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Partial Differential Equations and Fourier Analysis — A Short Introduction

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Preface

This short book is intended for a one-semester course for students in the sciences and engineering after they have taken one year of calculus and one term of ordinary differential equations. For universities on quarter systems, sections labelled "optional" can be omitted without loss of continuity. A course based on this book can be offered to sophomores and juniors.

Examples used in this book are drawn from traditional application areas such as physics and engineering, as well as from biology, music, finance and geophysics. I have tried, whenever appropriate, to emphasize physical motivation and have generally avoided theorems and proofs. I also have tried to teach solution techniques, which will be useful in a student's other courses, instead of the theory of partial differential equations.

I believe that the subjects of partial differential equations and Fourier analysis should be taught as early as feasible in an undergraduate's curriculum. Towards this end the book is written to present the subject matter as simply as possible. Ample worked examples are given at the end of the chapters as a further learning aid. Exercises are provided for the purpose of reinforcing standard techniques learned in class. Tricky problems, whose purpose is mainly to test the student's mental dexterity, are generally avoided.

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Chapter 1

Introduction

1.1 Review of Ordinary Differential Equations

This course is mainly about partial differential equations (PDEs). Previously you have studied ordinary differential equations (ODEs). We will review two common types of ordinary differential equations here. If you have no difficulty with these, you have no problem with the prerequisites for this course.

1.1.1 First and Second Order Equations

Example. *Population growth:*

$$\boxed{\frac{dN}{dt} = rN}\,, \tag{1.1}$$

where $N(t)$ is the population density of a species at time t. The above equation is simply a statement that the rate of population growth, $\frac{dN}{dt}$, is proportional to the population itself, with the proportionality constant r. To solve it, we move all the N's to one side and all the t's to the other side of the equation. (This process is called "separation of variables" in the ODE literature. We will not use this term here as it may get confused with a PDE solution method with the same name which we will discuss later.) Thus:

$$\frac{dN}{N} = rdt. \tag{1.2}$$

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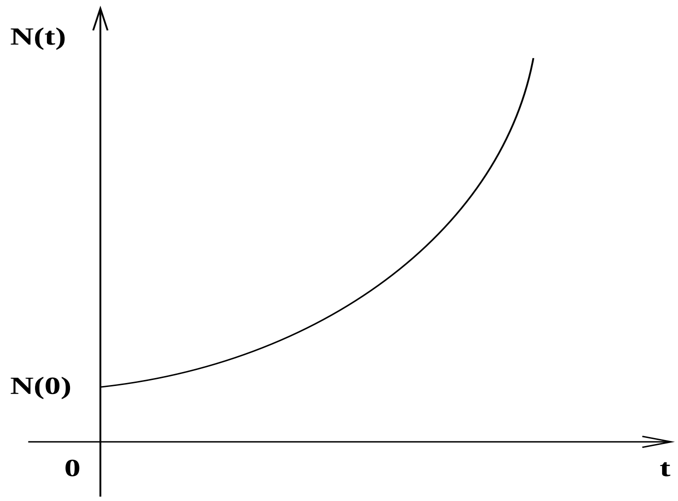


Figure 1.1: Solution to Equation 1.1

Integrating both sides yields:

$$\ell n N = rt + \text{ constant},$$

which can be rewritten as

$$N(t) = \text{ constant } \cdot e^{rt}.$$

In order for the left-hand side to equal to the right-hand side at $t = 0$, the "constant" in the second equation must be $N(0)$. Thus,

$$\boxed{N(t) = N(0)e^{rt}}\,. \tag{1.3}$$

Population grows exponentially from an initial value $N(0)$, with an e-folding time of r^{-1}. That is, $N(t)$ increases by a factor of e with every increment of r^{-1} in t. The solution is plotted in Figure 1.1.

Equation (1.1) is perhaps an unrealistic model for most population growths. Among other things, its solution implies that the population will grow indefinitely. A better model is given by the following equation:

$$\frac{dN}{dt} = rN \cdot (1 - N/k). \tag{1.4}$$

Try solving it using the same method. The solution is plotted in Figure 1.2 for $0 < N(0) < k$.

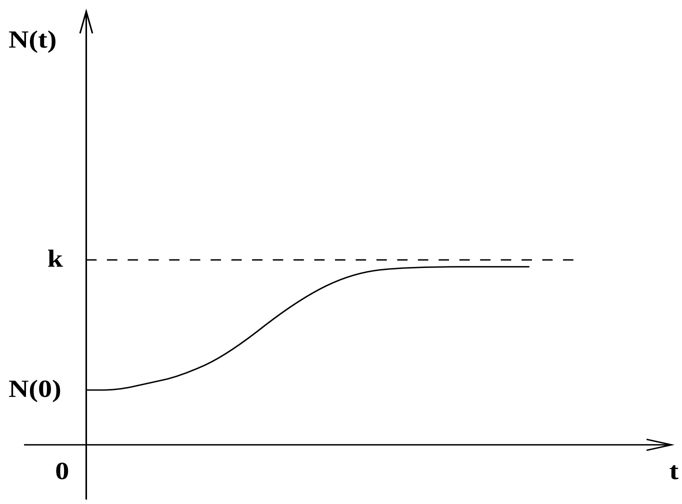


Figure 1.2: Solution to Equation 1.4

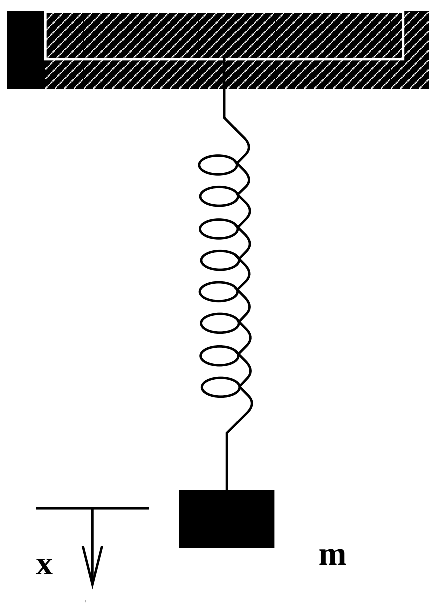


Figure 1.3: A spring of mass m suspended under gravity and in equilibrium. x is the displacement from the equilibrium position.

Example. *Harmonic Oscillator Equation:*

$$\boxed{m\frac{d^2x}{dt^2} + kx = 0}\,. \tag{1.5}$$

Here x is the vertical displacement from the equilibrium position of the spring, which has a mass m and a spring constant k. Equation (1.5) is a statement of Newton's Law of Motion: $m\frac{d^2x}{dt^2}$ is mass times acceleration. This is required to be equal to the spring's restoring force $-kx$. This force is assumed to be proportional to the displacement from equilibrium.

It is harder to solve a second order ODE, although this particular equation is so common that we were taught to try exponential solutions wherever we see linear equations with constant coefficients.

Solution by trial method: Try $e^{\alpha t}$.

Plugging the guess in Equation (1.5) for x then suggests that α must satisfy

$$m\alpha^2 + k = 0,$$

which means

$$\alpha = i\sqrt{k/m}, \quad \text{or } \alpha = -i\sqrt{k/m},$$

where $i \equiv \sqrt{-1}$ is the imaginary number. There are two different values of α which will make our trial solution satisfy equation (1.5). So the general solution should be a linear combination of the two:

$$\boxed{x(t) = c_1 e^{i\sqrt{k/m}t} + c_2 e^{-i\sqrt{k/m}t}}\,, \tag{1.6}$$

where c_1 and c_2 are two arbitrary (complex) constants.

If you do not like using complex notations (numbers that involve i), you can rewrite (1.6) in real notation, making use of the Euler's Identity, which we will derive a little later:

$$e^{i\theta} = \cos\theta + i\sin\theta. \tag{1.7}$$

Thus the rest solution (1.6) can be rewritten as

$$\boxed{x(t) = A\sin(\sqrt{k/m}t) + B\cos(\sqrt{k/m}t)}\,, \tag{1.8}$$

where A and B are some arbitrary real constants (since c_1 and c_2 were undetermined).

We can verify that (1.8) is indeed the solution to the harmonic oscillator equation (1.5) by noting, from calculus:

$$\frac{d}{dt}\sin\omega t = \omega\cos\omega t,\ \frac{d}{dt}\cos\omega t = -\omega\sin\omega t$$

so

$$\frac{d^2}{dt^2}\sin\omega t = \frac{d}{dt}(\omega\cos\omega t) = -\omega^2\sin\omega t$$

and

$$\frac{d^2}{dt^2}\cos\omega t = \frac{d}{dt}(-\omega\sin\omega t) = -\omega^2\cos\omega t.$$

Therefore, the sum (1.8) satisfies

$$\frac{d^2}{dt^2}x = -\omega^2 x, \tag{1.9}$$

which is the same as (1.5), provided that $\omega^2 = k/m$.

Euler's Identity:

Euler's Identity, as used in (1.7), deserves some comment. We will also need this identity later when we deal with Fourier series and transforms.

In calculus, we learned how to differentiate an exponential

$$\frac{d}{d\theta}e^{a\theta} = ae^{a\theta}.$$

Although you have always assumed a to be a real number, it does not make any difference if a is complex. So letting $a = i$, we find

$$\frac{d}{d\theta}e^{i\theta} = ie^{i\theta}$$

$$\frac{d^2}{d\theta^2}e^{i\theta} = \frac{d}{d\theta}(ie^{i\theta}) = i^2e^{i\theta} = -e^{i\theta}.$$

We have thus shown that the function

$$y(\theta) = e^{i\theta} \tag{1.10}$$

satisfies the second-order ODE:

$$\frac{d^2}{d\theta^2}y + y = 0. \tag{1.11}$$

$e^{i\theta}$ also happens to satisfy the *initial conditions*:

$$y(0) = 1, \ \frac{d}{d\theta}y(0) = i. \tag{1.12}$$

On the other hand, we have just verified in (1.9) that

$$y = A\sin\theta + B\cos\theta \tag{1.13}$$

also satisfies (1.11), which is the same as (1.9) if we replace t by θ and ω by 1. If we furthermore require the sum (1.13) to also satisfy the initial condition (1.12), we will find that $B = 1$ and $A = i$. Since,

$$y(\theta) = \cos\theta + i\sin\theta \tag{1.14}$$

satisfies the same ODE (1.11) and the same initial conditions (1.12) as (1.10), (1.14) and (1.10) must be the same by the Uniqueness Theorem for ODEs. What we have outlined is one way for proving the Euler Identity (1.7):

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}\ .$$

Solution from First Principles:

If you prefer to find the solution from first principles, i.e. not by guessing that it should be in the form of an exponential, it can be done by "reduction of order", although we normally do not bother to do it this way:

We recognize that Equation (1.5) is a type of ODE with its "independent variable missing". The method of reduction of order suggests that we let

$$p \equiv \frac{dx}{dt},$$

and write

$$\frac{d^2x}{dt^2} = \frac{dp}{dt} = \frac{dx}{dt}\frac{dp}{dx} = p\frac{dp}{dx}.$$

Treating p now as a function of x, Equation (1.5) becomes

$$p\frac{dp}{dx} + \omega^2 x = 0, \tag{1.15}$$

where we have used ω^2 for k/m.

Equation (1.15), a first order ODE, can be solved by the same method we used in Example 1:

Integrating $pdp + \omega^2 xdx = 0$ yields,

$$p^2 + \omega^2 x^2 = \omega^2 a^2.$$

with $\omega^2 a^2$ being an arbitrary constant of integration. From $p = \pm\omega\sqrt{a^2 - x^2}$, we have, since $p = \frac{d}{dt}x$,

$$\frac{dx}{dt} = \pm\alpha\sqrt{a^2 - x^2}.$$

This is again a first order ODE, which we solve as before.

Integrating both sides of

$$\frac{dx}{\sqrt{a^2 - x^2}} = \pm\omega dt$$

and using the integral formula:

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a} + b, \;\; b \text{ being a constant,}$$

we find

$$\sin^{-1}\frac{x}{a} = \pm\omega t + b.$$

Inverting, we find:

$$\frac{x}{a} = \sin(\pm\omega t + b),$$

which can (finally!) be rewritten as

$$x(t) = A\sin\omega t + B\cos\omega t \; . \tag{1.16}$$

1.2 Nonhomogeneous Ordinary Differential Equations

1.2.1 First-Order Equations:

A nonhomogeneous version of the example in (1.1) is

$$\boxed{\frac{dN}{dt} - rN = f(t),} \tag{1.17}$$

where $f(t)$ is a (known) specified function of t, independent of the "unknown" N. It is called the "inhomogeneous term" or the "forcing term". In the population growth example we discussed earlier, $f(t)$ can represent the rate of population growth of the species due to migration.

We are here concerned with the method of solution of (1.17) for any given $f(t)$. We proceed to multiply both sides of (1.17) by a yet-to-be-determined function $\mu(t)$, called the *integrating factor*:

$$\mu\frac{dN}{dt} - r\mu N = \mu f. \tag{1.18}$$

We choose $\mu(t)$ such that the product on the left-hand side of (1.18) is a perfect derivative, i.e.

$$\mu\frac{dN}{dt} - r\mu N = \frac{d}{dt}(\mu N). \tag{1.19}$$

If this can be done, then (1.10) would become:

$$\frac{d}{dt}(\mu N) = \mu f,$$

which can be integrated from $t = 0$ to t to yield:

$$\mu(t)N(t) - \mu(0)N(0) = \int_0^t \mu(t)f(t)dt. \tag{1.20}$$

The notation on the right-hand side of (1.20) is rather confusing. A better way is to use a different symbol, say, τ, in place of t as the dummy variable of integration. Then, (1.20) can be rewritten as

$$N(t) = N(0)\mu(0)\mu^{-1}(t) + \mu^{-1}(t)\int_0^t \mu(\tau)f(\tau)d\tau. \tag{1.21}$$

The remaining task is to find the integrating factor $\mu(t)$. In order for (1.19) to hold, we must have the right-hand side

$$\mu\frac{dN}{dt} + N\frac{d\mu}{dt}$$

equal the left-hand side, implying

$$\frac{d\mu}{dt} = -r\mu. \tag{1.22}$$

The solution to (1.22) is simply

$$\mu(t) = \mu(0)e^{-rt}. \tag{1.23}$$

Substituting (1.23) back into (1.21) then yields

$$\boxed{N(t) = N(0)e^{rt} + e^{rt}\int_0^t e^{-r\tau} f(\tau)d\tau}. \tag{1.24}$$

This then completes the solution of (1.17). If r is not a constant, but is a function t, the procedure remains the same up to, and including (1.22). The solution to (1.22) should now be

$$\mu(t) = \mu(0)e^{-\int_0^t r(t')dt'}. \tag{1.25}$$

The final solution is obtained by substituting (1.25) into (1.21).

Notice that the solution to the (linear) nonhomogeneous equation consists of two parts: a part satisfying the general homogeneous equation and a part that is a particular solution of the nonhomogeneous equation (the first and second terms on the right-hand side of (1.24) respectively). For some simple forcing functions $f(t)$, there is no need to use the general procedure of integrating factors if we can somehow guess a particular solution. For example, suppose we want to solve

$$\boxed{\frac{dN}{dt} - rN = 1}. \tag{1.26}$$

We write

$$N(t) = N_h(t) + N_p(t),$$

where $N_h(t)$ satisfies the homogeneous equation

$$\frac{dN_h}{dt} - rN_h = 0$$

and so is

$$N_h(t) = ke^{rt},$$

for some constant k. $N_p(t)$ is any solution of the Eq. (1.26). By the "method of judicious guessing", we try selecting a constant for $N_p(t)$. Upon substituting $N_p(t) = a$ into (1.26), we find the only possibility: $a = -\frac{1}{r}$. Thus the full solution is

$$N(t) = ke^{rt} - \frac{1}{r} = N(0)e^{rt} + \frac{1}{r}(e^{rt} - 1).$$

1.2.2 Second-Order Equations:

For our purpose here we will be using only the "method of judicious guessing" for linear second-order equations. The more general method of "variation of parameters" is too cumbersome for our limited purposes.

Example: Solve

$$\frac{d^2}{dt^2}x + \omega^2 x = 1. \tag{1.27}$$

We write the solution as a sum of a homogeneous solution x_h and a particular solution x_p, i.e.

$$x(t) = x_h(t) + x_p(t).$$

We already know that the homogeneous solution (to (1.5)) is of the form

$$x_h(t) = A\sin\omega t + B\cos\omega t. \tag{1.28}$$

We guess that a particular solution to (1.27) is a constant

$$x_p(t) = a. \tag{1.29}$$

Subsituting (1.29) into (1.27) then shows that constant is $1/\omega^2$. Thus the general solution to (1.27) is

$$\boxed{x(t) = A\sin\omega t + B\cos\omega t + 1/\omega^2}\,. \tag{1.30}$$

The arbitrary constants A and B are to be determined by initial conditions.

Example: Solve

$$\boxed{\frac{d^2}{dt^2}x + \omega^2 x = \sin\omega_0 t}\,. \tag{1.31}$$

Again we write the solution as a sum of the homogeneous solution and a particular solution. For the particular solution, we try:

$$x_p(t) = a\sin\omega_0 t. \tag{1.32}$$

Upon substitution of (1.32) into (1.31), we find

$$a = (\omega^2 - \omega_0^2)^{-1},$$

and so the general solution is

$$\boxed{x(t) = A\sin\omega t + B\cos\omega t + \frac{\sin\omega_0 t}{(\omega^2 - \omega_0^2)}.} \tag{1.33}$$

The solution in (1.33) is valid as is for $\omega_0 \neq \omega$. Some special treatment is helpful when the forcing frequency ω_0 approaches the natural frequency ω. Let us write

$$\omega_0 = \omega + \epsilon$$

and let $\epsilon \to 0$. The particular solution can be written as

$$\begin{aligned} x_p(t) &= \frac{\sin\omega_0 t}{(\omega^2 - \omega_0^2)} = \frac{\sin(\omega t + \epsilon t)}{\omega^2 - (\omega + \epsilon)^2} \\ &= \frac{\sin\omega t\cos\epsilon t + \cos\omega t\sin\epsilon t}{-2\omega\epsilon - \epsilon^2} \to -\frac{\cos\omega t}{2\omega}\cdot t - \frac{\sin\omega t}{2\omega\epsilon} \text{ as } \epsilon \to 0 \end{aligned}$$

Thus for the case of *resonance*, $\omega_0 = \omega$, the solution (1.33) becomes

$$x(t) = A'\sin\omega t + B\cos\omega t - \frac{1}{2\omega}t\cos\omega t,$$

where we have written $A = A' + \frac{1}{2\omega\epsilon}$, with A' being a (finite) arbitrary constant. The solution grows secularly in t.

1.3 Summary of ODE solutions

In this course we will be mostly dealing with linear differential equations with constant coefficients. For these, simply try an exponential solution. This is the easiest way. You are not expected to have to repeat each time the derivation given in the previous sections on why the exponentials are the right solutions to try. Just do the following:

(a) $\boxed{\frac{d}{dt}N = rN}$

Try $N(t) = ae^{\alpha t}$ and find $\alpha = r$ so the solution is

$$\boxed{N(t) = ae^{rt}}.$$

(b) $\boxed{\frac{d^2}{dt^2}x + \omega^2 x = 0}$

Try $x(t) = ae^{\alpha t}$ and find $\alpha = \pm i\omega$ so the complex solution is

$$\boxed{x(t) = a_1 e^{i\omega t} + a_2 e^{-i\omega t}},$$

and the real solution

$$x = A\cos\omega t + B\sin\omega t$$

1.4 Partial Derivatives

The ordinary differential equations we discussed in the last section describe functions of only one independent variable. For example, the unknown N in (1.1) is a function of t only, and Eq. (1.1) describes the rate of change of $N(t)$ with respect to t. It is not hard to imagine a physical situation where the population N depends not only on time t, but also on space x (more realistically on all three spatial dimensions, x, y, z). For a function of more than one independent variables, for example,

$$N = N(x,t),$$

we need to distinguish the derivative with respect to t from the derivative with respect to x. For this purpose, we define the *partial derivatives* in the following way.

The partial derivative of $N(x,t)$ with respect to t, denoted by $\frac{\partial}{\partial t}N(x,t)$, or $N_t(x,t)$ for short, is defined as the derivative of N with respect to t *holding all other independent variables—in this case x—constant*:

$$\boxed{\frac{\partial}{\partial t}N(x,t) = \lim_{\substack{\Delta t\to 0\\ x\text{ held constant}}} \frac{N(x,t+\Delta t)-N(x,t)}{\Delta t}}. \tag{1.34}$$

Similarly, the partial derivative of $N(x,t)$ with respect to x, denoted by $\frac{\partial}{\partial x}N(x,t)$, or $N_x(x,t)$ for short, is defined as:

$$\boxed{\frac{\partial}{\partial x}N(x,t) = \lim_{\substack{\Delta x\to 0\\ t\text{ held constant}}} \frac{N(x+\Delta x,t)-N(x,t)}{\Delta x}.} \tag{1.35}$$

Compare the partial derivatives with the ordinary derivatives:

$$\frac{d}{dt}N(t) = \lim_{\Delta t\to 0}\frac{N(t+\Delta t)-N(t)}{\Delta t},$$

and you will see that the partial derivative is the same as the ordinary derivative if you can just pretend that the other independent variables were constants.

Example: The (first) partial derivatives of

$$N(x,t) = x^3 + t^2 \tag{1.36}$$

are

$$N_t = 2t, \quad \text{and} \quad N_x = 3x^2. \tag{1.37}$$

When N is a function of x and t, its integral with respect to t is done by pretending that x is a constant, as in the following example.

Example: For $N(x,t)$ given by (1.36)

$$\int^t N(x,t)dt = x^3 t + \frac{1}{3}t^3 + f(x), \tag{1.38}$$

where $f(x)$ plays the role of a "constant of integration" with respect to t and is actually an arbitrary function of x. To verify this, we can take the (partial) derivative of (1.38) with respect to t and recover $N(x,t)$ in (1.36).

Example: The solution $N(x,t)$ of the PDE:

$$\frac{\partial}{\partial t}N = rN \tag{1.39}$$

for $t > 0$ is

$$N(x,t) = a(x)e^{rt}. \tag{1.40}$$

Here $a(x)$ plays the role of the "constant" in the solution to the ODE (1.1). Setting $t = 0$, we find

$$a(x) = N(x,0),$$

which is to be given by the initial distribution of the population.

1.5 Exercise I

Review of ordinary differential equations:

1. Find the most general solution to:

   (a) $\frac{d}{dt}N = rN + b$; r, b are constants,

   (b) $\frac{d^2}{dt^2}x + \beta\frac{d}{dt}x + \omega^2 x = 0$; β, ω^2 are constants,

   (c) $\frac{d^2}{dt^2}x + \omega^2 x = \cos\omega_0 t$, $\omega \neq \omega_0$,

   (d) $\frac{d^2}{dt^2}x + \omega_0^2 x = \cos\omega_0 t$.

2. Find the solution satisfying the specified initial conditions:

   (a) $\frac{d}{dt}N = rN + b$
   $N(0) = 0$.

   (b) $\frac{d^2}{dt^2}x + \beta\frac{d}{dt}x + \omega^2 x = 0$
   $x(0) = 1$, $\frac{d}{dt}x(0) = 0$.

   (c) $\frac{d^2}{dt^2}x + \omega^2 x = \cos\omega_0 t$, $\omega \neq \omega_0$
   $x(0) = 0$, $\frac{d}{dt}x(0) = 0$.

   (d) $\frac{d^2}{dt^2}x + \omega_0^2 x = \cos\omega_0 t$
   $x(0) = 0$, $\frac{d}{dt}x(0) = 0$.

1.6 Solutions to Exercise I

1. (a) $N(t) = N_h(t) + N_p(t)$, where $N_h(t)$ is the solution to the homogeneous equation and $N_p(t)$ is a particular solution to the non-homogeneous equation. For $N_p(t)$, we try a constant, i.e. $N_p(t) = c$. Substituting it into $\frac{d}{dt}N = rN + b$ yields $0 = rc + b$, implying $c = -b/r$.

The solution to the homogeneous equation $\frac{d}{dt}N = rN$ is: $N_h(t) = ae^{rt}$, where a is an arbitrary constant.

Combining:

$$N(t) = ae^{rt} - b/r.$$

(b) Try

$$x(t) = ae^{\alpha t}.$$

Substituting into the ODE yields:

$$\alpha^2 + \alpha\beta + \omega^2 = 0.$$

So $\alpha = \alpha_1$ or $\alpha = \alpha_2$, where

$$\alpha_1 \equiv -\frac{\beta}{2} + \sqrt{\frac{\beta^2}{4} - \omega^2} \quad \text{and} \quad \alpha_2 \equiv -\frac{\beta}{2} - \sqrt{\frac{\beta^2}{4} - \omega^2}$$

The general solution is

$$x(t) = a_1 e^{\alpha_1 t} + a_2 e^{\alpha_2 t}$$

(c) $x(t) = x_h(t) + x_p(t)$.

For $x_p(t)$, try $x_p(t) = a\cos\omega_0 t$. Substituting into the ODE yields

$$-\omega_0^2 a + \omega^2 a = 1.$$

So $a = 1/(\omega^2 - \omega_0^2)$. For $x_h(t)$, we know the general solution to the homogeneous ODE is

$$x_h(t) = A\sin\omega t + B\cos\omega t; A, B \text{ are arbitrary constants.}$$

The full solution is

$$x(t) = A\sin\omega t + B\cos\omega t + \cos\omega_0 t/(\omega^2 - \omega_0^2).$$

(d) This is the resonance case. Still try $x(t) = x_h(t) + x_p(t)$.

For $x_p(t)$, try $x_p(t) = at\sin\omega_0 t$. Substituting into the nonhomogeneous ODE yields

$$2a\omega_0\cos\omega_0 t - a\omega_0^2 t\sin\omega_0 t + \omega_0^2 at\sin\omega_0 t = \cos\omega_0 t.$$

Thus

$$2a\omega_0 = 1.$$

So $x_p(t) = t \sin \omega_0 t / 2\omega_0$. The general solution to the homogeneous ODE is:

$$x_h(t) = A \sin \omega_0 t + B \cos \omega_0 t.$$

The full solution is

$$x = A \sin \omega_0 t + B \cos \omega_0 t + t \sin \omega_0 t / 2\omega_0.$$

2. (a) $N(t) = ae^{rt} - b/r$

$$N(0) = a - b/r = 0 \text{ implies } a = b/r.$$

So,

$$N(t) = \frac{b}{r}(e^{rt} - 1)$$

(b) $x(t) = a_1 e^{\alpha_1 t} + a_2 e^{\alpha_2 t}$

$$x(0) = a_1 + a_2 = 1$$

$$\frac{d}{dt}x(0) = \alpha_1 a_1 + \alpha_2 a_2 = 0$$

Thus $a_1 = -\alpha_2/(\alpha_1 - \alpha_2)$, $a_2 = \alpha_1/(\alpha_1 - \alpha_2)$.
Finally

$$x(t) = \frac{1}{(\alpha_1 - \alpha_2)}[-\alpha_2 e^{\alpha_1 t} + \alpha_1 e^{\alpha_2 t}]$$

(c) $x(t) = A \sin \omega t + B \cos \omega t + \cos \omega_0 t / (\omega^2 - \omega_0^2)$

$$x(0) = B + 1/(\omega^2 - \omega_0^2)$$

$$\frac{d}{dt}x(0) = A\omega = 0$$

Thus $A = 0$, $B = -1/(\omega^2 - \omega_0^2)$ and

$$x(t) = \frac{1}{(\omega^2 - \omega_0^2)}[\cos \omega_0 t - \cos \omega t]$$

(d) $x(t) = A \sin \omega_0 t + B \cos \omega_0 t + t \sin \omega_0 t / \omega_0$

$$x(0) = B = 0$$

$$\frac{d}{dt}x(0) = A\omega_0 = 0.$$

Thus $A = 0$, $B = 0$, and

$$x(t) = t \sin \omega_0 t / \omega_0.$$

1.7 Exercise II

Partial derivatives:

1. Evaluate $\frac{\partial}{\partial x}u(x,y)$ for

   (a) $u(x,y) = e^{xy}$

   (b) $u(x,y) = (x+y)^2$

   (c) $u(x,y) = x^2 + y^2$

2. Evaluate $\int^y u(x,\eta)d\eta$ (as an indefinite integral) for $u(x,y)$ given by (a), (b) and (c) from Problem 1 above.

3. (a) Find the general solution $u(t)$ to the ODE

$$m\frac{d^2}{dt^2}u + ku = 0,$$

   where m and k are constants.

   (b) Find the general solution $u(x,y)$ to the PDE

$$\frac{\partial^2}{\partial x^2}u + (1+y^2)u = 0$$

1.8 Solutions to Exercise II

1. Evaluate $\frac{\partial}{\partial x}u(x,y)$ for

   $u(x,y) = e^{xy}$, $\frac{\partial}{\partial x}u = ye^{xy}$

   $u(x,y) = (x+y)^2$, $\frac{\partial}{\partial x}u = 2(x+y)$

   $u(x,y) = x^2 + y^2$, $\frac{\partial}{\partial x}u = 2x$.

2. Evaluate $\int^y u(x,\eta)d\eta$ from problem 1

   $\int^y e^{x\eta}d\eta = \frac{1}{x}e^{xy} + f(x)$

   $\int^y (x+\eta)^2 d\eta = \frac{(x+y)^3}{3} + g(x)$

   $\int^y (x^2+\eta^2)d\eta = x^2y + \frac{y^3}{3} + h(x)$,

   where f,g, and h are arbitrary functions of x.

3a. Find the general solution u(t) to $m\frac{d^2}{dt^2}u + ku = 0$, where m and k are constants.

   Try $u(t) = A\cos\alpha t + B\sin\alpha t$.

   Substitute into $m\frac{d^2}{dt^2}u + ku = 0$ to find $\alpha = \sqrt{k/m}$.

   So the general solution is $u(t) = A\cos\sqrt{k/m}t + B\sin\sqrt{k/m}t$ with A and B arbitrary constants.

3b. Find the general solution $u(x,y)$ to $\frac{\partial^2}{\partial x^2}u + (1+y^2)u = 0$.

   Treat y as a constant with respect to x-partial derivative.

   Try $u(x,y) = A(y)\cos(\alpha(y)x) + B(y)\sin(\alpha(y)x)$.

   Find $\alpha(y) = \sqrt{(1+y^2)}$.

   Thus we have $u(x,y) = A(y)\cos(\sqrt{(1+y^2)}x) + B(y)\sin(\sqrt{(1+y^2)}x)$, with A and B arbitrary functions of y.

Chapter 2

Physical Origins of Some PDEs

2.1 Introduction

In physical applications, PDEs are more ubiquitous than ODEs. This situation can be understood because physical quantities more often depend on space and time than on, say, time alone. A partial differential equation relates the variations of this physical quantity in time and in space. Of course, in mathematical abstraction, one does not need to assign the physical meaning of time to the symbol t, or space to the symbol x; one is simply concerned with the variations of the unknown with respect to more than one independent variable as governed by a PDE.

We will be dealing with first and second order PDEs in this course. In this chapter we will discuss the physical origin of these equations. This chapter is suitable for assigned casual reading.

2.2 Conservation Laws:

Many physical laws can be expressed as a *conservation law* of the form

$$u_t + q_x = 0. \tag{2.1}$$

Here $u(x,t)$ is the "concentration" of something under consideration and q is its "flux" in the x-direction. If the quantity under consideration is nonconservative, we need to add a "source or sink" term to the right-hand side of (2.1). We will provide physical examples of such terms in a moment.

In more than one space dimensions, q would be a vector $\mathbf{q}$, and (2.1) would be replaced by

$$u_t + \nabla \cdot \mathbf{q} = 0$$

for the gradient vector ∇. We will however not be concerned with more than one space dimension here; so you will not need to know vectors or divergence of a vector $(\nabla \cdot \mathbf{q})$.

If q in (2.1) is a function of u only and does not depend on its derivatives, then it can be written as

$$u_t + a(u)u_x = 0, \quad \text{where } a(u) \equiv \frac{dq}{du}. \tag{2.2}$$

(2.2) is a first order PDE. On the other hand, if the flux q depends on u_x, as is often the case for "down-gradient" fluxes, e.g.

$$q = -ku_x, \quad \text{for a constant } k,$$

(2.1) will become

$$u_t - ku_{xx} = 0, \tag{2.3}$$

which is a second-order PDE.

2.2.1 Diffusion of a tracer:

Let c be the concentration of a substance under consideration (in gm/volume). Let ρ be the density of the medium into which it is diffusing. For example, for the problem of diffusion of salt in water, c is the weight of salt per unit volume, and ρ is the weight of water per unit volume. Let $u = c/\rho$ be called the mass ratio of salt to water.

We now consider diffusion in one dimension, x. This can approximate the situation of diffusion in a thin tube of constant cross-sectional area A, with its axis oriented in the x-direction. The tube is filled with water and some salt is put into this tube initially. Since the tube is thin, we can ignore the diffusion of salt radially, assuming the salt quickly manages to diffuse radially to attain the same concentration throughout the cross-section. Its concentration varies mainly with x.

Consider the amount of salt between the sections x and $x + \Delta x$, a small distance apart. If Δx is so small that the variation of c with x can be ignored, that amount of salt contained is $cA\Delta x = \rho A\Delta x \cdot u$, since $A\Delta x$

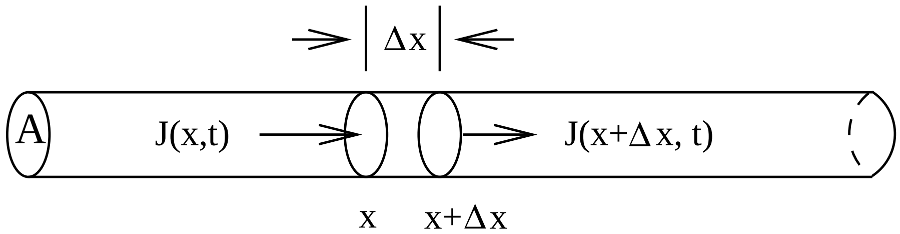


Figure 2.1: Diffusion in a tube of cross-sectional area A.

is the volume of the section under consideration. Since the mass of salt is conserved, we can state that the time rate of change of salt in this volume,

$$\frac{\partial}{\partial t}\rho A \Delta x u,$$

is equal to the flux of salt into the volume at x, minus the flux of salt out of the volume at $x + \Delta x$, i.e.

$$[J(x,t)A - J(x + \Delta x, t)A],$$

where $J(x,t)$ is the flux at x. It is defined as the time rate of salt flowing across x per unit area.

Thus the equation for the conservation of salt is:

$$\rho A \Delta x \frac{\partial}{\partial t} u = A[J(x,t) - J(x + \Delta t, t)]. \tag{2.4}$$

Dividing both sides by $A\Delta x$, we get

$$\rho \frac{\partial}{\partial t} u = -\frac{J(x + \Delta x, t) - J(x,t)}{\Delta x}. \tag{2.5}$$

We now take the limit as $\Delta x \to 0$, since the smaller Δx is, the better our previous approximation of assuming c to be constant between x and Δx is. The right-hand side of Equation (2.5) becomes,

$$-\lim_{\Delta x \to 0} \frac{J(x + \Delta x, t) - J(x,t)}{\Delta x} = -\frac{\partial J}{\partial x}(x,t).$$

Equation (2.5) is now in the form of a “conservation law”, Equation (2.1), if we write $q \equiv J/\rho$:

$$\frac{\partial}{\partial t} u = -\frac{\partial}{\partial x} q. \tag{2.6}$$

To relate the flux q to u, we use *Fick's law of diffusion*, which is obtained from experimental descriptions. It says:

The flux (of salt) is proportional to the negative gradient (of salt) (since salt always diffuses from a high concentration region to a low concentration region). Mathematically, we write this law as

$$J(x,t) = -k\frac{\partial}{\partial x}c = -k\rho\frac{\partial}{\partial x}u, \tag{2.7}$$

where the proportionality constant, k, is the coefficient of diffusivity for salt. Equation (2.6) finally becomes:

$$\boxed{\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}}\,. \tag{2.8}$$

It is a second-order PDE, because its highest derivative, $\frac{\partial^2 u}{\partial x^2}$, is second order. Equation (2.8) is called the *diffusion equation*. With a different value for k it can be applied to the problem of diffusion of pollutants in air (with ρ being the density of air and c the concentration of the pollutant). In biology and ecology, c could be the population of a species.

2.2.2 Advection of a tracer:

The diffusion of a tracer, say salt in water discussed in 2.2.1, is a macroscopic result of small-scale random molecular motion of salt and water molecules. This is a slow process. Advection, on the other hand, is a faster process if it occurs. If a pollutant of concentration c is put into a river whose water is flowing with speed V, the flux of that pollutant, defined as the time rate of the pollutant flowing across x per unit area, is

$$J(x,t) = V\cdot c, \tag{2.9}$$

because the pollutant is carried by the water with speed V across x. Letting $u = c/\rho$, and $q = J/\rho$, we have, from (2.6)

$$\frac{\partial}{\partial t}u = -\frac{\partial}{\partial x}q$$

or

$$\boxed{\frac{\partial}{\partial t}u + V\frac{\partial}{\partial x}u = 0}\,. \tag{2.10}$$

Equation (2.10) is a first-order PDE. It describes the advection of a tracer by a medium with velocity V.

If the time scales of advection and diffusion are comparable, we should include them both. In that case, the flux becomes

$$J(x,t) = V \cdot c - k\frac{\partial}{\partial x}c, \tag{2.11}$$

and the governing PDE becomes:

$$\boxed{\frac{\partial}{\partial t}u + V\frac{\partial}{\partial x}u = k\frac{\partial^2}{\partial x^2}u.} \tag{2.12}$$

Equation (2.9) is called the *advection-diffusion equation*.

2.2.3 Nonlinear advection:

The advection equation, (2.10), is *linear* if V does not depend on the unknown u. Otherwise it is *nonlinear*. A linear PDE is one where the unknown and its derivatives appear linearly, i.e. not multiplied by itself or its partial derivatives.

Nonlinear advection arises if the quantity u, whose conservation is being considered, is also related to the advecting velocity V. This is the case, for example, when we are considering the conservation of momentum ρu of the fluid, where ρ is the density of the fluid and u is the fluid's velocity in the x-direction. Since the momentum is advected by the fluid velocity u, the conservation law will look something like:

$$\rho\left[\frac{\partial}{\partial t}u + u\frac{\partial}{\partial x}u\right] = \text{ the forces acting.}$$

The simplest such equation is the nonlinear advection equation:

$$\boxed{\frac{\partial}{\partial t}u + u\frac{\partial}{\partial x}u = 0}. \tag{2.13}$$

If nonconservative forces, such as molecular diffusion (so-called viscosity) is included, one obtains the nonlinear counterpart to (2.12):

$$\boxed{\frac{\partial}{\partial t}u + u\frac{\partial}{\partial x}u = \nu\frac{\partial^2}{\partial x^2}u}. \tag{2.14}$$

Equation (2.14) is the famous *Burgers' equation*. Burgers' equation has been studied by mathematicians as a prototype model of the balance between nonlinear advection and viscosity in fluid flows (where ν is the coefficient of viscosity), although the equations governing real fluid flows are more complicated. [Burgers' equation does not include, among other things, the pressure forces.]

2.2.4 Heat conduction in a rod:

Consider a long cylindrical rod, insulated at the curved sides so that heat can flow in or out only through the ends. The rod is thin, so that the radial variation of temperature u can be ignored. We consider the variation of u with respect to x, measured along the axis of the cylinder, and time t, i.e.

$$u = u(x, t)$$

The time rate of change of heat energy in a small volume $A\Delta x$, situated between sections x and $x + \Delta x$, is

$$c\rho A\Delta x \frac{\partial u}{\partial t},$$

where c is the specific heat and ρ the density of the material of the rod (e.g. copper). This should be equal to the net flux of heat into the volume, i.e.

$$c\rho A\Delta x \frac{\partial u}{\partial t} = J(x, t)A - J(x + \Delta x, t)A. \tag{2.15}$$

where $J(x, t)$ is the flux of heat into the positive x direction across the section at x. Taking the limit as $\Delta x \to 0$, Equation (2.15) becomes

$$\frac{\partial u}{\partial t} = -\frac{\partial q}{\partial x}, \quad \text{where } q \equiv J/c\rho, \tag{2.16}$$

Equation (2.16) is in the form of a conservation law, (2.1).

To express the flux J in terms of the temperature u, we use Fourier's law of heat conduction: If there are differences in temperature in a conducting medium, the heat energy would flow from the hotter to the colder region. Therefore there is "down gradient" heat flux, proportional to the negative of the temperature gradient, i.e.

$$q(x, t) = -\alpha^2 \frac{\partial}{\partial x} u. \tag{2.17}$$

Here α^2 is the coefficient of thermal diffusivity. For a uniform α^2, Equation (2.7) becomes

$$\boxed{\frac{\partial}{\partial t} u = \alpha^2 \frac{\partial^2}{\partial x^2} u}\,. \tag{2.18}$$

Equation (2.9) is called the *heat equation*. It has the same form as the diffusion equation derived earlier.

2.2.5 Ubiquity of the Diffusion Equation

We have seen that the same diffusion equation arises from seemingly unrelated phenomena, from diffusion of salt and pollutants to conduction of heat. The ubiquity of the diffusion equation arises from (a) the ubiquity of the conservation law (2.1) and (b) the *Fickian flux-gradient relationship* (2.7):

$$J \propto -\frac{\partial c}{\partial x},$$

which is simply a statement that diffusion tends to transport matter or heat from high to low levels. This experimental law is a reflection of the underlying random molecular motion. Consider an imaginary interface at x separating water with no salt to the left and water with salt to the right. Suppose on average it is as likely, through their random motion, for molecules from the right to cross x to the left as it is for molecules from the left to go to the right. Since some of the molecules at the right are salt molecules, after a while some salt molecules have moved to the left while the water molecules from the left have taken their place at the right. The macroscopic result is that there is a *flux* of salt from the high salt concentration region to the low concentration region.

Similarly, let us consider the region of high temperature as consisting of molecules of higher vibrational energy. The energy can be transfered to other molecules upon collison. Thus, although it is equally likely for an energetic molecule to move to the left as it is for it to move to the right, when it moves to the left it has a higher probability of hitting a less energetic molecule, and transfering "heat" in the process. Therefore, macroscopically, heat flows from hot to cold regions.

Diverse phenomena in biology, ranging from animal and insect dispersal to the spread of diseases, can be modeled approximately by a Fickian flux-gradient relationship, and hence can be described by some sort of diffusion equation. (see Okubo (1980): *Diffusion and Ecological Problems: Mathematical Models*, Springer).

2.3 Random Walk

2.3.1 A drunken sailor

Consider the problem of predicting where a (microscopic) "drunken sailor" will be at time t if he walks randomly from an initial position at $x = 0$. For simiplicity let us consider here the one dimensional problem where he is constrained to walk straight along a narrow alley (assuming that the sailor can indeed walk straight!). He takes a step of size Δx in a time interval Δt, and it is equally likely for him to take that step in either direction.

Let $p(x,t)$ be the probability that we will find him at x in time t. In a previous time step $t - \Delta t$, he could be either at the neighboring locations $x - \Delta x$ or at $x + \Delta x$, with equal probability. Therefore we have

$$p(x,t) = \frac{1}{2}p(x - \Delta x, t - \Delta t) + \frac{1}{2}p(x + \Delta x, t - \Delta t). \tag{2.19}$$

(2.19) is simply a statement that the sailor could have arrived at x from either $x - \Delta x$ by taking a forward step (or lurch), or from $x + \Delta x$ by stepping backward, with equal likelihood. If we take Δx and Δt to be small in some sense, we can expand the right-hand side of (2.17) in a Taylor series about its two variables:

$$\begin{aligned} p(x - \Delta x, t - \Delta t) &\backsimeq p(x, t - \Delta t) + \frac{\partial p(x, t - \Delta t)}{\partial x}(-\Delta x) \\ &\quad + \frac{1}{2}\frac{\partial^2 p(x, t - \Delta t)}{\partial x^2}(\Delta x)^2 + \ldots \\ &\backsimeq p(x,t) + \frac{\partial p(x,t)}{\partial t}(-\Delta t) + \frac{1}{2}\frac{\partial^2 p(x,t)}{\partial t^2}(-\Delta t)^2 \\ &\quad + \frac{\partial p(x,t)}{\partial x}(-\Delta x) + \frac{\partial^2 p(x,t)}{\partial x \partial t}(-\Delta t)(-\Delta x) + \ldots \\ &\quad + \frac{1}{2}\frac{\partial^2 p(x,t)}{\partial x^2}(-\Delta x)^2 + \ldots\ . \end{aligned}$$

Similarly the Taylor series expansion for $p(x + \Delta x, t - \Delta t)$ is the same as for $p(x - \Delta x, t - \Delta t)$ except with $-\Delta x$ replaced by Δx. Therefore (2.19) becomes:

$$p(x,t) = p(x,t) - \frac{\partial}{\partial t}p(x,t)\Delta t + \frac{1}{2}\frac{\partial^2}{\partial t^2}p(x,t)(\Delta t)^2 + \frac{1}{2}\frac{\partial^2}{\partial x^2}p(x,t)(\Delta x)^2 + \ldots\ .$$

Upon taking the limit $\Delta t \to 0$, $\Delta x \to 0$, we find

$$\boxed{\frac{\partial}{\partial t}u(x,t) = D\frac{\partial^2}{\partial x^2}u(x,t)}\ , \tag{2.20}$$

where $u(x,t) \equiv p(x,t)/\Delta x$ is the *probability density*, which is finite no matter how small Δx is, and $D \equiv \lim_{\substack{\Delta x \to 0 \\ \Delta t \to 0}} \frac{(\Delta x)^2}{2\Delta t}$ is assumed to exist.

Equation (2.20) has the same form as the diffusion equation. Although we used the drunken sailor as an example, the derivation we have just outlined applies better to an ensemble of microscopic particles in random motion, where it makes more sense to take $\Delta x \to 0$, $\Delta t \to 0$, and to treat x and t as continuous variables.

2.3.2 Price of stocks as a random walk

In an efficient market, the price for a share of a company stock is an amount for which as many people wish to buy as to sell. Those wishing to buy are probably expecting the stock price to rise in the near future, while those willing to sell at that price are probably anticipating the stock to drop in price. In such a market, insider trading is curbed. Any information known at time t about the company which might impact its stock price is already reflected in the price at that time. It is equally likely for the price of that stock to rise a given amount Δx at a future time $t + \Delta t$ as it is to fall by the same amount. Consequently, the expectation of a change in price of a stock in a future time should behave like a random walk, and therefore should be governed by the same diffusion equation (2.20), with however a different value for the "diffusion coefficient" D depending on the "volatility" of each stock.

2.4 The Wave Equation

Let us consider as an example the problem of a vibrating (guitar) string. The string is stretched lengthwise with uniform tension T. To fix ideas, let us say that in its equilibrium position the string lies horizontally (in the x-direction), and we consider a vertical displacement $u(x,t)$ from this equilibrium position. We assume that these displacements are small, as compared to the equilibrium length of the string.

We consider a small section of the string between x and $x + \Delta x$. See Figure 2.2.

We apply Newton's law of motion:

$$ma = F$$

(mass times acceleration balancing force), to the vertical motion of this small section of the string. Its mass m is $\rho A \Delta x$, where ρ is the density of

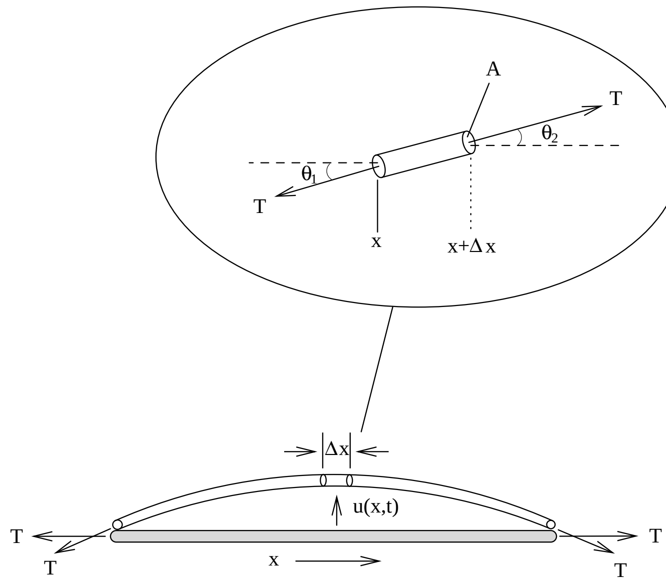


Figure 2.2: A stretched elastic string

the material of the string and A its cross-sectional area. The acceleration in the vertical direction is

$$a = \frac{\partial^2}{\partial t^2} u.$$

The force should be the vertical component of the tension, plus other forces such as gravity and air friction.

The net vertical component of tension is

$$\begin{aligned} & T\sin\theta_2 - T\sin\theta_1 \\ \cong\ & T[\theta_2 - \theta_1] \\ \cong\ & T[u_x(x+\Delta x, t) - u_x(x,t)], \end{aligned}$$

assuming that the angles θ_1 and θ_2 are small. Putting these all together, we have

$$\rho A \Delta x \frac{\partial^2}{\partial t^2} u = TA[u_x(x+\Delta x, t) - u_x(x,t)] + \rho A \Delta x \cdot f \tag{2.21}$$

where f represents all additional force per unit mass. Equation (2.21) is

$$\frac{\partial^2}{\partial t^2} u = \frac{T}{\rho} \frac{1}{\Delta x}[u_x(x+\Delta x, t) - u_x(x,t)] + f,$$

which becomes, as $\Delta x \to 0$:

$$\frac{\partial^2}{\partial t^2}u = c^2\frac{\partial^2}{\partial x^2}u + f, \tag{2.22}$$

where $c^2 \equiv T/\rho$.

The additional force f could represent gravity, in which case $f = -g$ ($g = 980$ cm/s^2), or a frictional force of the form: $f = -\gamma u_t$, where γ is a damping coefficient. In most of the examples to be considered, f will be ignored, and we will be dealing with the simple homogeneous wave equation

$$\boxed{\frac{\partial^2}{\partial t^2}u = c^2\frac{\partial^2}{\partial x^2}u}\,. \tag{2.23}$$

The wave equation is also quite ubiquitous. Equation (2.23) also governs the propagation of sound waves in the atmosphere, with c changed to the speed of sound. It also governs water waves travelling on the surface of shallow water, with c replaced by $\sqrt{gh}$ and h being the depth of the water.

2.5 Multiple Dimensions

Although our discussions have so far used one dimensional examples, extensions to two or three spatial dimensions are straightforward. These are indicated below:

1-D diffusion equation:

$$\frac{\partial}{\partial t}u = k\frac{\partial^2}{\partial x^2}u.$$

3-D diffusion equation:

$$\boxed{\frac{\partial}{\partial t}u = k\nabla^2 u}\,, \tag{2.24}$$

where $\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is the Laplacian operator.

1-D wave equation:

$$\frac{\partial^2}{\partial t^2}u = c^2\frac{\partial^2}{\partial x^2}u.$$

3-D wave equation:

$$\boxed{\frac{\partial^2}{\partial t^2}u = c^2\nabla^2 u}\,. \tag{2.25}$$

Essentially, in multi-dimensions, the Laplacian operator replaces the $\frac{\partial^2}{\partial x^2}$ term of the one-dimensional problem.

At steady state the heat equation in multi-dimensions is

$$\boxed{\nabla^2 u = 0}\,. \tag{2.26}$$

Equation (2.24) is called Laplace's equation. Its solution gives the steady state temperature distribution in multi-dimensions, for example, Laplace's equation also governs the distribution of electrostatic potential and the velocity potential in ideal fluid flows.

2.6 Types of second-order PDEs

There are three types of second-order PDEs, whose solutions have distinctly different behaviors. These are *parabolic*, *hyperbolic*, and *elliptic* PDEs. Parabolic equations are diffusion like, while hyperbolic equations are typified by the wave equation. Laplace's equation belongs to the category of elliptic PDEs. Namely:

$$\boxed{\begin{array}{ll} \dfrac{\partial}{\partial t}u = k\dfrac{\partial^2}{\partial x^2}u : & \text{Parabolic type} \\ \dfrac{\partial^2}{\partial t^2}u = c^2\dfrac{\partial^2}{\partial x^2}u : & \text{Hyperbolic type} \\ \dfrac{\partial^2}{\partial x^2}u + \dfrac{\partial^2}{\partial y^2}u = 0 : & \text{Elliptic type} \end{array}}$$

In general a second-order linear PDE in two independent variables (denoted by x and y) can be written in the form (with u_{xx} denoting $\frac{\partial^2}{\partial x^2}u$ etc.):

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G,$$

where A, B, C, D, E, F and G are given functions of x and y. This equation is said to be *parabolic* if $B^2 - 4AC = 0$, *hyperbolic* if $B^2 - 4AC > 0$, and *elliptic* if $B^2 - 4AC < 0$.

The equation is homogeneous if $G \equiv 0$.

2.7 Boundary Conditions

The mathematical specification of a problem involving a partial differential equation in space x and time t is incomplete without the imposition of *boundary conditions* and *initial conditions*.

Three different types of boundary conditions for second order PDEs:

A. *Dirichlet conditions*, where the value of the unknown u is specified at the spatial boundaries.

For example, suppose we want to solve for $u(x,t)$ for $0 < x < L$, the boundaries of the domain are at $x = 0$ and $x = L$. We specify $u = T_1$ at $x = 0$ and $u = T_2$ at $x = L$, this is a Dirichlet boundary condition, because here the values of the unknown u is specified at the boundaries $x = 0$ and $x = L$. If the values specified at the boundaries are zero, then we call this the *homogeneous* Dirichlet boundary condition. So

$$\boxed{u(0,t) = 0,\ u(L,t) = 0}$$

is a homogeneous Dirichlet boundary condition for a problem in $0 < x < L$.

For the heat conduction problem discussed in 2.2.3, this boundary condition is equivalent to specifying the value of the temperatures at the two ends of the rod.

B. *Neumann Condition*, where the value of the normal derivative of the unknown is specified at the boundaries. [The normal derivative is the derivative normal to the boundary. In a one-dimensional space x, u_x is the normal derivative of u.] For the above mentioned domain,

$$\boxed{u_x(0,t) = b_1,\ u_x(L,t) = b_2}$$

is a Neumann boundary condition if b_1 and b_2 are specified. If $b_1 = 0$ and $b_2 = 0$, we then have a homogeneous Neumann boundary condition.

For the heat conduction problem, the Neumann condition is equivalent to specifying the heat fluxes at the two ends of the rod. A zero flux represents the fact that the ends of the rod are insulated.

C. *Robin condition*, which involves the specification of a linear combination of u and its normal derivative. An example is

$$\boxed{\begin{aligned} ku_x(0,t) &= h[u(0,t) - b_1] \\ ku_x(L,t) &= h[u(L,t) - b_2] \end{aligned}}\,.$$

For the heat conduction problem, the above condition describes the heat fluxes at the ends of the rod as a difference of the rod temperature and the temperature b_1 and b_2 of the ambient medium with which the rod is in contact.

2.8 Initial Conditions

For the diffusion or heat equation, the PDE governs the diffusion of a tracer or the conduction of heat, whereas the boundary conditions tells us what is happening at the boundaries to affect the solution inside the domain of interest. The initial condition tells us the state from which the solution evolves. Without it, the mathematical specification of the problem is incomplete. Physically, we understand that two identical conducting rods with the same boundary conditions can evolve differently if they start with different initial temperatures. A completely specified problem can be, for example:

$$\begin{aligned}
&\text{PDE:} && u_t = k u_{xx},\ 0 < x < L,\ t > 0\\
&\text{BC:} && u(0,t) = 0,\ u(L,t) = 0,\ t > 0\\
&\text{IC:} && u(x,0) = f(x),\ 0 < x < L,\quad \text{where} f(x) \text{ is given.}
\end{aligned}$$

For the wave equation, which has a second-order derivative in time, we need two initial conditions. An example of a completely specified problem is:

$$\begin{aligned}
&\text{PDE:} && u_{tt} = c^2 u_{xx},\ 0 < x < L,\ t > 0\\
&\text{BC:} && u(0,t) = 0,\ u(L,t) = 0,\ t > 0\\
&\text{IC:} && u(x,0) = f(x),\ u_t(x,0) = g(x),\ 0 < x < L,
\end{aligned}$$

where $f(x)$ and $g(x)$ prescribed.

For the vibrating string problem, $f(x)$ represents the initial position of the string, while $g(x)$ represents the initial velocity with which the string is plucked. Both are needed to completely specify the problem. Physically, we understand that the note emitted by a vibrating guitar string is different if we gently displace it and then let go (with $g = 0$) than if we displace the string with a sudden pull.

For the Laplace equation, there is no need for initial conditions. Only boundary conditions are needed. Physically, Laplace's equation determines the temperature distribution at steady state and so this temperature should depend on the conditions at the boundaries only.

If we nevertheless insist on treating one of the independent variables (say y) in the Laplace equation as time-like, and specify "initial conditions" of the form

$$\begin{aligned}
u(x,0) &= f(x),\\
u_y(x,0) &= g(x),
\end{aligned}$$

the problem in many cases becomes unphysical ("ill-posed"). An example of an ill-posed problem is:

$$\begin{array}{ll} \text{PDE:} & \dfrac{\partial^2}{\partial x^2}u + \dfrac{\partial^2}{\partial y^2}u = 0,\ 0 < x < L,\ y > 0 \\ \text{BC:} & u(0, y) = 0,\ u(L, y) = 0 \\ \text{IC:} & u(x, 0) = f(x),\ u_y(x, 0) = g(x),\ 0 < x < L. \end{array}$$

2.9 Exercises

1. Classify the following PDEs and their associated boundary conditions:

   (a) PDE: $u_t - u_{xx} = \sin x$, $0 < x < \pi$, $t > 0$
   BC: $u(0, t) = 0$, $u(\pi, t) = 0$, $t > 0$

   (b) PDE: $u_{tt} - a(x)^2 u_{xx} = 0$, $t > 0$, $0 < x < L$ where $a(x)$ is a real and nonzero given function.
   BC: $u_x(0, t) = u_x(L, t) = 0$.

2. Consider the following heat equation

   $$u_t = \alpha^2 u_{xx},\ 0 < x < L,\ t > 0,\ \alpha = \text{constant}.$$

   for a rod of length L whose ends are maintained at temperature $u = 0$, i.e.

   $$u(x, t) = 0 \text{ at } x = 0 \text{ and at } x = L.$$

   Initially, at $t = 0$, the temperature distribution is given by

   $$u(x, 0) = \sin\left(\frac{\pi x}{L}\right),\ 0 < x < L.$$

   Solve for $u(x, t)$ for $t > 0$. [Hint: Assume that the solution can be written in the "separable" form:

   $$u(x, t) = T(t) \sin \frac{\pi x}{L}.$$

   Find $T(t)$ by substituting it into the PDE. Make sure that the initial condition and the boundary conditions are also satisfied.]

3. Same as problem # 1, except the initial condition is

   $$u(x, 0) = \sin \frac{\pi x}{L} + \frac{1}{4} \sin \frac{2\pi x}{L}.$$

   [Hint: Assume $u(x, t) = T_1(t) \sin \frac{\pi x}{L} + T_2(t) \sin \frac{2\pi x}{L}$.]

2.10 Solutions

1. (a) PDE: *parabolic*, nonhomogeneous, linear, second order, two independent variables

   BC: Homogeneous Dirichlet.

   (b) PDE: *Hyperbolic*, homogeneous, linear, second order, two independent variables.

   BC: Homogeneous Neumann.

2. Solve the heat equation,

   $u_t = \alpha^2 u_{xx}$, $0 < x < L$, $t > 0$

   with boundary conditions, $u(0,t) = u(L,t) = 0$ and initial condition $u(x,0) = \sin(\pi x/L)$.

   Assume that the solution can be written in the "separable" form: $u(x,t) = T(t)\sin(\pi x/L)$.

   Substitute into the PDE to find: $dT/dt + (\alpha\pi/L)^2 T = 0$.

   This is a first order ODE with general solution: $T(t) = ce^{-(\alpha\pi/L)^2 t}$.

   Now we use the initial condition to find: $c = 1$. Thus we have the solution:

   $u(x,t) = e^{-(\alpha\pi/L)^2 t}\sin(\pi x/L)$.

3. This problem is the same as problem 2 except that we now have the initial condition:

   $u(x,0) = sin(\pi x/L) + \sin(2\pi x/L)/4$.

   Following the hint we assume that the solution has the form $u(x,t) = u_1(x,t) + u_2(x,t)$ where $u_1(x,t) = T_1(t)\sin(\pi x/L)$ and $u_2(x,t) = T_2\sin(2\pi x/L)$.

   We can see that $u_1(x,t)$ is the solution we found in problem 2 and that in finding the solution $u_2(x,t)$ we do exactly as in problem 2, where the general solution becomes $T_2(t) = c_2 e^{-(2\alpha\pi/L)^2 t}$ with the constant $c_2 = 1/4$.

   Thus we have, after adding u_1 and u_2, the solution to problem 3 as:

$$u(x,t) = e^{-(\alpha\pi/L)^2 t}\sin(\pi x/L) + e^{-(2\alpha\pi/L)^2 t}\sin(2\pi x/L)/4.$$

Chapter 3

Method of Similarity Variables (Optional)

3.1 Introduction

We will start in Chapter 6 discussing standard, general methods for solving linear PDEs. There are also a few special methods which work for certain types of PDEs. The method of similarity variables is one of these, and works for the diffusion equation under certain circumstances. It will be discussed in this chapter. d'Alembert's method is a special method which works only for the wave equation. We will discuss that in Chapter 5. These special methods may appear to you as "tricks" which some smart mathematicians invented and which a student could probably never come up with if he or she were asked to do it from scratch. However, you are not being asked to invent them. We just want you to be exposed to them. If you do not like using the special methods, you can wait until a few chapters later, when the same problems will be solved again using the standard methods.

3.2 Similarity Variable

Let us consider the diffusion equation in an infinite domain

$$\text{PDE:}\quad \frac{\partial}{\partial t}u = D\frac{\partial^2}{\partial x^2}u,\ -\infty < x < \infty,\ t > 0. \tag{3.1}$$

To complete the specification of the problem we should also state what the boundary and initial conditions are. However, not all initial conditions are compatible with the method we are about to use. So what we are doing is to try this simple method of similarity solution and, if we find some solutions to (3.1), we find out what initial conditions and boundary conditions they satisfy.

We define a similarity variable

$$z \equiv \frac{x}{\sqrt{Dt}}, \tag{3.2}$$

and assume that the solution depends on this single variable only, i.e.,

$$u(x,t) = U(z). \tag{3.3}$$

The question, "How did you come up with (3.2)?" will be answered a little later. But first, let us see if (3.2) actually works.

Since

$$\frac{\partial z}{\partial x} = \frac{1}{\sqrt{Dt}}, \quad \frac{\partial z}{\partial t} = -\frac{xD}{2(Dt)^{3/2}} = -\frac{z}{2t},$$

we find, using a prime to denote ordinary differentiation with respect to the independent variable z,

$$\begin{aligned}
\frac{\partial}{\partial t}U(z) &= \frac{d}{dz}U(z)\frac{\partial z}{\partial t} = U'(z)[-\frac{z}{2t}] \\
\frac{\partial}{\partial x}U(z) &= \frac{d}{dz}U(z)\frac{\partial z}{\partial x} = U'(z)\left[\frac{1}{\sqrt{Dt}}\right] \\
\frac{\partial^2}{\partial x^2}U(z) &= U''(z)\left[\frac{1}{\sqrt{Dt}}\right]^2.
\end{aligned}$$

The diffusion equation (3.1) becomes

$$U''(z) + \frac{z}{2}U'(z) = 0. \tag{3.4}$$

Equation (3.4) is an ODE and can be solved using standard methods reviewed in Chapter 1. We first observe that (3.4) is actually a first order ODE for $W(z) \equiv U'(z)$. Namely,

$$\frac{d}{dz}W + \frac{z}{2}W = 0. \tag{3.5}$$

To solve (3.5), we move all the W's to one side and the z's to the other side:

$$\frac{dW}{W} = -\frac{z}{2}dz.$$

Integrating then yields

$$\ell n W = -\frac{z^2}{4} + \text{ constant},$$

and so

$$W(z) = ae^{-z^2/4}. \tag{3.6}$$

Once $W(z)$ is known, $U(z)$ can be found by a simple integration:

$$\begin{aligned} U(z) &= \int^z W(z')dz' \\ &= a\int^z e^{-z'^2/4}dz' + b. \end{aligned} \tag{3.7}$$

Converting z to $x/\sqrt{Dt}$ then yields the solution:

$$u(x,t) = U\left(\frac{x}{\sqrt{Dt}}\right) = a\int^{x/\sqrt{Dt}} e^{-z^2/4}dz + b. \tag{3.8}$$

Notice that there are two arbitrary constants, a and b, in the solution, which needs to be determined from the boundary and initial conditions. We could have rewritten (3.8) as

$$u(x,t) = a\int_{-\infty}^{x/\sqrt{Dt}} e^{-z^2/4}dz + U(-\infty), \tag{3.9}$$

or

$$u(x,t) = a\int_{\infty}^{x/\sqrt{Dt}} e^{-z^2/4}dz + U(\infty). \tag{3.10}$$

For example, suppose one of the boundary conditions is

$$u(x,t) \to 0 \quad \text{as} \quad x \to -\infty, \;\; t > 0. \tag{3.11}$$

We then know $U(-\infty) = 0$ and we should use (3.9) as our solution.

Let us now find out what IC (3.9) satisfies. To do this, we let t approach zero from above, and note that

$$\lim_{t\to 0^+} \frac{x}{\sqrt{Dt}} = \begin{cases} \infty & \text{if } x > 0 \\ -\infty & \text{if } x < 0. \end{cases}$$

Since $\int_{-\infty}^{-\infty} e^{-z^2/4} dz = 0$ and

$$\int_{-\infty}^{\infty} e^{-z^2/4} dz = \text{ some constant (actually } 2\sqrt{\pi}),$$

we have

$$u(x,0) = \begin{cases} 2\sqrt{\pi}a & \text{if } x > 0 \\ 0 & \text{if } x < 0. \end{cases} \tag{3.12}$$

Thus the solution (3.9) with $U(-\infty) = 0$:

$$u(x,t) = a \int_{-\infty}^{x/\sqrt{Dt}} e^{-z^2/4} dz$$

satisfies the PDE (3.1), the BC (3.11) as $x \to -\infty$, and boundedness as $x \to +\infty$, and the IC:

$$u(x,0) = \begin{cases} U_1, & \text{a constant for } x > 0 \\ 0 & \text{for } x < 0, \end{cases} \tag{3.13}$$

with a $= U_1/2\sqrt{\pi}$.

Although we are happy that we have found a solution to the diffusion equation, we are not satisfied with the fact that it works only for this very special initial condition (3.13). Can we find more solutions?

We notice, by differentiating the diffusion equation (3.1) with respect to t or x, that the derivatives of $u(x,t)$ also satisfy the same equation. For example: u_t and u_x both satisfy the diffusion equation because

$$\frac{\partial}{\partial t}(u_t) = D\frac{\partial^2}{\partial x^2}(u_t), \;\; \frac{\partial}{\partial t}(u_x) = D\frac{\partial^2}{\partial x^2}(u_x).$$

Consequently, since $U(z)$, as given by (3.8), is a solution to the equation, so should $\frac{\partial}{\partial t}U(z)$ or $\frac{\partial}{\partial x}U(z)$. Let us try

$$u(x,t) = \frac{\partial}{\partial x}U(z) = U'(z)\frac{\partial z}{\partial x} = \frac{1}{\sqrt{Dt}}W(z),$$

and so

$$u(x,t) = \frac{a}{\sqrt{Dt}} \cdot \exp\left\{-\frac{x^2}{4Dt}\right\}. \tag{3.14}$$

The solution in (3.14) satisfies the PDE, the BC that $u(x,t) \to 0$ as $x \to \pm\infty$ for $t > 0$, and the following special IC, obtained from (3.l4) by taking the limit $t \to 0^+$:

$$u(x,0) = \begin{cases} 0 & \text{if } x \neq 0 \\ \infty & \text{if } x = 0 \end{cases} \tag{3.15}$$

(3.15) is a rather special initial condition, but is nevertheless useful as a solution of the "drunken sailor" problem, to be discussed next, and also as a "fundamental solution," with which more general solutions can be constructed.

3.3 The "drunken sailor" problem solved

Solve:

$$\text{PDE:} \quad \frac{\partial}{\partial t}u = D\frac{\partial^2}{\partial x^2}u, \ -\infty < x < \infty, \ t > 0 \tag{3.16}$$

$$\text{BC:} \quad u(x,t) = 0 \ \text{ as } x \to \pm\infty \tag{3.17}$$

$$\text{IC:} \quad u(x,0) = \delta(x), \ -\infty < x < \infty. \tag{3.18}$$

The "delta function", $\delta(x)$, used in (3.18), is defined as

$$\delta(x) = \begin{cases} 0 & \text{if } x \neq 0 \\ \infty & \text{if } x = 0 \end{cases} \text{ and } \int_{-\infty}^{\infty} \delta(x)dx = 1.$$

It also is required to have the property that, for any smooth function $f(x)$

$$\int_{-\infty}^{\infty} f(x)\delta(x-a)dx = f(a).$$

This last equation can be derived by noting that $\delta(x-a)$ is zero for $x \neq a$, and so only the value of $f(x)$ at $x = a$ matters in the integrand:

$$\int_{-\infty}^{\infty} f(x)\delta(x-a)dx = f(a)\int_{-\infty}^{\infty} \delta(x-a)dx = f(a).$$

The problem just posed concerns a "drunken sailor" executing a random walk. He is initially located at $x = 0$. Let $p(x,t)$ be the probability of finding him subsequently at x a time t later. So we know

$$p(x,0) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x \neq 0. \end{cases} \tag{3.19}$$

In each time interval Δt, he takes a step Δx. He may take this step either forward or backward, with equal probability. Although his precise path is unpredictable, we can nevertheless try to find $p(x,t)$, the probability that he will end up at x at time t. The equation governing $p(x,t)$ in the case of small Δx and Δt has been derived in Chapter 2, and it obeys the diffusion equation (3.1). In (3.16), $u(x,t)$ is the *probability density*, $u(x,t) = p(x,t)/\Delta x$. So the initial probability (3.19) becomes, in terms of $u(x,t)$,

$$u(x,0) = \begin{cases} \infty & \text{if } x = 0 \\ 0 & \text{if } x \neq 0, \end{cases} \tag{3.20}$$

in the limit $\Delta x \to 0$. Hence the IC (3.18).

[Note also $\int_{-\infty}^{\infty} u(x,t)dx = \int_{-\infty}^{\infty} u(x,0)dx = 1$, which is true because the probability is 1 that we will always find the sailor somewhere in $-\infty < x < \infty$].

From the previous section we know that

$$u(x,t) = \frac{a}{\sqrt{Dt}} \exp\left\{-\frac{x^2}{4Dt}\right\}, \quad \text{for any constant } a, \tag{3.21}$$

satisfies the PDE (3.16) and the BC (3.17). It also is able to satisfy the IC because $u(x,0) = 0$ for $x \neq 0$ and $u(x,0) = \infty$ for $x = 0$. However, we still need to determine the constant a so that it satisfies the IC $u(x,0) = \delta(x)$ and not $u(x,0) = 2\delta(x)$, for example. Thus, we want (3.21) to satisfy

$$\int_{-\infty}^{\infty} u(x,0)dx = 1.$$

It is easier to perform this integration on (3.21) before taking the limit $t \to 0^+$.

$$\int_{-\infty}^{\infty} u(x,t) = \frac{a}{\sqrt{Dt}} \int_{-\infty}^{\infty} \exp\left\{-\frac{x^2}{4Dt}\right\} dx = a \int_{-\infty}^{\infty} e^{-z^2/4} dz = 2a\sqrt{\pi}.$$

Thus a must be given by $1/2\sqrt{\pi}$. Finally the solution to (3.16)–(3.18) is

$$\boxed{u(x,t) = \frac{1}{2}(\pi Dt)^{-1/2} \exp\left\{-\frac{x^2}{4Dt}\right\}}. \tag{3.22}$$

The behavior of the solution for various values of Dt is plotted in Figure 3.1. This behavior is very typical of diffusive processes: An initial concentration is smoothed as it spreads out into a wider and wider region as time increases. For the "drunken sailor" problem, we find that as time increases, the probability of finding him at his initial location $x = 0$ decreases while, as time goes by, it becomes more and more likely than initially to find him at locations further and further away from his initial spot.

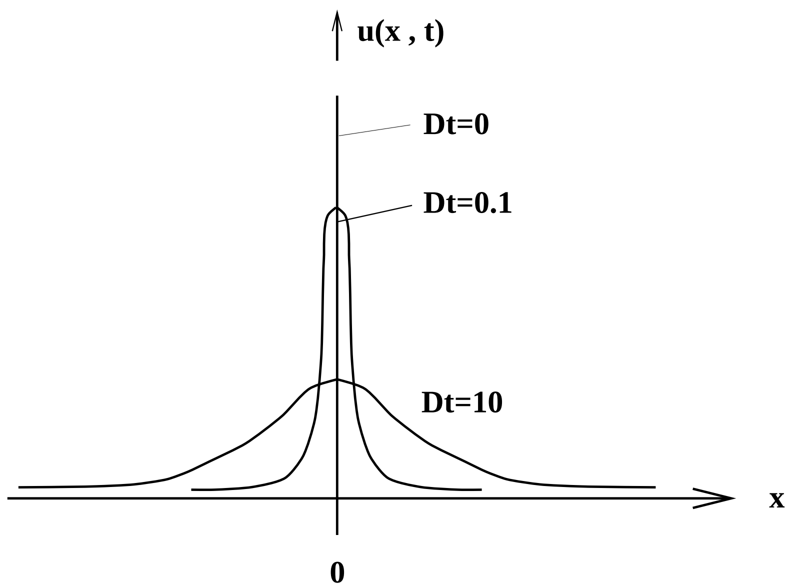


Figure 3.1: The solution $u(x,t)$ as a function of x for various values of Dt.

3.4 The fundamental solution

The solution to (3.16), (3.17), and (3.18) just obtained is called the *fundamental solution* for the diffusion equation in an infinite domain. It is so called because it can be used to construct the solution to the more general problem:

$$\text{PDE:} \quad \frac{\partial}{\partial t}u = D\frac{\partial^2}{\partial x^2}u, \ -\infty < x < \infty, \ t > 0 \tag{3.23}$$

$$\text{BC:} \quad u(x,t) \to 0 \text{ as } x \to \pm\infty, \ t > 0 \tag{3.24}$$

$$\text{IC:} \quad u(x,0) = f(x), \ -\infty < x < \infty, \tag{3.25}$$

where $f(x)$ is given.

We know

$$u_0(x - x_0, t) = \frac{1}{2\sqrt{\pi Dt}} \exp\left\{-\frac{(x-x_0)^2}{4Dt}\right\} \tag{3.26}$$

satisfies (3.23) and (3.24), with the IC:

$$u_0(x - x_0) = \delta(x - x_0). \tag{3.27}$$

The solution that satisfies the more general IC (3.25) can be constructed from u_0 as

$$u(x,t) = \int_{-\infty}^{\infty} f(x_0) u_0(x - x_0, t) dx_0. \tag{3.28}$$

It is easy to verify that the IC (3.25) is indeed satisfied by (3.28). Using (3.27), we have

$$\begin{aligned} u(x,0) &= \int_{-\infty}^{\infty} f(x_0)\delta(x - x_0) dx_0 \\ &= f(x), \end{aligned}$$

which is the same as (3.25). It is also easy to show that (3.28) satisfies the PDE and BC. (Please verify)

The general solution to (3.23), (3.24) and (3.25) will have to be left in the integral form

$$\boxed{u(x,t) = \frac{1}{2\sqrt{\pi D t}} \int_{-\infty}^{\infty} f(x_0) \exp\left\{ -\frac{(x - x_0)^2}{4Dt} \right\} dx_0}\,, \tag{3.29}$$

unless we are given some simple form for $f(x)$ that allows the integral to be evaluated explicitly.

3.5 Examples from Fluid Mechanics

3.5.1 The Rayleigh problem

The Rayleigh problem describes the viscous flow of a fluid above an infinite plate located at, say, $y = 0$. The fluid and the flat plate are moving with a uniform velocity U_1 in the x-direction for $t < 0$. For $t > 0$, the flat plate is suddenly stopped, dragging the fluid immediately above it to rest. Let u be the velocity of the fluid in the x direction. It satisfies the following PDE, BC and IC.

$$\text{PDE:} \quad \frac{\partial}{\partial t} u = \nu \frac{\partial^2}{\partial y^2} u, \;\; y > 0, \;\; t > 0. \tag{3.30}$$

$$\text{BC:} \quad u \to U_1 \;\; \text{as } y \to \infty, \;\; t < \infty \tag{3.31}$$

$$u = 0 \;\; \text{at } y = 0, \;\; t > 0 \tag{3.32}$$

$$\text{IC:} \quad u = U_1 \;\; \text{for} \;\; y > 0, \;\; t = 0 \tag{3.33}$$

In (3.30), ν is the coefficient of kinematic viscosity of the fluid. In terms of the similarity variable

$$z \equiv y/\sqrt{\nu t},$$

a similarity solution is (from (3.8))

$$u(y,t) = a \int^{y/\sqrt{\nu t}} e^{-z^2/4} dz + \text{ constant.}$$

Since we want the solution to be zero at $y = 0$ for $t > 0$, we set the lower limit of integration to zero. Thus

$$u(y,z) = a \int_0^{y/\sqrt{\nu t}} e^{-z^2/4} dz.$$

Since $u(\infty, t) = a \int_0^\infty e^{-z^2/4} dz = a\sqrt{\pi}$, we take $a = U_1/\sqrt{\pi}$, so that the BC $u(\infty, t) = U_1$ can be satisfied.

Thus,

$$\boxed{u(y,t) = \frac{U_1}{\sqrt{\pi}} \int_0^{y/\sqrt{\nu t}} e^{-z^2/4} dz = \frac{2U_1}{\sqrt{\pi}} \int_0^{y/2\sqrt{\nu t}} e^{-z'^2} dz'.} \tag{3.34}$$

This also satisfies the IC $u(y, 0) = U_1$ for $y > 0$, since

$$\begin{aligned} u(y,0) &= \frac{U_1}{\sqrt{\pi}} \int_0^\infty e^{-z^2/4} dz \quad \text{for } y > 0. \\ &= U_1. \end{aligned}$$

The integral in (3.34) is tabulated as the error function

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-z'^2} dz'.$$

(see, e.g. M. Abramowitz and L.A. Stegun: *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*). So we can rewrite Eq. (3.34) as

$$u(y;t)/U_1 = \operatorname{erf}\left(\frac{y}{2\sqrt{\nu t}}\right).$$

It is plotted as a function of its argument in Figure 3.2, which is also a plot of $u(y,t)/U_1$, vs $y/2\sqrt{\nu t}$.

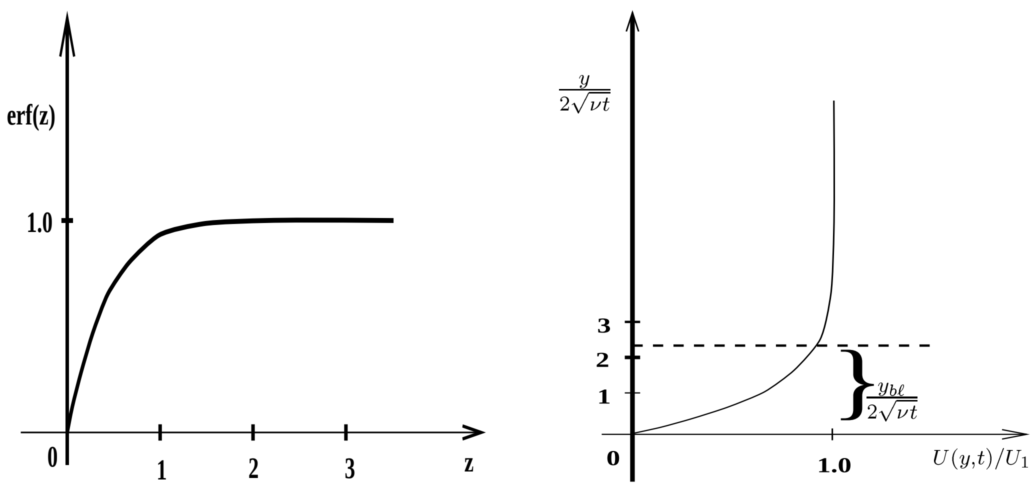


Figure 3.2: (a): The Error function; (b): The solution to the Rayleigh problem

We note that the error function is nearly 1 for most values of z except for z less than 2. In particular, we note that

$$\mathrm{erf}(2.4) = 0.999.$$

In terms of the solution $u(y,t)$, this shows that u approaches its far-field value of uniform flow U_1 for $y \geq y_{b\ell}$, where $y_{b\ell}$ is here given by

$$\frac{y_{b\ell}}{2\sqrt{\nu t}} \sim 2.4,$$

or

$$y_{b\ell} \sim 5\sqrt{\nu t}.$$

The quantity $y_{b\ell}$ is called the boundary layer thickness. There is a velocity deficit, i.e. $u(y,t)$ is significantly less than U_1, within the boundary layer defined by

$$0 \leq y \leq y_{b\ell}.$$

The boundary layer furthermore increases in thickness with time, so that the effect of the viscous drag by the boundary eventually spreads throughout the interior of the fluid as $t \to \infty$.

3.5.2 Diffusion of vorticity

Consider two uniform fluid streams initially adjacent to each other, with the velocity in the x-direction given by

$$u(y,0) = \begin{cases} +U_1 & \text{for } y > 0 \\ -U_1 & \text{for } y < 0. \end{cases}$$

Assuming that the flow is independent of x, the vorticity ω of the fluid is then given by

$$\omega(y,t) = -\frac{\partial}{\partial y} u(y,t).$$

Therefore initially there is a concentration of vorticity at $y = 0$:

$$\omega(y,0) = -\frac{\partial}{\partial y} u(y,0) = -2U_1 \delta(y),$$

where $\delta(y)$ is the delta function introduced earlier; it takes the value zero everywhere except at $y = 0$, where it is infinite. Since ω also satisfies the diffusion equation when there is fluid viscosity, the problem we want to solve is:

$$\begin{array}{ll} \text{PDE:} & \dfrac{\partial}{\partial t}\omega = \nu \dfrac{\partial^2}{\partial y^2}\omega, \ -\infty < y < \infty, \ t > 0 \\ \text{BC:} & \omega(y,t) \to 0 \text{ as } y \to \pm\infty \\ \text{IC:} & \omega(y,0) = -2U_1\delta(y), \ -\infty < y < \infty. \end{array}$$

The problem is the same as that discussed in section 3.3 except for the factor $-2U_1$. Thus

$$\boxed{\omega(y,t) = -\frac{U_1}{\sqrt{\pi\nu t}} \exp\left\{-\frac{y^2}{4\nu t}\right\}^2}.$$

The solution tells us that an initially concentrated vorticity at $y = 0$ is diffused by viscosity a distance of $y \sim \pm\sqrt{\nu t}$ as time increases. As the vorticity is spread out, its magnitude decreases. Eventually, when viscosity acts throughout the domain, $\omega \to 0$, as the flow becomes uniform, $u(y,t) \to 0$.

3.6 The Age of the Earth

In 1865, Lord Kelvin produced an estimate of the age of the earth by considering the earth as a warm, chemically inert sphere cooling through its surface. Kelvin knew at the time that the temperature of the earth is hotter within. Assuming that the hotter temperature was the temperature of the earth at an earlier time, he was able to deduce the age of the earth to be about a hundred million years based on measurements of the gradient of temperature near the surface. The mathematical formulation of the problem is as follows: Let $u(y,t)$ be the temperature of the earth at depth y and time t since the the surface of the earth first solidified.

$$\text{PDE:} \quad \frac{\partial}{\partial t}u = \alpha^2 \frac{\partial^2}{\partial y^2}u, \;\; y > 0, \;\; t > 0, \tag{3.35}$$

$$\text{BC:} \quad u(0,t) = u_s, \;\; t > 0 \tag{3.36}$$

$$\text{IC:} \quad u(y,0) = u_0, \;\; y > 0. \tag{3.37}$$

[Kelvin actually treated the earth as a sphere. However, since we are interested only in the variation with depth, we have simplified the problem to one dimension, following Korner: *Fourier Analysis*. The domain is artifically extended to $0 < y < \infty$, without problem, since the influence of the surface decays rapidly with depth. The fact that the core of the earth is not solid also does not matter for the same reason.]

In (3.36), u_s is the surface temperature at present and was assumed by Kelvin to be the same as in the past. The initial temperature of the earth, u_0, was taken to be the melting temperature of rock.

The problem to be solved is very similar to the Rayleigh problem considered in section 3.5. It is, in fact, identical for the new quantity ψ defined as

$$\psi(y,t) \equiv u(y,t) - u_s.$$

In terms of ψ the problem is

$$\begin{aligned} &\text{PDE:} && \frac{\partial}{\partial t}\psi = \alpha^2 \frac{\partial^2}{\partial y^2}\psi, \;\; 0 < y < \infty, \;\; t > 0 \\ &\text{BC:} && \psi(0,t) = 0, \;\; t > 0 \\ &\text{IC:} && \psi(y,0) = \psi_1 \equiv u_0 - u_s, y > 0. \end{aligned}$$

The solution was found, in (3.34), to be

$$\psi(y,t) = \frac{2\psi_1}{\sqrt{\pi}} \int_0^{y/2\sqrt{\alpha^2 t}} e^{-z^2} dz,$$

and so

$$u(y,t) = u_s + \frac{2(u_0 - u_s)}{\sqrt{\pi}} \int_0^{y/2\sqrt{\alpha^2 t}} e^{-z^2} dz. \tag{3.38}$$

The temperature gradient at the surface G, which can be measured at the present time t, can be evaluated from (3.38) as

$$G \equiv u_y(0,t) = \frac{(u_0 - u_s)}{(\pi\alpha^2 t)^{1/2}}. \tag{3.39}$$

Assuming that the earth is more or less homogeneous, its conductivity α^2 and the melting temperature u_0 can be determined by measuring samples of surface rock. The "age of the earth", t, can then be determined from (3.39) as

$$t = (u_0 - u_s)^2/(\pi\alpha^2 G^2). \tag{3.40}$$

From this formula Kelvin arrived at an estimate of a hundred million years as the age of the earth. This age estimate, although much higher than the previously believed age of a few thousand years, appeared to be too low for Darwin's theory of evolution, and this greatly troubled Darwin. The discovery of radioactivity near the end of the 19th century suggested a source of heat unknown to Kelvin that could replace heat lost from the surface. The current estimate of the age of the earth is about a few thousand million years, long enough for Darwin's theory to apply.

3.7 Summary

The method of similarity is a special solution method that almost requires one to "go backwards." One first finds a number of solutions and then sees which one satisfies the IC (and BCs). It appears to work for problems in infinite and semi-infinite domains where the physical problem does not contain a length or time scale. In such cases, if we seek to nondimensionalize the diffusion equation, we will be at a loss to find a typical length scale L to make x dimensionless and a typical time scale T to make t dimensionless. The only possible quantity, other than x, that has the *dimension* of length is $\sqrt{Dt}$, since D has the units of cm^2/sec. Making the bold move of nondimensionalizing x by $\sqrt{Dt}$ then leads to the similarity variable $z = x/\sqrt{Dt}$, and the assumption that the nondimensional quantity $u(x,t)/U_1$ should only depend on the nondimensional variable z, where U_1 is a typical value of u. Thus

$$u(x,t)/U_1 = F\left(\frac{x}{\sqrt{DT}}\right).$$

To find out if such a similarity solution works or not, we need to substitute it back into the diffusion equation and see if an ODE will result with $z = \frac{x}{\sqrt{Dt}}$

being its only independent variable. We have shown that this does indeed work. We further have found

$$F(z) = a \int^{z} e^{-z'^2} dz' + b,$$

which contains two arbitrary constants, a and b. These are to be determined by a combination of BC and IC.

The functional form of the solution can only satisfy a special form of ICs. Otherwise, we need to construct a different solution. We have shown that other solutions can be constructed by taking the partial derivatives of $F(z)$ with respect to x or t. These are no longer similarity solutions, because they do not only depend on z. Nevertheless, they are still solutions to the diffusion equation and may be appropriate for some (other) special IC's.

3.8 Exercises

1. Consider heat conduction in a very long rod (so long that we can consider it of infinite length). Initially, the temperature of the rod is known and is given by

$$\text{IC:} \qquad u(x,0) = \begin{cases} T_1, & \text{a constant, } x > 0 \\ 0, & x < 0. \end{cases}$$

Solve for $u(x,t)$ for $t > 0$, assuming that $u(x,t)$ satisfies the heat equation

$$u_t = \alpha^2 u_{xx}, \; -\infty < x < \infty, \; t > 0.$$

Impose appropriate BC's compatible with the IC.

2. As a variant of the Rayleigh problem, consider the case of a fluid at rest for $t \leq 0$ above a stationary flat plate located at $y = 0$. The flat plate then instantaneously starts moving at a constant speed of U_1 for $t > 0$. Solve

$$\begin{aligned} \text{PDE:} \quad & u_t = \nu u_{yy}, \; y > 0, \; t > 0 \\ \text{BC:} \quad & u(y,t) \to 0 \text{ as } y \to \infty, \; t > 0 \\ & u(0,t) = U_1 \;\text{ for } t > 0 \\ \text{IC:} \quad & u(y,0) = 0, \; y \geq 0. \end{aligned}$$

3.9 Solutions

1. The method of similarity variables yields the solution in (3.8)

$$u(x,t) = a \int_{-\infty}^{x/2\sqrt{\alpha^2 t}} e^{-z^2} dz + b.$$

We require, as a BC, $u(-\infty, t) = 0$. This implies that $b = 0$.

As $t \to 0^+$, the upper limit of the integral approaches infinity for $x > 0$. And since

$$\int_{-\infty}^{\infty} e^{-z^2} dz = \sqrt{\pi},$$

we want $a\sqrt{\pi} = T_1$.

Also as $t \to 0^+$, the upper limit of the integral approaches $-\infty$ for $x < 0$, so $u(x,0) = 0$ for $x < 0$. Thus, the solution satisfying the IC is

$$u(x,t) = T_1/\sqrt{\pi} \int_{-\infty}^{x/2\sqrt{\alpha^2 t}} e^{-z^2} dz.$$

2. The method of similarity variables yields the solution in (3.8), which can be rewritten as

$$u(y,t) = a \int_{0}^{y/2\sqrt{\nu t}} e^{-z^2} dz + b.$$

The BC that $u(0,t) = U_1$ implies that $b = U_1$. The BC that $u(\infty, t) = 0$ implies that

$$a = -b/\left(\int_0^{\infty} e^{-z^2} dz\right) = -U_1 2/\sqrt{\pi}.$$

so

$$u(y,t) = U_1(1 - \frac{2}{\sqrt{\pi}} \int_0^{y/2\sqrt{\nu t}} e^{-z^2} dz).$$

Let us check if this satisfies the IC. If it doesn't we would have failed and the solution is probably not of the similarity form. Luckily, it does satisfy the IC, since $y/2\sqrt{\nu t} \to \infty$ as $t \to 0^+$ for any $y > 0$, and $\frac{2}{\sqrt{\pi}} \int_0^{\infty} e^{-z^2} dz = 1$.

Chapter 4

Simple Plane-Wave Solutions

4.1 Introduction

As you can see from the examples discussed in the previous chapter, it is not easy to find the solution satisfying a given initial condition. In many situations, however, we are not particularly interested in the detailed evolution of the solution from a given initial condition. Then, it is much easier to find a solution to the PDE if we simply ignore the initial conditions for the moment and consider the solution to be in the form of a single wave component. This approach makes sense for wave problems where we are interested in the dependence of the frequency of oscillation on the wavelength, and for problems of forced oscillation where we know that eventually the solution will oscillate with a frequency imposed by the forcing. More general solutions can be constructed from these simple solutions by superposition, as will be discussed in later chapters.

4.2 Linear homogeneous equations

We know, from our experience from solving ODEs (see Chapter 1), that linear equations with constant coefficients have exponentials (or sines and cosines) as their solutions. Because of Euler's identity, derived in Chapter 1, sines and cosines are also exponentials (but with complex exponents). Therefore, we shall start by trying a solution of the form:

$$u(x,t) = Ae^{i\omega t}e^{ikx}, \tag{4.1}$$

for a linear PDE involving the two variables t and x, provided that the coefficients of the PDE are constants.

Here k can have the interpretation of the wave number (which is equal to 2π divided by the wavelength) and ω the frequency (which is equal to 2π divided by the period) provided that they are real.

4.2.1 Three different types of behaviors

One can very quickly deduce (using this simple method) that the three different types of second-order PDEs dicussed in Chapter 2 have qualitatively different behaviors.

(a) The wave equation:

$$\boxed{u_{tt} = c^2 u_{xx}}\,. \tag{4.2}$$

Substituting

$$u(x,t) = Ae^{i\omega t}e^{ikx}$$

into (4.2) yields

$$A(i\omega)^2 e^{i\omega t}e^{ikx} = c^2 A(ik)^2 e^{i\omega t}e^{ikx}.$$

On cancelling $Ae^{i\omega t}e^{ikx}$, we are left with

$$\omega^2 = c^2 k^2. \tag{4.3}$$

Equation (4.4) is called the *dispersion relationship* for Equation (4.2). It relates the frequency ω to the wavenumber k as:

$$\omega = \pm ck. \tag{4.4}$$

We thus obtain two solutions, one corresponding to the positive root and one to the negative root in (4.4):

$$\boxed{u_1(x,t) = A_1 e^{ik(x+ct)}}\,, \tag{4.5}$$

and

$$\boxed{u_2(x,t) = A_2 e^{ik(x-ct)}}\,. \tag{4.6}$$

A more general solution can be obtained by adding the two:

$$u(x,t) = u_1(x,t) + u_2(x,t).$$

The solution $u_1(x,t)$ has the property that it would appear to be unchanging to an observer travelling with speed c in the negative x-direction, i.e. $u_1(x,t) = \text{const.}$, if $x + ct = \text{const.}$ Similarly, $u_2(x,t)$ is invariant to an observer moving with speed c in the positive x-direction, i.e. $u_2(x,t) = \text{const.}$, if $x - ct = \text{const.}$ We interpret (4.5) as a sinusoidal wave whose phase moves to the left and (4.6) also as a sinusoidal wave whose phase moves to the right, both with speed c.

If you are not comfortable with complex exponents, you can either assume the solution is the *real part* of (4.5), or you can directly start with

$$u(x,t) = a\cos(\omega t + kx).$$

Either way, the same dispersion relationship as (4.4) results.

Note that the overall amplitude A is not determined by this method. It cannot be determined without a specification of the initial conditions.

(b) The diffusion equation:

$$\boxed{u_t = \alpha^2 u_{xx}}\,. \tag{4.7}$$

We again try

$$u(x,t) = Ae^{i\omega t}e^{ikx}.$$

Upon substitution into the diffusion equation, we get

$$Ai\omega e^{i\omega t}e^{ikx} = \alpha^2(ik)^2 Ae^{i\omega t}e^{ikx}$$

On cancelling $Ae^{i\omega t}e^{ikx}$, we find:

$$i\omega = -\alpha^2 k^2,$$

and, since $i^2 = -1$,

$$\omega = i\alpha^2 k^2. \tag{4.8}$$

Thus we find that the "frequency", ω, is imaginary. So instead of a wave oscillating in t, we have one that *decays* in time as

$$\boxed{u(x,t) = Ae^{-\alpha^2 k^2 t}e^{ikx}}\,. \tag{4.9}$$

The so-called e-folding time scale, $(\alpha^2 k^2)^{-1}$, is dependent on the wavenumber k so that waves with the shortest wavelengths (i.e. largest k) decay fastest with time. The physical interpretation is that the process of diffusion smoothes out the small scales.

(c) The Laplace equation:

$$\boxed{u_{xx} + u_{yy} = 0}\,. \tag{4.10}$$

Again, we try

$$u(x, y) = Ae^{ikx}e^{i\ell y}$$

and find

$$\ell^2 = -k^2.$$

Consequently

$$\ell = \pm ik.$$

Thus if k is real, i.e. the solution is sinusoidal in the x-direction, it must be exponential in the y-direction,

$$\boxed{u(x, y) = A_1 e^{-ky}e^{ikx} + A_2 e^{ky}e^{ikx}}\,. \tag{4.11}$$

On the other hand, if the solution is sinusoidal in the y-direction (i.e., ℓ real), it must be exponential in the x-direction

$$\boxed{u(x, y) = A_1 e^{i\ell y}e^{-\ell x} + A_2 e^{i\ell y}e^{\ell x}}\,. \tag{4.12}$$

Comment:

These simple solutions help shed light on the qualitatively different behaviors to the three types of second-order PDEs typified by the wave equation (hyperbolic type), the diffusion equation (parabolic type) and the Laplace equation (elliptic type) briefly mentioned in section 2.6. One can think of the hyperbolic equations as possessing wave solutions with two real wave speeds, the elliptic equation as having two imaginary wave speeds, and the parabolic equation as having one imaginary wave speed.

4.2.2 Rossby waves in the atmosphere

The PDE governing planetary-scale disturbances of small amplitude in the atmosphere can be found in many textbooks on atmospheric sciences (e.g. J.R. Holton: Introduction to Dynamic Meteorology) and will not be derived here. With x pointing eastward, y northward, and with t representing time, this PDE can be written as

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\Phi + \beta\frac{\partial}{\partial x}\Phi = 0, \tag{4.13}$$

where Φ is the pressure perturbation, U the speed of the eastward wind and β is the gradient in the north-south direction of the Coriolis parameter. We try the simple solution:

$$\Phi(x, y, t) = Ae^{i\omega t}e^{ikx}e^{i\ell y}, \tag{4.14}$$

and find

$$-i(\omega + Uk)(k^2 + \ell^2)\Phi + ik\beta\Phi = 0,$$

from which we obtain the dispersion relationship:

$$\omega/k = -U + \frac{\beta}{k^2 + \ell^2}.$$

There is only one root, and it is real. So the solution is oscillatory in time and has the form

$$\Phi(x, y, t) = Ae^{ik(x - c_{ph}t)}e^{i\ell y},$$

where

$$c_{ph} \equiv -\omega/k = U - \frac{\beta}{k^2 + \ell^2}$$

is the Rossby wave phase speed. It has the peculiar property, well-known to meteorologists, of propagation only to the west relative to the wind U. That is, its Doppler-shifted phase speed:

$$c_{ph} - U$$

is always negative. Furthermore, the longer Rossby waves (ones with smaller wavenumbers k and ℓ) propagate faster. This propagation is also faster for a planet with a more rapid rotation (larger β-values). All this information has been obtained without solving the full initial-value problem.

4.2.3 Laplace's equation in a circular disk

Consider the solution of Laplace's equation in a region bounded by a circle of radius a

$$\text{PDE:} \qquad \nabla^2 u = 0, \;\; r < a \tag{4.15}$$

$$\text{BCs:} \qquad u(r, \theta) = f(\theta) \;\; \text{at } r = a. \tag{4.16}$$

In polar coordinates, the Laplace operator can be written

$$\begin{aligned} \nabla^2 &= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \\ &= \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}, \end{aligned}$$

after we have made the transformation $x = r\cos\theta$ and $y = r\sin\theta$.

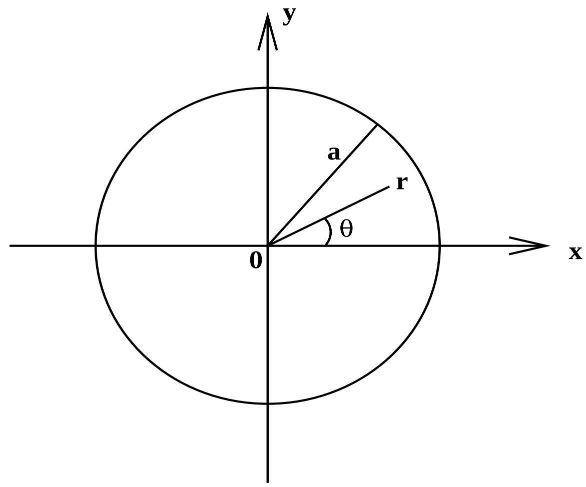


Figure 4.1: Polar coordinates.

The PDE is now rewritten

$$\boxed{r^2\frac{\partial^2}{\partial r^2}u + r\frac{\partial}{\partial r}u + \frac{\partial^2}{\partial\theta^2}u = 0}\,. \tag{4.17}$$

We shall first ignore the boundary condition at $r = a$ and proceed to find a solution to the PDE. The assumption $u(r,\theta) = e^{ikr}e^{i\ell\theta}$ would not work because the coefficients in the PDE (4.17) are not constants. However, since the coefficients are independent of θ, $e^{i\ell\theta}$ still provides the θ-dependence of $u(r,\theta)$. Thus we assume

$$u(r,\theta) = R(r)e^{i\ell\theta}. \tag{4.18}$$

Upon substitution into the PDE, we find:

$$r^2\frac{d^2}{dr^2}Re^{i\ell\theta} + r\frac{d}{dr}Re^{i\ell\theta} + (i\ell)^2Re^{i\ell\theta} = 0.$$

Cancelling out the common factor $e^{i\ell\theta}$, this becomes the ODE

$$r^2\frac{d^2}{dr^2}R + r\frac{d}{dr}R - \ell^2R = 0. \tag{4.19}$$

for $R(r)$. In ODE literature, Eq. (4.19) belongs to ODEs of "equi-dimensional type", sometimes also called an Euler or Cauchy equation. (That is, the second derivative term is multiplied by r^2, and the first derivative term by r, etc.) For this type of ODEs, one tries power solutions of the type:

$$R(r) = r^b. \tag{4.20}$$

Substituting the trial solution into the ODE yields

$$b(b-1) + b - \ell^2 = b^2 - \ell^2 = 0.$$

There are two roots to this equation:

$$b = b_1 \equiv \ell \text{ and } b = b_2 \equiv -\ell.$$

The general solution to the ODE is a linear combination of the two solutions:

$$R(r) = cr^\ell + dr^{-\ell}, \tag{4.21}$$

where c and d are arbitrary constants.

Some of the solutions in (4.21) blows up at the origin $r = 0$, and the other as $r \to \infty$. If ℓ is positive the solutions corresponding to b_2 will blow up at $r = 0$. We need in this case to set $d = 0$ if the PDE is to be solved in a region including the origin and we impose the "boundary condition" that the solution should be bounded in the domain. If ℓ is negative, we need to instead set $c = 0$. So the solutions to the PDE are given by

$$u(r, \theta) = c_\ell r^{|\ell|} e^{i\ell\theta}, \tag{4.22}$$

for constants c_ℓ.

We still have not specified ℓ. It is to be determined by the condition that the solution should be 2π periodic in θ. That is, as θ is increased by 2π, the solution should remain the same. Consequently, we require

$$e^{i\ell(\theta+2\pi)} = e^{i\ell\theta},$$

or, equivalently,

$$e^{i\ell 2\pi} = \cos(\ell 2\pi) + i\sin(\ell 2\pi) = 1.$$

This is true only if ℓ is an integer, because for $\ell = n$, an integer, $\sin(\ell 2\pi) = 0$ and $\cos(\ell 2\pi) = 1$.

The solution now becomes:

$$\boxed{u(r, \theta) = c_n r^{|n|} e^{in\theta}}\,, \tag{4.23}$$

where n can be any integer: $n = 0, \pm 1, \pm 2, \pm 3, \ldots$. Eq. (4.23) provides a family of a solutions to PDE which is bounded in a domain including the origin and 2π-periodic in θ.

We have in fact found an infinity of solutions to the PDE, each corresponding to a different value of n. So all these are solutions to the PDE:

$$\begin{aligned}
&c_0\\
&c_1 r^1 e^{i\theta}\\
&c_2 r^2 e^{2i\theta}\\
&c_3 r^3 e^{3i\theta}\\
&\vdots
\end{aligned}$$

Which one of these, or which combination of these, should be retained depends on the form of $u(r,\theta)$ specified at the boundary $r = a$. That is, on the exact form of $f(\theta)$ in (4.16):

(i) If $f(\theta) = 1$, then we should pick $c_0 = 1$, and the solution should be

$$u(r,\theta) = 1.$$

The solution to be Laplace's equation is a constant if a boundary value specified is a constant.

(ii) If $f(\theta) = \cos\theta$, the solution is

$$u(r,\theta) = (r/a)\cos\theta.$$

This result can be obtained in two ways. One can view $\cos\theta$ as the real part of $e^{i\theta}$, i.e.

$$f(\theta) = \mathrm{Re}\,\{e^{i\theta}\},$$

and we take it as understood that it is the real part of (4.23) that we should take as our (real) solution, i.e.

$$u(r,\theta) = \mathrm{Re}\,\{c_n r^{|n|} e^{in\theta}\}.$$

Therefore

$$u(a,\theta) = \mathrm{Re}\,\{c_n a^{|n|} e^{in\theta}\},$$

which should be equal to, by (4.16)

$$f(\theta) = \mathrm{Re}\,\{e^{i\theta}\}.$$

This implies that

$$c_n a^{|n|} = 1,\ \ n = 1.$$

Thus,

$$c_n = 1/a^{|n|} = 1/a$$

and

$$\begin{aligned} u(r,\theta) &= \mathrm{Re}\,\{(r/a)^1 e^{i\theta}\} \\ &= (r/a)\cos\theta. \end{aligned}$$

The second way of obtaining the same solution is to write, using Euler's identity,

$$f(\theta) = \cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta}).$$

In order for (4.23) to be equal to these two terms when $r = a$, we need to take the c_1-term and the c_{-1}-term and add them:

$$u(r,\theta) = c_1 r e^{i\theta} + c_{-1} r e^{-i\theta}.$$

At $r = a$, $u(a,\theta) = f(\theta) = \frac{1}{2}e^{i\theta} + \frac{1}{2}e^{-i\theta}$ and so $c_1 a = \frac{1}{2}$ and $c_{-1}a = \frac{1}{2}$. The final solution is

$$\begin{aligned} u(r,\theta) &= \frac{1}{2}(r/a)e^{i\theta} + \frac{1}{2}(r/a)e^{-i\theta} \\ &= (r/a)\cos\theta. \end{aligned}$$

(iii) If $f(\theta) = \cos\theta + 3\cos 5\theta$, the solution should be

$$u(r,\theta) = (r/a)\cos\theta + 3(r/a)^5\cos 5\theta.$$

(Please verify.)

The above solution can be written in the following form

$$u(r,\theta) = \sum_{n=-\infty}^{\infty} c_n r^{|n|} e^{in\theta},$$

with $c_n = 0$ except $n = \pm 1$ and ± 5,

$$c_1 = \frac{1}{2a},\ c_{-1} = \frac{1}{2a},\ c_5 = \frac{3}{2}a^{-5},\ c_{-5} = \frac{3}{2}a^{-5}.$$

The symbol, $\sum_{n=-\infty}^{\infty}$, indicates summation over all integer values of n for n from $-\infty$ to ∞. Of course with all but four of the c_n's nonzero, this sum has only 4 terms.

(iv) In general, if the boundary value is expressible as,

$$f(\theta) = \sum_{n=-\infty}^{\infty} f_n e^{in\theta},$$

for (complex) constants f_n, the solution is

$$u(r,\theta) = \sum_{n=-\infty}^{\infty} f_n \cdot (r/a)^{|n|} e^{in\theta}.$$

It can be verified that this solution satisfies the PDE and

$$u(a,\theta) = f(\theta).$$

Whether or not an arbitrary function $f(\theta)$ can be written in the form of such a sum (called the Fourier series) is a subject which we will take up later.

4.3 Further developments (optional)

The simplicity of the method we have presented in this chapter is a consequence of our decision to first ignore initial and/or boundary conditions. By focusing on a single wave component, we can learn quite a bit about the structure of the PDE under consideration. It is recommended that you always try this method first. These single wave solutions, however, do not always satisfy the given initial/boundary conditions. Nevertheless, we will show later that a rather general function can be constructed from a sum of single wave components; the solution to the PDE satisfying the general initial/boundary conditions can be constructed by adding up the single wave solutions obtained previously. We will indicate how this is done in this section. The justification will be left to later chapters.

4.3.1 The wave equation

Consider the wave equation on $-\infty < x < \infty$ with the more general initial conditions:

$$\text{PDE:} \quad u_{tt} = c^2 u_{xx} \tag{4.24}$$

$$\text{IC:} \quad u(x,0) = f(x),\ u_t(x,0) = g(x). \tag{4.25}$$

We have found, in (4.5) and (4.6), a simple solution involving a single wave $\sim e^{ikx}$:

$$\begin{aligned} u(x,t) &= (Ae^{ikct} + Be^{-ikct})e^{ikx} \\ &= Ae^{ik(x+ct)} + Be^{ik(x-ct)}. \end{aligned} \tag{4.26}$$

(4.26) would satisfy the initial conditions only if they are of the form $Ae^{ikx} + Be^{-ikx}$ for some constants A and B. To satisfy more general initial conditions, we superimpose over all possible k's, and take

$$u(x,t) = \sum_k A_k e^{ik(x+ct)} + \sum_k B_k e^{ik(x-ct)}, \tag{4.27}$$

where $\sum_k$ indicates summation over all possible values of the k, and the A_k and the B_k are arbitrary constants; they may be different for different k's.

Although we do not know at this point—without a knowledge of the theory of Fourier series—what $\sum_k A_k e^{ik(x+ct)}$ is, we can nevertheless say that it depends on x and t only through the combination $x + ct$. We denote it by $L(x + ct)$, i.e.

$$L(x+ct) = \sum_k A_k e^{ik(x+ct)}.$$

Similarly, we let

$$R(x-ct) = \sum_k B_k e^{ik(x-ct)}.$$

We shall assume that the summations converge and that the sums L and R possess second derivatives. Thus the solution can be written as

$$u(x,t) = L(x+ct) + R(x-ct). \tag{4.28}$$

It can be verified easily, by substituting (4.28) into the PDE (4.24) that it is indeed a solution to the PDE for any functions L and R, provided only that they are differentiable. To satisfy the initial conditions (4.25), we need

$$\begin{aligned} u(x,0) &= f(x) = L(x) + R(x) \\ u_t(x,0) &= g(x) = cL'(x) - cR'(x). \end{aligned}$$

These are two equations for two unknowns, and (amazingly!) $L(x)$ and $R(x)$ can thus be determined for arbitrary initial conditions. We will give more details to this solution when we discuss d'Alembert's solution later.

4.3.2 The diffusion equation

Consider the diffusion equation on $-\infty < x < \infty$ with a general initial condition:

$$\begin{aligned} \text{PDE:} \qquad & u_t = \alpha^2 u_{xx} \\ \text{IC:} \qquad & u(x,0) = f(x). \end{aligned}$$

Our simple solution (4.10):

$$u(x,t) = Ae^{-\alpha^2 k^2 t} e^{ikx} \tag{4.29}$$

can satisfy the initial condition only if

$$f(x) = Ae^{ikx}.$$

For more general IC, we will obtain a more general solution by adding up solutions of the form of (4.29) but with different k's, i.e.

$$u(x,t) = \sum_k A_k e^{-\alpha^2 k^2 t} e^{ikx}, \tag{4.30}$$

where the coefficients A_k are constants, which may be different for different k's, and $\sum_k$ denotes summation over all possible k's. The values of k allowed are to be determined by the boundary conditions (see the next few chapters). At $t = 0$, we have

$$u(x,0) = \sum_k A_k e^{ikx}.$$

In order for this to satisfy the IC

$$u(x,0) = f(x),$$

we are forced to require the decomposition

$$\sum_k A_k e^{ikx} = f(x). \tag{4.31}$$

To the extent that we can find coefficients A_k such that (4.31) holds, the initial-value problem is solved. This is the Fourier series representation of $f(x)$ which we will discuss in later chapters.

4.4 Forced oscillation

4.4.1 Example

In some problems, the frequency of solution oscillations is known from the frequency of the forcing. Let us consider the following example, due to Fourier, on determining the subsoil temperature at a depth y. We will first deal with the mathematical problem given by

$$\text{PDE:} \qquad \frac{\partial}{\partial t}u = \alpha^2 \frac{\partial^2}{\partial y^2}u, \;\; 0 < y < \infty \tag{4.32}$$

$$\text{BCs:} \qquad u(0,t) = e^{i\omega_0 t} \tag{4.33}$$

$$u(y,t) \text{ bounded as } y \to \infty. \tag{4.34}$$

[It is understood that the physical solution is to be ultimately obtained by taking the real part of u] We assume that the solution should have the same frequency of oscillation as the boundary forcing, viz. ω_0, which is known, and so we try:

$$u(y,t) = ae^{i\omega_0 t}e^{\lambda y}. \tag{4.35}$$

Substituting this trial solution into the PDE, we get

$$i\omega_0 u = \alpha^2\lambda^2 u.$$

Therefore,

$$\lambda^2 = i\omega_0/\alpha^2$$

so

$$\lambda = \pm\sqrt{i\omega_0}/\alpha.$$

To evaluate $\sqrt{i}$, recall Euler's identity

$$e^{i\theta} = \cos\theta + i\sin\theta$$

so the right-hand side is equal to i for $\theta = \pi/2$. We can write $i = e^{i\pi/2}$ and so

$$\sqrt{i} = e^{i\pi/4} = \cos\pi/4 + i\sin\pi/4 = \frac{1}{\sqrt{2}}(1+i).$$

Consequently,

$$\lambda = \pm(1+i)\sqrt{\omega_0/2}/\alpha,$$

and

$$u(y,t) = a_1 e^{i\omega_0 t}e^{(1+i)(\omega_0/2)^{1/2}y/\alpha} + a_2 e^{i\omega_0 t}e^{-(1+i)(\omega_0/2)^{1/2}y/\alpha},$$

where a_1 and a_2 are complex constants. The constant a_1 must be zero if the solution is to remain bounded as $y \to \infty$. And a_2 must equal to 1 if the boundary condition (4.33) at $y = 0$ is to be satisfied. Finally

$$u(y,t) = e^{i\omega_0 t} e^{-(1+i)(\omega_0/2)^{1/2} y/\alpha}. \tag{4.36}$$

The physical interpretation of this solution is discussed next.

4.4.2 Subsoil temperature

Fourier was interested in calculating the subsoil temperature as a consequence of conduction of heat from the surface. At the surface the temperature is known and is assumed to vary periodically on a daily and/or yearly basis. We can rewrite the solution in (4.36) as

$$u(y,t) = e^{-(\omega_0/2)^{1/2} y/\alpha} e^{i(\omega_0 t - (\omega_0/2)^{1/2} y/\alpha)}. \tag{4.37}$$

The first term's exponential decay implies that the effect underground of the temperature oscillation at the surface diminishes with increasing depth in general, but in particular, higher frequency oscillations (with larger ω_0 values) die off faster than lower frequency ones. (The rate of decay with depth is also dependent on the square root of the coefficient of theraml diffuse, α^2, which is about $0.02\mathrm{cm}^2/\mathrm{s}$ for soil.) In fact, Fourier found that daily variations become imperceptible at depths of a few centimeters, while seasonal variations can penetrate a few meters.

The second exponential factor in (4.37) is oscillatory in nature, since the complex exponential can be written in terms of sines and cosines using Euler's identity:

$$e^{i(\omega_0 t - (\omega_0/2)^{1/2} y/\alpha)} = \cos(\omega_0 t - (\omega_0/2)^{1/2} y/\alpha)$$
$$+ i \sin(\omega_0 t - (\omega_0/2)^{1/2} y/\alpha).$$

It is understood that we only consider the real part, which is

$$\cos(\omega_0 t - (\omega_0/2)^{1/2} y/\alpha).$$

It suggests a phase lag at a depth relative to the oscillation at the surface. Focusing on seasonal variations, for which $\omega_0 = 2\pi/(1 \text{ year})$, we find that the lag at a depth y is

$$(\pi/1\text{year})^{1/2} y/\alpha.$$

If this lag is π, then it will be winter at depth y when it is summer at the surface. Fourier found this depth to occur at about 4 meters and suggested it to be a good depth for cellars.

4.4.3 Why the earth does not have a corona

In section 4.2.2 we considered the horizontal propagation of Rossby waves in the earth's atmosphere. These waves also propagate vertically, from where they are forced, in the lower atmosphere, to the upper atmosphere, where the air is much thinner. J.G. Charney and P.G. Drazin noted in their famous 1961 paper on "Propagation of planetary-scale disturbances from the lower into the upper atmosphere" (Journal of Geophysical Research, volume 66, p. 83-109), that if the energy contained in the large-scale motions were to propagate upward with little attenuation, an atmospheric corona would, in all likelihood, be produced, similar to that in the sun. The energy produced by these disturbances near the surface was estimated to be of the order of 10^3 ergs cm^{-3}. If this energy were to travel upward and be converted into heat by friction or some other means at, say, 100km, where the density is diminished by a factor of 10^{-6}, it would raise the air temperature to about $100,000°$K, enough to produce a corona. Yet one is not seen. This motivated the authors to investigate the vertical propagation of the waves and why they are so effectively trapped in the lower atmosphere.

The three-dimensional PDE governing planetary-scale disturbances of small amplitude in the atmosphere is given by (see the reference quoted in section 4.2.2):

$$\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)\left[\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\Phi+\frac{f_0^2}{N^2}\left(\frac{\partial}{\partial z}-\frac{1}{H}\right)\frac{\partial}{\partial z}\Phi\right]+\beta\frac{\partial}{\partial x}\Phi=0. \tag{4.38}$$

Eq. (4.38) is the same as Eq. (4.13) except for the addition of one dimension (z) and the consequent appearance of the vertical derivative terms. Here f_0 is the Coriolis parameter, N is the frequency of buoyancy oscillation, and H is the density scale height. The density of the atmosphere in the lowest 100km is given by

$$\rho(z)=\rho(0)e^{-z/H}.$$

That is, the density decays exponentially, with a exponential scale height H, which is about 7km.

There is a boundary condition at the surface $z=0$, which specifies the frequency of the forced waves. What is nice about the method we are discussing in this chapter is that we need not worry about the BCs and ICs in the beginning. We will concentrate on finding a solution to the PDE of the form (cf. with (4.14)):

$$\Phi(x,y,z,t)=\phi(z)e^{i\omega t}e^{ikx}e^{i\ell y}. \tag{4.39}$$

When this is substituted into the PDE, we find

$$ik(U+\omega/k)\left[-(k^2+\ell^2)+\frac{f_0^2}{N^2}\left(\frac{d}{dz}-\frac{1}{H}\right)\frac{d}{dz}\right]\phi e^{i\omega t}e^{ikx}e^{i\ell y}$$
$$+ik\beta\phi e^{i\omega t}e^{ikx}e^{i\ell y}=0.$$

This can be simplified to the ODE

$$\frac{f_0^2}{N^2}\left(\frac{d}{dz}-\frac{1}{H}\right)\frac{d}{dz}\phi+\left[\frac{\beta}{U+\omega/k}-(k^2+\ell^2)\right]\phi=0. \tag{4.40}$$

Again, we assume exponential solutions to this equation with constant coefficients:

$$\phi(z)=Ae^{\mu z},$$

and find that

$$\mu=\frac{1}{2H}\pm im,$$

where the *vertical wave number*, m, is found to be given by

$$m^2=\frac{N^2}{f_0^2}\left[\frac{\beta}{U+\omega/k}-(k^2+\ell^2)\right]-\frac{1}{4H^2}. \tag{4.41}$$

Thus the solution is of the form:

$$\Phi(x,y,z,t)=Ae^{z/2H}e^{i\omega t}e^{i(kx+\ell y\pm mz)}. \tag{4.42}$$

The factor $e^{z/2H}$ in the solution reflects the fact that as the wave propagates upward into air of lower density, its amplitude increases as $1/\rho(z)^{1/2}$. The vertical wavenumber m is real if m^2 in (4.41) is positive. Otherwise, no vertical propagation is possible and the wave energy is trapped near the surface.

Most of the energy for planetary-scale wave disturbances comes from flow over continental-scale mountain ranges. These (forced) waves have a forced frequency of $\omega=\omega_0=0$. For these waves, we put $\omega=0$ in (4.40) and find that there is vertical propagation if

$$\frac{\beta}{U}>(k^2+\ell^2)+\frac{f_0^2}{4N^2H^2}. \tag{4.43}$$

During summer, the bulk of the atmosphere has easterly flows, with $U<0$. It is clear that the condition for vertical propagation (4.43) is violated. The solution for Φ decays exponentially with height for all waves. This result is consistent with the observational fact there is very little wave perturbation above the troposphere during summer.

During winter, the wind is westerly, i.e. $U > 0$, and so there is vertical propagation for those wavelengths that satisfy (4.43). These are the longest wavelengths whose horizontal wavenumbers, k and ℓ, are so small that they satisfy (4.43). However, since the earth's geometry determines the smallest wavenumbers (i.e. the longest wavelengths) that can exist in the earth's atmosphere, there would still be no vertical propagation if the westerly wind, U, is so strong that (4.43) is again violated even for the smallest possible k and ℓ. Charney and Drazin suggested that this is the case during winter, when the westerly jet in the stratosphere is very strong. Their theory however, cannot explain why there is no corona during autumn, when U is positive but weak. The explanation of this behavior comes from nonlinear theories. The equation Charney and Drazin used, Eq. (4.38), is only valid for small amplitude disturbances.

4.5 A comment

Finally, we mention one interesting note about the method we are using. The simple method involves assuming a single wave (in the form of e^{ikx} or $e^{i(kx+\ell y)}$) for our solution with a frequency ω (in the form of $e^{i\omega t}$). For forced problems, the boundary condition or a forcing term determines ω as the frequency of forcing. In free problems, ω is found to be related to the wavenumbers in a dispersion relation. The boundary conditions need not be considered in the beginning, but ultimately they are needed to fix the wavenumbers k and ℓ.

Initial-value problems are not studied in many geophysical applications. Many useful physical conclusions can already be drawn even without considering boundary and initial conditions, as we have illustrated in this chapter with examples.

The traditional, formal mathematical solution procedures for PDEs require the consideration of boundary conditions right from the beginning, and the solutions need to be written in their most general form, often involving obscure infinite summations or infinite integrals, in order to satisfy general initial conditions. We will develop these general procedures starting from Chapter 6. One should however always try the simple solution first. It may reveal a surprising amount of information about the PDE and the physical problem it describes.

4.6 Exercise I

1. Use exponential solutions, e.g.: $u(x,t) = e^{ikx+i\omega t}$ to answer the following questions:

   (a) *Sound waves:*
   $$\frac{\partial^2}{\partial t^2}u = (\gamma RT)\frac{\partial^2}{\partial x^2}u,$$
   where $\gamma = 1.4$ for air, $R = 287m^2/(s^2K)$ is the gas constant for air, and $T = 293K$ is the ambient temperature of air.

   What is the speed of sound in air?

   (b) *Shallow water waves:*
   $$\frac{\partial^2}{\partial t^2}u = gh\frac{\partial^2}{\partial x^2}u$$
   where $g = 9.8m/s^2$ is the gravitational acceleration and $h = 4m$ is the depth of the water.

   What is the speed of water waves?

   (c) *Rossby waves:*
   $$(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x})\frac{\partial^2}{\partial x^2}u + \beta\frac{\partial}{\partial x}u = 0$$
   Find the phase speed of the Rossby waves in terms of β, U and the wavenumber k. For what value of the wavenumber (in terms of β and U) does this wave becomes stationary (i.e. its phase speed equals zero)? Assuming $\beta = 1.6 \times 10^{-11}/(ms)$, $U = 20m/s$, what is the value of the wavelength $L \equiv \frac{2\pi}{k}$ for such a stationary Rossby wave?

2. *Stokes' second problem:* This is a classical problem in fluid mechanics of a viscous fluid above a flat plate located at $y = 0$ executing an oscillation with a known frequency ω_0. The mathematical problem for the fluid velocity $u(y,t)$ is
   $$\frac{\partial}{\partial t}u = \nu\frac{\partial^2}{\partial y^2}u, \quad 0 < u < \infty$$

where ν is the coefficient of viscosity, subject to the bounday conditions:

$$u(0,t) = Ue^{i\omega_0 t}$$
$$u(\infty,t) = \text{ bounded,}$$

U being the velocity amplitude of the flat plate. Try a solution of the form

$$u(y,t) = e^{i\omega_0 t} f(y)$$

(a) Find $f(y)$ and hence $u(y,t)$.

(b) Suppose the boundary condition at $y = 0$ is in real form, i.e. $u(0,t) = U\cos\omega_0 t$ (instead of the complex form $u(0,t) = Ue^{i\omega_0 t}$ used above). By taking the real part of the solution obtained above show that the solution is

$$u(y,t) = Ue^{-y\sqrt{\omega_0/2\nu}}\cos(\omega t - y\sqrt{\frac{\omega_0}{2\nu}}).$$

4.7 Solutions to Exercise I

1. (a) Try

$$u(x,t) = e^{ikx+i\omega t}.$$

Substituting into the PDE yields

$$\omega^2 = \gamma RTk^2.$$

So the phase speed is

$$\begin{aligned} c_{ph} &\equiv -\omega/k = \pm\sqrt{\gamma RT} = \pm\sqrt{1.4 \times 287 \times 293}m/s \\ &\cong \pm 340 m/s. \end{aligned}$$

(b) Try

$$u(x,t) = e^{ikx+i\omega t}.$$

Substituting into the PDE yeilds

$$\omega^2 = ghk^2.$$

So the phase speed is

$$\begin{aligned} c_{ph} &\equiv -\omega/k = \pm\sqrt{gh} = \pm\sqrt{9.8 \times 4}m/s \\ &= \pm 6.4 m/s \end{aligned}$$

(c) Try

$$u(x,t) = e^{ikx+i\omega t} = e^{ik(x-c_{ph}t)}.$$

Substituting into the PDE yields

$$(U - c_{ph})ik(-k^2) + \beta ik = 0.$$

So, $c_{ph} = U - \frac{\beta}{k^2}$ is the phase speed of the Rossby waves.

For stationary Rossby waves, $c_{ph} = 0$. This occurs for

$$k = \sqrt{\beta/U} = \sqrt{\frac{1.6 \times 10^{-11}}{20}} m^{-1}$$

$$L = \frac{2\pi}{k} = 2\pi\sqrt{\frac{2}{1.6}} \times 10^6 m = 7000km$$

2. (a) $\frac{\partial}{\partial t}u = \nu\frac{\partial^2}{\partial y^2}u$

$$u(0,t) = Ue^{i\omega_0 t}$$
$$u(y,t) = \text{ bounded as } y \to \infty$$

Assume

$$u(y,t) = e^{i\omega_0 t}f(y).$$

Substitute into the PDE:

$$i\omega_0 f(y) = \nu f''(y).$$

The solution to this ODE is:

$$f(y) = Ae^{-(1+i)y\sqrt{\omega_0/2\nu}} + Be^{(1+i)y\sqrt{\omega_0/2\nu}}$$

The solution which remains bounded as $y \to \infty$ must have $B = 0$. So

$$u(y,t) = Ae^{i\omega_0 t}e^{-(1+i)y\sqrt{\omega_0/2\nu}}$$

(b) The real solution is obtained by taking the real part of $u(y,t)$ in (a).

$$u(y,t) = Ue^{-y\sqrt{\omega_0/2\nu}}\cos(\omega_0 t - y\sqrt{\frac{\omega_0}{2\nu}}).$$

4.8 Exercise II

1. Consider the problem of heat conduction in a circular disk of radius a. The temperature at the boundary $r = a$ is specified to be

$$f(\theta) = \cos\theta + 3\cos 5\theta$$

Find the steady state temperature distribution for $r < a$. [Hint: Read section 4.2.3. The steady state temperature $u(r,\theta)$ satifies the Laplace's equation

$$\nabla^2 u = 0,\ \ r < a$$

where

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

Find $u(r,\theta)$ satisfying the above PDE and the boundary condition

$$u(a,\theta) = f(\theta)$$

$$u(0,\theta)\ \ \text{finite}$$

and

$$u(r,\theta + 2\pi) = u(r,\theta)]$$

2. (a) Find $u(x,y)$, the solution to Laplace's equation in a square

PDE: $\frac{\partial^2}{\partial x^2}u + \frac{\partial^2}{\partial y^2}u = 0$, $0 < x < 1$, $0 < y < 1$
BCs: $u(0,y) = 0$, $u(1,y) = 0$
$u(x,0) = \sin\pi x$, $u(x,1) = \sin\pi x$.

Hint: Assume the solution to be of the "separated" form:

$$u(x,y) = Y(y)X(x)$$

and take $X(x) = \sin\pi x$.

Substitute the assumed form into the PDE to find an ODE for $Y(y)$. Solve that ODE subject to the boundary conditions at $y = 0$ and $y = 1$.

(b) Same as in (a), except that the BCs are

BCs: $u(0,y) = 0$, $u(1,y) = 0$

$$u(x,0) = \sin \pi x + \sin 2\pi x,\ u(x,1) = \sin \pi x + \sin 2\pi x$$

Hint: Assume the solution to be of the form

$$u(x,y) = Y_1(y)X_1(x) + Y_2(y)X_2(x),$$

where $X_1(x) = \sin \pi x$ and $X_2(x) = \sin 2\pi x$. Find the ODE for $Y_1(y)$ and a different ODE for $Y_2(y)$.

4.9 Solutions to Exercise II

1. Assume solution of the form

$$u(r,\theta) = R(r)e^{i\ell\theta}$$

Find that R satisfies the ODE

$$r^2 \frac{d^2}{dr^2}R + r\frac{d}{dr}R - \ell^2 R = 0$$

The solution to this ODE is

$$R(r) = cr^{\ell} + dr^{-\ell}$$

The solution that is finite at $r = 0$ is

$$u(r,\theta) = c_\ell r^{|\ell|} e^{i\ell\theta}$$

The solution that is 2π-periodic in θ is when $\ell = n$, an integer. So the general solution is

$$u(r,\theta) = \sum_n c_n r^{|n|} e^{in\theta}$$

To satisfy the BC

$$u(a,\theta) = f(\theta),$$

we want

$$\begin{aligned} \sum_n c_n a^{|n|} e^{in\theta} &= f(\theta) \\ &= \cos\theta + 3\cos 5\theta \end{aligned}$$

Thus $c_1 a = \frac{1}{2}$, $c_{-1} a = \frac{1}{2}$, $c_5 a^5 = 3/2$, $c_{-5} a^5 = 3/2$, and all other c_n's $= 0$.

Finally

$$u(r,\theta) = (r/a)\cos\theta + 3(r/a)^5 \cos 5\theta$$

2. (a) Assume $u(x,y) = Y(y)\sin \pi x$

Note that the boundary conditions at $x = 0$ and $x = 1$ are automatically satisfied. Substitute this assumed form into the PDE then yields

$$Y''(y) - \pi^2 Y(y) = 0.$$

The solution to this ODE is

$$Y(y) = Ae^{\pi y} + Be^{-\pi y}$$

The boundary conditions are

$$Y(0) = 1,\ Y(1) = 1$$

so we must have

$$A + B = 1, \text{ and } Ae^{\pi} + Be^{-\pi} = 1$$

Thus

$$A = \frac{1 - e^{-\pi}}{e^{\pi} - e^{-\pi}} \text{ and } B = \frac{e^{\pi} - 1}{e^{\pi} - e^{-\pi}}$$

Finally:

$$\begin{aligned} u(x,y) &= \frac{\sin \pi x}{2\sinh \pi}[(1 - e^{-\pi})e^{\pi y} + (e^{\pi} - 1)e^{-\pi y}] \\ &= \frac{\sin \pi x}{\sinh \pi}[\sinh(\pi y) - \sinh(\pi(y-1))] \end{aligned}$$

(b) Assume solution of the form

$$u(x,y) = Y_1(x)\sin \pi x + Y_2(x)\sin 2\pi x$$

Substituting into the PDE yields

$$[Y_1''(y) - \pi^2 Y_1(y)]\sin \pi x + [Y_2''(y) - (2\pi)^2 Y_2(y)]\sin 2\pi x = 0$$

This is true for all x only if the terms in brackets are separately zero, i.e.

$$Y_1''(y) - \pi^2 Y_1(y) = 0$$
$$Y_2''(y) - (2\pi)^2 Y_2(y) = 0$$

The boundary conditions are

$$Y_1(0) = 1,\ Y_1(1) = 1$$

$$Y_2(0) = 1, \ Y_2(1) = 1$$

$Y_1(y)$ is the same as in (a), i.e.

$$Y_1(y) = \frac{1}{\sinh \pi}[\sinh(\pi y) - \sinh(\pi(y-1))]$$

$Y_2(y)$ is similarly obtained as

$$Y_2(y) = \frac{1}{\sinh 2\pi}[\sinh(2\pi y) - \sinh(2\pi(y-1)].$$

Chapter 5

d'Alembert's Solution

5.1 Introduction

Consider the wave equation in an infinite domain:

$$\begin{aligned}
&\text{PDE:} \quad u_{tt} = c^2 u_{xx}, \ -\infty < x < \infty, \ t > 0 && (5.1)\\
&\text{BCs:} \quad u(x,t) \to 0, \ x \to \pm\infty, \ t > 0 && (5.2)\\
&\text{ICs:} \quad u(x,0) = f(x), \ -\infty < x < \infty && (5.3)\\
&\qquad\quad\ u_t(x,0) = g(x), \ -\infty < x < \infty. && (5.4)
\end{aligned}$$

The problem can be solved using a Fourier transform, as we will show later. However, the first person to solve this problem was not Fourier (1768-1830), but d'Alembert (1746). d'Alembert had a simpler method, which works only on the wave equation. It also works whether or not the domain is infinite, but it can get complicated quickly when finite boundaries are included.

5.2 d'Alembert's approach

d'Alembert recognized that any function of either $x+ct$ or $x-ct$ will satisfy the wave equation. We can simply verify that any trial solution

$$\boxed{u(x,t) = L(x+ct) + R(x-ct)} \qquad (5.5)$$

satisfies the PDE (5.1) for any functions L and R (which have second derivatives, of course).

To show this, let us write

$$\begin{aligned}
\xi &\equiv x + ct\\
\eta &\equiv x - ct,
\end{aligned}$$

and

$$u(x,t) = L(\xi) + R(\eta).$$

We use a prime to denote ordinary differentiation with respect to the argument of the function, e.g.

$$L'(\xi) \equiv \frac{d}{d\xi}L(\xi),\ \ R'(\eta) \equiv \frac{d}{d\eta}R(\eta).$$

The partial derivatives are calculated in the following way:

$$\begin{aligned}
\frac{\partial}{\partial t}L(\xi) &= L'(\xi)\frac{\partial\xi}{\partial t} = cL'(\xi)\\
\frac{\partial}{\partial x}L(\xi) &= L'(\xi)\frac{\partial\xi}{\partial x} = L'(\xi)\\
\frac{\partial^2}{\partial t^2}L(\xi) &= \frac{\partial}{\partial t}(cL'(\xi)) = \frac{d}{d\xi}(cL'(\xi))\frac{\partial\xi}{\partial t} = c^2L''(\xi)\\
&= \frac{\partial}{\partial x}(L'(\xi)) = \frac{d}{d\xi}(L'(\xi))\frac{\partial\xi}{\partial x} = L''(\xi).
\end{aligned}$$

Similarly,

$$\begin{aligned}
\frac{\partial}{\partial t}R(\eta) &= R'(\eta)\frac{\partial\eta}{\partial t} = -cR'(\eta)\\
\frac{\partial}{\partial x}R(\eta) &= R'(\eta)\frac{\partial\eta}{\partial x} = R'(\eta)\\
\frac{\partial^2}{\partial t^2}R(\eta) &= \frac{\partial\eta}{\partial t}\frac{d}{d\eta}(-cR'(\eta)) = c^2R''(\eta)\\
\frac{\partial^2}{\partial x^2}R(\eta) &= \frac{\partial\eta}{\partial x}\frac{d}{d\eta}(R'(\eta)) = R''(\eta).
\end{aligned}$$

It is seen that both $L(\xi)$ and $R(\eta)$ satisfy the wave equations, i.e.

$$\left(\frac{\partial^2}{\partial t^2} - c^2\frac{\partial}{\partial x^2}\right)L(\xi) = 0,$$

and

$$\left(\frac{\partial^2}{\partial t^2} - c^2\frac{\partial^2}{\partial x^2}\right)R(\eta) = 0.$$

Therefore the combination, (5.5), also satisfies the wave equation.

The functional forms of L and R should be determined from initial conditions.

Since

$$u(x,t) = L(x+ct) + R(x-ct), \;\; u_t(x,t) = cL'(x+ct) - cR'(x-ct),$$

it follows from the initial conditions that,

$$u(x,0) = L(x) + R(x) = f(x), \;\; -\infty < x < \infty \tag{5.6}$$

$$u_t(x,0) = cL'(x) - cR'(x) = g(x), \;\; -\infty < x < \infty. \tag{5.7}$$

The second equation, (5.7), can be integrated with respect to x to yield

$$L(x) - R(x) = \frac{1}{c}\int_0^x g(\overline{x})d\overline{x} + K, \tag{5.8}$$

where K is an arbitrary constant. Adding (5.8) to (5.6) yields

$$\boxed{L(x) = \frac{1}{2}f(x) + \frac{1}{2c}\int_0^x g(\overline{x})d\overline{x} + \frac{K}{2}}\,, \tag{5.9}$$

and subtracting (5.8) from (5.6) yields

$$\boxed{R(x) = \frac{1}{2}f(x) - \frac{1}{2c}\int_0^x g(\overline{x})d\overline{x} - \frac{K}{2}}\,. \tag{5.10}$$

We change x to ξ in (5.9) and x to η in (5.10), and then add:

$$u(x,t) = L(\xi) + R(\eta).$$

Thus

$$\boxed{u(x,t) = \frac{1}{2}f(x+ct) + \frac{1}{2}f(x-ct) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\overline{x})d\overline{x}}\,. \tag{5.11}$$

This is the full solution to the wave equation in an infinite spatial domain.

5.3 Example

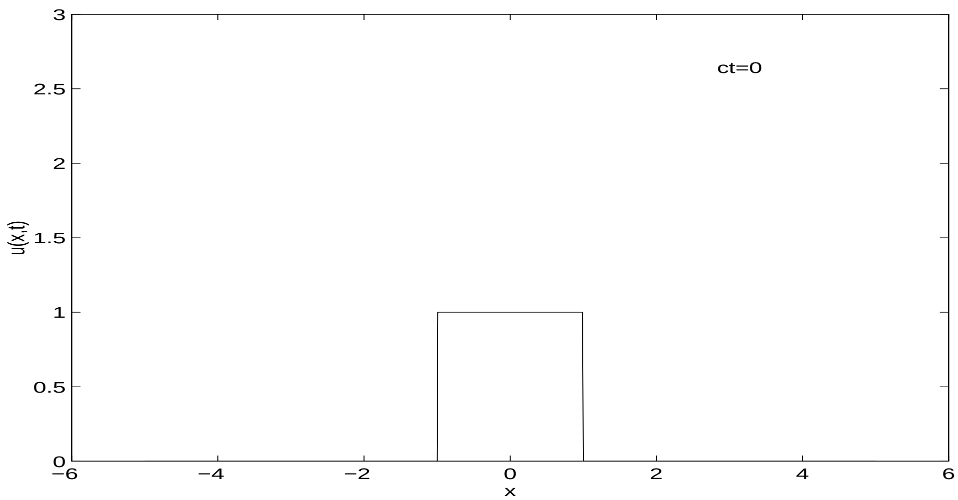
Solve:

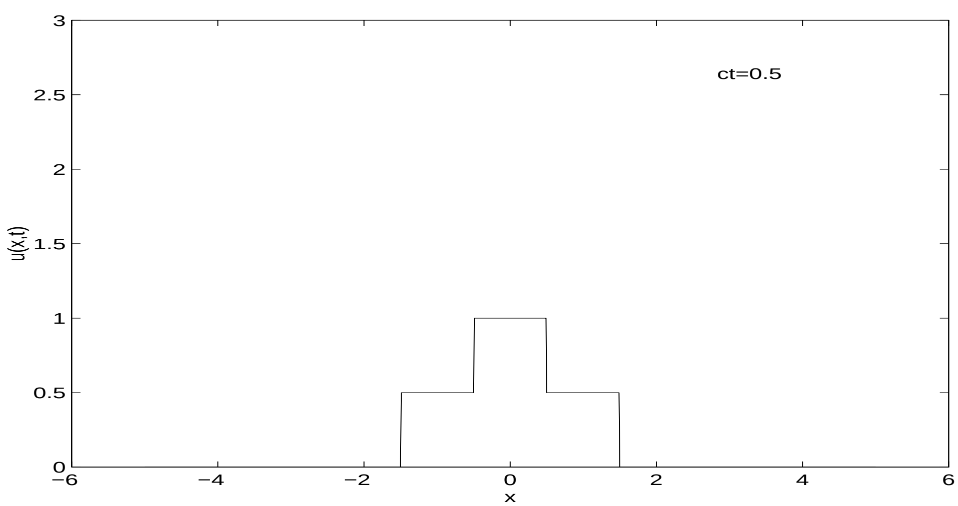
$$\begin{aligned}
\text{PDE:} \quad & u_{tt} = c^2 u_{xx}, \;\; -\infty < x < \infty, \;\; t > 0 \\
\text{ICs:} \quad & u(x,0) = f(x) = \begin{cases} 1, & -1 < x < 1 \\ 0, & \text{elsewhere} \end{cases} \\
& u_t(x,0) = 0, \;\; -\infty < x < \infty.
\end{aligned}$$

Since $g \equiv 0$, d'Alembert's solution is

$$u(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)].$$

We simply divide the initial shape in two halves. Let one half move to the left with speed c and the other half to the right with speed c. In the interval where the two halves overlap we add their amplitudes.





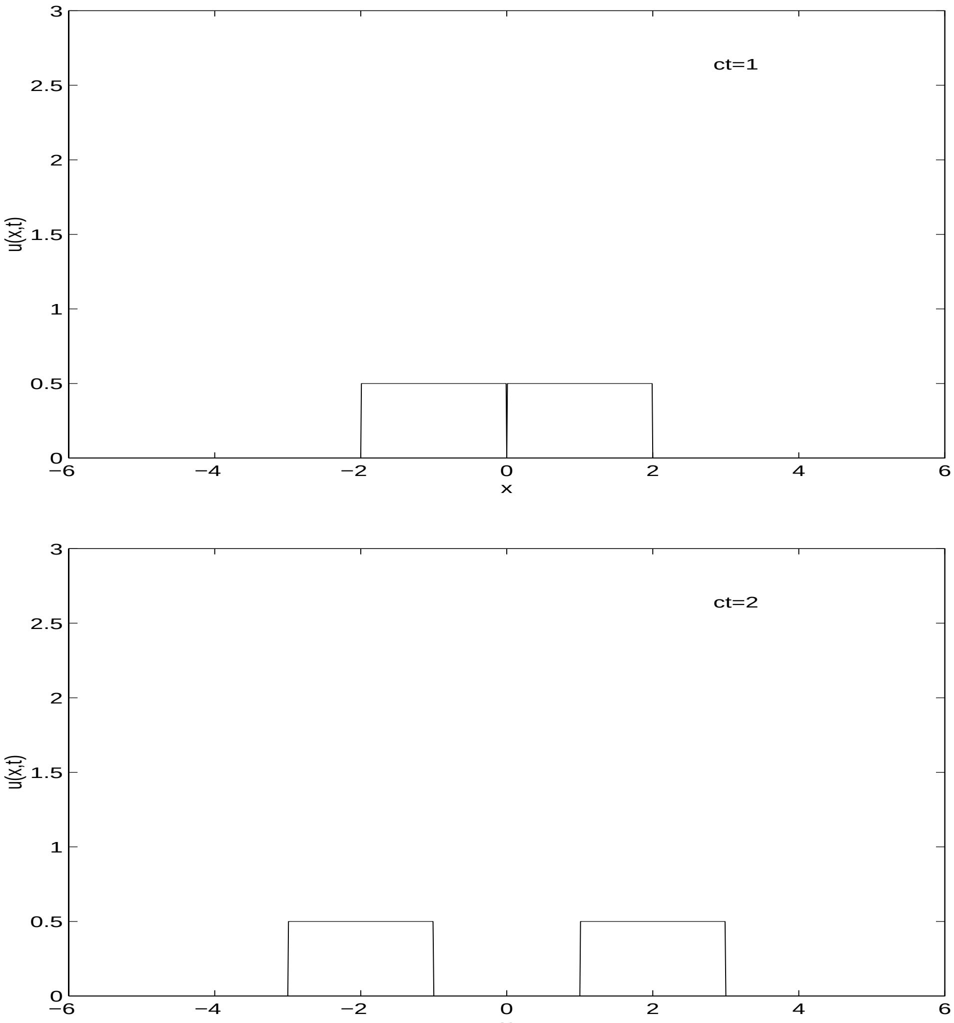


Figure 5.1. Wave propagation in an infinite domain.

5.4 Reflection

Consider the following problem in a semi-infinite domain:

$$\text{PDE:} \quad u_{tt} = c^2 u_{xx}, \ 0 < x < \infty, \ t > 0. \tag{5.12}$$

$$\text{BCs:} \quad u(0,t) = 0, \ u(x,t) \to 0 \ \text{ as } \ x \to \infty. \tag{5.13}$$

$$\text{ICs:} \quad u(x,0) = f(x), \ 0 < x < \infty. \tag{5.14}$$

$$u_t(x,0) = g(x), \ 0 < x < \infty. \tag{5.15}$$

We can proceed as before and easily show that

$$u(x,t) = L(x+ct) + R(x-ct)$$

satisfies the PDE (5.12). To satisfy the ICs, we have, as before (cf. (5.6) and (5.7)):

$$u(x,0) = L(x) + R(x) = f(x),\ 0 < x < \infty, \tag{5.16}$$

$$u_t(x,0) = cL'(x) - cR'(x) = g(x),\ 0 < x < \infty. \tag{5.17}$$

An important difference between these ((5.16), (5.17)) and the earlier ((5.6), (5.7)) is that (5.16) and (5.17) define $L(x)$ and $R(x)$ only for $x > 0$. We do not know $L(x)$ and $R(x)$ for $x < 0$; hence we do not know $R(x-ct)$ when $ct > x$.

For $ct < x$, we can proceed as before. Integrate (5.17) and add to or subtract from (5.16) to obtain (5.9) and (5.10).

So far we have not used the boundary conditions. These provide the missing information

$$u(0,t) = 0 = L(ct) + R(-ct),\ t > 0. \tag{5.18}$$

Equation (5.18) relates R with a negative argument to L with a positive argument:

$$R(\eta) = -L(-\eta),\ \eta < 0 \tag{5.19}$$

The solution can be summarized as follows. For $\xi = x + ct$, $\eta = x - ct$,

$$u(x,t) = L(\xi) + R(\eta), \tag{5.20}$$

where

$$L(x) = \frac{1}{2}f(x) + \frac{1}{2c}\int_0^x g(\overline{x})d\overline{x},\ x > 0, \tag{5.21}$$

and

$$R(x) = \frac{1}{2}f(x) - \frac{1}{2c}\int_0^x g(\overline{x})d\overline{x},\ x > 0. \tag{5.22}$$

In addition,

$$R(x) = -L(-x),\ x < 0. \tag{5.23}$$

If $g(x) \equiv 0$, the solution is simplified to:

$$u(x,t) = \begin{cases} \frac{1}{2}[f(x+ct) + f(x-ct)] & \text{for } x > ct \\ \frac{1}{2}[f(x+ct) - f(ct-x)] & \text{for } x < ct. \end{cases} \tag{5.24}$$

A simpler way of writing (5.24) is to define the initial condition for negative values of x by

$$f(x) = -f(-x) \quad \text{for } -\infty < x < 0 \tag{5.25}$$

and the solution in (5.24) can be rewritten as

$$u(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)] \tag{5.26}$$

for all $x > 0$ and $ct > 0$.

For $x > ct$, (5.26) agrees with (5.24). For $x < ct$, $f(x-ct)$ has a negative argument, and is now defined according to (5.25). With the change in sign dictated by (5.25), (5.26) is then the same as the second line in (5.24). Since (5.26) is the same as the solution in the infinite domain problem, we can pretend that there is no boundary at $x = 0$, provided we define $f(x)$ for negative x according to (5.26), find the solution in the infinite domain and ignore the final result for $x < 0$.

5.5 Example

$$\begin{aligned} &\text{PDE:} && u_{tt} = c^2 u_{xx},\ 0 < x < \infty,\ t > 0 \\ &\text{BCs:} && u(0,t) = 0,\ u(x,t) \to 0, \quad \text{as } x \to \infty \\ &\text{ICs:} && u(x,0) = f(x) = \begin{cases} 2 & \text{if } -1 < x - 4 < 1 \\ 0 & \text{elsewhere} \end{cases} \\ & && u_t(x,0) = 0,\ 0 < x < \infty. \end{aligned}$$

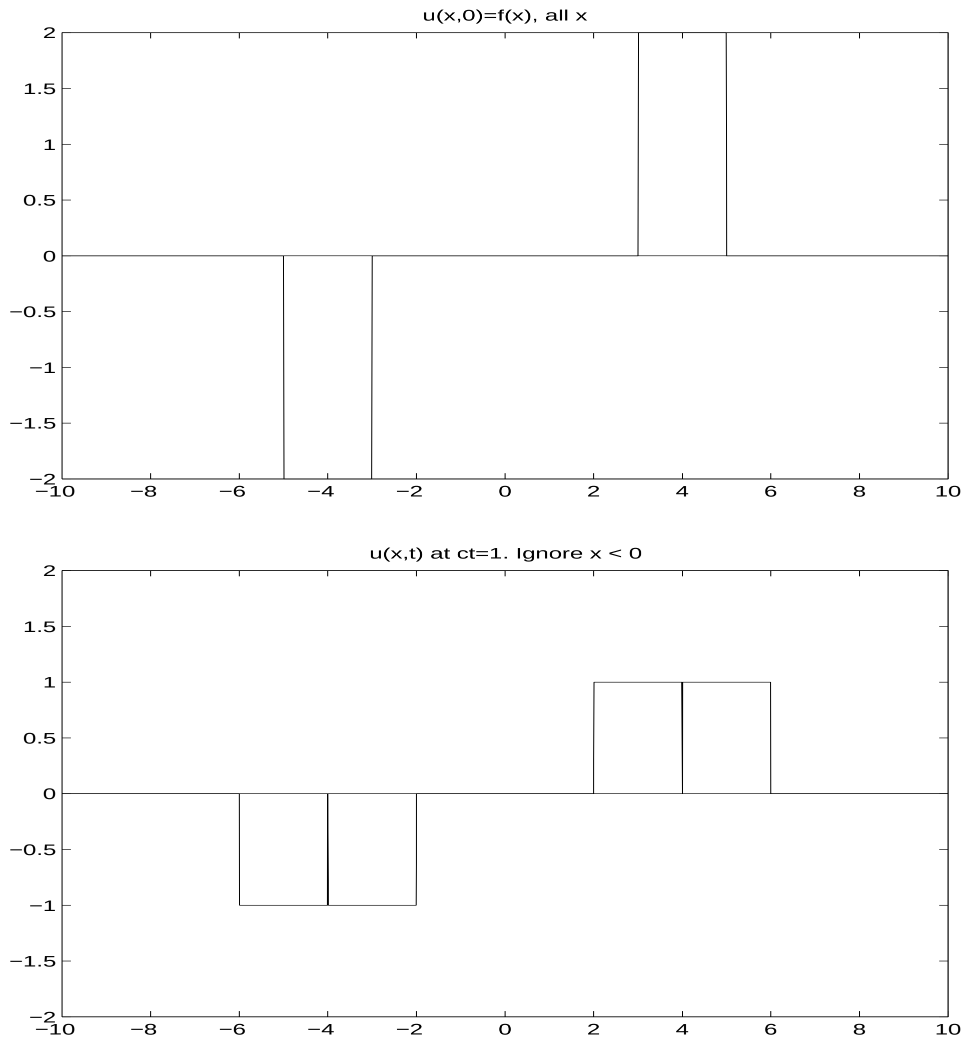
d'Alembert's solution is

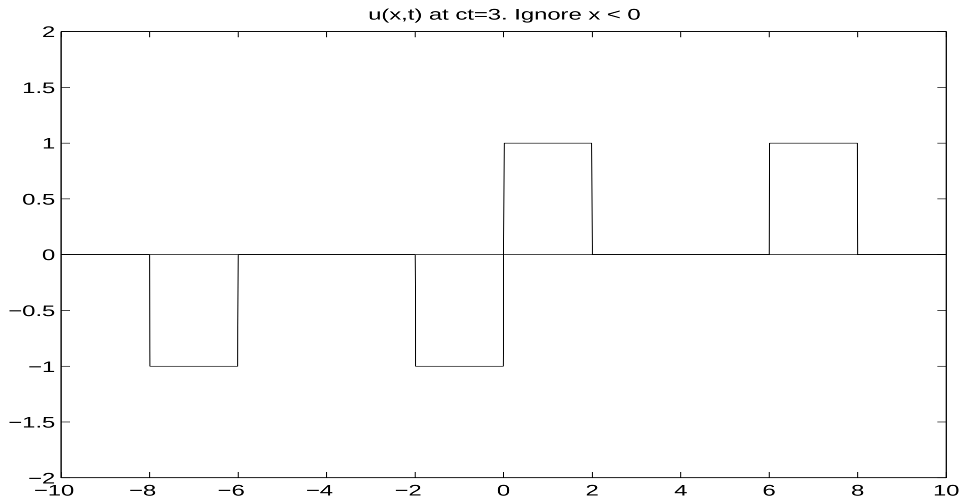
$$u(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)].$$

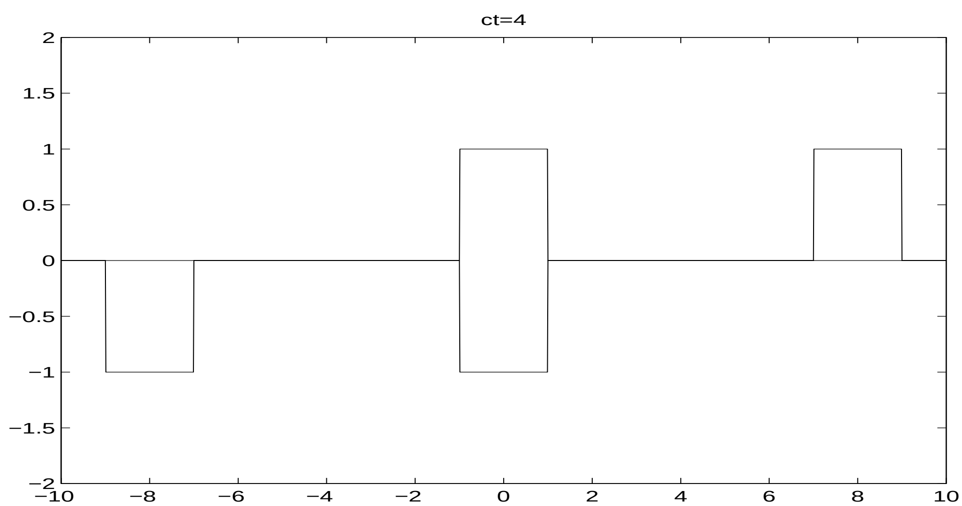
This solution is valid for all $x > 0$, $ct > 0$, provided that the initial condition is redefined so that

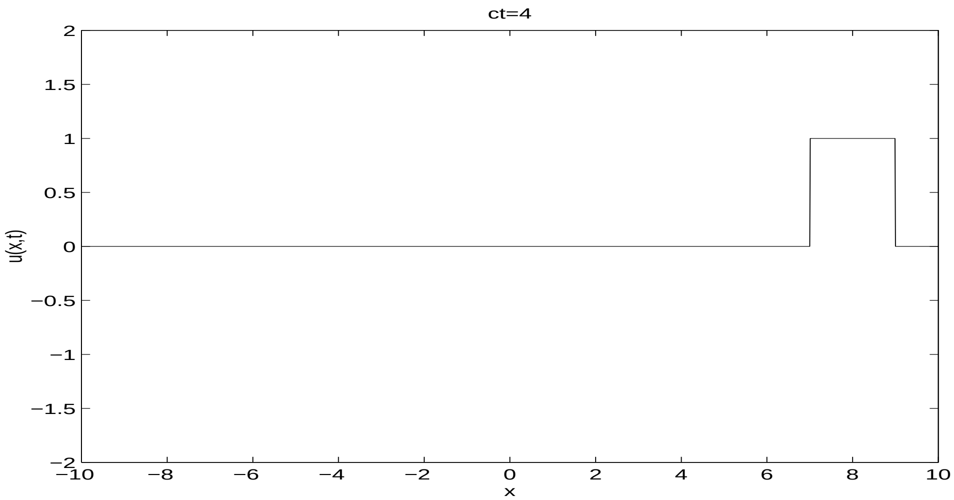
$$f(x) = -f(-x) \quad \text{for negative } x.$$

The initial condition is depicted in Figure 5.2(a). We now pretend this is an infinite domain problem and let the initial shape be split into two halves propagating to the left and right with speed c.

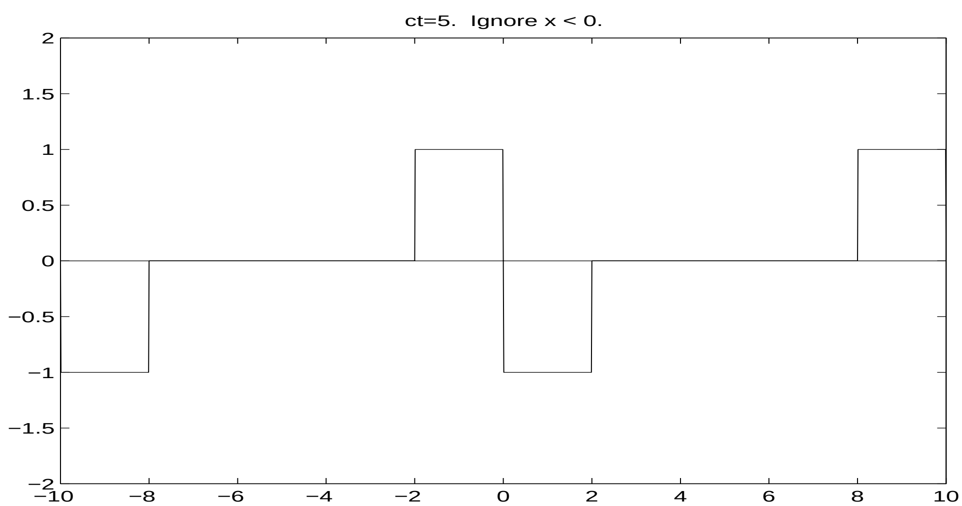



u(x,t) at ct=3. Ignore x < 0
2
1.5
1
0.5
0
−0.5
−1
−1.5
−2
−10
−8
−6
−4
−2
0
2
4
6
8
10


ct=4
2
1.5
1
0.5
0
−0.5
−1
−1.5
−2
−10
−8
−6
−4
−2
0
2
4
6
8
10



At this time part of the image initially at $x < 0$ moves to the real domain at $x > 0$, and cancels with the left going wave initially from $x > 0$.



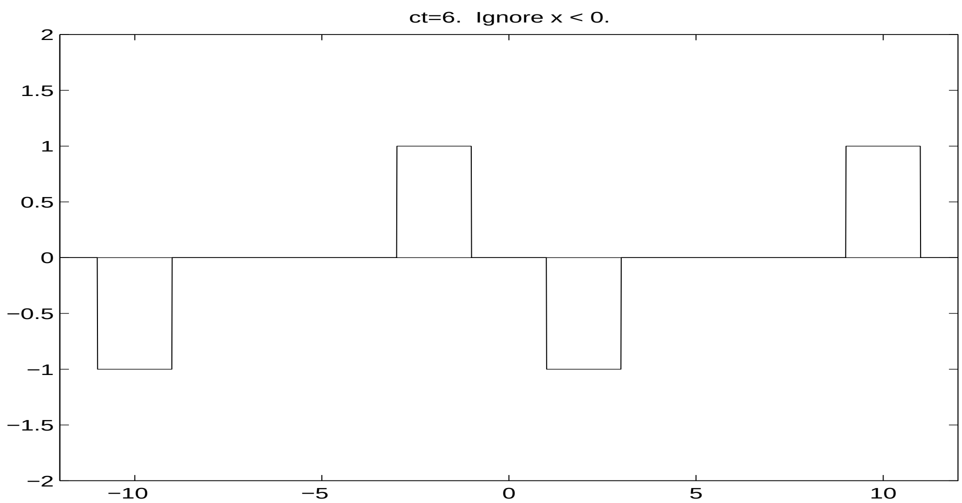


Figure 5.2. Wave propagation and reflection in a semi-infinite domain.

Chapter 6

Separation of Variables

6.1 Introduction

The method of separation of variables is a standard technique for solving linear PDEs in finite domains. Fourier series arise naturally from this method of solution.

6.2 An example of heat conduction in a rod:

Consider the problem of a copper rod of thermal diffusivity α^2 and of length L with a known initial temperature $u(x, 0) = f(x)$. For $t > 0$, the two ends of the rod are maintained at a constant temperature of 0° C. Find the temperature of the rod as a function of x and t.

The mathematical problem is specified by the partial differential equation (PDE) governing the heat conduction process, the boundary conditions (BCs), and the initial condition (IC):

$$\text{PDE:} \quad u_t = \alpha^2 u_{xx}, \quad 0 < x < L, \quad t > 0 \tag{6.1}$$

$$\text{BCs:} \quad u(0, t) = 0, \quad u(L, t) = 0, \quad t > 0 \tag{6.2}$$

$$\text{IC:} \quad u(x, 0) = f(x), \quad 0 < x < L. \tag{6.3}$$

(It is understood that u is expressed in units of °C.)

The boundary conditions we have imposed here are of Dirichlet type. We prefer to first make them homogeneous. If the boundary values were instead

$$u(0, t) = T_1, \quad u(L, t) = T_2,$$

we would first try to make them zero by defining a new unknown $\tilde{u}(x,t)$:

$$\tilde{u}(x,t) = u(x,t) - \left[T_1 + \frac{x}{L}(T_2 - T_1)\right],$$

so that the new problem for $\tilde{u}(x,t)$ has homogeneous boundary conditions:

PDE: $\tilde{u}_t = \alpha^2 \tilde{u}_{xx},\ 0 < x < L,\ t > 0$

BCs: $\tilde{u}(0,t) = 0,\ \tilde{u}(L,t) = 0,\ t > 0$

IC: $\tilde{u}(x,0) = f(x) - \left[T_1 + \frac{x}{L}(T_2 - T_1)\right] \equiv g(x),\ 0 < x < L.$

In the following section we will consider the system (6.1), (6.2), and (6.3), with the understanding that if the boundary conditions are not homogeneous, we can make them so with a redefinition of u and f.

6.3 Separation of variables:

- **Step 1:** We first assume the solution to the PDE (6.1) is of the "separable" form:

$$u(x,t) = T(t)X(x). \tag{6.4}$$

- **Step 2:** Substituting the assumed form (6.4) into Eq. (6.1) yields

$$\frac{d}{dt}T(t) \cdot X(x) = \alpha^2 T(t) \frac{d^2}{dx^2} X.$$

We divide both sides of the equation by $\alpha^2 T(t)X(x)$ to get

$$\frac{\frac{d}{dt}T(t)}{\alpha^2 T(t)} = \frac{\frac{d^2}{dx^2}X(x)}{X(x)}. \tag{6.5}$$

[Division by α^2 is not necessary, and will not make any difference to the procedure if this is not done.]

- **Step 3:** Notice that the left-hand side of Eq. (6.5) is a function of t only, while the right-hand side is a function of x only. The only way a function of t can be equal to a function of x is for each to equal to a constant. Let this *separation constant* be denoted by K.

So (6.5) becomes

$$\frac{d}{dt}T(t)/\alpha^2 T(t) = \frac{d^2}{dx^2}X(x)/X(x) = K. \tag{6.6}$$

This is actually *two* ordinary differential equations:

$$\frac{d^2}{dx^2}X(x) = KX(x), \tag{6.7}$$

and

$$\frac{d}{dt}T(t) = \alpha^2 KT(t). \tag{6.8}$$

- **Step 4:** We know how to solve Eq. (6.7) from Chapter 1 if K is negative. Let us solve this case first and later do the K positive case.

  Let $K = -\lambda^2 < 0$, where λ^2 is some positive constant. Eq. (5.7) becomes the harmonic oscillator equation:

  $$\frac{d^2}{dx^2}X(x) + \lambda^2 X(x) = 0, \tag{6.9}$$

  whose solution is, from Chapter 1:

  $$X(x) = A\sin\lambda x + B\cos\lambda x. \tag{6.10}$$

  [We can alternatively use the complex notation and write $X(x) = ae^{i\lambda x} + be^{-i\lambda x}$. In the present case it is more convenient, for the purpose of applying boundary condition, to use the real solution (6.10).]

  The constants A and B are (presumably) to be determined from the boundary conditions, which are, from (6.2) and (6.4):

  $$X(0) = 0, \quad X(L) = 0. \tag{6.11}$$

  From (6.10), we have

  $$X(0) = B,$$

  so the first boundary conditions demands that $B = 0$. Thus,

  $$X(x) = A\sin\lambda x, \tag{6.12}$$

  $$X(L) = A\sin\lambda L.$$

  The second boundary condition, $X(L) = 0$, then implies either

  $$A = 0 \quad \text{or} \quad \sin\lambda L = 0.$$

  The first possibility, $A = 0$, gives a trivial solution

  $$X(x) \equiv 0.$$

For nontrivial solutions, we need $\sin\lambda L = 0$, yielding the *eigenvalue*:

$$\lambda = \frac{n\pi}{L} \equiv \lambda_n, \quad n = 1, 2, 3, 4, 5, \dots \ . \tag{6.13}$$

[The nagative integer values of n doesnot give different solutions from the positive values because $A\sin(-\lambda_n x) = A' \sin(\lambda_n x)$, where $A' = -A$.]

The corresponding *eigenfunction* (from (6.12) and (6.13)) is:

$$X(x) = \sin\lambda_n x \equiv X_n(x). \tag{6.14}$$

[Note that we have set the arbitrary constant A to 1 in (6.14), without loss of generality, because we can always absorb a different A in $T(t)$. It is the *product*, $u(x,t) = T(t)X(x)$, that matters.]

It is easy to show that for $K > 0$, the solution to (6.7) is

$$X(x) = Ae^{\sqrt{K}x} + Be^{-\sqrt{K}x}.$$

$X(0) = 0$ implies

$$A + B = 0, \quad \text{or} \quad A = -B.$$

$X(L) = 0$ implies

$$Ae^{\sqrt{K}x} + Be^{-\sqrt{K}x} = 0,$$

or

$$B\left(-e^{\sqrt{K}L} + e^{-\sqrt{K}L}\right) = 0.$$

Since $K > 0$, $e^{\sqrt{K}L} > e^{-\sqrt{K}L}$, B should be zero. Thus $A = -B$ is also zero, leading to a trivial solution $X(x)$.

For $K = 0$, the ordinary differential equation for $X(x)$ becomes

$$\frac{d^2}{dx^2}X = 0.$$

Its solution is

$$X(x) = Ax + B.$$

Applying the boundary condition $X(0) = 0$ leads to $B = 0$. Applying $X(L) = 0$ then implies that $A = 0$. This again leads to the trivial solution $X(x)$.

Thus we conclude that for nontrivial solutions, K can only be negative, and furthermore K can only equal the following discrete values

$$K = -\lambda_n^2 = -\left(\frac{n\pi}{L}\right)^2, \quad n = 1, 2, 3, 4, \dots \ .$$

- **Step 5:** The $T(t)$ equation (6.8) now becomes

$$\frac{d}{dt}T = -\alpha^2\lambda_n^2 T(t). \tag{6.15}$$

We denote the solution for each value of n:

$$T(t) = T_n(t) = T_n(0)e^{-\alpha^2\lambda_n^2 t}. \tag{6.16}$$

- **Step 6:** We have in fact found an infinite number of solutions to the PDE (6.1), each satisfying the boundary conditions (6.2). They are of the form

$$\begin{aligned} u_n(x,t) &\equiv T_n(t)X_n(x) \\ &= T_n(0)e^{-\alpha^2\left(\frac{n\pi}{L}\right)^2 t}\sin\frac{n\pi x}{L}, \end{aligned} \tag{6.17}$$

each corresponding to a value of n, $n = 1, 2, 3, 4, \ldots$.

In (6.17), the $T_n(0)$'s are arbitrary constants, (presumably) to be determined from the initial condition (6.3). However, this is only feasible if the initial condition (6.3) is such a sine function. For example, if the initial condition is

$$u(x,0) = \sin\frac{\pi x}{L}, \tag{6.18}$$

we should then pick $n = 1$, and $T_1(0) = 1$. This leads to the solution

$$u(x,t) = u_1(x,t) = e^{-\alpha^2\left(\frac{\pi}{L}\right)^2 t}\sin\frac{\pi x}{L}. \tag{6.19}$$

You should now verify that (6.19) satisfies the PDE (6.18), and is therefore *the* solution we are looking for in this particular special case.

- **Step 7:** To satisfy more general initial conditions, we need to construct a more general solution. We do so by adding up all possible component solutions in (6.17). [This is referred to as the *principle of superposition.*]

We write:

$$\begin{aligned} u(x,t) &= u_1(x,t) + u_2(x,t) + u_3(x,t) + \ldots \\ &= \sum_{n=1}^{\infty} u_n(x,t) \qquad (6.20) \\ &= \sum_{n=1}^{\infty} T_n(0)e^{-(\alpha\pi/L)^2 t}\sin\frac{n\pi x}{L}. \qquad (6.21) \end{aligned}$$

You should check that the sum in (6.20) satisfies the PDE (6.1) and the boundary condition (6.2), presuming the infinite series converges.

- **Step 8:** We now use the more general solution (6.21) to satisfy the initial condition (6.3). To satisfy

$$u(x, 0) = f(x), \quad 0 < x < L.$$

we require

$$f(x) = \sum_{n=1}^{\infty} T_n(0) \sin \frac{n\pi x}{L}, \quad 0 < x < L, \tag{6.22}$$

The remaining task is to evaluate $T_n(0)$ given $f(x)$.

- **Step 9:** If we can express a function $f(x)$ in terms of what is now known as a *Fourier sine series*:

$$f(x) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{L}, \quad 0 < x < L, \tag{6.23}$$

then we can equate each term in (6.22) and (6.23) and find that

$$T_n(0) = a_n, \quad n = 1, 2, 3, \ldots$$

and the problem is then completely solved.

The question is, can we always represent an arbitrary function $f(x)$ in the form of a Fourier sine series (6.23)? We will leave this issue to the next chapter, where Fourier series will be discussed.

The French scientist Joseph Fourier faced these questions when he studied the heat conduction problem, much as we presented it here. Fourier claimed in 1807, when he presented his paper on heat conduction to the Paris Academy, that an arbitrary function $f(x)$ could indeed be expressed as a sum of sines in the form of (6.23). There was not much mathematical rigor in Fourier's arguments; he was probably motivated by his physical understanding of the heat conduction problem for which the general solution should be expressible in the form of (6.21). Setting $t = 0$ in this solution would then seen to "require" the initial arbitrary temperature distribution to be expressible as a sum of sines, as in (6.22). This assertion of Fourier's was ridiculed by the mathematician Lagrange at the time. We now know of course that Fourier was right: Any physically reasonable function $f(x)$ can be written in the form of a sum of sines (or cosines for that matter).

Assuming that (6.23) is true and the series converges, we can obtain the coefficient a_n of the Fourier sine series of $f(x)$ in the following manner.

Multiply both sides of Eq. (6.23) by $\sin \frac{m\pi x}{L}$, where m is any integer, and integrate over the domain:

$$\int_0^L f(x) \sin \frac{m\pi x}{L} dx = \sum_{n=1}^{\infty} a_n \int_0^L \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L} dx. \tag{6.24}$$

Note that in (6.24) we have switched the order of integration and summation. This is allowable if the series is uniformly convergent. In the next chapter, we will derive the so-called *orthogonality relationship* of sines which states

$$\frac{2}{L} \int_0^L \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L} dx = \delta mn \equiv \begin{cases} 1 & if \quad m = n \\ 0 & if \quad m \neq n. \end{cases} \tag{6.25}$$

Substituting (6.25) into (6.24), we find that only one term remains in the infinite sum on the right-hand side:

$$\sum_{n=1}^{\infty} a_n \frac{L}{2} \delta mn = \frac{L}{2} a_m. \tag{6.26}$$

Equating (6.26) to the left-hand side of (6.24), we obtain:

$$a_m = \frac{2}{L} \int_0^L f(x) \sin \frac{m\pi x}{L} dx, \quad \text{where } m = 1, 2, 3, 4, \ldots . \tag{6.27}$$

Since m is an arbitrary index, we can use any other symbol, including n. Thus (6.27) is the same as

$$a_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx, \quad n = 1, 2, 3, \ldots . \tag{6.28}$$

Either (6.27) or (6.28) can be used to generate the coefficients $a_1,\ a_2,\ a_3,\ a_4,\ \ldots.$

- **Step 10:** Finally, we have the solution which satisfies the PDE, the BCs, and the IC:

$$u(x,t) = \sum_{n=1}^{\infty} a_n e^{-(\alpha n\pi/L)^2 t} \sin\frac{n\pi x}{L}, \quad 0 < x < L$$
$$\text{where} \quad a_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx. \tag{6.29}$$

  You should try to verify *a posteri* that (6.29) does indeed satisfy (6.1), (6.2), and (6.3).

6.4 Physical interpretation of the solution:

The general solution (6.29) to the heat conduction problem, although complicated, has some simple physical interpretations.

It can be rewritten as

$$u(x,t) = \sum_{n=1}^{\infty} a_n e^{-n^2 t/t_e} \sin\frac{n\pi x}{L},$$

where $t_e \equiv (L/(\pi\alpha))^2$ is about an hour for a copper rod of length 2m (with $\alpha^2 = 1.16$ cm^2/s). The initial temperature distribution

$$u(x,0) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{L}$$

can be quite complicated and can consist of many sine modes. As time goes on, however, the small scales (i.e. the higher n sine modes) in the sum for $u(x,t)$ decay much faster than the larger scale modes. If a_1 and a_n are nonzero, we can look at the ratio

$$\frac{a_n e^{-n^2 t/t_e}}{a_1 e^{-t/t_e}}$$

of the coefficients of the nth mode and the first mode.

At $t = t_e$, about one hour later, the ratio gets smaller and smaller for increasing n because

$$\begin{aligned} e^{-n^2}/e^{-1} \quad &= 0.05 \quad \text{for } n = 2 \\ &3\times 10^{-4} \quad \text{for } n = 3 \\ &3\times 10^{-7} \quad \text{for } n = 4. \end{aligned}$$

Therefore, for most practical purposes, the full solution is dominated by the first term, the lowest sine mode:

$$u(x,t) \simeq a_1 e^{-t/t_e} \sin \frac{\pi x}{L} \quad \text{for } t \gtrsim t_e.$$

Eventually, even this lowest mode decays to zero, as the rod approaches a uniform zero temperature consistent with the temperature specified at the boundaries.

For small times, smaller scale modes are significant, if these were present in the initial temperature distribution. However, the smaller scales decay faster than the larger scales. We see a gradual *smoothing* of the solution. This is a common property of diffusion and heat conduction, which always tends to smooth out gradients that are present.

6.5 A vibrating string problem:

Consider a vibrating (guitar) string of length L—the length of the vibrating part being determined by where the player presses on the string. Let $c^2 = T/\rho$, where T is the tension on the string, which can be adjusted by the player, and ρ is the density of its material.

$$\text{PDE:} \quad u_{tt} = c^2 u_{xx}, \quad 0 < x < L, \quad t > 0 \tag{6.30}$$

$$\text{BCs:} \quad u(0,t) = 0, \quad u(L,t) = 0 \tag{6.31}$$

$$\text{IC:} \quad u(x,0) = f(x), \quad u_t(x,0) = g(x), \quad 0 < x < L. \tag{6.32}$$

A few words are in order about the initial conditions. Mathematically, since there is a second derivative in t in the governing PDE, there should be two initial conditions to completely specify the problem. We will see this as we proceed further with the solution of the problem and discover that there will be undetermined constants in the solution if we don't prescribe a second initial condition. This is unlike the case of the heat equation, which needs only a first order derivative in t be given.

In (6.32), $f(x)$ is the shape of the initial displacement and $g(x)$ is the shape of the initial velocity. We will see that the intensity of the sound and the spectrum of frequencies of sound generated depend on both initial conditions.

We again use the method of separation of variables to solve this problem. Since we have already discussed the ten steps of this method in detail in the previous section, there is no need to repeat every detail here again. You can follow this template in doing your homework and exam problems.

We first assume that the solution can be written in the separable form:

$$u(x,t) = T(t)X(x)$$

anticipating that we will ultimately superpose such solutions to satisfy initial conditions.

Substituting into the PDE yields

$$\frac{1}{c^2}\frac{d^2T}{dt^2}/T = \frac{d^2X}{dx^2}/X = -\lambda^2,$$

where $-\lambda^2$ is the separation constant. Solving the ordinary differential equation for X:

$$\frac{d^2X}{dx^2} + \lambda^2 X = 0,$$

subject to the boundary conditions:

$$X(0) = 0, \quad X(L) = 0,$$

yields the eigenvalues:

$$\lambda = \lambda_n = \frac{n\pi}{L}, \quad n = 1, 2, 3, \ldots$$

and the corresponding eigenfunction:

$$X(x) = X_n(x) = \sin\lambda_n x$$

(again scaling out any multiplicative factor)

The T-equation:

$$\frac{d^2T}{dt^2} = -c^2\lambda^2 T$$

can be solved for each value of $\lambda = \lambda_n$ to yield:

$$T(t) = T_n(t) = A_n\sin(\omega_n t) + B_n\cos(\omega_n t),$$

where $\omega_n = c\lambda_n = cn\pi/L$ is the *frequency* of oscillation and the constants A_n and B_n remain arbitrary.

We construct the general solution by superimposing all possible solutions of the form $T_n(t)X_n(x)$, to yield:

$$\boxed{\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} T_n(t)X_n(x) \\ &= \sum_{n=1}^{\infty}[A_n\sin(\omega_n t) + B_n\cos(\omega_n t)]\sin\frac{n\pi x}{L} \end{aligned}} . \tag{6.33}$$

To satisfy the initial conditions (6.32), we require

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}, \quad 0 < x < L, \tag{6.34}$$

and

$$u_t(x,0) = g(x) = \sum_{n=1}^{\infty} \frac{cn\pi}{L} A_n \sin\frac{n\pi x}{L}, \quad 0 < x < L. \tag{6.35}$$

(6.34) is a Fourier sine series for the initial displacement $f(x)$, and so (see (6.29))

$$\boxed{B_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx}\,. \tag{6.36}$$

(6.35) is a Fourier sine series for the initial velocity $g(x)$, and so

$$\boxed{A_n = \frac{2}{\pi nc}\int_0^L g(x)\sin\frac{n\pi x}{L}dx}\,. \tag{6.37}$$

The solution (6.33) is now completely specified. Convergence of the series solution remains to be considered.

Physical Interpretation:

Unlike the solution of the heat/diffusion equation, the solution to the wave equation does not decay in time. Instead there are *standing waves* set up between the two ends of the string. The gravest (fundamental) standing-mode, $\sin\frac{\pi x}{L}$, oscillates with a frequency $\omega_1 = c\lambda_1$, and the nth standing-mode, $\sin\frac{n\pi x}{L}$, oscillates with a frequency

$$\omega_n = c\lambda_n = \frac{n\pi c}{L} = \frac{n\pi}{L}\sqrt{\frac{T}{\rho}}.$$

i.e. n times the fundamental frequency w_1.

This property, that all frequencies generated by a vibrating string are integer multiples of the fundamental frequency, is what makes the sound of a violin or guitar pleasing to the human ear. This property is a consequence of the one space dimensionality of the vibrating string, and is not shared by two dimensional vibrating membranes (such as a drumhead), where the higher-order frequencies are not integer multiples of the fundamental one.

The frequencies produced by a vibrating string

$$\omega_n = \frac{n\pi}{L}\sqrt{\frac{T}{\rho}}$$

depend on a few physical parameters, as revealed by the solution. The higher the tension on the string, the higher the frequency; the denser the string material, the lower the frequency; and the longer the length of the vibrating part of the string, the lower the frequency. The latter part is controlled by the guitarist's placement of his (or her) finger when he clamps down on the string.

6.6 Exercises

1. Solve the following heat equation:

$$\begin{array}{ll} \text{PDE:} & u_{tt} = a^2 u_{xx}, \quad 0 < x < L, \quad t > 0 \\ \text{BCs:} & u(0,t) = 0, \quad u(L,t) = 0, \quad t > 0 \\ \text{IC:} & u(x,0) = \displaystyle\sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right), \quad 0 < x < L, \end{array}$$

   where the a_n's are known constants.

2. (a) Solve the following wave equation for a guitar string of density ρ under tension T:

$$\begin{array}{ll} \text{PDE:} & u_{tt} = \left(\dfrac{T}{\rho}\right) u_{xx}, \quad 0 < x < L, \quad t > 0 \\ \text{BCs:} & u(0,t) = 0, \quad u(L,t) = 0, \quad t > 0 \\ \text{IC:} & u(x,0) = a_1 \sin\dfrac{\pi x}{L}, \quad 0 < x < L \\ & u_t(x,0) = 0, \quad 0 < x < L. \end{array}$$

   (b) What is the frequency of oscillation of the string?

6.7 Solutions

1. Solve the following heat equation:

$$u_t = \alpha^2 u_{xx}, \; 0 < x < L, \; t > 0$$

with initial condition $u(x,0) = \sum_{n=0}^{\infty} a_n \sin(n\pi x/L)$,

and boundary conditions $u(0,t) = 0$, and $u(L,t) = 0$.

We use separation of variables. Since we have homogeneous Dirichlet BCs we use:

$u(x,t) = \sum_{n=0}^{\infty} T_n(t) \sin(n\pi x/L)$.

Plugging into the PDE, we are left with a first order ODE in time which has the solution:

$T_n(t) = A_n e^{-(n\pi/L)^2\alpha^2 t}$.

Using the initial condition we find:

$\sum_{n=0}^{\infty} A_n \sin(n\pi x/L) = \sum_{n=0}^{\infty} a_n \sin(n\pi x/L)$.

Thus $A_n = a_n$ and

$u(x,t) = \sum_{n=0}^{\infty} a_n e^{-(n\pi/L)^2\alpha^2 t} \sin(n\pi x/L)$.

2. Solve the following vibrating string problem:

   $u_{tt} = (T/\rho)u_{xx}$, $(T/\rho) = \text{constant}$,

   with initial conditions $u(x,0) = a_1 \sin(\pi x/L)$, $u_t(x,0) = 0$,

   and boundary conditions $u(0,t) = 0$, and $u(L,t) = 0$.

   We use separation of variables. Since we have homogeneous Dirichlet BC we find:

   $u(x,t) = \sum_{n=0}^{\infty} T_n(t) \sin(n\pi x/L)$.

   Plugging into the PDE we obtain a second order ODE in time, which has the solution:

   $T_n(t) = A_n \sin(\omega_n t) + B_n \cos(\omega_n t)$, where $\omega_n = (n\pi/L)(T/\rho)^{1/2}$.

   Using the initial conditions we find:

   $\sum_{n=0}^{\infty} A_n \sin(n\pi x/L) = 0$ and $\sum_{n=0}^{\infty} B_n \sin(n\pi x/L) = a_1 \sin(\pi x/L)$.

   Thus we have that: $A_n = 0$ for all n, $B_1 = a_1$ and $B_n = 0$ for $n \neq 1$,

   leaving us with:

$u(x,t) = a_1 \cos((\pi/L)(T/\rho)^{1/2} t) \sin(\pi x/L).$

The frequency of vibration is $(\pi/L)(T/\rho)^{1/2}$.

Chapter 7

Fourier Sine Series

7.1 Introduction

An important mathematical question raised by Joseph Fourier in 1807, arising from his practical work on heat conduction, is whether an arbitrary function $f(x)$ can be represented in the form of a "Fourier sine series":

$$\boxed{f(x) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{L},\ \ 0 < x < L.} \tag{7.1}$$

A second question is: suppose we *can* indeed represent $f(x)$ by a Fourier sine series of the form (7.1), how do we calculate the "Fourier sine coefficients", a_n's?

7.2 Finding the Fourier coefficients

Let us deal with the second question first. Suppose (7.1) holds. We multiply both sides by $\sin\frac{m\pi x}{L}$, where m is any integer, and then integrate both sides from 0 to L. Thus,

$$\int_0^L f(x)\sin\frac{m\pi x}{L}dx = \sum_{n=1}^{\infty} a_n \int_0^L \sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L}dx, \tag{7.2}$$

where we have interchanged the order of integration and summation. Using the trigonometric identity:

$$\sin a \sin b = \frac{1}{2}[\cos(a-b) - \cos(a+b)],$$

the integral on the right-hand side of Eq. (7.2) can be evaluated:

$$
\begin{aligned}
I_{mn} \equiv \int_0^L \sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L}dx &= \frac{1}{2}\int_0^L \cos\frac{(m-n)\pi x}{L}dx - \frac{1}{2}\int_0^L \cos\frac{(n+m)\pi x}{L}dx \\
&= \frac{1}{2}\frac{\sin((m-n)\pi x/L)}{(m-n)\pi/L}\Bigg|_0^L - \frac{1}{2}\frac{\sin((m+n)\pi x/L)}{(m+n)\pi/L}\Bigg|_0^L
\end{aligned}
$$

Thus, we obtain the so-called *orthogonality relationship* for sines,

$$
I_{mn} = \frac{L}{2}\delta_{mn}, \tag{7.3}
$$

where

$$
\delta_{mn} \equiv \begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m \neq n. \end{cases}
$$

[When $m = n$, we have

$$
\begin{aligned}
I_{mm} &= \int_0^L \left(\sin\frac{m\pi x}{L}\right)^2 dx = \frac{1}{2}\int_0^L \left(1 - \cos\frac{2m\pi x}{L}\right) dx \\
&= \frac{L}{2}.]
\end{aligned}
$$

Substituting (7.3) into (7.2), we find

$$
\int_0^L f(x)\sin\frac{m\pi x}{L}dx = \sum_{n=1}^{\infty} a_n \frac{L}{2}\delta_{mn} = \frac{L}{2}a_m.
$$

So for any specified integer m,

$$
a_m = \frac{2}{L}\int_0^L f(x)\sin\frac{m\pi x}{L}dx. \tag{7.4}
$$

(7.4) gives:

$$
\begin{aligned}
a_1 &= \frac{2}{L}\int_0^L f(x)\sin\frac{\pi x}{L}dx \\
a_2 &= \frac{2}{L}\int_0^L f(x)\sin\frac{2\pi x}{L}dx \\
a_3 &= \frac{2}{L}\int_0^L f(x)\sin\frac{3\pi x}{L}dx \\
&\vdots
\end{aligned}
$$

In particular, we can write

$$\boxed{a_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx,\ \ n = 1, 2, 3, \ldots} \tag{7.5}$$

and thus have completed our task of finding the Fourier sine series coefficients, a_n, in (7.1).

7.3 An Example:

Represent $f(x) = 100$ in the form of a Fourier sine series over the interval $0 < x < L$:

$$f(x) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{L},\ \ 0 < x < L.$$

The Fourier coefficients, a_n, are given by

$$\begin{aligned} a_n &= \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx \\ &= \frac{2}{L}\int_0^L 100\sin\frac{n\pi x}{L}dx \\ &= \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{400}{n\pi} & \text{if } n \text{ is odd.} \end{cases} \end{aligned}$$

So we say that in $0 < x < L$, we have

$$100 = \frac{400}{\pi}\left[\frac{\sin(\pi x/L)}{1} + \frac{\sin(3\pi x/L)}{3} + \frac{\sin(5\pi x/L)}{5} + \ldots\right]. \tag{7.6}$$

(7.6) is rather strange; it says that a constant, 100, can be represented by a sum of sines. Let us see what we will get if we add up the sines in the right-hand side of (7.6). In Figure 7.1, we plot one term in the sum (i.e. $\frac{400}{\pi}\sin(\pi x/L)$). In Figure 7.2, we plot two terms, i.e. $\frac{400}{\pi}[\sin(\pi x/L) + \frac{1}{3}\sin(3\pi x/L)]$. In Figure 7.3, we plot 3 terms, etc. By the time we have included enough terms, we see that the right-hand side of (7.6) approaches the constant value of 100 in the interior of the interval, $0 < x < L$. (Near the edges $x = 0$ and $x = L$, the oscillations get increasingly confined to the edges, where the sum of sines tries very hard to approach 100 in the interior of the domain, $0 < x < L$, while being identically zero at $x = 0$ and $x = L$. A discontinuity is created at the edges. There is also the so-called Gibbs phenomenon present near the edges, where just within the boundaries, there is an overshoot of the true value of 100, by 18%.]

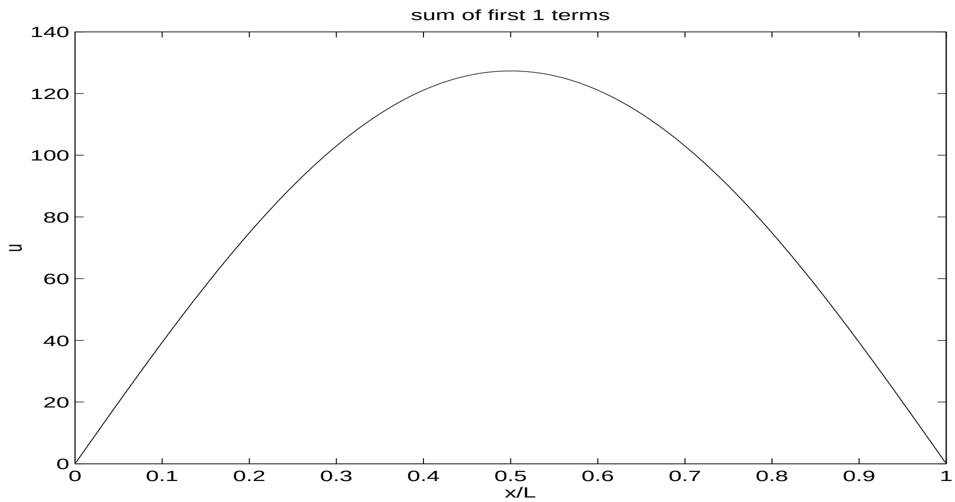


Figure 7.1: Plot of the first term in the Fourier sine expansion of 100.

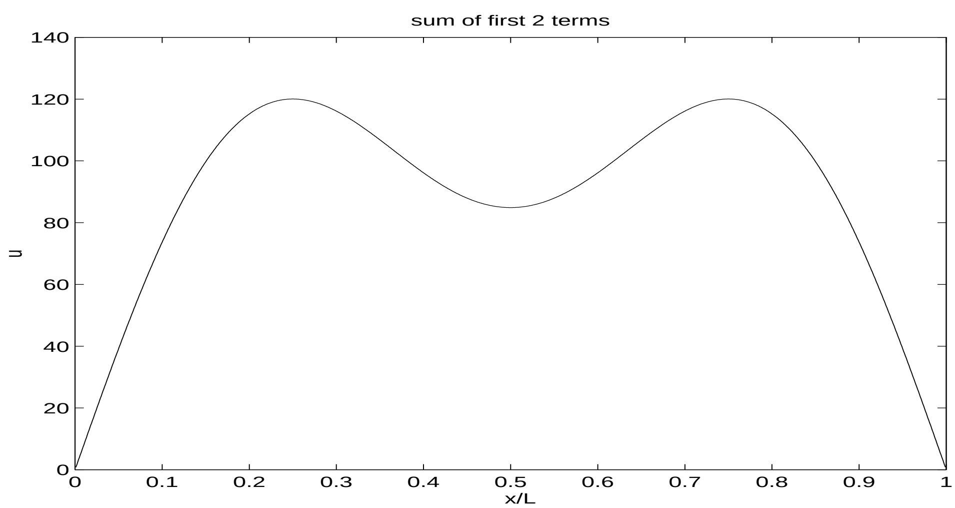


Figure 7.2: Plot of the sum of the first 2 terms in the Fourier sine expansion of 100.

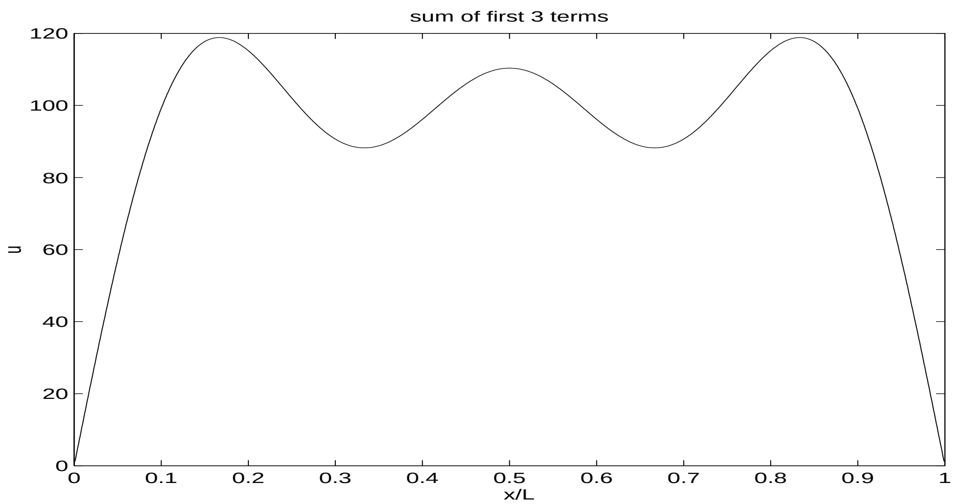


Figure 7.3: Plot of the sum of the first 3 terms in the Fourier sine expansion of 100.

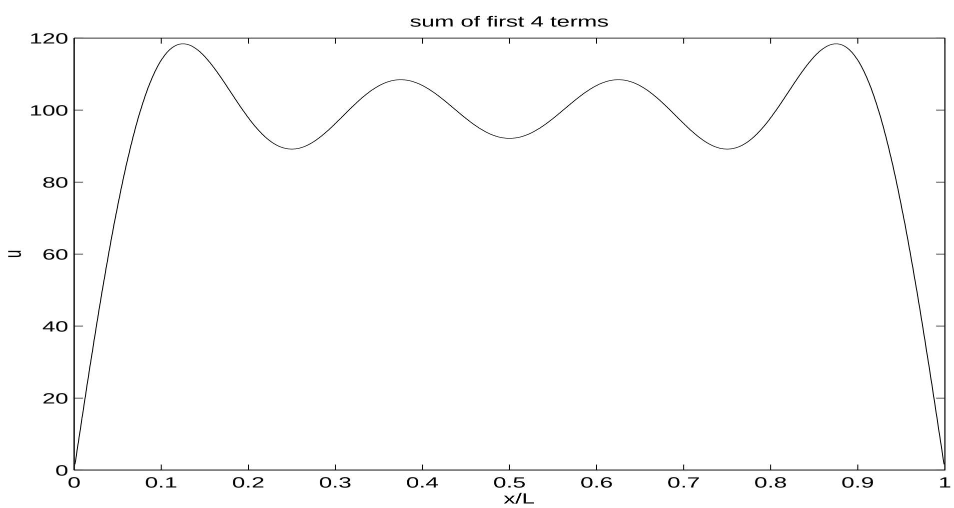


Figure 7.4: Plot of the sum of the first 100 terms in the Fourier sine expansion of 100.

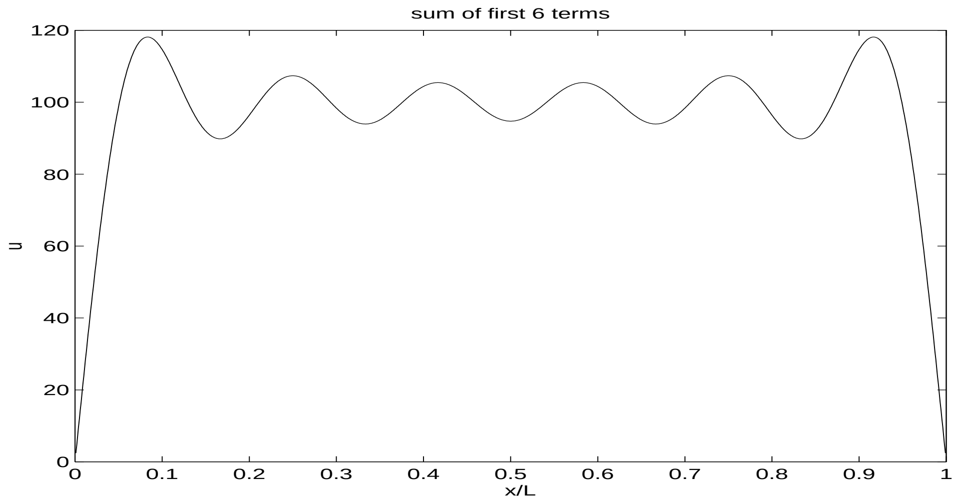


Figure 7.5: Plot of the sum of the first 6 terms in the Fourier sine expansion of 100.

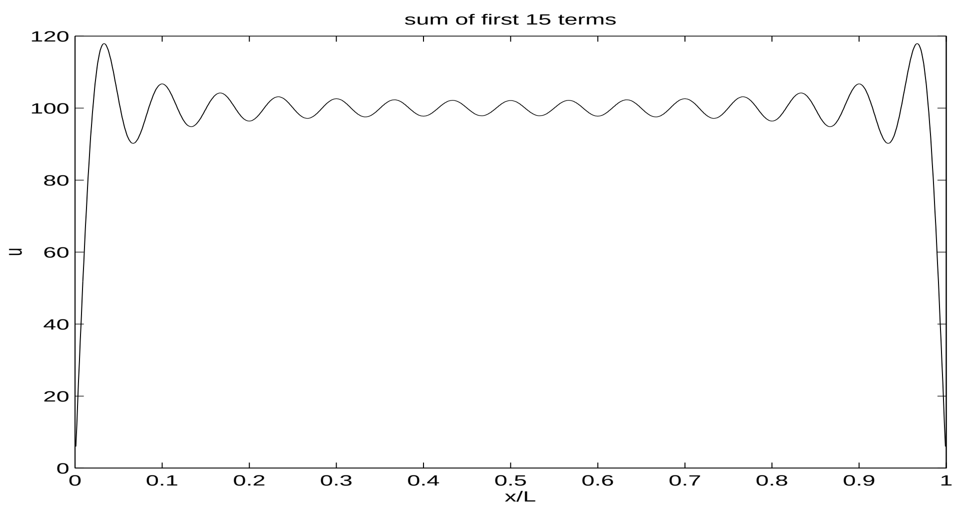


Figure 7.6: Plot of the sum of the first 15 terms in the Fourier sine expansion of 100.

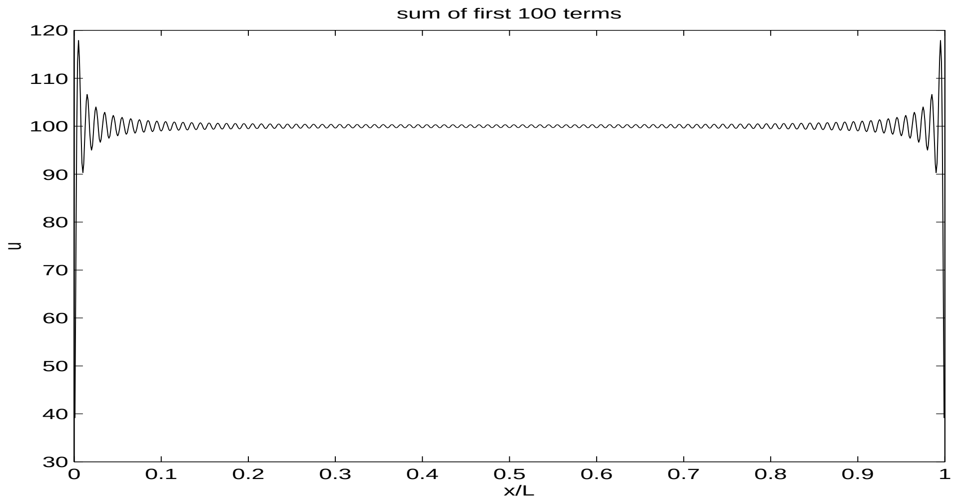


Figure 7.7: Plot of the sum of the first 100 terms in the Fourier sine expansion of 100.

7.4 Some comments:

What the example demonstrates is that the Fourier sine series can indeed represent $f(x)$ in the interval indicated. We can do this for other functions, and you will find that the Fourier sine series does a very good job in representing each of them. Actually, the most difficult function to represent by a Fourier sine series may be the one we have just done, $f(x) =$ constant in $0 < x < L$. This is because the sines all go to zero at $x = 0$ and $x = L$, but they have to add up to a nonzero constant slightly inside the boundaries. Many more terms in the sum are required to create this near discontinuity. For functions which are continuous and actually zero at the boundaries $x = 0$ and $x = L$, you will find that you do not need as many terms in the sum to give a good numerical representation of the original function.

It is not reasonable to expect that the sines can represent a function which blows up (i.e. attains infinite values) in the domain $0 < x < L$. Such unphysical functions are excluded in our consideration. The following mathematical result can be stated in a Theorem (a more general form is called Dirichlet's Theorem):

If $f(x)$ is a bounded function, which is continuous or piecewise continuous in a domain, the Fourier sine series representation of $f(x)$ converges to $f(x)$ for each point x in the domain where $f(x)$ is continuous. At those points where $f(x)$ jumps, the series converges to a value which is the average of the left- and right-hand limits of $f(x)$ at those points, where $f(x)$ is *discontinuous.*

[A piecewise continuous function is one which can take a finite number of finite jumps in the domain and be continuous elsewhere.]

7.5 A mathematical curiosity

If you are convinced that the function $f(x) = 100$ can be represented by a Fourier sine series in the interior of the domain:

$$100 = \frac{400}{\pi}\left(\sin\frac{\pi x}{L} + \frac{\sin\frac{3\pi x}{L}}{3} + \frac{\sin\frac{5\pi x}{L}}{5} + \dots\right), \ 0 < x < L,$$

then pick $x/L = \frac{1}{2}$ to get

$$100 = \frac{400}{\pi}(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \dots)$$

or

$$\pi = 4(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \dots). \tag{7.7}$$

This relationship between π and the odd integers was discovered by Leibniz in 1673 by a different route.

You may try adding up the right-hand side of (7.7) and see how many terms are needed to approximate π to the accuracy you want.

7.6 Representing the cosine by sines

When Fourier presented his work on heat conduction and Fourier series to the Paris Academy in 1807, neither Laplace nor Lagrange would accept his use of Fourier series. In particular, Laplace could not accept the fact that $\cos x$ could be represented using a sum of sines. Let us see if Fourier was right.

Let us try to represent

$$f(x) = \cos x$$

in the interval $0 < x < \pi$ by a sine series

$$f(x) = \sum_{n=1}^{\infty} a_n \sin nx.$$

Fourier's formula for the coefficients gives

$$a_n = \frac{2}{\pi}\int_0^{\pi} \cos x \sin nx \; dx, \;\; n = 1, 2, 3, \ldots .$$

Using the trigonometric identity

$$2 \cos a \sin b = \sin(a + b) - \sin(a - b),$$

the integral can be evaluated to yield:

$$a_n = \begin{cases} \frac{4n}{\pi(n^2-1)}, & n = \text{even} \\ 0, & n = \text{odd}. \end{cases}$$

Thus, we find

$$\begin{aligned} \cos x &= \sum_{n \; even} \frac{4n}{\pi(n^2 - 1)} \sin nx, \;\; 0 < x < \pi \\ &= \frac{8}{3\pi} \sin 2x + \frac{16}{15\pi} \sin 4x + \ldots , \;\; 0 < x < \pi. \end{aligned}$$

Try adding up as many terms as you can using a graphing calculator or computer. Does the sum approach $\cos x$ in the interval?

Comment: We can express a cosine in terms of a sine series only for half of its period. It is still true that a cosine cannot be expressed in terms of sines in its full period $-\pi < x < \pi$.

7.7 Application to the Heat Conduction Problem

PDE: $u_t = \alpha^2 u_{xx}, \;\; 0 < x < L, \;\; t > 0$

BCs: $u(0, t) = 0, \;\; u(L, t) = 0, \;\; t > 0$

IC: $u(x, 0) = f(x), \;\; 0 < x < L.$

In particular, we shall consider the case where $f(x) = 100$.

In Chapter 6, we found that the general solution to the PDE which satisfies the BCs is given by

$$u(x, t) = \sum_{n=1}^{\infty} T_n(0) e^{-\alpha^2 (\frac{n\pi}{L})t} \sin \frac{n\pi x}{L}, \;\; 0 < x < L. \tag{7.8}$$

where the constants $T_n(0)$ are yet to be determined from the IC.

Setting $t = 0$ in (7.8) and setting $u(x, 0) = f(x)$, we arrive at

$$f(x) = \sum_{n=1}^{\infty} T_n(0) \sin \frac{n\pi x}{L}, \ 0 < x < L, \tag{7.9}$$

Therefore $T_n(0)$ is the Fourier sine coefficient of $f(x)$ and is given by (7.4) as

$$T_n(0) = a_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx.$$

For $f(x) = 100$, we know that

$$a_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{400}{n\pi} & \text{if } n \text{ is odd.} \end{cases}$$

Thus finally, the solution to the above PDE problem is

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} a_n e^{-\alpha^2 (\frac{n\pi}{L})^2 t} \sin \frac{n\pi x}{L} \\ &= \sum_{k=1}^{N} \frac{400}{\pi} \frac{1}{(2k-1)} e^{-(2k-1)^2 (t/t_e)} \sin \frac{(2k-1)\pi x}{L}, \ 0 < x < L, \end{aligned} \tag{7.10}$$

where $t_e \equiv \left(\frac{L}{\alpha\pi}\right)^2$, and $N \to \infty$.

The solution in (7.10) is plotted in Figure 7.8 for different values of t/te. It turns out that unless t/te is very small, only a few terms are needed in the sum in (7.10). In Figure 7.9, we show that the solution can be represented to a high degree of accuracy by the first two terms:

$$u(x,t) \cong \frac{400}{\pi} e^{-t/te} \sin \frac{\pi n}{L} + \frac{400}{3\pi} e^{-9t/te} \sin \frac{3\pi x}{L}, \quad \text{for } t \gtrsim te.$$

Although we obtained this behavior already in Chapter 6, only now do we have the actual coefficients a_1 and a_3 etc calculated explicitly.

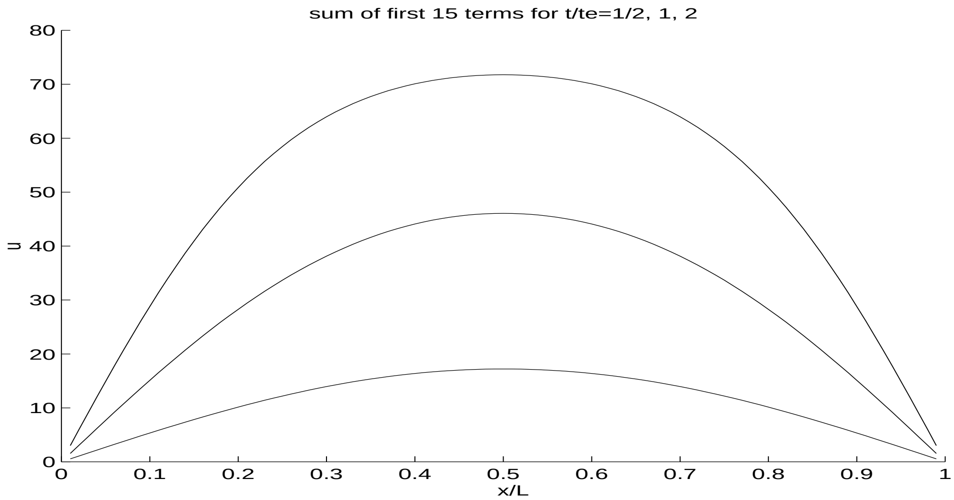


Figure 7.8: Plot of the sum of the first 15 terms in solution for $t/t_e = 1/2$, 1, 2.

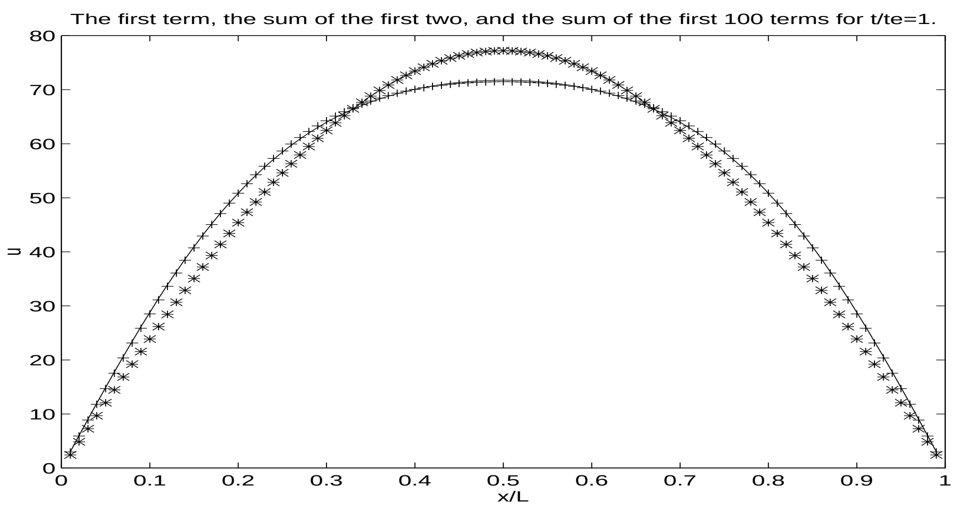


Figure 7.9: The first term, sum of the first two, and sum of the first 100 terms.

7.8 Exercises

1. Let $f(x)$ be given by

$$f(x) = \begin{cases} x, & 0 < x < L/2 \\ (L - x), & L/2 < x < L. \end{cases}$$

Represent $f(x)$ by a Fourier sine series of the form

$$f(x) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{L}, \quad 0 < x < L.$$

(a) Find a_n, $n = 1, 2, 3, \ldots$.

(b) Retain the first N terms as an approximation

$$f(x) \cong f_N(x) \equiv \sum_{n=1}^{N} a_n \sin \frac{n\pi x}{L}, \quad 0 < x < L.$$

Using a graphing calculator or a computer, plot $f_N(x)$ as a function of x/L for $N = 1$, 3, 5, 31, and (optional) 101.

[You will notice that $a_n = 0$ for even n's, so the actual number of nonzero terms in the sum is halved.]

2. The solution to

$$\begin{aligned} &\text{PDE:} \quad u_t = \alpha^2 u_{xx}, \quad 0 < x < L \\ &\text{BCs:} \quad u(0,t) = 0, \quad u(L,t) = 0 \\ &\text{IC:} \quad u(x,0) = f(x), \quad \text{where } f(x) \text{ is given in problem 1,} \end{aligned}$$

is given by

$$u(x,t) = \sum_{n=1}^{\infty} a_n e^{-n^2 (t/t_e)} \sin \frac{n\pi x}{L}, \quad 0 < x < L,$$

where $t_e = (L/\pi\alpha)^2$.

Again replace

$$\sum_{n=1}^{\infty} \text{ by } \sum_{n=1}^{N} \text{ as an approximation.}$$

Plot out the solution as a function of x/L for $t = \frac{1}{2}t_e$, t_e and $2t_e$. Use a large enough N so that your solution does not change noticeably when N is increased. You will find that you need only a few terms in the sum to get an accurate solution.

7.9 Solutions

Problem 1, figures 7.10-7.14

(a) We know that $f(x) = \sum_{n=1}^{\infty} a_n \sin(n\pi x/L)$. Thus $a_n = 2/L \int_0^L f(x) \sin(n\pi x/L) dx$ which upon integration yields;

$a_n = (4L/(n\pi)^2) \sin(n\pi/2)$.

$f(x) = \sum_{n=1}^{\infty} a_n \sin(n\pi x/L)$ where a_n is given above.

(b) see figures 7.10-7.14

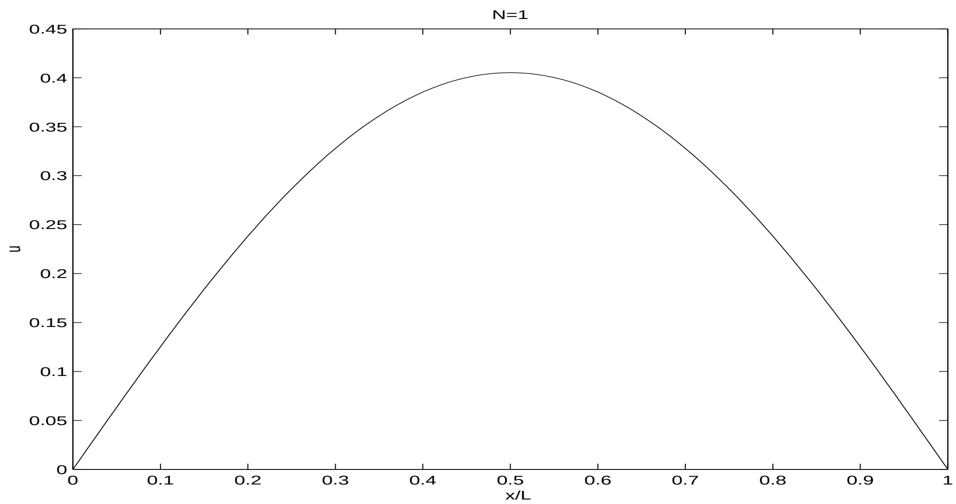


Figure 7.10: Sum of the first 1 terms in the Fourier sine expansion of x(L-x), L=1.

Problem 2, figures 7.15-7.17

$u(x,t) = \sum_{n=1}^{\infty} a_n e^{-n^2(t/t_e)} \sin(n\pi x/L)$ where a_n is given above.

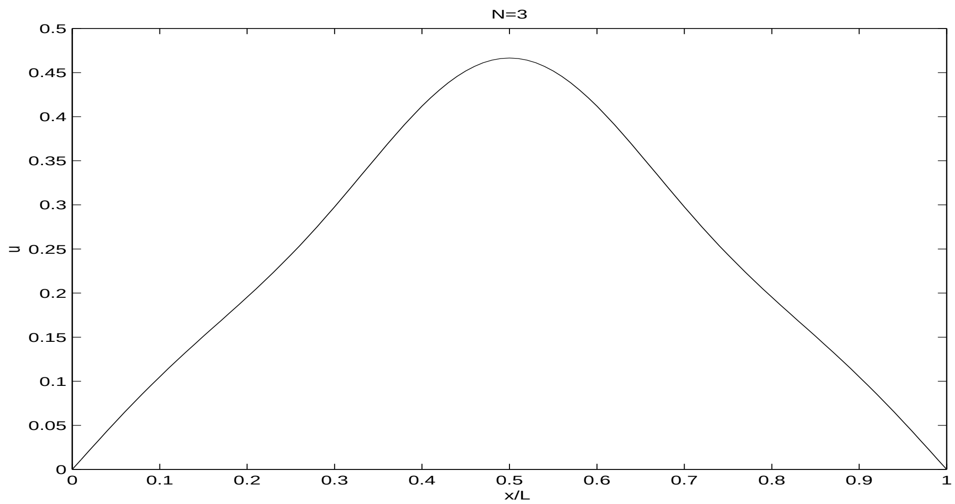


Figure 7.11: Sum of the terms up to N=3 in the Fourier sine expansion of x(L-x), L=1.

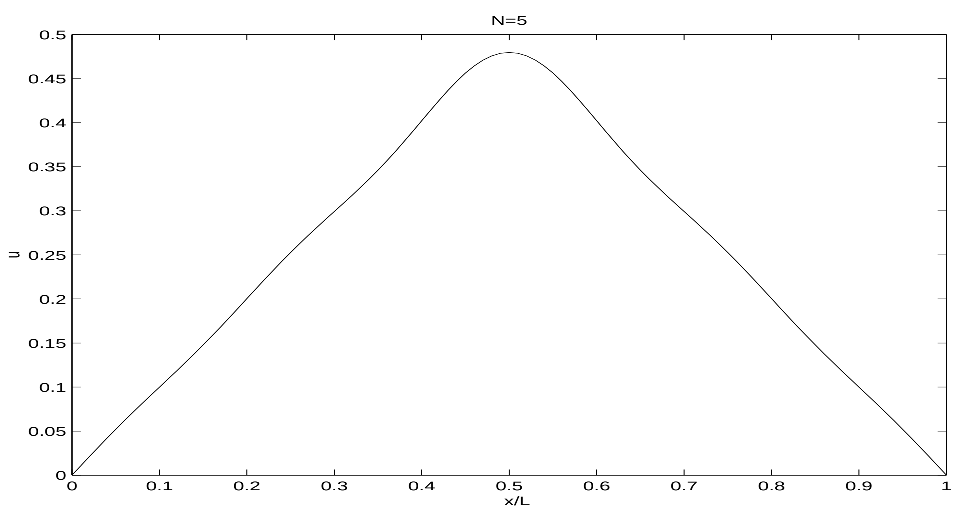


Figure 7.12: Sum of the terms up to N=5 in the Fourier sine expansion of x(L-x), L=1.

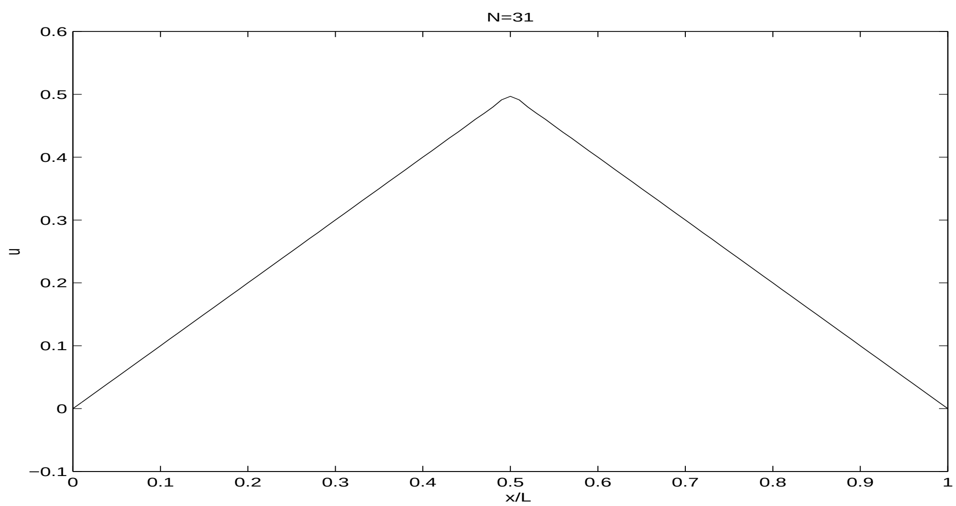


Figure 7.13: Sum of the terms up to N=31 in the Fourier sine expansion of x(L-x), L=1.

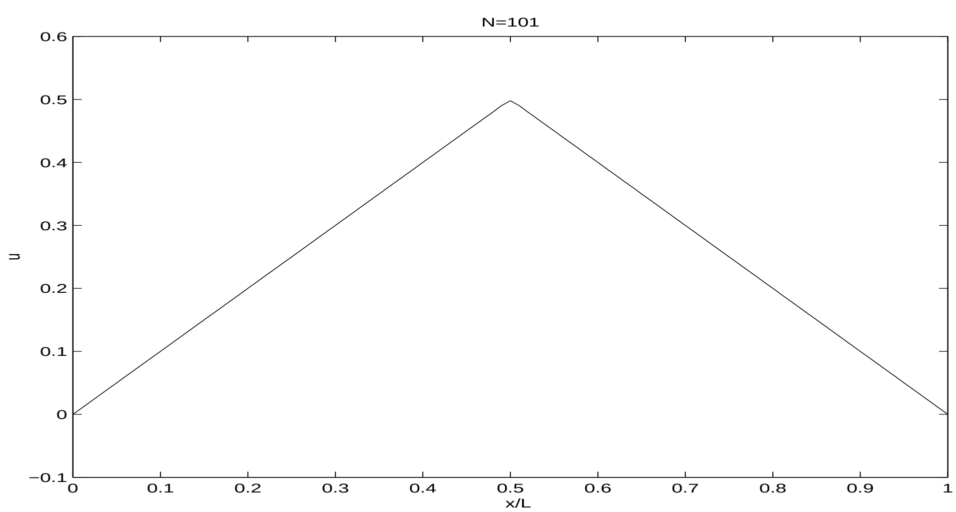


Figure 7.14: Sum of the terms up to N=101 in the Fourier sine expansion of x(L-x), L=1.

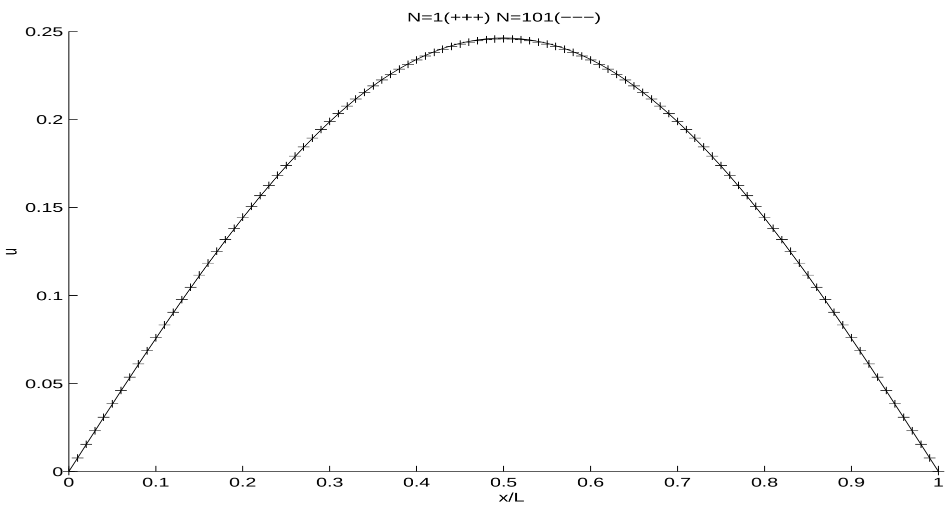


Figure 7.15: Sum of the terms for N=1 and N=101 in u(x,t) for t/te=1/2.

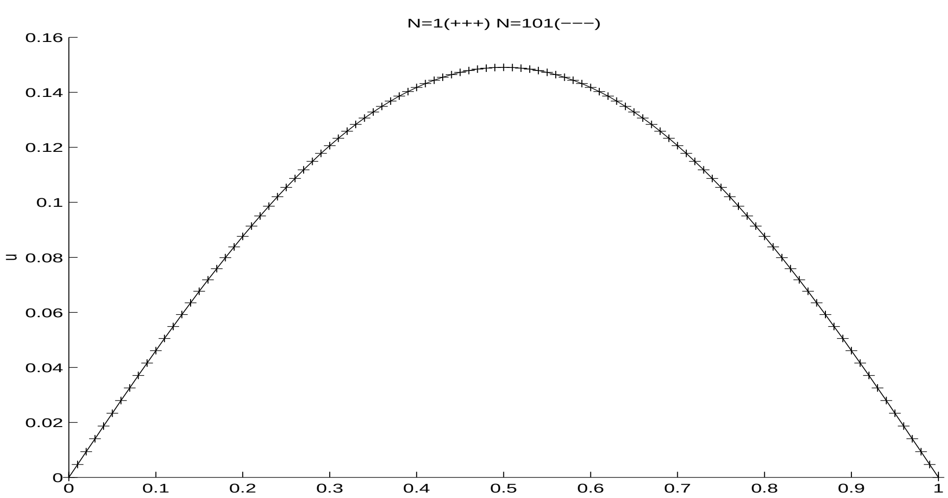


Figure 7.16: Sum of the terms for N=1 and N=101 in u(x,t) for t/te=1.

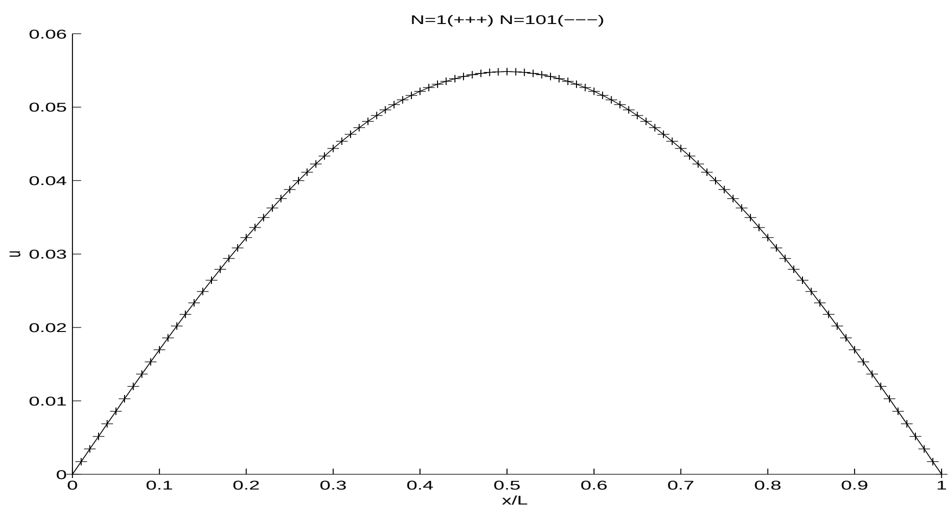


Figure 7.17: Sum of the terms for N=1 and N=101 in u(x,t) for t/te=2.

Chapter 8

Fourier Cosine Series

8.1 Introduction

In the previous chapter, function $f(x)$ were represented by a series of sines. It is also possible to express the same function alternatively in a series of cosines, in the form

$$f(x) = \sum_{n=0}^{\infty} b_n \cos\frac{n\pi x}{L}, \;\; 0 < x < L. \tag{8.1}$$

Here the summation starts from $n = 0$, because $\cos 0 = 1$ is not zero. We will delay the motivation for wanting to write $f(x)$ in this form until later. Here we discuss only *how* to find the Fourier cosine series coefficients b_n assuming that $f(x)$ can be represented in the form of (8.1).

8.2 Finding the Fourier coefficients

We multiply both sides of (8.1) by $\cos\frac{m\pi x}{L}$, where m is any integer, $m = 0$, 1, 2, 3, ..., and integrate from 0 to L:

$$\int_0^L f(x)\cos\frac{m\pi x}{L}dx = \sum_{n=0}^{\infty} b_n \int_0^L \cos\frac{m\pi x}{L}\cos\frac{n\pi x}{L}dx. \tag{8.2}$$

There is *an orthogonality* condition for the cosines which can be written as

$$I_{mn} \equiv \int_0^L \cos\frac{m\pi x}{L}\cos\frac{n\pi x}{L}dx = \begin{cases} 0 & \text{if } m \neq n \\ \frac{L}{2} & \text{if } m = n \neq 0 \\ L & \text{if } m = n = 0. \end{cases} \tag{8.3}$$

To show this, we note the trigonometric identity

$$\cos a \cos b = \frac{1}{2}\cos(a-b) + \frac{1}{2}\cos(a+b),$$

so

$$\begin{aligned}
\int_0^L \cos\frac{m\pi x}{L}\cos\frac{n\pi x}{L}dx &= \frac{1}{2}\int_0^L \cos\frac{(m-n)\pi x}{L}dx + \frac{1}{2}\int_0^L \cos\frac{(m+n)\pi x}{L}dx \\
&= \frac{\sin\frac{(m-n)\pi x}{L}}{2(m-n)\pi/L}\Bigg|_0^L + \frac{\sin\frac{(m+n)\pi x}{L}}{2(m+n)\pi/2}\Bigg|_0^L \\
&= 0 \quad \text{if } m \neq n.
\end{aligned}$$

When $n = m \neq 0$

$$\cos\frac{m\pi x}{L}\cos\frac{n\pi x}{L} = \frac{1}{2}\left(1 + \cos\frac{2m\pi x}{L}\right).$$

The integral from 0 to L of the first term, $\frac{1}{2}$, is $L/2$, while the integral of the second term, $-\frac{1}{2}\cos\frac{2m\pi x}{L}$, is zero. When $m = n = 0$,

$$\cos\frac{m\pi x}{L}\cos\frac{n\pi x}{L} = 1,$$

so its integral from 0 to L is L. Thus we have derived the identity in (8.3).

Substituting (8.3) into (8.2) then yields

$$\int_0^L f(x)\cos\frac{m\pi x}{L}dx = b_m I_{mn} = \begin{cases} b_0 L & \text{if } m = 0 \\ b_m \frac{L}{2} & \text{if } m \neq 0. \end{cases}$$

Thus we have the Fourier cosine series representation for $f(x)$ in the form

$$\boxed{f(x) = \sum_{n=0}^{\infty} b_n \cos\frac{n\pi x}{L},\ 0 < x < L},$$

where,

$$\begin{aligned}
b_0 &= \frac{1}{L}\int_0^L f(x)dx \\
b_n &= \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx,\ n = 1, 2, 3, 4, \dots\ . \qquad (8.4)
\end{aligned}$$

8.3 Application to PDE with Neumann Boundary Conditions

Consider heat conduction in a rod of length L whose initial temperature is given as

$$\boxed{\text{IC:} \quad u(x,0) = f(x), \ 0 < x < L}. \tag{8.5}$$

Find the evolution of $u(x,t)$ for $t > 0$ if the ends of the rod are insulated, i.e.

$$\boxed{\text{BCs:} \quad u_x(0,t) = 0, \ u_x(L,t) = 0, \ t > 0}. \tag{8.6}$$

Assume heat conduction is governed by the heat equation:

$$\boxed{\text{PDE:} \quad u_t = \alpha^2 u_{xx}, \ 0 < x < L}. \tag{8.7}$$

The usual method of separation of variables will lead us to the solution in the form:

$$u(x,t) = \sum_n T_n(t) X_n(x), \tag{8.8}$$

where the "eigenfunction", $X_n(x)$, satisfies

$$\frac{d^2}{dx^2} X_n(x) + \lambda_n^2 X_n(x) = 0. \tag{8.9}$$

The only difference between this case and the previous one in Chapter 3, section 2, is the boundary conditions. Here the Neumann condition. (8.6) implies

$$\frac{d}{dx} X_n(0) = 0, \ \frac{d}{dx} X_n(L) = 0. \tag{8.10}$$

Nontrivial solutions to (8.9) and (8.10) are

$$X_n(x) = \cos \lambda_n x \tag{8.11}$$

provided

$$\lambda_n = \frac{n\pi}{L}, \ n = 0, 1, 2, 3, \ldots .$$

The $T_n(t)$ satisfies, as in section 3.2:

$$\frac{d}{dt} T_n(t) + \alpha^2 \lambda_n^2 T_n(t) = 0. \tag{8.12}$$

So

$$T_n(t) = T_n(0)e^{-\alpha^2\lambda_n^2 t}. \tag{8.13}$$

The general solution, satisfying the PDE and the BCs, is

$$\boxed{u(x,t) = \sum_{n=0}^{\infty} T_n(0)e^{-\alpha^2\lambda_n^2 t}\cos\frac{n\pi x}{L}.} \tag{8.14}$$

To satisfy the IC, we require, at $t = 0$,

$$u(x,0) = f(x) = \sum_{n=0}^{\infty} T_n(0)\cos\frac{n\pi x}{L},\ 0 < x < L. \tag{8.15}$$

(8.15) implies that the constants, $T_n(0)$'s, are the Fourier cosine coefficients for $f(x)$. Thus

$$\boxed{\begin{aligned} &T_n(0) = b_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx,\ n = 1,2,3,\ldots \\ &T_0(0) = b_0 = \frac{1}{L}\int_0^L f(x)dx. \end{aligned}}$$

The problem is now completely solved, assuming $f(x)$ is given.

An Example:

Solve

PDE: $u_t = u_{xx}$, $0 < x < 1$, $t > 0$
BCs: $u_x(0,t) = 0$, $u_x(1,t) = 0$, $t > 0$
IC: $u(x,0) = x$, $0 < x < 1$.

Since the boundary conditions are homogeneous Neumann, try a cosine series expansion of the solution

$$u(x,t) = \sum_{n=0}^{\infty} T_n(t)X_n(x), \tag{8.16}$$

where $X_n(x) = \cos n\pi x$. Substituting the assumed form (7.16) into the PDE yields:

$$\frac{d}{dt}T_n(t) = -(n\pi)^2 T_n(t),\ n = 0,1,2,3,\ldots\ . \tag{8.17}$$

The solution of (8.17) is

$$T_n(t) = T_n(0)e^{-(n\pi)^2 t}.$$

Therefore,

$$u(x,t) = \sum_{n=0}^{\infty} T_n(0)e^{-(n\pi)^2 t}\cos n\pi x,\ \ 0 < x < 1.$$

To satisfy the IC, we require

$$x = \sum_{n=0}^{\infty} T_n(0)\cos n\pi x,\ \ 0 < x < 1.$$

So the $T_n(0)$'s are the Fourier cosine coefficients of the function x, and thus

$$\begin{aligned} T_n(0) &= \int_0^1 x dx = \frac{1}{2} \\ T_n(0) &= 2\int_0^1 x\cos n\pi x dx = \begin{cases} 0 & \text{if } n = \text{even} \\ -\frac{4}{(n\pi)^2} & \text{if } n = \text{odd.} \end{cases} \end{aligned}$$

Finally,

$$u(x,t) = \frac{1}{2} - \frac{4}{\pi^2}\sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{1}{n^2}e^{-(n\pi)^2 t}\cos n\pi x.$$

12.6 Example, Laplace's equation in a circular disk

Consider again the solution of Laplace's equation in a region bounded by a circle of radius a:

$$\text{PDE:}\nabla^2 u = 0,\ 0 \le r < a$$
$$\text{BC:}u(r,\theta) = f(\theta) \text{ for } r = a$$

and is periodic in θ with period 2π. In polar coordinates, the Laplace operator is (see Figure 4.1)

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2},$$

where θ is the angle and r is the radius.

Since $f(\theta)$ is periodic with period 2π, we can expand it in a periodic Fourier series:

$$f(\theta) = \sum_{n=-\infty}^{\infty} c_n e^{-in\theta},$$

with c_n given by (from 12.26)

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)e^{in\theta}d\theta.$$

Since the solution $u(r,\theta)$ should also be periodic with period 2π, it too can be expanded in a periodic Fourier series:

$$u(r,\theta) = \sum_{n=-\infty}^{\infty} R_n(r)e^{-in\theta}.$$

Substituting this assumed form for u into the PDE yields:

$$R_n''(r) + \frac{1}{r}R_n'(r) - \frac{n^2}{r^2}R_n(r) = 0.$$

As explained in Section 4.2.3, this ODE belongs to the "equi-dimensional type" and the solution is of the form r^b. Substituting this assumed form into the ODE we find that $b = \pm n$. Thus the solution is

$$R_n(r) = \alpha_n r^n + \beta_n r^{-n}$$

In order that $R_n(r)$ be finite at $r = 0$, we set $\beta_n = 0$ for $n > 0$ and set $\alpha_n = 0$ for $n < 0$. Also to satisfy the BC at $r = a$, we want $R_n(a) = c_n$. Thus

$$R_n(r) = c_n(r/a)^{|n|},\ n = 0, \pm 1, \pm 2, \pm 3, \ldots$$

Chapter 9

Eigenfunction Expansion

9.1 Introduction

The Fourier sine and cosine series discussed in the previous two chapters are special cases of a more general method of "eigenfunction expansions". In this method, the eigenfunctions $X_n(x)$ obtained from separation of variables are used to expand any function of x as a linear combination of the $X_n(x)$'s:

$$f(x) = \sum_n a_n X_n(x),\ 0 < x < L,$$

where the coefficients a_n of the expansion, are constants. For a function of x and t, the coefficients of expansion are in general function of t. Thus, $u(x,t)$ can be expanded as

$$u(x,t) = \sum_n a_n(t) X_n(x),\ 0 < x < L,$$

which we re-write as

$$u(x,t) = \sum_n T_n(t) X_n(x),\ 0 < x < L.$$

[We write $T_n(t)$ instead of $a_n(t)$ simply to be consistent with our previous notation]

What eigenfunctions $X_n(x)$ to use depends very much on the type of boundary conditions and the type of PDE we are given. Below we shall review different cases using the diffusion equation as an example.

9.2 Eigenfunctions and boundary conditions

To solve the diffusion equation

$$\begin{aligned} \text{PDE:} \quad & u_t = \alpha^2 u_{xx},\ 0 < x < L,\ t > 0 \\ \text{IC:} \quad & u(x,0) = f(x),\ 0 < x < L. \end{aligned} \tag{9.1}$$

Case A.

Step (1) If the boundary conditions are of the homogeneous Dirichlet type, i.e.:

$$u(0,t) = 0,\ u(L,t) = 0,\ t > 0 \tag{9.2}$$

use a sine series expansion of the form:

$$u(x,t) = \sum_{n=1}^{\infty} T_n(t) \sin \frac{n\pi x}{L},\ 0 < x < L. \tag{9.3}$$

Which automatically satisfies the boundary conditions.

Step (2) Substitute this assumed form into the PDE (9.1). Note the partial derivatives

$$u_t = \sum_{n=1}^{\infty} T_n'(t) \sin \frac{n\pi x}{L}$$

$$\alpha^2 u_{xx} = \alpha^2 \sum_{n=1}^{\infty} T_n(t) \left(- \left(\frac{n\pi}{L}\right)^2 \right) \sin \frac{n\pi x}{L}.$$

Therefore (9.1) becomes

$$\sum_{n=1}^{\infty} T_n'(t) \sin \frac{n\pi x}{L} = - \sum_{n=1}^{\infty} \alpha^2 \left(\frac{n\pi}{L}\right)^2 T_n(t) \sin \frac{n\pi x}{L},\ 0 < x < L.$$

The above equation is satisfied if

$$T_n'(t) = -\alpha^2 \left(\frac{n\pi}{L}\right)^2 T_n(t), \quad \text{for } n = 1, 2, 3, \dots . \tag{9.4}$$

The solution of (9.4) is

$$T_n(t) = T_n(0) e^{-\alpha^2 \left(\frac{n\pi}{L}\right)^2 t},$$

and so (9.3) implies

$$u(x,t) = \sum_{n=1}^{\infty} T_n(0) e^{-\alpha^2 \left(\frac{n\pi}{L}\right)^2 t} \sin \frac{n\pi x}{L}.$$

Step (3) To satisfy the IC, set

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} T_n(0) \sin \frac{n\pi x}{L}, \ 0 < x < L.$$

Thus the $T_n(0)$ are the Fourier sine series coefficients of the given function $f(x)$, i.e.

$$T_n(0) = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx. \tag{9.5}$$

Step (4) Find $T_n(0)$ by evaluating the integral in (9.5). This then completes the solution.

Case B.

Step (1) If the boundary conditions are of the homogeneous Neumann type, i.e.

$$\frac{\partial u}{\partial x} = 0 \text{ at } x = 0, \ \frac{\partial u}{\partial x} = 0 \text{ at } x = L, \tag{9.6}$$

use a cosine series expansion instead:

$$u(x,t) = \sum_{n=0}^{\infty} T_n(t) \cos \frac{n\pi x}{L}, \ 0 < x < L. \tag{9.7}$$

This assumed form automatically satisfies the boundary conditions.

Step (2) To satisfy the PDE, substitute (9.7) into (9.1)

$$\sum_{n=0}^{\infty} T_n'(t) \cos \frac{n\pi x}{L} = -\sum_{n=0}^{\infty} \alpha^2 \left(\frac{n\pi}{L}\right)^2 T_n(t) \cos \frac{n\pi x}{L}, \ 0 < x < L.$$

The above equation is satisfied if

$$T_n'(t) = -\alpha^2 \left(\frac{n\pi}{L}\right)^2 T_n(t), \quad \text{for } n = 0, 1, 2, 3, \dots . \tag{9.8}$$

The solution

$$T_n(t) = T_n(0)e^{-\alpha^2\left(\frac{n\pi}{L}\right)^2 t}.$$

Profides the full solution (9.7) is

$$u(x,t) = \sum_{n=0}^{\infty} T_n(0)e^{-\alpha^2\left(\frac{n\pi}{L}\right)^2 t} \cos\frac{n\pi x}{L},\ 0 < x < L. \tag{9.9}$$

Step (3) To satisfy IC, we must have

$$\begin{aligned} T_n(0) &= \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx,\ n = 1, 2, 3, \dots \\ T_n(0) &= \frac{1}{L}\int_0^L f(x)dx. \end{aligned} \tag{9.10}$$

Case C.

A more systematic way to proceed is to use the method of separation of variables. Here we shall emonstrate this method on a more general boundary condition (Robin condition):

$$\begin{aligned} \alpha_1\frac{\partial}{\partial x}u(0,t) + \beta_1 u(0,t) &= 0 \\ \alpha_2\frac{\partial}{\partial x}u(L,t) + \beta_2 u(L,t) &= 0. \end{aligned} \tag{9.11}$$

For $\alpha_1 = 0$, $\alpha_2 = 0$, (9.11) becomes the Dirichlet condition (9.2). For $\beta_1 = 0$, $\beta_2 = 0$, (9.11) becomes the Neumann condition (9.6). So what follows applies to both cases A and B.

Step (1) Assume a solution of the separated form

$$u(x,t) = T(t)X(x).$$

Substituting into the PDE (9.1) yields

$$T'(t)X(x) = \alpha^2 X''(x)T.$$

Divide by TX:

$$\frac{T'(x)}{T(t)} = \alpha^2\frac{X''(x)}{X(x)}.$$

Step (2) LHS being a function of t only and the RHS being a function of x only, the only way that the LHS can be equal to the RHS is for each to be equal to a constant. Let that constant be denoted by $-\alpha^2\lambda^2$, so

$$\frac{T'(t)}{T(t)} = \frac{\alpha^2 X''(x)}{X(x)} = -\alpha^2\lambda^2,$$

and

$$T'(t) = -\alpha^2\lambda^2 T(t) \tag{9.12}$$

$$X''(x) = -\lambda^2 X(x). \tag{9.13}$$

We now solve (9.13) subject to the boundary condition

$$\alpha_1 X'(0) + \beta_1 X(0) = 0 \tag{9.14}$$

$$\alpha_2 X'(L) + \beta_2 X(L) = 0. \tag{9.15}$$

The general solution to (9.13) is

$$X(x) = A\sin\lambda x + B cos\lambda x.$$

Boundary condition (9.14) implies that

$$\alpha_1\lambda A + \beta_1 B = 0. \tag{9.16}$$

Boundary condition (9.15) implies

$$\alpha_2\lambda(A\cos\lambda L - B\sin\lambda L) + \beta_2(A\sin\lambda L + B\cos\lambda L) = 0. \tag{9.17}$$

(9.16) yields $B = -\alpha_1\lambda A/\beta_1$, so (9.17) becomes:

$$A\{[\alpha_2\lambda\cos\lambda L + \beta_2\sin\lambda L] + \frac{\alpha_1\lambda}{\beta_1}[\alpha_2\lambda\sin\lambda L - \beta_2\cos\lambda L]\} = 0.$$

Step (3) For a nontrivial solution, we must have

$$\{[\alpha_2\lambda\cos\lambda L + \beta_2\sin\lambda L] + \frac{\alpha_1\lambda}{\beta_1}[\alpha_2\lambda\sin\lambda L - \beta_2\cos\lambda L]\} = 0. \tag{9.18}$$

(9.18) is called the eigenvalue equation. Its solutions are expected to be discrete

$$\lambda = \lambda_n,\ n = 0, 1, 2, 3, \ldots\ . \tag{9.19}$$

These are called eigenvalues. For each eigenvalue there is a corresponding eigenfunction

$$X(x) = X_n(x) = A_n \sin \lambda_n x + B_n \cos \lambda_n x. \tag{9.20}$$

The explicit forms of λ_n and X_n are obtainable for specific values of α_1, α_2, β_1 and β_2.

Step (4) The most general solution is obtained by superimposing all separated solutions $T_n(t)X_n(x)$:

$$u(x,t) = \sum_{n=0}^{\infty} T_n(t)X_n(x), \tag{9.21}$$

where $T_n(t)$ is determined from (9.12) to be

$$T_n(t) = T_n(0)e^{-\alpha^2\lambda_n^2 t}.$$

(9.21) already satisfies the boundary conditions and the PDE.

Step (5) To satisfy the initial condition, we require

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} T_n(0)X_n(x),$$

so

$$T_n(0) = \frac{1}{\int_0^L X_n^2 dx} \int_0^L f(x)X_n(x)dx. \tag{9.22}$$

This then completes the formal solution of this problem. (The derivation of the orthogonality relationship for the X_n's used in (9.22) will be derived in the next section. It does not need to be re-derived each time you solve a problem.)

9.3 Orthognality Condition for X_n

In (9.22) we have used the orthogonality relationship for the eigenfunctions X_n, subject to the general BCs (9.14) and (9.15). A derivation is given here. From (9.13), $X_n(x)$ is an eigenfunction corresponding to the eigenvalue λ_n, satisfying (9.13), (9.14) and (9.15):

$$\frac{d^2}{dx^2}X_n + \lambda_n^2 X_n(x) = 0, \; 0 < x < L \tag{9.23}$$

$$\alpha_1 X_n'(0) + \beta_1 X_n(0) = 0 \tag{9.24}$$
$$\alpha_2 X_n'(L) + \beta_2 X_n(L) = 0. \tag{9.25}$$

Let $X_m(x)$ be the eigenfunction corresponding to the eigenvalue λ_m, where m is not necessarily the same as n. So $X_m(x)$ satisfies

$$\frac{d^2}{dx^2}X_m + \lambda_m^2 X_m(x) = 0,\ 0 < x < L \tag{9.26}$$
$$\alpha_1 X_m'(0) + \beta_1 X_m(0) = 0 \tag{9.27}$$
$$\alpha_2 X_m'(L) + \beta_2 X_m(L) = 0. \tag{9.28}$$

Multiply (9.23) by X_m and (9.26) by X_n and subtract. We get

$$X_m \frac{d^2}{dx^2} X_n - X_n \frac{d^2}{dx^2} X_m = (\lambda_m^2 - \lambda_n^2) X_m X_n.$$

Integrate both sides from 0 to L:

$$\int_0^L (X_m \frac{d}{dx^2} X_n - X_n \frac{d^2}{dx^2} X_m) dx = (\lambda_m^2 - \lambda_n^2) \int_0^L X_m X_n dx. \tag{9.29}$$

The left-hand side of (9.29) can be evaluated through an integration by parts, to yield:

$$X_m \frac{d}{dx} X_n \Big|_0^L - X_n \frac{d}{dx} X_m \Big|_0^L,$$

which is zero when the boundary conditions (9.24), (9.25), (9.27) and (9.28) are applied. We therefore have, from (9.29):

$$(\lambda_m^2 - \lambda_n^2) \int_0^L X_m X_n dx = 0, \tag{9.30}$$

so

$$\int_0^L X_m X_n dx = 0 \text{ if } m \neq n. \tag{9.31}$$

(9.31) is the orthogonality relationship we were seeking.

Chapter 10

Nonhomogeneous Partial Differential Equations

10.1 Introduction

The method of eigenfunction expansions is used to solve nonhomogeneous partial differential equations.

Consider the following nonhomogeneous heat equation (with a given *heating* term $f(x,t)$) subject to the general boundary condition (which includes Dirichlet and Neumann as special cases):

$$\text{PDE:} \quad u_t = \alpha^2 u_{xx} + f(x,t),\ 0 < x < L,\ t > 0 \tag{10.1}$$

$$\text{BCs:} \quad \alpha_1 u_x(0,t) + \beta_1 u(0,t) = 0 \tag{10.2}$$

$$\alpha_2 u_x(L,t) + \beta_2 u(L,t) = 0$$

$$\text{IC:} \quad u(x,0) = \phi(x),\ 0 < x < L. \tag{10.3}$$

10.2 Eigenfunction expansion

Step 1: Find the eigenfunction of the homogeneous problem. That is, first (drop $f(x,t)$ and) solve the following homogeneous problem:

$$\text{PDE:} \quad u_t = \alpha^2 u_{xx},\ 0 < x < L,\ t > 0 \tag{10.4}$$

$$\text{BCs:} \quad \alpha_1 u_x(0,t) + \beta_1 u(0,t) = 0$$

$$\alpha_2 u_x(L,t) + \beta_2 u(L,t) = 0. \tag{10.5}$$

Write its solution in the form

$$u(x,t) = \sum_n T_n(t) X_n(x),$$

where the eigenfunctions, $X = X_n$ are determined by

$$\begin{cases} X''(x) + \lambda^2 x(x) = 0 \\ \alpha_1 X'(0) + \beta_1 X(0) = 0 \\ \alpha_2 X'(L) + \beta_2 X(L) = 0 \end{cases} \tag{10.6}$$

with the eigenvalues, $\lambda = \lambda_n$.
Do not work out $T_n(t)$ yet, since it will turn out that the $T_n(t)$ for the nonhomogeneous problem will be different than the $T_n(t)$ for the homogeneous problem.

Step 2: Expand the forcing term $f(x,t)$:

$$\boxed{f(x,t) = \sum_n f_n(t) X_n(x)}. \tag{10.7}$$

and expand the solution of the nonhomogeneous PDE in terms of these eigenfunction the same way:

$$\boxed{u(x,t) = \sum_n T_n(t) X_n(x)}. \tag{10.8}$$

Step 3: Substitute (10.7) and (10.8) into the PDE (10.1): Note that

$$\begin{aligned} u_t &= \sum_n T_n'(t) X_n(x) \\ u_{xx} &= \sum_n T_n(t) X_n''(x) = -\sum_n \lambda_n^2 T_n(t) X_n(x). \end{aligned}$$

Thus (10.1) becomes

$$\sum_n [T_n'(t) + \alpha^2 \lambda_n^2 T_n(t) - f_n(t)] X_n(x) = 0. \tag{10.9}$$

because X_n's are orthogonal, (10.9) implies that

$$T_n'(t) + \alpha^2 \lambda_n^2 T_n(t) - f_n(t) = 0. \tag{10.10}$$

Step 4: To satisfy the initial condition, we require

$$\boxed{u(x,0) = \phi(x) = \sum_n T_n(0)X_n(x)}, \tag{10.11}$$

yielding (see (9.22)):

$$T_n(0) = \frac{\int_0^L \phi(x)X_n(x)dx}{\int_0^L X_n^2(x)dx}. \tag{10.12}$$

Step 5: Solve the nonhomogeneous ODE (10.10) to get

$$\boxed{T_n(t) = T_n(0)e^{-\alpha^2\lambda_n^2 t} + \int_0^t e^{-\alpha^2\lambda_n^2(t-\tau)} f_n(\tau)d\tau}. \tag{10.13}$$

10.3 An example

Solve:

$$\begin{aligned} \text{PDE:} \quad & u_t = \alpha^2 u_{xx} + \sin(3\pi x),\ 0 < x < 1,\ t > 0 \\ \text{BCs:} \quad & u(0,t) = 0,\ u(1,t) = 0 \\ \text{IC:} \quad & u(x,0) = \sin(\pi x), 0 < x < 1. \end{aligned} \tag{10.14}$$

The eigenfunctions and eigenvalues of the homogeneous PDE are

$$\begin{aligned} X(x) &= X_n(x) = \sin \lambda_n x \\ \lambda &= \lambda_n = n\pi,\ n = 1, 2, 3, \ldots \end{aligned}$$

We will therefore use a sine series expansion of the solution:

$$u(x,t) = \sum_{n=1}^{\infty} T_n(t) \sin n\pi x,\ 0 < x < 1,$$

and the forcing term:

$$f(x,t) = \sin 3\pi x = \sum_{n=1}^{\infty} f_n \sin n\pi x,\ 0 < x < 1.$$

The latter means simply $f_n = 0$ except $f_3 = 1$. Substituting the expansions into the PDE, we have

$$T_n'(t) - \alpha^2(n\pi)^2 T_n = f_n,\ n = 1, 2, 3, \ldots .$$

For $n \neq 3$, this is

$$T_n'(t) - \alpha^2(n\pi)^2 T_n(t) = 0,$$

so

$$T_n(t) = T_n(0)e^{-\alpha^2 n^2 \pi^2 t}, \ n \neq 3.$$

For $n = 3$:

$$T_3'(t) - 9\pi^2\alpha^2 T_3(t) = 1.$$

The solution is:

$$T_3(t) = T_3(0)e^{-9\pi^2\alpha^2 t} + \frac{1}{(3\pi\alpha)^2}[1 - e^{-9\pi^2\alpha^2 t}].$$

To satisfy the initial condition, we require

$$\sin \pi x = \sum_{n=1}^{\infty} T_n(0) \sin n\pi x, \ 0 < x < 1,$$

so we take $T_n(0) = 0$ except $T_1(0) = 1$. Thus,

$$\begin{aligned}
T_1(t) &= e^{-\alpha^2\pi^2 t} \\
T_2(t) &= 0 \\
T_3(t) &= \frac{1}{(3\pi\alpha)^2}[1 - e^{-(3\pi\alpha)^2 t}] \\
T_4(t) &= 0 \\
&\vdots
\end{aligned}$$

The final solution is the two-term expansion

$$u(x,t) = e^{-(\alpha\pi)^2 t} \sin \pi x + \frac{1}{(3\pi\alpha)^2}[1 - e^{-(3\pi x)^2 t}] \sin(3\pi x).$$

Comments

For this simple problem where the forcing term $f(x,t)$ is a function of x only, there exists an alternative, perhaps simpler method. We write the solution as the sum of two parts, a steady state solution, $u_{\text{steady}}(x)$, and a transient solution, $u_{\text{transient}}(x,t)$. The steady state solution is to satisfy the steady state PDE, i.e. (10.14) without the time derivative term:

$$0 = \alpha^2 \frac{d^2}{dx^2} u_{\text{steady}} + \sin(3\pi x).$$

This yields

$$u_{\text{steady}}(x) = \frac{1}{(3\pi\alpha)^2}\sin(3\pi x).$$

The transient solution is found by substituting

$$u(x,t) = u_{\text{steady}}(x) + u_{\text{transient}}(x,t)$$

into the original PDE, (10.14). Thus $u_{\text{transient}}$ now satisfies a *homogeneous* PDE:

$$\begin{aligned}
\text{PDE:}\quad & \frac{\partial}{\partial t}u_{\text{transient}} = \alpha^2\frac{\partial^2}{\partial x^2}u_{\text{transient}} \\
\text{BC:}\quad & u_{\text{transient}}(0,t) = u_{\text{transient}}(1,t) = 0 \\
\text{IC:}\quad & u_{\text{transient}}(x,0) = \sin(\pi x) - u_{\text{steady}}(x).
\end{aligned}$$

The solution to this system is

$$u_{\text{transient}}(x,t) = e^{-(\alpha\pi)^2 t}\sin(\pi x) - \frac{1}{(3\pi\alpha)^2}e^{-(3\pi\alpha)^2 t}\sin(3\pi x).$$

Chapter 11

Collapsing Bridges

11.1 Introduction

As an application of nonhomogeneous PDEs, we consider the oscillations of suspension bridges under forcing. The forcing could come from wind, as in the case of the collapse of the Tacoma Narrow's Bridge in 1940, or as a result of a column of soldiers marching in cadence over a bridge, as in the collapse of the Broughton Bridge near Manchester, England in 1831.

These diasters have often been cited in textbooks on ordinary differential equations as examples of *resonance*, which happens when the frequency of forcing matches the natural frequency of oscillation of the bridge, with no discussion given on how the natural frequency is determined, or even where the ordinary differential equation used to model this phenomenon comes from. The modeling of bridge vibration by a partial differential equation, although still simple minded, is a big step forward in connecting to reality.

11.2 Marching soldiers on a bridge, a simple model

When a column of soldiers march in unison over a bridge, a vertical force

$$f(x,t)$$

is exerted on the bridge that is periodic in time t, with a period P determined by the time interval between steps. In this one dimensional problem, with x measured along the length of the bridge, we do not distinguish left-foot steps from right-foot steps. In reality these left-right steps create additional vibrations over the width of the bridge which, in some case, may be more important in causing collapse. This aspect of the problem can be handled by introducing another space dimension into the model, but will be ignored here.

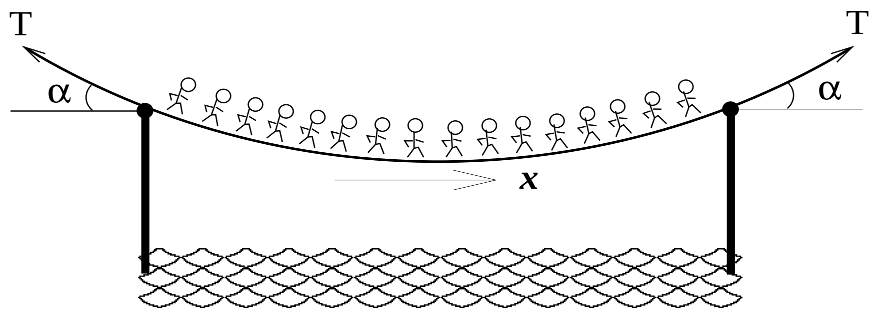


Figure 11.1 A schematic of our simple suspension bridge

Specifically, we will model the bridge as a "guitar string" of length L, suspended at only $x = 0$ and $x = L$. [We know of course that bridges do not behave like elastic strings. Nevertheless this simplification allows us to skip most of the structural mechanics that one needs to know and yet still retain most of the ingredients we need to illustrate the mathematical problem of resonance.] The tension along the bridge, T, is assumed to be uniform and is therefore equal to the force per unit area exerted on the suspension point $x = 0$ or $x = L$. Since the weight of the bridge is borne by these two suspension points, the vertical force exerted on each is half the weight of the bridge, and this should be equal to the projection of T in the vertical direction

$$T \sin\alpha = \frac{1}{2}(\rho LA)g/A = \frac{1}{2}\rho Lg,$$

where α is the angle from the horizonal to the tangent of the bridge at the suspension point, $g = 980\text{cm/sec}^2$, ρ is the density of the bridge material and A the cross section of the bridge. Let

$$c^2 \equiv T/\rho = \frac{1}{2}Lg/\sin\alpha. \tag{11.1}$$

The PDE governing the vibration of the "string" has been derived in Chapter 2. The system we need to solve is, with $u(x,t)$ being the vertical displacement of the bridge with respect to its equilibrium position:

$$\text{PDE:} \quad u_{tt} = c^2 u_{xx} + f(x,t), \ 0 < x < L, \ t > 0, \tag{11.2}$$

$$\text{BCs:} \quad u(0,t) = 0, \ u(L,t) = 0, \ t > 0, \tag{11.3}$$

$$\text{IC:} \quad u(x,0) = 0, \ u_t(x,0) = 0, \ 0 < x < L. \tag{11.4}$$

The simplest expression for the periodic force exerted by a column of marching soldiers is probably:

$$f(x,t) = a\sin(2\pi t/P)\sin(\pi x/L),\ 0 < x < L, \tag{11.5}$$

[Actually this is meant to be the force *anomaly*, that is, the difference between the force exerted by the marching soldiers and their static weight. This is why (11.5) can take on positive and negative values. The force due to the static weight of the soldiers, if it is a significant-fraction of the weight of the bridge, can be incorporated in the weight of the bridge in our earlier calculation of the tension T. Nevertheless, the parameter c^2 in (11.1) should not be affected, amazingly!].

11.3 Solution

We use the method of eigenfunction expansions to solve the nonhomogeneous system (11.2)-(11.4). Since the boundary conditions are of the homogeneous Dirichlet type, we use the sine series expansions:

$$u(x,t) = \sum_{n=1}^{\infty} T_n(t)\sin\frac{n\pi x}{L} \tag{11.6}$$

$$f(x,t) = \sum_{n=1}^{\infty} f_n(t)\sin\frac{n\pi x}{L}. \tag{11.7}$$

For the simple model of $f(x,t)$ in (11.5), $f_n = 0$ except

$$f_1(t) = a\sin(2\pi t/P).$$

Substituting (11.6) and (11.7) into (11.2) yields

$$\sum_{n=1}^{\infty}[T_n''(t) + c^2\left(\frac{n\pi}{L}\right)^2 T_n(t)]\sin\frac{n\pi x}{L} = \sum_{n=1}^{\infty} f_n(t)\sin\frac{n\pi x}{L}.$$

Therefore:

$$T_n''(t) + \omega_n^2 T_n(t) = f_n(t),\ n = 1,2,3,4,5\ldots, \tag{11.8}$$

for $\omega_n \equiv (cn\pi/L)$.

For $n > 1$, (11.8) is

$$T_n''(t) + \omega_n^2 T_n(t) = 0, \tag{11.9}$$

implying that

$$T_n(t) = A_n \sin \omega_n t + B_n \cos \omega_n t$$
$$= \frac{T_n'(0)}{\omega_n} \sin \omega_n t + T_n(0) \cos \omega_n t. \tag{11.10}$$

For $n = 1$, (11.8) is

$$T_1''(t) + \omega_1^2 T_1(t) = a \sin(2\pi t/P). \tag{11.11}$$

The solution to (11.11) consists of particular plus homogeneous solutions. The homogeneous solution is

$$A_1 \sin \omega_1 t + B_1 \cos \omega_1 t,$$

while the particular solution can be obtained by trying

$$D \sin(2\pi t/P)$$

and finding $D = a/(-(2\pi/P)^2 + \omega_1^2)$ upon substituting into (11.11). The solution for $n = 1$ is thus

$$T_1(t) = A_1 \sin \omega_1 t + B_1 \cos \omega_1 t + \frac{a \sin(2\pi t/P)}{\omega_1^2 - (2\pi/p)^2}$$
$$= \frac{T_1'(0)}{\omega_1} \sin \omega_1 t + T_1(0) \cos \omega_1 t + \frac{a}{\omega_1^2 - (2\pi/p)^2}[\sin(2\pi t/P) - \frac{(2\pi/P)}{\omega_1} \sin \omega_1 t].$$

Setting

$$u(x,t) = \sum_{n=1}^{\infty} [T_n(0) \cos \omega_n t + \frac{T_n'(0)}{\omega_n} \sin \omega_n t] \sin \frac{n\pi x}{L}$$
$$+ \frac{a}{\omega_1^2 - (2\pi/p)^2}[\sin(2\pi t/P) - \frac{(2\pi/P)}{\omega_1} \sin \omega_1 t] \sin \frac{\pi x}{L}. \tag{11.12}$$

Applying the ICs on (11.6), we have

$$u(x,0) = 0 = \sum_{n=1}^{\infty} T_n(0) \sin\left(\frac{n\pi x}{L}\right),\ 0 < x < L$$
$$u_t(x,0) = 0 = \sum_{n=1}^{\infty} T_n'(0) \sin \frac{n\pi x}{L},\ 0 < x < L.$$

These imply, $T_n(0) = 0$, $T_n'(0) = 0$, $n = 1, 2, 3, \ldots$. Finally, the solution (11.12) becomes the two term expression

$$u(x,t) = \frac{a}{\omega_1^2 - (2\pi/p)^2}[\sin(2\pi t/P) - \frac{(2\pi/P)}{\omega_1} \sin \omega_1 t] \sin \frac{\pi x}{L}. \tag{11.13}$$

11.4 Resonance

The solution (11.13) involves the interference of a forced frequency $(2\pi/P)$ with a fundamental frequency ω_1. When the two frequencies get close to each other, the numerator and the denominator of (11.13) both approach zero. Their ratio as $(2\pi/P) \to \omega_1$ is obtained by L'Hospital's rule to be

$$u(x,t) = a\left[\frac{-t\cos\omega_1 t}{2\omega_1} + \frac{\sin\omega_1 t}{2\omega_1^2}\right]\sin\left(\frac{\pi x}{L}\right). \tag{11.14}$$

The oscillation grows in amplitude linearly in time, leading, presumably, to the collapse of the bridge.

The fundamental frequency ω_1 of the bridge is given by

$$\omega_1 = c\pi/L = \pi\sqrt{\frac{1}{2}g/(L\sin\alpha)}.$$

Thus the period P of the forcing which could lead to resonance with this fundamental frequency is given by $2\pi/P = \omega_1$, or

$$P = 2\pi/\omega_1 = \sqrt{8L\sin\alpha/g}, \tag{11.15}$$

which is about 2.8 seconds for a bridge 10 meters long, if the bridge is loosely hung (i.e. $\alpha \sim 90°$). Since the period of forcing P (as measured by the time interval between the marching steps) is usually shorter, we conclude that such a bridge probably would not be resonantly forced by the column of soldiers.

If the bridge is stretched taut, the frequency of the first fundamental mode increases. If the bridge deck is nearly horizontal, say $\alpha \sim 10°$, the period of the first fundamental mode becomes

$$2\pi/\omega_1 = \sqrt{8L\sin\alpha/g} \sim 1.1 \text{ second}.$$

This is closer to the probable forcing frequency and resonance is more likely.

11.5 A different forcing function

Unlike ODE models of resonance, which assume some *given* natural frequency of the system, the PDE model discussed above determines the resonant frequency by the physical parameters of the bridge (via T/ρ) and by the x-shape of the forcing function $f(x,t)$. In the previous model, it was assumed that

$$f(x,t) = a\sin(2\pi t/P)\sin(\pi x/L),\ 0 < x < L.$$

So the forcing function has the shape of the first fundamental harmonic of the homogeneous system. Consequently, resonance occurs when the forcing frequency $2\pi/P$ equals the frequency ω_1 of this fundamental mode. If we had instead used

$$f(x,t) = a\sin(2\pi t/P)\sin(2\pi x/L),\ 0 < x < L$$

for our forcing function, resonance would have occurred when the forcing frequency $2\pi/P$ equalled the frequency ω_2 of the second fundamental mode.

This discussion points to the importance of modeling the forcing function realistically. A better model for $f(x,t)$ than (11.5) is probably,

$$f(x,t) = a\sin(2\pi t/P),\ 0 < x < L, \tag{11.16}$$

which assumes that the force exerted by the soldiers marching in unison is independent of where they are on the bridge. This seemingly simpler forcing function actually has a richer eigenfunction expansion:

$$f(x,t) = \sum_{n=1}^{\infty} f_n(t)\sin\frac{n\pi x}{L},\ 0 < x < L, \tag{11.17}$$

where

$$f_n(t) = \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{4a}{n\pi}\sin(2\pi t/P) & \text{if } n \text{ is odd.} \end{cases} \tag{11.18}$$

The solution to (11.2)-(11.4) now becomes

$$u(x,t) = \sum_{n=1}^{\infty} T_n(t)\sin\frac{n\pi x}{L}, \tag{11.19}$$

where

$$T_n(t) = 0$$

if n is even, and

$$T_n(t) = \frac{4a/\pi}{\omega_n^2 - (2\pi/p)^2}[\sin(2\pi t/P) - \frac{(2\pi/P)}{\omega_n}\sin\omega_n t]/n \tag{11.20}$$

if n is odd.

There are now chances for resonance whenever

$$P = 2\pi/\omega_n \text{ for some } n.$$

However, because the amplitude of $T_n(t)$ decreases with n, probably only the first two modes will have any real impact. To resonate the first harmonic mode, we must have $P \sim 1$ seconds. The next nonzero fundamental mode is the third one. To resonate with this mode, we need

$$P = 2\pi/\omega_3 = (2\pi/\omega_1)/3,$$

which is about 1/3 of a second. A column of soldiers running in unison with 1/3 of a second between steps may be able to induce an oscillation in the third mode. This is not as strong as the first mode if the first mode could be excited. This mode has the distinctive feature that the middle third of the bridge oscillates out of phase with the remaining two thirds of the bridge near the ends, even though the forcing is *independent* of x.

11.6 Tacoma Narrows Bridge

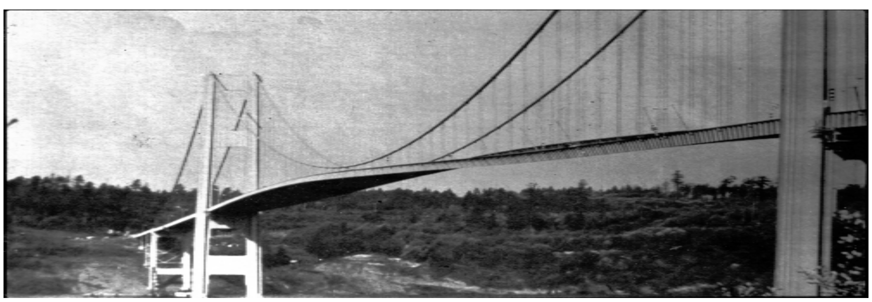


Figure 11.2 Twisting of Tacoma Narrows Bridge just prior to failure.

Even though numerous physics and mathematics textbooks attribute the 1940 collapse of Tacoma Narrows Bridge to "a resonance between the natural frequency of oscillation of the bridge and the frequency of wind-generated vortices that pushed and pulled alternately on the bridge structure" (D. Halliday and R. Resnick, *Fundamentals of Physics*, Wiley, New York, 1988, 3rd ed), that bridge probably did *not* collapse for this reason (see K.Y. Billah and R.H. Scanlan, 1991, *Am. J. Phys.,* **59** (2), 118–124). As observed by Professor Burt Farquharson of University of Washington, the wind speed at the time was 42 mph, giving a fequency of forcing by the vortex shedding mechanism of about 1 Hz. Professor Farquharson also observed that the frequency of the oscillation of the bridge just prior to its destruction was about 0.2 Hz. There was a mismatch of the two frequencies and consequently this simple resonance mechanism probably was not the cause of the bridge's collapse. The bridge collapsed due to a torsional (twisting) vibration as can be seen in old films and in Figure 8.2.

During its brief lifetime late in 1940, the bridge, under low-speed winds, did experience vertical modes of vibration which can probably be modeled by a model similar to the one presented here. However, the bridge endured this excited vibration *safely*. In fact, the bridge's nickname, "Galloping Gertie", was gained from such vertical motions under low wind. And this phenomenon occurred repeatedly since its opening day. Motorists crossing the bridge sometimes experienced "roller-coaster like" sensation as they watched cars ahead disappear from sight, then reappear, and tourists came from afar to experience it without worrying about their safety.

The above discussion points to the fact that although simple linear theories of forced resonance can perhaps explain the initial excitation of certain modes of oscillation, they cannot always be counted on to explain the final collapse of bridges, which is always a very nonlinear phenomenon.

11.7 Exercises

1. Consider the problem of a column of soldiers marching across a suspension bridge of length L. The marching is slightly out of step so the force exerted by the soldiers in the front of the column is opposite that in the rear. A simple model of the forcing term on the bridge is

$$f(x,t) = a\sin(2\pi t/P)\sin(2\pi x/L),\ 0 < x < L.$$

Solve:

$$\begin{aligned} \text{PDE:}\quad & u_{tt} = c^2 u_{xx} + f(x,t),\ 0 < x < L,\ t > 0 \\ \text{BCs:}\quad & u(0,t) = 0,\ u(L,t) = 0,\ t > 0 \\ \text{ICs:}\quad & u(x,0) = 0,\ u_t(x,0) = 0,\ 0 < x < L. \end{aligned}$$

Discuss the criteria condition for resonance and sketch the shape of the mode excited.

11.8 Solution

1. We wish to solve the nonhomogeneous wave equation with zero boundary and initial conditions where our forcing has the form:

$$f(x,t) = a\sin(2\pi t/P)\sin(2\pi x/L).$$

Our boundary conditions (in x) leads us to the expansion of $u(x,t) = \sum_{n=1}^{\infty} a_n(t)\sin(n\pi x/L)$. Substituting this into our PDE and equating like sine terms we find the equations:

$$a_2(t)'' + \omega_2^2 a_2(t) = a\sin(2\pi t/P) \quad \text{and} \quad a_n(t)'' + \omega_n^2 a_n(t) = 0 \text{ for } n \neq 2$$

where $\omega_n = nc\pi/L$. Solving the a_n equations with our initial conditions gives $a_n(t) = 0$ for $n \neq 2$, and leaves us with a single, second order inhomogeneous equation for $a_2(t)$.

The homogeneous solution is $y_h(x) = A\sin(\omega_2 t) + B\cos(\omega_2 t)$. Thus if $\omega_2 \neq 2\pi/P$, we guess a particular solution of the form:

$$y_p = C\sin(2\pi t/P) + D\cos(2\pi t/P).$$

Substituting into our equation for a_2 we find that $D = 0$ and $C = a/(\omega_2^2 - (2\pi/P)^2)$, which gives the solution:

$$u(x,t) = [A\cos(\omega_2 t) + B\sin(\omega_2 t) + a/(\omega_2^2 - (2\pi/P)^2)\sin(2\pi t/P)]\sin(2\pi x/L).$$

Now we use our initial conditions to find that $A = 0$ and $B = -2\pi a/(P\omega_2(\omega_2^2 - (2\pi/P)^2))$, which gives

$$u(x,t) = [-2\pi a/(P\omega_2(\omega_2^2 - (2\pi/P)^2))\sin(\omega_2 t) + a/(\omega_2^2 - (2\pi/P)^2\sin(2\pi t/P)]\sin(2\pi x/L)$$

Now if $c/L = 1/P$ or $\omega_2 = 2\pi/P$, we have resonance, and our inhomogeneity satisfies homogeneous equation, so we must guess a particular solution of the form:

$$y_p(t) = t(C\sin(\omega_2 t) + D\cos(\omega_2 t)).$$

Solving for the coefficients C and D, we find that $C = 0$ and $D = -a/(2\omega_2)$. Thus, when we have resonance, the solution is:

$$u(x,t) = [A\sin(\omega_2 t) + B\cos(\omega_2 t) - (a/(2\omega_2))t\cos(\omega_2 t)]\sin(2\pi x/L).$$

Now we use our initial conditions to find that $B = 0$ and $A = a/(2\omega_2^2)$, so our final solution is:

$$u(x,t) = [(a/(2\omega_2^2))\sin(\omega_2 t) - (a/(2\omega_2))t\cos(\omega_2 t)]\sin(2\pi x/L).$$

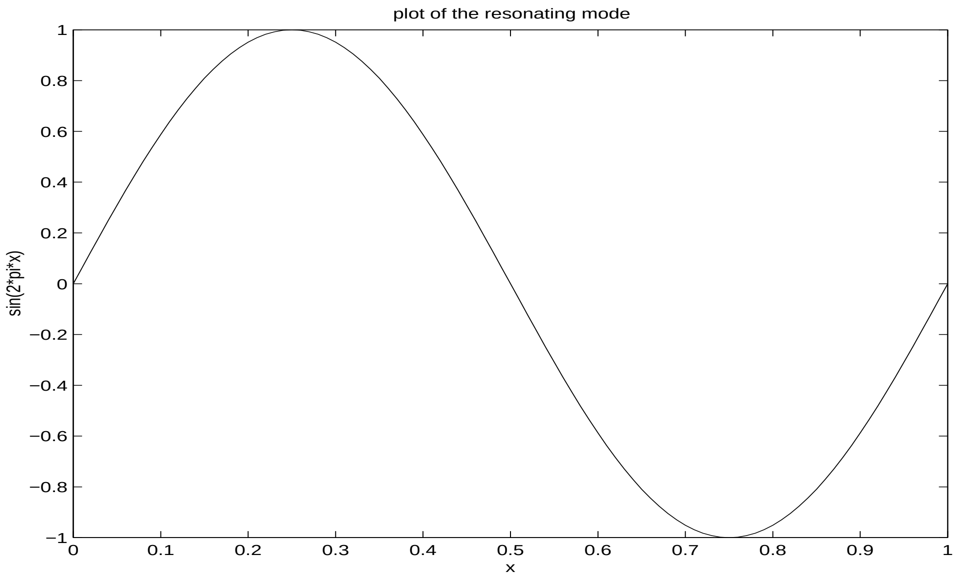


Figure 11.3. Resonating mode.

Chapter 12

Fourier Series

12.1 Introdution

We discussed the Fourier sine series in Chapter 7 and the Fourier cosine series in Chapter 8. Now, we shall combine the two to form a *periodic* Fourier series (or simply called the Fourier series). Before we do so however, we first try to motivate the need for such a series by looking for the eigenfunctions satisfying periodic boundary conditions.

12.2 Periodic Eigenfunctions

Consider heat conduction in a circular ring. Let us denote the circumference of the ring by $2L$. Denote any point on the ring by $x = 0$. Then the points $x = -L$ and $x = L$ are actually the same point. The problem is to be solved in the domain $-L < x < L$, subject to the boundary condition that the solution should be the same at $x = -L$ and $x = L$.

$$\begin{aligned} \text{PDE:} \quad & u_t = \alpha^2 u_{xx}, \ -L < x < L, \ t > 0 && (12.1)\\ \text{BCs:} \quad & u(-L,t) = u(L,t) && (12.2)\\ & u_x(-L,t) = u_x(L,t) && (12.3)\\ \text{IC:} \quad & u(x,0) = f(x), \ -L < x < L. && (12.4) \end{aligned}$$

As we will show, that these boundary conditions are sufficient to define a periodic function. That is, we can look for a solution for all x, $-\infty < x < \infty$, with the condition that it repeats itself with period $2L$, i.e.

$$\boxed{u(x,t) = u(x+2L,t)}\ . \qquad (12.5)$$

We will get the same result as (12.2) and (12.3).

We shall again use the method of separation of variables, and we first try

$$u(x,t) = T(t)X(x).$$

Substituting into the PDE yields

$$\frac{T'(t)}{\alpha^2 T(t)} = \frac{X''(x)}{X(x)} = -\lambda^2,$$

where $-\lambda^2$ is the separation constant. The eigenfunction $X(x) = X_n(x)$ is determined from

$$X''(x) + \lambda^2 X(x) = 0 \tag{12.6}$$

$$X(-L) = X(L),\ \ X'(-L) = X'(L). \tag{12.7}$$

The boundary conditions for X comes from (12.2) and (12.3). The solution to (12.6) is

$$X(x) = a\sin\lambda x + b\cos\lambda x, \tag{12.8}$$

so the first boundary condition in (12.7) implies

$$-a\sin\lambda L + b\cos\lambda L = a\sin\lambda L + b\cos\lambda L,$$

i.e.

$$2a\sin\lambda L = 0. \tag{12.9}$$

The second boundary condition

$$\lambda a\cos\lambda L + \lambda b\sin\lambda L = \lambda a\cos\lambda L - \lambda b\sin\lambda L,$$

is

$$2b\lambda\sin\lambda L = 0. \tag{12.10}$$

Both (12.9) and (12.10) can be satisfied if $\sin\lambda L = 0$, or by taking

$$\lambda = n\pi/L \equiv \lambda_n,\ \ n = 0, 1, 2, 3, \dots\ . \tag{12.11}$$

So the eigenfunction corresponding to λ_n is

$$X(x) = a_n\sin\lambda_n x + b_n\cos\lambda_n x \equiv X_n(x),\ \ n = 1, 2, 3, \dots\ . \tag{12.12}$$

Superposition over all n then yields the general solution of the PDE in the form

$$u(x,t) = \sum_n T_n(t)X_n(x).$$

Thus

$$\boxed{u(x,t) = \sum_{n=0}^{\infty}[a_n \sin\frac{n\pi x}{L} + b_n \cos\frac{n\pi x}{L}]e^{-(n\pi\alpha/L)^2 t}}. \tag{12.13}$$

To satisfy the IC, we need

$$\boxed{f(x) = \sum_{n=0}^{\infty}[a_n \sin\frac{n\pi x}{L} + b_n \cos\frac{n\pi x}{L}],\ -L < x < L}. \tag{12.14}$$

The right-hand side of (12.14) is the Fourier series expansion of $f(x)$ in the domain $-L < x < L$. It is periodic with period $2L$ in $-\infty < x < L$.

12.3 Fourier Series

We now return to the mathematical problem of representing an arbitrary, piecewise continuous, function $f(x)$ in a Fourier series of period $2L$,

$$f(x) = \sum_{n=0}^{\infty}[a_n \sin\frac{n\pi x}{L} + b_n \cos\frac{n\pi x}{L}],\ -L < x < L. \tag{12.15}$$

If $f(x)$ is itself periodic with period $2L$, then the representation is good for all x, $-\infty < x < \infty$. If $f(x)$ is not periodic outside the interval $-L < x < L$, or if $f(x)$ is not defined beyond this interval, the representation is good only in the restricted interval.

There are two ways to find the coefficients a_n and b_n of the Fourier series. The standard way is to use the orthogonality conditions between sines and cosines. They are, for integers m and n:

$$\int_{-L}^{L} \sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L}dx = \begin{cases} 0 & \text{if } m \neq n \\ L & \text{if } m = n \end{cases} \tag{12.16}$$

$$\int_{-L}^{L} \cos\frac{m\pi x}{L}\cos\frac{n\pi x}{L}dx = \begin{cases} 0 & \text{if } m \neq n \\ L & \text{if } m = n \neq 0 \\ 2L & \text{if } m = n = 0 \end{cases} \tag{12.17}$$

$$\int_{-L}^{L} \sin\frac{m\pi x}{L}\cos\frac{n\pi x}{L}dx = 0 \text{ for all } m \text{ and } n \tag{12.18}$$

Of the three orthogonality relations (12.16), (12.17) and (12.18), only the last one is really new. We derived the first two previously when the integration was over half the domain, from 0 to L. Since sines are odd functions of x and cosines are even function of x, the integrands in (12.16) and (12.17) are even in x. Thus,

$$\int_{-L}^{L} \sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L}dx = 2\int_{0}^{L} \sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L}dx$$
$$\int_{-L}^{L} \cos\frac{m\pi x}{L}\cos\frac{n\pi x}{L}dx = 2\int_{0}^{L} \cos\frac{m\pi x}{L}\cos\frac{n\pi x}{L}dx.$$

(12.16) then follows from our previous results, namely (6.7), and (12.17) follows from (8.3). The last identity, (12.18) follows from the fact that the integrand in (12.18) is odd in x and so the integral over positive and negative values of x yields zero.

Using these orthogonality relations, we can now obtain the coefficients a_n and b_n in the following way. Multiply both sides of (12.15) by $\sin\frac{m\pi x}{L}$ and integrate with respect to x from $-L$ to L to get

$$\begin{aligned}\int_{-L}^{L} f(x)\sin\frac{m\pi x}{L}dx &= \sum_{n=0}^{\infty}[a_n\int_{-L}^{L}\sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L}dx \\ &\quad + b_n\int_{-L}^{L}\sin\frac{m\pi x}{L}\cos\frac{n\pi x}{L}dx] \\ &= a_m L.\end{aligned}$$

so

$$a_m = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{m\pi x}{L}dx,\ m = 0, 1, 2, 3, \ldots\ .$$

Multiplying (12.15) by $\cos\frac{m\pi x}{L}$ and integrating with respect to x from $-L$ to L yields, similarly,

$$b_m = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx,\ m \neq 0$$
$$b_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx.$$

Summary: The Fourier series representation of a piecewise continuous function $f(x)$ in the interval $-L < x < L$ is given by

$$\boxed{f(x) = \sum_{n=0}^{\infty}[a_n\sin\frac{n\pi x}{L} + b_n\cos\frac{n\pi x}{L}],\ -L < x < L}, \tag{12.19}$$

where

$$\boxed{\begin{aligned} a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx \\ b_0 &= \frac{1}{2L}\int_{-L}^{L} f(x)dx \\ b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx,\ n = 1, 2, 3, \dots . \end{aligned}}$$

Notes: (i) If $f(x)$ is an odd function of x, i.e.

$$f(-x) = -f(x),\ -L < x < L,$$

then all the cosine coefficients b_n are zero, and the Fourier series (12.19) becomes a sine series:

$$f(x) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{L},\ -L < x < L,$$

where

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx = \frac{2}{L}\int_{0}^{L} f(x)\sin\frac{n\pi x}{L}dx$$

(ii) If $f(x)$ is an even function of x, i.e.

$$f(-x) = f(x),$$

then all the sine coefficients a_n are zero. The Fourier series (12.19) becomes a cosine series:

$$f(x) = \sum_{n=0}^{\infty} b_n \cos\frac{n\pi x}{L},\ -L < x < L,$$

where

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx = \frac{2}{L}\int_{0}^{2} f(x)\cos\frac{n\pi x}{L}dx$$

$$b_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx = \frac{1}{L}\int_{0}^{L} f(x)dx.$$

(iii) The above discussion suggests a second way for obtaining the coefficients of the Fourier series (12.19). Since any function $f(x)$ can be written as

$$f(x) = f_{sym}(x) + f_{anti}(x), \tag{12.20}$$

where $f_{sym}(x) \equiv \frac{1}{2}(f(x)+f(-x))$ is symmetric about $x=0$, and

$$f_{anti}(x) \equiv \frac{1}{2}(f(x)-f(-x))$$

is antisymmetric about $x=0$.

Now, the symmetric function can be represented by a cosine series in $-L<x<L$:

$$f_{sym}(x) = \sum_{n=0}^{\infty} b_n \cos\frac{n\pi x}{L}, \tag{12.21}$$

where, from (8.4),

$$b_n = \frac{2}{L}\int_0^L f_{sym}(x)\cos\frac{n\pi x}{L}dx = \frac{1}{L}\int_{-L}^{L} f_{sym}(x)\cos\frac{n\pi x}{L}dx = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx$$

and

$$b_0 = \frac{1}{L}\int_0^L f_{sym}(x)dx = \frac{1}{2L}\int_{-L}^{L} f_{sym}(x)dx = \frac{1}{2L}\int_{-L}^{L} f(x)dx.$$

Similarly, the antisymmetric function can be represented by a sine series in $-L<x<L$:

$$f_{anti}(x) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{L}, \tag{12.22}$$

where from (7.5),

$$\begin{aligned} a_n &= \frac{2}{L}\int_0^L f_{anti}(x)\sin\frac{n\pi x}{L}dx = \frac{1}{L}\int_{-L}^{L} f_{anti}(x)\sin\frac{n\pi x}{L}dx \\ &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx. \end{aligned}$$

Combining (12.21) and (12.22) into (12.20) then yields (12.19).

12.4 Examples

12.4.1

(a) Represent $f(x)=1$, as a Fourier sine series in $0<x<L$. We let

$$\begin{aligned} f_s(x) &\equiv \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{L}, \\ a_n &= \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx = \frac{2}{n\pi}[1-\cos n\pi]. \end{aligned}$$

The Fourier sine series representation, obtained by combining sines with coefficients determined above is an antisymmetric function and so looks like:

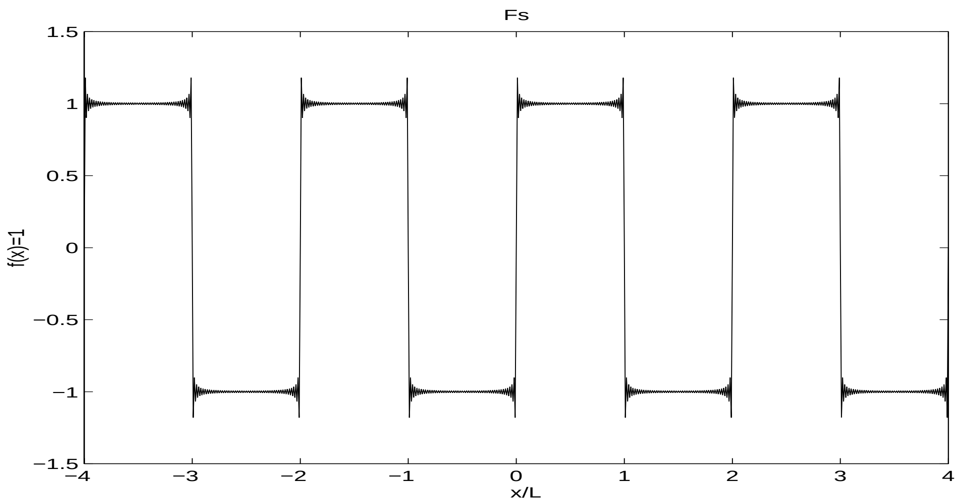


Figure 12.1 Fourier series representation of $f(x) = 1$, as a function of x/L; 50 terms used in the sum.

Thus, $f_s(x)$ looks like $f(x)$ only in the interval $0 < x < L$. It is antisymmetric about $x = 0$ and periodic with period $2L$.

(b) Represent $f(x) = 1$ as a Fourier cosine series in $0 < x < L$. We let

$$
\begin{aligned}
f_c(x) &\equiv \sum_{n=0}^{\infty} b_n \cos\frac{n\pi x}{L}, \\
b_n &= \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx = 0,\ n \neq 0 \\
b_0 &= \frac{1}{L}\int_0^L 1dx = 1.
\end{aligned}
$$

$f_c(x) = b_0$ is actually a one-term cosine series. With $b_0 = 1$ it is a perfect representation of $f(x)$ in $0 < x < L$.

(c) Represent $f(x) = 1$ as a Fourier series in $-L < x < L$. We let

$$f_{sc}(x) \equiv \sum_{n=0}^{\infty}[a_n \sin\frac{n\pi x}{L} + b_n \cos\frac{n\pi x}{L}],$$
$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx = 0$$
$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx = 0$$
$$b_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx = 1.$$

In this case $f_{sc}(x) = f_c(x) = f(x)$ in $-L < x < L$.

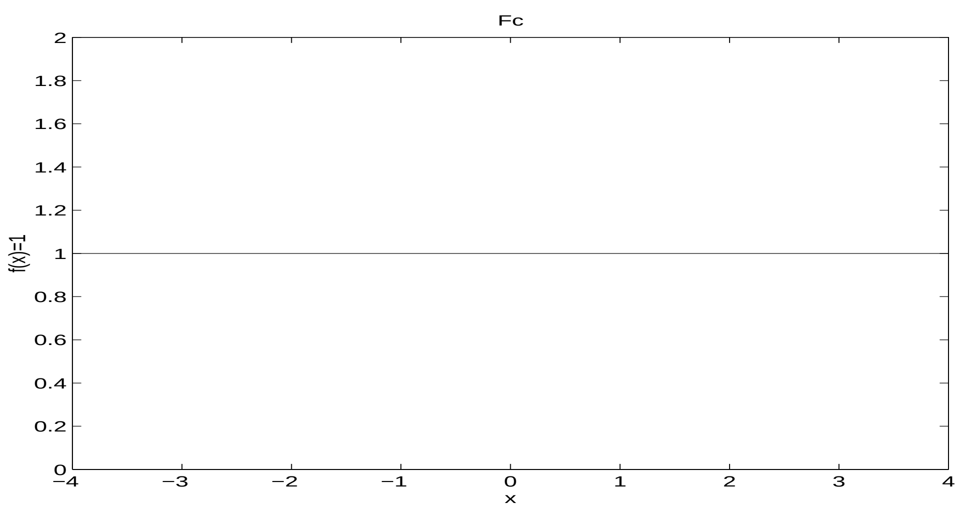


Figure 12.2 Fourier series representation of $f(x) = 1$.

12.4.2

(a) Represent $f(x) = 1 + x$ as a Fourier sine series in $0 < x < 1$.

The sine series representation, $f_s(x)$, is plotted in Figure 12.3, for all x.

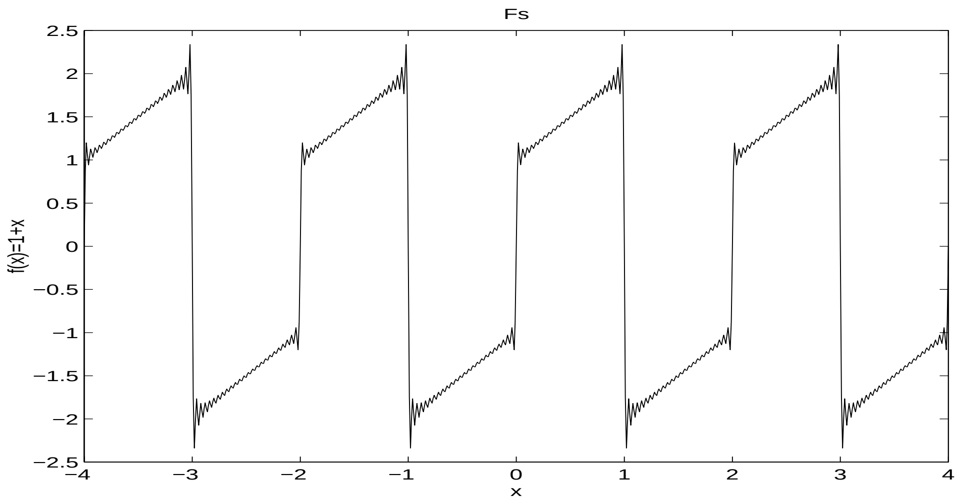


Figure 12.3 Fourier sine series representation of $f(x) = 1 + x$; 50 terms used in the sum.

(b) Represent $f(x) = 1 + x$ as a Fourier cosine series in $0 < x < 1$. The cosine series representation, $f_c(x)$, is plotted in Figure 12.4 for all x.

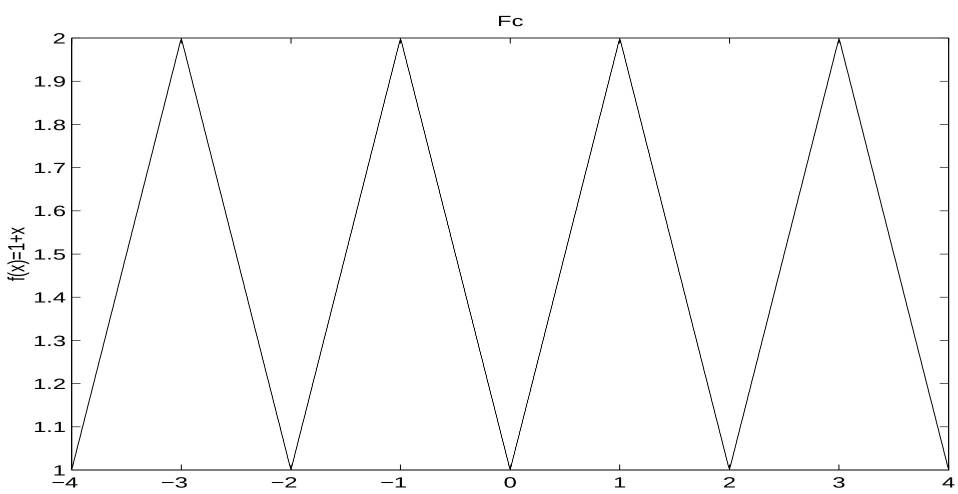


Figure 12.4 The Fourier cosine series representation of $f(x) = 1 + x$; using 50 terms in the sum.

(c) Represent $f(x) = 1 + x$ as a Fourier series in $-1 < x < 1$. $f_{sc}(x)$ is plotted in Figure 12.5 for all x. Notice that $f_{sc}(x)$ is different rom $f_c(x)$ or $f_s(x)$ beyond the interval $0 < x < 1$.

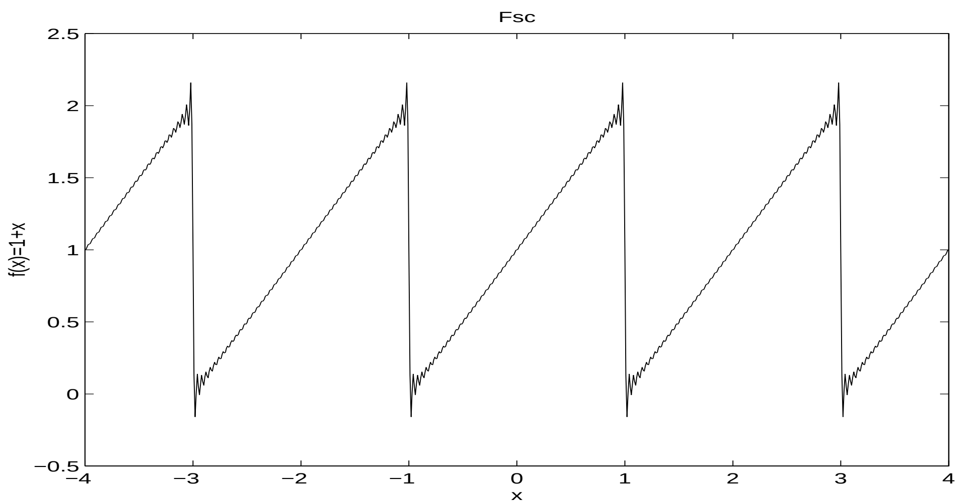


Figure 12.5 Fourier series representation of $f(x) = 1 + x$; 50 terms used in the sum.

12.4.3

(a) Represent $f(x) = e^x$ as a Fourier sine series in $0 < x < 1$. $f_s(x)$ is plotted for all x in Figure 12.6

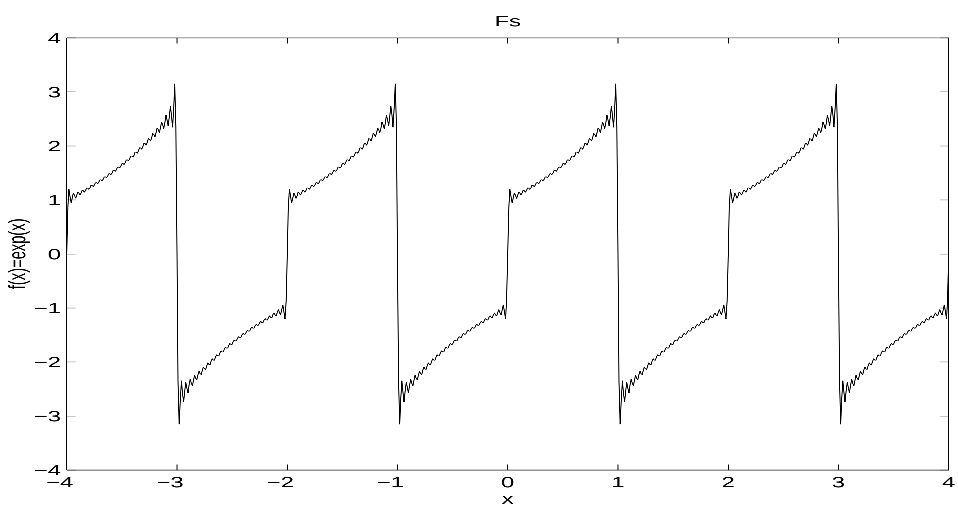


Figure 12.6 Fourier sine series representation of $f(x) = e^x$; 50 terms used in the sum.

(b) Represent $f(x) = e^x$ as a Fourier cosine series in $0 < x < 1$. $f_c(x)$ is plotted for all x in Figure 12.7

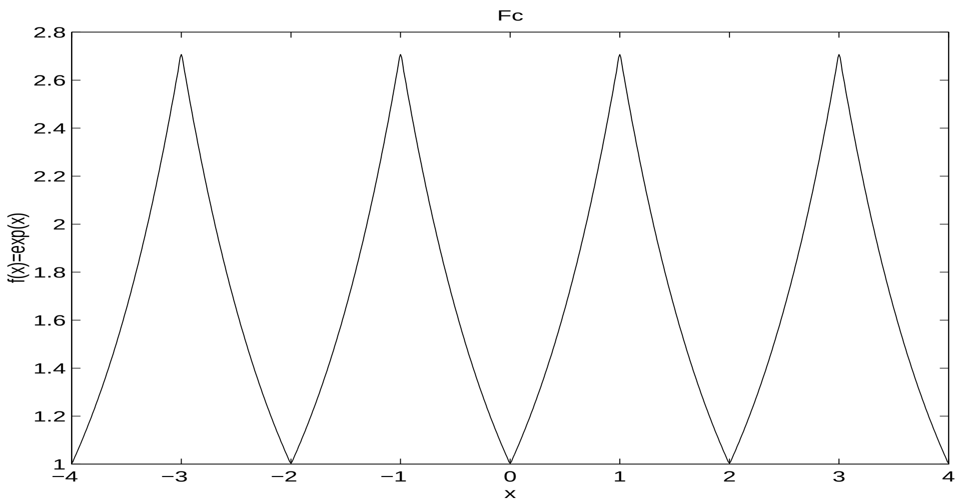


Figure 12.7 Fourier cosine series representation of $f(x) = e^x$; 50 terms used in the sum.

(c) Represent $f(x) = e^x$ as a Fourier series in $-1 < x < 1$. $f_{sc}(x)$ is plotted for all x in Figure 12.8.

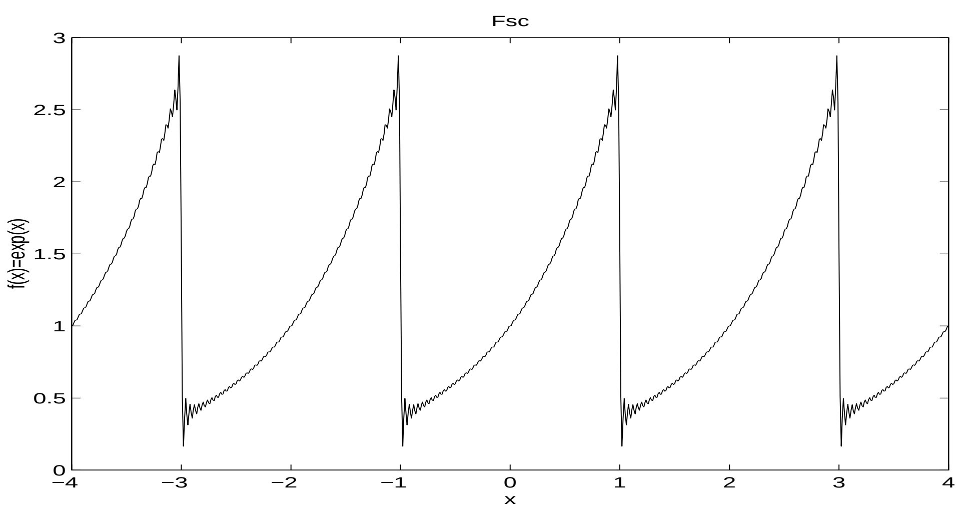


Figure 12.8 Fourier series representation of $f(x) = e^x$; 50 terms used in the sum.

12.5 Complex Fourier series

In this section we will discuss other forms of Fourier series (12.19).

In (12.19), the Fourier coefficients are

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx$$
$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx,\ n = 1, 2, 3, \ldots$$
$$b_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx.$$

If we change n to $-n$ in the above definition for a_n and b_n, we will find

$$a_{-n} = -a_n,\ b_{-n} = b_n.$$

Therefore we can rewrite the sum in (12.19) as

$$f(x) = \frac{1}{2}\sum_{n=-\infty}^{\infty}[a_n \sin\frac{n\pi x}{L} + b_n \cos\frac{n\pi x}{L}],\ -L < x < L, \tag{12.23}$$

where now, for $n = 0, \pm1, \pm2, \pm3, \ldots$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx,$$
$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx,$$

[Note that b_0 defined this way is twice as big as the definition obtained in (12.19). This is an advantage, since b_0 now has the same form as the rest of the b_n's.]

The form (12.23) is equivalent to (12.19) but is sometimes preferred because the coefficients are easier to remember.

(12.23) can be further written more compactly using the complex notation:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{-in\pi x/L},\ -L < x < L, \tag{12.24}$$

where

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(x)e^{in\pi x/L}dx.$$

To show this, we use Euler's identity

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

Changing θ to $-\theta$ gives

$$e^{-i\theta} = \cos\theta - i\sin\theta.$$

Adding yields

$$\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta}).$$

Subtracting yields

$$\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta}).$$

We now also rewrite $\sin\frac{n\pi x}{L}$ and $\cos\frac{n\pi x}{L}$ in terms of $e^{\pm in\pi x/L}$. (12.23) becomes

$$\begin{aligned} f(x) &= \frac{1}{2}\sum_{n=-\infty}^{\infty}[b_n\frac{1}{2}(e^{in\pi x/L} + e^{-in\pi x/L}) \\ &\qquad + a_n\frac{1}{2i}(e^{in\pi x/L} - e^{-in\pi x/L})] \\ &= \frac{1}{2}\sum_{n=-\infty}^{\infty}\frac{1}{2}(b_n - ia_n)e^{in\pi x/L} + \frac{1}{2}\sum_{n=-\infty}^{\infty}\frac{1}{2}(b_n + ia_n)e^{-in\pi x/L}. \end{aligned}$$

In the first sum we change n to $-n$, which is permitted since n is a dummy variable. We then see that the first sum is exactly the same as the second sum. Thus,

$$\boxed{f(x) = \sum_{n=-\infty}^{\infty} c_n e^{-in\pi x/L},\ -L < x < L}, \tag{12.25}$$

where

$$c_n = \frac{1}{2}(b_n + ia_n) = \frac{1}{2}\left\{\frac{1}{L}\int_{-L}^{L} f(x)(\cos\frac{n\pi x}{L} + i\sin\frac{n\pi x}{L})dx\right\}.$$

Thus,

$$\boxed{c_n = \frac{1}{2L}\int_{-L}^{L} f(x)e^{in\pi x/L}dx,\ n = 0,\ \pm1,\ \pm2,\ldots}. \tag{12.26}$$

The complex form (12.25) appears to be the most convenient to use.

12.6 Example, Laplace's equation in a circular disk

Consider again the solution of Laplace's equation in a region bounded by a circle of radius a:

$$\text{PDE:}\nabla^2 u = 0,\ 0 \le r < a$$
$$\text{BC:}u(r,\theta) = f(\theta) \text{ for } r = a$$

and is periodic in θ with period 2π. In polar coordinates, the Laplace operator is (see Figure 4.1)

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2},$$

where θ is the angle and r is the radius.

Since $f(\theta)$ is periodic with period 2π, we can expand it in a periodic Fourier series:

$$f(\theta) = \sum_{n=-\infty}^{\infty} c_n e^{-in\theta},$$

with c_n given by (from 12.26)

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta) e^{in\theta} d\theta.$$

Since the solution $u(r,\theta)$ should also be periodic with period 2π, it too can be expanded in a periodic Fourier series:

$$u(r,\theta) = \sum_{n=-\infty}^{\infty} R_n(r) e^{-in\theta}.$$

Substituting this assumed form for u into the PDE yields:

$$R_n''(r) + \frac{1}{r}R_n'(r) - \frac{n^2}{r^2}R_n(r) = 0.$$

As explained in Section 4.2.3, this ODE belongs to the "equi-dimensional type" and the solution is of the form r^b. Substituting this assumed form into the ODE we find that $b = \pm n$. Thus the solution is

$$R_n(r) = \alpha_n r^n + \beta_n r^{-n}$$

In order that $R_n(r)$ be finite at $r = 0$, we set $\beta_n = 0$ for $n > 0$ and set $\alpha_n = 0$ for $n < 0$. Also to satisfy the BC at $r = a$, we want $R_n(a) = c_n$. Thus

$$R_n(r) = c_n (r/a)^{|n|},\ n = 0, \pm 1, \pm 2, \pm 3, \ldots$$

Finally the solution to the PDE, satisfying all BCs, is:

$$u(r,\theta) = \sum_{n=-\infty}^{\infty} c_n (r/a)^{|n|} e^{-in\theta}$$

where

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) e^{in\theta} d\theta.$$

Chapter 13

Fourier Series, Fourier Transform and Laplace Transform

13.1 Introduction

In the previous chapter, we discussed the Fourier series

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{-in\pi x/L}, \ -L < x < L, \tag{13.1}$$

for $f(x)$ in the interval $-L < x < L$ where

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(x) e^{in\pi x/L} dx. \tag{13.2}$$

Previously, when sine and cosine series were discussed, we alluded to *Dirichlet's Theorem*, which tells us conditions under which Fourier series is a satisfactory representation of the original function $f(x)$. The full Dirichlet's Theorem is stated below.

13.2 Dirichlet Theorem

If $f(x)$ is a bounded and piecewise continuous function in $-L < x < L$, its Fourier series representation converges to $f(x)$ at each point x in the interval where $f(x)$ is continuous, and to the average of the left- and right-hand limits of $f(x)$ at those points where $f(x)$ is discontinuous. If $f(x)$ is periodic with period $2L$, the above statement applies throughout $-\infty < x < \infty$.

This theorem is easy to understand. The Fourier series, consisting of sines and cosines of period $2L$, has period $2L$. If $f(x)$ itself is also has the same period, then the Fourier series can be a good representation of $f(x)$ over the whole real axis $-\infty < x < \infty$. If on the other hand, $f(x)$ is either not periodic, or not defined beyond the interval $-L < x < L$, the Fourier series gives a good representation of $f(x)$ only in the stated interval. Beyond $-L < x < L$, the Fourier series is periodic and so simply repeats itself, but $f(x)$ may or may not be so. The above statements apply where $f(x)$ is continuous. When $f(x)$ takes a jump at a point, say x_0, the value of $f(x)$ at x_0 is not defined. The value which the Fourier seris of $f(x)$ converges to is the average of the value immediately to the left of x_0 and to the right of x_0, i.e. to $\lim_{\epsilon\to 0}\frac{1}{2}(f(x_0-\epsilon)+\frac{1}{2}f(x_0+\epsilon))$. We have already demonstrated this with the Fourier sine series. The same behavior applies to the full Fourier series.

13.3 Fourier integrals

Unless $f(x)$ is periodic, the Fourier series representation of $f(x)$ is an appropriate representation of $f(x)$ only over the interval $-L < x < L$. Question: Can we take $L \to \infty$, so as to obtain a good representation of $f(x)$ over the whole interval $-\infty < x < \infty$? The positive answer leads to the Fourier integral, and hence the Fourier transform, provided $f(x)$ is "integrable" over the whole domain.

We first rewrite (13.1) as a Riemann sum by letting

$$\Delta\omega = \pi/L \quad \text{and} \quad \omega_n = n\pi/L.$$

Thus, (13.1) becomes

$$f(x) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\Delta\omega F_n \cdot e^{-i\omega_n x}, \tag{13.3}$$

where

$$F_n \equiv (2Lc_n) = \int_{-L}^{L} f(x')e^{i\omega_n x'}dx'. \tag{13.4}$$

[We have changed the dummy variable in (13.4) from x to x', to avoid confusion later.]

In the limit $L \to \infty$, ω_n becomes ω, which takes on continuous values in $-\infty < \omega < \infty$. So $F_n \to F(\omega)$, where

$$F(\omega) = \int_{-\infty}^{\infty} f(x')e^{i\omega x'}dx', \tag{13.5}$$

and (13.3) becomes the integral:

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{-i\omega x}d\omega. \tag{13.6}$$

Substituting (13.5) into (13.6) leads to the *Fourier integral formula*:

$$\boxed{f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(x')e^{i\omega x'}dx'\right]e^{-i\omega x}d\omega}\,. \tag{13.7}$$

The validity of the formula (13.7) is subject to the integrability of the function $f(x)$ in (13.5). Furthermore, the left-hand side of (13.7) must be modified at points of discontinuity of $f(x)$ to be the average of the left- and right-hand limits at the discontinuity, because the Fourier series, upon which the (13.7) is based, has this property.

The formula shows that the operation of integrating $f(x)e^{i\omega x}$ over all x is "reversible", by multiplying it by $e^{-i\omega x}$ and integrating over all ω. (13.7) allows us to define Fourier transforms and inverse transforms.

13.4 Fourier transform and inverse transform

Let the Fourier transform of $f(x)$ be denoted by $\mathcal{F}[f(x)]$. We should use (13.5) for a definition of such an operation:

$$\boxed{F(\omega) = \mathcal{F}[f(x)] \equiv \int_{-\infty}^{\infty} f(x)e^{i\omega x}dx}\,. \tag{13.8}$$

Let $\mathcal{F}^{-1}$ be the inverse Fourier transform, which recovers the original function $f(x)$ from $F(\omega)$. (13.6) tells us that the inverse transform is given by

$$\boxed{f(x) = \mathcal{F}^{-1}[F(\omega)] \equiv \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{-i\omega x}d\omega}\,. \tag{13.9}$$

Equations (13.8) and (13.9) form the transform pair. Not too many integrals can be "reversed". Take for example

$$\int_{-\infty}^{\infty} f(x)dx.$$

Once integrated, the information about $f(x)$ is lost and cannot be recovered. Therefore the relationships such as (13.8) and (13.9) are rather special and have wide application in solving PDEs.

Note that our definition of the Fourier transform and inverse transform is not unique. One could, as in some textbooks, put the factor $\frac{1}{2\pi}$ in (13.8) instead of in (13.9), or split it as $\frac{1}{\sqrt{2\pi}}$ in (13.8) and $\frac{1}{\sqrt{2\pi}}$ in (13.9). The only thing that matters is that in the Fourier integral formula there is the factor $\frac{1}{2\pi}$ when the two integrals are both carried out. Also, in (13.7), we can change ω to $-\omega$ without changing the form of (13.7). So, the Fourier transform in (13.8) can alternatively be defined with a negative sign in front of ω in the exponent, and the inverse transform in (13.9) with a positive sign in front of ω in the exponent. It does not matter to the final result as long as the transform and inverse transform have opposite signs in front of ω in their exponents.

13.5 Laplace transform and inverse transform

The Laplace transform is often used to transform a function of time, $f(t)$ for $t > 0$. It is defined as

$$\boxed{\mathcal{L}[f(t)] = \int_{0}^{\infty} f(t)e^{-st}dt \equiv \tilde{f}(s)}\,. \tag{13.10}$$

Mathematically, it does not matter whether we denote our independent variable by the symbol t or x; nor does it matter whether we call t time and x space or vice versa. What does matter for Laplace transforms is the integration ranges only over a semi-infinite interval, $0 < t < \infty$. We do not consider what happens before $t = 0$. In fact, as we will see, we need to take $f(t) = 0$ for $t < 0$.

Functions which are zero for $t < 0$ are called one-sided functions. For one-sided $f(t)$, we see that the Laplace transform (13.10) is the same as the Fourier transform (13.8) if we replace x by t and ω by is. That is, from

(13.8)

$$
\begin{aligned}
F(is) &= \int_{-\infty}^{\infty} f(t)e^{-st}dt = \int_0^{\infty} f(t)e^{-st}dt \\
&\equiv \tilde{f}(s).
\end{aligned}
\tag{13.11}
$$

Since the Laplace transform is essentially the same as the Fourier transform, we can use the Fourier inverse transform (13.9) to recover $f(t)$ from its Laplace transform $\tilde{f}(s)$. Let

$$f(t) = \mathcal{L}^{-1}[\tilde{f}(s)].$$

Then the operation $\mathcal{L}^{-1}$, giving the inverse Laplace transform, must be defined by

$$f(t) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \tilde{f}(s)e^{st}ds. \tag{13.12}$$

This is because

$$
\begin{aligned}
f(t) = \mathcal{F}^{-1}[F(\omega)] &= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{-i\omega t}d\omega \\
&= \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \tilde{f}(s)e^{st}ds
\end{aligned}
$$

through the change from ω to is. Since the inverse Laplace transform involves an integration in the complex plane, it is usually not discussed in elementary mathematics courses which deal with real integrals. Tables of Laplace transforms and inverse transforms are used instead. Our purpose here is simply to point out the connection between Fourier and Laplace transform, and the origin of both in Fourier series.

Note: The inverse Laplace transform formula in (13.12) was obtained from the Fourier integral formula, which applies only to integrable functions. For "nonintegrable", but one-sided $f(t)$, (13.12) should be modified, with the limits of integration changed to $\alpha - i\infty$ and $\alpha + i\infty$, where α is some positive real constant bounding the exponential growth of f allowed.

To show this, suppose $f(t)$ is not integrable because it grows as $t \to \infty$. Suppose that for some positive α the product

$$g(t) \equiv f(t)e^{-\alpha t} \to 0 \text{ as } t \to \infty$$

and is thus integrable. We proceed to find the Laplace transform of $g(t)$:

$$\begin{aligned}\widetilde{g}(s) \equiv \mathcal{L}[g(t)] &= \int_0^\infty f(t)e^{-\alpha t}e^{-st}d\tau \\ &= \int_0^\infty f(t)e^{-(\alpha+s)t}dt.\end{aligned}$$

Thus if

$$\widetilde{f}(s) = \mathcal{L}[f(t)] = \int_0^\infty f(t)e^{-st}dt,$$

then

$$\widetilde{g}(s) = \widetilde{f}(\alpha + s).$$

The inverse of $\widetilde{g}(s)$ is

$$\begin{aligned}g(t) = \mathcal{L}^{-1}[\widetilde{g}(s)] &= \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \widetilde{g}(s)e^{st}ds \\ &= \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \widetilde{f}(\alpha + s)e^{st}ds \\ &= \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} \widetilde{f}(s')e^{-\alpha t}e^{s't}ds'\end{aligned}$$

where we have made the substitution $s' = \alpha + s$. Since $g(t) \equiv f(t)e^{-\alpha t}$, we obtain, on cancelling out the $e^{-\alpha t}$ factor:

$$f(t) = \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} \widetilde{f}(s')e^{s't}ds'.$$

This is a modified, and more general, formula for Laplace transform of a one-sided function $f(t)$, whether or not it decays as $t \to \infty$, as long as $f(t)e^{-\alpha t}$ is integrable.

$$\begin{aligned}\widetilde{f}(s) = \mathcal{L}[f(t)] &= \int_0^\infty f(t)e^{-st}dt, \ \text{Re } s \geq \alpha \\ f(t) = \mathcal{L}^{-1}[\widetilde{f}(s)] &= \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} \widetilde{f}(s)e^{st}ds.\end{aligned}$$

Chapter 14

Fourier Transform and Its Application to PDE

14.1 Introduction

We shall first practice taking the Fourier transform of some functions before applying it to solving PDEs in infinite domains.

14.2 Fourier transform of some simple functions

Example 1:

Take the Fourier transform of

$$f(x) = e^{-|x|}, \ -\infty < x < \infty.$$

$$\begin{aligned}
F(\omega) = \mathcal{F}[e^{-|x||}] &= \int_{-\infty}^{\infty} e^{-|x|} e^{i\omega x} dx \\
&= \int_{0}^{\infty} e^{-x+i\omega x} dx + \int_{-\infty}^{0} e^{x+i\omega x} dx \\
&= \frac{1}{-(1-i\omega)} e^{-(1-i\omega)x}|_{0}^{\infty} + \frac{1}{(1+i\omega)} e^{(1+i\omega)x}|_{-\infty}^{0} \\
&= \frac{1}{1-i\omega} + \frac{1}{1+i\omega} = \frac{2}{1+\omega^2}.
\end{aligned}$$

Note its decay as $\omega \to \pm\infty$.

Example 2:

Take the Fourier transform of

$$f(x) = e^x, \ -\infty < x < \infty.$$

$$\begin{aligned} F(\omega) = \mathcal{F}[e^x] &= \int_{-\infty}^{\infty} e^x e^{i\omega x} dx \\ &= \frac{1}{(1+i\omega)} e^{(1+i\omega)x}|_{-\infty}^{\infty}. \end{aligned}$$

The limit at $x = \infty$ blows up. We say the Fourier transform of e^x does not exist because the function $f(x) = e^x$ is not "integrable", i.e.

$$\int_{-\infty}^{\infty} |f(x)| dx = \int_{-\infty}^{\infty} e^x dx$$

does not have a finite value. [Draw a picture of e^x, and see that the area under the curve is infinitely large.]

Example 3:

Take the Fourier transform of

$$f(x) = e^{-x}, \ -\infty < x < \infty.$$

This funciton is also not integrable, and so its Fourier transform does not exist. [Show this.]

Example 4:

Take the Fourier transform of

$$f(x) = e^{-x^2}, \ -\infty < x < \infty.$$

The value of the function decreases rapidly when x is away from $x = 0$ in *both* the positive and negative x directions. There is a finite area under the curve e^{-x^2}, so this function is integrable. To find its Fourier transform, we need to perform the integral:

$$F(\omega) = \int_{-\infty}^{\infty} e^{-x^2} e^{i\omega x} dx.$$

You can either look up a table of integrals or by completing the square in the exponent to get

$$F(\omega) = \sqrt{\pi}e^{-\omega^2/4}.$$

[In case you are curious:

$$-x^2 + i\omega x = -(x - i\omega/2)^2 - \omega^2/4.$$

So $F(\omega) = e^{-\omega^2/4} \int_{-\infty}^{\infty} e^{-(x-i\omega)/2)^2} dx = e^{-\omega^2/4} \int_{-\infty}^{\infty} e^{-y^2} dy$, where we have made a change of variable $y = x - i\omega/2$ and also shifted the path of integration. The remaining integral is a standard one (Euler's integral) and is equal to $\sqrt{\pi}$. In general,

$$\int_{-\infty}^{\infty} e^{-(x+b)^2} dx = \sqrt{\pi}, \quad \text{and} \quad \int_{-\infty}^{\infty} e^{-p^2x^2 \pm qx} dx = \frac{\sqrt{\pi}}{p} e^{\frac{q^2}{4p^2}}, \ p > 0.]$$

Example 5:

Take the inverse transform of

$$F(\omega) = \sqrt{\pi}e^{-\omega^2/4}, \ -\infty < \omega < \infty.$$

$$\begin{aligned}
\mathcal{F}^{-1}[F(\omega)] &= \frac{\sqrt{\pi}}{2\pi} \int_{-\infty}^{\infty} e^{-\omega^2/4 - i\omega x} d\omega \\
&= \frac{\sqrt{\pi}}{2\pi} e^{-x^2} \int_{-\infty}^{\infty} e^{-(\omega/2 + ix)^2} d\omega \\
&= e^{-x^2}.
\end{aligned}$$

We have thus recovered the original function $f(x)$ in *Example 4.*

Example 6:

Take the inverse transform of

$$F(\omega) = \frac{2}{1+\omega^2}$$

$$\mathcal{F}^{-1}[F(\omega)] = \frac{2}{2\pi} \int_{-\infty}^{\infty} \frac{1}{1+\omega^2} e^{-i\omega x} d\omega.$$

This integral can be done very easily if you know complex variables and residue calculus. Otherwise, just rely on Tables of Integrals to tell you that

$$f(x) = e^{-|x|}.$$

We have thus recovered the $f(x)$ in *Example 1.*

14.3 Application to PDEs

The usual difficulty with PDEs is that the solution involves more than one independent variable. The transform method allows us to reduce one independent variable. We commonly try to transform the x-dependence through a Fourier transform, provided that the domain is infinite, i.e. $-\infty < x < \infty$. We sometimes use Laplace transform in t instead of or in addition to the Fourier transform in x, provided that $0 < t < \infty$.

Consider a function $u(x,t)$, with $-\infty < x < \infty$, $t > 0$. Let

$$\boxed{U(\omega,t) = \mathcal{F}[u(x,t)] = \int_{-\infty}^{\infty} u(x,t)e^{i\omega x}dx} \tag{14.1}$$

be the Fourier transform of $u(x,t)$ with respect to x. The original function $u(x,t)$ can then be recovered from the Fourier inverse transform:

$$\boxed{u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} U(\omega,t)e^{-i\omega x}d\omega}\ . \tag{14.2}$$

[Note that in (14.1) and (14.2) t plays no role; it may be regarded as arbitrary.] This is very similar to our previous method of writing the solution in the form of an eigenfunction expansion when the domain is finite. With (14.2), taking derivatives with respect to x is now very easy:

$$\begin{aligned}
u_x(x,t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} (-i\omega)U(\omega,t)e^{-i\omega x}d\omega \\
u_{xx}(x,t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} (-i\omega)^2U(\omega,t)e^{-i\omega x}d\omega \qquad (14.3) \\
u_t(x,t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} U_t(\omega,t)e^{-i\omega x}d\omega \\
u_{tt}(x,t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} U_{tt}(\omega,t)e^{-i\omega x}d\omega, \qquad (14.4)
\end{aligned}$$

provided of course that these integrals exist. At this point, there is no need to worry about these mathematical issues of integrability because we don't even know what $U(\omega,t)$ is yet.

14.4 Examples

11.4.1. The wave equation in an infinite domain

$$\begin{aligned} &\text{PDE:} \quad u_{tt} = c^2 u_{xx}, \ -\infty < x < \infty, \ t > 0 && (14.5)\\ &\text{BCs:} \quad u(x,t) \to 0 \ \text{ as } \ x \to \pm\infty && (14.6)\\ &\text{ICs:} \quad u(x,0) = f(x), && (14.7)\\ &\qquad\quad u_t(x,0) = 0, \ -\infty < x < \infty. && (14.8) \end{aligned}$$

We assume the solution to be of the form of an integral (14.2) which we substitute into the PDE (14.4). This yields, using (14.3),

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}(U_{tt}(\omega,t) + c^2\omega^2 U(\omega,t))e^{-i\omega x}d\omega = 0,$$

which is the same as

$$\mathcal{F}^{-1}[U_{tt} + c^2\omega^2 U] = 0, \tag{14.9}$$

so (by taking $\mathcal{F}$ of (14.9)):

$$U_{tt} + c^2\omega^2 U = 0. \tag{14.10}$$

This is an ODE; the partial derivatives $\frac{\partial^2}{\partial x^2}$ have been converted to $(-i\omega)^2$, an algebraic multiplication. The ODE in t is to be solved subject to the following ICs:

$$u_t(x,0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} U_t(\omega,0)e^{-i\omega x}d\omega = 0,$$

and

$$u(x,0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} U(\omega,0)e^{-i\omega x}d\omega = f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{-i\omega x}d\omega.$$

These imply:

$$U_t(\omega,0) = 0 \tag{14.11}$$

and

$$U(\omega,0) = F(\omega), \tag{14.12}$$

where the Fourier transform $F(\omega)$ of $f(x)$ is known if $f(x)$ is known.

The general solution to the ODE (14.10) is

$$U(\omega,t) = A(\omega)\sin(c\omega t) + B(\omega)\cos(c\omega t).$$

The ICs (14.10) and (14.11) can be used to determine the constants A and B to be $B(\omega) = F(\omega)$ and $A(\omega) = 0$. Thus,

$$U(\omega,t) = F(\omega)\cos(c\omega t). \tag{14.13}$$

We recover $u(x,t)$ by subsituting (14.13) back into (14.2).

$$\begin{aligned} u(x,t) = \mathcal{F}^{-1}[U(\omega,t)] &= \frac{1}{2\pi}\int_{-\infty}^{\infty} U(\omega,t)e^{-i\omega x}d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\cos(c\omega t)e^{-i\omega x}d\omega. \end{aligned} \tag{14.14}$$

Typically one cannot perform the integral explicitly unless $F(\omega)$ is known. In the particular case of the wave equation however, progress can be made by noting that

$$\cos(c\omega t) = \frac{1}{2}(e^{ic\omega t} + e^{-ic\omega t}),$$

and so (14.14) can be rewritten as

$$\begin{aligned} u(x,t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{2}F(\omega)e^{-i\omega(x-ct)}d\omega + \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{2}F(\omega)e^{-i\omega(x+ct)}d\omega \\ &= \frac{1}{2}f(x-ct) + \frac{1}{2}f(x+ct), \end{aligned} \tag{14.15}$$

since

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{-i\omega x}d\omega,$$

so

$$f(x-ct) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{-i\omega(x-ct)}d\omega$$

and

$$f(x+ct) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{-i\omega(x+ct)}d\omega.$$

The physical intepretation of the solution (14.15) to the wave equation (14.4) is that an initial displacement of $f(x)$ will split into two shapes for $t > 0$, each with half the amplitude of the original shape, one propagates to the left and one propagates to the right, both with speed c. The quantity c is therefore called the wave speed.

11.4.2. Diffusion equation in an infinite domain:

$$\begin{aligned}
&\text{PDE:} \quad u_t = \alpha^2 u_{xx}, \ -\infty < x < \infty, \ t > 0 && (14.16)\\
&\text{BCs:} \quad u(x,t) \to 0 \ \text{ as } \ x \to \pm\infty, && (14.17)\\
&\text{IC:} \quad u(x,0) = f(x), \ -\infty < x < \infty. && (14.18)
\end{aligned}$$

We assume a solution of the form of an integral (14.2) and substitute it into the PDE (14.16). This yields, using (14.3)

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}(U_t + \alpha^2\omega^2 U)e^{-i\omega x}d\omega = 0,$$

which implies

$$U_t + \alpha^2\omega^2 U = 0. \tag{14.19}$$

The ODE (14.19) is solved subject to the IC

$$U(\omega,0) = F(\omega), \tag{14.20}$$

which is obtained by taking the Fourier transform of (14.18).

The solution is

$$U(\omega,t) = A(\omega)e^{-\alpha^2\omega^2 t} = F(\omega)e^{-\alpha^2\omega^2 t}. \tag{14.21}$$

The final solution is obtained by substituting (14.21) into (14.2)

$$\boxed{u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{-\alpha^2\omega^2 t - i\omega x}d\omega.} \tag{14.22}$$

For the special case of

$$f(x) = ae^{-(x/L)^2}, \ -\infty < x < \infty,$$

we know from Section 14.2 that

$$F(\omega) = \mathcal{F}[f(x)] = aL\sqrt{\pi}e^{-(L\omega)^2/4}.$$

Then

$$u(x,t) = \frac{aL}{2\pi}\int_{-\infty}^{\infty}\sqrt{\pi}e^{-(L\omega)^2/4 - \alpha^2\omega^2 t - i\omega x}d\omega,$$

can be evaluated by completing squares

$$\begin{aligned} u(x,t) &= \frac{aL}{2\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-(\alpha^2 t + \frac{L^2}{4})\omega^2 - i\omega x} d\omega \\ &= \frac{aL}{\sqrt{4\alpha^2 t + L^2}} e^{-x^2/(4\alpha^2 t + L^2)}. \end{aligned} \tag{14.23}$$

The physical interpretation of the solution (14.23) is that an initial concentration near $x = 0$ an initial with width of approximately $2L$ spreads out into a wider and wider region while its amplitude at $x = 0$ decreases monotonically to zero. This is a typical behavior of solutions to the diffusion/heat equation. The underlying physical process reduces gradients and spreads any initial concentration/heat to wider regions.

14.5 The "drunken sailor" problem

In Chapter 3, the "drunken sailor" problem was solved using the method of similarity solutions. Here we shall solve it using Fourier transform:

$$\text{PDE:} \frac{\partial}{\partial t} u = D \frac{\partial^2}{\partial x^2} u, \ -\infty < x < \infty, \ t > 0 \tag{14.24}$$

$$\text{BC:} u(x,t) = 0 \text{ as } x \to \pm\infty \tag{14.25}$$

$$\text{IC:} u(x,0) = f(x) = \delta(x), \ -\infty < x < \infty. \tag{14.26}$$

By taking Fourier transform in x, we found, in (14.22), that the solution can be written as

$$u(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{-D\omega^2 t - i\omega x} d\omega, \tag{14.27}$$

where $F(\omega)$ is the Fourier transform of the initial distribution $u(x,0) = f(x)$. With $f(x) = \delta(x)$,

$$\begin{aligned} F(\omega) &= \int_{-\infty}^{\infty} f(x) e^{i\omega x} dx \\ &= \int_{-\infty}^{\infty} \delta(x) e^{i\omega x} dx = e^{i\omega 0} = 1. \end{aligned}$$

Thus

$$\begin{aligned} u(x,t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-D\omega^2 t - i\omega x} \\ &= (4\pi D t)^{-1/2} \exp\{-\frac{x^2}{4Dt}\}. \end{aligned} \tag{14.28}$$

This is the same answer as obtained previously in Chapter 3.

Also, since the delta function can be obtained from the limit

$$\lim_{L\to 0}\frac{1}{\sqrt{\pi}L}e^{-(x/L)^2}=\delta(x)$$

one can obtain the result in (14.28) by taking the limit of $L\to 0$ in (14.23), with $a=\frac{1}{\sqrt{\pi}L}$.

14.6 Wave equation in 3-D (optional)

$$\begin{aligned}
\text{PDE:}&\frac{\partial^2}{\partial t^2}u=c^2\nabla^2 u,\ \nabla^2\equiv\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\frac{\partial^2}{\partial x_3^2}\\
\text{BC:}&u\to 0 \text{ as } x^2+y^2+z^2\to\infty\\
\text{IC:}&u(\mathbf{x},t)=u_0(r)\\
&u_t(\mathbf{x},t)=0
\end{aligned}$$

where $r=(x_1^2+x_2^2+x_3^2)^{1/2}$. The initial u is assumed to have radial symmetry about the origin, and hence is a function of r only.

We apply Fourier transform to each space dimension by letting

$$\begin{aligned}
U(\boldsymbol{\lambda},t)&=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}u(\mathbf{x},t)e^{i(\lambda_1x_1+\lambda_2x_2+\lambda_3x_3)}dx_1dx_2dx_3\\
&=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}u(\mathbf{x},t)e^{i\boldsymbol{\lambda}\cdot\mathbf{x}}d\mathbf{x}.
\end{aligned}$$

It is understood that

$$d\mathbf{x}=dx_1dx_2dx_3,\mathbf{x}=(x_1,x_2,x_3),\boldsymbol{\lambda}=(\lambda_1,\lambda_2,\lambda_3).$$

If we take the 3-D transform of the PDE we will get

$$\frac{\partial^2}{\partial t^2}U=-c^2\lambda^2U,$$

where $\lambda^2\equiv\lambda_1^2+\lambda_2^2+\lambda_3^2$. The solution to the ODE is

$$U(\boldsymbol{\lambda},t)=A(\boldsymbol{\lambda})\cos c\lambda t+B(\boldsymbol{\lambda})\sin c\lambda t.$$

Applying the IC, we find $B(\boldsymbol{\lambda})=0$ and $A(\boldsymbol{\lambda})=U(\boldsymbol{\lambda},0)$, where

$$\begin{aligned}
U(\boldsymbol{\lambda},0)&=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}u_0(r)e^{i\boldsymbol{\lambda}\cdot\mathbf{x}}d\mathbf{x}\\
&=\int_0^{\infty}dr\int_0^{\pi}r^2\sin\theta d\theta\int_0^{2\pi}d\varphi u_0(r)e^{i\lambda r\cos\theta}
\end{aligned}$$

in spherical coordinates. [We have oriented the coordinate systems so that θ is the angle the vector $\mathbf{x}$ makes relative to a (fixed) vector $\boldsymbol{\lambda}$.]

$$\begin{aligned}
U(\boldsymbol{\lambda}, 0) &= 2\pi \int_0^\infty drr^2 u_0(r) \int_0^\pi d(-\cos\theta) e^{i\lambda r\cos\theta} \\
&= 2\pi \int_0^\infty drr^2 u_0(r)(e^{i\lambda r\cos\theta}/(-i\lambda r)) \Big|_0^\pi \\
&= 4\pi \int_0^\infty u_0(r)\frac{\sin\lambda r}{\lambda} rdr \equiv U_0(\lambda),
\end{aligned}$$

which is a function of the magnitude of $\boldsymbol{\lambda}$ only. Thus

$$U(\boldsymbol{\lambda}, t) = U_0(\lambda)\cos c\lambda t.$$

The inverse Fourier transform is

$$\begin{aligned}
u(\mathbf{x}, t) &= \frac{1}{(2\pi)^3} \int_{-\infty}^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty U_0(\lambda)\cos c\lambda t e^{-i\boldsymbol{\lambda}\cdot\mathbf{x}} d\boldsymbol{\lambda} \\
&= \frac{1}{(2\pi)^3} \int_0^\infty d\lambda \int_0^\pi 2\pi\sin\theta\lambda^2 U_0(\lambda)\cos(c\lambda t)e^{-i\lambda r\cos\theta} d\theta \\
&= \frac{2}{(2\pi)^2} \int_0^\infty \lambda d\lambda U_0(\lambda)\cos(c\lambda t)\sin\lambda r/r
\end{aligned}$$

Since $\sin\lambda r(\cos c\lambda t = \frac{1}{2}\sin\lambda(r-ct) + \frac{1}{2}\sin\lambda(r+ct)$ and

$$\begin{aligned}
u(\mathbf{x}, 0) = u_0(r) &= \frac{1}{(2\pi)^3} \int_{-\infty}^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty U_0(\lambda) e^{-i\boldsymbol{\lambda}\cdot\mathbf{x}} d\boldsymbol{\lambda} \\
&= \frac{2}{(2\pi)^2} \int_0^\infty \lambda d\lambda U_0(\lambda)\sin\lambda r/r,
\end{aligned}$$

we have

$$\begin{aligned}
ru(\mathbf{x}, t) &= \frac{1}{(2\pi)^2} \int_0^\infty \lambda d\lambda U_0(\lambda)\sin\lambda(r-ct) \\
&\quad + \frac{1}{(2\pi)^2} \int_0^\infty \lambda d\lambda U_0(\lambda)\sin\lambda(r+ct) \\
&= \frac{1}{2}(r-ct)u_0(r-ct) + \frac{1}{2}(r+ct)u_0(r+ct).
\end{aligned}$$