

Chapter 230

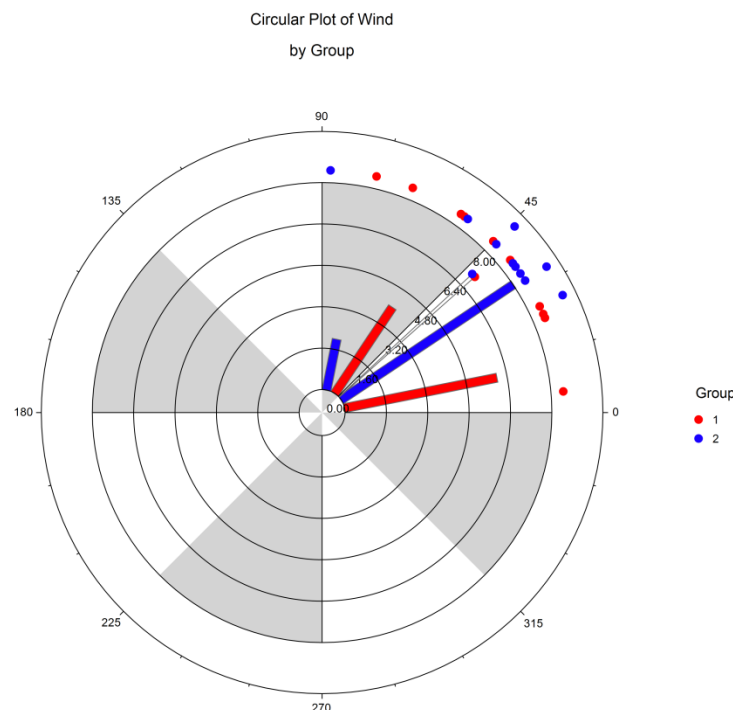
Circular Data Analysis

Introduction

This procedure computes summary statistics, generates rose plots and circular histograms, computes hypothesis tests appropriate for one, two, and several groups, and computes the circular correlation coefficient for circular data.

Angular data, recorded in degrees or radians, is generated in a wide variety of scientific research areas. Examples of angular (and cyclical) data include daily wind directions, ocean current directions, departure directions of animals, direction of bone-fracture plane, and orientation of bees in a beehive after stimuli.

The usual summary statistics, such as the sample mean and standard deviation, cannot be used with angular values. For example, consider the average of the angular values 1 and 359. The simple average is 180. But with a little thought, we might conclude that 0 is a better answer. Because of this and other problems, a special set of techniques have been developed for analyzing angular data. This procedure implements many of those techniques.



Technical Details

Suppose a sample of n angles a_1, a_2, \dots, a_n is to be summarized. It is assumed that these angles are in degrees. Fisher (1993) and Mardia & Jupp (2000) contain definitions of various summary statistics that are used for angular data. These results will be presented next. Let

$$C_p = \sum_{i=1}^n \cos(pa_i), \bar{C}_p = \frac{C_p}{n}, S_p = \sum_{i=1}^n \sin(pa_i), \bar{S}_p = \frac{S_p}{n},$$

$$R_p = \sqrt{C_p^2 + S_p^2}, \bar{R}_p = \frac{R_p}{n}$$

$$T_p = \begin{cases} \tan^{-1}\left(\frac{\bar{S}_p}{\bar{C}_p}\right) & \bar{C}_p > 0, \bar{S}_p > 0 \\ \tan^{-1}\left(\frac{\bar{S}_p}{\bar{C}_p}\right) + \pi & \bar{C}_p < 0 \\ \tan^{-1}\left(\frac{\bar{S}_p}{\bar{C}_p}\right) + 2\pi & \bar{S}_p < 0, \bar{C}_p > 0 \end{cases}$$

To interpret these quantities it may be useful to imagine that each angle represents a vector of length one in the direction of the angle. Suppose these individual vectors are arranged so that the beginning of the first vector is at the origin, the beginning of the second vector is at the end of the first, the beginning of the third vector is at the end of the second, and so on. We can then imagine a single vector \vec{a} that will stretch from the origin to the end of the last observation.

R_1 , called the *resultant length*, is the length of \vec{a} . \bar{R}_1 is the *mean resultant length* of \vec{a} . Note that \bar{R}_1 varies between zero and one and that a value of \bar{R}_1 near one implies that there was little variation in values of the angles.

The *mean direction*, θ , is a measure of the mean of the individual angles. θ is estimated by T_1 .

The *circular variance*, V , measures the variation in the angles about the mean direction. V varies from zero to one. The formula for V is

$$V = 1 - \bar{R}_1$$

The *circular standard deviation*, v , is defined as

$$v = \sqrt{-2 \ln(\bar{R}_1)}$$

The *circular dispersion*, used in the calculation of confidence intervals, is defined as

$$\delta = \frac{1 - T_2}{2\bar{R}_1^2}$$

The *skewness* is defined as

$$s = \frac{\bar{R}_2 \sin(T_2 - 2T_1)}{(1 - \bar{R}_1)^{3/2}}$$

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The *kurtosis* is defined as

$$k = \frac{\bar{R}_2 \cos(T_2 - 2T_1) - \bar{R}_1^4}{(1 - \bar{R}_1)^2}$$

Correction for Grouped Data

When the angles are grouped, a multiplicative correction for R may be necessary. The corrected value is given by

$$\bar{R}_p^* = g \bar{R}_p$$

where

$$g = \frac{\pi / J}{\sin(\pi / J)}$$

Here J is the number of equi-sized arcs. Thus, for monthly data, J would be 12.

Confidence Interval for the Mean Direction

Upton & Fingleton (1989) page 220 give a confidence interval for the mean direction when no distributional assumption is made as

$$T_1 \pm \sin^{-1}(z_{\alpha/2} \hat{\sigma})$$

where

$$\hat{\sigma} = \sqrt{\frac{n(1-H)}{4R^2}}$$

$$H = \frac{1}{n} \left\{ \cos(2T_1) \sum_{i=1}^n \cos(2a_i) + \sin(2T_1) \sum_{i=1}^n \sin(2a_i) \right\}$$

Circular Uniform Distribution

Uniformity refers to the situation in which all values around the circle are equally likely. The probability distribution on a circle with this property is the *circular uniform distribution*, or simply, the uniform distribution. The probability density function is given by

$$f(a) = \frac{1}{360}$$

The probability between any two points is given by

$$\Pr(a_1 < a_2 | a_1 \leq a_2, a_2 \leq a_1 + 2\pi) = \frac{a_2 - a_1}{360}$$

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Tests of Uniformity

Uniformity refers to the situation in which all values around the circle are equally likely. Occasionally, it is useful to perform a statistical test of whether a set of data do not follow the uniform distribution. Several tests of uniformity have been developed. Note that when any of the following tests are rejected, we can conclude that the data were not uniform. However, when the test is not rejected, we cannot conclude that the data follow the uniform distribution. Rather, we do not have enough evidence to reject the null hypothesis of uniformity.

Rayleigh Test

The Rayleigh test, discussed in Mardia & Jupp (2000) pages 94-95, is the score test and the likelihood ratio test for uniformity within the von Mises distribution family. The Rayleigh test statistic is $2n\bar{R}^2$. For large samples, the distribution of this statistic under uniformity is a chi-square with two degrees of freedom with an error of approximation of $O(n^{-1})$. A closer approximation to the chi-square with two degrees of freedom is achieved by the modified Rayleigh test. This test, which has an error of $O(n^{-2})$, is calculated as follows.

$$S^* = \left(1 - \frac{1}{2n}\right) 2n\bar{R}^2 + \frac{n\bar{R}^4}{2}$$

Modified Kuiper's Test

The modified Kuiper's test, Mardia & Jupp (2000) pages 99-103, was designed to test uniformity against any alternative. It measures the distance between the cumulative uniform distribution function and the empirical distribution function. It is accurate for samples as small as 8. The test statistic, V , is calculated as follows

$$V = V_n \left(\sqrt{n} + 0.155 + \frac{0.24}{\sqrt{n}} \right)$$

where

$$V_n = \max_{i=1 \text{ to } n} \left(\frac{a_{(i)}}{360} - \frac{i}{n} \right) - \min_{i=1 \text{ to } n} \left(\frac{a_{(i)}}{360} - \frac{i}{n} \right) + \frac{1}{n}$$

Published critical values of V are

<u>V</u>	<u>Alpha</u>
1.537	0.150
1.620	0.100
1.747	0.050
1.862	0.025
2.001	0.010

This table was used to create an interpolation formula from which the alpha values are calculated.

Watson Test

The following uniformity test is outlined in Mardia & Jupp pages 103-105. The test is conducted by calculating U^2 and comparing it to a table of values. If the calculated value is greater than the critical value, the null hypothesis of uniformity is rejected. Note that the test is only valid for samples of at least eight angles.

The calculation of U^2 is as follows

$$U^2 = \sum_{i=1}^n \left[u_{(i)} - \frac{i - \frac{1}{2}}{n} - \bar{u} + \frac{1}{2} \right]^2 + \frac{1}{12n}$$

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where

$$\bar{u} = \frac{\sum_{i=1}^n u_{(i)}}{n}, \quad u_{(i)} = \frac{a_{(i)}}{360}$$

$a_{(1)} \leq a_{(2)} \leq a_{(3)} \leq \dots \leq a_{(n)}$ are the sorted angles. Note that maximum likelihood estimates of κ and θ are used in the distribution function. Mardia & Jupp (2000) present a table of critical values that has been entered into NCSS. When a value of U^2 is calculated, the table is interpolated to determine its significance level.

Published critical values of U^2 are

U^2	Alpha
0.131	0.150
0.152	0.100
0.187	0.050
0.221	0.025
0.267	0.010

Von Mises Distributions

The *Von Mises distribution* takes the role in circular statistics that is held by the normal distribution in standard linear statistics. In fact, it is shaped like the normal distribution, except that its tails are truncated.

The probability density function is given by

$$f(a; \theta, \kappa) = \frac{1}{2\pi I_0(\kappa)} \exp[\kappa \cos(a - \theta)]$$

where $I_p(x)$ (the modified Bessel function of the first kind and order p) is defined by

$$I_p(x) = \sum_{r=0}^{\infty} \frac{1}{(r+p)!r!} \left(\frac{x}{2}\right)^{2r+p}, \quad p = 0, 1, 2, \dots$$

In particular

$$\begin{aligned} I_0(x) &= \sum_{r=0}^{\infty} \frac{1}{(r!)^2} \left(\frac{x}{2}\right)^{2r} \\ &= \frac{1}{2\pi} \int_0^{2\pi} e^{x \cos(\theta)} d\theta \end{aligned}$$

The parameter θ is the *mean direction* and the parameter κ is the *concentration parameter*.

The distribution is unimodal. It is symmetric about A . It appears as a normal distribution that is truncated at plus and minus 180 degrees. When κ is zero, the von Mises distribution reduces to the uniform distribution. As κ gets large, the von Mises distribution approaches the normal distribution.

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Point Estimation

The maximum likelihood estimate of θ is the sample mean direction. That is, $\hat{\theta} = T_1$.

The maximum likelihood of κ is the solution to

$$A_1(\kappa) = \bar{R}$$

where

$$A_1(x) = \frac{I_1(x)}{I_0(x)}.$$

That is, the MLE of κ is given by

$$\kappa^* = A_1^{-1}(\bar{R})$$

This can be approximated by (see Fisher (1993) page 88 and Mardia & Jupp (2000) pages 85-86)

$$\kappa^* = \begin{cases} 2\bar{R} + \bar{R}^3 + \frac{5\bar{R}^5}{6} & \bar{R} < 0.53 \\ -0.4 + 1.39\bar{R} + \frac{0.43}{1-\bar{R}} & 0.53 \leq \bar{R} < 0.85 \\ \frac{1}{3\bar{R} - 4\bar{R}^2 + \bar{R}^3} & \bar{R} \geq 0.85 \end{cases}$$

This estimate is very biased. This bias is corrected by using the following modified estimator.

$$\hat{\kappa} = \begin{cases} \max\left(\kappa^* - \frac{2}{n\kappa^*}, 0\right) & \kappa^* < 2 \\ \frac{(n-1)^3 \kappa^*}{n(n^2+1)} & \kappa^* \geq 2 \end{cases} \quad \begin{matrix} n \leq 15 \\ n > 15 \end{matrix}$$

Test for a Specified Mean Direction of Von Mises Data

There are several different hypothesis tests that have been proposed for testing $H_0: \theta = \theta_0$ versus $H_1: \theta \neq \theta_0$, where θ_0 is a specific value of the mean direction. The tests presented here require the additional assumption that the data follow the Von Mises distribution, at least approximately.

It will be useful to adopt the following notation.

$$\bar{C}^* = \frac{1}{n} \sum_{i=1}^n \cos(a_i - \theta_0)$$

$$\bar{S}^* = \frac{1}{n} \sum_{i=1}^n \sin(a_i - \theta_0)$$

$$\bar{R}^* = \sqrt{[\bar{S}^*]^2 + [\bar{C}^*]^2}$$

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Score Test

The score test, given by Mardia & Jupp (2000) page 123, is computed as

$$\chi_s^2 = \frac{n\hat{\kappa}}{A_1(\hat{\kappa})} (\bar{S}^*)^2$$

For large n , χ_s^2 follows the chi-square distribution with one degree of freedom.

Likelihood Ratio Test

The likelihood ratio test, given by Mardia & Jupp (2000) page 122, is computed as

$$\chi_L^2 = \begin{cases} \frac{4n[(\bar{R}^*)^2 - (\bar{C}^*)^2]}{2 - (\bar{C}^*)^2} & \text{if } n \geq 5 \text{ and } \bar{C}^* \leq 2/3 \\ \frac{2n^3}{n^2 + (n\bar{C}^*)^2 + 3n} \log\left(\frac{1 - (\bar{C}^*)^2}{1 - (\bar{R}^*)^2}\right) & \text{if } n \geq 5 \text{ and } \bar{C}^* > 2/3 \end{cases}$$

The test statistic, χ_L^2 , follows a chi-square distribution with one degree of freedom.

Watson & Williams Test

The Watson and Williams test, given by Mardia & Jupp (2000) page 123, is computed as

$$F = \frac{\bar{R}^* - \bar{C}^*}{(1 - \bar{R}^*) / (n-1)} \text{ if } \bar{C}^* \geq 5/6$$

The test statistic, F , follows an F distribution with one and $n-1$ degrees of freedom.

Stephens Test

This test, given by Fisher (1993) pages 93-94, is computed as

$$E = \frac{\sin(T_1 - \theta_0)}{\sqrt{1 / (n\hat{\kappa}\bar{R})}}$$

If $\hat{\kappa} \geq 2$, E follows the standard normal distribution.

Confidence Interval for Mean Direction assuming Von Mises

A general confidence interval for θ was given above. When the data can be assumed to follow a von Mises distribution, a more appropriate interval is given by Mardia & Jupp (2000) page 124 and Upton & Fingleton (1989) page 269. This confidence interval is given by

$$T_1 \pm \cos^{-1} \left(\sqrt{\frac{2n[2R^2 - nz_\alpha^2]}{R^2(4n - z_\alpha^2)}} \right) \quad \text{if } \bar{R} \leq 2/3$$

$$T_1 \pm \cos^{-1} \left(\frac{\sqrt{n^2 - (n^2 - R^2) \exp\left(\frac{z_\alpha^2}{n}\right)}}{R} \right) \quad \text{if } \bar{R} > 2/3$$

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Test for a Specified Concentration of Von Mises Data

Suppose you want to test a one-sided hypothesis concerning κ , given that the data come from a Von Mises distribution and that the mean direction parameter is unknown. Fisher (1993) page 95 suggests the following procedure when $\hat{\kappa} \geq 2$.

When testing $\kappa = \kappa_0$ versus $\kappa < \kappa_0$, reject the null hypothesis if

$$\bar{R} < 1 - \frac{\chi_{n-1;\alpha}^2}{2n} \left(\frac{1}{\kappa_0} + \frac{3}{8\kappa_0^2} \right)$$

When testing $\kappa = \kappa_0$ versus $\kappa > \kappa_0$, reject the null hypothesis if

$$\bar{R} > 1 - \frac{\chi_{n-1;1-\alpha}^2}{2n} \left(\frac{1}{\kappa_0} + \frac{3}{8\kappa_0^2} \right)$$

These tests are based on the result that

$$\frac{2n(1 - \bar{R})}{\frac{1}{\kappa_0} + \frac{3}{8\kappa_0^2}} \sim \chi_{n-1}^2$$

Confidence Interval for Concentration of Von Mises

An approximate confidence interval for κ when $\hat{\kappa} > 2$ was given by Mardia & Jupp (2000) pages 126-127 as

$$\left(\frac{1 + \sqrt{1 + 3b}}{4b}, \frac{1 + \sqrt{1 + 3d}}{4d} \right)$$

where

$$b = \frac{n(1 - \bar{R})}{\chi_{n-1,1-\alpha/2}^2}$$

$$d = \frac{n(1 - \bar{R})}{\chi_{n-1,\alpha/2}^2}$$

Goodness of Fit Tests for the Von Mises Distribution**Stephens Modified Watson's Test**

The following goodness-of-fit test, published by Lockhart & Stephens (1985) as a modification of the Watson test for the circle, is outlined in Fisher (1993) page 84. The test is conducted by calculating U^2 and comparing it to a table of values. If the calculated value is greater than the critical value, the null hypothesis of Von Misesness is rejected. Note that the test is only valid for samples of at least 20 angles.

The calculation of U^2 is as follows

$$U^2 = \sum_{i=1}^n \left[\hat{p}_{(i)} - \frac{2i-1}{2n} \right]^2 - n \left(\bar{\hat{p}} - \frac{1}{2} \right)^2 + \frac{1}{12n}$$

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where

$$\bar{\hat{p}} = \frac{\sum_{i=1}^n \hat{p}_{(i)}}{n}$$

$$\hat{p}_{(i)} = F_{\kappa}(a_{(i)} - T_1)$$

$a_{(1)} \leq a_{(2)} \leq a_{(3)} \leq \dots \leq a_{(n)}$ are the sorted angles and $F_{\kappa}(a - \theta)$ is the cumulative distribution function of the von Mises distribution. Note that maximum likelihood estimates of κ and θ are used in the distribution function. Lockhart & Stephens (1985) present a table of critical values that has been entered into NCSS. When a value of U^2 is calculated, the table is interpolated to determine its significance level.

Cox Test

Mardia & Jupp (2000) pages 142-143 present a von Mises goodness-of-fit test that was originally given by Cox (1975).

The test statistic, C , is distributed as a chi-squared variable with two degrees of freedom under the null hypothesis that the data follow the von Mises distribution. It is calculated as follows.

$$C = \frac{s_c^2}{nv_c(\hat{\kappa})} + \frac{s_s^2}{nv_s(\hat{\kappa})}$$

where

$$s_c = \sum_{i=1}^n \cos 2(a_i - T_1) - n\alpha_2(\hat{\kappa})$$

$$s_s = \sum_{i=1}^n \sin 2(a_i - T_1)$$

$$v_c(x) = \frac{1 + \alpha_4}{2} - \alpha_2^2 - \frac{[\alpha_1 / 2 + \alpha_3 / 2 - \alpha_1 \alpha_2]^2}{(1 + \alpha_2) / 2 - \alpha_1^2}$$

$$v_s(x) = \frac{\alpha_1 - \alpha_4}{2} - \frac{(\alpha_1 - \alpha_3)^2}{1 - \alpha_2}$$

Multi-Group Tests

Three multi-group tests are available for testing hypotheses about two or more groups. The nonparametric uniform-scores test tests whether the distributions of the groups are identical. The Watson-Williams F test tests whether a set of mean directions are equal given that the concentrations are unknown, but equal, given that the groups each follow a von Mises distribution. The concentration homogeneity test tests whether the concentration parameters are equal, given that the groups each follow a von Mises distribution.

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Mardia-Watson-Wheeler Uniform-Scores Test

Suppose you have g populations following any common distribution from which random samples are taken and you wish to test whether these distributions are equal. Fisher (1993) page 122 and Mardia & Jupp (2000) pages 156-157 present a nonparametric test that is calculated as follows

$$W_g = 2 \sum_{i=1}^g \frac{(C_{Ri}^2 + S_{Ri}^2)}{n_i}$$

where $C_{Ri} = \sum_{j=1}^{n_i} \cos(\gamma_{ij})$, $S_{Ri} = \sum_{j=1}^{n_i} \sin(\gamma_{ij})$, $n = \sum_{i=1}^g n_i$, and γ_{ij} are the circular ranks of the corresponding angles. The circular ranks are calculated using

$$\gamma_{ij} = \frac{2\pi r_{ij}}{n}$$

where the r_{ij} are the ranks of the corresponding a_{ij} .

If all n_i are greater than 10, the distribution of W_g is approximately distributed as a chi-square with $2g-2$ degrees of freedom.

Since ranks are used in this test, ties become an issue. We have adopted the strategy of applying average ranks. Note that little has been done to test the adoption of this strategy within the realm of circular statistics.

Watson-Williams High Concentration F Test

Suppose you have g Von Mises populations from which random samples are taken and you wish to test whether their mean directions are equal. That is, you wish to test the null hypothesis

$$H_0: \theta_1 = \theta_2 = \dots = \theta_g$$

Mardia & Jupp (2000) pages 134-135 present the Watson-William High-Concentration F Test that is calculated as follows

$$F_{WW} = \left(1 + \frac{3}{8\hat{\kappa}}\right) \frac{\left(\sum_{j=1}^g R_j - R\right) / (g-1)}{\left(n - \sum_{j=1}^g R_j\right) / (n-g)}$$

where $\hat{\kappa}$ is the maximum likelihood estimate of the concentration based on R and

$$R_j = \sqrt{C_j^2 + S_j^2}, C_j = \sum_{i=1}^{n_j} \cos(a_i), S_j = \sum_{i=1}^{n_j} \sin(a_i), R = \sqrt{C^2 + S^2}, C = \sum_{j=1}^g C_j, S = \sum_{j=1}^g S_j, \text{ and } n = \sum_{j=1}^g n_j.$$

The distribution of F_{WW} is approximately distributed as an F with $g-1$ and $n-1$ degrees of freedom when the assumptions that $\kappa_1 = \kappa_2 = \dots = \kappa_g$ and that the distributions are Von Mises are made. The approximation also requires that $\hat{\kappa} \geq 1$.

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Multi-Group Concentration Homogeneity Test

Suppose you have g groups from which random samples are taken and you wish to test whether the concentrations are equal. That is, you wish to test the null hypothesis

$$H_0: \kappa_1 = \kappa_2 = \dots = \kappa_g$$

Mardia & Jupp (2000) page 139 presents such a test. It is divided into three cases.

Case I. $\bar{R} < 0.45$

$U1$ is approximately distributed as a chi-square with $g-1$ degrees of freedom

$$U1 = \sum_{j=1}^g w_j f_j^2 - \frac{\left\{ \sum_{j=1}^g w_j f_j \right\}^2}{\sum_{j=1}^g w_j}$$

where $w_j = \frac{4(n_j - 4)}{3}$ and $f_j = \sin^{-1}(2\bar{R}_j \sqrt{3/8})$

Case II. $0.45 \leq \bar{R} \leq 0.70$

$U2$ is approximately distributed as a chi-square with $g-1$ degrees of freedom

$$U2 = \sum_{j=1}^g w_j h_j^2 - \frac{\left\{ \sum_{j=1}^g w_j h_j \right\}^2}{\sum_{j=1}^g w_j}$$

where $w_j = \frac{n_j - 3}{0.797449}$ and $h_j = \sinh^{-1}\left(\frac{\bar{R}_j - 1.089}{.258}\right)$

Case III. $\bar{R} > 0.70$

$U3$ is approximately distributed as a chi-square with $g-1$ degrees of freedom

$$U3 = \frac{1}{1+d} \left\{ \nu \log \left(\frac{n - \sum_{j=1}^g R_j}{\nu} \right) - \sum_{j=1}^g \nu_j \log \left(\frac{n_j - R_j}{\nu_j} \right) \right\}$$

where $\nu_j = n_j - 1$, $\nu = n - g$, and $d = \frac{1}{3(g-1)} \left\{ \sum_{j=1}^g \frac{1}{\nu_j} - \frac{1}{\nu} \right\}$.

Circular Correlation Measure

This section discusses a measure of the correlation between two circular variables presented by Jammalamadaka and SenGupta (2001). Suppose a sample of n pairs of angles $(a_{11}, a_{21}), (a_{12}, a_{22}), \dots, (a_{1n}, a_{2n})$ is available. The circular correlation coefficient is calculated as

$$r_c = \frac{\sum_{k=1}^n \sin(a_{1k} - T_{1,1}) \sin(a_{2k} - T_{2,1})}{\sqrt{\sum_{k=1}^n \sin^2(a_{1k} - T_{1,1}) \sum_{k=1}^n \sin^2(a_{2k} - T_{2,1})}}$$

where $T_{1,1}$ is the mean direction of the first circular variable and $T_{2,1}$ is the mean direction of second.

The significance of this correlation coefficient can be test using the fact the z_r is approximately distributed as a standard normal, where

$$z_r = r_c \sqrt{\frac{n \lambda_{20} \lambda_{02}}{\lambda_{22}}}$$

and

$$\lambda_{ij} = \frac{1}{n} \sum_{k=1}^n \sin^i(a_{1k} - T_{1,1}) \sin^j(a_{2k} - T_{2,1})$$

Data Structure

The data consist of one or more variables. Each variable contains a set of angular values. The rows may be separated into groups using the unique values of an optional grouping variable. An example of a dataset containing circular data is Circular1.S0. Missing values are entered as blanks (empty cells).

Procedure Options

This section describes the options available in this procedure.

Variables Tab

These options specify the variables that will be used in the analysis.

Data Variables

Data Variables

Specify one or more variables that contain the angular values. The values in these variables must be of the type specified in 'Data Type'.

If more than one variable is specified, the format of the reports depend on whether a 'Grouping Variable' is used. If a 'Grouping Variable' is specified, a separate set of reports is generated for each data variable. If no 'Grouping Variable' is specified, each of these variables are treated as a different group in a single set of reports.

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Data Type

Specify the type of circular data that is contained in the Data Variables. Note that all variables must be of the same data type. The possible data types are

- **Angle (0 to 360)**

Data are in the range 0 to 360 degrees. Negative values are converted to positive values by subtracting them from 360 (e.g. -20 becomes 340). Data outside 0 to 360 are converted to this range by subtracting (or adding) 360 until the value is in this range.

- **RADIAN (0 to 2 pi)**

Data are in the range 0 to 2pi radian. Negative values are converted to positive values by subtracting them from 2pi.

- **AXIAL (0 to 180)**

Data are bidirectional. Axial data are converted to angular data by multiplying by two. Axial data may be in the full 0-360 range.

- **Compass**

Text data representing the 16 points of the compass are entered. Values are converted into degrees using the recodes: N = 0, E = 90, S = 180, W = 270. Two and three letters may be used. For example, 'NNW' is north by north-west.

- **Time (0-24)**

Time of day values between 0 and 24 may be entered.

- **Weekday**

Integers representing the days of the week are entered. The relationship is 1 = Monday, 2 = Tuesday, ..., 7 = Sunday. The integers are converted to degrees using $1 = 180/7$, $2 = 180/7 + 360/7$, and so on.

- **Month of Year**

Integers representing the months of the year are entered. The relationship is 1 = January, 2 = February, ..., 12 = December. The integers are converted to degrees using $1 = 180/12$, $2 = 180/12 + 360/12$, and so on.

Grouping Variable

Grouping Variable

This optional variable separates the values of the Data Variables into groups. A separate analysis is then generated for each group.

Note that when a grouping variable is specified, the correlations are not generated.

Grouping Correction Factor

When the same data values occur repeatedly, a correction factor is suggested for the calculation of \bar{R} . This correction factor depends on the number of unique values, which is entered here. If '0' is entered, no correction factor is used.

Options – Hypothesized Values

Hypothesized Theta

This optional parameter specifies the hypothesized value of theta (mean direction) under the null hypothesis. A set of hypotheses tests are conducted to determine if the data support this hypothesized value.

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Note that this is a single-group test. If there are several groups, a separate test is provided for each group.

Hypothesized Kappa

This optional parameter specifies the hypothesized value of kappa (concentration) under the null hypothesis. A hypothesis test is conducted to determine if the data support this hypothesized value.

Note that this is a single-group test. If there are several groups, a separate test is provided for each group.

Options – Confidence Coefficient

Confidence Coefficient

Specify the value of confidence coefficient for the confidence intervals.

Reports Tab

The options on this panel control which reports and plots are displayed.

Select Reports

Summary Reports ... Correlations

Select these options to display the indicated reports.

Report Options

Show Notes

This option controls whether the available notes and comments that are displayed at the bottom of each report. This option lets you omit these notes to reduce the length of the output.

Precision

Specify the precision of numbers in the report. A single-precision number will show seven-place accuracy, while a double-precision number will show thirteen-place accuracy. Note that the reports were formatted for single precision. If you select double precision, some numbers may run into others. Also note that all calculations are performed in double precision regardless of which option you select here. This is for reporting purposes only.

Variable Names

This option lets you select whether to display only variable names, variable labels, or both.

Value Labels

This option applies to the *Group Variable(s)*. It lets you select whether to display data values, value labels, or both. Use this option if you want the output to automatically attach labels to the values (like 1=Yes, 2=No, etc.). See the section on specifying *Value Labels* elsewhere in this manual.

Report Options – Decimal Places

Mean and Probability Decimals

Specify the number of digits after the decimal point to display on the output of values of this type. Note that this option in no way influences the accuracy with which the calculations are done.

Enter 'All' to display all digits available. The number of digits displayed by this option is controlled by whether the PRECISION option is SINGLE or DOUBLE.

Plots Tab

The options on this panel control the appearance of the plots.

Select Plots

Rose Plot / Circular Histogram (Combined) and Rose Plots / Circular Histograms (Individual)

Select these options to display the indicated plots.

Format

Click the plot format button to change the plot settings (see the Window Options below).

Edit During Run

Checking this option will cause the bar chart format window to appear when the procedure is run. This allows you to modify the format of the graph with the actual data.

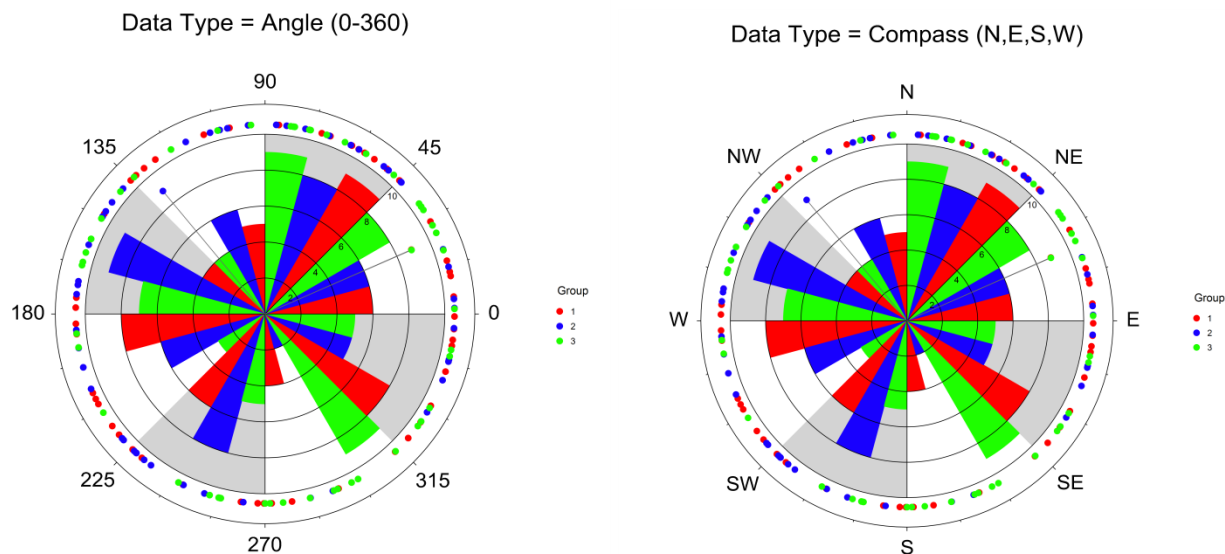
Rose Plot / Circular Histogram Window Options

This section describes the specific options available on the Rose Plot / Circular Histogram Format window, which is displayed when a Rose Plot / Circular Histogram Format button is clicked. Common options, such as axes, labels, legends, and titles are documented in the Graphics Components chapter.

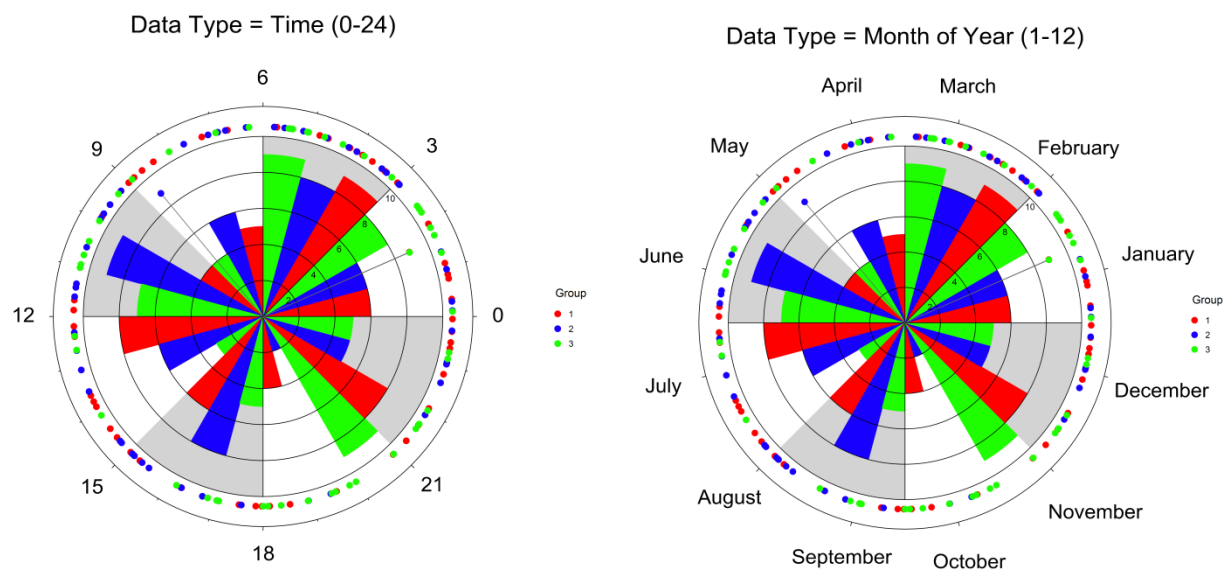
Rose Plot Tab

Data Type

The data type of the plot is specified independently of the data type specified on the Variables tab of the Circular Data Analysis procedure.

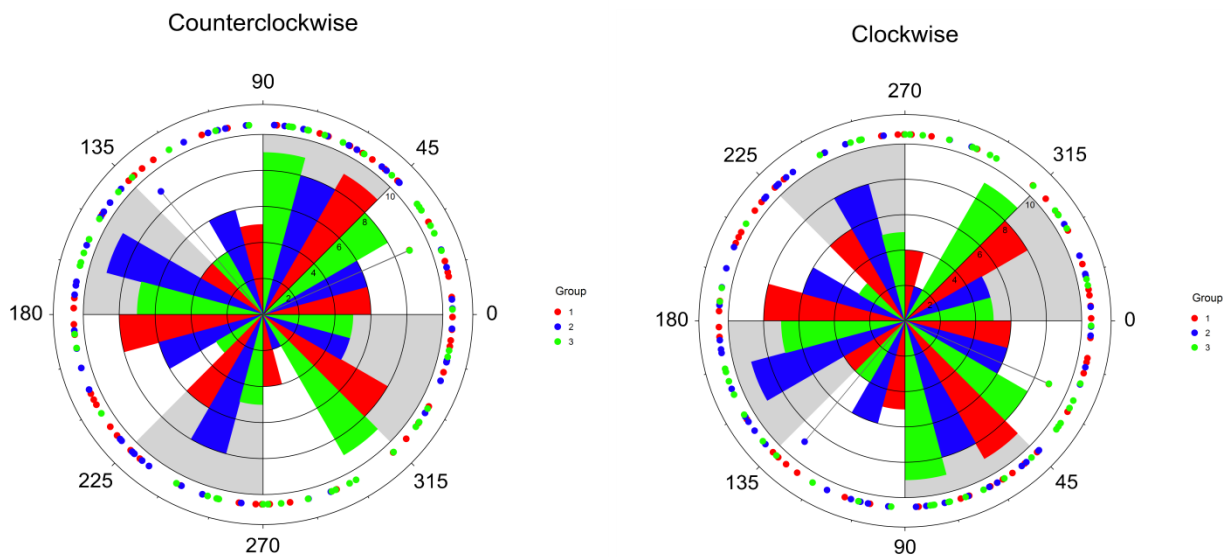


Circular Data Analysis



Direction

This option indicates whether the orientation of the plot is in a 'Clockwise' or 'Counter-Clockwise' direction.

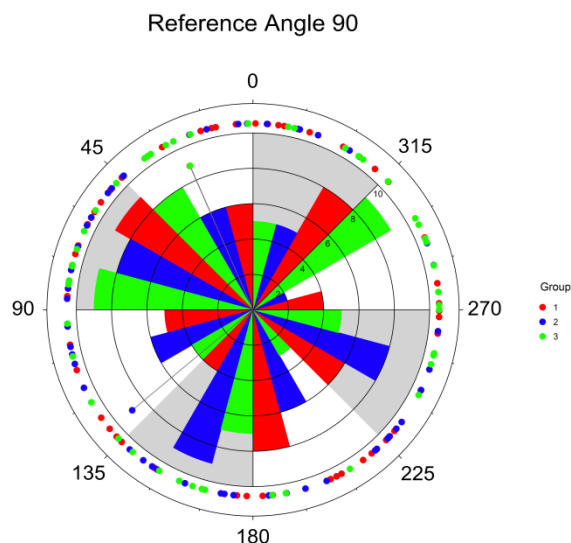
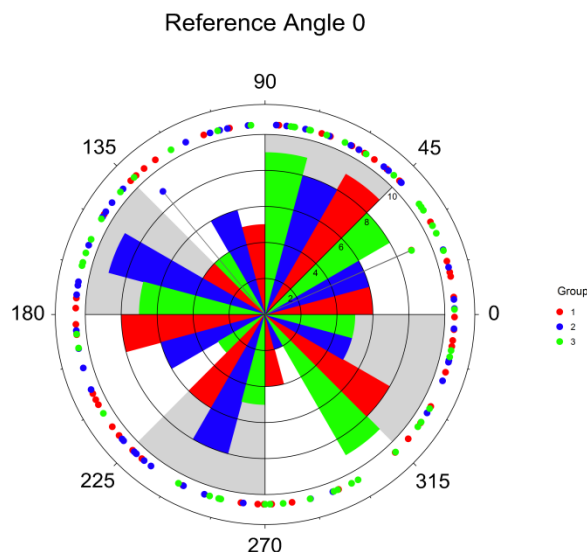


Circular Data Analysis

Reference Angle (Rotation)

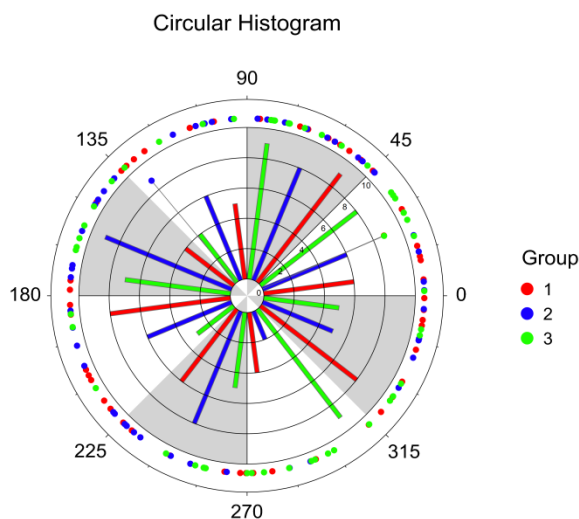
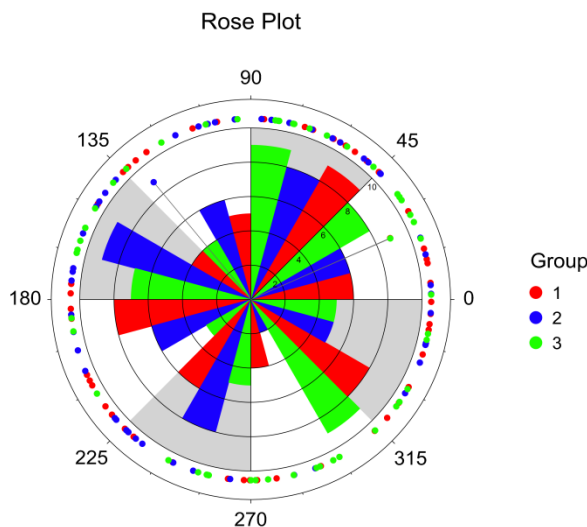
This option lets you indicate the position of 0 degrees by entering an offset angle. On the default circle, 0 degrees is on the right (east), 90 degrees is at the top (north), 180 degrees is on the left (west), and 270 degrees is at the bottom (south). This option lets you add an 'offset' to each angle which moves the position of 0 degrees around the circle.

The offset must be between 0 and 360 degrees.



Interior Objects

The two choices for plot styles are Rose Plot and Circular Histogram.



Circular Data Analysis

Group Display

When the data is grouped data, this option determines whether the petals within a bin are side-by-side, stacked upon each other, or overlaid.

Side-by-Side

The bin width is divided equally by the number of groups and the petals are laid out sequentially in the bin. Although the petals are narrower, they still encompass the points of the group that within the boundaries of the whole bin.

Stacked

A single petal in each bin is divided by the number of groups. Rose plots with the group display set to Stacked may be misleading because the proportional area is larger for the outside groups.

Overlaid

Each petal for each group is overlaid in each bin. Some degree of transparency is recommended when using the Overlaid group display. It is also difficult to distinguish groups when there are more than 2 or 3 groups.

Radius for Interior Objects

This option specifies the distance to the outer edge of the bins and petals of the rose plot or circular histogram.

Petal Width

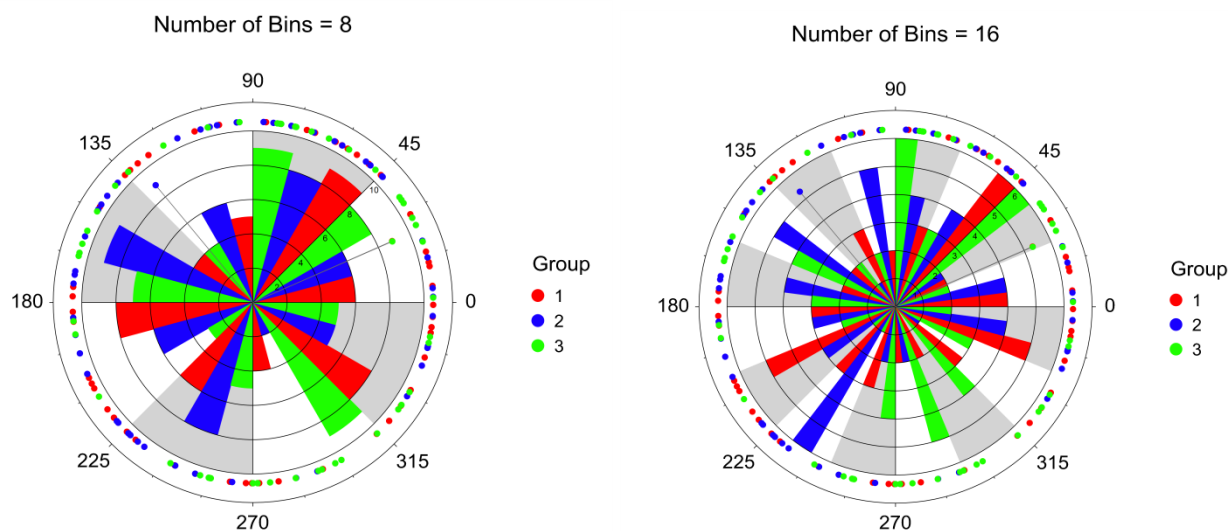
Specify the percent of the total width of each bin that is to be used for each petal.

Percent for Histogram Base

This is the percent of Radius for Interior Objects that is used for the base of the circular histogram.

Number of Bins

Specify the number of bins for the circle.



Circular Data Analysis

First Bin Start Angle

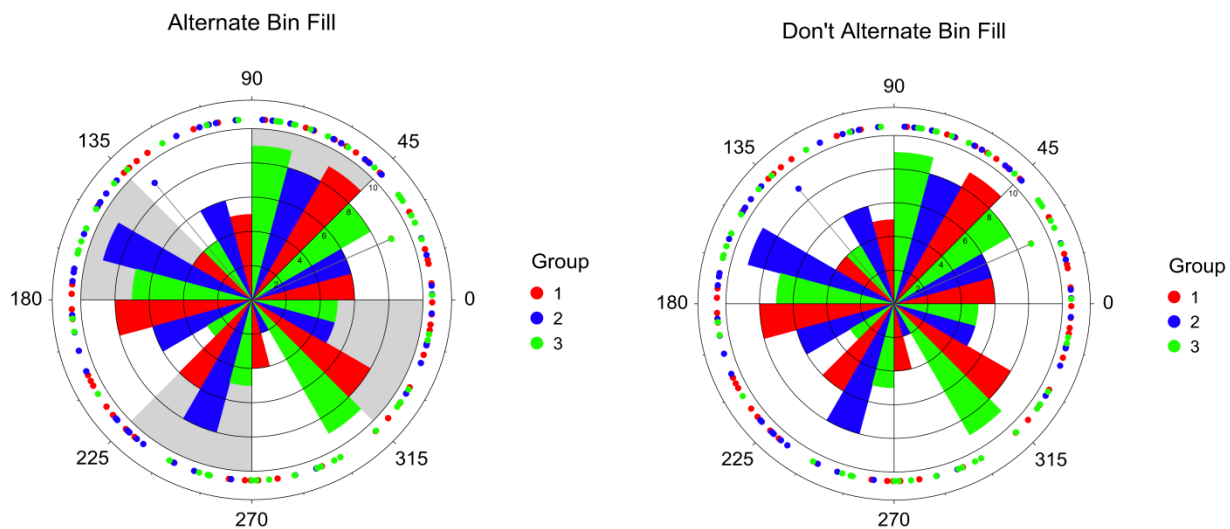
This permits the user to change the angle at which the binning begins. This is useful, for example when the Data Type is set to Compass, since this option can be used to center the bins on the directions. When Data Type is set to Compass, the recommended First Bin Start Angle is 45 for 4 bins, 22.5 for 8 bins, and 11.25 for 16 bins.

Number of Radial Axes

Select the number of axes that go from the center to the outer edge of the interior region. If this is set to the same number as the number of bins, these axes show the edges of the bins. The radial axes also begin at the First Bin Start Angle.

Alternate Bin Fill

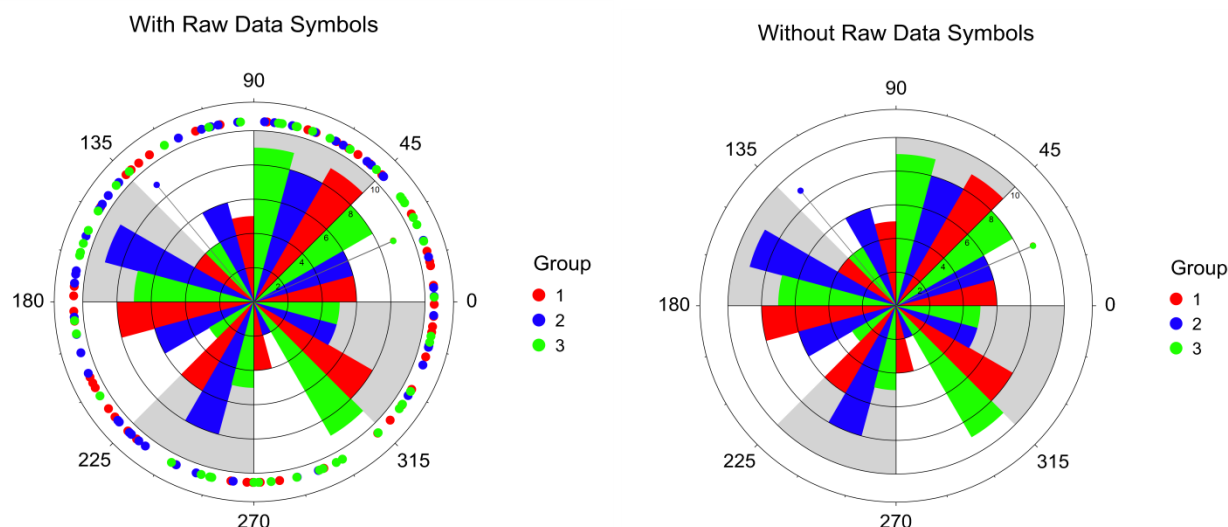
Check this box to show a background fill for each bin. The fills alternate beginning with Fill 1. When the number of bins is odd, the adjacent first and last bins will both have Fill 1.



Data & Means Tab

Raw Data Symbols

Check this box to show the raw data symbols.



Number of Raw Data Symbol Bins

Specify the number of bins for the raw data points. To use no binning set this to 0 or All Uniques.

First Bin Start Angle

This permits the user to change the angle at which the binning begins. This is useful, for example when the Data Type is set to Compass, since this option can be used to center the bins on the directions.

When Data Type is set to Compass, the recommended First Bin Start Angle is 45 for 4 bins, 22.5 for 8 bins, and 11.25 for 16 bins.

Radius for Raw Data Symbols

This is the distance from the center at which the symbols are shown.

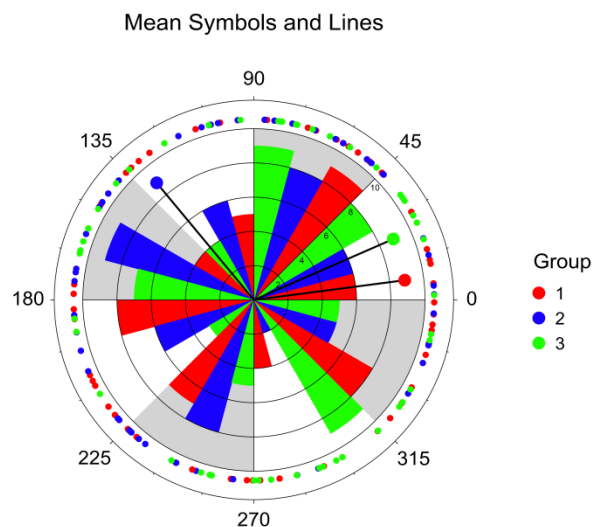
Width for Multiple Symbols at One Location

This specifies the width of the band that contains the symbols when there is more than one value at some locations.

Circular Data Analysis

Mean Symbols and Lines

Use these options to set up the visual representation of the circular means for each group.



References Tab

Direction References

The options in this section allow you to specify the tick marks and references going around the plot.

Magnitude References

The options in this section allow you to specify the tick marks and references going from the center to the outside of the plot.

Example 1 – Analysis of Circular Data

This section presents an example of how to run this procedure. The data are wind directions of two groups. The data are found in the Circular1 dataset.

You may follow along here by making the appropriate entries or load the completed template **Example 1** by clicking on Open Example Template from the File menu of the Circular Data Analysis window.

1 Open the Circular1 dataset.

- From the File menu of the NCSS Data window, select **Open Example Data**.
- Click on the file **Circular1.NCSS**.
- Click **Open**.

2 Open the Circular Data window.

- Using the Analysis menu or the Procedure Navigator, find and select the **Circular Data Analysis** procedure.
- On the menus, select **File**, then **New Template**. This will fill the procedure with the default template.

3 Specify the variables.

- On the Circular Data window, select the **Variables tab**. (This is the default.)
- Double-click in the **Data Variables** text box. This will bring up the variable selection window.
- Select **Wind** from the list of variables and then click **Ok**. “Wind” will appear in the Data Variables box.
- Double-click in the **Grouping Variable** text box. This will bring up the variable selection window.
- Select **Group** from the list of variables and then click **Ok**. “Group” will appear in the Paired Variables box.
- Set **Hypothesized Theta** to **40**.
- Set **Hypothesized Kappa** to **2**.

4 Run the procedure.

- From the Run menu, select **Run Procedure**. Alternatively, just click the green Run button,

The following reports and charts will be displayed in the Output window.

Summary Statistics Section

Group	Sample Size (N)	Mean Direction (Theta)	Mean Resultant Length (R bar)	Circular Variance (V)	Circular Standard Deviation (v)	Circular Dispersion (Delta)	Von Mises Concentration (Kappa)
1	10	41.5869	0.9324	0.0676	21.4299	0.1449	5.5452
2	10	42.6725	0.9599	0.0401	16.3991	0.0768	9.1850

Group

This is the group (or variable) presented on this line.

Sample Size

This is the number of nonmissing values in this group.

Mean Direction

This is estimated mean direction, T_1 .

Circular Data Analysis

Mean Resultant Length

This is the estimated mean resultant length, \bar{R}_1 . It is a measure of data concentration. An \bar{R}_1 close to zero implies low data concentration. An \bar{R}_1 close to one implies high data concentration.

Circular Variance

The circular variance, V , is a measure of variation in the data. Note that $V = 1 - \bar{R}_1$.

Circular Standard Deviation

The circular standard deviation is $v = \sqrt{-2 \ln(\bar{R}_1)}$. Note that it is not the square root of the circular variance.

Circular Dispersion

The circular dispersion, $\delta = \frac{1 - T_2}{2\bar{R}_1^2}$, is another measure of variation.

Von Mises Concentration

This is the estimated concentration parameter of the von Mises distribution, κ .

Mean Direction Section

Group	Sample Size (N)	Mean Direction (Theta)	Lower 95.0% Confidence Limit of Theta	Upper 95.0% Confidence Limit of Theta	Standard Error of Mean Direction
1	10	41.5869	27.9417	55.2321	6.8964
2	10	42.6725	32.7516	52.5934	5.0365

This report provides the large sample confidence interval for the mean direction as described by Upton & Fingleton (1989) page 220. Note that this interval does not require the assumption that the data come from the von Mises distribution.

Variation Statistics Section

Group	Sample Size (N)	Circular Variance (V)	Circular Standard Deviation (v)	Circular Dispersion (Delta)	Skewness (s)	Kurtosis (k)
1	10	0.0676	21.4299	0.1449	-0.0795	-1.7244
2	10	0.0401	16.3991	0.0768	-4.8582	5.4248

This report provides measures of data variation and dispersion which were defined in the Statistical Summary Report. It also provides measures of the skewness and kurtosis of the data.

Skewness

This is a measure of the skewness (lack of symmetry about the mean) in the data. Symmetric, unimodal datasets have a skewness value near zero.

Kurtosis

This is a measure of the kurtosis (peakedness) in the data. Von Mises datasets have a kurtosis near zero.

Von Mises Distribution Estimation Section

Group	Sample Size (N)	Mean Direction (Theta)	Lower 95.0% Confidence Limit of Theta	Upper 95.0% Confidence Limit of Theta	Von Mises Conc. (Kappa)	Lower 95.0% Confidence Limit of Kappa	Upper 95.0% Confidence Limit of Kappa
1	10	41.5869	29.1135	54.0603	5.5452	2.3222	8.4051
2	10	42.6725	33.2995	52.0455	9.1850	3.0227	13.2191

This report provides estimates and confidence intervals of the parameters (mean direction and concentration) of the von Mises distribution that best fits the data. Note that the von Mises distribution is a symmetric, unimodal distribution. You should check the rose plot or circular histogram to determine if the data are symmetric.

The formulas used in the estimation and confidence intervals were given earlier in this chapter. They come from Mardia & Jupp (2000).

Trigonometric Moments Section

Group	N	Mean Cos(a)	Mean Sin(a)	Mean Cos(2a)	Mean Sin(2a)	R bar	2R bar	Theta	2Theta
1	10	0.6974	0.6189	0.0903	0.7426	0.9324	0.7481	41.5869	83.0670
2	10	0.7057	0.6506	0.1085	0.8516	0.9599	0.8585	42.6725	82.7373

This report provides summary statistics that are used in other calculations.

Mean Cos(a)

$$\text{This is } \bar{C}_1 = \frac{1}{n} \sum_{i=1}^n \cos(a_i).$$

Mean Sin(a)

$$\text{This is } \bar{S}_1 = \frac{1}{n} \sum_{i=1}^n \sin(a_i).$$

Mean Cos(2a)

$$\text{This is } \bar{C}_2 = \frac{1}{n} \sum_{i=1}^n \cos(2a_i).$$

Mean Sin(2a)

$$\text{This is } \bar{S}_2 = \frac{1}{n} \sum_{i=1}^n \sin(2a_i).$$

R bar

$$\text{This is } \bar{R}_1 = \frac{1}{n} \sqrt{n(\bar{C}_1^2 + \bar{S}_1^2)}.$$

2R bar

$$\text{This is } \bar{R}_2 = \frac{1}{n} \sqrt{n(\bar{C}_2^2 + \bar{S}_2^2)}.$$

Circular Data Analysis

Theta, 2 Theta

This is calculated using the following formula with p set to 1 and then 2, respectively.

$$T_p = \begin{cases} \tan^{-1}\left(\frac{\bar{S}_p}{\bar{C}_p}\right) & \bar{C}_p > 0, \bar{S}_p > 0 \\ \tan^{-1}\left(\frac{\bar{S}_p}{\bar{C}_p}\right) + \pi & \bar{C}_p < 0 \\ \tan^{-1}\left(\frac{\bar{S}_p}{\bar{C}_p}\right) + 2\pi & \bar{S}_p < 0, \bar{C}_p > 0 \end{cases}$$

Multiple-Group Hypothesis Tests Section

Null Hypothesis (H0)	Test Name	Test Statistic	Prob Level	Reject H0 at 0.05 Level
Equal Distributions	Uniform Scores Test	6.7392	0.0344	Yes
Equal Directions	Watson-Williams F Test	0.0147	0.9047	No
Equal Concentrations	Concentration Homogeneity Test	0.5717	0.4496	No

Notes:

These statistics test various hypotheses about the parameters of von Mises distributions.

They require that each group follow the von Mises distribution.

The Uniform Scores test requires samples of at least 10.

The Watson-Williams F-test assumes that all kappa's are equal and that their average is > 1 .

This report provides tests for three hypotheses about the features of several von Mises datasets. That is, it provides a test of whether the distributions are identical, whether the mean directions are identical, and whether the concentrations are identical. These tests are documented in the Technical Details section of this chapter.

Two-Group Hypothesis Tests Section

First Group	Second Group	Equal Distributions		Equal Directions		Equal Concentrations	
		Test Statistic	Prob Level	Test Statistic	Prob Level	Test Statistic	Prob Level
1	2	6.7392	0.0344	0.0147	0.9047	0.5717	0.4496

Notes:

These statistics test various hypotheses about the parameters of von Mises distributions.

They require that each group follow the von Mises distribution.

Equal distributions tested by the Mardia-Watson-Wheeler uniform scores test. Requires all $N_i > 10$.

Equal directions tested by the Watson-Williams F test. Assumes Von Mises data with equal kappa's, all > 1 .

Equal concentrations tested by concentration homogeneity test. Assumes Von Mises data.

This report provides the same three tests as the Multiple-Group Hypothesis Tests Section, taken two groups at a time. It allows you to pinpoint where differences occur.

Circular Data Analysis

Tests for a Specified Mean Direction Assuming Von Mises Data – Test Statistic & Prob Levels

Tests for a Specified Mean Direction Assuming Von Mises Data - Test Statistics - Wind

Group	Sample Size (N)	Actual Mean Direction (Theta)	H0 Mean Direction (Theta0)	Score Test Z Value	Likelihood Ratio CS Value	Watson & Williams F Value	Stephens Test Z Value
1	10	41.5869	40.0000	0.0409	0.0470	0.0476	0.0397
2	10	42.6725	40.0000	0.1949	0.2266	0.2341	0.1917

Notes:

These procedures test whether the mean direction is equal to a specified value, when kappa (concentration) is unknown. They assume that the data follow the von Mises distribution.
 The Score Test requires a large sample size.
 The Likelihood Ratio Test requires a sample size of at least 5.
 The Watson & Williams Test requires a large value of kappa.
 The Stephens Test requires kappa to be greater than 2.

Tests for a Specified Mean Direction Assuming Von Mises Data - Probability Levels - Wind

Group	Sample Size (N)	Actual Mean Direction (Theta)	H0 Mean Direction (Theta0)	Score Test Prob Level	Likelihood Ratio Prob Level	Watson & Williams Prob Level	Stephens Test Prob Level
1	10	41.5869	40.0000	0.8398	0.8284	0.8321	0.8422
2	10	42.6725	40.0000	0.6589	0.6340	0.6400	0.6615

Notes:

This report gives the probability levels of the test statistics displayed in the previous report. Although the probability levels of four tests are given, you should use only one of these.

This section reports the results of four tests of the hypothesis that the mean direction of a particular group is equal to a specific value. These are two-sided tests. They were documented earlier in this chapter.

The first table gives the values of the test statistics. The second table gives the probability levels. The null hypothesis is rejected when the probability level is less than 0.05 (or some other appropriate cutoff).

Tests for a Specified Concentration Assuming Von Mises Data

Group	Sample Size (N)	Actual Concentration (Kappa)	H0 Concentration (Kappa0)	Chi-Square Value	Prob Level of (H1:Kappa < Kappa0)	Prob Level of (H1:Kappa > Kappa0)
1	10	5.5452	2.0000	2.2756	0.0137	0.9863
2	10	9.1850	2.0000	1.3518	0.0019	0.9981

Notes:

These statistics test whether the kappa (concentration) parameter is equal to the specified value. The tests require that the estimated kappa is > 2.

This section reports the results of two, one-sided tests of the hypothesis that the concentration parameter of each group is equal to a specific value. They were documented earlier in this chapter.

The first probability level is for testing the null hypothesis that kappa is greater than or equal to kappa0. The second probability level is for test the null hypothesis that kappa is less than or equal to kappa0.

Circular Data Analysis

Uniform Distribution Goodness-of-Fit Tests

Group	Sample Size (N)	Rayleigh's Test Statistic (S*)	Rayleigh's Test Prob Level	Kuiper's Test Statistic (V)	Kuiper's Test Prob Level	Watson's Test Statistic (U2)	Watson's Test Prob Level
1	10	20.2993	0.0000	2.7145	0.0000	0.5657	0.0001
2	10	21.7499	0.0000	2.8088	0.0000	0.6788	0.0000

Notes:

The tests in this report assess the goodness-of-fit of the uniform distribution.

The Rayleigh test requires samples of at least 20.

The Kuiper and Watson tests require samples of at least 8.

This section reports the results of three goodness-of-fit tests for the uniform distribution. They were documented earlier in this chapter.

These tests may be viewed as testing whether the data are distributed uniformly around the circle.

Von Mises Distribution Goodness-of-Fit Tests

Group	Sample Size (N)	Watson's Test Statistic (U2)	Watson's Test Prob Level	Cox's Test Statistic (S)	Cox's Test Prob Level
1	10	0.0340	0.5000	0.4030	0.8175
2	10	0.1282	0.0322	2.9309	0.2310

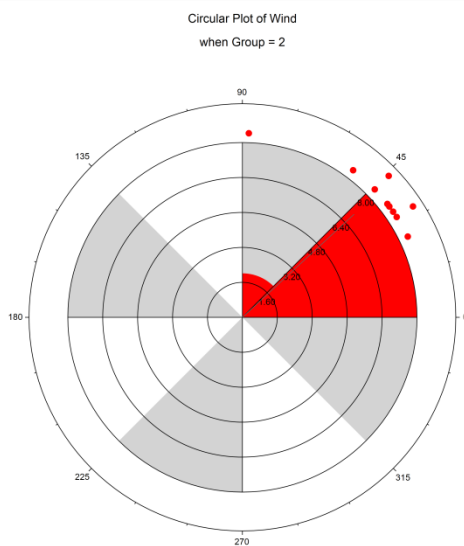
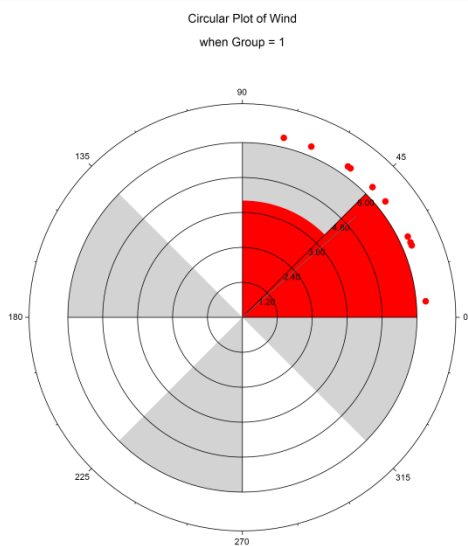
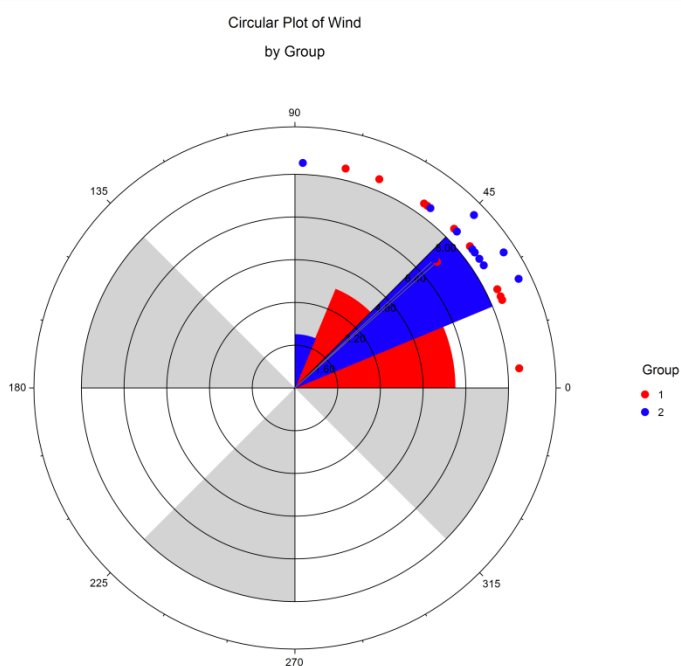
Notes:

The tests in this report assess the goodness-of-fit of the von Mises distribution.

Both tests require samples of at least 20.

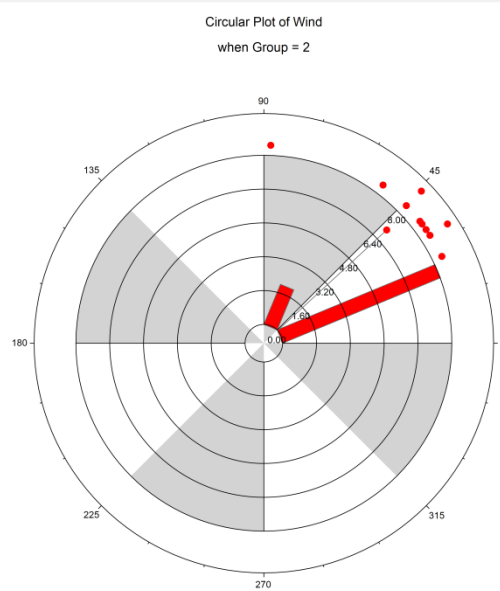
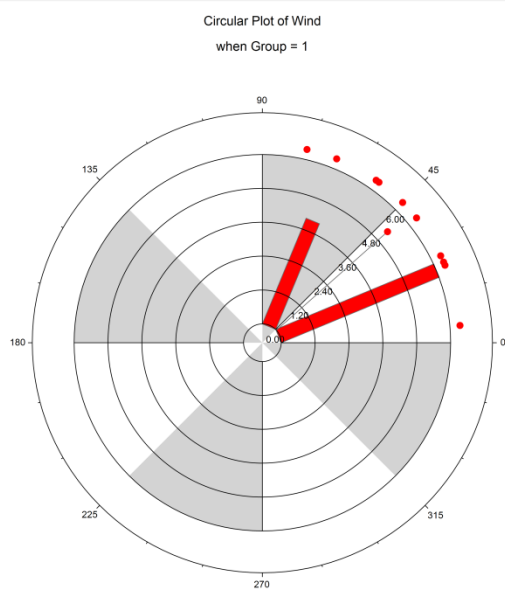
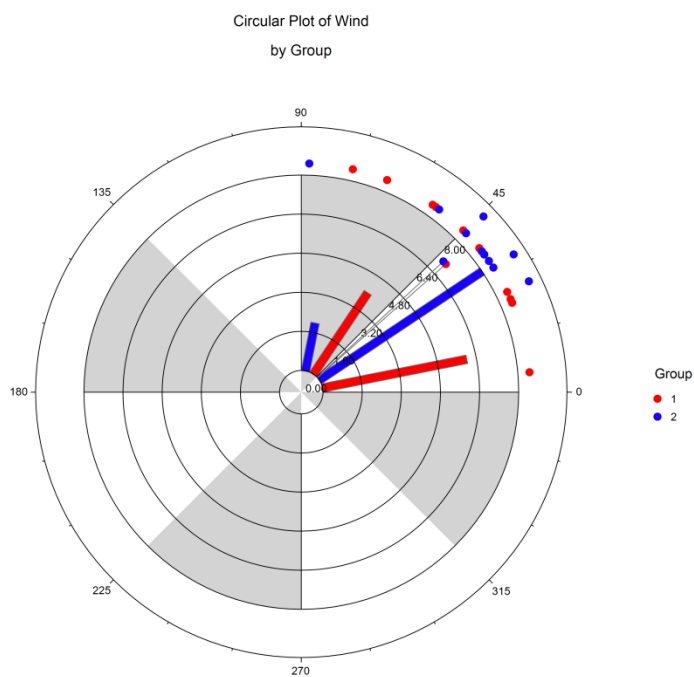
This section reports the results of two goodness-of-fit tests for the von Mises distribution. They were documented earlier in this chapter. Several hypothesis tests assume that the data follow a von Mises distribution. These tests allow you to check the accuracy of this assumption.

Rose Plots



These plots show the distribution of the data around the circle.

Circular Histograms



The circular histograms are generated by setting the Interior Objects on Plot to 'Circular Histogram'.