CS 506, Spring 2013, Exam 1 **Solution**

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Exam Distribution

Num Students	25
$90's$	10
$80's$	4
$70's$	6
$60's$	5
High:	100
Low:	60
Median:	85
Average:	81

1. Use both Venn diagram and algebraic method to prove each set equality.

(a)

$$A \cup (\overline{A} \cap B) = A \cup B$$

$A \cup (\overline{A} \cap B)$
$= (A \cup \overline{A}) \cap (A \cup B)$ Distributive
$= U \cap (A \cup B)$
$= A \cup B$

Venn Diagram is trivial and we won't bother for this solution.

(b)

$$A - (B \cup C) = (A - B) - C$$

$A - (B \cup C)$
$= A \cap \overline{(B \cup C)}$
$= A \cap (\overline{B} \cap \overline{C})$ DeMorgan
$= (A \cap \overline{B}) \cap \overline{C}$
$= (A - B) \cap \overline{C}$
$= (A - B) - C$

2. (a) Give an equivalent expression for the following proposition using only AND, OR, and NOT. Then use a truth table to show the two expressions are equivalent.

$$(A \rightarrow B)$$

$\neg A \vee B$

(b) Use algebraic manipulations to simplify the following expression. The simplified expression must use only AND, OR, and NOT operations.

$$(A \rightarrow B) \wedge (B \rightarrow A)$$

$(\neg A \vee B) \wedge (\neg B \vee A)$
$= (\neg A \wedge \neg B) \vee (\neg A \wedge A) \vee (B \wedge \neg B) \vee (B \wedge A)$
$= (\neg A \wedge \neg B) \vee (\text{FALSE}) \vee (\text{FALSE}) \vee (B \wedge A)$
$= (\neg A \wedge \neg B) \vee (A \wedge B)$

(c) Let the domain of x and y be all integers. Is the following proposition true or false? Give a reasoning.

$$\exists x \forall y, \text{ if } (x < y) \text{ then } (x^2 < y^2)$$

TRUE. Pick an $x > 0$. For every y, if $y > x > 0$, then obviously $y^2 > x^2$.

(d) Find negation of the above statement and simplify. Is the result true of false? Explain.

$\forall x \exists y, \neg(\text{ if } (x < y) \text{ then } (x^2 < y^2))$
$\forall x \exists y, (x < y) \wedge (x^2 \geq y^2)$

This is FALSE, because it is negation of the above which was proved true. It is also easy to give a direct reasoning why this is false. For any $x > 0$, there is no y such that $x < y$ and $x^2 \geq y^2$.

3. Use proof by contradiction or contrapositive proof, as appropriate, for each of the following.

   (a) Prove that if n^2 is not divisible by 9, then n is not divisible by 3. Assume domain of n is positive integers.

   Solution: We will prove the contrapositive form of the above statement:

   "If n is divisible by 3, then n^2 is divisible by 9."

   Suppose n is divisible by 3. Then $n = 3k$ for some integer k. Therefore, $n^2 = 9k^2$, which means n^2 is divisible by 9. □

   (b) Let the domain of x and y be positive real numbers. Suppose x is rational and y is irrational. Prove that $x + y$ is irrational.

   Solution: Proof by contradiction:
   Suppose to the contrary that $x + y$ is rational. Then $x + y = i/j$ for some integers i and j. And we know x is rational, so $x = m/n$ for some integers m and n. Therefore,

$$y = (x + y) - x = \frac{i}{j} - \frac{m}{n} = \frac{i * n - j * m}{j * n}.$$

   So y is rational, which is a contradiction. □

4. (a) Use simple induction (mathematical induction) to prove that any postage amount of n cents, $n \geq 28$, may be made using only 8-cent stamps and 5-cent stamps. That is, for every integer $n \geq 28$, there exist non-negative integers A and B such that,

$$n = 8A + 5B.$$

Solution: For the base, $n = 28$, we see $28 = 8 * 1 + 5 * 4$.
(Note that simple induction needs only one base case.)
Now suppose that for some $n \geq 28$, it is true that

$$n = 8A + 5B.$$

We will show this implies the truth for $n + 1$. That is we will prove how to get from the formulation for n to

$$n + 1 = 8C + 5D$$

for some non-negative integers C and D. So, suppose we are given the postage for $n = 8A + 5B$. We consider two cases:

- Case 1: $B \geq 3$. Then,
$$n + 1 = 8(A + 2) + 5(B - 3).$$
- Case 2: $B < 3$. That is, $B \leq 2$. Since $n \geq 28$, this implies that $A \geq 3$. (Otherwise, if $B \leq 2$ and $A \leq 2$, then n would be at most 26.) So, $A \geq 3$. Then,
$$n + 1 = 8(A - 3) + 5(B + 5).$$

$\square$

(b) Next, use strong induction to prove the same problem.

Solution: First, let us prove 8 base cases, $n = 28, 29, \cdots, 35$.

$28 = 8 * 1 + 5 * 4$
$29 = 8 * 3 + 5 * 1$
$30 = 8 * 0 + 5 * 6$
$31 = 8 * 2 + 5 * 3$
$32 = 8 * 4 + 5 * 0$
$33 = 8 * 1 + 5 * 5$
$34 = 8 * 3 + 5 * 2$
$35 = 8 * 0 + 5 * 7$

Now, to prove for any $n \geq 36$, suppose we have proven it for **all smaller postage values**, and in particular for $n-8$. (This is why we needed 8 bases cases.) That is, $n-8 \geq 36-8 = 28$. So by hypothesis, we know

$$n - 8 = 8A + 5B$$

for some non-negative integers A and B. Then, we add one 8-cent stamp to the postage for $n - 8$ to get n. So,

$$n = 8(A + 1) + 5B.$$

Note: As an alternative proof, we could have proved only 5 base cases (instead of 8) and then in the induction step, add one 5-cent stamp to $n - 5$ to get n. $\square$

5. Consider the relation $R = \{(1,1),(1,3),(2,2),(2,3),(3,1)\}$.

   (a) Does the relation satisfy each of following properties? Explain.

   - Reflexive?

     No, not reflexive. $(3,3) \notin R$.

   - Symmetric?

     Not symmetric. $(2,3) \in R$, but $(3,2) \notin R$.

   - Antisymmetric?

     Not antisymmetric. Both $(1,3) \in R$ and $(3,1) \in R$.

   - Transitive? (Decide this by directly examining the ordered pairs. Do NOT use matrix multiplication.)

     Not transitive. $(3,1) \in R$ and $(1,3) \in R$, but $(3,3) \notin R$.

   - Partial Order?

     No, because it is not reflexive. (Partial Order requires reflexive, antisymmetric, and transitive.)

   - Equivalence Relation?

     No, because it is not reflexive. (Equivalence relation requires reflexive, symmetric, and transitive.)

   (b) Show the matrix of this relation. Use matrix multiplication to decide if the relation is transitive. Explain.

   <u>Solution:</u>

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad A^2 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} X \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

   Not transitive because $A[3,3] = 0$ but $A^2[3,3] = 1$. □