2022 EXAMINATIONS

PART II – Fourth Year / MSc

MATHEMATICS & STATISTICS

MATH457/MATH557: Computationally Intensive Methods **3 hours**

You should answer ALL questions. There is a total of 100 marks (Q1. ??, Q2. ??, Q3. ??).

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Formula sheet

Standard distributions

The **normal distribution**, also known as the Gaussian distribution, with a parameters of μ and $\sigma > 0$ has a density of

$$f(y) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{1}{2\sigma^2}(y-\mu)^2\right\} \quad (-\infty < y < \infty).$$

A random variable $Y \sim \mathsf{N}(\mu, \sigma^2)$ has: $\mathbb{E}[Y] = \mu$ and $\mathsf{Var}[Y] = \sigma^2$.

The **exponential distribution** with a rate parameter of $\beta > 0$ has a density of

$$f(y) = \beta \exp(-\beta y) \quad (y > 0).$$

A random variable $Y \sim \mathsf{Exp}(\beta)$ has: $\mathbb{E}[Y] = 1/\beta$ and $\mathsf{Var}[Y] = 1/\beta^2$.

The **gamma distribution** with a shape parameter of $\alpha > 0$ and a rate parameter of $\beta > 0$ has a density of

$$f(y) = \frac{1}{\Gamma(\alpha)}\beta^\alpha y^{\alpha-1} \exp(-\beta y) \quad (y > 0).$$

A random variable $Y \sim \mathsf{Gamma}(\alpha, \beta)$ has: $\mathbb{E}[Y] = \alpha/\beta$ and $\mathsf{Var}[Y] = \alpha/\beta^2$.

The **beta distribution** with shape parameters of $\alpha > 0$ and $\beta > 0$ has a density of

$$f(y) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} y^{\alpha-1}(1-y)^{\beta-1} \quad (0 < y < 1).$$

A random variable $Y \sim \mathsf{Beta}(\alpha, \beta)$ has: $\mathbb{E}[Y] = \alpha/(\alpha+\beta)$ and

$$\mathsf{Var}[Y] = \frac{\alpha\beta}{(\alpha+\beta)^2(1+\alpha+\beta)}.$$

The **non-standardised univariate Student-t distribution** with parameters of μ and $\lambda > 0$ and ν degrees of freedom has a density of

$$f(y) = \frac{\Gamma(\nu/2 + 1/2)}{\sqrt{\nu\pi\lambda^2}\Gamma(\nu/2)} \left\{1 + \frac{(y-\mu)^2}{\nu\lambda^2}\right\}^{-\frac{\nu+1}{2}} \quad (-\infty < y < \infty).$$

A random variable $Y \sim \mathsf{t}_\nu(\mu, \lambda)$ has: $\mathbb{E}[Y] = \mu$ provided $\nu > 1$.

The **Poisson distribution** with a mean parameters of $\lambda > 0$ has a mass function of

$$f(y) = \frac{\lambda^y}{y!} \exp(-\lambda) \quad (y = 0, 1, 2, \dots).$$

A random variable $Y \sim \mathsf{Pois}(\lambda)$ has: $\mathbb{E}[Y] = \lambda$ and $\mathsf{Var}[Y] = \lambda$.

The **binomial distribution** with a probability parameters of $p \in [0,1]$ and a number of trials, n, has a mass function of

$$f(y) = \frac{n!}{y!(n-y)!} p^y (1-p)^{n-y} \quad (y = 0, 1, 2, \ldots, n).$$

A random variable $Y \sim \text{Bin}(n,p)$ has: $\mathbb{E}[Y] = np$ and $\text{Var}[Y] = np(1-p)$.

The **multivariate normal distribution**, also known as the multivariate Gaussian distribution, with expectation $\boldsymbol{\mu}$, a d-vector, and variance-covariance $\boldsymbol{\Sigma}$, a $d \times d$ positive-definite, symmetric matrix, has a density of

$$f(\mathbf{y}) = \frac{1}{(2\pi)^{d/2}\text{det}(\boldsymbol{\Sigma})^{1/2}} \exp\left\{-\frac{1}{2}(\mathbf{y}-\boldsymbol{\mu})^\top \boldsymbol{\Sigma}^{-1}(\mathbf{y}-\boldsymbol{\mu})\right\} \quad (\mathbf{y} \in \mathbb{R}^d).$$

Also, for some $d \times d$ positive-definite, symmetric matrix A and d-vector b, if $\mathbf{y}$ is a vector of dimension d with density

$$g(\mathbf{y}) \propto \exp(-\frac{1}{2}\mathbf{y}^\top A\mathbf{y} + \mathbf{y}^\top b),$$

then $\mathbf{Y} \sim \mathrm{N}_d(Ab, A^{-1})$, also sometimes written $\mathbf{Y} \sim \mathrm{MVN}(Ab, A^{-1})$.

Distribution quantiles

The 90%, 95%, 97.5%, 99% and 99.5% quantiles of the standard normal distribution are:

90%	95%	97.5%	99%	99.5%
1.282	1.645	1.960	2.326	2.576

Matrix inverse

The inverse of a general 2×2 matrix is as follows:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix},$$

provided that $ad - bc \neq 0$.

Gamma function

The gamma function is defined as $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}\exp(-x)\,\mathrm{d}x$, which is $(\alpha-1)!$ for $\alpha = 1, 2, 3, \ldots$. Also $\Gamma(1/2) = \sqrt{\pi}$.