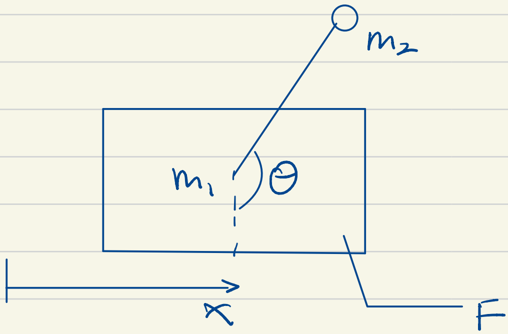
16-748 Problem Set 3

Problem # 1 a.

$$x = \begin{bmatrix} x \\ \theta \\ \dot{x} \\ \dot{\theta} \end{bmatrix}$$



Use Lagrangian:

$$L = T - V$$

$$T = \frac{1}{2}(m_1 + m_2)\dot{x}^2 + m_2 \cdot \dot{x} \cdot \dot{\theta} \cdot l \cdot \cos\theta + \frac{1}{2} \cdot m_2 \cdot l^2 \cdot \dot{\theta}^2$$

$$V = - m_2 \cdot g \cdot l \cdot \cos\theta$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = dt\left((m_1 + m_2)\dot{x} + m \cdot L \cdot \cos\theta \cdot \dot{\theta}\right)$$

$$= (m_1 + m_2)\ddot{x} - m_2 L \cdot \sin\theta \cdot \dot{\theta}^2 + m_2 \cdot L \cdot \cos\theta \cdot \ddot{\theta}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \underbrace{\frac{\partial L}{\partial x}}_{0} = F$$

$$(m_1 + m_2)\ddot{x} - m_2 \cdot L \cdot \sin\theta \cdot \dot{\theta}^2 + m_2 \cdot L \cdot \cos\theta \cdot \ddot{\theta} = F \quad (1)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0$$

$$\frac{d}{dt}\left(m_2 L^2 \cdot \dot{\theta} - m_2 . L \cdot \cos(\theta)\dot{x}\right) = m_2 \cdot L \cdot \sin(\theta)\dot{\theta}\dot{x} + m_2 g \cdot L \cdot \sin\theta$$

$$m_2 L^2 \ddot{\theta} - m_2 . L \cos\theta \ddot{x} + m_2 . L . \sin\theta \dot{x}\dot{\theta} - m_2 . L . \sin\theta \dot{\theta}\dot{x} - m_2 . g . L . \sin(\theta) = 0$$

$$m_2 . L . \ddot{\theta} - m_2 . L . \cos(\theta)\ddot{x} - m_2 g . L . \sin(\theta) = 0 \quad (2)$$

$$q = \begin{bmatrix} x \\ \theta \end{bmatrix}$$

$$H(q)\ddot{q} + C(q, \dot{q})\dot{q} + G(q) = Bu$$

$$H_q: \begin{bmatrix} m_1 + m_2 & m_2 \, l \cdot \cos\theta \\ m_2 . l . \cos\theta & m_2 \, l^2 \end{bmatrix} = \begin{bmatrix} 2 & \cos\theta \\ \cos\theta & 1 \end{bmatrix}$$

$$C(q, \dot{q}) = \begin{bmatrix} 0 & -m_2 . l \dot{\theta} \sin\theta \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\dot{\theta} . \sin\theta \\ 0 & 0 \end{bmatrix}$$

$$G(q) = \begin{bmatrix} 0 \\ m_2 . g . l \sin\theta \end{bmatrix} = \begin{bmatrix} 0 \\ \sin\theta \end{bmatrix}$$

$$B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} \dot{x} \\ \dot{\theta} \\ \ddot{x} \\ \ddot{\theta} \end{bmatrix}}_{\dot{x}} = \begin{bmatrix} \dot{q} \\ \ddot{q} \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} \dot{q} \\ \dfrac{B u - C(q, \dot{q})\dot{q} - G(q)}{H(q)} \end{bmatrix}}_{f(x, u)}$$

linearize the system around (x^*, u^*)

$$\dot{x} = f(x, u)$$

$$\approx \overset{\to 0}{f(x^*, u^*)} + \left.\frac{\partial f}{\partial x}\right|_{x^*, u^*} (x - x^*)$$

$$+ \left.\frac{\partial f}{\partial u}\right|_{x^*, u^*} (u - u^*)$$

$f(x^*, u^*)$ @ $(0, \pi, 0, 0) = 0$

$$\therefore \dot{x} = \left.\frac{\partial f}{\partial x}\right|_{x^*, u^*} (x - x^*) + \left.\frac{\partial f}{\partial u}\right|_{x^*, u^*} (u - u^*)$$

$$f = \begin{bmatrix} \dot q \\ H^{-1}(Bu - C(q,\dot q)\dot q - G(q)) \end{bmatrix}$$

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \boxed{} & & \boxed{} & \end{bmatrix}$$

$$\frac{\partial H^{-1}}{\partial q_i}\left(B u - \cancelto{0}{C(q,\dot q)\dot q} - G(q)\right)$$

$$+ H^{-1}\left(0 - \cancelto{0}{\frac{\partial C}{\partial q_i}\dot q} - \frac{\partial G(q)}{\partial q_i}\right)$$

$$\dot q = 0$$

$$\frac{\partial H^{-1}}{\partial q_i}\left(\cancelto{0}{\dots}\right)$$

$$+ H^{-1}(0 - C - 0)$$

$$\frac{\partial f}{\partial u} = \begin{bmatrix} 0 \\ 0 \\ H^{-1}B \end{bmatrix}$$

$$x = Ax + Bu$$

$$= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -H^{-1}\frac{\partial G}{\partial q} & & -H^{-1}C & \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ H^{-1}B \end{bmatrix} u$$

$$\frac{\partial G}{\partial q} = \begin{bmatrix} 0 & 0 \\ 0 & -M_2 \cdot g \cdot l \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$$

check controllability

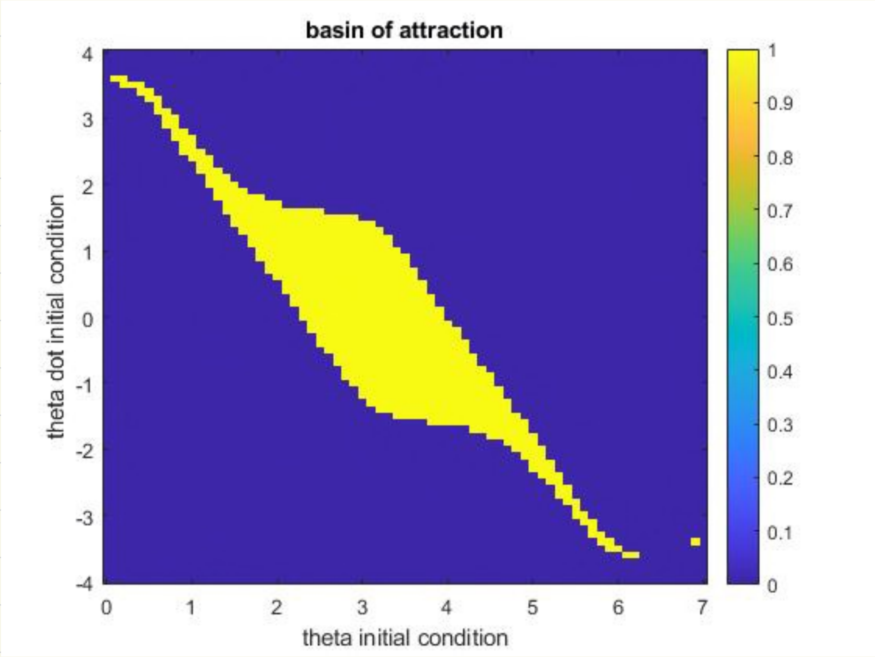
$$P = [B \quad AB \quad A^2B \quad A^3B]$$

$$\mathrm{Rank}(P) = 4 \quad \rightarrow \text{from MATLAB}$$

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{bmatrix}$$

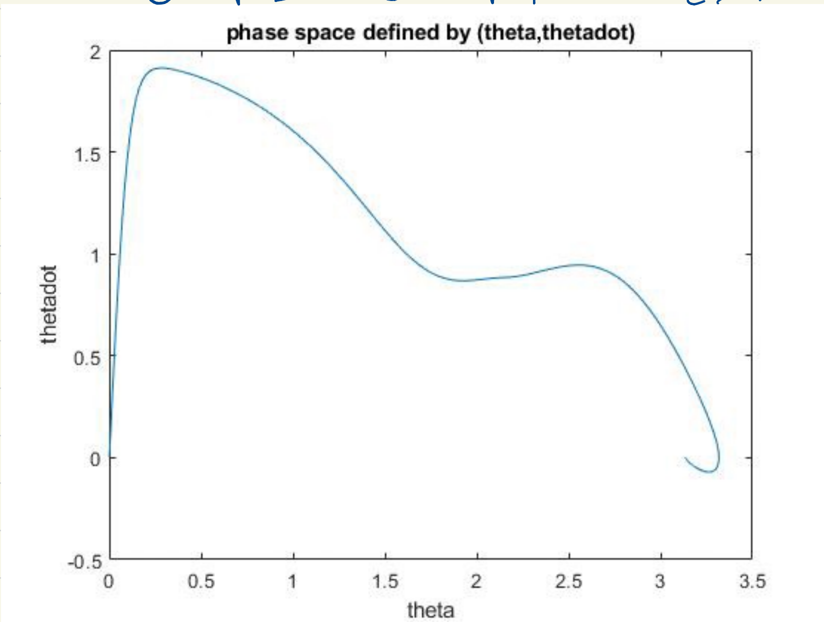
$$B = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

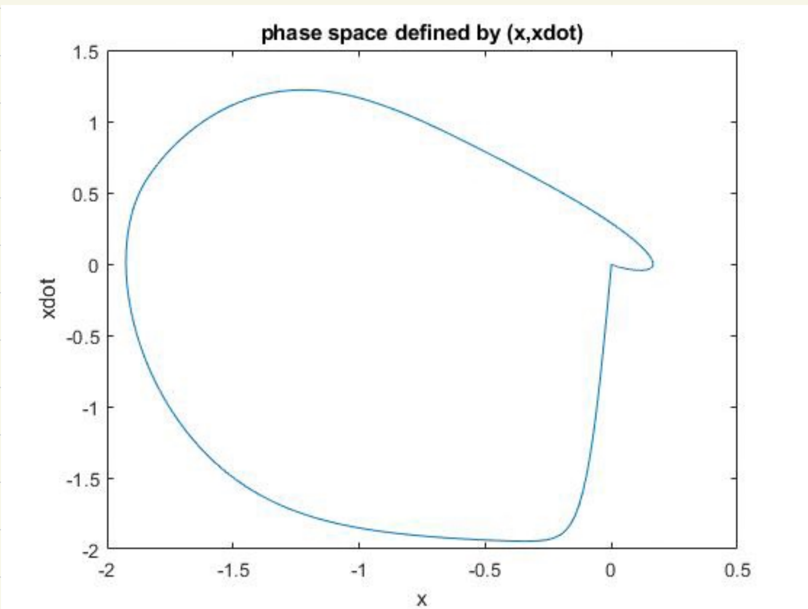
Basin of attraction



Problem 1b.

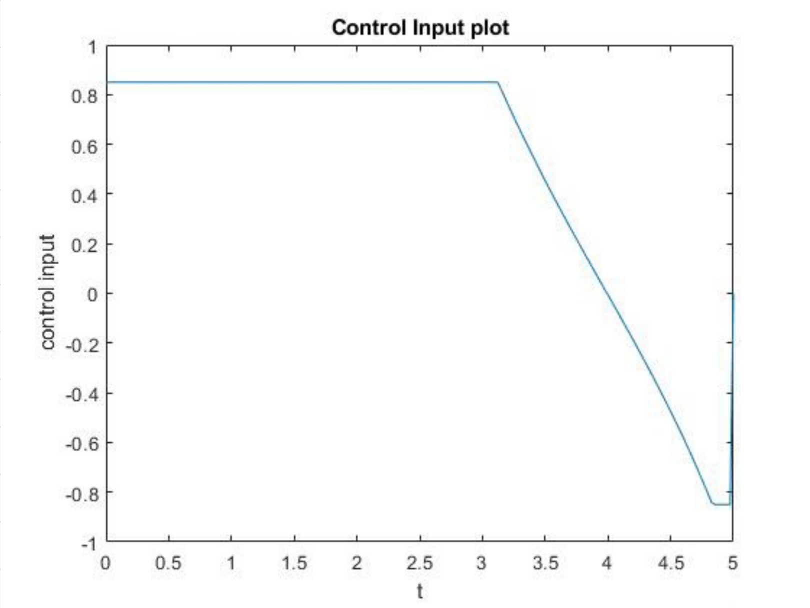
using $KE=10$ $KP=1$ $kd=1$

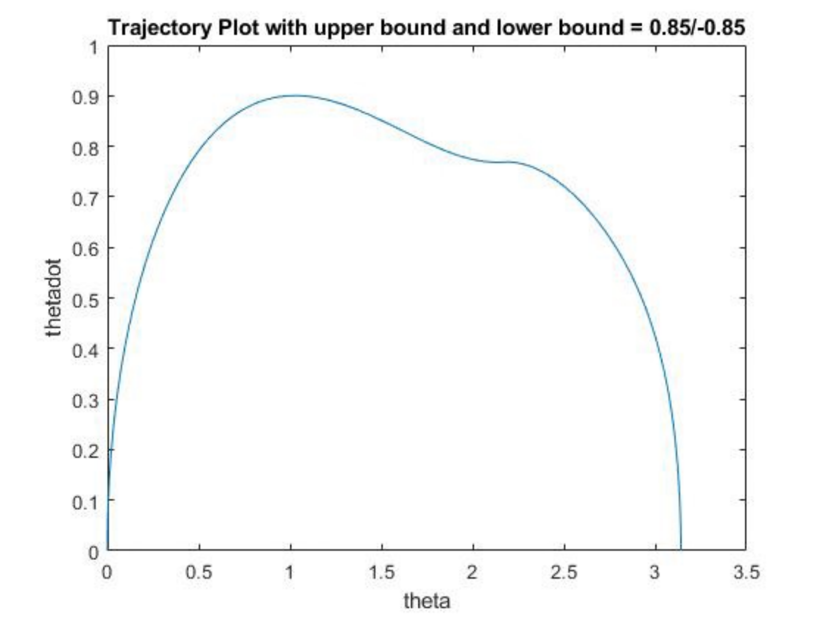




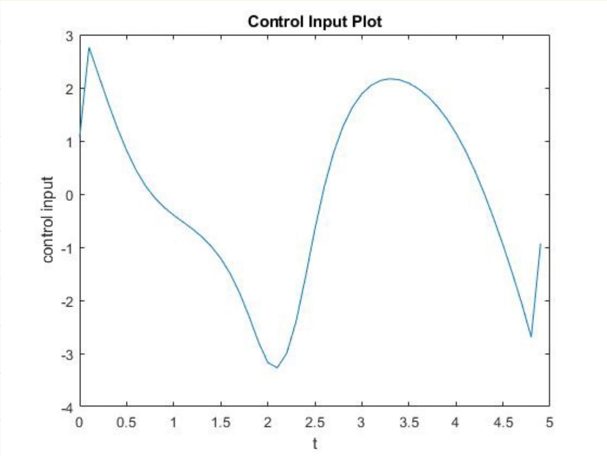
Problem 2

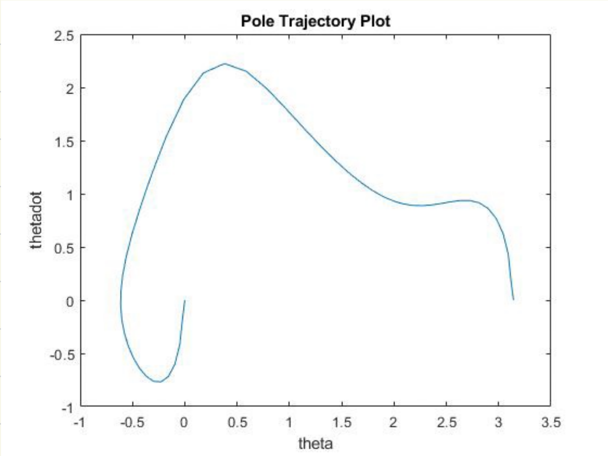
The minimum upper and lower bound is equal to $\boxed{\pm 0.85}$

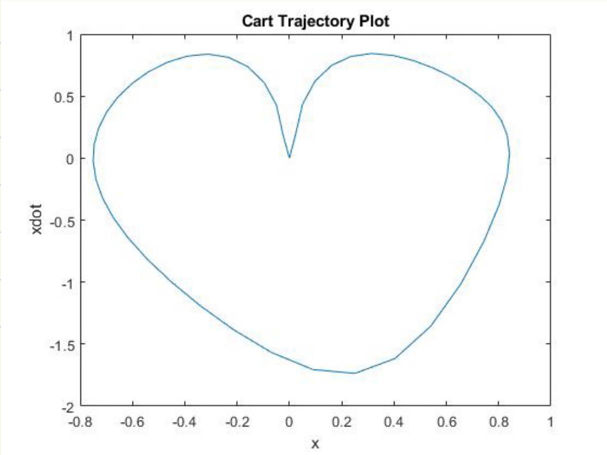




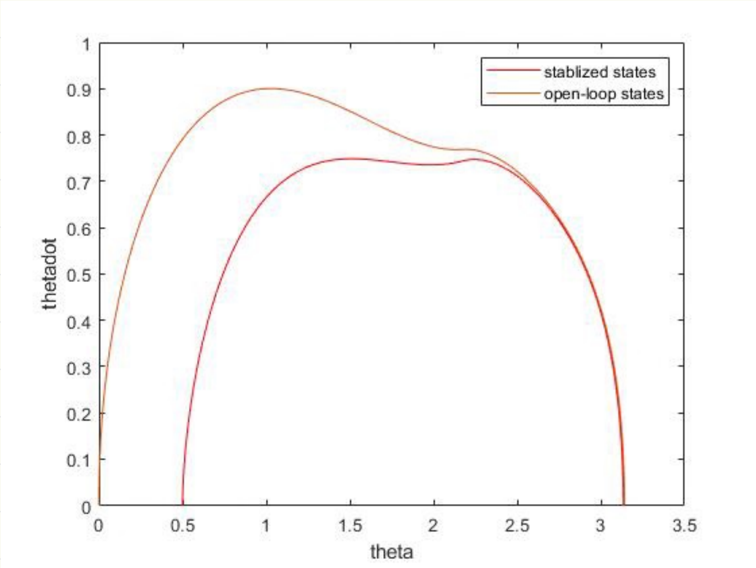
Problem 3


Control Input Plot
control input
t
3
2
1
0
-1
-2
-3
-4
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5

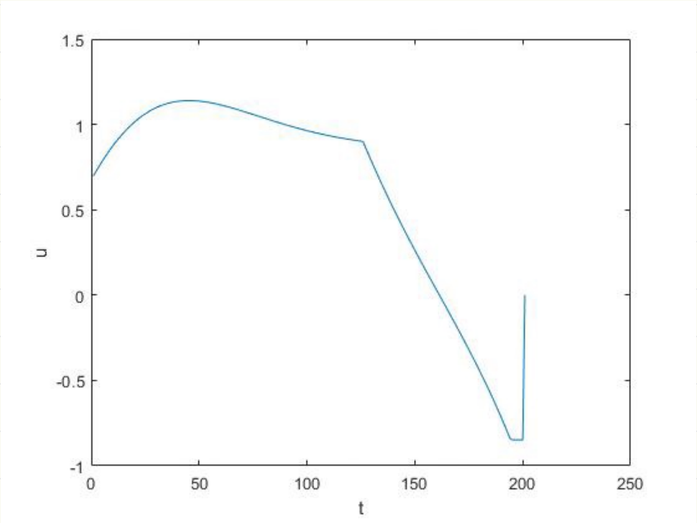

Pole Trajectory Plot
thetadot
theta
2.5
2
1.5
1
0.5
0
-0.5
-1
-1
-0.5
0
0.5
1
1.5
2
2.5
3
3.5


Cart Trajectory Plot
xdot
x
1
0.5
0
-0.5
-1
-1.5
-2
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1

Problem 4.



Open-loop States & stablized States



Input