT OUTPUT.

* EXAMPLE: SORTING PROBLEM

  INPUT: A SEQUENCE OF $n$ NUMBERS $\langle a_1, \ldots, a_n \rangle$

  OUTPUT: A PERMUTATION (REORDERING) OF THE INPUT SEQUENCE $\langle a'_1, \ldots, a'_n \rangle$ SUCH THAT $a'_1 \leq a'_2 \leq \ldots \leq a'_n$.

# ALGORITHMS

* $\Theta$-NOTATION

$\Theta(g(n)) = \{ f(n)$ : THERE EXIST POSITIVE CONSTANTS $c_1, c_2$ AND $n_0$ SUCH THAT $0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n)$ FOR ALL $n \geq n_0 \}$

EXAMPLE: $f(n) = \frac{1}{2} n^2 - 3n$

$c_1 = 1/14$, $c_2 = 1/2$ AND $n_0 = 7$ PROVIDE EVIDENCE THAT $\frac{1}{2} n^2 - 3n = \Theta(n^2)$

* O-NOTATION

$O(g(n)) = \{ f(n)$ : THERE EXIST POSITIVE CONSTANTS $c$ AND $n_0$ SUCH THAT $0 \leq f(n) \leq c g(n)$ FOR ALL $n \geq n_0 \}$

- ASYMPTOTIC UPPER BOUND

# ALGORITHMS

* $\Omega$-NOTATION

$\Omega(g(n)) = \{ f(n)$: THERE EXIST POSITIVE CONSTANTS $c$ AND $n_0$ SUCH THAT $0 \leq cg(n) \leq f(n)$ FOR ALL $n \geq n_0 \}$

- ASYMPTOTIC LOWER BOUND

* THEOREM

FOR ANY TWO FUNCTIONS $f(n)$ AND $g(n)$, $f(n) = \Theta(g(n))$ IF AND ONLY IF $f(n) = O(g(n))$ AND $f(n) = \Omega(g(n))$

* EXAMPLE

THE FACT THAT $an^2 + bn + c = \Theta(n^2)$ FOR ANY CONSTANTS $a, b,$ AND $c$ $(a > 0)$ IMMEDIATELY IMPLIES THAT $an^2 + bn + c = \Omega(n^2)$ AND $an^2 + bn + c = O(n^2)$

# BINARY SEARCH

* PROBLEM

LET $\langle a_1, a_2, \ldots, a_n \rangle$ BE A SEQUENCE OF REAL NUMBERS SUCH THAT $a_1 \leq a_2 \leq \ldots \leq a_n$. GIVEN A REAL NUMBER $z$, WE WANT TO FIND WHETHER $z$ APPEARS IN THE SEQUENCE, AND, IF IT DOES, TO FIND AN INDEX $i$ SUCH THAT $a_i = z$

* INPUT

A (A SORTED ARRAY IN THE RANGE 1 TO n)

z (THE SEARCH KEY)

* OUTPUT

POSITION (AN INDEX i SUCH THAT A[i] = z, OR 0 IF NO SUCH INDEX EXIST)

* ORDER IS IMPORTANT

# BINARY SEARCH

* ALGORITHM (PSEUDOCODE)

```
FIND(z, left, right)   {IT RETURNS INTEGER}
  if left = right then
      if A[left] = z then return left
      else return 0
  else
      middle ← ⌈½(left + right)⌉
      if z < A[middbe] then
          temp ← FIND(z, left, middbe-1)
      else
          temp ← FIND(z, middbe, right)
      return temp
```

* COMPLEXITY

THE NUMBER OF COMPARISONS IS O(logn)

# BINARY SEARCH

* BINARY SEARCH IN A CYCLIC SEQUENCE

(PROBLEM DESCRIPTION)

GIVEN A CYCLICALLY SORTED LIST, FIND THE POSITION OF THE MINIMAL ELEMENT IN THE LIST (IT IS ASSUMED THAT THIS POSITION IS UNIQUE)

* A SEQUENCE $\langle a_1, a_2, \ldots, a_n \rangle$ IS CYCLICALLY SORTED IF THE SMALLEST NUMBER IN THE SEQUENCE IS $a_i$ FOR SOME UNKNOWN $i$, AND THE SEQUENCE $\langle a_i, a_{i+1}, \ldots, a_n, a_1, \ldots, a_{i-1} \rangle$ IS SORTED IN INCREASING ORDER.

* THE ALGORITHM IS A VARIATION OF THE BINARY SEARCH.

* THE COMPLEXITY IS SIMILAR TO THE COMPLEXITY OF BINARY SEARCH.

# BINARY SEARCH

* ALGORITHM (PSEUDOCODE)

```
CYCLIC_FIND(left,right) { IT RETURNS INTEGER}
 if  left=right  then
     return left
 else
     middle ← ⌊½(left+right)⌋
     if  A[middle]<A[right] then
          temp← CYCLIC_FIND(left,middle)
     else
          temp← CYCLIC_FIND(middle+1,right)
     return  temp
```

# BINARY SEARCH

- BINARY SEARCH FOR A SPECIAL INDEX

  (PROBLEM DESCRIPTION)

  GIVEN A SORTED SEQUENCE OF DISTINCT INTEGERS $\langle a_1, \ldots, a_n \rangle$ DETERMINE WHETHER THERE EXISTS AN INDEX $i$ SUCH THAT $a_i = i$

- ALGORITHM

```
SPECIAL_FIND (left,right) {IT RETURNS INTEGER}
    if left=right then
        if A[left]=left then return left
        else return 0
    else
        middle ← ⌈1/2 (left+right)⌉
        if A[middle] < middle then
            temp ← SPECIAL_FIND(middle+1,right)
        else
            temp ← SPECIAL_FIND(left,middle)
        return temp
```

# THE STUTTERING-SUBSEQUENCE PROBLEM

* THE PROBLEM:

  GIVEN TWO SEQUENCES A AND B, FIND THE MAXIMAL VALUE OF $i$ SUCH THAT $B^i$ IS A SUBSEQUENCE OF A.

* GIVEN A SEQUENCE B, WE DEFINE $B^i$ TO BE THE SEQUENCE B WITH EACH CHARACTER APPEARING $i$ TIMES CONSECUTIVELY.

  EXAMPLE: $B = xyyz$ $B^2 = xxyyyyzz$

* SEQUENCE B (LENGTH $m$) IS SUBSEQUENCE OF THE SEQUENCE A (LENGTH $n$) IF WE CAN EMBED B INSIDE A IN THE SAME ORDER BUT WITH POSSIBLE HOLES.

* THE ALGORITHM THAT DETERMINES IF B IS SUBSEQUENCE OF A IS SIMPLE AND THE RUNNING TIME IS $O(n+m)$.

# INTERPOLATION SEARCH

* ALGORITHM (PSEUDOCODE)

```
I_FIND (z, left, right)   { IT RETURNS INTEGER}
    if  A[left] = z  then return left
    else if left = right or A[left] = A[right] then
         return 0
    else
         next_guess ← ⌈left + ((z - A[left])(right - left)) / (A[right] - A[left])⌉
         if  z < A[next_guess] then
              temp ← I_FIND(z, left, next_guess - 1)
         else
              temp ← I_FIND(z, next_guess, right)
         return temp
```

* COMPLEXITY

THE AVERAGE NUMBER OF COMPARISONS IS $O(\log\log n)$

# SORTING

* THE PROBLEM:

  GIVEN $n$ NUMBERS $a_1, a_2, \ldots, a_n$, ARRANGE THEM IN INCREASING ORDER.

* A SORTING ALGORITHM IS CALLED IN-PLACE IF NO ADDITIONAL WORKSPACE IS USED BESIDES THE INITIAL ARRAY THAT HOLDS THE ELEMENTS.

* DIFFERENT SORTING ALGORITHMS

| | |
|---|---|
| BUCKET SORT | MERGE SORT |
| COUNTING SORT | SELECTION SORT |
| HEAPSORT | QUICK SORT |
| INSERTION SORT | SHELL'S SORT |

* SORTING IS ONE OF THE MOST POPULAR PROBLEMS IN COMPUTER SCIENCE.

# INSERTION SORT

* ALGORITHM (PSEUDOCODE)

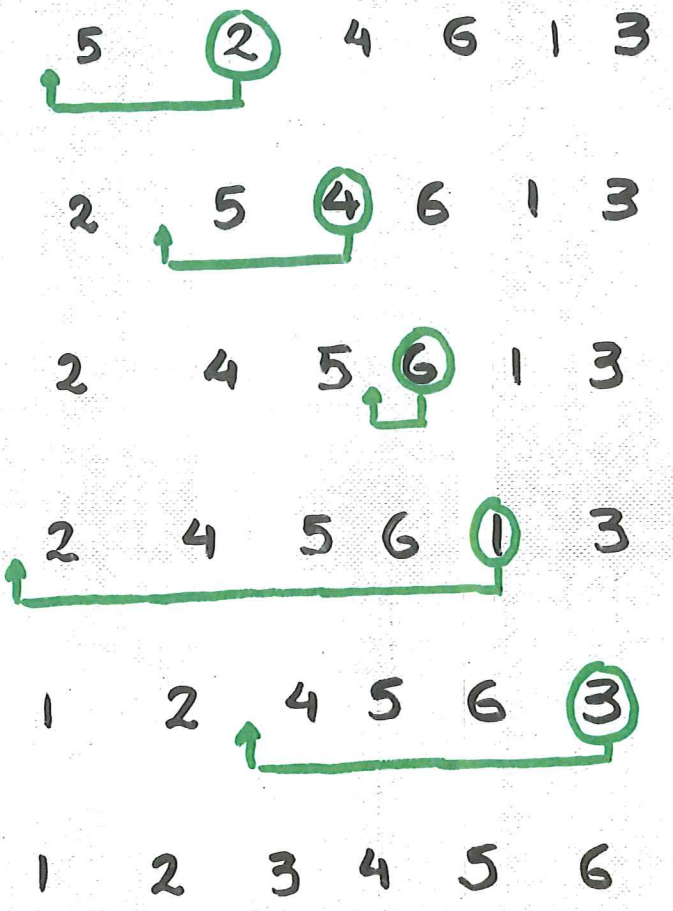
```
INSERTION_SORT(A)  {A IS THE ARRAY OF NUMBERS}
    for j ← 2 to length(A) do
        key ← A[j]
                          {INSERT A[j] INTO THE}
                          {SORTED SEQUENCE A[1..j-1]}
        i ← j-1
        while i>0 and A[i]>key do
            A[i+1] ← A[i]
            i ← i-1
        A[i+1] ← key
```

* BEST CASE

* WORST CASE

# INSERTION SORT

* EXAMPLE




* THE BEST CASE OCCURS WHEN THE ARRAY IS ALREADY SORTED.

* THE WORST CASE OCCURS WHEN THE ARRAY IS IN REVERSE SORTED ORDER.

# INSERTION SORT

* ANALYSIS OF THE ALGORITHM

| ALGORITHM | COST | TIMES |
|---|---|---|
| INSERTION_SORT(A) | | |
| for $j \leftarrow 2$ to length(A) do | $c_1$ | $n$ |
| $key \leftarrow A[j]$ | $c_2$ | $n-1$ |
| $i \leftarrow j-1$ | $c_3$ | $n-1$ |
| while $i>0$ and $A[i]>key$ do | $c_4$ | $\sum_{j=2}^{n} t_j$ |
| $A[i+1] \leftarrow A[i]$ | $c_5$ | $\sum_{j=2}^{n} (t_j-1)$ |
| $i \leftarrow i-1$ | $c_6$ | $\sum_{j=2}^{n} (t_j-1)$ |
| $A[i+1] \leftarrow key$ | $c_7$ | $n-1$ |

$t_j$: THE NUMBER OF TIMES THE WHILE LOOP TEST IS EXECUTED FOR THAT VALUE OF $j$

RUNNING TIME:

$$T(n) = c_1 n + c_2(n-1) + c_3(n-1) + c_4 \sum_{j=2}^{n} t_j + c_5 \sum_{j=2}^{n} (t_j-1) + c_6 \sum_{j=2}^{n} (t_j-1) + c_7(n-1)$$

# INSERTION SORT

## * ANALYSIS OF THE ALGORITHM

### - ARITHMETIC SERIES

$$\sum_{j=1}^{n} j = \frac{1}{2} n(n+1) \qquad \sum_{j=2}^{n} j = \frac{1}{2}(n+1)n - 1$$

$$\sum_{j=2}^{n} (j-1) = \frac{n(n-1)}{2}$$

### - BEST CASE ($t_j = 1$ for $j = 2, 3, \ldots, n$)

$$T(n) = c_1 n + c_2(n-1) + c_3(n-1) + c_4(n-1) + c_7(n-1)$$

$$= (c_1 + c_2 + c_3 + c_4 + c_7)n - (c_2 + c_3 + c_4 + c_7)$$

$$= an + b \quad \text{(LINEAR FUNCTION OF } n\text{)}$$

### - WORST CASE ($t_j = j$ for $j = 2, 3, \ldots, n$)

$$T(n) = c_1 n + c_2(n-1) + c_3(n-1) + c_4\left(\frac{n(n+1)}{2} - 1\right)$$

$$+ c_5\left(\frac{n(n-1)}{2}\right) + c_6\left(\frac{n(n-1)}{2}\right) + c_7(n-1)$$

$$= \left(\frac{c_4}{2} + \frac{c_5}{2} + \frac{c_6}{2}\right)n^2 + \left(c_1 + c_2 + c_3 + \frac{c_4}{2} - \frac{c_5}{2} - \frac{c_6}{2} + c_7\right)n$$

$$- (c_2 + c_3 + c_4 + c_7)$$

$$= an^2 + bn + c \quad \text{(QUADRATIC FUNCTION OF } n\text{)}$$

- AVERAGE CASE IS OFTEN ROUGHLY AS BAD AS THE WORST CASE

# BUCKET SORT

* ALGORITHM (PSEUDOCODE)

```
BUCKET_SORT(A)   {A[i]<1 AND A[i]>=0}
    n ← length(A)
    for i←1   to  n  do
        insert  A[i] into list  B[⌊n A[i]⌋]
    for i←0  to  n-1  do
        sort list  B[i] with insertion sort
    concatenate the lists B[0], B[1], ..., B[n-1]
    together in order
```

* EXAMPLE

| | A |
|---|---|
| 1 | .78 |
| 2 | .17 |
| 3 | .39 |
| 4 | .26 |
| 5 | .72 |
| 6 | .94 |
| 7 | .21 |
| 8 | .12 |
| 9 | .23 |
| 10 | .68 |

| | B |
|---|---|
| 0 | / |
| 1 | → .12 → .17 |
| 2 | → .21 → .23 → .26 |
| 3 | → .39 |
| 4 | / |
| 5 | / |
| 6 | → .68 |
| 7 | → .72 → .78 |
| 8 | / |
| 9 | → .94 |

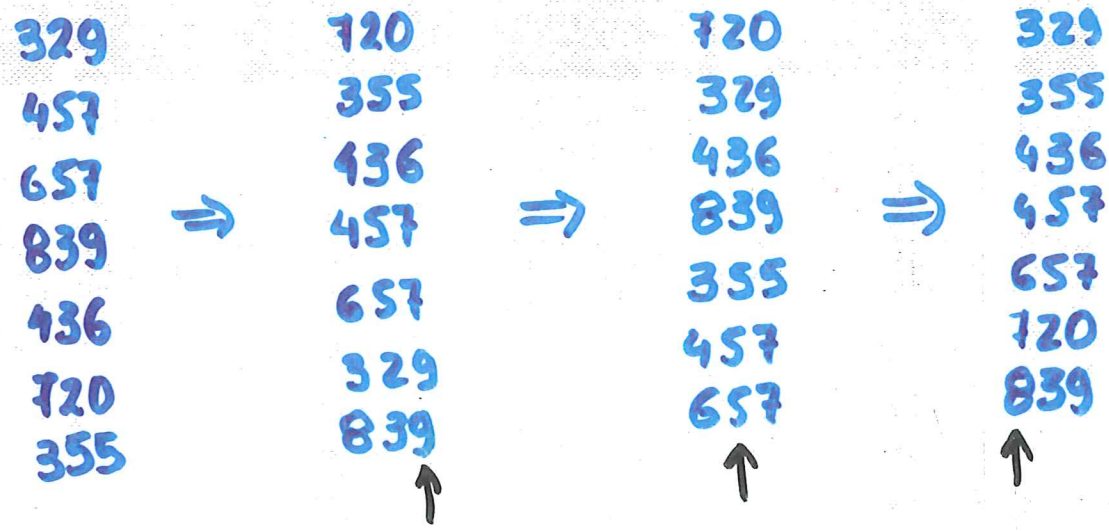
# RADIX SORT

* ALGORITHM (PSEUDOCODE)

```
RADIX_SORT(A,d)   {d: NUMBER OF DIGITS}
    for i ← 1 to d do
        use a stable sort to sort array A
        on digit i
```

- 1 IS THE LOWEST-ORDER DIGIT
- d IS THE HIGHEST-ORDER DIGIT

* EXAMPLE




* COMPLEXITY DEPENDS ON THE INTERMEDIATE SORTING ALGORITHM

# MERGESORT

* ALGORITHM (PSEUDOCODE)

```
M.SORT (left, right)
  if  right - left = 1  then
      if  A[left] > A[right] then swap(A[left], A[right])
  else if left ≠ right then
      middle ← ⌈1/2 (left + right)⌉
      M.SORT(left, middle - 1)
      M.SORT(middle, right)
      i ← left
      j ← middle
      k ← 0
      while (i ≤ middle-1) and (j ≤ right) do
          k ← k+1
          if A[i] ≤ A[j] then
              TEMP[k] ← A[i]
              i ← i+1
          else
              TEMP[k] ← A[j]
              j ← j+1
      if j > right then
          for t ← 0 to middle-1-i do
              A[right-t] ← A[middle-1-t]
      for t ← 0 to k-1 do
          A[left+t] ← TEMP[t]
```

# MERGESORT

* EXAMPLE

# MERGESORT

* COMPLEXITY

$T(n)$: NUMBER OF COMPARISONS THAT MERGESORT REQUIRES IN THE WORST CASE

n IS POWER OF 2

$$T(2n) = 2T(n) + O(n) \qquad T(2) = 1$$

$$\Downarrow$$

$$T(n) = O(n \log n)$$

* GENERAL DIVIDE AND CONQUER RELATION

$$T(n) = aT(n/b) + cn^k \qquad \left( \begin{matrix} n = b^m \\ T(1) = c \end{matrix} \right)$$

$$\Downarrow$$

$$T(n) = c \sum_{i=0}^{m} a^{m-i} b^{ik} = c a^m \sum_{i=0}^{m} \left( \frac{b^k}{a} \right)^i$$

$$T(n) = \begin{cases} O(n^{\log_b a}) & \text{if } a > b^k \\ O(n^k \log n) & \text{if } a = b^k \\ O(n^k) & \text{if } a < b^k \end{cases}$$

$a \geq 1$, $b \geq 2$, $c$ AND $k$ POSITIVE CONSTANTS

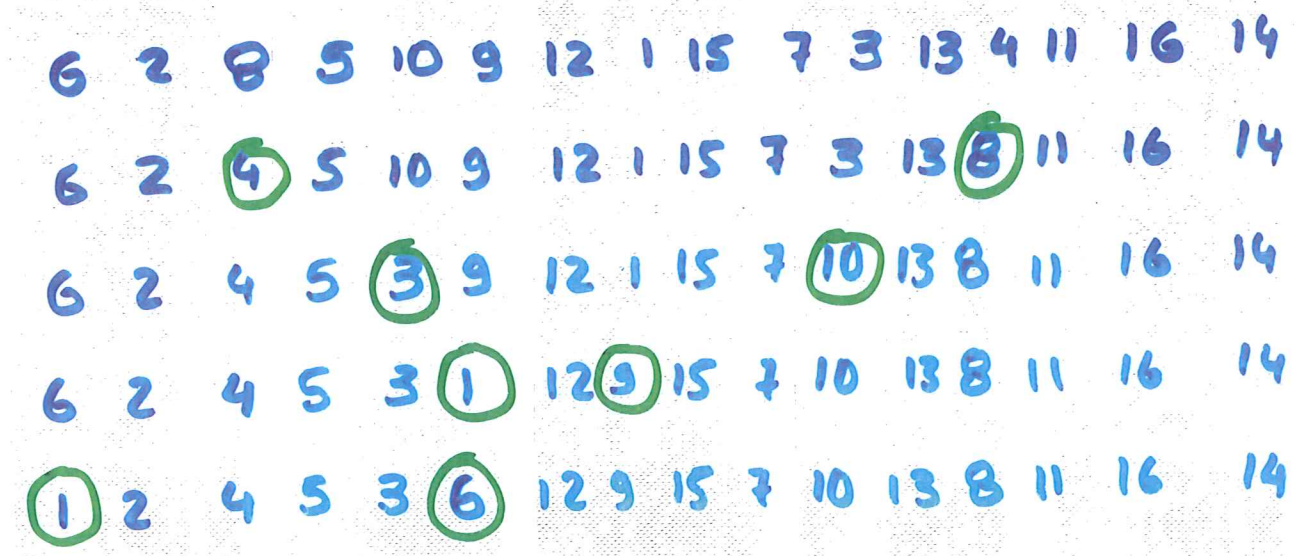
# QUICKSORT

* ALGORITHM (PSEUDOCODE)

```
Q_SORT(left,right)
    if left < right then
        PARTITION(A,left,right)
        Q_SORT(left, middle-1)
        Q_SORT(middle+1,right)

PARTITION(A,left,right) {IT SELECTS middle AND CHANGES A}
    pivot ← A[left]
    l ← left
    r ← right
    while l < r do
        while A[l]≤pivot and l≤right do
            l ← l+1
        while A[r]>pivot and r≥left do
            r ← r-1
        if l<r then
            exchange A[l] and A[r]
    middle ← r
    exchange A[left] with A[middle]
```

# QUICKSORT

## ↓ EXAMPLE OF PARTITIONING




## ↓ EXAMPLE OF THE ALGORITHM

# QUICKSORT

## * WORST-CASE PARTITIONING

THIS HAPPENS WHEN THE INPUT ARRAY IS COMPLETELY SORTED.

$$T(n) = T(n-1) + O(n) \quad \text{AND} \quad T(1) = O(1)$$

$$\Downarrow$$

$$T(n) = T(n-1) + O(n) = \sum_{k=1}^{n} O(k) = O\left(\sum_{k=1}^{n} k\right)$$

$$= O(n^2)$$

## * BEST-CASE PARTITIONING

THIS HAPPENS WHEN THE PARTITIONING PROCEDURE PRODUCES TWO REGIONS OF SIZE $n/2$.

$$T(n) = 2T(n/2) + O(n) \quad \text{AND} \quad T(2) = 1$$

$$\Downarrow$$

$$T(n) = O(n \log n)$$

## * BALANCED PARTITIONING

THE AVERAGE CASE RUNNING TIME OF QUICKSORT IS MUCH CLOSER TO THE BEST CASE THAN TO THE WORST CASE.

# QUICKSORT

* RANDOMIZED VERSIONS OF QUICKSORT

  - THE PIVOT IS SELECTED RANDOMLY.
  - EACH OF THE A[i] HAS THE SAME PROBABILITY OF BEING SELECTED (ASSUMPTION)

RUNNING TIME:

$$T(n) = n-1 + T(i-1) + T(n-i)$$ (ith SMALLEST ELEMENT IS THE PIVOT)

AVERAGE RUNNING TIME:

$$T(n) = n-1 + \frac{1}{n}\sum_{i=1}^{n}(T(i-1)+T(n-i))$$

$$= n-1 + \frac{2}{n}\sum_{i=0}^{n-1}T(i)$$

$$\Downarrow$$

$$T(n) = O(n\log n)$$

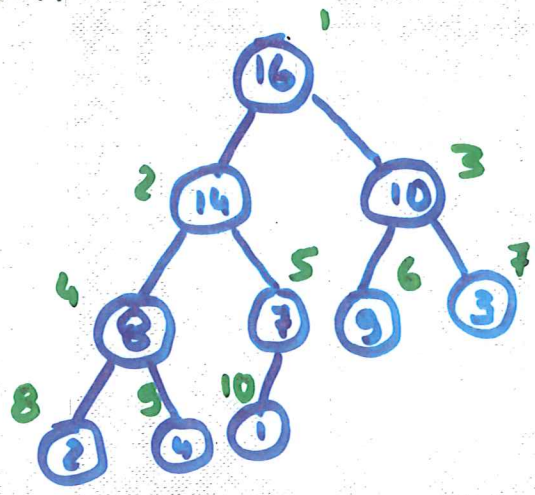
* PIVOT IS STORED IN A REGISTER (SPEED ISSUE).

* SELECTION OF THE BASE OF INDUCTION.

* SEVERAL VERSIONS OF QUICKSORT.

# HEAPSORT

* HEAP




| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|
| 16 | 14 | 10 | 8 | 7 | 9 | 3 | 2 | 4 | 1 |

* HEAP PROPERTY

- FOR EVERY NODE i OTHER THAN THE ROOT

  A[PARENT(i)] ≥ A[i]

* CODE

```
PARENT(i)
  return ⌊i/2⌋
LEFT(i)
  return 2i
RIGHT(i)
  return 2i+1
```

* THE HEIGHT OF A HEAP WITH n ELEMENTS IS $\Theta(\log n)$.

# HEAPSORT

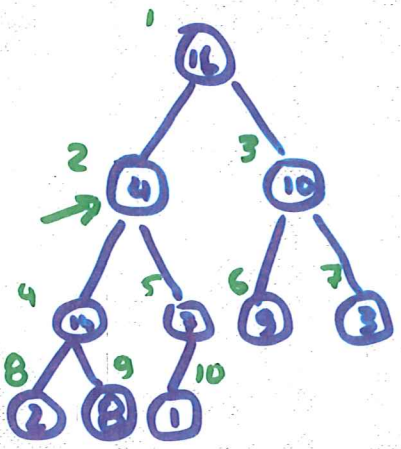
* ALGORITHM FOR HEAPIFY (PSEUDOCODE)

```
HEAPIFY(A,i)        {THE INPUTS ARE AN ARRAY}
                    {A AND AN INDEX i INTO THE}
                    {ARRAY.}
l ← LEFT(i)         {THE BINARY TREES ROOTED}
r ← RIGHT(i)        {AT LEFT(i) AND RIGHT(i) ARE}
                    {ASSUMED TO BE HEAPS.}
if l ≤ heap-size(A) and A[l] > A[i] then
    largest ← l
else
    largest ← i
if r ≤ heap-size(A) and A[r] > A[largest] then
    largest ← r
if largest ≠ i then
    exchange (A[i], A[largest])
    HEAPIFY(A, largest)
```

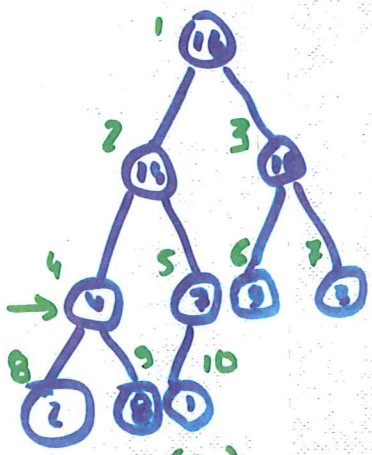
- heap-size(A): THE NUMBER OF ELEMENTS IN THE HEAP STORED WITHIN ARRAY A

# HEAPSORT

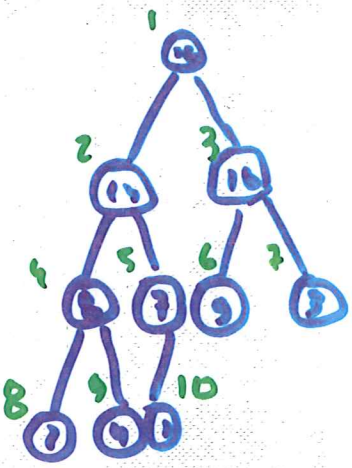
* EXAMPLE OF HEAPIFY



(a)



(b)



(c)

* RUNNING TIME OF HEAPIFY

$$T(n) = O(\log n)$$

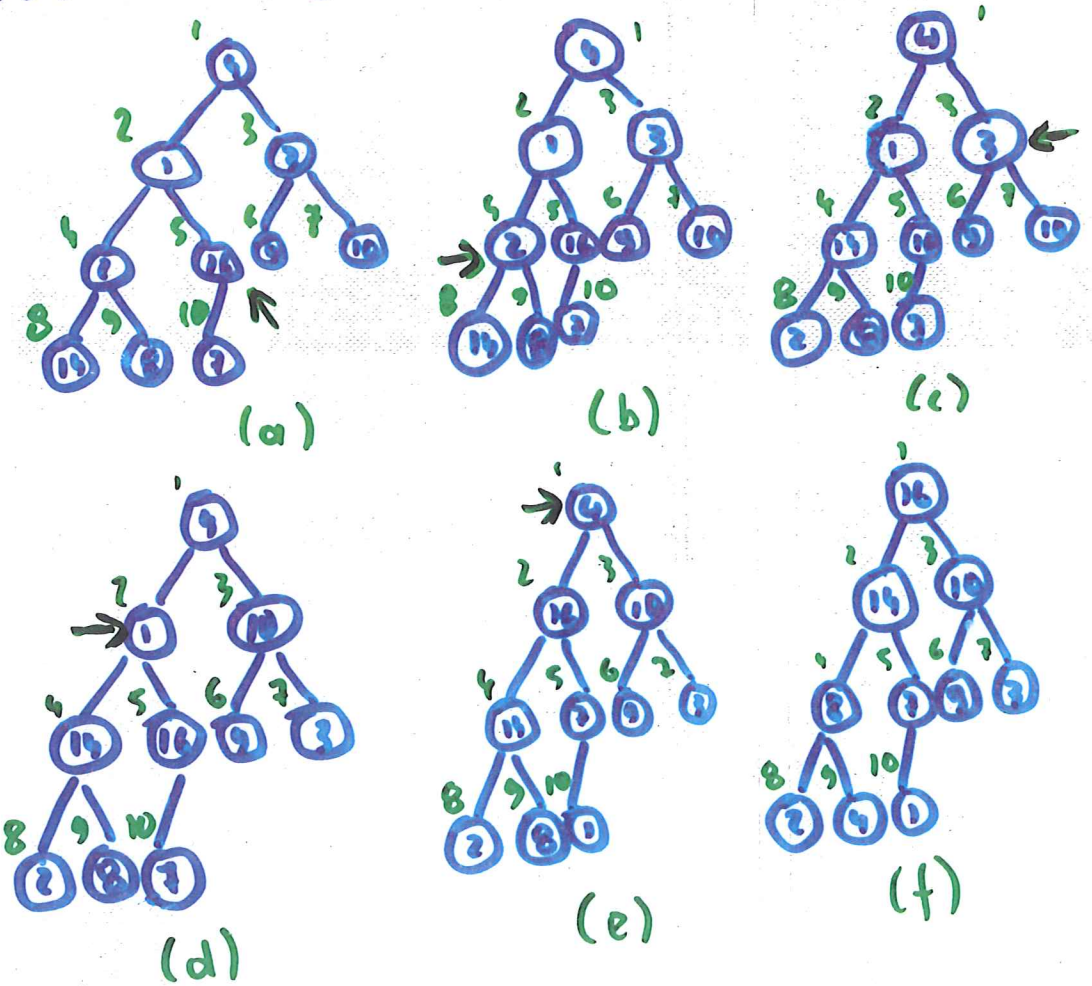
# HEAPSORT

## * BUILDING A HEAP

```
BUILD_HEAP(A)
    heap_size(A) ← length(A)
    for i ← ⌊length(A)/2⌋ downto 1 do
        HEAPIFY(A,i)
```

## * EXAMPLE OF BUILDING A HEAP

A → 4 1 3 2 16 9 10 14 8 7



# HEAPSORT

* ALGORITHM (PSEUDOCODE)

```
HEAPSORT(A)
    BUILD_HEAP(A)
    for i ← length(A) downto 2 do
        exchange(A[1], A[i])
        heap-size(A) ← heap-size(A)-1
        HEAPIFY(A, 1)
```

* THE HEAPSORT ALGORITHM TAKES TIME $O(n \log n)$ SINCE THE CALL TO BUILD_HEAP TAKES TIME $O(n)$ AND EACH OF THE $n-1$ CALLS TO HEAPIFY TAKES TIME $O(\log n)$.

* HEAPS ARE USEFUL FOR THE IMPLEMENTATION OF PRIORITY QUEUES.

# A LOWER BOUND FOR SORTING

* A LOWER BOUND FOR A PARTICULAR PROBLEM IS A PROOF THAT NO ALGORITHM CAN SOLVE THE PROBLEM BETTER.

* DECISION TREES CAN HELP.

* DECISION TREES ARE DEFINED AS BINARY TREES WITH TWO TYPES OF NODES (INTERNAL NODES AND LEAVES).

* EACH INTERNAL NODE IS ASSOCIATED WITH A QUERY.

* EACH LEAF IS ASSOCIATED WITH A POSSIBLE OUTPUT.

* THE WORST-CASE RUNNING TIME IS ASSOCIATED WITH THE HEIGHT OF THE TREE.

* EVERY DECISION TREE ALGORITHM FOR SORTING HAS HEIGHT $\Omega(n \log n)$.

# FINDING THE kth-SMALLEST ELEMENT

* PROBLEM

GIVEN A SEQUENCE S OF n ELEMENTS, AND AN INTEGER k SUCH THAT $1 \le k \le n$, FIND THE kth-SMALLEST ELEMENT IN S.

* ALGORITHM (PSEUDOCODE)

```
SELECT_K(left, right, k)   {IT RETURNS THE kth-}
                           {SMALLEST ELEMENT}
    if left=right then
        return left
    else
        PARTITION(A, left, right)   {IT IS SIMILAR TO}
                                    {THE PARTITION OF}
                                    {QUICKSORT}
        if middle-left+1 ≥ k then
            temp ← SELECT_K(left, middle, k)
        else
            temp ← SELECT_K(middle+1, right,
                            k-(middle-left+1))
        return temp
```

# DATA COMPRESSION

* DATA COMPRESSION IS AN IMPORTANT TECHNIQUE FOR SAVING STORAGE.

* EACH OF THE 26 ENGLISH CHARACTERS IS REPRESENTED BY A UNIQUE STRING OF BITS, CALLED THE ENCODING OF THE CHARACTER.

* EXAMPLE

  ASCII: A ↔ 1000001, B ↔ 1000010, ETC.

* PREFIX CONSTRAINT.

* PROBLEM

  GIVEN A TEXT, FIND AN ENCODING FOR THE CHARACTERS THAT SATISFIES THE PREFIX CONSTRAINT AND THAT MINIMIZES THE TOTAL NUMBER OF BITS NEEDED TO ENCODE THE TEXT.

# DATA COMPRESSION

* LENGTH OF THE FILE F COMPRESSED BY USING ENCODING E IS:

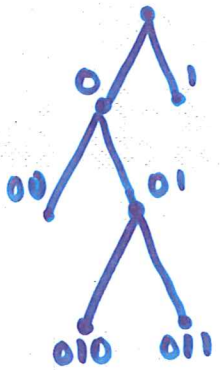
$$L(E,F) = \sum_{i=1}^{n} s_i f_i$$

$n$: NUMBER OF CHARACTERS

$s_i$: LENGTH OF THE BIT STRING THAT REPRESENTS THE CHARACTER $C_i$

$f_i$: FREQUENCY OF THE CHARACTER $C_i$

* TREE REPRESENTATION OF ENCODING




* HUFFMAN INVENTED A GREEDY ALGORITHM THAT CONSTRUCTS AN OPTIMAL PREFIX CODE CALLED A HUFFMAN CODE.

# DATA COMPRESSION

* HUFFMAN'S ALGORITHM (PSEUDOCODE)

```
HUFFMAN_ENCODING(S,f)   { S IS A STRING OF }
                        { CHARACTERS }
                        { f IS AN ARRAY OF }
                        { FREQUENCIES }
{ INSERT ALL CHARACTERS INTO A HEAP H ACCORDING }
{ TO THEIR FREQUENCIES }
    while H is not empty do
        if H contains only one character A then
            make A the root of the tree
        else
            pick two characters A and B with lowest
            frequencies and delete them from H.
            replace A and B with a new character C
            whose frequency is the sum of frequencies
            of A and B.
            insert C to H.
            make A and B children of C in the tree.
```

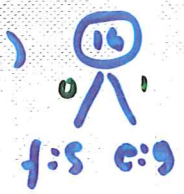
# DATA COMPRESSION

## * EXAMPLE OF THE HUFFMAN'S ALGORITHM

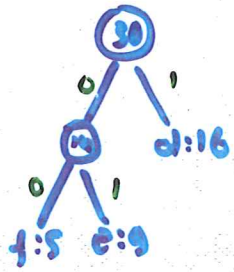
FREQUENCY

(a) f:5 e:9 c:12 b:13 d:16 a:45

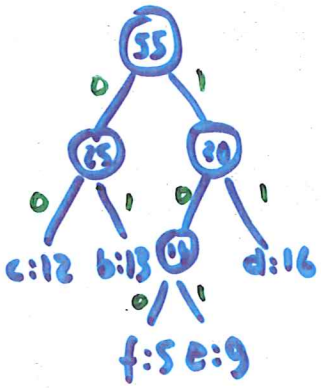
(b) c:12 b:13  d:16 a:45

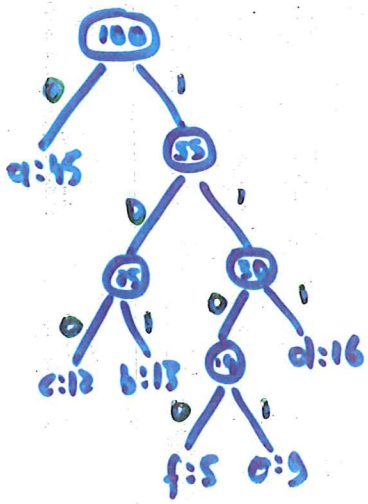

(c)  d:16  a:45



(d)   a:45



(e) a:45 


(f) 

# STRING MATCHING

* PROBLEM

GIVEN TWO STRINGS A AND B, FIND THE FIRST OCCURENCE(IF ANY) OF B IN A.

* STRAIGHTFORWARD STRING MATCHING

A: abaababababbababababbababaaa

B: ababbababaaa

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23

a b a a b a b a b b a b a b a b b a b a b a a

1: a b a b . . . . . .
2: a . . . . . .
3: a b . . . . . .
4: a b a b b . . . . . .
5: a . . . . . .
6: a b a b b a b a b a a
7: a . . . . . .
8: a b a . . . . . .
9: a . . . . . .
10: a . . . . . .
11: a b a b b . . . . . .
12: a . . . . . .
13: a b a b b a b a b a a

# STRING MATCHING

* A BETTER ALGORITHM.

* THE PREPROCESSING OF B IS THE ESSENCE OF THE IMPROVED ALGORITHM.

* ALGORITHM FOR PREPROCESSING OF B (PSEUDOCODE)

```
COMPUTE_NEXT (B,m)   {IF ASSIGNS VALUES TO THE}
                     {ELEMENTS OF THE ARRAY B}
    next[1] ← -1
    next[2] ← 0
    for i ← 3 to m do
        j ← next[i-1]+1
        while b[i-1] ≠ b[j] and j>0 do
            j ← next[j]+1
        next[i] ← j
```

* EXAMPLE

| i | : | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| B | : | a | b | a | b | b | a | b | a | b | a | a |
| next | : | -1 | 0 | 0 | 1 | 2 | 0 | 1 | 2 | 3 | 4 | 3 |

# STRING MATCHING

* MAIN ALGORITHM (PSEUDOCODE)

```
STRING-MATCH(A,n,B,m)
                    {IT RETURNS THE INDEX}
                    {START}
  j←1
  i←1
  START←0
  while START=0 AND i≤n do
     if B[j]=A[i] then
           j←j+1
           i←i+1
     else
           j←next[j]+1
           if  j=0   then
              j←1
              i←i+1
     if j=m+1  then  START=i-m
```

* THIS ALGORITHM IS KNOWN AS THE KNUTH, MORRIS, AND PRATT (KMP) ALGORITHM.

* THE KMP ALGORITHM RUNS IN TIME $O(m+n)$.

# SEQUENCE COMPARISONS

* PROBLEM

  FIND THE MINIMUM NUMBER OF EDIT STEPS (CHANGES) REQUIRED TO CHANGE ONE STRING INTO ANOTHER.

* MANY APPLICATIONS TO PROBLEMS IN MOLECULAR BIOLOGY.

* MAIN TECHNIQUE USED IS DYNAMIC PROGRAMMING.

* DYNAMIC PROGRAMMING IS USED IN CASES WHERE THE SOLUTION OF THE PROBLEM DEPENDS ON MANY SOLUTIONS OF SLIGHTLY SMALLER PROBLEMS.

* THE RUNNING TIME OF THE ALGORITHM THAT PROVIDES A SOLUTION IS $O(nm)$ ($n$ IS THE SIZE OF STRING A AND $m$ IS THE SIZE OF STRING B).

* THE DYNAMIC PROGRAMMING IS GENERALLY LESS EFFICIENT THAN THE DIVIDE-AND-CONQUER APPROACH.

# FINDING A MAJORITY

* PROBLEM

  GIVEN A SEQUENCE OF NUMBERS, FIND THE MAJORITY IN THE SEQUENCE OR DETERMINE THAT NONE EXISTS.

* THE MULTIPLICITY OF $a$ IN THE SEQUENCE A IS THE NUMBER OF TIMES $a$ APPEARS IN A.

* A NUMBER z IS A MAJORITY IN A IF ITS MULTIPLICITY IS GREATER THAN $n/2$ ($n$ IS THE NUMBER OF THE ELEMENTS IN THE SEQUENCE).

* SORTING IS A STRAIGHT FORWARD SOLUTION.

* THE FOLLOWING OBSERVATION MAY HELP:

  IF $a_i \neq a_j$ AND WE ELIMINATE BOTH OF THESE ELEMENTS FROM THE LIST, THEN THE MAJORITY IN THE ORIGINAL LIST REMAINS A MAJORITY IN THE NEW LIST.

# FINDING A MAJORITY

* ALGORITHM (PSEUDOCODE)

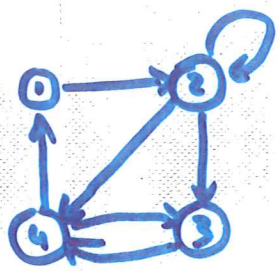
```
MAJORITY (A,n)    { IT RETURNS THE MAJORITY,
                    {IN A IF IT EXISTS, OR -1 }
                    {OTHERWISE.}

 c ← A[1]
 m ← 1

 for i←2 to n do
     if m=0 then
         c ← A[i]
         m ← 1
     else
         if c=A[i]  then m ← m+1
         else m ← m-1

 if m=0    then  maj ← -1
 else
     COUNT←0
     for i←1  to n do
         if A[i]=c then COUNT ← COUNT+1
     if COUNT > n/2 then maj←c
     else maj←-1
 return  maj
```

# GRAPHS

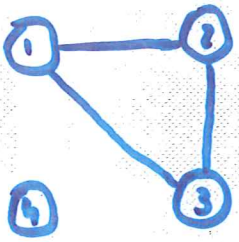
# GRAPHS

* A GRAPH $G=(V,E)$ CONSISTS OF A SET V OF VERTICES (ALSO CALLED NODES), AND A SET E OF EDGES.

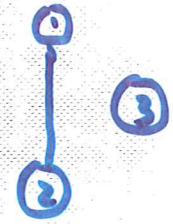
* EXAMPLES




(a)




(b)




(c)

* A GRAPH CAN BE DIRECTED, OR UNDIRECTED.

* THE EDGES IN A DIRECTED GRAPH ARE ORDERED PAIRS.

* THE EDGES IN A UNDIRECTED GRAPH ARE UNORDERED PAIRS.

* TREES ARE SIMPLE EXAMPLES OF GRAPHS.

* A MULTIGRAPH IS A GRAPH WITH POSSIBLY SEVERAL EDGES BETWEEN THE SAME PAIR OF VERTICES.

# GRAPHS

* THE DEGREE $d(v)$ OF A VERTEX $v$ IS THE NUMBER OF EDGES INCIDENT TO $v$.

* IN A DIRECTED GRAPH, WE HAVE THE INDEGREE AND OUTDEGREE OF A VERTEX $v$.

* THE INDEGREE OF A VERTEX $v$ IS THE NUMBER OF EDGES FOR WHICH $v$ IS THE HEAD.

* THE OUTDEGREE OF A VERTEX $v$ IS THE NUMBER OF EDGES FOR WHICH $v$ IS THE TAIL.

* IF $(u,v)$ IS AN EDGE IN A GRAPH $G=(V,E)$, WE SAY THAT VERTEX $v$ IS ADJACENT TO VERTEX $u$.

* A PATH OF LENGTH $k$ FROM A VERTEX $u$ TO A VERTEX $u'$ IN A GRAPH $G=(V,E)$ IS A SEQUENCE $\langle v_0, v_1, \ldots, v_k \rangle$ OF VERTICES SUCH THAT $u=v_0$, $u'=v_k$, AND $(v_{i-1}, v_i) \in E$ FOR $i=1,2,\ldots,k$.

* A PATH IS CALLED SIMPLE IF EACH VERTEX APPEARS IN IT AT MOST ONCE.

# GRAPHS

* VERTEX $u$ IS SAID TO BE REACHABLE FROM VERTEX $v$ IF THERE IS A PATH FROM $v$ TO $u$.

* A CIRCUIT IS A PATH WHOSE FIRST AND LAST VERTICES ARE THE SAME.

* A CIRCUIT IS CALLED SIMPLE (CYCLE) IF, EXCEPT FOR THE FIRST AND LAST VERTICES, NO VERTEX APPEARS MORE THAN ONCE.

* A GRAPH IS CALLED CONNECTED IF (IN ITS UNDIRECTED FORM) THERE IS A PATH FROM ANY VERTEX TO ANY OTHER VERTEX.

* A FOREST IS A GRAPH THAT DOES NOT CONTAIN A CYCLE (UNDIRECTED FORM OF THE GRAPH).

* A SUBGRAPH OF THE GRAPH $G=(V,E)$ IS A GRAPH $G'=(V', E')$ SUCH THAT $V' \subseteq V$ AND $E' \subseteq E$.

# GRAPHS

* A CONNECTED, ACYCLIC, UNDIRECTED GRAPH IS CALLED (FREE) TREE.

* A BIPARTITE GRAPH IS AN UNDIRECTED GRAPH $G=(V,E)$ IN WHICH $V$ CAN BE PARTITIONED INTO TWO SETS $V_1$ AND $V_2$ SUCH THAT $(u,v) \in E$ IMPLIES EITHER $u \in V_1$ AND $v \in V_2$ OR $u \in V_2$ AND $v \in V_1$. IN OTHER WORDS, ALL EDGES GO BETWEEN THE TWO SETS $V_1$ AND $V_2$.

* A DIRECTED ACYCLIC GRAPH IS CALLED DAG.

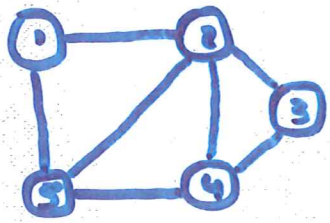
* A WEIGHTED GRAPH IS A GRAPH WITH WEIGHTS ASSOCIATED WITH THE EDGES.

* A CONNECTED GRAPH WHOSE ALL VERTICES HAVE EVEN DEGREES IS CALLED EULERIAN GRAPH.

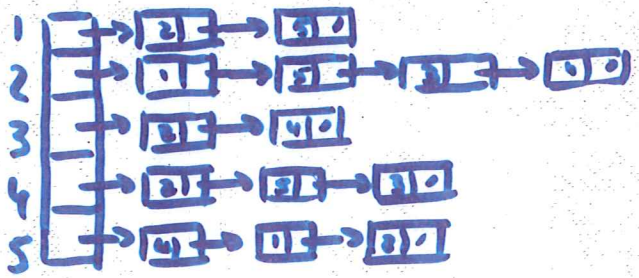
* DIFFERENT REPRESENTATIONS OF THE DIRECTED AND UNDIRECTED GRAPHS.

# GRAPHS

* TWO REPRESENTATIONS OF AN UNDIRECTED GRAPH.



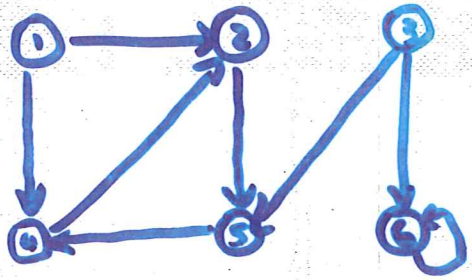
(A)



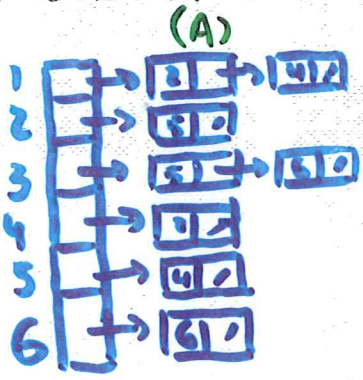
| | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| 1 | 0 | 1 | 0 | 0 | 1 |
| 2 | 1 | 0 | 1 | 1 | 1 |
| 3 | 0 | 1 | 0 | 1 | 0 |
| 4 | 0 | 1 | 1 | 0 | 1 |
| 5 | 1 | 1 | 0 | 1 | 0 |

(B)

* TWO REPRESENTATIONS OF A DIRECTED GRAPH.



(A)



| | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| 1 | 0 | 1 | 0 | 1 | 0 | 0 |
| 2 | 0 | 0 | 0 | 0 | 1 | 0 |
| 3 | 0 | 0 | 0 | 0 | 1 | 1 |
| 4 | 0 | 1 | 0 | 0 | 0 | 0 |
| 5 | 0 | 0 | 0 | 1 | 0 | 0 |
| 6 | 0 | 0 | 0 | 0 | 0 | 1 |

(B)

# GRAPHS

* GRAPH TRAVERSALS

  * DEPTH-FIRST SEARCH (DFS)

  * BREADTH-FIRST SEARCH (BFS)

* BREADTH-FIRST SEARCH (BFS)

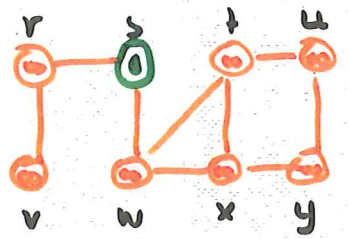
```
BFS(G,s)                                   {s IS THE SOURCE}
    for each vertex u ∈ V[G]-{s} do
        color[u] ← WHITE
        d[u] ← ∞                           {THE DISTANCE FROM}
        π[u] ← NIL                         {THE SOURCE s TO THE}
    color[s] ← GRAY                        {VERTEX u IS STORED}
    d[s] ← 0                               {IN d[u]}
    π[s] ← NIL
    Q ← {s}
    while Q ≠ ∅ do
        u ← head[Q]
        for each v ∈ Adj[u] do
            if color[v] = WHITE then
                color[v] ← GRAY
                d[v] ← d[u] + 1
                π[v] ← u
                ENQUEUE(Q,v)
        DEQUEUE(Q)
        color[u] ← BLACK
```

* THE COLOR OF EACH VERTEX $u \in V$ IS STORED IN color[u] AND THE PREDECESSOR IS STORED IN $\pi[u]$.

# GRAPHS

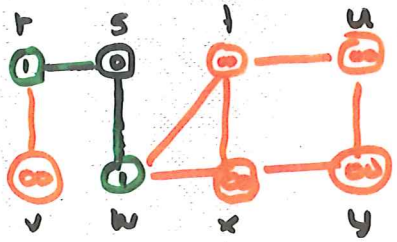
* EXAMPLE OF BFS

- WHITE → ○ - GRAY → ○ - BLACK → ○



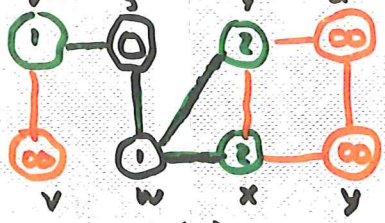

Q: [s] / 0

(a)



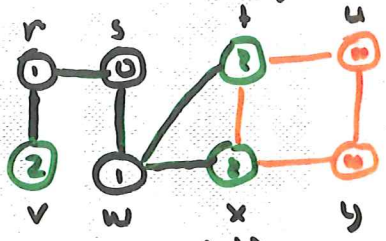

Q: [w | r] / 1 1

(b)



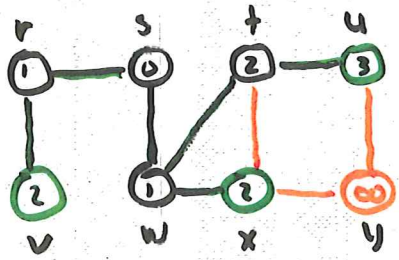

Q: [r | t | x] / 1 2 2

(c)



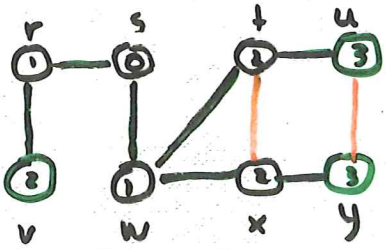

Q: [t | x | v] / 2 2 2

(d)



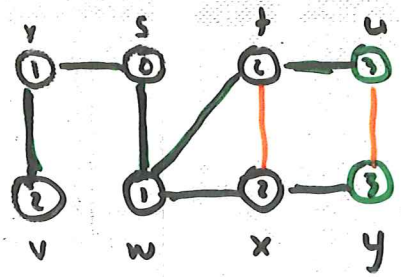

Q: [x | v | w] / 2 2 3

(e)



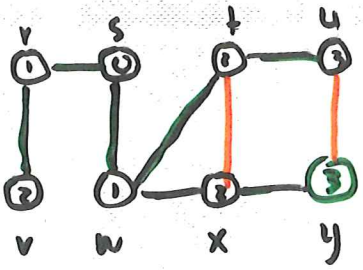

Q: [v | w | y] / 2 3 3

(f)



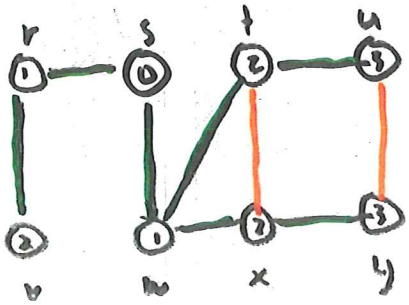

Q: [w | y] / 3 3

(g)




Q: [y] / 3

(h)




Q: ∅

(i)

# GRAPHS

* DEPTH-FIRST SEARCH (DFS)

- ALGORITHM (PSEUDOCODE)

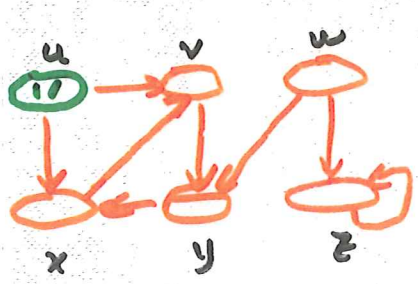
```
DFS (G)
    for each vertex u ∈ V[G] do
        color[u] ← WHITE
        π[u] ← NIL
    time ← 0
    for each vertex u ∈ V[G] do
        if color[u] = WHITE then
            DFS_VISIT(u)

DFS_VISIT(u)
    color[u] ← GRAY      {WHITE VERTEX u HAS JUS
    d[u] ← time ← time+1 {BEEN DISCOVERED.}
    for each v ∈ Adj[u] do {EXPLORE EDGE (u,v)}
        if color[v] = WHITE then
            π[v] ← u
            DFS_VISIT(v)
    color[u] ← BLACK     {BLACKEN u; IT IS FINISHED}
    f[u] ← time ← time+1
```

* THE TIME OF DISCOVERY OF THE VERTEX u IS STORED IN THE VARIABLE d[u]. THE END OF THE PROCESSING OF THE VERTEX u IS STORED IN THE VARIABLE f[u] (d[u] < f[u]).

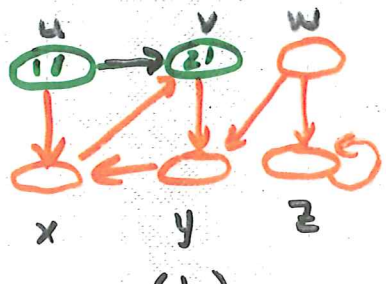
# GRAPHS

## * EXAMPLE OF THE DFS ALGORITHM

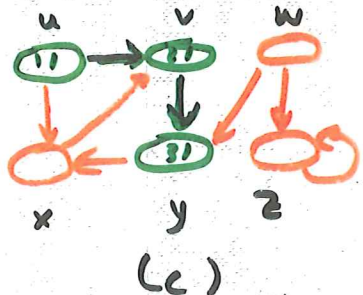
WHITE → —  GRAY → —  BLACK → —



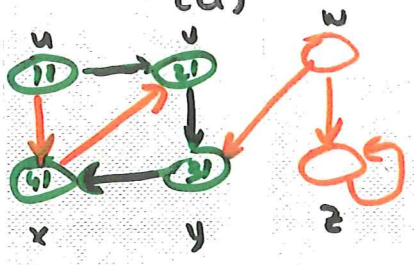

(a)



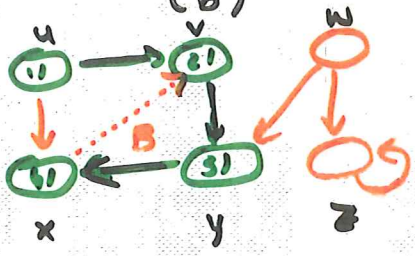

(b)



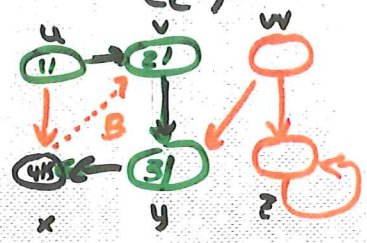

(c)



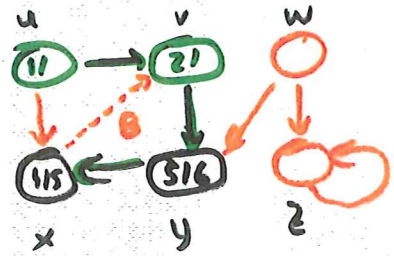

(d)



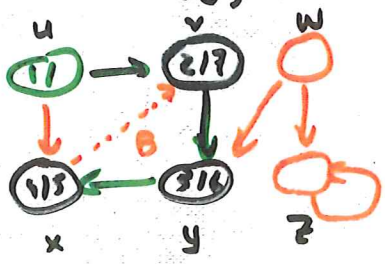

(e)



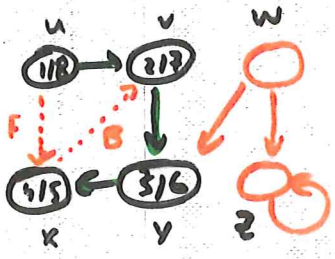

(f)



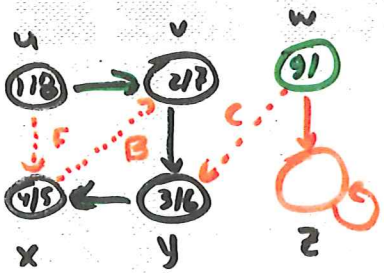

(g)



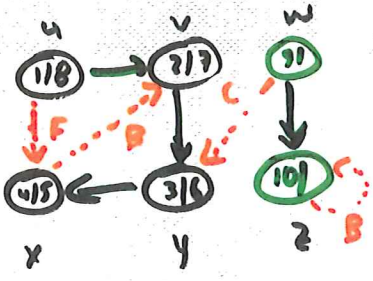

(h)



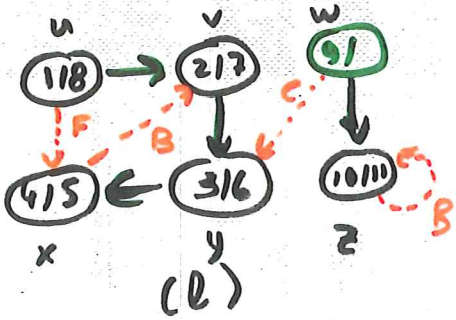

(i)



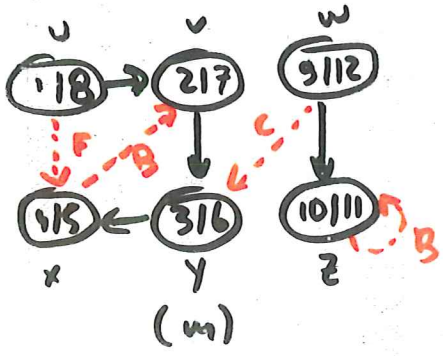

(j)




(k)




(l)

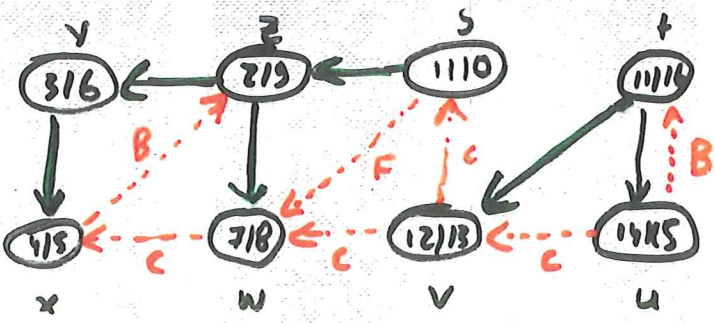



(m)

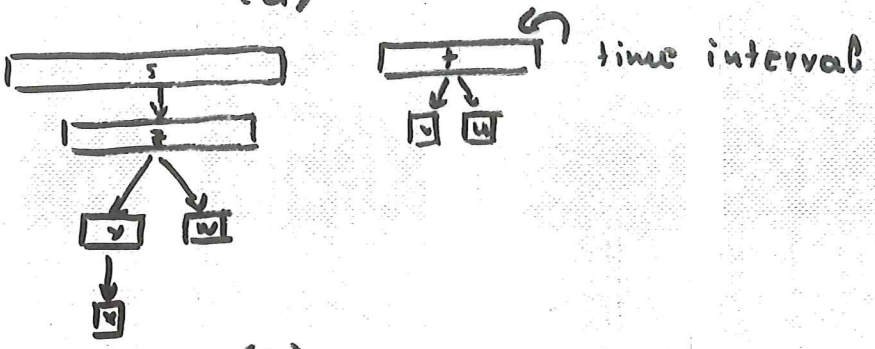
B: BACK EDGE
F: FORWARD EDGE
C: CROSS EDGE
1/2: IT IMPLIES $d[u] = 1$ AND $f[u] = 2$ FOR THE VERTEX u.

# GRAPHS

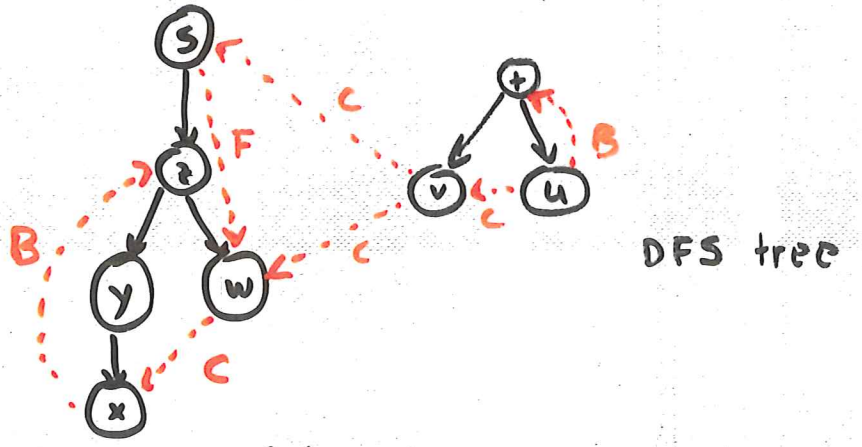
* ANOTHER EXAMPLE OF DFS




(a)




(b)




(c)

* THE RUNNING TIME OF DFS IS $\Theta(V+E)$.

* THE DFS ALGORITHM CAN BE MODIFIED TO CLASSIFY EDGES AS IT ENCOUNTERS THEM.

# GRAPHS

* DFS CAN BE USED TO FIND THE CONNECTED COMPONENTS OF A GRAPH.

* DFS CAN BE USED TO FIND CYCLES IN A GRAPH.

* DFS CAN BE USED FOR TOPOLOGICAL SORT.

* TOPOLOGICAL SORT OF A DAG $G=(V,E)$ IS A LINEAR ORDERING OF ALL ITS VERTICES SUCH THAT IF G CONTAINS AN EDGE $(u,v)$, THEN $u$ APPEARS BEFORE $v$ IN THE ORDERING.

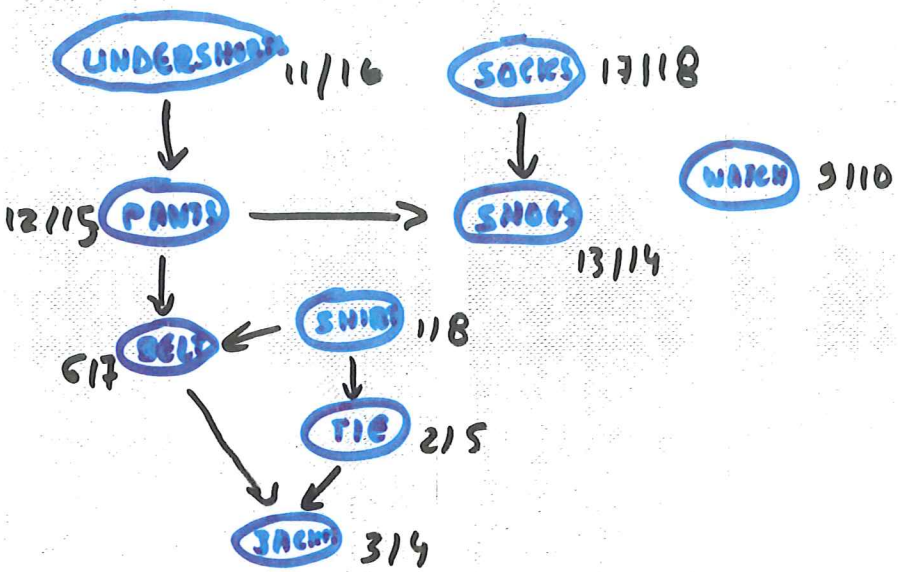
* TOPOLOGICAL SORT (PSEUDOCODE)

```
TOP_SORT(G)
  1. CALL DFS(G) TO COMPUTE FINISHING TIMES
  f[v] FOR EACH v.
  2. AS EACH VERTEX IS FINISHED, INSERT IT
  ONTO THE FRONT OF A LINKED LIST.
  3. RETURN THE LINKED LIST OF VERTICES.
```

# GRAPHS

* EXAMPLE OF TOPOLOGICAL SORT

PROFESSOR X TOPOLOGICALLY SORTS HIS CLOTHING WHEN GETTING DRESSED




(a)


(b)

* WE CAN PERFORM TOPOLOGICAL SORT IN TIME $\Theta(V+E)$, SINCE DEPTH-FIRST SEARCH TAKES $\Theta(V+E)$ TIME AND IT TAKES $O(1)$ TIME TO INSERT EACH OF THE $|V|$ VERTICES ONTO THE FRONT OF THE LINKED LIST.

# GRAPHS

* SINGLE-SOURCE SHORTEST PATHS

  - PROBLEM

    GIVEN A WEIGHTED, DIRECTED GRAPH $G=(V,E)$ AND A VERTEX $s$, FIND SHORTEST PATHS FROM $s$ TO ALL OTHER VERTICES OF $G$.

* SIMILAR PROBLEMS

  - SINGLE-DESTINATION SHORTEST-PATHS PROBLEM
  - SINGLE-PAIR SHORTEST-PATH PROBLEM
  - ALL-PAIRS SHORTEST-PATHS PROBLEM

* SEVERAL ALGORITHMS

  - DIJKSTRA'S ALGORITHM
  - BELLMAN-FORD ALGORITHM

* NEGATIVE-WEIGHT EDGES ARE POSSIBLE.

* THE USE OF RELAXATION IS POPULAR IN THESE ALGORITHMS.

# GRAPHS

* RELAXATION (PSEUDOCODE)

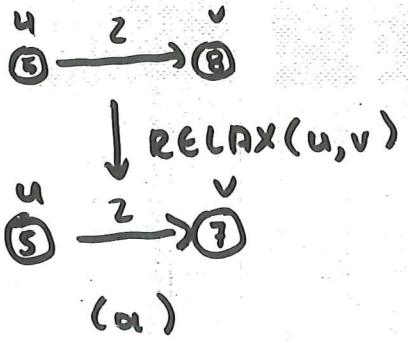
```
RELAX(u, v, w)
    if d[v] > d[u] + w(u,v) then
        d[v] ← d[u] + w(u,v)
        π[v] ← u
```

π[v]: v's PREDECESSOR FIELD

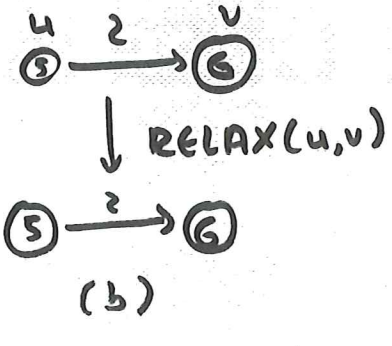
d[v]: SHORTEST-PATH ESTIMATE

w(u,v): WEIGHT

* RELAXATION OF AN EDGE (u,v)




(a)




(b)

* INITIALIZATION

```
INITIALIZE-SINGLE-SOURCE(G,s)
    for each vertex v ∈ V[G] do
        d[v] ← ∞
        π[v] ← NIL
    d[s] ← 0
```

# GRAPHS

* DIJKSTRA'S ALGORITHM (PSEUDOCODE)

```
DIJKSTRA(G, w, s)
    INITIALIZE_SINGLE_SOURCE(G,s)
    S ← ∅                    {S' IS A SET OF VERTICES WHOSE
    Q ← V[G]                 {FINAL SHORTEST-PATH WEIGHTS
                             {FROM THE SOURCE s HAVE ALREADY
                             {BEEN DETERMINED.}
    while Q ≠ 0 do
        u ← EXTRACT_MIN(Q)
        S' ← S' U {u}
        for each vertex v ∈ Adj[u] do
            RELAX(u,v,w)
```

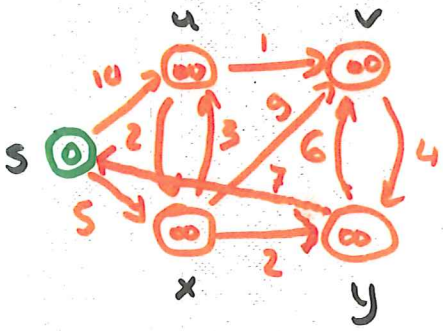
Q: PRIORITY QUEUE THAT CONTAINS ALL THE VERTICES IN $V-S'$, KEYED BY THEIR $d$ VALUES.

* DIJKSTRA'S ALGORITHM USES A GREEDY STRATEGY.

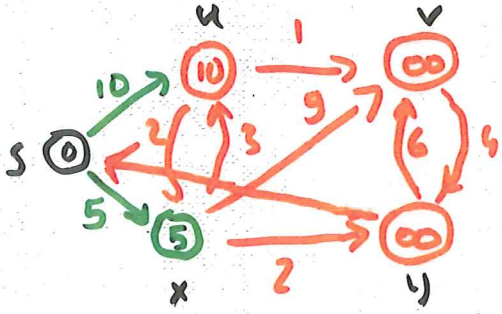
* THE RUNNING TIME IS $O(V^2+E) = O(V^2)$.

# GRAPHS

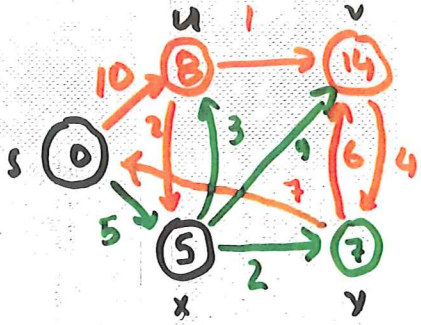
* EXAMPLE OF THE DIJKSTRA'S ALGORITHM



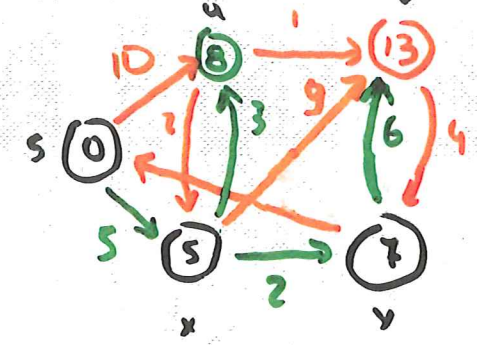

(a)



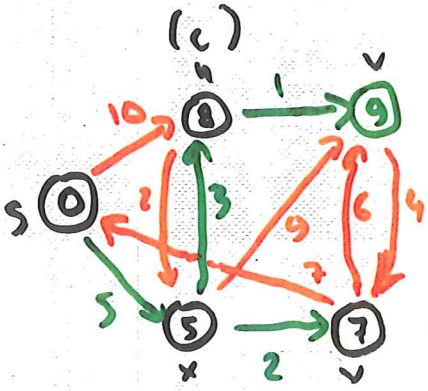

(b)



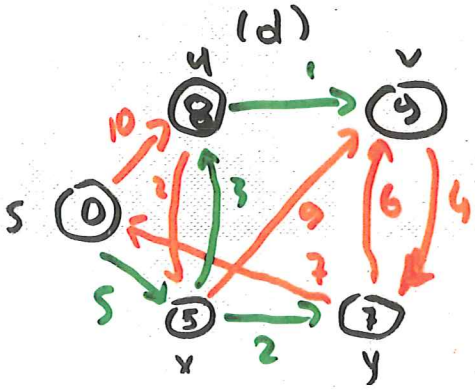

(c)




(d)




(e)




(f)

- IF EDGE $(u,v)$ IS GREEN, THEN $\pi[v] = u$.

- BLACK VERTICES ARE IN THE SET 'S'.

# GRAPHS

* BELLMAN-FORD ALGORITHM (PSEUDOCODE)

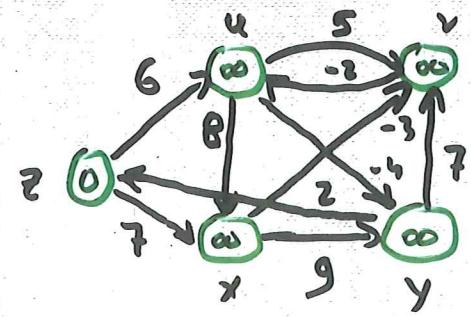
```
BELLMAN_FORD(G, w, s)
    INITIALIZE_SINGLE_SOURCE(G, s)
    for i ← 1 to |V[G] - 1| do
        for each edge (u,v) ∈ E[G] do
            RELAX(u, v, w)
    for each edge (u,v) ∈ E[G] do
        if d[v] > d[u] + w(u,v) then
            return FALSE
    return TRUE
```

* THE ALGORITHM RETURNS TRUE IF AND ONLY IF THE GRAPH CONTAINS NO NEGATIVE-WEIGHT CYCLES THAT ARE REACHABLE FROM THE SOURCE.

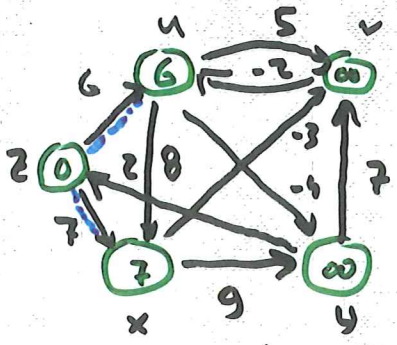
* THE BELLMAN-FORD ALGORITHM RUNS IN TIME $O(VE)$.

# GRAPHS

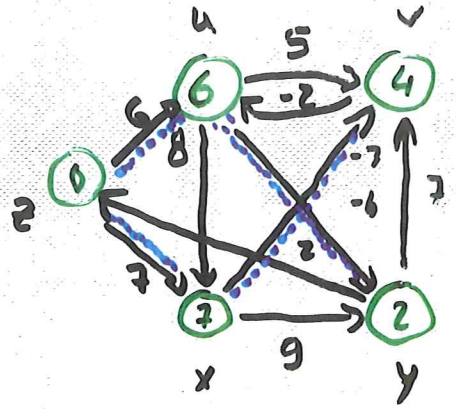
* EXAMPLE OF THE BELLMAN-FORD ALGORITHM.



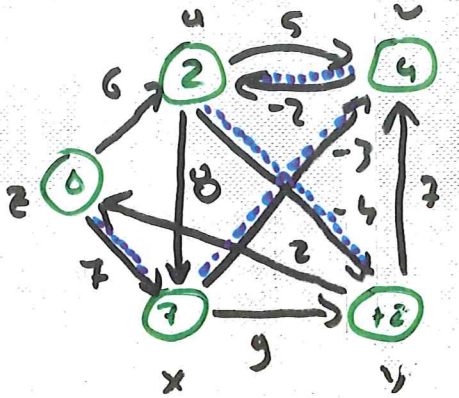

(a)



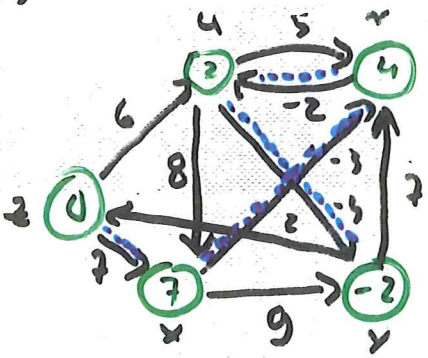

(b)




(c)




(d)




(e)

* EACH PASS RELAXES THE EDGES IN LEXICOGRAPHIC ORDER.