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The development of Algebraic K-theory before 1980

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Dedicated to Hyman Bass

Algebraic K-theory had its origins in many places, and it is a matter of taste to decide whether or not early papers were K-theory. We shall think of early results as pre-history, and only mention a few typical examples. One early example is the introduction in 1845 of Grassmann varieties by Cayley and Grassmann. This formed the core of the classifying space notions in K-theory.

Another example was the development (1870–1882) of the ideal class group of a Dedekind ring by Dedekind and Weber [**DW**], and its cousin the Picard group of an algebraic variety in the late 19th century [**PS**]. The Brauer group, introduced in the 1928 paper [**Br28**], and the Witt group, introduced in the 1937 paper [**Wi37**], also belong to this pre-historical era.

For various reasons, we will primarily limit ourselves to the period before 1980. In a few cases, we have followed tendrils which complete themes rooted firmly in the 1970's. Like all papers, this one has a finiteness obstruction: only finitely many people and results can be included. We apologize in advance to the large number of people whose important contributions we have skipped over.

The compendium *Reviews in K-theory* [**Mag**] was used frequently in preparing this article. Various other survey articles were very useful to us, including [**Mi66, Sw70, Ba75, Lo76, Va76, O88**]. We encourage interested readers to check those sources for more information, and textbooks like [**Rosen**] for technical information.

The pre-algebraic period

Algebraic K-theory had two beginnings, one in geometric topology and one in algebraic geometry. The first was Whitehead's construction of the Whitehead group $Wh(\pi)$ of a group π as an obstruction in homotopy theory. The second, from which the subject got its name, was Grothendieck's construction of the class group $K(X)$ of vector bundles on an algebraic variety X as a way to reformulate the Riemann-Roch theorem.

The work of J.H.C. Whitehead on simple homotopy theory [**Wh39**] [**Wh41**] [**Wh50**] formed one beginning of algebraic K-theory. Suppose given a homotopy

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equivalence f between two complexes (simplicial in 1939 and 1941, cellular in 1950). Whitehead found an obstruction, called the *torsion* $\tau(f)$, for f to be built up out of "simple" moves (expansions and contractions). Unlike the Reidemeister torsion of f, which is a real number, Whitehead constructed $\tau(f)$ as an element of a universal group, now called the *Whitehead group* $Wh(\pi)$, which depends only upon the fundamental group π. The thesis of Whitehead's student G. Higman [**H40**] contained some calculations of elements of $Wh(\pi)$ coming from units in $\mathbb{Z}\pi$. However, the group $Wh(\pi)$ proved to be intractable, and the subject was put aside until 1961.

In 1961, Milnor used Reidemeister torsion in [**Mi61**] to disprove the *Hauptvermutung*, the conjecture that two polyhedral decompositions of a space always have a common subdivision. The Hauptvermutung dated back to Poincaré's attempts (in 1895 and 1898) to prove that the Betti numbers (*i.e.,* the homology groups) of a polyhedron are independent of the polyhedral decomposition used to compute them. If the Hauptvermutung were true, one could easily fix the geometric gap (found by Heegard) in Poincaré's proof. J. W. Alexander had found a way around this gap in 1915, but the Hauptvermutung had remained open for over 50 years.

In [**Mi61**] Milnor coined the term *Whitehead group* for the quotient $GL(R)/E(R)$ of the general linear group $GL(R)$ by the subgroup $E(R)$ generated by the elementary matrices — the group we now call $K_1(R)$. Milnor also clarified that Whitehead's torsion invariant lies in the quotient of this group by $\pm\pi$. This quotient would not be called the *Whitehead group* $Wh(\pi)$ until [**Mi66**], but let us get back to the story.

The second origin of K-theory came from algebraic geometry, and was immediately influential. It was the key ingredient in Grothendieck's 1957 reformulation of the Riemann-Roch theorem [**BoSe**] [**Gr57**]. Here Grothendieck introduced a group $K(\mathcal{A})$ associated to a subcategory $\mathcal{A}$ of an abelian category. This was the real beginning of K-theory; Grothendieck chose the letter 'K' for "Klassen" (the German word for *classes*); today we write $K_0(\mathcal{A})$ for this group and call it the *Grothendieck group* of $\mathcal{A}$.

Grothendieck was interested in the abelian category $\mathbf{M}(X)$ of coherent sheaves on an algebraic variety X, together with its subcategory $\mathbf{P}(X)$ of locally free sheaves on X. He defined what we now call $K_0(X)$ and $G_0(X)$ as $K_0\mathbf{P}(X)$ and $K_0\mathbf{M}(X)$, respectively, and showed that these groups agree for nonsingular varieties. Then he showed that if Y is closed in X there is an exact localization sequence

$$G_0(Y) \to G_0(X) \to G_0(X-Y) \to 0.$$

Grothendieck also defined the γ-filtration on $K_0(X)$, and showed that the associated graded ring is isomorphic (up to torsion) to the Chow group $A^*(X)$, via Chern classes $c_i\colon K_0(X) \to A^i(X)$ [**Gr58**]. Moreover, the total Chern character induces a ring isomorphism $ch_X\colon K_0(X)\otimes\mathbb{Q} \cong A^*(X)\otimes\mathbb{Q}$. If $f\colon Y \to X$ is proper he showed that the formula

$$[\mathcal{F}] \mapsto \sum(-1)^i[R^i f_*(\mathcal{F})]$$

defines a covariant map $f_*\colon G_0(Y) \to G_0(X)$, and used this to formulate the *Grothendieck Riemann-Roch theorem*: if X and Y are nonsingular and T_X is the

Todd class of X then in $A^*(X) \otimes \mathbb{Q}$

$$ch_X(f_*(y)) \cdot T_X = f_*(ch_Y(y) \cdot T_Y), \quad y \in K_0(Y).$$

The formula $G_0(X) \cong G_0(X[t])$ was discovered by Serre in Fall 1957, and included in [**BoSe, p. 116**]. Here $X[t]$ denotes the product of X with the affine line $\mathrm{Spec}(\mathbb{Z}[t])$. For nonsingular X this meant that $K_0(X) \cong K_0(X[t])$. In particular, for $X = \mathrm{Spec}(k)$ this implies that $K_0(k[x_1, ..., x_n]) \cong K_0(k) \cong \mathbb{Z}$. In other words, every projective module P over $R = k[x_1, ..., x_n]$ satisfies $P \oplus R^m \cong R^n$ for some m and n.

The rise of topological K-theory (1959–1963)

In 1959, Michael Atiyah and Friedrich Hirzebruch had the idea of mimicking Grothendieck's construction for topological vector bundles on a compact Hausdorff space X. In [**AH59a**], they observed that the tensor product of vector bundles makes the Grothendieck group $K(X)$ into a commutative ring, and observed that Chern's original 1948 construction of Chern classes c_i (combined into the Chern character ch) defines a ring homomorphism $ch\colon K(X) \to H^*(X; \mathbb{Q})$, much as in Grothendieck's setting. They defined relative groups $K(X, Y)$ for closed subspaces Y, which are modules over $K(X)$. Using this module structure, they were able to prove a Riemann-Roch theorem for differentiable manifolds, also in [**AH59a**].

A systematic study of this method led Atiyah and Hirzebruch to write their foundational paper [**AH61**], in which they defined the higher groups $K^{-n}(X, Y) = \widetilde{K}(S^n(X/Y))$, $n \geq 0$. Observing that Bott's results on the periodicity of homotopy groups of the unitary group U could be expressed by an explicit "Bott periodicity" isomorphism $K^n(X) \cong K^{n+2}(X)$, they extended $K^n(X, Y)$ to all n using periodicity. Then they showed that the functors $K^n(X, Y)$ form an extraordinary cohomology theory, satisfying all the Eilenberg-Steenrod axioms except the dimension axiom. They were able to calculate the K-theory of certain homogeneous spaces using what we now call the Atiyah-Hirzebruch spectral sequence:

$$E_2^{pq} = H^p(X, K^q(\text{point})) \Longrightarrow K^{p+q}(X).$$

What made these new constructions so appealing was the power they brought to applications. We have already mentioned the Riemann-Roch theorem for differentiable manifolds above [**AH59a**]. In [**AH59b**] they were able to show that complex projective space $\mathbb{CP}^n$ cannot be embedded in $\mathbb{R}^N$ for $N = 4n - 2\alpha(n)$, where $\alpha(n)$ is the number of terms in the dyadic expansion of n. In [**AH62**] they showed that if X is a complex manifold then any cohomology class $u \in H^{2k}(X; \mathbb{Z})$ which is represented by a complex analytic subvariety must satisfy $Sq^3 u = 0$. This allowed them to disprove an overoptimistic conjecture of Hodge.

Ring homomorphisms $\psi^k\colon K(X) \to K(X)$, which we now call *Adams operations*, were constructed by J. Frank Adams in his paper [**A62**] (written in 1961). Using them, he was able to determine exactly how many linearly independent vector fields there are on S^n, which had been a famous outstanding problem. By further analyzing the effect of the ψ^k on the rings $K(X)$, Adams was able to describe the image of the Hopf-Whitehead J-homomorphism $J\colon \pi_i(SO_n) \to \pi_{n+i}(S^n)$ in terms of the denominator of the Bernoulli numbers $B_k/4k$. His ideas were based upon [**AH59a**], and discovered in 1961/1962, but his proof was not completely published until 1966 [**A63**].

At this point, the subject was off and running. But after 1962, the development of topological K-theory diverged somewhat from the development of algebraic K-theory. So let us put it aside and return to the algebraic side of things.

The structure of Projective Modules (1955–1962)

The notion of projective module was invented by Cartan and Eilenberg circa1953, and first appeared on p. 6 of their 1956 book [**CE**], as a tool for working with derived functors. This set off to a search for nontrivial examples of projective modules, and an investigation into their structure. Some results were contained in [**CE**]. On p. 11 we find the characterization of semisimple rings as rings for which every module is projective; later in the book (p. 111), this was viewed as the characterization of rings of global dimension 0.

For some rings, Cartan and Eilenberg could classify all (finitely generated) projective modules. If R is Dedekind (or more generally a *Prüfer domain*), it was pointed out (p. 14 and 134) that every projective module is a direct sum of ideals; this result was proven in Kaplansky's 1952 paper [**K52**], but in the language of torsionfree modules. This implied (p. 13) that all projective R-modules are free when R is a principal ideal domain (or more generally a Bezóut domain). Since the ring of integers in a number field is a Dedekind domain, this gave a classical source of non-free projective modules, discovered in the late 19th century.

On p. 157 we find the statement that if R is a local ring then all finitely generated projective R-modules are free. Kaplansky later showed [**K58**] that *all* projective modules over a local ring are free, as a consequence of the general result that any infinitely generated projective module is a direct sum of countably generated projective modules.

For other rings, the classification was much harder. In Serre's classic 1955 paper [**Se55, p.243**], he stated that it was unknown whether or not every projective R-module is free when $R = k[x_1, ..., x_N]$ is a polynomial ring over a field. This became known as the "Serre problem." Serre's 1957 result that $K_0(R) \cong \mathbb{Z}$ meant that every projective R-module P is *stably* free: $P \oplus R^m \cong R^n$ for some m and n. Using this, Seshadri [**Ssh58**] immediately solved the problem for $k[x, y]$, but for $n > 2$ this proved to be a notoriously hard problem. It was not completely solved (affirmatively) until 1976, by Quillen [**Q76**] and Suslin [**Su76**].

Serre's 1955 paper [**Se55**] had a more long-reaching effect: it set up a dictionary between projective modules and topological vector bundles. The analogy was strengthened by his 1958 paper [**Se58**], which showed that every projective R-module has the form $P \oplus R^n$, where the rank of P is at most the dimension of the maximal ideal spectrum of R. This dictionary gave rise to a flurry of examples of non-free projective modules in the period 1958–1962. These came from algebraic geometry [**BoSe, Se58**], algebra [**Ba61, Ba62**], group rings [**R59, Sw59, Sw60, R61**] and topological vector bundles [**Sw62**]. Two later papers also fit into in this trend: [**Fo69**] and [**Sw77**] constructed many examples of projective modules based upon known examples of topological vector bundles.

After 1962 people searched for projective $R[t]$-modules which were not *extended*, *i.e.,* of the form $P[t]$. Schanuel's example [**Ba62**] of a rank one projective $R[t]$-module that was not extended led to Horrock's criterion [**H64**], clarified in Murthy's papers [**Mu65**] [**Mu66**], and formed the basis of the notion of seminormality [**Tr70**]. Noticing that the rings involved were always singular, Bass conjectured in 1972 (problem IX of [**Ba72**]) that if R is a commutative regular ring then every projective $R[t]$-module is extended; Bass' conjecture is still open when $\dim(R) \geq 3$.

The rise of algebraic K-theory (1957–1964)

We have seen that Grothendieck's projective class group was immediately useful in *geometric* settings, for studying vector bundles in algebraic geometry and in topology. We now turn to the development of the Grothendieck group construction in a completely *algebraic* setting.

For now, let us write $K(R)$ for the Grothendieck group of finitely generated projective modules over a ring R. If R is a commutative ring, a good description of $K(R)$ could be read off from the results in [**BoSe**] for the scheme $X = \mathrm{Spec}(R)$: $K(R)$ is a commutative ring equipped with operations λ^k, and its associated γ-filtration is related to the Krull dimension of R.

The group $K(R)$ next appeared in the 1958 paper [**R59**] by Dock Rim. Writing $\Gamma[R]$ for the reduced group $K(R)/\mathbb{Z}$, Rim first restated an old result of Chevalley [**Ch36**] as saying that if R is Dedekind then $\Gamma[R]$ is the ideal class group of R. Then he considered the group ring $\mathbb{Z}\pi$ of a finite cyclic group π of order p. Combining his results on projective $\mathbb{Z}\pi$-modules with those of Reiner [**R57**], Rim then showed that $K(\mathbb{Z}\pi) \cong K(\mathbb{Z}[\zeta])$, where ζ is a primitive p-th root of unity.

Next came Swan's 1959 paper [**Sw59**], whose details appeared in [**Sw60**]. Here Grothendieck's methods were used in a deep way. In order to classify the (fin. gen.) projective modules over the group ring $\mathbb{Z}\pi$ of a finite group π, Swan considered the Grothendieck group $K(R\pi)$ of projective $R\pi$-modules (which he called $P(R\pi)$). Noting the analogy between this and the representation ring of π, Swan used induction techniques to prove that the group $K(R\pi)$ injects into the sum of the groups $K(R\pi')$ as π' ranges over all hyperelementary subgroups of π. To do this, he introduced the Grothendieck groups $G(R\pi)$ and $G'(R\pi)$, corresponding respectively to the classes of all (finitely generated) $R\pi$-modules, and all those which are torsionfree. If R is Dedekind, these Grothendieck groups are isomorphic.

Rim's second paper [**R61**] contained the germ of the Mayer-Vietoris sequence. Let B be a finite ring extension of a ring A, and consider the conductor ideal $I = \mathrm{ann}(B/A)$. In the special case when A is an order in a semisimple algebra (such as a group ring), and B is a maximal order, Rim showed that there is an exact sequence

$$\Gamma[A] \to \Gamma[B] \oplus \Gamma[A/I] \to \Gamma[B/I] \to 0.$$

In retrospect, Rim was unable to extend this result on the left (this was done in [**BaMur**]) because K_1 was not yet available; Rim had no extension on the right because A was one-dimensional. (The extension to the right uses K_{-1} and was done in [**Bass, p. 677**].)

Up to this point the Grothendieck group $K(R)$ had not become $K_0(R)$, because there was no analogue $K_1(R)$ of the topologist's $K^{-1}(X)$. This changed with the short 1962 announcement [**BaSch**] by Bass and Schanuel; the details appeared in the expanded version, published by Bass in [**Ba64**]. Motivated by Serre's dictionary with vector bundles, and the observation that the topologist's formula $K^1(X) = [X, U]$ has an algebraic sense, they wrote $K^0(R)$ for $K(R)$ and defined the group $K^1(R)$ to be the quotient $GL(R)/E(R)$. Like Milnor in [**Mi61**], they called K^1 the *Whitehead group*, explaining that they were "using notions visibly borrowed from Whitehead's theory of simple homotopy types" [**Wh50**]; these "notions" included Whitehead's theorem that $E(R)$ is the commutator subgroup of $GL(R)$.

They also introduced the stability problem: for which n are $GL_n(R)/E_n(R)$ isomorphic to $K_1(R)$? Dieudonné's theory [**D43**] of noncommutative determinants implied that $n = 1$ suffices for a division ring D: $K_1(D) = D^\times/[D^\times, D^\times]$. Bass introduced his *stable range* conditions in [**Ba64, p. 14**] in order to solve this problem. Looking forward, Bass' stable range results were improved in [**Bass**] and again in Vaserstein's work [**Va69**] [**Va71**]. A later milepost in the stability problem was Suslin's result [**Su77**] that $SL_3(F[x_1, ..., x_n])$ is generated by elementary matrices.

If $f: R \to S$ is a ring homomorphism, Bass and Schanuel also defined a relative group $K^0(f)$ fitting into an exact sequence

$$K^1(R) \to K^1(S) \to K^0(f) \to K^0(R) \to K^0(S).$$

Bass also defined the relative group $K^1(R, I) = GL(R, I)/E(R, I)$ of an ideal I in [**Ba64**], showing that it extended the above sequence for $f: R \to R/I$ by one term. This paper also defined products such as $K^0(R_1) \otimes K^1(R_2) \to K^1(R_1 \otimes R_2)$.

One important application of these results was given in [**Ba64**] (and announced in [**BaSch**]): if R is finitely generated as a module over $\mathbb{Z}$ then both $K^0(R)$ and $K^1(R)$ are finitely generated abelian groups. Consequently $GL_n(R)$ and $GL_n(R, I)$ are finitely generated groups for all n.

Bass, Heller and Swan published the companion paper [**BaHS**] in the same issue as [**Ba64**]. In it, they systematically studied the behavior of the new functor $K^1(R)$ under polynomial and related extensions. If R is left regular, they proved that $K^1(R) = K^1(R[t])$ and $K^1(R[t, t^{-1}]) \cong K^1(R) \oplus K^0(R)$. The map $K^1(R[t, t^{-1}]) \to K^0(R)$ was constructed for any ring R; its inverse is multiplication by $t \in K^1(R[t, t^{-1}])$.

A key step in their proof [**BaHS, p. 551**] was the introduction of a group $K^1(\mathcal{A})$ for any subcategory $\mathcal{A}$ of an abelian category, along with a devissage theorem for K^1. Alex Heller abstracted this in [**H65**] to introduce groups $K_1(S)$ and $K_0(S)$ for any symmetric monoidal category S. Bass then systematically developed Heller's ideas in his Tata Lecture notes [**BaRoy**].

In 1964, a consensus arose that one should write $K_0(R)$ and $K_1(R)$ instead of $K^0(R)$ and $K^1(R)$, to indicate that these new functors were covariant in the ring R. I believe that the first uses of this new notation were in two papers written in 1964: Heller's paper [**H65**] and Milnor's survey paper [**Mi66**] on Whitehead Torsion. Milnor's paper also introduced the notation $\widetilde{K}_0(R)$ for Rim's group $\Gamma[R]$.

Motivated by the classical unit/ideal class group sequences for Dedekind rings, Alex Heller turned to the construction of a localization sequence for $K_i(\mathbb{Z}\pi) \to K_i(\mathbb{Q}\pi)$. His paper [**HR64**] with Reiner had partial success, using the group $K_0\mathcal{B}$ of torsion $\mathbb{Z}\pi$-modules. In [**H65**], Heller extended Grothendieck's K_0 localization sequence [**Gr57**] for a Serre subcategory $\mathcal{B}$ of an abelian category $\mathcal{A}$ to an exact sequence (at least when $\mathcal{A}$ and $\mathcal{A}/\mathcal{B}$ are semisimple)

$$K_1\mathcal{A} \to K_1\mathcal{A}/\mathcal{B} \to K_0\mathcal{B} \to K_0\mathcal{A} \to K_0\mathcal{A}/\mathcal{B} \to 0.$$

It became apparent that the Bass-Heller-Swan group $K_1\mathcal{A}$ was not the appropriate group when $\mathcal{A}$ is abelian but not semisimple. This was confirmed by Leslie Roberts, whose 1968 thesis ([**Rob76**] [**Ger72, 5.2**]) showed that Bass' group $K^1(\mathcal{A})$ doesn't fit into a localization sequence with $K^1(\mathcal{B})$ when $\mathcal{A}$ is the category of vector bundles over an elliptic curve. But the way to repair $K_1\mathcal{A}$ did not become clear until 1972.

The foundations for K_0 and K_1 were now in place.

Applications to Topology (1963–1966)

The next few years were marked by a veritable explosion of applications of the new K-theory. We have already pointed out the use of the Whitehead group $Wh(\pi) = K_1(\mathbb{Z}\pi)/\pm\pi$ as an obstruction in simple homotopy theory since 1950, and its subsequent use by Milnor [**Mi61**] in disproving the Hauptvermutung in 1961.

Cobordisms also formed an important family of applications of $Wh(\pi)$. Recall that an h-cobordism (W, M, M') is a smooth closed manifold W whose boundary is the disjoint sum of two pieces, M and M', each of which are deformation retracts of W. If M is simply connected and $\dim(M) \geq 5$, Smale proved the *h-cobordism theorem* in 1962: W is diffeomorphic to the product $M \times I$ (in a way preserving M), so that M' is diffeomorphic to M. The first part of the following generalization was proven in 1963 by Barden, Mazur and Stallings; see [**Mi66**]. The second part was proven by J. Stallings [**St65**].

THE s-COBORDISM THEOREM. *If* $\dim(M) \geq 5$*, then an h-cobordism* (W, M, M') *is diffeomorphic to the product* $M \times I$ *if and only if the Whitehead torsion* $\tau(W, M) \in Wh(\pi_1 M)$ *vanishes.*

Moreover, Whitehead torsion gives a 1-1 correspondence between the elements of $Wh(\pi_1 M)$ *and diffeomorphism classes of h-cobordisms* (W, M, M') *of* M.

If π is finite, Bass had proven in [**Ba64**] that $Wh(\pi)$ is finitely generated, and computed its rank. It was implicit in Higman's paper [**H40**] that the torsion in $Wh(\pi)$ comes from $SL_2(\mathbb{Z}\pi)$. (Higman worked with abelian π; the general result came much later [**W74**].) In spite of a heroic effort by several people in the mid-1960's, examples of nonzero torsion proved to be elusive (see [**Mi66, p. 416**]), and not actually confirmed until the 1973 paper [**ADS**] by Alperin, Dennis and Stein. We refer the reader to Oliver's book [**O88**] for the rest of this intricate story.

An important application of K_0 was found by C. T. C. Wall in 1963, and published in [**W65**]. If X is a space dominated by a finite complex, he defined a generalized Euler characteristic $\chi(X)$ in $\widetilde{K}_0(\mathbb{Z}\pi)$. Wall showed that X is homotopy equivalent to a finite complex if and only if $\chi(X) = 0$. Naturally, we now call $\chi(X)$ *Wall's finiteness obstruction.*

L. Siebenmann [**Sie66**] and V. Golo [**Golo**] soon showed that similar obstructions exist in $\widetilde{K}_0(\mathbb{Z}\pi)$ to the problem of putting a boundary on an open manifold.

One of the most important K-theory problems in this era was to compute the group $\widetilde{K}_0(\mathbb{Z}\pi)$ containing these obstructions. When π is abelian, this was accomplished by Bass and Murthy in [**BaMur**]. Their idea was to decompose π as the product of a finite group π_0 and a free abelian group T and then write $\mathbb{Z}\pi = A[T]$ with $A = \mathbb{Z}\pi_0$. If B is the normalization of A, this could then be analyzed by extending Rim's conductor square sequence to K_1 and studying the units of $B/I[T]$.

When $\pi = \pi_0 \rtimes \mathbb{Z}$ is a semidirect product with $\mathbb{Z}$, Farrell and Hsiang [**FH70**] found a general formula for $K_1\mathbb{Z}\pi$ and $Wh(\pi)$, generalizing the Bass-Murthy formula.

The Congruence Subgroup Problem

The other important application of K-theory in the mid-1960's was the congruence subgroup problem. Let R be the ring of integers in a global field F. If I is an ideal of R, let $SL_n(I)$ denote the kernel of $SL_n(R) \to SL_n(R/I)$. Since R/I is a finite ring, $SL_n(I)$ has finite index in $SL_n(R)$. The *congruence subgroup problem* asks if every subgroup Γ of finite index contains a subgroup $SL_n(I)$ for some I, so that it corresponds to a subgroup of the finite group $SL_n(R/I)$. One restricts to $n \geq 3$ because it is classical, and known to Klein in the last century, that there are many other subgroups of finite index in $SL_2(\mathbb{Z})$.

If $n \geq 3$, we see K-theory appearing: the answer to the congruence subgroup problem for $SL_n(R)$ is 'yes' if and only if $SK_1(R, I) = 0$ for every nonzero ideal I of R. Indeed, for any nonzero I the group $E_n(R, I)$ also has finite index, and Bass' stability theorems in [**Ba64**] show that the quotient $SL_n(I)/E_n(R, I)$ is isomorphic to the subgroup $SK_1(R, I)$ of $K_1(R, I)$. Conversely, if $\Gamma \subset SL_n(R)$ is normal, has finite index and $n \geq 3$, Bass used stable range conditions for R to observe in [**Ba64**] that there is a unique ideal I such that $E_n(R, I) \subseteq \Gamma \subseteq SL_n(I)$.

For $R = \mathbb{Z}$, Mennicke [**Me65**] and Bass-Lazard-Serre [**BaLS**] proved that the congruence subgroup problem has an affirmative answer. Gradually the general problem was brought into focus. First Mennicke considered the coset classes of the subgroup $SL_2(R)$ of $SL_3(R)$, modulo $E_3(R)$. The class of a 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ only depends upon the first row $(a\ b)$ of the matrix, and Mennicke showed that this class is multiplicative in a and b. This led Milnor to introduce the universal Mennicke group generated by symbols $\left[\begin{smallmatrix} b \\ a \end{smallmatrix}\right]$, where $b \in I$ and $a \in 1 + I$ are relatively prime, subject to Mennicke's two relations; this universal group maps to $SK_1(R, I)$ by sending the Mennicke symbol $\left[\begin{smallmatrix} b \\ a \end{smallmatrix}\right]$ to the class of a 2×2 matrix with first row $(a\ b)$. Milnor observed that this could be used to bound $SK_1(R)$ when R is a ring of algebraic integers. Then Kubota used power residue symbols to produce characters on congruence subgroups (in the totally imaginary setting). Bass generalized Kubota's arguments, using stability theorems for K_1, to prove that the universal Mennicke group is isomorphic to $SK_1(R, I)$ for every Dedekind ring R.

The congruence problem was finally settled for all global fields in the 1967 paper [**BaMS**] by Bass, Milnor and Serre. The answer is 'yes' if and only if F has an embedding into $\mathbb{R}$. If there is no such embedding, then for each ideal I they give a formula for a number m so that the group μ_m of mth roots of unity lie in F and

$SK_1(R,I) \cong \mu_m$. Under this isomorphism, the Mennicke symbol $\left[\begin{smallmatrix} b \\ a \end{smallmatrix}\right]$ corresponds to Hasse's classical power residue symbol $\left(\frac{b}{a}\right)_m$.

The solution revealed an unexpected interaction between the relative groups $K_1(R,I)$ and explicit power reciprocity laws in F. This was to attract the attention of many people to K-theory, including H. Matsumoto [**Mat68**] and John Tate [**Ta70**].

Classical consolidation (1966–1968)

During the next few years, the theory was reorganized, consolidating many ideas into a few structural results. This would set the stage for the discovery of K_2, and form the backdrop for the appearance of the higher K-theory in the next few years.

The first book to appear was based on Bass' Tata lectures [**BaRoy**], written in 1966. Starting with a symmetrical monoidal category S, the relationship between $K_0(S)$ and $K_1(S)$ was systematically exposed. On the next page, we shall say more about the applications to $S = \mathbf{Az}$ and $S = \mathbf{Quad}$ in [**BaRoy**].

Bass and Murthy's paper [**BaMur**] also upgraded a lot of machinery into a form suitable for calculations. For example, they clarified the Heller-Reiner interpretation of the localization sequence for $A \to S^{-1}A$ using the category $\mathbf{M}_S(A)$ of S-torsion A-modules. For an integral extension $A \to B$ with conductor ideal I, they showed that excision holds for K_0 in the sense that $K_0(A,I) = K_0(B,I)$. This allowed them to extend Rim's Mayer-Vietoris sequence back as far as $K_1(A)$. They also proved that $K_1(A,I) \cong K_1(B,I)$ when B is a finite product, each factor of which is a quotient of A.

Swan's 1968 book [**Swan**], based on a Fall 1967 Chicago course, gave a very readable account of the theory, focussing on K_0 and K_1 of abelian categories and their additive subcategories. The lecture notes of Milnor's 1967 Princeton course were distributed widely, although they were not published until 1971 as [**Mi71**]. They contained a clear account of the Mayer-Vietoris sequence based upon Milnor's notion of patching projective modules.

Swan's book [**SwEv**] was also written in this period, and appeared in 1970. This book summarized the practical applications of algebraic K-theory to the representation theory of finite groups and orders, including the new induction techniques from Lam's thesis [**L68**].

However, the focal point of this era was the appearance of Bass' comprehensive book [**Bass**]. Based upon a 1966/1967 Columbia University course, and largely written in the summer of 1967, it provided a unified exposition of the subject. It also included Lam's induction techniques [**L68**] as well as an extended discussion of the relation between K_1 of a Dedekind ring and reciprocity laws. But most importantly, it contained a proof of the "fundamental theorem" for K_1: the Bass-Heller-Swan map ∂ and its splitting fit into a split exact sequence:

$$0 \to K_1(A) \to K_1(A[t]) \oplus K_1(A[t^{-1}]) \to K_1(A[t,t^{-1}]) \xrightarrow{\partial} K_0(A) \to 0.$$

This sequence first appeared in [**BaMur, 10.2**], except that exactness at the second term was not established there. Bass and Murthy gave a similar formula for

$\widetilde{K}_0(A) = \mathrm{Pic}(A)$ when A is 1-dimensional. It was further analyzed in [**Swan**], but the entire theorem was not made precise until [**Bass**].

In fact, the fundamental theorem gives a definition of the functor K_0 as a function LF of the functor $F = K_1$. Bass recognized that this process could be iterated to define negative K-groups $K_{-n}(R)$ as $L^n K_0(R)$ [**Bass, p. 677**]. To axiomatize the process, Bass introduced the notion of a *contracted functor*; the fact that K_1 is contracted implies that K_0 is also contracted. Independently, Karoubi [**Ka68**] gave a definition of K-groups $K^n\mathcal{A}$ $(n \geq 0)$ for any (idempotent complete) additive category $\mathcal{A}$, and proved that his $K^n\mathbf{P}(R)$ agree with Bass' $K_{-n}(R)$.

Quadratic Forms and L-theory

The study of quadratic forms has a long and venerable history, going back to Gauss, and focussed on the *Witt group* $W(k)$ of a field k, introduced by E. Witt in [**Wi37**]. Classical invariants include the discriminant (in $k^\times/k^{\times 2}$) and the Hasse invariant, lying in the Brauer group $Br(k)$. In the mid 1960's, the idea of replacing GL_n by the orthogonal and unitary groups led to the "hermitian" K-theory of hermitian forms (and/or quadratic forms), and this evolved into L-theory.

The algebraic part of the story begins in 1964, when Wall [**W64**] generalized the Hasse invariant to one lying in a larger group we now call the Brauer-Wall group $BW(k)$. This was generalized to any commutative ring R in Bass' Tata lectures [**BaRoy**]. Bass first formalized the notion of the category $\mathbf{Quad}(R)$ of quadratic forms over R and their isometries, and considered the groups $K_i\mathbf{Quad}(R)$. He observed that the Witt ring $W(R)$ of quadratic forms is the quotient of $K_0\mathbf{Quad}(R)$ by the image of a hyperbolic map $K_0(R) \to K_0\mathbf{Quad}(R)$. Bass also showed that $K_1\mathbf{Quad}(R)$ is related to the stable structure of the orthogonal groups $O_{2n}(R)$.

Next, Bass introduced the category $\mathbf{Az}(R)$ of Azumaya algebras over a commutative ring R, and studied $K_i\mathbf{Az}(R)$. The Brauer group $Br(R)$, defined by M. Auslander and O. Goldman in 1960, is a natural quotient of $K_0\mathbf{Az}(R)$. Bass defined the Brauer-Wall group $BW(R)$ as the corresponding quotient of $K_0\mathbf{Az}_2(R)$, where $\mathbf{Az}_2$ is the graded analogue of $\mathbf{Az}$. Finally, he showed that the Clifford algebra functor $\mathbf{Quad}(R) \to \mathbf{Az}_2$ induces Wall's invariant $W(R) \to BW(R)$.

The topological part of the story comes from surgery theory. Suppose M is a closed manifold with fundamental group π. The *surgery problem* asks when a map $\psi\colon M \to X$ can be made cobordant to a homotopy equivalence. For simply connected M, this problem was solved by Milnor, Kervaire, Browder and Novikov; one needs to lift the class of the normal bundle of M from $\widetilde{K}^0(M)$ to $\widetilde{K}^0(X)$.

In order to solve this problem when π is non-trivial (and X is nice), C. T. C. Wall [**W66**] observed that the middle homotopy group of ψ is a projective $\mathbb{Z}\pi$-module, equipped with a hermitian form Q. Wall then defined a 4-periodic sequence of Grothendieck groups of such forms; these groups were quickly christened $L_m(\pi)$, or $L_m(\mathbb{Z}\pi)$. Wall then proved that the obstruction to the surgery problem for ψ is the class of Q in $L_m(\pi)$, where $m = \dim(M)$.

Motivated by the topological applications, the subject rapidly evolved. Stability theorems were given by Bak [**B69**] and Bass' foundational paper [**Ba73**], which discussed $K_0\mathbf{Quad}^\epsilon(R)$ and $K_1\mathbf{Quad}^\epsilon(R)$ for a ring with involution R and $\epsilon \in R$

such that $\epsilon\bar{\epsilon} = 1$. It turned out, however, that Wall's 4-periodic obstruction groups $L_m^\epsilon(R)$ are subquotients of these groups [**W73, p. 286**], with $L_{m+2}^\epsilon(R) = L_m^{-\epsilon}(R)$. Much of the development of this subject was due to topologists, including Wall, Novikov, Hsiang, Cappell-Shaneson, Bak, Browder, and Ranicki.

In 1970, when Quillen's plus construction for higher K-theory was discovered, it was natural to consider the homotopy groups of the space $BO^\epsilon(R)^+$ as the *Hermitian* K-theory of R; see [**Ka73**]. However, it soon became apparent [**Lo76**] that for $m \geq 2$ these groups are less well related to the Wall groups $L_m^\epsilon(R)$. At this point these two approaches became disconnected from each other, and also from the development of algebraic K-theory. So let us return to our story.

K_2 arrives (1967–1971)

Steinberg [**St62**] gave a presentation of a universal central extension Δ of a Chevalley group G over a field k, noting that the center of Δ was a "multiplier" in the sense of Schur. When $G = SL_n$, we now write $St_n(k)$ for Δ and call it the *Steinberg group.* Steinberg also introduced 2-cocycles $k^\times \otimes k^\times \to St_n(k)$ (p. 121); we now write these cocycles as $\{x, y\}$ and call them *Steinberg symbols.* It was gradually realized that Steinberg's presentation gave a universal central extension $St_n(R)$ of the subgroup $E_n(R)$ of $GL_n(R)$ for any associative ring R (if $n \geq 5$).

As we have mentioned, the Bass-Milnor-Serre paper [**BaMS**] revealed an unexpected connection between the relative groups $K_1(R, I)$ and explicit power reciprocity laws in the field of fractions for R. For example, if R is a ring of integers in a number field, they observed that there is a central extension

$$1 \to K_1(R, I) \to \frac{SL_n(R)}{E_n(R, I)} \to E_n(R/I) \to 1.$$

In Spring 1967, Milnor gave a course at Princeton University, in which he defined K_2 of a ring and developed several of its basic properties. Writing $St(R)$ for the direct limit of the groups $St_n(R)$, Milnor defined $K_2(R)$ to be the kernel of the homomorphism $St(R) \to E(R)$. Milnor also wrote a letter to Steinberg explaining his results on $K_2(R)$. The contents of his letter were included in Steinberg's Yale course notes on Chevalley Groups [**St67, pp. 92ff.**], distributed in Fall 1967.

Milnor's course notes themselves were not published for several years, because of the unprecedented explosion in mathematics that they produced. When the book [**Mi71**] finally appeared, it incorporated many of the results found during 1967–1970.

First came the work of C. Moore [**Mor68**] (written in 1967). If F is a local field then the norm residue symbol maps $K_2(F)$ onto the finite group $\mu = \mu_n$ of roots of unity in F; Moore showed that its kernel V is divisible, so $K_2(F) \cong \mu_F \oplus V$. In fact, V is uniquely divisible; this was proven much later by Tate [**Ta77**] when $\mathrm{char}(F) = p$, and by Merkurjev [**Me83**] when $\mathrm{char}(F) = 0$.

Next, Matsumoto's 1968 thesis [**Mat68**] gave a presentation of $K_2(F)$ when F is a field: it is generated by the Steinberg symbols $\{x, y\}$, which are subject only to bilinearity and the relation $\{x, 1 - x\} = 1$ (which was discovered by Steinberg).

Let R be a Dedekind domain with field of fractions F. For each prime ideal $\wp$ of R, Bass had constructed transfer maps $K_1(R/\wp) \to SK_1(R)$ in [**Bass, p. 451**]. In 1969, Bass and Tate constructed the *tame symbols* $\lambda_\wp \colon K_2(F) \to K_1(R/\wp)$ and used reciprocity laws for $\lambda = \oplus\lambda_\wp$ similar to those in [**BaMS**] to construct the exact sequence

$$K_2(R) \to K_2(F) \xrightarrow{\lambda} \oplus_\wp K_1(R/\wp) \to SK_1(R) \to 0$$

[**Mi71, p. 123**] [**BaT**]. Their proof of exactness at $K_2(F)$ required R to have only countably many prime ideals, but that suffices for integers in global fields.

When R is the ring of integers in a number field F, the kernel of λ was called the *tame kernel* until 1972, when Quillen discovered [**Q73**] that the tame kernel was just $K_2(R)$. In 1971, Garland [**Gar71**] proved that the tame kernel $K_2(R)$ is finite. Since Tate was able to calculate the tame kernel for $F = \mathbb{Q}$ as well as the first six imaginary quadratic number fields [**BaT**], $K_2(F)$ was known for these fields.

In 1970/71, J. Tate [**Ta70**] discovered the norm residue symbols on $K_2(F)$ (one for each n with $1/n \in F$). If F has primitive n-th roots of unity, they give maps $K_2(F) \to Br(F)$; in general they take values in the étale cohomology group $H^2_{et}(F, \mu_n^{\otimes 2})$. Tate proved that $K_2(F)/n \to H^2_{et}(F, \mu_n^{\otimes 2})$ is an isomorphism for all global fields F, but the details were not published until several years later, in [**Ta76**]. The proof that this map is an isomorphism for all fields was discovered in 1981/82, by Merkurjev and Suslin [**Me81**] [**MS82**].

Birch [**Bir69**] and Tate [**Ta70**] conjectured that if R is the ring of integers in a totally real number field F then the order of the finite group $K_2(R)$ should be a certain number $w_2(F)$ times the value $\zeta_F(-1)$ of the Riemann Zeta function of F. The odd part of this conjecture was settled by A. Wiles in [**Wi90**]; the 2-primary part of this conjecture is still open, and is equivalent to part of the (2-adic) Main Conjecture of Iwasawa Theory; see [**RW97, A.1**].

In the 1970 paper [**Mi70**], Milnor observed that the Hasse invariant of quadratic forms over a field F could be factored through Tate's norm residue symbol $K_2(F)/2 \to Br(F)$. More precisely, if I denotes the kernel of $W(F) \to \mathbb{Z}/2$ then the Hasse invariant maps I^2 to $Br(F)$, and Milnor proved that it factors through an isomorphism $I^2/I^3 \cong K_2(F)/2$. This led him to introduce what we now call the *Milnor K-groups* of a field F, viz., a graded ring $K_*^M(F)$ which agrees with $K_*(F)$ for $* = 0, 1, 2$. Milnor constructed a canonical map from $K_n^M(F)/2$ onto I^n/I^{n+1} for each n, proved that it is an isomorphism for local and global fields, and asked (p. 332) if it is an isomorphism for every field F. If $\mathrm{char}(F) = 2$, this was solved positively by K. Kato in 1981 [**K82**].

The Milnor ring $K_*^M(F)$ of a field was systematically studied by Bass and Tate in [**BaT**]. For global fields, they were able to compute $K_n^M(F) \cong (\mathbb{Z}/2)^{r_1}$ for all $n \geq 3$, where r_1 is the number of embeddings (if any) of F into $\mathbb{R}$. They also constructed transfer maps $K_*^M(E) \to K_*^M(F)$ for certain finite field extensions $F \subset E$; the proof that transfer maps exist for all finite extensions was given in 1980 by Kato [**K80**]. After Kato distributed his 4-page proof at the 1980 Oberwolfach conference, Suslin immediately used Kato's transfer maps to construct a map $K_n(F) \to K_n^M(F)$, and prove that the natural map $K_n^M(F) \to K_n(F)$ is an injection modulo torsion [**Su80**].

Milnor also constructed higher norm residue symbols $K_n^M(F)/2 \xrightarrow{h_n} H^n_{et}(F,\mathbb{Z}/2)$ in [**Mi70**], and used Tate's results to prove that h_n is always an isomorphism for local and global fields. Milnor then stated (p. 340) that he did not know of any examples where it fails to be an isomorphism; this became known as the "Milnor conjecture" after the case $n=2$ was settled in 1981 by Merkurjev [**Me81**]. Jumping ahead, the case $n=3$ was solved in 1986 by Merkurjev-Suslin [**MS90**] and M. Rost. A proof that h_n is an isomorphism for all n was only discovered recently by V. Voevodsky [**Voe96**].

For rings other than fields, progress was slower. First, the group $K_2(R[\varepsilon])/K_2(R)$ was computed by Wilberd van der Kallen [**vdK71**]. Then Dennis and Stein [**DS73**] [**DS75**] constructed symbols $\langle a,b\rangle$ in $K_2(R)$ and showed that they form a complete set of generators for K_2 of a semilocal ring. If I is a radical ideal, they also showed that $K_2(R,I)$ is generated by the Dennis-Stein symbols $\langle a,b\rangle$ with a or b in I. A full set of relations for the Dennis-Stein symbols, and hence a presentation for $K_2(R,I)$ was later given by Maazen and Stienstra in [**MaSt**].

Higher K-theory arrives (1968–1972)

The search for higher K-groups had dominated much of the 1960's, as we have seen. It was clear that Milnor's definition of K_2 should form the basis of this search. In fact, Swan proposed the definition $K_3(R)=H_3(St(R);\mathbb{Z})$ in 1967 [**Swan, p.207**].

In 1968, Swan [**Sw68**] proposed a definition of higher groups $K_n(R)$, based upon the idea that free rings should have no higher K-theory. In 1969, Steve Gersten [**Ger69**] used the free ring functor F to produce a functorial "cotriple" resolution F^*R of any ring R; applying the functor $R\mapsto E(R)$ gives a simplicial group, and Gersten defined $K_n(R)=\pi_{n-2}E(F^*R)$ for $n\geq 2$. Swan [**Sw70, Sw72**] showed in 1970 that his definition agrees with Gersten's.

In 1968, Nobile and Villamayor [**NV68**] gave a definition of higher K-theory, essentially by defining the "loop space" of a ring. These ideas led Karoubi and Villamayor to define groups $K^{-n}(R)$ in [**KV69**] (whose details appeared in [**KV71**]). The drawback is that their group $K^{-1}(R)$ is the quotient of $K_1(R)=GL(R)/E(R)$ by the subgroup generated by the unipotent matrices, but it has the advantages that $K^{-n}(R)\cong K^{-n}(R[t])$ for every $n\geq 1$, and that long exact sequences (such as Mayer-Vietoris sequences) exist in many situations. Nowadays, we write $KV_n(R)$ for $K^{-n}(R)$ and call it the *Karoubi-Villamayor* K-theory of R.

Gersten [**Ger70**] gave a simpler version of Karoubi-Villamayor theory in 1969, using the notion of a homology theory associated to the functor GL from rings to groups. Then Rector remarked [**Glet**] [**R71, 2.6**] that to every ring R one could associate a simplicial ring ΔR with $\Delta R_n = R[t_0,\dots,t_n]/(\sum t_i = 1)$, and showed that the homotopy groups of the simplicial group $GL(\Delta R)$ also give the Karoubi-Villamayor groups. Rector's remark has formed the basis of all subsequent work in the subject.

Historically, the most important construction of higher K-theory was the plus construction, given by Quillen in 1969/70. In his work [**QΨ**] [**Q70**] on the Adams Conjecture, Quillen constructed a map from $BGL(\mathbb{F}_q)$ to the fiber $F\Psi^q$ of the

Adams operation $BU \xrightarrow{\psi^q - 1} BU$ for each finite field $\mathbb{F}_q$, and observed that the map is *acyclic* (induces isomorphisms on homology). Quillen observed more generally that if one started with any CW complex X then there is an acyclic map $X \to X^+$, unique up to homotopy, such that $\pi_1(X) \to \pi_1(X^+)$ is the quotient by the largest perfect subgroup. In particular, $BGL(\mathbb{F}_q)^+$ has the homotopy type of $F\Psi^q$.

Motivated by this, Quillen defined $K_n(R)$ as the homotopy group $\pi_n BGL(R)^+$ [**Q71, p. 50**]. Since the homotopy groups of $F\Psi^q$ could be read off from the known action of ψ^q on BU, this gave the calculation of $K_n(\mathbb{F}_q)$ for every finite field $\mathbb{F}_q$. Moreover, in light of the Kahn-Priddy theorem that $B\Sigma^+ \simeq \Omega^\infty S^\infty$, the inclusion of the symmetric groups Σ_n in $GL_n(\mathbb{Z})$ induces maps $\pi_n^s \to K_n(\mathbb{Z})$. Quillen showed in [**Qlet**] that these maps induced an injection of the cyclic groups $J(\pi_n O) \subset \pi_n^s$ into $K_n(\mathbb{Z})$ when $n \equiv 3 \pmod 4$.

Also in 1970, G. Segal was developing his infinite loop space machine which starts with a symmetric monoidal category S (or more generally a Γ-space) and produced an Ω-spectrum. (His paper [**Seg74**] only appeared in 1974.) Quillen showed in the 1971 preprint (now published as [**Q94**]) that if one takes S to be the category of fin. gen. projective R-modules and their isomorphisms then Segal's infinite loop space is $K_0(R) \times BGL(R)^+$. Of course, the same could be done with other infinite loop space machines, such as May's [**May72**]. This choice caused some confusion until 1977, when May and Thomason showed in [**MaTh**] that all reasonable infinite loop space machines agree.

Several other constructions of $K_n(R)$ appeared in 1971. One which was based upon the theory of buildings and upper triangular subgroups was given by I. Volodin [**Vo71**]. In an attempt to understand Volodin's construction, J. Wagoner [**Wag73**] came up with a similar construction. A construction along the lines of Swan's was given by F. Keune [**Keu71**].

In 1972, Quillen [**Q72, Q73**] defined the higher K-theory of an *exact* category, viz., a full subcategory $\mathcal{A}$ of an abelian category which is closed under extensions. To do so, he first constructed an auxiliary category $Q\mathcal{A}$ and defined $K_n(\mathcal{A}) = \pi_{n-1} BQ\mathcal{A}$. Theorem 1 of the announcement [**Q72**] stated that the loop space $\Omega BQ\mathbf{P}(R)$ is homotopy equivalent to $K_0(R) \times BGL(R)^+$, so that $K_n(R) \cong K_n\mathbf{P}(R)$. Awkwardly, the proof of this "$+ = Q$" theorem was not published until 1976 [**GQ76**].

Theorems 2–4 of [**Q72**] extended the fundamental structure theorems from classical K-theory (K_n for $n < 2$) to higher K-theory: devissage, resolution and localization. Theorem 5 stated that if R is regular then $K_*(R) \cong G_*(R)$, where $G_*(R)$ is the K-theory of the abelian category of finitely generated R-modules. Theorem 8 was the localization sequence for a Dedekind domain, and theorem 11 was the Fundamental Theorem:

$$G_n(R[t]) \cong G_n(R) \quad \text{and} \quad G_n(R[t,t^{-1}]) \cong G_n(R) \oplus G_{n-1}(R).$$

Now Gersten had given a spectral sequence [**Glet**] from $K_*(\Delta R)$ to the Karoubi-Villamayor groups $KV_*(R)$. If R is regular, Quillen's Fundamental Theorem implies [**Q72, p. 101**] that Gersten's spectral sequence degenerates, giving $K_*(R) \cong KV_*(R)$. Quillen concluded by defining the groups $G_*(X)$ of a quasi-projective scheme X, and giving a spectral sequence converging to $G_*(X)$, starting from the K-groups of the residue fields of X.

At this point in the rapid-fire sequence of events, Bass organized a two-week conference on algebraic K-theory. Held at the Battelle Memorial Institute in Seattle during August 28–September 8 of 1972, the relationship between the various K-theories became one of the focal points of the conference. As we have seen, things had been greatly clarified by Quillen's announcement [**Q72**]. Gersten used Roberts' thesis [**Rob76**] to show [**Ger72**] that the Bass-Heller-Swan group $K^1(\mathcal{A})$ differs from Quillen's $K_1(\mathcal{A})$ when $\mathcal{A}$ is the category of vector bundles over an elliptic curve.

The consensus at the Battelle Conference was that Quillen's K-theory of a free ring $\mathbb{Z}\{X\}$ measures the difference between the different constructions. A few months afterward, Gersten proved [**Ger74**] that $K_*(\mathbb{Z}\{X\})$ is trivial. Using this, Don Anderson proved in [**A73**] that Quillen's groups agree with the Gersten-Swan groups, and Wagoner showed that they also agree with his. A proof that Volodin's construction agrees with these was not given until much later, by Suslin [**Su81**].

Encouraged by Bass, Quillen wrote the foundational paper [**Q73**] during the first two months of 1973. It was a masterful piece of exposition, providing tools for calculations and elegant proofs of the structure results announced in [**Q72**]. (We will not elaborate on the details in order to focus on the flow of events.) At that time, Quillen discovered proofs of Gersten's conjecture and Bloch's Formula; we will discuss these results below.

Applications to Algebraic Geometry

The interaction of higher K-theory and algebraic geometry begins with Ken Brown's 1971 MIT thesis [**Br73**], which developed the cohomology theory of sheaves of spectra on a noetherian scheme X. Brown was motivated by Quillen's suggestion that such a theory might apply to K-theory. Gersten had recently constructed a functorial spectrum $\mathbb{K}(R)$ associated to a ring R; sheafifying gives such a sheaf spectra. Seizing upon this, Gersten announced in [**Ger73**] that if we set $K_n(X) = H^n(X, \mathbb{K})$ for regular separated schemes X then for $X = \mathrm{Spec}(R)$ we recover $K_n(X) = KV_n(R)$. (Quillen had already announced that $KV_n(R) = K_n(R)$ [**Q72**], but Gersten was unaware of this at the time.) Moreover, there is a spectral sequence converging to $K_*(X)$ whose E_2-terms are the Zariski cohomology $H^p(X, \mathcal{K}_q)$ of the sheaves associated to the presheaves sending $\mathrm{Spec}(R)$ to $K_q(R)$. The details were worked out by Brown and Gersten at the Battelle Conference, and appeared in [**BG73**].

Jouanolou [**J73**] also extended the functors KV_n from rings to quasi-projective schemes using what we now call "Jouanolou's device:" for each such X there is a ring R and a map $\mathrm{Spec}(R) \to X$ which is locally a vector bundle; one defines $KV_*(X) = KV_*(R)$. At the Battelle Conference it became clear that Quillen's definition $K_n(X) = K_n\mathbf{P}(X)$ agreed, at least for regular X, with the groups obtained using Jouanalou's device, as well as the Brown-Gersten definition [**BG73, Q73**].

In the other direction, the application of higher K-theory to the study of cycles in algebraic geometry was initiated by Spencer Bloch in [**Bl74**]. Bloch's preprint appeared just before the 1972 Battelle Conference, and was influenced by Gersten's results in [**Ger73**]. His main theorem, that the cohomology group $H^2(X, \mathcal{K}_2)$ is isomorphic to the Chow group $CH^2(X)$ on any regular surface, is now called *Bloch's formula*. One interesting conjecture made in [**Bl74**] was that if R is a (regular) local domain with quotient field F, then the following sequence is exact:

$$0 \to K_2(R) \to K_2(F) \xrightarrow{\lambda} \oplus_{ht\wp=1} k(\wp)^\times \to \mathcal{C}_2 \to 0,$$

where $k(\wp)$ denotes the residue field of $R_\wp$, and $\mathcal{C}_2$ is closely related (see p. 360) to the free abelian group on the height two prime ideals of R.

Gersten seized upon Bloch's formula, and made a series of striking conjectures about the higher K-theory of regular local rings, including what we now call *Gersten's conjecture*: $K_n(R)$ injects into $K_n(F)$ for every regular local ring R. These conjectures were presented at the Battelle Conference in 1972.

When writing up his foundational paper [**Q73**] in early 1973, Quillen discovered a proof of Gersten's Conjecture for the local rings of a nonsingular variety X over a field. His proof implied that Bloch's formula generalized to $H^p(X, \mathcal{K}_p) \cong CH^p(X)$ for all p. Quillen included this material in [**Q73**].

Looking ahead, Gersten's conjecture was proven for equicharacteristic DVR's by C. Sherman in [**Sh78**]. The general mixed characteristic case of Gersten's conjecture is still open today, in spite of some progress in the 1980's by Bloch, Gabber, Gillet, Levine, and Sherman.

Bloch-Ogus [**BlOgs**] soon showed that Quillen's trick for solving Gersten's conjecture could be used in other settings, such as de Rham cohomology. Gillet [**Gi80**] later used the ideas in the Bloch-Ogus paper to define Chern classes for higher K-theory, and to prove a very general Riemann-Roch theorem.

In 1978, Bloch [**Bl79**] used the cohomology of the sheaves $\mathcal{K}_n$ to describe the Abel-Jacobi map $CH^n(X) \to \mathrm{Alb}(X)$ of a smooth projective n-dimensional variety, and to give a different proof of Roitman's theorem that the Abel-Jacobi map induces an isomorphism between their torsion subgroups. In the 1980's, the use of K-cohomology to study cycles exploded, starting with [**CSS82**], but that is beyond the scope of this paper. The state-of-the-art as of 1979 is nicely described in Bloch's book [**Bl80**].

Homological methods

Most attempts to compute the groups $K_n(R)$ in the 1970's used homological methods. The idea was to use geometric methods to calculate either the homology of the groups $SL_n(R)$ or the homology of BQ_n for small subcategories Q_n of $Q\mathbf{P}(R)$. If R is the ring of integers in a number field, Quillen [**Q73$\mathbb{Z}$**] used these methods to show that the $K_n(R)$ are finitely generated, and Borel [**Bo72**] used them to compute the rank of the $K_n(R)$.

Motivated by these calculations, and also by his calculations in [**Ba64**], Bass posed this question at the Battelle conference: are the groups $K_n(R)$ finitely generated whenever R is either finitely generated as an abelian group, or regular and finitely generated as a ring? This question has become known as Bass' "finite generation conjecture," and is still open.

Here are two of the strongest results in this direction. If R is regular, finitely generated as a ring, and $\dim(R) \leq 1$, Quillen proved that the $K_n(R)$ are finitely generated in 1974; his proof was published in [**GQ82**]. If R is finite, or R is an order in a semi-simple $\mathbb{Q}$-algebra, Remi Kuku [**Ku86**] proved that each $K_n(R)$ is finite.

Several people used homological methods to prove stability theorems for higher K-groups, including Charney, Maazen, Vogtman, van der Kallen and Suslin.

It was sometimes possible to use homological methods to calculate K_3 and even K_4. In 1975, Lee and Szczarba [**LeeSz**] were able to calculate the homology of $SL_3(\mathbb{Z})$ and prove that $K_3(\mathbb{Z})$ is isomorphic to the cyclic group $\mathbb{Z}/48$. Evens and Friedlander computed K_3 and K_4 for rings of order p^2 in [**EF80**]. However, the difficulty of using this method quickly became overwhelming, as Staffeldt's 1977 thesis [**St79**] shows. After a ten year effort by Vic Snaith and his students culminated in the book [**ALS85**], the effort to use these techniques was effectively abandoned.

The Lichtenbaum Conjectures

Let F be a totally real number field F, and R its ring of integers. In 1972, S. Lichtenbaum [**Li73**] conjectured that the K-theory of R is related to the values of the zeta function $\zeta_F(s)$. More precisely, for each $i \geq 1$ he conjectured that the rational number $|\zeta_F(1-2i)|$ equals the ratio of the orders of the finite groups $K_{4i-2}(R)$ and $K_{4i-1}(R)$.

The first test of this conjecture was $R = \mathbb{Z}$ and $i = 1$. Since $\zeta_{\mathbb{Q}}(-1) = -1/12$ and $K_2(\mathbb{Z}) \cong \mathbb{Z}/2$, Lichtenbaum's conjecture predicted that $K_3(\mathbb{Z})$ has 24 elements. After Lee and Szczarba [**LeeSz**] showed that it has 48 elements, Lichtenbaum amended his conjecture to say that equality holds "up to a power of 2."

With hindsight, we know that the connection between the K-theory of R and the zeta function of F goes through the étale cohomology of R. To explain this connection, note that the Galois group G of $\bar{F}/F$ acts on the roots of unity μ_N and its twists $\mu_N^{\otimes i}$. For each i there is an integer $w_i(F)$ such that the groups $H^0_{et}(F, \mu_N^{\otimes i})$ are cyclic of order $w_i(F)$ for all large N. We have already seen that the number w_2 appears in the Birch-Tate conjecture that $\#K_2(R)$ is $w_2(F)|\zeta_F(-1)|$. Another connection was found by Bruno Harris and Graeme Segal, who proved in [**HS75**] that $K_{2i-1}(R)$ contains a cyclic summand whose order is $w_2(F)$ up to a factor of 2.

Lichtenbaum had already made several conjectures in number theory, including a conjecture that for all i and all primes ℓ, the ℓ-part of the integer $w_i(F)\zeta_F(i-1)$ equals the order of $H^1_{et}(R[1/\ell], \mu_{\ell^\nu}^{\otimes i})_{tors}$ for all large ν. In 1990, Wiles [**Wi90**] proved the Main Conjecture of Iwasawa Theory for odd primes ℓ, which implies this conjecture up to powers of two. The 2-part of this conjecture would follow from the Main Conjecture for the prime 2, which is still open.

Comparing these two circles of ideas, we may rephrase the second half of Lichtenbaum's conjectures as saying that for ν large, we have

$$2^{(?)} \cdot \frac{\#K_{2i-2}(R)}{\#K_{2i-1}(R)} = \prod_\ell \frac{\#H^2_{et}(R[1/\ell]; \mu_{\ell^\nu}^{\otimes i})}{\#H^1_{et}(R[1/\ell]; \mu_{\ell^\nu}^{\otimes i})}.$$

We now know [**RW97**] that "(?)" is the number r_1 of real embeddings of F. In fact, thanks to the work of Voevodsky in [**Voe96**], Rognes and Weibel [**RW97**] have recently affirmed the 2-primary part of the second half of Lichtenbaum's conjectures.

Quillen [**Q75**] generalized this second half, suggesting that for any regular ring R there should be a spectral sequence for each prime ℓ invertible in R, analogous to the Atiyah-Hirzebruch spectral sequence, but starting from the étale cohomology groups $H^p_{et}(R, \mathbb{Z}_\ell(q/2))$, whose abutment would coincide with the ℓ-adic completion of $K_n(R)$ at least in degrees $n > 1 + \dim(R)$. If R is the ring of integers in a number field (and ℓ is odd), such a spectral sequence would degenerate, yielding the cohomological formulas for $K_n(R)$ conjectured by Lichtenbaum.

Quillen made this conjecture explicit when F is an algebraically closed field of characteristic p: the groups $K_{2n}(F)$ should be uniquely divisible, and $K_{2n-1}(F)$ should be a divisible group whose torsion subgroup is $\mathbb{Q}/\mathbb{Z}[1/p]$. This was affirmed by Suslin in 1983 paper [**Su83**]; the case $p = 0$ was given in the sequel [**Su84**].

During the late 1970's, it gradually became clear that Quillen's conjectures should be attacked using K-theory with finite coefficients. The groups $K_*(R; \mathbb{Z}/\ell)$ were first introduced by W. Browder [**Br76**], who constructed products on them and showed that the Bott element $\beta \in K_2(R; \mathbb{Z}/\ell)$ plays a central role. For example, if a finite field $\mathbb{F}_q$ has $q \equiv 1 \pmod{\ell}$ then $K_*(\mathbb{F}_q; \mathbb{Z}/\ell)$ is the graded commutative ring $\mathbb{Z}/\ell[x, \beta]/(x^2 = 0)$, where $x \in K_1$ is the class of a unit generating $\mathbb{F}_q^\times$.

Soulé [**Sou79**] constructed étale chern classes $K_n(R; \mathbb{Z}/\ell^\nu) \to H^{2i-n}_{et}(R, \mu_{\ell^\nu}^{\otimes i})$, and showed that they are onto in many cases. To do this, he established a product formula, and applied it to elements in $K_*(R; \mathbb{Z}/\ell)$ of the form $\beta^i \cup x$ with $x \in K_j(R)$. Soulé showed that the orders of these elements are related to the numerator of the Bernoulli numbers $B_k/4k$, as predicted by Lichtenbaum's conjectures. For example, Soulé produced an element of order 691 in $K_{22}(\mathbb{Z})$.

Inspired by Soulé's calculations, Eric Friedlander invented an approximation to K-theory he dubbed "étale K-theory." Dwyer and Friedlander [**DF82**] constructed a spectral sequence of the kind described by Quillen, converging to étale K-theory. Later on, it was proven that étale K-theory of X is obtained from the ring $K_n(X; \mathbb{Z}/\ell^\nu)$ by inverting the Bott element, and that $K_n(X; \mathbb{Z}/\ell^\nu)$ agrees with étale K-theory for large n. The proof of these assertions was published in [**DFST**] and also in Thomason's masterpiece [**Th85**].

Backing up somewhat, we turn to the question of regulators. Let R be the ring of integers in a number field F. Borel had observed in [**Bo72**] that the rank d_i of the group $K_{2i-1}(R)$ equals the order of the zero of the zeta function $\zeta_F(s)$ at $s = 1 - i$ for each i. Indeed, the Hurewicz map $K_*(R) \to H_*(GL(R); \mathbb{R})$ induces a map $r_i \colon K_{2i-1}(R) \to \mathbb{R}^{d_i}$, which we now call the *Borel regulator*. In 1976, Borel showed [**Bo77**] that the image of r_i is a lattice whose covolume is a rational number times

$$\pi^{-d_i} \lim_{s \to 1-i} \zeta_F(s)(s + i - 1)^{d_i} = \pi^{-i[F:\mathbb{Q}]}\sqrt{D}\zeta_F(i).$$

Here d_i is the order of the pole of the zeta function $\zeta_F(s)$ at $s = 1 - i$, and is either r_2 or $r_1 + r_2$ depending on $i \pmod{2}$.

Next, Bloch gave an intense series of lectures on Borel's regulator at the University of California at Irvine, followed by some conjectures about possible generalizations to regulators for an elliptic curve E over a number field. Writing up these ideas led him to discover a regulator $K_2(E) \to \mathbb{R}^d$, which was included in the 1977 preprint [**Bl77**]; Bloch gave a talk on this at the 1978 Helsinki ICM.

In September 1979, Manin asked an undergraduate student in Moscow to give a report on Bloch's Irvine notes [**Bl77**]. That student was A. Beilinson, who was then 22 years old. Instead of giving Bloch's construction, he improved on it in [**Bei80**] by constructing what we now call *Beilinson's regulator*. This led Beilinson to make a series of conjectures in 1982 [**Bei82, Bei84**]; these are now called the *Beilinson conjectures*, and they have been a driving force in the development of algebraic K-theory in the 1980's and 1990's.

While the Beilinson conjectures are extremely interesting, they take us well beyond 1980 in our historical account. So we restrain ourselves from continuing past this point, and turn back to other developments which took place in the 1970's.

Symbol Calculations

From the beginning, there was an interest in constructing and detecting elements of higher K-theory. It was implicit from the start [**Ger72, p. 17**] that Quillen's theory should have products $K_i(A) \otimes K_j(B) \to K_{i+j}(A \otimes B)$. Such products were independently constructed by Loday [**Lo75**], using the space $BGL(R)^+$, and by Waldhausen [**Wa73**], who used an iterated Q-construction in the context of pairings of exact categories. In the published version of [**Wa73**], Waldhausen gave a proof that his products agree with Loday's; see [**We80**] for more details.

Products allowed one to form higher Steinberg symbols $\{x_1, \dots, x_n\}$ in $K_n(R)$, yielding the ring homomorphism $K_*^M(F) \to K_*(F)$ discussed above. Products were used to construct other elements in $K_n(R)$, such as the generalized Dennis-Stein symbols $\langle r_1, ..., r_n \rangle$, which are defined whenever $1 - \prod r_i$ is a unit in R [**Lo81**]. Applying cohomology to these products gives a bigraded ring $H^*(X, \mathcal{K}_*)$. Grayson [**Gr77**] used Waldhausen's product formalism to show that Bloch's formula $H^p(X, \mathcal{K}_p) \cong CH^p(X)$ is actually a ring isomorphism. Products were also used very effectively by Gillet [**Gi80**] in proving a version of the Riemann-Roch theorem for higher K-theory.

To show that these new elements were nonzero, people turned to Chern classes. Gersten had constructed Chern classes with values in Kähler differentials Ω_R^* as early as 1970 [**Glet**] [**Ger72, p. 238**]. These methods were used by Bloch in [**Bl78**] to construct Chern classes in crystalline cohomology, as part of an effort to relate K-theory to the slope spectral sequence in crystalline cohomology. Product formulas for the étale chern classes [**Sou79**] and for the classes in $H^*(X, \mathcal{K}_*)$ [**Su80**] showed that Chern classes on K_n typically could not detect elements of exponent $(n-1)!$.

In 1976, Keith Dennis [**D76**] discovered trace maps from $K_n(R)$ to the Hochschild homology group $HH_n(R)$, but never published the details. These Dennis trace maps were to play a major role in calculations during the 1980's.

Another aspect of the attempt to find generators and relations for $K_n(R)$ was the search for a Mayer-Vietoris sequence for higher K-theory associated to an "excision situation" where one had a ring map $R \to S$ sending an ideal I of R isomorphically onto an ideal of S. Around 1970, Swan [**Sw71**] had made the discovery that the relative groups $K_1(R,I)$ really depend upon R, so that there was no general extension of the Mayer-Vietoris sequence to K_2. Swan also gave a description for the kernel of the surjection $K_1(R,I) \to K_1(S,I)$. After his description was improved by Vorst [**Vo79**], S. Geller and C. Weibel [**GW83**] gave a complete description of the doubly relative term $K_1(R,S,I)$, *i.e.,* the preceding term in the appropriate long exact sequence.

The description of the relative group $K_2(R,I)$ followed a parallel path. In his 1970 thesis [**St71**], M. Stein gave one candidate for $K_2(R,I)$, the definition used in Milnor's book [**Mi71**]. Swan pointed out in [**Sw71**] that one should really define $K_2(R,I)$ to be the third group fitting into the natural long exact sequence for $K_*(R) \to K_*(R/I)$, and this relative group was a proper quotient of Stein's candidate. Keune and Loday independently found a description of $K_2(R,I)$ in 1977 [**Keu78, Lo78**]. Then in 1980, a description of the doubly relative group $K_2(R,R/J,I)$ was given by Keune [**Keu80**] and, again independently, by Guin-Waléry and Loday [**GWL**].

Computing the K-theory of nilpotent ideals, a problem posed by Swan at the Battelle Conference, was also an active topic in the 1970's. The calculation of $K_2(R,I)$ was mentioned above [**MaSt**]. Homological methods had limited success in characteristic p, highlighted by [**EF80**]. If R contains $\mathbb{Q}$, then each $K_n(R,I)$ is uniquely divisible, and rational homology methods had more success, including Soulé's calculation [**Sou80**] when $R = \mathbb{Q}[\varepsilon]$ and climaxing in Goodwillie's theorem [**Gw86**] that $K_n(R,I)$ is isomorphic to the cyclic homology group $HC_{n-1}(R,I)$. If R has residual characteristic p, Randy McCarthy [**Mc97**] has recently shown that $K_n(R,I)$ is isomorphic to the topological cyclic homology group $THC_{n-1}(R,I)$.

Pseudo-isotopies and $A(X)$

Having settled the h-cobordism question, topologists turned to the question of uniqueness of product structures on $M \times I$, and studied the topological group $\mathcal{P}(M)$ of *pseudo-isotopies* on M, *i.e.,* diffeomorphisms of $M \times I$ which are the identity on $M \times \{0\}$; a pseudo-isotopy which preserves each level $M \times \{t\}$ is an isotopy.

Cerf proved that if $\dim(M) > 5$ and M is simply connected then every pseudo-isotopy is homotopic to an isotopy. Hatcher and Wagoner [**HW73**] computed $\pi_0\mathcal{P}(M)$ when $\pi = \pi_1(M)$ is nontrivial; it is a sum of two terms, one of which is a quotient of $K_2(\mathbb{Z}\pi)$ they named $Wh_2(\pi)$. Hatcher also discovered a connection between $\pi_1\mathcal{P}(M)$ and K_2. This led him to construct a stable space $Wh^{\mathrm{PL}}(M)$ as the direct limit of the spaces $\Omega^2\mathcal{P}(M \times I^k)$ and prove that if $\dim(M) \gg n$ one has $\pi_n\mathcal{P}(M) \cong \pi_n Wh^{\mathrm{PL}}(M)$. (Unfortunately Hatcher stated the stable range incorrectly, but that is not part of this story.)

In 1975, Doug Anderson and W.-C. Hsiang [**AH75**] studied pseudo-isotopies of polyhedra, and showed that negative K-groups appear. For example, they showed that $\pi_0\mathcal{P}(S^{m+2}M) = K_{-m}(\mathbb{Z}\pi_1 M)$ if $m \geq 4$.

Loday [**Lo75**] introduced the *assembly map* $H_n(BG;\mathbb{KZ}) \to K_n(\mathbb{Z}\pi)$, where the domain is the generalized homology of BG with coefficients in the spectrum $\mathbb{KZ}$ for $K_*(\mathbb{Z})$, and observed that when $n = 2$ the cokernel is the Hatcher-Wagoner group $Wh_2(\pi)$. This led Loday and Waldhausen [**Wa78, p. 228**] to define the higher Whitehead groups $Wh_n(\pi)$, not in terms of $Wh^{\mathrm{PL}}(X)$ but as the third term in the natural long exact sequence

$$\cdots \to H_n(BG;\mathbb{KZ}) \to K_n(\mathbb{Z}\pi) \to Wh_n(\pi) \to H_{n-1}(BG;\mathbb{KZ}) \to \cdots$$

The study of pseudo-isotopies was revolutionized in 1976, when F. Waldhausen [**Wa78**] constructed an infinite loop space $A(X)$, functorial in X, and a natural map $A(X) \to Wh^{\mathrm{PL}}(X)$ which was later proven [**Wa87**] to give a decomposition:

$$A(X) \simeq \Omega^\infty S^\infty(X_+) \times Wh^{\mathrm{PL}}(X).$$

In addition, Waldhausen proved in [**Wa78**] that $\pi_* A(B\pi) \otimes \mathbb{Q} \cong K_*(\mathbb{Z}\pi) \otimes \mathbb{Q}$.

To do this, Waldhausen first defined the algebraic K-theory of a "ring up to homotopy," R, using a variation $B\widehat{GL}(R)^+$ of Quillen's $+$-construction. For example, if $\Omega' X$ is Kan's simplicial loop group for X, one can form the ring up to homotopy $Q[\Omega' X] = \Omega^\infty S^\infty(\Omega' X_+)$. Waldhausen defined $A(X)$ as the K-theory space of $Q[\Omega' X]$ [**Wa78, p. 42**].

Waldhausen also gave a K-theoretic construction of $A(X)$ in [**Wa78, p. 58**], a construction now called *Waldhausen K-theory* which we will describe shortly.

Here is one nice application that came out of Waldhausen's construction, showing one way that PL manifold theory is simpler than smooth manifold theory. Consider the two topological groups $\mathrm{Homeo}_{\mathrm{PL}}(D^n, \partial D^n)$ and $\mathrm{Diff}(D^n, \partial D^n)$ of PL-homeomorphisms, resp. diffeomorphisms, of the disk which fix the boundary. It is a well-known consequence of the "Alexander Trick" that $\mathrm{Homeo}_{\mathrm{PL}}(D^n, \partial D^n)$ is contractible. In contrast, Farrell and Hsiang [**FH76**] used Waldhausen's results and Borel's computation of $K_*(\mathbb{Z}) \otimes \mathbb{Q}$, together with an analysis of the assembly map, to compute the rational homotopy groups of $\mathrm{Diff}(D^n, \partial D^n)$; they are nonzero.

In 1978–1980, Dwyer, Hsiang and Staffeldt [**DHS80, HS82**] used rational homotopy theory to compute $\pi_n A(X) \otimes \mathbb{Q}$ when $\pi_1(X) = 0$. They stated their answer
in terms of classical invariant theory. Later on, it would be recognized as a harbinger of cyclic homology: $\pi_n A(X) \otimes \mathbb{Q} \cong K_n(\mathbb{Z}) \oplus HC_{n-1}(R_.)$, where $R_.$ is the differential graded algebra of singular chains on ΩX. The integral version of this result is given in [**CCGH**].

Also in [**Wa78**], Waldhausen described a stabilization process, using the K-theory of a simplicial ring, to construct groups $K_n^S(R, M)$, called the *stable K-theory* of R with coefficients in a bimodule M. Waldhausen constructed a natural map from $K_n^S(R, M)$ to the Hochschild homology group $HH_{n-1}(R, M)$ and showed that it is an isomorphism for $n = 1, 2$; Kassel [**Kas80**] showed that they differ for $n = 3$. Later, Dennis and Igusa showed in [**DI82**] that Hatcher's obstruction for $\pi_1 \mathcal{P}(X)$ is $K_2^S(\mathbb{Z}\pi, M)$, where $\pi = \pi_1(X)$ and $M = \pi_2(X)[\pi]$.

Looking ahead, Goodwillie conjectured that there existed a construction for rings like that used to construct Hochschild homology, and that it would be homotopy equivalent to stable K-theory for any ring. Bökstedt gave such a construction in [**Bö85**], calling it the *topological Hochschild Homology* $THH(R)$ of a ring. This played a key role in later computations such as [**BHM**]. Finally, Dundas and McCarthy showed in [**DM94**] that $THH(R; M) \simeq K^S(R, M)$ for any simplicial ring R and bimodule M.

Waldhausen K-theory

Waldhausen also gave a second construction of $A(X)$ in 1976. A preliminary version appeared in [**Wa73, p. 181ff**], and the announcement was published in [**Wa78, p. 58**]. However, Waldhausen's foundational paper [**Wa85**] containing the full details only appeared much later. For this construction, he introduced what we now call a *Waldhausen category, i.e.,* a category $\mathcal{A}$ with appropriate notions of cofibrations and weak equivalences w. Given $\mathcal{A}$, Waldhausen defined a simplicial category $wS_{.}\mathcal{A}$ and studied the space $K(\mathcal{A},w)=\Omega|wS_{.}\mathcal{A}|$. (The S in the notation is for G. Segal, who had earlier considered a similar construction; see [**Wa73, p. 181**].)

Waldhausen proved that $A(X)\simeq K(\mathcal{R}_f(X),h)$, where $\mathcal{R}_f(X)$ is the category of finite split retractions $Y\to X$ and h denotes homotopy equivalence over X. See [**Wa78, 5.7**]. This of course was the original point of the construction, and many of Waldhausen's structure theorems were based on the existence of a construction like the formation of the standard mapping cylinder, satisfying a "cylinder axiom."

Any exact category $\mathcal{A}$ is a Waldhausen category, where cofibration sequences are exact sequences and w is the class of isomorphisms. Waldhausen observed that $BQ\mathcal{A}\simeq|wS_{.}\mathcal{A}|$, so that $K_*(\mathcal{A},w)=\pi_*K(\mathcal{A},w)$ agrees with Quillen's $K_*(\mathcal{A})$.

A third important application of Waldhausen's construction is to the category $C_b(\mathcal{A})$ of bounded chain complexes in an exact category $\mathcal{A}$, where w is quasi-isomorphism; the classical mapping cylinder of chain complexes satisfies Waldhausen's cylinder axiom. In fact, Waldhausen's group $K_0(C_b(\mathcal{A}),w)$ recovers the definition of $K_0(\mathcal{A})$ given by Grothendieck in [**SGA6**].

Waldhausen realized early on that if $\mathcal{A}$ is the category of projective R-modules then $K(C_b(\mathcal{A}),w)\simeq K(\mathcal{A})$; Gillet proved more generally [**Gi80, p. 256**] that this formula holds for any (idempotent complete) exact category $\mathcal{A}$. (Cf. [**TT90, p. 279**].) Gillet applied this to the category $\mathcal{A}$ of vector bundles on a scheme X, showing that the Chern classes in [**Gi80**] are local Chern classes in the sense of [**SGA6**].

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