Formule iz Numeričke matematike

Lagrangeov interpolacijski polinom (LIP):

$$L_n(x) = \sum_{i=0}^{n} f(x_i) \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j}.$$

Ocjena greške kojom $L_n(x)$ aproksimira funkciju $f(x)$ u točki x:

$$|f(x) - L_n(x)| \leq \frac{M_{n+1}}{(n+1)!} |(x - x_0)(x - x_1) \ldots (x - x_n)|.$$

$$|f^{(n+1)}(x)| \leq M_{n+1}, \quad \forall x \in [a, b].$$

Lagrangeov interpolacijski polinom za ekvidistantne čvorove:

$$L_n(x) = (-1)^n \frac{q(q-1)\cdots(q-n)}{n!} \sum_{i=1}^{n} (-1)^i \binom{n}{i} \frac{f(x_i)}{q - x_i}, \quad q = \frac{x - x_0}{h}.$$

1. Newtonov interpolacijski polinom (NIP):

$$\begin{aligned} L_n(x) &= f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1) + \ldots \\ &\quad + f[x_0, x_1, \ldots, x_n](x - x_0)(x - x_1) \ldots (x - x_{n-1}). \end{aligned}$$

1. NIP za ekvidistantne čvorove:

$$L_n(x) = y_0 + q\Delta y_0 + \frac{q(q-1)}{2!} \Delta^2 y_0 + \ldots + \frac{q(q-1)\ldots(q-n+1)}{n!} \Delta^n y_0.$$

$$q := \frac{x - x_0}{h}, \quad y_i := f(x_i), \quad i = 0, \ldots, n.$$

2. Newtonov interpolacijski polinom (NIP):

$$\begin{aligned} L_n(x) &= f[x_n] + f[x_{n-1}, x_n](x - x_n) + f[x_{n-2}, x_{n-1}, x_n](x - x_n)(x - x_{n-1}) \\ &\quad + \ldots + f[x_0, x_1, \ldots, x_n](x - x_n)(x - x_{n-1}) \ldots (x - x_1). \end{aligned}$$

2. NIP za ekvidistantne čvorove:

$$L_n(x) = y_n + q\Delta y_{n-1} + \frac{q(q+1)}{2!} \Delta^2 y_{n-2} + \ldots + \frac{q(q+1)\ldots(q+n-1)}{n!} \Delta^n y_0.$$

gdje je

$$q = \frac{x - x_n}{h}.$$

Polinom najmanjih kvadrata:

$$\begin{array}{ccccccc} S_0 a_0 & + & S_1 a_1 & + \ldots + & S_m a_m & = & \sum_{i=0}^{n} y_i \\ S_1 a_0 & + & S_2 a_1 & + \ldots + & S_{m+1} a_m & = & \sum_{i=0}^{n} y_i x_i \\ & & & \vdots & & & \\ S_m a_0 & + & S_{m+1} a_1 & + \ldots + & S_{2m} a_m & = & \sum_{i=0}^{n} y_i x_i^m, \end{array}$$

gdje je

$$S_0 := n + 1, \qquad S_k := \sum_{i=0}^{n} x_i^k, \quad k = 1, 2, \ldots, 2m.$$

Trapezna formula:

$$T_n = h\left(\frac{y_0}{2} + y_1 + \ldots + y_{n-1} + \frac{y_n}{2}\right).$$

$$|R_n| \leq \frac{(b-a)^3}{12n^2} M_2.$$

Simpsonova formula:

$$S_n = \frac{h}{3} [y_0 + y_n + 4(y_1 + y_3 + \ldots + y_{n-1}) + 2(y_2 + y_4 + \ldots + y_{n-2})].$$

$$|R_n| \leq \frac{(b-a)^5}{180n^4} M_4.$$

Triosminsko pravilo:

$$I_n = \frac{3h}{8} [y_0 + y_{3m} + 2(y_3 + y_6 + \ldots + y_{3m-3}) + 3(y_1 + y_2 + y_4 + y_5 + \ldots + y_{3m-2} + y_{3m-1})].$$

$$h = \frac{b-a}{n} = \frac{b-a}{3m},$$

$$|R_n| \leq \frac{(b-a)^5}{80n^4} M_4$$

Rombergov algoritam:

$$T_{2^k m}^{(k)} := \frac{1}{4^k - 1} (4^k T_{2^k m}^{(k-1)} - T_{2^{k-1} m}^{(k-1)}), \quad \text{za} \quad m = n, 2n, 4n, 8n, \ldots, \quad k = 0, 1, 2, \ldots$$

$$\begin{array}{ccccc} T_n & & & & \\ T_{2n} & T_{2n}^{(1)} & & & \\ T_{4n} & T_{4n}^{(1)} & T_{4n}^{(2)} & & \\ T_{8n} & T_{8n}^{(1)} & T_{8n}^{(2)} & T_{8n}^{(3)} & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

$$T_{2n} = h(y_1 + y_3 + \ldots + y_{2n-1}) + \frac{1}{2} T_n, \quad \text{uz} \quad h = \frac{b-a}{2n}.$$

Gauss Legendreove formule:

$$\int_{-1}^{1} f(x)\, dx = \sum_{j=1}^{n} w_j f(x_j) + E_n(f).$$

Legendreovi polinomi:

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n$$

$$w_j = \frac{-2}{(n+1) P_n'(x_j) P_{n+1}(x_j)}, \quad j = 1, 2, \ldots, n.$$

$$E_n(f) = \frac{2^{2n+1} (n!)^4}{(2n+1)[(2n)!]^3} \cdot f^{(2n)}(\eta) \quad \text{za neki} \quad \eta \in [-1, 1].$$

$$\int_{a}^{b} f(x)\, dx = \frac{b-a}{2} \sum_{j=1}^{n} w_j f(x_j) + E_n^*(f).$$

gdje je

$$x_j = \frac{b+a}{2} + \frac{b-a}{2}x'_j, \quad E_n^*(f) = \left(\frac{b-a}{2}\right)^{2n+1} E_n(f).$$

i $\{x'_j\}$ su Gauss–Legendreovi čvorovi na intervalu $[-1, 1]$.
Gaussova metoda eliminacije:

$$A_p := [A, b], \quad a_{i,n+1} := b_i, \quad i = 1, 2, \ldots, n.$$

$$\begin{aligned} a_{ij}^{(0)} &:= a_{ij}, & & i = 1, \ldots, n, \\ & & & j = 1, \ldots, n+1; \\ a_{ij}^{(k)} &:= a_{ij}^{(k-1)} - a_{kj}^{(k-1)} \frac{a_{ik}^{(k-1)}}{a_{kk}^{(k-1)}}, \quad k = 1, \ldots, n-1, & & i = k+1, \ldots, n, \\ & & & j = k+1, \ldots, n+1. \end{aligned}$$

$$x_i = \frac{1}{a_{ii}^{(i-1)}} \left[a_{i,n+1}^{(i-1)} - \sum_{j=i+1}^{n} a_{ij}^{(i-1)} x_j \right], \quad i = n, n-1, \ldots, 1.$$

Gaussova metoda s parcijalnim pivotiranjem:

$$c_k = \max_{k \le i \le n} |a_{ik}^{(k-1)}|.$$

Gaussova metoda s kompletnim pivotiranjem:

$$c_k = \max_{k \le i, j \le n} |a_{ij}^{(k-1)}|.$$

Metoda Choleskog:

$$\begin{aligned} b_{i1} &= a_{i1}, \quad i \ge 1, \\ b_{ij} &= a_{ij} - \sum_{k=1}^{j-1} b_{ik} c_{kj}, \quad i \ge j > 1, \\ c_{1j} &= \frac{a_{1j}}{b_{11}}, \quad j > 1, \\ c_{ij} &= \frac{1}{b_{ii}} \left(a_{ij} - \sum_{k=1}^{i-1} b_{ik} c_{kj} \right), \quad 1 < i < j. \end{aligned}$$

$$y_1 = \frac{b_1}{b_{11}}, \qquad y_i = \frac{1}{b_{ii}} \left(b_i - \sum_{k=1}^{i-1} b_{ik} y_k \right), \quad i > 1.$$

$$\begin{aligned} x_n &= y_n, \\ x_i &= y_i - \sum_{k=i+1}^{n} c_{ik} x_k, \quad i = n-1, \ldots, 1. \end{aligned}$$

Vektorske norme:

$$\|x\|_1 = \sum_{i=1}^{n} |x_i|, \qquad \|x\|_2 = \left(\sum_{i=1}^{n} x_i^2 \right)^{\frac{1}{2}}, \qquad \|x\|_\infty = \max_{1 \le i \le n} |x_i|.$$

Matrične norme:

$$\|A\|_1 = \max_{1 \le j \le n} \sum_{i=1}^{n} |a_{ij}|, \quad \|A\|_2 = \sqrt{\rho(AA^*)}, \quad \|A\|_\infty = \max_{1 \le i \le n} \sum_{j=1}^{n} |a_{ij}|.$$

Iterativne metode:

$$Ax = b \Leftrightarrow x = Tx + d.$$

Ocjena greške:

$$\|\xi - x^{(k)}\| \le \frac{\|T\|}{1 - \|T\|} \|x^{(k)} - x^{(k-1)}\| < \varepsilon,$$

$$\|\xi - x^{(k)}\| \le \frac{\|T\|^k}{1 - \|T\|} \|x^{(1)} - x^{(0)}\| < \varepsilon.$$

Jacobijeva metoda:

$$\begin{aligned} x_i^{(k+1)} &= \frac{1}{a_{ii}} [b_i - \sum_{\substack{j=1 \\ j \ne i}}^{n} a_{ij} x_j^{(k)}], \quad i = 1, \ldots, n, \\ x_i^{(0)} &= \frac{b_i}{a_{ii}}. \end{aligned}$$

Gauss–Seidelova metoda:

$$x_i^{(k+1)} = -\frac{1}{a_{ii}} \left[\sum_{j=1}^{i-1} a_{ij} x_j^{(k+1)} + \sum_{j=i+1}^{n} a_{ij} x_j^{(k)} - b_i \right], \quad i = 1, \ldots, n.$$

Karakteristični polinom:

$$D(\lambda) = \det(\lambda I - A) = \lambda^n + p_1 \lambda^{n-1} + \ldots + p_n.$$

Metoda Danielvskog:

$$P = \begin{bmatrix} \sigma_1 & \sigma_2 & \cdots & \sigma_{n-1} & \sigma_n \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$

$$P = M_1^{-1} M_2^{-1} \ldots M_{n-2}^{-1} M_{n-1}^{-1} A M_{n-1} M_{n-2} \ldots M_2 M_1.$$

$$p_i = -\sigma_i, \quad i = 1, 2, \ldots, n.$$

Leverrierova metoda:

$$s_k = \lambda_1^k + \ldots + \lambda_n^k, \quad k = 1, 2, \ldots, n.$$

$$s_k + p_1 s_{k-1} + p_2 s_{k-2} + \ldots + p_{k-1} s_1 = -k p_k, \quad k = 1, 2, \ldots, n.$$

Metoda raspolavljanja:

$$\xi - a_n \le \frac{b-a}{2^n} \quad \text{(ocjena greške)}.$$

Metoda iteracije:

$$x_{n+1} = \varphi(x_n), \quad x_0 \in [a, b] \text{ proizvoljan}$$

Ocjena greške za metodu iteracije:

$$\begin{aligned} |\xi - x_n| &\le \frac{q^n}{1-q} |x_1 - x_0| < \varepsilon, \\ |\xi - x_n| &\le \frac{q}{1-q} |x_n - x_{n-1}| < \varepsilon. \end{aligned}$$

Newtonova metoda:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2, \ldots$$

$$f(x_0) f''(x_0) > 0$$

Ocjena greške za Newtonovu metodu:

$$|\xi - x_n| \le \frac{M_2}{2m_1} |x_n - x_{n-1}|^2 < \varepsilon.$$