TOTALNI DIFERENCIJAL

Totalni diferencijal I.reda

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$

Totalni diferencijal II.reda

$$d^2 z = \frac{\partial^2 z}{\partial x^2} dx^2 + 2 \frac{\partial^2 z}{\partial x \partial y} dx dy + \frac{\partial^2 z}{\partial y^2} dy^2$$

Derivacija kompozicije funkcija

I. $z = f(x,y)$
$x = x(t)$
$y = y(t)$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

II. $z = f(x,y)$
$x = x(u,v)$
$y = y(u,v)$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u}$$
$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial v}$$

Derivacija implicitno zadane funkcije:

I. $f(x,y) = 0$ -> funkcija jedne varijable

$$y' = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$$

II. $f(x,y,z) = 0$ -> funkcija dvije varijable $z = z(x,y)$

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial z}} \qquad \frac{\partial z}{\partial y} = -\frac{\frac{\partial f}{\partial y}}{\frac{\partial f}{\partial z}}$$

Ekstremi funkcija više varijabli

$z = f(x, y)$

1) Određivanje stacionarnih točaka (nužan uvjet ekstrema)
   riješiti sustav:
   $$\left.\begin{aligned}\frac{\partial f}{\partial x} = 0\\ \frac{\partial f}{\partial y} = 0\end{aligned}\right\} \Rightarrow T(x_0, y_0) \rightarrow stac.\, točka$$

2) Izračunavanje determinante Δ (dovoljan uvjet ekstrema)
   $\Delta = \begin{vmatrix} A & B \\ B & C \end{vmatrix}$ gdje je

   $$A = \frac{\partial^2 f}{\partial x^2}(x_0, y_0) \qquad B = \frac{\partial^2 f}{\partial x \partial y}(x_0, y_0) \qquad C = \frac{\partial^2 f}{\partial y^2}(x_0, y_0)$$

   Ako je Δ > 0 => u točki $T(x_0, y_0)$ je lokalni ekstrem i to:
   - lokalni minimum ako je A>0
   - lokalni maksimum ako je A<0

   Ako je Δ < 0 => u točki $T(x_0, y_0)$ nije ekstrem (sedlasta točka)
   Ako je Δ = 0 => nema odluke o ekstremu

Lokalni ekstremi funkcija triju varijabli

$u = f(x, y, z)$

1) $\frac{\partial u}{\partial x} = 0 \qquad \frac{\partial u}{\partial y} = 0 \qquad \frac{\partial u}{\partial z} = 0$ => stac. točke

2) $T(x_0, y_0, z_0)$ stac. točka

$$\begin{array}{cccc} & \Delta_1 & \Delta_2 & \Delta_3 \end{array}$$

$$\Delta = \begin{vmatrix} \frac{\partial^2 u}{\partial x^2}(T) & \frac{\partial^2 u}{\partial y \partial x}(T) & \frac{\partial^2 u}{\partial z \partial x}(T) \\ \frac{\partial^2 u}{\partial x \partial y}(T) & \frac{\partial^2 u}{\partial y^2}(T) & \frac{\partial^2 u}{\partial z \partial y}(T) \\ \frac{\partial^2 u}{\partial x \partial z}(T) & \frac{\partial^2 u}{\partial y \partial z}(T) & \frac{\partial^2 u}{\partial z^2}(T) \end{vmatrix}$$

Ako su Δ_1, Δ_2 i $\Delta_3 > 0$ => T je lokalni min.
Ako su $\Delta_1 < 0$, $\Delta_2 > 0$, $\Delta_3 < 0$ => T je lokalni maks.
U svim ostalim slučajevima, kad je Δ ≠ 0 nemamo ekstrem, a za Δ = 0 je potrebno dodatno ispitivanje.

Uvjetni ekstremi funkcija triju varijabli

a) Formirati tzv. Lagrangeovu funkciju

$$L(x, y, \lambda) = f(x, y) + \lambda \cdot \varphi(x, y)$$

b) Riješiti sustav:

$$\frac{\partial L}{\partial x}(x, y, \lambda) = 0$$

$$\frac{\partial L}{\partial y}(x, y, \lambda) = 0$$

$$\frac{\partial L}{\partial \lambda}(x, y, \lambda) = 0 \quad (\text{tj.}\, \varphi(x, y) = 0)$$

c) Za svako rješenje tog sustava izračunati determinantu $\Delta(x_0, y_0, \lambda_0)$

$$\Delta = \begin{vmatrix} \frac{\partial^2 L}{\partial x^2}(x_0, y_0, \lambda_0) & \frac{\partial^2 L}{\partial x \partial y}(x_0, y_0, \lambda_0) \\ \frac{\partial^2 L}{\partial x \partial y}(x_0, y_0, \lambda_0) & \frac{\partial^2 L}{\partial y^2}(x_0, y_0, \lambda_0) \end{vmatrix}$$

Ako je $\Delta(x_0, y_0, \lambda_0) > 0$ problem se dalje rješava kao u slučaju lok. ekstrema

Ako je $\Delta(x_0, y_0, \lambda_0) \leq 0$, izračunati determinantu:

$$\Delta = \begin{vmatrix} 0 & \frac{\partial \varphi}{\partial x}(x_0, y_0) & \frac{\partial \varphi}{\partial y}(x_0, y_0) \\ \frac{\partial \varphi}{\partial x}(x_0, y_0) & \frac{\partial^2 L}{\partial x^2}(x_0, y_0, \lambda_0) & \frac{\partial^2 L}{\partial x \partial y}(x_0, y_0, \lambda_0) \\ \frac{\partial \varphi}{\partial y}(x_0, y_0) & \frac{\partial^2 L}{\partial x \partial y}(x_0, y_0, \lambda_0) & \frac{\partial^2 L}{\partial y^2}(x_0, y_0, \lambda_0) \end{vmatrix}$$

Ako je $\Delta_1(x_0, y_0, \lambda_0) > 0$ => u $T(x_0, y_0)$ je maks.

Ako je $\Delta_1(x_0, y_0, \lambda_0) < 0$ => u $T(x_0, y_0)$ je min.

DVOSTRUKI INTEGRAL

$$\left.\begin{array}{l} S = \{(x,y)\epsilon\mathbb{R}^2: \quad a \le x \le b,\ \varphi_1(x) \le y \le \varphi_2(x)\} \\ \\ \varphi_1 \text{ i } \varphi_2 \text{ neprekidne na [a,b], } f(x,y) \text{ nepredkidna na S:} \\ \\ \iint\limits_S f(x,y)dxdy = \int\limits_a^b \left[\int\limits_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)dy\right]dx = \int\limits_a^b dx \int\limits_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)dy \end{array}\right\} \text{x je konst.}$$

$$\left.\begin{array}{l} S = \{(x,y)\epsilon\mathbb{R}^2: \quad c \le y \le d,\ \psi_1(x) \le y \le \psi_2(x)\} \\ \\ \psi_1 \text{ i } \psi_2 \text{ neprekidne na [c,d], } f(x,y) \text{ nepredkidna na S:} \\ \\ \iint\limits_S f(x,y)dxdy = \int\limits_c^d \left[\int\limits_{\psi_1(y)}^{\psi_2(y)} f(x,y)dx\right]dy = \int\limits_c^d dy \int\limits_{\psi_1(y)}^{\psi_2(y)} f(x,y)dx \end{array}\right\} \text{y je konst.}$$

Supstitucije

$x = x(u,v) \qquad y = y(u,v)$

$$\iint\limits_S f(x,y)dxdy = \iint\limits_{S\prime} f[x(u,v),y(u,v)] \cdot |J|dudv$$

a) polarne koordinate r i φ

$x = r\cos\varphi \qquad \begin{pmatrix} r=\sqrt{x^2+y^2} \\ tg\varphi=\frac{y}{x} \end{pmatrix}$

$y = r\sin\varphi$

$J = r$

b) eliptičke koordinate r i φ

$x = ar\cos\varphi \qquad \begin{pmatrix} r=\sqrt{\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2} \\ tg\varphi = \frac{ay}{bx} = \frac{\frac{y}{b}}{\frac{x}{a}} \end{pmatrix}$

$y = br\sin\varphi$

$J = abr$

Primjena dvostrukog integrala

a) Računanje površine

$$P = \iint_S dxdy$$

b) Izračunavanje volumena ograničenog neprekidnom plohom $z = f(x,y)$ iznad ograđenog prostora S u xy ravnini

$$S = \iint_S f(x,y)dxdy$$

c) Izračunavanje površine dijela glatke plohe $z = f(x,y)$ iznad ograđenog područja S u xy ravnini

$$S = \iint_S \sqrt{1 + (z'_x)^2 + \left(z'_y\right)^2} dxdy$$

TROSTRUKI INTEGRAL

$$V = \{(x,y,z): \quad a \le x \le b, \qquad \varphi_1(x) \le y \le \varphi_2(x), \;\; \psi_1(x,y) \le z \le \psi_2(x,y)\}$$

$$\iiint_V f(x,y,z)dxdydz = \iint_{V_{xy}} dxdy \int_{\psi_1(x,y)}^{\psi_2(x,y)} f(x,y,z)dz = \int_a^b dx \int_{\varphi_1(x)}^{\varphi_2(x)} dy \int_{\psi_1(x,y)}^{\psi_2(x,y)} f(x,y,z)dz$$

Supstitucije

$x = x(u,v,w) \qquad y(u,v,w) \qquad z(u,v,w)$

$$\iiint_V f(x,y,z)dxdydz = \iiint_{V\prime} f[x(u,v,w), y(u,v,w), z(u,v,w)] \cdot |J| dudvdw$$

a) cilindričke koordinate

$x = r\cos\varphi \qquad \varphi \epsilon [0,2\pi]$

$y = r\sin\varphi \qquad r\epsilon [0, +\infty >$

$z = z \qquad z\epsilon < -\infty, +\infty >$

$J = r$

b) sferne koordinate

$x = r\sin\theta\cos\varphi \qquad \varphi\epsilon[0,2\pi] \qquad \varphi = arctg\,\frac{y}{x}$

$y = r\sin\theta\sin\varphi \qquad \theta\epsilon[0,\pi] \qquad r = \sqrt{x^2 + y^2 + z^2}$

$z = r\cos\theta \qquad r\epsilon[0, +\infty > \qquad \cos\theta = \frac{\mathrm{z}}{\mathrm{r}}$

$J = r^2\sin\theta$

Primjena

a) volumen

$$V = \iiint_V dxdydz$$