MATH401/501 spec Exam - solutions

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1

```
## Quick confirmation that the density integrates to 1
ymax=10; ny=10^6; ymin=0.5/ny;
ys=seq(ymin,ymax,len=ny)
theta=2
fs=ys^5/theta^6*exp(-ys^2/theta^2)
dy=(ymax-ymin)/(ny-1)
dy*sum(fs)-0.5*(fs[1]+fs[ny])*dy ## Trapezium rule
```

```
## [1] 1
```

Note: If $X \sim Gam(3, \lambda)$ then $Y = X^{1/2}$ and $\theta = 1/\lambda^{1/2}$.

```
## Generate data
set.seed(12379)
thetastar=2; lambda=1/thetastar^2
n=100
xs=rgamma(n,shape=3,rate=lambda); ys=sqrt(xs)
S=round(sum(ys^2),1)
round(c(sum(ys),S,sum(exp(ys))),1)
```

```
## [1]  341.2 1248.0 4533.7
```

(a)

$$L(\theta;\mathbf{y}) \propto \theta^{-6n} \exp\left\{-\theta^{-2}\sum_{i=1}^{n} y_i^2\right\}$$

$$\ell(\theta;\mathbf{y}) = -6n\log\theta - \theta^{-2}\sum_{i=1}^{n} y_i^2 + \text{cst}$$

(4 marks)

$$u(\theta;\mathbf{y}) = \frac{-6n}{\theta} + \frac{2}{\theta^3}S,$$

where $S = \sum_{i=1}^{n} y_i^2$. *(2 marks)*

So, $\widehat{\theta} = (S/(3n))^{1/2}$. *(2 marks)*.

(b)

(i)

$$I_O(\theta;\mathbf{y}) = -\frac{6n}{\theta^2} + \frac{6}{\theta^4}S.$$

(4 marks)

(ii)

$\mathbb{E}\left[\sum_{i=1}^{n} Y_i^2\right] = n\theta^2 \times 3!/2 = 3n\theta^2$ *(2 marks)* so

$$I_E(\theta) = -\frac{6n}{\theta^2} + \frac{6}{\theta^4} \times 3n\theta^2 = \frac{12n}{\theta^2},$$

as required *(2 marks)*.

(c)

(i)

```r
thetahat=(S/(3*n))^0.5
IE=12*n/thetahat^2
SE=1/sqrt(IE)
c(thetahat,SE) ## for partial marks
```

```
## [1] 2.03960781 0.05887841
```

```r
CI=thetahat+qnorm(0.975)*c(-1,1)*SE
CI
```

```
## [1] 1.924208 2.155007
```

(4 marks)

(ii)

$E[Y^2] = 0.5\theta^2\Gamma(4) = 3\theta^2$, which is *monotonic* in θ since θ is positive, so the CI is

```r
3*CI^2
```

```
## [1] 11.10773 13.93217
```

(4 marks)

(d)

(i)

```r
theta0=2.2
siglevel=0.01
TWald=(thetahat-theta0)/SE
Tcrit=qnorm(1-siglevel/2)
pWald=2*pnorm(-abs(TWald)) ## only possible if online
c(TWald,Tcrit,pWald)
```

```
## [1] -2.724125975  2.575829304  0.006447191
```

Reject H_0 as $|T_{Wald}| \geq T_{crit}$. *(4 marks)*

(ii)

```
u=-6*n/theta0+2*S/(theta0^3)
IE0=12*n/theta0^2
Tscore=u^2/IE0
Tcrit=qchisq(1-siglevel,df=1)
pscore=pchisq(Tscore,df=1,lower.tail=FALSE) ## only possible if online
c(Tscore,Tcrit,pscore)
```

```
## [1] 5.9217267 6.6348966 0.0149553
```

Fail to reject H_0 as $T_{score} < T_{crit}$. *(4 marks)*

(iii)

```
u=-6*n/theta0+2*S/(theta0^3)
Tscore=u^2/IE
Tcrit=qchisq(1-siglevel,df=1)
pscore=pchisq(Tscore,df=1,lower.tail=FALSE) ## only possible if online
c(Tscore,Tcrit,pscore)
```

```
## [1] 5.08974853 6.63489660 0.02406768
```

Fail to reject H_0 *(4 marks)*

(e)

The score test using $I_E(\theta_0)$ does not require calculation of the MLE. *(2 marks)*

2

Note: $Y \sim Gamma(\alpha, \theta)$ and $Z \sim Weibull(2, \theta)$.

```
## Simulate some data
set.seed(123579)
n=150; m=50; alpha=3; theta=0.5
ys=rgamma(n,alpha,rate=theta);
zs=sqrt(rexp(m,rate=theta^2))
```

```r
S0=round(sum(log(ys)),2); S1=round(sum(ys),2); S2=round(sum(zs^2),2)
c(S0,S1,S2)
```

```
## [1] 240.28 925.64 190.05
```

```r
Q2log.like<-function(pars,ys,zs) {
  n=length(ys); m=length(zs); alpha=pars[1]; theta=pars[2]
  if ((alpha>0) && (theta>0)) {
    S0=sum(log(ys)); S1=sum(ys); S2=sum(zs^2)
    l=(n*alpha+2*m)*log(theta)-n*lgamma(alpha)+alpha*S0-theta*S1-theta^2*S2
  }
  else {
    l=-Inf
  }
  return(l)
}
```

(a)

$$L(\alpha,\theta;\mathbf{y}) \propto \frac{\theta^{n\alpha}}{\Gamma(\alpha)^n}\left(\prod_{i=1}^n y_i\right)^{\alpha} \exp\left\{-\theta\sum_{i=1}^n y_i\right\} \times \theta^{2m}\exp\left\{-\theta^2\sum_{i=1}^m z_i^2\right\}.$$

$$\implies \ell(\alpha,\theta;\mathbf{y},\mathbf{z}) = -n\log\Gamma(\alpha) + (n\alpha+2m)\log\theta + \alpha\sum_{i=1}^n \log y_i - \theta\sum_{i=1}^n y_i - \theta^2\sum_{i=1}^m z_i^2 + \text{cst.}$$

as required *(2 marks)*

$$\frac{\partial \ell}{\partial\theta} = \frac{\alpha n+2m}{\theta} - \sum_{i=1}^n y_i - 2\theta\sum_{i=1}^n z_i^2.$$

So

$$2\widehat{\theta}_\alpha^2 S_z + \widehat{\theta}_\alpha S_y - (\alpha n + 2m) = 0,$$

where $S_z = \sum_{i=1}^n z_i^2$ and $S_y = \sum_{i=1}^n y_i$. Thus

$$\widehat{\theta}_\alpha = \frac{-S_y + \sqrt{S_y^2 + 8S_z(\alpha n+2m)}}{4S_z},$$

where we have taken the positive root because θ is positive. *(6 marks)*

(b)

```r
ctl=list(fnscale=-1)
a=optim(c(1,1),Q2log.like,ys=ys,zs=zs,hessian=TRUE,control=ctl); a
```

```
## $par
## [1] 2.8447060 0.4760389
##
## $value
## [1] -274.3389
##
## $counts
## function gradient
##       73       NA
##
## $convergence
## [1] 0
##
## $message
## NULL
##
## $hessian
##            [,1]       [,2]
## [1,] -63.05874   315.1008
## [2,] 315.10076 -2704.3820
```

(i)

$$\frac{\partial^2 \ell}{\partial \theta^2} = -\frac{\alpha n + 2m}{\theta^2} - 2\sum_{i=1}^{m} z_i^2.$$

Substituting $\widehat{\alpha}$ for α, $\widehat{\theta}$ for θ etc, we obtain:

```r
alphahat=a$par[1]; thetahat=a$par[2]
-(n*alphahat+2*m)/(thetahat^2)-2*S2
```

```
## [1] -2704.352
```

(4 marks)

(ii)

Var($\widehat{\alpha}$) is the (1,1) entry of I_O^{-1}:

```
IO=-a$hessian
V=solve(IO)[1,1]; SE=sqrt(V) ## in-person exam: use (2x2) mx inverse formula
c(V,SE)
```

```
## [1] 0.03795814 0.19482848
```

```
CI=alphahat+c(-1,1)*qnorm(0.995)*SE
CI
```

```
## [1] 2.342861 3.346551
```

(6 marks)

(iii)

```
coeffs=c(1,-4)
MLElc=sum(a$par*coeffs)
Vlc=(t(coeffs) %*% solve(IO) %*% coeffs)[1,1] ## turn from 1x1 mx to scalar
SElc=sqrt(Vlc)
c(MLElc,SElc)
```

```
## [1] 0.9405504 0.1293750
```

```
CIlc=MLElc+c(-1,1)*qnorm(0.995)*SElc
CIlc
```

```
## [1] 0.6073024 1.2737984
```

The MLE is $\widehat{\alpha} - 4\widehat{\theta} = 0.9405504$. *(4 marks)* The variance of the linear combination is $V_{1,1} + 2 \times 1 \times (-4)V_{1,2} + 16V_{2,2} = 0.0167379$. *(4 marks)*

Which gives a CI of 0.6073024, 1.2737984. *(2 marks)*

(c)

```
alpha0=3.3
thetahatalpha0=sqrt(S1^2+8*S2*(alpha0*n+2*m))-S1
thetahatalpha0=thetahatalpha0/(4*S2)
```

From (a)(ii)

$$\widehat{\theta}_\alpha = \frac{-S_y + \sqrt{S_y^2 + 8S_z(\alpha n + 2m)}}{4S_z} = 0.5282233.$$

(4 marks)

So we need $\ell(3.3, 0.5282233)$:

```
l0=Q2log.like(c(alpha0,thetahatalpha0),ys,zs)
TWilks=2*(a$val-l0)
Tcrit=qchisq(0.975,df=1)
pWilks=pchisq(TWilks,df=1,lower.tail=FALSE) ## only if online (& is unnecessa
c(l0,TWilks,Tcrit,pWilks)
```

```
## [1] -276.86373341    5.04970686    5.02388619    0.02463011
```

Reject (just) H_0 as $T_{Wilks} \geq T_{crit}$.

(6 marks)

3

(a)

(i)

$$\mathbf{Y} \sim N_n(X\boldsymbol{\beta}, \sigma^2 I_n),$$

where I_n is the $n \times n$ identity matrix.

(ii)

$$\begin{aligned}
Var\left[(X^\top X)^{-1}X^\top \mathbf{Y}\right] &= (X^\top X)^{-1}X^\top Var\left[\mathbf{Y}\right]\left\{(X^\top X)^{-1}X^\top\right\}^\top \\
&= \sigma^2 (X^\top X)^{-1}X^\top I_d X(X^\top X)^{-1} \\
&= \sigma^2 (X^\top X)^{-1}
\end{aligned}$$

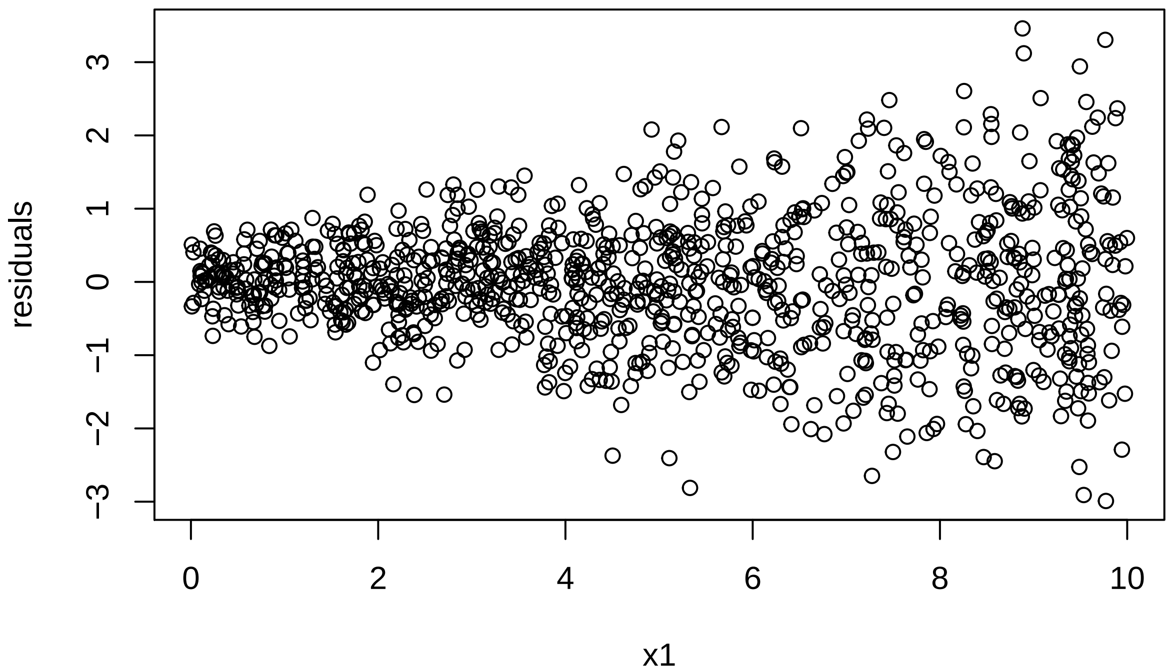
(6 marks) because $(X^\top X)^{-1}$ is symmetric *(2 marks)*.

(b)

```r
## Generate data
n=1000
set.seed(1235713)
x1=runif(n)*10
x3us=runif(n)
x2us=runif(n)/2+x3us/2
T21=x2us<0.5;T22=x2us>0.5
T31=x3us<0.2; T33=x3us>0.7; T32=!T31 & !T33
x2=rep("No",n); x2[T22]="Yes"
x3=rep("R",n); x3[T32]="G"; x3[T33]="B"
X=cbind(rep(1,n), x1, T22, T32, T33, T22*T32)
betas=c(6, 0.7,1.0, 2,3, -1.5)
sigma=0.8
etas=X %*% betas
Q3y=etas+sigma*rnorm(n)*(0.375+x1/8)
Mb=lm(Q3y~x1+x2+x3+x2*x3)
Sb=summary(Mb)
Sb$coeff
```

```
##                Estimate  Std. Error     t value     Pr(>|t|)
## (Intercept)  8.9584809 0.147068762  60.9135535 0.000000e+00
## x1           0.7133570 0.009844503  72.4624690 0.000000e+00
## x2Yes        0.9055976 0.148985487   6.0784284 1.728214e-09
## x3G         -0.9822243 0.147983942  -6.6373708 5.244480e-11
## x3R         -2.9402937 0.153737248 -19.1254481 1.157324e-69
## x2Yes:x3G   -1.5021881 0.169362111  -8.8696822 3.348565e-18
## x2Yes:x3R    0.1110128 0.273643059   0.4056846 6.850617e-01
```

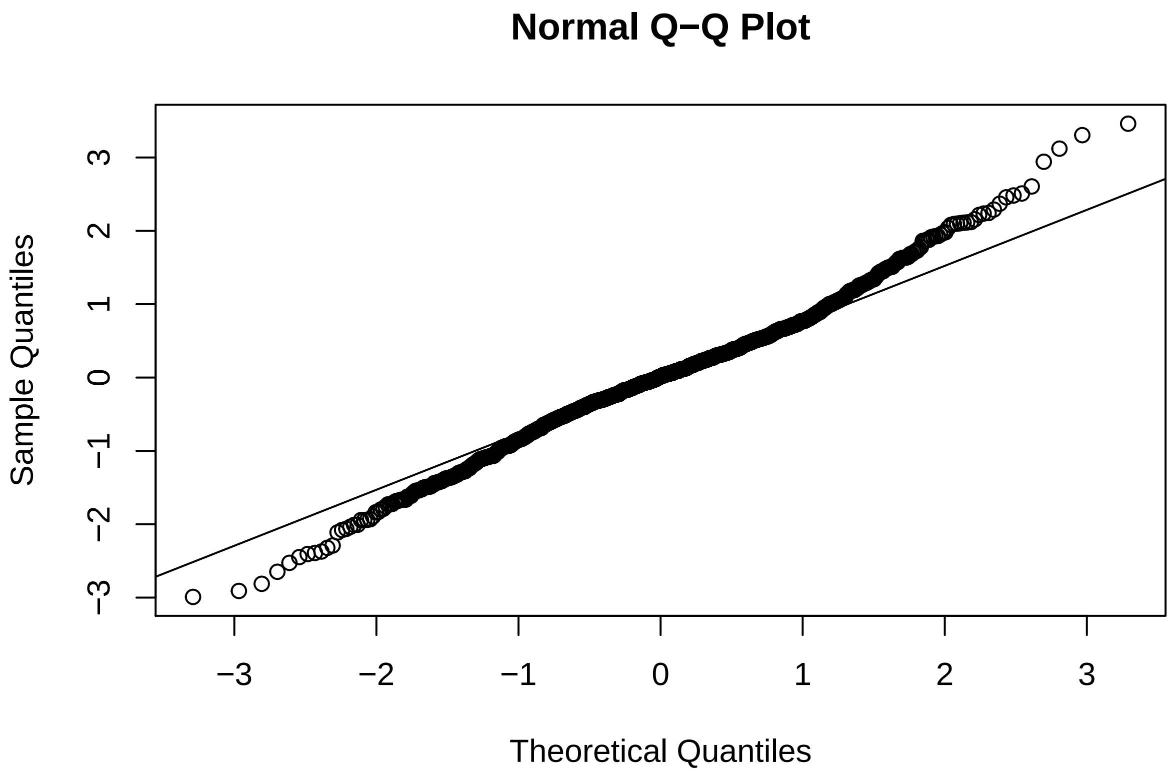
```r
resids=residuals(Mb)
plot(x1,resids,ylab="residuals")
```




```
dev.print(pdf,file="specFig1.pdf")
```

```
## pdf
##   2
```

```
qqnorm(resids)
qqline(resids)
```




```r
dev.print(pdf,file="specFig2.pdf")
```

```
## pdf
##   2
```

Note to student: data values here are slightly different to those appearing in the specimen exam. Answers are for the exam values.

(i)

For an increase of x_1 by 1, $\mathbb{E}[Y]$ increases by 0.71m (or y increases by 0.71m on average).

(ii)

When x_2 =No, y is 0.93m lower when x_3 =G than when x_3 =B, and is 2.87m lower when x_3 =R than when x_3=B *(2 marks)*. When x_2=Yes, y is 2.46m (0.93+1.53) lower when x_3 =G than when x_3 =B, and is 2.94m (2.868+0.069) lower (OR essentially the same discrepancy as when x_2=No) when x_3 =R than when x_3=B *(4 marks)*. Finally, when x_3 =B, y is 0.93m higher when x_2 =Yes than when x_2 =No. *(2 marks)* (lose two marks for inappropriate number of dp e.g. none or ≥ 4.

How to decide on dp?: look at the standard errors. For example, there is no point giving a precision which is a thousand times smaller than the uncertainty arising from the variability in the data. Depending on the context - the use to which the numbers will be put - it could also be crazy to give a precision which is worse than this uncertainty (or even the same size). Please use common sense.

(iii)

The residual variance increases with x_1 *(2 marks)*, this means that the overall distribution has fatter tails than a standard Gaussian, as is shown in the QQplot *(2 marks)*. Specify (something like) $\epsilon_i \sim N(0, \sigma^2 \exp(\gamma x_{1,i})$. Or, more generally, some monotonically increasing variance function $\sigma^2 f(x_3; \gamma)$. *(2 marks)* The log-likelihood is then

$$-n \log \sigma - \frac{1}{2}\sum_{i=1}^{n} \log f(x_{3,i};\gamma) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}\frac{(y_i - \eta_i)^2}{f(x_{3,i};\gamma)}.$$

(6 marks)

(c)

```
## Generate data
set.seed(321)
n=75
x1=runif(n)*3-1
x2=runif(n)*2-1.5
betas=c(-1,.2,-.06)
sigma=1
X=cbind(rep(1,n),x1,x2)
logity=X %*% betas+rnorm(n)*sigma
y=1/(1+exp(-logity))
Mc=lm(log(y/(1-y))~x1+x2)
Sc=summary(Mc)
Sc$coeff
```

```
##                Estimate Std. Error    t value     Pr(>|t|)
## (Intercept) -0.7776230  0.1546968 -5.0267546 3.516277e-06
## x1           0.1140109  0.1287506  0.8855178 3.788254e-01
## x2           0.1788088  0.1993881  0.8967876 3.728211e-01
```

(i)

Y must be between 0 and 1 *(2 marks)*, and the logit transform ensures this *(2 marks)*.

(ii)

```r
predlin=Sc$coeff[1,1]
pred=1/(1+exp(predlin))
pred
```

```
## [1] 0.6851676
```

(2 marks)

```r
CIlin=predlin+c(-1,1)*Sc$coeff[1,2]*qnorm(0.975)
CI=1/(1+exp(-CIlin))
CIlin
```

```
## [1] -1.0808233 -0.4744228
```

```r
CI
```

```
## [1] 0.2533503 0.3835700
```

(6 marks)

(iii)

$Y_i \sim Beta(a_i, b_i)$, *(2 marks)*. One sensible possibility is: $a_i = c\exp(\eta_i)$ and $b_i = c$, where $\eta_i = \mathbf{x}_i^\top \boldsymbol{\beta}$. Then $\mathbb{E}[Y_i] = \exp(\eta_i)/[1+\exp(\eta_i)]$. The point is that there is a linear predictor *(2 marks)*, the parameters have the right range - they are positive *(2 marks)* and the expectation increases monotonically with the linear predictor *(2 marks)*. In the particular example given, the extra parameter c controls the variance of Y_i. **NB**: the solution in the original version of this document, which I started to go through in the revision session, then queried, was indeed wrong.