CMSE 820 - Mathematical Foundations of Data Science

Chris Gerlach

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Question 1

Show that the kernel function associated with any reproducing kernel Hilbert space must be unique. Namely, given a real RKHS $\mathcal{H}$, if there are two kernels, k_1 and k_2, associated with $\mathcal{H}$ (with the reproducing property), then $k_1(x, y) = k_2(x, y)$ for all $x, y \in X$.

Let $\mathcal{H}$ be a real reproducing kernel Hilbert Space with two distinct kernels such that for all $x \in X$, $k_1(x, \cdot), k_2(x, \cdot) \in \mathcal{H}$. Additionally, for any $f \in \mathcal{H}$,

$$\langle f, k_1(x, \cdot)\rangle_{\mathcal{H}} = \langle f, k_2(x, \cdot)\rangle_{\mathcal{H}} = f(x).$$

Let $y \in X$. For $k_1(x, \cdot)$

$$\langle k_2(x, \cdot), k_1(y, \cdot)\rangle_{\mathcal{H}} = k_1(y, x)$$

and for $k_2(x, \cdot)$

$$\langle k_1(x, \cdot), k_2(y, \cdot)\rangle_{\mathcal{H}} = k_2(y, x).$$

Then

$$k_2(y, x) = \langle k_1(x, \cdot), k_2(y, \cdot)\rangle_{\mathcal{H}} = \langle k_2(y, \cdot), k_1(x, \cdot)\rangle_{\mathcal{H}} = k_1(x, y).$$

By symmetry of the kernel function, $k_1(x, y) = k_2(x, y)$.

Question 2

Let $k : \mathbb{X} \times \mathbb{X} \to \mathbb{R}$ be a positive semidefinite kernel, and let $f : \mathbb{X} \to [0, \infty)$ be an arbitrary function. Show that $\tilde{k}(x, y) = f(x)k(x, y)f(y)$ is also a positive semidefinite kernel.

Let $k : \mathbb{X}\times\mathbb{X} \to \mathbb{R}$ be a positive semidefinite kernel, and let $f : \mathbb{X} \to [0, \infty)$ be an arbitrary function. Let $x_{i}{}_{i=1}^{n} \in X$. Then

$$\begin{aligned}
\sum_{i=1}^{n}\sum_{j=1}^{n} c_i c_j \tilde{k}(x_i, x_j) &= \sum_{i=1}^{n}\sum_{j=1}^{n} c_i c_j f(x_i) k(x_i, x_j) f(x_j) \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} d_i d_j k(x, y) \geq 0
\end{aligned}$$

since k is positive semidefinite and where $d_k = c_k f(x_k)$ for $k = \{i, j\}$. Therefore, $\tilde{k}$ is positive semidefinite.

Question 3

Show that the kernel $k : [0,1] \times [0,1] \to \mathbb{R}$ given by $k(x,z) = \min\{x,z\}$ is positive semidefinite. Hint:

$$\min\{x,z\} = \int_0^1 I(t \le x) I(t \le z) dt,$$

where $I(t)$ is an indicator function of the interval $[0,t]$.

Let the kernel $k : [0,1] \times [0,1] \to \mathbb{R}$ be given by $k(x,z) = \min\{x,z\}$. Let $x_{i}{}_{i=1}^{n} \in X$.

$$\begin{aligned}
\sum_{i=1}^{n} \sum_{j=1}^{n} c_i c_j k(x_i, x_j) &= \sum_{i=1}^{n} \sum_{j=1}^{n} c_i c_j \int_0^1 I(t \le x_i) I(t \le x_j) dt \\
&= \int_0^1 \sum_{i=1}^{n} \sum_{j=1}^{n} c_i I(t \le x_i) c_j I(t \le x_j) dt \\
&= \int_0^1 \left(\sum_{i=1}^{n} c_i I(t \le x_i) \right)^2 dt \ge 0
\end{aligned}$$

Question 4

A function f over $[0,1]$ is said to be absolutely continuous if its derivative f' satisfies $\int_0^1 f'(x)dx \le \infty$, and we have $f(x) = f(0) + \int_0^x f'(z)dz$ for all $x \in [0,1]$. Now consider the set of functions,

$$\mathcal{H}_1[0,1] = \left\{ f : [0,1] \to \mathbb{R} \mid f(0) = 0, \text{ and } f \text{ is absolutely continuous with } f' \in L^2([0,1]) \right\}.$$

We define an inner product on this space via $\langle f, g \rangle_{\mathcal{H}_1} = \int_0^1 f'(z) g'(z) dz$, and claim that the resulting Hilbert space is an RKHS. You can prove it by the following steps:

Prove the representer of evaluation at $x \in [0,1]$ for $\mathcal{H}_1$ is $g_x(z) = \min\{x, z\}$. Namely, you need to prove $g_x \in \mathcal{H}_1$ and $\langle g_x, f\rangle_{\mathcal{H}_1} = f(x)$ for any $f \in \mathcal{H}_1$.

Let $g_x(z) = \min\{x, z\}$. Then we have the following:

1. $g_x : [0,1] \to \mathbb{R}$
2. $g_x(0) = \min\{x, 0\} = 0$ since $x \in [0,1]$
3. $g_x'(z) = \begin{cases} 0 & \text{if } z > x \\ 1 & \text{if } z \geq x \end{cases} \implies g_x' \in L^2([0,1])$
4. g_x is absolutely continuous because it is continuous and piecewise differantiable.

Thus, $g_x \in \mathcal{H}_1$. Furthermore, let $f \in \mathcal{H}_1$. Then

$$\langle g_x, f\rangle_{\mathcal{H}_1} = \int_0^1 g_x'(z) f'(z) dz = \int_0^x f'(z) dz = f(x) - f(0) = f(x)$$

by the Fundamental Theorem of Calculus and the properties of $\mathcal{H}_1$.

Define the evaluation functional δ_x as $\delta_x(f) = \langle f, g_x\rangle_{\mathcal{H}_1}$ and prove that it is continuous or bounded.

$$||\delta_x(f)|| = ||\langle f, g_x\rangle_{\mathcal{H}_1}|| = ||f(x)|| < \infty$$

since $f \in \mathcal{H}_1$.